Stacks Project

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CHAPTER 1

Introduction

1.1. Overview

Besides the book by Laumon and Moret-Bailly, see [**LMB00a**], and the work (in progress) by Fulton et al, we think there is a place for an open source textbook on algebraic stacks and the algebraic geometry that is needed to define them. The Stacks Project attempts to do this by building the foundations starting with commutative algebra and proceeding via the theory of schemes and algebraic spaces to a comprehensive foundation for the theory of algebraic stacks.

We expect this material to be read online as a key feature are the hyperlinks giving quick access to internal references spread over many different pages. If you use an embedded pdf or dvi viewer in your browser, the cross file links should work.

This project is a collaborative effort and we encourage you to help out. Please email any typos or errors you find while reading or any suggestions, additional material, or examples you have to stacks.project@gmail.com. You can download a tarball containing all source files, extract, run make, and use a dvi or pdf viewer locally. Please feel free to edit the LaTeX files and email your improvements.

1.2. Attribution

The scope of this work is such that it is a daunting task to attribute correctly and succinctly all of those mathematicians whose work has led to the development of the theory we try to explain here. We hope eventually to generate enough community interest to find contributors willing to write sections with historical remarks for each and every chapter.

Those who contributed to this work are listed on the title page of the book version of this work and online. Here we would like to name a selection of major contributions:

(1) Jarod Alper wrote Guide to Literature.
(2) Bhargav Bhatt wrote the initial version of Étale Morphisms of Schemes.
(3) Bhargav Bhatt wrote the initial version of More on Algebra, Section 12.8.
(4) Algebra, Section 7.25 and Injectives, Section 17.6 are from The CRing Project, courtesy of Akhil Mathew.
(5) Alex Perry wrote the material on projective modules, Mittag-Leffler modules, including the proof of Algebra, Theorem 7.89.5.
(6) Alex Perry wrote Formal Deformation Theory.
(7) Thibaut Pugin, Zachary Maddock and Min Lee took course notes which formed the basis for Étale Cohomology.
(8) David Rydh has contributed many helpful comments, pointed out several mistakes, helped out in an essential way with the material on residual gerbes, and was the originator for the material in More on Groupoids in Spaces, Sections 53.8 and 53.11.

1.3. Other chapters

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Commutative Algebra
(8) Brauer Groups
(9) Sites and Sheaves
(10) Homological Algebra
(11) Derived Categories
(12) More on Algebra
(13) Smoothing Ring Maps
(14) Simplicial Methods
(15) Sheaves of Modules
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(22) Constructions of Schemes
(23) Properties of Schemes
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(25) Coherent Cohomology
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CHAPTER 2

Conventions

2.1. Comments

The philosophy behind the conventions used in writing these documents is to choose those conventions that work.

2.2. Set theory

We use Zermelo-Fraenkel set theory with the axiom of choice. See [**Kun83**]. We do not use universes (different from SGA4). We do not stress set-theoretic issues, but we make sure everything is correct (of course) and so we do not ignore them either.

2.3. Categories

A category $\mathcal{C}$ consists of a set of objects and, for each pair of objects, a set of morphisms between them. In other words, it is what is called a ``small'' category in other texts. We will use ``big'' categories (categories whose objects form a proper class) as well, but only those that are listed in Categories, Remark 4.2.2.

2.4. Algebra

In these notes a ring is a commutative ring with a 1. Hence the category of rings has an initial object $\mathbf{Z}$ and a final object $\{0\}$ (this is the unique ring where $1 = 0$). Modules are assumed unitary. See [**Eis95**].

2.5. Notation

The natural integers are elements of $\mathbf{N} = \{1, 2, 3, \ldots\}$. The integers are elements of $\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. The field of rational numbers is denoted $\mathbf{Q}$. The field of real numbers is denoted $\mathbf{R}$. The field of complex numbers is denoted $\mathbf{C}$.

2.6. Other chapters

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(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
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(36) More on Groupoid Schemes
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CHAPTER 3

Set Theory

3.1. Introduction

We need some set theory every now and then. We use Zermelo-Fraenkel set theory with the axiom of choice (ZFC) as described in [**Kun83**] and [**Jec02**].

3.2. Everything is a set

Most mathematicians think of set theory as providing the basic foundations for mathematics. So how does this really work? For example, how do we translate the sentence ``X is a scheme'' into set theory? Well, we just unravel the definitions: A scheme is a locally ringed space such that every point has an open neighbourhood which is an affine scheme. A locally ringed space is a ringed space such that every stalk of the structure sheaf is a local ring. A ringed space is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ on it. A topological space is a pair (X, τ) consisting of a set X and a set of subsets $\tau \subset \mathcal{P}(X)$ satisfying the axioms of a topology. And so on and so forth.

So how, given a set S would we recognize whether it is a scheme? The first thing we look for is whether the set S is an ordered pair. This is defined (see [**Jec02**], page 7) as saying that S has the form $(a, b) := \{\{a\}, \{a, b\}\}$ for some sets a, b. If this is the case, then we would take a look to see whether a is an ordered pair (c, d). If so we would check whether $d \subset \mathcal{P}(c)$, and if so whether d forms the collection of sets for a topology on the set c. And so on and so forth.

So even though it would take a considerable amount of work to write a complete formula $\phi_{scheme}(x)$ with one free variable x in set theory that expresses the notion ``x is a scheme'', it is possible to do so. The same thing should be true for any mathematical object.

3.3. Classes

Informally we use the notion of a *class*. Given a formula $\phi(x, p_1, \ldots, p_n)$, we call

$$C = \{x : \phi(x, p_1, \ldots, p_n)\}$$

a *class*. A class is easier to manipulate than the formula that defines it, but it is not strictly speaking a mathematical object. For example, if R is a ring, then we may consider the class of all R-modules (since after all we may translate the sentence ``M is an R-module'' into a formula in set theory, which then defines a class). A *proper class* is a class which is not a set.

In this way we may consider the category of R-modules, which is a ``big'' category---in other words, it has a proper class of objects. Similarly, we may consider the ``big'' category of schemes, the ``big'' category of rings, etc.

3.4. Ordinals

A set T is *transitive* if $x \in T$ implies $x \subset T$. A set α is an *ordinal* if it is transitive and well-ordered by $\in$. In this case, we define $\alpha + 1 = \alpha \cup \{\alpha\}$, which is another ordinal called the *successor* of α. An ordinal α is called a *successor ordinal* if there exists an ordinal β such that $\alpha = \beta + 1$. The smallest ordinal is $\emptyset$ which is also denoted 0. If α is not 0, and not a successor ordinal, then α is called a *limit ordinal* and we have

$$\alpha = \bigcup\nolimits_{\gamma \in \alpha} \gamma.$$

The first limit ordinal is ω and it is also the first infinite ordinal. The collection of all ordinals is a proper class. It is well-ordered by $\in$ in the following sense: any nonempty set (or even class) of ordinals has a least element. Given a set A of ordinals, we define the *supremum* of A to be $\sup_{\alpha \in A} \alpha = \bigcup_{\alpha \in A} \alpha$. It is the least ordinal bigger or equal to all $\alpha \in A$. Given any well ordered set $(S, \geq)$, there is a unique ordinal α such that $(S, \geq) \cong (\alpha, \in)$; this is called the *order type* of the well ordered set.

3.5. The hierarchy of sets

We define, by transfinite induction, $V_0 = \emptyset$, $V_{\alpha+1} = P(V_\alpha)$ (power set), and for a limit ordinal α,

$$V_\alpha = \bigcup\nolimits_{\beta < \alpha} V_\beta.$$

Note that each V_α is a transitive set.

Lemma 3.5.1. *(See* [**Jec02**]*, Lemma 6.3.) Every set is an element of V_α for some ordinal α.*

In [**Kun83**, Chapter III] it is explained that this lemma is equivalent to the axiom of foundation. The *rank* of a set S is the least ordinal α such that $S \in V_\alpha$.

3.6. Cardinality

The *cardinality* of a set A is the least ordinal α such that there exists a bijection between A and α. We sometimes use the notation $\alpha = |A|$ to indicate this. We say an ordinal α is a *cardinal* if and only if it occurs as the cardinality of some set A---in other words, if $\alpha = |A|$. We use the greek letters κ, λ for cardinals. The first infinite cardinal is ω, and in this context it is denoted $\aleph_0$. A set is *countable* if its cardinality is $\leq \aleph_0$. If α is an ordinal, then we denote α^+ the least cardinal $> \alpha$. You can use this to define $\aleph_1 = \aleph_0^+$, $\aleph_2 = \aleph_1^+$, etc, and in fact you can define $\aleph_\alpha$ for any ordinal α by transfinite induction.

The *addition* of cardinals κ, λ is denoted $\kappa \oplus \lambda$; it is the cardinality of $\kappa \amalg \lambda$. The *multiplication* of cardinals κ, λ is denoted $\kappa \otimes \lambda$; it is the cardinality of $\kappa \times \lambda$. It is uninteresting since if κ and λ are infinite cardinals, then $\kappa \otimes \lambda = \max(\kappa, \lambda)$. The *exponentiation* of cardinals κ, λ is denoted κ^λ; it is the cardinality of the set of (set) maps from λ to κ. Given any set K of cardinals, the *supremum* of K is $\sup_{\kappa \in K} \kappa = \bigcup_{\kappa \in K} \kappa$, which is also a cardinal.

3.7. Cofinality

A *cofinal subset* S of a partially ordered set T is a subset $S \subset T$ such that $\forall t \in T \exists s \in S (t \leq s)$. Note that a subset of a well-ordered set is a well-ordered set (with induced ordering). Given an ordinal α, the *cofinality* $\mathrm{cf}(\alpha)$ of α is the least ordinal β which occurs as the order type of some cofinal subset of α. The cofinality of an ordinal is always a cardinal (this is clear from the definition). Hence alternatively we can define the cofinality of α as the least cardinality of a cofinal subset of α.

Lemma 3.7.1. *Suppose that $T_\beta = colim_{\alpha<\beta} T_\alpha$ is a colimit of sets indexed by ordinals less than a given ordinal β. Suppose that $\varphi : S \to T$ is a map of sets. Then φ lifts to a map into T_α for some $\alpha < \beta$ provided that β is not a limit of ordinals indexed by S, in other words, if β is an ordinal with $cf(\beta) > |S|$.*

Proof. For each element $s \in S$ pick a $\alpha_s < \beta$ and an element $t_s \in T_{\alpha_s}$ which maps to $\varphi(s)$ in T. By assumption $\alpha = \sup_{s\in S} \alpha_s$ is strictly smaller than β. Hence the map $\varphi_\alpha : S \to T_\alpha$ which assigns to s the image of t_s in T_α is a solution. □

The following is essentially Grothendieck's argument for the existence of ordinals with arbitrarily large cofinality which he used to prove the existence of enough injectives in certain abelian categories, see [**Gro57**].

Proposition 3.7.2. *Let κ be a cardinal. Then there exists an ordinal whose cofinality is bigger than κ.*

Proof. If κ is finite, then $\omega = \mathrm{cf}(\omega)$ works. Let us thus assume that κ is infinite. Consider the smallest ordinal α whose cardinality is strictly greater than κ. We claim that $\mathrm{cf}(\alpha) > \kappa$. Note that α is a limit ordinal, since if $\alpha = \beta + 1$, then $|\alpha| = |\beta|$ (because α and β are infinite) and this contradicts the minimality of α. (Of course α is also a cardinal, but we do not need this.) To get a contradiction suppose $S \subset \alpha$ is a cofinal subset with $|S| \leq \kappa$. For $\beta \in S$, i.e., $\beta < \alpha$, we have $|\beta| \leq \kappa$ by minimality of α. As α is a limit ordinal and S cofinal in α we obtain $\alpha = \bigcup_{\beta\in S} \beta$. Hence $|\alpha| \leq |S| \otimes \kappa \leq \kappa \otimes \kappa \leq \kappa$ which is a contradiction with our choice of α. □

3.8. Reflection principle

Some of this material is in the chapter of [**Kun83**] called ``Easy consistency proofs".

Let $\phi(x_1, \dots, x_n)$ be a formula of set theory. Let us use the convention that this notation implies that all the free variables in ϕ occur among $x_1, \dots, x_n$. Let M be a set. The formula $\phi^M(x_1, \dots, x_n)$ is the formula obtained from $\phi(x_1, \dots, x_n)$ by replacing all the $\forall x$ and $\exists x$ by $\forall x \in M$ and $\exists x \in M$, respectively. So the formula $\phi(x_1, x_2) = \exists x(x \in x_1 \wedge x \in x_2)$ is turned into $\phi^M(x_1, x_2) = \exists x \in M(x \in x_1 \wedge x \in x_2)$. The formula ϕ^M is called the *relativization of ϕ to M*.

Theorem 3.8.1. *See* [**Jec02**, Theorem 12.14] *or* [**Kun83**, Theorem 7.4]. *Suppose given $\phi_1(x_1, \dots, x_n), \dots, \phi_m(x_1, \dots, x_n)$ a* **finite** *collection of formulas of set theory. Let M_0 be a set. There exists a set M such that $M_0 \subset M$ and $\forall x_1, \dots, x_n \in M$, we have*

$$\forall i = 1, \dots, m,\ \phi_i^M(x_1, \dots, x_n) \Leftrightarrow \forall i = 1, \dots, m,\ \phi_i(x_1, \dots, x_n).$$

In fact we may take $M = V_\alpha$ for some limit ordinal α.

We view this theorem as saying the following: Given any $x_1, \dots, x_n \in M$ the formulas hold with the bound variables ranging through all sets if and only if they hold for the bound variables ranging through elements of V_α. This theorem is a meta-theorem because it deals with the formulas of set theory directly. It actually says that given the finite list of formulas $\phi_1, \dots, \phi_m$ with at most free variables $x_1, \dots, x_n$ the sentence

$$\forall M_0\ \exists M,\ M_0 \subset M\ \forall x_1, \dots, x_n \in M$$
$$\phi_1(x_1, \dots, x_n) \wedge \dots \wedge \phi_m(x_1, \dots, x_n) \leftrightarrow \phi_1^M(x_1, \dots, x_n) \wedge \dots \wedge \phi_m^M(x_1, \dots, x_n)$$

is provable in ZFC. In other words, whenever we actually write down a finite list of formulas ϕ_i, we get a theorem.

It is somewhat hard to use this theorem in ``ordinary mathematics" since the meaning of the formulas $\phi_i^M(x_1, \ldots, x_n)$ is not so clear! Instead, we will use the idea of the proof of the reflection principle to prove the existence results we need directly.

3.9. Constructing categories of schemes

We will discuss how to apply this to produce, given an initial set of schemes, a ``small" category of schemes closed under a list of natural operations. Before we do so, we introduce the size of a scheme. Given a scheme S we define

$$\text{size}(S) = \max(\aleph_0, \kappa_1, \kappa_2),$$

where we define the cardinal numbers κ_1 and κ_2 as follows:

(1) We let κ_1 be the cardinality of the set of affine opens of S.
(2) We let κ_2 be the supremum of all the cardinalities of all $\Gamma(U, \mathcal{O}_S)$ for all $U \subset S$ affine open.

Lemma 3.9.1. *For every cardinal κ, there exists a set A such that every element of A is a scheme and such that for every scheme S with $Size(S) \leq \kappa$, there is an element $X \in A$ such that $X \cong S$ (isomorphism of schemes).*

Proof. Omitted. Hint: think about how any scheme is isomorphic to a scheme obtained by glueing affines. □

We denote *Bound* the function which to each cardinal κ associates

$$Bound(\kappa) = \max\{\kappa^{\aleph_0}, \kappa^+\}. \tag{3.9.1.1}$$

We could make this function grow much more rapidly, e.g., we could set $Bound(\kappa) = \kappa^\kappa$, and the result below would still hold. For any ordinal α, we denote Sch_α the full subcategory of category of schemes whose objects are elements of V_α. Here is the result we are going to prove.

Lemma 3.9.2. *With notations size, Bound and Sch_α as above. Let S_0 be a set of schemes. There exists a limit ordinal α with the following properties:*

(1) *We have $S_0 \subset V_\alpha$; in other words, $S_0 \subset Ob(Sch_\alpha)$.*
(2) *For any $S \in Ob(Sch_\alpha)$ and any scheme T with $size(T) \leq Bound(size(S))$, there exists a scheme $S' \in Ob(Sch_\alpha)$ such that $T \cong S'$.*
(3) *For any countable diagram[1] category $\mathcal{J}$ and any functor $F : \mathcal{J} \to Sch_\alpha$, the limit $lim_\mathcal{J} F$ exists in Sch_α if and only if it exists in Sch and moreover, in this case, the natural morphism between them is an isomorphism.*
(4) *For any countable diagram category $\mathcal{J}$ and any functor $F : \mathcal{J} \to Sch_\alpha$, the colimit $colim_\mathcal{J} F$ exists in Sch_α if and only if it exists in Sch and moreover, in this case, the natural morphism between them is an isomorphism.*

Proof. We define, by transfinite induction, a function f which associates to every ordinal an ordinal as follows. Let $f(0) = 0$. Given $f(\alpha)$, we define $f(\alpha + 1)$ to be the least ordinal β such that the following hold:

(1) We have $\alpha + 1 \leq \beta$ and $f(\alpha) \leq \beta$.
(2) For any $S \in Ob(Sch_{f(\alpha)})$ and any scheme T with size$(T) \leq Bound(\text{size}(S))$, there exists a scheme $S' \in Ob(Sch_\beta)$ such that $T \cong S'$.

[1] Both the set of objects and the morphism sets are countable. In fact you can prove the lemma with $\aleph_0$ replaced by any cardinal whatsoever in (3) and (4).

(3) For any countable diagram category $\mathcal{I}$ and any functor $F : \mathcal{I} \to Sch_{f(\alpha)}$, if the limit $lim_{\mathcal{I}} F$ or the colimit $colim_{\mathcal{I}} F$ exists in Sch, then it is isomorphic to a scheme in Sch_β.

To see β exists, we argue as follows. Since $Ob(Sch_{f(\alpha)})$ is a set, we see that $\kappa = \sup_{S \in Ob(Sch_{f(\alpha)})} Bound(\text{size}(S))$ exists and is a cardinal. Let A be a set of schemes obtained starting with κ as in Lemma 3.9.1. There is a set $CountCat$ of countable categories such that any countable category is isomorphic to an element of $CountCat$. Hence in (3) above we may assume that $\mathcal{I}$ is an element in $CountCat$. This means that the pairs $(\mathcal{I}, F)$ in (3) range over a set. Thus, there exists a set B whose elements are schemes such that for every $(\mathcal{I}, F)$ as in (3), if the limit or colimit exists, then it is isomorphic to an element in B. Hence, if we pick any β such that $A \cup B \subset V_\beta$ and $\beta > \max\{\alpha + 1, f(\alpha)\}$, then (1)--(3) hold. Since every nonempty collection of ordinals has a least element, we see that $f(\alpha + 1)$ is well defined. Finally, if α is a limit ordinal, then we set $f(\alpha) = \sup_{\alpha' < \alpha} f(\alpha')$.

Pick β_0 such that $S_0 \subset V_{\beta_0}$. By construction $f(\beta) \geq \beta$ and we see that also $S_0 \subset V_{f(\beta_0)}$. Moreover, as f is nondecreasing, we see $S_0 \subset V_{f(\beta)}$ is true for any $\beta \geq \beta_0$. Next, choose any ordinal $\beta_1 > \beta_0$ with cofinality $\text{cf}(\beta_1) > \omega = \aleph_0$. This is possible since the cofinality of ordinals gets arbitrarily large, see Proposition 3.7.2. We claim that $\alpha = f(\beta_1)$ is a solution to the problem posed in the lemma.

The first property of the lemma holds by our choice of $\beta_1 > \beta_0$ above.

Since β_1 is a limit ordinal (as its cofinality is infinite), we get $f(\beta_1) = \sup_{\beta < \beta_1} f(\beta)$. Hence $\{f(\beta) \mid \beta < \beta_1\} \subset f(\beta_1)$ is a cofinal subset. Hence we see that

$$V_\alpha = V_{f(\beta_1)} = \bigcup\nolimits_{\beta < \beta_1} V_{f(\beta)}.$$

Now, let $S \in Ob(Sch_\alpha)$. We define $\beta(S)$ to be the least ordinal β such that $S \in Ob(Sch_{f(\beta)})$. By the above we see that always $\beta(S) < \beta_1$. Since $Ob(Sch_{f(\beta+1)}) \subset Ob(Sch_\alpha)$, we see by construction of f above that the second property of the lemma is satisfied.

Suppose that $\{S_1, S_2, \ldots\} \subset Ob(Sch_\alpha)$ is a countable collection. Consider the function $\omega \to \beta_1$, $n \mapsto \beta(S_n)$. Since the cofinality of β_1 is $> \omega$, the image of this function cannot be a cofinal subset. Hence there exists a $\beta < \beta_1$ such that $\{S_1, S_2, \ldots\} \subset Ob(Sch_{f(\beta)})$. It follows that any functor $F : \mathcal{I} \to Sch_\alpha$ factors through one of the subcategories $Sch_{f(\beta)}$. Thus, if there exists a scheme X that is the colimit or limit of the diagram F, then, by construction of f, we see X is isomorphic to an object of $Sch_{f(\beta+1)}$ which is a subcategory of Sch_α. This proves the last two assertions of the lemma. $\square$

Remark 3.9.3. The lemma above can also be proved using the reflection principle. However, one has to be careful. Namely, suppose the sentence $\phi_{scheme}(X)$ expresses the property ``X is a scheme'', then what does the formula $\phi_{scheme}^{V_\alpha}(X)$ mean? It is true that the reflection principle says we can find α such that for all $X \in V_\alpha$ we have $\phi_{scheme}(X) \leftrightarrow \phi_{scheme}^{V_\alpha}(X)$ but this is entirely useless. It is only by combining two such statements that something interesting happens. For example suppose $\phi_{red}(X, Y)$ expresses the property ``X, Y are schemes, and Y is the reduction of X'' (see Schemes, Definition 21.12.5). Suppose we apply the reflection principle to the pair of formulas $\phi_1(X, Y) = \phi_{red}(X, Y)$, $\phi_2(X) = \exists Y, \phi_1(X, Y)$. Then it is easy to see that any α produced by the reflection principle has the property that given $X \in Ob(Sch_\alpha)$ the reduction of X is also an object of Sch_α (left as an exercise).

Lemma 3.9.4. *Let S be an affine scheme. Let $R = \Gamma(S, \mathcal{O}_S)$. Then the size of S is equal to* $\max\{\aleph_0, |R|\}$.

Proof. There are at most $\max\{|R|, \aleph_0\}$ affine opens of $Spec(R)$. This is clear since any affine open $U \subset Spec(R)$ is a finite union of principal opens $D(f_1) \cup \ldots \cup D(f_n)$ and hence the number of affine opens is at most $\sup_n |R|^n = \max\{|R|, \aleph_0\}$, see [**Kun83**, Ch. I, 10.13]. On the other hand, we see that $\Gamma(U, \mathcal{O}) \subset R_{f_1} \times \ldots \times R_{f_n}$ and hence $|\Gamma(U, \mathcal{O})| \leq \max\{\aleph_0, |R_{f_1}|, \ldots, |R_{f_n}|\}$. Thus it suffices to prove that $|R_f| \leq \max\{\aleph_0, |R|\}$ which is omitted. $\square$

Lemma 3.9.5. *Let S be a scheme. Let $S = \bigcup_{i \in I} S_i$ be an open covering. Then $size(S) \leq \max\{|I|, \sup_i\{size(S_i)\}\}$.*

Proof. Let $U \subset S$ be any affine open. Since U is quasi-compact there exist finitely many elements $i_1, \ldots, i_n \in I$ and affine opens $U_i \subset U \cap S_i$ such that $U = U_1 \cup U_2 \cup \ldots \cup U_n$. Thus

$$|\Gamma(U, \mathcal{O}_U)| \leq |\Gamma(U_1, \mathcal{O})| \otimes \ldots \otimes |\Gamma(U_n, \mathcal{O})| \leq \sup_i\{\text{size}(S_i)\}$$

Moreover, it shows that the set of affine opens of S has cardinality less than or equal to the cardinality of the set

$$\coprod_{n \in \omega} \coprod_{i_1, \ldots, i_n \in I} \{\text{affine opens of } S_{i_1}\} \times \ldots \times \{\text{affine opens of } S_{i_n}\}.$$

Each of the sets inside the disjoint union has cardinality at most $\sup_i\{\text{size}(S_i)\}$. The index set has cardinality at most $\max\{|I|, \aleph_0\}$, see [**Kun83**, Ch. I, 10.13]. Hence by [**Jec02**, Lemma 5.8] the cardinality of the coproduct is at most $\max\{\aleph_0, |I|\} \otimes \sup_i\{\text{size}(S_i)\}$. The lemma follows. $\square$

Lemma 3.9.6. *Let $f : X \to S$, $g : Y \to S$ be morphisms of schemes. Then we have $size(X \times_S Y) \leq \max\{size(X), size(Y)\}$.*

Proof. Let $S = \bigcup_{k \in K} S_k$ be an affine open covering. Let $X = \bigcup_{i \in I} U_i$, $Y = \bigcup_{j \in J} V_j$ be affine open coverings with I, J of cardinality $\leq \text{size}(X), \text{size}(Y)$. For each $i \in I$ there exists a finite set K_i of $k \in K$ such that $f(U_i) \subset \bigcup_{k \in K_i} S_k$. For each $j \in J$ there exists a finite set K_j of $k \in K$ such that $g(V_j) \subset \bigcup_{k \in K_j} S_k$. Hence $f(X), g(Y)$ are contained in $S' = \bigcup_{k \in K'} S_k$ with $K' = \bigcup_{i \in I} K_i \cup \bigcup_{j \in J} K_j$. Note that the cardinality of K' is at most $\max\{\aleph_0, |I|, |J|\}$. Applying Lemma 3.9.5 we see that it suffices to prove that $\text{size}(f^{-1}(S_k) \times_{S_k} g^{-1}(S_k)) \leq \max\{\text{size}(X), \text{size}(Y)\}$ for $k \in K'$. In other words, we may assume that S is affine.

Assume S affine. Let $X = \bigcup_{i \in I} U_i$, $Y = \bigcup_{j \in J} V_j$ be affine open coverings with I, J of cardinality $\leq \text{size}(X), \text{size}(Y)$. Again by Lemma 3.9.5 it suffices to prove the lemma for the products $U_i \times_S V_j$. By Lemma 3.9.4 we see that it suffices to show that

$$|A \otimes_C B| \leq \max\{\aleph_0, |A|, |B|\}.$$

We omit the proof of this inequality. $\square$

Lemma 3.9.7. *Let S be a scheme. Let $f : X \to S$ be locally of finite type with X quasi-compact. Then $size(X) \leq size(S)$.*

Proof. We can find a finite affine open covering $X = \bigcup_{i=1,\ldots n} U_i$ such that each U_i maps into an affine open S_i of S. Thus by Lemma 3.9.5 we reduce to the case where both S and X are affine. In this case by Lemma 3.9.4 we see that it suffices to show

$$|A[x_1, \ldots, x_n]| \leq \max\{\aleph_0, |A|\}.$$

We omit the proof of this inequality. $\square$

In Algebra, Lemma 7.99.13 we will show that if $A \to B$ is an epimorphism of rings, then $|B| \leq |A|$. The analogue for schemes is the following lemma.

Lemma 3.9.8. *If $X \to Y$ is monomorphism of schemes, then $size(X) \leq size(Y)$.*

Proof. Let $Y = \bigcup_{j \in J} V_j$ be an affine open covering of Y with $|J| \leq \text{size}(Y)$. By Lemma 3.9.5 it suffices to bound the size of the inverse image of V_j in X. Hence we reduce to the case that Y is affine. As $X \to Y$ is a monomorphism the map $X \to Y$ is injective on underlying sets of points. For each $x \in X$ choose an affine open neighbourhood $U_x \subset X$. Then $U_x \to Y$ is a monomorphism too, and $X = \bigcup_{x \in X} U_x$ is an affine open covering whose index set has cardinality at most size(Y). Hence applying Lemma 3.9.5 again we see that we reduce to the case that both X and Y are affine. In this case the result follows from Lemma 3.9.4 and the lemma mentioned just above the statement of the lemma whose proof you are reading now. □

Lemma 3.9.9. *Let α be an ordinal as in Lemma 3.9.2 above. The category Sch_α satisfies the following properties:*

(1) *If $X, Y, S \in Ob(Sch_\alpha)$, then for any morphisms $f : X \to S$, $g : Y \to S$ the fibre product $X \times_S Y$ in Sch_α exists and is a fibre product in the category of schemes.*
(2) *Given any at most countable collection $S_1, S_2, \ldots$ of elements of $Ob(Sch_\alpha)$, the coproduct $\coprod_i S_i$ exists in $Ob(Sch_\alpha)$ and is a coproduct in the category of schemes.*
(3) *For any $S \in Ob(Sch_\alpha)$ and any open immersion $U \to S$, there exists a $V \in Ob(Sch_\alpha)$ with $V \cong U$.*
(4) *For any $S \in Ob(Sch_\alpha)$ and any closed immersion $T \to S$, there exists a $S' \in Ob(Sch_\alpha)$ with $S' \cong T$.*
(5) *For any $S \in Ob(Sch_\alpha)$ and any finite type morphism $T \to S$, there exists a $S' \in Ob(Sch_\alpha)$ with $S' \cong T$.*
(6) *Suppose S is a scheme which has an open covering $S = \bigcup_{i \in I} S_i$ such that there exists a $T \in Ob(Sch_\alpha)$ with (a) $size(S_i) \leq size(T)^{\aleph_0}$ for all $i \in I$, and (b) $|I| \leq size(T)^{\aleph_0}$. Then S is isomorphic to an object of Sch_α.*
(7) *For any $S \in Ob(Sch_\alpha)$ and any morphism $f : T \to S$ locally of finite type such that T can be covered by at most $size(S)^{\aleph_0}$ open affines, there exists a $S' \in Ob(Sch_\alpha)$ with $S' \cong T$. For example this holds if T can be covered by at most $|\mathbf{R}| = 2^{\aleph_0} = \aleph_0^{\aleph_0}$ open affines.*
(8) *For any $S \in Ob(Sch_\alpha)$ and any monomorphism $T \to S$, there exists a $S' \in Ob(Sch_\alpha)$ with $S' \cong T$.*
(9) *Suppose that $T \in Ob(Sch_\alpha)$ is affine. Write $R = \Gamma(T, \mathcal{O}_T)$. Then any of the following schemes is isomorphic to a scheme in Sch_α:*
    (a) *For any ideal $I \subset R$ with completion $R^* = lim_n R/I^n$, the scheme $Spec(R^*)$.*
    (b) *For any finite type R-algebra R', the scheme $Spec(R')$.*
    (c) *For any localization $S^{-1}R$, the scheme $Spec(S^{-1}R)$.*
    (d) *For any prime $\mathfrak{p} \subset R$, the scheme $Spec(\overline{\kappa(\mathfrak{p})})$.*
    (e) *For any subring $R' \subset R$, the scheme $Spec(R')$.*
    (f) *Any scheme of finite type over a ring of cardinality at most $|R|^{\aleph_0}$.*
    (g) *And so on.*

Proof. Statements (1) and (2) follow directly from the definitions. Statement (3) follows as the size of an open subscheme U of S is clearly smaller than or equal to the size of S. Statement (4) follows from (5). Statement (5) follows from (7). Statement (6) follows as the size of S is $\leq \max\{|I|, \sup_i \text{size}(S_i)\} \leq \text{size}(T)^{\aleph_0}$ by Lemma 3.9.5. Statement (7) follows from (6). Namely, for any affine open $V \subset T$ we have size$(V) \leq$ size(S) by Lemma 3.9.7. Thus, we see that (6) applies in the situation of (7). Part (8) follows from Lemma 3.9.8.

Statement (9) is translated, via Lemma 3.9.4, into an upper bound on the cardinality of the rings R^*, $S^{-1}R$, $\overline{\kappa(\mathfrak{p})}$, R', etc. Perhaps the most interesting one is the ring R^*. As a set, it is the image of a surjective map $R^{\mathbf{N}} \to R^*$. Since $|R^{\mathbf{N}}| = |R|^{\aleph_0}$, we see that it works by our choice of $Bound(\kappa) = \kappa^{\aleph_0}$. Phew! (The cardinality of the algebraic closure of a field is the same as the cardinality of the field, or it is $\aleph_0$.) $\square$

Remark 3.9.10. Let R be a ring. Suppose we consider the ring $\prod_{\mathfrak{p} \in Spec(R)} \kappa(\mathfrak{p})$. The cardinality of this ring is bounded by $|R|^{|R|}$, but is not bounded by $|R|^{\aleph_0}$ in general (for example if $R = \mathbf{C}[x]$). Thus the ``And so on" of Lemma 3.9.9 above should be taken with a grain of salt. Of course, if it ever becomes necessary to consider these rings in arguments pertaining to fppf/étale cohomology, then we can change the function *Bound* above into the function $\kappa \mapsto \kappa^\kappa$.

3.10. Sets with group action

Let G be a group. We denote *G-Sets* the ``big" category of G-sets. For any ordinal α, we denote *G-Sets*$_\alpha$ the full subcategory of *G-Sets* whose objects are in V_α. As a notion for size of a G-set we take size$(S) = \max\{\aleph_0, |G|, |S|\}$ (where $|G|$ and $|S|$ are the cardinality of the underlying sets). As above we use the function $Bound(\kappa) = \kappa^{\aleph_0}$.

Lemma 3.10.1. *With notations G, G-Sets$_\alpha$, size, and Bound as above. Let S_0 be a set of G-sets. There exists a limit ordinal α with the following properties:*

1. *We have $S_0 \cup \{{}_GG\} \subset Ob(G\text{-}Sets_\alpha)$.*
2. *For any $S \in Ob(G\text{-}Sets_\alpha)$ and any G-set T with size$(T) \leq Bound($size$(S))$, there exists a $S' \in Ob(G\text{-}Sets_\alpha)$ that is isomorphic to T.*
3. *For any countable diagram category $\mathcal{I}$ and any functor $F : \mathcal{I} \to G\text{-}Sets_\alpha$, the limit $lim_{\mathcal{I}} F$ and colimit $colim_{\mathcal{I}} F$ exist in G-Sets$_\alpha$ and are the same as in G-Sets.*

Proof. Omitted. Similar to but easier than the proof of Lemma 3.9.2 above. $\square$

Lemma 3.10.2. *Let α be an ordinal as in Lemma 3.10.1 above. The category G-Sets$_\alpha$ satisfies the following properties:*

1. *The G-set ${}_GG$ is an object of G-Sets$_\alpha$.*
2. *(Co)Products, fibre products, and pushouts exist in G-Sets$_\alpha$ and are the same as their counterparts in G-Sets.*
3. *Given an object U of G-Sets$_\alpha$, any G-stable subset $O \subset U$ is isomorphic to an object of G-Sets$_\alpha$.*

Proof. Omitted. $\square$

3.11. Coverings of a site

Suppose that $\mathcal{C}$ is a category (as in Categories, Definition 4.2.1) and that Cov$(\mathcal{C})$ is a proper class of coverings satisfying properties (1), (2), and (3) of Sites, Definition 9.6.2. We list them here:

1. If $V \to U$ is an isomorphism, then $\{V \to U\} \in$ Cov$(\mathcal{C})$.
2. If $\{U_i \to U\}_{i \in I} \in$ Cov$(\mathcal{C})$ and for each i we have $\{V_{ij} \to U_i\}_{j \in J_i} \in$ Cov$(\mathcal{C})$, then $\{V_{ij} \to U\}_{i \in I, j \in J_i} \in$ Cov$(\mathcal{C})$.

(3) If $\{U_i \to U\}_{i\in I} \in \text{Cov}(\mathcal{C})$ and $V \to U$ is a morphism of $\mathcal{C}$, then $U_i \times_U V$ exists for all i and $\{U_i \times_U V \to V\}_{i\in I} \in \text{Cov}(\mathcal{C})$.

For an ordinal α, we set $\text{Cov}(\mathcal{C})_\alpha = \text{Cov}(\mathcal{C}) \cap V_\alpha$. Given an ordinal α and a cardinal κ, we set $\text{Cov}(\mathcal{C})_{\kappa,\alpha}$ equal to the set of elements $\mathcal{U} = \{\varphi_i : U_i \to U\}_{i\in I} \in \text{Cov}(\mathcal{C})_\alpha$ such that $|I| \leq \kappa$.

We recall the following notion, see Sites, Definition 9.8.2. Two families of morphisms, $\{\varphi_i : U_i \to U\}_{i\in I}$ and $\{\psi_j : W_j \to U\}_{j\in J}$, with the same target of $\mathcal{C}$ are called *combinatorially equivalent* if there exist maps $\alpha : I \to J$ and $\beta : J \to I$ such that $\varphi_i = \psi_{\alpha(i)}$ and $\psi_j = \varphi_{\beta(j)}$. This defines an equivalence relation on families of morphisms having a fixed target.

Lemma 3.11.1. *With notations as above. Let $Cov_0 \subset Cov(\mathcal{C})$ be a set contained in $Cov(\mathcal{C})$. There exist a cardinal κ and a limit ordinal α with the following properties:*

(1) *We have $Cov_0 \subset Cov(\mathcal{C})_{\kappa,\alpha}$.*
(2) *The set of coverings $Cov(\mathcal{C})_{\kappa,\alpha}$ satisfies (1), (2), and (3) of Sites, Definition 9.6.2 (see above). In other words $(\mathcal{C}, Cov(\mathcal{C})_{\kappa,\alpha})$ is a site.*
(3) *Every covering in $Cov(\mathcal{C})$ is combinatorially equivalent to a covering in $Cov(\mathcal{C})_{\kappa,\alpha}$.*

Proof. To prove this, we first consider the set $\mathcal{S}$ of all sets of morphisms of $\mathcal{C}$ with fixed target. In other words, an element of $\mathcal{S}$ is a subset T of $\text{Arrows}(\mathcal{C})$ such that all elements of T have the same target. Given a family $\mathcal{U} = \{\varphi_i : U_i \to U\}_{i\in I}$ of morphisms with fixed target, we define $Supp(\mathcal{U}) = \{\varphi \in \text{Arrows}(\mathcal{C}) \mid \exists i \in I, \varphi = \varphi_i\}$. Note that two families $\mathcal{U} = \{\varphi_i : U_i \to U\}_{i\in I}$ and $\mathcal{V} = \{V_j \to V\}_{j\in J}$ are combinatorially equivalent if and only if $Supp(\mathcal{U}) = Supp(\mathcal{V})$. Next, we define $\mathcal{S}_\tau \subset \mathcal{S}$ to be the subset $\mathcal{S}_\tau = \{T \in \mathcal{S} \mid \exists\, \mathcal{U} \in \text{Cov}(\mathcal{C})\; T = Supp(\mathcal{U})\}$. For every element $T \in \mathcal{S}_\tau$, set $\beta(T)$ to equal the least ordinal β such that there exists a $\mathcal{U} \in \text{Cov}(\mathcal{C})_\beta$ such that $T = \text{Supp}(\mathcal{U})$. Finally, set $\beta_0 = \sup_{T\in S_\tau} \beta(T)$. At this point it follows that every $\mathcal{U} \in \text{Cov}(\mathcal{C})$ is combinatorially equivalent to some element of $\text{Cov}(\mathcal{C})_{\beta_0}$.

Let κ be the maximum of $\aleph_0$, the cardinality $|\text{Arrows}(\mathcal{C})|$,

$$\sup_{\{U_i\to U\}_{i\in I}\in\text{Cov}(\mathcal{C})_{\beta_0}} |I|, \quad \text{and} \quad \sup_{\{U_i\to U\}_{i\in I}\in\text{Cov}_0} |I|.$$

Since κ is an infinite cardinal, we have $\kappa \otimes \kappa = \kappa$. Note that obviously $\text{Cov}(\mathcal{C})_{\beta_0} = \text{Cov}(\mathcal{C})_{\kappa,\beta_0}$.

We define, by transfinite induction, a function f which associates to every ordinal an ordinal as follows. Let $f(0) = 0$. Given $f(\alpha)$, we define $f(\alpha + 1)$ to be the least ordinal β such that the following hold:

(1) We have $\alpha + 1 \leq \beta$ and $f(\alpha) \leq \beta$.
(2) If $\{U_i \to U\}_{i\in I} \in \text{Cov}(\mathcal{C})_{\kappa,f(\alpha)}$ and for each i we have $\{W_{ij} \to U_i\}_{j\in J_i} \in \text{Cov}(\mathcal{C})_{\kappa,f(\alpha)}$, then $\{W_{ij} \to U\}_{i\in I, j\in J_i} \in \text{Cov}(\mathcal{C})_{\kappa,\beta}$.
(3) If $\{U_i \to U\}_{i\in I} \in \text{Cov}(\mathcal{C})_{\kappa,\alpha}$ and $W \to U$ is a morphism of $\mathcal{C}$, then $\{U_i \times_U W \to W\}_{i\in I} \in \text{Cov}(\mathcal{C})_{\kappa,\beta}$.

To see β exists we note that clearly the collection of all coverings $\{W_{ij} \to U\}$ and $\{U_i \times_U W \to W\}$ that occur in (2) and (3) form a set. Hence there is some ordinal β such that V_β contains all of these coverings. Moreover, the index set of the covering $\{W_{ij} \to U\}$ has cardinality $\sum_{i\in I} |J_i| \leq \kappa \otimes \kappa = \kappa$, and hence these coverings are contained in $\text{Cov}(\mathcal{C})_{\kappa,\beta}$. Since every nonempty collection of ordinals has a least element we see that $f(\alpha+1)$ is well defined. Finally, if α is a limit ordinal, then we set $f(\alpha) = \sup_{\alpha'<\alpha} f(\alpha')$.

Pick an ordinal β_1 such that Arrows($\mathcal{C}$) $\subset V_{\beta_1}$, $\text{Cov}_0 \subset V_{\beta_0}$, and $\beta_1 \geq \beta_0$. By construction $f(\beta_1) \geq \beta_1$ and we see that the same properties hold for $V_{f(\beta_1)}$. Moreover, as f is nondecreasing this remains true for any $\beta \geq \beta_1$. Next, choose any ordinal $\beta_2 > \beta_1$ with cofinality $\text{cf}(\beta_2) > \kappa$. This is possible since the cofinality of ordinals gets arbitrarily large, see Proposition 3.7.2. We claim that the pair κ, $\alpha = f(\beta_2)$ is a solution to the problem posed in the lemma.

The first and third property of the lemma holds by our choices of κ, $\beta_2 > \beta_1 > \beta_0$ above.

Since β_2 is a limit ordinal (as its cofinality is infinite) we get $f(\beta_2) = \sup_{\beta<\beta_2} f(\beta)$. Hence $\{f(\beta) \mid \beta < \beta_2\} \subset f(\beta_2)$ is a confinal subset. Hence we see that

$$V_\alpha = V_{f(\beta_2)} = \bigcup\nolimits_{\beta<\beta_2} V_{f(\beta)}.$$

Now, let $\mathcal{U} \in \text{Cov}_{\kappa,\alpha}$. We define $\beta(\mathcal{U})$ to be the least ordinal β such that $\mathcal{U} \in \text{Cov}_{\kappa,f(\beta)}$. By the above we see that always $\beta(\mathcal{U}) < \beta_2$.

We have to show properties (1), (2), and (3) defining a site hold for the pair $(\mathcal{C}, \text{Cov}_{\kappa,\alpha})$. The first holds because by our choice of β_2 all arrows of $\mathcal{C}$ are contained in $V_{f(\beta_2)}$. For the third, we use that given a covering $\mathcal{U} = \{U_i \to U\}_{i\in I} \in \text{Cov}(\mathcal{C})_{\kappa,\alpha}$ we have $\beta(\mathcal{U}) < \beta_2$ and hence any base change of $\mathcal{U}$ is by construction of f contained in $\text{Cov}(\mathcal{C})_{\kappa,f(\beta+1)}$ and hence in $\text{Cov}(\mathcal{C})_{\kappa,\alpha}$.

Finally, for the second condition, suppose that $\{U_i \to U\}_{i\in I} \in \text{Cov}(\mathcal{C})_{\kappa,f(\alpha)}$ and for each i we have $\mathcal{W}_i = \{W_{ij} \to U_i\}_{j\in J_i} \in \text{Cov}(\mathcal{C})_{\kappa,f(\alpha)}$. Consider the function $I \to \beta_2$, $i \mapsto \beta(\mathcal{W}_i)$. Since the cofinality of β_2 is $> \kappa \geq |I|$ the image of this function cannot be a cofinal subset. Hence there exists a $\beta < \beta_1$ such that $\mathcal{W}_i \in \text{Cov}_{\kappa,f(\beta)}$ for all $i \in I$. It follows that the covering $\{W_{ij} \to U\}_{i\in I, j\in J_i}$ is an element of $\text{Cov}(\mathcal{C})_{\kappa,f(\beta+1)} \subset \text{Cov}(\mathcal{C})_{\kappa,\alpha}$ as desired. □

Remark 3.11.2. It is likely the case that, for some limit ordinal α, the set of coverings $\text{Cov}(\mathcal{C})_\alpha$ satisfies the conditions of the lemma. This is after all what an application of the reflection principle would appear to give (modulo caveats as described at the end of Section 3.8 and in Remark 3.9.3).

3.12. Abelian categories and injectives

The following lemma applies to the category of modules over a sheaf of rings on a site.

Lemma 3.12.1. *Suppose given a big category $\mathcal{A}$ (see Categories, Remark 4.2.2). Assume $\mathcal{A}$ is abelian and has enough injectives. See Homology, Definitions 10.3.12 and 10.20.4. Then for any given set of objects $\{A_s\}_{s\in S}$ of $\mathcal{A}$, there is an abelian subcategory $\mathcal{A}' \subset \mathcal{A}$ with the following properties:*

1. *$Ob(\mathcal{A}')$ is a set,*
2. *$Ob(\mathcal{A}')$ contains A_s for each $s \in S$,*
3. *$\mathcal{A}'$ has enough injectives, and*
4. *an object of $\mathcal{A}'$ is injective if and only if it is an injective object of $\mathcal{A}$.*

Proof. Omitted. □

3.13. Other chapters

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(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces

(7) Commutative Algebra
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CHAPTER 4

Categories

4.1. Introduction

Categories were first introduced in [**EL45**]. The category of categories (which is a proper class) is a 2-category. Similarly, the category of stacks forms a 2-category. If you already know about categories, but not about 2-categories you should read Section 4.25 as an introduction to the formal definitions later on.

4.2. Definitions

We recall the definitions, partly to fix notation.

Definition 4.2.1. A *category* $\mathcal{C}$ consists of the following data:

(1) A set of objects $Ob(\mathcal{C})$.
(2) For each pair $x, y \in Ob(\mathcal{C})$ a set of morphisms $Mor_{\mathcal{C}}(x, y)$.
(3) For each triple $x, y, z \in Ob(\mathcal{C})$ a composition map $Mor_{\mathcal{C}}(y, z) \times Mor_{\mathcal{C}}(x, y) \to Mor_{\mathcal{C}}(x, z)$, denoted $(\phi, \psi) \mapsto \phi \circ \psi$.

These data are to satisfy the following rules:

(1) For every element $x \in Ob(\mathcal{C})$ there exists a morphism $\text{id}_x \in Mor_{\mathcal{C}}(x, x)$ such that $\text{id}_x \circ \phi = \phi$ and $\psi \circ \text{id}_x = \psi$ whenever these compositions make sense.
(2) Composition is associative, i.e., $(\phi \circ \psi) \circ \chi = \phi \circ (\psi \circ \chi)$ whenever these compositions make sense.

It is customary to require all the morphism sets $Mor_{\mathcal{C}}(x, y)$ to be disjoint. In this way a morphism $\phi : x \to y$ has a unique *source* x and a unique *target* y. This is not strictly necessary, although care has to be taken in formulating condition (2) above if it is not the case. It is convenient and we will often assume this is the case. In this case we say that ϕ and ψ are *composable* if the source of ϕ is equal to the target of ψ, in which case $\phi \circ \psi$ is defined. An equivalent definition would be to define a category as a quintuple $(\text{Ob}, \text{Arrows}, s, t, \circ)$ consisting of a set of objects, a set of morphisms (arrows), source, target and composition subject to a long list of axioms. We will occasionally use this point of view.

Remark 4.2.2. Big categories. In some texts a category is allowed to have a proper class of objects. We will allow this as well in these notes but only in the following list of cases (to be updated as we go along). In particular, when we say: ``Let $\mathcal{C}$ be a category'' then it is understood that $Ob(\mathcal{C})$ is a set.

(1) The category *Sets* of sets.
(2) The category *Ab* of abelian groups.
(3) The category *Groups* of groups.
(4) Given a group G the category G-*Sets* of sets with a left G-action.
(5) Given a ring R the category Mod_R of R-modules.
(6) Given a field k the category of vector spaces over k.

(7) The category of rings.
(8) The category of schemes.
(9) The category *Top* of topological spaces.
(10) Given a topological space X the category $PSh(X)$ of presheaves of sets over X.
(11) Given a topological space X the category $Sh(X)$ of sheaves of sets over X.
(12) Given a topological space X the category $PAb(X)$ of presheaves of abelian groups over X.
(13) Given a topological space X the category $Ab(X)$ of sheaves of abelian groups over X.
(14) Given a small category $\mathcal{C}$ the category of functors from $\mathcal{C}$ to *Sets*.
(15) Given a category $\mathcal{C}$ the category of presheaves of sets over $\mathcal{C}$.
(16) Given a site $\mathcal{C}$ the category of sheaves of sets over $\mathcal{C}$.

One of the reason to enumerate these here is to try and avoid working with something like the ``collection" of ``big" categories which would be like working with the collection of all classes which I think definitively is a meta-mathematical object.

Remark 4.2.3. It follows directly from the definition any two identity morphisms of and object x of $\mathcal{A}$ are the same. Thus we may and will speak of *the* identity morphism id_x of x.

Definition 4.2.4. A morphism $\phi : x \to y$ is an *isomorphism* of the category $\mathcal{C}$ if there exists a morphism $\psi : y \to x$ such that $\phi \circ \psi = \text{id}_y$ and $\psi \circ \phi = \text{id}_x$.

An isomorphism ϕ is also sometimes called an *invertible* morphism, and the morphism ψ of the definition is called the *inverse* and denoted ϕ^{-1}. It is unique if it exists. Note that given an object x of a category $\mathcal{A}$ the set of invertible elements $\text{Aut}_{\mathcal{A}}(x)$ of $Mor_{\mathcal{A}}(x, x)$ forms a group under composition. This group is called the *automorphism* group of x in $\mathcal{A}$.

Definition 4.2.5. A *groupoid* is a category where every morphism is an isomorphism.

Example 4.2.6. A group G gives rise to a groupoid with a single object x and morphisms $Mor(x, x) = G$, with the composition rule given by the group law in G. Every groupoid with a single object is of this form.

Example 4.2.7. A set C gives rise to a groupoid $\mathcal{C}$ defined as follows: As objects we take $Ob(\mathcal{C}) := C$ and for morphisms we take $Mor(x, y)$ empty if $x \neq y$ and equal to $\{\text{id}_x\}$ if $x = y$.

Definition 4.2.8. A *functor* $F : \mathcal{A} \to \mathcal{B}$ between two categories $\mathcal{A}, \mathcal{B}$ is given by the following data:

(1) A map $F : Ob(\mathcal{A}) \to Ob(\mathcal{B})$.
(2) For every $x, y \in Ob(\mathcal{A})$ a map $F : Mor_{\mathcal{A}}(x, y) \to Mor_{\mathcal{B}}(F(x), F(y))$, denoted $\phi \mapsto F(\phi)$.

These data should be compatible with composition and identity morphisms in the following manner: $F(\phi \circ \psi) = F(\phi) \circ F(\psi)$ for a composable pair (ϕ, ψ) of morphisms of $\mathcal{A}$ and $F(\text{id}_x) = \text{id}_{F(x)}$.

Note that every category $\mathcal{A}$ has an *identity* functor $\text{id}_{\mathcal{A}}$. In addition, given a functor $G : \mathcal{B} \to \mathcal{C}$ and a functor $F : \mathcal{A} \to \mathcal{B}$ there is a *composition* functor $G \circ F : \mathcal{A} \to \mathcal{C}$ defined in an obvious manner.

Definition 4.2.9. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor.

(1) We say F is *faithful* if for any objects x, y of $Ob(\mathcal{A})$ the map
$$F : Mor_{\mathcal{A}}(x, y) \to Mor_{\mathcal{B}}(F(x), F(y))$$
is injective.
(2) If these maps are all bijective then F is called *fully faithful*.
(3) The functor F is called *essentially surjective* if for any object $y \in Ob(\mathcal{B})$ there exists an object $x \in Ob(\mathcal{A})$ such that $F(x)$ is isomorphic to y in $\mathcal{B}$.

Definition 4.2.10. A *subcategory* of a category $\mathcal{B}$ is a category $\mathcal{A}$ whose objects and arrows form subsets of the objects and arrows of $\mathcal{A}$ and such that source, target and composition in $\mathcal{A}$ agree with those of $\mathcal{B}$. We say $\mathcal{A}$ is a *full subcategory* of $\mathcal{B}$ if $Mor_{\mathcal{A}}(x, y) = Mor_{\mathcal{B}}(x, y)$ for all $x, y \in Ob(\mathcal{A})$. We say $\mathcal{A}$ is a *strictly full* subcategory of $\mathcal{B}$ if it is a full subcategory and given $x \in Ob(\mathcal{A})$ any object of $\mathcal{B}$ which is isomorphic to x is also in $\mathcal{A}$.

If $\mathcal{A} \subset \mathcal{B}$ is a subcategory then the identity map is a functor from $\mathcal{A}$ to $\mathcal{B}$. Furthermore a subcategory $\mathcal{A} \subset \mathcal{B}$ is full if and only if the inclusion functor is fully faithful. Note that given a category $\mathcal{B}$ the set of full subcategories of $\mathcal{B}$ is the same as the set of subsets of $Ob(\mathcal{B})$.

Remark 4.2.11. Suppose that $\mathcal{A}$ is a category. A functor F from $\mathcal{A}$ to *Sets* is a mathematical object (i.e., it is a set not a class or a formula of set theory, see Sets, Section 3.2) even though the category of sets is ``big''. Namely, the range of F on objects will be a set $F(Ob(\mathcal{A}))$ and then we may think of F as a functor between $\mathcal{A}$ and the full subcategory of the category of sets whose objects are elements of $F(Ob(\mathcal{A}))$.

Example 4.2.12. A homomorphism $p : G \to H$ of groups gives rise to a functor between the associated groupoids in Example 4.2.6. It is faithful (resp. fully faithful) if and only if p is injective (resp. an isomorphism).

Example 4.2.13. Given a category $\mathcal{C}$ and an object $X \in Ob(\mathcal{C})$ we define the *category of objects over* X, denoted $\mathcal{C}/X$ as follows. The objects of $\mathcal{C}/X$ are morphisms $Y \to X$ for some $Y \in Ob(\mathcal{C})$. Morphisms between objects $Y \to X$ and $Y' \to X$ are morphisms $Y \to Y'$ in $\mathcal{C}$ that make the obvious diagram commute. Note that there is a functor $p_X : \mathcal{C}/X \to \mathcal{C}$ which simply forgets the morphism. Moreover given a morphism $f : X' \to X$ in $\mathcal{C}$ there is an induced functor $F : \mathcal{C}/X' \to \mathcal{C}/X$ obtained by composition with f, and $p_X \circ F = p_{X'}$.

Example 4.2.14. Given a category $\mathcal{C}$ and an object $X \in Ob(\mathcal{C})$ we define the *category of objects under* X, denoted $X/\mathcal{C}$ as follows. The objects of $X/\mathcal{C}$ are morphisms $X \to Y$ for some $Y \in Ob(\mathcal{C})$. Morphisms between objects $X \to Y$ and $X \to Y'$ are morphisms $Y \to Y'$ in $\mathcal{C}$ that make the obvious diagram commute. Note that there is a functor $p_X : X/\mathcal{C} \to \mathcal{C}$ which simply forgets the morphism. Moreover given a morphism $f : X' \to X$ in $\mathcal{C}$ there is an induced functor $F : X/\mathcal{C} \to X'/\mathcal{C}$ obtained by composition with f, and $p_{X'} \circ F = p_X$.

Definition 4.2.15. Let $F, G : \mathcal{A} \to \mathcal{B}$ be functors. A *natural transformation*, or a *morphism of functors* $t : F \to G$, is a collection $\{t_x\}_{x \in Ob(\mathcal{A})}$ such that

(1) $t_x : F(x) \to G(x)$ is a morphism in the category $\mathcal{B}$, and
(2) for every morphism $\phi : x \to y$ of $\mathcal{A}$ the following diagram is commutative

$$\begin{tikzcd} F(x) \arrow[r, "t_x"] \arrow[d, "F(\phi)"'] & G(x) \arrow[d, "G(\phi)"] \\ F(y) \arrow[r, "t_y"] & G(y) \end{tikzcd}$$

Sometimes we use the diagram

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow t"] & \mathscr{B} \end{tikzcd}$$

to indicate that t is a morphism from F to G.

Note that every functor F comes with the *identity* transformation $\mathrm{id}_F : F \to F$. In addition, given a morphism of functors $t : F \to G$ and a morphism of functors $s : E \to F$ then the *composition* $t \circ s$ is defined by the rule

$$(t \circ s)_x = t_x \circ s_x : E(x) \to G(x)$$

for $x \in Ob(\mathscr{A})$. It is easy to verify that this is indeed a morphism of functors from E to G. In this way, given categories $\mathscr{A}$ and $\mathscr{B}$ we obtain a new category, namely the category of functors between $\mathscr{A}$ and $\mathscr{B}$.

Remark 4.2.16. This is one instance where the same thing does not hold if $\mathscr{A}$ is a ``big'' category. For example consider functors *Sets* $\to$ *Sets*. As we have currently defined it such a functor is a class and not a set. In other words, it is given by a formula in set theory (with some variables equal to specified sets)! It is not a good idea to try to consider all possible formulae of set theory as part of the definition of a mathematical object. The same problem presents itself when considering sheaves on the category of schemes for example. We will come back to this point later.

Definition 4.2.17. An *equivalence of categories* $F : \mathscr{A} \to \mathscr{B}$ is a functor such that there exists a functor $G : \mathscr{B} \to \mathscr{A}$ such that the compositions $F \circ G$ and $G \circ F$ are isomorphic to the identity functors $\mathrm{id}_{\mathscr{B}}$, respectively $\mathrm{id}_{\mathscr{A}}$. In this case we say that G is a *quasi-inverse* to F.

Lemma 4.2.18. *Let $F : \mathscr{A} \to \mathscr{B}$ be a fully faithful functor. Suppose for every $X \in Ob(\mathscr{B})$ given an object $j(X)$ of $\mathscr{A}$ and an isomorphism $i_X : X \to F(j(X))$. Then there is a unique functor $j : \mathscr{B} \to \mathscr{A}$ such that j extends the rule on objects, and the isomorphisms i_X define an isomorphism of functors $id_{\mathscr{B}} \to F \circ j$. Moreover, j and F are quasi-inverse equivalences of categories.*

Proof. This lemma proves itself. $\square$

Lemma 4.2.19. *A functor is an equivalence of categories if and only if it is both fully faithful and essentially surjective.*

Proof. Let $F : \mathscr{A} \to \mathscr{B}$ be essentially surjective and fully faithful. As by convention all categories are small and as F is essentially surjective we can, using the axiom of choice, choose for every $X \in Ob(\mathscr{B})$ an object $j(X)$ of $\mathscr{A}$ and an isomorphism $i_X : X \to F(j(X))$. Then we apply Lemma 4.2.18 using that F is fully faithful. $\square$

Definition 4.2.20. Let $\mathscr{A}$, $\mathscr{B}$ be categories. We define the *product category* to be the category $\mathscr{A} \times \mathscr{B}$ to be the category with objects $Ob(\mathscr{A} \times \mathscr{B}) = Ob(\mathscr{A}) \times Ob(\mathscr{B})$ and

$$Mor_{\mathscr{A} \times \mathscr{B}}((x, y), (x', y')) := Mor_{\mathscr{A}}(x, x') \times Mor_{\mathscr{B}}(y, y').$$

Composition is defined componentwise.

4.3. Opposite Categories and the Yoneda Lemma

Definition 4.3.1. Given a category $\mathcal{C}$ the *opposite category* $\mathcal{C}^{opp}$ is the category with the same objects as $\mathcal{C}$ but all morphisms reversed.

In other words $Mor_{\mathcal{C}^{opp}}(x, y) = Mor_{\mathcal{C}}(y, x)$. Composition in $\mathcal{C}^{opp}$ is the same as in $\mathcal{C}$ except backwards: if $\phi : y \to z$ and $\psi : x \to y$ in $\mathcal{C}^{opp}$ then $\phi \circ^{opp} \psi := \psi \circ \phi$.

Definition 4.3.2. Let $\mathcal{C}$, $\mathcal{S}$ be categories. A *contravariant* functor F from $\mathcal{C}$ to $\mathcal{S}$ is a functor $\mathcal{C}^{opp} \to \mathcal{S}$.

Concretely, a contravariant functor F is given by a map $F : Ob(\mathcal{C}) \to Ob(\mathcal{S})$ and for every morphism $\psi : x \to y$ in $\mathcal{C}$ a morphism $F(\psi) : F(y) \to F(x)$. These should satisfy the property that, given another morphism $\phi : y \to z$, we have $F(\phi \circ \psi) = F(\psi) \circ F(\phi)$ as morphisms $F(z) \to F(x)$. (Note the reverse of order.)

Definition 4.3.3. Let $\mathcal{C}$ be a category.

(1) A *presheaf of sets on* $\mathcal{C}$ or simply a *presheaf* is a contravariant functor F from $\mathcal{C}$ to *Sets*.
(2) The category of presheaves is denoted $PSh(\mathcal{C})$.

Of course the category of presheaves is a proper class.

Example 4.3.4. Functor of points. For any $U \in Ob(\mathcal{C})$ there is a contravariant functor

$$\begin{array}{rccc} h_U & : & \mathcal{C} & \longrightarrow & Sets \\ & & X & \longmapsto & Mor_{\mathcal{C}}(X, U) \end{array}$$

which takes an object X to the set $Mor_{\mathcal{C}}(X, U)$. In other words h_U is a presheaf. Given a morphism $f : X \to Y$ the corresponding map $h_U(f) : Mor_{\mathcal{C}}(Y, U) \to Mor_{\mathcal{C}}(X, U)$ takes ϕ to $\phi \circ f$. We will always denote this presheaf $h_U : \mathcal{C}^{opp} \to Sets$. It is called the *representable presheaf* associated to U If $\mathcal{C}$ is the category of schemes this functor is sometimes referred to as the *functor of points* of U.

Note that given a morphism $\phi : U \to V$ in $\mathcal{C}$ we get a corresponding natural transformation of functors $h(\phi) : h_U \to h_V$ defined simply by composing with the morphism $U \to V$. It is trivial to see that this turns composition of morphisms in $\mathcal{C}$ into composition of transformations of functors. In other words we get a functor

$$h : \mathcal{C} \longrightarrow \mathrm{Fun}(\mathcal{C}^{opp}, Sets) = PSh(\mathcal{C})$$

Note that the target is a ``big'' category, see Remark 4.2.2. On the other hand, h is an actual mathematical object (i.e. a set), compare Remark 4.2.11.

Lemma 4.3.5. *Yoneda lemma. Let $U, V \in Ob(\mathcal{C})$. Given any morphism of functors $s : h_U \to h_V$ there is a unique morphism $\phi : U \to V$ such that $h(\phi) = s$. In other words the functor h is fully faithful. More generally, given any contravariant functor F and any object U of $\mathcal{C}$ we have a natural bijection*

$$Mor_{PSh(\mathcal{C})}(h_U, F) \longrightarrow F(U), \quad s \longmapsto s_U(id_U).$$

Proof. Just take $\phi = s_U(\mathrm{id}_U) \in Mor_{\mathcal{C}}(U, V)$. □

Definition 4.3.6. A contravariant functor $F : \mathcal{C} \to Sets$ is said to be *representable* if it is isomorphic to the functor of points h_U for some object U of $\mathcal{C}$.

Choose an object U of $\mathcal{C}$ and an isomorphism $s : h_U \to F$. The Yoneda lemma guarantees that the pair (U, s) is unique up to unique isomorphism. The object U is called an object *representing* F.

4.4. Products of pairs

Definition 4.4.1. Let $x, y \in Ob(\mathscr{C})$, A *product* of x and y is an object $x \times y \in Ob(\mathscr{C})$ together with morphisms $p \in Mor_{\mathscr{C}}(x \times y, x)$ and $q \in Mor_{\mathscr{C}}(x \times y, y)$ such that the following universal property holds: for any $w \in Ob(\mathscr{C})$ and morphisms $\alpha \in Mor_{\mathscr{C}}(w, x)$ and $\beta \in Mor_{\mathscr{C}}(w, y)$ there is a unique $\gamma \in Mor_{\mathscr{C}}(w, x \times y)$ making the diagram

$$\begin{tikzcd}
w \arrow[dr, dashed, "\gamma"] \arrow[drr, "\beta"] \arrow[ddr, "\alpha"'] & & \\
& x \times y \arrow[r, "q"'] \arrow[d, "p"] & z \\
& x &
\end{tikzcd}$$

commute.

If a product exists it is unique up to unique isomorphism. This follows from the Yoneda lemma as the definition requires $x \times y$ to be an object of $\mathscr{C}$ such that

$$h_{x \times y}(w) = h_x(w) \times h_y(w)$$

functorially in w. In other words the product $x \times y$ is an object representing the functor $w \mapsto h_x(w) \times h_y(w)$.

Definition 4.4.2. We say the category $\mathscr{C}$ *has products of pairs of objects* if a product $x \times y$ exists for any $x, y \in Ob(\mathscr{C})$.

We use this terminology to distinguish this notion from the notion of ``having products'' or ``having finite products'' which usually means something else (in particular it always implies there exists a final object).

4.5. Coproducts of pairs

Definition 4.5.1. Let $x, y \in Ob(\mathscr{C})$, A *coproduct*, or *amalgamated sum* of x and y is an object $x \amalg y \in Ob(\mathscr{C})$ together with morphisms $i \in Mor_{\mathscr{C}}(x, x \amalg y)$ and $j \in Mor_{\mathscr{C}}(y, x \amalg y)$ such that the following universal property holds: for any $w \in Ob(\mathscr{C})$ and morphisms $\alpha \in Mor_{\mathscr{C}}(x, w)$ and $\beta \in Mor_{\mathscr{C}}(y, w)$ there is a unique $\gamma \in Mor_{\mathscr{C}}(x \amalg y, w)$ making the diagram

$$\begin{tikzcd}
& y \arrow[d, "j"] \arrow[ddr, "\beta"] & \\
x \arrow[r, "i"] \arrow[drr, "\alpha"'] & x \amalg y \arrow[dr, dashed, "\gamma"] & \\
& & w
\end{tikzcd}$$

commute.

If a coproduct exists it is unique up to unique isomorphism. This follows from the Yoneda lemma (applied to the opposite category) as the definition requires $x \amalg y$ to be an object of $\mathscr{C}$ such that

$$Mor_{\mathscr{C}}(x \amalg y, w) = Mor_{\mathscr{C}}(x, w) \times Mor_{\mathscr{C}}(y, w)$$

functorially in w.

Definition 4.5.2. We say the category $\mathscr{C}$ *has coproducts of pairs of objects* if a coproduct $x \amalg y$ exists for any $x, y \in Ob(\mathscr{C})$.

We use this terminology to distinguish this notion from the notion of ``having coproducts'' or ``having finite coproducts'' which usually means something else (in particular it always implies there exists an initial object in $\mathcal{C}$).

4.6. Fibre products

Definition 4.6.1. Let $x, y, z \in Ob(\mathcal{C})$, $f \in Mor_{\mathcal{C}}(x, y)$ and $g \in Mor_{\mathcal{C}}(z, y)$. A *fibre product* of f and g is an object $x \times_y z \in Ob(\mathcal{C})$ together with morphisms $p \in Mor_{\mathcal{C}}(x \times_y z, x)$ and $q \in Mor_{\mathcal{C}}(x \times_y z, z)$ making the diagram

$$\begin{tikzcd} x \times_y z \arrow[r, "q"] \arrow[d, "p"'] & z \arrow[d, "g"] \\ x \arrow[r, "f"] & y \end{tikzcd}$$

commute, and such that the following universal property holds: for any $w \in Ob(\mathcal{C})$ and morphisms $\alpha \in Mor_{\mathcal{C}}(w, x)$ and $\beta \in Mor_{\mathcal{C}}(w, z)$ with $f \circ \alpha = g \circ \beta$ there is a unique $\gamma \in Mor_{\mathcal{C}}(w, x \times_z y)$ making the diagram

$$\begin{tikzcd} w \arrow[dr, dashed, "\gamma"] \arrow[drr, "\beta"] \arrow[ddr, "\alpha"'] & & \\ & x \times_y z \arrow[r, "q"'] \arrow[d, "p"'] & z \arrow[d, "g"] \\ & x \arrow[r, "f"] & z \end{tikzcd}$$

commute.

If a fibre product exists it is unique up to unique isomorphism. This follows from the Yoneda lemma as the definition requires $x \times_y z$ to be an object of $\mathcal{C}$ such that

$$h_{x \times_y z}(w) = h_x(w) \times_{h_y(w)} h_z(w)$$

functorially in w. In other words the fibre product $x \times_y z$ is an object representing the functor $w \mapsto h_x(w) \times_{h_y(w)} h_z(w)$.

Definition 4.6.2. We say the category $\mathcal{C}$ *has fibre products* if the fibre product exists for any $f \in Mor_{\mathcal{C}}(x, z)$ and $g \in Mor_{\mathcal{C}}(y, z)$.

Definition 4.6.3. A morphism $f : x \to y$ of a category $\mathcal{C}$ is said to be *representable*, if and only if for every morphism $z \to y$ in $\mathcal{C}$ the fibre product $z \times_y x$ exists.

Lemma 4.6.4. *Let $\mathcal{C}$ be a category. Let $f : x \to y$, and $g : y \to z$ be representable. Then $g \circ f : x \to z$ is representable.*

Proof. Omitted. □

Lemma 4.6.5. *Let $\mathcal{C}$ be a category. Let $f : x \to y$ be representable. Let $y' \to y$ be a morphism of $\mathcal{C}$. Then the morphism $x' := x \times_y y' \to y'$ is representable also.*

Proof. Let $z \to y'$ be a morphism. The fibre product $x' \times_{y'} z$ is supposed to represent the functor

$$\begin{aligned} w \quad \mapsto \quad & h_{x'}(w) \times_{h_{y'}(w)} h_z(w) \\ = \quad & (h_x(w) \times_{h_y(w)} h_{y'}(w)) \times_{h_{y'}(w)} h_z(w) \\ = \quad & h_n(w) \times_{h_y(w)} h_z(w) \end{aligned}$$

which is representable by assumption. □

4.7. Examples of fibre products

In this section we list examples of fibre products and we describe them.

As a really trivial first example we observe that the category of sets has fibred products and hence every morphism is representable. Namely, if $f : X \to Y$ and $g : Z \to Y$ are maps of sets then we define $X \times_Y Z$ as the subset of $X \times Z$ consisting of pairs (x, z) such that $f(x) = g(z)$. The morphisms $p : X \times_Y Z \to X$ and $q : X \times_U Z \to Z$ are the projection maps $(x, z) \mapsto x$, and $(x, z) \mapsto z$. Finally, if $\alpha : W \to X$ and $\beta : W \to Z$ are morphisms such that $f \circ \alpha = g \circ \beta$ then the map $W \to X \times Y$, $w \mapsto (\alpha(w), \beta(w))$ obviously ends up in $X \times_Y Z$ as desired.

In many categories whose objects are sets endowed with certain types of algebraic structures the fibre product of the underlying sets also provides the fibre product in the category. For example, suppose that X, Y and Z above are groups and that f, g are homomorphisms of groups. Then the set-theoretic fibre product $X \times_Y Z$ inherits the structure of a group, simply by defining the product of two pairs by the formula $(x, z) \cdot (x', z') = (xx', zz')$. Here we list those categories for which a similar reasoning works.

(1) The category *Groups* of groups.
(2) The category *G-Sets* of sets endowed with a left G-action for some fixed group G.
(3) The category of rings.
(4) The category of R-modules given a ring R.

4.8. Fibre products and representability

In this section we work out fibre products in the category of contravariant functors from a category to the category of sets. This will later be superceded during the discussion of sites, presheaves, sheaves. Of some interest is the notion of a ``representable morphism'' between such functors.

Lemma 4.8.1. *Let $\mathcal{C}$ be a category. Let $F, G, H : \mathcal{C}^{opp} \to Sets$ be functors. Let $a : F \to G$ and $b : H \to G$ be transformations of functors. Then the fibre product $F \times_{a,G,b} H$ in the category $FUN(\mathcal{C}^{opp}, Sets)$ exists and is given by the formula*

$$(F \times_{a,G,b} H)(X) = F(X) \times_{a_X, G(X), b_X} H(X)$$

for any object X of $\mathcal{C}$.

Proof. Omitted. □

As a special case suppose we have a morphism $a : F \to G$, an object $U \in Ob(\mathcal{C})$ and an element $\xi \in G(U)$. According to the Yoneda Lemma 4.3.5 this gives a transformation $\xi : h_U \to G$. The fibre product in this case is described by the rule

$$(h_U \times_{\xi,G,a} F)(X) = \{(f, \xi') \mid f : X \to U,\ \xi' \in F(X),\ G(f)(\xi) = a_X(\xi')\}$$

If F, G are also representable, then this is the functor representing the fibre product, if it exists, see Section 4.6. The analogy with Definition 4.6.3 prompts us to define a notion of representable transformations.

Definition 4.8.2. Let $\mathcal{C}$ be a category. Let $F, G : \mathcal{C}^{opp} \to Sets$ be functors. We say a morphism $a : F \to G$ is *representable*, or that *F is relatively representable over G*, if for every $U \in Ob(\mathcal{C})$ and any $\xi \in G(U)$ the functor $h_U \times_G F$ is representable.

Lemma 4.8.3. *Let $\mathcal{C}$ be a category. Let $a : F \to G$ be a morphism of contravariant functors from $\mathcal{C}$ to Sets. If a is representable, and G is a representable functor, then F is representable.*

Proof. Omitted. □

Lemma 4.8.4. *Let $\mathcal{C}$ be a category. Let $F : \mathcal{C}^{opp} \to Sets$ be a functor. Assume $\mathcal{C}$ has products of pairs of objects and fibre products. The following are equivalent:*

(1) *The diagonal $F \to F \times F$ is representable.*
(2) *For every U in $\mathcal{C}$, and any $\xi \in F(U)$ the map $\xi : h_U \to F$ is representable.*

Proof. Suppose the diagonal is representable, and let U, ξ be given. Consider any $V \in Ob(\mathcal{C})$ and any $\xi' \in F(V)$. Note that $h_U \times h_V = h_{U\times V}$ is representable. Hence the fibre product of the maps $(\xi, \xi') : h_U \times h_V \to F \times F$ and $F \to F \times F$ is representable by assumption. This means there exists $W \in Ob(\mathcal{C})$, morphisms $W \to U$, $W \to V$ and $h_W \to F$ such that

$$\begin{tikzcd} h_W \arrow[r] \arrow[d] & F \arrow[d] \\ h_U \times h_V \arrow[r] & F \times F \end{tikzcd}$$

is cartesian. We leave it to the reader to see that this implies that $h_W = h_U \times_F h_V$ as desired.

Assume (2) holds. Consider any $V \in Ob(\mathcal{C})$ and any $(\xi, \xi') \in (F \times F)(V)$. We have to show that $h_V \times_{F\times F} F$ is representable. What we know is that $h_V \times_{\xi, F, \xi'} h_V$ is representable, say by W in $\mathcal{C}$ with corresponding morphisms $a, a' : W \to V$ (such that $\xi \circ a = \xi' \circ a'$). Consider $W' = W \times_{(a,a'), V\times V} V$. It is formal to show that W' represents $h_V \times_{F\times F} F$ because

$$h_{W'} = h_W \times_{h_V \times h_V} h_V = (h_V \times_{\xi, F, \xi'} h_V) \times_{h_V\times h_V} h_V = F \times_{F\times F} h_V.$$

□

4.9. Push outs

The dual notion to fibre products is that of push outs.

Definition 4.9.1. Let $x, y, z \in Ob(\mathcal{C})$, $f \in Mor_{\mathcal{C}}(y, x)$ and $g \in Mor_{\mathcal{C}}(y, z)$. A *push out* of f and g is an object $x \amalg_y z \in Ob(\mathcal{C})$ together with morphisms $p \in Mor_{\mathcal{C}}(x, x \amalg_y z)$ and $q \in Mor_{\mathcal{C}}(z, x \amalg_y z)$ making the diagram

$$\begin{tikzcd} y \arrow[r, "g"] \arrow[d, "f"'] & z \arrow[d, "q"] \\ x \arrow[r, "p"] & x \amalg_y z \end{tikzcd}$$

commute, and such that the following universal property holds: For any $w \in Ob(\mathcal{C})$ and morphisms $\alpha \in Mor_{\mathcal{C}}(x, w)$ and $\beta \in Mor_{\mathcal{C}}(z, w)$ with $\alpha \circ f = \beta \circ g$ there is a unique

$\gamma \in Mor_{\mathcal{C}}(x \amalg_z y, w)$ making the diagram

$$\begin{tikzcd}
y \arrow[r, "g"] \arrow[d, "f"'] & z \arrow[d, "q"] \arrow[ddr, "\beta"] & \\
x \arrow[r, "p"] \arrow[drr, "\alpha"'] & x \amalg_y z \arrow[dr, dashed, "\gamma"] & \\
& & w
\end{tikzcd}$$

commute.

It is possible and straightforward to prove the uniqueness of the triple $(x \amalg_y z, p, q)$ up to unique isomorphism (if it exists) by direct arguments. Another possibility is to think of the coproduct as the product in the opposite category, thereby getting this uniqueness for free from the discussion in Section 4.6.

4.10. Equalizers

Definition 4.10.1. Suppose that X, Y are objects of a category $\mathcal{C}$ and that $a, b : X \to Y$ are morphisms. We say a morphism $e : Z \to X$ is an *equalizer* for the pair (a, b) if $a \circ e = b \circ e$ and if (Z, e) satisfies the following universal property: For every morphism $t : W \to X$ in $\mathcal{C}$ such that $a \circ t = b \circ t$ there exists a unique morphism $s : W \to Z$ such that $t = e \circ s$.

As in the case of the fibre product above, equalizers when they exist are unique up to unique isomorphism. There is a straightforward generalization of this definition to the case where we have more than 2 morphisms.

4.11. Coequalizers

Definition 4.11.1. Suppose that X, Y are objects of a category $\mathcal{C}$ and that $a, b : X \to Y$ are morphisms. We say a morphism $c : Y \to Z$ is a *coequalizer* for the pair (a, b) if $c \circ a = c \circ b$ and if (Z, c) satisfies the following universal property: For every morphism $t : Y \to W$ in $\mathcal{C}$ such that $t \circ a = t \circ b$ there exists a unique morphism $s : Z \to W$ such that $t = s \circ c$.

As in the case of the push outs above, coequalizers when they exist are unique up to unique isomorphism, and this follows from the uniqueness of equalizers upon considering the opposite category. There is a straightforward generalization of this definition to the case where we have more than 2 morphisms.

4.12. Initial and final objects

Definition 4.12.1. Let $\mathcal{C}$ be a category.

(1) An object x of the category $\mathcal{C}$ is called an *initial* object if for every object y of $\mathcal{C}$ there is exactly one morphism $x \to y$.
(2) An object x of the category $\mathcal{C}$ is called a *final* object if for every object y of $\mathcal{C}$ there is exactly one morphism $y \to x$.

In the category of sets the empty set $\emptyset$ is an initial object, and in fact the only initial object. Also, any *singleton*, i.e., a set with one element, is a final object (so it is not unique).

4.13. Limits and colimits

Let $\mathcal{C}$ be a category. A *diagram* in $\mathcal{C}$ is simply a functor $M : \mathcal{I} \to \mathcal{C}$. We say that $\mathcal{I}$ is the *index category* or that M is an $\mathcal{I}$-diagram. We will use the notation M_i to denote the image of the object i of $\mathcal{I}$. Hence for $\phi : i \to i'$ a morphism in $\mathcal{I}$ we have $M(\phi) : M_i \to M_{i'}$.

Definition 4.13.1. A *limit* of the $\mathcal{I}$-diagram M in the category $\mathcal{C}$ is given by an object $lim_I\, M$ in $\mathcal{C}$ together with morphisms $p_i : lim_I\, M \to M_i$ such that

(1) for $\phi : i \to i'$ a morphism in $\mathcal{I}$ we have $p_{i'} = M(\phi) \circ p_i$, and
(2) for any object W in $\mathcal{C}$ and any family of morphisms $q_i : W \to M_i$ such that for all $\phi : i \to i'$ in $\mathcal{I}$ we have $q_{i'} = M(\phi) \circ q_i$ there exists a unique morphism $q : W \to lim_I\, M$ such that $q_i = p_i \circ q$ for every object i of $\mathcal{I}$.

Limits $(lim_I\, M, (p_i)_{i\in Ob(\mathcal{I})})$ are (if they exist) unique up to unique isomorphism by the uniqueness requirement in the definition. Products of pairs, fibred products, and equalizers are examples of limits. The limit over the empty diagram is a final object of $\mathcal{C}$. In the category of sets all limits exist. The dual notion is that of colimits.

Definition 4.13.2. A *colimit* of the $\mathcal{I}$-diagram M in the category $\mathcal{C}$ is given by an object $colim_I\, M$ in $\mathcal{C}$ together with morphisms $s_i : M_i \to colim_I\, M$ such that

(1) for $\phi : i \to i'$ a morphism in $\mathcal{I}$ we have $s_i = s_{i'} \circ M(\phi)$, and
(2) for any object W in $\mathcal{C}$ and any family of morphisms $t_i : M_i \to W$ such that for all $\phi : i \to i'$ in $\mathcal{I}$ we have $t_i = t_{i'} \circ M(\phi)$ there exists a unique morphism $t : colim_I\, M \to W$ such that $t_i = t \circ s_i$ for every object i of $\mathcal{I}$.

Colimits $(colim_I\, M, (s_i)_{i\in Ob(\mathcal{I})})$ are (if they exist) unique up to unique isomorphism by the uniqueness requirement in the definition. Coproducts of pairs, push outs, and coequalizers are examples of colimits. The colimit over an empty diagram is an initial object of $\mathcal{C}$. In the category of sets all colimits exist.

Remark 4.13.3. The index category of a (co)limit will never allowed to have a proper class of objects. In this project it means that it cannot be one of the categories listed in Remark 4.2.2

Remark 4.13.4. We often write $lim_i\, M_i$, $colim_i\, M_i$, $lim_{i\in\mathcal{I}}\, M_i$, or $colim_{i\in\mathcal{I}}\, M_i$ instead of the versions indexed by $\mathcal{I}$. Using this notation, and using the description of limits and colimits of sets in Section 4.14 below, we can say the following. Let $M : \mathcal{I} \to \mathcal{C}$ be a diagram.

(1) The object $lim_i\, M_i$ if it exists satisfies the following property
$$Mor_{\mathcal{C}}(W, lim_i\, M_i) = lim_i\, Mor_{\mathcal{C}}(W, M_i)$$
where the limit on the right takes place in the category of sets.
(2) The object $colim_i\, M_i$ if it exists satisfies the following property
$$Mor_{\mathcal{C}}(colim_i\, M_i, W) = lim_{i\in\mathcal{I}^{opp}}\, Mor_{\mathcal{C}}(M_i, W)$$
where on the right we have the limit over the opposite category with value in the category of sets.

By the Yoneda lemma (and its dual) this formula completely determines the limit, respectively the colimit.

As an application of the notions of limits and colimits we define products and coproducts.

Definition 4.13.5. Suppose that I is a set, and suppose given for every $i \in I$ an object M_i of the category $\mathcal{C}$. A *product* $\prod_{i \in I} M_i$ is by definition $lim_{\mathcal{I}} M$ (if it exists) where $\mathcal{I}$ is the category having only identities as morphisms and having the elements of I as objects.

An important special case is where $I = \emptyset$ in which case the product is a final object of the category. The morphisms $p_i : \prod M_i \to M_i$ are called the *projection morphisms*.

Definition 4.13.6. Suppose that I is a set, and suppose given for every $i \in I$ an object M_i of the category $\mathcal{C}$. A *coproduct* $\coprod_{i \in I} M_i$ is by definition $colim_{\mathcal{I}} M$ (if it exists) where $\mathcal{I}$ is the category having only identities as morphisms and having the elements of I as objects.

An important special case is where $I = \emptyset$ in which case the product is an initial object of the category. Note that the coproduct comes equipped with morphisms $M_i \to \coprod M_i$. These are sometimes called the *coprojections*.

Lemma 4.13.7. *Suppose that $M : \mathcal{I} \to \mathcal{C}$, and $N : \mathcal{J} \to \mathcal{C}$ are diagrams whose colimits exist. Suppose $H : \mathcal{I} \to \mathcal{J}$ is a functor, and suppose $t : M \to N \circ H$ is a transformation of functors. Then there is a unique morphism*

$$\theta : colim_{\mathcal{I}} M \longrightarrow colim_{\mathcal{J}} N$$

such that all the diagrams

$$\begin{tikzcd} M_i \arrow[r] \arrow[d, "t_{H(i)}"'] & colim_{\mathcal{I}} M \arrow[d, "\theta"] \\ N_{H(i)} \arrow[r] & colim_{\mathcal{J}} N \end{tikzcd}$$

commute.

Proof. Omitted. □

Lemma 4.13.8. *Suppose that $M : \mathcal{I} \to \mathcal{C}$, and $N : \mathcal{J} \to \mathcal{C}$ are diagrams whose limits exist. Suppose $H : \mathcal{I} \to \mathcal{J}$ is a functor, and suppose $t : N \circ H \to M$ is a transformation of functors. Then there is a unique morphism*

$$\theta : lim_{\mathcal{J}} N \longrightarrow lim_{\mathcal{I}} M$$

such that all the diagrams

$$\begin{tikzcd} lim_{\mathcal{J}} N \arrow[r] \arrow[d, "\theta"] & N_{H(i)} \arrow[d, "t_{H(i)}"'] \\ lim_{\mathcal{I}} M \arrow[r] & M_i \end{tikzcd}$$

commute.

Proof. Omitted. □

Lemma 4.13.9. *Let $\mathcal{I}$, $\mathcal{J}$ be index categories. Let $M : \mathcal{I} \times \mathcal{J} \to \mathcal{C}$ be a functor. We have*

$$colim_i \, colim_j \, M_{i,j} = colim_{i,j} \, M_{i,j} = colim_j \, colim_i \, M_{i,j}$$

provided all the indicated colimits exist. Similar for limits.

Proof. Omitted. □

Lemma 4.13.10. *Let $M : \mathcal{I} \to \mathcal{C}$ be a diagram. Write $I = Ob(\mathcal{I})$ and $A = Arrow(\mathcal{I})$. Denote $s, t : A \to I$ the source and target maps. Suppose that $\prod_{i \in I} M_i$ and $\prod_{a \in A} M_{t(a)}$ exist. Suppose that the equalizer of*

$$\prod\nolimits_{i \in I} M_i \underset{\psi}{\overset{\phi}{\rightrightarrows}} \prod\nolimits_{a \in A} M_{t(a)}$$

exists, where the morphisms are determined by their components as follows: $p_a \circ \psi = a \circ p_{s(a)}$ and $p_a \circ \phi = p_{t(a)}$. Then this equalizer is the limit of the diagram.

Proof. Omitted. ☐

Lemma 4.13.11. *Let $M : \mathcal{I} \to \mathcal{C}$ be a diagram. Write $I = Ob(\mathcal{I})$ and $A = Arrow(\mathcal{I})$. Denote $s, t : A \to I$ the source and target maps. Suppose that $\coprod_{i \in I} M_i$ and $\coprod_{a \in A} M_{s(a)}$ exist. Suppose that the coequalizer of*

$$\coprod\nolimits_{a \in A} M_{s(a)} \underset{\psi}{\overset{\phi}{\rightrightarrows}} \coprod\nolimits_{i \in I} M_i$$

exists, where the morphisms are determined by their components as follows: The component $M_{s(a)}$ maps via ψ to the component $M_{t(a)}$ via the morphism a. The component $M_{s(a)}$ maps via ϕ to the component $M_{s(a)}$ by the identity morphism. Then this coequalizer is the colimit of the diagram.

Proof. Omitted. ☐

4.14. Limits and colimits in the category of sets

Not only do limits and colimits exist in *Sets* but they are also easy to describe. Namely, let $M : \mathcal{I} \to Sets$, $i \mapsto M_i$ be a diagram of sets. Denote $I = Ob(\mathcal{I})$. The limit is described as

$$lim_{\mathcal{I}} M = \{(m_i)_{i \in I} \in \prod\nolimits_{i \in I} M_i \mid \forall \phi : i \to i' \text{ in } \mathcal{I}, M(\phi)(m_i) = m_{i'}\}.$$

So we think of an element of the limit as a compatible system of elements of all the sets M_i.

On the other hand, the colimit is

$$colim_{\mathcal{I}} M = (\coprod\nolimits_{i \in I} M_i)/\sim$$

where the equivalence relation $\sim$ is the equivalence relation generated by setting $m_i \sim m_{i'}$ if $m_i \in M_i$, $m_{i'} \in M_{i'}$ and $M(\phi)(m_i) = m_{i'}$ for some $\phi : i \to i'$. In other words, $m_i \in M_i$ and $m_{i'} \in M_{i'}$ are equivalent if there is a chain of morphisms in $\mathcal{I}$

$$\begin{tikzcd}
& i_1 \arrow[dl] \arrow[dr] & & i_3 \arrow[dl] & & i_{2n-1} \arrow[dr] & \\
i = i_0 & & i_2 & & \ldots & & i_{2n} = i'
\end{tikzcd}$$

and elements $m_{i_j} \in M_{i_j}$ mapping to each other under the maps $M_{i_{2k-1}} \to M_{i_{2k-2}}$ and $M_{i_{2k-1}} \to M_{i_{2k}}$ induced from the maps in $\mathcal{I}$ above.

This is not a very pleasant type of object to work with. But if the diagram is filtered then it is much easier to describe. We will explain this in Section 4.17.

4.15. Connected limits

A (co)limit is called connected if its index category is connected.

Definition 4.15.1. We say that a category $\mathcal{I}$ is *connected* if the equivalence relation generated by $x \sim y \Leftrightarrow Mor_{\mathcal{I}}(x, y) \neq \emptyset$ has at most one equivalence class.

The following in some vague sense characterizes connected nonempty limits.

Lemma 4.15.2. *Let $\mathcal{C}$ be a category. Let X be an object of $\mathcal{C}$. Let $M : \mathcal{I} \to \mathcal{C}/X$ be a diagram in the category of objects over X. If the index category $\mathcal{I}$ is nonempty and connected and the limit of M exists in $\mathcal{C}/X$, then the limit of the composition $\mathcal{I} \to \mathcal{C}/X \to \mathcal{C}$ exists and is the same.*

Proof. Let $M \to X$ be an object representing the limit in $\mathcal{C}/X$. Consider the functor

$$W \longmapsto lim_i\ Mor_{\mathcal{C}}(W, M_i).$$

Let (φ_i) be an element of the set on the right. Since each M_i comes equipped with a morphism $s_i : M_i \to X$ we get morphisms $f_i = s_i \circ \varphi_i : W \to X$. But as $\mathcal{I}$ is connected we see that all f_i are equal. Since $\mathcal{I}$ is nonempty there is at least one f_i. Hence this common value $W \to X$ defines the structure of an object of W in $\mathcal{C}/X$ and (φ_i) defines is an element of $lim_i\ Mor_{\mathcal{C}/X}(W, M_i)$. Thus we obtain a unique morphism $\phi : W \to M$ such that φ_i is the composition of ϕ with $M \to M_i$ as desired. □

Lemma 4.15.3. *Let $\mathcal{C}$ be a category. Let X be an object of $\mathcal{C}$. Let $M : \mathcal{I} \to X/\mathcal{C}$ be a diagram in the category of objects under X. If the index category $\mathcal{I}$ is nonempty and connected and the colimit of M exists in $X/\mathcal{C}$, then the colimit of the composition $\mathcal{I} \to X/\mathcal{C} \to \mathcal{C}$ exists and is the same.*

Proof. Omitted. Hint: This lemma is dual to Lemma 4.15.2. □

Lemma 4.15.4. *Let $\mathcal{I}$ be a category. Let $\mathcal{I}'$ be a subcategory of $\mathcal{I}$, see Definition 4.2.10. Assume that*

(1) *for every $x \in Ob(\mathcal{I})$ there exists a $x' \in Ob(\mathcal{I}')$ and a morphism $x' \to x$ in $\mathcal{I}$,*
(2) *for any $x \in Ob(\mathcal{I})$, $x', x'' \in Ob(\mathcal{I}')$ and morphisms $x' \to x$, $x'' \to x$ in $\mathcal{I}$ there exists $n \geq 0$ and a commutative diagram*

$$\begin{tikzcd}
x' \arrow[r] \arrow[drrr] & x_1 \arrow[drr] & x_2 \arrow[l] \arrow[r] \arrow[dr] & \ldots \arrow[r] & x_{2n-1} \arrow[dl] & x_{2n} \arrow[l] \arrow[r] \arrow[dll] & x'' \arrow[dlll] \\
 & & & x & & &
\end{tikzcd}$$

where all the horizontal morphisms are in $\mathcal{I}'$, and the vertical arrows in $\mathcal{I}$.

Then for every diagram $M : \mathcal{I} \to \mathcal{C}$ the limit $lim_{\mathcal{I}} M$ exists if and only if $lim_{\mathcal{I}'} M$ exists and if so these limits agree.

Proof. Omitted. □

Lemma 4.15.5. *Let $F : \mathcal{I} \to \mathcal{I}'$ be a functor. Assume*

(1) *the fibre categories (see Definition 4.29.2) of $\mathcal{I}$ over $\mathcal{I}'$ are all nonempty and connected, and*
(2) *for every morphism $\alpha' : x' \to y'$ in $\mathcal{I}'$ there exist a morphism $\alpha : x \to y$ in $\mathcal{I}$ such that $F(\alpha) = \alpha'$.*

Then for every diagram $M : \mathcal{I}' \to \mathcal{C}$ the colimit $colim_{\mathcal{I}} M \circ F$ exists if and only if $colim_{\mathcal{I}'} M$ exists and if so these colimits agree.

Proof. It suffices to show that for any object T of $\mathcal{C}$ we have

$$lim_{\mathcal{I}} Mor_{\mathcal{C}}(M_{F(i)}, T) = lim_{\mathcal{I}'} Mor_{\mathcal{C}}(M_{i'}, T)$$

If $(g_{i'})_{i' \in Ob(\mathcal{I}')}$ is an element of the right hand side, then setting $f_i = g_{F(i)}$ we obtain an element $(f_i)_{i \in Ob(\mathcal{I})}$ of the left hand side. Conversely, let $(f_i)_{i \in Ob(\mathcal{I})}$ be an element of the left hand side. Note that on each (nonempty connected) fibre category $\mathcal{I}_{i'}$ the functor $M \circ F$ is constant with value $M_{i'}$. Hence the morphisms f_i for $i \in Ob(\mathcal{I})$ with $F(i) = i'$ are all the same and determine a well defined morphism $g_{i'} : M_{i'} \to T$. By assumption (2) the collection $(g_{i'})_{i' \in Ob(\mathcal{I}')}$ defines an element of the right hand side. □

4.16. Finite limits and colimits

A *finite* (co)limit is a (co)limit whose diagram category is finite, i.e., the diagram category has finitely many objects and finitely many morphisms. A (co)limit is called *nonempty* if the index category is nonempty. A (co)limit is called *connected* if the index category is connected, see Definition 4.15.1. It turns out that there are ``enough'' finite diagram categories.

Lemma 4.16.1. *Let $\mathcal{I}$ be a category with*

1. *$Ob(\mathcal{I})$ is finite, and*
2. *there exist finitely many morphisms $f_1, \ldots, f_m \in Arrows(\mathcal{I})$ such that every morphism of $\mathcal{I}$ is a composition $f_{j_1} \circ f_{j_2} \circ \ldots \circ f_{j_k}$.*

Then there exists a functor $F : \mathcal{J} \to \mathcal{I}$ such that

- (a) *$\mathcal{J}$ is a finite category, and*
- (b) *for any diagram $M : \mathcal{I} \to \mathcal{C}$ the (co)limit of M over $\mathcal{I}$ exists if and only if the (co)limit of $M \circ F$ over $\mathcal{J}$ exists and in this case the (co)limits are canonically isomorphic.*

Moreover, $\mathcal{J}$ is connected (resp. nonempty) if and only if $\mathcal{I}$ is so.

Proof. Say $Ob(\mathcal{I}) = \{x_1, \ldots, x_n\}$. Denote $s, t : \{1, \ldots, m\} \to \{1, \ldots, n\}$ the functions such that $f_j : x_{s(j)} \to x_{t(j)}$. We set $Ob(\mathcal{J}) = \{y_1, \ldots, y_n, z_1, \ldots, z_n\}$ Besides the identity morphisms we introduce morphisms $g_j : y_{s(j)} \to z_{t(j)}$, $j = 1, \ldots, m$ and morphisms $h_i : y_i \to z_i$, $i = 1, \ldots, n$. Since all of the nonidentity morphisms in $\mathcal{J}$ go from an x to a y there are no compositions to define and no associativity to check. Set $F(y_i) = F(z_i) = x_i$. Set $F(g_j) = f_j$ and $F(h_i) = \text{id}_{x_i}$. It is clear that F is a functor. It is clear that $\mathcal{J}$ is finite. It is clear that $\mathcal{J}$ is connected, resp. nonempty if and only if $\mathcal{I}$ is so.

Let $M : \mathcal{I} \to \mathcal{C}$ be a diagram. Consider an object W of $\mathcal{C}$ and morphisms $q_i : W \to M(x_i)$ as in Definition 4.13.1. Then by taking $q_i : W \to M(F(y_i)) = M(F(z_i)) = M(x_i)$ we obtain a family of maps as in Definition 4.13.1 for the diagram $M \circ F$. Conversely, suppose we are given maps $qy_i : W \to M(F(y_i))$ and $qz_i : W \to M(F(z_i))$ as in Definition 4.13.1 for the diagram $M \circ F$. Since

$$M(F(h_i)) = \text{id} : M(F(y_i)) = M(x_i) \longrightarrow M(x_i) = M(F(z_i))$$

we conclude that $qy_i = qz_i$ for all i. Set q_i equal to this common value. The compatibility of $q_{s(j)} = qy_{s(j)}$ and $q_{t(j)} = qz_{t(j)}$ with the morphism $M(f_j)$ guarantees that the family q_i is compatible with all morphisms in $\mathcal{I}$ as by assumption every such morphism is a composition of the morphisms f_j. Thus we have found a canonical bijection

$$lim_{B\in Ob(\mathcal{J})} Mor_{\mathcal{C}}(W, M(F(B))) = lim_{A\in Ob(\mathcal{I})} Mor_{\mathcal{C}}(W, M(A))$$

which implies the statement on limits in the lemma. The statement on colimits is proved in the same way (proof omitted). □

Lemma 4.16.2. *Let $\mathcal{C}$ be a category. The following are equivalent:*

(1) *Nonempty connected finite limits exist in $\mathcal{C}$.*
(2) *Equalizers and fibre products exist in $\mathcal{C}$.*

Proof. Since equalizers and fibre products are finite nonempty connected limits we see that (2) implies (1). For the converse, let $\mathcal{I}$ be a finite nonempty connected diagram category. Let $F : \mathcal{J} \to \mathcal{I}$ be the functor of diagram categories constructed in the proof of Lemma 4.16.1. Then we see that we may replace $\mathcal{I}$ by $\mathcal{J}$. The result is that we may assume that $Ob(\mathcal{I}) = \{x_1, \ldots, x_n\} \amalg \{y_1, \ldots, y_m\}$ with $n, m \geq 1$ such that all nonidentity morphisms in $\mathcal{I}$ are morphisms $f : x_i \to y_j$ for some i and j.

Suppose that $n > 1$. Since $\mathcal{I}$ is connected there exist indices i_1, i_2 and j_0 and morphisms $a : x_{i_1} \to y_{j_0}$ and $b : x_{i_2} \to y_{j_0}$. Consider the category

$$\mathcal{I}' = \{x\} \amalg \{x_1, \ldots, \hat{x}_{i_1}, \ldots, \hat{x}_{i_2}, \ldots x_n\} \amalg \{y_1, \ldots, y_m\}$$

with

$$Mor_{\mathcal{I}'}(x, y_j) = Mor_{\mathcal{I}}(x_{i_1}, y_j) \amalg Mor_{\mathcal{I}}(x_{i_2}, y_j)$$

and all other morphism sets the same as in $\mathcal{I}$. For any functor $M : \mathcal{I} \to \mathcal{C}$ we can construct a functor $M' : \mathcal{I}' \to \mathcal{C}$ by setting

$$M'(x) = M(x_{i_1}) \times_{M(a), M(y_j), M(b)} M(x_{i_2})$$

and for a morphism $f' : x \to y_j$ corresponding to, say, $f : x_{i_1} \to y_j$ we set $M'(f) = M(f) \circ \mathrm{pr}_1$. Then the functor M has a limit if and only if the functor M' has a limit (proof omitted). Hence by induction we reduce to the case $n = 1$.

If $n = 1$, then the limit of any $M : \mathcal{I} \to \mathcal{C}$ is the successive equalizer of pairs of maps $x_1 \to y_j$ hence exists by assumption. □

Lemma 4.16.3. *Let $\mathcal{C}$ be a category. The following are equivalent:*

(1) *Nonempty finite limits exist in $\mathcal{C}$.*
(2) *Products of pairs and equalizers exist in $\mathcal{C}$.*
(3) *Products of pairs and fibre products exist in $\mathcal{C}$.*

Proof. Since products of pairs, fibre products, and equalizers are limits with nonempty index categories we see that (1) implies both (2) and (3). Assume (2). Then finite nonempty products and equalizers exist. Hence by Lemma 4.13.10 we see that finite nonempty limits exist, i.e., (1) holds. Assume (3). If $a, b : A \to B$ are morphisms of $\mathcal{C}$, then the equalizer of a, b is

$$(A \times_{a,B,b} A) \times_{(pr_1, pr_2), A\times A, \Delta} A.$$

Thus (3) implies (2), and the lemma is proved. □

Lemma 4.16.4. *Let $\mathcal{C}$ be a category. The following are equivalent:*

(1) *Finite limits exist in $\mathcal{C}$.*
(2) *Finite products and equalizers exist.*
(3) *The category has a final object and fibred products exist.*

Proof. Since products of pairs, fibre products, equalizers, and final objects are limits over finite index categories we see that (1) implies both (2) and (3). By Lemma 4.13.10 above we see that (2) implies (1). Assume (3). Note that the product $A \times A$ is the fibre product over the final object. If $a, b : A \to B$ are morphisms of $\mathcal{C}$, then the equalizer of a, b is

$$(A \times_{a,B,b} A) \times_{(pr_1,pr_2),A\times A,\Delta} A.$$

Thus (3) implies (2) and the lemma is proved. □

Lemma 4.16.5. *Let $\mathcal{C}$ be a category. The following are equivalent:*

(1) *Nonempty connected finite colimits exist in $\mathcal{C}$.*
(2) *Coequalizers and push outs exist in $\mathcal{C}$.*

Proof. Omitted. Hint: This is dual to Lemma 4.16.2. □

Lemma 4.16.6. *Let $\mathcal{C}$ be a category. The following are equivalent:*

(1) *Nonempty finite colimits exist in $\mathcal{C}$.*
(2) *Coproducts of pairs and coequalizers exist in $\mathcal{C}$.*
(3) *Coproducts of pairs and push outs exist in $\mathcal{C}$.*

Proof. Omitted. Hint: This is the dual of Lemma 4.16.3. □

Lemma 4.16.7. *Let $\mathcal{C}$ be a category. The following are equivalent:*

(1) *finite colimits exist in $\mathcal{C}$,*
(2) *finite coproducts and coequalizers exist in $\mathcal{C}$, and*
(3) *$\mathcal{C}$ has an initial object and pushouts exist.*

Proof. Omitted. Hint: This is dual to Lemma 4.16.4. □

4.17. Filtered colimits

Colimits are easier to compute or describe when they are over a filtered diagram. Here is the definition.

Definition 4.17.1. We say that a diagram $M : \mathcal{I} \to \mathcal{C}$ is *directed*, or *filtered* if the following conditions hold:

(1) the category $\mathcal{I}$ has at least one object,
(2) for every pair of objects x, y of $\mathcal{I}$ there exists an object z and morphisms $x \to z$, $y \to z$, and
(3) for every pair of objects x, y of $\mathcal{I}$ and every pair of morphisms $a, b : x \to y$ of $\mathcal{I}$ there exists a morphism $c : y \to z$ of $\mathcal{I}$ such that $M(c \circ a) = M(c \circ b)$ as morphisms in $\mathcal{C}$.

We say that an index category $\mathcal{I}$ is *directed*, or *filtered* if $\text{id} : \mathcal{I} \to \mathcal{I}$ is filtered (in other words you erase the M in part (3) above.)

We observe that any diagram with filtered index category is filtered, and this is how filtered colimits usually come about. In fact, if $M : \mathcal{I} \to \mathcal{C}$ is a filtered diagram, then we can factor M as $\mathcal{I} \to \mathcal{I}' \to \mathcal{C}$ where $\mathcal{I}'$ is a filtered index category[1] such that $colim_{\mathcal{I}} M$ exists if and only if $colim_{\mathcal{I}'} M'$ exists in which case the colimits are canonically isomorphic.

[1]Namely, let $\mathcal{I}'$ have the same objects as $\mathcal{I}$ but where $Mor_{\mathcal{I}'}(x, y)$ is the quotient of $Mor_{\mathcal{I}}(x, y)$ by the equivalence relation which identifies $a, b : x \to y$ if $M(a) = M(b)$.

Suppose that $M : \mathcal{I} \to \mathit{Sets}$ is a filtered diagram. In this case we may describe the equivalence relation in the formula

$$colim_{\mathcal{I}} M = (\coprod\nolimits_{i\in I} M_i)/\sim$$

simply as follows

$$m_i \sim m_{i'} \Leftrightarrow \exists i'', \phi : i \to i'', \phi' : i' \to i'', M(\phi)(m_i) = M(\phi')(m_{i'}).$$

In other words, two elements are equal in the colimit if and only if the ``eventually become equal''.

Lemma 4.17.2. *Let $\mathcal{I}$ and $\mathcal{J}$ be index categories. Assume that $\mathcal{I}$ is filtered and $\mathcal{J}$ is finite. Let $M : \mathcal{J}\times\mathcal{I} \to Sets$, $(i, j) \mapsto M_{i,j}$ be a diagram of diagrams of sets. In this case*

$$colim_i\, lim_j\, M_{i,j} = lim_j\, colim_i\, M_{i,j}.$$

In particular, colimits over $\mathcal{I}$ commute with finite products, fibre products, and equalizers of sets.

Proof. Omitted. □

Instead of giving the easy proof of the lemma we give a counter example to the case where $\mathcal{J}$ is infinite. Namely, let $\mathcal{I}$ consist of $\mathbf{N} = \{1, 2, 3, \ldots\}$ with a unique morphism $i \to i'$ whenever $i \leq i'$. Let $\mathcal{J}$ consist of the discrete category $\mathbf{N} = \{1, 2, 3, \ldots\}$ (only morphisms are identities). Let $M_{i,j} = \{1, 2, \ldots , i\}$ with obvious inclusion maps $M_{i,j} \to M_{i',j}$ when $i \leq i'$. In this case $colim_i\, M_{i,j} = \mathbf{N}$ and hence

$$lim_j\, colim_i\, M_{i,j} = \prod\nolimits_j \mathbf{N} = \mathbf{N}^{\mathbf{N}}$$

On the other hand $lim_j\, M_{i,j} = \prod_j M_{i,j}$ and hence

$$colim_i\, lim_j\, M_{i,j} = \bigcup\nolimits_i \{1, 2, \ldots , i\}^{\mathbf{N}}$$

which is smaller than the other limit.

Lemma 4.17.3. *Let $\mathcal{I}$ be an index category, i.e., a category. Assume*

1. *for every pair of morphisms $a : w \to x$ and $b : w \to y$ in $\mathcal{I}$ there exists an object z and morphisms $c : x \to z$ and $d : y \to z$ such that $c \circ a = d \circ b$, and*
2. *for every pair of morphisms $a, b : x \to y$ there exists a morphism $c : y \to z$ such that $c \circ a = c \circ b$.*

Then $\mathcal{I}$ is a (possibly empty) union of disjoint filtered index categories $\mathcal{I}_j$.

Proof. If $\mathcal{I}$ is the empty category, then the lemma is true. Otherwise, we define a relation on objects of $\mathcal{I}$ by saying that $x \sim y$ if there exists a z and morphisms $x \to z$ and $y \to z$. This is an equivalence relation by the first assumption of the lemma. Hence $Ob(\mathcal{I})$ is a disjoint union of equivalence classes. Let $\mathcal{I}_j$ be the full subcategories corresponding to these equivalence classes. The rest is clear from the definitions. □

Lemma 4.17.4. *Let $\mathcal{I}$ be an index category satisfying the hypotheses of Lemma 4.17.3 above. Then colimits over $\mathcal{I}$ commute with fibre products and equalizers in sets (and more generally with connected finite nonempty limits).*

Proof. By Lemma 4.17.3 we may write $\mathcal{I} = \coprod \mathcal{I}_j$ with each $\mathcal{I}_j$ filtered. By Lemma 4.17.2 we see that colimits of $\mathcal{I}_j$ commute with equalizers and fibred products. Thus it suffices to show that equalizers and fibre products commute with coproducts in the category of sets (including empty coproducts). In other words, given a set J and sets A_j, B_j, C_j and set maps $A_j \to B_j$, $C_j \to B_j$ for $j \in J$ we have to show that

$$(\coprod\nolimits_{j\in J} A_j) \times_{(\coprod_{j\in J} B_j)} (\coprod\nolimits_{j\in J} C_j) = \coprod\nolimits_{j\in J} A_j \times_{B_j} C_j$$

and given $a_j, a'_j : A_j \to B_j$ that

$$\text{Equalizer}(\coprod\nolimits_{j\in J} a_j, \coprod\nolimits_{j\in J} a'_j) = \coprod\nolimits_{j\in J} \text{Equalizer}(a_j, a'_j)$$

This is true even if $J = \emptyset$. Details omitted. □

Definition 4.17.5. Let $\mathcal{I}, \mathcal{J}$ be filtered index categories. Let $H : \mathcal{I} \to \mathcal{J}$ be a functor. We say *$\mathcal{I}$ is cofinal in $\mathcal{J}$* if

(1) for all $y \in Ob(\mathcal{J})$ there exists a $x \in Ob(\mathcal{I})$ and a morphism $y \to H(x)$, and
(2) for all $x_1, x_2 \in Ob(\mathcal{I})$ and any $\varphi : H(x_1) \to H(x_2)$ there exists $x_{12} \in Ob(\mathcal{I})$ and morphisms $x_1 \to x_{12}$, $x_2 \to x_{12}$ such that

$$\begin{tikzcd} & H(x_{12}) & \\ H(x_1) \arrow[ur] \arrow[rr, "\varphi"] & & H(x_2) \arrow[ul] \end{tikzcd}$$

commutes.

Lemma 4.17.6. *Let $\mathcal{I}, \mathcal{J}$ be filtered index categories. Let $H : \mathcal{I} \to \mathcal{J}$ be a functor. Assume $\mathcal{I}$ is cofinal in $\mathcal{J}$. Then for every diagram $M : \mathcal{J} \to \mathcal{C}$ we have a canonical isomorphism*

$$colim_{\mathcal{I}} M \circ H = colim_{\mathcal{J}} M$$

if either side exists.

Proof. Omitted. □

4.18. Cofiltered limits

Limits are easier to compute or describe when they are over a cofiltered diagram. Here is the definition.

Definition 4.18.1. We say that a diagram $M : \mathcal{I} \to \mathcal{C}$ is *codirected* or *cofiltered* if the following conditions hold:

(1) the category $\mathcal{I}$ has at least one object,
(2) for every pair of objects x, y of $\mathcal{I}$ there exists an object z and morphisms $z \to x$, $z \to y$, and
(3) for every pair of objects x, y of $\mathcal{I}$ and every pair of morphisms $a, b : x \to y$ of $\mathcal{I}$ there exists a morphism $c : w \to x$ of $\mathcal{I}$ such that $M(a \circ c) = M(b \circ c)$ as morphisms in $\mathcal{C}$.

We say that an index category $\mathcal{I}$ is *codirected*, or *cofiltered* if id $: \mathcal{I} \to \mathcal{I}$ is cofiltered (in other words you erase the M in part (3) above.)

We observe that any diagram with cofiltered index category is cofiltered, and this is how this situation usually occurs.

Here is an example of why cofiltered limits of sets are ``easier'' than general ones: If $M : \mathcal{I} \to Sets$ is a cofiltered diagram, and all the M_i are finite nonempty, then $lim_i M_i$ is nonempty. The same does not hold for a general limit of finite nonempty sets.

4.19. Limits and colimits over partially ordered sets

A special case of diagrams is given by systems over partially ordered sets.

Definition 4.19.1. Let $(I, \geq)$ be a partially ordered set. Let $\mathcal{C}$ be a category.

(1) A *system over I in $\mathcal{C}$*, sometimes called a *inductive system over I in $\mathcal{C}$* is given by objects M_i of $\mathcal{C}$ and for every $i \leq i'$ a morphism $f_{ii'} : M_i \to M_{i'}$ such that $f_{ii} = \text{id}$ and such that $f_{ii''} = f_{i'i''} \circ f_{ii'}$ whenever $i \leq i' \leq i''$.
(2) An *inverse system over I in $\mathcal{C}$*, sometimes called a *projective system over I in $\mathcal{C}$* is given by objects M_i of $\mathcal{C}$ and for every $i \geq i'$ a morphism $f_{ii'} : M_i \to M_{i'}$ such that $f_{ii} = \text{id}$ and such that $f_{ii''} = f_{i'i''} \circ f_{ii'}$ whenever $i \geq i' \geq i''$. (Note reversal of inequalities.)

We will say $(M_i, f_{ii'})$ is a (inverse) system over I to denote this. The maps $f_{ii'}$ are sometimes called the *transition maps*.

In other words a system over I is just a diagram $M : \mathcal{I} \to \mathcal{C}$ where $\mathcal{I}$ is the category with objects I and a unique arrow $i \to i'$ if and only $i \leq i'$. And an inverse system is a diagram $M : \mathcal{I}^{opp} \to \mathcal{C}$. From this point of view we could take (co)limits of any (inverse) system over I. However, it is customary to take *only colimits of systems over I* and *only limits of inverse systems over I*. More precisely: Given a system $(M_i, f_{ii'})$ over I the colimit of the system $(M_i, f_{ii'})$ is defined as

$$colim_{i \in I} M_i = colim_{\mathcal{I}} M,$$

i.e., as the colimit of the corresponding diagram. Given a inverse system $(M_i, f_{ii'})$ over I the limit of the inverse system $(M_i, f_{ii'})$ is defined as

$$lim_{i \in I} M_i = lim_{\mathcal{I}^{opp}} M,$$

i.e., as the limit of the corresponding diagram.

Definition 4.19.2. With notation as above. We say the system (resp. inverse system) $(M_i, f_{ii'})$ is a *directed system* (resp. *directed inverse system*) if the partially ordered set I is *directed*: I is nonempty and for all $i_1, i_2 \in I$ there exists $i \in I$ such that $i_1 \leq i$ and $i_2 \leq i$.

In this case the colimit is sometimes (unfortunately) called the ``direct limit''. We will not use this last terminology. It turns out that diagrams over a filtered category are no more general than directed systems in the following sense.

Lemma 4.19.3. *Let $\mathcal{I}$ be a filtered index category. There exists a directed partially ordered set $(I, \geq)$ and a system $(x_i, \varphi_{ii'})$ over I in $\mathcal{I}$ with the following properties:*

(1) *For every category $\mathcal{C}$ and every diagram $M : \mathcal{I} \to \mathcal{C}$ with values in $\mathcal{C}$, denote $(M(x_i), M(\varphi_{ii'}))$ the corresponding system over I. If $colim_{i \in I} M(x_i)$ exists then so does $colim_{\mathcal{I}} M$ and the transformation*
$$\theta : colim_{i \in I} M(x_i) \longrightarrow colim_{\mathcal{I}} M$$
of Lemma 4.13.7 is an isomorphism.
(2) *For every category $\mathcal{C}$ and every diagram $M : \mathcal{I}^{opp} \to \mathcal{C}$ in $\mathcal{C}$, denote $(M(x_i), M(\varphi_{ii'}))$ the corresponding inverse system over I. If $lim_{i \in I} M(x_i)$ exists then so does $lim_{\mathcal{I}} M$ and the transformation*
$$\theta : lim_{\mathcal{I}^{opp}} M \longrightarrow lim_{i \in I} M(x_i)$$
of Lemma 4.13.8 is an isomorphism.

Proof. Consider quadruples $(S, A, x, \{f_s\}_{s\in S})$ with the following properties

1. S is a finite set of objects of $\mathcal{I}$,
2. A is a finite set of arrows of $\mathcal{I}$ such that each $a \in A$ is an arrow $a : s(a) \to t(a)$ with $s(a), t(a) \in S$,
3. x is an object of $\mathcal{I}$, and
4. $f_s : s \to x$ is a morphism of $\mathcal{I}$ such that for all $a \in A$ we have $f_{t(a)} \circ a = f_{s(a)}$.

Given such a quadruple $i = (S, A, x, \{f_s\}_{s\in S})$ we denote $S_i = S$, $A_i = A$, $x_i = x$, and $f_{s,i} = f_s$ for $s \in S_i$. We also set $\tilde{S}_i = S_i \cup \{x_i\}$ and $\tilde{A}_i = A_i \cup \{f_{s,i}, s \in S_i\}$. Let I be the set of all such quadruples. We define a relation on I by the rule

$$i \leq i' \Leftrightarrow \tilde{S}_i \subset S_{i'} \text{ and } \tilde{A}_i \subset A_{i'}$$

It is obviously a partial ordering on I. Note that if $i \leq i'$, then there is a given morphism $\varphi_{ii'} : x_i \to x_{i'}$ namely $f_{x_i,i'}$ because $x_i \in S_{i'}$. Hence we have a system over I in $\mathcal{I}$ by taking $(x_i, \varphi_{ii'})$. We claim that this system satisfies all the conditions of the lemma.

First we show that I is a directed partially ordered set. Note that I is nonempty since $(\{x\}, \emptyset, x, \{\text{id}_x\})$ is a quadruple where x is any object of $\mathcal{I}$, and $\mathcal{I}$ is not empty according to Definition 4.17.1. Suppose that $i, i' \in I$. Consider the set of objects $S = S_i \cup S_{i'} \cup \{x_i, x_{i'}\}$ of $\mathcal{I}$. This is a finite set. According to Definition 4.17.1 and a simple induction argument there exists an object x' of $\mathcal{I}$ such that for each $s \in S$ there is a morphism $f'_s : s \to x'$. Consider the set of arrows $A = A_i \cup A_{i'} \cup \{f_{s,i}, s \in S_i\} \cup \{f_{s,i'}, s \in S_{i'}\}$. This is a finite set of arrows whose source and target are elements of S. According to Definition 4.17.1 and a simple induction argument there exists a morphism $f : x' \to x$ such that for all $a \in A$ we have

$$f \circ f'_{t(a)} \circ a = f \circ f'_{s(a)}$$

as morphisms into x. Hence we see that $(S, A, x, \{f \circ f'_s\}_{s\in S})$ is a quadruple which is $\geq i$ and $\geq i'$ in the partial ordering defined above. This proves I is directed.

Next, we prove the statement about colimits. Let $\mathcal{C}$ be a category. Let $M : \mathcal{I} \to \mathcal{C}$ be a functor. Denote $(M(x_i), M(\varphi_{ii'}))$ the corresponding system over I. Below we will write $M_i = M(x_i)$ for clarity. Assume $K = colim_{i\in I} M(x_i)$ exists. We will verify that K is also the colimit of the diagram M. Recall that for every object x of $\mathcal{I}$ the quadruple $i_x = (\{x\}, \emptyset, x, \{\text{id}_x\})$ is an element of I. By definition of a colimit there is a morphism

$$M(x) = M_{i_x} \longrightarrow K$$

Let $\varphi : x \to x'$ be a morphism of $\mathcal{I}$. The quadruples i_x, $i_{x'}$ and

$$i_\varphi = (\{x, x'\}, \{\text{id}_x, \text{id}_{x'}, \varphi\}, x', \{\varphi, \text{id}_{x'}\})$$

are elements of I. Moreover, $i_x \leq i_\varphi$ and $i_{x'} \leq i_\varphi$. Thus the diagram

$$\begin{tikzcd}
M(x) = M_{i_x} \arrow[r] \arrow[dr] & M(x') = M_{i_\varphi} \arrow[d] & M(x') = M_{i_{x'}} \arrow[l] \arrow[dl] \\
& K &
\end{tikzcd}$$

is commutative in $\mathcal{C}$. Since the left pointing horizontal arrow is the identity morphism on $M(x')$ by our definition of $\varphi_{i_{x'}i_\varphi}$ we see that the morphisms $M(x) \to K$ so defined satisfy condition (1) of Definition 4.13.2.

Finally we have to verify condition (2) of Definition 4.13.2. Suppose that W is an object of $\mathcal{C}$ and suppose that we are given morphisms $w_x : M(x) \to W$ such that for all morphisms a of $\mathcal{J}$ we have $w_{s(a)} = w_{t(a)} \circ a$. In this case, set $w_i = w_{x_i}$ for a quadruple $i = (S_i, A_i, x_i, \{f_{s,i}\}_{s \in S_i})$. Note that the condition on the maps w_x in particular guarantees that $w_{i'} = w_i \circ M(\varphi_{ii'})$ if $i \leq i'$ in I. Because K is the colimit of the system $(M(x_i), M(\varphi_{ii'})$ we obtain a unique morphism $K \to W$ compatible with the maps w_i and the given morphisms $M_i \to K$. This proves the statement about colimits of the lemma.

We omit the proof of the statement about limits. (Hint: You can change it into a statement about colimits by considering the opposite category of $\mathcal{C}$.) $\square$

4.20. Essentially constant systems

Let $M : \mathcal{J} \to \mathcal{C}$ be a diagram in a category $\mathcal{C}$. Assume the index category $\mathcal{J}$ is filtered. In this case there are three successively stronger notions which pick out an object X of $\mathcal{C}$. The first is just

$$X = colim_{i \in \mathcal{J}} M_i.$$

Then X comes equipped with projection morphisms $M_i \to X$. A stronger condition would be to require that X is the colimit and that there exists an $i \in \mathcal{J}$ and a morphism $X \to M_i$ such that the composition $X \to M_i \to X$ is id_X. A stronger condition is the following.

Definition 4.20.1. Let $M : \mathcal{J} \to \mathcal{C}$ be a diagram in a category $\mathcal{C}$.

(1) Assume the index category $\mathcal{J}$ is filtered. We say M *is essentially constant* with *value* X if $X = colim_i M_i$ and there exists an $i \in \mathcal{J}$ and a morphism $X \to M_i$ such that
   (a) $X \to M_i \to X$ is id_X, and
   (b) for all j there exist k and morphisms $i \to k$ and $j \to k$ such that the morphism $M_j \to M_k$ equals the composition $M_j \to X \to M_i \to M_k$.

(2) Assume the index category $\mathcal{J}$ is cofiltered. We say M is *essentially constant* with *value* X if $X = lim_i M_i$ and there exists an $i \in \mathcal{J}$ and a morphism $M_i \to X$ such that
   (a) $X \to M_i \to X$ is id_X, and
   (b) for all j there exist k and morphisms $k \to i$ and $k \to j$ such that the morphism $M_k \to M_j$ equals the composition $M_k \to M_i \to X \to M_j$.

Which of the two versions is meant will be clear from context. If there is any confusion we will distinguish between these by saying that the first version means M is essentially constant as an *ind-object*, and in the second case we will say it is essentially constant as an *pro-object*. This terminology is further explained in Remarks 4.20.3 and 4.20.4. In fact we will often use the terminology ``essentially constant system" which formally speaking is only defined for systems over directed partially ordered sets.

Definition 4.20.2. Let $\mathcal{C}$ be a category. A directed system $(M_i, f_{ii'})$ is an *essentially constant system* if M viewed as a functor $I \to \mathcal{C}$ defines an essentially constant diagram. A directed inverse system $(M_i, f_{ii'})$ is an *essentially constant inverse system* if M viewed as a functor $I^{opp} \to \mathcal{C}$ defines an essentially constant inverse diagram.

If $(M_i, f_{ii'})$ is an essentially constant system and the morphisms $f_{ii'}$ are monomorphisms, then for all $i \leq i'$ sufficiently large the morphisms $f_{ii'}$ are isomorphisms. In general this need not be the case however. An example is the system

$$\mathbf{Z}^2 \to \mathbf{Z}^2 \to \mathbf{Z}^2 \to \ldots$$

with maps given by $(a, b) \mapsto (a + b, 0)$. This system is essentially constant with value $\mathbf{Z}$. A non-example is to let $M = \bigoplus_{n\geq 0} \mathbf{Z}$ and to let $S : M \to M$ be the shift operator $(a_0, a_1, \ldots) \mapsto (a_1, a_2, \ldots)$. In this case the system $M \to M \to M \to \ldots$ with transition maps S has colimit 0, and a map $0 \to M$ but the system is not essentially constant.

Remark 4.20.3. Let $\mathcal{C}$ be a category. There exists a big category Ind-$\mathcal{C}$ of *ind-objects of* $\mathcal{C}$. Namely, if $F : \mathcal{I} \to \mathcal{C}$ and $G : \mathcal{J} \to \mathcal{C}$ are filtered diagrams in $\mathcal{C}$, then we can define

$$Mor_{\text{Ind-}\mathcal{C}}(F, G) = lim_i\, colim_j\, Mor_{\mathcal{C}}(F(i), G(j)).$$

There is a canonical functor $\mathcal{C} \to$ Ind-$\mathcal{C}$ which maps X to the *constant system* on X. This is a fully faithful embedding. In this language one sees that a diagram F is essentially constant if and only F is isomorphic to a constant system. If we ever need this material, then we will formulate this into a lemma and prove it here.

Remark 4.20.4. Let $\mathcal{C}$ be a category. There exists a big category Pro-$\mathcal{C}$ of *pro-objects* of $\mathcal{C}$. Namely, if $F : \mathcal{I} \to \mathcal{C}$ and $G : \mathcal{J} \to \mathcal{C}$ are cofiltered diagrams in $\mathcal{C}$, then we can define

$$Mor_{\text{Pro-}\mathcal{C}}(F, G) = lim_j\, colim_i\, Mor_{\mathcal{C}}(F(i), G(j)).$$

There is a canonical functor $\mathcal{C} \to$ Pro-$\mathcal{C}$ which maps X to the *constant system* on X. This is a fully faithful embedding. In this language one sees that a diagram F is essentially constant if and only F is isomorphic to a constant system. If we ever need this material, then we will formulate this into a lemma and prove it here.

Lemma 4.20.5. *Let $\mathcal{C}$ be a category. Let $M : \mathcal{I} \to \mathcal{C}$ be a diagram with filtered (resp. cofiltered) index category $\mathcal{I}$. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. If M is essentially constant as an ind-object (resp. pro-object), then so is $F \circ M : \mathcal{I} \to \mathcal{D}$.*

Proof. If X is a value for M, then it follows immediately from the definition that $F(X)$ is a value for $F \circ M$. ☐

Lemma 4.20.6. *Let $\mathcal{C}$ be a category. Let $M : \mathcal{I} \to \mathcal{C}$ be a diagram with filtered index category $\mathcal{I}$. The following are equivalent*

(1) *M is an essentially constant ind-object, and*
(2) *$X = colim_i\, M_i$ exists and for any W in $\mathcal{C}$ the map*

$$colim_i\, Mor_{\mathcal{C}}(W, M_i) \longrightarrow Mor_{\mathcal{C}}(W, X)$$

is bijective.

Proof. Assume (2) holds. Then $\text{id}_X \in Mor_{\mathcal{C}}(X, X)$ comes from a morphism $X \to M_i$ for some i, i.e., $X \to M_i \to X$ is the identity. Then both maps

$$Mor_{\mathcal{C}}(W, X) \longrightarrow colim_i\, Mor_{\mathcal{C}}(W, M_i) \longrightarrow Mor_{\mathcal{C}}(W, X)$$

are bijective for all W where the first one is induced by the morphism $X \to M_i$ we found above, and the composition is the identity. This means that the composition

$$colim_i\, Mor_{\mathcal{C}}(W, M_i) \longrightarrow Mor_{\mathcal{C}}(W, X) \longrightarrow colim_i\, Mor_{\mathcal{C}}(W, M_i)$$

is the identity too. Setting $W = M_j$ and starting with id_{M_j} in the colimit, we see that $M_j \to X \to M_i \to M_k$ is equal to $M_j \to M_k$ for some k large enough. This proves (1) holds. The proof of (1) $\Rightarrow$ (2) is omitted. ☐

Lemma 4.20.7. *Let $\mathcal{C}$ be a category. Let $M : \mathcal{I} \to \mathcal{C}$ be a diagram with cofiltered index category $\mathcal{I}$. The following are equivalent*

(1) *M is an essentially constant pro-object, and*

(2) *$X = lim_i\, M_i$ exists and for any W in $\mathcal{C}$ the map*
$$colim_i\, Mor_{\mathcal{C}}(M_i, W) \longrightarrow Mor_{\mathcal{C}}(X, W)$$
is bijective.

Proof. Assume (2) holds. Then $\text{id}_X \in Mor_{\mathcal{C}}(X, X)$ comes from a morphism $M_i \to X$ for some i, i.e., $X \to M_i \to X$ is the identity. Then both maps
$$Mor_{\mathcal{C}}(X, W) \longrightarrow colim_i\, Mor_{\mathcal{C}}(M_i, W) \longrightarrow Mor_{\mathcal{C}}(X, W)$$
are bijective for all W where the first one is induced by the morphism $M_i \to X$ we found above, and the composition is the identity. This means that the composition
$$colim_i\, Mor_{\mathcal{C}}(M_i, W) \longrightarrow Mor_{\mathcal{C}}(X, W) \longrightarrow colim_i\, Mor_{\mathcal{C}}(M_i, W)$$
is the identity too. Setting $W = M_j$ and starting with id_{M_j} in the colimit, we see that $M_k \to M_i \to X \to M_j$ is equal to $M_k \to M_j$ for some k large enough. This proves (1) holds. The proof of (1) ⇒ (2) is omitted. □

4.21. Exact functors

Definition 4.21.1. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor.

(1) Suppose all finite limits exist in $\mathcal{A}$. We say F is *left exact* if it commutes with all finite limits.
(2) Suppose all finite colimits exist in $\mathcal{A}$. We say F is *right exact* if it commutes with all finite colimits.
(3) We say F is *exact* if it is both left and right exact.

Lemma 4.21.2. *Let $F : \mathcal{A} \to \mathcal{B}$ be a functor. Suppose all finite limits exist in $\mathcal{A}$, see Lemma 4.16.4. The following are equivalent:*

(1) *F is left exact,*
(2) *F commutes with finite products and equalizers, and*
(3) *F transforms a final object of $\mathcal{A}$ into a final object of $\mathcal{B}$, and commutes with fibre products.*

Proof. Lemma 4.13.10 shows that (2) implies (1). Suppose (3) holds. The fibre product over the final object is the product. If $a, b : A \to B$ are morphisms of $\mathcal{A}$, then the equalizer of a, b is
$$(A \times_{a,B,b} A) \times_{(pr_1, pr_2), A\times A, \Delta} A.$$
Thus (3) implies (2). Finally (1) implies (3) because the empty limit is a final object, and fibre products are limits. □

4.22. Adjoint functors

Definition 4.22.1. Let $\mathcal{C}$, $\mathcal{D}$ be categories. Let $u : \mathcal{C} \to \mathcal{D}$ and $v : \mathcal{D} \to \mathcal{C}$ be functors. We say that u is a *left adjoint* of v, or that v is a *right adjoint* to u if there are bijections
$$Mor_{\mathcal{D}}(u(X), Y) \longrightarrow Mor_{\mathcal{C}}(X, v(Y))$$
functorial in $X \in Ob(\mathcal{C})$, and $Y \in Ob(\mathcal{D})$.

In other words, this means that there is an isomorphism of functors $\mathcal{C}^{opp} \times \mathcal{D} \to Sets$ from $Mor_{\mathcal{D}}(u(-), -)$ to $Mor_{\mathcal{C}}(-, v(-))$.

Lemma 4.22.2. *Let u be a left adjoint to v as in Definition 4.22.1.*

(1) *Suppose that* $M : \mathcal{I} \to \mathcal{C}$ *is a diagram, and suppose that* $colim_{\mathcal{I}} M$ *exists in* $\mathcal{C}$. *Then* $u(colim_{\mathcal{I}} M) = colim_{\mathcal{I}} u \circ M$. *In other words, u commutes with (representable) colimits.*
(2) *Suppose that* $M : \mathcal{I} \to \mathcal{D}$ *is a diagram, and suppose that* $lim_{\mathcal{I}} M$ *exists in* $\mathcal{D}$. *Then* $v(lim_{\mathcal{I}} M) = lim_{\mathcal{I}} v \circ M$. *In other words v commutes with representable limits.*

Proof. A morphism from a colimit into an object is the same as a compatible system of morphisms from the constituents of the limit into the object, see Remark 4.13.4. So

$$\begin{aligned} Mor_{\mathcal{D}}(u(colim_{i \in \mathcal{I}} M_i), Y) &= Mor_{\mathcal{C}}(colim_{i \in \mathcal{I}} M_i, v(Y)) \\ &= lim_{i \in \mathcal{I}^{opp}} Mor_{\mathcal{C}}(M_i, v(Y)) \\ &= lim_{i \in \mathcal{I}^{opp}} Mor_{\mathcal{D}}(u(M_i), Y) \end{aligned}$$

proves that $u(colim_{i \in \mathcal{I}} M_i)$ is the colimit we are looking for. A similar argument works for the other statement. ☐

Lemma 4.22.3. *Let u be a left adjoint of v as in Definition 4.22.1.*
(1) *If* $\mathcal{C}$ *has finite colimits, then u is right exact.*
(2) *If* $\mathcal{D}$ *has finite limits, then v is left exact.*

Proof. Obvious from the definitions and Lemma 4.22.2. ☐

4.23. Monomorphisms and Epimorphisms

Definition 4.23.1. Let $\mathcal{C}$ be a category, and let $f : X \to Y$ be a morphism of $\mathcal{C}$.
(1) We say that f is a *monomorphism* if for every object W and every pair of morphisms $a, b : W \to X$ such that $f \circ a = f \circ b$ we have $a = b$.
(2) We say that f is an *epimorphism* if for every object W and every pair of morphisms $a, b : Y \to W$ such that $a \circ f = b \circ f$ we have $a = b$.

Example 4.23.2. In the category of sets the monomorphisms correspond to injective maps and the epimorphisms correspond to surjective maps.

4.24. Localization in categories

The basic idea of this section is given a category $\mathcal{C}$ and a set of arrows to construct a functor $F : \mathcal{C} \to S^{-1}\mathcal{C}$ such that all elements of S become invertible in $S^{-1}\mathcal{C}$ and such that F is universal among all functors with this property. References for this section are [**GZ67**, Chapter I, Section 2] and [**Ver96**, Chapter II, Section 2].

Definition 4.24.1. Let $\mathcal{C}$ be a category. A set of arrows S of $\mathcal{C}$ is called a *left multiplicative system* if it has the following properties:

LMS1 The identity of every object of $\mathcal{C}$ is in S and the composition of two composable elements of S is in S.
LMS2 Every solid diagram

$$\begin{tikzcd} X \arrow[r, "g"'] \arrow[d, "t"'] & Y \arrow[d, dotted, "s"] \\ Z \arrow[r, dotted, "f"] & W \end{tikzcd}$$

with $t \in S$ can be completed to a commutative dotted square with $s \in S$.
LMS3 For every pair of morphisms $f, g : X \to Y$ and $t \in S$ with target X such that $f \circ t = g \circ t$ there exists a $s \in S$ with source Y such that $s \circ f = s \circ g$.

A set of arrows S of $\mathcal{C}$ is called a *right multiplicative system* if it has the following properties:

RMS1 The identity of every object of $\mathcal{C}$ is in S and the composition of two composable elements of S is in S.

RMS2 Every solid diagram

$$\begin{tikzcd} X \arrow[r, dotted, "g"] \arrow[d, dotted, "t"'] & Y \arrow[d, "s"] \\ Z \arrow[r, "f"] & W \end{tikzcd}$$

with $s \in S$ can be completed to a commutative dotted square with $t \in S$.

RMS3 For every pair of morphisms $f, g : X \to Y$ and $s \in S$ with source Y such that $s \circ f = s \circ g$ there exists a $t \in S$ with target X such that $f \circ t = g \circ t$.

A set of arrows S of $\mathcal{C}$ is called a *multiplicative system* if it is both a left multiplicative system and a right multiplicative system. In other words, this means that MS1, MS2, MS3 hold, where MS1 = LMS1 = RMS1, MS2 = LMS2 + RMS2, and MS3 = LMS3 + RMS3.

These conditions are useful to construct the categories $S^{-1}\mathcal{C}$ as follows.

Left calculus of fractions. Let $\mathcal{C}$ be a category and let S be a left multiplicative system. We define a new category $S^{-1}\mathcal{C}$ as follows (we verify this works in the proof of Lemma 4.24.2):

(1) We set $Ob(S^{-1}\mathcal{C}) = Ob(\mathcal{C})$.
(2) Morphisms $X \to Y$ of $S^{-1}\mathcal{C}$ are given by pairs $(f : X \to Y', s : Y \to Y')$ with $s \in S$ up to equivalence. (Think of this as $s^{-1}f : X \to Y$.)
(3) Two pairs $(f_1 : X \to Y_1, s_1 : Y \to Y_1)$ and $(f_2 : X \to Y_2, s_2 : Y \to Y_2)$ are said to be equivalent if there exists a third pair $(f_3 : X \to Y_3, s_3 : Y \to Y_3)$ and morphisms $u : Y_1 \to Y_3$ and $v : Y_2 \to Y_3$ of $\mathcal{C}$ fitting into the commutative diagram

$$\begin{tikzcd} & Y_1 \arrow[d, "u"] & \\ X \arrow[ur, "f_1"] \arrow[r, "f_3"] \arrow[dr, "f_2"'] & Y_3 & Y \arrow[ul, "s_1"'] \arrow[l, "s_3"'] \arrow[dl, "s_2"] \\ & Y_2 \arrow[u, "v"'] & \end{tikzcd}$$

(4) The composition of the equivalence classes of the pairs $(f : X \to Y', s : Y \to Y')$ and $(g : Y \to Z', t : Z \to Z')$ is defined as the equivalence class of a pair $(h \circ f : X \to Z'', u \circ t : Z \to Z'')$ where h and $u \in S$ are chosen to fit into a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "s"'] & Z' \arrow[d, "u"] \\ Y' \arrow[r, "h"] & Z'' \end{tikzcd}$$

which exists by assumption.

Lemma 4.24.2. *Let $\mathcal{C}$ be a category and let S be a left multiplicative system.*

(1) *The relation on pairs defined above is an equivalence relation.*
(2) *The composition rule given above is well defined on equivalence classes.*
(3) *Composition is associative and hence $S^{-1}\mathcal{C}$ is a category.*

Proof. Proof of (1). Let us say two pairs $p_1 = (f_1 : X \to Y_1, s_1 : Y \to Y_1)$ and $p_2 = (f_2 : X \to Y_2, s_2 : Y \to Y_2)$ are elementary equivalent if there exists a morphism $a : Y_1 \to Y_2$ of $\mathcal{C}$ such that $a \circ f_1 = f_2$ and $a \circ s_1 = s_2$. Diagram:

$$\begin{tikzcd} X \arrow[r, "f_1"'] \arrow[d, equal] & Y_1 \arrow[d, "a"] & Y \arrow[l, "s_1"'] \arrow[d, equal] \\ X \arrow[r, "f_2"] & Y_2 & Y \arrow[l, "s_2"] \end{tikzcd}$$

Let us denote this property by saying $p_1 E p_2$. Note that pEp and $aEb, bEc \Rightarrow aEc$. Part (1) claims that the relation $p \sim p' \Leftrightarrow \exists q : pEq \wedge p'Eq$ is an equivalence relation. A simple formal argument, using the properties of E above shows that it suffices to prove $p_2 E p_1, p_2 E p_3 \Rightarrow p_1 \sim p_2$. Thus suppose that we are given a commutative diagram

$$\begin{tikzcd} & Y_1 & \\ X \arrow[ur, "f_1"] \arrow[r, "f_3"] \arrow[dr, "f_2"'] & Y_3 \arrow[u, "a_{31}"] \arrow[d, "a_{32}"] & Y \arrow[ul, "s_1"'] \arrow[l, "s_3"'] \arrow[dl, "s_2"] \\ & Y_2 & \end{tikzcd}$$

with $s_i \in S$. First we apply LMS2 to get a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "s_3"'] \arrow[d, "s_1"'] & Y_3 \arrow[d, dotted, "s_{34}"] \\ Y_1 \arrow[r, dotted, "a_{14}"] & Y_4 \end{tikzcd}$$

with $s_{34} \in S$. Then we have $s_{34} \circ s_2 = a_{14} \circ a_{31} \circ s_2$. Hence by LMS3 there exists a morphism $s_{44} : Y_4 \to Y_4'$, $s_{44} \in S$ such that $s_{44} \circ s_{34} = s_{44} \circ a_{14} \circ a_{31}$. Hence after replacing Y_4 by Y_4', a_{14} by $s_{44} \circ a_{14}$, and s_{24} by $s_{44} \circ s_{24}$ we may assume that $s_{34} = a_{14} \circ a_{31}$. Next, we apply LMS2 to get a commutative diagram

$$\begin{tikzcd} Y_3 \arrow[r, "s_{34}"'] \arrow[d, "a_{32}"'] & Y_4 \arrow[d, dotted, "s_{45}"] \\ Y_2 \arrow[r, dotted, "a_{25}"] & Y_5 \end{tikzcd}$$

with $s_{45} \in S$. Thus we obtain a pair $p_5 = (s_{45} \circ s_{34} \circ f_3 : X \to Y_5, s_{45} \circ s_{34} \circ s_3 : Y \to Y_5)$ and the morphisms $s_{45} \circ a_{14} : Y_1 \to Y_5$ and $a_{25} : Y_2 \to Y_5$ show that indeed $p_1 E p_5$ and $p_2 E p_5$ as desired.

Proof of (2). Let $p = (f : X \to Y', s : Y \to Y')$ and $q = (g : Y \to Z', t : Z \to Z')$ be pairs as in the definition of composition above. To compose we have to choose a diagram

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "s"'] & Z' \arrow[d, "u_2"] \\ Y' \arrow[r, "h_2"] & Z_2 \end{tikzcd}$$

We first show that the equivalence class of the pair $r_2 = (h_2 \circ f : X \to Z_2, u_2 \circ t : Z \to Z_2)$ is independent of the choice of (Z_2, h_2, u_2). Namely, suppose that (Z_3, h_3, u_3) is another choice with corresponding composition $r_3 = (h_3 \circ f : X \to Z_3, u_3 \circ t : Z \to Z_3)$. Then by LMS2 we can choose a diagram

$$\begin{tikzcd} Z' \arrow[r, "u_3"'] \arrow[d, "u_2"'] & Z_3 \arrow[d, "u_{34}"] \\ Z_2 \arrow[r, "h_{24}"] & Z_4 \end{tikzcd}$$

with $u_{34} \in S$. Hence we obtain a pair $r_4 = (h_{24} \circ h_2 \circ f : X \to Z_4, u_{34} \circ u_3 \circ t : Z \to Z_4)$ and the morphisms $h_{24} : Z_2 \to Z_4$ and $u_{34} : Z_3 \to Z_4$ show that we have $r_2 E r_4$ and $r_3 E r_4$ as desired. Thus it now makes sense to define $p \circ q$ as the equivalence class of all possible pairs r obtained as above.

To finish the proof of (2) we have to show that given pairs p_1, p_2, q such that $p_1 E p_2$ then $p_1 \circ q = p_2 \circ q$ and $q \circ p_1 = q \circ p_2$ whenever the compositions make sense. To do this, write $p_1 = (f_1 : X \to Y_1, s_1 : Y \to Y_1)$ and $p_2 = (f_2 : X \to Y_2, s_2 : Y \to Y_2)$ and let $a : Y_1 \to Y_2$ be a morphism of $\mathcal{C}$ such that $f_2 = a \circ f_1$ and $s_2 = a \circ s_1$. First assume that $q = (g : Y \to Z', t : Z \to Z')$. In this case choose a commutative diagram as the one on the left

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "s_2"'] & Z' \arrow[d, "u"] \\ Y_2 \arrow[r, "h"] & Z'' \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "s_1"'] & Z' \arrow[d, "u"] \\ Y_1 \arrow[r, "h \circ a"] & Z'' \end{tikzcd}$$

which implies the diagram on the right is commutative as well. Using these diagrams we see that both compositions are the equivalence class of $(h \circ a \circ f_1 : X \to Z'', u \circ t : Z \to Z'')$. Thus $p_1 \circ q = p_2 \circ q$. The proof of the other case, in which we have to show $q \circ p_1 = q \circ p_2$, is omitted.

Proof of (3). We have to prove associativity of composition. Consider a solid diagram

$$\begin{tikzcd} & & & Z \arrow[d] \\ & & Y \arrow[r] \arrow[d] & Z' \arrow[d, dotted] \\ & X \arrow[r] \arrow[d] & Y' \arrow[r, dotted] \arrow[d, dotted] & Z'' \arrow[d, dotted] \\ W \arrow[r] & W' \arrow[r, dotted] & Y'' \arrow[r, dotted] & Z''' \end{tikzcd}$$

which gives rise to three composable pairs. Using LMS2 we can choose the dotted arrows making the squares commutative and such that the vertical arrows are in S. Then it is clear that the composition of the three pairs is the equivalence class of the pair $(W \to Z''', Z \to Z''')$ gotten by composing the horizontal arrows on the bottom row and the vertical arrows on the right column. $\square$

We can ``write any finite collection of morphisms with the same target as fractions with common denominator".

Lemma 4.24.3. *Let $\mathcal{C}$ be a category and let S be a left multiplicative system of morphisms of $\mathcal{C}$. Given any finite collection $g_i : X_i \to Y$ of morphisms of $S^{-1}\mathcal{C}$ we can find an element $s : Y \to Y'$ of S and $f_i : X_i \to Y'$ such that g_i is the equivalence class of the pair $(f_i : X_i \to Y', s : Y \to Y')$.*

Proof. For each i choose a representative $(X_i \to Y_i, s_i : Y \to Y_i)$. The lemma follows if we can find a morphism $s : Y \to Y'$ in S such that for each i there is a morphism $a_i : Y_i \to Y'$ with $a_i \circ s_i = s$. If we have two indices $i = 1, 2$, then we can do this by completing the square

$$\begin{tikzcd} Y \arrow[r, "s_2"'] \arrow[d, "s_1"'] & Y_2 \arrow[d, "t_2"] \\ Y_1 \arrow[r, "a_1"] & Y' \end{tikzcd}$$

with $t_2 \in S$ as is possible by Definition 4.24.1. Then $s = t_2 \circ s_1 \in S$ works. If we have $n > 2$ morphisms, then we use the above trick to reduce to the case of $n - 1$ morphisms, and we win by induction. $\square$

There is an easy characterization of equality of morphisms if they have the same denominator.

Lemma 4.24.4. *Let $\mathcal{C}$ be a category and let S be a left multiplicative system of morphisms of $\mathcal{C}$. Let $A, B : X \to Y$ be morphisms of $S^{-1}\mathcal{C}$ which are the equivalence classes of $(f : X \to Y', s : Y \to Y')$ and $(g : X \to Y', s : Y \to Y')$. Then $A = B$ if and only if there exists a morphism $a : Y' \to Y''$ with $a \circ s \in S$ and such that $a \circ f = a \circ g$.*

Proof. The equality of A and B means that there exists a commutative diagram

$$\begin{tikzcd} & Y' \arrow[d, "u"] & \\ X \arrow[ur, "f"] \arrow[r, "h"] \arrow[dr, "g"'] & Z & Y \arrow[ul, "s"'] \arrow[l, "t"'] \arrow[dl, "s"] \\ & Y' \arrow[u, "v"'] & \end{tikzcd}$$

with $t \in S$. In particular $u \circ s = v \circ s$. Hence by LMS3 there exists a $s' : Z \to Y''$ in S such that $s' \circ u = s' \circ v$. Setting a equal to this common value does the job. $\square$

Remark 4.24.5. Let $\mathcal{C}$ be a category. Let S be a left multiplicative system. Given an object Y of $\mathcal{C}$ we denote Y/S the category whose objects are $s : Y \to Y'$ with $s \in S$ and whose morphisms are commutative diagrams

$$\begin{tikzcd} & Y \arrow[dl, "s"'] \arrow[dr, "t"] & \\ Y' \arrow[rr, "a"] & & Y'' \end{tikzcd}$$

where $a : Y' \to Y''$ is arbitrary. We claim that the category Y/S is filtered (see Definition 4.17.1). Namely, LMS1 implies that $\mathrm{id}_Y : Y \to Y$ is in Y/S hence Y/S is nonempty. LMS2 implies that given $s_1 : Y \to Y_1$ and $s_2 : Y \to Y_2$ we can find a diagram

$$\begin{tikzcd} Y \arrow[r, "s_2"'] \arrow[d, "s_1"'] & Y_2 \arrow[d, "t"] \\ Y_1 \arrow[r, "a"] & Y_3 \end{tikzcd}$$

with $t \in S$. Hence $s_1 : Y \to Y_1$ and $s_2 : Y \to Y_2$ both map to $t \circ s_2 : Y \to Y_3$ in Y/S. Finally, given two morphisms a, b from $s_1 : Y \to Y_1$ to $s_2 : Y \to Y_2$ in S/Y we see that $a \circ s_1 = b \circ s_1$ hence by LMS3 there exists a $t : Y_2 \to Y_3$ such that $t \circ a = t \circ b$. Now the combined results of Lemmas 4.24.3 and 4.24.4 tell us that

$$Mor_{S^{-1}\mathcal{C}}(X, Y) = colim_{(s : Y \to Y') \in Y/S} Mor_{\mathcal{C}}(X, Y') \tag{4.24.5.1}$$

This formula expressing morphisms in $S^{-1}\mathcal{C}$ as a filtered colimit of morphisms in $\mathcal{C}$ is occasionally useful.

Lemma 4.24.6. *Let $\mathcal{C}$ be a category and let S be a left multiplicative system of morphisms of $\mathcal{C}$.*

1. *The rules $X \mapsto X$ and $(f : X \to Y) \mapsto (f : X \to Y, id_Y : Y \to Y)$ define a functor $Q : \mathcal{C} \to S^{-1}\mathcal{C}$.*
2. *For any $s \in S$ the morphism $Q(s)$ is an isomorphism in $S^{-1}\mathcal{C}$.*
3. *If $G : \mathcal{C} \to \mathcal{D}$ is any functor such that $G(s)$ is invertible for every $s \in S$, then there exists a unique functor $H : S^{-1}\mathcal{C} \to \mathcal{D}$ such that $H \circ Q = G$.*

Proof. Parts (1) and (2) are clear. To see (3) just set $H(X) = G(X)$ and set $H((f : X \to Y', s : Y \to Y')) = H(s)^{-1} \circ H(f)$. Details omitted. ☐

Lemma 4.24.7. *Let $\mathcal{C}$ be a category and let S be a left multiplicative system of morphisms of $\mathcal{C}$. The localization functor $Q : \mathcal{C} \to S^{-1}\mathcal{C}$ commutes with finite colimits.*

Proof. This is clear from (4.24.5.1), Remark 4.13.4, and Lemma 4.17.2. ☐

Lemma 4.24.8. *Let $\mathcal{C}$ be a category. Let S be a left multiplicative system. If $f : X \to Y$, $f' : X' \to Y'$ are two morphisms of $\mathcal{C}$ and if*

$$\begin{tikzcd} Q(X) \arrow[r, "a"'] \arrow[d, "Q(f)"'] & Q(X') \arrow[d, "Q(f')"] \\ Q(Y) \arrow[r, "b"] & Q(Y') \end{tikzcd}$$

is a commutative diagram in $S^{-1}\mathcal{C}$, then there exists a morphism $f'' : X'' \to Y''$ in $\mathcal{C}$ and a commutative diagram

$$\begin{tikzcd} X \arrow[r, "g"'] \arrow[d, "f"'] & X'' \arrow[d, "f''"] & X' \arrow[l, "s"'] \arrow[d, "f'"] \\ Y \arrow[r, "h"] & Y'' & Y' \arrow[l, "t"] \end{tikzcd}$$

in $\mathcal{C}$ with $s, t \in S$ and $a = s^{-1}g$, $b = t^{-1}h$.

Proof. We choose maps and objects in the following way: First write $a = s^{-1}g$ for some $s : X' \to X''$ in S and $h : X \to X''$. By LMS2 we can find $t : Y' \to Y''$ in S and

$f'' : X'' \to Y''$ such that

$$\begin{tikzcd} X \arrow[r, "g"'] \arrow[d, "f"'] & X'' \arrow[d, "f''"] & X' \arrow[l, "s"'] \arrow[d, "f'"] \\ Y & Y'' & Y' \arrow[l, "t"'] \end{tikzcd}$$

commutes. Now in this diagram we are going to repeatedly change our choice of

$$X'' \xrightarrow{f''} Y'' \xleftarrow{t} Y'$$

by postcomposing both t and f'' by a morphism $d : Y'' \to Y'''$ with the property that $d \circ t \in S$. According to Remark 4.24.5 we may after such a replacement assume that there exists a morphism $h : Y \to Y''$ such that $b = t^{-1}h$. At this point we have everything as in the lemma except that we don't know that the left square of the diagram commutes. However, we do know that $Q(f''g) = Q(hf)$ in $S^{-1}\mathcal{D}$ because the right square commutes, the outer square commutes in $S^{-1}\mathcal{D}$ by assumption, and because $Q(s), Q(t)$ are isomorphisms. Hence using Lemma 4.24.4 we can find a morphism $d : X''' \to X''$ in S (!) such that $df''g = dhf$. Hence we make one more replacement of the kind described above and we win. $\square$

Right calculus of fractions. Let $\mathcal{C}$ be a category and let S be a right multiplicative system. We define a new category $S^{-1}\mathcal{C}$ as follows (we verify this works in the proof of Lemma 4.24.9):

(1) We set $Ob(S^{-1}\mathcal{C}) = Ob(\mathcal{C})$.
(2) Morphisms $X \to Y$ of $S^{-1}\mathcal{C}$ are given by pairs $(f : X' \to Y, s : X' \to X)$ with $s \in S$ up to equivalence. (Think of this as $fs^{-1} : X \to Y$.)
(3) Two pairs $(f_1 : X_1 \to Y, s_1 : X_1 \to X)$ and $(f_2 : X_2 \to Y, s_2 : X_2 \to X)$ are said to be equivalent if there exists a third pair $(f_3 : X_3 \to Y, s_3 : X_3 \to X)$ and morphisms $u : X_3 \to X_1$ and $v : X_3 \to X_2$ of $\mathcal{C}$ fitting into the commutative diagram

$$\begin{tikzcd} & X_1 \arrow[dl, "s_1"'] \arrow[dr, "f_1"] & \\ X & X_3 \arrow[l, "s_3"'] \arrow[r, "f_3"] \arrow[u, "u"'] \arrow[d, "v"] & Y \\ & X_2 \arrow[ul, "s_2"] \arrow[ur, "f_2"'] & \end{tikzcd}$$

(4) The composition of the equivalence classes of the pairs $(f : X' \to Y, s : X' \to X)$ and $(g : Y' \to Z, t : Y' \to Y)$ is defined as the equivalence class of a pair $(g \circ h : X'' \to Z, s \circ u : X'' \to X)$ where h and $u \in S$ are chosen to fit into a commutative diagram

$$\begin{tikzcd} X'' \arrow[r, "h"] \arrow[d, "u"'] & Y' \arrow[d, "t"] \\ X' \arrow[r, "f"] & Y \end{tikzcd}$$

which exists by assumption.

Lemma 4.24.9. *Let $\mathcal{C}$ be a category and let S be a right multiplicative system.*

(1) *The relation on pairs defined above is an equivalence relation.*
(2) *The composition rule given above is well defined on equivalence classes.*

(3) *Composition is associative and hence $S^{-1}\mathcal{C}$ is a category.*

Proof. This lemma is dual to Lemma 4.24.2. It follows formally from that lemma by replacing $\mathcal{C}$ by its opposite category in which S is a left multiplicative system. $\square$

We can ``write any finite collection of morphisms with the same source as fractions with common denominator".

Lemma 4.24.10. *Let $\mathcal{C}$ be a category and let S be a right multiplicative system of morphisms of $\mathcal{C}$. Given any finite collection $g_i : X \to Y_i$ of morphisms of $S^{-1}\mathcal{C}$ we can find an element $s : X' \to X$ of S and $f_i : X' \to Y_i$ such that g_i is the equivalence class of the pair $(f_i : X' \to Y_i, s : X' \to X)$.*

Proof. This lemma is the dual of Lemma 4.24.3 and follows formally from that lemma by replacing all categories in sight by their opposites. $\square$

There is an easy characterization of equality of morphisms if they have the same denominator.

Lemma 4.24.11. *Let $\mathcal{C}$ be a category and let S be a right multiplicative system of morphisms of $\mathcal{C}$. Let $A, B : X \to Y$ be morphisms of $S^{-1}\mathcal{C}$ which are the equivalence classes of $(f : X' \to Y, s : X' \to X)$ and $(g : X' \to Y, s : X' \to X)$. Then $A = B$ if and only if there exists a morphism $a : X'' \to X'$ with $s \circ a \in S$ and such that $f \circ a = g \circ a$.*

Proof. This is dual to Lemma 4.24.4. $\square$

Remark 4.24.12. Let $\mathcal{C}$ be a category. Let S be a right multiplicative system. Given an object X of $\mathcal{C}$ we denote S/X the category whose objects are $s : X' \to X$ with $s \in S$ and whose morphisms are commutative diagrams

$$\begin{tikzcd} X' \arrow[rr, "a"'] \arrow[dr, "s"'] & & X'' \arrow[dl, "t"] \\ & X & \end{tikzcd}$$

where $a : X' \to X''$ is arbitrary. The category S/X is cofiltered (see Definition 4.18.1). (This is dual to the corresponding statement in Remark 4.24.5.) Now the combined results of Lemmas 4.24.10 and 4.24.11 tell us that

$$Mor_{S^{-1}\mathcal{C}}(X, Y) = colim_{(s : X' \to X) \in (S/X)^{opp}} Mor_{\mathcal{C}}(X', Y) \tag{4.24.12.1}$$

This formula expressing morphisms in $S^{-1}\mathcal{C}$ as a filtered colimit of morphisms in $\mathcal{C}$ is occasionally useful.

Lemma 4.24.13. *Let $\mathcal{C}$ be a category and let S be a right multiplicative system of morphisms of $\mathcal{C}$.*

(1) *The rules $X \mapsto X$ and $(f : X \to Y) \mapsto (f : X \to Y, id_X : X \to X)$ define a functor $Q : \mathcal{C} \to S^{-1}\mathcal{C}$.*
(2) *For any $s \in S$ the morphism $Q(s)$ is an isomorphism in $S^{-1}\mathcal{C}$.*
(3) *If $G : \mathcal{C} \to \mathcal{D}$ is any functor such that $G(s)$ is invertible for every $s \in S$, then there exists a unique functor $H : S^{-1}\mathcal{C} \to \mathcal{D}$ such that $H \circ Q = G$.*

Proof. This lemma is the dual of Lemma 4.24.6 and follows formally from that lemma by replacing all categories in sight by their opposites. $\square$

Lemma 4.24.14. *Let $\mathcal{C}$ be a category and let S be a right multiplicative system of morphisms of $\mathcal{C}$. The localization functor $Q : \mathcal{C} \to S^{-1}\mathcal{C}$ commutes with finite limits.*

Proof. This is clear from (4.24.12.1), Remark 4.13.4, and Lemma 4.17.2. $\square$

Lemma 4.24.15. *Let $\mathcal{C}$ be a category. Let S be a right multiplicative system. If $f : X \to Y$, $f' : X' \to Y'$ are two morphisms of $\mathcal{C}$ and if*

$$\begin{tikzcd} Q(X) \arrow[r, "a"'] \arrow[d, "Q(f)"'] & Q(X') \arrow[d, "Q(f')"] \\ Q(Y) \arrow[r, "b"] & Q(Y') \end{tikzcd}$$

is a commutative diagram in $S^{-1}\mathcal{C}$, then there exists a morphism $f'' : X'' \to Y''$ in $\mathcal{C}$ and a commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & X'' \arrow[l, "s"'] \arrow[r, "g"'] \arrow[d, "f''"] & X' \arrow[d, "f'"] \\ Y & Y'' \arrow[l, "t"] \arrow[r, "h"] & Y' \end{tikzcd}$$

in $\mathcal{C}$ with $s, t \in S$ and $a = gs^{-1}$, $b = ht^{-1}$.

Proof. This lemma is dual to Lemma 4.24.8 but we can also prove it directly as follows. We choose maps and objects in the following way: First write $b = ht^{-1}$ for some $t : Y'' \to Y$ in S and $h : Y'' \to Y'$. By RMS2 we can find $s : X'' \to X$ in S and $f'' : X'' \to Y''$ such that

$$\begin{tikzcd} X \arrow[d, "f"'] & X'' \arrow[l, "s"'] \arrow[d, "f''"] & X' \arrow[d, "f'"] \\ Y & Y'' \arrow[l, "t"] \arrow[r, "h"] & Y' \end{tikzcd}$$

commutes. Now in this diagram we are going to repeatedly change our choice of

$$X \xleftarrow{s} X'' \xrightarrow{f''} Y''$$

by precomposing both s and f'' by a morphism $d : X''' \to X''$ with the property that $s \circ d \in S$. According to Remark 4.24.12 we may after such a replacement assume that there exists a morphism $g : X'' \to X'$ such that $a = gs^{-1}$. At this point we have everything as in the lemma except that we don't know that the right square of the diagram commutes. However, we do know that $Q(f'g) = Q(hf'')$ in $S^{-1}\mathcal{D}$ because the left square commutes, the outer square commutes in $S^{-1}\mathcal{D}$ by assumption, and because $Q(s), Q(t)$ are isomorphisms. Hence using Lemma 4.24.11 we can find a morphism $d : X''' \to X''$ in S (!) such that $f'gd = hf''d$. Hence we make one more replacement of the kind described above and we win. $\square$

Multiplicative systems and two sided calculus of fractions. If S is a multiplicative system then left and right calculus of fractions given canonically isomorphic categories.

Lemma 4.24.16. *Let $\mathcal{C}$ be a category and let S be a multiplicative system. The category of left fractions and the category of right fractions $S^{-1}\mathcal{C}$ are canonically isomorphic.*

Proof. Denote $\mathcal{C}_{left}$, $\mathcal{C}_{right}$ the two categories of fractions. By the universal properties of Lemmas 4.24.6 and 4.24.13 we obtain functors $\mathcal{C}_{left} \to \mathcal{C}_{right}$ and $\mathcal{C}_{right} \to \mathcal{C}_{left}$. By the uniqueness of these functors they are each others inverse. $\square$

Definition 4.24.17. Let $\mathcal{C}$ be a category and let S be a multiplicative system. We say S is *saturated* if, in addition to MS1, MS2, MS3 we also have

MS4 Given three composable morphisms f, g, h, if $fg, gh \in S$, then $g \in S$.

Note that a saturated multiplicative system contains all isomorphisms. Moreover, if f, g, h are composable morphisms in a category and fg, gh are isomorphisms, then g is an isomorphism (because then g has both a left and a right inverse, hence is invertible).

Lemma 4.24.18. *Let $\mathcal{C}$ be a category and let S be a multiplicative system. Denote $Q : \mathcal{S} \to S^{-1}\mathcal{C}$ the localization functor. The set*

$$\hat{S} = \{f \in Arrows(\mathcal{C}) \mid Q(f) \text{ is an isomorphism}\}$$

is equal to

$$S' = \{f \in Arrows(\mathcal{C}) \mid \text{there exist } g, h \text{ such that } gf, fh \in S\}$$

and is the smallest saturated multiplicative system containing S. In particular, if S is saturated, then $\hat{S} = S$.

Proof. It is clear that $S \subset S' \subset \hat{S}$ because elements of S' map to morphisms in $S^{-1}\mathcal{C}$ which have both left and right inverses. Note that S' satisfies MS4, and that $\hat{S}$ satisfies MS1. Next, we prove that $S' = \hat{S}$.

Let $f \in \hat{S}$. Let $s^{-1}g = ht^{-1}$ be the inverse morphism in $S^{-1}\mathcal{C}$. (We may use both left fractions and right fractions to describe morphisms in $S^{-1}\mathcal{C}$, see Lemma 4.24.16.) The relation $\mathrm{id}_X = s^{-1}gf$ in $S^{-1}\mathcal{C}$ means there exists a commutative diagram

$$\begin{tikzcd}
& X' \arrow[d, "u"] & \\
X \arrow[ur, "gf"] \arrow[r, "f'"] \arrow[dr, "\mathrm{id}_X"'] & X'' & X \arrow[ul, "s"'] \arrow[l, "s'"'] \arrow[dl, "\mathrm{id}_X"] \\
& X \arrow[u, "v"] &
\end{tikzcd}$$

for some morphisms f', u, v and $s' \in S$. Hence $ugf = s' \in S$. Similarly, using that $\mathrm{id}_Y = fht^{-1}$ one proves that $fhw \in S$ for some w. We conclude that $f \in S'$. Thus $S' = \hat{S}$. Provided we prove that $S' = \hat{S}$ is a multiplicative system it is now clear that this implies that $S' = \hat{S}$ is the smallest saturated system containing S.

Our remarks above take care of MS1 and MS4, so to finish the proof of the lemma we have to show that LMS2, RMS2, LMS3, RMS3 hold for $\hat{S}$. Let us check that LMS2 holds for $\hat{S}$. Suppose we have a solid diagram

$$\begin{tikzcd}
X \arrow[r, "g"'] \arrow[d, "t"'] & Y \arrow[d, dotted, "s"] \\
Z \arrow[r, dotted, "f"] & W
\end{tikzcd}$$

with $t \in \hat{S}$. Pick a morphism $a : Z \to Z'$ such that $at \in S$. Then we can use LMS2 for S to find a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "g"'] \arrow[d, "t"'] & Y \arrow[dd, "s"] \\
Z \arrow[d, "a"'] & \\
Z' \arrow[r, "f'"] & W
\end{tikzcd}$$

and setting $f = f' \circ a$ we win. The proof of RMS2 is dual to this. Finally, suppose given a pair of morphisms $f, g : X \to Y$ and $t \in \hat{S}$ with target X such that $ft = gt$. Then we pick a morphism b such that $tb \in S$. Then $ftb = gtb$ which implies by LMS3 for S that there exists an $s \in S$ with source Y such that $sf = sg$ as desired. The proof of RMS3 is dual to this. $\square$

4.25. Formal properties

In this section we discuss some formal properties of the 2-category of categories. This will lead us to the definition of a (strict) 2-category later.

Let us denote $Ob(Cat)$ the class of all categories. For every pair of categories $\mathcal{A}, \mathcal{B} \in Ob(Cat)$ we have the ``small'' category of functors $\text{Fun}(\mathcal{A}, \mathcal{B})$. Composition of transformation of functors such as

$$\begin{tikzcd}
\mathcal{A} \arrow[r, bend left=50, "F''"] \arrow[r, "F'" description] \arrow[r, bend right=50, "F"'] \arrow[r, phantom, shift left=3, "\Downarrow t'"] \arrow[r, phantom, shift right=3, "\Downarrow t"] & \mathcal{B}
\end{tikzcd}
\text{ composes to }
\begin{tikzcd}
\mathcal{A} \arrow[r, bend left, "F''"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Downarrow t\circ t'"] & \mathcal{B}
\end{tikzcd}$$

is called *vertical* composition. We will use the usual symbol $\circ$ for this. Next, we will define *horizontal* composition. In order to do this we explain a bit more of the structure at hand.

Namely for every triple of categories $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ there is a composition law

$$\circ : Ob(\text{Fun}(\mathcal{B}, \mathcal{C})) \times Ob(\text{Fun}(\mathcal{A}, \mathcal{B})) \longrightarrow Ob(\text{Fun}(\mathcal{A}, \mathcal{C}))$$

coming from composition of functors. This composition law is associative, and identity functors act as units. In other words -- forgetting about transformations of functors -- we see that Cat forms a category. How does this structure interact with the morphisms between functors?

Well, given $t : F \to F'$ a transformation of functors $F, F' : \mathcal{A} \to \mathcal{B}$ and a functor $G : \mathcal{B} \to \mathcal{C}$ we can define a transformation of functors $G \circ F \to G \circ F'$. We will denote this transformation ${}_Gt$. It is given by the formula $({}_Gt)_x = G(t_x) : G(F(x)) \to G(F'(x))$ for all $x \in \mathcal{A}$. In this way composition with G becomes a functor

$$\text{Fun}(\mathcal{A}, \mathcal{B}) \longrightarrow \text{Fun}(\mathcal{A}, \mathcal{C}).$$

To see this you just have to check that ${}_G(\text{id}_F) = \text{id}_{G \circ F}$ and that ${}_G(t_1 \circ t_2) = {}_Gt_1 \circ {}_Gt_2$. Of course we also have that ${}_{\text{id}_\mathcal{A}}t = t$.

Similarly, given $s : G \to G'$ a transformation of functors $G, G' : \mathcal{B} \to \mathcal{C}$ and $F : \mathcal{A} \to \mathcal{B}$ a functor we can define s_F to be the transformation of functors $G \circ F \to G' \circ F$ given by

$(s_F)_x = s_{F(x)} : G(F(x)) \to G'(F(x))$ for all $x \in \mathcal{A}$. In this way composition with F becomes a functor

$$\mathrm{Fun}(\mathcal{B}, \mathcal{C}) \longrightarrow \mathrm{Fun}(\mathcal{A}, \mathcal{C}).$$

To see this you just have to check that $(\mathrm{id}_G)_F = \mathrm{id}_{G \circ F}$ and that $(s_1 \circ s_2)_F = s_{1,F} \circ s_{2,F}$. Of course we also have that $s_{\mathrm{id}_{\mathcal{B}}} = s$.

These constructions satisfy the additional properties

$${}_{G_1}({}_{G_2}t) = {}_{G_1 \circ G_2}t, \ (s_{F_1})_{F_2} = s_{F_1 \circ F_2}, \text{ and } {}_H(s_F) = ({}_H s)_F$$

whenever these make sense. Finally, given functors $F, F' : \mathcal{A} \to \mathcal{B}$, and $G, G' : \mathcal{B} \to \mathcal{C}$ and transformations $t : F \to F'$, and $s : G \to G'$ the following diagram is commutative

$$\begin{tikzcd}
G \circ F \arrow[r, "{}_G t"] \arrow[d, "s_F"'] & G \circ F' \arrow[d, "s_{F'}"] \\
G' \circ F \arrow[r, "{}_{G'} t"'] & G' \circ F'
\end{tikzcd}$$

in other words ${}_{G'}t \circ s_F = s_{F'} \circ {}_G t$. To prove this we just consider what happens on any object $x \in Ob(\mathcal{A})$:

$$\begin{tikzcd}
G(F(x)) \arrow[r, "G(t_x)"] \arrow[d, "s_{F(x)}"'] & G(F'(x)) \arrow[d, "s_{F'(x)}"] \\
G'(F(x)) \arrow[r, "G'(t_x)"'] & G'(F'(x))
\end{tikzcd}$$

which is commutative because s is a transformation of functors. This compatibility relation allows us to define horizontal composition.

Definition 4.25.1. Given a diagram as in the left hand side of:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\Downarrow t"] & \mathcal{B} \arrow[r, bend left, "G"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow s"] & \mathcal{C}
\end{tikzcd} \text{ gives } \begin{tikzcd}
\mathcal{A} \arrow[r, bend left, "G \circ F"] \arrow[r, bend right, "G' \circ F'"'] \arrow[r, phantom, "\Downarrow s \star t"] & \mathcal{C}
\end{tikzcd}$$

we define the *horizontal* composition $s \star t$ to be the transformation of functors ${}_{G'}t \circ s_F = s_{F'} \circ {}_G t$.

Now we see that we may recover our previously constructed transformations ${}_G t$ and s_F as ${}_G t = \mathrm{id}_G \star t$ and $s_F = s \star \mathrm{id}_F$. Furthermore, all of the rules we found above are consequences of the properties stated in the lemma that follows.

Lemma 4.25.2. *The horizontal and vertical compositions have the following properties*

(1) $\circ$ *and* $\star$ *are associative,*
(2) *the identity transformations* id_F *are units for* $\circ$*,*
(3) *the identity transformations of the identity functors* $\mathrm{id}_{\mathrm{id}_{\mathcal{A}}}$ *are units for* $\star$ *and* $\circ$*, and*

(4) *given a diagram*

$$\begin{tikzcd}[column sep=huge]
\mathscr{A} \arrow[r, bend left=50, "F"] \arrow[r, "F'"'] \arrow[r, bend right=50, "F''"'] \arrow[r, phantom, shift left=3, "\Downarrow t"] \arrow[r, phantom, shift right=3, "\Downarrow t'"] & \mathscr{B} \arrow[r, bend left=50, "G"] \arrow[r, "G'"'] \arrow[r, bend right=50, "G''"'] \arrow[r, phantom, shift left=3, "\Downarrow s"] \arrow[r, phantom, shift right=3, "\Downarrow s'"] & \mathscr{C}
\end{tikzcd}$$

we have $(s' \circ s) \star (t' \circ t) = (s' \star t') \circ (s \star t)$.

Proof. The last statement turns using our previous notation into the following equation

$$s'_{F''} \circ {}_{G'}t' \circ s_{F'} \circ {}_{G}t = (s' \circ s)_{F''} \circ {}_{G}(t' \circ t).$$

According to our result above applied to the middle composition we may rewrite the left hand side as $s'_{F''} \circ s_{F''} \circ {}_{G}t' \circ {}_{G}t$ which is easily shown to be equal to the right hand side. □

Another way of formulating condition (4) of the lemma is that composition of functors and horizontal composition of transformation of functors gives rise to a functor

$$(\circ, \star) : \mathrm{Fun}(\mathscr{B}, \mathscr{C}) \times \mathrm{Fun}(\mathscr{A}, \mathscr{B}) \longrightarrow \mathrm{Fun}(\mathscr{A}, \mathscr{C})$$

whose source is the product category, see Definition 4.2.20.

4.26. 2-categories

We will give a definition of (strict) 2-categories as they appear in the setting of stacks. Before you read this take a look at Section 4.25 and Example 4.27.2. Basically, you take this example and you write out all the rules satisfied by the objects, 1-morphisms and 2-morphisms in that example.

Definition 4.26.1. A (strict) 2-*category* $\mathscr{C}$ consists of the following data

(1) A set of objects $Ob(\mathscr{C})$.
(2) For each pair $x, y \in Ob(\mathscr{C})$ a category $Mor_{\mathscr{C}}(x, y)$. The objects of $Mor_{\mathscr{C}}(x, y)$ will be called 1-*morphisms* and denoted $F : x \to y$. The morphisms between these 1-morphisms will be called 2-*morphisms* and denoted $t : F' \to F$. The composition of 2-morphisms in $Mor_{\mathscr{C}}(x, y)$ will be called *vertical* composition and will be denoted $t \circ t'$ for $t : F' \to F$ and $t' : F'' \to F'$.
(3) For each triple $x, y, z \in Ob(\mathscr{C})$ a functor

$$(\circ, \star) : Mor_{\mathscr{C}}(y, z) \times Mor_{\mathscr{C}}(x, y) \longrightarrow Mor_{\mathscr{C}}(x, z).$$

The image of the pair of 1-morphisms (F, G) on the left hand side will be called the *composition* of F and G, and denoted $F \circ G$. The image of the pair of 2-morphisms (t, s) will be called the *horizontal* composition and denoted $t \star s$.

These data are to satisfy the following rules:

(1) The set of objects together with the set of 1-morphisms endowed with composition of 1-morphisms forms a category.
(2) Horizontal composition of 2-morphisms is associative.
(3) The identity 2-morphism $\mathrm{id}_{\mathrm{id}_x}$ of the identity 1-morphism id_x is a unit for horizontal composition.

This is obviously not a very pleasant type of object to work with. On the other hand, there are lots of examples where it is quite clear how you work with it. The only example we have so far is that of the 2-category whose objects are a given collection of categories, 1-morphisms are functors between these categories, and 2-morphisms are natural transformations of functors, see Section 4.25. As far as this text is concerned all 2-categories will be sub 2-categories of this example. Here is what it means to be a sub 2-category.

Definition 4.26.2. Let $\mathscr{C}$ be a 2-category. A *sub 2-category* $\mathscr{C}'$ of $\mathscr{C}$, is given by a subset $Ob(\mathscr{C}')$ of $Ob(\mathscr{C})$ and sub categories $Mor_{\mathscr{C}'}(x, y)$ of the categories $Mor_{\mathscr{C}}(x, y)$ for all $x, y \in Ob(\mathscr{C}')$ such that these, together with the operations $\circ$ (composition 1-morphisms), $\circ$ (vertical composition 2-morphisms), and $\star$ (horizontal composition) form a 2-category.

Remark 4.26.3. Big 2-categories. In many texts a 2-category is allowed to have a class of objects (but hopefully a ``class of classes'' is not allowed). We will allow these ``big'' 2-categories as well, but only in the following list of cases (to be updated as we go along):

(1) The 2-category of categories *Cat*.
(2) The $(2, 1)$-category of categories *Cat*.
(3) The 2-category of groupoids *Groupoids*.
(4) The $(2, 1)$-category of groupoids *Groupoids*.
(5) The 2-category of fibred categories over a fixed category.
(6) The $(2, 1)$-category of fibred categories over a fixed category.

Note that in each case the class of objects of the 2-category $\mathscr{C}$ is a proper class, but for all objects $x, y \in Ob(C)$ the category $Mor_{\mathscr{C}}(x, y)$ is ``small'' (according to our conventions).

The notion of equivalence of categories that we defined in Section 4.2 extends to the more general setting of 2-categories as follows.

Definition 4.26.4. Two objects x, y of a 2-category are *equivalent* if there exist 1-morphisms $F : x \to y$ and $G : y \to x$ such that $F \circ G$ is 2-isomorphic to id_y and $G \circ F$ is 2-isomorphic to id_x.

Sometimes we need to say what it means to have a functor from a category into a 2-category.

Definition 4.26.5. Let $\mathscr{A}$ be a category and let $\mathscr{C}$ be a 2-category.

(1) A *functor* from an ordinary category into a 2-category will ignore the 2-morphisms unless mentioned otherwise. In other words, it will be a ``usual'' functor into the category formed out of 2-category by forgetting all the 2-morphisms.
(2) A *weak functor*, or a *pseudo functor* φ from $\mathscr{A}$ into the 2-category $\mathscr{C}$ is given by the following data
   (a) a map $\varphi : Ob(\mathscr{A}) \to Ob(\mathscr{C})$,
   (b) for every pair $x, y \in Ob(\mathscr{A})$, and every morphism $f : x \to y$ a 1-morphism $\varphi(f) : \varphi(x) \to \varphi(y)$,
   (c) for every $x \in Ob(A)$ a 2-morphism $\alpha_x : \mathrm{id}_{\varphi(x)} \to \varphi(\mathrm{id}_x)$, and
   (d) for every pair of composable morphisms $f : x \to y$, $g : y \to z$ of $\mathscr{A}$ a 2-morphism $\alpha_{g,f} : \varphi(g \circ f) \to \varphi(g) \circ \varphi(f)$.

   These data are subject to the following conditions:
   (a) the 2-morphisms α_x and $\alpha_{g,f}$ are all isomorphisms,
   (b) for any morphism $f : x \to y$ in $\mathscr{A}$ we have $\alpha_{\mathrm{id}_y, f} = \alpha_y \star \mathrm{id}_{\varphi(f)}$:

$$\begin{tikzcd} \varphi(x) \arrow[r, bend left, "\varphi(f)"] \arrow[r, bend right, "\varphi(f)"'] \arrow[r, phantom, "\Downarrow \mathrm{id}_{\varphi(f)}"] & \varphi(y) \arrow[r, bend left, "\mathrm{id}_y"] \arrow[r, bend right, "\varphi(\mathrm{id}_y)"'] \arrow[r, phantom, "\Downarrow \alpha_y"] & \varphi(y) \end{tikzcd} = \begin{tikzcd} \varphi(x) \arrow[r, bend left, "\varphi(f)"] \arrow[r, bend right, "\varphi(\mathrm{id}_y) \circ \varphi(f)"'] \arrow[r, phantom, "\Downarrow \alpha_{f, \mathrm{id}_y}"] & \varphi(y) \end{tikzcd}$$

(c) for any morphism $f : x \to y$ in $\mathcal{A}$ we have $\alpha_{f,\mathrm{id}_x} = \mathrm{id}_{\varphi(f)} \star \alpha_x$,
(d) for any triple of composable morphisms $f : w \to x$, $g : x \to y$, and $h : y \to z$ of $\mathcal{A}$ we have

$$(\mathrm{id}_{\varphi(h)} \star \alpha_{g,f}) \circ \alpha_{h,g\circ f} = (\alpha_{h,g} \star \mathrm{id}_{\varphi(f)}) \circ \alpha_{h\circ g,f}$$

in other words the following diagram with objects 1-morphisms and arrows 2-morphisms commutes

$$\begin{tikzcd} \varphi(h \circ g \circ f) \arrow[rr, "\alpha_{h\circ g,f}"'] \arrow[d, "\alpha_{h,g\circ f}"'] & & \varphi(h \circ g) \circ \varphi(f) \arrow[d, "\alpha_{h,g}\star \mathrm{id}_{\varphi(f)}"] \\ \varphi(h) \circ \varphi(g \circ f) \arrow[rr, "\mathrm{id}_{\varphi(h)}\star\alpha_{g,f}"] & & \varphi(h) \circ \varphi(g) \circ \varphi(f) \end{tikzcd}$$

Again this is not a very workable notion, but it does sometimes come up. There is a theorem that says that any pseudo-functor is isomorphic to a functor. Finally, there are the notions of *functor between* 2*-categories*, and *pseudo functor between* 2*-categories*. This last notion leads us into 3-category territory. We would like to avoid having to define this at almost any cost!

4.27. (2, 1)-categories

Some 2-categories have the property that all 2-morphisms are isomorphisms. These will play an important role in the following, and they are easier to work with.

Definition 4.27.1. A (strict) (2, 1)-*category* is a 2-category in which all 2-morphisms are isomorphisms.

Example 4.27.2. The 2-category *Cat*, see Remark 4.26.3, can be turned into a (2, 1)-category by only allowing isomorphisms of functors as 2-morphisms.

In fact, more generally any 2-category $\mathcal{C}$ produces a (2, 1)-category by considering the sub 2-category $\mathcal{C}'$ with the same objects and 1-morphisms but whose 2-morphisms are the invertible 2-morphisms of $\mathcal{C}$. In this situation we will say ``*let* $\mathcal{C}'$ *be the* (2, 1)*-category associated to* $\mathcal{C}$" or similar. For example, the (2, 1)-category of groupoids means the 2-category whose objects are groupoids, whose 1-morphisms are functors and whose 2-morphisms are isomorphisms of functors. Except that this is a bad example as a transformation between functors between groupoids is automatically an isomorphism!

Remark 4.27.3. Thus there are variants of the construction of Example 4.27.2 above where we look at the 2-category of groupoids, or categories fibred in groupoids over a fixed category, or stacks. And so on.

4.28. 2-fibre products

In this section we introduce 2-fibre products. Suppose that $\mathcal{C}$ is a 2-category. We say that a diagram

$$\begin{tikzcd} w \arrow[r] \arrow[d] & y \arrow[d] \\ x \arrow[r] & z \end{tikzcd}$$

2-commutes if the two 1-morphisms $w \to y \to z$ and $w \to x \to z$ are 2-isomorphic. In a 2-category it is more natural to ask for 2-commutativity of diagrams than for actually commuting diagrams. (Indeed, some may say that we should not work with strict 2-categories at all, and in a ``weak" 2-category the notion of a commutative diagram of 1-morphisms does not even make sense.) Correspondingly the notion of a fibre product has to be adjusted.

Let $\mathcal{C}$ be a 2-category. Let $x, y, z \in Ob(\mathcal{C})$ and $f \in Mor_{\mathcal{C}}(x, z)$ and $g \in Mor_{\mathcal{C}}(y, z)$. In order to define the 2-fibre product of f and g we are going to look at 2-commutative diagrams

$$\begin{tikzcd} w \arrow[r, "a"'] \arrow[d, "b"'] & x \arrow[d, "f"] \\ y \arrow[r, "g"] & z. \end{tikzcd}$$

Now in the case of categories, the fibre product is a final object in the category of such diagrams. Correspondingly a 2-fibre product is a final object in a 2-category (see definition below). The *2-category of 2-commutative diagrams* is the 2-category defined as follows:

(1) Objects are quadruples (w, a, b, ϕ) as above where ϕ is an invertible 2-morphism $\phi : f \circ a \to g \circ b$,
(2) 1-morphisms from (w', a', b', ϕ') to (w, a, b, ϕ) are given by $(k : w' \to w, \alpha : a' \to a \circ k, \beta : b' \to b \circ k)$ such that

$$\begin{tikzcd} f \circ a' \arrow[rr, "\mathrm{id}_f \star \alpha"'] \arrow[d, "\phi'"'] & & f \circ a \circ k \arrow[d, "\phi \star \mathrm{id}_k"] \\ f \circ b' \arrow[rr, "\mathrm{id}_f \star \beta"] & & f \circ b \circ k \end{tikzcd}$$

is commutative,
(3) given a second 1-morphism $(k', \alpha', \beta') : (w'', a'', b'', \phi'') \to (w', \alpha', \beta', \phi')$ the composition of 1-morphisms is given by the rule

$$(k, \alpha, \beta) \circ (k', \alpha', \beta') = (k \circ k', (\alpha \star \mathrm{id}_{k'}) \circ \alpha', (\beta \star \mathrm{id}_{k'}) \circ \beta'),$$

(4) a 2-morphism between 1-morphisms (k_i, α_i, β_i), $i = 1, 2$ with the same is given by a 2-morphism $\delta : k_1 \to k_2$ such that

$$\begin{tikzcd} a' \arrow[r, "\alpha_1"'] \arrow[dr, "\alpha_2"'] & a \circ k_1 \arrow[d, "\mathrm{id}_a \star \delta"] \\ & a \circ k_2 \end{tikzcd} \qquad \begin{tikzcd} b \circ k_1 \arrow[d, "\mathrm{id}_b \star \delta"'] & b' \arrow[l, "\beta_1"] \arrow[dl, "\beta_2"] \\ b \circ k_2 & \end{tikzcd}$$

commute,
(5) vertical composition of 2-morphisms is given by vertical composition of the morphisms δ in $\mathcal{C}$, and
(6) horizontal composition of the diagram

$$\begin{tikzcd}[column sep=huge] (w'', a'', b'', \phi'') \arrow[r, bend left, "{(k'_1, \alpha'_1, \beta'_1)}"] \arrow[r, bend right, "{(k'_2, \alpha'_2, \beta'_2)}"'] \arrow[r, phantom, "\Downarrow \delta'"] & (w', a', b', \phi') \arrow[r, bend left, "{(k_1, \alpha_1, \beta_1)}"] \arrow[r, bend right, "{(k_2, \alpha_2, \beta_2)}"'] \arrow[r, phantom, "\Downarrow \delta"] & (w, a, b, \phi) \end{tikzcd}$$

is given by the diagram

$$\begin{tikzcd}[column sep=huge] (w'', a'', b'', \phi'') \arrow[r, bend left, "{(k_1 \circ k'_1, (\alpha_1 \star \mathrm{id}_{k'_1}) \circ \alpha'_1, (\beta_1 \star \mathrm{id}_{k'_1}) \circ \beta'_1)}"] \arrow[r, bend right, "{(k_2 \circ k'_2, (\alpha_2 \star \mathrm{id}_{k'_2}) \circ \alpha'_2, (\beta_2 \star \mathrm{id}_{k'_2}) \circ \beta'_2)}"'] \arrow[r, phantom, "\Downarrow \delta \star \delta'"] & (w, a, b, \phi) \end{tikzcd}$$

Note that if $\mathcal{C}$ is actually a $(2, 1)$-category, the morphisms α and β in (2) above are automatically also isomorphisms[2]. In addition the 2-category of 2-commutative diagrams is also a $(2, 1)$-category if $\mathcal{C}$ is a $(2, 1)$-category.

Definition 4.28.1. A *final object* of a $(2, 1)$-category $\mathcal{C}$ is an object x such that

(1) for every $y \in Ob(\mathcal{C})$ there is a morphism $y \to x$, and
(2) every two morphisms $y \to x$ are isomorphic by a unique 2-morphism.

Likely, in the more general case of 2-categories there are different flavours of final objects. We do not want to get into this and hence we only define 2-fibre products in the $(2, 1)$-case.

Definition 4.28.2. Let $\mathcal{C}$ be a $(2, 1)$-category. Let $x, y, z \in Ob(\mathcal{C})$ and $f \in Mor_{\mathcal{C}}(x, z)$ and $g \in Mor_{\mathcal{C}}(y, z)$. A *2-fibre product of f and g* is a final object in the category of 2-commutative diagrams described above. If a 2-fibre product exists we will denote it $x \times_z y \in Ob(\mathcal{C})$, and denote the required morphisms $p \in Mor_{\mathcal{C}}(x \times_z y, x)$ and $q \in Mor_{\mathcal{C}}(x \times_z y, y)$ making the diagram

$$\begin{tikzcd} x \times_y z \arrow[r, "p"] \arrow[d, "q"'] & x \arrow[d, "f"] \\ y \arrow[r, "g"] & z \end{tikzcd}$$

2-commute and we will denote the given invertible 2-morphism exhibiting this by $\psi : f \circ p \to g \circ q$.

Thus the following universal property holds: for any $w \in Ob(\mathcal{C})$ and morphisms $a \in Mor_{\mathcal{C}}(w, x)$ and $b \in Mor_{\mathcal{C}}(w, y)$ with a given 2-isomorphism $\phi : f \circ a \to g \circ b$ there is a $\gamma \in Mor_{\mathcal{C}}(w, x \times_z y)$ making the diagram

$$\begin{tikzcd} w \arrow[rrd, "a"] \arrow[rd, dashed, "\gamma"] \arrow[rdd, "b"'] & & \\ & x \times_y z \arrow[r, "p"'] \arrow[d, "q"'] & x \arrow[d, "f"] \\ & y \arrow[r, "g"] & z \end{tikzcd}$$

2-commute such that for suitable choices of $a \to p \circ \gamma$ and $b \to q \circ \gamma$ the diagram

$$\begin{tikzcd} f \circ a \arrow[r] \arrow[d, "\phi"'] & f \circ p \circ \gamma \arrow[d, "\psi \star \mathrm{id}_\gamma"] \\ g \circ b \arrow[r] & g \circ q \circ \gamma \end{tikzcd}$$

commutes. Moreover γ is unique up to isomorphism. Of course the exact properties are finer than this. All of the cases of 2-fibre products that we will need later on come from the following example of 2-fibre products in the 2-category of categories.

Example 4.28.3. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be categories. Let $F : \mathcal{A} \to \mathcal{C}$ and $G : \mathcal{B} \to \mathcal{C}$ be functors. We define a category $\mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ as follows:

[2] In fact it seems in the 2-category case that one could define another 2-category of 2-commutative diagrams where the direction of the arrows α, β is reversed, or even where the direction of only one of them is reversed. This is why we restrict to $(2, 1)$-categories later on.

(1) an object of $\mathscr{A} \times_{\mathscr{C}} \mathscr{B}$ is a triple (A, B, f), where $A \in Ob(\mathscr{A})$, $B \in Ob(\mathscr{B})$, and $f : F(A) \to G(B)$ is an isomorphism in $\mathscr{C}$,
(2) a morphism $(A, B, f) \to (A', B', f')$ is given by a pair (a, b), where $a : A \to A'$ is a morphism in $\mathscr{A}$, and $b : B \to B'$ is a morphism in $\mathscr{B}$ such that the diagram

$$\begin{tikzcd} F(A) \arrow[r, "f"] \arrow[d, "F(a)"] & G(B) \arrow[d, "G(b)"] \\ F(A') \arrow[r, "f'"] & G(B') \end{tikzcd}$$

is commutative.

Moreover, we define functors $p : \mathscr{A} \times_{\mathscr{C}} \mathscr{B} \to \mathscr{A}$ and $q : \mathscr{A} \times_{\mathscr{C}} \mathscr{B} \to \mathscr{B}$ by setting

$$p(A, B, f) = A, \quad q(A, B, f) = B,$$

in other words, these are the forgetful functors. We define a transformation of functors $\psi : F \circ p \to G \circ q$. On the object $\xi = (A, B, f)$ it is given by $\psi_\xi = f : F(p(\xi)) = F(A) \to G(B) = G(q(\xi))$.

Lemma 4.28.4. *In the* $(2,1)$*-category of categories* 2*-fibre products exist and are given by the construction of Example 4.28.3.*

Proof. Let us check the universal property: let $\mathscr{W}$ be a category, let $a : \mathscr{W} \to \mathscr{A}$ and $b : \mathscr{W} \to \mathscr{B}$ be functors, and let $t : F \circ a \to G \circ b$ be an isomorphism of functors.

Consider the functor $\gamma : \mathscr{W} \to \mathscr{A} \times_{\mathscr{C}} \mathscr{B}$ given by $W \mapsto (a(W), b(W), t_W)$. (Check this is a functor omitted.) Moreover, consider $\alpha : a \to p \circ \gamma$ and $\beta : b \to q \circ \gamma$ obtained from the identities $p \circ \gamma = a$ and $q \circ \gamma = b$. Then it is clear that (γ, α, β) is a morphism from (W, a, b, t) to $(\mathscr{A} \times_{\mathscr{C}} \mathscr{B}, p, q, \psi)$.

Let $(k, \alpha', \beta') : (W, a, b, t) \to (\mathscr{A} \times_{\mathscr{C}} \mathscr{B}, p, q, \psi)$ be a second such morphism. For an object W of $\mathscr{W}$ let us write $k(W) = (a_k(W), b_k(W), t_{k,W})$. Hence $p(k(W)) = a_k(W)$ and so on. The map α' corresponds to functorial maps $\alpha' : a(W) \to a_k(W)$. Since we are working in the $(2,1)$-category of categories, in fact each of the maps $a(W) \to a_k(W)$ is an isomorphism. We can use these (and their counterparts $b(W) \to b_k(W)$) to get isomorphisms

$$\delta_W : \gamma(W) = (a(W), b(W), t_W) \longrightarrow (a_k(W), b_k(W), t_{k,W}) = k(W).$$

It is straightforward to show that δ defines a 2-isomorphism between γ and k in the 2-category of 2-commutative diagrams as desired. $\square$

Remark 4.28.5. Let $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ be categories. Let $F : \mathscr{A} \to \mathscr{C}$ and $G : \mathscr{B} \to \mathscr{C}$ be functors. Another, slightly more symmetrical, construction of a 2-fibre product $\mathscr{A} \times_{\mathscr{C}} \mathscr{B}$ is as follows. An object is a quintuple (A, B, C, a, b) where A, B, C are objects of $\mathscr{A}, \mathscr{B}, \mathscr{C}$ and where $a : F(A) \to C$ and $b : G(B) \to C$ are isomorphisms. A morphism $(A, B, C, a, b) \to (A', B', C', a', b')$ is given by a triple of morphisms $A \to A', B \to B', C \to C'$ compatible with the morphisms a, b, a', b'. We can prove directly that this leads to a 2-fibre product. However, it is easier to observe that the functor $(A, B, C, a, b) \mapsto (A, B, b^{-1} \circ a)$ gives an equivalence from the category of quintuples to the category constructed in Example 4.28.3.

Lemma 4.28.6. *Let*

$$\begin{tikzcd} & \mathcal{Y} \arrow[d, "I"'] \arrow[dr, "K"] & \\ \mathcal{X} \arrow[r, "H"] \arrow[dr, "L"'] & \mathcal{Z} \arrow[dr, "M"] & \mathcal{B} \arrow[d, "G"] \\ & \mathcal{A} \arrow[r, "F"] & \mathcal{C} \end{tikzcd}$$

be a 2*-commutative diagram of categories. A choice of isomorphisms* $\alpha : G \circ K \to M \circ I$ *and* $\beta : M \circ H \to F \circ L$ *determines a morphism*

$$\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \longrightarrow \mathcal{A} \times_{\mathcal{C}} \mathcal{B}$$

of 2*-fibre products associated to this situation.*

Proof. Just use the functor

$$(X, Y, \phi) \longmapsto (L(X), K(Y), \alpha_Y^{-1} \circ M(\phi) \circ \beta_X^{-1})$$

on objects and

$$(a, b) \longmapsto (L(a), K(b))$$

on morphisms. □

Lemma 4.28.7. *Assumptions as in Lemma 4.28.6.*

1. *If K and L are faithful then the morphism* $\mathcal{Y} \times_{\mathcal{Z}} \mathcal{X} \to \mathcal{B} \times_{\mathcal{C}} \mathcal{A}$ *is faithful.*
2. *If K and L are fully faithful and M is faithful then the morphism* $\mathcal{Y} \times_{\mathcal{Z}} \mathcal{X} \to \mathcal{B} \times_{\mathcal{C}} \mathcal{A}$ *is fully faithful.*
3. *If K and L are equivalences and M is fully faithful then the morphism* $\mathcal{Y} \times_{\mathcal{Z}} \mathcal{X} \to \mathcal{B} \times_{\mathcal{C}} \mathcal{A}$ *is an equivalence.*

Proof. Let (X, Y, ϕ) and (X', Y', ϕ') be objects of $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$. Set $Z = H(X)$ and identify it with $I(Y)$ via ϕ. Also, identify $M(Z)$ with $F(L(X))$ via α_X and identify $M(Z)$ with $G(K(Y))$ via β_Y. Similarly for $Z' = H(X')$ and $M(Z')$. The map on morphisms is the map

$$\begin{tikzcd} Mor_{\mathcal{X}}(X, X') \times_{Mor_{\mathcal{Z}}(Z, Z')} Mor_{\mathcal{Y}}(Y, Y') \arrow[d] \\ Mor_{\mathcal{A}}(L(X), L(X')) \times_{Mor_{\mathcal{C}}(M(Z), M(Z'))} Mor_{\mathcal{B}}(K(Y), K(Y')) \end{tikzcd}$$

Hence parts (1) and (2) follow. Moreover, if K and L are equivalences and M is fully faithful, then any object (A, B, ϕ) is in the essential image for the following reasons: Pick X, Y such that $L(X) \cong A$ and $K(Y) \cong B$. Then the fully faithfulness of M guarantees that we can find an isomorphism $H(X) \cong I(Y)$. Some details omitted. □

Lemma 4.28.8. *Let*

$$\begin{tikzcd} \mathcal{A} \arrow[dr] & & \mathcal{C} \arrow[dl] \arrow[dr] & & \mathcal{E} \arrow[dl] \\ & \mathcal{B} & & \mathcal{D} & \end{tikzcd}$$

be a diagram of categories and functors. Then there is a canonical isomorphism

$$(\mathcal{A} \times_{\mathcal{B}} \mathcal{C}) \times_{\mathcal{D}} \mathcal{E} \cong \mathcal{A} \times_{\mathcal{B}} (\mathcal{C} \times_{\mathcal{D}} \mathcal{E})$$

of categories.

Proof. Just use the functor

$$((A, C, \phi), E, \psi) \longmapsto (A, (C, E, \psi), \phi)$$

if you know what I mean. $\square$

Henceforth we do not write the parentheses when dealing with fibred products of more than 2 categories.

Lemma 4.28.9. *Let*

$$\begin{tikzcd} \mathcal{A} \arrow[dr] & & \mathcal{C} \arrow[dl] \arrow[dr] & & \mathcal{E} \arrow[dl] \\ & \mathcal{B} \arrow[dr, "F"'] & & \mathcal{D} \arrow[dl, "G"] & \\ & & \mathcal{F} & & \end{tikzcd}$$

be a commutative diagram of categories and functors. Then there is a canonical functor

$$pr_{02} : \mathcal{A} \times_{\mathcal{B}} \mathcal{C} \times_{\mathcal{D}} \mathcal{E} \longrightarrow \mathcal{A} \times_{\mathcal{F}} \mathcal{E}$$

of categories.

Proof. If we write $\mathcal{A} \times_{\mathcal{B}} \mathcal{C} \times_{\mathcal{D}} \mathcal{E}$ as $(\mathcal{A} \times_{\mathcal{B}} \mathcal{C}) \times_{\mathcal{D}} \mathcal{E}$ then we can just use the functor

$$((A, C, \phi), E, \psi) \longmapsto (A, E, G(\psi) \circ F(\phi))$$

if you know what I mean. $\square$

Lemma 4.28.10. *Let*

$$\mathcal{A} \to \mathcal{B} \leftarrow \mathcal{C} \leftarrow \mathcal{D}$$

be a diagram of categories and functors. Then there is a canonical isomorphism

$$\mathcal{A} \times_{\mathcal{B}} \mathcal{C} \times_{\mathcal{C}} \mathcal{D} \cong \mathcal{A} \times_{\mathcal{C}} \mathcal{D}$$

of categories.

Proof. Omitted. $\square$

We claim that this means you can work with these 2-fibre products just like with ordinary fibre products. Here are some further lemmas that actually come up later.

Lemma 4.28.11. *Let*

$$\begin{tikzcd} \mathcal{C}_3 \arrow[r] \arrow[d] & \mathcal{S} \arrow[d, "\Delta"] \\ \mathcal{C}_1 \times \mathcal{C}_2 \arrow[r, "G_1 \times G_2"] & \mathcal{S} \times \mathcal{S} \end{tikzcd}$$

be a 2-fibre product of categories. Then there is a canonical isomorphism $\mathcal{C}_3 \cong \mathcal{C}_1 \times_{G_1, \mathcal{S}, G_2} \mathcal{C}_2$.

Proof. We may assume that $\mathcal{C}_3$ is the category $(\mathcal{C}_1 \times \mathcal{C}_2) \times_{\mathcal{S}\times\mathcal{S}} \mathcal{S}$ constructed in Example 4.28.3. Hence an object is a triple $((X_1, X_2), S, \phi)$ where $\phi = (\phi_1, \phi_2) : (G_1(X_1), G_2(X_2)) \to (S, S)$ is an isomorphism. Thus we can associate to this the triple $(X_1, X_2, \phi_2 \circ \phi_1^{-1})$. Conversely, if (X_1, X_2, ψ) is an object of $\mathcal{C}_1 \times_{G_1,\mathcal{S},G_2} \mathcal{C}_2$, then we can associate to this the triple $((X_1, X_2), G_1(X_1), (\mathrm{id}_{G_1(X_1)}, \psi))$. We claim these constructions given mutually inverse functors. We omit describing how to deal with morphisms and show they are mutually inverse. ☐

Lemma 4.28.12. *Let*

$$\begin{tikzcd} \mathcal{C}' \arrow[r] \arrow[d] & \mathcal{S} \arrow[d, "\Delta"] \\ \mathcal{C} \arrow[r, "G_1\times G_2"] & \mathcal{S}\times\mathcal{S} \end{tikzcd}$$

be a 2*-fibre product of categories. Then there is a canonical isomorphism*

$$\mathcal{C}' \cong (\mathcal{C} \times_{G_1,\mathcal{S},G_2} \mathcal{C}) \times_{(p,q),\mathcal{C}\times\mathcal{C},\Delta} \mathcal{C}.$$

Proof. An object of the right hand side is given by $((C_1, C_2, \phi), C_3, \psi)$ where $\phi : G_1(C_1) \to G_2(C_2)$ is an isomorphism and $\psi = (\psi_1, \psi_2) : (C_1, C_2) \to (C_3, C_3)$ is an isomorphism. Hence we can associate to this the triple $(C_3, G_1(C_1), (G_1(\psi_1^{-1}), \varphi^{-1} \circ G_2(\psi_2^{-1})))$ which is an object of $\mathcal{C}'$. Details omitted. ☐

Lemma 4.28.13. *Let $\mathcal{A} \to \mathcal{C}$, $\mathcal{B} \to \mathcal{C}$ and $\mathcal{C} \to \mathcal{D}$ be functors between categories. Then the diagram*

$$\begin{tikzcd} \mathcal{A} \times_{\mathcal{C}} \mathcal{B} \arrow[r] \arrow[d] & \mathcal{A} \times_{\mathcal{D}} \mathcal{B} \arrow[d] \\ \mathcal{C} \arrow[r, "\Delta_{\mathcal{C}/\mathcal{D}}"] & \mathcal{C} \times_{\mathcal{D}} \mathcal{C} \end{tikzcd}$$

is a 2*-fibre product diagram.*

Proof. Omitted. ☐

Lemma 4.28.14. *Let*

$$\begin{tikzcd} \mathcal{U} \arrow[r] \arrow[d] & \mathcal{V} \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{Y} \end{tikzcd}$$

be a 2*-fibre product. Then the diagram*

$$\begin{tikzcd} \mathcal{U} \arrow[r] \arrow[d] & \mathcal{U} \times_{\mathcal{V}} \mathcal{U} \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \end{tikzcd}$$

is 2*-cartesian.*

Proof. This is a purely 2-category theoretic statement, valid in any $(2,1)$-category with 2-fibre products. Explicitly, it follows from the following chain of equivalences:

$$\begin{aligned}\mathcal{X} \times_{(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})} (\mathcal{U} \times_{\mathcal{V}} \mathcal{U}) &= \mathcal{X} \times_{(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})} ((\mathcal{X} \times_{\mathcal{Y}} \mathcal{V}) \times_{\mathcal{V}} (\mathcal{X} \times_{\mathcal{Y}} \mathcal{V})) \\ &= \mathcal{X} \times_{(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})} (\mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \times_{\mathcal{Y}} \mathcal{V}) \\ &= \mathcal{X} \times_{\mathcal{Y}} \mathcal{V} = \mathcal{U}\end{aligned}$$

see Lemmas 4.28.8 and 4.28.10. □

4.29. Categories over categories

In this section we have a functor $p : \mathcal{S} \to \mathcal{C}$. We think of $\mathcal{S}$ as being on top and of $\mathcal{C}$ as being at the bottom. To make sure that everybody knows what we are talking about we define the 2-category of categories over $\mathcal{C}$.

Definition 4.29.1. Let $\mathcal{C}$ be a category. The *2-category of categories over $\mathcal{C}$* is the sub 2-category of *Cat* defined as follows:

(1) Its objects will be functors $p : \mathcal{S} \to \mathcal{C}$.
(2) Its 1-morphisms $(\mathcal{S}, p) \to (\mathcal{S}', p')$ will be functors $G : \mathcal{S} \to \mathcal{S}'$ such that $p' \circ G = p$.
(3) Its 2-morphisms $t : G \to H$ for $G, H : (\mathcal{S}, p) \to (\mathcal{S}', p')$ will be morphisms of functors such that $p'(t_x) = \mathrm{id}_{p(x)}$ for all $x \in Ob(\mathcal{S})$.

In this situation we will denote

$$Mor_{Cat/\mathcal{C}}(\mathcal{S}, \mathcal{S}')$$

the category of 1-morphisms between $(\mathcal{S}, p)$ and $(\mathcal{S}', p')$

Since we have defined this as a sub 2-category of *Cat* we do not have to check any of the axioms. Rather we just have to check things such as ``vertical composition of 2-morphisms over $\mathcal{C}$ gives another 2-morphism over $\mathcal{C}$". This is clear.

Analogously to the fibre of a map of spaces, we have the notion of a fibre category, and some notions of lifting associated to this situation.

Definition 4.29.2. Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a category over $\mathcal{C}$.

(1) The *fibre category* over an object $U \in Ob(\mathcal{C})$ is the category $\mathcal{S}_U$ with objects

$$Ob(\mathcal{S}_U) = \{x \in Ob(\mathcal{S}) : p(x) = U\}$$

and morphisms

$$Mor_{\mathcal{S}_U}(x, y) = \{\phi \in Mor_{\mathcal{S}}(x, y) : p(\phi) = \mathrm{id}_U\}.$$

(2) A *lift* of an object $U \in Ob(\mathcal{C})$ is an object $x \in Ob(\mathcal{S})$ such that $p(x) = U$, i.e., $x \in Ob(\mathcal{S}_U)$. We will also sometime say that *x lies over U*.
(3) Similarly, a *lift* of a morphism $f : V \to U$ in $\mathcal{C}$ is a morphism $\phi : y \to x$ in $\mathcal{S}$ such that $p(\phi) = f$. We sometimes say that *ϕ lies over f*.

There are some observations we could make here. For example if $F : (\mathcal{S}, p) \to (\mathcal{S}', p')$ is a 1-morphism of categories over $\mathcal{C}$, then F induces functors of fibre categories $F : \mathcal{S}_U \to \mathcal{S}'_U$. Similarly for 2-morphisms.

Here is the obligatory lemma describing the 2-fibre product in the $(2,1)$-category of categories over $\mathcal{C}$.

Lemma 4.29.3. *Let $\mathcal{C}$ be a category. The $(2,1)$-category of categories over $\mathcal{C}$ has 2-fibre products. Suppose that $f : \mathcal{X} \to \mathcal{S}$ and $g : \mathcal{Y} \to \mathcal{S}$ are morphisms of categories over $\mathcal{C}$. An explicit 2-fibre product $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ is given by the following description*

(1) *an object of $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ is a quadruple (U, x, y, f), where $U \in Ob(\mathcal{C})$, $x \in Ob(\mathcal{X}_U)$, $y \in Ob(\mathcal{Y}_U)$, and $f : F(x) \to G(y)$ is an isomorphism in $\mathcal{S}_U$,*
(2) *a morphism $(U, x, y, f) \to (U', x', y', f')$ is given by a pair (a, b), where $a : x \to x'$ is a morphism in $\mathcal{X}$, and $b : y \to y'$ is a morphism in $\mathcal{Y}$ such that*
   (a) *a and b induced the same morphism $U \to U'$, and*
   (b) *the diagram*

$$\begin{tikzcd} F(x) \arrow[r, "f"] \arrow[d, "F(a)"] & G(y) \arrow[d, "G(b)"] \\ F(x') \arrow[r, "f'"] & G(y') \end{tikzcd}$$

   is commutative.

The functors $p : \mathcal{X} \times_{\mathcal{S}} \mathcal{Y} \to \mathcal{X}$ and $q : \mathcal{X} \times_{\mathcal{S}} \mathcal{Y} \to \mathcal{Y}$ are the forgetful functors in this case. The transformation $\psi : F \circ p \to G \circ q$ is given on the object $\xi = (U, x, y, f)$ by $\psi_\xi = f : F(p(\xi)) = F(x) \to G(y) = G(q(\xi))$.

Proof. Let us check the universal property: let $p_{\mathcal{W}} : \mathcal{W} \to \mathcal{C}$ be a category over $\mathcal{C}$, let $X : \mathcal{W} \to \mathcal{X}$ and $Y : \mathcal{W} \to \mathcal{Y}$ be functors over $\mathcal{C}$, and let $t : F \circ X \to G \circ Y$ be an isomorphism of functors over $\mathcal{C}$. The desired functor $\gamma : \mathcal{W} \to \mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ is given by $W \mapsto (p_{\mathcal{W}}(W), X(W), Y(W), t_W)$. Details omitted; compare with Lemma 4.28.4. $\square$

Lemma 4.29.4. *Let $\mathcal{C}$ be a category. Let $f : \mathcal{X} \to \mathcal{S}$ and $g : \mathcal{Y} \to \mathcal{S}$ be morphisms of categories over $\mathcal{C}$. For any object U of $\mathcal{C}$ we have the following identity of fibre categories*

$$\left(\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}\right)_U = \mathcal{X}_U \times_{\mathcal{S}_U} \mathcal{Y}_U$$

Proof. Omitted. $\square$

4.30. Fibred categories

A very brief discussion of fibred categories is warranted.

Let $p : \mathcal{S} \to \mathcal{C}$ be a category over $\mathcal{C}$. Given an object $x \in \mathcal{S}$ with $p(x) = U$, and given a morphism $f : V \to U$, we can try to take some kind of ``fibre product $V \times_U x$'' (or a *base change* of $x$ via $V \to U$). Namely, a morphism from an object $z \in \mathcal{S}$ into ``$V \times_U x$'' should be given by a pair (φ, g), where $\varphi : z \to x$, $g : p(z) \to V$ such that $p(\varphi) = f \circ g$. Pictorially:

$$\begin{tikzcd} z \arrow[r, dash] \arrow[d, squiggly, "p"] & ? \arrow[r] \arrow[d, squiggly, "p"] & x \arrow[d, squiggly, "p"] \\ p(Z) \arrow[r] & V \arrow[r, "f"] & U \end{tikzcd}$$

If such a morphism $V \times_U x \to x$ exists then it is called a strongly cartesian morphism.

Definition 4.30.1. Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a category over $\mathcal{C}$. A *strongly cartesian morphism*, or more precisely a *strongly $\mathcal{C}$-cartesian morphism* is a morphism $\varphi : y \to x$ of $\mathcal{S}$ such that for every $z \in Ob(\mathcal{S})$ the map

$$Mor_{\mathcal{S}}(z, y) \longrightarrow Mor_{\mathcal{S}}(z, x) \times_{Mor_{\mathcal{C}}(p(z), p(x))} Mor_{\mathcal{C}}(p(z), p(y)),$$

given by $\psi \longmapsto (\varphi \circ \psi, p(\psi))$ is bijective.

Note that by the Yoneda Lemma 4.3.5, given $x \in Ob(\mathcal{S})$ lying over $U \in Ob(\mathcal{C})$ and the morphism $f : V \to U$ of $\mathcal{C}$, if there is a strongly cartesian morphism $\varphi : y \to x$ with $p(\varphi) = f$, then (y, φ) is unique up to unique isomorphism. This is clear from the definition above, as the functor

$$z \longmapsto Mor_{\mathcal{S}}(z, x) \times_{Mor_{\mathcal{C}}(p(z),U)} Mor_{\mathcal{C}}(p(z), V)$$

only depends on the data $(x, U, f : V \to U)$. Hence we will sometimes use $V \times_U x \to x$ or $f^*x \to x$ to denote a strongly cartesian morphism which is a lift of f.

Lemma 4.30.2. *Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a category over $\mathcal{C}$.*

(1) *The composition of two strongly cartesian morphisms is strongly cartesian.*
(2) *Any isomorphism of $\mathcal{S}$ is strongly cartesian.*
(3) *Any strongly cartesian morphism φ such that $p(\varphi)$ is an isomorphism, is an isomorphism.*

Proof. Proof of (1). Let $\varphi : y \to x$ and $\psi : z \to y$ be strongly cartesian. Let t be an arbitrary object of $\mathcal{S}$. Then we have

$$\begin{aligned}
& Mor_{\mathcal{S}}(t, z) \\
&= Mor_{\mathcal{S}}(t, y) \times_{Mor_{\mathcal{C}}(p(t),p(y))} Mor_{\mathcal{C}}(p(t), p(z)) \\
&= Mor_{\mathcal{S}}(t, x) \times_{Mor_{\mathcal{C}}(p(t),p(x))} Mor_{\mathcal{C}}(p(t), p(y)) \times_{Mor_{\mathcal{C}}(p(t),p(y))} Mor_{\mathcal{C}}(p(t), p(z)) \\
&= Mor_{\mathcal{S}}(t, x) \times_{Mor_{\mathcal{C}}(p(t),p(x))} Mor_{\mathcal{C}}(p(t), p(z))
\end{aligned}$$

hence $x \to z$ is strongly cartesian.

Proof of (2). Let $y \to x$ be an isomorphism. Then $p(y) \to p(x)$ is an isomorphism too. Hence $Mor_{\mathcal{C}}(p(z), p(y)) \to Mor_{\mathcal{C}}(p(z), p(x))$ is a bijection. Hence $Mor_{\mathcal{S}}(z, x) \times_{Mor_{\mathcal{C}}(p(z),p(x))} Mor_{\mathcal{C}}(p(z), p(y))$ is just bijective to $Mor_{\mathcal{S}}(z, x)$. Hence the displayed map of Definition 4.30.1 is a bijection as $y \to x$ is an isomorphism, and we conclude that $x \to y$ is strongly cartesian.

Proof of (3). Assume $\varphi : y \to x$ is strongly cartesian with $p(\varphi) : p(y) \to p(x)$ an isomorphism. Applying the definition with $z = x$ shows that $(\mathrm{id}_x, p(\varphi)^{-1})$ comes from a unique morphism $\chi : x \to y$. We omit the verification that χ is the inverse of φ. □

Lemma 4.30.3. *Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a category over $\mathcal{C}$. Let $x \to y$ and $z \to y$ be morphisms of $\mathcal{S}$. Assume*

(1) *$x \to y$ is strongly cartesian,*
(2) *$p(x) \times_{p(y)} p(z)$ exists, and*
(3) *there exists a strongly cartesian morphism $a : w \to z$ in $\mathcal{S}$ with $p(w) = p(x) \times_{p(y)} p(z)$ and $p(a) = pr_2 : p(x) \times_{p(y)} p(z) \to p(z)$.*

Then the fibre product $x \times_y z$ exists and is isomorphic to w.

Proof. Since $x \to y$ is strongly cartesian there exists a unique morphism $b : w \to x$ such that $p(b) = \mathrm{pr}_1$. To see that w is the fibre product we compute

$$
\begin{aligned}
& Mor_{\mathcal{S}}(t, w) \\
&= Mor_{\mathcal{S}}(t, z) \times_{Mor_{\mathcal{C}}(p(t), p(z))} Mor_{\mathcal{C}}(p(t), p(w)) \\
&= Mor_{\mathcal{S}}(t, z) \times_{Mor_{\mathcal{C}}(p(t), p(z))} (Mor_{\mathcal{C}}(p(t), p(x)) \times_{Mor_{\mathcal{C}}(p(t), p(y))} Mor_{\mathcal{C}}(p(t), p(z))) \\
&= Mor_{\mathcal{S}}(t, z) \times_{Mor_{\mathcal{C}}(p(t), p(y))} Mor_{\mathcal{C}}(p(t), p(x)) \\
&= Mor_{\mathcal{S}}(t, z) \times_{Mor_{\mathcal{S}}(t, y)} Mor_{\mathcal{S}}(t, y) \times_{Mor_{\mathcal{C}}(p(t), p(y))} Mor_{\mathcal{C}}(p(t), p(x)) \\
&= Mor_{\mathcal{S}}(t, z) \times_{Mor_{\mathcal{S}}(t, y)} Mor_{\mathcal{S}}(t, x)
\end{aligned}
$$

as desired. The first equality holds because $a : w \to z$ is strongly cartesian and the last equality holds because $x \to y$ is strongly cartesian. $\square$

Definition 4.30.4. Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a category over $\mathcal{C}$. We say $\mathcal{S}$ is a *fibred category over* $\mathcal{C}$ if given any $x \in Ob(\mathcal{S})$ lying over $U \in Ob(\mathcal{C})$ and any morphism $f : V \to U$ of $\mathcal{C}$, there exists a strongly cartesian morphism $f^*x \to x$ lying over f.

Assume $p : \mathcal{S} \to \mathcal{C}$ is a fibred category. For every $f : V \to U$ and $x \in Ob(\mathcal{S}_U)$ as in the definition we may choose a strongly cartesian morphism $f^*x \to x$ lying over f. By the axiom of choice we may choose $f^*x \to x$ for all $f : V \to U = p(x)$ simultaneously. We claim that for every morphism $\phi : x \to x'$ in $\mathcal{S}_U$ and $f : V \to U$ there is a unique morphism $f^*\phi : f^*x \to f^*x'$ in $\mathcal{S}_V$ such that

$$
\begin{tikzcd}
f^*x \arrow[r, "f^*\phi"'] \arrow[d] & f^*x' \arrow[d] \\
x \arrow[r, "\phi"] & x'
\end{tikzcd}
$$

commutes. Namely, the arrow exists and is unique because $f^*x' \to x'$ is strongly cartesian. The uniqueness of this arrow guarantees that f^* (now also defined on morphisms) is a functor $f^* : \mathcal{S}_U \to \mathcal{S}_V$.

Definition 4.30.5. Assume $p : \mathcal{S} \to \mathcal{C}$ is a fibred category.

(1) A *choice of pullbacks*[3] for $p : \mathcal{S} \to \mathcal{C}$ is given by a choice of a strongly cartesian morphism $f^*x \to x$ lying over f for any morphism $f : V \to U$ of $\mathcal{C}$ and any $x \in Ob(\mathcal{S}_U)$.
(2) Given a choice of pullbacks, for any morphism $f : V \to U$ of $\mathcal{C}$ the functor $f^* : \mathcal{S}_U \to \mathcal{S}_V$ described above is called a *pullback functor* (associated to the choices $f^*x \to x$ made above).

Of course we may always assume our choice of pullbacks has the property that $\mathrm{id}_U^*x = x$, although in practice this is a useless property without imposing further assumptions on the pullbacks.

Lemma 4.30.6. *Assume $p : \mathcal{S} \to \mathcal{C}$ is a fibred category. Assume given a choice of pullbacks for $p : \mathcal{S} \to \mathcal{C}$.*

[3]This is probably nonstandard terminology. In some texts this is called a ``cleavage'' but it conjures up the wrong image. Maybe a ``cleaving'' would be a better word. A related notion is that of a ``splitting'', but in many texts a ``splitting'' means a choice of pullbacks such that $g^* f^* = (f \circ g)^*$ for any composable pair of morphisms. Compare also with Definition 4.33.2.

(1) *For any pair of composable morphisms $f : V \to U$, $g : W \to V$ there is a unique isomorphism*
$$\alpha_{g,f} : (f \circ g)^* \longrightarrow g^* \circ f^*$$
as functors $\mathcal{S}_U \to \mathcal{S}_W$ such that for every $y \in Ob(\mathcal{S}_U)$ the following diagram commutes
$$\begin{tikzcd} g^* f^* y \arrow[r] & f^* y \arrow[d] \\ (f \circ g)^* y \arrow[r] \arrow[u, "(\alpha_{g,f})_y"] & y \end{tikzcd}$$
(2) *If $f = id_U$, then there is a canonical isomorphism $\alpha_U : id \to (id_U)^*$ as functors $\mathcal{S}_U \to \mathcal{S}_U$.*
(3) *The quadruple $(U \mapsto \mathcal{S}_U, f \mapsto f^*, \alpha_{g,f}, \alpha_U)$ defines a pseudo functor from $\mathcal{C}^{opp}$ to the $(2,1)$-category of categories, see Definition 4.26.5.*

Proof. In fact, it is clear that the commutative diagram of part (1) uniquely determines the morphism $(\alpha_{g,f})_y$ in the fibre category $\mathcal{S}_W$. It is an isomorphism since both the morphism $(f \circ g)^* y \to y$ and the morphism and the composition $g^* f^* y \to f^* y \to Y$ are strongly cartesian morphisms lifting $f \circ g$ (see discussion following Definition 4.30.1 and Lemma 4.30.2). In the same way, since $\mathrm{id}_x : x \to x$ is clearly strongly cartesian over id_U (with $U = p(x)$) we see that there exists an isomorphism $(\alpha_U)_x : x \to (\mathrm{id}_U)^* x$. (Of course we could have assumed beforehand that $f^* x = x$ whenever f is an identity morphism, but it is better for the sake of generality not to assume this.) We omit the verification that $\alpha_{g,f}$ and α_U so obtained are transformations of functors. We also omit the verification of (3). □

Lemma 4.30.7. *Let $\mathcal{C}$ be a site. Let $\mathcal{S}_1$, $\mathcal{S}_2$ be categories over $\mathcal{C}$. Suppose that $\mathcal{S}_1$ and $\mathcal{S}_2$ are equivalent as categories over $\mathcal{C}$. Then $\mathcal{S}_1$ is fibred over $\mathcal{C}$ if and only if $\mathcal{S}_2$ is fibred over $\mathcal{C}$.*

Proof. Let $F : \mathcal{S}_1 \to \mathcal{S}_2$, $G : \mathcal{S}_2 \to \mathcal{S}_1$ be functors over $\mathcal{C}$, and let $i : F \circ G \to \mathrm{id}_{\mathcal{S}_2}$, $j : G \circ F \to \mathrm{id}_{\mathcal{S}_1}$ be isomorphisms of functors over $\mathcal{C}$. We claim that in this case f maps strongly cartesian morphisms to strongly cartesian morphisms. Namely, suppose that $\varphi : y \to x$ is strongly cartesian in $\mathcal{S}_1$. Set $f : V \to U$ equal to $p_1(\varphi)$. Suppose that $z' \in Ob(\mathcal{S}_2)$, with $W = p_2(z')$, and we are given $g : W \to V$ and $\psi' : z' \to F(x)$ such that $p(\psi') = f \circ g$. Then
$$\psi = j \circ G(\psi') : G(z') \to G(F(x)) \to x$$
is a morphism in $\mathcal{S}_1$ with $p(\psi) = f \circ g$. Hence by assumption there exists a unique morphism $\xi : G(z') \to y$ lying over g such that $\psi = \varphi \circ \xi$. This in turn gives a morphism
$$\xi' = F(\xi) \circ i^{-1} : z' \to F(G(z')) \to F(y)$$
lying over g with $\psi' = F(\varphi) \circ \xi'$. We omit the verification that ξ' is unique. □

The conclusion from Lemma 4.30.7 is that equivalences map strongly cartesian morphisms to strongly cartesian morphisms. But this may not be the case for an arbitrary functor between fibred categories over $\mathcal{C}$. Hence we define the 2-category of fibred categories as follows.

Definition 4.30.8. Let $\mathcal{C}$ be a category. The 2-*category of fibred categories over $\mathcal{C}$* is the sub 2-category of the 2-category of categories over $\mathcal{C}$ (see Definition 4.29.1) defined as follows:

(1) Its objects will be fibred categories $p : \mathcal{S} \to \mathcal{C}$.

(2) Its 1-morphisms $(\mathcal{S}, p) \to (\mathcal{S}', p')$ will be functors $G : \mathcal{S} \to \mathcal{S}'$ such that $p' \circ G = p$ and such that G maps strongly cartesian morphisms to strongly cartesian morphisms.
(3) Its 2-morphisms $t : G \to H$ for $G, H : (\mathcal{S}, p) \to (\mathcal{S}', p')$ will be morphisms of functors such that $p'(t_x) = \mathrm{id}_{p(x)}$ for all $x \in Ob(\mathcal{S})$.

In this situation we will denote

$$Mor_{Fib/\mathcal{C}}(\mathcal{S}, \mathcal{S}')$$

the category of 1-morphisms between $(\mathcal{S}, p)$ and $(\mathcal{S}', p')$

Note the condition on 1-morphisms. Note also that this is a true 2-category and not a $(2,1)$-category. Hence when taking 2-fibre products we first pass to the associated $(2,1)$-category.

Lemma 4.30.9. *Let $\mathcal{C}$ be a category. The $(2,1)$-category of fibred categories over $\mathcal{C}$ has 2-fibre products, and they are described as in Lemma 4.29.3.*

Proof. Basically what one has to show here is that given $f : \mathcal{X} \to \mathcal{S}$ and $g : \mathcal{Y} \to \mathcal{S}$ morphisms of fibred categories over $\mathcal{C}$, then the category $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ described in Lemma 4.29.3 is fibred. Let us show that $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ has plenty of strongly cartesian morphisms. Namely, suppose we have (U, x, y, ϕ) an object of $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$. And suppose $f : V \to U$ is a morphism in $\mathcal{C}$. Choose strongly cartesian morphisms $a : f^*x \to x$ in $\mathcal{X}$ lying over f and $b : f^*y \to y$ in $\mathcal{Y}$ lying over f. By assumption $F(a)$ and $G(b)$ are strongly cartesian. Since $\phi : F(x) \to G(y)$ is an isomorphism, by the uniqueness of strongly cartesian morphisms we find a unique isomorphism $f^*\phi : F(f^*x) \to G(f^*y)$ such that $G(b) \circ f^*\phi = \phi \circ G(a)$. In other words $(G(a), G(b)) : (V, f^*x, f^*y, f^*\phi) \to (U, x, y, \phi)$ is a morphism in $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$. We omit the verification that this is a strongly cartesian morphism (and that these are in fact the only strongly cartesian morphisms). $\square$

Lemma 4.30.10. *Let $\mathcal{C}$ be a category. Let $U \in Ob(\mathcal{C})$. If $p : \mathcal{S} \to \mathcal{C}$ is a fibred category and p factors through $p' : \mathcal{S} \to \mathcal{C}/U$ then $p' : \mathcal{S} \to \mathcal{C}/U$ is a fibred category.*

Proof. Suppose that $\varphi : x' \to x$ is strongly cartesian with respect to p. We claim that φ is strongly cartesian with respect to p' also. Set $g = p'(\varphi)$, so that $g : V'/U \to V/U$ for some morphisms $f : V \to U$ and $f' : V' \to U$. Let $z \in Ob(\mathcal{S})$. Set $p'(z) = (W \to U)$. To show that φ is strongly cartesian for p' we have to show

$$Mor_{\mathcal{S}}(z, x') \longrightarrow Mor_{\mathcal{S}}(z, x) \times_{Mor_{\mathcal{C}/U}(W/U, V/U)} Mor_{\mathcal{C}/U}(W/U, V'/U),$$

given by $\psi' \longmapsto (\varphi \circ \psi', p'(\psi'))$ is bijective. Suppose given an element (ψ, h) of the right hand side, then in particular $g \circ h = p(\psi)$, and by the condition that φ is strongly cartesian we get a unique morphism $\psi' : z \to x'$ with $\psi = \varphi \circ \psi'$ and $p(\psi') = h$. OK, and now $p'(\psi') : W/U \to V/U$ is a morphism whose corresponding map $W \to V$ is h, hence equal to h as a morphism in $\mathcal{C}/U$. Thus ψ' is a unique morphism $z \to x'$ which maps to the given pair (ψ, h). This proves the claim.

Finally, suppose given $g : V'/U \to V/U$ and x with $p'(x) = V/U$. Since $p : \mathcal{S} \to \mathcal{C}$ is a fibred category we see there exists a strongly cartesian morphism $\varphi : x' \to x$ with $p(\varphi) = g$. By the same argument as above it follows that $p'(\varphi) = g : V'/U \to V/U$. And as seen above the morphism φ is strongly cartesian. Thus the conditions of Definition 4.30.4 are satisfied and we win. $\square$

Lemma 4.30.11. *Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category. Let $x \to y$ and $z \to y$ be morphisms of $\mathcal{S}$ with $x \to y$ strongly cartesian. If $p(x) \times_{p(y)} p(z)$ exists, then $x \times_y z$ exists, $p(x \times_y z) = p(x) \times_{p(y)} p(z)$, and $x \times_y z \to z$ is strongly cartesian.*

Proof. Pick a strongly cartesian morphism $\text{pr}_2^*z \to z$ lying over $\text{pr}_2 : p(x) \times_{p(y)} p(z) \to p(z)$. Then $\text{pr}_2^*z = x \times_y z$ by Lemma 4.30.3. $\square$

4.31. Inertia

Given a fibred categories $p : \mathcal{S} \to \mathcal{C}$ and $p' : \mathcal{S}' \to \mathcal{C}$ over a category $\mathcal{C}$ and a 1-morphism $F : \mathcal{S} \to \mathcal{S}'$ we have the diagonal morphism

$$\Delta = \Delta_{\mathcal{S}/\mathcal{S}'} : \mathcal{S} \longrightarrow \mathcal{S} \times_{\mathcal{S}'} \mathcal{S}$$

in the $(2,1)$-category of fibred categories over $\mathcal{C}$.

Lemma 4.31.1. *Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ and $p' : \mathcal{S}' \to \mathcal{C}$ be fibred categories. Let $F : \mathcal{S} \to \mathcal{S}'$ be a 1-morphism of fibred categories over $\mathcal{C}$. Consider the category $\mathcal{I}_{\mathcal{S}/\mathcal{S}'}$ over $\mathcal{C}$ whose*

(1) *objects are pairs (x, α) where $x \in Ob(\mathcal{S})$ and $\alpha : x \to x$ is an automorphism with $F(\alpha) = id$,*
(2) *morphisms $(x, \alpha) \to (y, \beta)$ are given by morphisms $\phi : x \to y$ such that*

$$\begin{tikzcd} x \arrow[r, "\phi"] \arrow[d, "\alpha"'] & y \arrow[d, "\beta"] \\ x \arrow[r, "\phi"] & y \end{tikzcd}$$

commutes, and
(3) *the functor $\mathcal{I}_{\mathcal{S}/\mathcal{S}'} \to \mathcal{C}$ is given by $(x, \alpha) \mapsto p(x)$.*

Then

(1) *there is an equivalence*

$$\mathcal{I}_{\mathcal{S}/\mathcal{S}'} \longrightarrow \mathcal{S} \times_{\Delta, (\mathcal{S} \times_{\mathcal{S}'} \mathcal{S}), \Delta} \mathcal{S}$$

in the $(2,1)$-category of categories over $\mathcal{C}$, and
(2) *$\mathcal{I}_{\mathcal{S}/\mathcal{S}'}$ is a fibred category over $\mathcal{C}$.*

Proof. Note that (2) follows from (1) by Lemma 4.30.9. Thus it suffices to prove (1). We will use without further mention the construction of the 2-fibre product from Lemma 4.30.9. In particular an object of $\mathcal{S} \times_{\Delta, (\mathcal{S} \times_{\mathcal{S}'} \mathcal{S}), \Delta} \mathcal{S}$ is a triple $(x, y, (\iota, \kappa))$ where x and y are objects of $\mathcal{S}$, and $(\iota, \kappa) : (x, x, \text{id}_{F(x)}) \to (y, y, \text{id}_{F(y)})$ is an isomorphism in $\mathcal{S} \times_{\mathcal{S}'} \mathcal{S}$. This just means that $\iota, \kappa : x \to y$ are isomorphisms and that $F(\iota) = F(\kappa)$. Consider the functor

$$I_{\mathcal{S}/\mathcal{S}'} \longrightarrow \mathcal{S} \times_{\Delta, (\mathcal{S} \times_{\mathcal{S}'} \mathcal{S}), \Delta} \mathcal{S}$$

which to an object (x, α) of the left hand side assigns the object $(x, x, (\alpha, \text{id}_x))$ of the right hand side and to a morphism ϕ of the left hand side assigns the morphism (ϕ, ϕ) of the right hand side. We claim that a quasi-inverse to that morphism is given by the functor

$$\mathcal{S} \times_{\Delta, (\mathcal{S} \times_{\mathcal{S}'} \mathcal{S}), \Delta} \mathcal{S} \longrightarrow I_{\mathcal{S}/\mathcal{S}'}$$

which to an object $(x, y, (\iota, \kappa))$ of the left hand side assigns the object $(x, \kappa^{-1} \circ \iota)$ of the right hand side and to a morphism $(\phi, \phi') : (x, y, (\iota, \kappa)) \to (z, w, (\lambda, \mu))$ of the left hand side assigns the morphism ϕ. Indeed, the endo-functor of $I_{\mathcal{S}/\mathcal{S}'}$ induced by composing the two functors above is the identity on the nose, and the endo-functor induced on $\mathcal{S} \times_{\Delta, (\mathcal{S} \times_{\mathcal{S}'} \mathcal{S}), \Delta} \mathcal{S}$ is isomorphic to the identity via the natural isomorphism

$$(\iota^{-1} \circ \kappa, \kappa \circ \iota^{-1} \circ \kappa) : (x, x, (\kappa^{-1} \circ \iota, \text{id}_x)) \longrightarrow (x, y, (\iota, \kappa)).$$

Some details omitted. $\square$

Definition 4.31.2. Let $\mathcal{C}$ be a category.

(1) Let $F : \mathcal{S} \to \mathcal{S}'$ be a 1-morphism of fibred categories over $\mathcal{C}$. The *relative inertia of $\mathcal{S}$ over $\mathcal{S}'$* is the fibred category $\mathcal{I}_{\mathcal{S}/\mathcal{S}'} \to \mathcal{C}$ of Lemma 4.31.1.
(2) By the *inertia fibred category $\mathcal{I}_{\mathcal{S}}$ of $\mathcal{S}$* we mean $\mathcal{I}_{\mathcal{S}} = \mathcal{I}_{\mathcal{S}/\mathcal{C}}$.

Note that there are canonical 1-morphisms

$$\mathcal{I}_{\mathcal{S}/\mathcal{S}'} \longrightarrow \mathcal{S} \quad \text{and} \quad \mathcal{I}_{\mathcal{S}} \longrightarrow \mathcal{S} \tag{4.31.2.1}$$

of fibred categories over $\mathcal{C}$. In terms of the description of Lemma 4.31.1 these simply map the object (x, α) to the object x and the morphism $\phi : (x, \alpha) \to (y, \beta)$ to the morphism $\phi : x \to y$. There is also a *neutral section*

$$e : \mathcal{S} \to \mathcal{I}_{\mathcal{S}/\mathcal{S}'} \quad \text{and} \quad e : \mathcal{S} \to \mathcal{I}_{\mathcal{S}} \tag{4.31.2.2}$$

defined by the rules $x \mapsto (x, \mathrm{id}_x)$ and $(\phi : x \to y) \mapsto \phi$. This is a right inverse to (4.31.2.1). Given a 2-commutative square

$$\begin{tikzcd} \mathcal{S}_1 \arrow[r, "G"'] \arrow[d, "F_1"'] & \mathcal{S}_2 \arrow[d, "F_2"] \\ \mathcal{S}'_1 \arrow[r, "G'"] & \mathcal{S}'_2 \end{tikzcd}$$

there is a *functoriality map*

$$\mathcal{I}_{\mathcal{S}_1/\mathcal{S}'_1} \longrightarrow \mathcal{I}_{\mathcal{S}_2/\mathcal{S}'_2} \quad \text{and} \quad \mathcal{I}_{\mathcal{S}_1} \longrightarrow \mathcal{I}_{\mathcal{S}_2} \tag{4.31.2.3}$$

defined by the rules $(x, \alpha) \mapsto (G(x), G(\alpha))$ and $\phi \mapsto G(\phi)$. In particular there is always a comparison map

$$\mathcal{I}_{\mathcal{S}/\mathcal{S}'} \longrightarrow \mathcal{I}_{\mathcal{S}} \tag{4.31.2.4}$$

and all the maps above are compatible with this.

Lemma 4.31.3. *Let $F : \mathcal{S} \to \mathcal{S}'$ be a 1-morphism of categories fibred over a category $\mathcal{C}$. Then the diagram*

$$\begin{tikzcd} \mathcal{I}_{\mathcal{S}/\mathcal{S}'} \arrow[r, "(4.31.2.4)"'] \arrow[d, "F \circ (4.31.2.1)"'] & \mathcal{I}_{\mathcal{S}} \arrow[d, "(4.31.2.3)"] \\ \mathcal{S}' \arrow[r, "e"] & \mathcal{I}_{\mathcal{S}'} \end{tikzcd}$$

is a 2-fibre product.

Proof. Omitted. $\square$

4.32. Categories fibred in groupoids

In this section we explain how to think about categories in groupoids and we see how they are basically the same as functors with values in the (2, 1)-category of groupoids.

Definition 4.32.1. We say that $\mathcal{S}$ is *fibred in groupoids* over $\mathcal{C}$ if the following two conditions hold:

(1) For every morphism $f : V \to U$ in $\mathcal{C}$ and every lift x of U there is a lift $\phi : y \to x$ of f with target x.
(2) For every pair of morphisms $\phi : y \to x$ and $\psi : z \to x$ and any morphism $f : p(z) \to p(y)$ such that $p(\phi) \circ f = p(\psi)$ there exists a unique lift $\chi : z \to y$ of f such that $\phi \circ \chi = \psi$.

Condition (2) phrased differently says that applying the functor p gives a bijection between the sets of dotted arrows in the following commutative diagram below:

$$
\begin{tikzcd}
y \arrow[r] & x \\
z \arrow[u, dashed] \arrow[ur]
\end{tikzcd}
\qquad
\begin{tikzcd}
p(y) \arrow[r] & p(x) \\
p(z) \arrow[u, dashed] \arrow[ur]
\end{tikzcd}
$$

Another way to think about the second condition is the following. Suppose that $g : W \to V$ and $f : V \to U$ are morphisms in $\mathcal{C}$. Let $x \in Ob(\mathcal{S}_U)$. By the first condition we can lift f to $\phi : y \to x$ and then we can lift g to $\psi : z \to y$. Instead of doing this two step process we can directly lift $g \circ f$ to $\gamma : z' \to x$. This gives the solid arrows in the diagram

(4.32.1.1)
$$
\begin{tikzcd}
z' \arrow[d, dashed, shift left] \arrow[rrd, "\gamma"] & & \\
z \arrow[u, dashed, shift left] \arrow[r, "\psi"] \arrow[d, squiggly, "p"] & y \arrow[r, "\phi"] \arrow[d, squiggly, "p"] & x \arrow[d, squiggly, "p"] \\
W \arrow[r, "g"] & V \arrow[r, "f"] & U
\end{tikzcd}
$$

where the squiggly arrows represent not morphisms but the functor p. Applying the second condition to the arrows $\phi \circ \psi$, γ and id_W we conclude that there is a unique morphism $\chi : z \to z'$ in $\mathcal{S}_W$ such that $\gamma \circ \chi = \phi \circ \psi$. Similarly there is a unique morphism $z' \to z$. The uniqueness implies that the morphisms $z' \to z$ and $z \to z'$ are mutually inverse, in other words isomorphisms.

It should be clear from this discussion that a category fibred in groupoids is very closely related to a fibred category. Here is the result.

Lemma 4.32.2. *Let $p : \mathcal{S} \to \mathcal{C}$ be a functor. The following are equivalent*

(1) *$p : \mathcal{S} \to \mathcal{C}$ is a category fibred in groupoids, and*
(2) *all fibre categories are groupoids and $\mathcal{S}$ is a fibred category over $\mathcal{C}$.*

Moreover, in this case every morphism of $\mathcal{S}$ is strongly cartesian. In addition, given $f^*x \to x$ lying over f for all $f : V \to U = p(x)$ the data $(U \mapsto \mathcal{S}_U, f \mapsto f^*, \alpha_{f,g}, \alpha_U)$ constructed in Lemma 4.30.6 defines a pseudo functor from $\mathcal{C}^{opp}$ in to the $(2,1)$-category of groupoids.

Proof. Assume $p : \mathcal{S} \to \mathcal{C}$ is fibred in groupoids. To show all fibre categories $\mathcal{S}_U$ for $U \in Ob(\mathcal{C})$ are groupoids, we must exhibit for every $f : y \to x$ in $\mathcal{S}_U$ an inverse morphism. The diagram on the left (in $\mathcal{S}_U$) is mapped by p to the diagram on the right:

$$
\begin{tikzcd}
y \arrow[r, "f"] & x \\
x \arrow[u, dashed] \arrow[ur, "id_x"']
\end{tikzcd}
\qquad
\begin{tikzcd}
U \arrow[r, "id_U"] & U \\
U \arrow[u, dashed] \arrow[ur, "id_U"']
\end{tikzcd}
$$

Since only id_U makes the diagram on the right commute, there is a unique $g : x \to y$ making the diagram on the left commute, so $fg = id_x$. By a similar argument there is a unique $h : y \to x$ so that $gh = id_y$. Then $fgh = f : y \to x$. We have $fg = id_x$, so $h = f$. Condition (2) of Definition 4.32.1 says exactly that every morphism of $\mathcal{S}$ is strongly cartesian. Hence condition (1) of Definition 4.32.1 implies that $\mathcal{S}$ is a fibred category over $\mathcal{C}$.

Conversely, assume all fibre categories are groupoids and $\mathcal{S}$ is a fibred category over $\mathcal{C}$. We have to check conditions (1) and (2) of Definition 4.32.1. The first condition follows trivially. Let $\phi : y \to x$, $\psi : z \to x$ and $f : p(z) \to p(y)$ such that $p(\phi) \circ f = p(\psi)$ be as in condition (2) of Definition 4.32.1. Write $U = p(x)$, $V = p(y)$, $W = p(z)$, $p(\phi) = g : V \to U$, $p(\psi) = h : W \to U$. Choose a strongly cartesian $g^*x \to x$ lying over g. Then we get a morphism $i : y \to g^*x$ in $\mathcal{S}_V$, which is therefore an isomorphism. We also get a morphism $j : z \to g^*x$ corresponding to the pair (ψ, f) as $g^*x \to x$ is strongly cartesian. Then one checks that $\chi = i^{-1} \circ j$ is a solution.

We have seen in the proof of (1) $\Rightarrow$ (2) that every morphism of $\mathcal{S}$ is strongly cartesian. The final statement follows directly from Lemma 4.30.6. □

Lemma 4.32.3. *Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category. Let $\mathcal{S}'$ be the subcategory of $\mathcal{S}$ defined as follows*

(1) *$Ob(\mathcal{S}') = Ob(\mathcal{S})$, and*
(2) *for $x, y \in Ob(\mathcal{S}')$ the set of morphisms between x and y in $\mathcal{S}'$ is the set of of strongly cartesian morphisms between x and y in $\mathcal{S}$.*

Let $p' : \mathcal{S}' \to \mathcal{C}$ be the restriction of p to $\mathcal{S}'$. Then $p' : \mathcal{S}' \to \mathcal{C}$ is fibred in groupoids.

Proof. Note that the construction makes sense since by Lemma 4.30.2 the identity morphism of any object of $\mathcal{S}$ is strongly cartesian, and the composition of strongly cartesian morphisms is strongly cartesian. The first lifting property of Definition 4.32.1 follows from the condition that in a fibred category given any morphism $f : V \to U$ and x lying over U there exists a strongly cartesian morphism $\varphi : y \to x$ lying over f. Let us check the second lifting property of Definition 4.32.1 for the category $p' : \mathcal{S}' \to \mathcal{C}$ over $\mathcal{C}$. To do this we argue as in the discussion following Definition 4.32.1. Thus in Diagram 4.32.1.1 the morphisms ϕ, ψ and γ are strongly cartesian morphisms of $\mathcal{S}$. Hence γ and $\phi \circ \psi$ are strongly cartesian morphisms of $\mathcal{S}$ lying over the same arrow of $\mathcal{C}$ and having the same target in $\mathcal{S}$. By the discussion following Definition 4.30.1 this means these two arrows are isomorphic as desired (here we use also that any isomorphism in $\mathcal{S}$ is strongly cartesian, by Lemma 4.30.2 again). □

Example 4.32.4. A homomorphism of groups $p : G \to H$ gives rise to a functor $p : \mathcal{S} \to \mathcal{C}$ as in Example 4.2.12. This functor $p : \mathcal{S} \to \mathcal{C}$ is fibred in groupoids if and only if p is surjective. The fibre category $\mathcal{S}_U$ over the (unique) object $U \in Ob(\mathcal{C})$ is the category associated to the kernel of p as in Example 4.2.6.

Given $p : \mathcal{S} \to \mathcal{C}$, we can ask: if the fibre category $\mathcal{S}_U$ is a groupoid for all $U \in Ob(\mathcal{C})$, must $\mathcal{S}$ be fibred in groupoids over $\mathcal{C}$? We can see the answer is no as follows. Start with a category fibred in groupoids $p : \mathcal{S} \to \mathcal{C}$. Altering the morphisms in $\mathcal{S}$ which do not map to the identity morphism on some object does not alter the categories $\mathcal{S}_U$. Hence we can violate the existence and uniqueness conditions on lifts. One example is the functor from Example 4.32.4 when $G \to H$ is not surjective. Here is another example.

Example 4.32.5. Let $Ob(\mathcal{C}) = \{A, B, T\}$ and $Mor_{\mathcal{C}}(A, B) = \{f\}$, $Mor_{\mathcal{C}}(B, T) = \{g\}$, $Mor_{\mathcal{C}}(A, T) = \{h\} = \{gf\}$, plus the identity morphism for each object. See the diagram below for a picture of this category. Now let $Ob(\mathcal{S}) = \{A', B', T'\}$ and $Mor_{\mathcal{S}}(A', B') = \emptyset$, $Mor_{\mathcal{S}}(B', T') = \{g'\}$, $Mor_{\mathcal{S}}(A', T') = \{h'\}$, plus the identity morphisms. The functor $p : \mathcal{S} \to \mathcal{C}$ is obvious. Then for every $U \in Ob(\mathcal{C})$, $\mathcal{S}_U$ is the category with one object and the identity morphism on that object, so a groupoid, but the morphism $f : A \to B$ cannot be lifted. Similarly, if we declare $Mor_{\mathcal{S}}(A', B') = \{f'_1, f'_2\}$ and $Mor_{\mathcal{S}}(A', T') = \{h'\} =$

$\{g'f_1'\} = \{g'f_2'\}$, then the fibre categories are the same and $f : A \to B$ in the diagram below has two lifts.

$$\begin{tikzcd} B' \arrow[r, "g'"] & T' \\ A' \arrow[u, dashed, "??"] \arrow[ur, "h'"'] \end{tikzcd} \quad \text{above} \quad \begin{tikzcd} B \arrow[r, "g"] & T \\ A \arrow[u, "f"] \arrow[ur, "gf=h"'] \end{tikzcd}$$

Later we would like to make assertions such as ``any category fibred in groupoids over $\mathcal{C}$ is equivalent to a split one", or ``any category fibred in groupoids whose fibre categories are setlike is equivalent to a category fibred in sets". The notion of equivalence depends on the 2-category we are working with.

Definition 4.32.6. Let $\mathcal{C}$ be a category. The *2-category of categories fibred in groupoids over $\mathcal{C}$* is the sub 2-category of the 2-category of fibred categories over $\mathcal{C}$ (see Definition 4.30.8) defined as follows:

(1) Its objects will be categories $p : \mathcal{S} \to \mathcal{C}$ fibred in groupoids.
(2) Its 1-morphisms $(\mathcal{S}, p) \to (\mathcal{S}', p')$ will be functors $G : \mathcal{S} \to \mathcal{S}'$ such that $p' \circ G = p$ (since every morphism is strongly cartesian G automatically preserves them).
(3) Its 2-morphisms $t : G \to H$ for $G, H : (\mathcal{S}, p) \to (\mathcal{S}', p')$ will be morphisms of functors such that $p'(t_x) = \mathrm{id}_{p(x)}$ for all $x \in Ob(\mathcal{S})$.

Note that every 2-morphism is automatically an isomorphism! Hence this is actually a $(2, 1)$-category and not just a 2-category. Here is the obligatory lemma on 2-fibre products.

Lemma 4.32.7. *Let $\mathcal{C}$ be a category. The 2-category of categories fibred in groupoids over $\mathcal{C}$ has 2-fibre products, and they are described as in Lemma 4.29.3.*

Proof. By Lemma 4.30.9 the fibre product as described in Lemma 4.29.3 is a fibred category. Hence it suffices to prove that the fibre categories are groupoids, see Lemma 4.32.2. By Lemma 4.29.4 it is enough to show that the 2-fibre product of groupoids is a groupoid, which is clear (from the construction in Lemma 4.28.4 for example). $\square$

Lemma 4.32.8. *Let $p : \mathcal{S} \to \mathcal{C}$ and $p' : \mathcal{S}' \to \mathcal{C}$ be categories fibred in groupoids, and suppose that $G : \mathcal{S} \to \mathcal{S}'$ is a functor over $\mathcal{C}$.*

(1) *Then G is faithful (resp. fully faithful, resp. an equivalence) if and only if for each $U \in Ob(\mathcal{C})$ the induced functor $G_U : \mathcal{S}_U \to \mathcal{S}'_U$ is faithful (resp. fully faithful, resp. an equivalence).*
(2) *If G is an equivalence, then G is an equivalence in the 2-category of categories fibred in groupoids over $\mathcal{C}$.*

Proof. Let x, y be objects of $\mathcal{S}$ lying over the same object U. Consider the commutative diagram

$$\begin{tikzcd} Mor_{\mathcal{S}}(x, y) \arrow[rr, "G"'] \arrow[dr, "p"'] & & Mor_{\mathcal{S}'}(G(x), G(y)) \arrow[dl, "p'"] \\ & Mor_{\mathcal{C}}(U, U) & \end{tikzcd}$$

From this diagram it is clear that if G is faithful (resp. fully faithful) then so is each G_U.

Suppose G is an equivalence. For every object x' of $\mathcal{S}'$ there exists an object x of $\mathcal{S}$ such that $G(x)$ is isomorphic to x'. Suppose that x' lies over U' and x lies over U. Then there is an isomorphism $f : U' \to U$ in $\mathcal{C}$, namely, p' applied to the isomorphism $x' \to G(X)$. By the axioms of a category fibred in groupoids there exists an arrow $f^*x \to x$ of $\mathcal{S}$ lying over f. Hence there exists an isomorphism $\alpha : x' \to G(f^*x)$ such that $p'(\alpha) = \mathrm{id}_{U'}$ (this time by the axioms for $\mathcal{S}'$). All in all we conclude that for every object x' of $\mathcal{S}'$ we can choose a pair $(o_{x'}, \alpha_{x'})$ consisting of an object $o_{x'}$ of $\mathcal{S}$ and an isomorphism $\alpha_{x'} : x' \to G(o_{x'})$ with $p(\alpha_{x'}) = \mathrm{id}_{p'(x')}$. From this point on we proceed as usual (see proof of Lemma 4.2.19) to produce an inverse functor $F : \mathcal{S}' \to \mathcal{S}$, by taking $x' \mapsto o_{x'}$ and $\varphi' : x' \to y'$ to the unique arrow $\varphi_{\varphi'} : o_{x'} \to o_{y'}$ with $\alpha_{x'}^{-1} \circ G(\varphi_{\varphi'}) \circ \alpha_{y'} = \varphi'$. With these choices F is a functor over $\mathcal{C}$. We omit the verification that $G \circ F$ and $F \circ G$ are 2-isomorphic (in the 2-category of categories fibred in groupoids over $\mathcal{C}$).

Suppose that G_U is faithful (resp. fully faithful) for all $U \in Ob(\mathcal{C})$. To show that G is faithful (resp. fully faithful) we have to show for any objects $x, y \in Ob(\mathcal{S})$ that G induces an injection (resp. bijection) between $Mor_{\mathcal{S}}(x, y)$ and $Mor_{\mathcal{S}'}(G(x), G(y))$. Set $U = p(x)$ and $V = p(y)$. It suffices to prove that G induces an injection (resp. bijection) between morphism $x \to y$ lying over f to morphisms $G(x) \to G(y)$ lying over f for any morphism $f : U \to V$. Now fix $f : U \to V$. Denote $f^*y \to y$ a pullback. Then also $G(f^*y) \to G(y)$ is a pullback. The set of morphisms from x to y lying over f is bijective to the set of morphisms between x and f^*y lying over id_U. (By the second axiom of a category fibred in groupoids.) Similarly the set of morphisms from $G(x)$ to $G(y)$ lying over f is bijective to the set of morphisms between $G(x)$ and $G(f^*y)$ lying over id_U. Hence the fact that G_U is faithful (resp. fully faithful) gives the desired result.

Finally suppose for all G_U is an equivalence for all U, so it is fully faithful and essentially surjective. We have seen this implies G is fully faithful, and thus to prove it is an equivalence we have to prove that it is essentially surjective. This is clear, for if $z' \in Ob(\mathcal{S}')$ then $z' \in Ob(\mathcal{S}'_U)$ where $U = p'(z')$. Since G_U is essentially surjective we know that z' is isomorphic, in $\mathcal{S}'_U$, to an object of the form $G_U(z)$ for some $z \in Ob(\mathcal{S}_U)$. But morphisms in $\mathcal{S}'_U$ are morphisms in $\mathcal{S}'$ and hence z' is isomorphic to $G(z)$ in $\mathcal{S}'$. □

Lemma 4.32.9. *Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ and $p' : \mathcal{S}' \to \mathcal{C}$ be categories fibred in groupoids. Let $G : \mathcal{S} \to \mathcal{S}'$ be a functor over $\mathcal{C}$. Then G is fully faithful if and only if the diagonal*

$$\Delta_G : \mathcal{S} \longrightarrow \mathcal{S} \times_{G, \mathcal{S}', G} \mathcal{S}$$

is an equivalence.

Proof. By Lemma 4.32.8 it suffices to look at fibre categories over an object U of $\mathcal{C}$. An object of the right hand side is a triple (x, x', α) where $\alpha : G(x) \to G(x')$ is a morphism in $\mathcal{S}'_U$. The functor Δ_G maps the object x of $\mathcal{S}_U$ to the triple $(x, x, \mathrm{id}_{G(x)})$. Note that (x, x', α) is in the essential image of Δ_G if and only if $\alpha = G(\beta)$ for some morphism $\beta : x \to x'$ in $\mathcal{S}_U$ (details omitted). Hence in order for Δ_G to be an equivalence, every α has to be the image of a morphism $\beta : x \to x'$, and also every two distinct morphisms $\beta, \beta' : x \to x'$ have to given distinct morphisms $G(\beta), G(\beta')$. This proves one direction of the lemma. We omit the proof of the other direction. □

Lemma 4.32.10. *Let $\mathcal{C}$ be a category. Let $\mathcal{S}_i$, $i = 1, 2, 3, 4$ be categories fibred in groupoids over $\mathcal{C}$. Suppose that $\varphi : \mathcal{S}_1 \to \mathcal{S}_2$ and $\psi : \mathcal{S}_3 \to \mathcal{S}_4$ are equivalences over $\mathcal{C}$. Then*

$$Mor_{Cat/\mathcal{C}}(\mathcal{S}_2, \mathcal{S}_3) \longrightarrow Mor_{Cat/\mathcal{C}}(\mathcal{S}_1, \mathcal{S}_4), \quad \alpha \longmapsto \psi \circ \alpha \circ \varphi$$

is an equivalence of categories.

Proof. This is a generality and holds in any 2-category. □

Lemma 4.32.11. *Let $\mathcal{C}$ be a category. If $p : \mathcal{S} \to \mathcal{C}$ is fibred in groupoids, then so is the inertia fibred category $\mathcal{I}_\mathcal{S} \to \mathcal{C}$.*

Proof. Clear from the construction in Lemma 4.31.1 or by using (from the same lemma) that $I_\mathcal{S} \to \mathcal{S} \times_{\Delta, \mathcal{S} \times_\mathcal{C} \mathcal{S}, \Delta} \mathcal{S}$ is an equivalence and appealing to Lemma 4.32.7. $\square$

Lemma 4.32.12. *Let $\mathcal{C}$ be a category. Let $U \in Ob(\mathcal{C})$. If $p : \mathcal{S} \to \mathcal{C}$ is a category fibred in groupoids and p factors through $p' : \mathcal{S} \to \mathcal{C}/U$ then $p' : \mathcal{S} \to \mathcal{C}/U$ is fibred in groupoids.*

Proof. We have already seen in Lemma 4.30.10 that p' is a fibred category. Hence it suffices to prove the fibre categories are groupoids, see Lemma 4.32.2. For $V \in Ob(\mathcal{C})$ we have

$$\mathcal{S}_V = \coprod\nolimits_{f : V \to U} \mathcal{S}_{(f : V \to U)}$$

where the left hand side is the fibre category of p and the right hand side is the disjoint union of the fibre categories of p'. Hence the result. $\square$

Lemma 4.32.13. *Let $p : \mathcal{S} \to \mathcal{C}$ be a category fibred in groupoids. Let $x \to y$ and $z \to y$ be morphisms of $\mathcal{S}$. If $p(x) \times_{p(y)} p(z)$ exists, then $x \times_y z$ exists and $p(x \times_y z) = p(x) \times_{p(y)} p(z)$.*

Proof. Follows from Lemma 4.30.11. $\square$

Lemma 4.32.14. *Let $\mathcal{C}$ be a category. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $\mathcal{C}$. There exists a factorization $\mathcal{X} \to \mathcal{X}' \to \mathcal{Y}$ by 1-morphisms of categories fibred in groupoids over $\mathcal{C}$ such that $\mathcal{X} \to \mathcal{X}'$ is an equivalence over $\mathcal{C}$ and such that $\mathcal{X}'$ is a category fibred in groupoids over $\mathcal{Y}$.*

Proof. Denote $p : \mathcal{X} \to \mathcal{C}$ and $q : \mathcal{Y} \to \mathcal{C}$ the structure functors. We construct $\mathcal{X}'$ explicitly as follows. An object of $\mathcal{X}'$ is a quadruple (U, x, y, f) where $x \in Ob(\mathcal{X}_U)$, $y \in Ob(\mathcal{Y}_U)$ and $f : F(x) \to y$ is an isomorphism in $\mathcal{Y}_U$. A morphism $(a, b) : (U, x, y, f) \to (U', x', y', f')$ is given by $a : x \to x'$ and $b : y \to y'$ with $p(a) = q(b)$ and such that $f' \circ F(a) = b \circ f$. In other words $\mathcal{X}' = \mathcal{X} \times_{F, \mathcal{Y}, \mathrm{id}} \mathcal{Y}$ with the construction of the 2-fibre product from Lemma 4.29.3. By Lemma 4.32.7 we see that $\mathcal{X}'$ is a category fibred in groupoids over $\mathcal{C}$ and that $\mathcal{X}' \to \mathcal{Y}$ is a morphism of categories over $\mathcal{C}$. As functor $\mathcal{X} \to \mathcal{X}'$ we take $x \mapsto (p(x), x, F(x), \mathrm{id}_{F(x)})$ on objects and $(a : x \to x') \mapsto (a, F(a))$ on morphisms. It is clear that the composition $\mathcal{X} \to \mathcal{X}' \to \mathcal{Y}$ equals F. We omit the verification that $\mathcal{X} \to \mathcal{X}'$ is an equivalence of fibred categories over $\mathcal{C}$.

Finally, we have to show that $\mathcal{X}' \to \mathcal{Y}$ is a category fibred in groupoids. Let $b : y' \to y$ be a morphism in $\mathcal{Y}$ and let (U, x, y, f) be an object of $\mathcal{X}'$ lying over y. Because $\mathcal{X}$ is fibred in groupoids over $\mathcal{C}$ we can find a morphism $a : x' \to x$ lying over $U' = q(y') \to q(y) = U$. Since $\mathcal{Y}$ is fibred in groupoids over $\mathcal{C}$ and since both $F(x') \to F(x)$ and $y' \to y$ lie over the same morphism $U' \to U$ we can find $f' : F(x') \to y'$ lying over $\mathrm{id}_{U'}$ such that $f \circ F(a) = b \circ f'$. Hence we obtain $(a, b) : (U', x', y', f') \to (U, x, y, f)$. This verifies the first condition (1) of Definition 4.32.1. To see (2) let $(a, b) : (U', x', y', f') \to (U, x, y, f)$ and $(a', b') : (U'', x'', y'', f'') \to (U, x, y, f)$ be morphisms of $\mathcal{X}'$ and let $b'' : y' \to y''$ be a morphism of $\mathcal{Y}$ such that $b' \circ b'' = b$. We have to show that there exists a unique morphism $a'' : x' \to x''$ such that $f'' \circ F(a'') = b'' \circ f'$ and such that $(a', b') \circ (a'', b'') = (a, b)$. Because $\mathcal{X}$ is fibred in groupoids we know there exists a unique morphism $a'' : x' \to x''$ such that $a' \circ a'' = a$ and $p(a'') = q(b'')$. Because $\mathcal{Y}$ is fibred in groupoids we see that $F(a'')$ is the unique morphism $F(x') \to F(x'')$ such that $F(a') \circ F(a'') = F(a)$ and $q(F(a'')) = q(b'')$. The relation $f'' \circ F(a'') = b'' \circ f'$ follows from this and the given relations $f \circ F(a) = b \circ f'$ and $f \circ F(a') = b' \circ f''$. $\square$

Lemma 4.32.15. *Let $\mathcal{C}$ be a category. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $\mathcal{C}$. Assume we have a 2-commutative diagram*

$$\begin{tikzcd} \mathcal{X}' \arrow[dr, "f"'] & \mathcal{X} \arrow[l, "a"'] \arrow[r, "b"] \arrow[d, "F"] & \mathcal{X}'' \arrow[dl, "g"] \\ & \mathcal{Y} & \end{tikzcd}$$

where a and b are equivalences of categories over $\mathcal{C}$ and f and g are categories fibred in groupoids. Then there exists an equivalence $h : \mathcal{X}'' \to \mathcal{X}'$ of categories over $\mathcal{Y}$ such that $h \circ b$ is 2-isomorphic to a as 1-morphisms of categories over $\mathcal{C}$. If the diagram above actually commutes, then we can arrange it so that $h \circ b$ is 2-isomorphic to a as 1-morphisms of categories over $\mathcal{Y}$.

Proof. We will show that both $\mathcal{X}'$ and $\mathcal{X}''$ over $\mathcal{Y}$ are equivalent to the category fibred in groupoids $\mathcal{X} \times_{F,\mathcal{Y},\mathrm{id}} \mathcal{Y}$ over $\mathcal{Y}$, see proof of Lemma 4.32.14. Choose a quasi-inverse $b^{-1} : \mathcal{X}'' \to \mathcal{X}$ in the 2-category of categories over $\mathcal{C}$. Since the right triangle of the diagram is 2-commutative we see that

$$\begin{tikzcd} \mathcal{X} \arrow[d, "F"'] & \mathcal{X}'' \arrow[l, "b^{-1}"'] \arrow[d, "g"] \\ \mathcal{Y} & \mathcal{Y} \arrow[l] \end{tikzcd}$$

is 2-commutative. Hence we obtain a 1-morphism $c : \mathcal{X}'' \to \mathcal{X} \times_{F,\mathcal{Y},\mathrm{id}} \mathcal{Y}$ by the universal property of the 2-fibre product. Moreover c is a morphism of categories over $\mathcal{Y}$ (!) and an equivalence (by the assumption that b is an equivalence, see Lemma 4.28.7). Hence c is an equivalence in the 2-category of categories fibred in groupoids over $\mathcal{Y}$ by Lemma 4.32.8.

We still have to construct a 2-isomorphism between $c \circ b$ and the functor $d : \mathcal{X} \to \mathcal{X} \times_{F,\mathcal{Y},\mathrm{id}} \mathcal{Y}$, $x \mapsto (p(x), x, F(x), \mathrm{id}_{F(x)})$ constructed in the proof of Lemma 4.32.14. Let $\alpha : F \to g \circ b$ and $\beta : b^{-1} \circ b \to \mathrm{id}$ be 2-isomorphisms between 1-morphisms of categories over $\mathcal{C}$. Note that $c \circ b$ is given by the rule

$$x \mapsto (p(x), b^{-1}(b(x)), g(b(x)), \alpha_x \circ F(\beta_x))$$

on objects. Then we see that

$$(\beta_x, \alpha_x) : (p(x), x, F(x), \mathrm{id}_{F(x)}) \longrightarrow (p(x), b^{-1}(b(x)), g(b(x)), \alpha_x \circ F(\beta_x))$$

is a functorial isomorphism which gives our 2-morphism $d \to b \circ c$. Finally, if the diagram commutes then α_x is the identity for all x and we see that this 2-morphism is a 2-morphism in the 2-category of categories over $\mathcal{Y}$. $\square$

4.33. Presheaves of categories

In this section we compare the notion of fibred categories with the closely related notion of a ``presheaf of categories''. The basic construction is explained in the following example.

Example 4.33.1. Let $\mathcal{C}$ be a category. Suppose that $F : \mathcal{C}^{opp} \to \textit{Cat}$ is a functor to the 2-category of categories, see Definition 4.26.5. For $f : V \to U$ in $\mathcal{C}$ we will suggestively write $F(f) = f^*$ for the functor from $F(U)$ to $F(V)$. From this we can construct a fibred category $\mathcal{S}_F$ over $\mathcal{C}$ as follows. Define

$$Ob(\mathcal{S}_F) = \{(U, x) \mid U \in Ob(\mathcal{C}), x \in Ob(F(U))\}.$$

For $(U, x), (V, y) \in Ob(\mathcal{S}_F)$ we define

$$\begin{aligned} Mor_{\mathcal{S}_F}((V, y), (U, x)) &= \{(f, \phi) \mid f \in Mor_{\mathcal{C}}(V, U), \phi \in Mor_{F(V)}(y, f^*x)\} \\ &= \coprod\nolimits_{f \in Mor_{\mathcal{C}}(V,U)} Mor_{F(V)}(y, f^*x) \end{aligned}$$

In order to define composition we use that $g^* \circ f^* = (f \circ g)^*$ for a pair of composable morphisms of $\mathcal{C}$ (by definition of a functor into a 2-category). Namely, we define the composition of $\psi : z \to g^*y$ and $\phi : y \to f^*x$ to be $g^*(\phi) \circ \psi$. The functor $p_F : \mathcal{S}_F \to \mathcal{C}$ is given by the rule $(U, x) \mapsto U$. Let us check that this is indeed a fibred category. Given $f : V \to U$ in $\mathcal{C}$ and (U, x) a lift of U, then we claim $(f, id_{f^*x}) : (V, f^*x) \to (U, x)$ is a strongly cartesian lift of f. We have to show a h in the diagram on the left determines (h, ν) on the right:

$$\begin{tikzcd} V \arrow[r, "f"] & U \\ W \arrow[u, dashed, "h"] \arrow[ur, "g"'] & \end{tikzcd} \qquad \begin{tikzcd} (V, f^*x) \arrow[r, "{(f,\mathrm{id}_{f^*x})}"] & (U, x) \\ (W, z) \arrow[u, dashed, "{(h,\nu)}"] \arrow[ur, "{(g,\psi)}"'] & \end{tikzcd}$$

Just take $\nu = \psi$ which works because $f \circ h = g$ and hence $g^*x = h^*f^*x$. Moreover, this is the only lift making the diagram (on the right) commute.

Definition 4.33.2. Let $\mathcal{C}$ be a category. Suppose that $F : \mathcal{C}^{opp} \to Cat$ is a functor to the 2-category of categories. We will write $p_F : \mathcal{S}_F \to \mathcal{C}$ for the fibred category constructed in Example 4.33.1. A *split fibred category* is a fibred category isomorphic (!) over $\mathcal{C}$ to one of these categories $\mathcal{S}_F$.

Lemma 4.33.3. *Let $\mathcal{C}$ be a category. Let $\mathcal{S}$ be a fibred category over $\mathcal{C}$. Then $\mathcal{S}$ is split if and only if for some choice of pullbacks (see Definition 4.30.5) the pullback functors $(f \circ g)^*$ and $g^* \circ f^*$ are equal.*

Proof. This is immediate from the definitions. □

Lemma 4.33.4. *Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category. There exists a functor $F : \mathcal{C} \to Cat$ such that $\mathcal{S}$ is equivalent to $\mathcal{S}_F$ in the 2-category of fibred categories over $\mathcal{C}$. In other words, every fibred category is equivalent to a split one.*

Proof. Let us make a choice of pullbacks (see Definition 4.30.5). By Lemma 4.30.6 we get pullback functors f^* for every morphism f of $\mathcal{C}$.

We construct a new category $\mathcal{S}'$ as follows. The objects of $\mathcal{S}'$ are pairs (x, f) consisting of a morphism $f : V \to U$ of $\mathcal{C}$ and an object x of $\mathcal{S}$ over U, i.e., $x \in Ob(\mathcal{S}_U)$. The functor $p' : \mathcal{S}' \to \mathcal{C}$ will map the pair (x, f) to the source of the morphism f, in other words $p'(x, f : V \to U) = V$. A morphism $\varphi : (x_1, f_1 : V_1 \to U_1) \to (x_2, f_2 : V_2 \to U_2)$ is given by a pair (φ, g) consisting of a morphism $g : V_1 \to V_2$ and a morphism $\varphi : f_1^*x_1 \to f_2^*x_2$ with $p(\varphi) = g$. It is no problem to define the composition law: $(\varphi, g) \circ (\psi, h) = (\varphi \circ \psi, g \circ h)$ for any pair of composable morphisms. There is a natural functor $\mathcal{S} \to \mathcal{S}'$ which simply maps x over U to the pair (x, id_x).

At this point we need to check that p' makes $\mathcal{S}'$ into a fibred category over $\mathcal{C}$, and we need to check that $\mathcal{S} \to \mathcal{S}'$ is an equivalence of categories over $\mathcal{C}$ which maps strongly cartesian morphisms to strongly cartesian morphisms. We omit the verifications.

Finally, we can define pullback functors on $\mathcal{S}'$ by setting $g^*(x, f) = (x, f \circ g)$ on objects if $g : V' \to V$ and $f : V \to U$. On morphisms $(\varphi, \mathrm{id}_V) : (x_1, f_1) \to (x_2, f_2)$ between morphisms in $\mathcal{S}'_V$ we set $g^*(\varphi, \text{id}_V) = (g^*\varphi, \text{id}_{V'})$ where we use the unique identifications $g^* f_i^* x_i = (f_i \circ g)^* x_i$ from Lemma 4.30.6 to think of $g^*\varphi$ as a morphism from $(f_1 \circ g)^* x_1$ to $(f_2 \circ g)^* x_2$. Clearly, these pullback functors g^* have the property that $g_1^* \circ g_2^* = (g_2 \circ g_1)^*$, in other words $\mathcal{S}'$ is split as desired. □

4.34. Presheaves of groupoids

In this section we compare the notion of categories fibred in groupoids with the closely related notion of a ``presheaf of groupoids''. The basic construction is explained in the following example.

Example 4.34.1. This example is the analogue of Example 4.33.1, for ``presheaves of groupoids'' instead of ``presheaves of categories''. The output will be a category fibred in groupoids instead of a fibred category. Suppose that $F : \mathcal{C}^{opp} \to \text{Groupoids}$ is a functor to the category of groupoids, see Definition 4.26.5. For $f : V \to U$ in $\mathcal{C}$ we will suggestively write $F(f) = f^*$ for the functor from $F(U)$ to $F(V)$. We construct a category $\mathcal{S}_F$ fibred in groupoids over $\mathcal{C}$ as follows. Define

$$Ob(\mathcal{S}_F) = \{(U, x) \mid U \in Ob(\mathcal{C}), x \in Ob(F(U))\}.$$

For $(U, x), (V, y) \in Ob(\mathcal{S}_F)$ we define

$$\begin{aligned} Mor_{\mathcal{S}_F}((V, y), (U, x)) &= \{(f, \phi) \mid f \in Mor_{\mathcal{C}}(V, U), \phi \in Mor_{F(V)}(y, f^* x)\} \\ &= \coprod\nolimits_{f \in Mor_{\mathcal{C}}(V,U)} Mor_{F(V)}(y, f^* x) \end{aligned}$$

In order to define composition we use that $g^* \circ f^* = (f \circ g)^*$ for a pair of composable morphisms of $\mathcal{C}$ (by definition of a functor into a 2-category). Namely, we define the composition of $\psi : z \to g^* y$ and $\phi : y \to f^* x$ to be $g^*(\phi) \circ \psi$. The functor $p_F : \mathcal{S}_F \to \mathcal{C}$ is given by the rule $(U, x) \mapsto U$. The condition that $F(U)$ is a groupoid for every U guarantees that $\mathcal{S}_F$ is fibred in groupoids over $\mathcal{C}$, as we have already seen in Example 4.33.1 that $\mathcal{S}_F$ is a fibred category, see Lemma 4.32.2. But we can also prove conditions (1), (2) of Definition 4.32.1 directly as follows: (1) Lifts of morphisms exist since given $f : V \to U$ in $\mathcal{C}$ and (U, x) an object of $\mathcal{S}_F$ over U, then $(f, id_{f^* x}) : (V, f^* x) \to (U, x)$ is a lift of f. (2) Suppose given solid diagrams as follows

$$\begin{tikzcd} V \arrow[r, "f"] & U \\ W \arrow[u, dashed, "h"] \arrow[ru, "g"'] & \end{tikzcd} \qquad \begin{tikzcd} (V, y) \arrow[r, "{(f,\phi)}"] & (U, x) \\ (W, z) \arrow[u, dashed, "{(h,\nu)}"] \arrow[ru, "{(g,\psi)}"'] & \end{tikzcd}$$

Then for the dotted arrows we have $\nu = (h^* \phi)^{-1} \circ \psi$ so given h there exists a ν which is unique by uniqueness of inverses.

Definition 4.34.2. Let $\mathcal{C}$ be a category. Suppose that $F : \mathcal{C}^{opp} \to \textit{Groupoids}$ is a functor to the 2-category of groupoids. We will write $p_F : \mathcal{S}_F \to \mathcal{C}$ for the category fibred in groupoids constructed in Example 4.34.1. A *split category fibred in groupoids* is a category fibred in groupoids isomorphic (!) over $\mathcal{C}$ to one of these categories $\mathcal{S}_F$.

Lemma 4.34.3. *Let $p : \mathcal{S} \to \mathcal{C}$ be a category fibred in groupoids. There exists a functor $F : \mathcal{C} \to \textit{Groupoids}$ such that $\mathcal{S}$ is equivalent to $\mathcal{S}_F$ over $\mathcal{C}$. In other words, every category fibred in groupoids is equivalent to a split one.*

Proof. Make a choice of pullbacks (see Definition 4.30.5). By Lemmas 4.30.6 and 4.32.2 we get pullback functors f^* for every morphism f of $\mathcal{C}$.

We construct a new category $\mathcal{S}'$ as follows. The objects of $\mathcal{S}'$ are pairs (x, f) consisting of a morphism $f : V \to U$ of $\mathcal{C}$ and an object x of $\mathcal{S}$ over U, i.e., $x \in Ob(\mathcal{S}_U)$. The functor $p' : \mathcal{S}' \to \mathcal{C}$ will map the pair (x, f) to the source of the morphism f, in other words $p'(x, f : V \to U) = V$. A morphism $\varphi : (x_1, f_1 : V_1 \to U_1) \to (x_2, f_2 : V_2 \to U_2)$ is given by a pair (φ, g) consisting of a morphism $g : V_1 \to V_2$ and a morphism $\varphi : f_1^* x_1 \to f_2^* x_2$ with $p(\varphi) = g$. It is no problem to define the composition law: $(\varphi, g) \circ (\psi, h) = (\varphi \circ \psi, g \circ h)$ for any pair of composable morphisms. There is a natural functor $\mathcal{S} \to \mathcal{S}'$ which simply maps x over U to the pair (x, id_x).

At this point we need to check that p' makes $\mathcal{S}'$ into a category fibred in groupoids over $\mathcal{C}$, and we need to check that $\mathcal{S} \to \mathcal{S}'$ is an equivalence of categories over $\mathcal{C}$. We omit the verifications.

Finally, we can define pullback functors on $\mathcal{S}'$ by setting $g^*(x, f) = (x, f \circ g)$ on objects if $g : V' \to V$ and $f : V \to U$. On morphisms $(\varphi, \mathrm{id}_V) : (x_1, f_1) \to (x_2, f_2)$ between morphisms in $\mathcal{S}'_V$ we set $g^*(\varphi, \mathrm{id}_V) = (g^*\varphi, \mathrm{id}_{V'})$ where we use the unique identifications $g^* f_i^* x_i = (f_i \circ g)^* x_i$ from Lemma 4.32.2 to think of $g^*\varphi$ as a morphism from $(f_1 \circ g)^* x_1$ to $(f_2 \circ g)^* x_2$. Clearly, these pullback functors g^* have the property that $g_1^* \circ g_2^* = (g_2 \circ g_1)^*$, in other words $\mathcal{S}'$ is split as desired. $\square$

We will see an alternative proof of this lemma in Section 4.38.

4.35. Categories fibred in sets

Definition 4.35.1. A category is called *discrete* if the only morphisms are the identity morphisms.

A discrete category has only one interesting piece of information: its set of objects. Thus we sometime confuse discrete categories with sets.

Definition 4.35.2. Let $\mathcal{C}$ be a category. A *category fibred in sets*, or a *category fibred in discrete categories* is a category fibred in groupoids all of whose fibre categories are discrete.

We want to clarify the relationship between categories fibred in sets and presheaves (see Definition 4.3.3). To do this it makes sense to first make the following definition.

Definition 4.35.3. Let $\mathcal{C}$ be a category. The 2-*category of categories fibred in sets over* $\mathcal{C}$ is the sub 2-category of the category of categories fibred in groupoids over $\mathcal{C}$ (see Definition 4.32.6) defined as follows:

(1) Its objects will be categories $p : \mathcal{S} \to \mathcal{C}$ fibred in sets.
(2) Its 1-morphisms $(\mathcal{S}, p) \to (\mathcal{S}', p')$ will be functors $G : \mathcal{S} \to \mathcal{S}'$ such that $p' \circ G = p$ (since every morphism is strongly cartesian G automatically preserves them).
(3) Its 2-morphisms $t : G \to H$ for $G, H : (\mathcal{S}, p) \to (\mathcal{S}', p')$ will be morphisms of functors such that $p'(t_x) = \mathrm{id}_{p(x)}$ for all $x \in Ob(\mathcal{S})$.

Note that every 2-morphism is automatically an isomorphism. Hence this 2-category is actually a $(2, 1)$-category. Here is the obligatory lemma on the existence of 2-fibre products.

Lemma 4.35.4. *Let $\mathcal{C}$ be a category. The 2-category of categories fibred in sets over $\mathcal{C}$ has 2-fibre products. More precisely, the 2-fibre product described in Lemma 4.29.3 returns a category fibred in sets if one starts out with such.*

Proof. Omitted. □

Example 4.35.5. This example is the analogue of Examples 4.33.1 and 4.34.1 for presheaves instead of ``presheaves of categories". The output will be a category fibred in sets instead of a fibred category. Suppose that $F : \mathcal{C}^{opp} \to Sets$ is a presheaf. For $f : V \to U$ in $\mathcal{C}$ we will suggestively write $F(f) = f^* : F(U) \to F(V)$. We construct a category $\mathcal{S}_F$ fibred in sets over $\mathcal{C}$ as follows. Define

$$Ob(\mathcal{S}_F) = \{(U, x) \mid U \in Ob(\mathcal{C}), x \in Ob(F(U))\}.$$

For $(U, x), (V, y) \in Ob(\mathcal{S}_F)$ we define

$$Mor_{\mathcal{S}_F}((V, y), (U, x)) = \{f \in Mor_{\mathcal{C}}(V, U) \mid f^*x = y\}$$

Composition is inherited from composition in $\mathcal{C}$ which works as $g^* \circ f^* = (f \circ g)^*$ for a pair of composable morphisms of $\mathcal{C}$. The functor $p_F : \mathcal{S}_F \to \mathcal{C}$ is given by the rule $(U, x) \mapsto U$. As every fibre category $\mathcal{S}_{F,U}$ is discrete with underlying set $F(U)$ and we have already see in Example 4.34.1 that $\mathcal{S}_F$ is a category fibred in groupoids, we conclude that $\mathcal{S}_F$ is fibred in sets.

Lemma 4.35.6. *Let $\mathcal{C}$ be a category. The only 2-morphisms between categories fibred in sets are identities. In other words, the 2-category of categories fibred in sets is a category. Moreover, there is an equivalence of categories*

$$\left\{ \begin{array}{c} \text{the category of presheaves} \\ \text{of sets over } \mathcal{C} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{the category of categories} \\ \text{fi}\mathcal{C} \end{array} \right\}$$

The functor from left to right is the construction $F \to \mathcal{S}_F$ discussed in Example 4.35.5. The functor from right to left assigns to $p : \mathcal{S} \to \mathcal{C}$ the presheaf of objects $U \mapsto Ob(\mathcal{S}_U)$.

Proof. The first assertion is clear, as the only morphisms in the fibre categories are identities.

Suppose that $p : \mathcal{S} \to \mathcal{C}$ is fibred in sets. Let $f : V \to U$ be a morphism in $\mathcal{C}$ and let $x \in Ob(\mathcal{S}_U)$. Then there is exactly one choice for the object f^*x. Thus we see that $(f \circ g)^*x = g^*(f^*x)$ for f, g as in Lemma 4.32.2. It follows that we may think of the assignments $U \mapsto Ob(\mathcal{S}_U)$ and $f \mapsto f^*$ as a presheaf on $\mathcal{C}$. □

Here is an important example of a category fibred in sets.

Example 4.35.7. Let $\mathcal{C}$ be a category. Let $X \in Ob(\mathcal{C})$. Consider the representable presheaf $h_X = Mor_{\mathcal{C}}(-, X)$ (see Example 4.3.4). On the other hand, consider the category $p : \mathcal{C}/X \to \mathcal{C}$ from Example 4.2.13. The fibre category $(\mathcal{C}/X)_U$ has as objects morphisms $h : U \to X$, and only identities as morphisms. Hence we see that under the correspondence of Lemma 4.35.6 we have

$$h_X \longleftrightarrow \mathcal{C}/X.$$

In other words, the category $\mathcal{C}/X$ is canonically equivalent to the category $\mathcal{S}_{h_X}$ associated to h_X in Example 4.35.5.

For this reason it is tempting to define a ``representable" object in the 2-category of categories fibred in groupoids to be a category fibred in sets whose associated presheaf is representable. However, this is would not be a good definition for use since we prefer to have a notion which is invariant under equivalences. To make this precise we study exactly which categories fibred in groupoids are equivalent to categories fibred in sets.

4.36. Categories fibred in setoids

Definition 4.36.1. Let us call a category a *setoid*[4] if it is a groupoid where every object has exactly one automorphism: the identity.

If C is a set with an equivalence relation $\sim$, then we can make a setoid $\mathcal{C}$ as follows: $Ob(\mathcal{C}) = C$ and $Mor_{\mathcal{C}}(x, y) = \emptyset$ unless $x \sim y$ in which case we set $Mor_{\mathcal{C}}(x, y) = \{1\}$. Transitivity of $\sim$ means that we can compose morphisms. Conversely any setoid category defines an equivalence relation on its objects (isomorphism) such that you recover the category (up to unique isomorphism -- not equivalence) from the procedure just described.

Discrete categories are setoids. For any setoid $\mathcal{C}$ there is a canonical procedure to make a discrete category equivalent to it, namely one replaces $Ob(\mathcal{C})$ by the set of isomorphism classes (and adds identity morphisms). In terms of sets endowed with an equivalence relation this corresponds to taking the quotient by the equivalence relation.

Definition 4.36.2. Let $\mathcal{C}$ be a category. A *category fibred in setoids* is a category fibred in groupoids all of whose fibre categories are setoids.

Below we will clarify the relationship between categories fibred in setoids and categories fibred in sets.

Definition 4.36.3. Let $\mathcal{C}$ be a category. The 2-*category of categories fibred in setoids over* $\mathcal{C}$ is the sub 2-category of the category of categories fibred in groupoids over $\mathcal{C}$ (see Definition 4.32.6) defined as follows:

(1) Its objects will be categories $p : \mathcal{S} \to \mathcal{C}$ fibred in setoids.
(2) Its 1-morphisms $(\mathcal{S}, p) \to (\mathcal{S}', p')$ will be functors $G : \mathcal{S} \to \mathcal{S}'$ such that $p' \circ G = p$ (since every morphism is strongly cartesian G automatically preserves them).
(3) Its 2-morphisms $t : G \to H$ for $G, H : (\mathcal{S}, p) \to (\mathcal{S}', p')$ will be morphisms of functors such that $p'(t_x) = \mathrm{id}_{p(x)}$ for all $x \in Ob(\mathcal{S})$.

Note that every 2-morphism is automatically an isomorphism. Hence this 2-category is actually a $(2, 1)$-category.
Here is the obligatory lemma on the existence of 2-fibre products.

Lemma 4.36.4. *Let $\mathcal{C}$ be a category. The 2-category of categories fibred in setoids over $\mathcal{C}$ has 2-fibre products. More precisely, the 2-fibre product described in Lemma 4.29.3 returns a category fibred in setoids if one starts out with such.*

Proof. Omitted. $\square$

Lemma 4.36.5. *Let $\mathcal{C}$ be a category. Let $\mathcal{S}$ be a category over $\mathcal{C}$.*

(1) *If $\mathcal{S} \to \mathcal{S}'$ is an equivalence over $\mathcal{C}$ with $\mathcal{S}'$ fibred in sets over $\mathcal{C}$, then*
    (a) *$\mathcal{S}$ is fibred in setoids over $\mathcal{C}$, and*
    (b) *for each $U \in Ob(\mathcal{C})$ the map $Ob(\mathcal{S}_U) \to Ob(\mathcal{S}'_U)$ identifies the target as the set of isomorphism classes of the source.*
(2) *If $p : \mathcal{S} \to \mathcal{C}$ is a category fibred in setoids, then there exists a category fibred in sets $p' : \mathcal{S}' \to \mathcal{C}$ and an equivalence $can : \mathcal{S} \to \mathcal{S}'$ over $\mathcal{C}$.*

Proof. Let us prove (2). An object of the category $\mathcal{S}'$ will be a pair (U, ξ), where $U \in Ob(\mathcal{C})$ and ξ is an isomorphism class of objects of $\mathcal{S}_U$. A morphism $(U, \xi) \to (V, \psi)$ is given by a morphism $x \to y$, where $x \in \xi$ and $y \in \psi$. Here we identify two morphisms $x \to y$ and

[4] A set on steroids!?

$x' \to y'$ if they induce the same morphism $U \to V$, and if for some choices of isomorphisms $x \to x'$ in $\mathcal{S}_U$ and $y \to y'$ in $\mathcal{S}_V$ the compositions $x \to x' \to y'$ and $x \to y \to y'$ agree. By construction there are surjective maps on objects and morphisms from $\mathcal{S} \to \mathcal{S}'$. We define composition of morphisms in $\mathcal{S}'$ to be the unique law that turns $\mathcal{S} \to \mathcal{S}'$ into a functor. Some details omitted. $\square$

Thus categories fibred in setoids are exactly the categories fibred in groupoids which are equivalent to categories fibred in sets. Moreover, an equivalence of categories fibred in sets is an isomorphism by Lemma 4.35.6.

Lemma 4.36.6. *Let $\mathcal{C}$ be a category. The construction of Lemma 4.36.5 part (2) gives a functor*

$$F : \left\{ \begin{matrix} \textit{the 2-category of categories} \\ \textit{fi}\mathcal{C} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \textit{the category of categories} \\ \textit{fi}\mathcal{C} \end{matrix} \right\}$$

(see Definition 4.26.5). This functor is an equivalence in the following sense:

1. *for any two 1-morphisms $f, g : \mathcal{S}_1 \to \mathcal{S}_2$ with $F(f) = F(g)$ there exists a unique 2-isomorphism $f \to g$,*
2. *for any morphism $h : F(\mathcal{S}_1) \to F(\mathcal{S}_2)$ there exists a 1-morphism $f : \mathcal{S}_1 \to \mathcal{S}_2$ with $F(f) = h$, and*
3. *any category fibred in sets $\mathcal{S}$ is equal to $F(\mathcal{S})$.*

In particular, defining $F_i \in PSh(\mathcal{C})$ by the rule $F_i(U) = Ob(\mathcal{S}_{i,U})/\cong$, we have

$$Mor_{Cat/\mathcal{C}}(\mathcal{S}_1, \mathcal{S}_2)\Big/\text{2-isomorphism} = Mor_{PSh(\mathcal{C})}(F_1, F_2)$$

More precisely, given any map $\phi : F_1 \to F_2$ there exists a 1*-morphism $f : \mathcal{S}_1 \to \mathcal{S}_2$ which induces ϕ on isomorphism classes of objects and which is unique up to unique* 2*-isomorphism.*

Proof. By Lemma 4.35.6 the target of F is a category hence the assertion makes sense. The construction of Lemma 4.36.5 part (2) assigns to $\mathcal{S}$ the category fibred in sets whose value over U is the set of isomorphism classes in $\mathcal{S}_U$. Hence it is clear that it defines a functor as indicated. Let $f, g : \mathcal{S}_1 \to \mathcal{S}_2$ with $F(f) = F(g)$ be as in (1). For each object U of $\mathcal{C}$ and each object x of $\mathcal{S}_{1,U}$ we see that $f(x) \cong g(x)$ by assumption. As $\mathcal{S}_2$ is fibred in setoids there exists a unique isomorphism $t_x : f(x) \to g(x)$ in $\mathcal{S}_{2,U}$. Clearly the rule $x \mapsto t_x$ gives the desired 2-isomorphism $f \to g$. We omit the proofs of (2) and (3). To see the final assertion use Lemma 4.35.6 to see that the right hand side is equal to $Mor_{Cat(\mathcal{C})}(F(\mathcal{S}_1), F(\mathcal{S}_2))$ and apply (1) and (2) above. $\square$

Here is another characterization of categories fibred in setoids among all categories fibred in groupoids.

Lemma 4.36.7. *Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a category fibred in groupoids. The following are equivalent:*

1. *$p : \mathcal{S} \to \mathcal{C}$ be a category fibred in setoids, and*
2. *the canonical* 1*-morphism $\mathcal{I}_\mathcal{S} \to \mathcal{S}$, see (4.31.2.1), is an equivalence (of categories over $\mathcal{C}$).*

Proof. Assume (2). The category $\mathcal{I}_\mathcal{S}$ has objects (x, α) where $x \in \mathcal{S}$, say with $p(x) = U$, and $\alpha : x \to x$ is a morphism in $\mathcal{S}_U$. Hence if $\mathcal{I}_\mathcal{S} \to \mathcal{S}$ is an equivalence over $\mathcal{C}$ then every pair of objects (x, α), (x, α') are isomorphic in the fibre category of $\mathcal{I}_\mathcal{S}$ over U. Looking at the definition of morphisms in $\mathcal{I}_\mathcal{S}$ we conclude that α, α' are conjugate in the group of automorphisms of x. Hence taking $\alpha' = \mathrm{id}_x$ we conclude that every automorphism of x is equal to the identity. Since $\mathcal{S} \to \mathcal{C}$ is fibred in groupoids this implies that $\mathcal{S} \to \mathcal{C}$ is fibred in setoids. We omit the proof of (1) $\Rightarrow$ (2). $\square$

Lemma 4.36.8. *Let $\mathcal{C}$ be a category. The construction of Lemma 4.36.6 which associates to a category fibred in setoids a presheaf is compatible with products, in the sense that the presheaf associated to a 2-fibre product $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ is the fibre product of the presheaves associated to $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$.*

Proof. Let $U \in Ob(\mathcal{C})$. The lemma just says that

$$Ob((\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z})_U)/\cong \quad \text{equals} \quad Ob(\mathcal{X}_U)/\cong \times_{Ob(\mathcal{Y}_U)/\cong} Ob(\mathcal{Z}_U)/\cong$$

the proof of which we omit. (But note that this would not be true in general if the category $\mathcal{Y}_U$ is not a setoid.) $\square$

4.37. Representable categories fibred in groupoids

Here is our definition of a representable category fibred in groupoids. As promised this is invariant under equivalences.

Definition 4.37.1. Let $\mathcal{C}$ be a category. A category fibred in groupoids $p : \mathcal{S} \to \mathcal{C}$ is called *representable* if there exists an object X of $\mathcal{C}$ and an equivalence $j : \mathcal{S} \to \mathcal{C}/X$ (in the 2-category of groupoids over $\mathcal{C}$).

The usual abuse of notation is to say that X *represents* $\mathcal{S}$ and not mention the equivalence j. We spell out what this entails.

Lemma 4.37.2. *Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a category fibred in groupoids.*

(1) *$\mathcal{S}$ is representable if and only if the following conditions are satisfied:*
   (a) *$\mathcal{S}$ is fibred in setoids, and*
   (b) *the presheaf $U \mapsto Ob(\mathcal{S}_U)/\cong$ is representable.*
(2) *If $\mathcal{S}$ is representable the pair (X, j), where j is the equivalence $j : \mathcal{S} \to \mathcal{C}/X$ is uniquely determined up to isomorphism.*

Proof. The first assertion follows immediately from Lemma 4.36.5. For the second, suppose that $j' : \mathcal{S} \to \mathcal{C}/X'$ is a second such pair. Choose a 1-morphism $t' : \mathcal{C}/X' \to \mathcal{S}$ such that $j' \circ t' \cong \mathrm{id}_{\mathcal{C}/X'}$ and $t' \circ j' \cong \mathrm{id}_{\mathcal{S}}$. Then $j \circ t' : \mathcal{C}/X' \to \mathcal{C}/X$ is an equivalence. Hence it is an isomorphism, see Lemma 4.35.6. Hence by the Yoneda Lemma 4.3.5 (via Example 4.35.7 for example) it is given by an isomorphism $X' \to X$. $\square$

Lemma 4.37.3. *Let $\mathcal{C}$ be a category. Let $\mathcal{X}$, $\mathcal{Y}$ be categories fibred in groupoids over $\mathcal{C}$. Assume that $\mathcal{X}$, $\mathcal{Y}$ are representable by objects X, Y of $\mathcal{C}$. Then*

$$Mor_{Cat/\mathcal{C}}(\mathcal{X}, \mathcal{Y}) \Big/ \textit{2-isomorphism} = Mor_{\mathcal{C}}(X, Y)$$

More precisely, given $\phi : X \to Y$ there exists a 1-morphism $f : \mathcal{X} \to \mathcal{Y}$ which induces ϕ on isomorphism classes of objects and which is unique up to unique 2-isomorphism.

Proof. By Example 4.35.7 we have $\mathcal{C}/X = \mathcal{S}_{h_X}$ and $\mathcal{C}/Y = \mathcal{S}_{h_Y}$. By Lemma 4.36.6 we have

$$Mor_{Cat/\mathcal{C}}(\mathcal{X}, \mathcal{Y}) \Big/ \text{2-isomorphism} = Mor_{PSh(\mathcal{C})}(h_X, h_Y)$$

By the Yoneda Lemma 4.3.5 we have $Mor_{PSh(\mathcal{C})}(h_X, h_Y) = Mor_{\mathcal{C}}(X, Y)$. $\square$

4.38. Representable 1-morphisms

Let $\mathcal{C}$ be a category. In this section we explain what it means for a 1-morphism between categories fibred in groupoids over $\mathcal{C}$ to be representable. Note that the 2-category of categories fibred in groupoids over $\mathcal{C}$ is a ``full'' sub 2-category of the 2-category of categories over $\mathcal{C}$ (see Definition 4.32.6). Hence if $\mathcal{S}$, $\mathcal{S}'$ are fibred in groupoids over $\mathcal{C}$ then

$$Mor_{Cat/\mathcal{C}}(\mathcal{S}, \mathcal{S}')$$

denotes the category of 1-morphisms in this 2-category (see Definition 4.29.1). These are all groupoids, see remarks following Definition 4.32.6. Here is the 2-category analogue of the Yoneda lemma.

Lemma 4.38.1. *(2-Yoneda lemma) Let $\mathcal{S} \to \mathcal{C}$ be fibred in groupoids. Let $U \in Ob(\mathcal{C})$. The functor*

$$Mor_{Cat/\mathcal{C}}(\mathcal{C}/U, \mathcal{S}) \longrightarrow \mathcal{S}_U$$

given by $G \mapsto G(id_U)$ is an equivalence.

Proof. Make a choice of pullbacks for $\mathcal{S}$ (see Definition 4.30.5). We define a functor

$$\mathcal{S}_U \longrightarrow Mor_{Cat/\mathcal{C}}(\mathcal{C}/U, \mathcal{S})$$

as follows. Given $x \in Ob(\mathcal{S}_U)$ the associated functor is

(1) on objects: $(f : V \to U) \mapsto f^*x$, and
(2) on morphisms: the arrow $(g : V'/U \to V/U)$ maps to the composition

$$(f \circ g)^*x \xrightarrow{(\alpha_{g,f})_x} g^*f^*x \to f^*x$$

where $\alpha_{g,f}$ is as in Lemma 4.32.2.

We omit the verification that this is an inverse to the functor of the lemma. □

Remark 4.38.2. We can use the 2-Yoneda lemma to give an alternative proof of Lemmas 4.34.3. Let $p : \mathcal{S} \to \mathcal{C}$ be a category fibred in groupoids. We define a contravariant functor F from $\mathcal{C}$ to the category of groupoids as follows: for $U \in Ob(\mathcal{C})$ let

$$F(U) = Mor_{Cat/\mathcal{C}}(\mathcal{C}/U, \mathcal{S}).$$

If $f : U \to V$ the induced functor $\mathcal{C}/U \to \mathcal{C}/V$ induces the morphism $F(f) : F(V) \to F(U)$. Clearly F is a functor. Let $\mathcal{S}'$ be the associated category fibred in groupoids from Example 4.34.1. There is an obvious functor $G : \mathcal{S}' \to \mathcal{S}$ over $\mathcal{C}$ given by taking the pair (U, x), where $U \in Ob(\mathcal{C})$ and $x \in F(U)$, to $x(\mathrm{id}_U) \in \mathcal{S}$. Now Lemma 4.38.1 implies that for each U,

$$G_U : \mathcal{S}'_U = F(U) = Mor_{Cat/\mathcal{C}}(\mathcal{C}/U, \mathcal{S}) \to \mathcal{S}_U$$

is an equivalence, and thus G equivalence between $\mathcal{S}$ and $\mathcal{S}'$ by Lemma 4.32.8.

Let $\mathcal{C}$ be a category. Let $\mathcal{X}$, $\mathcal{Y}$ be categories fibred in groupoids over $\mathcal{C}$. Let $U \in Ob(\mathcal{C})$. Let $F : \mathcal{X} \to \mathcal{Y}$ and $G : \mathcal{C}/U \to \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $\mathcal{C}$. We want to describe the 2-fibre product

$$\begin{tikzcd}
(\mathcal{C}/U) \times_{\mathcal{Y}} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "F"] \\
\mathcal{C}/U \arrow[r, "G"] & \mathcal{Y}
\end{tikzcd}$$

Let $y = G(\mathrm{id}_U) \in \mathcal{Y}_U$. Make a choice of pullbacks for $\mathcal{Y}$ (see Definition 4.30.5). Then G is isomorphic to the functor $(f : V \to U) \mapsto f^*y$, see Lemma 4.38.1 and its proof. We may think of an object of $(\mathcal{C}/U) \times_{\mathcal{Y}} \mathcal{X}$ as a quadruple $(V, f : V \to U, x, \phi)$, see Lemma 4.29.3. Using the description of G above we may think of ϕ as an isomorphism $\phi : f^*y \to F(x)$ in $\mathcal{Y}_V$.

Lemma 4.38.3. *In the situation above the fibre category of $(\mathcal{C}/U) \times_{\mathcal{Y}} \mathcal{X}$ over an object $f : V \to U$ of $\mathcal{C}/U$ is the category described as follows:*

1. *objects are pairs (x, ϕ), where $x \in Ob(\mathcal{X}_V)$, and $\phi : f^*y \to F(x)$ is a morphism in $\mathcal{Y}_V$,*
2. *the set of morphisms between (x, ϕ) and (x', ϕ') is the set of morphisms $\psi : x \to x'$ in $\mathcal{X}_V$ such that $F(\psi) = \phi' \circ \phi^{-1}$.*

Proof. See discussion above. □

Lemma 4.38.4. *Let $\mathcal{C}$ be a category. Let $\mathcal{X}$, $\mathcal{Y}$ be categories fibred in groupoids over $\mathcal{C}$. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism. Let $G : \mathcal{C}/U \to \mathcal{Y}$ be a 1-morphism. Then*

$$(\mathcal{C}/U) \times_{\mathcal{Y}} \mathcal{X} \longrightarrow \mathcal{C}/U$$

is a category fibred in groupoids.

Proof. We have already seen in Lemma 4.32.7 that the composition

$$(\mathcal{C}/U) \times_{\mathcal{Y}} \mathcal{X} \longrightarrow \mathcal{C}/U \longrightarrow \mathcal{C}$$

is a category fibred in groupoids. Then the lemma follows from Lemma 4.32.12. □

Definition 4.38.5. Let $\mathcal{C}$ be a category. Let $\mathcal{X}$, $\mathcal{Y}$ be categories fibred in groupoids over $\mathcal{C}$. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism. We say F is *representable*, or that $\mathcal{X}$ *is relatively representable over* $\mathcal{Y}$, if for every $U \in Ob(\mathcal{C})$ and any $G : \mathcal{C}/U \to \mathcal{X}$ the category fibred in groupoids

$$(\mathcal{C}/U) \times_{\mathcal{Y}} \mathcal{X} \longrightarrow \mathcal{C}/U$$

is representable over $\mathcal{C}/U$.

Lemma 4.38.6. *Let $\mathcal{C}$ be a category. Let $\mathcal{X}$, $\mathcal{Y}$ be categories fibred in groupoids over $\mathcal{C}$. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism. If F is representable then every one of the functors*

$$F_U : \mathcal{X}_U \longrightarrow \mathcal{Y}_U$$

between fibre categories is faithful.

Proof. Clear from the description of fibre categories in Lemma 4.38.3 and the characterization of representable fibred categories in Lemma 4.37.2. □

Lemma 4.38.7. *Let $\mathcal{C}$ be a category. Let $\mathcal{X}$, $\mathcal{Y}$ be categories fibred in groupoids over $\mathcal{C}$. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism. Make a choice of pullbacks for $\mathcal{Y}$. Assume*

1. *each functor $F_U : \mathcal{X}_U \longrightarrow \mathcal{Y}_U$ between fibre categories is faithful, and*
2. *for each U and each $y \in \mathcal{Y}_U$ the presheaf*

$$(f : V \to U) \longmapsto \{(x, \phi) \mid x \in \mathcal{X}_V, \phi : f^*y \to F(x)\}/\cong$$

is a representable presheaf on $\mathcal{C}/U$.

Then F is representable.

Proof. Clear from the description of fibre categories in Lemma 4.38.3 and the characterization of representable fibred categories in Lemma 4.37.2. □

Before we state the next lemma we point out that the 2-category of categories fibred in groupoids is a $(2,1)$-category, and hence we know what it means to say that it has a final object (see Definition 4.28.1). And it has a final object namely $\mathrm{id} : \mathcal{C} \to \mathcal{C}$. Thus we define 2-*products* of categories fibred in groupoids over $\mathcal{C}$ as the 2-fibred products

$$\mathcal{X} \times \mathcal{Y} := \mathcal{X} \times_{\mathcal{C}} \mathcal{Y}.$$

With this definition in place the following lemma makes sense.

Lemma 4.38.8. *Let $\mathcal{C}$ be a category. Let $\mathcal{S} \to \mathcal{C}$ be a category fibred in groupoids. Assume $\mathcal{C}$ has products of pairs of objects and fibre products. The following are equivalent:*

(1) *The diagonal $\mathcal{S} \to \mathcal{S} \times \mathcal{S}$ is representable.*
(2) *For every U in $\mathcal{C}$, any $G : \mathcal{C}/U \to \mathcal{S}$ is representable.*

Proof. Suppose the diagonal is representable, and let U, G be given. Consider any $V \in Ob(\mathcal{C})$ and any $G' : \mathcal{C}/V \to \mathcal{S}$. Note that $\mathcal{C}/U \times \mathcal{C}/V = \mathcal{C}/U \times V$ is representable. Hence the fibre product

$$\begin{tikzcd} (\mathcal{C}/U \times V) \times_{(\mathcal{S}\times\mathcal{S})} \mathcal{S} \arrow[r] \arrow[d] & \mathcal{S} \arrow[d] \\ \mathcal{C}/U \times V \arrow[r, "{(G,G')}"] & \mathcal{S} \times \mathcal{S} \end{tikzcd}$$

is representable by assumption. This means there exists $W \to U \times V$ in $\mathcal{C}$, such that

$$\begin{tikzcd} \mathcal{C}/W \arrow[r] \arrow[d] & \mathcal{S} \arrow[d] \\ \mathcal{C}/U \times \mathcal{C}/V \arrow[r] & \mathcal{S} \times \mathcal{S} \end{tikzcd}$$

is cartesian. This implies that $\mathcal{C}/W \cong \mathcal{C}/U \times_{\mathcal{S}} \mathcal{C}/V$ (see Lemma 4.28.11) as desired.

Assume (2) holds. Consider any $V \in Ob(\mathcal{C})$ and any $(G, G') : \mathcal{C}/V \to \mathcal{S} \times \mathcal{S}$. We have to show that $\mathcal{C}/V \times_{\mathcal{S}\times\mathcal{S}} \mathcal{S}$ is representable. What we know is that $\mathcal{C}/V \times_{G,\mathcal{S},G'} \mathcal{C}/V$ is representable, say by $a : W \to V$ in $\mathcal{C}/V$. The equivalence

$$\mathcal{C}/W \to \mathcal{C}/V \times_{G,\mathcal{S},G'} \mathcal{C}/V$$

followed by the second projection to $\mathcal{C}/V$ gives a second morphism $a' : W \to V$. Consider $W' = W \times_{(a,a'),V\times V} V$. There exists an equivalence

$$\mathcal{C}/W' \cong \mathcal{C}/V \times_{\mathcal{S}\times\mathcal{S}} \mathcal{S}$$

namely

$$\begin{aligned} \mathcal{C}/W' &\cong \mathcal{C}/W \times_{(\mathcal{C}/V \times \mathcal{C}/V)} \mathcal{C}/V \\ &\cong \left(\mathcal{C}/V \times_{(G,\mathcal{S},G')} \mathcal{C}/V\right) \times_{(\mathcal{C}/V \times \mathcal{C}/V)} \mathcal{C}/V \\ &\cong \mathcal{C}/V \times_{(\mathcal{S}\times\mathcal{S})} \mathcal{S} \end{aligned}$$

(for the last isomorphism see Lemma 4.28.12) which proves the lemma. □

Biographical notes: Parts of this have been taken from Vistoli's notes [**Vis**].

4.39. Other chapters

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces

(7) Commutative Algebra
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CHAPTER 5

Topology

5.1. Introduction

Basic topology will be explained in this document. A reference is [**Eng77**].

5.2. Basic notions

The following notions are considered basic and will not be defined, and or proved. This does not mean they are all necessarily easy or well known.

(1) X is a *topological space*,
(2) $x \in X$ is a *point*,
(3) $x \in X$ is a *closed point*,
(4) $f : X_1 \to X_2$ is *continuous*,
(5) a *neighbourhood of* $x \in X$ is any subset $E \subset X$ which contains an open subset that contains x,
(6) $\mathcal{U} : U = \bigcup_{i \in I} U_i$ is an *open covering of* U (note: we allow any U_i to be empty and we even allow, in case U is empty, the empty set for I),
(7) the open covering $\mathcal{V}$ is a *refinement* of the open covering $\mathcal{U}$ (if $\mathcal{V} : V = \bigcup_{j \in J} V_j$ and $\mathcal{U} : U = \bigcup_{i \in I} U_i$ this means each V_j is completely contained in one of the U_i),
(8) $\{E_i\}_{i \in I}$ *is a fundamental system of neighbourhoods of* x *in* X,
(9) a topological space X is called *Hausdorff* or *separated* if and only if for every distinct pair of points $x, y \in X$ there exist disjoint opens $U, V \subset X$ such that $x \in U$, $y \in V$,
(10) the *product* of two topological spaces,
(11) the *fibre product* $X \times_Y Z$ of a pair of continuous maps $f : X \to Y$ and $g : Z \to Y$,
(12) etc.

5.3. Bases

Definition 5.3.1. Let X be a topological space. A collection of subsets $\mathcal{B}$ of X is called a *base for the topology on* X or a *basis for the topology on* X if the following conditions hold:

(1) Every element $B \in \mathcal{B}$ is open in X.
(2) For every open $U \subset X$ and every $x \in U$, there exists an element $B \in \mathcal{B}$ such that $x \in B \subset U$.

Lemma 5.3.2. *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Let $\mathcal{U} : U = \bigcup_i U_i$ be an open covering of $U \subset X$. There exists an open covering $U = \bigcup V_j$ which is a refinement of $\mathcal{U}$ such that each V_j is an element of the basis $\mathcal{B}$.*

Proof. Omitted. □

5.4. Connected components

Definition 5.4.1. Let X be a topological space.

(1) We say X is *connected* if whenever $X = T_1 \coprod T_2$ with $T_i \subset X$ open and closed, then either $T_1 = \emptyset$ or $T_2 = \emptyset$.
(2) We say $T \subset X$ is a *connected component* of X if T is a maximal connected subset of X.

The empty space is connected.

Lemma 5.4.2. *Let $f : X \to Y$ be a continuous map of topological spaces. If $E \subset X$ is a connected subset, then $f(E) \subset Y$ is connected as well.*

Proof. Omitted. □

Lemma 5.4.3. *Let X be a topological space. If $T \subset X$ is connected, then so is its closure. Each point of X is contained in a connected component. Connected components are always closed, but not necessarily open.*

Proof. Let $\overline{T}$ be the closure of the connected subset T. Suppose $\overline{T} = T_1 \coprod T_2$ with $T_i \subset \overline{T}$ open and closed. Then $T = (T \cap T_1) \coprod (T \cap T_2)$. Hence T equals one of the two, say $T = T_1 \cap T$. Thus clearly $\overline{T} \subset T_1$ as desired.

Pick a point $x \in X$. Consider the set A of connected subsets $x \in T_\alpha \subset X$. Note that A is nonempty since $\{x\} \in A$. There is a partial ordering on A coming from inclusion: $\alpha \leq \alpha' \Leftrightarrow T_\alpha \subset T_{\alpha'}$. Choose a maximal totally ordered subset $A' \subset A$, and let $T = \bigcup_{\alpha \in A'} T_\alpha$. We claim that T is connected. Namely, suppose that $T = T_1 \coprod T_2$ is a disjoint union of two open and closed subsets of T. For each $\alpha \in A'$ we have either $T_\alpha \subset T_1$ or $T_\alpha \subset T_2$, by connectedness of T_α. Suppose that for some $\alpha_0 \in A'$ we have $T_{\alpha_0} \not\subset T_1$ (say, if not we're done anyway). Then, since A' is totally ordered we see immediately that $T_\alpha \subset T_2$ for all $\alpha \in A'$. Hence $T = T_2$.

To get an example where connected components are not open, just take an infinite product $\prod_{n \in \mathbf{N}} \{0, 1\}$ with the product topology. This is a totally disconnected space so connected components are singletons, which are not open. □

Lemma 5.4.4. *Let $f : X \to Y$ be a continuous map of topological spaces. Assume that*

(1) *all fibres of f are nonempty and connected, and*
(2) *a set $T \subset Y$ is closed if and only if $f^{-1}(T)$ is closed.*

Then f induces a bijection between the sets of connected components of X and Y.

Proof. Let $T \subset Y$ be a connected component. Note that T is closed, see Lemma 5.4.3. The lemma follows if we show that $p^{-1}(T)$ is connected because any connected subset of X maps into a connected component of Y by Lemma 5.4.2. Suppose that $p^{-1}(T) = Z_1 \coprod Z_2$ with Z_1, Z_2 closed. For any $t \in T$ we see that $p^{-1}(\{t\}) = Z_1 \cap p^{-1}(\{t\}) \coprod Z_2 \cap p^{-1}(\{t\})$. By (1) we see $p^{-1}(\{t\})$ is connected we conclude that either $p^{-1}(\{t\}) \subset Z_1$ or $p^{-1}(\{t\}) \subset Z_2$. In other words $T = T_1 \coprod T_2$ with $p^{-1}(T_i) = Z_i$. By (2) we conclude that T_i is closed in Y. Hence either $T_1 = \emptyset$ or $T_2 = \emptyset$ as desired. □

Lemma 5.4.5. *Let $f : X \to Y$ be a continuous map of topological spaces. Assume that (a) f is open, (b) all fibres of f are nonempty and connected. Then f induces a bijection between the sets of connected components of X and Y.*

Proof. This is a special case of Lemma 5.4.4. □

Definition 5.4.6. A topological space is *totally disconnected* if the connected components are all singletons.

A discrete space is totally disconnected. A totally disconnected space need not be discrete, for example $\mathbf{Q} \subset \mathbf{R}$ is totally disconnected but not discrete.

Definition 5.4.7. A topological space X is called *locally connected* if every point $x \in X$ has a fundamental system of connected neighbourhoods.

Lemma 5.4.8. *Let X be a topological space. If X is locally connected, then*

(1) *any open subset of X is locally connected, and*
(2) *the connected components of X are open.*

So also the connected components of open subsets of X are open. In particular, every point has a fundamental system of open connected neighbourhoods.

Proof. Omitted. □

5.5. Irreducible components

Definition 5.5.1. Let X be a topological space.

(1) We say X is *irreducible*, if X is not empty, and whenever $X = Z_1 \cup Z_2$ with Z_i closed, we have $X = Z_1$ or $X = Z_2$.
(2) We say $Z \subset X$ is an *irreducible component* of X if Z is a maximal irreducible subset of X.

An irreducible space is obviously connected.

Lemma 5.5.2. *Let $f : X \to Y$ be a continuous map of topological spaces. If $E \subset X$ is an irreducible subset, then $f(E) \subset Y$ is irreducible as well.*

Proof. Omitted. □

Lemma 5.5.3. *Let X be a topological space. If $T \subset X$ is irreducible so is its closure in X. Any irreducible component of X is closed. Every point of X is contained in some irreducible component of X.*

Proof. Let $\overline{T}$ be the closure of the irreducible subset T. If $\overline{T} = Z_1 \cup Z_2$ with $Z_i \subset \overline{T}$ closed, then $T = (T \cap Z_1) \cup (T \cap Z_2)$ and hence T equals one of the two, say $T = Z_1 \cap T$. Thus clearly $\overline{T} \subset Z_1$ as desired.

Pick a point $x \in X$. Consider the set A of irreducible subsets $x \in T_\alpha \subset X$. Note that A is nonempty since $\{x\} \in A$. There is a partial ording on A coming from inclusion: $\alpha \leq \alpha' \Leftrightarrow T_\alpha \subset T_{\alpha'}$. Choose a maximal totally ordered subset $A' \subset A$, and let $T = \bigcup_{\alpha \in A'} T_\alpha$. We claim that T is irreducible. Namely, suppose that $T = Z_1 \cup Z_2$ is a union of two closed subsets of T. For each $\alpha \in A'$ we have either $T_\alpha \subset Z_1$ or $T_\alpha \subset Z_2$, by irreducibility of T_α. Suppose that for some $\alpha_0 \in A'$ we have $T_{\alpha_0} \not\subset Z_1$ (say, if not we're done anyway). Then, since A' is totally ordered we see immediately that $T_\alpha \subset Z_2$ for all $\alpha \in A'$. Hence $T = Z_2$. □

A singleton is irreducible. Thus if $x \in X$ is a point then the closure $\overline{\{x\}}$ is an irreducible closed subset of X.

Definition 5.5.4. Let X be a topological space.

(1) Let $Z \subset X$ be an irreducible closed subset. A *generic point* of Z is a point $\xi \in Z$ such that $Z = \overline{\{\xi\}}$.

(2) The space X is called *Kolmogorov*, if for every $x, x' \in X$, $x \neq x'$ there exists a closed subset of X which contains exactly one of the two points.
(3) The space X is called *sober* if every irreducible closed subset has a unique generic point.

A space X is Kolmogorov if for $x_1, x_2 \in X$ we have $x_1 = x_2$ if and only if $\overline{\{x_1\}} = \overline{\{x_2\}}$. Hence we see that a sober topological space is Kolmogorov.

Lemma 5.5.5. *Let X be a topological space. If X has an open covering $X = \bigcup X_i$ with X_i sober (resp. Kolmogorov), then X is sober (resp. Kolmogorov).*

Proof. Omitted. □

Example 5.5.6. Recall that a topological space X is Hausdorff iff for every distinct pair of points $x, y \in X$ there exist disjoint opens $U, V \subset X$ such that $x \in U$, $y \in V$. In this case X is irreducible if and only if X is a singleton. Similarly, any subset of X is irreducible if and only if it is a singleton. Hence a Hausdorff space is sober.

Lemma 5.5.7. *Let $f : X \to Y$ be a continuous map of topological spaces. Assume that (a) Y is irreducible, (b) f is open, and (c) there exists a dense collection of points $y \in Y$ such that $f^{-1}(y)$ is irreducible. Then X is irreducible.*

Proof. Suppose $Y = Z_1 \cup Z_2$ with Z_i closed. Consider the open sets $U_1 = Z_1 \setminus Z_2 = Y \setminus Z_2$ and $U_2 = Z_2 \setminus Z_1 = Y \setminus Z_2$. To get a contradiction assume that U_1 and U_2 are both nonempty. By (b) we see that $f(U_i)$ is open. By (a) we have X irreducible and hence $f(U_1) \cap f(U_2) \neq \emptyset$. By (c) there is a point y which corresponds to a point of this intersection such that the fibre $X_y = f^{-1}(y)$ is irreducible. Then $X_y \cap U_1$ and $X_y \cap U_2$ are nonempty disjoint open subsets of X_y which is a contradiction. □

Lemma 5.5.8. *Let $f : X \to Y$ be a continuous map of topological spaces. Assume that (a) f is open, and (b) for every $y \in Y$ the fibre $f^{-1}(y)$ is irreducible. Then f induces a bijection between irreducible components.*

Proof. We point out that assumption (b) implies that f is surjective (see Definition 5.5.1). Let $T \subset Y$ be an irreducible component. Note that T is closed, see Lemma 5.5.3. The lemma follows if we show that $p^{-1}(T)$ is irreducible because any irreducible subset of X maps into an irreducible component of Y by Lemma 5.5.2. Note that $p^{-1}(T) \to T$ satisfies the assumptions of Lemma 5.5.7. Hence we win. □

5.6. Noetherian topological spaces

Definition 5.6.1. A topological space is called *Noetherian* if the descending chain condition holds for closed subsets of X. A topological space is called *locally Noetherian* if every point has a neighbourhood which is Noetherian.

Lemma 5.6.2. *Let X be a Noetherian topological space.*
(1) *Any subset of X with the induced topology is Noetherian.*
(2) *The space X has finitely many irreducible components.*
(3) *Each irreducible component of X contains a nonempty open of X.*

Proof. Let $T \subset X$ be a subset of X. Let $T_1 \supset T_2 \supset \ldots$ be a descending chain of closed subsets of T. Write $T_i = T \cap Z_i$ with $Z_i \subset X$ closed. Consider the descending chain of closed subsets $Z_1 \supset Z_1 \cap Z_2 \supset Z_1 \cap Z_2 \cap Z_3 \ldots$ This stabilizes by assumption and hence the original sequence of T_i stabilizes. Thus T is Noetherian.

Let A be the set of closed subsets of X which do not have finitely many irreducible components. Assume that A is not empty to arrive at a contradiction. The set A is partially ordered by inclusion: $\alpha \leq \alpha' \Leftrightarrow Z_\alpha \subset Z_{\alpha'}$. By the descending chain condition we may find a smallest element of A, say Z. As Z is not a finite union of irreducible components, it is not irreducible. Hence we can write $Z = Z' \cup Z''$ and both are strictly smaller closed subsets. By construction $Z' = \bigcup Z'_i$ and $Z'' = \bigcup Z''_j$ are finite unions of their irreducible components. Hence $Z = \bigcup Z'_i \cup \bigcup Z''_j$ is a finite union of irreducible closed subsets. After removing redundant members of this expression, this will be the decomposition of Z into its irreducible components, a contradiction.

Let $Z \subset X$ be an irreducible component of X. Let $Z_1, \ldots, Z_n$ be the other irreducible components of X. Consider $U = Z \setminus (Z_1 \cup \ldots \cup Z_n)$. This is not empty since otherwise the irreducible space Z would be contained in one of the other Z_i. Because $X = Z \cup Z_1 \cup \ldots Z_n$ (see Lemma 5.5.3), also $U = X \setminus (Z_1 \cup \ldots \cup Z_n)$ and hence open in X. Thus Z contains a nonempty open of X. □

Lemma 5.6.3. *Let $f : X \to Y$ be a continuous map of topological spaces.*

(1) *If X is Noetherian, then $f(X)$ is Noetherian.*
(2) *If X is locally Noetherian and f open, then $f(X)$ is locally Noetherian.*

Proof. In case (1), suppose that $Z_1 \supset Z_2 \supset Z_2 \supset \ldots$ is a descending chain of closed subsets of $f(X)$ (as usual with the induced topology as a subset of Y). Then $f^{-1}(Z_1) \supset f^{-1}(Z_2) \supset f^{-1}(Z_3) \supset \ldots$ is a descending chain of closed subsets of X. Hence this chain stabilizes. Since $f(f^{-1}(Z_i)) = Z_i$ we conclude that $Z_1 \supset Z_2 \supset Z_2 \supset \ldots$ stabilizes also. In case (2), let $y \in f(X)$. Choose $x \in X$ with $f(x) = y$. By assumption there exists a neighbourhood $E \subset X$ of x which is Noetherian. Then $f(E) \subset f(X)$ is a neighbourhood which is Noetherian by part (1). □

Lemma 5.6.4. *Let X be a topological space. Let $X_i \subset X$, $i = 1, \ldots, n$ be a finite collection of subsets. If each X_i is Noetherian (with the induced topology), then $\bigcup_{i=1,\ldots,n} X_i$ is Noetherian (with the induced topology).*

Proof. Omitted. □

Example 5.6.5. Any Noetherian topological space has a closed point (combine Lemmas 5.9.6 and 5.9.9). Let $X = \{1, 2, 3, \ldots\}$. Define a topology on X with opens $\emptyset$, $\{1, 2, \ldots, n\}$, $n \geq 1$ and X. Thus X is a locally Noetherian topological space, without any closed points. This space cannot be the underlying topological space of a locally Noetherian scheme, see Properties, Lemma 23.5.8.

Lemma 5.6.6. *Let X be a locally Noetherian topological space. Then X is locally connected.*

Proof. Let $x \in X$. Let E be a neighbourhood of x. We have to find a connected neighbourhood of x contained in E. By assumption there exists a neighbourhood E' of x which is Noetherian. Then $E \cap E'$ is Noetherian, see Lemma 5.6.2. Let $E \cap E' = Y_1 \cup \ldots \cup Y_n$ be the decomposition into irreducible components, see Lemma 5.6.2. Let $E'' = \bigcup_{x \in Y_i} Y_i$. This is a connected subset of $E \cap E'$ containing x. It contains the open $E \cap E' \setminus (\bigcup_{x \notin Y_i} Y_i)$ of $E \cap E'$ and hence it is a neighbourhood of x in X. This proves the lemma. □

5.7. Krull dimension

Definition 5.7.1. Let X be a topological space.

(1) A *chain of irreducible closed subsets* of X is a sequence $Z_0 \subset Z_1 \subset \ldots \subset Z_n \subset X$ with Z_i closed irreducible and $Z_i \neq Z_{i+1}$ for $i = 0, \ldots, n-1$.
(2) The *length* of a chain $Z_0 \subset Z_1 \subset \ldots \subset Z_n \subset X$ of irreducible closed subsets of X is the integer n.
(3) The *dimension* or more precisely the *Krull dimension* $\dim(X)$ of X is the element of $\{\infty, 0, 1, 2, 3, \ldots\}$ defined by the formula:

$$\dim(X) = \sup\{\text{lengths of chains of irreducible closed subsets}\}$$

(4) Let $x \in X$. The *Krull dimension of X at x* is defined as

$$\dim_x(X) = \min\{\dim(U), x \in U \subset X \text{ open}\}$$

the minimum of $\dim(U)$ where U runs over the open neighbourhoods of x in X.

Note that if $U' \subset U \subset X$ are open then $\dim(U') \leq \dim(U)$. Hence if $\dim_x(X) = d$ then x has a fundamental system of open neighbourhoods U with $\dim(U) = \dim_x(X)$.

Example 5.7.2. The Krull dimension of the usual Euclidean space $\mathbf{R}^n$ is 0.

Example 5.7.3. Let $X = \{s, \eta\}$ with open sets given by $\{\emptyset, \{\eta\}, \{s, \eta\}\}$. In this case a maximal chain of irreducible closed subsets is $\{s\} \subset \{s, \eta\}$. Hence $\dim(X) = 1$. It is easy to generalize this example to get a $(n+1)$-element topological space of Krull dimension n.

Definition 5.7.4. Let X be a topological space. We say that X is *equidimensional* if every irreducible component of X has the same dimension.

5.8. Codimension and catenary spaces

Definition 5.8.1. Let X be a topological space. We say X is *catenary* if for every pair of irreducible closed subsets $T \subset T'$ there exist a maximal chain of irreducible closed subsets

$$T = T_0 \subset T_1 \subset \ldots \subset T_e = T'$$

and every such chain has the same length.

Lemma 5.8.2. *Let X be a topological space. The following are equivalent:*

(1) *X is catenary,*
(2) *X has an open covering by catenary spaces.*

Moreover, in this case any locally closed subspace of X is catenary.

Proof. Suppose that X is catenary and that $U \subset X$ is an open subset. The rule $T \mapsto \overline{T}$ defines a bijective inclusion preserving map between the closed irreducible subsets of U and the closed irreducible subsets of X which meet U. Using this the lemma easily follows. Details omitted. □

Definition 5.8.3. Let X be a topological space. Let $Y \subset X$ be an irreducible closed subset. The *codimension* of Y in X is the supremum of the lengths e of chains

$$Y = Y_0 \subset Y_1 \subset \ldots \subset Y_e \subset X$$

of irreducible closed subsets in X starting with Y. We will denote this $\text{codim}(Y, X)$.

Lemma 5.8.4. *Let X be a topological space. Let $Y \subset X$ be an irreducible closed subset. Let $U \subset X$ be an open subset such that $Y \cap U$ is nonempty. Then*

$$\mathit{codim}(Y, X) = \mathit{codim}(Y \cap U, U)$$

Proof. Follows from the observation made in the proof of Lemma 5.8.2. □

Example 5.8.5. Let $X = [0, 1]$ be the unit interval with the following topology: The sets $[0, 1]$, $(1 - 1/n, 1]$ for $n \in \mathbf{N}$, and $\emptyset$ are open. So the closed sets are $\emptyset$, $\{0\}$, $[0, 1 - 1/n]$ for $n > 1$ and $[0, 1]$. This is clearly a Noetherian topological space. But the irreducible closed subset $Y = \{0\}$ has infinite codimension $\text{codim}(Y, X) = \infty$. To see this we just remark that all the closed sets $[0, 1 - 1/n]$ are irreducible.

Lemma 5.8.6. *Let X be a topological space. The following are equivalent:*

(1) *X is catenary, and*
(2) *for pair of irreducible closed subsets $Y \subset Y'$ we have $codim(Y, Y') < \infty$ and for every triple $Y \subset Y' \subset Y''$ of irreducible closed subsets we have*

$$codim(Y, Y'') = codim(Y, Y') + codim(Y', Y'').$$

Proof. Omitted. □

5.9. Quasi-compact spaces and maps

The phrase ``compact" will be reserved for Hausdorff topological spaces. And many spaces occuring in algebraic geometry are not Hausdorff.

Definition 5.9.1. Quasi-compactness.

(1) We say that a topological space X is *quasi-compact* if every open covering of X has a finite refinement.
(2) We say that a continuous map $f : X \to Y$ is *quasi-compact* if the inverse image $f^{-1}(V)$ of every quasi-compact open $V \subset Y$ is quasi-compact.
(3) We say a subset $Z \subset X$ is *retrocompact* if the inclusion map $Z \to X$ is quasi-compact.

In many texts on topology a space is called *compact* if it is quasi-compact and Hausdorff; and in other texts the Hausdorff condition is omitted. To avoid confusion in algebraic geometry we use the term quasi-compact. Note that the notion of quasi-compactness of a map is very different from the notion of a ``proper map" in topology, since there one requires the inverse image of any (quasi-)compact subset of the target to be (quasi-)compact, whereas in the definition above we only consider quasi-compact *open* sets.

Lemma 5.9.2. *A composition of quasi-compact maps is quasi-compact.*

Proof. Omitted. □

Lemma 5.9.3. *A closed subset of a quasi-compact topological space is quasi-compact.*

Proof. Omitted. □

The following is really a reformulation of the quasi-compact property.

Lemma 5.9.4. *Let X be a quasi-compact topological space. If $\{Z_\alpha\}_{\alpha \in A}$ is a collection of closed subsets such that the intersection of each finite subcollection is nonempty, then $\bigcap_{\alpha \in A} Z_\alpha$ is nonempty.*

Proof. Omitted. □

Lemma 5.9.5. *Let $f : X \to Y$ be a continuous map of topological spaces.*

(1) *If X is quasi-compact, then $f(X)$ is quasi-compact.*
(2) *If f is quasi-compact, then $f(X)$ is retrocompact.*

Proof. If $f(X) = \bigcup V_i$ is an open covering, then $X = \bigcup f^{-1}(V_i)$ is an open covering. Hence if X is quasi-compact then $X = f^{-1}(V_{i_1}) \cup \ldots \cup f^{-1}(V_{i_n})$ for some $i_1, \ldots, i_n \in I$ and hence $f(X) = V_{i_1} \cup \ldots \cup V_{i_n}$. This proves (1). Assume f is quasi-compact, and let $V \subset Y$ be quasi-compact open. Then $f^{-1}(V)$ is quasi-compact, hence by (1) we see that $f(f^{-1}(V)) = f(X) \cap V$ is quasi-compact. Hence $f(X)$ is retrocompact. □

Lemma 5.9.6. *Let X be a topological space. Assume that*

(1) *X is nonempty,*
(2) *X is quasi-compact, and*
(3) *X is Kolmogorov.*

Then X has a closed point.

Proof. Consider the set

$$\mathcal{T} = \{Z \subset X \mid Z = \overline{\{x\}} \text{ for some } x \in X\}$$

of all closures of singletons in X. It is nonempty since X is nonempty. Make $\mathcal{T}$ into a partially ordered set using the relation of inclusion. Suppose Z_α, $\alpha \in A$ is a totally ordered subset of $\mathcal{T}$. By Lemma 5.9.4 we see that $\bigcap_{\alpha \in A} Z_\alpha \neq \emptyset$. Hence there exists some $x \in \bigcap_{\alpha \in A} Z_\alpha$ and we see that $Z = \overline{\{x\}} \in \mathcal{T}$ is a lower bound for the family. By Zorn's lemma there exists a minimal element $Z \in \mathcal{T}$. As X is Kolmogorov we conclude that $Z = \{x\}$ for some x and $x \in X$ is a closed point. □

Lemma 5.9.7. *Let X be a topological space. Assume*

(1) *X is quasi-compact,*
(2) *X has a basis for the topology consisting of quasi-compact opens, and*
(3) *the intersection of two quasi-compact opens is quasi-compact.*

For any $x \in X$ the connected component of X containing x is the intersection of all open and closed subsets of X containing x.

Proof. Let T be the connected component containing x. Let $S = \bigcap_{\alpha \in A} Z_\alpha$ be the intersection of all open and closed subsets Z_α of X containing x. Note that S is closed in X. Note that any finite intersection of Z_α's is a Z_α. Because T is connected and $x \in T$ we have $T \subset S$. It suffices to show that S is connected. If not, then there exists a disjoint union decomposition $S = B \coprod C$ with B and C open and closed in S. In particular, B and C are closed in X, and so quasi-compact by Lemma 5.9.3 and assumption (1). By assumption (2) there exist quasi-compact opens $U, V \subset X$ with $B = S \cap U$ and $C = S \cap V$ (details omitted). Then $U \cap V \cap S = \emptyset$. Hence $\bigcap_\alpha U \cap V \cap Z_\alpha = \emptyset$. By assumption (3) the intersection $U \cap V$ is quasi-compact. By Lemma 5.9.4 for some $\alpha \in A$ we have $U \cap V \cap Z_\alpha = \emptyset$. Hence $Z_\alpha = U \cap Z_\alpha \coprod V \cap Z_\alpha$ is a decomposition into two open pieces, hence $U \cap Z_\alpha$ and $V \cap Z_\alpha$ are open and closed in X. Thus, if $x \in B$ say, then we see that $S \subset U \cap Z_\alpha$ and we conclude that $C = \emptyset$. □

Lemma 5.9.8. *Let X be a topological space. Assume*

(1) *X is quasi-compact,*
(2) *X has a basis for the topology consisting of quasi-compact opens, and*
(3) *the intersection of two quasi-compact opens is quasi-compact.*

For a subset $T \subset X$ the following are equivalent:

(a) *T is an intersection of open and closed subsets of X, and*
(b) *T is closed in X and is a union of connected components of X.*

Proof. It is clear that (a) implies (b). Assume (b). Let $x \in X$, $x \not\in T$. Let $x \in C \subset X$ be the connected component of X containing x. By Lemma 5.9.7 we see that $C = \bigcap V_\alpha$ is the intersection of all open and closed subsets V_α of X which contain C. In particular, any pairwise intersection $V_\alpha \cap V_\beta$ occurs as a V_α. As T is a union of connected components of X we see that $C \cap T = \emptyset$. Hence $T \cap \bigcap V_\alpha = \emptyset$. Since T is quasi-compact as a closed subset of a quasi-compact space (see Lemma 5.9.3) we deduce that $T \cap V_\alpha = \emptyset$ for some α, see Lemma 5.9.4. For this α we see that $U_\alpha = X \setminus V_\alpha$ is an open and closed subset of X which contains T and not x. The lemma follows. □

Lemma 5.9.9. *Let X be a Noetherian topological space.*

(1) *The space X is quasi-compact.*
(2) *Any subset of X is retrocompact.*

Proof. Suppose $X = \bigcup U_i$ is an open covering of X indexed by the set I which does not have a refinement by a finite open covering. Choose $i_1, i_2, \ldots$ elements of I inductively in the following way: If $X \neq U_{i_1} \cup \ldots \cup U_{i_n}$ then choose i_{n+1} such that $U_{i_{n+1}}$ is not contained in $U_{i_1} \cup \ldots \cup U_{i_n}$. Thus we see that $X \supset (X \setminus U_{i_1}) \supset (X \setminus U_{i_1} \cup U_{i_2}) \supset \ldots$ is a strictly decreasing infinite sequence of closed subsets. This contradicts the fact that X is Noetherian. This proves the first assertion. The second assertion is now clear since every subset of X is Noetherian by Lemma 5.6.2. □

Lemma 5.9.10. *A quasi-compact locally Noetherian space is Noetherian.*

Proof. The conditions imply immediately that X has a finite covering by Noetherian subsets, and hence is Noetherian by Lemma 5.6.4. □

5.10. Constructible sets

Definition 5.10.1. Let X be a topological space. Let $E \subset X$ be a subset of X.

(1) We say E is *constructible*[1] in X if E is a finite union of subsets of the form $U \cap V^c$ where $U, V \subset X$ are open and retrocompact in X.
(2) We say E is *locally constructible* in X if there exists an open covering $X = \bigcup V_i$ such that each $E \cap V_i$ is construcible in V_i.

Lemma 5.10.2. *The collection of constructible sets is closed under finite intersections, finite unions and complements.*

Proof. Note that if U_1, U_2 are open and retrocompact in X then so is $U_1 \cup U_2$ because the union of two quasi-compact subsets of X is quasi-compact. It is also true that $U_1 \cap U_2$ is retrocompact. Namely, suppose $U \subset X$ is quasi-compact open, then $U_2 \cap U$ is quasi-compact because U_2 is retrocompact in X, and then we conclude $U_1 \cap (U_2 \cap U)$ is quasi-compact because U_1 is retrocompact in X. From this it is formal to show that the complement of a constructible set is constructible, that finite unions of constructibles are constructible, and that finite intersections of constructibles are constructible. □

Lemma 5.10.3. *Let $f : X \to Y$ be a continuous map of topological spaces. If the inverse image of every retrocompact open subset of Y is retrocompact in X, then inverse images of constructible sets are constructible.*

Proof. This is true because $f^{-1}(U \cap V^c) = f^{-1}(U) \cap f^{-1}(V)^c$, combined with the definition of constructible sets. □

[1] In the second edition of EGA I [**GD71**] this was called a ``globally constructible'' set and a the terminology ``constructible'' was used for what we call a locally constructible set.

Lemma 5.10.4. *Let $U \subset X$ be open. For a constructible set $E \subset X$ the intersection $E \cap U$ is constructible in U.*

Proof. Suppose that $V \subset X$ is retrocompact open in X. It suffices to show that $V \cap U$ is retrocompact in U by Lemma 5.10.3. To show this let $W \subset U$ be open and quasi-compact. Then W is open and quasi-compact in X. Hence $V \cap W = V \cap U \cap W$ is quasi-compact as V is retrocompact in X. $\square$

Lemma 5.10.5. *Let X be a topological space. Let $E \subset X$ be a subset. Let $X = V_1 \cup \ldots \cup V_m$ be a finite covering by retrocompact opens. Then E is constructible in X if and only if $E \cap V_j$ is constructible in V_j for each $j = 1, \ldots, m$.*

Proof. If E is constructible in X, then by Lemma 5.10.4 we see that $E \cap V_j$ is construcible in V_j for all j. Conversely, suppose that $E \cap V_j$ is constructible in V_j for each $j = 1, \ldots, m$. Then E is a finite union of sets of the form $E' = U' \cap (V_j \setminus V')$ where U', V' are open and retrocompact subsets of V_j. Note that U' and V' are also open and retrocompact in X (as a composition of quasi-compact maps is quasi-compact, see Lemma 5.9.2). Since $E' = U' \cap (V')^c$ where the complement is in X we win. $\square$

Lemma 5.10.6. *Let X be a topological space. Suppose that $Z \subset X$ is irreducible. Let $E \subset X$ be a finite union of locally closed subsets (e.g. E is constructible). The following are equivalent*

(1) *The intersection $E \cap Z$ contains an open dense subset of Z.*
(2) *The intersection $E \cap Z$ is dense in Z.*

If Z has a generic point ξ, then this is also equivalent to

(3) *We have $\xi \in E$.*

Proof. Write $E = \bigcup U_i \cap Z_i$ as the finite union of intersections of open sets U_i and closed sets Z_i. Suppose that $E \cap Z$ is dense in Z. Note that the closure of $E \cap Z$ is the union of the closures of the intersections $U_i \cap Z_i \cap Z$. Hence we see that $U_i \cap Z_i \cap Z$ is dense in Z for some $i = i_0$. As Z is closed we have either $Z \cap Z_i = Z$ or $Z \cap Z_i$ is not dense, hence we conclude $Z \subset Z_{i_0}$. Then $U_{i_0} \cap Z_{i_0} \cap Z = U_{i_0} \cap Z$ is an open not empty subset of Z. Because Z is irreducible, it is open dense. The converse is obvious.

Suppose that $\xi \in Z$ is a generic point. Of course if (1) $\Leftrightarrow$ (2) holds, then $\xi \in E$. Conversely, if $\xi \in E$, then $\xi \in U_i \cap Z_i$ for some $i = i_0$. Clearly this implies $Z \subset Z_{i_0}$ and hence $U_{i_0} \cap Z_{i_0} \cap Z = U_{i_0} \cap Z$ is an open not empty subset of Z. We conclude as before. $\square$

5.11. Constructible sets and Noetherian spaces

Lemma 5.11.1. *Let X be a Noetherian topological space. Constructible sets in X are finite unions of locally closed subsets of X.*

Proof. This follows immediately from Lemma 5.9.9. $\square$

Lemma 5.11.2. *Let $f : X \to Y$ be a continuous map of Noetherian topological spaces. If $E \subset Y$ is constructible in Y, then $f^{-1}(E)$ is constructible in X.*

Proof. Follows immediately from Lemma 5.11.1 and the definition of a continuous map. $\square$

Lemma 5.11.3. *Let X be a Noetherian topological space. Let $E \subset X$ be a subset. The following are equivalent*

(1) *E is constructible in X, and*

(2) *for every irreducible closed $Z \subset X$ the intersection $E \cap Z$ either contains a nonempty open of Z or is not dense in Z.*

Proof. Assume E is constructible and $Z \subset X$ irreducible closed. Then $E \cap Z$ is constructible in Z by Lemma 5.11.2. Hence $E \cap Z$ is a finite union of nonempty locally closed subsets T_i of Z. Clearly if none of the T_i is open in Z, then $E \cap Z$ is not dense in Z. In this way we see that (1) implies (2).

Conversely, assume (2) holds. Consider the set $\mathcal{S}$ of closed subsets Y of X such that $E \cap Y$ is not constructible in Y. If $\mathcal{S} \neq \emptyset$, then it has a smallest element Y as X is Noetherian. Let $Y = Y_1 \cup \ldots \cup Y_r$ be the decomposition of Y into its irreducible components, see Lemma 5.6.2. If $r > 1$, then each $Y_i \cap E$ is constructible in Y_i and hence a finite union of locally closed subsets of Y_i. Thus $E \cap Y$ is a finite union of locally closed subsets of Y too and we conclude that $E \cap Y$ is constructible in Y by Lemma 5.11.1. This is a contradication and so $r = 1$. If $r = 1$, then Y is irreducible, and by assumption (2) we see that $E \cap Y$ either (a) contains an open V of Y or (b) is not dense in Y. In case (a) we see, by minimality of Y, that $E \cap (Y \setminus V)$ is a finite union of locally closed subsets of $Y \setminus V$. Thus $E \cap Y$ is a finite union of locally closed subsets of Y and is constructible by Lemma 5.11.1. This is a contradication and so we must be in case (b). In case (b) we see that $E \cap Y = E \cap Y'$ for some proper closed subset $Y' \subset Y$. By minimality of Y we see that $E \cap Y'$ is a finite union of locally closed subsets of Y' and we see that $E \cap Y' = E \cap Y$ is a finite union of locally closed subsets of Y and is constructible by Lemma 5.11.1. This contradication finishes the proof of the lemma. $\square$

Lemma 5.11.4. *Let X be a Noetherian topological space. Let $x \in X$. Let $E \subset X$ be constructible in X. The following are equivalent*

(1) *E is a neighbourhood of x, and*
(2) *for every irreducible closed subset Y of X which contains x the intersection $E \cap Y$ is dense in Y.*

Proof. It is clear that (1) implies (2). Assume (2). Consider the set $\mathcal{S}$ of closed subsets Y of X containing x such that $E \cap Y$ is not a neighbourhood of x in Y. If $\mathcal{S} \neq \emptyset$, then it has a smallest element Y as X is Noetherian. Let $Y = Y_1 \cup \ldots \cup Y_r$ be the decomposition of Y into its irreducible components, see Lemma 5.6.2. If $r > 1$, then each $Y_i \cap E$ is a neighbourhood of x in Y_i by minimality of Y. Thus $E \cap Y$ is a neighbourhood of x in Y. This is a contradication and so $r = 1$. If $r = 1$, then Y is irreducible, and by assumption (2) we see that $E \cap Y$ is dense in Y. Thus $E \cap Y$ contains an open V of Y, see Lemma 5.11.3. If $x \in V$ then $E \cap Y$ is a neighbourhood of x in Y which is a contradiction. If $x \notin V$, then $Y' = Y \setminus V$ is a proper closed subset of Y containing x. By minimality of Y we see that $E \cap Y'$ contains an open neighbourhood $V' \subset Y'$ of x in Y'. But then $V' \cup V$ is an open neighbourhood of x in Y contained in E, a contradiction. This contradication finishes the proof of the lemma. $\square$

Lemma 5.11.5. *Let X be a Noetherian topological space. Let $E \subset X$ be a subset. The following are equivalent*

(1) *E is open in X, and*
(2) *for every irreducible closed subset Y of X the intersection $E \cap Y$ is either empty or contains a nonempty open of Y.*

Proof. This follows formally from Lemmas 5.11.3 and 5.11.4. $\square$

5.12. Characterizing proper maps

We include a section discussing the notion of a proper map in usual topology. It turns out that in topology, the notion of being proper is the same as the notion of being universally closed, in the sense that any base change is a closed morphism (not just taking products with spaces). The reason for doing this is that in algebraic geometry we use this notion of universal closedness as the basis for our definition of properness.

Lemma 5.12.1. *(Tube lemma.) Let X and Y be topological spaces. Let $A \subset X$ and $B \subset Y$ be quasi-compact subsets. Let $A \times B \subset W \subset X \times Y$ with W open in $X \times Y$. Then there exists opens $A \subset U \subset X$ and $B \subset V \subset Y$ such that $U \times V \subset W$.*

Proof. For every $a \in A$ and $b \in B$ there exist opens $U_{(a,b)}$ of X and $V_{(a,b)}$ of Y such that $(a, b) \in U_{(a,b)} \times V_{(a,b)} \subset W$. Fix b and we see there exist a finite number $a_1, \ldots, a_n$ such that $A \subset U_{(a_1,b)} \cup \ldots \cup U_{(a_n,b)}$. Hence $A \times \{b\} \subset (U_{(a_1,b)} \cup \ldots \cup U_{(a_n,b)}) \times (V_{(a_1,b)} \cup \ldots \cup V_{(a_n,b)}) \subset W$. Thus for every $b \in B$ there exists opens $U_b \subset X$ and $V_b \subset Y$ such that $A \times \{b\} \subset U_b \times V_b \subset W$. As above there exist a finite number $b_1, \ldots, b_m$ such that $B \subset V_{b_1} \cup \ldots \cup V_{b_m}$. Then we win because $A \times B \subset (U_{b_1} \cap \ldots \cap U_{b_m}) \times (V_{b_1} \cup \ldots \cup V_{b_m})$. $\square$

The notation in the following definition may be slightly different from what you are used to.

Definition 5.12.2. Let $f : X \to Y$ be a continuous map between topological spaces.

(1) We say that the map f is *closed* iff the image of every closed subset is closed.
(2) We say that the map f is *proper*[2] iff the map $Z \times X \to Z \times Y$ is closed for any topological space Z.
(3) We say that the map f is *quasi-proper* iff the inverse image $f^{-1}(V)$ of every quasi-compact $V \subset Y$ is quasi-compact.
(4) We say that f is *universally closed* iff the map $f' : Z \times_Y X \to Z$ is closed for any map $g : Z \to Y$.

The following lemma is useful later.

Lemma 5.12.3. *A topological space X is quasi-compact if and only if the projection map $Z \times X \to Z$ is closed for any topological space Z.*

Proof. (See also remark below.) If X is not quasi-compact, there exists an open covering $X = \bigcup_{i \in I} U_i$ such that no finite number of U_i cover X. Let Z be the subset of the power set $\mathcal{P}(I)$ of I consisting of I and all nonempty finite subsets of I. Define a topology on Z with as a basis for the topology the following sets:

(1) All subsets of $Z \setminus \{I\}$.
(2) The empty set.
(3) For every finite subset K of I the set $U_K := \{J \subset I \mid J \in Z,\ K \subset J\}$).

It is left to the reader to verify this is the basis for a topology. Consider the subset of $Z \times X$ defined by the formula

$$M = \{(J, x) \mid J \in Z,\ x \in \bigcap\nolimits_{i \in J} U_i^c)\}$$

If $(J, x) \notin M$, then $x \in U_i$ for some $i \in J$. Hence $U_{\{i\}} \times U_i \subset Z \times X$ is an open subset containing (J, x) and not intersecting M. Hence M is closed. The projection of M to Z is $Z - \{I\}$ which is not closed. Hence $Z \times X \to Z$ is not closed.

[2] This is the terminology used in [**Bou71**]. Usually this is what is called ``universally closed'' in the literature. Thus our notion of proper does not involve any separation conditions.

Assume X is quasi-compact. Let Z be a topological space. Let $M \subset Z \times X$ be closed. Let $z \in Z$ be a point which is not in $\mathrm{pr}_1(M)$. By the Tube Lemma 5.12.1 there exists an open $U \subset Z$ such that $U \times X$ is contained in the complement of M. Hence $\mathrm{pr}_1(M)$ is closed. □

Remark 5.12.4. Lemma 5.12.3 is a combination of [**Bou71**, I, p. 75, Lemme 1] and [**Bou71**, I, p. 76, Corrolaire 1].

Theorem 5.12.5. *Let $f : X \to Y$ be a continuous map between topological spaces. The following condition is equivalent.*

(1) *The map f is quasi-proper and closed.*
(2) *The map f is proper.*
(3) *The map f is universally closed.*
(4) *The map f is closed and $f^{-1}(y)$ is quasi-compact for any $y \in Y$.*

Proof. (See also the remark below.) If the map f satisfies (1), it automatically satisfies (4) because any single point is quasi-compact.

Assume map f satisfies (4). We will prove it is universally closed, i.e., (3) holds. Let $g : Z \to Y$ be a continuous map of topological spaces and consider the diagram

$$\begin{tikzcd} Z \times_Y X \arrow[r, "g'"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Z \arrow[r, "g"] & Y \end{tikzcd}$$

During the proof we will use that $Z \times_Y X \to Z \times X$ is a homeomorphism onto its image, i.e., that we may identify $Z \times_Y X$ with the corresponding subset of $Z \times X$ with the induced topology. The image of $f' : X \times_Y Z \to Z$ is $\mathrm{Im}(f') = \{z : g(z) \in f(X)\}$. Because $f(X)$ is closed, we see that $\mathrm{Im}(f')$ is a closed subspace of Z. Consider a closed subset $P \subset X \times_Y Z$. Let $z \in Z$, $z \not\in f'(P)$. If $z \not\in \mathrm{Im}(f')$, then $Z \setminus \mathrm{Im}(f')$ is an open neighbourhood which avoids $f'(P)$. If z is in $\mathrm{Im}(f')$ then $(f')^{-1}\{z\} = \{z\} \times f^{-1}\{g(z)\}$ and $f^{-1}\{g(z)\}$ is quasi-compact by assumption. Because P is a closed subset of $Z \times_Y X$, we have a closed P' of $Z \times X$ such that $P = P' \cap Z \times_Y X$. Since $(f')^{-1}\{z\}$ is a subset of $P^c = P'^c \cup (Z \times_Y X)^c$, we see that $(f')^{-1}\{z\}$ is disjoint from $(Z \times_Y X)^c$. Hence $(f')^{-1}\{z\}$ is contained in P'^c. We may apply the Tube Lemma 5.12.1 to $(f')^{-1}\{z\} = \{z\} \times f^{-1}\{g(z)\} \subset (P')^c \subset Z \times X$. This gives $U \times V$ containing $(f')^{-1}\{z\}$ where U and V are open sets in X and Z respectively and $U \times V$ has empty intersection with P'. Hence z is contained in V and V has empty intersection with the image of P. As a result, the map f is universally closed.

The implication (3) $\Rightarrow$ (2) is trivial. Namely, given any topological space Z consider the projection morphism $g : Z \times Y \to Y$. Then it is easy to see that f' is the map $Z \times X \to Z \times Y$, in other words that $(Z \times Y) \times_Y X = Z \times X$. (This identification is a purely categorical property having nothing to do with topological spaces per se.)

Assume f satisfies (2). We will prove it satisfies (1). Note that f is closed as f can be identified with the map $\{pt\} \times X \to \{pt\} \times Y$ which is assumed closed. Choose any quasi-compact subset $K \subset Y$. Let Z be any topological space. Because $Z \times X \to Z \times Y$ is closed we see the map $Z \times f^{-1}(K) \to Z \times K$ is closed (if T is closed in $Z \times f^{-1}(K)$, write $T = Z \times f^{-1}(K) \cap T'$ for some closed $T' \subset Z \times X$). Because K is quasi-compact, $K \times Z \to Z$ is closed by Lemma 5.12.3. Hence the composition $Z \times f^{-1}(K) \to Z \times K \to Z$ is closed and therefore $f^{-1}(K)$ must be quasi-compact by Lemma 5.12.3 again. □

Remark 5.12.6. Here are some references to the literature. In [**Bou71**, I, p. 75, Theorem 1] you can find: (2) ⇔ (4). In [**Bou71**, I, p. 77, Proposition 6] you can find: (2) ⇒ (1). Of course, trivially we have (1) ⇒ (4). Thus (1), (2) and (4) are equivalent. Fan Zhou claimed and proved that (3) and (4) are equivalent; let me know if you find a reference in the literature.

5.13. Jacobson spaces

Definition 5.13.1. Let X be a topological space. Let X_0 be the set of closed points of X. We say that X is *Jacobson* if every closed subset $Z \subset X$ is the closure of $Z \cap X_0$.

Let X be a Jacobson space and let X_0 be the set of closed points of X with the induced topology. Clearly, the definition implies that the morphism $X_0 \to X$ induces a bijection between the closed subsets of X_0 and the closed subsets of X. Thus many properties of X are inherted by X_0. For example, the Krull dimensions of X and X_0 are the same.

Lemma 5.13.2. *Let X be a topological space. Let X_0 be the set of closed points of X. Suppose that for every irreducible closed subset $Z \subset X$ the intersection $X_0 \cap Z$ is dense in Z. Then X is Jacobson.*

Proof. Let $Z \subset X$ be closed. According to Lemma 5.5.3 we have $Z = \bigcup Z_i$ with Z_i irreducible and closed. Thus is $X_0 \cap Z_i$ is dense in each Z_i, then $X_0 \cap Z$ is dense in Z. □

Lemma 5.13.3. *Let X be a sober, Noetherian topological space. If X is not Jacobson, then there exists a non-closed point $\xi \in X$ such that $\{\xi\}$ is locally closed.*

Proof. Assume X is sober, Noetherian and not Jacobson. By Lemma 5.13.2 there exists an irreducible closed subset $Z \subset X$ which is not the closure of its closed points. Since X is Noetherian we may assume Z is minimal with this property. Let $\xi \in Z$ be the unique generic point (here we use X is sober). Note that the closed points are dense in $\overline{\{z\}}$ for any $z \in Z$, $z \neq \xi$ by minimality of Z. Hence the closure of the set of closed points of Z is a closed subset containing all $z \in Z$, $z \neq \xi$. Hence $\{\xi\}$ is locally closed as desired. □

Lemma 5.13.4. *Let X be a topological space. Let $X = \bigcup U_i$ be an open covering. Then X is Jacobson if and only if each U_i is Jacobson. Moreover, in this case $X_0 = \bigcup U_{i,0}$.*

Proof. Let X be a topological space. Let X_0 be the set of closed points of X. Let $U_{i,0}$ be the set of closed points of U_i. Then $X_0 \cap U_i \subset U_{i,0}$ but equality may not hold in general.

First, assume that each U_i is Jacobson. We claim that in this case $X_0 \cap U_i = U_{i,0}$. Namely, suppose that $x \in U_{i,0}$, i.e., x is closed in U_i. Let $\overline{\{x\}}$ be the closure in X. Consider $\overline{\{x\}} \cap U_j$. If $x \notin U_j$, then $\overline{\{x\}} \cap U_j = \emptyset$. If $x \in U_j$, then $U_i \cap U_j \subset U_j$ is an open subset of U_j containing x. Let $T' = U_j \setminus U_i \cap U_j$ and $T = \{x\} \coprod T'$. Then T, T' are closed subsets of U_j and T contains x. As U_j is Jacobson we see that the closed points of U_j are dense in T. Because $T = \{x\} \coprod T'$ this can only be the case if x is closed in U_j. Hence $\overline{\{x\}} \cap U_j = \{x\}$. We conlude that $\overline{\{x\}} = \{x\}$ as desired.

Let $Z \subset X$ be a closed subset (still assuming each U_i is Jacobson). Since now we know that $X_0 \cap Z \cap U_i = U_{i,0} \cap Z$ are dense in $Z \cap U_i$ it follows immediately that $X_0 \cap Z$ is dense in Z.

Conversely, assume that X is Jacobson. Let $Z \subset U_i$ be closed. Then $X_0 \cap \overline{Z}$ is dense in $\overline{Z}$. Hence also $X_0 \cap Z$ is dense in Z, because $\overline{Z} \setminus Z$ is closed. As $X_0 \cap U_i \subset U_{i,0}$ we see that $U_{i,0} \cap Z$ is dense in Z. Thus U_i is Jacobson as desired. □

Lemma 5.13.5. *Let X be Jacobson. The following types of subsets $T \subset X$ are Jacobson:*

1. *Open subspaces.*
2. *Closed subspaces.*
3. *Locally closed subspaces.*
4. *Finite unions of locally closed subspaces.*
5. *Constructible sets.*
6. *Any subset $T \subset X$ which locally on X is a finite union of locally closed subsets.*

In each of these cases closed points of T are closed in X.

Proof. Let X_0 be the set of closed points of X. For any subset $T \subset X$ we let $(*)$ denote the property:

$(*)$ For every closed subset $Z \subset T$ the set $Z \cap X_0$ is dense in Z.

Note that always $X_0 \cap T \subset T_0$. Hence property $(*)$ implies that T is Jacobson. In addition it clearly implies that every closed point of T is closed in X.

Let $U \subset X$ be an open subset. Suppose $Z \subset U$ is closed. Then $X_0 \cap \overline{Z}$ is dense in $\overline{Z}$. Hence $X_0 \cap Z$ is dense in Z, because $\overline{Z} \setminus Z$ is closed. Thus $(*)$ holds.

Let $Z \subset X$ be a closed subset. Since closed subsets of Z are the same as closed subsets of X contained in Z property $(*)$ is immediate.

Let $T \subset X$ be locally closed. Write $T = U \cap Z$ for some open $U \cap X$ and some closed $Z \subset X$. Note that closed subsets of T are the same thing as closed subsets of U which happen to be contained in Z. Hence $(*)$ holds for T because we proved it for U above.

Suppose $T_i \subset X$, $i = 1, \ldots, n$ are locally closed subsets. Let $T = T_1 \cup \ldots \cup T_n$. Suppose $Z \subset T$ is closed. Then $Z_i = Z \cap T_i$ is closed in T_i. By $(*)$ for T_i we see that $Z_i \cap X_0$ is dense in Z_i. Clearly this implies that $X_0 \cap Z$ is dense in Z, and property $(*)$ holds for T.

The case of constructible subsets is subsumed in the case of finite unions of locally closed subsets, see Definition 5.10.1.

The condition of the last assertion means that there exists an open covering $X = \bigcup U_i$ such that each $T \cap U_i$ is a finite union of locally closed subsets of U_i. We conclude that T is Jacobson by Lemma 5.13.4 and the case of a finite union of locally closed subsets dealt with above. It is formal to deduce $(*)$ for T from $(*)$ for all the inclusions $T \cap U_i \subset U_i$ and the assertions $X_0 = \bigcup U_{i,0}$ and $T_0 = \bigcup (T \cap U_i)_0$ from Lemma 5.13.4. □

Lemma 5.13.6. *A finite Kolmogorov Jacobson space is discrete.*

Proof. By induction on the number of points. The lemma holds if the space is empty. If X is a non-empty finite Kolmogorov space, choose a closed point $x \in X$, see Lemma 5.9.6. Then $U = X \setminus \{x\}$ is a finite Jacobson space, see Lemma 5.13.5. By induction U is a finite discrete space, hence all its points are closed. By Lemma 5.13.5 all the points of U are also closed in X and we win. □

Lemma 5.13.7. *Suppose X is a Jacobson topological space. Let X_0 be the set of closed points of X. There is a bijective, inclusion preserving correspondence*

$$\{\text{constructible subsets of } X\} \leftrightarrow \{\text{constructible subsets of } X_0\}$$

given by $E \mapsto E \cap X_0$. This correspondence preserves the subset of retrocompact open subsets, as well as complements of these.

Proof. Obvious from Lemma 5.13.5 above. □

Lemma 5.13.8. *Suppose X is a Jacobson topological space. Let X_0 be the set of closed points of X. There is a bijective, inclusion preserving correspondence*

$$\{fiX\} \leftrightarrow \{fiX_0\}$$

given by $E \mapsto E \cap X_0$. This correspondence preserves the subsets of locally closed, of open and of closed subsets.

Proof. Obvious from Lemma 5.13.5 above. □

5.14. Specialization

Definition 5.14.1. Let X be a toplogical space.

(1) If $x, x' \in X$ then we say x is a *specialization* of x', or x' is a *generalization* of x if $x \in \overline{\{x'\}}$. Notation: $x' \rightsquigarrow x$.
(2) A subset $T \subset X$ is *stable under specialization* if for all $x' \in T$ and every specialization $x' \rightsquigarrow x$ we have $x \in T$.
(3) A subset $T \subset X$ is *stable under generalization* if for all $x \in T$ and every generalization x' of x we have $x' \in T$.

Lemma 5.14.2. *Let X be a toplogical space.*

(1) *Any closed subset of X is stable under specialization.*
(2) *Any open subset of X is stable under generalization.*
(3) *A subset $T \subset X$ is stable under specialization if and only if the complement T^c is stable under generalization.*

Proof. Omitted. □

Definition 5.14.3. Let $f : X \to Y$ be a continuous map of topological spaces.

(1) We say that *specializations lift along f* or that f is *specializing* if given $y' \rightsquigarrow y$ in Y and any $x' \in X$ with $f(x') = y'$ there exists a specialization $x' \rightsquigarrow x$ of x' in X such that $f(x) = y$.
(2) We say that *generalizations lift along f* or that f is *generalizing* if given $y' \rightsquigarrow y$ in Y and any $x \in X$ with $f(x) = y$ there exists a generalization $x' \rightsquigarrow x$ of x in X such that $f(x') = y'$.

Lemma 5.14.4. *Suppose $f : X \to Y$ and $g : Y \to Z$ are continuous maps of topological spaces. If specializations lift along both f and g then specializations lift along $g \circ f$. Similarly for ``generalizations lift along''.*

Proof. Omitted. □

Lemma 5.14.5. *Let $f : X \to Y$ be a continuous map of topological spaces.*

(1) *If specializations lift along f, and if $T \subset X$ is stable under specialization, then $f(T) \subset Y$ is stable under specialization.*
(2) *If generalizations lift along f, and if $T \subset X$ is stable under generalization, then $f(T) \subset Y$ is stable under generalization.*

Proof. Omitted. □

Lemma 5.14.6. *Let $f : X \to Y$ be a continuous map of topological spaces.*

(1) *If f is closed then specializations lift along f.*
(2) *If f is open, X is a Noetherian topological space, each irreducible closed subset of X has a generic point, and Y is Kolmogorov then generalizations lift along f.*

Proof. Assume f is closed. Let $y' \rightsquigarrow y$ in Y and any $x' \in X$ with $f(x') = y'$ be given. Consider the closed subset $T = \overline{\{x'\}}$ of X. Then $f(T) \subset Y$ is a closed subset, and $y' \in f(T)$. Hence also $y \in f(T)$. Hence $y = f(x)$ with $x \in T$, i.e., $x' \rightsquigarrow x$.

Assume f is open, X Noetherian, every irreducible closed subset of X has a generic point, and Y is Kolmogorov. Let $y' \rightsquigarrow y$ in Y and any $x \in X$ with $f(x) = y$ be given. Consider $T = f^{-1}(\{y'\}) \subset X$. Take an open neighbourhood $x \in U \subset X$ of x. Then $f(U) \subset Y$ is open and $y \in f(U)$. Hence also $y' \in f(U)$. In other words, $T \cap U \neq \emptyset$. This proves that $x \in \overline{T}$. Since X is Noetherian, T is Noetherian (Lemma 5.6.2). Hence it has a decomposition $T = T_1 \cup \ldots \cup T_n$ into irreducible components. Then correspondingly $\overline{T} = \overline{T_1} \cup \ldots \cup \overline{T_n}$. By the above $x \in \overline{T_i}$ for some i. By assumption there exists a generic point $x' \in \overline{T_i}$, and we see that $x' \rightsquigarrow x$. As $x' \in \overline{T}$ we see that $f(x') \in \overline{\{y'\}}$. Note that $f(\overline{T_i}) = f(\overline{\{x'\}}) \subset \overline{\{f(x')\}}$. If $f(x') \neq y'$, then since Y is Kolmogorov $f(x')$ is not a generic point of the irreducible closed subset $\overline{\{y'\}}$ and the inclusion $\overline{\{f(x')\}} \subset \overline{\{y'\}}$ is strict, i.e., $y' \notin f(\overline{T_i})$. This contradicts the fact that $f(T_i) = \{y'\}$. Hence $f(x') = y'$ and we win. $\square$

Lemma 5.14.7. *Suppose that $s, t : R \to U$ and $\pi : U \to X$ are continuous maps of topological spaces such that*

1. *π is open,*
2. *U is sober,*
3. *s, t have finite fibres,*
4. *generalizations lift along s, t,*
5. *$(t, s)(R) \subset U \times U$ is an equivalence relation on U and X is the quotient of U by this equivalence relation (as a set).*

Then X is Kolmogorov.

Proof. Properties (3) and (5) imply that a point x corresponds to an finite equivalence class $\{u_1, \ldots, u_n\} \subset U$ of the equivalence relation. Suppose that $x' \in X$ is a second point corresponding to the equivalence class $\{u'_1, \ldots, u'_m\} \subset U$. Suppose that $u_i \rightsquigarrow u'_j$ for some i, j. Then for any $r' \in R$ with $s(r') = u'_j$ by (4) we can find $r \rightsquigarrow r'$ with $s(r) = u_i$. Hence $t(t) \rightsquigarrow t(r')$. Since $\{u'_1, \ldots, u'_m\} = t(s^{-1}(\{u'_j\}))$ we conclude that every element of $\{u'_1, \ldots, u'_m\}$ is the specialization of an element of $\{u_1, \ldots, u_n\}$. Thus $\overline{\{u_1\}} \cup \ldots \cup \overline{\{u_n\}}$ is a union of equivalence classes, hence of the form $\pi^{-1}(Z)$ for some subset $Z \subset X$. By (1) we see that Z is closed in X and in fact $Z = \overline{\{x\}}$ because $\pi(\overline{\{u_i\}}) \subset \overline{\{x\}}$ for each i. In other words, $x \rightsquigarrow x'$ if and only if some lift of x in U specializes to some lift of x' in U, if and only if every lift of x' in U is a specialization of some lift of x in U.

Suppose that both $x \rightsquigarrow x'$ and $x' \rightsquigarrow x$. Say x corresponds to $\{u_1, \ldots, u_n\}$ and x' corresponds to $\{u'_1, \ldots, u'_m\}$ as above. Then, by the resuls of the preceding paragraph, we can find a sequence

$$\ldots \rightsquigarrow u'_{j_3} \rightsquigarrow u_{i_3} \rightsquigarrow u'_{j_2} \rightsquigarrow u_{i_2} \rightsquigarrow u'_{j_1} \rightsquigarrow u_{i_1}$$

which must repeat, hence by (2) we conclude that $\{u_1, \ldots, u_n\} = \{u'_1, \ldots, u'_m\}$, i.e., $x = x'$. Thus X is Kolmogorov. $\square$

Lemma 5.14.8. *Let $f : X \to Y$ be a morphism of topological spaces. Suppose that Y is a sober topological space, and f is surjective. If either specializations or generalizations lift along f, then* $\dim(X) \geq \dim(Y)$.

Proof. Assume specializations lift along f. Let $Z_0 \subset Z_1 \subset \ldots Z_e \subset Y$ be a chain of irreducible closed subsets of X. Let $\xi_e \in X$ be a point mapping to the generic point of

Z_e. By assumption there exists a specialization $\xi_e \rightsquigarrow \xi_{e-1}$ in X such that ξ_{e-1} maps to the generic point of Z_{e-1}. Continuing in this manner we find a sequence of specializations

$$\xi_e \rightsquigarrow \xi_{e-1} \rightsquigarrow \ldots \rightsquigarrow \xi_0$$

with ξ_i mapping to the generic point of Z_i. This clearly implies the sequence of irreducible closed subsets

$$\overline{\{\xi_0\}} \subset \overline{\{\xi_1\}} \subset \ldots \overline{\{\xi_e\}}$$

is a chain of length e in X. The case when generalizations lift along f is similar. □

Lemma 5.14.9. *Let X be a Noetherian sober topological space. Let $E \subset X$ be a subset of X.*

1. *If E is constructible and stable under specialization, then E is closed.*
2. *If E is constructible and stable under generalization, then E is open.*

Proof. Let E be constructible and stable under generalization. Let $Y \subset X$ be an irreducible closed subset with generic point $\xi \in Y$. If $E \cap Y$ is nonempty, then it contains ξ (by stability under generalization) and hence is dense in Y, hence it contains a nonempty open of Y, see Lemma 5.11.3. Thus E is open by Lemma 5.11.5. This proves (2). To prove (1) apply (2) to the complement of E in X. □

5.15. Submersive maps

Definition 5.15.1. Let $f : X \to Y$ be a continuous map of topological spaces. We say f is *submersive*[3] if f is surjective and for any $T \subset Y$ we have T is open or closed if and only if $f^{-1}(T)$ is so.

Another way to express the second condition is that Y has the quotient topology relative to the map $X \to Y$. Here is an example where this holds.

Lemma 5.15.2. *Let $f : X \to Y$ be surjective, open, continuous map of topological spaces. Let $T \subset Y$ be a subset. Then*

1. $f^{-1}(\overline{T}) = \overline{f^{-1}(T)}$,
2. *$T \subset Y$ is closed if and only $f^{-1}(T)$ is closed,*
3. *$T \subset Y$ is open if and only $f^{-1}(T)$ is open, and*
4. *$T \subset Y$ is locally closed if and only $f^{-1}(T)$ is locally closed.*

In particular we see that f is submersive.

Proof. It is clear that $\overline{f^{-1}(T)} \subset f^{-1}(\overline{T})$. If $x \in X$, and $x \not\in \overline{f^{-1}(T)}$, then there exists an open neighbourhood $x \in U \subset X$ with $U \cap f^{-1}(T) = \emptyset$. Since f is open we see that $f(U)$ is an open neighbourhood of $f(x)$ not meeting T. Hence $x \not\in f^{-1}(\overline{T})$. This proves (1). Part (2) is an easy consequences of this. Part (3) is obvious from the fact that f is open. For (4), if $f^{-1}(T)$ is locally closed, then $f^{-1}(T) \subset \overline{f^{-1}(T)} = f^{-1}(\overline{T})$ is open, and hence by (3) applied to the map $f^{-1}(\overline{T}) \to \overline{T}$ we see that T is open in $\overline{T}$, i.e., T is locally closed. □

[3]This is very different from the notion of a submersion between differential manifolds!

5.16. Dimension functions

It scarcely makes sense to consider dimension functions unless the space considered is sober (Definition 5.5.4). Thus the definition below can be improved by considering the sober topological space associated to X. Since the underlying topological space of a scheme is sober we do not bother with this improvement.

Definition 5.16.1. Let X be a topological space.

(1) Let $x, y \in X$, $x \neq y$. Suppose $x \rightsquigarrow y$, that is y is a specialization of x. We say y is an *immediate specialization* of x if there is no $z \in X \setminus \{x, y\}$ with $x \rightsquigarrow z$ and $z \rightsquigarrow y$.

(2) A map $\delta : X \to \mathbf{Z}$ is called a *dimension function*[4] if
   (a) whenever $x \rightsquigarrow y$ and $x \neq y$ we have $\delta(x) > \delta(y)$, and
   (b) for every immediate specialization $x \rightsquigarrow y$ in X we have $\delta(x) = \delta(y) + 1$.

It is clear that if δ is a dimension function, then so is $\delta + t$ for any $t \in \mathbf{Z}$. Here is a fun lemma.

Lemma 5.16.2. *Let X be a topological space. If X is sober and has a dimension function, then X is catenary. Moreover, for any $x \rightsquigarrow y$ we have*

$$\delta(x) - \delta(y) = \mathit{codim}\left(\overline{\{y\}}, \overline{\{x\}}\right).$$

Proof. Suppose $Y \subset Y' \subset X$ are irreducible closed subsets. Let $\xi \in Y$, $\xi' \in Y'$ be their generic points. Then we see immediately from the definitions that $\text{codim}(Y, Y') \leq \delta(\xi) - \delta(\xi') < \infty$. In fact the first inequality is an equality. Namely, suppose

$$Y = Y_0 \subset Y_1 \subset \ldots \subset Y_e = Y'$$

is any maximal chain of irreducible closed subsets. Let $\xi_i \in Y_i$ denote the generic point. Then we see that $\xi_i \rightsquigarrow \xi_{i+1}$ is an immediate specialization. Hence we see that $e = \delta(\xi) - \delta(\xi')$ as desired. This also proves the last statement of the lemma. $\square$

Lemma 5.16.3. *Let X be a topological space. Let δ, δ' be two dimension functions on X. If X is locally Noetherian and sober then $\delta - \delta'$ is locally constant on X.*

Proof. Let $x \in X$ be a point. We will show that $\delta - \delta'$ is constant in a neighbourhood of x. We may replace X by an open neighbourhood of x in X which is Noetherian. Hence we may assume X is Noetherian and sober. Let $Z_1, \ldots, Z_r$ be the irreducible components of X passing through x. (There are finitely many as X is Noetherian, see Lemma 5.6.2.) Let $\xi_i \in Z_i$ be the generic point. Note $Z_1 \cup \ldots \cup Z_r$ is a neighbourhood of x in X (not necessarily closed). We claim that $\delta - \delta'$ is constant on $Z_1 \cup \ldots \cup Z_r$. Namely, if $y \in Z_i$, then

$$\delta(x) - \delta(y) = \delta(x) - \delta(\xi_i) + \delta(\xi_i) - \delta(y) = -\text{codim}(\overline{\{x\}}, Z_i) + \text{codim}(\overline{\{y\}}, Z_i)$$

by Lemma 5.16.2. Similarly for δ'. Whence the result. $\square$

Lemma 5.16.4. *Let X be locally Noetherian, sober and catenary. Then any point has an open neighbourhood $U \subset X$ which has a dimension function.*

[4]This is likely nonstandard notation. This notion is usually introduced only for (locally) Noetherian schemes, in which case condition (a) is implied by (b).

Proof. We will use repeatedly that an open subspace of a catenary space is catenary, see Lemma 5.8.2 and that a Noetherian topological space has finitely many irreducible components, see Lemma 5.6.2. In the proof of Lemma 5.16.3 we saw how to construct such a function. Namely, we first replace X by a Noetherian open neighbourhood of x. Next, we let $Z_1, \ldots, Z_r \subset X$ be the irreducible components of X. Let

$$Z_i \cap Z_j = \bigcup Z_{ijk}$$

be the decomposition into irreducible components. We replace X by

$$X \setminus \left(\bigcup\nolimits_{x \notin Z_i} Z_i \cup \bigcup\nolimits_{x \notin Z_{ijk}} Z_{ijk}\right)$$

so that we may assume $x \in Z_i$ for all i and $x \in Z_{ijk}$ for all i, j, k. For $y \in X$ choose any i such that $y \in Z_i$ and set

$$\delta(y) = -\text{codim}(\overline{\{x\}}, Z_i) + \text{codim}(\overline{\{y\}}, Z_i).$$

We claim this is a dimension function. First we show that it is well defined, i.e., independent of the choice of i. Namely, suppose that $y \in Z_{ijk}$ for some i, j, k. Then we have (using Lemma 5.8.6)

$$\begin{aligned}
\delta(y) &= -\text{codim}(\overline{\{x\}}, Z_i) + \text{codim}(\overline{\{y\}}, Z_i) \\
&= -\text{codim}(\overline{\{x\}}, Z_{ijk}) - \text{codim}(Z_{ijk}, Z_i) + \text{codim}(\overline{\{y\}}, Z_{ijk}) + \text{codim}(Z_{ijk}, Z_i) \\
&= -\text{codim}(\overline{\{x\}}, Z_{ijk}) + \text{codim}(\overline{\{y\}}, Z_{ijk})
\end{aligned}$$

which is symmetric in i and j. We omit the proof that it is a dimension function. □

Remark 5.16.5. Combining Lemmas 5.16.3 and 5.16.4 we see that on a catenary, locally Noetherian, sober topological space the obstruction to having a dimension function is an element of $H^1(X, \mathbf{Z})$.

5.17. Nowhere dense sets

Definition 5.17.1. Let X be a topological space.

(1) Given a subset $T \subset X$ the *interior* of T is the largest open subset of X contained in T.
(2) A subset $T \subset X$ is called *nowhere dense* if the closure of T has empty interior.

Lemma 5.17.2. *Let X be a topological space. The union of a finite number of nowhere dense sets is a nowhere dense set.*

Proof. Omitted. □

Lemma 5.17.3. *Let X be a topological space. Let $U \subset X$ be an open. Let $T \subset U$ be a subset. If T is nowhere dense in U, then T is nowhere dense in X.*

Proof. Assume T is nowhere dense in U. Suppose that $x \in X$ is an interior point of the closure $\overline{T}$ of T in X. Say $x \in V \subset \overline{T}$ with $V \subset X$ open in X. Note that $\overline{T} \cap U$ is the closure of T in U. Hence the interior of $\overline{T} \cap U$ being empty implies $V \cap U = \emptyset$. Thus x cannot be in the closure of U, a fortiori cannot be in the closure of T, a contradiction. □

Lemma 5.17.4. *Let X be a topological space. Let $X = \bigcup U_i$ be an open covering. Let $T \subset X$ be a subset. If $T \cap U_i$ is nowhere dense in U_i for all i, then T is nowhere dense in X.*

Proof. Omitted. (Hint: closure commutes with intersecting with opens.) □

Lemma 5.17.5. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $T \subset X$ be a subset. If f identifies X with a closed subset of Y and T is nowhere dense in X, then also $f(T)$ is nowhere dense in Y.*

Proof. Omitted. □

Lemma 5.17.6. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $T \subset Y$ be a subset. If f is open and T is a closed nowhere dense subset of Y, then also $f^{-1}(T)$ is a closed nowhere dense subset of X. If f is surjective and open, then T is closed nowhere dense if and only if $f^{-1}(T)$ is closed nowhere dense.*

Proof. Omitted. (Hint: In the first case the interior of $f^{-1}(T)$ maps into the interior of T, and in the second case the interior of $f^{-1}(T)$ maps onto the interior of T.) □

5.18. Miscellany

Recall that a neighbourhood of a point need not be open.

Definition 5.18.1. A topological space X is called *locally quasi-compact*[5] if every point has a fundamental system of quasi-compact neighbourhoods.

The following lemma applies to the underlying topological space associated to a quasi-separated scheme.

Lemma 5.18.2. *Let X be a topological space which*

(1) *has a basis of the topology consisting of quasi-compact opens, and*
(2) *has the property that the intersection of any two quasi-compact opens is quasi-compact.*

Then

(1) *any quasi-compact open $U \subset X$ has a cofinal system of open coverings $\mathcal{U} : U = \bigcup_{j \in J} U_j$ with J finite and all $U_j \cap U_{j'}$ quasi-compact for all $j, j' \in J$*
(2) *add more here.*

Proof. Omitted. □

Definition 5.18.3. Let X be a topological space. We say $x \in X$ is an *isolated point* of X if $\{x\}$ is open in X.

5.19. Other chapters

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Commutative Algebra
(8) Brauer Groups
(9) Sites and Sheaves
(10) Homological Algebra
(11) Derived Categories
(12) More on Algebra
(13) Smoothing Ring Maps
(14) Simplicial Methods
(15) Sheaves of Modules
(16) Modules on Sites
(17) Injectives
(18) Cohomology of Sheaves
(19) Cohomology on Sites
(20) Hypercoverings

[5]This may not be standard notation. Alternative notions used in the literature are: (1) Every point has some quasi-compact neighbourhood, and (2) Every point has a closed quasi-compact neighbourhood. A scheme has the property that every point has a fundamental system of open quasi-compact neighbourhoods.

(21) Schemes
(22) Constructions of Schemes
(23) Properties of Schemes
(24) Morphisms of Schemes
(25) Coherent Cohomology
(26) Divisors
(27) Limits of Schemes
(28) Varieties
(29) Chow Homology
(30) Topologies on Schemes
(31) Descent
(32) Adequate Modules
(33) More on Morphisms
(34) More on Flatness
(35) Groupoid Schemes
(36) More on Groupoid Schemes
(37) Étale Morphisms of Schemes
(38) Étale Cohomology
(39) Crystalline Cohomology
(40) Algebraic Spaces
(41) Properties of Algebraic Spaces
(42) Morphisms of Algebraic Spaces
(43) Decent Algebraic Spaces
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CHAPTER 6

Sheaves on Spaces

6.1. Introduction

Basic properties of sheaves on topological spaces will be explained in this document. A reference is [**God73**].

This will be superceded by the discussion of sheaves over sites later in the documents. But perhaps it makes sense to briefly define some of the notions here.

6.2. Basic notions

The following notions are considered basic and will not be defined, and or proved. This does not mean they are all necessarily easy or well known.

(1) Let X be a topological space. The phrase: ``Let $U = \bigcup_{i \in I} U_i$ be an open covering" means the following: I is a set, and for each $i \in I$ we are given an open subset $U_i \subset X$. Furthermore U is the union of the U_i. It is allowed to have $I = \emptyset$ in which case there are no U_i and $U = \emptyset$. It is also allowed, in case $I \neq \emptyset$ to have any or all of the U_i be empty.
(2) etc, etc.

6.3. Presheaves

Definition 6.3.1. Let X be a topological space.

(1) A *presheaf $\mathcal{F}$ of sets on X* is a rule which assigns to each open $U \subset X$ a set $\mathcal{F}(U)$ and to each inclusion $V \subset U$ a map $\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ such that $\rho^U_U = \text{id}_{\mathcal{F}(U)}$ and whenever $W \subset V \subset U$ we have $\rho^U_W = \rho^V_W \circ \rho^U_V$.
(2) A *morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves of sets on X* is a rule which assigns to each open $U \subset X$ a map of sets $\varphi : \mathcal{F}(U) \to \mathcal{G}(U)$ compatible with restriction maps, i.e., whenever $V \subset U \subset X$ are open the diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\varphi"] \arrow[d, "\rho^U_V"] & \mathcal{G}(U) \arrow[d, "\rho^U_V"] \\ \mathcal{F}(V) \arrow[r, "\varphi"] & \mathcal{G}(V) \end{tikzcd}$$

commutes.
(3) The category of presheaves of sets on X will be denoted $PSh(X)$.

The elements of the set $\mathcal{F}(U)$ are called the *sections* of $\mathcal{F}$ over U. For every $V \subset U$ the map $\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ is called the *restricton map*. We will use the notation $s|_V := \rho^U_V(s)$ if $s \in \mathcal{F}(U)$. This notation is consistent with the notion of restriction of functions from topology because if $W \subset V \subset U$ and s is a section of $\mathcal{F}$ over U then $s|_W = (s|_V)|_W$ by the property of the restriction maps expressed in the definition above.

Another notation that is often used is to indicate sections over an open U by the symbol $\Gamma(U, -)$ or by $H^0(U, -)$. In other words, the following equalities are tautological

$$\Gamma(U, \mathcal{F}) = \mathcal{F}(U) = H^0(U, \mathcal{F}).$$

In this chapter we will not use this notation, but in others we will.

Definition 6.3.2. Let X be a topological space. Let A be a set. The *constant presheaf with value* A is the presheaf that assigns the set A to every open $U \subset X$, and such that all restriction mappings are id_A.

6.4. Abelian presheaves

In this section we briefly point out some features of the category of presheaves that allow one to define presheaves of abelian groups.

Example 6.4.1. Let X be a topological space X. Consider a rule $\mathcal{F}$ that associates to every open subset a singleton set. Since every set has a unique map into a singleton set, there exist unique restriction maps ρ^U_V. The resulting structure is a presheaf of sets. It is a final object in the category of presheaves of sets, by the property of singleton sets mentioned above. Hence it is also unique up to unique isomorphism. We will sometimes write $*$ for this presheaf.

Lemma 6.4.2. *Let X be a topological space. The category of presheaves of sets on X has products (see Categories, Definition 4.13.5). Moreover, the set of sections of the product $\mathcal{F} \times \mathcal{G}$ over an open U is the product of the sets of sections of $\mathcal{F}$ and $\mathcal{G}$ over U.*

Proof. Namely, suppose $\mathcal{F}$ and $\mathcal{G}$ are presheaves of sets on the topological space X. Consider the rule $U \mapsto \mathcal{F}(U) \times \mathcal{G}(U)$, denoted $\mathcal{F} \times \mathcal{G}$. If $V \subset U \subset X$ are open then define the restriction mapping

$$(\mathcal{F} \times \mathcal{G})(U) \longrightarrow (\mathcal{F} \times \mathcal{G})(V)$$

by mapping $(s, t) \mapsto (s|_V, t|_V)$. Then it is immediately clear that $\mathcal{F} \times \mathcal{G}$ is a presheaf. Also, there are projection maps $p : \mathcal{F} \times \mathcal{G} \to \mathcal{F}$ and $q : \mathcal{F} \times \mathcal{G} \to \mathcal{G}$. We leave it to the reader to show that for any third presheaf $\mathcal{H}$ we have $Mor(\mathcal{H}, \mathcal{F}\times\mathcal{G}) = Mor(\mathcal{H}, \mathcal{F})\times Mor(\mathcal{H}, \mathcal{G})$. $\square$

Recall that if $(A, + : A\times A \to A, - : A \to A, 0 \in A)$ is an abelian group, then the zero and the negation maps are uniquely determined by the addition law. In other words, it makes sense to say ``let $(A, +)$ be an abelian group''.

Lemma 6.4.3. *Let X be a topological space. Let $\mathcal{F}$ be a presheaf of sets. Consider the following types of structure on $\mathcal{F}$:*

1. *For every open U the structure of an abelian group on $\mathcal{F}(U)$ such that all restriction maps are abelian group homomorphisms.*
2. *A map of presheaves $+ : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$, a map of presheaves $- : \mathcal{F} \to \mathcal{F}$ and a map $0 : * \to \mathcal{F}$ (see Example 6.4.1) satisfying all the axioms of $+, -, 0$ in a usual abelian group.*
3. *A map of presheaves $+ : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$, a map of presheaves $- : \mathcal{F} \to \mathcal{F}$ and a map $0 : * \to \mathcal{F}$ such that for each open $U \subset X$ the quadruple $(\mathcal{F}(U), +, -, 0)$ is an abelian group,*
4. *A map of presheaves $+ : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ such that for every open $U \subset X$ the map $+ : \mathcal{F}(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ defines the structure of an abelian group.*

There are natural bijections between the collections of types of data (1) - (4) above.

Proof. Omitted. $\square$

The lemma says that to give an abelian group object $\mathcal{F}$ in the category of presheaves is the same as giving a presheaf of sets $\mathcal{F}$ such that all the sets $\mathcal{F}(U)$ are endowed with the structure of an abelian group and such that all the restriction mappings are group homomorphisms. For most algebra structures we will take this approach to (pre)sheaves of such objects, i.e., we will define a (pre)sheaf of such objects to be a (pre)sheaf $\mathcal{F}$ of sets all of whose sets of sections $\mathcal{F}(U)$ are endowed with this structure compatibly with the restriction mappings.

Definition 6.4.4. Let X be a topological space.

(1) A *presheaf of abelian groups on X* or an *abelian presheaf over X* is a presheaf of sets $\mathcal{F}$ such that for each open $U \subset X$ the set $\mathcal{F}(U)$ is endowed with the structure of an abelian group, and such that all restriction maps ρ^U_V are homomorphisms of abelian groups, see Lemma 6.4.3 above.
(2) A *morphism of abelian presheaves over X* $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves of sets which induces a homomorphism of abelian groups $\mathcal{F}(U) \to \mathcal{G}(U)$ for every open $U \subset X$.
(3) The category of presheaves of abelian groups on X is denoted $PAb(X)$.

Example 6.4.5. Let X be a topological space. For each $x \in X$ suppose given an abelian group M_x. For $U \subset X$ open we set

$$\mathcal{F}(U) = \bigoplus\nolimits_{x \in U} M_x.$$

We denote a typical element in this abelian group by $\sum_{i=1}^n m_{x_i}$, where $x_i \in U$ and $m_{x_i} \in M_{x_i}$. (Of course we may always choose our representation such that $x_1, \ldots, x_n$ are pairwise distinct.) We define for $V \subset U \subset X$ open a restriction mapping $\mathcal{F}(U) \to \mathcal{F}(V)$ by mapping an element $s = \sum_{i=1}^n m_{x_i}$ to the element $s|_V = \sum_{x_i \in V} m_{x_i}$. We leave it to the reader to verify that this is a presheaf of abelian groups.

6.5. Presheaves of algebraic structures

Let us clarify the definition of presheaves of algebraic structures. Suppose that $\mathcal{C}$ is a category and that $F : \mathcal{C} \to Sets$ is a faithful functor. Typically F is a ``forgetful" functor. For an object $M \in Ob(\mathcal{C})$ we often call $F(M)$ the *underlying set* of the object M. If $M \to M'$ is a morphism in $\mathcal{C}$ we call $F(M) \to F(M')$ the *underlying map of sets*. In fact, we will often not distinguish between an object and its underlying set, and similarly for morphisms. So we will say a map of sets $F(M) \to F(M')$ is a *morphism of algebraic structures*, if it is equal to $F(f)$ for some morphism $f : M \to M'$ in $\mathcal{C}$.

In analogy with Definition 6.4.4 above a ``presheaf of objects of $\mathcal{C}$" could be defined by the following data:

(1) a presheaf of sets $\mathcal{F}$, and
(2) for every open $U \subset X$ a choice of an object $A(U) \in Ob(\mathcal{C})$

subject to the following conditions (using the phraseology above)

(1) for every open $U \subset X$ the set $\mathcal{F}(U)$ is the underlying set of $A(U)$, and
(2) for every $V \subset U \subset X$ open the map of sets $\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ is a morphism of algebraic structures.

In other words, for every $V \subset U$ open in X the restriction mappings ρ^U_V is the image $F(\alpha^U_V)$ for some unique morphism $\alpha^U_V : A(U) \to A(V)$ in the category $\mathcal{C}$. The uniqueness is forced by the condition that F is faithful; it also implies that $\alpha^U_W = \alpha^V_W \circ \alpha^U_V$ whenever $W \subset V \subset U$ are open in X. The system $(A(-), \alpha^U_V)$ is what we will define as a presheaf with values in

$\mathcal{C}$ on X, compare Sites, Definition 9.2.2. We recover our presheaf of sets $(\mathcal{F}, \rho^U_V)$ via the rules $\mathcal{F}(U) = F(A(U))$ and $\rho^U_V = F(\alpha^U_V)$.

Definition 6.5.1. Let X be a topological space. Let $\mathcal{C}$ be a category.

(1) A *presheaf $\mathcal{F}$ on X with values in $\mathcal{C}$* is given by a rule which assigns to every open $U \subset X$ an object $\mathcal{F}(U)$ of $\mathcal{C}$ and to each inclusion $V \subset U$ a morphism $\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ in $\mathcal{C}$ such that whenever $W \subset V \subset U$ we have $\rho^U_W = \rho^V_W \circ \rho^U_V$.
(2) A *morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves with value in $\mathcal{C}$* is given by a morphism $\varphi : \mathcal{F}(U) \to \mathcal{G}(U)$ in $\mathcal{C}$ compatible with restriction morphisms.

Definition 6.5.2. Let X be a topological space. Let $\mathcal{C}$ be a category. Let $F : \mathcal{C} \to \textit{Sets}$ be a faithful functor. Let $\mathcal{F}$ be a presheaf on X with values in $\mathcal{C}$. The presheaf of sets $U \mapsto F(\mathcal{F}(U))$ is called the *underlying presheaf of sets of $\mathcal{F}$*.

It is customary to use the same letter $\mathcal{F}$ to denote the underlying presheaf of sets, and this makes sense according to our discussion preceding Definition 6.5.1. In particular, the phrase ``let $s \in \mathcal{F}(U)$'' or ``let s be a section of $\mathcal{F}$ over U'' signifies that $s \in F(\mathcal{F}(U))$.

This notation and these definitions apply in particular to: *Presheaves of (not necessarily abelian) groups, rings, modules over a fixed ring, vector spaces over a fixed field,* etc and *morphisms between these*.

6.6. Presheaves of modules

Suppose that $\mathcal{O}$ is a presheaf of rings on X. We would like to define the notion of a presheaf of $\mathcal{O}$-modules over X. In analogy with Definition 6.4.4 we are tempted to define this as a sheaf of sets $\mathcal{F}$ such that for every open $U \subset X$ the set $\mathcal{F}(U)$ is endowed with the structure of an $\mathcal{O}(U)$-module compatible with restriction mappings (of $\mathcal{F}$ and $\mathcal{O}$). However, it is customary (and equivalent) to define it as in the following definition.

Definition 6.6.1. Let X be a topological space, and let $\mathcal{O}$ be a presheaf of rings on X.

(1) A *presheaf of $\mathcal{O}$-modules* is given by an abelian presheaf $\mathcal{F}$ together with a map of presheaves of sets
$$\mathcal{O} \times \mathcal{F} \longrightarrow \mathcal{F}$$
such that for every open $U \subset X$ the map $\mathcal{O}(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ defines the structure of an $\mathcal{O}(U)$-module structure on the abelian group $\mathcal{F}(U)$.
(2) A *morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves of $\mathcal{O}$-modules* is a morphism of abelian presheaves $\varphi : \mathcal{F} \to \mathcal{G}$ such that the diagram
$$\begin{tikzcd} \mathcal{O} \times \mathcal{F} \arrow[r] \arrow[d, "\mathrm{id}\times\varphi"'] & \mathcal{F} \arrow[d, "\varphi"] \\ \mathcal{O} \times \mathcal{G} \arrow[r] & \mathcal{G} \end{tikzcd}$$
commutes.
(3) The set of $\mathcal{O}$-module morphisms as above is denoted $\textit{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$.
(4) The category of presheaves of $\mathcal{O}$-modules is denoted $\textit{PMod}(\mathcal{O})$.

Suppose that $\mathcal{O}_1 \to \mathcal{O}_2$ is a morphism of presheaves of rings on X. In this case, if $\mathcal{F}$ is a presheaf of $\mathcal{O}_2$-modules then we can think of $\mathcal{F}$ as a presheaf of $\mathcal{O}_1$-modules by using the composition
$$\mathcal{O}_1 \times \mathcal{F} \to \mathcal{O}_2 \times \mathcal{F} \to \mathcal{F}.$$

We sometimes denote this by $\mathcal{F}_{\mathcal{O}_1}$ to indicate the restriction of rings. We call this the *restriction of* $\mathcal{F}$. We obtain the restriction functor

$$PMod(\mathcal{O}_2) \longrightarrow PMod(\mathcal{O}_1)$$

On the other hand, given a presheaf of $\mathcal{O}_1$-modules $\mathcal{G}$ we can construct a presheaf of $\mathcal{O}_2$-modules $\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}$ by the rule

$$\left(\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}\right)(U) = \mathcal{O}_2(U) \otimes_{\mathcal{O}_1(U)} \mathcal{G}(U)$$

The index p stands for ``presheaf'' and not ``point''. This presheaf is called the tensor product presheaf. We obtain the *change of rings* functor

$$PMod(\mathcal{O}_1) \longrightarrow PMod(\mathcal{O}_2)$$

Lemma 6.6.2. *With X, $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{F}$ and $\mathcal{G}$ as above there exists a canonical bijection*

$$Hom_{\mathcal{O}_1}(\mathcal{G}, \mathcal{F}_{\mathcal{O}_1}) = Hom_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}, \mathcal{F})$$

In other words, the restriction and change of rings functors are adjoint to each other.

Proof. This follows from the fact that for a ring map $A \to B$ the restriction functor and the change of ring functor are adjoint to each other. $\square$

6.7. Sheaves

In this section we explain the sheaf condition.

Definition 6.7.1. Let X be a topological space.

(1) A *sheaf $\mathcal{F}$ of sets on X* is a presheaf of sets which satsifies the following additional property: Given any open covering $U = \bigcup_{i \in I} U_i$ and any collection of sections $s_i \in \mathcal{F}(U_i)$, $i \in I$ such that $\forall i, j \in I$

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$$

there exists a unique section $s \in \mathcal{F}(U)$ such that $s_i = s|_{U_i}$ for all $i \in I$.

(2) A *morphism of sheaves of sets* is simply a morphism of presheaves of sets.

(3) The category of sheaves of sets on X is denoted $Sh(X)$.

Remark 6.7.2. There is always a bit of confusion as to whether it is necessary to say something about the set of sections of a sheaf over the empty set $\emptyset \subset X$. It is necessary, and we already did if you read the definition right. Namely, note that the empty set is covered by the empty open covering, and hence the ``collection of section s_i'' from the definition above actually form an element of the empty product which is the final object of the category the sheaf has values in. In other words, if you read the definition right you automatically deduce that $\mathcal{F}(\emptyset) =$ *a final object*, which in the case of a sheaf of sets is a singleton. If you do not like this argument, then you can just require that $\mathcal{F}(\emptyset) = \{*\}$.

In particular, this condition will then ensure that if $U, V \subset X$ are open and *disjoint* then

$$\mathcal{F}(U \cup V) = \mathcal{F}(U) \times \mathcal{F}(V).$$

(Because the fibre product over a final object is a product.)

Example 6.7.3. Let X, Y be topological spaces. Consider the rule $\mathcal{F}$ wich associates to the open $U \subset X$ the set

$$\mathcal{F}(U) = \{f : U \to Y \mid f \text{ is continuous}\}$$

with the obvious restriction mappings. We claim that $\mathcal{F}$ is a sheaf. To see this suppose that $U = \bigcup_{i \in I} U_i$ is an open covering, and $f_i \in \mathcal{F}(U_i)$, $i \in I$ with $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i, j \in I$. In this case define $f : U \to Y$ by setting $f(u)$ equal to the value of $f_i(u)$ for any $i \in I$ such that $u \in U_i$. This is well defined by assumption. Moreover, $f : U \to Y$ is a map such that its restriction to U_i agrees with the continuous map U_i. Hence clearly f is continuous!

We can use the result of the example to define constant sheaves. Namely, suppose that A is a set. Endow A with the discrete topology. Let $U \subset X$ be an open subset. Then we have

$$\{f : U \to A \mid f \text{ continuous}\} = \{f : U \to A \mid f \text{ locally constant}\}.$$

Thus the rule which assigns to an open all locally constant maps into A is a sheaf.

Definition 6.7.4. Let X be a topological space. Let A be a set. The *constant sheaf with value A* denoted $\underline{A}$, or $\underline{A}_X$ is the sheaf that assigns to an open $U \subset X$ the set of all locally constant maps $U \to A$ with restriction mappings given by restrictions of functions.

Example 6.7.5. Let X be a topological space. Let $(A_x)_{x \in X}$ be a family of sets A_x indexed by points $x \in X$. We are going to construct a sheaf of sets Π from this data. For $U \subset X$ open set

$$\Pi(U) = \prod\nolimits_{x \in U} A_x.$$

For $V \subset U \subset X$ open define a restriction mapping by the following rule: An element $s = (a_x)_{x \in U} \in \Pi(U)$ restricts to $s|_V = (a_x)_{x \in V}$. It is obvious that this defines a presheaf of sets. We claim this is a sheaf. Namely, let $U = \bigcup U_i$ be an open covering. Suppose that $s_i \in \Pi(U_i)$ are such that s_i and s_j agree over $U_i \cap U_j$. Write $s_i = (a_{i,x})_{x \in U_i}$. The compatibility condition implies that $a_{i,x} = a_{j,x}$ in the set A_x whenever $x \in U_i \cap U_j$. Hence there exists a unique element $s = (a_x)_{x \in U}$ in $\Pi(U) = \prod_{x \in U} A_x$ with the property that $a_x = a_{i,x}$ whenever $x \in U_i$ for some i. Of course this element s has the property that $s|_{U_i} = s_i$ for all i.

Example 6.7.6. Let X be a topological space. Suppose for each $x \in X$ we are given an abelian group M_x. Consider the presheaf $\mathcal{F} : U \mapsto \bigoplus_{x \in U} M_x$ defined in Example 6.4.5. This is not a sheaf in general. For example, if X is an infinite set with the discrete topology, then the sheaf condition would imply that $\mathcal{F}(X) = \prod_{x \in X} \mathcal{F}(\{x\})$ but by definition we have $\mathcal{F}(X) = \bigoplus_{x \in X} M_x = \bigoplus_{x \in X} \mathcal{F}(\{x\})$. And an infinite direct sum is in general different from an infinite direct product.

However, if X is a topological space such that every open of X is quasi-compact, then $\mathcal{F}$ *is* a sheaf. This is left as an exercise to the reader.

6.8. Abelian sheaves

Definition 6.8.1. Let X be a topological space.

(1) An *abelian sheaf on X* or *sheaf of abelian groups on X* is an abelian presheaf on X such that the underlying presheaf of sets is a sheaf.
(2) The category of sheaves of abelian groups is denoted $Ab(X)$.

Let X be a topological space. In the case of an abelian presheaf $\mathcal{F}$ the sheaf condition with regards to an open covering $U = \bigcup U_i$ is often expressed by saying that the complex of abelian groups

$$0 \to \mathcal{F}(U) \to \prod\nolimits_i \mathcal{F}(U_i) \to \prod\nolimits_{(i_0,i_1)} \mathcal{F}(U_{i_0} \cap U_{i_1})$$

is exact. The first map is the usual one, whereas the second maps the element $(s_i)_{i\in I}$ to the element

$$(s_{i_0}|_{U_{i_0}\cap U_{i_1}} - s_{i_1}|_{U_{i_0}\cap U_{i_1}})_{(i_0,i_1)} \in \prod\nolimits_{(i_0,i_1)} \mathcal{F}(U_{i_0} \cap U_{i_1})$$

6.9. Sheaves of algebraic structures

Let us clarify the definition of sheaves of certain types of structures. First, let us reformulate the sheaf condition. Namely, suppose that $\mathcal{F}$ is a presheaf of sets on the topological space X. The sheaf condition can be reformulated as follows. Let $U = \bigcup_{i\in I} U_i$ be an open covering. Consider the diagram

$$\mathcal{F}(U) \longrightarrow \prod\nolimits_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod\nolimits_{(i_0,i_1)\in I\times I} \mathcal{F}(U_{i_0} \cap U_{i_1})$$

Here the left map is defined by the rule $s \mapsto \prod_{i\in I} s|_{U_i}$. The two maps on the right are the maps

$$\prod\nolimits_i s_i \mapsto \prod\nolimits_{(i_0,i_1)} s_{i_0}|_{U_{i_0}\cap U_{i_1}} \text{ resp. } \prod\nolimits_i s_i \mapsto \prod\nolimits_{(i_0,i_1)} s_{i_1}|_{U_{i_0}\cap U_{i_1}}.$$

The sheaf condition exactly says that the left arrow is the equalizer of the right two. This generalizes immediately to the case of presheaves with values in a category as long as the category has products.

Definition 6.9.1. Let X be a topological space. Let $\mathcal{C}$ be a category with products. A presheaf $\mathcal{F}$ with values in $\mathcal{C}$ on X is a *sheaf* if for every open covering the diagram

$$\mathcal{F}(U) \longrightarrow \prod\nolimits_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod\nolimits_{(i_0,i_1)\in I\times I} \mathcal{F}(U_{i_0} \cap U_{i_1})$$

is an equalizer diagram in the category $\mathcal{C}$.

Suppose that $\mathcal{C}$ is a category and that $F : \mathcal{C} \to Sets$ is a faithful functor. A good example to keep in mind is the case where $\mathcal{C}$ is the category of abelian groups and F is the forgetful functor. Consider a presheaf $\mathcal{F}$ with values in $\mathcal{C}$ on X. We would like to reformulate the condition above in terms of the underlying presheaf of sets (Definition 6.5.2). Note that the underlying presheaf of sets is a sheaf of sets if and only if all the diagrams

$$F(\mathcal{F}(U)) \longrightarrow \prod\nolimits_{i\in I} F(\mathcal{F}(U_i)) \rightrightarrows \prod\nolimits_{(i_0,i_1)\in I\times I} F(\mathcal{F}(U_{i_0} \cap U_{i_1}))$$

of sets -- after applying the forgetful functor F -- are equalizer diagrams! Thus we would like $\mathcal{C}$ to have products and equalizers and we would like F to commute with them. This is equivalent to the condition that $\mathcal{C}$ has limits and that F commutes with them, see Categories, Lemma 4.13.10. But this is not yet good enough (see Example 6.9.4); we also need F to *reflect isomorphisms*. This property means that given a morphism $f : A \to A'$ in $\mathcal{C}$, then f is an isomorphism if (and only if) $F(f)$ is a bijection.

Lemma 6.9.2. *Suppose the category $\mathcal{C}$ and the functor $F : \mathcal{C} \to Sets$ have the following properties:*

(1) *F is faithful,*
(2) *$\mathcal{C}$ has limits and F commutes with them, and*
(3) *the functor F reflects isomorphisms.*

Let X be a topological space. Let $\mathcal{F}$ be a presheaf with values in $\mathcal{C}$. Then $\mathcal{F}$ is a sheaf if and only if the underlying presheaf of sets is a sheaf.

Proof. Assume that $\mathcal{F}$ is a sheaf. Then $\mathcal{F}(U)$ is the equalizer of the diagram above and by assumption we see $F(\mathcal{F}(U))$ is the equalizer of the corresponding diagram of sets. Hence $F(\mathcal{F})$ is a sheaf of sets.

Assume that $F(\mathcal{F})$ is a sheaf. Let $E \in Ob(\mathcal{C})$ be the equalizer of the two parrallel arrows in Definition 6.9.1. We get a canonical morphism $\mathcal{F}(U) \to E$, simply because $\mathcal{F}$ is a presheaf. By assumption, the induced map $F(\mathcal{F}(U)) \to F(E)$ is an isomorphism, because $F(E)$ is the equalizer of the corresponding diagram of sets. Hence we see $\mathcal{F}(U) \to E$ is an isomorphism by condition (3) of the lemma. $\square$

The lemma in particular applies to *sheaves of groups, rings, algebras over a fixed ring, modules over a fixed ring, vector spaces over a fixed field,* etc. In other words, these are presheaves of groups, rings, modules over a fixed ring, vector spaces over a fixed field, etc such that the underlying presheaf of sets is a sheaf.

Example 6.9.3. Let X be a topological space. For each open $U \subset X$ consider the $\mathbf{R}$-algebra $\mathcal{C}^0(U) = \{ f : U \to \mathbf{R} \mid f \text{ is continuous}\}$. There are obvious restriction mappings that turn this into a presheaf of $\mathbf{R}$-algebras over X. By Example 6.7.3 it is a sheaf of sets. Hence by the Lemma 6.9.2 it is a sheaf of $\mathbf{R}$-algebras over X.

Example 6.9.4. Consider the category of topological spaces *Top*. There is a natural faithful functor $Top \to Sets$ which commutes with products and equalizers. But it does not reflect isomorphisms. And, in fact it turns out that the analogue of Lemma 6.9.2 is wrong. Namely, suppose $X = \mathbf{N}$ with the discrete topology. Let A_i, for $i \in \mathbf{N}$ be a discrete topological space. For any subset $U \subset \mathbf{N}$ define $\mathcal{F}(U) = \prod_{i \in U} A_i$ with the discrete topology. Then this is a presheaf of topological spaces whose underlying presheaf of sets is a sheaf, see Example 6.7.5. However, if each A_i has at least two elements, then this is not a sheaf of topological spaces according to Definition 6.9.1. The reader may check that putting the *product topology* on each $\mathcal{F}(U) = \prod_{i \in U} A_i$ does lead to a sheaf of topological spaces over X.

6.10. Sheaves of modules

Definition 6.10.1. Let X be a topological space. Let $\mathcal{O}$ be a sheaf of rings on X.

1. A *sheaf of $\mathcal{O}$-modules* is a presheaf of $\mathcal{O}$-modules $\mathcal{F}$, see Definition 6.6.1, such that the underlying presheaf of abelian groups $\mathcal{F}$ is a sheaf.
2. A *morphism of sheaves of $\mathcal{O}$-modules* is a morphism of presheaves of $\mathcal{O}$-modules.
3. Given sheaves of $\mathcal{O}$-modules $\mathcal{F}$ and $\mathcal{G}$ we denote $Hom_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$ the set of morphism of sheaves of $\mathcal{O}$-modules.
4. The category of sheaves of $\mathcal{O}$-modules is denoted $Mod(\mathcal{O})$.

This definition kind of makes sense even if $\mathcal{O}$ is just a presheaf of rings, allthough we do not know any examples where this is useful, and we will avoid using the terminology ``sheaves of $\mathcal{O}$-modules" in case $\mathcal{O}$ is not a sheaf of rings.

6.11. Stalks

Let X be a topological space. Let $x \in X$ be a point. Let $\mathcal{F}$ be a presheaf of sets on X. The *stalk of $\mathcal{F}$ at x* is the set

$$\mathcal{F}_x = colim_{x \in U} \mathcal{F}(U)$$

where the colimit is over the set of open neighbourhoods U of x in X. The set of open neighbourhoods is (partially) ordered by (reverse) inclusion: We say $U \geq U' \Leftrightarrow U \subset U'$. The transition maps in the system are given by the restriction maps of $\mathcal{F}$. See Categories, Section 4.19 for notation and terminology regarding (co)limits over systems. Note that the colimit is a directed colimit. Thus it is easy to describe $\mathcal{F}_x$. Namely,

$$\mathcal{F}_x = \{(U, s) \mid x \in U, s \in \mathcal{F}(U)\}/\sim$$

with equivalence relation given by $(U, s) \sim (U', s')$ if and only if $s|_{U \cap U'} = s'|_{U \cap U'}$. By abuse of notation we will often denote (U, s), s_x, or even s the corresponding element in $\mathcal{F}_x$. Also we will say $s = s'$ in $\mathcal{F}_x$ for two local sections of $\mathcal{F}$ defined in an open neighbourhod of x to denote that they have the same image in $\mathcal{F}_x$.

An obvious consequence of this definition is that for any open $U \subset X$ there is a canonical map

$$\mathcal{F}(U) \longrightarrow \prod\nolimits_{x \in U} \mathcal{F}_x$$

defined by $s \mapsto \prod_{x \in U}(U, s)$. Think about it!

Lemma 6.11.1. *Let $\mathcal{F}$ be a sheaf of sets on the topological space X. For every open $U \subset X$ the map*

$$\mathcal{F}(U) \longrightarrow \prod\nolimits_{x \in U} \mathcal{F}_x$$

is injective.

Proof. Suppose that $s, s' \in \mathcal{F}(U)$ map to the same element in every stalk $\mathcal{F}_x$ for all $x \in U$. This means that for every $x \in U$, there exists an open $V^x \subset U$, $x \in V^x$ such that $s|_{V^x} = s'|_{V^x}$. But then $U = \bigcup_{x \in U} V^x$ is an open covering. Thus by the uniqueness in the sheaf condition we see that $s = s'$. □

Definition 6.11.2. Let X be a topological space. A presheaf of sets $\mathcal{F}$ on X is *separated* if for every open $U \subset X$ the map $\mathcal{F}(U) \to \prod_{x \in U} \mathcal{F}_x$ is injective.

Another observation is that the construction of the stalk $\mathcal{F}_x$ is functorial in the presheaf $\mathcal{F}$. In other words, it gives a functor

$$PSh(X) \longrightarrow Sets, \ \mathcal{F} \longmapsto \mathcal{F}_x.$$

This functor is called the *stalk functor*. Namely, if $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves, then we define $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ by the rule $(U, s) \mapsto (U, \varphi(s))$. To see that this works we have to check that if $(U, s) = (U', s')$ in $\mathcal{F}_x$ then also $(U, \varphi(s)) = (U', \varphi(s'))$ in $\mathcal{G}_x$. This is clear since φ is compatible with the restriction mappings.

Example 6.11.3. Let X be a topological space. Let A be a set. Denote temporarily A_p the constant presheaf with value A (p for presheaf -- not for point). There is a canonical map of presheaves $A_p \to \underline{A}$ into the constant sheaf with value A. For evey point we have canonical bijections $A = (A_p)_x = \underline{A}_x$, where the second map is induced by functoriality from the map $A_p \to \underline{A}$.

Example 6.11.4. Suppose $X = \mathbf{R}^n$ with the Euclidean topology. Consider the presheaf of $\mathcal{C}^\infty$ functions on X, denoted $\mathcal{C}^\infty_{\mathbf{R}^n}$. In other words, $\mathcal{C}^\infty_{\mathbf{R}^n}(U)$ is the set of $\mathcal{C}^\infty$-functions $f : U \to \mathbf{R}$. As in Example 6.7.3 it is easy to show that this is a sheaf. In fact it is a sheaf of $\mathbf{R}$-vector spaces.

Next, let $x \in X = \mathbf{R}^n$ be a point. How do we think of an element in the stalk $\mathcal{C}^\infty_{\mathbf{R}^n,x}$? Such an element is given by a $\mathcal{C}^\infty$-function f whose domain contains x. And a pair of such functions f, g determine the same element of the stalk if they agree in a neighbourhood of x. In other words, an element if $\mathcal{C}^\infty_{\mathbf{R}^n,x}$ is the same thing as what is sometimes called a *germ of a $\mathcal{C}^\infty$-function at x.*

Example 6.11.5. Let X be a topological space. Let A_x be a set for each $x \in X$. Consider the sheaf $\mathcal{F} : U \mapsto \prod_{x \in U} A_x$ of Example 6.7.5. We would just like to point out here that the stalk $\mathcal{F}_x$ of $\mathcal{F}$ at x is in general *not* equal to the set A_x. Of course there is a map $\mathcal{F}_x \to A_x$, but that is in general the best you can say. For example, if each neighbourhood of x has infinitely many points, and each $A_{x'}$ has exactly two elements, then $\mathcal{F}_x$ has infinitely many elements. (Left to the reader.) On the other hand, if every neighbourhood of x contains a point y such that $A_y = \emptyset$, then $\mathcal{F}_x = \emptyset$.

6.12. Stalks of abelian presheaves

We first deal with the case of abelian groups as a model for the general case.

Lemma 6.12.1. *Let X be a topological space. Let $\mathcal{F}$ be a presheaf of abelian groups on X. There exists a unique structure of an abelian group on $\mathcal{F}_x$ such that for every $U \subset X$ open, $x \in U$ the map $\mathcal{F}(U) \to \mathcal{F}_x$ is a group homomorphism. Moreover,*

$$\mathcal{F}_x = colim_{x \in U} \mathcal{F}(U)$$

holds in the category of abelian groups.

Proof. We define addition of a pair of elements (U, s) and (V, t) as the pair $(U \cap V, s|_{U \cap V} + t|_{U \cap V})$. The rest is easy to check. □

What is crucial in the proof above is that the partially ordered set of open neighbourhoods is a directed system (compare Categories, Definition 4.19.2). Namely, the coproduct of two abelian groups A, B is the direct sum $A \oplus B$, whereas the coproduct in the category of sets is the disjoint union $A \coprod B$, showing that colimits in the category of abelian groups do not agree with colimits in the category of sets in general.

6.13. Stalks of presheaves of algebraic structures

The proof of Lemma 6.12.1 will work for any type of algebraic structure such that directed colimits commute with the forgetful functor.

Lemma 6.13.1. *Let $\mathcal{C}$ be a category. Let $F : \mathcal{C} \to Sets$ be a functor. Assume that*

1. *F is faithful, and*
2. *directed colimits exist in $\mathcal{C}$ and F commutes with them.*

Let X be a topological space. Let $x \in X$. Let $\mathcal{F}$ be a presheaf with values in $\mathcal{C}$. Then

$$\mathcal{F}_x = colim_{x \in U} \mathcal{F}(U)$$

exists in $\mathcal{C}$. Its underlying set is equal to the stalk of the underlying presheaf of sets of $\mathcal{F}$. Furthermore, the construction $\mathcal{F} \mapsto \mathcal{F}_x$ is a functor from the category of presheaves with values in $\mathcal{C}$ to $\mathcal{C}$.

Proof. Omitted. □

By the very definition, all the morphisms $\mathcal{F}(U) \to \mathcal{F}_x$ are morphisms in the category $\mathcal{C}$ which (after applying the forgetful functor F) turn into the corresponding maps for the underlying sheaf of sets. As usual we will not distinguish between the morphism in $\mathcal{C}$ and the underlying map of sets, which is permitted since F is faithful.

This lemma applies in particular to: *Presheaves of (not necessarily abelian) groups, rings, modules over a fixed ring, vector spaces over a fixed field.*

6.14. Stalks of presheaves of modules

Lemma 6.14.1. *Let X be a topological space. Let $\mathcal{O}$ be a presheaf of rings on X Let $\mathcal{F}$ be a presheaf $\mathcal{O}$-modules. Let $x \in X$. The canonical map $\mathcal{O}_x \times \mathcal{F}_x \to \mathcal{F}_x$ coming from the multiplication map $\mathcal{O} \times \mathcal{F} \to \mathcal{F}$ defines a $\mathcal{O}_x$-module structure on the abelian group $\mathcal{F}_x$.*

Proof. Omitted. □

Lemma 6.14.2. *Let X be a topological space. Let $\mathcal{O} \to \mathcal{O}'$ be a morphism of presheaves of rings on X Let $\mathcal{F}$ be a presheaf $\mathcal{O}$-modules. Let $x \in X$. We have*

$$\mathcal{F}_x \otimes_{\mathcal{O}_x} \mathcal{O}'_x = (\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{O}')_x$$

as $\mathcal{O}'_x$-modules.

Proof. Omitted. □

6.15. Algebraic structures

In this section we mildly formalize the notions we have encountered in the sections above.

Definition 6.15.1. A *type of algebraic structure* is given by a category $\mathcal{C}$ and a functor $F : \mathcal{C} \to \textit{Sets}$ with the following properties

(1) F is faithful,
(2) $\mathcal{C}$ has limits and F commutes with limits,
(3) $\mathcal{C}$ has filtered colimits and F commutes with them, and
(4) F reflects isomorphisms.

We make this definition to point out the properties we will use in a number of arguments below. But we will not actually study this notion in any great detail, since we are prohibited from studying ``big'' categories by convention, except for those listed in Categories, Remark 4.2.2. Among those the following have the required properties.

Lemma 6.15.2. *The following categories, endowed with the obvious forgetful functor, define types of algebraic structures:*

(1) *The category of pointed sets.*
(2) *The category of abelian groups.*
(3) *The category of groups.*
(4) *The category of monoids.*
(5) *The category of rings.*
(6) *The category of R-modules for a fixed ring R.*
(7) *The category of Lie algebras over a fixed field.*

Proof. Omitted. □

From now on we will think of a (pre)sheaf of algebraic structures and their stalks, in terms of the underlying (pre)sheaf of sets. This is allowable by Lemmas 6.9.2 and 6.13.1.

In the rest of this section we point out some results on algebraic structures that will be useful in the future.

Lemma 6.15.3. *Let $(\mathcal{C}, F)$ be a type of algebraic structure.*

(1) *$\mathcal{C}$ has a final object 0 and $F(0) = \{*\}$.*

(2) *$\mathcal{C}$ has products and $F(\prod A_i) = \prod F(A_i)$.*
(3) *$\mathcal{C}$ has fibre products and $F(A \times_B C) = F(A) \times_{F(B)} F(C)$.*
(4) *$\mathcal{C}$ has equalizers, and if $E \to A$ is the equalizer of $a, b : A \to B$, then $F(E) \to F(A)$ is the equalizer of $F(a), F(b) : F(A) \to F(B)$.*
(5) *$A \to B$ is a monomorphism if and only if $F(A) \to F(B)$ is injective.*
(6) *if $F(a) : F(A) \to F(B)$ is surjective, then a is an epimorphism.*
(7) *given $A_1 \to A_2 \to A_3 \to \ldots$, then $colim\, A_i$ exists and $F(colim\, A_i) = colim\, F(A_i)$, and more generally for any filtered colimit.*

Proof. Omitted. The only interesting statement is (5) which follows because $A \to B$ is a monomorphism if and only if $A \to A \times_B A$ is an isomorphism, and then applying the fact that F reflects isomorphisms. □

Lemma 6.15.4. *Let $(\mathcal{C}, F)$ be a type of algebraic structure. Suppose that $A, B, C \in Ob(\mathcal{C})$. Let $f : A \to B$ and $g : C \to B$ be morphisms of $\mathcal{C}$. If $F(g)$ is injective, and $Im(F(f)) \subset Im(F(g))$, then f factors as $f = g \circ t$ for some morphism $t : A \to C$.*

Proof. Consider $A \times_B C$. The assumptions imply that $F(A \times_B C) = F(A) \times_{F(B)} F(C) = F(A)$. Hence $A = A \times_B C$ because F reflects isomorphisms. The result follows. □

Example 6.15.5. The lemma will be applied often to the following situation. Suppose that we have a diagram

$$\begin{tikzcd} A \arrow[r] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

in $\mathcal{C}$. Suppose $C \to D$ is injective on underlying sets, and suppose that the composition $A \to B \to D$ has image on underlying sets in the image of $C \to D$. Then we get a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

in $\mathcal{C}$.

Example 6.15.6. Let $F : \mathcal{C} \to Sets$ be a type of algebraic structures. Let X be a topological space. Suppose that for every $x \in X$ we are given an object $A_x \in \text{ob}(\mathcal{C})$. Consider the presheaf Π with values in $\mathcal{C}$ on X defined by the rule $\Pi(U) = \prod_{x \in U} A_x$ (with obvious restriction mappings). Note that the associated presheaf of sets $U \mapsto F(\Pi(U)) = \prod_{x \in U} F(A_x)$ is a sheaf by Example 6.7.5. Hence Π is a sheaf of algebraic structures of type $(\mathcal{C}, F)$. This gives many examples of sheaves of abelian groups, groups, rings, etc.

6.16. Exactness and points

In any category we have the notion of epimorphism, monomorphism, isomorphism, etc.

Lemma 6.16.1. *Let X be a topological space. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of sets on X.*

(1) *The map φ is a monomorphism in the category of sheaves if and only if for all $x \in X$ the map $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective.*
(2) *The map φ is an epimorphism in the category of sheaves if and only if for all $x \in X$ the map $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is surjective.*

(3) *The map φ is a isomorphism in the category of sheaves if and only if for all $x \in X$ the map $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is bijective.*

Proof. Omitted. □

It follows that in the category of sheaves of sets the notions epimorphism and monomorphism can be described as follows.

Definition 6.16.2. Let X be a topological space.

(1) A presheaf $\mathcal{F}$ is called a *subpresheaf* of a presheaf $\mathcal{G}$ if $\mathcal{F}(U) \subset \mathcal{G}(U)$ for all open $U \subset X$ such that the restriction maps of $\mathcal{G}$ induce the restriction maps of $\mathcal{F}$. If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, then $\mathcal{F}$ is called a *subsheaf* of $\mathcal{G}$. We sometimes indicate this by the notation $\mathcal{F} \subset \mathcal{G}$.
(2) A morphism of presheaves of sets $\varphi : \mathcal{F} \to \mathcal{G}$ on X is called *injective* if and only if $\mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all U open in X.
(3) A morphism of presheaves of sets $\varphi : \mathcal{F} \to \mathcal{G}$ on X is called *surjective* if and only if $\mathcal{F}(U) \to \mathcal{G}(U)$ is surjective for all U open in X.
(4) A morphism of sheaves of sets $\varphi : \mathcal{F} \to \mathcal{G}$ on X is called *injective* if and only if $\mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all U open in X.
(5) A morphism of sheaves of sets $\varphi : \mathcal{F} \to \mathcal{G}$ on X is called *surjective* if and only if for every open U of X and every section s of $\mathcal{F}(U)$ there exists an open covering $U = \bigcup U_i$ such that $s|_{U_i}$ is in the image of $\mathcal{F}(U_i) \to \mathcal{G}(U)$ for all i.

Lemma 6.16.3. *Let X be a topological space.*

(1) *Epimorphisms (resp. monomorphisms) in the category of presheaves are exactly the surjective (resp. injective) maps of presheaves.*
(2) *Epimorphisms (resp. monomorphisms) in the category of sheaves are exactly the surjective (resp. injective) maps of sheaves, and are exactly those maps with are surjective (resp. injective) on all the stalks.*
(3) *The sheafification of a surjective (resp. injective) morphism of presheaves of sets is surjective (resp. injective).*

Proof. Omitted. □

Lemma 6.16.4. *let X be a topological space. Let $(\mathcal{C}, F)$ be a type of algebraic structure. Suppose that $\mathcal{F}$, $\mathcal{G}$ are sheaves on X with values in $\mathcal{C}$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of the underlying sheaves of sets. If for all points $x \in X$ the map $\mathcal{F}_x \to \mathcal{G}_x$ is a morphism of algebraic structures, then φ is a morphism of sheaves of algebraic structures.*

Proof. Let U be an open subset of X. Consider the diagram of (underlying) sets

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r] \arrow[d] & \prod_{x \in U} \mathcal{F}_x \arrow[d] \\
\mathcal{G}(U) \arrow[r] & \prod_{x \in U} \mathcal{G}_x
\end{tikzcd}$$

By assumption, and previous results, all but the left vertical arrow are morphisms of algebraic structures. In addition the bottom horizontal arrow is injective, see Lemma 6.11.1. Hence we conclude by Lemma 6.15.4, see also Example 6.15.5 □

Short exact sequences of abelian sheaves, etc will be discussed in the chapter on sheaves of modules. See Modules, Section 15.3.

6.17. Sheafification

In this section we explain how to get the sheafification of a presheaf on a topological space. We will use stalks to describe the sheafification in this case. This is different from the general procedure described in Sites, Section 9.10, and perhaps somewhat easier to understand.

The basic construction is the following. Let $\mathcal{F}$ be a presheaf of sets $\mathcal{F}$ on a topological space X. For every open $U \subset X$ we define

$$\mathcal{F}^\#(U) = \{(s_u) \in \prod\nolimits_{u \in U} \mathcal{F}_u \text{ such that } (*)\}$$

where $(*)$ is the property:

$(*)$ For every $u \in U$, there exists an open neighbourhood $u \in V \subset U$, and a section $\sigma \in \mathcal{F}(V)$ such that for all $v \in V$ we have $s_v = (V, \sigma)$ in $\mathcal{F}_v$.

Note that $(*)$ is a condition for each $u \in U$, and that given $u \in U$ the truth of this condition depends only on the values s_v for v in any open neighbourhood of u. Thus it is clear that, if $V \subset U \subset X$ are open, the projection maps

$$\prod\nolimits_{u \in U} \mathcal{F}_u \longrightarrow \prod\nolimits_{v \in V} \mathcal{F}_v$$

maps elements of $\mathcal{F}^\#(U)$ into $\mathcal{F}^\#(V)$. In other words, we get the structure of a presheaf of sets on $\mathcal{F}^\#$.

Furthermore, the map $\mathcal{F}(U) \to \prod_{u \in U} \mathcal{F}_u$ described in Section 6.11 clearly has image in $\mathcal{F}^\#(U)$. In addition, if $V \subset U \subset X$ are open then we have the following commutative diagram

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r] \arrow[d] & \mathcal{F}^\#(U) \arrow[r] \arrow[d] & \prod_{u \in U} \mathcal{F}_u \arrow[d] \\
\mathcal{F}(V) \arrow[r] & \mathcal{F}^\#(V) \arrow[r] & \prod_{v \in V} \mathcal{F}_v
\end{tikzcd}$$

where the vertical maps are induced from the restriction mappings. Thus we see that there is a canonical morphism of presheaves $\mathcal{F} \to \mathcal{F}^\#$.

In Example 6.7.5 we saw that the rule $\Pi(\mathcal{F}) : U \mapsto \prod_{u \in U} \mathcal{F}_u$ is a sheaf, with obvious restriction mappings. And by construction $\mathcal{F}^\#$ is a subpresheaf of this. In other words, we have morphisms of presheaves

$$\mathcal{F} \to \mathcal{F}^\# \to \Pi(\mathcal{F}).$$

In addition the rule that associates to $\mathcal{F}$ the sequence above is clearly functorial in the presheaf $\mathcal{F}$. This notation will be used in the proofs of the lemmas below.

Lemma 6.17.1. *The presheaf $\mathcal{F}^\#$ is a sheaf.*

Proof. It is probably better for the reader to find their own explanation of this than to read the proof here. In fact the lemma is true for the same reason as why the presheaf of continuous function is a sheaf, see Example 6.7.3 (and this analogy can be made precise using the ``espace étalé").

Anyway, let $U = \bigcup U_i$ be an open covering. Suppose that $s_i = (s_{i,u})_{u \in U_i} \in \mathcal{F}^\#(U_i)$ such that s_i and s_j agree over $U_i \cap U_j$. Because $\Pi(\mathcal{F})$ is a sheaf, we find an element $s = (s_u)_{u \in U}$ in $\prod_{u \in U} \mathcal{F}_u$ restricting to s_i on U_i. We have to check property $(*)$. Pick $u \in U$. Then $u \in U_i$ for some i. Hence by $(*)$ for s_i, there exists a V open, $u \in V \subset U_i$ and a $\sigma \in \mathcal{F}(V)$ such that $s_{i,v} = (V, \sigma)$ in $\mathcal{F}_v$ for all $v \in V$. Since $s_{i,v} = s_v$ we get $(*)$ for s. $\square$

Lemma 6.17.2. *Let X be a topological space. Let $\mathcal{F}$ be a presheaf of sets on X. Let $x \in X$. Then $\mathcal{F}_x = \mathcal{F}^\#_x$.*

Proof. The map $\mathcal{F}_x \to \mathcal{F}^\#_x$ is injective, since already the map $\mathcal{F}_x \to \Pi(\mathcal{F})_x$ is injective. Namely, there is a canonical map $\Pi(\mathcal{F})_x \to \mathcal{F}_x$ which is a left inverse to the map $\mathcal{F}_x \to \Pi(\mathcal{F})_x$, see Example 6.11.5. To show that it is surjective, suppose that $\overline{s} \in \mathcal{F}^\#_x$. We can find an open neighbourhood U of x such that $\overline{s}$ is the equivalence class of (U, s) with $s \in \mathcal{F}^\#(U)$. By definition, this means there exists an open neighbourhood $V \subset U$ of x and a section $\sigma \in \mathcal{F}(V)$ such that $s|_V$ is the image of σ in $\Pi(\mathcal{F})(V)$. Clearly the class of (V, σ) defines an element of $\mathcal{F}_x$ mapping to $\overline{s}$. $\square$

Lemma 6.17.3. *Let $\mathcal{F}$ be a presheaf of sets on X. Any map $\mathcal{F} \to \mathcal{G}$ into a sheaf of sets factors uniquely as $\mathcal{F} \to \mathcal{F}^\# \to \mathcal{G}$.*

Proof. Clearly, there is a commutative diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F}^\# \arrow[r] \arrow[d] & \Pi(\mathcal{F}) \arrow[d] \\ \mathcal{G} \arrow[r] & \mathcal{G}^\# \arrow[r] & \Pi(\mathcal{G}) \end{tikzcd}$$

So it suffices to prove that $\mathcal{G} = \mathcal{G}^\#$. To see this it suffices to prove, for every point $x \in X$ the map $\mathcal{G}_x \to \mathcal{G}^\#_x$ is bijective, by Lemma 6.16.1. And this is Lemma 6.17.2 above. $\square$

This lemma really says that there is an adjoint pair of functors: $i : Sh(X) \to PSh(X)$ (inclusion) and $\# : PSh(X) \to Sh(X)$ (sheafification). The formula is that

$$Mor_{PSh(X)}(\mathcal{F}, i(\mathcal{G})) = Mor_{Sh(X)}(\mathcal{F}^\#, \mathcal{G})$$

which says that sheafification is a left adjoint of the inclusion functor. See Categories, Section 4.22.

Example 6.17.4. See Example 6.11.3 for notation. The map $A_p \to \underline{A}$ induces a map $A_p^\# \to \underline{A}$. It is easy to see that this is an isomorphism. In words: The sheafification of the constant presheaf with value A is the constant sheaf with value A.

Lemma 6.17.5. *Let X be a topological space. A presheaf $\mathcal{F}$ is separated (see Definition 6.11.2) if and only if the canonical map $\mathcal{F} \to \mathcal{F}^\#$ is injective.*

Proof. This is clear from the construction of $\mathcal{F}^\#$ in this section. $\square$

6.18. Sheafification of abelian presheaves

The following strange looking lemma is likely unnecessary, but very convenient to deal with sheafification of presheaves of algebraic structures.

Lemma 6.18.1. *Let X be a topological space. Let $\mathcal{F}$ be a presheaf of sets on X. Let $U \subset X$ be open. There is a canonical fibre product diagram*

$$\begin{tikzcd} \mathcal{F}^\#(U) \arrow[r] \arrow[d] & \Pi(\mathcal{F})(U) \arrow[d] \\ \prod_{x \in U} \mathcal{F}_x \arrow[r] & \prod_{x \in U} \Pi(\mathcal{F})_x \end{tikzcd}$$

where the maps are the following:

(1) *The left vertical map has components* $\mathcal{F}^\#(U) \to \mathcal{F}^\#_x = \mathcal{F}_x$ *where the equality is Lemma 6.17.2.*
(2) *The top horizontal map comes from the map of presheaves* $\mathcal{F} \to \Pi(\mathcal{F})$ *described in Section 6.17.*
(3) *The right vertical map has obvious component maps* $\Pi(\mathcal{F})(U) \to \Pi(\mathcal{F})_x$.
(4) *The bottom horizontal map has components* $\mathcal{F}_x \to \Pi(\mathcal{F})_x$ *which come from the map of presheaves* $\mathcal{F} \to \Pi(\mathcal{F})$ *described in Section 6.17.*

Proof. It is clear that the diagram commutes. We have to show it is a fibre product diagram. The bottom horizontal arrow is injective since all the maps $\mathcal{F}_x \to \Pi(\mathcal{F})_x$ are injective (see beginning proof of Lemma 6.17.2). A section $s \in \Pi(\mathcal{F})(U)$ is in $\mathcal{F}^\#$ if and only if $(*)$ holds. But $(*)$ says that around every point the section s comes from a section of $\mathcal{F}$. By definition of the stalk functors, this is equivalent to saying that the value of s in every stalk $\Pi(\mathcal{F})_x$ comes from an element of the stalk $\mathcal{F}_x$. Hence the lemma. □

Lemma 6.18.2. *Let X be a topological space. Let $\mathcal{F}$ be an abelian presheaf on X. Then there exists a unique structure of abelian sheaf on $\mathcal{F}^\#$ such that $\mathcal{F} \to \mathcal{F}^\#$ is a morphism of abelian presheaves. Moreover, the following adjointness property holds*

$$Mor_{PAb(X)}(\mathcal{F}, i(\mathcal{G})) = Mor_{Ab(X)}(\mathcal{F}^\#, \mathcal{G}).$$

Proof. Recall the sheaf of sets $\Pi(\mathcal{F})$ defined in Section 6.17. All the stalks $\mathcal{F}_x$ are abelian groups, see Lemma 6.12.1. Hence $\Pi(\mathcal{F})$ is a sheaf of abelian groups by Example 6.15.6. Also, it is clear that the map $\mathcal{F} \to \Pi(\mathcal{F})$ is a morphism of abelian presheaves. If we show that condition $(*)$ of Section 6.17 defines a subgroup of $\Pi(\mathcal{F})(U)$ for all open subsets $U \subset X$, then $\mathcal{F}^\#$ canonically inherits the structure of abelian sheaf. This is quite easy to do by hand, and we leave it to the reader to find a good simple argument. The argument we use here, which generalizes to presheaves of algebraic structures is the following: Lemma 6.18.1 show that $\mathcal{F}^\#(U)$ is the fibre product of a diagram of abelian groups. Thus $\mathcal{F}^\#$ is an abelian subgroup as desired.

Note that at this point $\mathcal{F}^\#_x$ is an abelian group by Lemma 6.12.1 and that $\mathcal{F}_x \to \mathcal{F}^\#_x$ is a bijection (Lemma 6.17.2) and a homomorphism of abelian groups. Hence $\mathcal{F}_x \to \mathcal{F}^\#_x$ is an isomorphism of abelian groups. This will be used below without further mention.

To prove the adjointness property we use the adjointness property of sheafification of presheaves of sets. For example if $\psi : \mathcal{F} \to i(\mathcal{G})$ is morphism of presheaves then we obtain a morphism of sheaves $\psi' : \mathcal{F}^\# \to \mathcal{G}$. What we have to do is to check that this is a morphism of abelian sheaves. We may do this for example by noting that it is true on stalks, by Lemma 6.17.2, and then using Lemma 6.16.4 above. □

6.19. Sheafification of presheaves of algebraic structures

Lemma 6.19.1. *Let X be a topological space. Let $(\mathcal{C}, F)$ be a type of algebraic structure. Let $\mathcal{F}$ be a presheaf with values in $\mathcal{C}$ on X. Then there exists a sheaf $\mathcal{F}^\#$ with values in $\mathcal{C}$ and a morphism $\mathcal{F} \to \mathcal{F}^\#$ of presheaves with values in $\mathcal{C}$ with the following properties:*

(1) *The map $\mathcal{F} \to \mathcal{F}^\#$ identifies the underlying sheaf of sets of $\mathcal{F}^\#$ with the sheafification of the underlying presheaf of sets of $\mathcal{F}$.*
(2) *For any morphism $\mathcal{F} \to \mathcal{G}$, where $\mathcal{G}$ is a sheaf with values in $\mathcal{C}$ there exists a unique factorization $\mathcal{F} \to \mathcal{F}^\# \to \mathcal{G}$.*

Proof. The proof is the same as the proof of Lemma 6.18.2, with repeated application of Lemma 6.15.4 (see also Example 6.15.5). The main idea however, is to define $\mathcal{F}^\#(U)$ as the fibre product in $\mathcal{C}$ of the diagram

$$\begin{tikzcd} & \Pi(\mathcal{F})(U) \arrow[d] \\ \prod_{x\in U}\mathcal{F}_x \arrow[r] & \prod_{x\in U}\Pi(\mathcal{F})_x \end{tikzcd}$$

compare Lemma 6.18.1. □

6.20. Sheafification of presheaves of modules

Lemma 6.20.1. *Let X be a topological space. Let $\mathcal{O}$ be a presheaf of rings on X Let $\mathcal{F}$ be a presheaf $\mathcal{O}$-modules. Let $\mathcal{O}^\#$ be the sheafification of $\mathcal{O}$. Let $\mathcal{F}^\#$ be the sheafification of $\mathcal{F}$ as a presheaf of abelian groups. There exists a map of sheaves of sets*

$$\mathcal{O}^\# \times \mathcal{F}^\# \longrightarrow \mathcal{F}^\#$$

which makes the diagram

$$\begin{tikzcd} \mathcal{O}\times\mathcal{F} \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\ \mathcal{O}^\#\times\mathcal{F}^\# \arrow[r] & \mathcal{F}^\# \end{tikzcd}$$

commute and which makes $\mathcal{F}^\#$ into a sheaf of $\mathcal{O}^\#$-modules. In addition, if $\mathcal{G}$ is a sheaf of $\mathcal{O}^\#$-modules, then any morphism of presheaves of $\mathcal{O}$-modules $\mathcal{F} \to \mathcal{G}$ (into the restriction of $\mathcal{G}$ to a $\mathcal{O}$-module) factors uniquely as $\mathcal{F} \to \mathcal{F}^\# \to \mathcal{G}$ where $\mathcal{F}^\# \to \mathcal{G}$ is a morphism of $\mathcal{O}^\#$-modules.

Proof. Omitted. □

This actually means that the functor $i : Mod(\mathcal{O}^\#) \to PMod(\mathcal{O})$ (combining restriction and including sheaves into presheaves) and the sheafification functor of the lemma $^\# :$ $PMod(\mathcal{O}) \to Mod(\mathcal{O}^\#)$ are adjoint. In a formula

$$Mor_{PMod(\mathcal{O})}(\mathcal{F}, i\mathcal{G}) = Mor_{Mod(\mathcal{O}^\#)}(\mathcal{F}^\#, \mathcal{G})$$

Let X be a topological space. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a morphism of sheaves of rings on X. In Section 6.6 we defined a restriction functor and a change of rings functor on presheaves of modules associated to this situation.

If $\mathcal{F}$ is a sheaf of $\mathcal{O}_2$-modules then the restriction $\mathcal{F}_{\mathcal{O}_1}$ of $\mathcal{F}$ is clearly a sheaf of $\mathcal{O}_1$-modules. We obtain the restriction functor

$$Mod(\mathcal{O}_2) \longrightarrow Mod(\mathcal{O}_1)$$

On the other hand, given a sheaf of $\mathcal{O}_1$-modules $\mathcal{G}$ the presheaf of $\mathcal{O}_2$-modules $\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}$ is in general not a sheaf. Hence we define the *tensor product sheaf* $\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G}$ by the formula

$$\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G} = (\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G})^\#$$

as the sheafification of our construction for presheaves. We obtain the *change of rings* functor

$$Mod(\mathcal{O}_1) \longrightarrow Mod(\mathcal{O}_2)$$

Lemma 6.20.2. *With X, $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{F}$ and $\mathcal{G}$ as above there exists a canonical bijection*

$$Hom_{\mathcal{O}_1}(\mathcal{G}, \mathcal{F}_{\mathcal{O}_1}) = Hom_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G}, \mathcal{F})$$

In other words, the restriction and change of rings functors are adjoint to each other.

Proof. This follows from Lemma 6.6.2 and the fact that $Hom_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G}, \mathcal{F}) = Hom_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}, \mathcal{F})$ because $\mathcal{F}$ is a sheaf. $\square$

Lemma 6.20.3. *Let X be a topological space. Let $\mathcal{O} \to \mathcal{O}'$ be a morphism of sheaves of rings on X Let $\mathcal{F}$ be a sheaf $\mathcal{O}$-modules. Let $x \in X$. We have*

$$\mathcal{F}_x \otimes_{\mathcal{O}_x} \mathcal{O}'_x = (\mathcal{F} \otimes_{\mathcal{O}} \mathcal{O}')_x$$

as $\mathcal{O}'_x$-modules.

Proof. Follows directly from Lemma 6.14.2 and the fact that taking stalks commutes with sheafification. $\square$

6.21. Continuous maps and sheaves

Let $f : X \to Y$ be a continuous map of topological spaces. We will define the push forward and pull back functors for presheaves and sheaves.

Let $\mathcal{F}$ be a presheaf of sets on X. We define the *pushforward* of $\mathcal{F}$ by the rule

$$f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$$

for any open $V \subset Y$. Given $V_1 \subset V_2 \subset Y$ open the restriction map is given by the commutativity of the diagram

$$\begin{tikzcd}
f_*\mathcal{F}(V_2) \arrow[r, equal] \arrow[d] & \mathcal{F}(f^{-1}(V_2)) \arrow[d, "\text{restriction for } \mathcal{F}"] \\
f_*\mathcal{F}(V_1) \arrow[r, equal] & \mathcal{F}(f^{-1}(V_1))
\end{tikzcd}$$

It is clear that this defines a presheaf of sets. The construction is clearly functorial in the presheaf $\mathcal{F}$ and hence we obtain a functor

$$f_* : PSh(X) \longrightarrow PSh(Y).$$

Lemma 6.21.1. *Let $f : X \to Y$ be a continuous map. Let $\mathcal{F}$ be a sheaf of sets on X. Then $f_*\mathcal{F}$ is a sheaf on Y.*

Proof. This immediately follows from the fact that if $V = \bigcup V_j$ is an open covering in Y, then $f^{-1}(V) = \bigcup f^{-1}(V_j)$ is an open covering in X. $\square$

As a consequence we obtain a functor

$$f_* : Sh(X) \longrightarrow Sh(Y).$$

This is compatible with composition in the following strong sense.

Lemma 6.21.2. *Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps of topological spaces. The functors $(g \circ f)_*$ and $g_* \circ f_*$ are equal (on both presheaves and sheaves of sets).*

Proof. This is because $(g \circ f)_*\mathcal{F}(W) = \mathcal{F}((g \circ f)^{-1}W)$ and $(g_* \circ f_*)\mathcal{F}(W) = \mathcal{F}(f^{-1}g^{-1}W)$ and $(g \circ f)^{-1}W = f^{-1}g^{-1}W$. $\square$

Let $\mathcal{G}$ be a presheaf of sets on Y. The *pullback presheaf* $f_p\mathcal{G}$ of a given presheaf $\mathcal{G}$ is defined as the left adjoint of the pushforward f_* on presheaves. In other words it should be a presheaf $f_p\mathcal{G}$ on X such that

$$Mor_{PSh(X)}(f_p\mathcal{G}, \mathcal{F}) = Mor_{PSh(Y)}(\mathcal{G}, f_*\mathcal{F}).$$

By the Yoneda lemma this determines the pullback uniquely. It turns out that it actually exists.

Lemma 6.21.3. *Let $f : X \to Y$ be a continuous map. There exists a functor $f_p : PSh(Y) \to PSh(X)$ which is right adjoint to f_*. For a presheaf $\mathcal{G}$ it is determined by the rule*

$$f_p\mathcal{G}(U) = colim_{f(U)\subset V}\mathcal{G}(V)$$

where the colimit is over the collection of open neighbourhoods V of $f(U)$ in Y. The colimits are over directed partially ordered sets. (The restriction mappings of $f_p\mathcal{G}$ are explained in the proof.)

Proof. The colimit is over the partially ordered set consisting of open subset $V \subset Y$ which contain $f(U)$ with ordering by reverse inclusion. This is a directed partially ordered set, since if V, V' are in it then so is $V \cap V'$. Furthermore, if $U_1 \subset U_2$, then every open neighbourhood of $f(U_2)$ is an open neighbourhood of $f(U_1)$. Hence the system defining $f_p\mathcal{G}(U_2)$ is a subsystem of the one defining $f_p\mathcal{G}(U_1)$ and we obtain a restiction map (for example by applying the generalities in Categories, Lemma 4.13.7).

Note that the construction of the colimit is clearly functorial in $\mathcal{G}$, and similarly for the restriction mappings. Hence we have defined f_p as a functor.

A small useful remark is that there exists a canonical map $\mathcal{G}(U) \to f_p\mathcal{G}(f^{-1}(U))$, because the system of open neighbourhoods of $f(f^{-1}(U))$ contains the element U. This is compatible with restriction mappings. In other words, there is a canonical map $i_{\mathcal{G}} : \mathcal{G} \to f_*f_p\mathcal{G}$.

Let $\mathcal{F}$ be a presheaf of sets on X. Suppose that $\psi : f_p\mathcal{G} \to \mathcal{F}$ is a map of presheaves of sets. The corresponding map $\mathcal{G} \to f_*\mathcal{F}$ is the map $f_*\psi \circ i_{\mathcal{G}} : \mathcal{G} \to f_*f_p\mathcal{G} \to f_*\mathcal{F}$.

Another small useful remark is that there exists a canonical map $c_{\mathcal{F}} : f_pf_*\mathcal{F} \to \mathcal{F}$. Namely, let $U \subset X$ open. For every open neighbourhood $V \supset f(U)$ in Y there exists a map $f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V)) \to \mathcal{F}(U)$, namely the restriction map on $\mathcal{F}$. And this is certainly compatible wrt restriction mappings between values of $\mathcal{F}$ on f^{-1} of varying opens containing $f(U)$. Thus we obtain a canonical map $f_pf_*\mathcal{F}(U) \to \mathcal{F}(U)$. Another trivial verification show that these maps are compatible with restrictions and define a map $c_{\mathcal{F}}$ of presheaves of sets.

Suppose that $\varphi : \mathcal{G} \to f_*\mathcal{F}$ is a map of presheaves of sets. Consider $f_p\varphi : f_p\mathcal{G} \to f_pf_*\mathcal{F}$. Postcomposing with $c_{\mathcal{F}}$ gives the desired map $c_{\mathcal{F}}\circ f_p\varphi : f_p\mathcal{G} \to \mathcal{F}$. We omit the verification that this construction is inverse to the construction in the other direction given above. $\square$

Lemma 6.21.4. *Let $f : X \to Y$ be a continuous map. Let $x \in X$. Let $\mathcal{G}$ be a presheaf of sets on Y. There is a canonical bijection of stalks $(f_p\mathcal{G})_x = \mathcal{G}_{f(x)}$.*

Proof. This you can see as follows

$$\begin{aligned}
(f_p\mathcal{G})_x &= colim_{x\in U} f_p\mathcal{G}(U) \\
&= colim_{x\in U} colim_{f(U)\subset V}\mathcal{G}(V) \\
&= colim_{f(x)\in V}\mathcal{G}(V) \\
&= \mathcal{G}_{f(x)}
\end{aligned}$$

Here we have used Categories, Lemma 4.13.9, and the fact that any V open in Y containing $f(x)$ occurs in the third description above. Details omitted. $\square$

Let $\mathcal{G}$ be a sheaf of sets on Y. The *pullback sheaf* $f^{-1}\mathcal{G}$ is defined by the formula

$$f^{-1}\mathcal{G} = (f_p\mathcal{G})^\#.$$

Sheafification is a left adjoint to the inclusion of sheaves in presheaves, and f_p is a left adjoint to f_* on presheaves. As a formal consequence we obtain that f^{-1} is a left adjoint of pushforward on sheaves. In other words,

$$Mor_{Sh(X)}(f^{-1}\mathcal{G}, \mathcal{F}) = Mor_{Sh(Y)}(\mathcal{G}, f_*\mathcal{F}).$$

The formal argument is given in the setting of abelian sheaves in the next section.

Lemma 6.21.5. *Let $x \in X$. Let $\mathcal{G}$ be a sheaf of sets on Y. There is a canonical bijection of stalks $(f^{-1}\mathcal{G})_x = \mathcal{G}_{f(x)}$.*

Proof. This is a combination of Lemmas 6.17.2 and 6.21.4. $\square$

Lemma 6.21.6. *Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps of topological spaces. The functors $(g \circ f)^{-1}$ and $f^{-1} \circ g^{-1}$ are canonically isomorphic. Similarly $(g \circ f)_p \cong f_p \circ g_p$ on presheaves.*

Proof. To see this use that adjoint functors are unique up to unique isomorphism, and Lemma 6.21.2. $\square$

Definition 6.21.7. Let $f : X \to Y$ be a continuous map. Let $\mathcal{F}$ be a sheaf of sets on X and let $\mathcal{G}$ be a sheaf of sets on Y. An *f-map* $\xi : \mathcal{G} \to \mathcal{F}$ is a collection of maps $\xi_V : \mathcal{G}(V) \to \mathcal{F}(f^{-1}(V))$ indexed by open subsets $V \subset Y$ such that

$$\begin{tikzcd}
\mathcal{G}(V) \arrow[r, "\xi_V"'] \arrow[d, "\text{restriction of } \mathcal{G}"'] & \mathcal{F}(f^{-1}V) \arrow[d, "\text{restriction of } \mathcal{F}"] \\
\mathcal{G}(V') \arrow[r, "\xi_{V'}"] & \mathcal{F}(f^{-1}V')
\end{tikzcd}$$

commutes for all $V' \subset V \subset Y$ open.

Lemma 6.21.8. *Let $f : X \to Y$ be a continuous map. Let $\mathcal{F}$ be a sheaf of sets on X and let $\mathcal{G}$ be a sheaf of sets on Y. There are canonical bijections between the following three sets:*

(1) *The set of maps $\mathcal{G} \to f_*\mathcal{F}$.*
(2) *The set of maps $f^{-1}\mathcal{G} \to \mathcal{F}$.*
(3) *The set of f-maps $\xi : \mathcal{G} \to \mathcal{F}$.*

Proof. We leave the easy verification to the reader. $\square$

It is sometimes convenient to think about f-maps instead of maps between sheaves either on X or on Y. We define composition of f-maps as follows.

Definition 6.21.9. Suppose that $f : X \to Y$ and $g : Y \to Z$ are continuous maps of topological spaces. Suppose that $\mathcal{F}$ is a sheaf on X, $\mathcal{G}$ is a sheaf on Y, and $\mathcal{H}$ is a sheaf on

Z. Let $\varphi : \mathcal{G} \to \mathcal{F}$ be an f-map. Let $\psi : \mathcal{H} \to \mathcal{G}$ be an g-map. The *composition of φ and ψ* is the $(g \circ f)$-map $\varphi \circ \psi$ defined by the commutativity of the diagrams

$$\begin{tikzcd} \mathcal{H}(W) \arrow[rr, "(\varphi\circ\psi)_W"'] \arrow[dr, "\psi_W"'] & & \mathcal{F}(f^{-1}g^{-1}W) \\ & \mathcal{G}(g^{-1}W) \arrow[ur, "\varphi_{g^{-1}W}"'] & \end{tikzcd}$$

We leave it to the reader to verify that this works. Another way to think about this is to think of $\varphi \circ \psi$ as the composition

$$\mathcal{H} \xrightarrow{\psi} g_*\mathcal{G} \xrightarrow{g_*\varphi} g_*f_*\mathcal{F} = (g \circ f)_*\mathcal{F}$$

Now, doesn't it seem that thinking about f-maps is somehow easier?

Finally, given a continuous map $f : X \to Y$, and an f-map $\varphi : \mathcal{G} \to \mathcal{F}$ there is a natural map on stalks

$$\varphi_x : \mathcal{G}_{f(x)} \longrightarrow \mathcal{F}_x$$

for all $x \in X$. The image of a representative (V, s) of an element in $\mathcal{G}_{f(x)}$ is mapped to the element in $\mathcal{F}_x$ with representative $(f^{-1}V, \varphi_V(s))$. We leave it to the reader to see that this is well defined. Another way to state it is that it is the unique map such that all diagrams

$$\begin{tikzcd} \mathcal{F}(f^{-1}V) \arrow[r] & \mathcal{F}_x \\ \mathcal{G}(V) \arrow[r] \arrow[u, "\varphi_V"] & \mathcal{G}_{f(x)} \arrow[u, "\varphi_x"'] \end{tikzcd}$$

(for $x \in V \subset Y$ open) commute.

Lemma 6.21.10. *Suppose that $f : X \to Y$ and $g : Y \to Z$ are continuous maps of topological spaces. Suppose that $\mathcal{F}$ is a sheaf on X, $\mathcal{G}$ is a sheaf on Y, and $\mathcal{H}$ is a sheaf on Z. Let $\varphi : \mathcal{G} \to \mathcal{F}$ be an f-map. Let $\psi : \mathcal{H} \to \mathcal{G}$ be an g-map. Let $x \in X$ be a point. The map on stalks $(\varphi \circ \psi)_x : \mathcal{H}_{g(f(x))} \to \mathcal{F}_x$ is the composition*

$$\mathcal{H}_{g(f(x))} \xrightarrow{\psi_{f(x)}} \mathcal{G}_{f(x)} \xrightarrow{\varphi_x} \mathcal{F}_x$$

Proof. Immediate from Definition 6.21.9 and the definition of the map on stalks above. $\square$

6.22. Continuous maps and abelian sheaves

Let $f : X \to Y$ be a continuous map. We claim there are functors

$$\begin{aligned} f_* : PAb(X) &\longrightarrow PAb(Y) \\ f_* : Ab(X) &\longrightarrow Ab(Y) \\ f_p : PAb(Y) &\longrightarrow PAb(X) \\ f^{-1} : Ab(Y) &\longrightarrow Ab(X) \end{aligned}$$

with similar properties to their counterparts in Section 6.21. To see this we argue in the following way.

Each of the functors will be constructed in the same way as the corresponding functor in Section 6.21. This works because all the colimits in that section are directed colimits (but we will work through it below).

First off, given an abelian presheaf $\mathcal{F}$ on X and an abelian presheaf $\mathcal{G}$ on Y we define

$$\begin{aligned} f_*\mathcal{F}(V) &= \mathcal{F}(f^{-1}(V)) \\ f_p\mathcal{G}(U) &= colim_{f(U)\subset V}\mathcal{G}(V) \end{aligned}$$

as abelian groups. The restriction mappings are the same as the restriction mappings for presheaves of sets (and they are all homomorphisms of abelian groups).

The assignments $\mathcal{F} \mapsto f_*\mathcal{F}$ and $\mathcal{G} \to f_p\mathcal{G}$ are functors on the categories of presheaves of abelian groups. This is clear, as (for example) a map of abelian presheaves $\mathcal{G}_1 \to \mathcal{G}_2$ gives rise to a map of directed systems $\{\mathcal{G}_1(V)\}_{f(U)\subset V} \to \{\mathcal{G}_2(V)\}_{f(U)\subset V}$ all of whose maps are homomorphisms and hence gives rise to a homomorphism of abelian groups $f_p\mathcal{G}_1(U) \to f_p\mathcal{G}_2(U)$.

The functors f_* and f_p are adjoint on the category of presheaves of abelian groups, i.e., we have

$$Mor_{PAb(X)}(f_p\mathcal{G}, \mathcal{F}) = Mor_{PAb(Y)}(\mathcal{G}, f_*\mathcal{F}).$$

To prove this, note that the map $i_\mathcal{G} : \mathcal{G} \to f_*f_p\mathcal{G}$ from the proof of Lemma 6.21.3 is a map of abelian presheaves. Hence if $\psi : f_p\mathcal{G} \to \mathcal{F}$ is a map of abelian presheaves, then the corresponding map $\mathcal{G} \to f_*\mathcal{F}$ is the map $f_*\psi \circ i_\mathcal{G} : \mathcal{G} \to f_*f_p\mathcal{G} \to f_*\mathcal{F}$ is also a map of abelian presheaves. For the other direction we point out that the map $c_\mathcal{F} : f_pf_*\mathcal{F} \to \mathcal{F}$ from the proof of Lemma 6.21.3 is a map of abelian presheaves as well (since it is made out of restriction mappings of $\mathcal{F}$ which are all homomorphisms). Hence given a map of abelian presheaves $\varphi : \mathcal{G} \to f_*\mathcal{F}$ the map $c_\mathcal{F} \circ f_p\varphi : f_p\mathcal{G} \to \mathcal{F}$ is a map of abelian presheaves as well. Since these constructions $\psi \mapsto f_*\psi$ and $\varphi \mapsto c_\mathcal{F} \circ f_p\varphi$ are inverse to each other as constructions on maps of presheaves of sets we see they are also inverse to each other on maps of abelian presheaves.

If $\mathcal{F}$ is an abelian sheaf on Y, then $f_*\mathcal{F}$ is an abelian sheaf on X. This is true because of the definition of an abelian sheaf and because this is true for sheaves of sets, see Lemma 6.21.1. This defines the functor f_* on the category of abelian sheaves.

We define $f^{-1}\mathcal{G} = (f_p\mathcal{G})^\#$ as before. Adjointness of f_* and f^{-1} follows formally as in the case of presheaves of sets. Here is the argument:

$$\begin{aligned} Mor_{Ab(X)}(f^{-1}\mathcal{G}, \mathcal{F}) &= Mor_{PAb(X)}(f_p\mathcal{G}, \mathcal{F}) \\ &= Mor_{PAb(Y)}(\mathcal{G}, f_*\mathcal{F}) \\ &= Mor_{Ab(Y)}(\mathcal{G}, f_*\mathcal{F}) \end{aligned}$$

Lemma 6.22.1. *Let $f : X \to Y$ be a continuous map.*

(1) *Let $\mathcal{G}$ be an abelian presheaf on Y. Let $x \in X$. The bijection $\mathcal{G}_{f(x)} \to (f_p\mathcal{G})_x$ of Lemma 6.21.4 is an isomorphism of abelian groups.*
(2) *Let $\mathcal{G}$ be an abelian sheaf on Y. Let $x \in X$. The bijection $\mathcal{G}_{f(x)} \to (f^{-1}\mathcal{G})_x$ of Lemma 6.21.5 is an isomorphism of abelian groups.*

Proof. Omitted. □

Given a continuous map $f : X \to Y$ and sheaves of abelian groups $\mathcal{F}$ on X, $\mathcal{G}$ on Y, the notion of an *f-map $\mathcal{G} \to \mathcal{F}$ of sheaves of abelian groups* makes sense. We can just define it exactly as in Definition 6.21.7 (replacing maps of sets with homomorphisms of abelian groups) or we can simply say that it is the same as a map of abelian sheaves $\mathcal{G} \to f_*\mathcal{F}$. We will use this notion freely in the following. The group of f-maps between $\mathcal{G}$ and $\mathcal{F}$ will be in canonical bijection with the groups $Mor_{Ab(X)}(f^{-1}\mathcal{G}, \mathcal{F})$ and $Mor_{Ab(Y)}(\mathcal{G}, f_*\mathcal{F})$.

Composition of f-maps is defined in exactly the same manner as in the case of f-maps of sheaves of sets. In addition, given an f-map $\mathscr{G} \to \mathscr{F}$ as above, the induced maps on stalks

$$\varphi_x : \mathscr{G}_{f(x)} \longrightarrow \mathscr{F}_x$$

are abelian group homomorphisms.

6.23. Continuous maps and sheaves of algebraic structures

Let $(\mathscr{C}, F)$ be a type of algebraic structure. For a topological space X let us introduce the notation:

(1) $PSh(X, \mathscr{C})$ will be the category of presheaves with values in $\mathscr{C}$.
(2) $Sh(X, \mathscr{C})$ will be the category of sheaves with values in $\mathscr{C}$.

Let $f : X \to Y$ be a continuous map of topological spaces. The same arguments as in the previous section show there are functors

$$\begin{aligned} f_* : PSh(X, \mathscr{C}) &\longrightarrow PSh(Y, \mathscr{C}) \\ f_* : Sh(X, \mathscr{C}) &\longrightarrow Sh(Y, \mathscr{C}) \\ f_p : PSh(Y, \mathscr{C}) &\longrightarrow PSh(X, \mathscr{C}) \\ f^{-1} : Sh(Y, \mathscr{C}) &\longrightarrow Sh(X, \mathscr{C}) \end{aligned}$$

constructed in the same manner and with the same properties as the functors constructed for abelian (pre)sheaves. In particular there are commutative diagrams

$$\begin{tikzcd} PSh(X, \mathscr{C}) \arrow[r, "f_*"] \arrow[d, "F"] & PSh(Y, \mathscr{C}) \arrow[d, "F"] \\ PSh(X) \arrow[r, "f_*"] & PSh(Y) \end{tikzcd} \qquad \begin{tikzcd} Sh(X, \mathscr{C}) \arrow[r, "f_*"] \arrow[d, "F"] & Sh(Y, \mathscr{C}) \arrow[d, "F"] \\ Sh(X) \arrow[r, "f_*"] & Sh(Y) \end{tikzcd}$$

$$\begin{tikzcd} PSh(Y, \mathscr{C}) \arrow[r, "f_p"] \arrow[d, "F"] & PSh(X, \mathscr{C}) \arrow[d, "F"] \\ PSh(Y) \arrow[r, "f_p"] & PSh(X) \end{tikzcd} \qquad \begin{tikzcd} Sh(Y, \mathscr{C}) \arrow[r, "f^{-1}"] \arrow[d, "F"] & Sh(X, \mathscr{C}) \arrow[d, "F"] \\ Sh(Y) \arrow[r, "f^{-1}"] & Sh(X) \end{tikzcd}$$

The main formulas to keep in mind are the following

$$\begin{aligned} f_*\mathscr{F}(V) &= \mathscr{F}(f^{-1}(V)) \\ f_p\mathscr{G}(U) &= colim_{f(U) \subset V} \mathscr{G}(V) \\ f^{-1}\mathscr{G} &= (f_p\mathscr{G})^\# \\ (f_p\mathscr{G})_x &= \mathscr{G}_{f(x)} \\ (f^{-1}\mathscr{G})_x &= \mathscr{G}_{f(x)} \end{aligned}$$

Each of these formulas has the property that they hold in the category $\mathscr{C}$ and that upon taking underlying sets we get the corresponding formula for presheaves of sets. In addition we have the adjointness properties

$$\begin{aligned} Mor_{PSh(X,\mathscr{C})}(f_p\mathscr{G}, \mathscr{F}) &= Mor_{PSh(Y,\mathscr{C})}(\mathscr{G}, f_*\mathscr{F}) \\ Mor_{Sh(X,\mathscr{C})}(f^{-1}\mathscr{G}, \mathscr{F}) &= Mor_{Sh(Y,\mathscr{C})}(\mathscr{G}, f_*\mathscr{F}). \end{aligned}$$

To prove these, the main step is to construct the maps

$$i_{\mathcal{G}} : \mathcal{G} \longrightarrow f_* f_p \mathcal{G}$$

and

$$c_{\mathcal{F}} : f_p f_* \mathcal{F} \longrightarrow \mathcal{F}$$

which occur in the proof of Lemma 6.21.3 as morphisms of presheaves with values in $\mathcal{C}$. This may be safely left to the reader since the constructions are exactly the same as in the case of presheaves of sets.

Given a continuous map $f : X \to Y$ and sheaves of algebraic structures $\mathcal{F}$ on X, $\mathcal{G}$ on Y, the notion of an *f-map $\mathcal{G} \to \mathcal{F}$ of sheaves of algebraic structures* makes sense. We can just define it exactly as in Definition 6.21.7 (replacing maps of sets with morphisms in $\mathcal{C}$) or we can simply say that it is the same as a map of sheaves of algebraic structures $\mathcal{G} \to f_*\mathcal{F}$. We will use this notion freely in the following. The set of f-maps between $\mathcal{G}$ and $\mathcal{F}$ will be in canonical bijection with the sets $Mor_{Sh(X,\mathcal{C})}(f^{-1}\mathcal{G}, \mathcal{F})$ and $Mor_{Sh(Y,\mathcal{C})}(\mathcal{G}, f_*\mathcal{F})$.

Composition of f-maps is defined in exactly the same manner as in the case of f-maps of sheaves of sets. In addition, given an f-map $\mathcal{G} \to \mathcal{F}$ as above, the induced maps on stalks

$$\varphi_x : \mathcal{G}_{f(x)} \longrightarrow \mathcal{F}_x$$

are homomorphisms of algebraic structures.

Lemma 6.23.1. *Let $f : X \to Y$ be a continuous map of topological spaces. Suppose given sheaves of algebraic structures $\mathcal{F}$ on X, $\mathcal{G}$ on Y. Let $\varphi : \mathcal{G} \to \mathcal{F}$ be an f-map of underlying sheaves of sets. If for every $V \subset Y$ open the map of sets $\varphi_V : \mathcal{G}(V) \to \mathcal{F}(f^{-1}V)$ is the effect of a morphism in $\mathcal{C}$ on underlying sets, then φ comes from a unique f-morphism between sheaves of algebraic structures.*

Proof. Omitted. □

6.24. Continuous maps and sheaves of modules

The case of sheaves of modules is more complicated. The reason is that the natural setting for defining the pullback and pushforward functors, is the setting of ringed spaces, which we will define below. First we state a few obvious lemmas.

Lemma 6.24.1. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a presheaf of rings on X. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules. There is a natural map of underlying presheaves of sets*

$$f_*\mathcal{O} \times f_*\mathcal{F} \longrightarrow f_*\mathcal{F}$$

which turns $f_*\mathcal{F}$ into a presheaf of $f_*\mathcal{O}$-modules. This construction is functorial in $\mathcal{F}$.

Proof. Let $V \subset Y$ is open. We define the map of the lemma to be the map

$$f_*\mathcal{O}(V) \times f_*\mathcal{F}(V) = \mathcal{O}(f^{-1}V) \times \mathcal{F}(f^{-1}V) \to \mathcal{F}(f^{-1}V) = f_*\mathcal{F}(V).$$

Here the arrow in the middle is the multiplication map on X. We leave it to the reader to see this is compatible with restriction mappings and defines a structure of $f_*\mathcal{O}$-module on $f_*\mathcal{F}$. □

Lemma 6.24.2. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a presheaf of rings on Y. Let $\mathcal{G}$ be a presheaf of $\mathcal{O}$-modules. There is a natural map of underlying presheaves of sets*

$$f_p\mathcal{O} \times f_p\mathcal{G} \longrightarrow f_p\mathcal{G}$$

which turns $f_p\mathcal{G}$ into a presheaf of $f_p\mathcal{O}$-modules. This construction is functorial in $\mathcal{G}$.

Proof. Let $U \subset X$ is open. We define the map of the lemma to be the map

$$\begin{aligned} f_p\mathcal{O}(U) \times f_p\mathcal{G}(U) &= colim_{f(U)\subset V}\mathcal{O}(V) \times colim_{f(U)\subset V}\mathcal{G}(V) \\ &= colim_{f(U)\subset V}(\mathcal{O}(V) \times \mathcal{G}(V)) \\ &\to colim_{f(U)\subset V}\mathcal{G}(V) \\ &= f_p\mathcal{G}(U). \end{aligned}$$

Here the arrow in the middle is the multiplication map on Y. The second equality holds because directed colimits commute with finite limits, see Categories, Lemma 4.17.2. We leave it to the reader to see this is compatible with restriction mappings and defines a structure of $f_p\mathcal{O}$-module on $f_p\mathcal{G}$. □

Let $f : X \to Y$ be a continuous map. Let $\mathcal{O}_X$ be a presheaf of rings on X and let $\mathcal{O}_Y$ be a presheaf of rings on Y. So at the moment we have defined functors

$$\begin{aligned} f_* : PMod(\mathcal{O}_X) &\longrightarrow PMod(f_*\mathcal{O}_X) \\ f_p : PMod(\mathcal{O}_Y) &\longrightarrow PMod(f_p\mathcal{O}_Y) \end{aligned}$$

These satisfy some compatibilities as follows.

Lemma 6.24.3. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a presheaf of rings on Y. Let $\mathcal{G}$ be a presheaf of $\mathcal{O}$-modules. Let $\mathcal{F}$ be a presheaf of $f_p\mathcal{O}$-modules. Then*

$$Mor_{PMod(f_p\mathcal{O})}(f_p\mathcal{G}, \mathcal{F}) = Mor_{PMod(\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}).$$

Here we use Lemmas 6.24.2 and 6.24.1, and we think of $f_*\mathcal{F}$ as an $\mathcal{O}$-module via the map $i_{\mathcal{O}} : \mathcal{O} \to f_*f_p\mathcal{O}$ (defined first in the proof of Lemma 6.21.3).

Proof. Note that we have

$$Mor_{PAb(X)}(f_p\mathcal{G}, \mathcal{F}) = Mor_{PAb(Y)}(\mathcal{G}, f_*\mathcal{F}).$$

according to Section 6.22. So what we have to prove is that under this correspondence, the subsets of module maps correspond. In addition, the correspondence is determined by the rule

$$\left(\psi : f_p\mathcal{G} \to \mathcal{F}\right) \longmapsto \left(f_*\psi \circ i_{\mathcal{G}} : \mathcal{G} \to f_*f_p\mathcal{G} \to f_*\mathcal{F}\right)$$

Hence, using the functoriality of the pushforward we see that it suffices to prove that the map $i_{\mathcal{G}} : \mathcal{G} \to f_*f_p\mathcal{G}$ is compatible with module structure, which we leave to the reader. □

Lemma 6.24.4. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a presheaf of rings on X. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules. Let $\mathcal{G}$ be a presheaf of $f_*\mathcal{O}$-modules. Then*

$$Mor_{PMod(\mathcal{O})}(\mathcal{O} \otimes_{p,f_pf_*\mathcal{O}} f_p\mathcal{G}, \mathcal{F}) = Mor_{PMod(f_*\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}).$$

Here we use Lemmas 6.24.2 and 6.24.1, and we use the map $c_{\mathcal{O}} : f_pf_*\mathcal{O} \to \mathcal{O}$ in the definition of the tensor product.

Proof. This follows from the equalities

$$\begin{aligned} Mor_{PMod(\mathcal{O})}(\mathcal{O} \otimes_{p,f_pf_*\mathcal{O}} f_p\mathcal{G}, \mathcal{F}) &= Mor_{PMod(f_pf_*\mathcal{O})}(f_p\mathcal{G}, \mathcal{F}_{f_pf_*\mathcal{O}}) \\ &= Mor_{PMod(f_*\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}). \end{aligned}$$

which is a combination of Lemmas 6.6.2 and 6.24.3. □

Lemma 6.24.5. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a sheaf of rings on X. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. The pushforward $f_*\mathcal{F}$, as defined in Lemma 6.24.1 is a sheaf of $f_*\mathcal{O}$-modules.*

Proof. Obvious from the definition and Lemma 6.21.1. $\square$

Lemma 6.24.6. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a sheaf of rings on Y. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}$-modules. There is a natural map of underlying presheaves of sets*

$$f^{-1}\mathcal{O} \times f^{-1}\mathcal{G} \longrightarrow f^{-1}\mathcal{G}$$

which turns $f^{-1}\mathcal{G}$ into a sheaf of $f^{-1}\mathcal{O}$-modules.

Proof. Recall that f^{-1} is defined as the composition of the functor f_p and sheafification. Thus the lemma is a combination of Lemma 6.24.2 and Lemma 6.20.1. $\square$

Let $f : X \to Y$ be a continuous map. Let $\mathcal{O}_X$ be a sheaf of rings on X and let $\mathcal{O}_Y$ be a sheaf of rings on Y. So now we have defined functors

$$\begin{array}{rcl} f_* : Mod(\mathcal{O}_X) & \longrightarrow & Mod(f_*\mathcal{O}_X) \\ f^{-1} : Mod(\mathcal{O}_Y) & \longrightarrow & Mod(f^{-1}\mathcal{O}_Y) \end{array}$$

These satisfy some compatibilities as follows.

Lemma 6.24.7. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a sheaf of rings on Y. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}$-modules. Let $\mathcal{F}$ be a sheaf of $f^{-1}\mathcal{O}$-modules. Then*

$$Mor_{Mod(f^{-1}\mathcal{O})}(f^{-1}\mathcal{G}, \mathcal{F}) = Mor_{Mod(\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}).$$

Here we use Lemmas 6.24.6 and 6.24.5, and we think of $f_*\mathcal{F}$ as an $\mathcal{O}$-module by restriction via $\mathcal{O} \to f_*f^{-1}\mathcal{O}$.

Proof. Argue by the equalities

$$\begin{array}{rcl} Mor_{Mod(f^{-1}\mathcal{O})}(f^{-1}\mathcal{G}, \mathcal{F}) & = & Mor_{Mod(f_p\mathcal{O})}(f_p\mathcal{G}, \mathcal{F}) \\ & = & Mor_{Mod(\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}). \end{array}$$

where the second is Lemmas 6.24.3 and the first is by Lemma 6.20.1. $\square$

Lemma 6.24.8. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a sheaf of rings on X. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. Let $\mathcal{G}$ be a sheaf of $f_*\mathcal{O}$-modules. Then*

$$Mor_{Mod(\mathcal{O})}(\mathcal{O} \otimes_{f^{-1}f_*\mathcal{O}} f^{-1}\mathcal{G}, \mathcal{F}) = Mor_{Mod(f_*\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}).$$

Here we use Lemmas 6.24.6 and 6.24.5, and we use the canonical map $f^{-1}f_*\mathcal{O} \to \mathcal{O}$ in the definition of the tensor product.

Proof. This follows from the equalities

$$\begin{array}{rcl} Mor_{Mod(\mathcal{O})}(\mathcal{O} \otimes_{f^{-1}f_*\mathcal{O}} f^{-1}\mathcal{G}, \mathcal{F}) & = & Mor_{Mod(f^{-1}f_*\mathcal{O})}(f^{-1}\mathcal{G}, \mathcal{F}_{f^{-1}f_*\mathcal{O}}) \\ & = & Mor_{Mod(f_*\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}). \end{array}$$

which are a combination of Lemma 6.20.2 and 6.24.7. $\square$

Let $f : X \to Y$ be a continuous map. Let $\mathcal{O}_X$ be a (pre)sheaf of rings on X and let $\mathcal{O}_Y$ be a (pre)sheaf of rings on Y. So at the moment we have defined functors

$$\begin{array}{rcl} f_* : PMod(\mathcal{O}_X) & \longrightarrow & PMod(f_*\mathcal{O}_X) \\ f_* : Mod(\mathcal{O}_X) & \longrightarrow & Mod(f_*\mathcal{O}_X) \\ f_p : PMod(\mathcal{O}_Y) & \longrightarrow & PMod(f_p\mathcal{O}_Y) \\ f^{-1} : Mod(\mathcal{O}_Y) & \longrightarrow & Mod(f^{-1}\mathcal{O}_Y) \end{array}$$

Clearly, usually the pair of functors (f_*, f^{-1}) on sheaves of modules are not adjoint, because their target categories do not match. Namely, as we saw above, it works only if by some miracle the sheaves of rings $\mathcal{O}_X, \mathcal{O}_Y$ satisfy the relations $\mathcal{O}_X = f^{-1}\mathcal{O}_Y$ and $\mathcal{O}_Y = f_*\mathcal{O}_X$. This is almost never true in practice. We interrupt the discussion to define the correct notion of morphism for which a suitable adjoint pair of functors on sheaves of modules exists.

6.25. Ringed spaces

Let X be a topological space and let $\mathcal{O}_X$ be a sheaf of rings on X. We are supposed to think of the sheaf of rings $\mathcal{O}_X$ as a sheaf of functions on X. And if $f : X \to Y$ is a ``suitable'' map, then by composition a function on Y turns into a function on X. Thus there should be a natural f-map from $\mathcal{O}_Y$ to $\mathcal{O}_X$ See Definition 6.21.7, and the remarks in previous sections for terminology. For a precise example, see Example 6.25.2 below. Here is the relevant abstract definition.

Definition 6.25.1. A *ringed space* is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ on X. A *morphism of ringed spaces* $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair consisting of a continuous map $f : X \to Y$ and an f-map of sheaves of rings $f^\sharp : \mathcal{O}_Y \to \mathcal{O}_X$.

Example 6.25.2. Let $f : X \to Y$ be a continuous map of topological spaces. Consider the sheaves of continuous real valued functions $\mathcal{C}^0_X$ on X and $\mathcal{C}^0_Y$ on Y, see Example 6.9.3. We claim that there is a natural f-map $f^\sharp : \mathcal{C}^0_Y \to \mathcal{C}^0_X$ associated to f. Namely, we simply define it by the rule

$$\begin{array}{ccc} \mathcal{C}^0_Y(V) & \longrightarrow & \mathcal{C}^0_X(f^{-1}V) \\ h & \longmapsto & h \circ f \end{array}$$

Stricly speaking we should write $f^\sharp(h) = h \circ f|_{f^{-1}(V)}$. It is clear that this is a family of maps as in Definition 6.21.7 and compatible with the $\mathbf{R}$-algebra structures. Hence it is an f-map of sheaves of $\mathbf{R}$-algebras, see Lemma 6.23.1.

Of course there are lots of other situations where there is a canonical morphism of ringed spaces associated to a geometrical type of morphism. For example, if M, N are $\mathcal{C}^\infty$-manifolds and $f : M \to N$ is a infinitely differentiable map, then f induces a canonical morphism of ringed spaces $(M, \mathcal{C}^\infty_M) \to (N, \mathcal{C}^\infty_N)$. The construction (which is identical to the above) is left to the reader.

It may not be completely obvious how to compose morphisms of ringed spaces hence we spell it out here.

Definition 6.25.3. Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $(g, g^\sharp) : (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ be morphisms of ringed spaces. Then we define the *composition of morphisms of ringed spaces* by the rule

$$(g, g^\sharp) \circ (f, f^\sharp) = (g \circ f, f^\sharp \circ g^\sharp).$$

Here we use composition of f-maps defined in Definition 6.21.9.

6.26. Morphisms of ringed spaces and modules

We have now introduced enough notation so that we are able to define the pullback and pushforward of modules along a morphism of ringed spaces.

Definition 6.26.1. Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces.

(1) Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We define the *pushforward* of $\mathcal{F}$ as the sheaf of $\mathcal{O}_Y$-modules which as a sheaf of abelian groups equals $f_*\mathcal{F}$ and with module structure given by the restriction via $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of the module structure given in Lemma 6.24.5.

(2) Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_Y$-modules. We define the *pullback* $f^*\mathcal{G}$ to be the sheaf of $\mathcal{O}_X$-modules defined by the formula
$$f^*\mathcal{F} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{F}$$
where the ring map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ is the map corresponding to $f^\sharp$, and where the module structure is given by Lemma 6.24.6.

Thus we have defined functors
$$\begin{aligned} f_* : Mod(\mathcal{O}_X) &\longrightarrow Mod(\mathcal{O}_Y) \\ f^* : Mod(\mathcal{O}_Y) &\longrightarrow Mod(\mathcal{O}_X) \end{aligned}$$
The final result on these functors is that they are indeed adjoint as expected.

Lemma 6.26.2. *Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_Y$-modules. There is a canonical bijection*
$$Hom_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) = Hom_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}).$$
In other words: the functor f^* is the left adjoint to f_*.

Proof. This follows from the work we did before:
$$\begin{aligned} Hom_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) &= Mor_{Mod(\mathcal{O}_X)}(\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{G}, \mathcal{F}) \\ &= Mor_{Mod(f^{-1}\mathcal{O}_Y)}(f^{-1}\mathcal{G}, \mathcal{F}_{f^{-1}\mathcal{O}_Y}) \\ &= Hom_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}). \end{aligned}$$
Here we use Lemmas 6.20.2 and 6.24.7. $\square$

Lemma 6.26.3. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of ringed spaces. The functors $(g \circ f)_*$ and $g_* \circ f_*$ are equal. There is a canonical isomorphism of functors $(g \circ f)^* \cong f^* \circ g^*$.*

Proof. The result on pushforwards is a consequence of Lemma 6.21.2 and our definitions. The result on pullbacks follows from this by the same argument as in the proof of Lemma 6.21.6. $\square$

Given a morphism of ringed spaces $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, and a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, a sheaf of $\mathcal{O}_Y$-modules $\mathcal{G}$ on Y, the notion of an *f-map $\varphi : \mathcal{G} \to \mathcal{F}$ of sheaves of modules* makes sense. We can just define it as an f-map $\varphi : \mathcal{G} \to \mathcal{F}$ of abelian sheaves such that for all open $V \subset Y$ the map
$$\mathcal{G}(V) \longrightarrow \mathcal{F}(f^{-1}V)$$
is an $\mathcal{O}_Y(V)$-module map. Here we think of $\mathcal{F}(f^{-1}V)$ as an $\mathcal{O}_Y(V)$-module via the map $f^\sharp_V : \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}V)$. The set of f-maps between $\mathcal{G}$ and $\mathcal{F}$ will be in canonical bijection with the sets $Mor_{Mod(\mathcal{O}_X)}(f^*\mathcal{G}, \mathcal{F})$ and $Mor_{Mod(\mathcal{O}_Y)}(\mathcal{G}, f_*\mathcal{F})$. See above.

Composition of f-maps is defined in exactly the same manner as in the case of f-maps of sheaves of sets. In addition, given an f-map $\mathcal{G} \to \mathcal{F}$ as above, and $x \in X$ the induced map on stalks

$$\varphi_x : \mathcal{G}_{f(x)} \longrightarrow \mathcal{F}_x$$

is an $\mathcal{O}_{Y,f(x)}$-module map where the $\mathcal{O}_{Y,f(x)}$-module structure on $\mathcal{F}_x$ comes from the $\mathcal{O}_{X,x}$-module structure via the map $f_x^\sharp : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$. Here is a related lemma.

Lemma 6.26.4. *Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_Y$-modules. Let $x \in X$. Then*

$$f^*\mathcal{G}_x = \mathcal{F}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}, f_x^\sharp} \mathcal{O}_{X,x}$$

as $\mathcal{O}_{X,x}$-modules.

Proof. This follows from Lemma 6.20.3 and the identification of the stalks of pullback sheaves at x with the corresponding stalks at $f(x)$. See the formulae in Section 6.23 for example. $\square$

6.27. Skyscraper sheaves and stalks

Definition 6.27.1. Let X be a topological space.

(1) Let $x \in X$ be a point. Denote $i_x : \{x\} \to X$ the inclusion map. Let A be a set and think of A as a sheaf on the one point space $\{x\}$. We call $i_{x,*}A$ the *skyscraper sheaf at x with value A*.
(2) If in (1) above A is an abelian group then we think of $i_{x,*}A$ as a sheaf of abelian groups on X.
(3) If in (1) above A is an algebraic structure then we think of $i_{x,*}A$ as a sheaf of algeberaic structures.
(4) If $(X, \mathcal{O}_X)$ is a ringed space, then we think of $i_x : \{x\} \to X$ as a morphism of ringed spaces $(\{x\}, \mathcal{O}_{X,x}) \to (X, \mathcal{O}_X)$ and if A is a $\mathcal{O}_{X,x}$-module, then we think of $i_{x,*}A$ as a sheaf of $\mathcal{O}_X$-modules.
(5) We say a sheaf of sets $\mathcal{F}$ is a *skyscraper sheaf* if there exists an point x of X and a set A such that $\mathcal{F} \cong i_{x,*}A$.
(6) We say a sheaf of abelian groups $\mathcal{F}$ is a *skyscraper sheaf* if there exists an point x of X and an abelian group A such that $\mathcal{F} \cong i_{x,*}A$ as sheaves of abelian groups.
(7) We say a sheaf of algebraic structures $\mathcal{F}$ is a *skyscraper sheaf* if there exists an point x of X and an algebraic structure A such that $\mathcal{F} \cong i_{x,*}A$ as sheaves of algebraic structures.
(8) If $(X, \mathcal{O}_X)$ is a ringed space and $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules, then we say $\mathcal{F}$ is a *skyscraper sheaf* if there exists a point $x \in X$ and a $\mathcal{O}_{X,x}$-module A such that $\mathcal{F} \cong i_{x,*}A$ as sheaves of $\mathcal{O}_X$-modules.

Lemma 6.27.2. *Let X be a topological space, $x \in X$ a point, and A a set. For any point $x' \in X$ the stalk of the skyscraper sheaf at x with value A at x' is*

$$(i_{x,*}A)_{x'} = \begin{cases} A & \text{if} \quad x' \in \overline{\{x\}} \\ \{*\} & \text{if} \quad x' \notin \overline{\{x\}} \end{cases}$$

A similar description holds for the case of abelian groups, algebraic structures and sheaves of modules.

Proof. Omitted. $\square$

Lemma 6.27.3. *Let X be a topological space, and let $x \in X$ a point. The functors $\mathcal{F} \mapsto \mathcal{F}_x$ and $A \mapsto i_{x,*}A$ are adjoint. In a formula*

$$Mor_{Sets}(\mathcal{F}_x, A) = Mor_{Sh(X)}(\mathcal{F}, i_{x,*}A).$$

A similar satement holds for the case of abelian groups, algebraic structures. In the case of sheaves of modules we have

$$Hom_{\mathcal{O}_{X,x}}(\mathcal{F}_x, A) = Hom_{\mathcal{O}_X}(\mathcal{F}, i_{x,*}A).$$

Proof. Omitted. Hint: The stalk functor can be seen as the pullback functor for the morphism $i_x : \{x\} \to X$. Then the adjointness follows from adjointness of i_x^{-1} and $i_{x,*}$ (resp. i_x^* and $i_{x,*}$ in the case of sheaves of modules. □

6.28. Limits and colimits of presheaves

Let X be a topological space. Let $\mathcal{I} \to PSh(X)$, $i \mapsto \mathcal{F}_i$ be a diagram.

(1) Both $lim_i\, \mathcal{F}_i$ and $colim_i\, \mathcal{F}_i$ exist.
(2) For any open $U \subset X$ we have

$$(lim_i\, \mathcal{F}_i)(U) = lim_i\, \mathcal{F}_i(U)$$

and

$$(colim_i\, \mathcal{F}_i)(U) = colim_i\, \mathcal{F}_i(U).$$

(3) Let $x \in X$ be a point. In general the stalk of $lim_i\, \mathcal{F}_i$ at x is not equal to the limit of the stalks. But if the diagram category is finite then it is the case. In other words, the stalk functor is left exact (see Categories, Definition 4.21.1).
(4) Let $x \in X$. We always have

$$(colim_i\, \mathcal{F}_i)_x = colim_i\, \mathcal{F}_{i,x}.$$

The proofs are all easy.

6.29. Limits and colimits of sheaves

Let X be a topological space. Let $\mathcal{I} \to Sh(X)$, $i \mapsto \mathcal{F}_i$ be a diagram.

(1) Both $lim_i\, \mathcal{F}_i$ and $colim_i\, \mathcal{F}_i$ exist.
(2) The inclusion functor $i : Sh(X) \to PSh(X)$ commutes with limits. In other words, we may compute the limit in the category of sheaves as the limit in the category of presheaves. In particular, for any open $U \subset X$ we have

$$(lim_i\, \mathcal{F}_i)(U) = lim_i\, \mathcal{F}_i(U).$$

(3) The inclusion functor $i : Sh(X) \to PSh(X)$ does not commute with colimits in general (not even with finite colimits -- think surjections). The colimit is computed as the sheafification of the colimit in the category of presheaves:

$$colim_i\, \mathcal{F}_i = \left(U \mapsto colim_i\, \mathcal{F}_i(U)\right)^{\#}.$$

(4) Let $x \in X$ be a point. In general the stalk of $lim_i\, \mathcal{F}_i$ at x is not equal to the limit of the stalks. But if the diagram category is finite then it is the case. In other words, the stalk functor is left exact.
(5) Let $x \in X$. We always have

$$(colim_i\, \mathcal{F}_i)_x = colim_i\, \mathcal{F}_{i,x}.$$

(6) The sheafification functor $^{\#} : PSh(X) \to Sh(X)$ commutes with all colimits, and with finite limits. But it does not commute with all limits.

The proofs are all easy. Here is an example of what is true for directed colimits of sheaves.

Lemma 6.29.1. *Let X be a topological space. Let I be a directed partially ordered set. Let $(\mathcal{F}_i, \varphi_{ii'})$ be a system of sheaves of sets over I, see Categories, Section 4.19. Let $U \subset X$ be an open subset. Consider the canonical map*

$$\Psi : colim_i\, \mathcal{F}_i(U) \longrightarrow \left(colim_i\, \mathcal{F}_i\right)(U)$$

1. *If all the transition maps are injective then Ψ is injective for any open U.*
2. *If U is quasi-compact, then Ψ is injective.*
3. *If U is quasi-compact and all the transition maps are injective then Ψ is an isomorphism.*
4. *If U has a cofinal system of open coverings $\mathcal{U} : U = \bigcup_{j \in J} U_j$ with J finite and $U_j \cap U_{j'}$ quasi-compact for all $j, j' \in J$, then Ψ is bijective.*

Proof. Assume all the transition maps are injective. In this case the presheaf $\mathcal{F}' : V \mapsto colim_i\, \mathcal{F}_i(V)$ is separated (see Definition 6.11.2). By the discussion above we have $(\mathcal{F}')^\# = colim_i\, \mathcal{F}_i$. By Lemma 6.17.5 we see that $\mathcal{F}' \to (\mathcal{F}')^\#$ is injective. This proves (1).

Assume U is quasi-compact. Suppose that $s \in \mathcal{F}_i(U)$ and $s' \in \mathcal{F}_{i'}(U)$ give rise to elements on the left hand side which have the same image under Ψ. Since U is quasi-compact this means there exists a finite open covering $U = \bigcup_{j=1,\ldots,m} U_j$ and for each j an index $i_j \in I$, $i_j \geq i$, $i_j \geq i'$ such that $\varphi_{ii_j}(s) = \varphi_{i'i_j}(s')$. Let $i'' \in I$ be $\geq$ than all of the i_j. We conclude that $\varphi_{ii''}(s)$ and $\varphi_{i'i''}(s)$ agree on the opens U_j for all j and hence that $\varphi_{ii''}(s) = \varphi_{i'i''}(s)$. This proves (2).

Assume U is quasi-compact and all transition maps injective. Let s be an element of the target of Ψ. Since U is quasi-compact there exists a finite open covering $U = \bigcup_{j=1,\ldots,m} U_j$, for each j an index $i_j \in I$ and $s_j \in \mathcal{F}_{i_j}(U_j)$ such that $s|_{U_j}$ comes from s_j for all j. Pick $i \in I$ which is $\geq$ than all of the i_j. By (1) the sections $\varphi_{i_ji}(s_j)$ agree over the overlaps $U_j \cap U_{j'}$. Hence they glue to a section $s' \in \mathcal{F}_i(U)$ which maps to s under Ψ. This proves (3).

Assume the hypothesis of (4). Let s be an element of the target of Ψ. By assumption there exists a finite open covering $U = \bigcup_{j=1,\ldots,m} U_j$, with $U_j \cap U_{j'}$ quasi-compact for all $j, j' \in J$ and for each j an index $i_j \in I$ and $s_j \in \mathcal{F}_{i_j}(U_j)$ such that $s|_{U_j}$ is the image of s_j for all j. Since $U_j \cap U_{j'}$ is quasi-compact we can apply (2) and we see that there exists an $i_{jj'} \in I$, $i_{jj'} \geq i_j$, $i_{jj'} \geq i_{j'}$ such that $\varphi_{i_ji_{jj'}}(s_j)$ and $\varphi_{i_{j'}i_{jj'}}(s_{j'})$ agree over $U_j \cap U_{j'}$. Choose an index $i \in I$ wich is bigger or equal than all the $i_{jj'}$. Then we see that the sections $\varphi_{i_ji}(s_j)$ of $\mathcal{F}_i$ glue to a section of $\mathcal{F}_i$ over U. This section is mapped to the element s as desired. $\square$

Example 6.29.2. Let $X = \{s_1, s_2, \xi_1, \xi_2, \xi_3, \ldots\}$ as a set. Declare a subset $U \subset X$ to be open if $s_1 \in U$ or $s_2 \in U$ implies U contains all of the ξ_i. Let $U_n = \{\xi_n, \xi_{n+1}, \ldots\}$, and let $j_n : U_n \to X$ be the inclusion map. Set $\mathcal{F}_n = j_{n,*}\underline{\mathbf{Z}}$. There are transition maps $\mathcal{F}_n \to \mathcal{F}_{n+1}$. Let $\mathcal{F} = colim\, \mathcal{F}_n$. Note that $\mathcal{F}_{n,\xi_m} = 0$ if $m < n$ because $\{\xi_m\}$ is an open subset of X which misses U_n. Hence we see that $\mathcal{F}_{\xi_n} = 0$ for all n. On the other hand the stalk $\mathcal{F}_{s_i}$, $i = 1, 2$ is the colimit

$$M = colim_n \prod\nolimits_{m \geq n} \mathbf{Z}$$

which is not zero. We conclude that the sheaf $\mathcal{F}$ is the direct sum of the skyscraper sheaves with value M at the closed points s_1 and s_2. Hence $\Gamma(X, \mathcal{F}) = M \oplus M$. On the other hand, the reader can verify that $colim_n\, \Gamma(X, \mathcal{F}_n) = M$. Hence some condition is necessary in part (4) of Lemma 6.29.1 above.

6.30. Bases and sheaves

Sometimes there exists a basis for the topology consisting of opens that are easier to work with than general opens. For convenience we give here some definitions and simple lemmas in order to facilitate working with (pre)sheaves in such a situation.

Definition 6.30.1. Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X.

(1) A *presheaf $\mathcal{F}$ of sets on $\mathcal{B}$* is a rule which assigns to each $U \in \mathcal{B}$ a set $\mathcal{F}(U)$ and to each inclusion $V \subset U$ of elements of $\mathcal{B}$ a map $\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ such that whenever $W \subset V \subset U$ in $\mathcal{B}$ we have $\rho^U_W = \rho^V_W \circ \rho^U_V$.
(2) A *morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves of sets on $\mathcal{B}$* is a rule which assigns to each element $U \in \mathcal{B}$ a map of sets $\varphi : \mathcal{F}(U) \to \mathcal{G}(U)$ compatible with restriction maps.

As in the case of usual presheaves we use the terminology of sections, restrictions of sections, etc. In particular, we may define the *stalk* of $\mathcal{F}$ at a point $x \in X$ by the colimit

$$\mathcal{F}_x = colim_{U \in \mathcal{B}, x \in U} \mathcal{F}(U).$$

As in the case of the stalk of a presheaf on X this limit is directed. The reason is that the collection of $U \in \mathcal{B}$, $x \in U$ is a fundamental system of open neighbourhoods of x.

It is easy to make examples to show that the notion of a presheaf on X is very different from the notion of a presheaf on a basis for the topology on X. This does not happen in the case of sheaves. A much more useful notion therefore, is the following.

Definition 6.30.2. Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X.

(1) A *sheaf $\mathcal{F}$ of sets on $\mathcal{B}$* is a presheaf of sets on $\mathcal{B}$ which satisfies the following additional property: Given any $U \in \mathcal{B}$, and any covering $U = \bigcup_{i \in I} U_i$ with $U_i \in \mathcal{B}$, and any coverings $U_i \cap U_j = \bigcup_{k \in I_{ij}} U_{ijk}$ with $U_{ijk} \in \mathcal{B}$ the sheaf condition holds:
   (∗∗) For any collection of sections $s_i \in \mathcal{F}(U_i)$, $i \in I$ such that $\forall i, j \in I, \forall k \in I_{ij}$
   $$s_i|_{U_{ijk}} = s_j|_{U_{ijk}}$$
   there exists a unique section $s \in \mathcal{F}(U)$ such that $s_i = s|_{U_i}$ for all $i \in I$.
(2) A *morphism of sheaves of sets on $\mathcal{B}$* is simply a morphism of presheaves of sets.

First we explain that it suffices to check the sheaf condition (∗∗) on a cofinal system of coverings. In the situation of the definition, suppose $U \in \mathcal{B}$. Let us temporarily denote $\text{Cov}_{\mathcal{B}}(U)$ the set of all coverings of U by elements of $\mathcal{B}$. Note that $\text{Cov}_{\mathcal{B}}(U)$ is partially ordered by refinement. A subset $C \subset \text{Cov}_{\mathcal{B}}(U)$ is a cofinal system, if for every $\mathcal{U} \in \text{Cov}_{\mathcal{B}}(U)$ there exists a covering $\mathcal{V} \in C$ which refines $\mathcal{U}$.

Lemma 6.30.3. *With notation as above. For each $U \in \mathcal{B}$, let $C(U) \subset Cov_{\mathcal{B}}(U)$ be a cofinal system. For each $U \in \mathcal{B}$, and each $\mathcal{U} : U = \bigcup U_i$ in $C(U)$, let coverings $\mathcal{U}_{ij} : U_i \cap U_j = \bigcup U_{ijk}$, $U_{ijk} \in \mathcal{B}$ be given. Let $\mathcal{F}$ be a presheaf of sets on $\mathcal{B}$. The following are equivalent*

(1) *The presheaf $\mathcal{F}$ is a sheaf on $\mathcal{B}$.*
(2) *For every $U \in \mathcal{B}$ and every covering $\mathcal{U} : U = \bigcup U_i$ in $C(U)$ the sheaf condition (∗∗) holds (for the given coverings $\mathcal{U}_{ij}$).*

Proof. We have to show that (2) implies (1). Suppose that $U \in \mathcal{B}$, and that $\mathcal{U} : U = \bigcup_{i \in I} U_i$ is an arbitrary covering by elements of $\mathcal{B}$. Because the system $C(U)$ is cofinal we can find an element $\mathcal{V} : U = \bigcup_{j \in J} V_j$ in $C(U)$ which refines $\mathcal{U}$. This means there exists a map $\alpha : J \to I$ such that $V_j \subset U_{\alpha(i)}$.

Note that if $s, s' \in \mathcal{F}(U)$ are sections such that $s|_{U_i} = s'|_{U_i}$, then

$$s|_{V_j} = (s|_{U_{\alpha(j)}})|_{V_j} = (s'|_{U_{\alpha(j)}})|_{V_j} = s'|_{V_j}$$

for all j. Hence by the uniqueness in $(**)$ for the covering $\mathcal{V}$ we conclude that $s = s'$. Thus we have proved the uniqueness part of $(**)$ for our arbitrary covering $\mathcal{U}$.

Suppose furthermore that $U_i \cap U_{i'} = \bigcup_{k \in I_{ii'}} U_{ii'k}$ are arbitrary coverings by $U_{ii'k} \in \mathcal{B}$. Let us try to prove the existence part of $(**)$ for the system $(\mathcal{U}, \mathcal{U}_{ij})$. Thus let $s_i \in \mathcal{F}(U_i)$ and suppose we have

$$s_i|_{U_{ijk}} = s_{i'}|_{U_{ii'k}}$$

for all i, i', k. Set $t_j = s_{\alpha(i)}|_{V_j}$, where $\mathcal{V}$ and α are as above.

There is one small kink in the argument here. Namely, let $\mathcal{V}_{jj'} : V_j \cap V_{j'} = \bigcup_{l \in J_{jj'}} V_{jj'l}$ be the covering given to us by the statement of the lemma. It is not a priori clear that

$$t_j|_{V_{jj'l}} = t_{j'}|_{V_{jj'l}}$$

for all j, j', l. To see this, note that we do have

$$t_j|_W = t_{j'}|_W \text{ for all } W \in \mathcal{B}, W \subset V_{jj'l} \cap U_{\alpha(j)\alpha(j')k}$$

for all $k \in I_{\alpha(j)\alpha(j')}$, by our assumption on the family of elements s_i. And since $V_j \cap V_{j'} \subset U_{\alpha(j)} \cap U_{\alpha(j')}$ we see that $t_j|_{V_{jj'l}}$ and $t_{j'}|_{V_{jj'l}}$ agree on the members of a covering of $V_{jj'l}$ by elements of $\mathcal{B}$. Hence by the uniqueness part proved above we finally deduce the desired equality of $t_j|_{V_{jj'l}}$ and $t_{j'}|_{V_{jj'l}}$. Then we get the existence of an element $t \in \mathcal{F}(U)$ by property $(**)$ for $(\mathcal{V}, \mathcal{V}_{jj'})$.

Again there is a small snag. We know that t restricts to t_j on V_j but we do not yet know that t restricts to s_i on U_i. To conclude this note that the sets $U_i \cap V_j$, $j \in J$ cover U_i. Hence also the sets $U_{i\alpha(j)k} \cap V_j$, $j \in J$, $k \in I_{i\alpha(j)}$ cover U_i. We leave it to the reader to see that t and s_i restrict to the same section of $\mathcal{F}$ on any $W \in \mathcal{B}$ which is contained in one of the open sets $U_{i\alpha(j)k} \cap V_j$, $j \in J$, $k \in I_{i\alpha(j)}$. Hence by the uniqueness part seen above we win. □

Lemma 6.30.4. *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Assume that for every pair $U, U' \in \mathcal{B}$ we have $U \cap U' \in \mathcal{B}$. For each $U \in \mathcal{B}$, let $C(U) \subset Cov_{\mathcal{B}}(U)$ be a cofinal system. Let $\mathcal{F}$ be a presheaf of sets on $\mathcal{B}$. The following are equivalent*

1. *The presheaf $\mathcal{F}$ is a sheaf on $\mathcal{B}$.*
2. *For every $U \in \mathcal{B}$ and every covering $\mathcal{U} : U = \bigcup U_i$ in $C(U)$ and for every family of sections $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ there exists a unique section $s \in \mathcal{F}(U)$ which restricts to s_i on U_i.*

Proof. This is a reformulation of Lemma 6.30.3 above in the special case where the coverings $\mathcal{U}_{ij}$ each consist of a single element. But also this case is much easier and is an easy exercise to do directly. □

Lemma 6.30.5. *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Let $U \in \mathcal{B}$. Let $\mathcal{F}$ be a sheaf of sets on $\mathcal{B}$. The map*

$$\mathcal{F}(U) \to \prod_{x \in U} \mathcal{F}_x$$

identifies $\mathcal{F}(U)$ with the elements $(s_x)_{x \in U}$ with the property

($*$) *For any $x \in U$ there exists a $V \in \mathcal{B}$, $x \in V$ and a section $\sigma \in \mathcal{F}(V)$ such that for all $y \in V$ we have $s_y = (V, \sigma)$ in $\mathcal{F}_y$.*

Proof. First note that the map $\mathcal{F}(U) \to \prod_{x \in U} \mathcal{F}_x$ is injective by the uniqueness in the sheaf condition of Definition 6.30.2. Let (s_x) be any element on the right hand side which satisfies ($*$). Clearly this means we can find a covering $U = \bigcup U_i$, $U_i \in \mathcal{B}$ such that $(s_x)_{x \in U_i}$ comes from certain $\sigma_i \in \mathcal{F}(U_i)$. For every $y \in U_i \cap U_j$ the sections σ_i and σ_j agree in the stalk $\mathcal{F}_y$. Hence there exists an element $V_{ijy} \in \mathcal{B}$, $y \in V_{ijy}$ such that $\sigma_i|_{V_{ijy}} = \sigma_j|_{V_{ijy}}$. Thus the sheaf condition ($**$) of Definition 6.30.2 applies to the system of σ_i and we obtain a section $s \in \mathcal{F}(U)$ with the desired property. $\square$

Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. There is a natural restriction functor from the category of sheaves of sets on X to the category of sheaves of sets on $\mathcal{B}$. It turns out that this is an equivalence of categories. In down to earth terms this means the following.

Lemma 6.30.6. *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Let $\mathcal{F}$ be a sheaf of sets on $\mathcal{B}$. There exists a unique sheaf of sets $\mathcal{F}^{ext}$ on X such that $\mathcal{F}^{ext}(U) = \mathcal{F}(U)$ for all $U \in \mathcal{B}$ compatibly with the restriction mappings.*

Proof. We first construct a presheaf $\mathcal{F}^{ext}$ with the desired property. Namely, for an arbitrary open $U \subset X$ we define $\mathcal{F}^{ext}(U)$ as the set of elements $(s_x)_{x \in U}$ such that ($*$) of Lemma 6.30.5 holds. It is clear that there are restriction mappings that turn $\mathcal{F}^{ext}$ into a presheaf of sets. Also, by Lemma 6.30.5 we see that $\mathcal{F}(U) = \mathcal{F}^{ext}(U)$ whenever U is an element of the basis $\mathcal{B}$. To see $\mathcal{F}^{ext}$ is a sheaf one may argue as in the proof of Lemma 6.17.1. $\square$

Note that we have

$$\mathcal{F}_x = \mathcal{F}^{ext}_x$$

in the situation of the lemma. This is so because the collection of elements of $\mathcal{B}$ containing x forms a fundamental system of open neighbourhoods of x.

Lemma 6.30.7. *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Denote $Sh(\mathcal{B})$ the category of sheaves on $\mathcal{B}$. There is an equivalence of categories*

$$Sh(X) \longrightarrow Sh(\mathcal{B})$$

which assigns to a sheaf on X its restriction to the members of $\mathcal{B}$.

Proof. The inverse functor in given in Lemma 6.30.6 above. Checking the obvious functorialities is left to the reader. $\square$

This ends the discussion of sheaves of sets on a basis $\mathcal{B}$. Let $(\mathcal{C}, F)$ be a type of algebraic structure. At the end of this section we would like to point out that the constructions above work for sheaves with values in $\mathcal{C}$. Let us briefly define the relevant notions.

Definition 6.30.8. Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Let $(\mathcal{C}, F)$ be a type of algebraic structure.

(1) A *presheaf $\mathcal{F}$ with values in $\mathcal{C}$ on $\mathcal{B}$* is a rule which assigns to each $U \in \mathcal{B}$ an object $\mathcal{F}(U)$ of $\mathcal{C}$ and to each inclusion $V \subset U$ of elements of $\mathcal{B}$ a morphism $\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ in $\mathcal{C}$ such that whenever $W \subset V \subset U$ in $\mathcal{B}$ we have $\rho^U_W = \rho^V_W \circ \rho^U_V$.
(2) A *morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves with values in $\mathcal{C}$ on $\mathcal{B}$* is a rule which assigns to each element $U \in \mathcal{B}$ a morphism of algebraic structures $\varphi : \mathcal{F}(U) \to \mathcal{G}(U)$ compatible with restriction maps.

(3) Given a presheaf $\mathcal{F}$ with values in $\mathcal{C}$ on $\mathcal{B}$ we say that $U \mapsto F(\mathcal{F}(U))$ is the underlying presheaf of sets.
(4) A *sheaf $\mathcal{F}$ with values in $\mathcal{C}$ on $\mathcal{B}$* is a presheaf with values in $\mathcal{C}$ on $\mathcal{B}$ whose underlying presheaf of sets is a sheaf.

At this point we can define the *stalk* at $x \in X$ of a presheaf with values in $\mathcal{C}$ on $\mathcal{B}$ as the directed colimit

$$\mathcal{F}_x = colim_{U \in \mathcal{B}, x \in U} \mathcal{F}(U).$$

It exists as an object of $\mathcal{C}$ because of our assumptions on $\mathcal{C}$. Also, we see that the underlying set of $\mathcal{F}_x$ is the stalk of the underlying presheaf of sets on $\mathcal{B}$.

Note that Lemmas 6.30.3, 6.30.4 and 6.30.5 refer to the sheaf property which we have defined in terms of the associated presheaf of sets. Hence they generalize without change to the notion of a presheaf with values in $\mathcal{C}$. The analogue of Lemma 6.30.6 need some care. Here it is.

Lemma 6.30.9. *Let X be a topological space. Let $(\mathcal{C}, F)$ be a type of algebraic structure. Let $\mathcal{B}$ be a basis for the topology on X. Let $\mathcal{F}$ be a sheaf with values in $\mathcal{C}$ on $\mathcal{B}$. There exists a unique sheaf $\mathcal{F}^{ext}$ with values in $\mathcal{C}$ on X such that $\mathcal{F}^{ext}(U) = \mathcal{F}(U)$ for all $U \in \mathcal{B}$ compatibly with the restriction mappings.*

Proof. By the conditions imposed on the pair $(\mathcal{C}, F)$ it suffices to come up with a presheaf $\mathcal{F}^{ext}$ which does the correct thing on the level of underlying presheaves of sets. Thus our first task is to construct a suitable object $\mathcal{F}^{ext}(U)$ for all open $U \subset X$. We could do this by imitating Lemma 6.18.1 in the setting of presheaves on $\mathcal{B}$. However, a slightly different method (but basically equivalent) is the following: Define it as the directed colimit

$$\mathcal{F}^{ext}(U) := colim_{\mathcal{U}} FIB(\mathcal{U})$$

over all coverings $\mathcal{U} : U = \bigcup_{i \in I} U_i$ by $U_i \in \mathcal{B}$ of the fibre product

$$\begin{tikzcd} FIB(\mathcal{U}) \arrow[r] \arrow[d] & \prod_{x \in U} \mathcal{F}_x \arrow[d] \\ \prod_{i \in I} \mathcal{F}(U_i) \arrow[r] & \prod_{i \in I} \prod_{x \in U_i} \mathcal{F}_x \end{tikzcd}$$

By the usual arguments, see Lemma 6.15.4 and Example 6.15.5 it suffices to show that this construction on underlying sets is the same as the definition using $(**)$ above. Details left to the reader. $\square$

Note that we have

$$\mathcal{F}_x = \mathcal{F}^{ext}_x$$

as objects in $\mathcal{C}$ in the situation of the lemma. This is so because the collection of elements of $\mathcal{B}$ containing x forms a fundamental system of open neighbourhoods of x.

Lemma 6.30.10. *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Let $(\mathcal{C}, F)$ be a type of algebraic structure. Denote $Sh(\mathcal{B}, \mathcal{C})$ the category of sheaves with values in $\mathcal{C}$ on $\mathcal{B}$. There is an equivalence of categories*

$$Sh(X, \mathcal{C}) \longrightarrow Sh(\mathcal{B}, \mathcal{C})$$

which assigns to a sheaf on X its restriction to the members of $\mathcal{B}$.

Proof. The inverse functor in given in Lemma 6.30.9 above. Checking the obvious functorialities is left to the reader. $\square$

Finally we come to the case of (pre)sheaves of modules on a basis. We will use the easy fact that the category of presheaves of sets on a basis has products and that they are described by taking products of values on elements of the bases.

Definition 6.30.11. Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Let $\mathcal{O}$ be a presheaf of rings on $\mathcal{B}$.

(1) A *presheaf of $\mathcal{O}$-modules $\mathcal{F}$ on $\mathcal{B}$* is a presheaf of abelian groups on $\mathcal{B}$ together with a morphism of presheaves of sets $\mathcal{O} \times \mathcal{F} \to \mathcal{F}$ such that for all $U \in \mathcal{B}$ the map $\mathcal{O}(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ turns the group $\mathcal{F}(U)$ into an $\mathcal{O}(U)$-module.
(2) A *morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves of $\mathcal{O}$-modules on $\mathcal{B}$* is a morphism of abelian presheaves on $\mathcal{B}$ which induces an $\mathcal{O}(U)$-module homomorphism $\mathcal{F}(U) \to \mathcal{G}(U)$ for every $U \in \mathcal{B}$.
(3) Suppose that $\mathcal{O}$ is a sheaf of rings on $\mathcal{B}$. A *sheaf $\mathcal{F}$ of $\mathcal{O}$-modules on $\mathcal{B}$* is a presheaf of $\mathcal{O}$-modules on $\mathcal{B}$ whose underlying presheaf of abelain groups is a sheaf.

We can define the *stalk* at $x \in X$ of a presheaf of $\mathcal{O}$-modules on $\mathcal{B}$ as the directed colimit

$$\mathcal{F}_x = colim_{U \in \mathcal{B}, x \in U} \mathcal{F}(U).$$

It is a $\mathcal{O}_x$-module.

Note that Lemmas 6.30.3, 6.30.4 and 6.30.5 refer to the sheaf property which we have defined in terms of the associated presheaf of sets. Hence they generalize without change to the notion of a presheaf of $\mathcal{O}$-modules. The analogue of Lemma 6.30.6 is as follows.

Lemma 6.30.12. *Let X be a topological space. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{B}$. Let $\mathcal{B}$ be a basis for the topology on X. Let $\mathcal{F}$ be a sheaf with values in $\mathcal{C}$ on $\mathcal{B}$. Let $\mathcal{O}^{ext}$ be the sheaf of rings on X extending $\mathcal{O}$ and let $\mathcal{F}^{ext}$ be the abelian sheaf on X extending $\mathcal{F}$, see Lemma 6.30.9. There exists a canonical map*

$$\mathcal{O}^{ext} \times \mathcal{F}^{ext} \longrightarrow \mathcal{F}^{ext}$$

which agrees with the given map over elements of $\mathcal{B}$ and which endows $\mathcal{F}^{ext}$ with the structure of an $\mathcal{O}^{ext}$-module.

Proof. It suffices to construct the multiplication map on the level of presheaves of sets. Perhaps the easiest way to see this is to prove directly that if $(f_x)_{x \in U}$, $f_x \in \mathcal{O}_x$ and $(m_x)_{x \in U}$, $m_x \in \mathcal{F}_x$ satisfy $(*)$, then the element $(f_x m_x)_{x \in U}$ also satisfies $(*)$. Then we get the desired result, because in the proof of Lemma 6.30.6 we construct the extension in terms of families of elements of stalks satisfying $(*)$. □

Note that we have

$$\mathcal{F}_x = \mathcal{F}^{ext}_x$$

as $\mathcal{O}_x$-modules in the situation of the lemma. This is so because the collection of elements of $\mathcal{B}$ containing x forms a fundamental system of open neighbourhoods of x, or simply because it is true on the underlying sets.

Lemma 6.30.13. *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Let $\mathcal{O}$ be a sheaf of rings on X. Denote $Mod(\mathcal{O}|_{\mathcal{B}})$ the category of sheaves of $\mathcal{O}|_{\mathcal{B}}$-modules on $\mathcal{B}$. There is an equivalence of categories*

$$Mod(\mathcal{O}) \longrightarrow Mod(\mathcal{O}|_{\mathcal{B}})$$

which assigns to a sheaf of $\mathcal{O}$-modules on X its restriction to the members of $\mathcal{B}$.

Proof. The inverse functor in given in Lemma 6.30.12 above. Checking the obvious functorialities is left to the reader. □

Finally, we address the question of the relationship of this with continuous maps. This is now very easy thanks to the work above. First we do the case where there is a basis on the target given.

Lemma 6.30.14. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $(\mathcal{C}, F)$ be a type of algebraic structures. Let $\mathcal{F}$ be a sheaf with values in $\mathcal{C}$ on X. Let $\mathcal{G}$ be a sheaf with values in $\mathcal{C}$ on Y. Let $\mathcal{B}$ be a basis for the topology on Y. Suppose given for every $V \in \mathcal{B}$ a morphism*

$$\varphi_V : \mathcal{G}(V) \longrightarrow \mathcal{F}(f^{-1}V)$$

of $\mathcal{C}$ compatible with restriction mappings. Then there is a unique f-map (see Definition 6.21.7 and discussion of f-maps in Section 6.23) $\varphi : \mathcal{G} \to \mathcal{F}$ recovering φ_V for $V \in \mathcal{B}$.

Proof. This is trivial because the collection of maps amounts to a morphism between the restrictions of $\mathcal{G}$ and $f_*\mathcal{F}$ to $\mathcal{B}$. By Lemma 6.30.10 this is the same as giving a morphism from $\mathcal{G}$ to $f_*\mathcal{F}$, which by Lemma 6.21.8 is the same as an f-map. See also Lemma 6.23.1 and the discussion preceding it for how to deal with the case of sheaves of algebraic structures. □

Here is the analogue for ringed spaces.

Lemma 6.30.15. *Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_Y$-modules. Let $\mathcal{B}$ be a basis for the topology on Y. Suppose given for every $V \in \mathcal{B}$ a $\mathcal{O}_Y(V)$-module map*

$$\varphi_V : \mathcal{G}(V) \longrightarrow \mathcal{F}(f^{-1}V)$$

(where $\mathcal{F}(f^{-1}V)$ has a module structure using $f^\sharp_V : \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}V)$) compatible with restriction mappings. Then there is a unique f-map (see discussion of f-maps in Section 6.26) $\varphi : \mathcal{G} \to \mathcal{F}$ recovering φ_V for $V \in \mathcal{B}$.

Proof. Same as the proof of the corresponding lemma for sheaves of algebraic structures above. □

Lemma 6.30.16. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $(\mathcal{C}, F)$ be a type of algebraic structures. Let $\mathcal{F}$ be a sheaf with values in $\mathcal{C}$ on X. Let $\mathcal{G}$ be a sheaf with values in $\mathcal{C}$ on Y. Let $\mathcal{B}_Y$ be a basis for the topology on Y. Let $\mathcal{B}_X$ be a basis for the topology on X. Suppose given for every $V \in \mathcal{B}_Y$, and $U \in \mathcal{B}_X$ such that $f(U) \subset V$ a morphism*

$$\varphi_V^U : \mathcal{G}(V) \longrightarrow \mathcal{F}(U)$$

of $\mathcal{C}$ compatible with restriction mappings. Then there is a unique f-map (see Definition 6.21.7 and the discussion of f-maps in Section 6.23) $\varphi : \mathcal{G} \to \mathcal{F}$ recovering φ_V^U as the composition

$$\mathcal{G}(V) \xrightarrow{\varphi_V} \mathcal{F}(f^{-1}(V)) \xrightarrow{restr.} \mathcal{F}(U)$$

for every pair (U, V) as above.

Proof. Let us first proves this for sheaves of sets. Fix $V \subset Y$ open. Pick $s \in \mathcal{G}(V)$. We are going to construct an element $\varphi_V(s) \in \mathcal{F}(f^{-1}V)$. We can define a value $\varphi(s)_x$ in the stalk $\mathcal{F}_x$ for every $x \in f^{-1}V$ by picking a $U \in \mathcal{B}_X$ with $x \in U \subset f^{-1}V$ and setting $\varphi(s)_x$ equal to the equivalence class of $(U, \varphi_V^U(s))$ in the stalk. Clearly, the family $(\varphi(s)_x)_{x \in f^{-1}V}$ satisfies condition $(*)$ because the maps φ_V^U for varying U are compatible with restrictions in the sheaf $\mathcal{F}$. Thus, by the proof of Lemma 6.30.6 we see that $(\varphi(s)_x)_{x \in f^{-1}V}$ corresponds to a unique element $\varphi_V(s)$ of $\mathcal{F}(f^{-1}V)$. Thus we have defined a set map $\varphi_V : \mathcal{G}(V) \to \mathcal{F}(f^{-1}V)$. The compatibility between φ_V and φ_V^U follows from Lemma 6.30.5.

We leave it to the reader to show that the construction of φ_V is compatible with restriction mappings as we vary $v \in \mathcal{B}_Y$. Thus we may apply Lemma 6.30.14 above to ``glue" them to the desired f-map.

Finally, we note that the map of sheaves of sets so constructed satisfies the property that the map on stalks

$$\mathcal{G}_{f(x)} \longrightarrow \mathcal{F}_x$$

is the colimit of the system of maps φ_V^U as $V \in \mathcal{B}_Y$ varies over those elements that contain $f(x)$ and $U \in \mathcal{B}_X$ varies over those elements that contain x. In particular, if $\mathcal{G}$ and $\mathcal{F}$ are the underlying sheaves of sets of sheaves of algebraic structures, then we see that the maps on stalks is a morphism of algebraic structures. Hence we conclude that the associated map of sheaves of underlying sets $f^{-1}\mathcal{G} \to \mathcal{F}$ satisfies the assumptions of Lemma 6.23.1. We conclude that $f^{-1}\mathcal{G} \to \mathcal{F}$ is a morphism of sheaves with values in $\mathcal{C}$. And by adjointness this means that φ is an f-map of sheaves of algebraic structures. □

Lemma 6.30.17. *Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_Y$-modules. Let $\mathcal{B}_Y$ be a basis for the topology on Y. Let $\mathcal{B}_X$ be a basis for the topology on X. Suppose given for every $V \in \mathcal{B}_Y$, and $U \in \mathcal{B}_X$ such that $f(U) \subset V$ a $\mathcal{O}_Y(V)$-module map*

$$\varphi_V^U : \mathcal{G}(V) \longrightarrow \mathcal{F}(U)$$

compatible with restriction mappings. Here the $\mathcal{O}_Y(V)$-module structure on $\mathcal{F}(U)$ comes from the $\mathcal{O}_X(U)$-module structure via the map $f_V^\sharp : \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}V) \to \mathcal{O}_X(U)$. Then there is a unique f-map of sheaves of modules (see Definition 6.21.7 and the discussion of f-maps in Section 6.26) $\varphi : \mathcal{G} \to \mathcal{F}$ recovering φ_V^U as the composition

$$\mathcal{G}(V) \xrightarrow{\varphi_V} \mathcal{F}(f^{-1}(V)) \xrightarrow{restrc.} \mathcal{F}(U)$$

for every pair (U, V) as above.

Proof. Similar to the above and omitted. □

6.31. Open immersions and (pre)sheaves

Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset U into X. In Section 6.21 we have defined functors j_* and j^{-1} such that j_* is right adjoint to j^{-1}. It turns out that for an open immersion there is a left adjoint for j^{-1}, which we will denote $j_!$. First we point out that j^{-1} has a particularly simple description in the case of an open immersion.

Lemma 6.31.1. *Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset U into X.*

1. *Let $\mathcal{G}$ be a presheaf of sets on X. The presheaf $j_p\mathcal{G}$ (see Section 6.21) is given by the rule $V \mapsto \mathcal{G}(V)$ for $V \subset U$ open.*
2. *Let $\mathcal{G}$ be a sheaf of sets on X. The sheaf $j^{-1}\mathcal{G}$ is given by the rule $V \mapsto \mathcal{G}(V)$ for $V \subset U$ open.*

(3) *For any point $u \in U$ and any sheaf $\mathcal{G}$ on X we have a canonical identification of stalks*

$$j^{-1}\mathcal{G}_u = (\mathcal{G}|_U)_u = \mathcal{G}_u.$$

(4) *On the category of presheaves of U we have $j_p j_* = id$.*

(5) *On the category of sheaves of U we have $j^{-1} j_* = id$.*

The same description holds for (pre)sheaves of abelian groups, (pre)sheaves of algebraic structures, and (pre)sheaves of modules.

Proof. The colimit in the definition of $j_p\mathcal{G}(V)$ is over collection of all $W \subset X$ open such that $V \subset W$ ordered by reverse inclusion. Hence this has a largest element, namely V. This proves (1). And (2) follows because the assignment $V \mapsto \mathcal{G}(V)$ for $V \subset U$ open is clearly a sheaf if $\mathcal{G}$ is a sheaf. Assertion (3) follows from (2) since the collection of open neighbourhoods of u which are contained in U is cofinal in the collection of all open neighbourhoods of u in X. Parts (4) and (5) follow by computing $j^{-1}j_*\mathcal{F}(V) = j_*\mathcal{F}(V) = \mathcal{F}(V)$.

The exact same arguments work for (pre)sheaves of abelian groups and (pre)sheaves of algebraic structures. □

Definition 6.31.2. Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset.

(1) Let $\mathcal{G}$ be a presheaf of sets, abelian groups or algebraic structures on X. The presheaf $j_p\mathcal{G}$ described in Lemma 6.31.1 is called the *restriction of $\mathcal{G}$ to U* and denoted $\mathcal{G}|_U$.

(2) Let $\mathcal{G}$ be a sheaf of sets on X, abelian groups or algebraic structures on X. The sheaf $j^{-1}\mathcal{G}$ is called the *restriction of $\mathcal{G}$ to U* and denoted $\mathcal{G}|_U$.

(3) If $(X, \mathcal{O})$ is a ringed space, then the pair $(U, \mathcal{O}|_U)$ is called the *open subspace of $(X, \mathcal{O})$ associated to U*.

(4) If $\mathcal{G}$ is a presheaf of $\mathcal{O}$-modules then $\mathcal{G}|_U$ together with the multiplication map $\mathcal{O}|_U \times \mathcal{G}|_U \to \mathcal{G}|_U$ (see Lemma 6.24.6) is called the *restriction of $\mathcal{G}$ to U*.

We leave a definition of the restriction of presheaves of modules to the reader. Ok, so in this section we will discuss a left adjoint to the restriction functor. Here is the definition in the case of (pre)sheaves of sets.

Definition 6.31.3. Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset.

(1) Let $\mathcal{F}$ be a presheaf of sets on U. We define the *extension of $\mathcal{F}$ by the empty set $j_{p!}\mathcal{F}$* to be the presheaf of sets on X defined by the rule

$$j_{p!}\mathcal{F}(V) = \begin{cases} \emptyset & \text{if} \quad V \not\subset U \\ \mathcal{F}(V) & \text{if} \quad V \subset U \end{cases}$$

with obvious restriction mappings.

(2) Let $\mathcal{F}$ be a sheaf of sets on U. We define the *extension of $\mathcal{F}$ by the empty set $j_!\mathcal{F}$* to be the sheafification of the presheaf $j_{p!}\mathcal{F}$.

Lemma 6.31.4. *Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset.*

(1) *The functor $j_{p!}$ is a left adjoint to the restriction functor j_p (see Lemma 6.31.1).*

(2) *The functor $j_!$ is a left adjoint to restriction, in a formula*

$$Mor_{Sh(X)}(j_!\mathcal{F}, \mathcal{G}) = Mor_{Sh(U)}(\mathcal{F}, j^{-1}\mathcal{G}) = Mor_{Sh(U)}(\mathcal{F}, \mathcal{G}|_U)$$

bifunctorially in $\mathcal{F}$ and $\mathcal{G}$.

(3) *Let $\mathcal{F}$ be a sheaf of sets on U. The stalks of the sheaf $j_!\mathcal{F}$ are described as follows*

$$j_!\mathcal{F}_x = \begin{cases} \emptyset & if \quad x \notin U \\ \mathcal{F}_x & if \quad x \in U \end{cases}$$

(4) *On the category of presheaves of U we have $j_p j_{p!} = id$.*

(5) *On the category of sheaves of U we have $j^{-1}j_! = id$.*

Proof. To map $j_{p!}\mathcal{F}$ into $\mathcal{G}$ it is enough to map $\mathcal{F}(V) \to \mathcal{G}(V)$ whenever $V \subset U$ compatibly with restriction mappings. And by Lemma 6.31.1 the same description holds for maps $\mathcal{F} \to \mathcal{G}|_U$. The adjointness of $j_!$ and restriction follows from this and the properties of sheafification. The identification of stalks is obvious from the definition of the extension by the empty set and the definition of a stalk. Statements (4) and (5) follow by computing the value of the sheaf on any open of U. □

Note that if $\mathcal{F}$ is a sheaf of abelian groups on U, then in general $j_!\mathcal{F}$ as defined above, is not a sheaf of abelian groups, for example because some of its stalks are empty (hence not abelian groups for sure). Thus we need to modify the definition of $j_!$ depending on the type of sheafs we consider. The reason for choosing the empty set in the definition of the extension by the empty set, is that it is the initial object in the category of sets. Thus in the case of abelian groups we use 0 (and more generally for sheaves with values in any abelian category).

Definition 6.31.5. Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset.

(1) Let $\mathcal{F}$ be an abelian presheaf on U. We define the *extension $j_{p!}\mathcal{F}$ of $\mathcal{F}$ by* 0 to be the abelian presheaf on X defined by the rule

$$j_{p!}\mathcal{F}(V) = \begin{cases} 0 & \text{if} \quad V \not\subset U \\ \mathcal{F}(V) & \text{if} \quad V \subset U \end{cases}$$

with obvious restriction mappings.

(2) Let $\mathcal{F}$ be an abelian sheaf on U. We define the *extension $j_!\mathcal{F}$ of $\mathcal{F}$ by* 0 to be the sheafification of the abelian presheaf $j_{p!}\mathcal{F}$.

(3) Let $\mathcal{C}$ be a category having an initial object e. Let $\mathcal{F}$ be a presheaf on U with values in $\mathcal{C}$. We define the *extension $j_{p!}\mathcal{F}$ of $\mathcal{F}$ by e* to be the presheaf on X with values in $\mathcal{C}$ defined by the rule

$$j_{p!}\mathcal{F}(V) = \begin{cases} e & \text{if} \quad V \not\subset U \\ \mathcal{F}(V) & \text{if} \quad V \subset U \end{cases}$$

with obvious restriction mappings.

(4) Let $(\mathcal{C}, F)$ be a type of algebraic structure such that $\mathcal{C}$ has an initial object e. Let $\mathcal{F}$ be a sheaf of algebraic structures on U (of the give type). We define the *extension $j_!\mathcal{F}$ of $\mathcal{F}$ by e* to be the sheafification of the presheaf $j_{p!}\mathcal{F}$ defined above.

(5) Let $\mathcal{O}$ be a presheaf of rings on X. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}|_U$-modules. In this case we define the *extension by* 0 to be the presheaf of $\mathcal{O}$-modules which is equal to $j_{p!}\mathcal{F}$ as an abelian presheaf endowed with the multiplication map $\mathcal{O} \times j_{p!}\mathcal{F} \to j_{p!}\mathcal{F}$.

(6) Let $\mathcal{O}$ be a sheaf of rings on X. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}|_U$-modules. In this case we define the *extension by* 0 to be the $\mathcal{O}$-module which is equal to $j_!\mathcal{F}$ as an abelian sheaf endowed with the multiplication map $\mathcal{O} \times j_!\mathcal{F} \to j_!\mathcal{F}$.

It is true that one can define $j_!$ in the setting of sheaves of algebraic structures (see below). However, it depends on the type of algebraic structures involved what the resulting object is. For example, if $\mathcal{O}$ is a sheaf of rings on U, then $j_{!,rings}\mathcal{O} \neq j_{!,abelian}\mathcal{O}$ since the initial object in the category of rings is $\mathbf{Z}$ and the initial object in the category of abelian groups is 0. In particular the functor $j_!$ *does not commute with taking underlying sheaves of sets*, in contrast to what we have seen sofar! We separate out the case of (pre)sheaves of abelian groups, (pre)sheaves of algebraic structures and (pre)sheaves of modules as usual.

Lemma 6.31.6. *Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset. Consider the functors of restriction and extension by* 0 *for abelian (pre)sheaves.*

(1) *The functor $j_{p!}$ is a left adjoint to the restriction functor j_p (see Lemma 6.31.1).*
(2) *The functor $j_!$ is a left adjoint to restriction, in a formula*

$$Mor_{Ab(X)}(j_!\mathcal{F}, \mathcal{G}) = Mor_{Ab(U)}(\mathcal{F}, j^{-1}\mathcal{G}) = Mor_{Ab(U)}(\mathcal{F}, \mathcal{G}|_U)$$

bifunctorially in $\mathcal{F}$ and $\mathcal{G}$.
(3) *Let $\mathcal{F}$ be an abelian sheaf on U. The stalks of the sheaf $j_!\mathcal{F}$ are described as follows*

$$j_!\mathcal{F}_x = \begin{cases} 0 & \text{if} \quad x \notin U \\ \mathcal{F}_x & \text{if} \quad x \in U \end{cases}$$

(4) *On the category of abelian presheaves of U we have $j_p j_{p!} = id$.*
(5) *On the category of abelian sheaves of U we have $j^{-1} j_! = id$.*

Proof. Omitted. □

Lemma 6.31.7. *Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset. Let $(\mathcal{C}, F)$ be a type of algebraic structure such that $\mathcal{C}$ has an initial object e. Consider the functors of restriction and extension by e for (pre)sheaves of algebraic structure defined above.*

(1) *The functor $j_{p!}$ is a left adjoint to the restriction functor j_p (see Lemma 6.31.1).*
(2) *The functor $j_!$ is a left adjoint to restriction, in a formula*

$$Mor_{Sh(X,\mathcal{C})}(j_!\mathcal{F}, \mathcal{G}) = Mor_{Sh(U,\mathcal{C})}(\mathcal{F}, j^{-1}\mathcal{G}) = Mor_{Sh(U,\mathcal{C})}(\mathcal{F}, \mathcal{G}|_U)$$

bifunctorially in $\mathcal{F}$ and $\mathcal{G}$.
(3) *Let $\mathcal{F}$ be a sheaf on U. The stalks of the sheaf $j_!\mathcal{F}$ are described as follows*

$$j_!\mathcal{F}_x = \begin{cases} e & \text{if} \quad x \notin U \\ \mathcal{F}_x & \text{if} \quad x \in U \end{cases}$$

(4) *On the category of presheaves of algebraic structures on U we have $j_p j_{p!} = id$.*
(5) *On the category of sheaves of algebraic structures on U we have $j^{-1} j_! = id$.*

Proof. Omitted. □

Lemma 6.31.8. *Let $(X, \mathcal{O})$ be a ringed space. Let $j : (U, \mathcal{O}|_U) \to (X, \mathcal{O})$ be an open subspace. Consider the functors of restriction and extension by* 0 *for (pre)sheaves of modules defined above.*

(1) *The functor $j_{p!}$ is a left adjoint to restriction, in a formula*

$$Mor_{PMod(\mathcal{O})}(j_{p!}\mathcal{F}, \mathcal{G}) = Mor_{PMod(\mathcal{O}|_U)}(\mathcal{F}, \mathcal{G}|_U)$$

bifunctorially in $\mathcal{F}$ and $\mathcal{G}$.

(2) *The functor $j_!$ is a left adjoint to restriction, in a formula*

$$Mor_{Mod(\mathcal{O})}(j_!\mathcal{F}, \mathcal{G}) = Mor_{Mod(\mathcal{O}|_U)}(\mathcal{F}, \mathcal{G}|_U)$$

bifunctorially in $\mathcal{F}$ and $\mathcal{G}$.

(3) *Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules on U. The stalks of the sheaf $j_!\mathcal{F}$ are described as follows*

$$j_!\mathcal{F}_x = \begin{cases} 0 & \text{if} \quad x \notin U \\ \mathcal{F}_x & \text{if} \quad x \in U \end{cases}$$

(4) *On the category of sheaves of $\mathcal{O}|_U$-modules on U we have $j^{-1}j_! = id$.*

Proof. Omitted. $\square$

Note that by the lemmas above, both the functors j_* and $j_!$ are fully faithful embeddings of the category of sheaves on U into the category of sheaves on X. It is only true for the functor $j_!$ that one can easily describe the essential image of this functor.

Lemma 6.31.9. *Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset. The functor*

$$j_! : Sh(U) \longrightarrow Sh(X)$$

is fully faithful. Its essential image consists exactly of those sheaves $\mathcal{G}$ such that $\mathcal{G}_x = \emptyset$ for all $x \in X \setminus U$.

Proof. Fully faithfullness follows formally from $j^{-1}j_! = \text{id}$. We have seen that any sheaf in the image of the functor has the property on the stalks mentioned in the lemma. Conversely, suppose that $\mathcal{G}$ has the indicated property. Then it is easy to check that

$$j_!j^{-1}\mathcal{G} \to \mathcal{G}$$

is an isomorphism on all stalks and hence an isomorphism. $\square$

Lemma 6.31.10. *Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset. The functor*

$$j_! : Ab(U) \longrightarrow Ab(X)$$

is fully faithful. Its essential image consists exactly of those sheaves $\mathcal{G}$ such that $\mathcal{G}_x = 0$ for all $x \in X \setminus U$.

Proof. Omitted. $\square$

Lemma 6.31.11. *Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset. Let $(\mathcal{C}, F)$ be a type of algebraic structure such that $\mathcal{C}$ has an initial object e. The functor*

$$j_! : Sh(U, \mathcal{C}) \longrightarrow Sh(X, \mathcal{C})$$

is fully faithful. Its essential image consists exactly of those sheaves $\mathcal{G}$ such that $\mathcal{G}_x = e$ for all $x \in X \setminus U$.

Proof. Omitted. $\square$

Lemma 6.31.12. *Let $(X, \mathcal{O})$ be a ringed space. Let $j : (U, \mathcal{O}|_U) \to (X, \mathcal{O})$ be an open subspace. The functor*

$$j_! : Mod(\mathcal{O}|_U) \longrightarrow Mod(\mathcal{O})$$

is fully faithful. Its essential image consists exactly of those sheaves $\mathcal{G}$ such that $\mathcal{G}_x = 0$ for all $x \in X \setminus U$.

Proof. Omitted. □

Remark 6.31.13. Let $j : U \to X$ be an open immersion of topological spaces as above. Let $x \in X$, $x \notin U$. Let $\mathcal{F}$ be a sheaf of sets on U. Then $\mathcal{F}_x = \emptyset$ by Lemma 6.31.4. Hence $j_!$ does not transform a final object of $Sh(U)$ into a final object of $Sh(X)$ unless $U = X$. According to our conventions in Categories, Section 4.21 this means that the functor $j_!$ is not left exact as a functor between the categories of sheaves of sets. It will be shown later that $j_!$ on abelian sheaves is exact, see Modules, Lemma 15.3.5.

6.32. Closed immersions and (pre)sheaves

Let X be a topological space. Let $i : Z \to X$ be the inclusion of a closed subset Z into X. In Section 6.21 we have defined functors i_* and i^{-1} such that i_* is right adjoint to i^{-1}.

Lemma 6.32.1. *Let X be a topological space. Let $i : Z \to X$ be the inclusion of a closed subset Z into X. Let $\mathcal{F}$ be a sheaf of sets on Z. The stalks of $i_*\mathcal{F}$ are described as follows*

$$i_*\mathcal{F}_x = \begin{cases} \{*\} & \text{if} \quad x \notin Z \\ \mathcal{F}_x & \text{if} \quad x \in Z \end{cases}$$

where $\{*\}$ denotes a singleton set. Moreover, $i^{-1}i_* = id$ on the category of sheaves of sets on Z. Moreover, the same holds for abelian sheaves on Z, resp. sheaves of algebraic structures on Z where $\{*\}$ has to be replaced by 0, resp. a final object of the category of algebraic structures.

Proof. If $x \notin Z$, then there exist arbitrarily small open neighbourhoods U of x which do not meet Z. Because $\mathcal{F}$ is a sheaf we have $\mathcal{F}(i^{-1}(U)) = \{*\}$ for any such U, see Remark 6.7.2. This proves the first case. The second case comes from the fact that for $z \in Z$ any open neighbourhood of z is of the form $Z \cap U$ for some open U of X. For the statement that $i^{-1}i_* = \text{id}$ consider the canonical map $i^{-1}i_*\mathcal{F} \to \mathcal{F}$. This is an isomorphism on stalks (see above) and hence an isomorphism.

For sheaves of abelian groups, and sheaves of algebraic structures you argue in the same manner. □

Lemma 6.32.2. *Let X be a topological space. Let $i : Z \to X$ be the inclusion of a closed subset. The functor*

$$i_* : Sh(Z) \longrightarrow Sh(X)$$

is fully faithful. Its essential image consists exactly of those sheaves $\mathcal{G}$ such that $\mathcal{G}_x = \{*\}$ for all $x \in X \setminus Z$.

Proof. Fully faithfullness follows formally from $i^{-1}i_* = \text{id}$. We have seen that any sheaf in the image of the functor has the property on the stalks mentioned in the lemma. Conversely, suppose that $\mathcal{G}$ has the indicated property. Then it is easy to check that

$$\mathcal{G} \to i_*i^{-1}\mathcal{G}$$

is an isomorphism on all stalks and hence an isomorphism. □

Lemma 6.32.3. *Let X be a topological space. Let $i : Z \to X$ be the inclusion of a closed subset. The functor*
$$i_* : Ab(Z) \longrightarrow Ab(X)$$
is fully faithful. Its essential image consists exactly of those sheaves $\mathcal{G}$ such that $\mathcal{G}_x = 0$ for all $x \in X \setminus Z$.

Proof. Omitted. $\square$

Lemma 6.32.4. *Let X be a topological space. Let $i : Z \to X$ be the inclusion of a closed subset. Let $(\mathcal{C}, F)$ be a type of algebraic structure with final object 0. The functor*
$$i_* : Sh(Z, \mathcal{C}) \longrightarrow Sh(X, \mathcal{C})$$
is fully faithful. Its essential image consists exactly of those sheaves $\mathcal{G}$ such that $\mathcal{G}_x = 0$ for all $x \in X \setminus Z$.

Proof. Omitted. $\square$

Remark 6.32.5. Let $i : Z \to X$ be a closed immersion of topological spaces as above. Let $x \in X$, $x \notin Z$. Let $\mathcal{F}$ be a sheaf of sets on Z. Then $(i_*\mathcal{F})_x = \{*\}$ by Lemma 6.32.1. Hence if $\mathcal{F} = * \amalg *$, where $*$ is the singleton sheaf, then $i_*\mathcal{F}_x = \{*\} \neq i_*(*)_x \amalg i_*(*)_x$ because the latter is a two point set. According to our conventions in Categories, Section 4.21 this means that the functor i_* is not right exact as a functor between the categories of sheaves of sets. In particular, it cannot have a right adjoint, see Categories, Lemma 4.22.3.

On the other hand, we will see later (see Modules, Lemma 15.6.3) that i_* on abelian sheaves is exact, and does have a right adjoint, namely the functor that associates to an abelian sheaf on X the sheaf of sections supported in Z.

Remark 6.32.6. We have not discussed the relationship between closed immersions and ringed spaces. This is because the notion of a closed immersion of ringed spaces is best discussed in the setting of quasi-coherent sheaves, see Modules, Section 15.13.

6.33. Glueing sheaves

In this section we glue sheaves defined on the members of a covering of X. We first deal with maps.

Lemma 6.33.1. *Let X be a topological space. Let $X = \bigcup U_i$ be an open covering. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of sets on X. Given a collection*
$$\varphi_i : \mathcal{F}|_{U_i} \longrightarrow \mathcal{G}|_{U_i}$$
of maps of sheaves such that for all $i, j \in I$ the maps φ_i, φ_j restrict to the same map $\mathcal{F}|_{U_i \cap U_j} \to \mathcal{G}|_{U_i \cap U_j}$ then there exists a unique map of sheaves
$$\varphi : \mathcal{F} \longrightarrow \mathcal{G}$$
whose restriction to each U_i agrees with φ_i.

Proof. Omitted. $\square$

The previous lemma implies that given two sheaves $\mathcal{F}$, $\mathcal{G}$ on the topological space X the rule
$$U \longmapsto Mor_{Sh(U)}(\mathcal{F}|_U, \mathcal{G}|_U)$$
defines a sheaf. This is a kind of *internal hom sheaf*. It is seldom used in the setting of sheaves of sets, and more usually in the setting of sheaves of modules, see Modules, Section 15.19.

Let X be a topological space. Let $X = \bigcup_{i \in I} U_i$ be an open covering. For each $i \in I$ let $\mathcal{F}_i$ be a sheaf of sets on U_i. For each pair $i, j \in I$, let

$$\varphi_{ij} : \mathcal{F}_i|_{U_i \cap U_j} \longrightarrow \mathcal{F}_j|_{U_i \cap U_j}$$

be an isomorphism of sheaves of sets. Assume in addition that for every triple of indices $i, j, k \in I$ the following diagram is commutative

$$\begin{tikzcd} \mathcal{F}_i|_{U_i \cap U_j \cap U_k} \arrow[rr, "\varphi_{ik}"] \arrow[dr, "\varphi_{ij}"'] & & \mathcal{F}_k|_{U_i \cap U_j \cap U_k} \\ & \mathcal{F}_j|_{U_i \cap U_j \cap U_k} \arrow[ur, "\varphi_{jk}"'] & \end{tikzcd}$$

We will call such a collection of data $(\mathcal{F}_i, \varphi_{ij})$ a *glueing data for sheaves of sets with respect to the covering* $X = \bigcup U_i$.

Lemma 6.33.2. *Let X be a topological space. Let $X = \bigcup_{i \in I} U_i$ be an open covering. Given any glueing data $(\mathcal{F}_i, \varphi_{ij})$ for sheaves of sets with respect to the covering $X = \bigcup U_i$ there exists a sheaf of sets $\mathcal{F}$ on X together with isomorphisms*

$$\varphi_i : \mathcal{F}|_{U_i} \to \mathcal{F}_i$$

such that the diagrams

$$\begin{tikzcd} \mathcal{F}|_{U_i \cap U_j} \arrow[r, "\varphi_i"'] \arrow[d, "id"'] & \mathcal{F}_i|_{U_i \cap U_j} \arrow[d, "\varphi_{ij}"] \\ \mathcal{F}|_{U_i \cap U_j} \arrow[r, "\varphi_j"] & \mathcal{F}_j|_{U_i \cap U_j} \end{tikzcd}$$

are commutative.

Proof. Actually we can write a formula for the set of sections of $\mathcal{F}$ over an open $W \subset X$. Namely, we define

$$\mathcal{F}(W) = \{(s_i)_{i \in I} \mid s_i \in \mathcal{F}_i(W \cap U_i), \varphi_{ij}(s_i|_{W \cap U_i \cap U_j}) = s_j|_{W \cap U_i \cap U_j}\}.$$

Restriction mappings for $W' \subset W$ are defined by the restricting each of the s_i to $W' \cap U_i$. The sheaf condition for $\mathcal{F}$ follows immediately from the sheaf condition for each of the $\mathcal{F}_i$.

We still have to prove that $\mathcal{F}|_{U_i}$ maps isomorphically to $\mathcal{F}_i$. Let $W \subset U_i$. In this case the condition in the definition of $\mathcal{F}(W)$ implies that $s_j = \varphi_{ij}(s_i|_{W \cap U_j})$. And the commutativity of the diagrams in the definition of a glueing data assures that we may start with *any* section $s \in \mathcal{F}_i(W)$ and obtain a compatible collection by setting $s_i = s$ and $s_j = \varphi_{ij}(s_i|_{W \cap U_j})$. Thus the lemma follows. $\square$

Lemma 6.33.3. *Let X be a topological space. Let $X = \bigcup U_i$ be an open covering. Let $(\mathcal{F}_i, \varphi_{ij})$ be a glueing data of sheaves of abelian groups, resp. sheaves of algebraic structures, resp. sheaves of $\mathcal{O}$-modules for some sheaf of rings $\mathcal{O}$ on X. Then the construction in the proof of Lemma 6.33.2 above leads to a sheaf of abelian groups, resp. sheaf of algebraic structures, resp. sheaf of $\mathcal{O}$-modules.*

Proof. This is true because in the construction the set of sections $\mathcal{F}(W)$ over an open W is given as the equalizer of the maps

$$\prod_{i \in I} \mathcal{F}_i(W \cap U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}_i(W \cap U_i \cap U_j)$$

And in each of the cases envisioned this equalizer gives an object in the relevant category whose underlying set is the object considered in the cited lemma. □

Lemma 6.33.4. *Let X be a topological space. Let $X = \bigcup_{i \in I} U_i$ be an open covering. The functor which associates to a sheaf of sets $\mathcal{F}$ the following collection of glueing data*

$$(\mathcal{F}|_{U_i}, (\mathcal{F}|_{U_i})|_{U_i \cap U_j} \to (\mathcal{F}|_{U_j})|_{U_i \cap U_j})$$

with respect to the covering $X = \bigcup U_i$ defines an equivalence of categories between $Sh(X)$ and the category of glueing data. A similar statement holds for abelian sheaves, resp. sheaves of algebraic structures, resp. sheaves of $\mathcal{O}$-modules.

Proof. The functor is fully faithful by Lemma 6.33.1 and essentially surjective (via an explicitly given quasi-inverse functor) by Lemma 6.33.2. □

This lemma means that if the sheaf $\mathcal{F}$ was constructed from the glueing data $(\mathcal{F}_i, \varphi_{ij})$ and if $\mathcal{G}$ is a sheaf on X, then a morphism $f : \mathcal{F} \to \mathcal{G}$ is given by a collection of morphisms of sheaves

$$f_i : \mathcal{F}_i \longrightarrow \mathcal{G}|_{U_i}$$

compatible with the glueing maps φ_{ij}. Similarly, to give a morphism of sheaves $g : \mathcal{G} \to \mathcal{F}$ is the same as giving a collection of morphisms of sheaves

$$g_i : \mathcal{G}|_{U_i} \longrightarrow \mathcal{F}_i$$

compatible with the glueing maps φ_{ij}.

6.34. Other chapters

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CHAPTER 7

Commutative Algebra

7.1. Introduction

Basic commutative algebra will be explained in this document. A reference is [**Mat70**].

7.2. Conventions

A ring is commutative with 1. The zero ring is a ring. In fact it is the only ring that does not have a prime ideal. The Kronecker symbol δ_{ij} will be used. If $R \to S$ is a ring map and $\mathfrak q$ a prime of S, then we use the notation "$\mathfrak p = R \cap \mathfrak q$" to indicate the prime which is the inverse image of $\mathfrak q$ under $R \to S$ even if R is not a subring of S and even if $R \to S$ is not injective.

7.3. Basic notions

The following notions are considered basic and will not be defined, and or proved. This does not mean they are all necessarily easy or well known.

(1) R is a *ring*,
(2) $x \in R$ is *nilpotent*,
(3) $x \in R$ is a *zero-divisor*,
(4) $x \in R$ is a *unit*,
(5) $e \in R$ is an *idempotent*,
(6) an idempotent $e \in R$ is called *trivial* if $e = 1$ or $e = 0$,
(7) $\varphi : R_1 \to R_2$ is a *ring homomorphism*,
(8) $\varphi : R_1 \to R_2$ is *of finite presentation*, or R_2 *is a finitely presented* R_1*-algebra*, see Definition 7.6.1,
(9) $\varphi : R_1 \to R_2$ is *of finite type*, or R_2 *is a finitely type* R_1*-algebra*, see Definition 7.6.1,
(10) $\varphi : R_1 \to R_2$ is *finite*, or R_2 *is a finite* R_1*-algebra*,
(11) R is a *(integral) domain*,
(12) R is *reduced*,
(13) R is *Noetherian*,
(14) R is a *principal ideal domain* or a *PID*,
(15) R is a *Euclidean domain*,
(16) R is a *unique factorization domain* or a *UFD*,
(17) R is a *discrete valuation ring* or a *dvr*,
(18) K is a *field*,
(19) $K \subset L$ is a *field extension*,
(20) $K \subset L$ is an *algebraic field extension*,
(21) $\{t_i\}_{i \in I}$ is a *transcendence basis* for L over K,
(22) the *transcendence degree* $\text{trdeg}(L/K)$ of L over K,
(23) the field k is *algebraically closed*,

(24) if $K \subset L$ is algebraic, and $K \to k$ a field map, then there exists a map $L \to k$ extending the map on K,
(25) $I \subset R$ is an *ideal*,
(26) $I \subset R$ is *radical*,
(27) if I is an ideal then we have its *radical* $\sqrt{I}$,
(28) $I \subset R$ is *nilpotent* means that $I^n = 0$ for some $n \in \mathbf{N}$,
(29) $I \subset R$ is *locally nilpotent* means that every element of I is nilpotent,
(30) $\mathfrak{p} \subset R$ is a *prime ideal*,
(31) if $\mathfrak{p} \subset R$ is prime and if $I, J \subset R$ are ideal, and if $IJ \subset \mathfrak{p}$, then $I \subset \mathfrak{p}$ or $J \subset \mathfrak{p}$.
(32) $\mathfrak{m} \subset R$ is a *maximal ideal*,
(33) any nonzero ring has a maximal ideal,
(34) the *Jacobson radical* of R is $\text{rad}(R) = \bigcap_{\mathfrak{m} \subset R} \mathfrak{m}$ the intersection of all the maximal ideals of R,
(35) the ideal (T) *generated* by a subset $T \subset R$,
(36) the *quotient ring* R/I,
(37) an ideal I in the ring R is prime if and only if R/I is a domain,
(38) an ideal I in the ring R is maximal if and only if the ring R/I is a field,
(39) if $\varphi : R_1 \to R_2$ is a ring homomorphism, and if $I \subset R_2$ is an ideal, then $\varphi^{-1}(I)$ is an ideal of R_1,
(40) if $\varphi : R_1 \to R_2$ is a ring homomorphism, and if $I \subset R_1$ is an ideal, then $\varphi(I) \cdot R_2$ (sometimes denoted $I \cdot R_2$, or IR_2) is the ideal of R_2 generated by $\varphi(I)$,
(41) if $\varphi : R_1 \to R_2$ is a ring homomorphism, and if $\mathfrak{p} \subset R_2$ is a prime ideal, then $\varphi^{-1}(\mathfrak{p})$ is a prime ideal of R_1,
(42) M is an *R-module*,
(43) for $m \in M$ the *annihilator* $I = \{f \in R \mid fm = 0\}$ of m in R,
(44) $N \subset M$ is an *R-submodule*,
(45) M is an *Noetherian R-module*,
(46) M is a *finite R-module*,
(47) M is a *finitely generated R-module*,
(48) M is a *finitely presented R-module*,
(49) M is a *free R-module*,
(50) if $0 \to K \to L \to M \to 0$ is a short exact sequence of R-modules and K, M are free, then L is free,
(51) if $N \subset M \subset L$ are R-modules, then $L/M = (L/N)/(M/N)$,
(52) S is a *multiplicative subset of* R,
(53) the *localization* $R \to S^{-1}R$ of R,
(54) if R is a ring and S is a multiplicative subset of R then $S^{-1}R$ is the zero ring if and only if S contains 0,
(55) if R is a ring and if the multiplicative subset S consists completely of nonzero divisors, then $R \to S^{-1}R$ is injective,
(56) if $\varphi : R_1 \to R_2$ is a ring homomorphism, and S is a multiplicative subsets of R_1, then $\varphi(S)$ is a multiplicative subset of R_2,
(57) if S, S' are multiplicative subsets of R, and if SS' denotes the set of products $SS' = \{r \in R \mid \exists s \in S, \exists s' \in S', r = ss'\}$ then SS' is a multiplicative subset of R,
(58) if S, S' are multiplicative subsets of R, and if $\overline{S}$ denotes the image of S in $(S')^{-1}R$, then $(SS')^{-1}R = \overline{S}^{-1}((S')^{-1}R)$,
(59) the *localization* $S^{-1}M$ of the R-module M,

(60) the functor $M \mapsto S^{-1}M$ preserves injective maps, surjective maps, and exactness,
(61) if S, S' are multiplicative subsets of R, and if M is an R-module, then $(SS')^{-1}M = S^{-1}((S')^{-1}M)$,
(62) if R is a ring, I and ideal of R and S a multiplicative subset of R, then $S^{-1}I$ is an ideal of $S^{-1}R$, and we have $S^{-1}R/S^{-1}I = \overline{S}^{-1}(R/I)$, where $\overline{S}$ is the image of S in R/I,
(63) if R is a ring, and S a multiplicative subset of R, then any ideal I' of $S^{-1}R$ is of the form $S^{-1}I$, where one can take I to be the inverse image of I' in R,
(64) if R is a ring, M an R-module, and S a multiplicative subset of R, then any submodule N' of $S^{-1}M$ is of the form $S^{-1}N$ for some submodule $N \subset M$, where one can take N to be the inverse image of N' in M,
(65) if $S = \{1, f, f^2, \ldots\}$ then $R_f = S^{-1}R$, and $M_f = S^{-1}M$,
(66) if $S = R \setminus \mathfrak{p} = \{x \in R \mid x \notin \mathfrak{p}\}$ for some prime ideal $\mathfrak{p}$, then it is customary to denote $R_\mathfrak{p} = S^{-1}R$ and $M_\mathfrak{p} = S^{-1}M$,
(67) a *local ring* is a ring with exactly one maximal ideal,
(68) a *semi-local ring* is a ring with finitely many maximal ideals,
(69) if $\mathfrak{p}$ is a prime in R, then $R_\mathfrak{p}$ is a local ring with maximal ideal $\mathfrak{p}R_\mathfrak{p}$,
(70) the *residue field*, denoted $\kappa(\mathfrak{p})$, of the prime $\mathfrak{p}$ in the ring R is the quotient $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} = (R \setminus \mathfrak{p})^{-1}R/\mathfrak{p}$,
(71) given R and M_1, M_2 the *tensor product* $M_1 \otimes_R M_2$,
(72) etc.

7.4. Snake lemma

The snake lemma and its variants are discussed in the setting of abelian categories in Homology, Section 10.3.

Lemma 7.4.1. *Suppose given a commutative diagram*

$$\begin{tikzcd} & X \arrow[r] \arrow[d, "\alpha"] & Y \arrow[r] \arrow[d, "\beta"] & Z \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & U \arrow[r] & V \arrow[r] & W & \end{tikzcd}$$

of abelian groups with exact rows, then there is a canonical exact sequence

$$Ker(\alpha) \to Ker(\beta) \to Ker(\gamma) \to Coker(\alpha) \to Coker(\beta) \to Coker(\gamma)$$

Moreover, if $X \to Y$ is injective, then the first map is injective, and if $V \to W$ is surjective, then the last map is surjective.

Proof. The map $\partial : \mathrm{Ker}(\gamma) \to \mathrm{Coker}(\alpha)$ is defined as follows. Take $z \in \mathrm{Ker}(\gamma)$. Choose $y \in Y$ mapping to z. Then $\beta(y) \in V$ maps to zero in W. Hence $\beta(y)$ is the image of some $u \in U$. Set $\partial z = \overline{u}$ the class of u in the cokernel of α. Proof of exactness is omitted. □

7.5. Finite modules and finitely presented modules

Just some basic notation and lemmas.

Definition 7.5.1. Let R be a ring. Let M be an R-module

(1) We say M is a *finite R-module*, or a *finitely generated R-module* if there exist $n \in \mathbf{N}$ and $x_1, \ldots, x_n \in M$ such that every element of M is a R-linear combination of the x_i. Equivalently, this means there exists a surjection $R^{\oplus n} \to M$ for some $n \in \mathbf{N}$.
(2) We say M is a *finitely presented R-module* or an *R-module of finite presentation* if there exist integers $n, m \in \mathbf{N}$ and an exact sequence

$$R^{\oplus m} \longrightarrow R^{\oplus n} \longrightarrow M \longrightarrow 0$$

Informally this means that M is finitely generated and that the module of relations among these generators is finitely generated as well. A choice of an exact sequence as in the definition is called a *presentation* of M.

Lemma 7.5.2. *Let R be a ring. Let $\alpha : R^{\oplus n} \to M$ and $\beta : N \to M$ be module maps. If $Im(\alpha) \subset Im(\beta)$, then there exists an R-module map $\gamma : R^{\oplus n} \to N$ such that $\alpha = \beta \circ \gamma$.*

Proof. Let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ be the ith basis vector of $R^{\oplus n}$. Let $x_i \in N$ be an element with $\alpha(e_i) = \beta(x_i)$ which exists by assumption. Set $\gamma(a_1, \ldots, a_n) = \sum a_i x_i$. By construction $\alpha = \beta \circ \gamma$. □

Lemma 7.5.3. *Let M be an R-module of finite presentation. For any surjection $\alpha : R^{\oplus n} \to M$ the kernel of α is a finitely generated R-module.*

Proof. Choose a presentation

$$R^{\oplus l} \to R^{\oplus m} \to M \to 0$$

Let $K = \text{Ker}(\alpha)$. By Lemma 7.5.2 there exists a map $R^{\oplus m} \to R^{\oplus n}$ such that the solid diagram

$$\begin{tikzcd} & R^{\oplus l} \arrow[r] \arrow[d, dotted] & R^{\oplus m} \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "id"] & 0 \\ 0 \arrow[r] & K \arrow[r] & R^{\oplus n} \arrow[r, "\alpha"] & M \arrow[r] & 0 \end{tikzcd}$$

commutes. This produces the dotted arrow. By the snake lemma (Lemma 7.4.1) we see that we get an isomorphism

$$\text{Coker}(R^{\oplus l} \to K) \cong \text{Coker}(R^{\oplus m} \to R^{\oplus n})$$

In particular we conclude that $\text{Coker}(R^{\oplus l} \to K)$ is a finite R-module. Hence there are finitely many elements of K which together with the images of the basis vectors of $R^{\oplus l}$ generate K, i.e., K is finitely generated. □

Lemma 7.5.4. *Let R be a ring. Let*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be a short exact sequence of R-modules.

(1) *If M_1 and M_3 are finite R-modules, then M_2 is a finite R-module.*
(2) *If M_1 and M_3 are finitely presented R-modules, then M_2 is a finitely presented R-module.*
(3) *If M_2 is a finite R-module, then M_3 is a finite R-module.*
(4) *If M_2 is a finitely presented R-module and M_1 is a finite R-module, then M_3 is a finitely presented R-module.*
(5) *If M_3 is a finitely presented R-module and M_2 is a finite R-module, then M_1 is a finite R-module.*

Proof. We prove part (5). Assume M_3 is finitely presented and M_2 finite. Let $\alpha : R^{\oplus n} \to M_2$ be a surjection. Then we can find $k_1, \dots, k_m \in R^{\oplus n}$ which generate the kernel of the composition $R^{\oplus n} \to M_2 \to M_3$. Then $\alpha(k_1), \dots, \alpha(k_m)$ generate M_1 as a submodule of M_2. The proofs of the other parts are omitted. □

Lemma 7.5.5. *Let R be a ring, and let M be a finite R-module. There exists a filtration by R-submodules*

$$0 = M_0 \subset M_1 \subset \dots \subset M_n = M$$

such that each quotient M_i/M_{i-1} is isomorphic to R/I_i for some ideal I_i of R.

Proof. By induction on the number of generators of M. Let $x_1, \dots, x_r \in M$ be a minimal number of generators. Let $M' = Rx_1 \subset M$. Then M/M' has $r - 1$ generators and the induction hypothesis applies. And clearly $M' \cong R/I_1$ with $I_1 = \{f \in R \mid fx_1 = 0\}$. □

Lemma 7.5.6. *Let $R \to S$ be a ring map. Let M be an S-module. If M is finite as an R-module, then M is finite as an S-module.*

Proof. In fact, any R-generating set of M is also an S-generating set of M, since the R-module structure is induced by the image of R in S. □

7.6. Ring maps of finite type and of finite presentation

Definition 7.6.1. Let $R \to S$ be a ring map.

(1) We say $R \to S$ is of *finite type*, or that *S is a finite type R-algebra* if there exists an $n \in \mathbf{N}$ and an surjection of R-algebras $R[x_1, \dots, x_n] \to S$.
(2) We say $R \to S$ is of *finite presentation* if there exist integers $n, m \in \mathbf{N}$ and polynomials $f_1, \dots, f_m \in R[x_1, \dots, x_n]$ and an isomorphism of R-algebras $R[x_1, \dots, x_n]/(f_1, \dots, f_m) \cong S$.

Informally this means that S is finitely generated as an R-algebra and that the ideal of relations among the generators is finitely generated. A choice of a surjection $R[x_1, \dots, x_n] \to S$ as in the definition is sometimes called a *presentation* of S.

Lemma 7.6.2. *The notions finite type and finite presentation have the following permanence properties.*

(1) *A composition of ring maps of finite type is of finite type.*
(2) *A composition of ring maps of finite presentation is of finite presentation.*
(3) *Given $R \to S' \to S$ with $R \to S$ of finite type, then $S' \to S$ is of finite type.*
(4) *Given $R \to S' \to S$, with $R \to S$ of finite presentation, and $R \to S'$ of finite type, then $S' \to S$ is of finite presentation.*

Proof. We only prove the last assertion. Write $S = R[x_1, \dots, x_n]/(f_1, \dots, f_m)$ and $S' = R[y_1, \dots, y_a]/I$. Say that the class $\bar{y}_i$ of y_i maps to h_i mod $(f_1, \dots, f_m)$ in S. Then it is clear that $S' = S[x_1, \dots, x_n]/(f_1, \dots, f_m, h_1 - \bar{y}_1, \dots, h_m - \bar{y}_m)$. □

Lemma 7.6.3. *Let $R \to S$ be a ring map of finite presentation. For any surjection $\alpha : R[x_1, \dots, x_n] \to S$ the kernel of α is a finitely generated ideal in $R[x_1, \dots, x_n]$.*

Proof. Write $S = R[y_1, \dots, y_m]/(f_1, \dots, f_k)$. Choose $g_i \in R[y_1, \dots, y_m]$ which are lifts of $\alpha(x_i)$. Then we see that $S = R[x_i, y_j]/(f_j, x_i - g_i)$. Choose $h_j \in R[x_1, \dots, x_n]$ such that $\alpha(h_j)$ corresponds to y_j mod $(f_1, \dots, f_k)$. Consider the map $\psi : R[x_i, y_j] \to R[x_i]$, $x_i \mapsto x_i$, $y_j \mapsto h_j$. Then the kernel of α is the image of $(f_j, x_i - g_i)$ under ψ and we win. □

Lemma 7.6.4. *Let $R \to S$ be a ring map. Let M be an S-module. Assume $R \to S$ is of finite type and M is finitely presented as an R-module. Then M is finitely presented as an S-module.*

Proof. This is similar to the proof of part (4) of Lemma 7.6.2. We may assume $S = R[x_1, \ldots, x_n]/J$. Choose $y_1, \ldots, y_m \in M$ which generate M as an R-module and choose relations $\sum a_{ij} y_j = 0$, $i = 1, \ldots, t$ which generate the kernel of $R^{\oplus n} \to M$. For any $i = 1, \ldots, n$ and $j = 1, \ldots, m$ write

$$x_i y_j = \sum a_{ijk} y_k$$

for some $a_{ijk} \in R$. Consider the S-module N generated by $y_1, \ldots, y_m$ subject to the relations $\sum a_{ij} y_j = 0$, $i = 1, \ldots, t$ and $x_i y_j = \sum a_{ijk} y_k$, $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Then N has a presentation

$$S^{\oplus nm+t} \longrightarrow S^{\oplus m} \longrightarrow M \longrightarrow 0$$

By construction there is a surjective map $\varphi : N \to M$. To finish the proof we show φ is injective. Suppose $z = \sum b_j y_j \in N$ for some $b_j \in S$. We may think of b_j as a polynomial in $x_1, \ldots, x_n$ with coefficients in R. By applying the relations of the form $x_i y_j = \sum a_{ijk} y_k$ we can inductively lower the degree of the polynomials. Hence we see that $z = \sum c_j y_j$ for some $c_j \in R$. Hence if $\varphi(z) = 0$ then the vector $(c_1, \ldots, c_m)$ is an R-linear combination of the vectors $(a_{i1}, \ldots, a_{im})$ and we conclude that $z = 0$ as desired. □

7.7. Finite ring maps

Definition 7.7.1. Let $\varphi : R \to S$ be a ring map. We say $\varphi : R \to S$ is *finite* if if S is finite as an R-module.

Lemma 7.7.2. *Let $R \to S$ be a finite ring map. Let M be an S-module. Then M is finite as an R-module if and only if M is finite as an S-module.*

Proof. One of the implications follows from Lemma 7.5.6. To see the other assume that M is finite as an S-module. Pick $x_1, \ldots, x_n \in S$ which generate S as an R-module. Pick $y_1, \ldots, y_m \in M$ which generate M as an S-module. Then $x_i y_j$ generate M as an R-module. □

Lemma 7.7.3. *Suppose that $R \to S$ and $S \to T$ are finite ring maps. Then $R \to T$ is finite.*

Proof. If t_i generate T as an S-module and s_j generate S as an R-module, then $t_i s_j$ generate T as an R-module. (Also follows from Lemma 7.7.2.) □

Lemma 7.7.4. *Let $R \to S$ be a finite and finitely presented ring map. Let M be an S-module. Then M is finitely presented as an R-module if and only if M is finitely presented as an S-module.*

Proof. One of the implications follows from Lemma 7.6.4. To see the other assume that M is finitely presented as an S-module. Pick a presentation

$$S^{\oplus m} \longrightarrow S^{\oplus n} \longrightarrow M \longrightarrow 0$$

As S is finite as an R-module, the kernel of $S^{\oplus n} \to M$ is a finite R-module. Thus from Lemma 7.5.4 we see that it suffices to prove that S is finitely presented as an R-module.

Pick $x_1, \ldots, x_n \in S$ which generate S as an R-module. Write $x_i x_j = \sum a_{ijk} x_k$ for some $a_{ijk} \in R$. Let $J = \mathrm{Ker}(R[X_1, \ldots, X_n] \to S)$ where $R[X_1, \ldots, X_n] \to S$ is the R-algebra map determined by $X_i \mapsto x_i$. Let $g_{ij} = X_i X_j - \sum a_{ijk} X_k$ which is an element of J. Let $I = (g_{ij})$ so that $I \subset J$. By Lemma 7.6.3 there exist finitely many $g_1, \ldots, g_N \in J$ such that

$J = (g_1, \ldots, g_N)$. For every index $l \in \{1, \ldots, N\}$ we can write $g_l = h_l \bmod I$ for some $h_l \in J$ which has degree ≤ 1 in $X_1, \ldots, X_n$. (Details omitted; hint: use the g_{ij} get rid of the monomial of highest degree in g_l and use induction.) Write $h_l = a_{l0} + \sum a_{li} X_i$ for some $a_{li} \in R$. Then S has the following presentation

$$R^{\oplus N} \longrightarrow R^{\oplus n+1} \longrightarrow M \longrightarrow 0$$

as an R-module where the first arrow maps the lth basis vector to $(a_{l0}, a_{l1}, \ldots, a_{ln})$ and the second arrow maps $(a_0, a_1, \ldots, a_n)$ to $a_0 + \sum a_i x_i$. □

7.8. Colimits

Some of the material in this section overlaps with the general discussion on colimits in Categories, Sections 4.13 -- 4.19.

Definition 7.8.1. A *partially ordered set* is a set I together with a relation $\leq$ which is associative (if $i \leq j$ and $j \leq k$ then $i \leq k$) and reflexive ($i \leq i$ for all $i \in I$). A *directed set* $(I, \leq)$ is a partially ordered set $(I, \leq)$ such that I is not empty and such that $\forall i, j \in I$, there exists $k \in I$ with $i \leq k, j \leq k$.

It is customary to drop the $\leq$ from the notation when talking about a partially ordered set. This is the same as the notion defined in Categories, Section 4.19.

Definition 7.8.2. Let $(I, \leq)$ be a partially ordered set. A *system (M_i, μ_{ij}) of R-modules over I* consists of a family of R-modules $\{M_i\}_{i \in I}$ indexed by I and a family of R-module maps $\{\mu_{ij} : M_i \to M_j\}_{i \leq j}$ such that for all $i \leq j \leq k$

(7.8.2.1) $$\mu_{ii} = id_{M_i}$$

(7.8.2.2) $$\mu_{ik} = \mu_{jk} \circ \mu_{ij}$$

We say (M_i, μ_{ij}) is a *directed system* if I is a directed set.

This is the same as the notion defined in Categories, Definition 4.19.1 and Section 4.19. We refer to Categories, Definition 4.13.2 for the definition of a colimit of a diagram/system in any category.

Lemma 7.8.3. *Let (M_i, μ_{ij}) be a system of R-modules over the partially ordered set I. The colimit of the system (M_i, μ_{ij}) is the quotient R-module $(\bigoplus_{i \in I} M_i)/Q$ where Q is the R-submodule generated by all elements*

$$\iota_i(x_i) - \iota_j(\mu_{ij}(x_i))$$

where $\iota_i : M_i \to \bigoplus_{i \in I} M_i$ is the natural inclusion. We denote the colimit $M = colim_i M_i$. We denote $\pi : \bigoplus_{i \in I} M_i \to M$ the projection map and $\phi_i = \pi \circ \iota_i : M_i \to M$.

Proof. This lemma is a special case of Categories, Lemma 4.13.11 but we will also prove it directly in this case. Namely, note that $\phi_i = \phi_j \circ \mu_{ij}$ in the above construction. To show the pair (M, ϕ_i) is the colimit we have to show it satisfies the universal property: for any other such pair (Y, ψ_i) with $\psi_i : M_i \to Y$, $\psi_i = \psi_j \circ \mu_{ij}$, there is a unique R-module homomorphism $g : M \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} M_i \arrow[rr, "\mu_{ij}"] \arrow[dr, "\phi_i"] \arrow[ddr, "\psi_i"'] & & M_j \arrow[dl, "\phi_j"'] \arrow[ddl, "\psi_j"] \\ & M \arrow[d, "g"] & \\ & Y & \end{tikzcd}$$

And this is clear because we can define g by taking the map ψ_i on the summand M_i in the direct sum $\bigoplus M_i$. ☐

Lemma 7.8.4. *Let (M_i, μ_{ij}) be a system of R-modules over the partially ordered set I. Assume that I is directed. The colimit of the system (M_i, μ_{ij}) is canonically isomorphic to the module M defined as follows:*

(1) *as a set let*
$$M = \left(\coprod\nolimits_{i \in I} M_i\right) / \sim$$
where for $m \in M_i$ and $m' \in M_{i'}$ we have
$$m \sim m' \Leftrightarrow \mu_{ij}(m) = \mu_{i'j}(m') \text{ for some } j \geq i, i'$$
(2) *as an abelian group for $m \in M_i$ and $m' \in M_{i'}$ we define the sum of the classes of m and m' in M to be the class of $\mu_{ij}(m) + \mu_{i'j}(m')$ where $j \in I$ is any index with $i \leq j$ and $i' \leq j$, and*
(3) *as an R-module define for $m \in M_i$ and $x \in R$ the product of x and the class of m in M to be the class of xm in M.*

The canonical maps $\phi_i : M_i \to M$ are induced by the canonical maps $M_i \to \coprod_{i \in I} M_i$.

Proof. Omitted. Compare with Categories, Section 4.17. ☐

Lemma 7.8.5. *Let (M_i, μ_{ij}) be a directed system. Let $M = colim\, M_i$ with $\mu_i : M_i \to M$, then $\mu_i(x_i) = 0$ for $x_i \in M_i$ if and only if there exists $j \geq i$ such that $\mu_{ij}(x_i) = 0$.*

Proof. This is clear from the description of the directed colimit in Lemma 7.8.4. ☐

Example 7.8.6. Consider the partially ordered set $I = \{a, b, c\}$ with $a \prec b$ and $a \prec c$ and no other strict inequalities. A system $(M_a, M_b, M_c, \mu_{ab}, \mu_{ac})$ over I consists of three R-modules M_a, M_b, M_c and two R-module homomorphisms $\mu_{ab} : M_a \to M_b$ and $\mu_{ac} : M_a \to M_c$. The colimit of the system is just

$$M := colim_{i \in I} M_i = \mathrm{Coker}(M_a \to M_b \oplus M_c)$$

where the map is $\mu_{ab} \oplus -\mu_{ac}$. Thus the kernel of the canonical map $M_a \to M$ is $\mathrm{Ker}(\mu_{ab}) + \mathrm{Ker}(\mu_{ac})$. And the kernel of the canonical map $M_b \to M$ is the image of $\mathrm{Ker}(\mu_{ac})$ under the map μ_{ab}. Hence clearly the result of Lemma 7.8.5 is false for general systems.

Definition 7.8.7. Let (M_i, μ_{ij}), (N_i, ν_{ij}) be systems of R-modules over the same partially ordered set I. A *homomorphism of systems* Φ from (M_i, μ_{ij}) to (N_i, ν_{ij}) is by definition a family of R-module homomorphisms $\phi_i : M_i \to N_i$ such that $\phi_j \circ \mu_{ij} = \nu_{ij} \circ \phi_i$ for all $i \leq j$.

This is the same notion as a transformation of functors between the associated diagrams $M : I \to \text{Mod}_R$ and $N : I \to \text{Mod}_R$, in the language of categories. The following lemma is a special case of Categories, Lemma 4.13.7.

Lemma 7.8.8. *Let (M_i, μ_{ij}), (N_i, ν_{ij}) be systems of R-modules over the same partially ordered set. A morphism of systems $\Phi = (\phi_i)$ from (M_i, μ_{ij}) to (N_i, ν_{ij}) induces a unique homomorphism*

$$colim\, \phi_i : colim\, M_i \longrightarrow colim\, N_i$$

such that

$$\begin{tikzcd} M_i \arrow[r] \arrow[d, "\phi_i"'] & colim\, M_i \arrow[d, "colim\, \phi_i"] \\ N_i \arrow[r] & colim\, N_i \end{tikzcd}$$

commutes for all $i \in I$.

Proof. Write $M = colim\, M_i$ and $N = colim\, N_i$ and $\phi = colim\, \phi_i$ (as yet to be constructed). We will use the explicit description of M and N in Lemma 7.8.3 without further mention. The condition of the lemma is equivalent to the condition that

$$\begin{tikzcd} \bigoplus_{i \in I} M_i \arrow[r] \arrow[d, "\bigoplus \phi_i"'] & M \arrow[d, "\phi"] \\ \bigoplus_{i \in I} N_i \arrow[r] & N \end{tikzcd}$$

commutes. Hence it is clear that if ϕ exists, then it is unique. To see that ϕ exists, it suffices to show that the kernel of the upper horizontal arrow is mapped by $\bigoplus \phi_i$ to the kernel of the lower horizontal arrow. To see this, let $j \leq k$ and $x_j \in M_j$. Then

$$(\bigoplus \phi_i)(x_j - \mu_{jk}(x_j)) = \phi_j(x_j) - \phi_k(\mu_{jk}(x_j)) = \phi_j(x_j) - \nu_{jk}(\phi_i(x_j))$$

which is in the kernel of the lower horizontal arrow as required. $\square$

Lemma 7.8.9. *Let I be a directed partially ordered set. Let (L_i, λ_{ij}), (M_i, μ_{ij}), and (N_i, ν_{ij}) be systems of R-modules over I. Let $\varphi_i : L_i \to M_i$ and $\psi_i : M_i \to N_i$ be morphisms of systems over I. Assume that for all $i \in I$ the sequence of R-modules*

$$L_i \xrightarrow{\varphi_i} M_i \xrightarrow{\psi_i} N_i$$

is a complex with homology H_i. Then the R-modules H_i form a system over I, the sequence of R-modules

$$colim_i\, L_i \xrightarrow{\varphi} colim_i\, M_i \xrightarrow{\psi} colim_i\, N_i$$

is a complex as well, and denoting H its homology we have

$$H = colim_i\, H_i.$$

Proof. We are going to repeatedly use the description of colimits over I as in Lemma 7.8.4 without further mention. Let $h \in H$. Since $H = \ker(\varphi)/\text{Im}(\psi)$ we see that h is the class mod $\text{Im}(\psi)$ of an element $[m]$ in $\text{Ker}(\psi) \subset colim_i\, M_i$. Choose an i such that $[m]$ comes from an element $m \in M_i$. Choose a $j \geq i$ such that $\nu_{ij}(\psi_i(m)) = 0$ which is possible since $[m] \in \text{Ker}(\psi)$. After replacing i by j and m by $\mu_{ij}(m)$ we see that we may assume $m \in \text{Ker}(\psi_i)$. This shows that the map $colim_i\, H_i \to H$ is surjective.

Suppose that $h_i \in H_i$ has image zero on H. Since $H_i = \mathrm{Ker}(\psi_i)/\mathrm{Im}(\varphi_i)$ we may represent h_i by an element $m \in \mathrm{Ker}(\psi_i) \subset M_i$. The assumption on the vanishing of h_i in H means that the class of m in $colim_i\, M_i$ lies in the image of φ. Hence there exists an $j \geq i$ and a $l \in L_j$ such that $\varphi_j(l) = \mu_{ij}(m)$. Clearly this shows that the image of h_i in H_j is zero. This proves the injectivity of $colim_i\, H_i \to H$. □

Example 7.8.10. Taking colimits is not exact in general. Consider the partially ordered set $I = \{a, b, c\}$ with $a \prec b$ and $a \prec c$ and no other strict inequalities, as in Example 7.8.6. Consider the map of systems $(0, \mathbf{Z}, \mathbf{Z}, 0, 0) \to (\mathbf{Z}, \mathbf{Z}, \mathbf{Z}, 1, 1)$. From the description of the colimit in Example 7.8.6 we see that the associated map of colimits is not injective, even though the map of systems is injective on each object. Hence the result of Lemma 7.8.9 is false for general systems.

Lemma 7.8.11. *Let $\mathcal{I}$ be an index category satisfying the assumptions of Categories, Lemma 4.17.3. Then taking colimits of diagrams of abelian groups over $\mathcal{I}$ is exact (i.e., the analogue of Lemma 7.8.9 holds in this situation).*

Proof. By Categories, Lemma 4.17.3 we may write $\mathcal{I} = \coprod_{j \in J} \mathcal{I}_j$ with each $\mathcal{I}_j$ a filtered category, and J possibly empty. By Categories, Lemma 4.19.3 taking colimits over the index categories $\mathcal{I}_j$ is the same as taking the colimit over some directed partially ordered set. Hence Lemma 7.8.9 applies to these colimits. This reduces the problem to showing that coproducts in the category of R-modules over the set J are exact. In other words, exact sequences $L_j \to M_j \to N_j$ of R modules we have to show that

$$\bigoplus\nolimits_{j \in J} L_j \longrightarrow \bigoplus\nolimits_{j \in J} M_j \longrightarrow \bigoplus\nolimits_{j \in J} N_j$$

is exact. This can be verified by hand, and holds even if J is empty. □

For purposes of reference, we define what it means to have a relation between elements of a module.

Definition 7.8.12. Let R be a ring. Let M be an R-module. Let $n \geq 0$ and $x_i \in M$ for $i = 1, \ldots, n$. A *relation* between $x_1, \ldots, x_n$ in M is a sequence of elements $f_1, \ldots, f_n \in R$ such that $\sum_{i=1,\ldots,n} f_i x_i = 0$.

Lemma 7.8.13. *Let R be a ring and let M be an R-module. Then M is the colimit of a directed system (M_i, μ_{ij}) of R-modules with all M_i finitely presented R-modules.*

Proof. Consider any finite subset $S \subset M$ and any finite collection of relations E among the elements of S. So each $s \in S$ corresponds to $x_s \in M$ and each $e \in E$ consists of a vector of elements $f_{e,s} \in R$ such that $\sum f_{e,s} x_s = 0$. Let $M_{S,E}$ be the cokernel of the map

$$R^{\#E} \longrightarrow R^{\#S}, \quad (g_e)_{e \in E} \longmapsto (\sum g_e f_{e,s})_{s \in S}.$$

There are canonical maps $M_{S,E} \to M$. If $S \subset S'$ and if the elements of E correspond, via this map, to relations in E', then there is an obvious map $M_{S,E} \to M_{S',E'}$ commuting with the maps to M. Let I be the set of pairs (S, E) with ordering by inclusion as above. It is clear that the colimit of this directed system is M. □

7.9. Localization

Definition 7.9.1. Let R be a ring, S a subset of R. We say S is a *multiplicative subset of R* is $1 \in S$ and S is closed under multiplication, i.e., $s, s' \in S \Rightarrow ss' \in S$.

Given a ring A and a multiplicative subset S, we define a relation on $A \times S$ as follows:

$$(x, s) \sim (y, t) \iff \exists u \in S, \text{ such that } (xt - ys)u = 0$$

It is easily checked that this is an equivalence relation. Let x/s (or $\frac{x}{s}$) be the equivalence class of (x, s) and $S^{-1}A$ be the set of all equivalence classes. Define addition and multiplication in $S^{-1}A$ as follows:

$$x/s + y/t = (xt + ys)/st \tag{7.9.1.1}$$

$$x/s \cdot y/t = xy/st \tag{7.9.1.2}$$

One can check that $S^{-1}A$ becomes a ring under these operations.

Definition 7.9.2. This ring is called the *localization of A with respect to S*.

We have a natural ring map from A to its localization $S^{-1}A$,

$$A \longrightarrow S^{-1}A, \quad x \longmapsto x/1$$

which is sometimes called the *localization map*. In general the localization map is not injective, unless S contains no zero divisors. For, if $x/1 = 0$, then there is a $u \in S$ such that $xu = 0$ in A and hence $x = 0$ since there are no zero divisors in S. The localization of a ring has the following universal property.

Proposition 7.9.3. *Let $f : A \to B$ be a ring map that sends every element in S to a unit of B. Then there is a unique homomorphism $g : S^{-1}A \to B$ such that the following diagram commutes.*

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr] & & B \\ & S^{-1}A \arrow[ur, "g"'] & \end{tikzcd}$$

Proof. Existence. We define a map g as follows. For $x/s \in S^{-1}A$, let $g(x/s) = f(x)f(s)^{-1} \in B$. It is easily checked from the definition that this is a well-defined ring map. And it is also clear that this makes the diagram commutative.

Uniqueness. We now show that if $g' : S^{-1}A \to B$ satisfies $g'(x/1) = f(x)$, then $g = g'$. Hence $f(s) = g'(s/1)$ for $s \in S$ by the commutativity of the diagram. But then $g'(1/s)f(s) = 1$ in B, which implies that $g'(1/s) = f(s)^{-1}$ and hence $g'(x/s) = g'(x/1)g'(1/s) = f(x)f(s)^{-1} = g(x/s)$. □

Lemma 7.9.4. *The localization $S^{-1}A$ is the zero ring if and only if $0 \in S$.*

Proof. If $0 \in S$, any pair $(a, s) \sim (0, 1)$ by definition. If $0 \notin S$, then clearly $1/1 \neq 0/1$ in $S^{-1}A$. □

Lemma 7.9.5. *Let R be a ring. Let $S \subset R$ be a multiplicative subset. The category of $S^{-1}R$-modules is equivalent to the category of R-modules N with the property that every $s \in S$ acts as an automorphism on N.*

Proof. The functor which defines the equivalence associates to an $S^{-1}R$-module M the same module but now viewed as an R-module via the localization map $R \to S^{-1}R$. Conversely, if N is an R-module, such that every $s \in S$ acts via an automorphism s_N, then we can think of N as an $S^{-1}R$-module by letting x/s act via $x_N \circ s_N^{-1}$. We omit the verification that these two functors are quasi-inverse to each other. □

The notion of localization of a ring can be generalized to the localization of a module. Let A be a ring, S a multiplicative subset of A and M an A-module. We define a relation on $M \times S$ as follows

$$(m, s) \sim (n, t) \iff \exists u \in S, \text{ such that } (mt - ns)u = 0$$

This is clearly an equivalence relation. Denote by m/s (or $\frac{m}{s}$) be the equivalence class of (m, s) and $S^{-1}M$ be the set of all equivalence classes. Define the addition and scalar multiplication as follows

$$m/s + n/t = (mt + ns)/st \tag{7.9.5.1}$$

$$m/s \cdot n/t = mn/st \tag{7.9.5.2}$$

It is clear that this makes $S^{-1}M$ an $S^{-1}A$ module.

Definition 7.9.6. The $S^{-1}A$-module $S^{-1}M$ is called the *localization* of M at S.

Note that there is an A-module map $M \to S^{-1}M$, $m \mapsto m/1$ which is sometimes called the *localization map*. It satisfies the following universal property.

Lemma 7.9.7. *Let R be a ring. Let $S \subset R$ a multiplicative subset. Let M, N be R-modules. Assume all the elements of S act as automorphisms on N. Then the canonical map*

$$Hom_R(S^{-1}M, N) \longrightarrow Hom_R(M, N)$$

induced by the localisation map, is an isomorphism.

Proof. It is clear that the map is well-defined and R-linear. Injectivity: Let $\alpha \in Hom_R(S^{-1}M, N)$ and take an arbitrary element $m/s \in S^{-1}M$. Then, since $s \cdot \alpha(m/s) = \alpha(m/1)$, we have $\alpha(m/s) = s^{-1}(\alpha(m/1))$, so α is completely determined by what it does on the image of M in $S^{-1}M$. Surjectivity: Let $\beta : M \to N$ be a given R-linear map. We need to show that it can be "extended" to $S^{-1}M$. Define a map of sets

$$M \times S \to N, \quad (m, s) \mapsto s^{-1}(m)$$

Clearly, this map respects the equivalence relation from above, so it descends to a well-defined map $\alpha : S^{-1}M \to N$. It remains to show that this map is R-linear, so take $r, r' \in R$ as well as $s, s' \in S$ and $m, m' \in M$. Then

$$\begin{aligned} \alpha(r \cdot m/s + r' \cdot m'/s') &= \alpha((r \cdot s' \cdot m + r' \cdot s \cdot m')/(ss')) \\ &= (ss')^{-1}(\beta(r \cdot s' \cdot m + r' \cdot s \cdot m') \\ &= (ss')^{-1}(r \cdot s'\beta(m) + r' \cdot s\beta(m') \\ &= r\alpha(m/s) + r'\alpha(m'/s') \end{aligned}$$

and we win. □

Example 7.9.8. Let A be a ring and let M be an A-module. Here are some important examples of localizations.

(1) Given $\mathfrak{p}$ a prime ideal of A consider $S = A \setminus \mathfrak{p}$. It is immediately checked that S is a multiplicative set. In this case we denote $A_{\mathfrak{p}}$ and $M_{\mathfrak{p}}$ the localization of A and M with respect to S respectively. These are called the *localization of A, resp. M at $\mathfrak{p}$*.
(2) Let $f \in A$. Consider $S = \{1, f, f^2, \cdots\}$. This is clearly a multiplicative subset of A. In this case we denote A_f (resp. M_f) the localization $S^{-1}A$ (resp. $S^{-1}M$). This is called the *localization of A, resp. M with respect to f*. Note that $A_f = 0$ if and only if f is nilpotent in A.

(3) Let $S = \{f \in A \mid f \text{ is not a zerodivisor in } A\}$. This is a multiplicative subset of A. In this case the ring $Q(A) = S^{-1}A$ is called either the *total quotient ring*, or the *total ring of fractions* of A.

Lemma 7.9.9. *Let R be a ring. Let $S \subset R$ be a multiplicative subset. Let M be an R-module. Then*

$$S^{-1}M = colim_{f \in S} M_f$$

where the partial ordering on S is given by $f \geq f' \Leftrightarrow f = f'f''$ for some $f' \in R$ in which case the map $M_{f'} \to M_f$ is given by $m/(f')^e \mapsto m(f'')^e/f^e$.

Proof. Omitted. Hint: Use the universal property of Lemma 7.9.7. □

In the following paragraph, let A denote a ring, and M, N denote modules over A.

If S and S' are multiplicative sets of A, then it is clear that

$$SS' = \{ss' : s \in S, \ s' \in S'\}$$

is also a multiplicative set of A. Then the following holds.

Proposition 7.9.10. *Let $\overline{S}$ be the image of S in $S'^{-1}A$, then $(SS')^{-1}A$ is isomorphic to $\overline{S}^{-1}(S'^{-1}A)$.*

Proof. The map sending $x \in A$ to $x/1 \in (SS'^{-1})A$ induces a map sending $x/s \in S'^{-1}A$ to $x/s \in (SS'^{-1})A$, by universal property. The image of the elements in $\overline{S}$ are invertible in $(SS'^{-1})A$. By the universal property we get a map $f : \overline{S}^{-1}(S'^{-1}A) \to (SS'^{-1})A$ which maps $(x/t')/(s/s')$ to $(x/t') \cdot (s/s')^{-1}$.

On the other hand, the map from A to $\overline{S}^{-1}(S'^{-1}A)$ sending $x \in A$ to $(x/1)/(1/1)$ also induces a map $g : (SS'^{-1})A \to \overline{S}^{-1}(S'^{-1}A)$ which sends x/ss' to $(x/s')/(s/1)$, by the universal property again. It is immediately checked that f and g are inverse to each other, hence they are both isomorphisms. □

For the module M we have

Proposition 7.9.11. *View $S'^{-1}M$ as an A-module, then $S^{-1}(S'^{-1}M)$ is isomorphic to $(SS')^{-1}M$.*

Proof. Note that given a A-module M, we have not proved any universal property for $S^{-1}M$. Hence we cannot reason as in the preceding proof; we have to construct the isomorphism explicitly.

We define the maps as follows

$$f : S^{-1}(S'^{-1}M) \longrightarrow (SS')^{-1}M, \quad \frac{x/s'}{s} \mapsto x/ss'$$
$$g : (SS')^{-1}M \longrightarrow S^{-1}(S'^{-1}M), \quad x/t \mapsto \frac{x/s'}{s} \text{ for some } s \in S, s' \in S', \text{ and } t = ss'$$

We have to check that these homomorphisms are well-defined, that is, independent the choice of the fraction. This is easily checked and it is also straightforward to show that they are inverse to each other. □

If $u : M \to N$ is an A homomorphism, then the localization indeed induces a well-defined $S^{-1}A$ homomorphism $S^{-1}u : S^{-1}M \to S^{-1}N$ which sends x/s to $u(x)/s$. It is immediately checked that this construction is functorial, so that S^{-1} is actually a functor from the category of A-modules to the category of $S^{-1}A$-modules. Moreover this functor is exact, as we show in the following proposition.

Proposition 7.9.12. *Let $L \xrightarrow{u} M \xrightarrow{v} N$ is an exact sequence of R modules. Then $S^{-1}L \to S^{-1}M \to S^{-1}N$ is also exact.*

Proof. First it is clear that $S^{-1}L \to S^{-1}M \to S^{-1}N$ is a complex since localization is a functor. Next suppose that x/s maps to zero in $S^{-1}N$ for some $x/s \in S^{-1}M$. Then by definition there is a $t \in S$ such that $v(xt) = v(x)t = 0$ in M, which means $xt \in \mathrm{Ker}(v)$. By the exactness of $L \to M \to N$ we have $xt = u(y)$ for some y in L. Then x/s is the image of y/st. This proves the exactness. $\square$

Lemma 7.9.13. *Localization respects quotients, i.e. if N is a submodule of M, then $S^{-1}(M/N) \simeq (S^{-1}M)/(S^{-1}N)$.*

Proof. From the exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$$

we have

$$0 \longrightarrow S^{-1}N \longrightarrow S^{-1}M \longrightarrow S^{-1}(M/N) \longrightarrow 0$$

The corollary then follows. $\square$

If, in the preceding Corollary, we take $N = I$ and $M = A$ for an ideal I of A, we see that $S^{-1}A/S^{-1}I \simeq S^{-1}(A/I)$ as A-modules. The next proposition shows that they are isomorphic as rings.

Proposition 7.9.14. *Let I be an ideal of A, S a multiplicative set of A. Then $S^{-1}I$ is an ideal of $S^{-1}A$ and $\overline{S}^{-1}(A/I)$ is isomorphic to $S^{-1}A/S^{-1}I$, where $\overline{S}$ is the image of S in A/I.*

Proof. The fact that $S^{-1}I$ is an ideal is clear since I itself is an ideal. Define

$$f : S^{-1}A \longrightarrow \overline{S}^{-1}(A/I), \quad x/s \mapsto \overline{x}/\overline{s}$$

where $\overline{x}$ and $\overline{s}$ are the images of x and s in A/I. We shall keep similar notations in this proof. This map is well-defined by the universal property of $S^{-1}A$, and $S^{-1}I$ is contained in the kernel of it, therefore it induces a map

$$\overline{f} : S^{-1}A/S^{-1}I \longrightarrow \overline{S}^{-1}(A/I), \quad \overline{x/s} \mapsto \overline{x}/\overline{s}$$

On the other hand, the map $A \to S^{-1}A/S^{-1}I$ sending x to $\overline{x/1}$ induces a map $A/I \to S^{-1}A/S^{-1}I$ sending $\overline{x}$ to $\overline{x/1}$. The image of $\overline{S}$ is invertible in $S^{-1}A/S^{-1}I$, thus induces a map

$$g : \overline{S}^{-1}(A/I) \longrightarrow S^{-1}A/S^{-1}I, \quad \frac{\overline{x}}{\overline{s}} \mapsto \overline{x/s}$$

by the universal property. It is then clear that $\overline{f}$ and g are inverse to each other, hence are both isomorphisms. $\square$

We now consider how submodules behave in localization.

Lemma 7.9.15. *Any submodule N' of $S^{-1}M$ is of the form $S^{-1}N$ for some $N \subset M$. Indeed one can take N to be the inverse image of N' in M.*

Proof. Let N be the inverse image of N' in M. Then one can see that $S^{-1}N \supset N'$. To show they are equal, take x/s in $S^{-1}N$, where $s \in S$ and $x \in N$. This yields that $x/1 \in N'$. Since N' is an $S^{-1}R$-submodule we have $x/s = x/1 \cdot 1/s \in N'$. This finishes the proof. $\square$

Taking $M = A$ and $N = I$ an ideal of A, we have the following corollary, which can be viewed as a converse of the first part of Proposition 7.9.14.

Lemma 7.9.16. *Each ideal I' of $S^{-1}A$ takes the form $S^{-1}I$, where one can take I to be the inverse image of I' in A.*

The next lemma concerns the spectrum and localization. FIXME: This should be moved later in the manuscript.

Lemma 7.9.17. *Let S be a multiplicative set of A. Then the map*

$$f : Spec(S^{-1}A) \longrightarrow Spec(A)$$

induced by the canonical ring map $A \to S^{-1}A$ is a homeomorphism onto its image and $Im(f) = \{\mathfrak{p} \in Spec(R) : \mathfrak{p} \cap S = \emptyset\}$.

Proof. Denote the localization map by $\varphi : A \to S^{-1}A$. We first show that $\mathrm{Im}(f) = \{\mathfrak{p} \in Spec(R) : \mathfrak{p} \cap S = \emptyset\}$. For any ideal $\mathfrak{q}$ of $S^{-1}A$, $\varphi^{-1}(\mathfrak{q}) \cap S = 0$. Otherwise if $x \neq 0 \in \varphi^{-1}(\mathfrak{q}) \cap S$, then $x/1 \in \mathfrak{q}$. But $x \in S$, hence $x/1$ is invertible in $S^{-1}A$ which is impossible since $\mathfrak{q}$ is a prime ideal. For any prime ideal $\mathfrak{p}$ in A which does not meet S, $S^{-1}\mathfrak{p}$ is an ideal in $S^{-1}A$. Moreover it is a prime ideal. This is because $S^{-1}A/S^{-1}\mathfrak{p}$ is isomorphic to $\overline{S}^{-1}(A/\mathfrak{p})$ and the localization of an integral domain is again an integral domain.

We still have to show that this map is open, i.e. we have to show that the image of a standard open set is again open. For any $x/s \in S^{-1}A$, we claim that the image of $D(x/s)$ is $D(x) \cap \mathrm{Im}(f)$. First if $x/s \notin S^{-1}\mathfrak{p}$ for some prime ideal $\mathfrak{p}$ of A, then $x \notin \mathfrak{p}$. Conversely, if $x \notin \mathfrak{p}$ and $\mathfrak{p}$ does not meet S, then $x/s \notin S^{-1}\mathfrak{p}$. This is due to that fact that $\mathfrak{p} \cap S = \emptyset$.

Thus f is indeed an homeomorphism onto its image. □

7.10. Internal Hom

If R is a ring, and M, N are R-modules, then

$$Hom_R(M, N) = \{\varphi : M \to N\}$$

is the set of R-linear maps from M to N. This set comes with the structure of an abelian group by setting $(\varphi + \psi)(m) = \varphi(m) + \psi(m)$, as usual. In fact, $Hom_R(M, N)$ is also an R-module via the rule $(x\varphi)(m) = x\varphi(m) = \varphi(xm)$.

Given maps $a : M \to M'$ and $b : N \to N'$ of R-modules, we can pre-compose and post-compose homomorphisms by a and b. This leads to the following commutative diagram

$$\begin{tikzcd}
Hom_R(M', N) \arrow[r, "b\circ-"'] \arrow[d, "-\circ a"'] & Hom_R(M', N') \arrow[d, "-\circ a"] \\
Hom_R(M, N') \arrow[r, "b\circ-"] & Hom_R(M, N)
\end{tikzcd}$$

In fact, the maps in this diagram are R-module maps. Thus Hom_R defines an additive functor

$$\mathrm{Mod}_R^{opp} \times \mathrm{Mod}_R \longrightarrow \mathrm{Mod}_R, \quad (M, N) \longmapsto Hom_R(M, N)$$

Lemma 7.10.1. *Exactness and Hom_R. Let R be a ring.*

(1) *Let $M_1 \to M_2 \to M_3 \to 0$ be a complex of R-modules. Then $M_1 \to M_2 \to M_3 \to 0$ is exact if and only if $0 \to Hom_R(M_3, N) \to Hom_R(M_2, N) \to Hom_R(M_1, N)$ is exact for all R-modules N.*

(2) *Let $0 \to M_1 \to M_2 \to M_3$ be a complex of R-modules. Then $0 \to M_1 \to M_2 \to M_3$ is exact if and only if $0 \to Hom_R(N, M_1) \to Hom_R(N, M_2) \to Hom_R(N, M_1)$ is exact for all R-modules N.*

Proof. Omitted. $\square$

Lemma 7.10.2. *Let R be a ring. Let M be a finitely presented R-module. Let N be an R-module. Let $f \in R$. Then $Hom_R(M, N)_f \cong Hom_{R_f}(M_f, N_f)$.*

Proof. Choose a presentation

$$\bigoplus_{j=1,\ldots,m} R \longrightarrow \bigoplus_{i=1,\ldots,n} R \to M \to 0.$$

By Lemma 7.10.1 this gives an exact sequence

$$0 \to \mathscr{H}om_R(M, N) \to \bigoplus_{i=1,\ldots,n} N \longrightarrow \bigoplus_{j=1,\ldots,m} N.$$

Inverting f we get an exact sequence

$$0 \to (\mathscr{H}om_R(M, N))_f \to \bigoplus_{i=1,\ldots,n} N_f \longrightarrow \bigoplus_{j=1,\ldots,m} N_f$$

and the result follows since M_f sits in an exact sequence

$$\bigoplus_{j=1,\ldots,m} R_f \longrightarrow \bigoplus_{i=1,\ldots,n} R_f \to M_f \to 0$$

which induces (by Lemma 7.10.1) the exact sequence

$$0 \to Hom_{R_f}(M_f, N_f) \to \bigoplus_{i=1,\ldots,n} N_f \longrightarrow \bigoplus_{j=1,\ldots,m} N_f$$

which is the same as the one above. $\square$

7.11. Tensor products

Definition 7.11.1. Let R be a ring, M, N, P be three R-modules. A mapping $f : M \times N \to P$ (where $M \times N$ is viewed only as Cartesian product of two R-modules) is said to be *R-bilinear* if for each $x \in M$ the mapping $y \mapsto f(x, y)$ of N into P is R-linear, and for each $y \in N$ the mapping $x \mapsto f(x, y)$ is also R-linear.

Lemma 7.11.2. *Let M, N be R-modules. Then there exists a pair (T, g) where T is an R-module, and $g : M \times N \to T$ an R-bilinear mapping, with the following universal property: For any R-module P and any R-bilinear mapping $f : M \times N \to P$, there exists a unique R-linear mapping $\tilde{f} : T \to P$ such that $f = \tilde{f} \circ g$. In other words, the following diagram commutes:*

$$\begin{tikzcd}
M \times N \arrow[rr, "f"] \arrow[dr] & & P \\
& T \arrow[ur, "f'"'] &
\end{tikzcd}$$

Moreover, if (T, g) and (T', g') are two pairs with this property, then there exists a unique isomorphism $j : T \to T'$ such that $j \circ g = g'$.

The R-module T which satisfies the above universal property is called the *tensor product* of R-modules M and N, denoted as $M \otimes_R N$.

We first prove the existence of such R-module T. Let M, N be R-modules. Let T be the quotient module P/Q, where P is the free R-module $R^{M\times N}$ and Q is the R-module generated by all elements of the following types: $(x \in M, y \in N)$

$$(x + x', y) - (x, y) - (x', y) \tag{7.11.2.1}$$

$$(x, y + y') - (x, y) - (x, y') \tag{7.11.2.2}$$

$$(ax, y) - a \cdot (x, y) \tag{7.11.2.3}$$

$$(x, ay) - a \cdot (x, y) \tag{7.11.2.4}$$

Let $\pi : M\times N \to T$ denote the natural map. This map is R-bilinear, as implied by the above relations when we check the bilinearity conditions. Denote the image $\pi(x, y) = x \otimes y$, then these elements generate T. Now let $f : M \times N \to P$ be an R-bilinear map, then we can define $f' : T \to P$ by extending the mapping $f'(x \otimes y) = f(x, y)$. Clearly $f = f' \circ \pi$. Moreover, f' is uniquely determined by the value on the generating sets $\{x \otimes y : x \in M, y \in N\}$. Suppose there is another pair (T', g') satisfying the same properties. Then there is a unique $j : T \to T'$ and also $j' : T' \to T$ such that $g' = j \circ g$, $g = j' \circ g'$. But then both the maps $(j \circ j') \circ g$ and g satisfies the universal properties, so by uniqueness they are equal, and hence $j' \circ j$ is identity on T. Similarly $(j' \circ j) \circ g' = g'$ and $j \circ j'$ is identity on T'. So j is an isomorphism.

Lemma 7.11.3. *Let M, N, P be R-modules, then the bilinear maps*

$$(x, y) \mapsto y \otimes x \tag{7.11.3.1}$$

$$(x + y, z) \mapsto x \otimes z + y \otimes z \tag{7.11.3.2}$$

$$(r, x) \mapsto rx \tag{7.11.3.3}$$

induce unique isomorphisms

$$M \otimes_R N \to N \otimes_R M, \tag{7.11.3.4}$$

$$(M \oplus N) \otimes_R P \to (M \otimes_R P) \oplus (N \otimes_R P), \tag{7.11.3.5}$$

$$R \otimes_R M \to M \tag{7.11.3.6}$$

Proof. Omitted. □

We may generalize the tensor product of two R-modules to finitely many R-modules, and set up a correspondence between the multi-tensor product with multilinear mappings. Using almost the same construction one can prove that:

Lemma 7.11.4. *Let $M_1, \dots, M_r$ be R-modules. Then there exists a pair (T, g) consisting of an R-module T and an R-multilinear mapping: $g : M_1 \times \dots \times M_r \to T$ with the universal property: For any R-multilinear mapping $f : M_1 \times \dots \times M_r \to P$ there exists a unique R-homomorphism $f' : T \to P$ such that $f' \circ g = f$.*
Such a module T is unique up to isomorphism, i.e. if (T, g) and (T', g') are two such pairs, then there exists a unique isomorphism $j : T' \to T$ with $j \circ g = g'$. We denote $T = M_1 \otimes_R \dots \otimes_R M_r$.

Proof. Omitted. □

Lemma 7.11.5. *The homomorphisms*

$$(M \otimes_R N) \otimes_R P \to M \otimes_R N \otimes_R P \to M \otimes_R (N \otimes_R P) \tag{7.11.5.1}$$

such that $f((x \otimes y) \otimes z) = x \otimes y \otimes z$ and $g(x \otimes y \otimes z) = x \otimes (y \otimes z)$, $x \in M, y \in N, z \in P$ are well-defined and are isomorphisms.

Proof. We shall prove f is well-defined and is an isomorphism, and this proof carries analogously to g. Fix any $z \in P$, then the mapping $(x, y) \mapsto x \otimes y \otimes z$, $x \in M, y \in N$, is R-bilinear in x and y, and hence induces homomorphism $f_z : M \otimes N \to M \otimes N \otimes P$ $f_z(x \otimes y) = x \otimes y \otimes z$. Then consider $(M \otimes N) \times P \to M \otimes N \otimes P$ given by $(w, z) \mapsto f_z(w)$. The map is R-bilinear and thus induces $f : (M \otimes_R N) \otimes_R P \to M \otimes_R N \otimes_R P$ and $f((x \otimes y) \otimes z) = x \otimes y \otimes z$. To construct the inverse, we note that the map $\pi : M \times N \times P \to (M \otimes N) \otimes P$ is R-trilinear. Therefore, it induces an R-linear map $h : M \otimes N \otimes P \to (M \otimes N) \otimes P$ which agrees with the universal property. Here we see that $h(x \otimes y \otimes z) = (x \otimes y) \otimes z$. From the explicit expression of f and h, $f \circ h$ and $h \circ f$ are identity maps of $M \otimes N \otimes P$ and $(M \otimes N) \otimes P$ respectively, hence f is our desired isomorphism. $\square$

Doing induction we see that this extends to multi-tensor products. Combined with Lemma 7.11.3 we see that the tensor product operation on the category of R-modules is associative, commutative and distributive.

Definition 7.11.6. An abelian group N is called an (A, B)-*bimodule* if it is both an A-module and a B-module, and the actions $A \to End(M)$ and $B \to End(M)$ are compatible in the sense that $(ax)b = a(xb)$ for all $a \in A, b \in B, x \in N$. Usually we denote it as ${}_AN_B$.

Lemma 7.11.7. *For A-module M, B-module P and (A, B)-bimodule N, the modules $(M \otimes_A N) \otimes_B P$ and $M \otimes_A (N \otimes_B P)$ can both be given (A, B)-bimodule structure, and moreover*

$$(M \otimes_A N) \otimes_B P \cong M \otimes_A (N \otimes_B P). \tag{7.11.7.1}$$

Proof. A priori $M \otimes_A N$ is an A-module, but we can give it a B-module structure by letting

$$(x \otimes y)b = x \otimes yb, \quad x \in M, y \in N, b \in B$$

Thus $M \otimes_A N$ becomes an (A, B)-bimodule. Similarly for $N \otimes_B P$, and thus for $(M \otimes_A N) \otimes_B P$ and $M \otimes_A (N \otimes_B P)$. Therefore by the above lemma, these two modules are isomorphic as both as A-module and B-module via the same mapping. $\square$

Lemma 7.11.8 (Tensor products commute with colimits)**.** *Let (M_i, μ_{ij}) be a system over the partially ordered set I. Let N be an R-module. Then*

$$colim(M_i \otimes N) \cong (colim\, M_i) \otimes N. \tag{7.11.8.1}$$

Moreover, the isomorphism is induced by the homomorphisms $\mu_i \otimes 1 : M_i \otimes N \to M \otimes N$ where $M = colim_i\, M_i$ with natural maps $\mu_i : M_i \to M$.

Proof. Let $P = colim(M_i \otimes N)$, $M = colim\, M_i$. Then for all $i \leq j$, the following diagram commutes:

$$\begin{tikzcd}
M_i \otimes N \arrow[r, "\mu_i \otimes 1"'] \arrow[d, "\mu_{ij} \otimes 1"'] & M \otimes N \arrow[d, "\mathrm{id}"] \\
M_j \otimes N \arrow[r, "\mu_j \otimes 1"] & M \otimes N
\end{tikzcd}$$

By Lemma 7.8.8, these maps induce a unique homomorphism $\psi : P \to M \otimes N$, with $\lambda_i : M_i \otimes N \to P$ given by $\lambda_i = \pi \circ (\iota_i \otimes 1)$.

To construct the inverse map, for each $i \in I$, there is the canonical R-bilinear mapping $g_i : M_i \times N \to M_i \otimes N$. This induces a unique mapping $\widehat{\phi} : M \times N \to P$ such that

$\widehat{\phi}\circ(\mu_i\times 1) = \lambda_i\circ g_i$. It is R-bilinear. Thus it induces an R-linear mapping $\phi : M\otimes N \to P$. From the commutative diagram below:

$$\begin{tikzcd}
M_i \times N \arrow[r, "g_i"] \arrow[d, "\mu_i\times \mathrm{id}"] & M_i \otimes N \arrow[r, "\mathrm{id}"'] \arrow[d, "\lambda_i"'] & M_i \otimes N \arrow[d, "\mu_i\otimes \mathrm{id}"'] \arrow[dr, "\lambda_i"] & \\
M \times N \arrow[r, "\widehat{\phi}"] & P \arrow[r, "\psi"] & M \otimes N \arrow[r, "\phi"] & P
\end{tikzcd}$$

we see that $\psi \circ \widehat{\phi} = g$, the canonical R-bilinear mapping $g : M \times N \to M \otimes N$. So $\psi \circ \phi$ is identity on $M \otimes N$. From the right-hand square and triangle, $\phi \circ \psi$ is also identity on P. $\square$

Exactness Properties. We first make a basic observation relating tensor products and the functor Hom:

Lemma 7.11.9. *For any three R-modules M, N, P,*

(7.11.9.1) $$Hom_R(M \otimes_R N, P) \cong Hom_R(M, Hom_R(N, P))$$

Proof. An R-linear map $\hat{f} \in Hom_R(M \otimes_R N, P)$ corresponds to an R-bilinear map $f : M \times N \to P$. For each $x \in M$ the mapping $y \mapsto f(x, y)$ is R-linear by the universal property. Thus f corresponds to a map $\phi_f : M \to Hom_R(N, P)$. This map is R-linear since

$$\phi_f(ax + y)(z) = f(ax + y, z) = af(x, z) + f(y, z) = (a\phi_f(x) + \phi_f(y))(z),$$

$\forall y \in N$ and $\forall a \in R, x, z \in M$. Conversely, any $f \in Hom_R(M, Hom_R(N, P))$ defines an R-bilinear map $M \times N \to P$, namely $(x, y) \mapsto f(x)(y)$. So this is a natural one-to-one correspondence between the two modules. $\square$

Lemma 7.11.10. *Let*

(7.11.10.1) $$M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$$

be an exact sequence of R-modules and homomorphisms, and let N be any R-module. Then the sequence

(7.11.10.2) $$M_1 \otimes N \xrightarrow{f\otimes 1} M_2 \otimes N \xrightarrow{g\otimes 1} M_3 \otimes N \to 0$$

is exact. In other words, the functor $-\otimes_R N$ is right exact, in the sense that tensoring each term in the original right exact sequence preserves the exactness.

Proof. We apply the functor $Hom(-, Hom(N, P))$ to the first exact sequence. We obtain

$$0 \to Hom(M_3, Hom(N, P)) \to Hom(M_2, Hom(N, P)) \to Hom(M_1, Hom(N, P))$$

By Lemma 7.11.9, we have

$$0 \to Hom(M_3 \otimes N, P) \to Hom(M_2 \otimes N, P) \to Hom(M_1 \otimes N, P)$$

Using the pullback property again, we arrive at the desired exact sequence. $\square$

Remark 7.11.11. However, tensor product does NOT preserve exact sequences in general. In other words, if $M_1 \to M_2 \to M_3$ is exact, then it is not necessarily true that $M_1 \otimes N \to M_2 \otimes N \to M_3 \otimes N$ is exact for arbitrary R-module N.

Example 7.11.12. Consider the injective map $2 : \mathbf{Z} \to \mathbf{Z}$ viewed as a map of $\mathbf{Z}$-modules. Let $N = \mathbf{Z}/2$. Then the induced map $\mathbf{Z} \otimes \mathbf{Z}/2 \to \mathbf{Z} \otimes \mathbf{Z}/2$ is NOT injective. This is because for $x \otimes y \in \mathbf{Z} \otimes \mathbf{Z}/2$,

$$(2 \otimes 1)(x \otimes y) = 2x \otimes y = x \otimes 2y = x \otimes 0 = 0$$

Therefore the induced map is the zero map while $\mathbf{Z} \otimes N \neq 0$.

Remark 7.11.13. For R-modules N, if the functor $- \otimes_R N$ is exact, i.e. tensoring with N preserves all exact sequences, then N is said to be *flat* R-module. We will discuss this later.

Lemma 7.11.14. *Let R be a ring. Let M and N be R-modules.*

(1) *If N and M are finite, then so is $M \otimes_R N$.*
(2) *If N and M are finitely presented, then so is $M \otimes_R N$.*

Proof. Suppose M is finite. Then choose a presentation $0 \to K \to R^{\otimes n} \to M \to 0$. This gives an exact sequence $K \otimes_R N \to N^{\oplus n} \to M \otimes_R N \to 0$ by Lemma 7.11.10 above. We conclude that if N is finite too then $M \otimes_R N$ is a quotient of a finite module, hence finite, see Lemma 7.5.4. Similarly, if both N and M are finitely presented, then we see that K is finite and that $M \otimes_R N$ is a quotient of the finitely presented module $N^{\oplus n}$ by a finite module, namely $K \otimes_R N$, and hence finitely presented, see Lemma 7.5.4. $\square$

Lemma 7.11.15. *Let M be an R-module. Then the $S^{-1}R$ modules $S^{-1}M$ and $S^{-1}R \otimes_R M$ are canonically isomorphic, and the unique isomorphism $f : S^{-1}R \otimes_R M \to S^{-1}M$ is given by*

$$f((a/s) \otimes m) = am/s, \forall a \in R, m \in M, s \in S \tag{7.11.15.1}$$

Proof. Obviously, the map $f' : S^{-1}R \times M \to S^{-1}M$ given by $f((am, s)) = am/s$ is bilinear, and thus by the universal property, this map induces a unique $S^{-1}R$-module homomorphism $f : S^{-1}R \otimes_R M \to S^{-1}M$ given as in the above lemma. Actually every element in $S^{-1}M$ is of the form m/s, $m \in M, s \in S$ and every element in $S^{-1}R \otimes_R M$ is of the form $1/s \otimes m$. To see the latter fact, write an element in $S^{-1}R \otimes_R M$ as

$$\sum_k \frac{a_k}{s_k} \otimes m_k = \sum_k \frac{a_k t_k}{s} \otimes m_k = \frac{1}{s} \otimes \sum_k a_k t_k m_k = \frac{1}{s} \otimes m$$

Where $m = \sum_k a_k t_k m_k$. Then it is obvious that f is surjective, and if $f(\frac{1}{s} \otimes m) = m/s = 0$ then there exists $t' \in S$ with $tm = 0$ in M. Then we have

$$\frac{1}{s} \otimes m = \frac{1}{st} \otimes tm = \frac{1}{st} \otimes 0 = 0$$

Therefore f is injective. $\square$

Lemma 7.11.16. *Let M, N be R-modules, then there is a canonical $S^{-1}R$-module isomorphism $f : S^{-1}M \otimes_{S^{-1}R} S^{-1}N \to S^{-1}(M \otimes_R N)$, given by*

$$f((m/s) \otimes (n/t) = (m \otimes n)/st$$

Proof. We may use Lemma 7.11.7 and Lemma 7.11.15 repeatedly to see that these two $S^{-1}R$-modules are isomorphic, noting that $S^{-1}R$ is an $(R, S^{-1}R)$-bimodule:

$$S^{-1}(M \otimes_R N) \cong S^{-1}R \otimes_R (M \otimes_R N) \tag{7.11.16.1}$$

$$\cong S^{-1}M \otimes_R N \tag{7.11.16.2}$$

$$\cong (S^{-1}M \otimes_{S^{-1}R} S^{-1}R) \otimes_R N \tag{7.11.16.3}$$

$$\cong S^{-1}M \otimes_{S^{-1}R} (S^{-1}R \otimes_R N) \tag{7.11.16.4}$$

$$\cong S^{-1}M \otimes_{S^{-1}R} S^{-1}N \tag{7.11.16.5}$$

This isomorphism is easily seen to be the one stated in the lemma. □

Let $\varphi : R \to S$ be a ring map. Given a S-module N we obtain an R-module N_R by the rule $r \cdot n = \varphi(r)n$. This is sometimes called the *restriction* of N to R.

Lemma 7.11.17. *Let $R \to S$ be a ring map. The functors $Mod_S \to Mod_R$, $N \mapsto N_R$ (restriction) and $Mod_R \to Mod_S$, $M \mapsto M \otimes_R S$ (base change) are adjoint functors. In a formula*

$$Hom_R(M, N_R) = Hom_S(M \otimes_R S, N)$$

Proof. If $\alpha : M \to N_R$ is an R-module map, then we define $\alpha' : M \otimes_R S \to N$ by the rule $\alpha'(m \otimes s) = s\alpha(m)$. If $\beta : M \otimes_R S \to N$ is an S-module map, we define $\beta' : M \to N_R$ by the rule $\beta'(m) = \beta(m \otimes 1)$. We omit the verification that these constructions are mutually inverse. □

7.12. Tensor algebra

Let R be a ring. Let M be an R-module. We define the *tensor algebra of M over R* to be the noncommutative R-algebra

$$\mathrm{T}(M) = \mathrm{T}_R(M) = \bigoplus_{n \geq 0} \mathrm{T}^n(M)$$

with $\mathrm{T}^0(M) = R$, $\mathrm{T}^1(M) = M$, $\mathrm{T}^2(M) = M \otimes_R M$, $\mathrm{T}^3(M) = M \otimes_R M \otimes_R M$, and so on. Multiplication is defined by the rule that on pure tensors we have

$$(x_1 \otimes x_2 \otimes \ldots \otimes x_n) \cdot (y_1 \otimes y_2 \otimes \ldots \otimes y_m) = x_1 \otimes x_2 \otimes \ldots \otimes x_n \otimes y_1 \otimes y_2 \otimes \ldots \otimes y_m$$

and we extend this by linearity.

We define the *exterior algebra $\wedge(M)$ of M over R* to be the quotient of $\mathrm{T}(M)$ by the two sided ideal generated by the elements $x \otimes x \in \mathrm{T}^2(M)$. The image of a pure tensor $x_1 \otimes \ldots \otimes x_n$ in $\wedge^n(M)$ is denoted $x_1 \wedge \ldots \wedge x_n$. These elements generate $\wedge^n(M)$, they are R-linear in each x_i and they are zero when two of the x_i are equal (i.e., alternating). The multiplication on $\wedge(M)$ is graded commutative, i.e., $x \wedge y = -y \wedge x$.

An example of this is when $M = Rx_1 \oplus \ldots \oplus Rx_n$ is a finite free module. In this case $\wedge(M)$ is free over R with basis the elements

$$x_{i_1} \wedge \ldots \wedge x_{i_r}$$

with $0 \leq r \leq n$ and $1 \leq i_1 < i_2 < \ldots < i_r \leq n$.

We define the *symmetric algebra $Sym(M)$ of M over R* to be the quotient of $\mathrm{T}(M)$ by the two sided ideal generated by the elements $x \otimes y - y \otimes x \in \mathrm{T}^2(M)$. The image of a pure tensor $x_1 \otimes \ldots \otimes x_n$ in $\mathrm{Sym}^n(M)$ is denoted just $x_1 \ldots x_n$. These elements generate $\mathrm{Sym}^n(M)$, these are R-linear in each x_i and $x_1 \ldots x_n = x'_1 \ldots x'_n$ if the sequence of elements $x_1, \ldots, x_n$ is a permutation of the sequence $x'_1, \ldots, x'_n$. Thus we see that $\text{Sym}(M)$ is commutative.

An example of this is when $M = Rx_1 \oplus \ldots \oplus Rx_n$ is a finite free module. In this case $\text{Sym}(M) = R[x_1, \ldots, x_n]$ is a polynomial algebra.

Lemma 7.12.1. *Let R be a ring. Let M be an R-module. If M is a free R-module, so is each symmetric and exterior power.*

Proof. Omitted, but see above for the finite free case. □

Lemma 7.12.2. *Let R be a ring. Let $M_2 \to M_1 \to M \to 0$ be an exact sequence of R-modules. There are exact sequences*

$$M_2 \otimes_R Sym^{n-1}(M_1) \to Sym^n(M_1) \to Sym^n(M) \to 0$$

and similarly

$$M_2 \otimes_R \wedge^{n-1}(M_1) \to \wedge^n(M_1) \to \wedge^n(M) \to 0$$

Proof. Omitted. □

Lemma 7.12.3. *Let R be a ring. Let M be an R-module. Let x_i, $i \in I$ be a given system of generators of M as an R-module. Let $n \geq 2$. There exists a canonical exact sequence*

$$\bigoplus_{1 \leq j_1 < j_2 \leq n} \bigoplus_{i_1, i_2 \in I} T^{n-2}(M) \oplus \bigoplus_{1 \leq j_1 < j_2 \leq n} \bigoplus_{i \in I} T^{n-2}(M) \to T^n(M) \to \wedge^n(M) \to 0$$

where the pure tensor $m_1 \otimes \ldots \otimes m_{n-2}$ in the first summand maps to

$$m_1 \otimes \ldots \otimes x_{i_1} \otimes \ldots \otimes x_{i_2} \otimes \ldots \otimes m_{n-2} + m_1 \otimes \ldots \otimes x_{i_2} \otimes \ldots \otimes x_{i_1} \otimes \ldots \otimes m_{n-2}$$

and $m_1 \otimes \ldots \otimes m_{n-2}$ in the second summand maps to

$$m_1 \otimes \ldots \otimes x_i \otimes \ldots \otimes x_i \otimes \ldots \otimes m_{n-2}$$

There is also a canonical exact sequence

$$\bigoplus_{1 \leq j_1 < j_2 \leq n} \bigoplus_{i_1, i_2 \in I} T^{n-2}(M) \to T^n(M) \to Sym^n(M) \to 0$$

where the pure tensor $m_1 \otimes \ldots \otimes m_{n-2}$ maps to

$$m_1 \otimes \ldots \otimes x_{i_1} \otimes \ldots \otimes x_{i_2} \otimes \ldots \otimes m_{n-2} - m_1 \otimes \ldots \otimes x_{i_2} \otimes \ldots \otimes x_{i_1} \otimes \ldots \otimes m_{n-2}$$

Proof. Omitted. □

Lemma 7.12.4. *Let R be a ring. Let M_i be a directed system of R-modules. Then $colim_i\, T(M) = T(colim_i\, M_i)$ and similarly for the symmetric and exterior algebras.*

Proof. Omitted. □

7.13. Base change

We formally introduce base change in algebra as follows.

Definition 7.13.1. Let $\varphi : R \to S$ be a ring map. Let M be an S-module. Let $R \to R'$ be any ring map. The *base change* of φ by $R \to R'$ is the ring map $R' \to S \otimes_R R'$. In this situation we often write $S' = S \otimes_R R'$. The *base change* of the S-module M is the S'-module $M \otimes_R R'$.

If $S = R[x_i]/(f_j)$ for some collection of variables x_i, $i \in I$ and some collection of polynomials $f_j \in R[x_i]$, $j \in J$, then $S \otimes_R R' = R'[x_i]/(f'_j)$, where $f'_j \in R'[x_i]$ is the image of f_j under the map $R[x_i] \to R'[x_i]$ induced by $R \to R'$. This simple remark is the key to understanding base change.

Lemma 7.13.2. *Let $R \to S$ be a ring map. Let M be an S-module. Let $R \to R'$ be a ring map and let $S' = S \otimes_R R'$ and $M' = M \otimes_R R'$ be the base changes.*

1. *If M is a finite S-module, then the base change M' is a finite S'-module.*
2. *If M is an S-module finite presentation, then the base change M' is an S'-module of finite presentation.*
3. *If $R \to S$ is of finite type, then the base change $R' \to S'$ is of finite type.*
4. *If $R \to S$ is of finite presentation, then the base change $R' \to S'$ is of finite presentation.*

Proof. Proof of (1). Take a surjective, R-linear map $R^{\oplus n} \to M \to 0$. By Lemma 7.11.3 and 7.11.10 the result after tensoring with R' is a surjection $R'^{\oplus n} \to M' \to 0$, so M' is a finitely generated R'-module. Proof of (2). Take a presentation $R^{\oplus m} \to R^{\oplus n} \to M \to 0$. By Lemma 7.11.3 and 7.11.10 the result after tensoring with R' gives a finite presentation $R'^{\oplus m} \to R'^{\oplus n} \to M' \to 0$, of the R'-module M'. Proof of (3). This follows by the remark preceding the lemma as we can take I to be finite by assumption. Proof of (4). This follows by the remark preceding the lemma as we can take I and J to be finte by assumption. □

7.14. Miscellany

The proofs in this section should not refer to any results except those from the section on basic notions, Section 7.3.

Lemma 7.14.1. *Let $R \to R'$ be a ring map and let $I \subset R$ be a locally nilpotent ideal. Then IR' is a locally nilpotent ideal of R'.*

Proof. This follows from the fact that if $x, y \in R'$ are nilpotent, then $x + y$ is nilpotent too. Namely, if $x^n = 0$ and $y^m = 0$, then $(x + y)^{n+m-1} = 0$. □

Lemma 7.14.2. *Let R be a ring, I and J two ideals and $\mathfrak{p}$ a prime ideal containing the product IJ. Then $\mathfrak{p}$ contains I or J.*

Proof. Asumme the contrary and take $x \in I \setminus \mathfrak{p}$ and $y \in I \setminus \mathfrak{p}$. Their product is an element of $IJ \subseteq \mathfrak{p}$, which contradicts the assumption that $\mathfrak{p}$ was prime. □

Lemma 7.14.3 (Prime avoidance)**.** *Let R be a ring. Let $I_i \subset R$, $i = 1, \ldots, r$, and $J \subset R$ be ideals. Assume*

1. *$J \not\subset I_i$ for $i = 1, \ldots, r$, and*
2. *all but two of I_i are prime ideals.*

Then there exists an $x \in J$, $x \notin I_i$ for all i.

Proof. The result is true for $r = 1$. If $r = 2$, then let $x, y \in J$ with $x \notin I_1$ and $y \notin I_2$. We are done unless $x \in I_2$ and $y \in I_1$. Then the element $x + y$ cannot be in I_1 (since that would mean $x + y - y \in I_1$) and it also cannot be in I_2.

For $r \geq 3$, assume the result holds for $r - 1$. After renumbering we may assume that I_r is prime. We may also assume there are no inclusions among the I_i. Pick $x \in J$, $x \notin I_i$ for all $i = 1, \ldots, r - 1$. If $x \notin I_r$ we are done. So assume $x \in I_r$. If $JI_1 \ldots I_{r-1} \subset I_r$ then $J \subset I_r$ (by Lemma 7.14.2) a contradiction. Pick $y \in JI_1 \ldots I_{r-1}$, $y \notin I_r$. Then $x + y$ works. □

Lemma 7.14.4. *(Chinese remainder.) Let R be a ring.*

(1) *If $I_1, \ldots, I_r$ are ideals such that $I_a + I_b = R$ when $a \neq b$, then $I_1 \cap \ldots \cap I_r = I_1 I_2 \ldots I_r$ and $R/(I_1 I_2 \ldots I_r) \cong R/I_1 \times \ldots \times R/I_r$.*
(2) *If $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ are pairwise distinct maximal ideals then $\mathfrak{m}_a + \mathfrak{m}_b = R$ for $a \neq b$ and the above applies.*

Proof. Let us first prove $I_1 \cap \ldots \cap I_r = I_1 \ldots I_r$ as this will also imply the injectivity of the induced ring homomorphism $R/(I_1 \ldots I_r) \to R/I_1 \times \ldots \times R/I_r$. The inclusion $I_1 \cap \ldots \cap I_r \supseteq I_1 \ldots I_r$ is always fulfilled since ideals are closed under multiplication with arbitrary ring elements. To prove the other inclusion, we claim that the ideals

$$I_1 \ldots \hat{I}_i \ldots I_r, \quad i = 1, \ldots, r$$

generate the ring R. We prove this by induction on r. It holds when $r = 2$. If $r > 2$, then we see that R is the sum of the ideals $I_1 \ldots \hat{I}_i \ldots I_{r-1}$, $i = 1, \ldots, r-1$. Hence I_r is the sum of the ideals $I_1 \ldots \hat{I}_i \ldots I_r$, $i = 1, \ldots, r-1$. Applying the same argument with the reverse ordering on the ideals we see that I_1 is the sum of the ideals $I_1 \ldots \hat{I}_i \ldots I_r$, $i = 2, \ldots, r$. Since $R = I_1 + I_r$ by assumption we see that R is the sum of the ideals displayed above. Therefore we can find elements $a_i \in I_1 \ldots \hat{I}_i \ldots I_r$ such that their sum is one. Multiplying this equation by an element of $I_1 \cap \ldots \cap I_r$ gives the other inclusion. It remains to show that the canonical map $R/(I_1 \ldots I_r) \to R/I_1 \times \ldots \times R/I_r$ is surjective. For this, consider its action on the equation $1 = \sum_{i=1}^r a_i$ we derived above. On the one hand, a ring morphism sends 1 to 1 and on the other hand, the image of any a_i is zero in R/I_j for $j \neq i$. Therefore, the image of a_i in R/I_i is the identity. So given any element $(\bar{b_1}, \ldots, \bar{b_r}) \in R/I_1 \times \ldots \times R/I_r$, the element $\sum_{i=1}^r a_i \cdot b_i$ is an inverse image in R.

To see (2), by the very definition of being distinct maximal ideals, we have $\mathfrak{m}_a + \mathfrak{m}_b = R$ for $a \neq b$ and so the above applies. □

Lemma 7.14.5. *(Nakayama's lemma.) Let R be a ring, let M be an R-module, and let $I \subset R$ be an ideal.*

(1) *If M is finite, and $IM = M$, then there exists a $f = 1 + i \in 1 + I$ such that $fM = 0$.*
(2) *If M is finite, $IM = M$, and $I \subset rad(R)$ then $M = 0$.*
(3) *If $IM = M$, I is nilpotent, then $M = 0$.*

Proof. Proof of (1). Write $M = \sum Rx_j$, $j = 1, \ldots, r$. Write $x_j = \sum i_{jj'} x_{j'}$ with $i_{jj'} \in I$. In other words $\sum (\delta_{jj'} - i_{jj'}) x_{j'} = 0$. Hence the determinant f of the $r \times r$ matrix $(\delta_{jj'} - i_{jj'})$ is a solution. The other parts are easy. □

Lemma 7.14.6. *Let R be a ring. Let $n \geq m$. Let $A = (a_{ij})$ be an $n \times m$ matrix with coefficients in R. Let $I \subset R$ be the ideal generated by the $m \times m$ minors of A.*

(1) *For any $f \in I$ there exists a $m \times n$ matrix B such that $BA = f 1_{m \times m}$.*
(2) *If $f \in R$ and $BA = f 1_{m \times m}$ for some $m \times m$ matrix B, then $f^m \in I$.*

Proof. For $I \subset \{1, \ldots, n\}$ with $|I| = m$ denote E_I the $m \times n$ matrix of the projection

$$R^{\oplus n} = \bigoplus\nolimits_{i \in \{1, \ldots, n\}} R \longrightarrow \bigoplus\nolimits_{i \in I} R$$

and set $A_I = E_I A$, i.e., A_I is the $m \times m$ matrix whose rows are the rows of A with indices in I. Let B_I be the adjugate (transpose of cofactor) matrix to A_I, i.e., such that $A_I B_I = B_I A_I = \det(A_I)$. If $f \in I$ then we can write $f = \sum c_I \det(A_I)$ for some $c_I \in R$. Set $B = \sum c_I B_I E_I$ to see that (1) holds.

If $f 1_{m\times m} = BA$ then by the Cauchy-Binet formula we have $f^m = \sum b_I \det(A_I)$ where b_I is the determinant of the $m \times m$ matrix whose columns are the columns of B with indices in I. $\square$

7.15. Cayley-Hamilton

Lemma 7.15.1. *Let R be a ring. Let $A = (a_{ij})$ be an $n \times n$ matrix with coefficients in R. Let $P(x) \in R[x]$ be the characteristic polynomial of A (defined as $\det(xid_{n\times n} - A)$). Then $P(A) = 0$ in $Mat(n \times n, R)$.*

Proof. We reduce the question to the well-known Cayley-Hamilton theorem from linear algebra in several steps:

1. If $\phi : S \to R$ is a ring morphism and b_{ij} are inverse images of the a_{ij} under this map, then it suffices to show the statement for S and (b_{ij}) since ϕ is a ring morphism.
2. If $\psi : R \hookrightarrow S$ is an injective ring morphism, it clearly suffices to show the result for S and the a_{ij} considered as elements of S.
3. Thus we may first reduce to the case $R = \mathbb{Z}[X_{ij}]$, $a_{ij} = X_{ij}$ of a polynomial ring and then further to the case $R = \mathbb{Q}(X_{ij})$ where we may finally apply Cayley-Hamilton.

$\square$

Lemma 7.15.2. *Let R be a ring. Let M be a finite R-module. Let $\varphi : M \to M$ be an endomorphism. Then there exists a monic polynomial $P \in R[T]$ such that $P(\varphi) = 0$ as an endomorphism of M.*

Proof. Choose a surjective R-module map $R^{\oplus n} \to M$, given by $(a_1, \ldots, a_n) \mapsto \sum a_i x_i$ for some generators $x_i \in M$. Choose $(a_{i1}, \ldots, a_{in}) \in R^{\oplus n}$ such that $\varphi(x_i) = \sum a_{ij} x_j$. In other words the diagram

$$\begin{tikzcd} R^{\oplus n} \arrow[r] \arrow[d, "A"'] & M \arrow[d, "\varphi"] \\ R^{\oplus n} \arrow[r] & M \end{tikzcd}$$

is commutative where $A = (a_{ij})$. By Lemma 7.15.1 there exists a monic polynomial P such that $P(A) = 0$. Then it follows that $P(\varphi) = 0$. $\square$

Lemma 7.15.3. *Let R be a ring. Let $I \subset R$ be an ideal. Let M be a finite R-module. Let $\varphi : M \to M$ be an endomorphism such that $\varphi(M) \subset IM$. Then there exists a monic polynomial $P = t^n + a_1 t^{n-1} + \ldots + a_n \in R[T]$ such that $a_j \in I^j$ and $P(\varphi) = 0$ as an endomorphism of M.*

Proof. Choose a surjective R-module map $R^{\oplus n} \to M$, given by $(a_1, \ldots, a_n) \mapsto \sum a_i x_i$ for some generators $x_i \in M$. Choose $(a_{i1}, \ldots, a_{in}) \in I^{\oplus n}$ such that $\varphi(x_i) = \sum a_{ij} x_j$. In other words the diagram

$$\begin{tikzcd} R^{\oplus n} \arrow[r] \arrow[d, "A"'] & M \arrow[d, "\varphi"] \\ I^{\oplus n} \arrow[r] & M \end{tikzcd}$$

is commutative where $A = (a_{ij})$. By Lemma 7.15.1 the polynomial $P(t) = \det(t\mathrm{id}_{n\times n} - A)$ has all the desired properties. $\square$

As a fun example application we prove the following surprising lemma.

Lemma 7.15.4. *Let R be a ring. Let M be a finite R-module. Let $\varphi : M \to M$ be a surjective R-module map. Then φ is an isomorphism.*

First proof. Write $R' = R[x]$ and think of M as a finite R'-module with x acting via φ. Set $I = (x) \subset R'$. By our assumption that φ is surjective we have $IM = M$. Hence we may apply Lemma 7.15.3 to M as an R' module, the ideal I and the endomorphism id_M. We conclude that $(1 + a_1 + \ldots + a_n)\mathrm{id}_M = 0$ with $a_j \in I$. Write $a_j = b_j(x)x$ for some $b_j(x) \in R[x]$. Translating back into φ we see that $\mathrm{id}_M = -(\sum_{j=1,\ldots,n} b_j(\varphi)\varphi$ and hence φ is invertible. □

Second proof. We perform induction on the number of generators of M over R. If M is generated by one element, then $M \cong R/I$ for some ideal $I \subset R$. In this case we may replace R by R/I so that $M = R$. In this case $\varphi : R \to R$ is given by multiplication on M by an element $r \in R$. The surjectivity of φ forces r invertible, since φ must hit 1, which implies that φ is invertible.

Now assume that we have proven the lemma in the case of modules generated by $n - 1$ elements, and are examining a module M generated by n elements. Let A mean the ring $R[t]$, and regard the module M as an A-module by letting t act via φ; since M is finite over R, it is finite over $R[t]$ as well, and since we're trying to prove φ injective, a set-theoretic property, we might as well prove the endomorphism $t : M \to M$ over A injective. We have reduced our problem to the case our endomorphism is multiplication by an element of the ground ring. Let $M' \subset M$ denote the sub-A-module generated by the first $n - 1$ of the generators of M, and consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] \arrow[d, "\varphi|_{M'}"] & M \arrow[r] \arrow[d, "\varphi"] & M/M' \arrow[r] \arrow[d, "\varphi \bmod M'"] & 0 \\
0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M/M' \arrow[r] & 0,
\end{tikzcd}$$

where the restriction of φ to M' and the map induced by φ on the quotient M/M' are well-defined since φ is multiplication by an element in the base, and M' and M/M' are A-modules in their own right. By the case $n = 1$ the the map $M/M' \to M/M'$ is an isomorphism. A diagram chase implies that $\varphi|_{M'}$ is surjective hence by induction $\varphi|_{M'}$ is an isomorphism. This forces the middle column to be an isomorphism by the snake lemma. □

7.16. The spectrum of a ring

We arbitrarily decide that the spectrum of a ring as a topological space is part of the algebra chapter, whereas an affine scheme is part of the chapter on schemes.

Definition 7.16.1. Let R be a ring.

(1) The *spectrum* of R is the set of prime ideals of R. It is usually denoted $Spec(R)$.
(2) Given a subset $T \subset R$ we let $V(T) \subset Spec(R)$ be the set of primes containing T, i.e., $V(T) = \{\mathfrak{p} \in Spec(R) \mid \forall f \in T, f \in \mathfrak{p}\}$.
(3) Given an element $f \in R$ we let $D(f) \subset Spec(R)$ be the set of primes not containing f.

Lemma 7.16.2. *Let R be a ring.*

(1) *The spectrum of a ring R is empty if and only if R is the zero ring.*

(2) *Every nonzero ring has a maximal ideal.*
(3) *Every nonzero ring has a minimal prime ideal.*
(4) *Given an ideal $I \subset R$ and a prime ideal $I \subset \mathfrak{p}$ there exists a prime $I \subset \mathfrak{q} \subset \mathfrak{p}$ such that $\mathfrak{q}$ is minimal over I.*
(5) *If $T \subset R$, and if (T) is the ideal generated by T in R, then $V((T)) = V(T)$.*
(6) *If I is an ideal and $\sqrt{I}$ is its radical, see basic notion (27), then $V(I) = V(\sqrt{I})$.*
(7) *Given an ideal I of R we have $\sqrt{I} = \bigcap_{I \subset \mathfrak{p}} \mathfrak{p}$.*
(8) *If I is an ideal then $V(I) = \emptyset$ if and only if I is the unit ideal.*
(9) *If I, J are ideals of R then $V(I) \cup V(J) = V(I \cap J)$.*
(10) *If $(I_a)_{a\in A}$ is a set of ideals of R then $\cap_{a\in A} V(I_a) = V(\cup_{a\in A} I_a)$.*
(11) *If $f \in R$, then $D(f) \coprod V(f) = Spec(R)$.*
(12) *If $f \in R$ then $D(f) = \emptyset$ if and only if f is nilpotent.*
(13) *If $f = uf'$ for some unit $u \in R$, then $D(f) = D(f')$.*
(14) *If $I \subset R$ is an ideal, and $\mathfrak{p}$ is a prime of R with $\mathfrak{p} \notin V(I)$, then there exists an $f \in R$ such that $\mathfrak{p} \in D(f)$, and $D(f) \cap V(I) = \emptyset$.*
(15) *If $f, g \in R$, then $D(fg) = D(f) \cap D(g)$.*
(16) *If $f_i \in R$ for $i \in I$, then $\bigcup_{i\in I} D(f_i)$ is the complement of $V(\{f_i\}_{i\in I})$ in $Spec(R)$.*
(17) *If $f \in R$ and $D(f) = Spec(R)$, then f is a unit.*

Proof.

(1) This is a direct consequence of (2) and (3).
(2) Let $\mathfrak{A}$ be the set of all proper ideals of R. This set is ordered by inclusion and is non-empty, since $(0) \in \mathfrak{A}$ is a proper ideal. Let A be a totally ordered subset of R. $\bigcup_{I\in A} I$ is in fact an ideal. Since $1 \notin I$ for all $I \in A$, the union does not contain 1 and thus is proper. Hence $\bigcup_{I\in A} I$ is in $\mathfrak{A}$ and is an upper bound for the set A. Thus by Zorn's lemma $\mathfrak{A}$ has a maximal element, which is the sought-after maximal ideal.
(3) Since R is nonzero, it contains a maximal ideal which is a prime ideal. Thus the set $\mathfrak{A}$ of all prime ideals of R is nonempty. $\mathfrak{A}$ is ordered by reverse-inclusion. Let A be a totally ordered subset of $\mathfrak{A}$. It's pretty clear that $J = \bigcap_{I\in A} I$ is in fact an ideal. Not so clear, however, is that it is prime. Let $xy \in J$. Then $xy \in I$ for all $I \in A$. Now let $B = \{I \in A | y \in I\}$. Let $K = \bigcap_{I\in B} I$. Since A is totally ordered, either $K = J$ (and we're done, since then $y \in J$) or $K \supset J$ and for all $I \in A$ such that I is properly contained in K, we have $y \notin I$. But that means that for all those $I, x \in I$, since they are prime. Hence $x \in J$. In either case, J is prime as desired. Hence by Zorn's lemma we get a maximal element which in this case is a minimal prime ideal.
(4) This is the same exact argument as (3) except you only consider prime ideals contained in $\mathfrak{p}$ and containing I.
(5) (T) is the smallest ideal containing T. Hence if $T \subset I$, some ideal, then $(T) \subset I$ as well. Hence if $I \in V(T)$, then $I \in V((T))$ as well. The other inclusion is obvious.
(6) Since $I \subset \sqrt{I}$, $V(\sqrt{I}) \subset V(I)$. Now let $\mathfrak{p} \in V(I)$. Let $x \in \sqrt{I}$. Then $x^n \in I$ for some n. Hence $x^n \in \mathfrak{p}$. But since $\mathfrak{p}$ is prime, a boring induction argument gets you that $x \in \mathfrak{p}$. Hence $\sqrt{I} \subset \mathfrak{p}$ and $\mathfrak{p} \in V(\sqrt{I})$.
(7) Let $f \in R \setminus \sqrt{I}$. Then $f^n \notin I$ for all n. Hence $S = \{1, f, f^2, \dots\}$ is a multiplicative subset, not containing 0. Take a prime ideal $\bar{\mathfrak{p}} \subset S^{-1}R$ containing $S^{-1}I$. Then the pull-back $\mathfrak{p}$ in R of $\bar{\mathfrak{p}}$ is a prime ideal containing I that does not intersect S. This shows that $\bigcap_{I\subset\mathfrak{p}} \mathfrak{p} \subset \sqrt{I}$. Now if $a \in \sqrt{I}$, then $a^n \in I$ for

some n. Hence if $I \subset \mathfrak{p}$, then $a^n \in \mathfrak{p}$. But since $\mathfrak{p}$ is prime, we have $a \in \mathfrak{p}$. Thus the equality is shown.

(8) I is not the unit ideal iff I is contained in some maximal ideal (to see this, apply (2) to the ring R/I) which is therefore prime.

(9) If $\mathfrak{p} \in V(I) \cup V(J)$, then $I \subset \mathfrak{p}$ or $J \subset \mathfrak{p}$ which means that $I \cap J \subset \mathfrak{p}$. Now if $I \cap J \subset \mathfrak{p}$, then $IJ \subset \mathfrak{p}$ and hence either I of J is in $\mathfrak{p}$, since $\mathfrak{p}$ is prime.

(10) $\mathfrak{p} \in \bigcap_{a\in A} V(I_a) \Leftrightarrow I_a \subset \mathfrak{p}, \forall a \in A \Leftrightarrow \mathfrak{p} \in V(\cup_{a\in A} I_a)$

(11) If $\mathfrak{p}$ is a prime ideal and $f \in R$, then either $f \in \mathfrak{p}$ or $f \notin \mathfrak{p}$ (strictly) which is what the disjoint union says.

(12) If $a \in R$ is nilpotent, then $a^n = 0$ for some n. Hence $a^n \in \mathfrak{p}$ for any prime ideal. Thus $a \in \mathfrak{p}$ as can be shown by induction and $D(f) = \emptyset$. Now, as shown in (7), if $a \in R$ is not nilpotent, then there is a prime ideal that does not contain it.

(13) $f \in \mathfrak{p} \Leftrightarrow uf \in \mathfrak{p}$, since u is invertible.

(14) If $\mathfrak{p} \notin V(I)$, then $\exists f \in I \setminus \mathfrak{p}$. Then $f \notin \mathfrak{p}$ so $\mathfrak{p} \in D(f)$. Also if $\mathfrak{q} \in D(f)$, then $f \notin \mathfrak{q}$ and thus I is not contained in $\mathfrak{q}$. Thus $D(f) \cap V(I) = \emptyset$.

(15) If $fg \in \mathfrak{p}$, then $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$. Hence if $f \notin \mathfrak{p}$ and $g \notin \mathfrak{p}$, then $fg \notin \mathfrak{p}$. Since $\mathfrak{p}$ is an ideal, if $fg \notin \mathfrak{p}$, then $f \notin \mathfrak{p}$ and $g \notin \mathfrak{p}$.

(16) $\mathfrak{p} \in \bigcup_{i\in I} D(f_i) \Leftrightarrow \exists i \in I, f_i \notin \mathfrak{p} \Leftrightarrow \mathfrak{p} \in Spec(R) \setminus V(\{f_i\}_{i\in I})$

(17) If $D(f) = Spec(R)$, then $V(f) = \emptyset$ and hence $fR = R$, so f is a unit.

□

The lemma implies that the subsets $V(T)$ from Definition 7.16.1 form the closed subsets of a topology on $Spec(R)$. And it also shows that the sets $D(f)$ are open and form a basis for this topology.

Definition 7.16.3. Let R be a ring. The topology on $Spec(R)$ whose closed sets are the sets $V(T)$ is called the *Zariski* topology. The open subsets $D(f)$ are called the *standard opens* of $Spec(R)$.

It should be clear from context whether we consider $Spec(R)$ just as a set or as a topological space.

Lemma 7.16.4. *Suppose that $\varphi : R \to R'$ is a ring homomorphism. The induced map*

$$Spec(\varphi) : Spec(R') \longrightarrow Spec(R), \quad \mathfrak{p}' \longmapsto \varphi^{-1}(\mathfrak{p}')$$

is continuous for the Zariski topologies. In fact, for any element $f \in R$ we have $Spec(\varphi)^{-1}(D(f)) = D(\varphi(f))$.

Proof. It is basic notion (41) that $\mathfrak{p} := \varphi^{-1}(\mathfrak{p}')$ is indeed a prime ideal of R. The last assertion of the lemma follows directly from the definitions, and implies the first. □

If $\varphi' : R' \to R''$ is a second ring homomorphism then the composition

$$Spec(R') \longrightarrow Spec(R') \longrightarrow Spec(R'')$$

equals $Spec(\varphi' \circ \varphi)$. In other words, $Spec$ is a contravariant functor from the category of rings to the category of topological spaces.

Lemma 7.16.5. *Let R be a ring. Let $S \subset R$ be a multiplicative subset. The map $R \to S^{-1}R$ induces via the functoriality of Spec a homeomorphism*

$$Spec(S^{-1}R) \longrightarrow \{\mathfrak{p} \in Spec(R) \mid S \cap \mathfrak{p} = \emptyset\}$$

where the topology on the right hand side is that induced from the Zariski topology on $Spec(R)$. The inverse map is given by $\mathfrak{p} \mapsto S^{-1}\mathfrak{p}$.

Proof. Denote the right hand side of the arrow of the lemma by D. Choose a prime $\mathfrak{p}' \subset S^{-1}R$ and let $\mathfrak{p}$ the inverse image of $\mathfrak{p}'$ in R. Since $\mathfrak{p}'$ does not contain 1 we see that $\mathfrak{p}$ does not contain any element of S. Hence $\mathfrak{p} \in D$ and we see that the image is contained in D. Let $\mathfrak{p} \in D$. By assumption the image $\overline{S}$ does not contain 0. By basic notion (54) $\overline{S}^{-1}(R/\mathfrak{p})$ is not the zero ring. By basic notion (62) we see $S^{-1}R/S^{-1}\mathfrak{p} = \overline{S}^{-1}(R/\mathfrak{p})$ is a domain, and hence $S^{-1}\mathfrak{p}$ is a prime. The equality of rings also shows that the inverse image of $S^{-1}\mathfrak{p}$ in R is equal to $\mathfrak{p}$, because $R/\mathfrak{p} \to \overline{S}^{-1}(R/\mathfrak{p})$ is injective by basic notion (55). This proves that the map $Spec(S^{-1}R) \to Spec(R)$ is bijective onto D with inverse as given. It is continuous by Lemma 7.16.4. Finally, let $D(g) \subset Spec(S^{-1}R)$ be a standard open. Write $g = h/s$ for some $h \in R$ and $s \in S$. Since g and $h/1$ differ by a unit we have $D(g) = D(h/1)$ in $Spec(S^{-1}R)$. Hence by Lemma 7.16.4 and the bijectivity above the image of $D(g) = D(h/1)$ is $D \cap D(h)$. This proves the map is open as well. □

Lemma 7.16.6. *Let R be a ring. Let $f \in R$. The map $R \to R_f$ induces via the functoriality of Spec a homeomorphism*

$$Spec(R_f) \longrightarrow D(f) \subset Spec(R).$$

The inverse is given by $\mathfrak{p} \mapsto \mathfrak{p} \cdot R_f$.

Proof. This is a special case of Lemma 7.16.5 above. □

It is not the case that every ``affine open'' of a spectrum is a standard open. See Example 7.24.4.

Lemma 7.16.7. *Let R be a ring. Let $I \subset R$ be an ideal. The map $R \to R/I$ induces via the functoriality of Spec a homeorphism*

$$Spec(R/I) \longrightarrow V(I) \subset Spec(R).$$

The inverse is given by $\mathfrak{p} \mapsto \mathfrak{p}/I$.

Proof. It is immediate that the image is contained in $V(I)$. On the other hand, if $\mathfrak{p} \in V(I)$ then $\mathfrak{p} \supset I$ and we may consider the ideal $\mathfrak{p}/I \subset R/I$. Using basic notion (51) we see that $(R/I)/(\mathfrak{p}/I) = R/\mathfrak{p}$ is a domain and hence $\mathfrak{p}/I$ is a prime ideal. From this it is immediately clear that the image of $D(f+I)$ is $D(f) \cap V(I)$, and hence the map is a homeomorphism. □

Remark 7.16.8. A fundamental commutative diagram associated to $\varphi : R \to S$, $\mathfrak{q} \subset S$ and $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$ is the following

$$\begin{tikzcd}
\kappa(\mathfrak{q}) = S_\mathfrak{q}/\mathfrak{q}S_\mathfrak{q} & S_\mathfrak{q} \arrow[l] & S \arrow[l] \arrow[r] & S/\mathfrak{q} \arrow[r] & \kappa(\mathfrak{q}) = \text{f.f.}(S/\mathfrak{q}) \\
\kappa(\mathfrak{p}) \otimes_R S = S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p} \arrow[u] & S_\mathfrak{p} \arrow[l] \arrow[u] & S \arrow[l] \arrow[r] \arrow[u] & S/\mathfrak{p}S \arrow[r] \arrow[u] & (R \setminus \mathfrak{p})^{-1}S/\mathfrak{p}S \arrow[u] \\
\kappa(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \arrow[u] & R_\mathfrak{p} \arrow[l] \arrow[u] & R \arrow[l] \arrow[r] \arrow[u] & R/\mathfrak{p} \arrow[r] \arrow[u] & \kappa(\mathfrak{p}) = \text{f.f.}(R/\mathfrak{p}) \arrow[u]
\end{tikzcd}$$

In this diagram the arrows on the outer left and outer right columns are identical. The horizontal maps induce on the associated spectrums always a homeomorphism onto the image. The lower two rows of the diagram make sense without assuming $\mathfrak{q}$ exists. The lower squares induce fibre squares of topological spaces. This diagram shows that $\mathfrak{p}$ is in the image of the map on Spec if and only if $S \otimes_R \kappa(\mathfrak{p})$ is not the zero ring.

Lemma 7.16.9. *Let $\varphi : R \to S$ be a ring map. Let $\mathfrak{p}$ be a prime of R. The following are equivalent*

1. $\mathfrak{p}$ *is in the image of* $Spec(S) \to Spec(R)$,
2. $S \otimes_R \kappa(\mathfrak{p}) \neq 0$,
3. $S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p} \neq 0$,
4. $(S/\mathfrak{p}S)_\mathfrak{p} \neq 0$, *and*
5. $\mathfrak{p} = \varphi^{-1}(\mathfrak{p}S)$.

Proof. We have already seen the equivalence of the first two in Remark 7.16.8. The others are just reformulations of this. □

Lemma 7.16.10. *Let R be a ring. The space $Spec(R)$ is quasicompact.*

Proof. It suffices to prove that any covering of $Spec(R)$ by standard opens can be refined by a finite covering. Thus suppose that $Spec(R) = \cup D(f_i)$ for a set of elements $\{f_i\}_{i\in I}$ of R. This means that $\cap V(f_i) = \emptyset$. According to Lemma 7.16.2 this means that $V(\{f_i\}) = \emptyset$. According to the same lemma this means that the ideal generated by the f_i is the unit ideal of R. This means that we can write 1 as a *finite* sum: $1 = \sum_{i\in J} r_i f_i$ with $J \subset I$ finite. And then it follows that $Spec(R) = \cup_{i\in J} D(f_i)$. □

Lemma 7.16.11. *Let R be a ring. The topology on $X = Spec(R)$ has the following properties:*

1. *X is quasi-compact,*
2. *X has a basis for the topology consisting of quasi-compact opens, and*
3. *the intersection of any two quasi-compact opens is quasi-compact.*

Proof. The spectrum of a ring is quasi-compact, see Lemma 7.16.10. It has a basis for the topology consisting of the standard opens $D(f) = Spec(R_f)$ (Lemma 7.16.6) which are quasi-compact by the first remark. The intersection of two standard opens is quasi-compact as $D(f) \cap D(g) = D(fg)$. Given any two quasi-compact opens $U, V \subset X$ we may write $U = D(f_1) \cup \ldots \cup D(f_n)$ and $V = D(g_1) \cup \ldots \cup D(g_m)$. Then $U \cap V = \bigcup D(f_i g_j)$ which is quasi-compact. □

7.17. Local rings

Local rings are the bread and butter of algebraic geometry.

Definition 7.17.1. A *local ring* is a ring with exactly one maximal ideal. The maximal ideal is often denoted $\mathfrak{m}_R$ in this case. We often say ``let $(R, \mathfrak{m}, \kappa)$ be a local ring'' to indicate that R is local, $\mathfrak{m}$ is its unique maximal ideal and $\kappa = R/\mathfrak{m}$ is its residue field. A *local homomorphism of local rings* is a ring map $\varphi : R \to S$ such that R and S are local rings and such that $\varphi(\mathfrak{m}_R) \subset \mathfrak{m}_S$. If it is given that R and S are local rings, then the phrase ``*local ring map* $\varphi : R \to S$'' means that φ is a local homomorphism of local rings.

A field is a local ring. Any ring map between fields is a local homomorphism of local rings.

Lemma 7.17.2. *Let R be a ring. The following are equivalent:*

1. *R is a local ring,*
2. *$Spec(R)$ has exactly one closed point,*
3. *R has a maximal ideal $\mathfrak{m}$ and every element of $R \setminus \mathfrak{m}$ is a unit, and*
4. *R is not the zero ring and every $x \in R$ is either invertible or $1 - x$ is invertible.*

Proof. Let R be a ring, and $\mathfrak{m}$ a maximal ideal. If $x \in R \setminus \mathfrak{m}$, and x is not a unit then there is a maximal ideal $\mathfrak{m}'$ containing x. Hence R has at least two maximal ideals. Conversely, if $\mathfrak{m}'$ is another maximal ideal, then choose $x \in \mathfrak{m}'$, $x \notin \mathfrak{m}$. Clearly x is not a unit. This proves the equivalence of (1) and (3). The equivalence (1) and (2) is tautological. If R is local then (4) holds since x is either in $\mathfrak{m}$ or not. If (4) holds, and $\mathfrak{m}$, $\mathfrak{m}'$ are distinct maximal ideals then we may choose $x \in R$ such that $x \bmod \mathfrak{m}' = 0$ and $x \bmod \mathfrak{m} = 1$ by the Chinese remainder theorem (Lemma 7.14.4). This element x is not invertible and neither is $1 - x$ which is a contradiction. Thus (4) and (1) are equivalent. ☐

The localization $R_{\mathfrak{p}}$ of a ring R at a prime $\mathfrak{p}$ is a local ring with maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$. Namely, the quotient $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ is the fraction field of the domain $R/\mathfrak{p}$ and every element of $R_{\mathfrak{p}}$ which is not contained in $\mathfrak{p}R_{\mathfrak{p}}$ is invertible.

Lemma 7.17.3. *Let $\varphi : R \to S$ be a ring map. Assume R and S are local rings. The following are equivalent:*

1. *φ is a local ring map,*
2. *$\varphi(\mathfrak{m}_R) \subset \mathfrak{m}_S$, and*
3. *$\varphi^{-1}(\mathfrak{m}_S) = \mathfrak{m}_R$.*

Proof. Conditions (1) and (2) are equivalent by definition. If (3) holds then (2) holds. Conversely, if (2) holds, then $\varphi^{-1}(\mathfrak{m}_S)$ is a prime ideal containg the maximal ideal $\mathfrak{m}_R$, hence $\varphi^{-1}(\mathfrak{m}_S) = \mathfrak{m}_R$. ☐

Let $\varphi : R \to S$ be a ring map. Let $\mathfrak{q} \subset S$ be a prime and set $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$. Then the induced ring map $R_{\mathfrak{p}} \to S_{\mathfrak{q}}$ is a local ring map.

7.18. Open and closed subsets of spectra

It turns out that open and closed subsets of a spectrum correspond to idempotents of the ring.

Lemma 7.18.1. *Let R be a ring. Let $e \in R$ be an idempotent. In this case*
$$Spec(R) = D(e) \coprod D(1-e).$$

Proof. Note that an idempotent e of a domain is either 1 or 0. Hence we see that
$$\begin{aligned} D(e) &= \{\mathfrak{p} \in Spec(R) \mid e \notin \mathfrak{p}\} \\ &= \{\mathfrak{p} \in Spec(R) \mid e \neq 0 \text{ in } \kappa(\mathfrak{p})\} \\ &= \{\mathfrak{p} \in Spec(R) \mid e = 1 \text{ in } \kappa(\mathfrak{p})\} \end{aligned}$$
Similarly we have
$$\begin{aligned} D(1-e) &= \{\mathfrak{p} \in Spec(R) \mid 1 - e \notin \mathfrak{p}\} \\ &= \{\mathfrak{p} \in Spec(R) \mid e \neq 1 \text{ in } \kappa(\mathfrak{p})\} \\ &= \{\mathfrak{p} \in Spec(R) \mid e = 0 \text{ in } \kappa(\mathfrak{p})\} \end{aligned}$$
Since the image of e in any residue field is either 1 or 0 we deduce that $D(e)$ and $D(1-e)$ cover all of $Spec(R)$. ☐

Lemma 7.18.2. *Let R_1 and R_2 be rings. Let $R = R_1 \times R_2$. The maps $R \to R_1$, $(x, y) \mapsto x$ and $R \to R_2$, $(x, y) \mapsto y$ induce continuous maps $Spec(R_1) \to Spec(R)$ and $Spec(R_2) \to Spec(R)$. The induced map*
$$Spec(R_1) \coprod Spec(R_2) \longrightarrow Spec(R)$$

is a homeomorphism. In other words, the spectrum of $R = R_1 \times R_2$ is the disjoint union of the spectrum of R_1 and the spectrum of R_2.

Proof. Write $1 = e_1 + e_2$ with $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Note that e_1 and $e_2 = 1 - e_1$ are idempotents. We leave it to the reader to show that $R_1 = R_{e_1}$ is the localization of R at e_1. Similarly for e_2. Thus the statement of the lemma follows from Lemma 7.18.1 combined with Lemma 7.16.6. □

We reprove the following lemma later after introducing a glueing lemma for functions. See Section 7.20.

Lemma 7.18.3. *Let R be a ring. For each $U \subset Spec(R)$ which is open and closed there exists a unique idempotent $e \in R$ such that $U = D(e)$. This induces a 1-1 correspondence between open and closed subsets $U \subset Spec(R)$ and idempotents $e \in R$.*

First proof of Lemma 7.18.3. Let $U \subset Spec(R)$ be open and closed. Since U is closed it is quasi-compact by Lemma 7.16.10, and similarly for its complement. Write $U = \bigcup_{i=1}^n D(f_i)$ as a finite union of standard opens. Similarly, write $Spec(R) \setminus U = \bigcup_{j=1}^m D(g_j)$ as a finite union of standard opens. Since $\emptyset = D(f_i) \cap D(g_j) = D(f_i g_j)$ we see that $f_i g_j$ is nilpotent by Lemma 7.16.2. Let $I = (f_1, \ldots, f_n) \subset R$ and let $J = (g_1, \ldots, g_m) \subset R$. Note that $V(J)$ equals U, that $V(I)$ equals the complement of U, so $Spec(R) = V(I) \coprod V(J)$. By the remark on nilpotency above, we see that $(IJ)^N = (0)$ for some sufficiently large integer N. Since $\bigcup D(f_i) \cup \bigcup D(g_j) = Spec(R)$ we see that $I + J = R$, see Lemma 7.16.2. By raising this equation to the $2N$th power we conclude that $I^N + J^N = R$. Write $1 = x + y$ with $x \in I^N$ and $y \in J^N$. Then $1 = (x+y)^2 = x^2 + y^2$ because $I^N J^N = (0)$. Then $z = x - x^2 \in I^N \cap J^N$. Thus $zx = 0$ and $z^2 = 0$. Hence $(x - z) - (x - z)^2 = x - x^2 - z = 0$. In other words, $e = x - z$ is an idempotent contained in $I^N \subset I$, and the idempotent $e' = 1 - e = y + z$ is contained in $J^N \subset J$. This shows that the idempotent e maps to 1 in every residue field $\kappa(\mathfrak{p})$ for $\mathfrak{p} \in V(J)$ and that e maps to 0 in $\kappa(\mathfrak{p})$ for every $\mathfrak{p} \in V(I)$.

To see uniqueness suppose that e_1, e_2 are distinct idempotents in R. We have to show there exists a prime $\mathfrak{p}$ such that $e_1 \in \mathfrak{p}$ and $e_2 \notin \mathfrak{p}$, or conversely. Write $e_i' = 1 - e_i$. If $e_1 \neq e_2$, then $0 \neq e_1 - e_2 = e_1(e_2 + e_2') - (e_1 + e_1')e_2 = e_1 e_2' - e_1' e_2$. Hence either the idempotent $e_1 e_2' \neq 0$ or $e_1' e_2 \neq 0$. An idempotent is not nilpotent, and hence we find a prime $\mathfrak{p}$ such that either $e_1 e_2' \notin \mathfrak{p}$ or $e_1' e_2 \notin \mathfrak{p}$, by Lemma 7.16.2. It is easy to see this gives the desired prime. □

Lemma 7.18.4. *Let R be a ring. Then $Spec(R)$ is connected if and only if R has no non-trivial idempotents.*

Proof. Obvious from Lemma 7.18.3 above. □

Lemma 7.18.5. *Let R be a ring. Let I be a finitely generated ideal. Assume that $I = I^2$. Then $V(I)$ is open and closed in $Spec(R)$, and $R/I \cong R_e$ for some idempotent $e \in R$.*

Proof. By Nakayama's Lemma 7.14.5 there exists an element $f = 1 + i$, $i \in I$ in R such that $fI = 0$. It follows that $V(I) = D(f)$ by a simple argument. Also, $0 = fi = i + i^2$, and hence $f^2 = 1 + i + i + i^2 = 1 + i = f$, so f is an idempotent. Consider the canonical map $R \to R_f$. It is surjective since $x/f^n = x/f = xf/f^2 = xf/f = x/1$ in R_f. Any element of I is in the kernel since $fI = 0$. If $x \mapsto 0$ in R_f, then $f^n x = 0$ for some $n > 0$ and hence $(1 + i)x = 0$ hence $x \in I$. □

7.19. Connected components of spectra

Connected components of spectra are not as easy to understand as one may think at first. This is because we are used to the topology of locally connected spaces, but the spectrum of a ring is in general not locally connected.

Lemma 7.19.1. *Let R be a ring. Let $T \subset Spec(R)$ be a subset of the spetrum. The following are equivalent*

1. *T is closed and is a union of connected components of $Spec(R)$,*
2. *T is an intersection of open and closed subsets of $Spec(R)$, and*
3. *$T = V(I)$ where $I \subset R$ is an ideal generated by idempotents.*

Moreover, the ideal in (3) if it exists is unique.

Proof. By Lemma 7.16.11 and Topology, Lemma 5.9.8 we see that (1) and (2) are equivalent. Assume (2) and write $T = \bigcap U_\alpha$ with $U_\alpha \subset Spec(R)$ open and closed. Then $U_\alpha = D(e_\alpha)$ for some idempotent $e_\alpha \in A$ by Lemma 7.18.3. Then setting $I = (1 - e_\alpha)$ we see that $T = V(I)$, i.e., (3) holds. Finally, assume (3). Write $T = V(I)$ and $I = (e_\alpha)$ for some collection of idempotents e_α. Then it is clear that $T = \bigcap V(e_\alpha) = \bigcap D(1 - e_\alpha)$.

Suppose that I is an ideal generated by idempotents. Let $e \in R$ be an idempotent such that $V(I) \subset V(e)$. Then by Lemma 7.16.2 we see that $e^n \in I$ for some $n \geq 1$. As e is an idempotent this means that $e \in I$. Hence we see that I is generated by exactly those idempotents e such that $T \subset V(e)$. In other words, the ideal I is completely determined by the closed subset T which proves uniqueness. ☐

Lemma 7.19.2. *Let R be a ring. A connected component of $Spec(R)$ is of the form $V(I)$, where I is an ideal generated by idempotents such that every idempotent of R either maps to 0 or 1 in R/I.*

Proof. Let $\mathfrak{p}$ be a prime of R. By Lemma 7.16.11 we have see that the hypotheses of Topology, Lemma 5.9.7 are satisfied for the topological space $Spec(R)$. Hence the connected component of $\mathfrak{p}$ in $Spec(R)$ is the intersection of open and closed subsets of $Spec(R)$ containing $\mathfrak{p}$. Hence it equals $V(I)$ where I is generated by the idempotents $e \in R$ such that e maps to 0 in $\kappa(\mathfrak{p})$, see Lemma 7.18.3. Any idempotent e which is not in this collection clearly maps to 1 in R/I. ☐

7.20. Glueing functions

In this section we show that given an open covering

$$Spec(R) = \bigcup\nolimits_{i=1}^{n} D(f_i)$$

by standard opens, and given an element $h_i \in R_{f_i}$ for each i such that $h_i = h_j$ as elements of $R_{f_i f_j}$ then there exists a unique $h \in R$ such that the image of h in R_{f_i} is h_i. This result can be interpreted in two ways:

1. The rule $D(f) \mapsto R_f$ is a sheaf of rings on the standard opens, see Sheaves, Section 6.30.
2. If we think of elements of R_f as the ``algebraic'' or ``regular'' functions on $D(f)$, then these glue as would continuous, resp. differentiable functions on a topological, resp. differentiable manifold.

At the end of this section we use this result to reprove the lemma describing open and closed subsets in terms of idempotents.

Lemma 7.20.1. *Let R be a ring, and let $f_1, f_2, \dots f_n \in R$ generate the unit ideal in R. Then the following sequence is exact:*

$$0 \longrightarrow R \longrightarrow \bigoplus\nolimits_i R_{f_i} \longrightarrow \bigoplus\nolimits_{i,j} R_{f_i f_j}$$

where the maps $\alpha : R \longrightarrow \bigoplus_i R_{f_i}$ and $\beta : \bigoplus_i R_{f_i} \longrightarrow \bigoplus_{i,j} R_{f_i f_j}$ are defined as

$$\alpha(x) = \left(\frac{x}{1}, \dots, \frac{x}{1}\right) \text{ and } \beta\left(\frac{x_1}{f_1}, \dots, \frac{x_n}{f_n}\right) = \left(\frac{x_i}{f_i} - \frac{x_j}{f_j} \text{ in } R_{f_i f_j}\right).$$

Proof. We first show that α is injective, and then that the image of α equals the kernel of β. Assume there exists $x \in R$ such that $\alpha(x) = (0, \dots, 0)$. Then $\frac{x}{1} = 0$ in R_{f_i} for all i. This means, for all i, there exists a number n_i such that

$$f_i^{n_i} x = 0$$

Since the f_i generate R, we can pick a_i so

$$1 = \sum\nolimits_{i=1}^n a_i f_i$$

Then for all $M \geq \sum n_i$, we have

$$1^M = \left(\sum a_i f_i\right)^M,$$

where each term has a factor of at least $f_i^{n_i}$ for some i. Therefore,

$$x = 1x = 1^M x = \left(\sum a_i f_i\right)^M x = 0.$$

Thus, if $\alpha(x) = 0$, $x = 0$ and α is injective. We check that the image of α equals the kernel of β. First, note that for $x \in R$,

$$\beta(\alpha(x)) = \beta\left(\frac{x}{1}, \dots, \frac{x}{1}\right) = \left(\frac{x}{1} - \frac{x}{1} \text{ in } R_{f_i f_j}\right) = 0.$$

Therefore, the image of α is in the kernel of β, and it remains only to verify that if

$$\beta\left(\frac{x_1}{f_1}, \dots, \frac{x_n}{f_n}\right) = 0,$$

then there exists $x \in R$ so that for all i,

$$\frac{x}{1} = \frac{x_i}{f_i}$$

Assume we have $x_1, \dots, x_n$ such that

$$\beta\left(\frac{x_1}{f_1}, \dots, \frac{x_n}{f_n}\right) = 0.$$

Then, for all pairs i, j, there exists an n_{ij} such that

$$f_i^{n_{ij}} f_j^{n_{ij}} (f_j x_i - f_i x_j) = 0$$

Choosing N so $N \geq n_{ij}$ for all i, j, we see that

$$f_i^N f_j^N (f_j x_i - f_i x_j) = 0$$

Define elements $\widetilde{x_i}$ and $\widetilde{f_i}$ as follows:

$$\widetilde{f_i} = f_i^{N+1}, \qquad \widetilde{x_i} = f_i^N x_i.$$

Notice that

$$\frac{\widetilde{x}_i}{\widetilde{f}_i} = \frac{x_i}{f_i}.$$

Also, we can use this to rewrite the above equation to get the following equality, for all i, j,

$$\widetilde{f}_j \widetilde{x}_i = \widetilde{f}_i \widetilde{x}_j.$$

Since $f_1, \ldots, f_n$ generate R, we clearly have that $\widetilde{f}_1, \ldots, \widetilde{f}_n$ also generate R. Therefore, there exist $a_1, \ldots, a_n$ in R so that

$$1 = \sum\nolimits_{i=1}^n a_i \widetilde{f}_i$$

Therefore, we finally conclude that for all i,

$$\frac{x_i}{f_i} = \frac{\widetilde{x}_i}{\widetilde{f}_i} = \sum\nolimits_{j=1}^n \frac{a_j \widetilde{f}_j \widetilde{x}_i}{\widetilde{f}_i} = \sum\nolimits_{j=1}^n \frac{a_j \widetilde{f}_i \widetilde{x}_j}{\widetilde{f}_i} = \frac{\sum_{j=1}^n a_j \widetilde{x}_j}{1}.$$

Thus, we have

$$\alpha\left(\sum\nolimits_{j=1}^n a_j \widetilde{x}_j\right) = \left(\frac{x_1}{f_1}, \ldots, \frac{x_n}{f_n}\right),$$

as required. There the sequence is exact. □

Lemma 7.20.2. *Let R be a ring. Let $f_1, \ldots, f_n$ be elements of R generating the unit ideal. Let M be an R-module. The sequence*

$$0 \to M \xrightarrow{\alpha} \bigoplus\nolimits_{i=1}^n M_{f_i} \xrightarrow{\beta} \bigoplus\nolimits_{i,j=1}^n M_{f_i f_j}$$

is exact, where $\alpha(m) = (m/1, \ldots, m/1)$ and $\beta(m_1/f_1^{e_1}, \ldots, m_n/f_n^{e_n}) = (m_i/f_i^{e_i} - m_j/f_j^{e_j})_{(i,j)}$.

Proof. The same as the proof of Lemma 7.20.1. □

Second proof of Lemma 7.18.3. Having assured ourselves (Lemma 7.20.1) that for generators $f_1, \ldots, f_n$ for the unit ideal of a ring R the sequence

$$0 \to R \to \bigoplus\nolimits_{i=1}^n R_{f_i} \to \bigoplus\nolimits_{i,j} R_{f_i f_j}$$

is exact, we now provide an alternate proof of the surjectivity of the map from idempotents e of R to open and closed subsets of $Spec(R)$ presented in Lemma 7.18.3. Let $U \subset Spec(R)$ be open and closed, and W be its complement. We can write U and V as unions of standard opens such that $U = \bigcup_{i=1}^n D(f_i)$ and $W = \bigcup_{j=1}^m D(g_j)$. Since $Spec(R) = \bigcup D(f_i) \cup \bigcup D(g_j)$, we observe that the collection $\{f_i; g_j\}$ must generate the unit ideal in R by Lemma 7.16.2. So the following sequence is exact.

(7.20.2.1)

$$0 \to R \xrightarrow{\alpha} \bigoplus\nolimits_{i=1}^n R_{f_i} \oplus \bigoplus\nolimits_{j=1}^m R_{g_j} \to \bigoplus\nolimits_{i_1,i_2} R_{f_{i_1} f_{i_2}} \oplus \bigoplus\nolimits_{i,j} R_{f_i g_j} \oplus \bigoplus\nolimits_{j_1,j_2} R_{g_{j_1} g_{j_2}}$$

However, notice that for any pair i, j, $D(f_i) \cap D(g_j) = \emptyset$ since $D(f_i) \subset U$ and $D(g_j) \subset W$). From part (15) of Lemma 7.16.2 we recall that $D(f_i g_j) = D(f_i) \cap D(g_j) = \emptyset$. Therefore by Lemma 7.16.5 $Spec(R_{f_i g_j}) = D(f_i g_j) = \emptyset$, implying that $R_{f_i g_j}$ is the zero ring for each pair i, j by part (3) of Lemma 7.16.2. Consider the element $(1, \ldots, 1, 0, \ldots, 0) \in \bigoplus_{i=1}^n R_{f_i} \oplus \bigoplus_{j=1}^m R_{g_j}$ whose coordinates are 1 in each R_{f_i} and 0 in each R_{g_j}. This is sent to 0 under the map

$$\beta : \bigoplus\nolimits_{i=1}^n R_{f_i} \oplus \bigoplus\nolimits_{j=1}^m R_{g_j} \to \bigoplus\nolimits_{i_1,i_2} R_{f_{i_1} f_{i_2}} \oplus \bigoplus\nolimits_{j_1,j_2} R_{g_{j_1} g_{j_2}}$$

so by the exactness of the sequence (7.20.2.1), there must be some element of R whose image under α is $(1, \ldots, 1, 0, \ldots, 0)$. Call it e. We see that $\alpha(e^2) = \alpha(e)^2 = (1, \ldots, 1, 0, \ldots, 0) = \alpha(e)$. Since α is injective, $e = e^2$ in R and e is an idempotent of R. We claim that $U = D(e)$. Notice that for arbitrary j, the map $R \to R_{g_j}$ maps e to 0. Therefore there must be some positive integer k_j such that $g_j^{k_j}(e - 0) = 0$ in R. Multiplying by e as necessary, we see that $(g_j e)^{k_j} = 0$, so $g_j e$ is nilpotent in R. By Lemma 7.16.2 $D(g_j) \cap D(e) = D(g_j e) = \emptyset$. So since $V = \bigcup D(g_j)$, $D(e) \cap V = \emptyset$ and $D(e) \subset U$. Furthermore, for arbitrary i, the map $R \to R_{f_i}$ maps e to 1, so there must be some l_i such that $f_i^{l_i}(e - 1) = 0$ in R. Hence $f_i^{l_i} e = f_i^{l_i}$. Suppose $\mathfrak{p} \in Spec(R)$ contains e, then $\mathfrak{p}$ contains $f_i^{l_i} e = f_i^{l_i}$, and since $\mathfrak{p}$ is prime, $f_i \in \mathfrak{p}$. So $V(e) \subset V(f_i)$, implying that $D(f_i) \subset D(e)$. Therefore $U = \bigcup D(f_i) \subset D(e)$, and $U = D(e)$. Therefore any open and closed subset of $Spec(R)$ is the standard open of an idempotent as desired. $\square$

The following we have already seen above, but we state it explicitly here for convenience.

Lemma 7.20.3. *Let R be a ring. If $Spec(R) = U \amalg V$ with both U and V open then $R \cong R_1 \times R_2$ with $U \cong Spec(R_1)$ and $V \cong Spec(R_2)$ via the maps in Lemma 7.18.2. Moreover, both R_1 and R_2 are localizations as well as quotients of the ring R.*

Proof. By Lemma 7.18.3 we have $U = D(e)$ and $V = D(1 - e)$ for some idempotent e. By Lemma 7.20.1 we see that $R \cong R_e \times R_{1-e}$ (since clearly $R_{e(1-e)} = 0$ so the glueing condition is trivial; of course it is trivial to prove the product decomposition directly in this case). The lemma follows. $\square$

Lemma 7.20.4. *Let R be a ring. Let $f_1, \ldots, f_n \in R$. Let M be an R-module. Then $M \to \bigoplus M_{f_i}$ is injective if and only if*

$$M \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} M, \quad m \longmapsto (f_1 m, \ldots, f_n m)$$

is injective.

Proof. The map $M \to \bigoplus M_{f_i}$ is injective if and only if for all $m \in M$ and $e_1, \ldots, e_n \geq 1$ such that $f_i^{e_i} m = 0$, $i = 1, \ldots, n$ we have $m = 0$. This clearly implies the displayed map is injective. Conversely, suppose the displayed map is injective and $m \in M$ and $e_1, \ldots, e_n \geq 1$ are such that $f_i^{e_i} m = 0$, $i = 1, \ldots, n$. If $e_i = 1$ for all i, then we immediately conclude that $m = 0$ from the injectivity of the displayed map. Next, we prove this holds for any such data by induction on $e = \sum e_i$. The base case is $e = n$, and we have just dealt with this. If some $e_i > 1$, then set $m' = f_i m$. By induction we see that $m' = 0$. Hence we see that $f_i m = 0$, i.e., we may take $e_i = 1$ which decreases e and we win. $\square$

7.21. More glueing results

In this section we put a number of standard results of the form: if something is true for all members of a standard open covering then it is true. In fact, it often suffices to check things on the level of local rings as in the following lemma.

Lemma 7.21.1. *Let R be a ring.*

1. *For an element x of an R-module M the following are equivalent*
   (a) *$x = 0$,*
   (b) *x maps to zero in $M_{\mathfrak{p}}$ for all $\mathfrak{p} \in Spec(R)$,*
   (c) *x maps to zero in $M_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ of R.*

In other words, the map $M \to \prod_{\mathfrak{m}} M_{\mathfrak{m}}$ is injective.

(2) *Given an R-module M the following are equivalent*
   (a) *M is zero,*
   (b) *$M_{\mathfrak{p}}$ is zero for all $\mathfrak{p} \in Spec(R)$,*
   (c) *$M_{\mathfrak{m}}$ is zero for all maximal ideals $\mathfrak{m}$ of R.*

(3) *Given a complex $M_1 \to M_2 \to M_3$ of R-modules the following are equivalent*
   (a) *$M_1 \to M_2 \to M_3$ is exact,*
   (b) *for every prime $\mathfrak{p}$ of R the localization $M_{1,\mathfrak{p}} \to M_{2,\mathfrak{p}} \to M_{3,\mathfrak{p}}$ is exact,*
   (c) *for every maximal ideal $\mathfrak{m}$ of R the localization $M_{1,\mathfrak{m}} \to M_{2,\mathfrak{m}} \to M_{3,\mathfrak{m}}$ is exact.*

(4) *Given a map $f : M \to M'$ of R-modules the following are equivalent*
   (a) *f is injective,*
   (b) *$f_{\mathfrak{p}} : M_{\mathfrak{p}} \to M'_{\mathfrak{p}}$ is injective for all primes $\mathfrak{p}$ of R,*
   (c) *$f_{\mathfrak{m}} : M_{\mathfrak{m}} \to M'_{\mathfrak{m}}$ is injective for all maximal ideals $\mathfrak{m}$ of R.*

(5) *Given a map $f : M \to M'$ of R-modules the following are equivalent*
   (a) *f is surjective,*
   (b) *$f_{\mathfrak{p}} : M_{\mathfrak{p}} \to M'_{\mathfrak{p}}$ is surjective for all primes $\mathfrak{p}$ of R,*
   (c) *$f_{\mathfrak{m}} : M_{\mathfrak{m}} \to M'_{\mathfrak{m}}$ is surjective for all maximal ideals $\mathfrak{m}$ of R.*

(6) *Given a map $f : M \to M'$ of R-modules the following are equivalent*
   (a) *f is bijective,*
   (b) *$f_{\mathfrak{p}} : M_{\mathfrak{p}} \to M'_{\mathfrak{p}}$ is bijective for all primes $\mathfrak{p}$ of R,*
   (c) *$f_{\mathfrak{m}} : M_{\mathfrak{m}} \to M'_{\mathfrak{m}}$ is bijective for all maximal ideals $\mathfrak{m}$ of R.*

Proof. Let $x \in M$ as in (1). Let $I = \{f \in R \mid fx = 0\}$. It is easy to see that I is an ideal (it is the annihilator of x). Condition (1)(c) means that for all maximal ideals $\mathfrak{m}$ there exists an $f \in R \setminus \mathfrak{m}$ such that $fx = 0$. In other words, $V(I)$ does not contain a closed point. By Lemma 7.16.2 we see I is the unit ideal. Hence x is zero, i.e., (1)(a) holds. This proves (1).

Part (2) follows by applying (1) to all elements of M simultaneously.

Proof of (3). Let H be the homology of the sequence, i.e., $H = \text{Ker}(M_2 \to M_3)/\text{Im}(M_1 \to M_2)$. By Proposition 7.9.12 we have that $H_{\mathfrak{p}}$ is the homology of the sequence $M_{1,\mathfrak{p}} \to M_{2,\mathfrak{p}} \to M_{3,\mathfrak{p}}$. Hence (3) is a consequence of (2).

Parts (4) and (5) are special cases of (3). Part (6) follows formally on combining (4) and (5). ☐

Lemma 7.21.2. *Let R be a ring. Let M be an R-module. Let S be an R-algebra. Suppose that $f_1, \ldots, f_n$ is a finite list of elements of R such that $\bigcup D(f_i) = Spec(R)$ in other words $(f_1, \ldots, f_n) = R$.*

(1) *If each $M_{f_i} = 0$ then $M = 0$.*
(2) *If each M_{f_i} is a finite R_{f_i}-module, then M is a finite R-module.*
(3) *If each M_{f_i} is a finitely presented R_{f_i}-module, then M is a finitely presented R-module.*
(4) *Let $M \to N$ be a map of R-modules. If $M_{f_i} \to N_{f_i}$ is an isomorphism for each i then $M \to N$ is an isomorphism.*
(5) *Let $0 \to M'' \to M \to M' \to 0$ be a complex of R-module. If $0 \to M''_{f_i} \to M_{f_i} \to M'_{f_i} \to 0$ is exact for each i, then $0 \to M'' \to M \to M' \to 0$ is exact.*
(6) *If each R_{f_i} is Noetherian, then R is Noetherian.*
(7) *If each S_{f_i} is a finite type R-algebra, so is S.*

(8) *If each S_{f_i} is of finite presentation over R, so is S.*

Proof. We prove each of the parts in turn.

(1) By Proposition 7.9.10 this implies $M_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in Spec(R)$, so we conclude by Lemma 7.21.1.
(2) For each i take a finite generating set X_i of M_{f_i}. Without loss of generality, we may assume that the elements of X_i are in the image of the localisation map $M \to M_{f_i}$, so we take a finite set Y_i of preimages of the elements of X_i in M. Let Y be the union of these sets. This is still a finite set. Consider the obvious R-linear map $R^Y \to M$ sending the basis element e_y to y. By assumption this map is surjective after localizing at an arbitrary prime ideal $\mathfrak{p}$ of R, so it surjective by Lemma 7.21.1 and M is finitely generated.
(3) By (2) we have a short exact sequence

$$0 \to K \to R^n \to M \to 0$$

Since finite presentation does not depend on the chosen presentation (Lemma 7.5.3) and localisation is an exact functor K_{f_i} is finitely generated for all $1 \leq i \leq n$. By 2. this implies that K is a finitely generated R-module and therefore M is finitely presented.
(4) By Proposition 7.9.10 the assumption implies that the induced morphism on localisations at all prime ideals is an isomorphism, so we conclude by Lemma 7.21.1.
(5) By Proposition 7.9.10 the assumption implies that the induced sequence of localisations at all prime ideals is short exact, so we conclude by Lemma 7.21.1.
(6) We will show that every ideal of R has a finite generating set: For this, let $I \subseteq R$ be an arbitrary ideal. By Proposition 7.9.12 each $I_{f_i} \subseteq R_{f_i}$ is an ideal. These are all finitely generated by assumption, so we conclude by (2).
(7) For each i take a finite generating set X_i of S_{f_i}. Without loss of generality, we may assume that the elements of X_i are in the image of the localisation map $S \to S_{f_i}$, so we take a finite set Y_i of preimages of the elements of X_i in S. Let Y be the union of these sets. This is still a finite set. Consider the algebra homomorphism $R[X_y]_{y \in Y} \to S$ induced by Y. Since it is an algebra homomorphism, the image T is an R-submodule of the R-module S, so we can consider the quotient module S/T. By assumption, this is zero if we localise at the f_i, so it is zero by item 1. and therefore S is an R-algebra of finite type.
(8) By the previous item, there exists a surjective R-algebra homomorphism $R[X_1, ..., X_n] \to S$. Let K be the kernel of this map. This is an ideal in $R[X_1, ..X_n]$, finitely generated in each localisation at f_i. Since the f_i generate the unit ideal in R, they also generate the unit ideal in $R[X_1, ..., X_n]$, so an application of (2) finishes the proof.

□

Lemma 7.21.3. *Let $R \to S$ be a ring map. Suppose that $g_1, \ldots, g_m$ is a finite list of elements of S such that $\bigcup D(g_j) = Spec(S)$ in other words $(g_1, \ldots, g_m) = S$.*

(1) *If each S_{g_i} is of finite type over R, then S is of finite type over R.*
(2) *If each S_{g_i} is of finite presentation over R, then S is of finite presentation over R.*

Proof. Omitted. □

The following lemma is better stated and proved in the more general context of flat descent. However, it makes sense to state it here since it fits well with the above.

Lemma 7.21.4. *Let R be a ring. Let $f_1, \ldots, f_n \in R$ be elements which generate the unit ideal in R. Suppose we are given the following data:*

(1) *For each i an R_{f_i}-module M_i.*
(2) *For each pair i, j an $R_{f_i f_j}$-module isomorphism $\psi_{ij} : (M_i)_{f_j} \to (M_j)_{f_i}$.*

which satisfy the ``cocycle condition'' that all the diagrams

$$\begin{tikzcd} (M_i)_{f_j f_k} \arrow[rr, "\psi_{ik}"] \arrow[dr, "\psi_{ij}"'] & & (M_k)_{f_i f_j} \\ & (M_j)_{f_i f_k} \arrow[ur, "\psi_{jk}"'] & \end{tikzcd}$$

commute (for all triples i, j, k). Given this data define

$$M = Ker\left(\bigoplus\nolimits_{1 \leq i \leq n} M_i \longrightarrow \bigoplus\nolimits_{1 \leq i, j \leq n} (M_i)_{f_i}\right)$$

where $(m_1, \ldots, m_n)$ maps to the element whose (i, j)th entry is $m_i/1 - \psi_{ji}(m_j/1)$. Then the natural map $M \to M_i$ identifies M_i with M_{f_i}. Moreover $\psi_{ij}(m/1) = m/1$ for all $m \in M$ (with obvious notation).

Proof. Omitted. $\square$

7.22. Total rings of fractions

We can apply the glueing results above to prove something about total rings of fractions $Q(R)$. Namely, Lemma 7.22.2 below.

Lemma 7.22.1. *Let R be a ring. Let $S \subset R$ be a multiplicative subset consisting of nonzero divisors. Then $Q(R) \cong Q(S^{-1}R)$. In particular $Q(R) \cong Q(Q(R))$.*

Proof. If $x \in S^{-1}R$ is a nonzero divisor, and $x = r/f$ for some $r \in R$, $f \in S$, then r is a nonzero divisor in R. Whence the lemma. $\square$

Lemma 7.22.2. *Let R be a ring. Assume that R has finitely many minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$, and that $\mathfrak{q}_1 \cup \ldots \cup \mathfrak{q}_t$ is the set of zero divisors of R. Then the total ring of fractions $Q(R)$ (Example 7.9.8) is equal to $R_{\mathfrak{q}_1} \times \ldots \times R_{\mathfrak{q}_t}$.*

Proof. There are natural maps $Q(R) \to R_{\mathfrak{q}_i}$ since any nonzero divisor is contained in $R \setminus \mathfrak{q}_i$. Hence a natural map $Q(R) \to R_{\mathfrak{q}_1} \times \ldots \times R_{\mathfrak{q}_t}$. For any nonminimal prime $\mathfrak{p} \subset R$ we see that $\mathfrak{p} \not\subset \mathfrak{q}_1 \cup \ldots \cup \mathfrak{q}_t$ by Lemma 7.14.3. Hence $Spec(Q(R)) = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_t\}$ (as subsets of $Spec(R)$, see Lemma 7.16.5). Therefore $Spec(Q(R))$ is a finite discrete set and it follows that $Q(R) = A_1 \times \ldots \times A_t$ with $Spec(A_i) = \{q_i\}$, see Lemma 7.20.3. Moreover A_i is a local ring, which is a localization of R. Hence $A_i \cong R_{\mathfrak{q}_i}$. $\square$

7.23. Irreducible components of spectra

We show that irreducible components of the spectrum of a ring correspond to the minimal primes in the ring.

Lemma 7.23.1. *Let R be a ring.*

(1) *For a prime $\mathfrak{p} \subset R$ the closure of $\{\mathfrak{p}\}$ in the Zariski topology is $V(\mathfrak{p})$. In a formula $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$.*

(2) *The irreducible closed subsets of* $Spec(R)$ *are exactly the subsets* $V(\mathfrak{p})$*, with* $\mathfrak{p} \subset R$ *a prime.*
(3) *The irreducible components (see Topology, Definition 5.5.1) of* $Spec(R)$ *are exactly the subsets* $V(\mathfrak{p})$*, with* $\mathfrak{p} \subset R$ *a minimal prime.*

Proof. Note that if $\mathfrak{p} \in V(I)$, then $I \subset \mathfrak{p}$. Hence, clearly $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$. In particular $V(\mathfrak{p})$ is the closure of a singleton and hence irreducible. The second assertion implies the third. To show the second, let $V(I) \subset Spec(R)$ with I a radical ideal. If I is not prime, then choose $a, b \in R$, $a, b \notin I$ with $ab \in I$. In this case $V(I, a) \cup V(I, b) = V(I)$, but neither $V(I, b) = V(I)$ nor $V(I, a) = V(I)$, by Lemma 7.16.2. Hence $V(I)$ is not irreducible. □

In other words, this lemma shows that every irreducible closed subset of $Spec(R)$ is of the form $V(\mathfrak{p})$ for some prime $\mathfrak{p}$. Since $V(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$ we see that each irreducible closed subset has a unique generic point, see Topology, Definition 5.5.4. In particular, $Spec(R)$ is a sober topological space.

Lemma 7.23.2. *Let* R *be a ring. Let* $\mathfrak{p} \subset R$ *be a prime.*
(1) *the set of irreducible closed subsets of* $Spec(R)$ *passing through* $\mathfrak{p}$ *is in one-to-one correspondence with primes* $\mathfrak{q} \subset R_{\mathfrak{p}}$.
(2) *The set of irreducible components of* $Spec(R)$ *passing through* $\mathfrak{p}$ *is in one-to-one correspondence with minimal primes* $\mathfrak{q} \subset R_{\mathfrak{p}}$.

Proof. Omitted. □

Lemma 7.23.3. *Let* $\mathfrak{p}$ *be a minimal prime of a ring* R*. Every element of the maximal ideal of* $R_{\mathfrak{p}}$ *is nilpotent. If* R *is reduced then* $R_{\mathfrak{p}}$ *is a field.*

Proof. If some element x of $\mathfrak{p}R_{\mathfrak{p}}$ is not nilpotent, then $D(x) \neq \emptyset$, see Lemma 7.16.2. This contradicts the minimality of $\mathfrak{p}$. If R is reduced, then $\mathfrak{p}R_{\mathfrak{p}} = 0$ and hence it is a field. □

Lemma 7.23.4. *Let* R *be a ring. Let* $\mathfrak{p}$ *be a minimal prime of* R*. Let* $W \subset Spec(R)$ *be a quasi-compact open not containing the point* $\mathfrak{p}$*. Then there exists an* $f \in R$*,* $f \notin \mathfrak{p}$ *such that* $D(f) \cap W = \emptyset$*.*

Proof. Since W is quasi-compact we may write it as a finite union of standard affine opens $D(g_i)$, $i = 1, \ldots, n$. Since $\mathfrak{p} \notin W$ we have $g_i \in \mathfrak{p}$ for all i. By Lemma 7.23.3 above each g_i is nilpotent in $R_{\mathfrak{p}}$. Hence we can find an $f \in R$, $f \notin \mathfrak{p}$ such that for all i we have $fg_i^{n_i} = 0$ for some $n_i > 0$. Then $D(f)$ works. □

Lemma 7.23.5. *Let* R *be a ring. The following are equivalent*
(1) *there are no nontrivial inclusions between its prime ideals,*
(2) *every prime ideal is minimal,*
(3) *every prime ideal is a maximal ideal,*
(4) *every quasi-compact open of* $Spec(R)$ *is also closed, and*
(5) $Spec(R)$ *is totally disconnected.*

Proof. It is clear that (1), (2), and (3) are equivalent. The implication (2) $\Rightarrow$ (4) follows from Lemma 7.23.4. Assume (4) holds. Let $\mathfrak{p}, \mathfrak{p}'$ be distinct primes of R. Then we can choose an $f \in \mathfrak{p}'$, $f \notin \mathfrak{p}$. Then $\mathfrak{p}' \notin D(f)$ and $\mathfrak{p} \in D(f)$. By (4) the open $D(f)$ is also closed. Hence $\mathfrak{p}$ and $\mathfrak{p}'$ cannot be in the same connected component of $Spec(R)$ and we see (5) holds. Finally, if (5) holds then there cannot be any specializations between points of $Spec(R)$ and we see that (1) holds. □

Lemma 7.23.6. *Let R be a reduced ring. Then R is a subring of a product of fields. In fact, $R \subset \prod_{\mathfrak{p}\ minimal} R_{\mathfrak{p}}$ is such an embedding.*

Proof. This is clear from Lemma 7.23.3 above and the fact that $\bigcap_{\mathfrak{p}} \mathfrak{p} = (0)$ in a reduced ring, see Lemma 7.16.2. $\square$

7.24. Examples of spectra of rings

In this section we put some examples of spectra.

Example 7.24.1. In this example we describe $X = Spec(\mathbf{Z}[x]/(x^2 - 4))$. Let $\mathfrak{p}$ be an arbitrary prime in X. Let $\phi : \mathbf{Z} \to \mathbf{Z}[x]/(x^2 - 4)$ be the natural ring map. Then, $\phi^{-1}(\mathfrak{p})$ is a prime in $\mathbf{Z}$. If $\phi^{-1}(\mathfrak{p}) = (2)$, then since $\mathfrak{p}$ contains 2, it corresponds to a prime ideal in $\mathbf{Z}[x]/(x^2 - 4, 2) \cong (\mathbf{Z}/2\mathbf{Z})[x]/(x^2)$ via the map $\mathbf{Z}[x]/(x^2 - 4) \to \mathbf{Z}[x]/(x^2 - 4, 2)$. Any prime in $(\mathbf{Z}/2\mathbf{Z})[x]/(x^2)$ corresponds to a prime in $(\mathbf{Z}/2\mathbf{Z})[x]$ containing (x^2). Such primes will then contain x. Since $(\mathbf{Z}/2\mathbf{Z}) \cong (\mathbf{Z}/2\mathbf{Z})[x]/(x)$ is a field, (x) is a maximal ideal. Since any prime contains (x) and (x) is maximal, the ring contains only one prime (x). Thus, in this case, $\mathfrak{p} = (2, x)$. Now, if $\phi^{-1}(\mathfrak{p}) = (q)$ for $q > 2$, then since $\mathfrak{p}$ contains q, it corresponds to a prime ideal in $\mathbf{Z}[x]/(x^2 - 4, q) \cong (\mathbf{Z}/q\mathbf{Z})[x]/(x^2 - 4)$ via the map $\mathbf{Z}[x]/(x^2 - 4) \to \mathbf{Z}[x]/(x^2 - 4, q)$. Any prime in $(\mathbf{Z}/q\mathbf{Z})[x]/(x^2 - 4)$ corresponds to a prime in $(\mathbf{Z}/q\mathbf{Z})[x]$ containing $(x^2 - 4) = (x - 2)(x + 2)$. Hence, these primes must contain either $x - 2$ or $x + 2$. Since $(\mathbf{Z}/q\mathbf{Z})[x]$ is a PID, all nonzero primes are maximal, and so there are precisely 2 primes in $(\mathbf{Z}/q\mathbf{Z})[x]$ containing $(x-2)(x+2)$, namely $(x-2)$ and $(x+2)$. In conclusion, there exist two primes $(q, x - 2)$ and $(q, x + 2)$ since $2 \neq -2 \in \mathbf{Z}/(q)$. Finally, we treat the case where $\phi^{-1}(\mathfrak{p}) = (0)$. Notice that $\mathfrak{p}$ corresponds to a prime ideal in $\mathbf{Z}[x]$ that contains $(x^2-4) = (x-2)(x+2)$. Hence, $\mathfrak{p}$ contains either $(x-2)$ or $(x+2)$. Hence, $\mathfrak{p}$ corresponds to a prime in $\mathbf{Z}[x]/(x-2)$ or one in $\mathbf{Z}[x]/(x+2)$ that intersects $\mathbf{Z}$ only at 0, by assumption. Since $\mathbf{Z}[x]/(x - 2) \cong \mathbf{Z}$ and $\mathbf{Z}[x]/(x - 2) \cong \mathbf{Z}$, this means that $\mathfrak{p}$ must correspond to 0 in one of these rings. Thus, $\mathfrak{p} = (x - 2)$ or $\mathfrak{p} = (x + 2)$ in the original ring.

Example 7.24.2. In this example we describe $X = Spec(\mathbf{Z}[x])$. Fix $\mathfrak{p} \in X$. Let $\phi : \mathbf{Z} \to \mathbf{Z}[x]$ and notice that $\phi^{-1}(\mathfrak{p}) \in Spec(\mathbf{Z})$. If $\phi^{-1}(\mathfrak{p}) = (q)$ for q a prime number $q > 0$, then it $\mathfrak{p}$ corresponds to a prime in $(\mathbf{Z}/(q))[x]$, which must be generated by a polynomial that is irreducible in $(\mathbf{Z}/(q))[x]$. If we choose a representative of this polynomial with minimal degree, then it will also be irreducible in $\mathbf{Z}[x]$. Hence, in this case $\mathfrak{p} = (q, f_q)$ where f_q is an irreducible polynomial in $\mathbf{Z}[x]$ that is irreducible when viewed in $(\mathbf{Z}/(q)[x])$. Now, assume that $\phi^{-1}(\mathfrak{p}) = (0)$. In this case, $\mathfrak{p}$ must be generated by nonconstant polynomials which, since $\mathfrak{p}$ is prime, may be assumed to be irreducible in $\mathbf{Z}[x]$. By Gauss' lemma, these polynomials are also irreducible in $\mathbf{Q}[x]$. Since $\mathbf{Q}[x]$ is a Euclidean domain, if there are at least two distinct irreducibles f, g generating $\mathfrak{p}$, then $1 = af + bg$ for $a, b \in \mathbf{Q}[x]$. Multiplying through by a common denominator, we see that $m = \bar{a}f + \bar{b}g$ for $\bar{a}, \bar{b} \in \mathbf{Z}[x]$ and nonzero $m \in \mathbf{Z}$. This is a contradiction. Hence, $\mathfrak{p}$ is generated by one irreducible polynomial in $\mathbf{Z}[x]$.

Example 7.24.3. In this example we describe $X = Spec(k[x, y])$ when k is an arbitrary field. Clearly (0) is prime, and any principal ideal generated by an irreducible polynomial will also be a prime since $k[x, y]$ is a unique factorization domain. Now assume $\mathfrak{p}$ is an element of X that is not principal. Since $k[x, y]$ is a Noetherian UFD, the prime ideal $\mathfrak{p}$ can be generated by a finite number of irreducible polynomials $(f_1, \ldots, f_n)$. Now, I claim that if f, g are irreducible polynomials in $k[x, y]$ that are not associates, then $(f, g) \cap k[x] \neq 0$. To do this, it is enough to show that f and g are relatively prime when viewed in $k(x)[y]$.

In this case, $k(x)[y]$ is a Euclidean domain, so by applying the Euclidean algorithm and clearing denominators, we obtain $p = af + bg$ for $p, a, b \in k[x]$. Thus, assume this is not the case, that is, that some nonunit $h \in k(x)[y]$ divides both f and g. Then, by Gauss's lemma, for some $a, b \in k(x)$ we have $ah|f$ and $bh|g$ for $ah, bh \in k[x]$ since $\mathrm{Frac}(k[x]) = k(x)$. By irreducibility, $ah = f$ and $bh = g$ (since $h \notin k(x)$). So, back in $k(x)[y]$, f, g are associates, as $\frac{a}{b}g = f$. Since $k(x) = \mathrm{Frac}(k[x])$, we can write $g = \frac{r}{s}f$ for elements $r, s \in k[x]$ sharing no common factors. This implies that $sg = rf$ in $k[x, y]$ and so s must divide f since $k[x, y]$ is a UFD. Hence, $s = 1$ or $s = f$. If $s = f$, then $r = g$, implying $f, g \in k[x]$ and thus must be units in $k(x)$ and relatively prime in $k(x)[y]$, contradicting our hypothesis. If $s = 1$, then $g = rf$, another contradiction. Thus, we must have f, g relatively prime in $k(x)[y]$, a Euclidean domain. Thus, we have reduced to the case $\mathfrak{p}$ contains some irreducible polynomial $p \in k[x] \subseteq k[x, y]$. By the above, $\mathfrak{p}$ corresponds to a prime in the ring $k[x, y]/(p) = k(\alpha)[y]$, where α is an element algebraic over k with minimum polynomial p. This is a PID, and so any prime ideal corresonds to (0) or an irreducible polynomial in $k(\alpha)[y]$. Thus, $\mathfrak{p}$ is of the form (p) or (p, f) where f is a polynomial in $k[x, y]$ that is irreducible in the quotient $k[x, y]/(p)$.

Example 7.24.4. Consider the ring

$$R = \{f \in \mathbf{Q}[z] \text{ with } f(0) = f(1)\}.$$

Consider the map

$$\varphi : \mathbf{Q}[A, B] \to R$$

defined by $\varphi(A) = z^2 - z$ and $\varphi(B) = z^3 - z^2$. It is easily checked that $(A^3 - B^2 + AB) \subseteq \ker(\varphi)$ and that $A^3 - B^2 + AB$ is irreducible. Assume that φ is surjective; then since R is an integral domain (it is a subring of an integral domain), $\ker(\phi)$ must be a prime ideal of $\mathbf{Q}[A, B]$. The prime ideals which contain $(A^3 - B^2 + AB)$ are $(A^3 - B^2 + AB)$ itself and any maximal ideal (f, g) with $f, g \in \mathbf{Q}[A, B]$ such that f is irreducible mod g. But R is not a field, so the kernel must be $(A^3 - B^2 + AB)$; hence φ gives an isomorphism $R \to \mathbf{Q}[A, B]/(A^3 - B^2 + AB)$.

To see that φ is surjective, we must express any $f \in R$ as a $\mathbf{Q}$-coefficient polynomial in $A(z) = z^2 - z$ and $B(z) = z^3 - z^2$. Note the relation $zA(z) = B(z)$. Let $a = f(0) = f(1)$. Then $z(z-1)$ must divide $f(z) - a$, so we can write $f(z) = z(z-1)g(z) + a = A(z)g(z) + a$. If $\deg(g) < 2$, then $h(z) = c_1 z + c_0$ and $f(z) = A(z)(c_1 z + c_0) + a = c_1 B(z) + c_0 A(z) + a$, so we are done. If $\deg(g) \geq 2$, then by the polynomial division algorithm, we can write $g(z) = A(z)h(z) + b_1 z + b_0$ ($\deg(h) \leq \deg(g) - 2$), so $f(z) = A(z)^2 h(z) + b_1 B(z) + b_0 A(z)$. Applying division to $h(z)$ and iterating, we obtain an expression for $f(z)$ as a polynomial in $A(z)$ and $B(z)$; hence φ is surjective.

Now let $a \in \mathbf{Q}$, $a \neq 0, \frac{1}{2}, 1$ and consider

$$R_a = \{f \in \mathbf{Q}[z, \frac{1}{z-a}] \text{ with } f(0) = f(1)\}.$$

This is a finitely generated $\mathbf{Q}$-algebra as well: it is easy to check that the functions $z^2 - z$, $z^3 - z$, and $\frac{a^2-a}{z-a} + z$ generate R_a as an $\mathbf{Q}$-algebra. We have the following inclusions:

$$R \subset R_a \subset \mathbf{Q}[z, \frac{1}{z-a}], \qquad R \subset \mathbf{Q}[z] \subset \mathbf{Q}[z, \frac{1}{z-a}].$$

Recall (Lemma 7.16.5) that for a ring T and a multiplicative subset $S \subset T$, the ring map $T \to S^{-1}T$ induces a map on spectra $Spec(S^{-1}T) \to Spec(T)$ which is a homeomorphism onto the subset

$$\{\mathfrak{p} \in Spec(T) \mid S \cap \mathfrak{p} = \varnothing\} \subseteq Spec(T).$$

When $S = \{1, f, f^2, \dots\}$ for some $f \in T$, this is the open set $D(f) \subset T$. We now verify a corresponding property for the ring map $R \to R_a$: we will show that the map $\theta : Spec(R_a) \to Spec(R)$ induced by inclusion $R \subset R_a$ is a homeomorphism onto an open subset of $Spec(R)$ by verifying that θ is an injective local homeomorphism. We do so with respect to an open cover of $Spec(R_a)$ by two distinguished opens, as we now describe. For any $r \in \mathbf{Q}$, let $\mathrm{ev}_r : R \to \mathbf{Q}$ be the homomorphism given by evaluation at r. Note that for $r = 0$ and $r = 1 - a$, this can be extended to a homomorphism $\mathrm{ev}'_r : R_a \to \mathbf{Q}$ (the latter because $\frac{1}{z-a}$ is well-defined at $z = 1 - a$, since $a \neq \frac{1}{2}$). However, ev_a does not extend to R_a. Write $\mathfrak{m}_r = \ker(\mathrm{ev}_r)$; it is easy to check that

$$\mathfrak{m}_0 = (z^2 - z, z^3 - z),$$

$$\mathfrak{m}_a = ((z - 1 + a)(z - a), (z^2 - 1 + a)(z - a)), \text{ and}$$

$$\mathfrak{m}_{1-a} = ((z - 1 + a)(z - a), (z - 1 + a)(z^2 - a)).$$

(To do so, note that the right-hand sides are clearly contained in the left-hand sides. Then check that the right-hand sides are maximal ideals by writing the generators in terms of A and B, and viewing R as $\mathbf{Q}[A, B]/(A^3 - B^2 + AB)$.) Note that $\mathfrak{m}_a$ is not in the image of θ: we have $(z^2 - 1 + a)(z - a) - (z - 1 + a)(z - a) = (z^2 - z)(z - a)$ is in $\mathfrak{m}_a$, so $z^2 - z = \frac{(z^2-z)(z-a)}{z-a}$ is in $\mathfrak{m}_a R_a$. Hence no ideal I of R_a can satisfy $I \cap R = \mathfrak{m}_\mathrm{a}$, as such an I would have to contain $z^2 - z$, which is in R but not in $\mathfrak{m}_a$. The distinguished open set $D((z - 1 + a)(z - a)) \subset Spec(R)$ is equal to the complement of the closed set $\{\mathfrak{m}_a, \mathfrak{m}_{1-a}\}$. Then check that $R_{(z-1+a)(z-a)} = (R_a)_{(z-1+a)(z-a)}$; calling this localized ring R', then, it follows that the map $R \to R'$ factors as $R \to R_a \to R'$. By Lemma 7.16.5, then, these maps express $Spec(R') \subseteq Spec(R_a)$ and $Spec(R') \subseteq Spec(R)$ as open subsets; hence $\theta : Spec(R_a) \to Spec(R)$, when restricted to $D((z - 1 + a)(z - a))$, is a homeomorphism onto an open subset. Similarly, θ restricted to $D((z^2 + z + 2a - 2)(z - a)) \subseteq Spec(R_a)$ is a homeomorphism onto the open subset $D((z^2+z+2a-2)(z-a)) \subseteq Spec(R)$. Depending on whether $z^2 + z + 2a - 2$ is irreducible or not over $\mathbf{Q}$, this former distinguished open set has complement equal to one or two closed points along with the closed point $\mathfrak{m}_a$. Furthermore, the ideal in R_a generated by the elements $(z^2+z+2a-a)(z-a)$ and $(z-1+a)(z-a)$ is all of R_a, so these two distinguished open sets cover $Spec(R_a)$. Hence in order to show that θ is a homeomorphism onto $Spec(R) - \{\mathfrak{m}_a\}$, it suffices to show that these one or two points can never equal $\mathfrak{m}_{1-a}$. And this is indeed the case, since $1 - a$ is a root of $z^2 + z + 2a - 2$ if and only of $a = 0$ or $a = 1$, both of which do not occur.

Despite this homeomorphism which mimics the behavior of a localization at an element of R, while $\mathbf{Q}[z, \frac{1}{z-a}]$ is the localization of $\mathbf{Q}[z]$ at the maximal ideal $(z - a)$, the ring R_a is *not* a localization of R: Any localization $S^{-1}R$ results in more units than the original ring R. The units of R are $\mathbf{Q}^\times$, the units of $\mathbf{Q}$. If $\frac{f}{(z-a)^k}$ is a unit in R_a ($f \in R$ and $k \geq 0$ an integer), then we have

$$\frac{f}{(z-a)^k} \cdot \frac{g}{(z-a)^\ell} = 1$$

for some $g \in R$ and some integer $\ell \geq 0$. Since R is an integral domain, this is equivalent to

$$fg = (z - a)^{k+\ell}.$$

But $(z-a)^{k+\ell}$ is only an element of R if $k=\ell=0$; hence f,g are units in R as well. Hence R_a has no more units than R does, and thus cannot be a localization of R.

We used the fact that $a \neq 0,1$ to ensure that $\frac{1}{z-a}$ makes sense at $z=0,1$. We used the fact that $a \neq 1/2$ in a few places: (1) In order to be able to talk about the kernel of ev_{1-a} on R_a, which ensures that $\mathfrak{m}_{1-a}$ is a point of R_a (i.e., that R_a is missing just one point of R). (2) At the end in order to conclude that $(z-a)^{k+\ell}$ can only be in R for $k=\ell=0$; indeed, if $a=1/2$, then this is in R as long as $k+\ell$ is even. Hence there would indeed be more units in R_a than in R, and R_a could possibly be a localization of R.

7.25. A meta-observation about prime ideals

This section is taken from the CRing project. Let R be a ring and let $S \subset R$ be a multiplicative subset. A consequence of Lemma 7.16.5 is that an ideal $I \subset R$ maximal with respect to the property of not intersecting S is prime. The reason is that $I = R \cap \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of the ring $S^{-1}R$. It turns out that for many properties of ideals, the maximal ones are prime. A general method of seeing this was developed in [**LR08**]. In this section, we digress to explain this phenomenon.

Let R be a ring. If I is an ideal of R and $a \in R$, we define

$$(I : a) = \{x \in R \mid xa \in I\}.$$

More generally, if $J \subset R$ is an ideal, we define

$$(I : J) = \{x \in R \mid xJ \subset I\}.$$

Lemma 7.25.1. *Let R be a ring. For a principal ideal $J \subset R$, and for any ideal $I \subset J$ we have $I = J(I : J)$.*

Proof. Say $J = (a)$. Then $(I : J) = (I : a)$. Since $I \subset J$ we see that any $y \in I$ is of the form $y = xa$ for some $x \in (I : a)$. Hence $I \subset J(I : J)$. Conversely, if $x \in (I : a)$, then $xJ = (xa) \subset I$, which proves the other inclusion. $\square$

Let $\mathcal{F}$ be a collection of ideals of R. We are interested in conditions that will guarantee that the maximal elements in the complement of $\mathcal{F}$ are prime.

Definition 7.25.2. Let R be a ring. Let $\mathcal{F}$ be a set of ideals of R. We say $\mathcal{F}$ is an *Oka family* if $R \in \mathcal{F}$ and whenever $I \subset R$ is an ideal and $(I : a), (I, a) \in \mathcal{F}$ for some $a \in R$, then $I \in \mathcal{F}$.

Let us give some examples of Oka families. The first example is the basic example discussed in the introduction to this section.

Example 7.25.3. Let R be a ring and let S be a multiplicative subset of R. We claim that $\mathcal{F} = \{I \subset R \mid I \cap S \neq \emptyset\}$ is an Oka family. Namely, suppose that $(I : a), (I, a) \in \mathcal{F}$ for some $a \in R$. Then pick $s \in (I, a) \cap S$ and $s' \in (I : a) \cap S$. Then $ss' \in I \cap S$ and hence $I \in \mathcal{F}$. Thus $\mathcal{F}$ is an Oka family.

Example 7.25.4. Let R be a ring, $I \subset R$ an ideal, and $a \in R$. If $(I : a)$ is generated by $a_1, \ldots, a_n$ and (I, a) is generated by $a, b_1, \ldots, b_m$ with $b_1, \ldots, b_m \in I$, then I is generated by $aa_1, \ldots, aa_n, b_1, \ldots, b_m$. To see this, note that if $x \in I$, then $x \in (I, a)$ is a linear combination of $a, b_1, \ldots, b_m$, but the coefficient of a must lie in $(I : a)$. As a result, we deduce that the family of finitely generated ideals is an Oka family.

Example 7.25.5. Let us show that the family of principal ideals of a ring R is an Oka family. Indeed, suppose $I \subset R$ is an ideal, $a \in R$, and (I, a) and $(I : a)$ are principal. Note that $(I : a) = (I : (I, a))$. Setting $J = (I, a)$, we find that J is principal and $(I : J)$ is too. By Lemma 7.25.1 we have $I = J(I : J)$. Thus we find in our situation that since $J = (I, a)$ and $(I : J)$ are principal, I is principal.

Example 7.25.6. Let R be a ring. Let κ be an infinite cardinal. The family of ideals which can be generated by at most κ elements is an Oka family. The argument is analogous to the argument in Example 7.25.4 and is omitted.

Proposition 7.25.7. *If $\mathcal{F}$ is an Oka family of ideals, then any maximal element of the complement of $\mathcal{F}$ is prime.*

Proof. Suppose $I \notin \mathcal{F}$ is maximal with respect to not being in $\mathcal{F}$ but I is not prime. Note that $I \neq R$ because $R \in \mathcal{F}$. Since I is not prime we can find $a, b \in R - I$ with $ab \in I$. it follows that $(I, a) \neq I$ and $(I : a)$ contains $b \notin I$ so also $(I : a) \neq I$. Thus $(I : a), (I, a)$ both strictly contain I, so they must belong to $\mathcal{F}$. By the Oka condition, we have $I \in \mathcal{F}$, a contradiction. ☐

At this point we are able to turn most of the examples above into a lemma about prime ideals in a ring.

Lemma 7.25.8. *Let R be a ring. Let S be a multiplicative subset of R. An ideal $I \subset R$ which is maximal with respect to the property that $I \cap S = \emptyset$ is prime.*

Proof. This is the example discussed in the introduction to this section. For an alternative proof, combine Example 7.25.3 with Proposition 7.25.7. ☐

Lemma 7.25.9. *Let R be a ring.*

(1) *An ideal $I \subset R$ maximal with respect to not being finitely generated is prime.*
(2) *If every prime ideal of R is finitely generated, then every ideal of R is finitely generated*[1].

Proof. The first assertion is an immediate consequence of Example 7.25.4 and Proposition 7.25.7. For the second, suppose that there exists an ideal $I \subset R$ which is not finitely generated. The union of a totally ordered chain $\{I_\alpha\}$ of ideals that are not finitely generated is not finitely generated; indeed, if $I = \bigcup I_\alpha$ were generated by $a_1, \ldots, a_n$, then all the generators would belong to some I_α and would consequently generate it. By Zorn's lemma, there is an ideal maximal with respect to being not finitely generated. By the first part this ideal is prime. ☐

Lemma 7.25.10. *Let R be a ring.*

(1) *An ideal $I \subset R$ maximal with respect to not being principal is prime.*
(2) *If every prime ideal of R is principal, then every ideal of R is principal.*

Proof. This first part follows from Example 7.25.5 and Proposition 7.25.7. For the second, suppose that there exists an ideal $I \subset R$ which is not principal. The union of a totally ordered chain $\{I_\alpha\}$ of ideals that not principal is not principal; indeed, if $I = \bigcup I_\alpha$ were generated by a, then a would belong to some I_α and a would generate it. By Zorn's lemma, there is an ideal maximal with respect to not being principal. This ideal is necessarily prime by the first part. ☐

[1]Later we will say that R is Noetherian.

Lemma 7.25.11. *Let R be a ring.*

(1) *An ideal maximal among the ideals which do not contain a zero divisor is prime.*
(2) *If every nonzero prime ideal in R contains a non-zerodivisor, then R is a domain.*

Proof. Consider the set S of nonzerodivisors. It is a multiplicative subset of R. Hence any ideal maximal with respect to not intersecting S is prime, see Lemma 7.25.8. Thus, if every nonzero prime ideal contains a nonzero divisor, then (0) is prime, i.e., R is a domain. □

Remark 7.25.12. Let R be a ring. Let κ be an infinite cardinal. By applying Example 7.25.6 and Proposition 7.25.7 we see that any ideal maximal with respect to the property of not being generated by κ elements is prime. This result is not so useful because there exists a ring for which every prime ideal of R can be generated by $\aleph_0$ elements, but some ideal cannot. Namely, let k be a field, let T be a set whose cardinality is greater than $\aleph_0$ and let

$$R = k[\{x_n\}_{n\geq 1}, \{z_{t,n}\}_{t\in T, n\geq 0}]/(x_n^2, z_{t,n}^2, x_n z_{t,n} - z_{t,n-1})$$

This is a local ring with unique prime ideal $\mathfrak{m} = (x_n)$. But the ideal $(z_{t,n})$ cannot be generated by countably many elements.

7.26. Images of ring maps of finite presentation

In this section we prove some results on the topology of maps $Spec(S) \to Spec(R)$ induced by ring maps $R \to S$, mainly Chevalley's Theorem. In order to do this we will use the notions of constructible sets, quasi-compact sets, retrocompact sets, and so on which are defined in Topology, Section 5.9.

Lemma 7.26.1. *Let $U \subset Spec(R)$ be open. The following are equivalent:*

(1) *U is retrocompact in $Spec(R)$,*
(2) *U is quasi-compact, and*
(3) *U is a finite union of standard opens.*

Proof. The implication (2)⇒(3) is immediate from the fact that standard opens form a basis for the topology. Each standard open is homeomorphic to the spectrum of a ring and hence quasi-compact, by Lemmas 7.16.10 and 7.16.6. Hence a finite union of standard opens is quasi-compact as well. To finish it suffices to show that a finite union $\bigcup_{i=1\ldots n} D(f_i)$ is retrocompact in $Spec(R)$. In order to do this it suffices to show that $(\bigcup_{i=1\ldots n} D(f_i)) \cap (\bigcup_{j=1\ldots m} D(g_j))$ is quasi-compact, which is clear because it equals $\bigcup_{i,j} D(f_i g_j)$. □

Lemma 7.26.2. *Let $\varphi : R \to S$ be a ring map. The induced continuous map $f : Spec(S) \to Spec(R)$ is quasi-compact. For any constructible set $E \subset Spec(R)$ the inverse image $f^{-1}(E)$ is constructible in $Spec(S)$.*

Proof. We first show that the inverse image of any quasi-compact open $U \subset Spec(R)$ is quasi-compact. By Lemma 7.26.1 we may write U as a finite open of standard opens. Thus by Lemma 7.16.4 we see that $f^{-1}(U)$ is a finite union of standard opens. Hence $f^{-1}(U)$ is quasi-compact by Lemma 7.26.1 again. The second assertion now follows from Topology, Lemma 5.10.3. □

Lemma 7.26.3. *Let R be a ring and let $T \subset Spec(R)$ be constructible. Then there exists a ring map $R \to S$ of finite presentation such that T is the image of $Spec(S)$ in $Spec(R)$.*

Proof. Let $T \subset Spec(R)$ be constructible. The spectrum of a finite product of rings is the disjoint union of the spectra, see Lemma 7.18.2. Hence if $T = T_1 \cup T_2$ and the result holds for T_1 and T_2, then the result holds for T. In particular we may assume that $T =$

$U \cap V^c$, where $U, V \subset Spec(R)$ are retrocompact open. By Lemma 7.26.1 we may write $T = (\bigcup D(f_i)) \cap (\bigcup D(g_j))^c = \bigcup \big(D(f_i) \cap V(g_1, \ldots, g_m)\big)$. In fact we may assume that $T = D(f) \cap V(g_1, \ldots, g_m)$ (by the argument on unions above). In this case T is the image of the map $R \to (R/(g_1, \ldots, g_m))_f$, see Lemmas 7.16.6 and 7.16.7. □

Lemma 7.26.4. *Let R be a ring. Let f be an element of R. Let $S = R_f$. Then the image of a constructible subset of $Spec(S)$ is constructible in $Spec(R)$.*

Proof. We repeatedly use Lemma 7.26.1 without mention. Let U, V be quasi-compact open in $Spec(S)$. We will show that the image of $U \cap V^c$ is constructible. Under the identification $Spec(S) = D(f)$ of Lemma 7.16.6 the sets U, V correspond to quasi-compact opens U', V' of $Spec(R)$. Hence it suffices to show that $U' \cap (V')^c$ is constructible in $Spec(R)$ which is clear. □

Lemma 7.26.5. *Let R be a ring. Let I be a finitely generated ideal of R. Let $S = R/I$. Then the image of a constructible of $Spec(S)$ is constructible in $Spec(R)$.*

Proof. If $I = (f_1, \ldots, f_m)$, then we see that $V(I)$ is the complement of $\bigcup D(f_i)$, see Lemma 7.16.2. Hence it is constructible, by Lemma 7.26.1. Denote the map $R \to S$ by $f \mapsto \overline{f}$. We have to show that if $\overline{U}, \overline{V}$ are retrocompact opens of $Spec(S)$, then the image of $\overline{U} \cap \overline{V}^c$ in $Spec(R)$ is constructible. By Lemma 7.26.1 we may write $\overline{U} = \bigcup D(\overline{g_i})$. Setting $U = \bigcup D(g_i)$ we see $\overline{U}$ has image $U \cap V(I)$ which is constructible in $Spec(R)$. Similarly the image of $\overline{V}$ equals $V \cap V(I)$ for some retrocompact open V of $Spec(R)$. Hence the image of $\overline{U} \cap \overline{V}^c$ equals $U \cap V(I) \cap V^c$ as desired. □

Lemma 7.26.6. *Let R be a ring. The map $Spec(R[x]) \to Spec(R)$ is open, and the image of any standard open is a quasi-compact open.*

Proof. It suffices to show that the image of a standard open $D(f)$, $f \in R[x]$ is quasi-compact open. The image of $D(f)$ is the image of $Spec(R[x]_f) \to Spec(R)$. Let $\mathfrak{p} \subset R$ be a prime ideal. Let $\overline{f}$ be the image of f in $\kappa(\mathfrak{p})[x]$. Recall, see Lemma 7.16.9, that $\mathfrak{p}$ is in the image if and only if $R[x]_f \otimes_R \kappa(\mathfrak{p}) = \kappa(\mathfrak{p})[x]_{\overline{f}}$ is not the zero ring. This is exactly the condition that f does not map to zero in $\kappa(\mathfrak{p})[x]$, in other words, that some coefficient of f is not in $\mathfrak{p}$. Hence we see: if $f = a_d x^d + \ldots a_0$, then the image of $D(f)$ is $D(a_d) \cup \ldots \cup D(a_0)$. □

We prove a property of characteristic polynomials which will be used below.

Lemma 7.26.7. *Let $R \to A$ be a ring homomorphism. Assume $A \cong R^{\oplus n}$ as an R-module. Let $f \in A$. The multiplication map $m_f : A \to A$ is R-linear and hence has a characteristic polynomial $P(T) = T^n + r_{n-1}T^{n-1} + \cdots + r_0 \in R[T]$. For any prime $\mathfrak{p} \in Spec(R)$, f acts nilpotently on $A \otimes_R \kappa(\mathfrak{p})$ if and only if $\mathfrak{p} \in V(r_0, \ldots, r_{n-1})$.*

Proof. This follows quite easily once we prove that the characteristic polynomial $\bar{P}(T) \in \kappa(\mathfrak{p})[T]$ of the multiplication map $m_{\bar{f}} : A \otimes_R \kappa(\mathfrak{p}) \to A \otimes_R \kappa(\mathfrak{p})$ which multiplies elements of $A \otimes_R \kappa(\mathfrak{p})$ by $\bar{f}$, the image of f viewed in $\kappa(\mathfrak{p})$, is just the image of $P(T)$ in $\kappa(\mathfrak{p})[T]$. Let (a_{ij}) be the matrix of the map m_f with entries in R, using a basis $e_1, \ldots, e_n$ of A as an R-module. Then, $A \otimes_R \kappa(\mathfrak{p}) \cong (R \otimes_R \kappa(\mathfrak{p}))^{\oplus n} = \kappa(\mathfrak{p})^n$, which is an n-dimensional vector space over $\kappa(\mathfrak{p})$ with basis $e_1 \otimes 1, \ldots, e_n \otimes 1$. The image $\bar{f} = f \otimes 1$, and so the multiplication map $m_{\bar{f}}$ has matrix $(a_{ij} \otimes 1)$. Thus, the characteristic polynomial is precisely the image of $P(T)$.

From linear algebra, we know that a linear transformation acts nilpotently on an n-dimensional vector space if and only if the characteristic polynomial is T^n (since the characteristic polynomial divides some power of the minimal polynomial). Hence, f acts nilpotently on $A \otimes_R \kappa(\mathfrak{p})$ if and only if $\bar{P}(T) = T^n$. This occurs if and only if $r_i \in \mathfrak{p}$ for all $0 \leq i \leq n-1$, that is when $\mathfrak{p} \in V(r_0, \ldots, r_{n-1})$. □

Lemma 7.26.8. *Let R be a ring. Let $f, g \in R[x]$ be polynomials. Assume the leading coefficient of g is a unit of R. There exists elements $r_i \in R$, $i = 1 \ldots, n$ such that the image of $D(f) \cap V(g)$ in $Spec(R)$ is $\bigcup_{i=1,\ldots,n} D(r_i)$.*

Proof. Write $g = ux^d + a_{d-1}x^{d-1} + \ldots a_0$, where d is the degree of g, and hence $u \in R^*$. Consider the ring $A = R[x]/(g)$. It is, as an R-module, finite free with basis the images of $1, x, \ldots, x^{d-1}$. Consider multiplication by (the image of) f on A. This is an R-module map. Hence we can let $P(T) \in R[T]$ be the characteristic polynomial of this map. Write $P(T) = T^d + r_{d-1}T^{d-1} + \ldots r_0$. We claim that $r_0, \ldots, r_{d-1}$ have the desired property. We will use below the property of characteristic polynomials that

$$\mathfrak{p} \in V(r_0, \ldots, r_{d-1}) \Leftrightarrow \text{multiplication by } f \text{ is nilpotent on } A \otimes_R \kappa(\mathfrak{p}).$$

This was proved in Lemma 7.26.7 above.

Suppose $\mathfrak{q} \in D(f) \cap V(g)$, and let $\mathfrak{p} = \mathfrak{q} \cap R$. Then there is a nonzero map $A \otimes_R \kappa(\mathfrak{p}) \to \kappa(\mathfrak{q})$ which is compatible with multiplication by f. And f acts as a unit on $\kappa(\mathfrak{q})$. Thus we conclude $\mathfrak{p} \notin V(r_0, \ldots, r_{d-1})$.

On the other hand, suppose that $r_i \notin \mathfrak{p}$ for some prime $\mathfrak{p}$ of R and some $0 \leq i \leq d-1$. Then multiplication by f is not nilpotent on the algebra $A \otimes_R \kappa(\mathfrak{p})$. Hence there exists a maximal ideal $\overline{\mathfrak{q}} \subset A \otimes_R \kappa(\mathfrak{p})$ not containing the image of f. The inverse image of $\overline{\mathfrak{q}}$ in $R[x]$ is an element of $D(f) \cap V(g)$ mapping to $\mathfrak{p}$. □

Theorem 7.26.9. *Chevalley's Theorem. Suppose that $R \to S$ is of finite presentation. The image of a constructible subset of $Spec(S)$ in $Spec(R)$ is constructible.*

Proof. Write $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. We may factor $R \to S$ as $R \to R[x_1] \to R[x_1, x_2] \to \ldots \to R[x_1, \ldots, x_{n-1}] \to S$. Hence we may assume that $S = R[x]/(f_1, \ldots, f_m)$. In this case we factor the map as $R \to R[x] \to S$, and by Lemma 7.26.5 we reduce to the case $S = R[x]$. By Lemma 7.26.1 suffices to show that if $T = (\bigcup_{i=1\ldots n} D(f_i)) \cap V(g_1, \ldots, g_m)$ for $f_i, g_j \in R[x]$ then the image in $Spec(R)$ is constructible. Since finite unions of constructible sets are constructible, it suffices to deal with the case $n = 1$, i.e., when $T = D(f) \cap V(g_1, \ldots, g_m)$.

Note that if $c \in R$, then we have

$$Spec(R) = V(c) \coprod D(c) = Spec(R/(c)) \coprod Spec(R_c)),$$

and correspondingly $Spec(R[x]) = V(c) \coprod D(c) = Spec(R/(c)[x]) \coprod Spec(R_c[x]))$. The intersection of $T = D(f) \cap V(g_1, \ldots, g_m)$ with each part still has the same shape, with f, g_i replaced by their images in $R/(c)[x]$, respectively $R_c[x]$. Note that the image of T in $Spec(R)$ is the union of the image of $T \cap V(c)$ and $T \cap D(c)$. Using Lemmas 7.26.4 and 7.26.5 it suffices to prove the images of both parts are constructible in $Spec(R/(c))$, respectively $Spec(R_c)$.

Let us assume we have $T = D(f) \cap V(g_1, \ldots, g_m)$ as above, with $\deg(g_1) \leq \deg(g_2) \leq \ldots \leq \deg(g_m)$. We are going to use descending induction on m, and on the degrees of the g_i. Let $d = \deg(g_1)$, i.e., $g_1 = cx^{d_1} + l.o.t$ with $c \in R$ not zero. Cutting R up into the pieces $R/(c)$ and R_c we either lower the degree of g_1 (and this is covered by induction) or we reduce to the case where c is invertible. If c is invertible, and $m > 1$, then write $g_2 = c'x^{d_2} + l.o.t.$ In this case consider $g_2' = g_2 - (c'/c)x^{d_2-d_1}g_1$. Since the ideals $(g_1, g_2, \ldots, g_m)$ and $(g_1, g_2', g_3, \ldots, g_m)$ are equal we see that $T = D(f) \cap V(g_1, g_2', g_3 \ldots, g_m)$. But here the degree of g_2' is strictly less than the degree of g_2 and hence this case is covered by induction.

The bases case for the induction above are the cases (a) $T = D(f) \cap V(g)$ where the leading coefficient of g is invertible, and (b) $T = D(f)$. These two cases are dealt with in Lemmas 7.26.8 and 7.26.6. □

7.27. More on images

In this section we collect a few additional lemmas concerning the image on *Spec* for ring maps. See also Section 7.36 for example.

Lemma 7.27.1. *Let $R \subset S$ be an inclusion of domains. Assume that $R \to S$ is of finite type. There exists a nonzero $f \in R$, and a nonzero $g \in S$ such that $R_f \to S_{fg}$ is of finite presentation.*

Proof. By induction on the number of generators of S over R.

Suppose that S is generated by a single element over R. Then $S = R[x]/\mathfrak{q}$ for some prime ideal $\mathfrak{q} \subset R[x]$. If $\mathfrak{q} = (0)$ there is nothing to prove. If $\mathfrak{q} \neq (0)$, then let $g \in \mathfrak{q}$ be an element with minimal degree in x. Since $K[x] = f.f.(R)[x]$ is a PID we see that g is irreducible over K and that $f.f.(S) = K[x]/(g)$. Write $g = a_d x^d + \ldots + a_0$ with $a_i \in R$ and $a_d \neq 0$. After inverting a_d in R we may assume that g is monic. Hence we see that $R \to R[x]/(g) \to S$ with the last map surjective. But $R[x]/(g) = R \oplus Rx \oplus \ldots \oplus Rx^{d-1}$ maps injectively into $f.f.(S) = K[x]/(g) = K \oplus Kx \oplus \ldots \oplus Kx^{d-1}$. Thus $S \cong R[x]/(g)$ is finitely presented.

Suppose that S is generated by $n > 1$ elements over R. Say $x_1, \ldots, x_n \in S$ generate S. Denote $S' \subset S$ the subring generated by $x_1, \ldots, x_{n-1}$. By induction hypothesis we see that there exist $f \in R$ and $g \in S'$ nonzero such that $R_f \to S'_{fg}$ is of finite presentation. Next we apply the induction hypothesis to $S'_{fg} \to S_{fg}$ to see that there exist $f' \in S'_{fg}$ and $g' \in S_{fg}$ such that $S'_{fgf'} \to S_{fgf'g'}$ is of finite presentation. We leave it to the reader to conclude. □

Lemma 7.27.2. *Let $R \to S$ be a finite type ring map. Denote $X = Spec(R)$ and $Y = Spec(S)$. Write $f : Y \to X$ the induced map of spectra. Let $E \subset Y = Spec(S)$ be a constructible set. If a point $\xi \in X$ is in $f(E)$, then $\overline{\{\xi\}} \cap f(E)$ contains an open dense subset of $\overline{\{\xi\}}$.*

Proof. Let $\xi \in X$ be a point of $f(E)$. Choose a point $\eta \in E$ mapping to ξ. Let $\mathfrak{p} \subset R$ be the prime corresponding to ξ and let $\mathfrak{q} \subset S$ be the prime corresponding to η. Consider the diagram

$$\begin{tikzcd}
\eta \arrow[r] \arrow[d, mapsto] & E \cap Y' \arrow[r] \arrow[d] & Y' = Spec(S/\mathfrak{q}) \arrow[r] \arrow[d] & Y \arrow[d] \\
\xi \arrow[r] & f(E) \cap X' \arrow[r] & X' = Spec(R/\mathfrak{p}) \arrow[r] & X
\end{tikzcd}$$

By Lemma 7.26.2 the set $E \cap Y'$ is constructible in Y'. It follows that we may replace X by X' and Y by Y'. Hence we may assume that $R \subset S$ is an inclusion of domains, ξ is the generic point of X, and η is the generic point of Y. By Lemma 7.27.1 combined with Chevalley's theorem (Theorem 7.26.9) we see that there exist dense opens $U \subset X$, $V \subset Y$

such that $f(V) \subset U$ and such that $f : V \to U$ maps constructible sets to constructible sets. Note that $E \cap V$ is constructible in V, see Topology, Lemma 5.10.4. Hence $f(E \cap V)$ is constructible in U and contains ξ. By Topology, Lemma 5.10.6 we see that $f(E \cap V)$ contains a dense open $U' \subset U$. $\square$

At the end of this section we present a few more results on images of maps on Spectra that have nothing to do with constructible sets.

Lemma 7.27.3. *Let $\varphi : R \to S$ be a ring map. The following are equivalent:*

1. *The map $Spec(S) \to Spec(R)$ is surjective.*
2. *For any radical ideal $I \subset R$ the inverse image of IS in R is equal to I.*
3. *For every prime $\mathfrak p$ of R the inverse image of $\mathfrak pS$ in R is $\mathfrak p$.*

In this case the same is true after any base change: Given a ring map $R \to R'$ the ring map $R' \to R' \otimes_R S$ has the equivalent properties (1), (2), (3) also.

Proof. The implication (2) $\Rightarrow$ (3) is immediate. If $I \subset R$ is a radical ideal, then Lemma 7.16.2 guarantees that $I = \bigcap_{I \subset \mathfrak p} \mathfrak p$. Hence (3) $\Rightarrow$ (2). By Lemma 7.16.9 we have $\mathfrak p = \varphi^{-1}(\mathfrak pS)$ if and only if $\mathfrak p$ is in the image. Hence (1) $\Leftrightarrow$ (3). Thus (1), (2), and (3) are equivalent.

Assume (1) holds. Let $R \to R'$ be a ring map. Let $\mathfrak p' \subset R'$ be a prime ideal lying over the prime $\mathfrak p$ of R. To see that $\mathfrak p'$ is in the image of $Spec(R' \otimes_R S) \to Spec(R')$ we have to show that $(R' \otimes_R S) \otimes_{R'} \kappa(\mathfrak p')$ is not zero, see Lemma 7.16.9. But we have

$$(R' \otimes_R S) \otimes_{R'} \kappa(\mathfrak p') = S \otimes_R \kappa(\mathfrak p) \otimes_{\kappa(\mathfrak p)} \kappa(\mathfrak p')$$

which is not zero as $S \otimes_R \kappa(\mathfrak p)$ is not zero by assumption and $\kappa(\mathfrak p) \to \kappa(\mathfrak p')$ is an extension of fields. $\square$

Lemma 7.27.4. *Let R be a domain. Let $\varphi : R \to S$ be a ring map. The following are equivalent:*

1. *The ring map $R \to S$ is injective.*
2. *The image $Spec(S) \to Spec(R)$ contains a dense set of points.*
3. *There exists a prime ideal $\mathfrak q \subset S$ whose inverse image in R is (0).*

Proof. Let K be the field of fractions of the domain R. Assume that $R \to S$ is injective. Since localization is exact we see that $K \to S \otimes_R K$ is injective. Hence there is a prime mapping to (0) by Lemma 7.16.9.

Note that (0) is dense in $Spec(R)$, so that the last condition implies the second.

Suppose the second condition holds. Let $f \in R$, $f \neq 0$. As R is a domain we see that $V(f)$ is a proper closed subset of R. By assumption there exists a prime $\mathfrak q$ of S such that $\varphi(f) \notin \mathfrak q$. Hence $\varphi(f) \neq 0$. Hence $R \to S$ is injective. $\square$

Lemma 7.27.5. *Let $R \subset S$ be an injective ring map. Then $Spec(S) \to Spec(R)$ hits all the minimal primes of $Spec(R)$.*

Proof. Let $\mathfrak p \subset R$ be a minimal prime. In this case $R_{\mathfrak p}$ has a unique prime ideal. Hence it suffices to show that $S_{\mathfrak p}$ is not zero. And this follows from the fact that localization is exact, see Proposition 7.9.12. $\square$

Lemma 7.27.6. *Let $R \to S$ be a ring map. The following are equivalent:*

1. *The kernel of $R \to S$ consists of nilpotent elements.*
2. *The minimal primes of R are in the image of $Spec(S) \to Spec(R)$.*

(3) *The image of $Spec(S) \to Spec(R)$ is dense in $Spec(R)$.*

Proof. Let $I = \text{Ker}(R \to S)$. Note that $\sqrt{(0)} = \bigcap_{\mathfrak q \subset S} \mathfrak q$, see Lemma 7.16.2. Hence $\sqrt{I} = \bigcap_{\mathfrak q \subset S} R \cap \mathfrak q$. Thus $V(I) = V(\sqrt{I})$ is the closure of the image of $Spec(S) \to Spec(R)$. This shows that (1) is equivalent to (3). It is clear that (2) implies (3). Finally, assume (1). We may replace R by R/I and S by S/IS without affecting the topology of the spectra and the map. Hence the implication (1) $\Rightarrow$ (2) follows from Lemma 7.27.5 above. $\square$

7.28. Noetherian rings

A ring R is *Noetherian* if any ideal of R is finitely generated. This is clearly equivalent to the ascending chain condition for ideals of R. By Lemma 7.25.9 it suffices to check that every prime ideal of R is finitely generated.

Lemma 7.28.1. *Any finitely generated ring over a Noetherian ring is Noetherian. Any localization of a Noetherian ring is Noetherian.*

Proof. The statement on localizations follows from the fact that any ideal $J \subset S^{-1}R$ is of the form $I \cdot S^{-1}R$. Any quotient R/I of a Noetherian ring R is Noetherian because any ideal $\overline{J} \subset R/I$ is of the form J/I for some ideal $I \subset J \subset R$. Thus it suffices to show that if R is Noetherian so is $R[X]$. Suppose $J_1 \subset J_2 \subset \ldots$ is an ascending chain of ideals in $R[X]$. Consider the ideals $I_{i,d}$ defined as the ideal of elements of R which occur as leading coefficients of degree d polynomials in J_i. Clearly $I_{i,d} \subset I_{i',d'}$ whenever $i \leq i'$ and $d \leq d'$. By the ascending chain condition in R there are at most finitely many distinct ideals among all of the $I_{i,d}$. (Hint: Any infinite set of elements of $\mathbf{N} \times \mathbf{N}$ contains an increasing infinite sequence.) Take i_0 so large that $I_{i,d} = I_{i_0,d}$ for all $i \geq i_0$ and all d. Suppose $f \in J_i$ for some $i \geq i_0$. By induction on the degree $d = \deg(f)$ we show that $f \in J_{i_0}$. Namely, there exists a $g \in J_{i_0}$ whose degree is d and which has the same leading coefficient as f. By induction $f - g \in J_{i_0}$ and we win. $\square$

Lemma 7.28.2. *If R is a Noetherian ring, then so is the formal power series ring $R[[x_1, \ldots, x_n]]$.*

Proof. Since $R[[x_1, \ldots, x_{n+1}]] \cong R[[x_1, \ldots, x_n]][[x_{n+1}]]$ it suffices to prove the statement that $R[[x]]$ is Noetherian if R is Noetherian. Let $I \subset R[[x]]$ be a ideal. We have to show that I is a finitely generated ideal. For each integer d denote $I_d = \{a \in R \mid ax^d + \text{h.o.t.} \in I\}$. Then we see that $I_0 \subset I_1 \subset \ldots$ stabilizes as R is Noetherian. Choose d_0 such that $I_{d_0} = I_{d_0+1} = \ldots$. For each $d \leq d_0$ choose elements $f_{d,j} \in I \cap (x^d)$, $j = 1, \ldots, n_d$ such that if we write $f_{d,j} = a_{d,j}x^d + \text{h.o.t}$ then $I_d = (a_{d,j})$. Denote $I' = (\{f_{d,j}\}_{d=0,\ldots,d_0, j=1,\ldots,n_d})$. Then it is clear that $I' \subset I$. Pick $f \in I$. First we may choose $c_{d,i} \in R$ such that

$$f - \sum c_{d,i} f_{d,i} \in (x^{d_0+1}) \cap I.$$

Next, we can choose $c_{i,1} \in R$, $i = 1, \ldots, n_{d_0}$ such that

$$f - \sum c_{d,i} f_{d,i} - \sum c_{i,1} x f_{d_0,i} \in (x^{d_0+2}) \cap I.$$

Next, we can choose $c_{i,2} \in R$, $i = 1, \ldots, n_{d_0}$ such that

$$f - \sum c_{d,i} f_{d,i} - \sum c_{i,1} x f_{d_0,i} - \sum c_{i,2} x^2 f_{d_0,i} \in (x^{d_0+3}) \cap I.$$

And so on. In the end we see that

$$f = \sum c_{d,i} f_{d,i} + \sum\nolimits_i (\sum\nolimits_e c_{i,e} x^e) f_{d_0,i}$$

is contained in I' as desired. $\square$

The following lemma, although easy, is useful because finite type $\mathbf{Z}$-algebras come up quite often in a technique called ``absolute Noetherian reduction''.

Lemma 7.28.3. *Any finite type algebra over a field is Noetherian. Any finite type algebra over $\mathbf{Z}$ is Noetherian.*

Proof. This is immediate from the above and the fact that $\mathbf{Z}$ is a Noetherian ring because it is a principal ideal domain. ☐

Lemma 7.28.4. *Let R be a Noetherian ring.*

(1) *Any finite R-module is of finite presentation.*
(2) *Any finite type R-algebra is of finite presentation over R.*

Proof. Let M be a finite R-module. By Lemma 7.5.5 we can find a finite filtration of M whose succesive quotients are of the form R/I. Since any ideal is finitely generated, each of the quotients R/I is finitely presented. Hence M is finitely presented by Lemma 7.5.4. This proves (1). To see (2) note that any ideal of $R[x_1, \ldots, x_n]$ is finitely generated by Lemma 7.28.1 above. ☐

Lemma 7.28.5. *If R is a Noetherian ring then $Spec(R)$ is a Noetherian topological space, see Topology, Definition 5.6.1.*

Proof. This is because any closed subset of $Spec(R)$ is uniquely of the form $V(I)$ with I a radical ideal, see Lemma 7.16.2. And this correspondence is inclusion reversing. Thus the result follows from the definitions. ☐

Lemma 7.28.6. *If R is a Noetherian ring then $Spec(R)$ has finitely many irreducible components. In other words R has finitely many minimal primes.*

Proof. By Lemma 7.28.5 and Topology, Lemma 5.6.2 we see there are finitely many irreducible components. By Lemma 7.23.1 these correspond to minimal primes of R. ☐

Lemma 7.28.7. *Let k be a field and let R be a Noetherian k-algebra. If $k \subset K$ is a finitely generated field extension the $K \otimes_k R$ is Noetherian.*

Proof. Write $K = S^{-1}B$ where B is a finite type k-algebra, and $S \subset B$ is a multiplicative subset. Then we have $K \otimes_k R = S^{-1}(B \otimes_k R)$. Hence $K \otimes_k R$ is a localization of the finite type R-algebra $B \otimes_k R$ which is Noetherian by Lemma 7.28.1. ☐

Here is fun lemma that is sometimes useful.

Lemma 7.28.8. *Any surjective endomorphism of a Noetherian ring is an isomorphism.*

Proof. If $f : R \to R$ were such an endomorphism but not injective, then

$$\mathrm{Ker}(f) \subset \mathrm{Ker}(f \circ f) \subset \mathrm{Ker}(f \circ f \circ f) \subset \ldots$$

would be a strictly increasing chain of ideals. ☐

7.29. Curiosity

Lemma 7.20.3 explains what happens if $V(I)$ is open for some ideal $I \subset R$. But what if $Spec(S^{-1}R)$ is closed in $Spec(R)$? The next two lemmas give a partial answer. For more information see Section 7.100.

Lemma 7.29.1. *Let R be a ring. Let $S \subset R$ be a multiplicative subset. Assume the image of the map $Spec(S^{-1}R) \to Spec(R)$ is closed. Then $S^{-1}R \cong R/I$ for some ideal $I \subset R$.*

Proof. Let $I = \text{Ker}(R \to S^{-1}R)$ so that $V(I)$ contains the image. Say the image is the closed subset $V(I') \subset Spec(R)$ for some ideal $I' \subset R$. So $V(I') \subset V(I)$. For $f \in I'$ we see that $f/1 \in S^{-1}R$ is contained in every prime ideal. Hence f^n maps to zero in $S^{-1}R$ for some $n \geq 1$ (Lemma 7.16.2). Hence $V(I') = V(I)$. Then this implies every $g \in S$ is invertible mod I. Hence we get ring maps $R/I \to S^{-1}R$ and $S^{-1}R \to R/I$. The first map is injective by choice of I. The second is the map $S^{-1}R \to S^{-1}(R/I) = R/I$ which has kernel $S^{-1}I$ because localization is exact. Since $S^{-1}I = 0$ we see also the second map is injective. Hence $S^{-1}R \cong R/I$. ☐

Lemma 7.29.2. *Let R be a ring. Let $S \subset R$ be a multiplicative subset. Assume the image of the map $Spec(S^{-1}R) \to Spec(R)$ is closed. If R is Noetherian, or $Spec(R)$ is a Noetherian topological space, or S is finitely generated as a monoid, then $R \cong S^{-1}R \times R'$ for some ring R'.*

Proof. By Lemma 7.29.1 we have $S^{-1}R \cong R/I$ for some ideal $I \subset R$. By Lemma 7.20.3 it suffices to show that $V(I)$ is open. If R is Noetherian then $Spec(R)$ is a Noetherian topological space, see Lemma 7.28.5. If $Spec(R)$ is a Noetherian topological space, then the complement $Spec(R) \setminus V(I)$ is quasi-compact, see Topology, Lemma 5.9.9. Hence there exist finitely many $f_1, \ldots, f_n \in I$ such that $V(I) = V(f_1, \ldots, f_n)$. Since each f_i maps to zero in $S^{-1}R$ there exists a $g \in S$ such that $gf_i = 0$ for $i = 1, \ldots, n$. Hence $D(g) = V(I)$ as desired. In case S is finitely generated as a monoid, say S is generated by $g_1, \ldots, g_m$, then $S^{-1}R \cong R_{g_1 \ldots g_m}$ and we conclude that $V(I) = D(g_1 \ldots g_m)$. ☐

7.30. Hilbert Nullstellensatz

Theorem 7.30.1. *(Hilbert Nullstellensatz) Let k be a field.*

(1) *For any maximal ideal $\mathfrak{m} \subset k[x_1, \ldots, x_n]$ the field extension $k \subset \kappa(\mathfrak{m})$ is finite.*
(2) *Any radical ideal $I \subset k[x_1, \ldots, x_n]$ is the intersection of maximal ideals containing it.*

The same is true in any finite type k-algebra.

Proof. It is enough to prove part (1) of the theorem for the case of a polynomial algebra $k[x_1, \ldots, x_n]$, because any finitely generated k-algebra is a quotient of such a polynomial algebra. We prove this by induction on n. The case $n = 0$ is clear. Suppose that $\mathfrak{m}$ is a maximal ideal in $k[x_1, \ldots, x_n]$. Let $\mathfrak{p} \subset k[x_n]$ be the intersection of $\mathfrak{m}$ with $k[x_n]$.

If $\mathfrak{p} \neq (0)$, then $\mathfrak{p}$ is maximal and generated by an irreducible monic polynomial P (because of the Euclidean algorithm in $k[x_n]$). Then $k' = k[x_n]/\mathfrak{p}$ is a finite field extension of k and contained in $\kappa(\mathfrak{m})$. In this case we get a surjection

$$k'[x_1, \ldots, x_{n-1}] \to k'[x_1, \ldots, x_n] = k' \otimes_k k[x_1, \ldots, x_n] \longrightarrow \kappa(\mathfrak{m})$$

and hence we see that $\kappa(\mathfrak{m})$ is a finite extension of k' by induction hypothesis. Thus $\kappa(\mathfrak{m})$ is finite over k as well.

If $\mathfrak{p} = (0)$ we consider the ring extension $k[x_n] \subset k[x_1, \ldots, x_n]/\mathfrak{m}$. This is a finitely generated ring extension, hence of finite presentation by Lemmas 7.28.3 and 7.28.4. Thus the image of $Spec(k[x_1, \ldots, x_n]/\mathfrak{m})$ in $Spec(k[x_n])$ is constructible by Theorem 7.26.9. Since the image contains (0) we conclude that it contains a standard open $D(f)$ for some $f \in k[x_n]$ nonzero. Since clearly $D(f)$ is infinite we get a contradiction with the assumption that $k[x_1, \ldots, x_n]/\mathfrak{m}$ is a field (and hence has a spectrum consisting of one point).

To prove part (2) let $I \subset R$ be radical, with R of finite type over k. Let $f \in R$, $f \notin I$. Pick a maximal ideal $\mathfrak{m}'$ in the nonzero ring $R_f/IR_f = (R/I)_f$. Let $\mathfrak{m} \subset R$ be the inverse image of $\mathfrak{m}'$ in R. We see that $I \subset \mathfrak{m}$ and $f \notin \mathfrak{m}$. If we show that $\mathfrak{m}$ is a maximal ideal of R, then we are done. We clearly have

$$k \subset R/\mathfrak{m} \subset \kappa(\mathfrak{m}').$$

By part (1) the field extension $k \subset \kappa(\mathfrak{m}')$ is finite. By elementary field theory we conclude that $R/\mathfrak{m}$ is a field. □

Lemma 7.30.2. *Let R be a ring. Let K be a field. If $R \subset K$ and K is of finite type over R, then there exists a $f \in R$ such that R_f is a field, and $R_f \subset K$ is a finite field extension.*

Proof. By Lemma 7.27.2 there exist a nonempty open $U \subset Spec(R)$ contained in the image $\{(0)\}$ of $Spec(K) \to Spec(R)$. Choose $f \in R$, $f \neq 0$ such that $D(f) \subset U$, i.e., $D(f) = \{(0)\}$. Then R_f is a domain whose spectrum has exactly one point and R_f is a field. Then K is a finitely generated algebra over the field R_f and hence a finite field extension of R_f by the Hilbert Nullstellensatz above. □

7.31. Jacobson rings

Let R be a ring. The closed points of $Spec(R)$ are the maximal ideals of R. Often rings which occur naturally in algebraic geometry have lots of maximal ideals. For example finite type algebras over a field or over $\mathbf{Z}$. We will show that these are examples of Jacobson rings.

Definition 7.31.1. Let R be a ring. We say that R is a *Jacobson ring* if every radical ideal I is the intersection of the maximal ideals containing it.

Lemma 7.31.2. *Any algebra of finite type over a field is Jacobson.*

Proof. This follows from Theorem 7.30.1 and Definition 7.31.1. □

Lemma 7.31.3. *Let R be a ring. If every prime ideal of R is the intersection of the maximal ideals containing it, then R is Jacobson.*

Proof. This is immediately clear from the fact that every radical ideal $I \subset R$ is the intersection of the primes containing it. See Lemma 7.16.2. □

Lemma 7.31.4. *A ring R is Jacobson if and only if $Spec(R)$ is Jacobson, see Topology, Definition 5.13.1.*

Proof. Suppose R is Jacobson. Let $Z \subset Spec(R)$ be a closed subset. We have to show that the set of closed points in Z is dense in Z. Let $U \subset Spec(R)$ be an open such that $U \cap Z$ is nonempty. We have to show $Z \cap U$ contains a closed point of $Spec(R)$. We may assume $U = D(f)$ as standard opens form a basis for the topology on $Spec(R)$. According to Lemma 7.16.2 we may assume that $Z = V(I)$, where I is a radical ideal. We see also that $f \notin I$. By assumption, there exists a maximal ideal $\mathfrak{m} \subset R$ such that $I \subset \mathfrak{m}$ but $f \notin \mathfrak{m}$. Hence $\mathfrak{m} \in D(f) \cap V(I) = U \cap Z$ as desired.

Conversely, suppose that $Spec(R)$ is Jacobson. Let $I \subset R$ be a radical ideal. Let $J = \cap_{I \subset \mathfrak{m}} \mathfrak{m}$ be the intersection of the maximal ideals containing I. Clearly J is radical, $V(J) \subset V(I)$, and $V(J)$ is the smallest closed subset of $V(I)$ containing all the closed points of $V(I)$. By assumption we see that $V(J) = V(I)$. But Lemma 7.16.2 shows there is a bijection between Zariski closed sets and radical ideals, hence $I = J$ as desired. □

Lemma 7.31.5. *Let R be a ring. If R is not Jacobson there exist a prime $\mathfrak{p} \subset R$, an element $f \in R$ such that the following hold*

(1) $\mathfrak{p}$ *is not a maximal ideal,*
(2) $f \notin \mathfrak{p}$,
(3) $V(\mathfrak{p}) \cap D(f) = \{\mathfrak{p}\}$, *and*
(4) $(R/\mathfrak{p})_f$ *is a field.*

On the other hand, if R is Jacobson, then for any pair $(\mathfrak{p}, f)$ such that (1) and (2) hold the set $V(\mathfrak{p}) \cap D(f)$ is infinite.

Proof. Assume R is not Jacobson. By Lemma 7.31.4 this means there exists an closed subset $T \subset Spec(R)$ whose set $T_0 \subset T$ of closed points is not dense in T. Choose an $f \in R$ such that $T_0 \subset V(f)$ but $T \not\subset V(f)$. Note that $T \cap D(f)$ is homeomorphic to $Spec((R/I)_f)$ if $T = V(I)$, see Lemmas 7.16.7 and 7.16.6. As any ring has a maximal ideal (Lemma 7.16.2) we can choose a closed point t of space $T \cap D(f)$. Then t corresponds to a prime ideal $\mathfrak{p} \subset R$ which is not maximal (as $t \notin T_0$). Thus (1) holds. By construction $f \notin \mathfrak{p}$, hence (2). As t is a closed point of $T \cap D(f)$ we see that $V(\mathfrak{p}) \cap D(f) = \{\mathfrak{p}\}$, i.e., (3) holds. Hence we conclude that $(R/\mathfrak{p})_f$ is a domain whose spectrum has one point, hence (4) holds (for example combine Lemmas 7.17.2 and 7.23.3).

Conversely, suppose that R is Jacobson and $(\mathfrak{p}, f)$ satisfy (1) and (2). If $V(\mathfrak{p}) \cap V(f) = \{\mathfrak{p}, \mathfrak{q}_1, \ldots, \mathfrak{q}_t\}$ then $\mathfrak{p} \neq \mathfrak{q}_i$ implies there exists an element $g \in R$ such that $g \notin \mathfrak{p}$ but $g \in \mathfrak{q}_i$ for all i. Hence $V(\mathfrak{p}) \cap D(fg) = \{\mathfrak{p}\}$ which is impossible since each locally closed subset of $Spec(R)$ contains at least one closed point as $Spec(R)$ is a Jacobson topological space. □

Lemma 7.31.6. *The ring $\mathbf{Z}$ is a Jacobson ring. More generally, let R be a ring such that*

(1) *R is a domain,*
(2) *R is Noetherian,*
(3) *any nonzero prime ideal is a maximal ideal, and*
(4) *R has infinitely many maximal ideals.*

Then R is a Jacobson ring.

Proof. Let R satisfy (1), (2), (3) and (4). The statement means that $(0) = \bigcap_{\mathfrak{m} \subset R} \mathfrak{m}$. Since R has infinitely many maximal ideals it suffices to show that any nonzero $x \in R$ is contained in at most finitely many maximal ideals, in other words that $V(x)$ is finite. By Lemma 7.16.7 we see that $V(x)$ is homeomorphic to $Spec(R/xR)$. By assumption (3) every prime of R/xR is minimal and hence corresponds to an irreducible component of $Spec(R)$ (Lemma 7.23.1). As R/xR is Noetherian, the topological space $Spec(R/xR)$ is Noetherian (Lemma 7.28.5) and has finitely many irreducible components (Topology, Lemma 5.6.2). Thus $V(x)$ is finite as desired. □

Example 7.31.7. Let A be an infinite set. For each $\alpha \in A$, let k_α be a field. We claim that $R = \prod_{\alpha \in A} k_\alpha$ is Jacobson. First, note that any element $f \in R$ has the form $f = ue$, with $u \in R$ a unit and $e \in R$ an idempotent (left to the reader). Hence $D(f) = D(e)$, and $R_f = R_e = R/(1-e)$ is a quotient of R. Actually, any ring with this property is Jacobson. Namely, say $\mathfrak{p} \subset R$ is a prime ideal and $f \in R$, $f \notin \mathfrak{p}$. We have to find a maximal ideal $\mathfrak{m}$ of R such that $\mathfrak{p} \subset \mathfrak{m}$ and $f \notin \mathfrak{m}$. Because R_f is a quotient of R we see that any maximal ideal of R_f corresponds to a maximal ideal of R not containing f. Hence the result follows by choosing a maximal ideal of R_f containing $\mathfrak{p}R_f$.

Example 7.31.8. A domain R with finitely many maximal ideals $\mathfrak{m}_i$, $i = 1, \ldots, n$ is not a Jacobson ring, except when it is a field. Namely, in this case (0) is not the intersection of the maximal ideals $(0) \neq \mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \ldots \cap \mathfrak{m}_n \supset \mathfrak{m}_1 \cdot \mathfrak{m}_2 \cdot \ldots \cdot \mathfrak{m}_n \neq 0$. In particular a discrete valuation ring, or any local ring with at least two prime ideals is not a Jacobson ring.

Lemma 7.31.9. *Let $R \to S$ be a ring map. Let $\mathfrak{m} \subset R$ be a maximal ideal. Let $\mathfrak{q} \subset S$ be a prime ideal lying over $\mathfrak{m}$ such that $\kappa(\mathfrak{m}) \subset \kappa(\mathfrak{q})$ is an algebraic field extension. Then $\mathfrak{q}$ is a maximal ideal of S.*

Proof. Consider the diagram

$$\begin{tikzcd} S \arrow[r] & S/\mathfrak{q} \arrow[r] & \kappa(\mathfrak{q}) \\ R \arrow[r] \arrow[u] & R/\mathfrak{m} \arrow[u] & \end{tikzcd}$$

We see that $\kappa(\mathfrak{m}) \subset S/\mathfrak{q} \subset \kappa(\mathfrak{q})$. Because the field extension $\kappa(\mathfrak{m}) \subset \kappa(\mathfrak{q})$ is algebraic, any ring between $\kappa(\mathfrak{m})$ and $\kappa(\mathfrak{q})$ is a field (by elementary field theory). Thus $S/\mathfrak{q}$ is a field, and a posteriori equal to $\kappa(\mathfrak{q})$. $\square$

Lemma 7.31.10. *Suppose that k is a field and suppose that V is a nonzero vector space over k. Asssume the dimension of V (which is a cardinal number) is smaller than the cardinality of k. Then for any linear operator $T : V \to V$ there exists some monic polynomial $P(t) \in k[t]$ such that $P(T)$ is not invertible.*

Proof. If not then V inherits the structure of a vector space over the field $k(t)$. But the dimension of $k(t)$ over k is at least the cardinality of k for example due to the fact that the elements $\frac{1}{t-\lambda}$ are k-linearly independent. $\square$

Here is another version of Hilbert's Nullstellensatz.

Theorem 7.31.11. *Let k be a field. Let S be a k-algebra generated over k by the elements $\{x_i\}_{i\in I}$. Assume the cardinality of I is smaller than the cardinality of k. Then*

(1) *for all maximal ideals $\mathfrak{m} \subset S$ the field extension $k \subset \kappa(\mathfrak{m})$ is algebraic, and*
(2) *S is a Jacobson ring.*

Proof. If I is finite then the result follows from the Hilbert Nullstellensatz, Theorem 7.30.1. In the rest of the proof we assume I is infinite. It suffices to prove the result for $\mathfrak{m} \subset k[\{x_i\}_{i\in I}]$ maximal in the polynomial ring on variables x_i, since S is a quotient of this. As I is infinite the set of monomials $x_{i_1}^{e_1} \ldots x_{i_r}^{e_r}$, $i_1, \ldots, i_r \in I$ and $e_1, \ldots, e_r \geq 0$ has cardinality at most equal to the cardinality of I. Because the cardinality of $I \times \ldots \times I$ is the cardinality of I, and also the cardinality of $\bigcup_{n\geq 0} I^n$ has the same cardinality. (If I is finite, then this is not true and in that case this proof only works if k is uncountable.)

To arrive at a contradiction pick $T \in \kappa(\mathfrak{m})$ transcendental over k. Note that the k-linear map $T : \kappa(\mathfrak{m}) \to \kappa(\mathfrak{m})$ given by multiplication by T has the property that $P(T)$ is invertible for all monic polynomials $P(t) \in k[t]$. Also, $\kappa(\mathfrak{m})$ has dimension at most the cardinality of I over k since it is a quotient of the vector space $k[\{x_i\}_{i\in I}]$ over k (whose dimension is $\#I$ as we saw above). This is impossible by Lemma 7.31.10.

To show that S is Jacobson we argue as follows. If not then there exists a prime $\mathfrak{q} \subset S$ and an element $f \in S$, $f \not\in \mathfrak{q}$ such that $\mathfrak{q}$ is not maximal and $(S/\mathfrak{q})_f$ is a field, see Lemma 7.31.5. But note that $(S/\mathfrak{q})_f$ is generated by at most $\#I + 1$ elements. Hence the field extension $k \subset (R/\mathfrak{q})_f$ is algebraic (by the first part of the proof). This implies that $\kappa(\mathfrak{q})$ is an algebraic extension of k hence $\mathfrak{q}$ is maximal by Lemma 7.31.9. This contradiction finishes the proof. $\square$

Lemma 7.31.12. *Let k be a field. Let S be a k-algebra. For any field extension $k \subset K$ whose cardinality is larger than the cardinality of S we have*

(1) *for every maximal ideal* $\mathfrak{m}$ *of* S_K *the field* $\kappa(\mathfrak{m})$ *is algebraic over* K*, and*
(2) S_K *is a Jacobson ring.*

Proof. Choose $k \subset K$ such that the cardinality of K is greater than the cardinality of S. Since the elements of S generate the K-algebra S_K we see that Theorem 7.31.11 applies. $\square$

Example 7.31.13. The trick in the proof of Theorem 7.31.11 really does not work if k is a countable field and I is countable too. Let k be a countable field. Let x be a variable, and let $k(x)$ be the field of rational functions in x. Consider the polynomial algebra $R = k[x, \{x_f\}_{f\in k[x]-\{0\}}]$. Let $I = (\{fx_f - 1\}_{f\in k[x]-\{0\}})$. Note that I is a proper ideal in R. Choose a maximal ideal $I \subset \mathfrak{m}$. Then $k \subset R/\mathfrak{m}$ is isomorphic to $k(x)$, and is not algebraic over k.

Lemma 7.31.14. *Let* R *be a Jacobson ring. Let* $f \in R$*. The ring* R_f *is Jacobson and maximal ideals of* R_f *correspond to maximal ideals of* R*.*

Proof. By Topology, Lemma 5.13.5 we see that $D(f) = Spec(R_f)$ is Jacobson and that closed points of $D(f)$ correspond to closed points in $Spec(R)$ which happen to lie in $D(f)$. Thus we win by Lemma 7.31.4. $\square$

Example 7.31.15. Here is a simple example that shows Lemma 7.31.14 to be false if R is not Jacobson. Consider the ring $R = \mathbf{Z}_{(2)}$, i.e., the localization of $\mathbf{Z}$ at the prime (2). The localization of R at the element 2 is isomorphic to $\mathbf{Q}$, in a formula: $R_2 \cong \mathbf{Q}$. Clearly the map $R \to R_2$ maps the closed point of $Spec(\mathbf{Q})$ to the generic point of $Spec(R)$.

Example 7.31.16. Here is a simple example that shows Lemma 7.31.14 is false if R is Jacobson but we localize at infinitely many elements. Namely, let $R = \mathbf{Z}$ and consider the localization $(R \setminus \{0\})^{-1}R \cong \mathbf{Q}$ of R at the set of all nonzero elements. Clearly the map $\mathbf{Z} \to \mathbf{Q}$ maps the closed point of $Spec(\mathbf{Q})$ to the generic point of $Spec(\mathbf{Z})$.

Lemma 7.31.17. *Let* R *be a Jacobson ring. Let* $I \subset R$ *be an ideal. The ring* R/I *is Jacobson and maximal ideals of* R/I *correspond to maximal ideals of* R*.*

Proof. The proof is the same as the proof of Lemma 7.31.14. $\square$

Proposition 7.31.18. *Let* R *be a Jacobson ring. Let* $R \to S$ *be a ring map of finite type. Then*

(1) *The ring* S *is Jacobson.*
(2) *The map* $Spec(S) \to Spec(R)$ *transforms closed points to closed points.*
(3) *For* $\mathfrak{m}' \subset S$ *maximal lying over* $\mathfrak{m} \subset R$ *the field extension* $\kappa(\mathfrak{m}) \subset \kappa(\mathfrak{m}')$ *is finite.*

Proof. Let $A \to B \to C$ be finite type ring maps. Suppose $Spec(C) \to Spec(B)$ and $Spec(B) \to Spec(A)$ map closed points to closed points, and induce finite residue field extensions on residue fields at closed points. Then so does $Spec(C) \to Spec(A)$. Thus it is clear that if we factor $R \to S$ as $R \to S' \to S$ for some finite type R-algebra S', then it suffices to prove the lemma for $R \to S'$ and then $S' \to S$. Writing $S = R[x_1, \ldots, x_n]/I$ we see that it suffices to prove the lemma in the cases $S = R[x]$ and $S = R/I$. The case $S = R/I$ is Lemma 7.31.17.

The case $S = R[x]$. Take an irreducible closed subset $Z \subset Spec(R[x])$. In other words $Z = V(\mathfrak{q})$ for some prime $\mathfrak{q} \subset R[x]$. Set $\mathfrak{p} = \mathfrak{q} \cap R$. Let $U \subset Spec(R[x])$ be open such that $U \cap Z \neq \emptyset$. We have to find a closed point in $U \cap Z$. In fact, we will find

($*$) a closed point y of $U \cap Z$ which maps to a closed point x of $Spec(R)$ such that additionally $\kappa(x) \subset \kappa(y)$ is finite.

To do this we may assume $U = D(f)$ for some $f \in R[x]$. In this case $U \cap V(\mathfrak{q}) \neq \emptyset$ means $f \notin \mathfrak{q}$. Consider the diagram

$$\begin{tikzcd} R[x] \arrow[r] & R/\mathfrak{p}[x] \\ R \arrow[r] \arrow[u] & R/\mathfrak{p} \arrow[u] \end{tikzcd}$$

It suffices to solve the problem on the right hand side of this diagram. Thus we see we may assume R is Jacobson, a domain and $\mathfrak{p} = (0)$.

In case $\mathfrak{q} = (0)$, write $f = a_d x^d + \ldots + a_0$. We see that not all a_i are zero. Take any maximal ideal $\mathfrak{m}$ of R such that $a_i \notin \mathfrak{m}$ for some i (here we use R is Jacobson). Next, choose a maximal ideal $\overline{\mathfrak{m}}' \subset (R/\mathfrak{m})[x]$ not containing the image of f (possible because $\kappa(\mathfrak{m})[x]$ is Jacobson). Then the inverse image $\mathfrak{m}' \subset R[x]$ defines a closed point of $U \cap Z$ and maps to $\mathfrak{m}$. Also, by construction $\kappa(\mathfrak{m}) \subset \kappa(\mathfrak{m}')$ is finite. Thus we have shown ($*$) in this case.

In case $\mathfrak{q} \neq (0)$, let K be the fraction field of R. Write $\mathfrak{q}K[x] = (g)$ for some irreducible $g \in K[x]$. Clearing denominators, we may assume that $g \in R[x]$, and hence in $\mathfrak{q}$. Write $g = b_e x^e + \ldots + b_0$, $b_i \in R$ with $b_e \neq 0$. The maps $R \to R_{b_e}$ and $R[x] \to R[x]_{b_e}$ satisfies the conclusion of the lemma, by Lemma 7.31.14 and moreover induce isomorphisms on residue fields. Hence, in order to prove ($*$), we may replace R by R_{b_e} and assume that g is monic. In this case we see that $R[x]/\mathfrak{q}$ is a quotient of the finite free R-module $R[x]/(g) = R \oplus Rx \oplus \ldots \oplus Rx^{e-1}$. But on the other hand we have $R[x]/(g) \subset K[x]/(g) = K[x]/\mathfrak{q}K[x]$. Hence $\mathfrak{q} = (g)$, and $Z = V(\mathfrak{q}) = V(g)$. At this point, by Lemma 7.26.8 the image of $D(f) \cap V(g)$ in $Spec(R)$ is $D(r_1) \cup \ldots \cup D(r_d)$ for some $r_i \in R$ (of course it is nonempty). Take any maximal ideal $\mathfrak{m} \subset R$ in this image (possible because R is Jacobson) and take any prime $\mathfrak{m}' \subset R[x]$ corresponding to a point of $D(f) \cap V(g)$ lying over $\mathfrak{m}$. Note that the residue field extension $\kappa(\mathfrak{m}) \subset \kappa(\mathfrak{m}')$ is finite (because $g \in \mathfrak{m}'$). By Lemma 7.31.9 we see that $\mathfrak{m}'$ is a closed point. This proves ($*$) in this case.

At this point we are done. Namely, ($*$) implies that $Spec(R[x])$ is Jacobson (via Lemma 7.31.4). Also, if Z is a singleton closed set, then ($*$) implies that $Z = \{\mathfrak{m}'\}$ with $\mathfrak{m}'$ lying over a maximal ideal $\mathfrak{m} \subset R$ such that $\kappa(\mathfrak{m}) \subset \kappa(\mathfrak{m}')$ is finite. □

Lemma 7.31.19. *Any finite type algebra over* $\mathbf{Z}$ *is Jacobson.*

Proof. Combine Lemma 7.31.6 and Proposition 7.31.18. □

Lemma 7.31.20. *Let $R \to S$ be a finite type ring map of Jacobson rings. Denote $X = Spec(R)$ and $Y = Spec(S)$. Write $f : Y \to X$ the induced map of spectra. Let $E \subset Y = Spec(S)$ be a constructible set. Denote with a subscript $_0$ the set of closed points of a topological space.*

(1) *We have $f(E)_0 = f(E_0) = X_0 \cap f(E)$.*
(2) *A point $\xi \in X$ is in $f(E)$ if and only if $\overline{\{\xi\}} \cap f(E_0)$ is dense in $\overline{\{\xi\}}$.*

Proof. We have a commutative diagram of continuous maps

$$\begin{tikzcd} E \arrow[r] \arrow[d] & Y \arrow[d] \\ f(E) \arrow[r] & X \end{tikzcd}$$

Suppose $x \in f(E)$ is closed in $f(E)$. Then $f^{-1}(\{x\}) \cap E$ is closed in E. Hence $f^{-1}(\{x\}) \cap E$ is constructible, nonempty in Y. By Topology, Lemma 5.13.5, the intersection $Y_0 \cap f^{-1}(\{x\}) \cap E$ is not empty. Thus there exists $y \in Y_0$ mapping to x. Since clearly $y \in E_0$ we see that $x \in f(E_0)$. This proves that $f(E)_0 \subset f(E_0)$. Proposition 7.31.18 implies that $f(E_0) \subset X_0 \cap f(E)$. The inclusion $X_0 \cap f(E) \subset f(E)_0$ is trivial. This proves the first assertion.

Suppose that $\xi \in f(E)$. According to Lemma 7.27.2 the set $f(E) \cap \overline{\{\xi\}}$ contains a dense open subset of $\overline{\{\xi\}}$. Since X is Jacobson we conclude that $f(E) \cap \overline{\{\xi\}}$ contains a dense set of closed points, see Topology, Lemma 5.13.5. We conclude by part (1) of the lemma.

On the other hand, suppose that $\overline{\{\xi\}} \cap f(E_0)$ is dense in $\overline{\{\xi\}}$. By Lemma 7.26.3 there exists a ring map $S \to S'$ of finite presentation such that E is the image of $Y' := Spec(S') \to Y$. Then E_0 is the image of Y'_0 by the first part of the lemma applied to the ring map $S \to S'$. Thus we may assume that $E = Y$ by replacing S by S'. Suppose ξ corresponds to $\mathfrak{p} \subset R$. Consider the diagram

$$\begin{tikzcd} S \arrow[r] & S/\mathfrak{p}S \\ R \arrow[r] \arrow[u] & R/\mathfrak{p} \arrow[u] \end{tikzcd}$$

This diagram and the density of $f(Y_0) \cap V(\mathfrak{p})$ in $V(\mathfrak{p})$ shows that the morphism $R/\mathfrak{p} \to S/\mathfrak{p}S$ satisfies condition (2) of Lemma 7.27.4. Hence we conclude there exists a prime $\overline{\mathfrak{q}} \subset S/\mathfrak{p}S$ mapping to (0). In other words the inverse image $\mathfrak{q}$ of $\overline{\mathfrak{q}}$ in S maps to $\mathfrak{p}$ as desired. □

The conclusion of the lemma above is that we can read off the image of f from the set of closed points of the image. This is a little nicer in case the map is of finite presentation because then we know that images of constructibles are constructible. Before we state it we introduce some notation. Denote Constr(X) the set of constructible Let $R \to S$ be a ring map. Denote $X = Spec(R)$ and $Y = Spec(S)$. Write $f : Y \to X$ the induced map of spectra. Denote with a subscript $_0$ the set of closed points of a topological space.

Lemma 7.31.21. *With notation as above. Assume that R is a Noetherian Jacobson ring. Further assume $R \to S$ is of finite type. There is a commutative diagram*

$$\begin{tikzcd} Constr(Y) \arrow[r, "E \mapsto E_0"] \arrow[d, "E \mapsto f(E)"] & Constr(Y_0) \arrow[d, "E \mapsto f(E)"] \\ Constr(X) \arrow[r, "E \mapsto E_0"] & Constr(X_0) \end{tikzcd}$$

where the horizontal arrows are the bijections from Topology, Lemma 5.13.7.

Proof. Since $R \to S$ is of finite type, it is of finite presentation, see Lemma 7.28.4. Thus the image of a constructible set in X is constructible in Y by Chevalley's theorem (Theorem 7.26.9). Combined with Lemma 7.31.20 above the lemma follows. □

To illustrate the use of Jacobson rings, we give the following two examples.

Example 7.31.22. Let k be a field. The space $Spec(k[x, y]/(xy))$ has two irreducible components: namely the x-axis and the y-axis. As a generalization, let

$$R = k[x_{11}, x_{12}, x_{21}, x_{22}, y_{11}, y_{12}, y_{21}, y_{22}]/\mathfrak{a},$$

where $\mathfrak{a}$ is the ideal in $k[x_{11}, x_{12}, x_{21}, x_{22}, y_{11}, y_{12}, y_{21}, y_{22}]$ generated by the entries of the 2×2 product matrix

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix},$$

we shall also describe $Spec(R)$

To prove the statement about $Spec(k[x, y]/(xy))$ we argue as follows. If $\mathfrak{p} \subset k[x, y]$ is any ideal containing xy, then either x or y would be contained in $\mathfrak{p}$. Hence the minimal such prime ideals are just (x) and (y). In case k is algebraically closed, the max-Spec of these components can then be visualized as the point sets of y and x axis.

For the generalization, note that we may identify the closed points of the spectrum of $k[x_{11}, x_{12}, x_{21}, x_{22}, y_{11}, y_{12}, y_{21}, y_{22}])$ with the space of matrices

$$\left\{ (X, Y) \in Mat(2, k) \times Mat(2, k) \mid X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, Y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \right\}$$

at least if k is algebraically closed. Now define a group action of $GL(2, k) \times GL(2, k) \times GL(2, k)$ on the space of matrices $\{(X, Y)\}$ by

$$(g_1, g_2, g_3) \times (X, Y) \mapsto ((g_1 X g_2^{-1}, g_2 Y g_3^{-1})).$$

Here, also observe that the algebraic set

$$GL(2, k) \times GL(2, k) \times GL(2, k) \subseteq Mat(2, k) \times Mat(2, k) \times Mat(2, k)$$

is irreducible since it is the max spectrum of the domain

$$k[x_{11}, x_{12}, \ldots, z_{21}, z_{22}, (x_{11}x_{22} - x_{12}x_{21})^{-1}, (y_{11}y_{22} - y_{12}y_{21})^{-1}, (z_{11}z_{22} - z_{12}z_{21})^{-1}].$$

Since the image of irreducible an algebraic set is still irreducible, it suffices to classify the orbits of the set $\{(X, Y) \in Mat(2, k) \times Mat(2, k) | XY = 0\}$ and take their closures. From standard linear algebra, we are reduced to the following three cases:

(1) $\exists (g_1, g_2)$ such that $g_1 X g_2^{-1} = I_{2\times 2}$. Then Y is necessarily 0, which as an algebraic set is invariant under the group action. It follows that this orbit is contained in the irreducible algebraic set defined by the prime ideal $(y_{11}, y_{12}, y_{21}, y_{22})$. Taking the closure, we see that $(y_{11}, y_{12}, y_{21}, y_{22})$ is actually a component.

(2) $\exists (g_1, g_2)$ such that

$$g_1 X g_2^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

This case occurs if and only if X is a rank 1 matrix, and furthermore, Y is killed by such an X if and only if

$$x_{11}y_{11} + x_{12}y_{21} = 0; \quad x_{11}y_{12} + x_{12}y_{22} = 0;$$

$$x_{21}y_{11} + x_{22}y_{21} = 0; \quad x_{21}y_{12} + x_{22}y_{22} = 0.$$

Fix a rank 1 X, such non zero Y's satisfying the above equations form an irreducible algebraic set for the following reason($Y = 0$ is contained the the previous case): $0 = g_1 X g_2^{-1} g_2 Y$ implies that

$$g_2 Y = \begin{pmatrix} 0 & 0 \\ y'_{21} & y'_{22} \end{pmatrix}.$$

With a further $GL(2,k)$-action on the right by g_3, $g_2 Y$ can be brought into

$$g_2 Y g_3^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and thus such Y's form an irreducible algebraic set isomorphic to the image of $GL(2,k)$ under this action. Finally, notice that the ``rank 1" condition for X's forms an open dense subset of the irreducible algebraic set $\det X = x_{11}x_{22} - x_{12}x_{21} = 0$. It now follows that all the five equations define an irreducible component $(x_{11}y_{11}+x_{12}y_{21}, x_{11}y_{12}+x_{12}y_{22}, x_{21}y_{11}+x_{22}y_{21}, x_{21}y_{12}+x_{22}y_{22}, x_{11}x_{22}-x_{12}x_{21})$, in open subset of the space of pairs of nonzero matrices. It can be shown that the pair of equations $\det X = 0$, $\det Y = 0$ cuts $Spec(R)$ in an irreducible component with the above locus an open dense subset.

(3) $\exists (g_1, g_2)$ such that $g_1 X g_2^{-1} = 0$, or equivalently, $X = 0$. Then Y can be arbitrary and this component is thus defined by $(x_{11}, x_{12}, x_{21}, x_{22})$.

Example 7.31.23. For another example, consider $R = k[\{t_{ij}\}_{i,j=1}^n]/\mathfrak{a}$, where $\mathfrak{a}$ is the ideal generated by the entries of the product matrix $T^2 - T$, $T = (t_{ij})$. From linear algebra, we know that under the $GL(n,k)$-action defined by $g, T \mapsto gTg^{-1}$, T is classified by the its rank and each T is conjugate to some $\mathrm{diag}(1, \ldots, 1, 0, \ldots, 0)$, which has r 1's and $n - r$ 0's. Thus each orbit of such a $\mathrm{diag}(1, \ldots, 1, 0, \ldots, 0)$ under the group action forms an irreducible component and every idempotent matrix is contained in one such orbit. Next we will show that any two different orbits are necessarily disjoint. For this purpose we only need to cook up polynomial functions that take different values on different orbits. In characteristic 0 cases, such a function can be taken to be $f(t_{ij}) = trace(T) = \sum_{i=1}^n t_{ii}$. In positive characteristic cases, things are slightly more tricky since we might have $trace(T) = 0$ even if $T \neq 0$. For instance, $char = 3$

$$trace \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} = 3 = 0$$

Anyway, these components can be separated using other functions. For instance, in the characteristic 3 case, $tr(\wedge^3 T)$ takes value 1 on the components corresponding to $diag(1,1,1)$ and 0 on other components.

7.32. Finite and integral ring extensions

Trivial lemmas concerning finite and integral ring maps. We recall the definition.

Definition 7.32.1. Let $\varphi : R \to S$ be a ring map.

(1) An element $s \in S$ is *integral over R* if there exists a monic polynomial $P(x) \in R[x]$ such that $P^\varphi(s) = 0$, where $P^\varphi(x) \in S[x]$ is the image of P under $\varphi : R[x] \to S[x]$.
(2) The ring map φ is *integral* if every $s \in S$ is integral over R.

Lemma 7.32.2. *Let $\varphi : R \to S$ be a ring map. Let $y \in S$. If there exists a finite R-submodule M of S such that $1 \in M$ and $yM \subset M$, then y is integral over R.*

Proof. Let $x_1 = 1 \in M$ and $x_i \in M$, $i = 2, \ldots, n$ be a finite set of elements generating M as an R-module. Write $yx_i = \sum \varphi(a_{ij})x_j$ for some $a_{ij} \in R$. Let $P(T) \in R[T]$ be the characteristic polynomial of the $n \times n$ matrix $A = (a_{ij})$. By Lemma 7.15.1 we see $P(A) = 0$. By construction the map $\pi : R^n \to M$, $(a_1, \ldots, a_n) \mapsto \sum \varphi(a_i)x_i$ commutes with $A : R^n \to R^n$ and multiplication by y. In a formula $\pi(Av) = y\pi(v)$. Thus $P(y) = P(y) \cdot 1 = P(y) \cdot x_1 = P(y) \cdot \pi((1, 0, \ldots, 0)) = \pi(P(A)(1, 0, \ldots, 0)) = 0$. □

Lemma 7.32.3. *A finite ring extension is integral.*

Proof. Let $R \to S$ be finite. Let $y \in S$. Apply Lemma 7.32.2 to $M = S$ to see that y is integral over R. □

Lemma 7.32.4. *Let $\varphi : R \to S$ be a ring map. Let $s_1, \ldots, s_n$ be a finite set of elements of S. In this case s_i is integral over R for all $i = 1, \ldots, n$ if and only if there exists an R-subalgebra $S' \subset S$ finite over R containing all of the s_i.*

Proof. If each s_i is integral, then the subalgebra generated by $\varphi(R)$ and the s_i is finite over R. Namely, if s_i satisfies a monic equation of degree d_i over R, then this subalgebra is generated as an R-module by the elements $s_1^{e_1} \ldots s_n^{e_n}$ with $0 \leq e_i \leq d_i - 1$. Conversely, suppose given a finite R-subalgebra S' containing all the s_i. Then all of the s_i are integral by Lemma 7.32.3. □

Lemma 7.32.5. *Let $R \to S$ be a ring map. The following are equivalent*

(1) *$R \to S$ is finite,*
(2) *$R \to S$ is integral and of finite type, and*
(3) *there exist $x_1, \ldots, x_n \in S$ which generate S as an algebra over R such that each x_i is integral over R.*

Proof. Clear from Lemma 7.32.4. □

Lemma 7.32.6. *Suppose that $R \to S$ and $S \to T$ are integral ring maps. Then $R \to T$ is integral.*

Proof. Let $t \in T$. Let $P(x) \in S[x]$ be a monic polynomial such that $P(t) = 0$. Apply Lemma 7.32.4 to the finite set of coefficients of P. Hence t is integral over some subalgebra $S' \subset S$ finite over R. Apply Lemma 7.32.4 again to find a subalgebra $T' \subset T$ finite over S' and containing t. Lemma 7.7.3 applied to $R \to S' \to T'$ shows that T' is finite over R. The integrality of t over R now follows from Lemma 7.32.3. □

Lemma 7.32.7. *Let $R \to S$ be a ring homomorphism. The set*

$$S' = \{s \in S \mid s \text{ is integral over } R\}$$

is an R-subalgebra of S.

Proof. This is clear from Lemmas 7.32.4 and 7.32.3. □

Definition 7.32.8. Let $R \to S$ be a ring map. The ring $S' \subset S$ of elements integral over R, see Lemma 7.32.7, is called the *integral closure* of R in S. If $R \subset S$ we say that R is *integrally closed* in S if $R = S'$.

In particular, we see that $R \to S$ is integral if and only if the integral closure of R in S is all of S.

Lemma 7.32.9. *Integral closure commutes with localization: If $A \to B$ is a ring map, and $S \subset A$ is a multiplicative subset, then the integral closure of $S^{-1}A$ in $S^{-1}B$ is $S^{-1}B'$, where $B' \subset B$ is the integral closure of A in B.*

Proof. Since localization is exact we see that $S^{-1}B' \subset S^{-1}B$. Suppose $x \in B'$ and $f \in S$. Then $x^d + \sum_{i=1,\ldots,d} a_i x^{d-i} = 0$ in B for some $a_i \in A$. Hence also

$$(x/f)^d + \sum\nolimits_{i=1,\ldots,d} a_i/f^i (x/f)^{d-i} = 0$$

in $S^{-1}B$. In this way we see that $S^{-1}B'$ is contained in the integral closure of $S^{-1}A$ in $S^{-1}B$. Conversely, suppose that $x/f \in S^{-1}B$ is integral over $S^{-1}A$. Then we have

$$(x/f)^d + \sum\nolimits_{i=1,\ldots,d} (a_i/f_i)(x/f)^{d-i} = 0$$

in $S^{-1}B$ for some $a_i \in A$ and $f_i \in S$. This means that

$$(f' f_1 \ldots f_d x)^d + \sum\nolimits_{i=1,\ldots,d} f^i (f')^i f_1^i \ldots f_i^{i-1} \ldots f_d^i a_i (f' f_1 \ldots f_d x)^{d-i} = 0$$

for a suitable $f' \in S$. Hence $f' f_1 \ldots f_d x \in B'$ and thus $x/f \in S^{-1}B'$ as desired. $\square$

Lemma 7.32.10. *Let $\varphi : R \to S$ be a ring map. Let $x \in S$. The following are equivalent:*

(1) *x is integral over R, and*
(2) *for every $\mathfrak{p} \in R$ the element $x \in S_{\mathfrak{p}}$ is integral over $R_{\mathfrak{p}}$.*

Proof. It is clear that (1) implies (2). Assume (1). Consider the R-algebra $S' \subset S$ generated by $\varphi(R)$ and x. Let $\mathfrak{p}$ be a prime ideal of R. Then we know that $x^d + \sum_{i=1,\ldots,d} \varphi(a_i) x^{d-i} = 0$ in $S_{\mathfrak{p}}$ for some $a_i \in R_{\mathfrak{p}}$. Hence we see, by looking at which denominators occur, that for some $f \in R$, $f \not\in \mathfrak{p}$ we have $a_i \in R_f$ and $x^d + \sum_{i=1,\ldots,d} \varphi(a_i) x^{d-i} = 0$ in S_f. This implies that S'_f is finite over R_f. Since $\mathfrak{p}$ was arbitrary and $Spec(R)$ is quasi-compact (Lemma 7.16.10) we can find finitely many elements $f_1, \ldots, f_n \in R$ which generate the unit ideal of R such that S'_{f_i} is finite over R. Hence we conclude from Lemma 7.21.2 that S' is finite over R. Hence x is integral over R by Lemma 7.32.4. $\square$

Lemma 7.32.11. *Let $R \to S$ and $R \to R'$ be ring maps. Set $S' = R' \otimes_R S$.*

(1) *If $R \to S$ is integral so is $R' \to S'$.*
(2) *If $R \to S$ is finite so is $R' \to S'$.*

Proof. We prove (1). Let $s_i \in S$ be generators for S over R. Each of these satisfies a monic polynomial equation P_i over R. Hence the elements $1 \otimes s_i \in S'$ generate S' over R' and satisfy the corresponding polynomial P_i' over R'. Since these elements generate S' over R' we see that S' is integral over R'. Proof of (2) omitted. $\square$

Lemma 7.32.12. *Let $R \to S$ be a ring map. Let $f_1, \ldots, f_n \in R$ generate the unit ideal.*

(1) *If each $R_{f_i} \to S_{f_i}$ is integral, so is $R \to S$.*
(2) *If each $R_{f_i} \to S_{f_i}$ is finite, so is $R \to S$.*

Proof. Proof of (1). Let $s \in S$. Consider the ideal $I \subset R[x]$ of polynomials P such that $P(s) = 0$. Let $J \subset R$ denote the ideal (!) of leading coefficients of elements of I. By assumption and clearing denominators we see that $f_i^{n_i} \in J$ for all i and certain $n_i \geq 0$. Hence J contains 1 and we see s is integral over R. Proof of (2) omitted. $\square$

Lemma 7.32.13. *Let $A \to B \to C$ be ring maps.*

(1) *If $A \to C$ is integral so is $B \to C$.*

(2) *If $A \to C$ is finite so is $B \to C$.*

Proof. Omitted. ☐

Lemma 7.32.14. *Let $A \to B \to C$ be ring maps. Let B' be the integral closure of A in B, let C' be the integral closure of B' in C. Then C' is the integral closure of A in C.*

Proof. Omitted. ☐

Lemma 7.32.15. *Suppose that $R \to S$ is an integral ring extension with $R \subset S$. Then $\varphi : Spec(S) \to Spec(R)$ is surjective.*

Proof. Let $\mathfrak{p} \subset R$ be a prime ideal. We have to show $\mathfrak{p}S_{\mathfrak{p}} \neq S_{\mathfrak{p}}$, see Lemma 7.16.9. The localization $R_{\mathfrak{p}} \to S_{\mathfrak{p}}$ is injective (as localization is exact) and integral by Lemma 7.32.9 or 7.32.11. Hence we may replace R, S by $R_{\mathfrak{p}}$, $S_{\mathfrak{p}}$ and we may assume R is local with maximal ideal $\mathfrak{m}$ and it suffices to show that $\mathfrak{m}S \neq S$. Suppose $1 = \sum f_i s_i$ with $f_i \in \mathfrak{m}$ and $s_i \in S$ in order to get a contradiction. Let $R \subset S' \subset S$ be such that $R \to S'$ is finite and $s_i \in S'$, see Lemma 7.32.4. The equation $1 = \sum f_i s_i$ implies that the finite R-module S' satisfies $S' = \mathfrak{m}S'$. Hence by Nakayama's Lemma 7.14.5 we see $S' = 0$. Contradiction. ☐

Lemma 7.32.16. *Let R be a ring. Let K be a field. If $R \subset K$ and K is integral over R, then R is a field and K is an algebraic extension. If $R \subset K$ and K is finite over R, then R is a field and K is a finite algebraic extension.*

Proof. Assume that $R \subset K$ is integral. By Lemma 7.32.15 above we see that $Spec(R)$ has 1 point. Since clearly R is a domain we see that $R = R_{(0)}$ is a field. The other assertions are immediate from this. ☐

Lemma 7.32.17. *Let k be a field. Let S be a k-algebra over k.*

(1) *If S is a domain and finite dimensional over k, then S is a field.*
(2) *If S is integral over k and a domain, then S is a field.*
(3) *If S is integral over k then every prime of S is a maximal ideal (see Lemma 7.23.5 for more consequences).*

Proof. The statement on primes follows from the statement ``integral + domain $\Rightarrow$ field''. Let S integral over k and assume S is a domain, Take $s \in S$. By Lemma 7.32.4 we may find a finite dimensional k-subalgebra $k \subset S' \subset S$. containing s. Hence S is a field if we can prove the first statement. Assume S finite dimensional over k and a domain. Pick $s \in S$. Since S is a domain the multiplication map $s : S \to S$ is surjective by dimension reasons. Hence there exists an element $s_1 \in S$ such that $ss_1 = 1$. So S is a field. ☐

Lemma 7.32.18. *Suppose $R \to S$ is integral. Let $\mathfrak{q}, \mathfrak{q}' \in Spec(S)$ be distinct primes having the same image in $Spec(R)$. Then neither $\mathfrak{q} \subset \mathfrak{q}'$ nor $\mathfrak{q}' \subset \mathfrak{q}$.*

Proof. Let $\mathfrak{p} \subset R$ be the image. By Remark 7.16.8 the primes $\mathfrak{q}, \mathfrak{q}'$ correspond to ideals in $S \otimes_R \kappa(\mathfrak{p})$. Thus the lemma follows from Lemma 7.32.17. ☐

Lemma 7.32.19. *Suppose $R \to S$ is finite. Then the fibres of $Spec(S) \to Spec(R)$ are finite.*

Proof. By the discussion in Remark 7.16.8 the fibres are the spectra of the rings $S \otimes_R \kappa(\mathfrak{p})$. As $R \to S$ is finite, these fibre rings are finite over $\kappa(\mathfrak{p})$ hence Noetherian by Lemma 7.28.1. By Lemma 7.32.18 every prime of $S \otimes_R \kappa(\mathfrak{p})$ is a minimal prime. Hence by Lemma 7.28.6 there are at most finitely many. ☐

Lemma 7.32.20. *Let $R \to S$ be a ring map such that S is integral over R. Let $\mathfrak{p} \subset \mathfrak{p}' \subset R$ be primes. Let $\mathfrak{q}$ be a prime of S mapping to $\mathfrak{p}$. Then there exists a prime $\mathfrak{q}'$ with $\mathfrak{q} \subset \mathfrak{q}'$ mapping to $\mathfrak{p}'$.*

Proof. We may replace R by $R/\mathfrak{p}$ and S by $S/\mathfrak{q}$. This reduces us to the situation of having an integral extension of domains $R \subset S$ and a prime $\mathfrak{p}' \subset R$. By Lemma 7.32.15 we win. □

The property expressed in the lemma above is called the ``going up property" for the ring map $R \to S$, see Definition 7.36.1.

Lemma 7.32.21. *Let R be a ring. Let $x, y \in R$ be nonzero divisors. Let $R[x/y] \subset R_{xy}$ be the R-subalgebra generated by x/y, and similarly for the subalgebras $R[y/x]$ and $R[x/y, y/x]$. If R is integrally closed in R_x or R_y, then the sequence*

$$0 \to R \xrightarrow{(-1,1)} R[x/y] \oplus R[y/x] \xrightarrow{(1,1)} R[x/y, y/x] \to 0$$

is a short exact sequence of R-modules.

Proof. Since $x/y \cdot y/x = 1$ it is clear that the map $R[x/y] \oplus R[y/x] \to R[x/y, y/x]$ is surjective. Let $\alpha \in R[x/y] \cap R[y/x]$. To show exactness in the middle we have to prove that $\alpha \in R$. By assumption we may write

$$\alpha = a_0 + a_1 x/y + \ldots + a_n (x/y)^n = b_0 + b_1 y/x + \ldots + b_m (y/x)^m$$

for some $n, m \geq 0$ and $a_i, b_j \in R$. Pick some $N > \max(n, m)$. Consider the finite R-submodule M of R_{xy} generated by the elements

$$(x/y)^N, (x/y)^{N-1}, \ldots, x/y, 1, y/x, \ldots, (y/x)^{N-1}, (y/x)^N$$

We claim that $\alpha M \subset M$. Namely, it is clear that $(x/y)^i (b_0 + b_1 y/x + \ldots + b_m (y/x)^m) \in M$ for $0 \leq i \leq N$ and that $(y/x)^i (a_0 + a_1 x/y + \ldots + a_n (x/y)^n) \in M$ for $0 \leq i \leq N$. Hence α is integral over R by Lemma 7.32.2. Note that $\alpha \in R_x$, so if R is integrally closed in R_x then $\alpha \in R$ as desired. □

7.33. Normal rings

We first introduce the notion of a normal domain, and then we introduce the (very general) notion of a normal ring.

Definition 7.33.1. A domain R is called *normal* if it is integrally closed in its field of fractions.

Lemma 7.33.2. *Let $R \to S$ be a ring map. If S is a normal domain, then the integral closure of R in S is a normal domain.*

Proof. Omitted. □

The following notion is occasionally useful when studying normality.

Definition 7.33.3. Let R be a domain.

(1) An element g of the fraction field of R is called *almost integral over R* if there exists an element $r \in R$, $r \neq 0$ such that $rg^n \in R$ for all $n \geq 0$.
(2) The domain R is called *completely normal* if every almost integral element of the fraction field of R is contained in R.

The following lemma shows that a Noetherian domain is normal if and only if it is completely normal.

Lemma 7.33.4. *Let R be a domain with fraction field K. If $u, v \in K$ are almost integral over R, then so are $u + v$ and uv. Any element $g \in K$ which is integral over R is almost integral over R. If R is Noetherian then the converse holds as well.*

Proof. If $ru^n \in R$ for all $n \geq 0$ and $v^n r' \in R$ for all $n \geq 0$, then $(uv)^n rr'$ and $(u+v)^n rr'$ are in R for all $n \geq 0$. Hence the first assertion. Suppose $g \in K$ is integral over R. In this case there exists an $d > 0$ such that the ring $R[g]$ is generated by $1, g, \ldots, g^d$ as an R-module. Let $r \in R$ be a common denominator of the elements $1, g, \ldots, g^d \in K$. It is follows that $rR[g] \subset R$, and hence g is almost integral over R.

Suppose R is Noetherian and $g \in K$ is almost integral over R. Let $r \in R$, $r \neq 0$ be as in the definition. Then $R[g] \subset \frac{1}{r}R$ as an R-module. Since R is Noetherian this implies that $R[g]$ is finite over R. Hence g is integral over R, see Lemma 7.32.3. $\square$

Lemma 7.33.5. *Any localization of a normal domain is normal.*

Proof. Let R be a normal domain, and let $S \subset R$ be a multiplicative subset. Suppose g is an element of the fraction field of R which is integral over $S^{-1}R$. Let $P = x^d + \sum_{j<d} a_j x^j$ be a polynomial with $a_i \in S^{-1}R$ such that $P(g) = 0$. Choose $s \in S$ such that $sa_i \in R$ for all i. Then sg satisfies the monic polynomial $x^d + \sum_{j<d} s^{d-j} a_j x^j$ which has coefficients $s^{d-j} a_j$ in R. Hence $sg \in R$ because R is normal. Hence $g \in S^{-1}R$. $\square$

Lemma 7.33.6. *A principal ideal domain is normal.*

Proof. Let R be a principal ideal domain. Let $g = a/b$ be an element of the fraction field of R integral over R. Because R is a principal ideal domain we may divide out a common factor of a and b and assume $(a, b) = R$. In this case, any equation $(a/b)^n + r_{n-1}(a/b)^{n-1} + \ldots + r_0 = 0$ with $r_i \in R$ would imply $a^n \in (b)$. This contradicts $(a, b) = R$ unless b is a unit in R. $\square$

Lemma 7.33.7. *Let R be a domain with fraction field K. Suppose $f = \sum \alpha_i x^i$ is an element of $K[x]$.*

(1) *If f is integral over $R[x]$ then all α_i are integral over $R[x]$, and*
(2) *If f is almost integral over $R[x]$ then all α_i are almost integral over $R[x]$.*

Proof. We first prove the second statement. Write $f = \alpha_d x^d + \ldots + \alpha_r x^r$ with $\alpha_r \neq 0$. By assumption there exists $h = b_d x^d + \ldots + b_s x^s \in R[x]$, $b_s \neq 0$ such that $f^n h \in R[x]$ for all $n \geq 0$. This implies that $b_s \alpha_r^n \in R$ for all $n \geq 0$. Hence α_r is almost integral over R. Since the set of almost integral elements form a subring we deduce that $f - \alpha_r x^r = \alpha_d x^d + \ldots + \alpha_{r-1} x^{r-1}$ is almost integral over $R[x]$. By induction on $d - r$ we win.

In order to prove the first statement we will use absolute Noetherian reduction. Namely, write $\alpha_i = a_i/b_i$ and let $P(t) = t^d + \sum_{j<d} f_j t^j$ be a polynomial with coefficients $f_j \in R[x]$ such that $P(g) = 0$. Let $f_j = \sum f_{ji} x^i$. Consider the subring $R_0 \subset R$ generated by the finite list of elements a_i, b_i, f_{ji} of R. It is a domain; let K_0 be its field of fractions. Since R_0 is a finite type $\mathbf{Z}$-algebra it is Noetherian, see Lemma 7.28.3. It is still the case that $g \in K_0[x]$ is integral over $R_0[x]$, because all the identities in R among the elements a_i, b_i, f_{ji} also hold in R_0. By Lemma 7.33.4 the element g is almost integral over $R_0[x]$. By the first part of the lemma, the elements α_i are almost integral over R_0. And since R_0 is Noetherian, the are integral over R_0, see Lemma 7.33.4. Of course, then they are integral over R. $\square$

Lemma 7.33.8. *Let R be a normal domain. Then $R[x]$ is a normal domain.*

Proof. The result is true if R is a field K because $K[x]$ is a euclidean domain and hence a principal ideal domain and hence normal by Lemma 7.33.6. Let g be an element of the fraction field of $R[x]$ which is integral over $R[x]$. Because g is integral over $K[x]$ where K is the fraction field of R we may write $g = \alpha_d x^d + \alpha_{d-1}x^{d-1} + \ldots + \alpha_0$ with $\alpha_i \in K$. By Lemma 7.33.7 the elements α_i are integral over R and hence are in R. $\square$

Lemma 7.33.9. *Let R be a domain. The following are equivalent:*

(1) *The domain R is a normal domain,*
(2) *for every prime $\mathfrak p \subset R$ the local ring $R_{\mathfrak p}$ is a normal domain, and*
(3) *for every maximal ideal $\mathfrak m$ the ring $R_{\mathfrak m}$ is a normal domain.*

Proof. This follows easily from the fact that for any domain R we have

$$R = \bigcap\nolimits_{\mathfrak m} R_{\mathfrak m}$$

inside the fraction field of R. Namely, if g is an element of the right hand side then the ideal $I = \{x \in R \mid xg \in R\}$ is not contained in any maximal ideal $\mathfrak m$, whence $I = R$. $\square$

Lemma 7.33.9 shows that the following definition is compatible with Definition 7.33.1. (It is the definition from EGA -- see [**DG67**, IV, 5.13.5 and 0, 4.1.4].)

Definition 7.33.10. A ring R is called *normal* if for every prime $\mathfrak p \subset R$ the localization $R_{\mathfrak p}$ is a normal domain (see Definition 7.33.1).

Note that a normal ring is a reduced ring, as R is a subring of the product of its localizations at all primes (see for example Lemma 7.21.1).

Lemma 7.33.11. *A normal ring is integrally closed in its total ring of fractions.*

Proof. Let R be a normal ring. Let $x \in Q(R)$ be an element of the total ring of fractions of R integral over R. Set $I = \{f \in R, fx \in R\}$. Let $\mathfrak p \subset R$ be a prime. As $R \subset R_{\mathfrak p}$ is flat we see that $R_{\mathfrak p} \subset Q(R) \otimes R_{\mathfrak p}$. As $R_{\mathfrak p}$ is a normal domain we see that $x \otimes 1$ is an element of $R_{\mathfrak p}$. Hence we can find $a, f \in R$, $f \not\in \mathfrak p$ such that $x \otimes 1 = a \otimes 1/f$. This means that $fx - a$ maps to zero in $Q(R) \otimes_R R_{\mathfrak p} = Q(R)_{\mathfrak p}$, which in turn means that there exists an $f' \in R$, $f' \not\in \mathfrak p$ such that $f'fx = f'a$ in R. In other words, $ff' \in I$. Thus I is an ideal which isn't contained in any of the prime ideals of R, i.e., $I = R$ and $x \in R$. $\square$

Lemma 7.33.12. *A localization of a normal ring is a normal ring.*

Proof. Omitted. $\square$

Lemma 7.33.13. *Let R be a normal ring. Then $R[x]$ is a normal ring.*

Proof. Let $\mathfrak q$ be a prime of $R[x]$. Set $\mathfrak p = R \cap \mathfrak q$. Then we see that $R_{\mathfrak p}[x]$ is a normal domain by Lemma 7.33.8. Hence $(R[x])_{\mathfrak q}$ is a normal domain by Lemma 7.33.5. $\square$

Lemma 7.33.14. *Let R be a ring. Assume R is reduced and has finitely many minimal primes. Then the following are equivalent:*

(1) *R is a normal ring,*
(2) *R is integrally closed in its total ring of fractions, and*
(3) *R is a finite product of normal domains.*

Proof. Let $\mathfrak q_1, \ldots, \mathfrak q_t$ be the minimal primes of R. By Lemma 7.22.2 we have $Q(R) = R_{\mathfrak q_1} \times \ldots \times R_{\mathfrak q_t}$, and by Lemma 7.23.3 each factor is a field. Denote $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ the ith idempotent of $Q(R)$.

If R is integrally closed in $Q(R)$, then it contains in particular the idempotents e_i, and we see that R is a product of t domains (see Sections 7.19 and 7.20). Hence it is clear that R is a finite product of normal domains.

If R is normal, then it is clear that $e_i \in R_{\mathfrak p}$ for every prime ideal $\mathfrak p$ of R. Hence we see that R contains the elements e_i (see proof of Lemma 7.33.9). We conclude that R is a product of t domains as before. Each of these t domains is normal by Lemma 7.33.9 and the assumption that R is a normal ring. Hence it follows that R is a finite product of normal domains.

We omit the verification that (3) implies (1) and (2). □

Lemma 7.33.15. *Let $(R_i, \varphi_{ii'})$ be a directed system (Categories, Definition 7.8.2) of rings. If each R_i is a normal ring so is $R = colim_i R_i$.*

Proof. Let $\mathfrak p \subset R$ be a prime ideal. Set $\mathfrak p_i = R_i \cap \mathfrak p$ (usual abuse of notation). Then we see that $R_{\mathfrak p} = colim_i (R_i)_{\mathfrak p_i}$. Since each $(R_i)_{\mathfrak p_i}$ is a normal domain we reduce to proving the statement of the lemma for normal domains. If $a, b \in R$ and a/b satisfies a monic polynomial $P(T) \in R[T]$, then we can find a (sufficiently large) $i \in I$ such that a, b, P all come from objects a_i, b_i, P_i over R_i. Since R_i is normal we see $a_i/b_i \in R_i$ and hence also $a/b \in R$. □

7.34. Going down for integral over normal

We first play around a little bit with the notion of elements integral over an ideal, and then we prove the theorem referred to in the section title.

Definition 7.34.1. Let $\varphi : R \to S$ be a ring map. Let $I \subset R$ be an ideal. We say an element $g \in S$ is *integral over I* if there exists a monic polynomial $P = x^d + \sum_{j<d} a_j x^j$ with coefficients $a_j \in I^{d-j}$ such that $P^\varphi(g) = 0$ in S.

This is mostly used when $\varphi = \text{id}_R : R \to R$. In this case the set I' of elements integral over I is called the *integral closure of I*. We will see that I' is an ideal of R (and of course $I \subset I'$).

Lemma 7.34.2. *Let $\varphi : R \to S$ be a ring map. Let $I \subset R$ be an ideal. Let $A = \sum I^n t^n \subset R[t]$ be the subring of the polynomial ring generated by $R \oplus It \subset R[t]$. An element $s \in S$ is integral over I if and only if the element $st \in S[t]$ is integral over A.*

Proof. Suppose st is integral over A. Let $P = x^d + \sum_{j<d} a_j x^j$ be a monic polynomial with coefficients in A such that $P^\varphi(st) = 0$. Let $a'_j \in A$ be the degree $d - j$ part of a_i, in other words $a'_j = a''_j t^{d-j}$ with $a''_j \in I^{d-j}$. For degree reasons we still have $(st)^d + \sum_{j<d} \varphi(a''_j) t^{d-j} (st)^j = 0$. Hence we see that s is integral over I.

Suppose that s is integral over I. Say $P = x^d + \sum_{j<d} a_j x^j$ with $a_j \in I^{d-j}$. The we immediately find a polynomial $Q = x^d + \sum_{j<d} (a_j t^{d-j}) x^j$ with coefficients in A which proves that st is integral over A. □

Lemma 7.34.3. *Let $\varphi : R \to S$ be a ring map. Let $I \subset R$ be an ideal. The set of elements of S which are integral over I form a R-submodule of S. Furthermore, if $s \in S$ is integral over R, and s' is integral over I, then ss' is integral over I.*

Proof. Closure under addition is clear from the characterization of Lemma 7.34.2. Any element $s \in S$ which is integral over R corresponds to the degree 0 element s of $S[x]$ which is integral over A (because $R \subset A$). Hence we see that multiplication by s on $S[x]$ preserves the property of being integral over A, by Lemma 7.32.7. □

Lemma 7.34.4. *Suppose $\varphi : R \to S$ is integral. Suppose $I \subset R$ is an ideal. Then every element of IS is integral over I.*

Proof. Immediate from Lemma 7.34.3. □

Lemma 7.34.5. *Let R be a domain with field of fractions K. Let $n, m \in \mathbf{N}$ and $a_0, \ldots, a_{n-1}, b_0, \ldots, b_{m-1} \in R$. If the polynomial $x^n + a_{n-1}x^{n-1} + \ldots + a_0$ divides the polynomial $x^m + b_{m-1}x^{m-1} + \ldots + b_0$ in $K[x]$ then*

(1) *$a_0, \ldots, a_{n-1}$ are integral over the subring of R generated by $b_0, \ldots, b_{m-1}$, and*
(2) *each a_i lies in $\sqrt{(b_0, \ldots, b_m)}$.*

Proof. Let $K \supset R$ be the fraction field of R. Let $L \supset K$ be a field extension such that we can write $x^m + b_{m-1}x^{m-1} + \ldots + b_0 = \prod_{i=1}^m (x - \beta_i)$ with $\beta_i \in L$. Each β_i is integral over the subring generated by $b_0, \ldots, b_{m-1}$. Since each a_i is a homogeneous polynomial in $\beta_1, \ldots, \beta_m$ we deduce the same for the a_i.

Choose $c_0, \ldots, c_{m-n-1} \in K$ such that

$$x^m + b_{m-1}x^{m-1} + \ldots + b_0 = \\ (x^n + a_{n-1}x^{n-1} + \ldots + a_0)(x^{m-n} + c_{m-n-1}x^{m-n-1} + \ldots + c_0).$$

By the first part we see that the elements c_i are integral over R. Let R' be the sub R-algebra of K generated by $c_0, \ldots, c_{m-n-1}$. By Lemmas 7.32.15 and 7.27.3 we see that $R \cap \sqrt{(b_0, \ldots, b_m)R'} = \sqrt{(b_0, \ldots, b_m)}$. Thus we may replace R by R' and assume $c_i \in R$. Dividing out the radical $\sqrt{(b_0, \ldots, b_m)}$ we get a reduced ring $\overline{R}$. We have to show that the images $\overline{a_i} \in \overline{R}$ are zero. And in $\overline{R}[x]$ we have the relation

$$x^m = x^m + \overline{b_{m-1}}x^{m-1} + \ldots + \overline{b_0} = \\ (x^n + \overline{a_{n-1}}x^{n-1} + \ldots + \overline{a_0})(x^{m-n} + \overline{c_{m-n-1}}x^{m-n-1} + \ldots + \overline{c_0}).$$

It is easy to see that this implies $\overline{a_i} = 0$ for all i. For example one can see this by localizing at all the minimal primes, see Lemma 7.23.6. □

Lemma 7.34.6. *Let $R \subset S$ be an inclusion of domains. Assume R is normal. Let $g \in S$ be integral over R. Then the minimal polynomial of g has coefficients in R.*

Proof. Let $P = x^m + b_{m-1}x^{m-1} + \ldots + b_0$ be a polynomial with coefficients in R such that $P(g) = 0$. Let $Q = x^n + a_{n-1}x^{n-1} + \ldots + a_0$ be the minimal polynomial for g over the fraction field K of R. Then Q divides P in $K[x]$. By Lemma 7.34.5 we see the a_i are integral over R. Since R is normal this means they are in R. □

Proposition 7.34.7. *Let $R \subset S$ be an inclusion of domains. Assume R is normal and S integral over R. Let $\mathfrak{p} \subset \mathfrak{p}' \subset R$ be primes. Let $\mathfrak{q}'$ be a prime of S with $\mathfrak{p}' = R \cap \mathfrak{q}'$. Then there exists a prime $\mathfrak{q}$ with $\mathfrak{q} \subset \mathfrak{q}'$ such that $\mathfrak{p} = R \cap \mathfrak{q}$. In other words: the going down property holds for $R \to S$, see Definition 7.36.1.*

Proof. Let $\mathfrak{p}$, $\mathfrak{p}'$ and $\mathfrak{q}'$ be as in the statement. We have to show there is a prime $\mathfrak{q}$, $\mathfrak{q} \subset \mathfrak{q}'$ such that $R \cap \mathfrak{q} = \mathfrak{p}$. This is the same as finding a prime of $S_{\mathfrak{q}'}$ mapping to $\mathfrak{p}$. According to Lemma 7.16.9 we have to show that $\mathfrak{p}S_{\mathfrak{q}'} \cap R = \mathfrak{p}$. Pick $z \in \mathfrak{p}S_{\mathfrak{q}'} \cap R$. We may write $z = y/g$ with $y \in \mathfrak{p}S$ and $g \in S$, $g \not\in \mathfrak{q}'$. Written differently we have $zg = y$.

By Lemma 7.34.4 there exists a monic polynomial $P = x^m + b_{m-1}x^{m-1} + \ldots + b_0$ with $b_i \in \mathfrak{p}$ such that $P(y) = 0$.

By Lemma 7.34.6 the minimal polynomial of g over K has coefficients in R. Write it as $Q = x^n + a_{n-1}x^{n-1} + \ldots + a_0$. Note that not all a_i, $i = n-1, \ldots, 0$ are in $\mathfrak p$ since that would imply $g^n = \sum_{j<n} a_j g^j \in \mathfrak p S \subset \mathfrak p' S \subset \mathfrak q'$ which is a contradiction.

Since $y = zg$ we see immediately from the above that $Q' = x^n + za_{n-1}x^{n-1} + \ldots + z^n a_0$ is the minimal polynomial for y. Hence Q' divides P and by Lemma 7.34.5 we see that $z^j a_{n-j} \in \sqrt{(b_0, \ldots, b_{m-1})} \subset \mathfrak p$, $j = 1, \ldots, n$. Because not all a_i, $i = n-1, \ldots, 0$ are in $\mathfrak p$ we conclude $z \in \mathfrak p$ as desired. □

7.35. Flat modules and flat ring maps

One often used result is that if $M = colim_{i \in \mathcal{I}} M_i$ is a colimit of R-modules and if N is another then

$$M \otimes N = colim_{i \in \mathcal{I}} M_i \otimes_R N,$$

see Lemma 7.11.8. This property is usually expressed by saying that $\otimes$ *commutes with colimits*. Another often used result is that if $0 \to N_1 \to N_2 \to N_3 \to 0$ is an exact sequence and if M is any R-module, then

$$M \otimes_R N_1 \to M \otimes_R N_2 \to M \otimes_R N_3 \to 0$$

is still exact, see Lemma 7.11.10. Both of these properties tell us that the functor $N \mapsto M \otimes_R N$ *is right exact*. See Categories, Section 4.21 and Homology, Section 10.5. An R-module M is flat if this functor is also left exact. Here is the precise definition.

Definition 7.35.1. Let R be a ring.

(1) An R-module M is called *flat* if whenever $N_1 \to N_2 \to N_3$ is an exact sequence of R-modules the sequence $M \otimes_R N_1 \to M \otimes_R N_1 \to M \otimes_R N_1$ is exact as well.
(2) An R-module M is called *faithfully flat* if the complex of R-modules $N_1 \to N_2 \to N_3$ is exact if and only if the sequence $M \otimes_R N_1 \to M \otimes_R N_1 \to M \otimes_R N_1$ is exact.
(3) A ring map $R \to S$ is called *flat* if S is flat as an R-module.
(4) A ring map $R \to S$ is called *faithfully flat* if S is faithfully flat as an R-module.

Lemma 7.35.2. *Let R be a ring. Let $\{M_i, \varphi_{ii'}\}$ be a directed system of flat R-modules. Then $colim_i M_i$ is a flat R-module.*

Proof. This follows as $\otimes$ commutes with colimits and because directed colimits are exact, see Lemma 7.8.9. □

Lemma 7.35.3. *A composition of (faithfully) flat ring maps is (faithfully) flat. If $R \to R'$ is flat, and M' is a flat R'-module, then M' is a flat R-module.*

Proof. Omitted. □

Lemma 7.35.4. *Let M be an R-module. The following are equivalent:*

(1) *M is flat over R.*
(2) *for every injection of R-modules $N \subset N'$ the map $N \otimes_R M \to N' \otimes_R M$ is injective.*
(3) *for every ideal $I \subset R$ the map $I \otimes_R M \to R \otimes_R M = M$ is injective.*
(4) *for every finitely generated ideal $I \subset R$ the map $I \otimes_R M \to R \otimes_R M = M$ is injective.*

Proof. The implications (1) implies (2) implies (3) implies (4) are all trivial. Thus we prove (4) implies (1). Suppose that $N_1 \to N_2 \to N_3$ is exact. Let $K_2 = \text{Ker}(N_2 \to N_3)$. It is clear that the surjection $N_1 \to K$ induces a surjection $N_1 \otimes_R M \to K_2 \otimes_R M$. Hence it suffices to show $K_2 \otimes_R M \to N_2 \otimes_R M$ is injective.

Let $x \in \text{Ker}(K_2 \otimes_R M \to N_2 \otimes_R M)$. We have to show that x is zero. Write $x = \sum_{i=1,\ldots,r} k_i \otimes m_i$. By Lemma 7.8.13 we can find a finitely generated module N, a map $N \to N_2$, and elements $n_i \in N$, $i = 1, \ldots, r$ such that (a) n_i maps to k_i, (b) the element $y = \sum_i n_i \otimes m_i$ maps to zero in $N \otimes_R M$. Let $K \subset N$ be the submodule generated by the n_i. It suffices to show that y is zero as an element of $K' \otimes_R M$.

We do this by induction on the minimal number of generators of N. If this number is > 1 then we can find a short exact sequence $0 \to N' \to N \to N'' \to 0$ such that N' and N'' are finitely generated with a smaller number of generators. By induction the element y maps to zero in $K'' \otimes_R M$ with K'' the image of K in N''. And by the right exactness of $\otimes$ we see that y comes from some element of $K' \otimes_R M$ where K' is the intersection of K with N'. Again by induction we see that $y' = 0$.

The base case of the induction above is when N is generated by 1 element. In other words $N = R/I$, and then $y = \sum g_i \otimes m_i$, Let $J = (g_1, \ldots, g_r) \subset R$. By right exactness, we see that $R/I \otimes_R M = M/IM$. Our assumption is that y is zero in $R/I \otimes_R M = M/IM$ in other words $\sum g_i m_i \in IM$, in other words $\sum g_i m_i = \sum h_j m'_j$ for suitable $h_j \in I$. We may replace I by the finitely generated ideal (h_j) without modifying the assumptions. In this case we have $K = J + I/I$

$$K \otimes_R M = (J + I) \otimes_R M/I \otimes_R M = (J + I)M/IM$$

the first equality by right exactness and the second by assumption on M. Thus y is zero in $K \otimes_R M$ as desired. □

Lemma 7.35.5. *Let $\{R_i, \varphi_{ii'}\}$ be a system of rings of the directed partially ordered set I. Let $R = colim_i R_i$. Let M be an R-module such that M is flat as an R_i-module for all i. Then M is flat as an R-module.*

Proof. Let $\mathfrak{a} \subset R$ be a finitely generated ideal. By Lemma 7.35.4 it suffices to show that $\mathfrak{a} \otimes_R M \to M$ is injective. We can find an $i \in I$ and a finitely generated ideal $\mathfrak{a}' \subset R_i$ such that $\mathfrak{a} = \mathfrak{a}'R$. Then $\mathfrak{a} = colim_{i' \geq i} \mathfrak{a}'R_{i'}$. Hence the map $\mathfrak{a} \otimes_R M \to M$ is the colimit of the maps

$$\mathfrak{a}'R_{i'} \otimes_{R_{i'}} M \longrightarrow M$$

which are all injective by assumption. Since $\otimes$ commutes with colimits and since colimits over I are exact by Lemma 7.8.9 we win. □

Lemma 7.35.6. *Suppose that M is flat over R, and that $R \to R'$ is a ring map. Then $M \otimes_R R'$ is flat over R'.*

Proof. For any R'-module N we have a canonical isomorphism $N \otimes_{R'} (R' \otimes_R M) = N \otimes_R M$. Hence the exactness of $- \otimes_{R'} (R' \otimes_R M)$ follows from the exactness of $- \otimes_R M$. □

Lemma 7.35.7. *Let $R \to R'$ be a faithfully flat ring map. Let M be a module over R, and set $M' = R' \otimes_R M$. Then M is flat over R if and only if M' is flat over R'.*

Proof. By Lemma 7.35.6 we see that if M is flat then M' is flat. For the converse, suppose that M' is flat. Let $N_1 \to N_2 \to N_3$ be an exact sequence of R-modules. We want to show that $N_1 \otimes_R M \to N_2 \otimes_R M \to N_3 \otimes_R M$ is exact. We know that $N_1 \otimes_R R' \to$

$N_2 \otimes_R R' \to N_3 \otimes_R R'$ is exact, because $R \to R'$ is flat. Flatness of M' implies that $N_1 \otimes_R R' \otimes_{R'} M' \to N_2 \otimes_R R' \otimes_{R'} M' \to N_3 \otimes_R R' \otimes_{R'} M'$ is exact. We may write this as $N_1 \otimes_R M \otimes_R R' \to N_2 \otimes_R M \otimes_R R' \to N_3 \otimes_R M \otimes_R R'$. Finally, faithfull flatness implies that $N_1 \otimes_R M \to N_2 \otimes_R M \to N_3 \otimes_R M$ is exact. $\square$

Lemma 7.35.8. *Let R be a ring. Let $S \to S'$ be a faithfully flat map of R-algebras. Let M be a module over S, and set $M' = S' \otimes_S M$. Then M is flat over R if and only if M' is flat over R.*

Proof. Let $N \to N'$ be an injection of R-modules. By the faithful flatness of $S \to S'$ we have

$$\mathrm{Ker}(N \otimes_R M \to N' \otimes_R M) \otimes_S S' = \mathrm{Ker}(N \otimes_R M' \to N' \otimes_R M')$$

Hence the equivalence of the lemma follows from the second characterization of flatness in Lemma 7.35.4. $\square$

Lemma 7.35.9. *Let $R \to S$ be a ring map. Let M be an S-module. If M is flat as an R-module and faithfully flat as an S-module, then $R \to S$ is flat.*

Proof. Let $N_1 \to N_2 \to N_3$ be an exact sequence of R-modules. By assumption $N_1 \otimes_R M \to N_2 \otimes_R M \to N_3 \otimes_R M$ is exact. We may write this as

$$N_1 \otimes_R S \otimes_S M \to N_2 \otimes_R S \otimes_S M \to N_3 \otimes_R S \otimes_S M.$$

By faithful flatness of M over S we conclude that $N_1 \otimes_R S \to N_2 \otimes_R S \to N_3 \otimes_R S$ is exact. Hence $R \to S$ is flat. $\square$

Let R be a ring. Let M be an R-module. Let $\sum f_i x_i = 0$ be a relation in M. We say the relation $\sum f_i x_i$ is *trivial* if there exist an integer $m \geq 0$, elements $y_j \in M$, $j = 1, \ldots, m$, and elements $a_{ij} \in R$, $i = 1, \ldots, n$, $j = 1, \ldots, m$ such that

$$x_i = \sum\nolimits_j a_{ij} y_j, \forall i, \quad \text{and} \quad 0 = \sum\nolimits_i f_i a_{ij}, \forall j.$$

Lemma 7.35.10. *(Equational criterion of flatness.) A module M over R is flat if and only if every relation in M is trivial.*

Proof. Assume M is flat and let $\sum f_i x_i$ be a relation. Let $I = (f_1, \ldots, f_n)$, and let $K = \ker(R^n \to I)$. So we have the short exact sequence $0 \to K \to R^n \to I \to 0$. Then $\sum f_i \otimes x_i$ is an element of $I \otimes_R M$ which maps to zero in $R \otimes_R M = M$. By flatness $\sum f_i \otimes x_i$ is zero in $I \otimes_R M$. Thus there exists an element of $K \otimes_R M$ mapping to $\sum e_i \otimes m_i \in R^n \otimes_R M$. Write this element as $\sum k_j \otimes y_j$ and then write the image of k_j in R^n as $\sum a_{ij} e_i$ to get the result.

Assume every relation is trivial, let I be a finitely generated ideal, and let $x = \sum f_i \otimes x_i$ be an element of $I \otimes_R M$ mapping to zero in $R \otimes_R M = M$. This just means exactly that $\sum h_i x_i$ is a relation in M. And the fact that it is trivial implies easily that x is zero, because

$$x = \sum f_i \otimes x_i = \sum f_i \otimes (\sum a_{ij} y_j) = \sum (\sum f_i a_{ij}) \otimes y_j = 0$$

$\square$

Lemma 7.35.11. *Suppose that R is a ring, $0 \to M'' \to M' \to M \to 0$ a short exact sequence, and N an R-module. If M is flat then $N \otimes_R M'' \to N \otimes_R M'$ is injective, i.e., the sequence*

$$0 \to N \otimes_R M'' \to N \otimes_R M' \to N \otimes_R M \to 0$$

is a short exact sequence.

Proof. Let $R^{(I)} \to N$ be a surjection from a free module onto N with kernel K. The result follows by a simple diagram chase from the following diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & M'' \otimes_R N & \to & M' \otimes_R N & \to & M \otimes_R N & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & (M'')^{(I)} & \to & (M')^{(I)} & \to & M^{(I)} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & M'' \otimes_R N & \to & M' \otimes_R N & \to & M \otimes_R N & \to & 0 \\
 & & & & & & \uparrow & & \\
 & & & & & & 0 & &
\end{array}$$

with exact rows and columns. The middle row is exact because tensoring with the free module $R^{(I)}$ is exact. $\square$

Lemma 7.35.12. *Suppose that $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of R-modules. If M' and M'' are flat so is M. If M and M'' are flat so is M'.*

Proof. We will use the criterion that a module N is R flat if for every ideal the map $N \otimes_R I \to N$ is injective, see Lemma 7.35.4. Consider an ideal $I \subset R$. Consider the diagram

$$\begin{array}{ccccccccc}
0 & \to & M' & \to & M & \to & M'' & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & M' \otimes_R I & \to & M \otimes_R I & \to & M'' \otimes_R I & \to & 0
\end{array}$$

with exact rows. This immediately proves the first assertion. The second follows because if M'' is flat then the lower left horizontal arrow is injective by Lemma 7.35.11. $\square$

Lemma 7.35.13. *Let R be a ring. Let M be an R-module. The following are equivalent*

(1) *M is faithfully flat, and*
(2) *M is flat and for all R-module homomorpisms $\alpha : N \to N'$ we have $\alpha = 0$ if and only if $\alpha \otimes id_M = 0$.*

Proof. If M is faithfully flat, then $0 \to \text{Ker}(\alpha) \to N \to 0$ is exact if and only if the same holds after tensoring with M. This proves (1) implies (2). For the other, assume (2). Let $N_1 \to N_2 \to N_3$ be a complex, and assume the complex $N_1 \otimes_R M \to N_2 \otimes_R M \to N_3 \otimes_R M$ is exact. Take $x \in \text{Ker}(N_2 \to N_3)$, and consider the map $\alpha : R \to N_2/\text{Im}(N_1)$, $r \mapsto rx + \text{Im}(N_1)$. By the exactness of the complex $- \otimes_R M$ we see that $\alpha \otimes \text{id}_M$ is zero. By assumption we get that α is zero. Hence x is in the image of $N_1 \to N_2$. $\square$

Lemma 7.35.14. *Let M be a flat R-module. The following are equivalent:*

(1) *M is faithfully flat,*
(2) *for all $\mathfrak{p} \in Spec(R)$ the tensor product $M \otimes_R \kappa(\mathfrak{p})$ is nonzero, and*
(3) *for all maximal ideals $\mathfrak{m}$ of R the tensor product $M \otimes_R \kappa(\mathfrak{m}) = M/\mathfrak{m}M$ is nonzero.*

Proof. Assume M faithfully flat. Since $R \to \kappa(\mathfrak{p})$ is not zero we deduce that $M \to M \otimes_R \kappa(\mathfrak{p})$ is not zero, see Lemma 7.35.13.

Conversely assume that M is flat and that $M/\mathfrak{m}M$ is never zero. Suppose that $N_1 \to N_2 \to N_3$ is a complex and suppose that $N_1 \otimes_R M \to N_2 \otimes_R M \to N_3 \otimes_R M$ is exact. Let H be the cohomology of the complex, so $H = \text{Ker}(N_2 \to N_3)/\text{Im}(N_1 \to N_2)$. By flatness we see that $H \otimes_R M = 0$. Take $x \in H$ and let $I = \{f \in R \mid fx = 0\}$ be its annihilator.

Since $R/I \subset H$ we get $M/IM \subset H \otimes_R M = 0$ by flatness of M. If $I \neq R$ we may choose a maximal ideal $I \subset \mathfrak{m} \subset R$. This immediately gives a contradiction. □

Lemma 7.35.15. *Let $R \to S$ be a flat ring map. The following are equivalent:*

(1) *$R \to S$ is faithfully flat,*
(2) *the induced map on Spec is surjective, and*
(3) *any closed point $x \in Spec(R)$ is in the image of the map $Spec(S) \to Spec(R)$.*

Proof. This follows quickly from Lemma 7.35.14, because we saw in Remark 7.16.8 that $\mathfrak{p}$ is in the image if and only if the ring $S \otimes_R \kappa(\mathfrak{p})$ is nonzero. □

Lemma 7.35.16. *A flat local ring homomorphism of local rings is faithfully flat.*

Proof. Immediate from Lemma 7.35.15. □

Lemma 7.35.17. *Let $R \to S$ be flat. Let $\mathfrak{p} \subset \mathfrak{p}'$ be primes of R. Let $\mathfrak{q}' \subset S$ be a prime of S mapping to $\mathfrak{p}'$. Then there exists a prime $\mathfrak{q} \subset \mathfrak{q}'$ mapping to $\mathfrak{p}$.*

Proof. Namely, consider the flat local ring map $R_{\mathfrak{p}'} \to S_{\mathfrak{q}'}$. By Lemma 7.35.16 above this is faithfully flat. By Lemma 7.35.15 there is a prime mapping to $\mathfrak{p}R_{\mathfrak{p}'}$. The inverse image of this prime in S does the job. □

The property of $R \to S$ described in the lemma is called the ``going down property''. See Definition 7.36.1.

Lemma 7.35.18. *If $R \to S$ is a faithfully flat ring map then for every R-module M the map $M \to S \otimes_R M$, $x \mapsto 1 \otimes x$ is injective.*

Proof. This is true because the base change $S \otimes_R M \to S \otimes_R S \otimes_R M$ by the faithfully flat ring map $R \to S$ is injective: It has a section, namely $s \otimes s' \otimes m \mapsto ss' \otimes m$. □

We finish with some remarks on flatness and localization.

Lemma 7.35.19. *Let R be a ring. Let $S \subset R$ be a multiplicative subset.*

(1) *The localization $S^{-1}R$ is a flat R-algebra.*
(2) *If M is a $S^{-1}R$-module, then M is a flat R-module if and only if M is a flat $S^{-1}R$-module.*
(3) *Suppose M is an R-module. Then M is a flat R-module if and only if $M_{\mathfrak{p}}$ is a flat $R_{\mathfrak{p}}$-module for all primes $\mathfrak{p}$ of R.*
(4) *Suppose M is an R-module. Then M is a flat R-module if and only if $M_{\mathfrak{m}}$ is a flat $R_{\mathfrak{m}}$-module for all maximal ideals $\mathfrak{m}$ of R.*
(5) *Suppose $R \to A$ is a ring map, M is an A-module, and $g_1, \ldots, g_m \in A$ are elements generating the unit ideal of A. Then M is flat over R if and only if each localization M_{g_i} is flat over R.*
(6) *Suppose $R \to A$ is a ring map, and M is an A-module. Then M is a flat R-module if and only if the localization $M_{\mathfrak{q}}$ is a flat $R_{\mathfrak{p}}$-module (with $\mathfrak{p}$ the prime of R lying under $\mathfrak{q}$) for all primes $\mathfrak{q}$ of A.*
(7) *Suppose $R \to A$ is a ring map, and M is an A-module. Then M is a flat R-module if and only if the localization $M_{\mathfrak{m}}$ is a flat $R_{\mathfrak{p}}$-module (with $\mathfrak{p} = R \cap \mathfrak{m}$) for all maximal ideals $\mathfrak{m}$ of A.*

Proof. Let us prove the last statement of the lemma. In the proof we will use repeatedly that localization is exact and commutes with tensor product, see Sections 7.9 and 7.11.

Suppose $R \to A$ is a ring map, and M is an A-module. Assume that $M_\mathfrak{m}$ is a flat $R_\mathfrak{p}$-module for all maximal ideals $\mathfrak{m}$ of A (with $\mathfrak{p} = R \cap \mathfrak{m}$). Let $I \subset R$ be an ideal. We have to show the map $I \otimes_R M \to M$ is injective. We can think of this as a map of A-modules. By assumption the localization $(I \otimes_R M)_\mathfrak{m} \to M_\mathfrak{m}$ is injective because $(I \otimes_R M)_\mathfrak{m} = I_\mathfrak{p} \otimes_{R_\mathfrak{p}} M_\mathfrak{m}$. Hence the kernel of $I \otimes_R M \to M$ is zero by Lemma 7.21.1. Hence M is flat over R.

Conversely, assume M is flat over R. Pick a prime $\mathfrak{q}$ of A lying over the prime $\mathfrak{p}$ of R. Suppose that $I \subset R_\mathfrak{p}$ is an ideal. We have to show that $I \otimes_{R_\mathfrak{p}} M_\mathfrak{q} \to M_\mathfrak{q}$ is injective. We can write $I = J_\mathfrak{p}$ for some ideal $J \subset R$. Then the map $I \otimes_{R_\mathfrak{p}} M_\mathfrak{q} \to M_\mathfrak{q}$ is just the localization (at $\mathfrak{q}$) of the map $J \otimes_R M \to M$ which is injective. Since localization is exact we see that $M_\mathfrak{q}$ is a flat $R_\mathfrak{p}$-module.

This proves (7) and (6). The other statements follow in a straightforward way from the last statement (proofs omitted). □

7.36. Going up and going down

Suppose $\mathfrak{p}, \mathfrak{p}'$ are primes of the ring R. Let $X = Spec(R)$ with the Zariski topology. Denote $x \in X$ the point corresponding to $\mathfrak{p}$ and $x' \in X$ the point corresponding to $\mathfrak{p}'$. Then we have:

$$x' \rightsquigarrow x \Leftrightarrow \mathfrak{p}' \subset \mathfrak{p}.$$

In words: x is a specialization of x' if and only if $\mathfrak{p}' \subset \mathfrak{p}$. See Topology, Section 5.14 for terminology and notation.

Definition 7.36.1. Let $\varphi : R \to S$ be a ring map.

(1) We say a $\varphi : R \to S$ satisfies *going up* if given primes $\mathfrak{p} \subset \mathfrak{p}'$ in R and a prime $\mathfrak{q}$ in S lying over $\mathfrak{p}$ there exists a prime $\mathfrak{q}'$ of S such that (a) $\mathfrak{q} \subset \mathfrak{q}'$, and (b) $\mathfrak{q}'$ lies over $\mathfrak{p}'$.
(2) We say a $\varphi : R \to S$ satisfies *going down* if given primes $\mathfrak{p} \subset \mathfrak{p}'$ in R and a prime $\mathfrak{q}'$ in S lying over $\mathfrak{p}'$ there exists a prime $\mathfrak{q}$ of S such that (a) $\mathfrak{q} \subset \mathfrak{q}'$, and (b) $\mathfrak{q}$ lies over $\mathfrak{p}$.

Sofar we have see the following cases of this:

(1) An integral ring map satisfies going up, see Lemma 7.32.20.
(2) As a special case finite ring maps satisfy going up.
(3) As a special case quotient maps $R \to R/I$ satisfy going up.
(4) A flat ring map satisfies going down, see Lemma 7.35.17
(5) As a special case any localization satisfies going down.
(6) An extension $R \subset S$ of domains, with R normal and S integral over R satisfies going down, see Proposition 7.34.7.

Here is another case where going down holds.

Lemma 7.36.2. *Let $R \to S$ be a ring map. If the induced map $\varphi : Spec(S) \to Spec(R)$ is open, then $R \to S$ satisfies going down.*

Proof. Suppose that $\mathfrak{p} \subset \mathfrak{p}' \subset R$ and $\mathfrak{q}' \subset S$ lies over $\mathfrak{p}'$. As φ is open, for every $g \in S$, $g \notin \mathfrak{q}'$ we see that $\mathfrak{p}$ is in the image of $D(g) \subset Spec(S)$. In other words $S_g \otimes_R \kappa(\mathfrak{p})$ is not zero. Since $S_{\mathfrak{q}'}$ is the directed colimit of these S_g this implies that $S_{\mathfrak{q}'} \otimes_R \kappa(\mathfrak{p})$ is not zero, see Lemmas 7.9.9 and 7.11.8. Hence $\mathfrak{p}$ is in the image of $Spec(S_{\mathfrak{q}'}) \to Spec(R)$ as desired. □

Lemma 7.36.3. *Let $R \to S$ be a ring map.*

(1) *$R \to S$ satisfies going down if and only if generalizations lift along the map $Spec(S) \to Spec(R)$, see Topology, Definition 5.14.3.*
(2) *$R \to S$ satisfies going up if and only if specializations lift along the map $Spec(S) \to Spec(R)$, see Topology, Definition 5.14.3.*

Proof. Omitted. □

Lemma 7.36.4. *Suppose $R \to S$ and $S \to T$ are ring maps satisfying going down. Then so does $R \to T$. Similarly for going up.*

Proof. According to Lemma 7.36.3 this follows from Topology, Lemma 5.14.4 □

Lemma 7.36.5. *Let $R \to S$ be a ring map. Let $T \subset Spec(R)$ be the image of $Spec(S)$. If T is stable under specialization, then T is closed.*

Proof. We give two proofs.

First proof. Let $\mathfrak{p} \subset R$ be a prime ideal such that the corresponding point of $Spec(R)$ is in the closure of T. This means that for ever $f \in R$, $f \notin \mathfrak{p}$ we have $D(f) \cap T \neq \emptyset$. Note that $D(f) \cap T$ is the image of $Spec(S_f)$ in $Spec(R)$. Hence we conclude that $S_f \neq 0$. In other words, $1 \neq 0$ in the ring S_f. Since $S_{\mathfrak{p}}$ is the directed limit of the rings S_f we conclude that $1 \neq 0$ in $S_{\mathfrak{p}}$. In other words, $S_{\mathfrak{p}} \neq 0$ and considering the image of $Spec(S_{\mathfrak{p}}) \to Spec(S) \to Spec(R)$ we see there exists a $\mathfrak{p}' \in T$ with $\mathfrak{p}' \subset \mathfrak{p}$. As we assumed T closed under specialization we conclude $\mathfrak{p}$ is a point of T as desired.

Second proof. Let $I = \mathrm{Ker}(R \to S)$. We may replace R by R/I. In this case the ring map $R \to S$ is injective. By Lemma 7.27.5 all the minimal primes of R are contained in the image T. Hence if T is stable under specialization then it contains all primes. □

Lemma 7.36.6. *Let $R \to S$ be a ring map. The following are equivalent:*

(1) *Going up holds for $R \to S$, and*
(2) *the map $Spec(S) \to Spec(R)$ is closed.*

Proof. It is a general fact that specializations lift along a closed map of topological spaces, see Topology, Lemma 5.14.6. Hence the second condition implies the first.

Assume that going up holds for $R \to S$. Let $V(I) \subset Spec(S)$ be a closed set. We want to show that the image of $V(I)$ in $Spec(R)$ is closed. The ring map $S \to S/I$ obviously satisfies going up. Hence $R \to S \to S/I$ satisfies going up, by Lemma 7.36.4. Replacing S by S/I it suffices to show the image T of $Spec(S)$ in $Spec(R)$ is closed. By Topology, Lemmas 5.14.2 and 5.14.5 this image is stable under specialization. Thus the result follows from Lemma 7.36.5. □

Lemma 7.36.7. *Let R be a ring. Let $E \subset Spec(R)$ be a constructible subset.*

(1) *If E is stable under specialization, then E is closed.*
(2) *If E is stable under generalization, then E is open.*

Proof. The first assertion follows from Lemma 7.36.5 combined with Lemma 7.26.3. The second follows because the complement of a constructible set is constructible (see Topology, Lemma 5.10.2), the first part of the lemma and Topology, Lemma 5.14.2. □

Proposition 7.36.8. *Let $R \to S$ be flat and of finite presentation. Then $Spec(R) \to Spec(S)$ is open. More generally this holds for any ring map $R \to S$ of finite presentation which satisfies going down.*

Proof. Assume that $R \to S$ has finite presentation and satisfies going down. It suffices to prove that the image of a standard open $D(f)$ is open. Since $S \to S_f$ satsifies going down as well, we see that $R \to S_f$ satisfies going down. Thus after replacing S by S_f we see it suffices to prove the image is open. By Chevalley's theorem (Theorem 7.26.9) the image is a constructible set E. And E is stable under generalization because $R \to S$ satisfies going down, see Topology, Lemmas 5.14.2 and 5.14.5. Hence E is open by Lemma 7.36.7. $\square$

Lemma 7.36.9. *Let k be a field, and let R, S be k-algebras. Let $S' \subset S$ be a sub k-algebra, and let $f \in S' \otimes_k R$. In the commutative diagram*

$$\begin{tikzcd} Spec((S \otimes_k R)_f) \arrow[rr] \arrow[dr] & & Spec((S' \otimes_k R)_f) \arrow[dl] \\ & Spec(R) & \end{tikzcd}$$

the images of the diagonal arrows are the same.

Proof. Let $\mathfrak p \subset R$ be in the image of the south-west arrow. This means (Lemma 7.16.9) that

$$(S' \otimes_k R)_f \otimes_R \kappa(\mathfrak p) = (S' \otimes_k \kappa(\mathfrak p))_f$$

is not the zero ring, i.e., $S' \otimes_k \kappa(\mathfrak p)$ is not the zero ring and the image of f in it is not nilpotent. The ring map $S' \otimes_k \kappa(\mathfrak p) \to S \otimes_k \kappa(\mathfrak p)$ is injective. Hence also $S \otimes_k \kappa(\mathfrak p)$ is not the zero ring and the image of f in it is not nilpotent. Hence $(S \otimes_k R)_f \otimes_R \kappa(\mathfrak p)$ is not the zero ring. Thus (Lemma 7.16.9) we see that $\mathfrak p$ is in the image of the south-east arrow as desired. $\square$

Lemma 7.36.10. *Let k be a field. Let R and S be k-algebras. The map $Spec(S \otimes_k R) \to Spec(R)$ is open.*

Proof. Let $f \in R \otimes_k S$. It suffices to prove that the image of the standard open $D(f)$ is open. Let $S' \subset S$ be a finite type k-subalgebra such that $f \in S' \otimes_k R$. The map $R \to S' \otimes_k R$ is flat and of finite presentation, hence the image U of $Spec((S' \otimes_k R)_f) \to Spec(R)$ is open by Proposition 7.36.8. By Lemma 7.36.9 this is also the image of $D(f)$ and we win. $\square$

Here is a tricky lemma that is sometimes useful.

Lemma 7.36.11. *Let $R \to S$ be a ring map. Let $\mathfrak p \subset R$ be a prime. Assume that*

1. *there exists a unique prime $\mathfrak q \subset S$ lying over $\mathfrak p$, and*
2. *either*
   - (a) *going up holds for $R \to S$, or*
   - (b) *going down holds for $R \to S$ and there is at most one prime of S above every prime of R.*

Then $S_{\mathfrak p} = S_{\mathfrak q}$.

Proof. Consider any prime $\mathfrak q' \subset S$ which corresponds to a point of $Spec(S_{\mathfrak p})$. This means that $\mathfrak p' = R \cap \mathfrak q'$ is contained in $\mathfrak p$. Here is a picture

$$\begin{tikzcd} \mathfrak q' \arrow[r, dash] \arrow[d, dash] & ? \arrow[r, dash] \arrow[d, dash] & S \arrow[d, dash] \\ \mathfrak p' \arrow[r, dash] & \mathfrak p \arrow[r, dash] & R \end{tikzcd}$$

Assume (1) and (2)(a). By going up there exists a prime $\mathfrak{q}'' \subset S$ with $\mathfrak{q}' \subset \mathfrak{q}''$ and $\mathfrak{q}''$ lying over $\mathfrak{p}$. By the uniqueness of $\mathfrak{q}$ we conclude that $\mathfrak{q}'' = \mathfrak{q}$. In other words $\mathfrak{q}'$ defines a point of $Spec(S_{\mathfrak{q}})$.

Assume (1) and (2)(b). By going down there exists a prime $\mathfrak{q}'' \subset \mathfrak{q}$ lying over $\mathfrak{p}'$. By the uniqueness of primes lying over $\mathfrak{p}'$ we see that $\mathfrak{q}' = \mathfrak{q}''$. In other words $\mathfrak{q}'$ defines a point of $Spec(S_{\mathfrak{q}})$.

In both cases we conclude that the map $Spec(S_{\mathfrak{q}}) \to Spec(S_{\mathfrak{p}})$ is bijective. Clearly this means all the elements of $S - \mathfrak{q}$ are all invertible in $S_{\mathfrak{p}}$, in other words $S_{\mathfrak{p}} = S_{\mathfrak{q}}$. □

7.37. Transcendence

We recall the standard definitions.

Definition 7.37.1. Let $k \subset K$ be a field extension.

(1) A collection of elements $\{x_i\}_{i \in I}$ of K is called *algebraically independent* over k if the map

$$k[X_i; i \in I] \longrightarrow K$$

which maps X_i to x_i is injective.

(2) The field of fractions of a polynomial ring $k[x_i; i \in I]$ is denoted $k(x_i; i \in I)$.

(3) A *purely transcendental extension* of k is any field extension $k \subset K$ isomorphic to the field of fractions of a polynomial ring over k.

(4) A *transcendence basis* of K/k is a collection of elements $\{x_i\}_{i \in I}$ which are algebraically independent over k and such that the extension $k(x_i; i \in I) \subset K$ is algebraic.

Lemma 7.37.2. *Let $k \subset K$ be a field extension. A transcendence basis of K over k exists. Any two transcendence bases have the same cardinality.*

Proof. Omitted. Good exercise. □

Definition 7.37.3. Let $k \subset K$ be a field extension. The *transcendence degree* of K over k is the cardinality of a transcendence basis of K over k. It is denoted $\text{trdeg}_k(K)$.

Lemma 7.37.4. *Let $k \subset K \subset L$ be field extensions. Then*

$$trdeg_k(L) = trdeg_K(L) + trdeg_k(K).$$

Proof. Omitted. □

7.38. Algebraic elements of field extensions

Let $k \subset K$ be a field extension. Let $\alpha \in K$. Then we have the following possibilities:

(1) The element α is transcendental over k.

(2) The element α is algebraic over k. Denote $P(T) \in k[T]$ its *minimal polynomial*. This is a monic polynomial $P(T) = T^d + a_1 T^{d-1} + \ldots + a_d$ with coefficients in k. It is irreducible and $P(\alpha) = 0$. These properties uniquely determine P, and the integer d is called the *degree of α over k*. There are two subcases:

  (a) The polynomial dP/dT is not identically zero. This is equivalent to the condition that $P(T) = \prod_{i=1,\ldots,d}(T - \alpha_i)$ for pairwise distinct elements $\alpha_1, \ldots, \alpha_d$ in the algebraic closure of k. In this case we say that α is *separable* over k.

(b) The dP/dT is identically zero. In this case the characteristic p of k is > 0, and P is actually a polynomial in T^p. Clearly there exists a largest power $q = p^e$ such that P is a polynomial in T^q. Then the element α^q is separable over k.

Definition 7.38.1. Algebraic field extensions.

(1) A field extension $k \subset K$ is called *algebraic* if every element of K is algebraic over k.
(2) An algebraic extension $k \subset k'$ is called *separable* if every $\alpha \in k'$ is separable over k.
(3) An algebraic extension $k \subset k'$ is called *purely inseparable* if the characteristic of k is $p > 0$ and for every element $\alpha \in k'$ there exists a power q of p such that $\alpha^q \in k$.
(4) An algebraic extension $k \subset k'$ is called *normal* if for every $\alpha \in k'$ the minimal polynomial $P(T) \in k[T]$ of α over k splits completely into linear factors over k'.
(5) An algebraic extension $k \subset k'$ is called *Galois* if it is separable and normal.

Here are some well-known lemmas on field extensions.

Lemma 7.38.2. *Let $k \subset k'$ be an algebraic field extension. There exists a unique subfield $k \subset (k')_{sep} \subset k'$ such that $k \subset (k')_{sep}$ is separable, and $(k')_{sep} \subset k'$ is purely inseparable.*

Proof. The Lemma only makes sense if the characteristic of k is $p > 0$. Given an α in k' let q_α be the smallest power of p such that α^{q_α} is separable over k (see discussion at the beginning of this section). Then let $(k')_{sep}$ be the subfield of k' generated by these elements. Details omitted. □

Definition 7.38.3. Let $k \subset k'$ be a finite field extension. Let $k \subset (k')_{sep} \subset k'$ be the subfield found in Lemma 7.38.2.

(1) The integer $[(k')_{sep} : k]$ is called the *separable degree* of the extension.
(2) The integer $[k' : (k')_{sep}]$ is called the *inseparable degree*, or the *degree of inseparability* of the extension.

Lemma 7.38.4. *Let $k \subset k'$ be a normal algebraic field extension. There exists a unique subfields $k \subset (k')_{sep} \subset k'$ and $k \subset (k')_{insep} \subset k'$ such that*

(1) *$k \subset (k')_{sep}$ is separable, and $(k')_{sep} \subset k'$ is purely inseparable,*
(2) *$k \subset (k')_{insep}$ is purely inseparable, and $(k')_{sep} \subset k'$ is separable,*
(3) *$k' = (k')_{sep} \otimes_k (k')_{insep}$.*

Proof. We found the subfield $(k')_{sep}$ in Lemma 7.38.2. The subfield $(k')_{insep} = (k')^{\mathrm{Aut}(k'/k)}$. Details omitted. □

Lemma 7.38.5. *Let $k \subset k'$ be a finite separable field extension. Then there exists an $\alpha \in k'$ such that $k' = k(\alpha)$.*

Proof. Omitted. □

Definition 7.38.6. Let $k \subset K$ be a field extension.

(1) The *algebraic closure of k in K* is the subfield k' of K consisting of elements of K which are algebraic over k.
(2) We say k is *algebraically closed in K* if every element of K which is algebraic over k is contained in k.

Lemma 7.38.7. *Let $k \subset K$ be a finitely generated field extension. The algebraic closure of k in K is finite over k.*

Proof. Let $x_1, \ldots, x_r \in K$ be a transcendence basis for K over k. Then $n = [K : k(x_1, \ldots, x_r)] < \infty$. Suppose that $k \subset k' \subset K$ with k'/k finite. In this case $[k'(x_1, \ldots, x_r) : k(x_1, \ldots, x_r)] = [k' : k] < \infty$. Hence

$$[k' : k] = [k'(x_1, \ldots, x_r) : k(x_1, \ldots, x_r)] < [K : k(x_1, \ldots, x_r)] = n.$$

In other words, the degrees of finite subextensions are bounded and the lemma follows. □

Lemma 7.38.8. *Let K be a field of characteristic $p > 0$. Let $K \subset L$ be a separable algebraic extension. Let $\alpha \in L$. If the coefficients of the minimal polynomial of α over K are pth powers in K then α is a pth power in L.*

Proof. We may assume that $K \subset L$ is finite Galois. Let $P(T) = T^d + \sum_{i=1}^d a_i T^{d-i}$ be the minimal polynomial and assume $a_i = b_i^p$. The polynomial $Q(T) = T^d + \sum_{i=1}^d b_i T^{d-i}$ is a separable irreducible polynomial as well. Let $L \subset L'$ be the field extension obtained by adjoining a single root β of Q to L. This is a separable extension on the one hand, and a purely inseparable extension on the other hand, since clearly β^p is equal to a conjugate of α. Hence $L = L'$ which means a conjugate of α is a pth power. Hence α is a pth power. □

7.39. Separable extensions

In this section we talk about separability for nonalgebraic field extensions. This is closely related to the concept of geometrically reduced algebras, see Definition 7.40.1.

Definition 7.39.1. Let $k \subset K$ be a field extension.

(1) We say K is *separably generated over k* if there exists a transcendence basis $\{x_i; i \in I\}$ of K/k such that the extension $k(x_i; i \in I) \subset K$ is a separable algebraic extension.
(2) We say K is *separable over k* if for every subextension $k \subset K' \subset K$ with K' finitely generated over k, the extension $k \subset K'$ is separably generated.

With this awkward definition it is not clear that a separably generated field extension is itself separable. It will turn out that this is the case, see Lemma 7.41.2.

Lemma 7.39.2. *Let $k \subset K$ be a separable field extension. For any subextension $k \subset K' \subset K$ the field extension $k \subset K'$ is separable.*

Proof. This is direct from the definition. □

Lemma 7.39.3. *Let $k \subset K$ be a separably generated, and finitely generated field extension. Set $r = trdeg_k(K)$. Then there exist elements $x_1, \ldots, x_{r+1}$ of K such that*

(1) *$x_1, \ldots, x_r$ is a transcendence basis of K over k,*
(2) *$K = k(x_1, \ldots, x_{r+1})$, and*
(3) *x_{r+1} is separable over $k(x_1, \ldots, x_r)$.*

Proof. Combine the definition with Lemma 7.38.5. □

Lemma 7.39.4. *Let $k \subset K$ be a finitely generated field extension. There exists a diagram*

$$\begin{array}{ccc} K & \longrightarrow & K' \\ \uparrow & & \uparrow \\ k & \longrightarrow & k' \end{array}$$

where $k \subset k'$, $K \subset K'$ are finite purely inseparable field extensions such that $k' \subset K'$ is a separably generated field extension.

Proof. This lemma is only interesting when the characteristic of k is $p > 0$. Choose $x_1, \ldots, x_r$ a transcendence basis of K over k. As K is finitely generated over k the extension $k(x_1, \ldots, x_r) \subset K$ is finite. Let $k(x_1, \ldots, x_r) \subset K_{sep} \subset K$ be the subextension found in Lemma 7.38.2. If $K = K_{sep}$ then we are done. We will use induction on $d = [K : K_{sep}]$.

Assume that $d > 1$. Choose a $\beta \in K$ with $\alpha = \beta^p \in K_{sep}$ and $\beta \notin K_{sep}$. Let $P = T^d + a_1 T^{d-1} + \ldots + a_d$ be the minimal polynomial of α over $k(x_1, \ldots, x_r)$. Let $k \subset k'$ be a finite purely inseparable extension obtained by adjoining pth roots such that each a_i is a pth power in $k'(x_1^{1/p}, \ldots, x_r^{1/p})$. Such an extension exists; details omitted. Let L be a field fitting into the diagram

$$\begin{tikzcd} K \arrow[r] & L \\ k(x_1, \ldots, x_r) \arrow[u] \arrow[r] & k'(x_1^{1/p}, \ldots, x_r^{1/p}) \arrow[u] \end{tikzcd}$$

and such that L is the compositum of K and $k'(x_1^{1/p}, \ldots, x_r^{1/p})$. Let

$$k'(x_1^{1/p}, \ldots, x_r^{1/p}) \subset L_{sep} \subset L$$

be the subextesion found in Lemma 7.38.2. Then it is clear that L_{sep} is the compositum of K_{sep} and $k'(x_1^{1/p}, \ldots, x_r^{1/p})$. The element $\alpha \in L_{sep}$ has a minimal polynomial P all of whose coefficients are pth powers in $k'(x_1^{1/p}, \ldots, x_r^{1/p})$. By Lemma 7.38.8 we see that $\alpha = (\alpha')^p$ for some $\alpha' \in L_{sep}$. Clearly, this means that β maps to $\alpha' \in L_{sep}$. In other words, we get the tower of fields

$$\begin{tikzcd} K \arrow[r] & L \\ K_{sep}(\beta) \arrow[u] \arrow[r] & L_{sep} \arrow[u] \\ K_{sep} \arrow[u] \arrow[r] & L_{sep} \arrow[u, equal] \\ k(x_1, \ldots, x_r) \arrow[u] \arrow[r] & k'(x_1^{1/p}, \ldots, x_r^{1/p}) \arrow[u] \\ k \arrow[u] \arrow[r] & k' \arrow[u] \end{tikzcd}$$

Thus this construction leads to a new situation with $[L : L_{sep}] < [K : K_{sep}]$. By induction we can find $k' \subset k''$ and $L \subset L'$ as in the lemma for the extension $k' \subset L$. Then the extensions $k \subset k''$ and $K \subset L''$ work for the extension $k \subset K$. This proves the lemma. $\square$

7.40. Geometrically reduced algebras

The main result on geometrically reduced algebras is Lemma 7.41.3. We suggest the reader skip to the lemma after reading the definition.

Definition 7.40.1. Let k be a field. Let S be a k-algebra. We say S is *geometrically reduced over k* if for every field extension $k \subset K$ the K-algebra $K \otimes_k S$ is reduced.

Let k be a field and let S be a reduced k algebra. To check that S is geometrically reduced it will suffice to check that $\overline{k} \otimes_k S$ is reduced (where $\overline{k}$ denotes the algebraic closure of k). In fact it is enough to check this for finite purely inseparable field extensions $k \subset k'$. See Lemma 7.41.3.

Lemma 7.40.2. *Elementary properties of geometrically reducedness. Let k be a field. Let S be a k-algebra.*

(1) *If S is geometrically reduced over k so is every k-subalgebra.*
(2) *If all finitely generated k-subalgebras of S are geometrically reduced, then S is geometrically reduced.*
(3) *A directed colimit of geometrically reduced k-algebras is geometrically reduced.*
(4) *If S is geometrically reduced over k, then any localization of S is geometrically reduced over k.*

Proof. Omitted. The second and third property follow from the fact that tensor product commutes with colimits. □

Lemma 7.40.3. *Let k be a field. If R is geometrically reduced over k, and $S \subset R$ is a multiplicative subset, then the localization $S^{-1}R$ is geometrically reduced over k. If R is geometrically reduced over k, then $R[x]$ is geometrically reduced over k.*

Proof. Omitted. Hints: A localization of a reduced ring is reduced, and localization commutes with tensor products. □

In the proofs of the following lemmas we will repeatedly use the following observation: Suppose that $R' \subset R$ and $S' \subset S$ are inclusions of k-algebras. Then the map $R' \otimes_k S' \to R \otimes_k S$ is injective.

Lemma 7.40.4. *Let k be a field. Let R, S be k-algebras.*

(1) *If $R \otimes_k S$ is nonreduced, then there exist finitely generated subalgebras $R' \subset R$, $S' \subset S$ such that $R' \otimes_k S'$ is not reduced.*
(2) *If $R \otimes_k S$ contains a nonzero zero divisor, then there exist finitely generated subalgebras $R' \subset R$, $S' \subset S$ such that $R' \otimes_k S'$ contains a nonzero zero divisor.*
(3) *If $R \otimes_k S$ contains a nontrivial idempotent, then there exist finitely generated subalgebras $R' \subset R$, $S' \subset S$ such that $R' \otimes_k S'$ contains a nontrivial idempotent.*

Proof. Suppose $z \in R \otimes_k S$ is nilpotent. We may write $z = \sum_{i=1,\ldots,n} x_i \otimes y_i$. Thus we may take R' the k-subalgebra generated by the x_i and S' the k-subalgebra generated by the y_i. The second and third statements are proved in the same way. □

Lemma 7.40.5. *Let k be a field. Let S be a geometrically reduced k-algebra. Let R be any reduced k-algebra. Then $R \otimes_k S$ is reduced.*

Proof. By Lemma 7.40.4 we may assume that R is of finite type over k. Then R, as a reduced Noetherian ring, embeds into a finite product of fields (see Lemmas 7.22.2, 7.28.6, and 7.23.3). Hence we may assume R is a finite product of fields. In this case the reducedness follows from the definition. □

Lemma 7.40.6. *Let k be a field. Let S be a reduced k-algebra. Let $k \subset K$ be either a separable field extension, or a separably generated field extension. Then $K \otimes_k S$ is reduced.*

Proof. Assume $k \subset K$ is separable. By Lemma 7.40.4 we may assume that S is of finite type over k and K is finitely generated over k. Then S embeds into a finite product of fields, namely its total ring of fractions (see Lemmas 7.23.3 and 7.22.2). Hence we may actually assume that S is a domain. We choose $x_1, \ldots, x_{r+1} \in K$ as in Lemma 7.39.3. Let $P \in k(x_1, \ldots, x_r)[T]$ be the minimal polynomial of x_{r+1}. It is a separable polynomial. It is easy to see that $k[x_1, \ldots, x_r] \otimes_k S = S[x_1, \ldots, x_r]$ is a domain. This implies $k(x_1, \ldots, x_r) \otimes_k S$ is a domain as it is a localization of $S[x_1, \ldots, x_r]$. The ring extension $k(x_1, \ldots, x_r) \otimes_k S \subset K \otimes_k S$ is generated by a single element x_{r+1} with a single equation, namely P. Hence $K \otimes_k S$ embeds into $f.f.(k(x_1, \ldots, x_n) \otimes_k S)[T]/(P)$. Since P is separable this is a finite product of fields and we win.

At this point we do not yet know that a separably generated field extension is separable, so we have to prove the lemma in this case also. To do this suppose that $\{x_i\}_{i \in I}$ is a separating transcendence basis for K over k. For any finite set of elements $\lambda_j \in K$ there exists a finite subset $T \subset I$ such that $k(\{x_i\}_{i \in T}) \subset k(\{x_i\}_{i \in T} \cup \{\lambda_j\})$ is finite separable. Hence we see that K is a directed colimit of finitely generated and separably generated extensions of k. Thus the argument of the preceding paragraph applies to this case as well. ☐

Lemma 7.40.7. *Let k be a field and let S be a k-algebra. Assume that S is reduced and that $S_{\mathfrak p}$ is geometrically reduced for every minimal prime $\mathfrak p$ of S. Then S is geometrically reduced.*

Proof. Since S is reduced the map $S \to \prod_{\mathfrak p \text{ minimal}} S_{\mathfrak p}$ is injective, see Lemma 7.23.6. If $k \subset K$ is a field extension, then the maps

$$S \otimes_k K \to (\prod S_{\mathfrak p}) \otimes_k K \to \prod S_{\mathfrak p} \otimes_k K$$

are injective: the first as $k \to K$ is flat and the second by inspection because K is a free k-module. As $S_{\mathfrak p}$ is geometrically reduced the ring on the right is reduced. Thus we see that $S \otimes_k K$ is reduced as a subring of a reduced ring. ☐

7.41. Separable extensions, continued

In this section we continue the discussion started in Section 7.39. Let p be a prime number and let k be a field of characteristic p. In this case we write $k^{1/p}$ for the extension of k gotten by adjoining pth roots of all the elements of k to k. (In other words it is the subfield of an algebraic closure of k generated by the pth roots of elements of k.)

Lemma 7.41.1. *Let k be a field of characteristic $p > 0$. Let $k \subset K$ be a field extension. The following are equivalent:*

(1) *K is separable over k,*
(2) *the ring $K \otimes_k k^{1/p}$ is reduced, and*
(3) *K is geometrically reduced over k.*

Proof. The implication (1) ⇒ (3) follows from Lemma 7.40.6. The implication (3) ⇒ (2) is immediate.

Assume (2). Let $k \subset L \subset K$ be a subextension such that L is a finitely generated field extension of k. We have to show that we can find a separating transcendence basis of L. The assumption implies that $L \otimes_k k^{1/p}$ is reduced. Let $x_1, \ldots, x_r$ be a transcendence basis of L over k such that the degree of inseparability of the finite extension $k(x_1, \ldots, x_r) \subset L$ is minimal. If L is separable over $k(x_1, \ldots, x_r)$ then we win. Assume this is not the case to get a contradiction. Then there exists an element $\alpha \in L$ which is not separable over $k(x_1, \ldots, x_r)$. Let $P(T) \in k(x_1, \ldots, x_r)[T]$ be its minimal polynomial. Because α is not separable actually P is a polynomial in T^p. Clear denominators to get an irreducible polynomial

$$G(X_1, \ldots, X_r, T) \in k[X_1, \ldots, X_r, T]$$

such that $G(x_1, \ldots, x_r, \alpha) = 0$ in L. Note that this means $k[X_1, \ldots, X_r, T]/(G) \subset L$. We claim that dG/dX_i is not identically zero for at least one i. Namely, if this was not the case, then G is actually a polynomial in $X_1^p, \ldots, X_r^p, T^p$ and hence $G^{1/p} \in k^{1/p}[X_1, \ldots, X_r, T]$ would map to a nonzero nilpotent element of $k^{1/p} \otimes_k L$! Thus, after renumbering, we may assume that dG/dX_1 is not zero. Then we see that x_1 is separably algebraic over $k(x_2, \ldots, x_r, \alpha)$, and that $x_2, \ldots, x_r, \alpha$ is a transcendence basis of L over k. This means that the degree of inseparability of the finite extension $k(x_2, \ldots, x_r, \alpha) \subset L$ is less than the degree of inseparability of the finite extension $k(x_1, \ldots, x_r) \subset L$, which is a contradiction. □

Lemma 7.41.2. *A separably generated field extension is separable.*

Proof. Combine Lemma 7.40.6 with Lemma 7.41.1. □

In the following lemma we will use the notion of the perfect closure which is defined in Definition 7.42.5.

Lemma 7.41.3. *Let k be a field. Let S be a k-algebra. The following are equivalent:*

(1) *$k' \otimes_k S$ is reduced for every finite purely inseparable extension k' of k,*
(2) *$k^{1/p} \otimes_k S$ is reduced,*
(3) *$k^{perf} \otimes_k S$ is reduced, where k^{perf} is the perfect closure of k,*
(4) *$\overline{k} \otimes_k S$ is reduced, where $\overline{k}$ is the algebraic closure of k, and*
(5) *S is geometrically reduced over k.*

Proof. Note that any finite purely inseparable extension $k \subset k'$ embeds in k^{perf}. Moreover, $k^{1/p}$ embeds into k^{perf} which embeds into $\overline{k}$. Thus it is clear that (5) ⇒ (4) ⇒ (3) ⇒ (2) and that (3) ⇒ (1).

We prove that (1) ⇒ (5). Assume $k' \otimes_k S$ is reduced for every finite purely inseparable extension k' of k. Let $k \subset K$ be an extension of fields. We have to show that $K \otimes_k S$ is reduced. By Lemma 7.40.4 we reduce to the case where $k \subset K$ is a finitely generated field extension. Choose a diagram

$$\begin{tikzcd} K \arrow[r] & K' \\ k \arrow[u] \arrow[r] & k' \arrow[u] \end{tikzcd}$$

as in Lemma 7.39.4. By assumption $k' \otimes_k S$ is reduced. By Lemma 7.40.6 it follows that $K' \otimes_k S$ is reduced. Hence we conclude that $K \otimes_k S$ is reduced as desired.

Finally we prove that (2) ⇒ (5). Assume $k^{1/p} \otimes_k S$ is reduced. Then S is reduced. Moreover, for each localization $S_{\mathfrak p}$ at a minimal prime $\mathfrak p$, the ring $k^{1/p} \otimes_k S_{\mathfrak p}$ is a localization of $k^{1/p} \otimes_k S$ hence is reduced. But $S_{\mathfrak p}$ is a field by Lemma 7.23.3, hence $S_{\mathfrak p}$ is geometrically reduced by Lemma 7.41.1. It follows from Lemma 7.40.7 that S is geometrically reduced. □

7.42. Perfect fields

Here is the definition.

Definition 7.42.1. Let k be a field. We say k is *perfect* if every field extension of k is separable over k.

Lemma 7.42.2. *A field k is perfect if and only if it is a field of characteristic 0 or a field of characteristic $p > 0$ such that every element has a pth root.*

Proof. The characteristic zero case is clear. Assume the characteristic of k is $p > 0$. If k is perfect, then all the field extensions where we adjoin a pth root of an element of k have to be trivial, hence every element of k has a pth root. Conversely if every element has a pth root, then $k = k^{1/p}$ and every field extension of k is separable by Lemma 7.41.1. $\square$

Lemma 7.42.3. *Let $k \subset K$ be a finitely generated field extension. There exists a diagram*

$$\begin{tikzcd} K \arrow[r] & K' \\ k \arrow[u] \arrow[r] & k' \arrow[u] \end{tikzcd}$$

where $k \subset k'$, $K \subset K'$ are finite purely inseparable field extensions such that $k' \subset K'$ is a separable field extension. In this situation we can assume that $K' = k'K$ is the compositum, and also that $K' = (k' \otimes_k K)_{red}$.

Proof. By Lemma 7.39.4 we can find such a diagram with $k' \subset K'$ separably generated. By Lemma 7.41.2 this implies that K' is separable over k'. The compositum $k'K$ is a subextension of $k' \subset K'$ and hence $k' \subset k'K$ is separable by Lemma 7.39.2. The ring $(k' \otimes_k K)_{red}$ is a domain as for some $n \gg 0$ the map $x \mapsto x^{p^n}$ maps it into K. Hence it is a field by Lemma 7.32.17. Thus $(k' \otimes_k K)_{red} \to K'$ maps it isomorphically onto $k'K$. $\square$

Lemma 7.42.4. *For every field k there exists a purely inseparable extension $k \subset k'$ such that k' is perfect. The field extension $k \subset k'$ is unique up to unique isomorphism.*

Proof. If the characteristic of k is zero, then $k' = k$ is the unique choice. Assume the characteristic of k is $p > 0$. For every $n > 0$ there exists a unique algebraic extension $k \subset k^{1/p^n}$ such that (a) every element $\lambda \in k$ has a p^nth root in k^{1/p^n} and (b) for every element $\mu \in k^{1/p^n}$ we have $\mu^{p^n} \in k$. Namely, consider the ring map $k \to k^{1/p^n} = k$, $x \mapsto x^{p^n}$. This is injective and satisfies (a) and (b). It is clear that $k^{1/p^n} \subset k^{1/p^{n+1}}$ as extensions of k via the map $y \mapsto y^p$. Then we can take $k' = \bigcup k^{1/p^n}$. Some details omitted. $\square$

Definition 7.42.5. Let k be a field. The field extension $k \subset k'$ of Lemma 7.42.4 is called the *perfect closure* of k. Notation $k \subset k^{perf}$.

Note that if $k \subset k'$ is any algebraic purely inseparable extension, then $k' \subset k^{perf}$. Namely, $(k')^{perf}$ is isomorphic to k^{perf} by the uniqueness of Lemma 7.42.4.

Lemma 7.42.6. *Let k be a perfect field. Any reduced k algebra is geometrically reduced over k. Let R, S be k-algebras. Assume both R and S are reduced. Then the k-algebra $R \otimes_k S$ is reduced.*

Proof. The first statement follows from Lemma 7.41.3. For the second statement use the first statement and Lemma 7.40.5. $\square$

7.43. Geometrically irreducible algebras

Let $k \subset k'$ be an algebraic purely inseparable field extension. Then for any k-algebra R the ring map $R \to k' \otimes_k R$ induces a homeomorphism of spectra. The reason for this is the following slightly more general Lemma 7.43.2 below. The second part of this lemma is often applied when $R \subset S$ and there exists a prime number p such that $pR = 0$ and S is generated over R by elements x such that $x^{p^f} \in R$ for some $f = f(x) \geq 0$.

Lemma 7.43.1. *Let $n, m > 0$ be two integers. There exist $a, b \geq 0$ such that setting $N = n^a m^b$ we have $(x + y)^N \in \mathbf{Z}[x^n, nx, y^m, my]$.*

Proof. Let $k = nm$. Then it suffices to prove that we can find a power $N = k^c$ such that $(x + y)^N \in \mathbf{Z}[x^k, kx, y^k, ky]$. Write

$$(x + y)^N = \sum\nolimits_{i,j \geq 0, i+j=N} \binom{N}{i, j} x^i y^j$$

The condition means that

$$k^{r+r'} \mid \binom{N}{i, j}$$

where $i = qk + r$ with $r \in \{0, \ldots, k-1\}$ and $j = q'k + r'$ with $r' \in \{0, \ldots, k-1\}$. Choose $N = k^{k+1}$. Then the fact that $i + j = N$ is divisible by k implies that either $r = r' = 0$ if k divides both i and j, or else $r + r' = k$. The choice of N works: If k divides both i and j then the result is clear. If not then we write because $\binom{N}{i,j} = \frac{N}{i} \cdot \binom{N-1}{i-1}$ and N/i is divisible by k^k because as $r \not= 0$ we see that $\gcd(N, i)$ divides k in this case. □

Lemma 7.43.2. *Let $\varphi : R \to S$ be a ring map. If*

1. *for any $x \in S$ there exists $n > 0$ such that x^n is in the image of φ, and*
2. *for any $x \in Ker(\varphi)$ there exists $n > 0$ such that $x^n = 0$,*

then φ induces a homeomorphism on spectra. If

- (a) *S is generated as an R-algebra by elements x such that there exists an $n > 0$ with $x^n \in \varphi(R)$ and $nx \in \varphi(R)$, and*
- (b) *the kernel of φ is generated by nilpotent elements,*

then (1) and (2) hold, and for any ring map $R \to R'$ the ring map $R' \to R' \otimes_R S$ also satisfies (a), (b), (1), and (2) and in particular induces a homeomorphism on spectra.

Proof. Assume (1) and (2). Let $\mathfrak{q}, \mathfrak{q}'$ be primes of S lying over the same prime ideal $\mathfrak{p}$ of R. Suppose $x \in S$ with $x \in \mathfrak{q}$, $x \not\in \mathfrak{q}'$. Then $x^n \in \mathfrak{q}$ and $x^n \not\in \mathfrak{q}'$ for all $n > 0$. If $x^n = \varphi(y)$ with $y \in R$ for some $n > 0$ then

$$x^n \in \mathfrak{q} \Rightarrow y \in \mathfrak{p} \Rightarrow x^n \in \mathfrak{q}'$$

which is a contradication. Hence there does not exist an x as above and we conclude that $\mathfrak{q} = \mathfrak{q}'$, i.e., the map on spectra is injective. By assumption (2) the kernel $I = \text{Ker}(\varphi)$ is contained in every prime, hence $Spec(R) = Spec(R/I)$ as topological spaces. As the induced map $R/I \to S$ is integral by assumption (1) Lemma 7.32.15 shows that $Spec(S) \to Spec(R/I)$ is surjective. Combining the above we see that $Spec(S) \to Spec(R)$ is bijective. If $x \in S$ is arbitrary, and we pick $y \in R$ such that $\varphi(y) = x^n$ for some $n > 0$, then we see that the open $D(x) \subset Spec(S)$ corresponds to the open $D(y) \subset Spec(R)$ via the bijection above. Hence we see that the map $Spec(S) \to Spec(R)$ is a homeomorphism.

Assume (a) and (b). Note that (b) is equivalent to (2). We claim that for any $x \in S$ there exists an integer $n > 0$ such that $x^n, nx \in \varphi(R)$. By assumption (a) it suffices to show that the set of elements with this property forms a R-sub algebra. Suppose $x, y \in S$ and $n, m > 0$ such that $x^n, y^m, nx, my \in \varphi(R)$. Then $(xy)^{nm}, nmxy \in \varphi(R)$ and we see that xy satisfies the condition. Note that $nm(x + y) \in \varphi(R)$ and that there exists an integer $N = n^a m^b$, $a, b \geq 0$ such that $(x + y)^N \in \varphi(R)$, see Lemma 7.43.1. Thus $x + y$ satisfies the condition and the claim is proved. In particular it follows from the first part of the lemma that $Spec(S) \to Spec(R)$ is a homeomorphism. In particular it is surjective which is a property preserved under any base change, see Lemma 7.27.3. Therefore for any $R \to R'$ the kernel of the ring map $R' \to R' \otimes_R S$ consists of nilpotent elements, see Lemma 7.27.6, in other words (b) holds for $R' \to R' \otimes_R S$. Finally, it is clear that (a) is preserved under base change which finishes the proof. □

Lemma 7.43.3. *Let $R \to S$ be a ring map. Assume*

(a) *$Spec(R)$ is irreducible,*
(b) *$R \to S$ is flat,*
(c) *$R \to S$ is of finite presentation,*
(d) *the fibre rings $S \otimes_R \kappa(\mathfrak{p})$ have irreducible spectra for a dense collection of primes $\mathfrak{p}$ of R.*

Then $Spec(S)$ is irreducible. This is true more generally with (b) + (c) replaced by ``the map $Spec(S) \to Spec(R)$ is open".

Proof. The assumptions (b) and (c) imply that the map on spectra is open, see Proposition 7.36.8. Hence the lemma follows from Topology, Lemma 5.5.7. □

Lemma 7.43.4. *Let k be a separably algebraically closed field. Let R, S be k-algebras. If R, S have a unique minimal prime, so does $R \otimes_k S$.*

Proof. Let $k \subset \overline{k}$ be a perfect closure, see Definition 7.42.5. By assumption $\overline{k}$ is algebraically closed. The ring maps $R \to R \otimes_k \overline{k}$ and $S \to S \otimes_k \overline{k}$ and $R \otimes_k S \to (R \otimes_k S) \otimes_k \overline{k} = (R \otimes_k \overline{k}) \otimes_{\overline{k}} (S \otimes_k \overline{k})$ satisfy the assumptions of Lemma 7.43.2. Hence we may assume k is algebraically closed.

We may replace R and S by their reductions. Hence we may assume that R and S are domains. By Lemma 7.42.6 we see that $R \otimes_k S$ is reduced. Hence its spectrum is reducible if and only if it contains a nonzero zero divisor. By Lemma 7.40.4 we reduce to the case where R and S are domains of finite type over k algebraically closed.

Note that the ring map $R \to R \otimes_k S$ is of finite presentation and flat. Moreover, for every maximal ideal $\mathfrak{m}$ of R we have $(R \otimes_k S) \otimes_R R/\mathfrak{m} \cong S$ because $k \cong R/\mathfrak{m}$ by the Hilbert Nullstellensatz Theorem 7.30.1. Moreover, the set of maximal ideals is dense in the spectrum of R since $Spec(R)$ is Jacobson, see Lemma 7.31.2. Hence we see that Lemma 7.43.3 applies to the ring map $R \to R \otimes_k S$ and we conclude that the spectrum of $R \otimes_k S$ is irreducible as desired. □

Lemma 7.43.5. *Let k be a field. Let R be a k-algebra. The following are equivalent*

(1) *for every field extension $k \subset k'$ the spectrum of $R \otimes_k k'$ is irreducible, and*
(2) *for every finite separable field extension $k \subset k'$ the spectrum of $R \otimes_k k'$ is irreducible.*

Proof. Let $k \subset k^{perf}$ be a perfect closure of k, see Definition 7.42.5. By Lemma 7.43.2 we may replace R by $(R \otimes_k k^{perf})_{reduction}$ and k by k^{perf} (some details omitted). Hence we may assume that R is geometrically reduced over k.

Assume R is geometrically reduced over k. For any extension of fields $k \subset k'$ we see irreducibility of the spectrum of $R \otimes_k k'$ is equivalent to $R \otimes_k k'$ being a domain. Assume (2). Let $k \subset \overline{k}$ be a separable algebraic closure of k. Using Lemma 7.40.4 we see that (2) is equivalent to $R \otimes_k \overline{k}$ being a domain. For any field extension $k \subset k'$, there exists a field extension $\overline{k} \subset \overline{k}'$ with $k' \subset \overline{k}'$. By Lemma 7.43.4 we see that $R \otimes_k \overline{k}'$ is a domain. If $R \otimes_k k'$ is not a domain, then also $R \otimes_k \overline{k}'$ is not a domain, contradiction. □

Definition 7.43.6. Let k be a field. Let S be a k-algebra. We say S is *geometrically irreducible over k* if for every field extension $k \subset k'$ the spectrum of $R \otimes_k k'$ is irreducible[2].

By Lemma 7.43.5 it suffices to check this for finite separable field extensions $k \subset k'$.

Lemma 7.43.7. *Let k be a field. Let R be a k-algebra. If k is separably algebraically closed then R is geometrically irreducible over k if and only if the spectrum of R is irreducible.*

Proof. Immediate from the remark following Definition 7.43.6. □

Lemma 7.43.8. *Let k be a field. Let S be a k-algebra.*

(1) *If S is geometrically irreducible over k so is every k-subalgebra.*
(2) *If all finitely generated k-subalgebras of S are geometrically irreducible, then S is geometrically irreducible.*
(3) *A directed colimit of geometrically irreducible k-algebras is geometrically irreducible.*

Proof. Let $S' \subset S$ be a subalgebra. Then for any extension $k \subset k'$ the ring map $S' \otimes_k k' \to S \otimes_k k'$ is injective also. Hence (1) follows from Lemma 7.27.5 (and the fact that the image of an irreducible space under a continuous map is irreducible). The second and third property follow from the fact that tensor product commutes with colimits. □

Lemma 7.43.9. *Let k be a field. Let S be a geometrically irreducible k-algebra. Let R be any k-algebra. The map*

$$Spec(R \otimes_k S) \longrightarrow Spec(R)$$

induces a bijection on irreducible components.

Proof. Recall that irreducible components correspond to minimal primes (Lemma 7.23.1). As $R \to R \otimes_k S$ is flat we see by going down (Lemma 7.35.17) that any minimal prime of $R \otimes_k S$ lies over a minimal prime of R. Conversely, if $\mathfrak{p} \subset R$ is a (minimal) prime then

$$R \otimes_k S/\mathfrak{p}(R \otimes_k S) = (R/\mathfrak{p}) \otimes_k S \subset f.f.(R/\mathfrak{p}) \otimes_k S$$

by flatness of $R \to R \otimes_k S$. The ring $f.f.(R/\mathfrak{p}) \otimes_k S$ has irreducible spectrum by assumption. It follows that $R \otimes_k S/\mathfrak{p}(R \otimes_k S)$ has a single minimal prime (Lemma 7.27.5). In other words, the inverse image of the irreducible set $V(\mathfrak{p})$ is irreducible. Hence the lemma follows. □

Let us make some remarks on the notion of geometrically irreducible field extensions.

Lemma 7.43.10. *Let $k \subset K$ be a field extension. If k is algebraically closed in K, then K is geometrically irreducible over k.*

[2] An irreducible space is nonempty.

Proof. Let $k \subset k'$ be a finite separable extension, say generated by $\alpha \in k'$ over k (see Lemma 7.38.5). Let $P = T^d + a_1 T^{d-1} + \ldots + a_d \in k[T]$ be the minimal polynomial of α. Then $K \otimes_k k' \cong K[T]/(P)$. The only way the spectrum of $K[T]/(P)$ can be reducible is if P is reducible in $K[T]$. Say $P = P_1 P_2$ is a nontrivial factorization of P into monic polynomials. Let $b_1, \ldots, b_t \in K$ be the coefficients of P_1. Then we see that b_i is algebraic over k by Lemma 7.34.5. Hence the lemma follows. ☐

Lemma 7.43.11. *Let $k \subset K$ be a field extension. Consider the subextension $k \subset k' \subset K$ such that $k \subset k'$ is separable algebraic and $k' \subset K$ maximal with this property. Then K is geometrically irreducible over k'. If K/k is a finitely generated field extension, then $[k' : k] < \infty$.*

Proof. Let $k'' \subset K$ be the algebraic closure of k in K. By Lemma 7.43.10 we see that K is geometrically irreducible over k''. Since $k' \subset k''$ is purely inseparable we see from Lemma 7.43.2 that also the extension $k' \subset K$ is geometrically irreducible (some details omitted). If $k \subset K$ is finitely generated, then k' is finite over k by Lemma 7.38.7. ☐

Lemma 7.43.12. *Let $k \subset K$ be an extension of fields. Let $k \subset \overline{k}$ be a separable algebraic closure. Then $Gal(\overline{k}/k)$ acts transitively on the primes of $\overline{k} \otimes_k K$.*

Proof. Let $k \subset k' \subset K$ be the subextension found in Lemma 7.43.11. Note that as $k \subset \overline{k}$ is integral all the prime ideals of $\overline{k} \otimes_k K$ and $\overline{k} \otimes_k k'$ are maximal, see Lemma 7.32.18. In particular the residue field of any prime ideal of $\overline{k} \otimes_k k'$ is isomorphic to $\overline{k}$. Hence the prime ideals of $\overline{k} \otimes_k k'$ correspond one to one to elements of $Hom_k(k', \overline{k})$ with $\sigma \in Hom_k(k' \overline{k})$ corresponding to the kernel $\mathfrak{p}_\sigma$ of $1 \otimes \sigma : \overline{k} \otimes_k k' \to \overline{k}$. In particular Gal$(\overline{k}/k)$ acts transitively on this set. Finally, since K is geometrically irreducible over k' we see that there is a unique prime of $\overline{k} \otimes_k K$ lying over each $\mathfrak{p}_\sigma$ since the set of these primes is the set of primes in the ring

$$(\overline{k} \otimes_k K) \otimes_{(\overline{k} \otimes_k k'), 1 \otimes \sigma} \kappa(\mathfrak{p}_\sigma) = \overline{k} \otimes_{\overline{k}} (K \otimes_{k'} \overline{k}) = K \otimes_{k', \sigma} \overline{k}$$

Thus the lemma holds. ☐

7.44. Geometrically connected algebras

Lemma 7.44.1. *Let k be a separably algebraically closed field. Let R, S be k-algebras. If $Spec(R)$, and $Spec(S)$ are connected, then so is $Spec(R \otimes_k S)$.*

Proof. Recall that $Spec(R)$ is connected if and only if R has no nontrivial idempotents, see Lemma 7.18.4. Hence, by Lemma 7.40.4 we may assume R and S are of finite type over k. In this case R and S are Noetherian, and have finitely many minimal primes, see Lemma 7.28.6. Thus we may argue by induction on $n + m$ where n, resp. m is the number of irreducible components of $Spec(R)$, resp. $Spec(S)$. Of course the case where either n or m is zero is trivial. If $n = m = 1$, i.e., $Spec(R)$ and $Spec(S)$ both have one irreducible component, then the result holds by Lemma 7.43.4. Suppose that $n > 1$. Let $\mathfrak{p} \subset R$ be a minimal prime corresponding to the irreducible closed subset $T \subset Spec(R)$. Let $I \subset R$ be such that $T' = V(I) \subset Spec(R)$ is the closure of the complement of T. Note that this means that $T' = Spec(R/I)$ (Lemma 7.16.7) has $n - 1$ irreducible components. Then $T \cup T' = Spec(R)$, and $T \cap T' = V(\mathfrak{p} + I) = Spec(R/(\mathfrak{p} + I))$ is not empty as $Spec(R)$ is assumed connected. The inverse image of T in $Spec(R \otimes_k S)$ is $Spec(R/\mathfrak{p} \otimes_k S)$, and the inverse of T' in $Spec(R \otimes_k S)$ is $Spec(R/I \otimes_k S)$. By induction these are both connected.

The inverse image of $T \cap T'$ is $Spec(R/(\mathfrak{p}+I) \otimes_k S)$ which is nonempty. Hence $Spec(R \otimes_k S)$ is connected. $\square$

Lemma 7.44.2. *Let k be a field. Let R be a k-algebra. The following are equivalent*

(1) *for every field extension $k \subset k'$ the spectrum of $R \otimes_k k'$ is connected, and*
(2) *for every finite separable field extension $k \subset k'$ the spectrum of $R \otimes_k k'$ is connected.*

Proof. For any extension of fields $k \subset k'$ the connectivity of the spectrum of $R \otimes_k k'$ is equivalent to $R \otimes_k k'$ having no nontrivial idempotents, see Lemma 7.18.4. Assume (2). Let $k \subset \overline{k}$ be a separable algebraic closure of k. Using Lemma 7.40.4 we see that (2) is equivalent to $R \otimes_k \overline{k}$ having no nontrivial idempotents. For any field extension $k \subset k'$, there exists a field extension $\overline{k} \subset \overline{k}'$ with $k' \subset \overline{k}'$. By Lemma 7.44.1 we see that $R \otimes_k \overline{k}'$ has no nontrivial idempotents. If $R \otimes_k k'$ has a nontrivial idempotent, then also $R \otimes_k \overline{k}'$, contradiction. $\square$

Definition 7.44.3. Let k be a field. Let S be a k-algebra. We say S is *geometrically connected over k* if for every field extension $k \subset k'$ the spectrum of $R \otimes_k k'$ is connected.

By Lemma 7.44.2 it suffices to check this for finite separable field extensions $k \subset k'$.

Lemma 7.44.4. *Let k be a field. Let R be a k-algebra. If k is separably algebraically closed then R is geometrically connected over k if and only if the spectrum of R is connected.*

Proof. Immediate from the remark following Definition 7.44.3. $\square$

Lemma 7.44.5. *Let k be a field. Let S be a k-algebra.*

(1) *If S is geometrically connected over k so is every k-subalgebra.*
(2) *If all finitely generated k-subalgebras of S are geometrically connected, then S is geometrically connected.*
(3) *A directed colimit of geometrically irreducible k-algebras is geometrically connected.*

Proof. This follows from the characterization of connectedness in terms of the nonexistence of nontrivial idempotents. The second and third property follow from the fact that tensor product commutes with colimits. $\square$

The following lemma will be superceded by the more general Varieties, Lemma 28.5.4.

Lemma 7.44.6. *Let k be a field. Let S be a geometrically connected and nonzero k-algebra. Let R be any k-algebra. The map*

$$R \longrightarrow R \otimes_k S$$

induces a bijection on idempotents, and the map

$$Spec(R \otimes_k S) \longrightarrow Spec(R)$$

induces a bijection on connected components.

Proof. The second assertion follows from the first combined with Lemma 7.19.2. By Lemmas 7.44.5 and 7.40.4 we may assume that R and S are of finite type over k. Then we see that also $R \otimes_k S$ is of finite type over k. Note that in this case all the rings are Noetherian and hence their spectra have finitely many connected components (since they have finitely many irreducible components, see Lemma 7.28.6). In particular, all connected components in question are open! Hence via Lemma 7.20.3 we see that the first statement of the lemma in this case is equivalent to the second. Let's prove this. As the algebra S is geometrically connected and nonzero we see that all fibres of $X = Spec(R \otimes_k S) \to Spec(R) = Y$ are connected and nonempty. Also, as $R \to R \otimes_k S$ is flat of finite presentation the map $X \to Y$ is open (Proposition 7.36.8). Topology, Lemma 5.4.5 shows that $X \to Y$ induces bijection on connected components. □

7.45. Geometrically integral algebras

Definition 7.45.1. Let k be a field. Let S be a k-algebra. We say S is *geometrically integral over k* if for every field extension $k \subset k'$ the ring of $R \otimes_k k'$ is a domain.

Any question about geometrically integral algebras can be translated in a question about geometrically reduced and irreducible algebras.

Lemma 7.45.2. *Let k be a field. Let S be a k-algebra. In this case S is geometrically integral over k if and only if S is geometrically irreducible as well as geometrically reduced over k.*

Proof. Omitted. □

7.46. Valuation rings

Here are some definitions.

Definition 7.46.1. Valuation rings.

(1) Let K be a field. Let A, B be local rings contained in K. We say that B *dominates* A if $A \subset B$ and $\mathfrak{m}_A = A \cap \mathfrak{m}_B$.
(2) Let A be a ring. We say A is a *valuation ring* if A is a local domain, not a field, and if A is maximal for the relation of domination among local rings contained in the fraction field of A.
(3) Let A be a valuation ring with fraction field K. If $R \subset K$ is a subring of K, then we say A is *centered* on R if $R \subset A$.

Lemma 7.46.2. *Let K be a field. Let $A \subset K$ be a local subring which is not a field. Then there exists a valuation ring with fraction field K dominating A.*

Proof. During this proof, and during this proof only, the phrase ``local ring'' will mean ``local ring, not a field''. We consider the collection of local subrings of K as a partially ordered set using the relation of domination. Suppose that $\{A_i\}_{i \in I}$ is a totally ordered collection of local subrings of K. Then $B = \bigcup A_i$ is a local subring which dominates all of the A_i. Hence by Zorn's Lemma, it suffices to show that if $A \subset K$ is a local ring whose fraction field is not K, then there exists a local ring $B \subset K$, $B \neq A$ dominating A.

Pick $t \in K$ which is not in the fraction field of A. If t is transcendental over A, then $A[t] \subset K$ and hence $A[t]_{(t,\mathfrak{m})} \subset K$ is a local ring dominating A. Suppose t is algebraic over A. Then for some $a \in A$ the element at is integral over A. In this case the subring $A' \subset K$ generated by A and ta is finite over A. By Lemma 7.32.15 there exists a prime ideal $\mathfrak{m}' \subset A'$ lying over $\mathfrak{m}$. Then $A'_{\mathfrak{m}'}$ clearly dominates A and we win. □

Lemma 7.46.3. *Let A be a valuation ring with maximal ideal $\mathfrak{m}$ and fraction field K. Let $x \in K$. Then either $x \in A$ or $x^{-1} \in A$ or both.*

Proof. Assume that x is not in A. Let A' denote the subring of A generated by A and x. Since A is a valuation ring we see that there is no prime of A' lying over $\mathfrak{m}$. Hence we can write $1 = \sum_{i=0}^{d} t_i x^i$ with $t_i \in \mathfrak{m}$. This implies that $(1 - t_0)(x^{-1})^d - \sum t_i (x^{-1})^{d-i} = 0$. In particular we see that x^{-1} is integral over A. Thus the subring A'' of K generated by A and x^{-1} is finite over A and we see there exists a prime ideal $\mathfrak{m}'' \subset A''$ lying over $\mathfrak{m}$ by Lemma 7.32.15. Since A is a valuation ring we conclude that $A = (A'')_{\mathfrak{m}''}$ and hence $x^{-1} \in A$. □

Lemma 7.46.4. *Let $A \subset K$ be a local domain contained in a field K. Assume that A is not a field, and for every $x \in K$ either $x \in A$ or $x^{-1} \in A$ or both. Then A is a valuation ring.*

Proof. Suppose that A' is a local ring contained in K which dominates A. Let $x \in A'$. We have to show that $x \in A$. If not, then $x^{-1} \in A$, and of course $x^{-1} \in \mathfrak{m}_A$. But then $x^{-1} \in \mathfrak{m}_{A'}$ which contradicts $x \in A'$. □

Lemma 7.46.5. *Let $K \subset L$ be an extension of fields. If $B \subset L$ is a valuation ring, then $A = K \cap B$ is either a field or a valuation ring.*

Proof. Omitted. Hint: Combine Lemmas 7.46.3 and 7.46.4. □

Lemma 7.46.6. *Let A be a valuation ring. Then A is a normal domain.*

Proof. Suppose x is in the field of fractions of A and integral over A, say $x^d + \sum_{i<d} a_i x^i = 0$. By Lemma 7.46.4 either $x \in A$ (and we're done) or $x^{-1} \in A$. In the second case we see that $x = -\sum a_i x^{i-d} \in A$ as well. □

An *totally ordered abelian group* is a pair $(\Gamma, \geq)$ consisting of an abelian group Γ endowed with a total ordering $\geq$ such that $\gamma \geq \gamma' \Rightarrow \gamma + \gamma'' \geq \gamma' + \gamma''$ for all $\gamma, \gamma', \gamma'' \in \Gamma$.

Lemma 7.46.7. *Let A be a valuation ring with field of fractions K. Set $\Gamma = K^*/A^*$ (with group law written additively). For $\gamma, \gamma' \in \Gamma$ define $\gamma \geq \gamma'$ if and only if $\gamma - \gamma'$ is in the image of $A - \{0\} \to \Gamma$. Then $(\Gamma, \geq)$ is a totally ordered abelian group.*

Proof. Omitted, but follows easily from Lemma 7.46.3 above. □

Definition 7.46.8. Let A be a valuation ring.

(1) The totally ordered abelian group $(\Gamma, \geq)$ is called the *value group* of the valuation ring A.
(2) The map $v : A - \{0\} \to \Gamma$ and also $v : K^* \to \Gamma$ is called the *valuation* associated to A.
(3) The valuation ring A is called a *discrete valuation ring* if $\Gamma \cong \mathbf{Z}$.

Note that if $\Gamma \cong \mathbf{Z}$ then there is a unique such isomorphism such that $1 \geq 0$. If the isomorphism is chose in this way, then the ordering becomes the usual ordering of the integers.

Lemma 7.46.9. *Let A be a valuation ring. The valution $v : A - \{0\} \to \Gamma_{\geq 0}$ has the following properties:*

(1) $v(a) = 0 \Leftrightarrow a \in A^*$,
(2) $v(ab) = v(a) + v(b)$,
(3) $v(a + b) \geq min(v(a), v(b))$.

Proof. Omitted. □

Lemma 7.46.10. *Let $(\Gamma, \geq)$ be a totally ordered abelian group. Let K be a field. Let $v : K^* \to \Gamma$ be a homomorphism of abelian groups. Then*

$$A = \{x \in K \mid x = 0 \text{ or } v(x) \geq 0\}$$

is a valuation ring with value group $Im(v) \subset \Gamma$, with maximal ideal

$$\mathfrak{m} = \{x \in K \mid x = 0 \text{ or } v(x) > 0\}$$

and with group of units

$$A^* = \{x \in K^* \mid v(x) = 0\}.$$

Proof. Omitted. □

Let $(\Gamma, \geq)$ be a totally ordered abelian group. An *ideal of* Γ is a subset $I \subset \Gamma$ such that all elements of I are ≥ 0 and $\gamma \in I$, $\gamma' \geq \gamma$ implies $\gamma' \in I$. We say that such an ideal is *prime* if $\gamma + \gamma' \in I, \gamma, \gamma' \geq 0 \Rightarrow \gamma \in I$ or $\gamma' \in I$.

Lemma 7.46.11. *Let A be a valuation ring. Ideals in A correspond $1-1$ with ideals of Γ. This bijection is inclusion preserving, and maps prime ideals to prime ideals.*

Proof. Omitted. □

Lemma 7.46.12. *A valuation ring is Noetherian if and only if it is a discrete valuation ring.*

Proof. Suppose A is a discrete valutation ring with valuation $v : A \setminus \{0\} \to \mathbf{Z}$ normalized so that $\mathrm{Im}(v) \subset \mathbf{Z}_{\geq 0}$. By Lemma 7.46.11 the ideals of A are the subsets $I_n = \{0\} \cup v^{-1}(\mathbf{Z}_{\geq n})$. It is clear that any element $x \in A$ with $v(x) = n$ generates I_n. Hence A is a PID so certainly Noetherian.

Suppose A is a Noetherian valuation ring with value group Γ. By Lemma 7.46.11 we see the ascending chain condition holds for ideals in Γ. In particular, by considering the subsets $\gamma + \Gamma_{\geq 0}$ with $\gamma > 0$ we see there exists a smallest element γ_0 which is bigger than 0. Let $\gamma \in \Gamma$ be an element $\gamma > 0$. Consider the sequence of elements $\gamma, \gamma - \gamma_0, \gamma - 2\gamma_0$, etc. By the ascending chain condition these cannot all be > 0. Let $\gamma - n\gamma_0$ be the last one ≥ 0. By minimality of γ_0 we see that $0 = \gamma - n\gamma_0$. Hence Γ is a cyclic group as desired. □

Lemma 7.46.13. *Let $(R, \mathfrak{m})$ be a local domain with fraction field K. Let $R \subset A \subset K$ be a valuation ring which dominates R. Then*

$$A = colim\, R[\tfrac{I}{a}]$$

is a directed colimit of affine blowups $R \to R[\frac{I}{a}]$ with the following two properties

(1) *$a \in I \subset \mathfrak{m}$,*
(2) *I is finitely generated, and*
(3) *the fibre ring of $R \to R[\frac{I}{a}]$ at $\mathfrak{m}$ is not zero.*

Proof. Consider a finite subset $E \subset A$. Say $E = \{e_1, \ldots, e_n\}$. Choose a nonzero $a \in R$ such that we can write $e_i = f_i/a$ for all $i = 1, \ldots, n$. Set $I = (f_1, \ldots, f_n, a)$. We claim that $R[\frac{I}{a}] \subset A$. This is clear as an element of $R[\frac{I}{a}]$ can be represented as a polynomial in the elements e_i. The lemma follows immediately from this observation. □

7.47. More Noetherian rings

Lemma 7.47.1. *Let R be a Noetherian ring. Any finite R-module is of finite presentation. Any submodule of a finite R-module is finite. The ascending chain condition holds for R-submodules of a finite R-module.*

Proof. We first show that any submodule N of a finite R-module M is finite. We do this by induction on the number of generators of M. If this number is 1, then $N = J/I \subset M = R/I$ for some ideals $I \subset J \subset R$. Thus the definition of Noetherian implies the result. If the number of generators of M is greater than 1, then we can find a short exact sequence $0 \to M' \to M \to M'' \to 0$ where M' and M'' have fewer generators. Note that setting $N' = M' \cap N$ and $N'' = \mathrm{Im}(N \to M'')$ gives a similar short exact sequence for N. Hence the result follows from the induction hypothesis since the number of generators of N is at most the number of generators of N' plus the number of generators of N''.

To show that M is finitely presented just apply the previous result to the kernel of a presentation $R^n \to M$.

It is well known and easy to prove that the ascending chain condition for R-submodules of M is equivalent to the condition that every submodule of M is a finite R-module. We omit the proof. $\square$

Definition 7.47.2. Let R be a ring. Let $I \subset R$ be an ideal. We say I is *locally nilpotent* if for every $x \in I$ there exists an $n \in \mathbf{N}$ such that $x^n = 0$. We say I is *nilpotent* if there exists an $n \in \mathbf{N}$ such that $I^n = 0$.

Lemma 7.47.3. *Let R be a Noetherian ring. Let I, J be ideals of R. Suppose $J \subset \sqrt{I}$. Then $J^n \subset I$ for some n. In particular, in a Noetherian ring the notions of "locally nilpotent ideal" and "nilpotent ideal" coincide.*

Proof. Say $J = (f_1, \ldots, f_s)$. By assumption $f_i^{d_i} \in I$. Take $n = d_1 + d_2 + \ldots + d_s + 1$. $\square$

Lemma 7.47.4. *(Artin-Rees lemma) Suppose that R is Noetherian, $I \subset R$ an ideal. Let $N \subset M$ be finite R-modules. There exists a constant $c > 0$ such that $I^n M \cap N = I^{n-c}(I^c M \cap N)$.*

Proof. Consider the ring $S = R \oplus I \oplus I^2 \oplus \ldots = \bigoplus_{n \geq 0} I^n$. Convention: $I^0 = R$. Multiplication maps $I^n \times I^m$ into I^{n+m} by multiplication in R. Note that if $I = (f_1, \ldots, f_t)$ then S is a quotient of the Noetherian ring $R[X_1, \ldots, X_t]$. The map just sends the monomial $X_1^{e_1} \ldots X_t^{e_t}$ to $f_1^{e_1} \ldots f_t^{e_t}$. Thus S is Noetherian. Similarly, consider the module $M \oplus IM \oplus I^2 M \oplus \ldots = \bigoplus_{n \geq 0} I^n M$. This is a finitely generated S-module. Namely, if $x_1, \ldots, x_r$ generate M over R, then they also generate $\bigoplus_{n \geq 0} I^n M$ over S. Next, consider the submodule $\bigoplus_{n \geq 0} I^n M \cap N$. This is an S-submodule, as is easily verified. By Lemma 7.47.1 it is finitely generated as an S-module, say by $\xi_j \in \bigoplus_{n \geq 0} I^n M \cap N$, $j = 1, \ldots, s$. We may assume by decomposing each ξ_j into its homogeneous pieces that each $\xi_j \in I^{d_j} M \cap N$ for some d_j. Set $c = \max\{d_j\}$. Then for all $n \geq c$ every element in $I^n M \cap N$ is of the form $\sum h_j \xi_j$ with $h_j \in I^{n-d_j}$. The lemma now follows from this and the trivial observation that $I^{n-d_j}(I^{d_j} M \cap N) \subset I^{n-c}(I^c M \cap N)$. $\square$

Lemma 7.47.5. *Suppose that $0 \to K \to M \xrightarrow{f} N$ is an exact sequence of finitely generated modules over a Noetherian ring R. Let $I \subset R$ be an ideal. Then there exists a c such that $f^{-1}(I^n N) = K + I^{n-c} f^{-1}(I^c N)$ for all $n \geq c$.*

Proof. Apply Lemma 7.47.4 to $\text{Im}(f) \subset N$ and note that $f : I^{n-c}M \to I^{n-c}f(M)$ is surjective. $\square$

Lemma 7.47.6. *Let R be a Noetherian local ring. Let $I \subset R$ be a proper ideal. Let M be a finite R-module. Then $\bigcap_{n\geq 0} I^n M = 0$.*

Proof. Let $N = \bigcap_{n\geq 0} I^n M$. Then $N = I^n M \cap N$ for all $n \geq 0$. By the Artin-Rees Lemma 7.47.4 we see that $N = I^n M \cap N \subset IN$ for some suitably large n. By Nakayama's Lemma 7.14.5 we see that $N = 0$. $\square$

Lemma 7.47.7. *Let R be a Noetherian ring. Let $I \subset R$ be an ideal. Let M be a finite R-module. Let $N = \bigcap_n I^n M$. For every prime $\mathfrak{p}$, $I \subset \mathfrak{p}$ there exists a $f \in R$, $f \notin \mathfrak{p}$ such that $N_f = 0$.*

Proof. Let $x_1, \ldots, x_n$ be generators for the module N, see Lemma 7.47.1. For every prime $\mathfrak{p}$, $I \subset \mathfrak{p}$ we see that the image of N in the localization $M_{\mathfrak{p}}$ is zero, by Lemma 7.47.6. Hence we can find $g_i \in R$, $g_i \notin \mathfrak{p}$ such that x_i maps to zero in N_{g_i}. Thus $N_{g_1 g_2 \ldots g_n} = 0$. $\square$

Remark 7.47.8. Lemma 7.47.6 in particular implies that $\bigcap_n I^n = (0)$ when $I \subset R$ is a non-unit ideal in a Noetherian local ring R. More generally, let R be a Noetherian ring and $I \subset R$ an ideal. Suppose that $f \in \bigcap_{n\in\mathbf{N}} I^n$. Then Lemma 7.47.7 says that for every prime ideal $I \subset \mathfrak{p}$ there exists a $g \in R$, $g \notin \mathfrak{p}$ such that f maps to zero in R_g. In algebraic geometry we express this by saying that ``f is zero in an open neighbourhood of the closed set $V(I)$ of $Spec(R)$''.

Lemma 7.47.9. *(Artin-Tate) Let R be a Noetherian ring. Let S be a finitely generated R-algebra. If $T \subset S$ is an R-subalgebra such that S is finitely generated as a T-module, then T is a finite type over R.*

Proof. Choose elements $x_1, \ldots, x_n \in S$ which generate S as an R-algebra. Choose $y_1, \ldots, y_m$ in S which generate S as a T-module. Thus there exist $a_{ij} \in T$ such that $x_i = \sum a_{ij} y_j$. There also exist $b_{ijk} \in T$ such that $y_i y_j = \sum b_{ijk} y_k$. Let $T' \subset T$ be the sub R-algebra generated by a_{ij} and b_{ijk}. This is a finitely generated R-algebra, hence Noetherian. Consider the algebra

$$S' = T'[Y_1, \ldots, Y_m]/(Y_i Y_j - \sum b_{ijk} Y_k).$$

Note that S' is finite over T', namely as a T'-module it is generated by the classes of $1, Y_1, \ldots, Y_m$. Consider the T'-algebra homomorphism $S' \to S$ which maps Y_i to y_i. Because $a_{ij} \in T'$ we see that x_j is in the image of this map. Thus $S' \to S$ is surjective. Therefore S is finite over T' as well. Since T' is Noetherian and we conclude that $T \subset S$ is finite over T' and we win. $\square$

Lemma 7.47.10. *Let R be a ring. Let $\alpha : A \to B$ and $\gamma : C \to B$ be R-algebra maps. Assume*

(1) *$A \to B$ is surjective, and*
(2) *B is finite over C.*

Then the fibre product ring $T = \{(a, c) \mid \alpha(a) = \gamma(c)\}$ is of finite type over R.

Proof. Note that there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] & A \arrow[r] & B \arrow[r] & 0 \\
0 \arrow[r] & I \arrow[r] \arrow[u] & T \arrow[r] \arrow[u] & C \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

with exact rows. Choose $y_1, \ldots, y_n$ which are generators for B as a C-module. Choose $x_1, \ldots, x_n \in A$ mapping to y_i. Then it is clear that $1, x_1, \ldots, x_n$ are generators for A as a T-module. By the diagram the map $T \to A \times C$ is injective, and by what we just said the ring $A \times C$ is finite as a T-module (because it is the direct sum of the finite modules A and C). Hence the lemma follows from the Artin-Tate Lemma 7.47.9. □

7.48. Length

Definition 7.48.1. Let R be a ring. For any R-module M we define the *length* of M over R by the formula

$$\text{length}_R(M) = \sup\{n \mid \exists\, 0 = M_0 \subset M_1 \subset \ldots \subset M_n = M,\ M_i \neq M_{i+1}\}.$$

In other words it is the supremum of the lengths of chains of submodules. There is an obvious notion of when a chain of submodules is a refinement of another. This gives a partial ordering on the collection of all chains of submodules, with the smallest chain having the shape $0 = M_0 \subset M_1 = M$ if M is not zero. We note the obvious fact that if the length of M is finite, then every chain can be refined to a maximal chain. But it is not as obvious that all maximal chains have the same length (as we will see later).

Lemma 7.48.2. *Let R be a ring. Let M be an R-module. If $\text{length}_R(M) < \infty$ then M is a finite R-module.*

Proof. Omitted. □

Lemma 7.48.3. *If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of modules over R then the length of M is the sum of the lengths of M' and M''.*

Proof. Given filtrations of M' and M'' of lengths n', n'' it is easy to make a corresponding filtration of M of length $n' + n''$. Thus we see that $\text{length}_R M \geq \text{length}_R M' + \text{length}_R M''$. Conversely, given a filtration $M_0 \subset M_1 \subset \ldots \subset M_n$ of M consider the induced filtrations $M_i' = M_i \cap M'$ and $M_i'' = \text{Im}(M_i \to M'')$. Let n' (resp. n'') be the number of steps in the filtration $\{M_i'\}$ (resp. $\{M_i''\}$). If $M_i' = M_{i+1}'$ and $M_i'' = M_{i+1}''$ then $M_i = M_{i+1}$. Hence we conclude that $n' + n'' \geq n$. Combined with the earlier result we win. □

Lemma 7.48.4. *Let R be a local ring with maximal ideal $\mathfrak{m}$. Let M be an R-module.*

(1) *If M is a finite and $\mathfrak{m}^n M \neq 0$ for all $n \geq 0$, then $\text{length}_R(M) = \infty$.*
(2) *If M has finite length then $\mathfrak{m}^n M = 0$ for some n.*

Proof. Assume M is finite and $\mathfrak{m}^n M \neq 0$ for all $n \geq 0$. By NAK, Lemma 7.14.5 all the steps in the filtration $0 \subset \mathfrak{m}^n M \subset \mathfrak{m}^{n-1} M \subset \ldots \subset \mathfrak{m} M \subset M$ are distinct. Hence the length is infinite, i.e., (1) holds. Combine (1) and Lemma 7.48.2 to see (2). □

Lemma 7.48.5. *Let $R \to S$ be a ring map. Let M be an S-module. We always have $\text{length}_R(M) \geq \text{length}_S(M)$. If $R \to S$ is surjective then equality holds.*

Proof. A filtration of M by S-submodules gives rise a filtration of M by R-submodules. This proves the inequality. And if $R \to S$ is surjective, then any R-submodule of M is automatically a S-submodule. Hence equality in this case. □

Lemma 7.48.6. *Let R be a ring with maximal ideal $\mathfrak{m}$. Suppose that M is an R-module with $\mathfrak{m} M = 0$. Then the length of M as an R-module agrees with the dimension of M as a $R/\mathfrak{m}$ vector space. The length is finite if and only if M is a finite R-module.*

Proof. The first part is a special case of Lemma 7.48.5. Thus the length is finite if and only if M has a finite basis as a $R/\mathfrak{m}$-vector space if and only if M has a finite set of generators as an R-module. □

Lemma 7.48.7. *Let R be a ring. Let M be an R-module. Let $S \subset R$ be a multiplicative subset. Then $length_R(M) \geq length_{S^{-1}R}(S^{-1}M)$.*

Proof. Any submodule $N' \subset S^{-1}M$ is of the form $S^{-1}N$ for some R-submodule $N \subset M$, by Lemma 7.9.15. The lemma follows. □

Lemma 7.48.8. *Let R be a ring with finitely generated maximal ideal $\mathfrak{m}$. (For example R Noetherian.) Suppose that M is a finite R-module with $\mathfrak{m}^n M = 0$ for some n. Then $length_R(M) < \infty$.*

Proof. Consider the filtration $0 = \mathfrak{m}^n M \subset \mathfrak{m}^{n-1}M \subset \ldots \subset \mathfrak{m}M \subset M$. All of the subquotients are finitely generated R-modules to which Lemma 7.48.6 applies. We conclude by additivity, see Lemma 7.48.3. □

Definition 7.48.9. Let R be a ring. Let M be an R-module. We say M is *simple* if $M \neq 0$ and every submodule of M is either equal to M or to 0.

Lemma 7.48.10. *Let R be a ring. Let M be an R-module. The following are equivalent:*

(1) *M is simple,*
(2) *$length_R(M) = 1$, and*
(3) *$M \cong R/\mathfrak{m}$ for some maximal ideal $\mathfrak{m} \subset R$.*

Proof. Let $\mathfrak{m}$ be a maximal ideal of R. By Lemma 7.48.6 the module $R/\mathfrak{m}$ has length 1. The equivalence of the first two assertions is tautological. Suppose that M is simple. Choose $x \in M$, $x \neq 0$. As M is simple we have $M = R \cdot x$. Let $I \subset R$ be the annihilator of x, i.e., $I = \{f \in R \mid fx = 0\}$. The map $R/I \to M$, $f \bmod I \mapsto fx$ is an isomorphism, hence R/I is a simple R-module. Since $R/I \neq 0$ we see $I \neq R$. Let $I \subset \mathfrak{m}$ be a maximal ideal containing I. If $I \neq \mathfrak{m}$, then $\mathfrak{m}/I \subset R/I$ is a nontrivial submodule contradicting the simplicity of R/I. Hence we see $I = \mathfrak{m}$ as desired. □

Lemma 7.48.11. *Let R be a ring. Let M be a finite length R-module. Let $\ell = length_R(M)$. Choose any maximal chain of submodules*

$$0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_n = M$$

with $M_i \neq M_{i-1}$, $i = 1, \ldots, n$. Then

(1) *$n = \ell$,*
(2) *each M_i/M_{i-1} is simple,*
(3) *each M_i/M_{i-1} is of the form $R/\mathfrak{m}_i$ for some maximal ideal $\mathfrak{m}_i$,*
(4) *given a maximal ideal $\mathfrak{m} \subset R$ we have*

$$\#\{i \mid \mathfrak{m}_i = \mathfrak{m}\} = length_{R_\mathfrak{m}}(M_\mathfrak{m}).$$

Proof. If M_i/M_{i-1} is not simple then we can refine the filtration and the filtration is not maximal. Thus we see that M_i/M_{i-1} is simple. By Lemma 7.48.10 the modules M_i/M_{i-1} have length 1 and are of the form $R/\mathfrak{m}_i$ for some maximal ideals $\mathfrak{m}_i$. By additivity of length, Lemma 7.48.3, we see $n = \ell$. Since localization is exact, we see that

$$0 = (M_0)_\mathfrak{m} \subset (M_1)_\mathfrak{m} \subset (M_2)_\mathfrak{m} \subset \ldots \subset (M_n)_\mathfrak{m} = M_\mathfrak{m}$$

is a filtration of $M_{\mathfrak m}$ with successive quotients $(M_i/M_{i-1})_{\mathfrak m}$. Thus the last statement follows directly from the fact that given maximal ideals $\mathfrak m$, $\mathfrak m'$ of R we have

$$(R/\mathfrak m')_{\mathfrak m} \cong \begin{cases} 0 & \text{if } \mathfrak m \neq \mathfrak m', \\ R_{\mathfrak m}/\mathfrak m R_{\mathfrak m} & \text{if } \mathfrak m = \mathfrak m' \end{cases}$$

This we leave to the reader. □

Lemma 7.48.12. *Let A be a local ring with maximal ideal $\mathfrak m$. Let B be a semi-local ring with maximal ideals $\mathfrak m_i$, $i = 1, \ldots, n$. Suppose that $A \to B$ is a homomorphism such that each $\mathfrak m_i$ lies over $\mathfrak m$ and such that*

$$[\kappa(\mathfrak m_i) : \kappa(\mathfrak m)] < \infty.$$

Let M be a B-module of finite length. Then

$$length_A(M) = \sum\nolimits_{i=1,\ldots,n} [\kappa(\mathfrak m_i) : \kappa(\mathfrak m)] length_{B_{\mathfrak m_i}}(M_{\mathfrak m_i}),$$

in particular $length_A(M) < \infty$.

Proof. Choose a maximal chain

$$0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_n = M$$

by B-submodules as in Lemma 7.48.11. Then each quotient M_i/M_{i-1} is isomorphic to $\kappa(\mathfrak m_{j(i)})$ for some $j(i) \in \{1, \ldots, n\}$. Moreover $\text{length}_A(\kappa(\mathfrak m_i)) = [\kappa(\mathfrak m_i) : \kappa(\mathfrak m)]$ by Lemma 7.48.6. The lemma follows by additivity of lengths (Lemma 7.48.3). □

Lemma 7.48.13. *Let $A \to B$ be a flat local homomorphism of local rings. Then for any A-module M we have*

$$length_A(M) length_B(B/\mathfrak m_A B) = length_B(M \otimes_A B).$$

In particular, if $length_B(B/\mathfrak m_A B) < \infty$ then M has finite length if and only if $M \otimes_A B$ has finite length.

Proof. The ring map $A \to B$ is faithfully flat by Lemma 7.35.16. Hence if $0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$ is a chain of length n in M, then the corresponding chain $0 = M_0 \otimes_A B \subset M_1 \otimes_A B \subset \ldots \subset M_n \otimes_A B = M \otimes_A B$ has length n also. This proves $\text{length}_A(M) = \infty \Rightarrow \text{length}_B(M \otimes_A B) = \infty$. Next, assume $\text{length}_A(M) < \infty$. In this case we see that M has a filtration of length $\ell = \text{length}_A(M)$ whose quotients are $A/\mathfrak m_A$. Arguing as above we see that $M \otimes_A B$ has a filtration of length ℓ whose quotients are isomorphic to $B \otimes_A A/\mathfrak m_A = B/\mathfrak m_A B$. Thus the lemma follows. □

Lemma 7.48.14. *Let $A \to B \to C$ be flat local homomorphisms of local rings. Let M be an A-module of finite length. Then*

$$length_B(B/\mathfrak m_A B) length_C(C/\mathfrak m_B C) = length_C(C/\mathfrak m_A C)$$

Proof. Follows from Lemma 7.48.13 applied to the ring map $B \to C$ and the B-module $M = B/\mathfrak m_A B$ □

7.49. Artinian rings

Artinian rings, and especially local Artinian rings, play an important role in algebraic geometry, for example in deformation theory.

Definition 7.49.1. A ring R is *Artinian* if it satisfies the descending chain condition for ideals.

Lemma 7.49.2. *Suppose R is a finite dimensional algebra over a field. Then R is Artinian.*

Proof. The descending chain condition for ideals obviously holds. $\square$

Lemma 7.49.3. *If R is Artinian then R has only finitely many maximal ideals.*

Proof. Suppose that $\mathfrak{m}_i$, $i = 1, 2, 3, \ldots$ are maximal ideals. Then $\mathfrak{m}_1 \supset \mathfrak{m}_1 \cap \mathfrak{m}_2 \supset \mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \mathfrak{m}_3 \supset \ldots$ is an infinite descending sequence (because by the Chinese remainder theorem all the maps $R \to \bigoplus_{i=1}^n R/\mathfrak{m}_i$ are surjective). $\square$

Lemma 7.49.4. *Let R be Artinian. The radical $rad(R)$ of R is a nilpotent ideal.*

Proof. Denote the radical I. Note that $I \supset I^2 \supset I^3 \supset \ldots$ is a descending sequence. Thus $I^n = I^{n+1}$ for some n. Set $J = \{x \in R \mid xI^n = 0\}$. We have to show $J = R$. If not, choose an ideal $J' \neq J$, $J \subset J'$ minimal (possible by the Artinian property). Then $J' = J + Rx$ for some $x \in R$. By NAK, Lemma 7.14.5, we have $IJ' \subset J$. Hence $xI^{n+1} \subset xI \cdot I^n \subset J \cdot I^n = 0$. Since $I^{n+1} = I^n$ we conclude $x \in J$. Contradiction. $\square$

Lemma 7.49.5. *Let R be a ring. Let $I \subset R$ be a locally nilpotent ideal. Then $R \to R/I$ induces a bijection on idempotents.*

First proof of Lemma 7.49.5. As I is locally nilpotent it is contained in every prime ideal. Hence $Spec(R/I) = V(I) = Spec(R)$. Hence the lemma follows from Lemma 7.18.3. $\square$

Second proof of Lemma 7.49.5. First assume I is nilpotent. Suppose $\overline{e} \in R/I$ is an idempotent. We have to lift $\overline{e}$ to an idempotent of R. Choose a lift $e \in R$ such that $x = e^2 - e \in I^k$ for some $k \geq 1$. Let $e' = e - (2e - 1)x = 3e^2 - 2e^3$, which is another lift of $\overline{e}$. Then

$$(e')^2 - e' = (4e^2 - 4e - 3)(e^2 - e)^2 \in I^{2k}$$

by a simple computation. Hence e' is an idempotent in R/I^{2k}. By succesively improving the approximation as above we reach a stage where $I^k = 0$, and we win.

Next, suppose I is locally nilpotent. Let $\overline{e} \in R/I$ be an idempotent. Let $f \in R$ be any element lifting $\overline{e}$. Denote $R' \subset R$ the $\mathbf{Z}$-subalgebra of R generated by f. Denote $I' = R' \cap I$. Since R' is Noetherian, see Lemma 7.28.3 we see that I' is nilpotent, see Lemma 7.47.3. On the other hand we have $R'/I' \subset R/I$ and hence the image $\overline{f} \in R'/I'$ of f is an idempotent. Thus by the first part of the proof we see that we can find an idempotent $e \in R'$ which is a lift of $\overline{f}$. Then $e \in R$ is also a lift of $\overline{e}$ in R/I. $\square$

Lemma 7.49.6. *Let A be a possibly noncommutative algebra. Let $e \in A$ be an element such that $x = e^2 - e$ is nilpotent. Then there exists an idempotent of the form $e' = e + x(\sum a_{i,j} e^i x^j) \in A$ with $a_{i,j} \in \mathbf{Z}$.*

Proof. Consider the ring $R_n = \mathbf{Z}[e]/((e^2 - e)^n)$. It is clear that if we can prove the result for each R_n then the lemma follows. In R_n consider the ideal $I = (e^2 - e)$ and apply Lemma 7.49.5. $\square$

Lemma 7.49.7. *Any ring with finitely many maximal ideals and locally nilpotent radical is the product of its localizations at its maximal ideals. Also, all primes are maximal.*

Proof. Let R be a ring with finitely many maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$. Let $I = \bigcap_{i=1}^n \mathfrak{m}_i$ be the radical of R. Assume I is locally nilpotent. Let $\mathfrak{p}$ be a prime ideal of R. Since every prime contains every nilpotent element of R we see $\mathfrak{p} \supset \mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_n$. Since $\mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_n \supset \mathfrak{m}_1 \ldots \mathfrak{m}_n$ we conclude $\mathfrak{p} \supset \mathfrak{m}_1 \ldots \mathfrak{m}_n$. Hence $\mathfrak{p} \supset \mathfrak{m}_i$ for some i, and so $\mathfrak{p} = \mathfrak{m}_i$. By the Chinese remainder theorem (Lemma 7.14.4) we have $R/I \cong \bigoplus R/\mathfrak{m}_i$ which is a product of fields. Hence by Lemma 7.49.5 there are idempotents e_i, $i = 1, \ldots, n$ with $e_i \bmod \mathfrak{m}_j = \delta_{ij}$. Hence $R = \prod Re_i$, and each Re_i is a ring with exactly one maximal ideal. □

Lemma 7.49.8. *A ring R is Artinian if and only if it has finite length as a module over itself. Any such ring R is both Artinian and Noetherian, any prime ideal of R is a maximal ideal, and R is equal to the (finite) product of its localizations at its maximal ideals.*

Proof. If R has finite length over itself then it satisfies both the ascending chain condition and the descending chain condition for ideals. Hence it is both Noetherian and Artinian. Any Artinian ring is equal to product of its localizations at maximal ideals by Lemmas 7.49.3, 7.49.4, and 7.49.7.

Suppose that R is Artinian. We will show R has finite length over itself. It suffices to exhibit a chain of submodules whose succesive quotients have finite length. By what we said above we may assume that R is local, with maximal ideal $\mathfrak{m}$. By Lemma 7.49.4 we have $\mathfrak{m}^n = 0$ for some n. Consider the sequence $0 = \mathfrak{m}^n \subset \mathfrak{m}^{n-1} \subset \ldots \subset \mathfrak{m} \subset R$. By Lemma 7.48.6 the length of each subquotient $\mathfrak{m}^j/\mathfrak{m}^{j+1}$ is the dimension of this as a vector space over $\kappa(\mathfrak{m})$. This has to be finite since otherwise we would have an infinite descending chain of subvector spaces which would correspond to an infinite descending chain of ideals in R. □

7.50. Homomorphisms essentially of finite type

Some simple remarks on localizations of finite type ring maps.

Definition 7.50.1. Let $R \to S$ be a ring map.

(1) We say that $R \to S$ is *essentially of finite type* if S is the localization of an R-algebra of finite type.
(2) We say that $R \to S$ is *essentially of finite presentation* if S is the localization of an R-algebra of finite presentation.

Lemma 7.50.2. *The class of ring maps which are essentially of finite type is preserved under composition. Similarly for essentially of finite presentation.*

Proof. Omitted. □

Lemma 7.50.3. *Let $R \to S$ be a ring map. Assume S is an Artinian local ring with maximal ideal $\mathfrak{m}$. Then*

(1) *$R \to S$ is finite if and only if $R \to S/\mathfrak{m}$ is finite,*
(2) *$R \to S$ is of finite type if and only if $R \to S/\mathfrak{m}$ is of finite type.*
(3) *$R \to S$ is essentially of finite type if and only if the composition $R \to S/\mathfrak{m}$ is essentially of finite type.*

Proof. If $R \to S$ is finite, then $R \to S/\mathfrak{m}$ is finite by Lemma 7.7.3. Conversely, assume $R \to S/\mathfrak{m}$ is finite. As S has finite length over itself (Lemma 7.49.8) we can choose a filtration

$$0 \subset I_1 \subset \ldots \subset I_n = S$$

by ideals such that $I_i/I_{i-1} \cong S/\mathfrak{m}$ as S-modules. Thus S has a filtration by R-submodules I_i such that each successive quotient is a finite R-module. Thus S is a finite S-module by Lemma 7.5.4.

If $R \to S$ is of finite type, then $R \to S/\mathfrak{m}$ is of finite type by Lemma 7.6.2. Conversely, assume that $R \to S/\mathfrak{m}$ is of finite type. Choose $f_1, \ldots, f_n \in S$ which map to generators of $S/\mathfrak{m}$. Then $A = R[x_1, \ldots, x_n] \to S$, $x_i \mapsto f_i$ is a ring map such that $A \to S/\mathfrak{m}$ is surjective (in particular finite). Hence $A \to S$ is finite by part (1) and we see that $R \to S$ is of finite type by Lemma 7.6.2.

If $R \to S$ is essentially of finite type, then $R \to S/\mathfrak{m}$ is essentially of finite type by Lemma 7.50.2. Conversely, assume that $R \to S/\mathfrak{m}$ is essentially of finite type. Suppose $S/\mathfrak{m}$ is the localization of $R[x_1, \ldots, x_n]/I$. Choose $f_1, \ldots, f_n \in S$ whose congruence classes modulo $\mathfrak{m}$ correspond to the congruence classes of $x_1, \ldots, x_n$ modulo I. Consider the map $R[x_1, \ldots, x_n] \to S$, $x_i \mapsto f_i$ with kernel J. Set $A = R[x_1, \ldots, x_n]/J \subset S$ and $\mathfrak{p} = A \cap \mathfrak{m}$. Note that $A/\mathfrak{p} \subset S/\mathfrak{m}$ is equal to the image of $R[x_1, \ldots, x_n]/I$ in $S/\mathfrak{m}$. Hence $\kappa(\mathfrak{p}) = S/\mathfrak{m}$. Thus $A_{\mathfrak{p}} \to S$ is finite by part (1). We conclude that S is essentially of finite type by Lemma 7.50.2. □

7.51. K-groups

Let R be a ring. We will introduce two abelian groups associated to R. The first of the two is denoted $K'_0(R)$ and has the following properties:

(1) For every finite R-module M there is given an element $[M]$ in $K'_0(R)$,
(2) for every short exact sequence $0 \to M' \to M \to M'' \to 0$ we have the relation $[M] = [M'] + [M'']$,
(3) the group $K'_0(R)$ is generated by the elements $[M]$, and
(4) all relations in $K'_0(R)$ are $\mathbf{Z}$-linear combinations of the relations coming from exact sequences as above.

The actual construction is a bit more annoying since one has to take care that the collection of all finitely generated R-modules is a proper class. However, this problem can be overcome by taking as set of generators of the group $K'_0(R)$ the elements $[R^n/K]$ where n ranges over all integers and K ranges over all submodules $K \subset R^n$. The generators of for the subgroup of relations imposed on these elements will be the relations coming from short exact sequences whose terms are of the form R^n/K. The element $[M]$ is defined by choosing n and K such that $M \cong R^n/K$ and putting $[M] = [R^n/K]$. Details left to the reader.

Lemma 7.51.1. *If R is an Artinian local ring then the length function defines a natural abelian group homomorphism $length_R : K'_0(R) \to \mathbf{Z}$.*

Proof. The length of any finite R-module is finite, because it is the quotient of R^n which has finite length by Lemma 7.49.8. And the length function is additive, see Lemma 7.48.3. □

The second of the two is denoted $K_0(R)$ and has the following properties:

(1) For every finite projective R-module M there is given an element $[M]$ in $K_0(R)$,
(2) for every short exact sequence $0 \to M' \to M \to M'' \to 0$ of finite projective R-modules we have the relation $[M] = [M'] + [M'']$,

(3) the group $K_0(R)$ is generated by the elements $[M]$, and
(4) all relations in $K_0(R)$ are $\mathbf{Z}$-linear combinations of the relations coming from exact sequences as above.

The construction of this group is done as above.

We note that there is an obvious map $K_0(R) \to K_0'(R)$ which is not an isomorphism in general.

Example 7.51.2. Note that if $R = k$ is a field then we clearly have $K_0(k) = K_0'(k) \cong \mathbf{Z}$ with the isomorphism given by the dimension function (which is also the length function).

Example 7.51.3. Let k be a field. Then $K_0(k[x]) = K_0'(k[x]) = \mathbf{Z}$.

Since $R = k[x]$ is a principal ideal domain, any finite projective R-module is free. In a short exact sequence of modules

$$0 \to M' \to M \to M'' \to 0$$

we have $rank(M) = rank(M') + rank(M'')$, which gives $K_0(k[x]) = Z$.

As for K_0', the structure theorem for modules of a PID says that any finitely generated R-module is of the form $M = R^r \times R/(d_1) \times \ldots \times R/(d_k)$. Consider the short exact sequence

$$0 \to (d_i) \to R \to R/(d_i) \to 0$$

Since the ideal (d_i) is isomorphic to R as a module (it is free with generator d_i), in $K_0'(R)$ we have $[(d_i)] = [R]$. Then $[R/(d_i)] = [(d_i)] - [R] = 0$. From this it follows that any torsion part ``disappears" in K_0'. Again the rank of the free part determines that $K_0'(k[x]) = Z$, and the canonical homomorphism from K_0 to K_0' is an isomorphism.

Example 7.51.4. Let k be a field. Let $R = \{f \in k[x] \mid f(0) = f(1)\}$, compare Example 7.24.4. In this case $K_0(R) \cong k^* \oplus \mathbf{Z}$, but $K_0'(R) = \mathbf{Z}$.

Lemma 7.51.5. *Let $R = R_1 \times R_2$. Then $K_0(R) = K_0(R_1) \times K_0(R_2)$ and $K_0'(R) = K_0'(R_1) \times K_0'(R_2)$*

Proof. Omitted. □

Lemma 7.51.6. *Let R be an Artinian local ring. The map $length_R : K_0'(R) \to \mathbf{Z}$ of Lemma 7.51.1 is an isomorphism.*

Proof. Omitted. □

Lemma 7.51.7. *Let R be a local ring. Every finite projective R-module is finite free. The map $rank_R : K_0(R) \to \mathbf{Z}$ defined by $[M] \to rank_R(M)$ is well defined and an isomorphism.*

Proof. Let P be a finite projective R-module. The n generators of P give a surjection $R^n \to P$, and since P is projective it follows that $R^n \cong P \oplus Q$ for some projective module Q.

If $\mathfrak{m} \subset R$ is the maximal ideal, then $P/\mathfrak{m}$ and $Q/\mathfrak{m}$ are $R/\mathfrak{m}$-vector spaces, with $P/\mathfrak{m} \oplus Q/\mathfrak{m} \cong (R/\mathfrak{m})^n$. Say that $\dim P = p$, $\dim Q = q$, so $p + q = n$.

Choose elements $a_1, \ldots, a_p$ in P and $b_1, \ldots, b_q$ in Q lying above bases for $P/\mathfrak{m}$ and $Q/\mathfrak{m}$. The homomorphism $R^n \to P \oplus Q \cong R^n$ given by $(r_1, \ldots, r_n) \mapsto r_1 a_1 + \ldots + r_p a_p + r_{p+1} b_1 + \ldots + r_n b_q$ is a matrix A which is invertible over $R/\mathfrak{m}$. Let B be a matrix over R lying over the inverse of A in $R/\mathfrak{m}$. $AB = I + M$, where M is a matrix whose entries all lie in $\mathfrak{m}$. Thus $\det AB = 1 + x$, for $x \in \mathfrak{m}$, so AB is invertible, so A is invertible.

The homomorphism $R^p \to P$ given by $(r_1, \dots, r_p) \mapsto r_1 a_1 + \dots + r_p a_p$ inherits injectivity and surjectivity from A. Hence, $P \cong R^p$.

Next we show that the rank of a finite projective module over R is well defined: if $P \cong R^\alpha \cong R^\beta$, then $\alpha = \beta$. This is immediate in the vector space case, and so it is true in the general module case as well, by dividing out the maximal ideal on both sides. If $0 \to R^\alpha \to R^\beta \to R^\gamma \to 0$ is exact, the sequence splits, so $R^\beta \cong R^\alpha \oplus R^\gamma$, so $\beta = \alpha + \gamma$.

So far we have seen that the map $\text{rank}_R : K_0(R) \to \mathbf{Z}$ is a well-defined homomorphism. It is surjective because $\text{rank}_R[R] = 1$. It is injective because the element of $K_0(R)$ with rank $\pm\alpha$ is uniquely $\pm[R^\alpha]$. $\square$

Lemma 7.51.8. *Let R be a local Artinian ring. There is a commutative diagram*

$$\begin{tikzcd} K_0(R) \arrow[r] \arrow[d, "rank_R"'] & K'_0(R) \arrow[d, "length_R"] \\ \mathbf{Z} \arrow[r, "length_R(R)"] & \mathbf{Z} \end{tikzcd}$$

where the vertical maps are isomorphisms by Lemmas 7.51.6 and 7.51.7.

Proof. By induction on the rank of M. Suppose $[M] \in K_0(R)$. Then M is a finite projective R-module over a local ring, so M is free; $M \cong R^n$ for some n. The claim is that $\text{rank}(M)\text{length}_R(R) = \text{length}_R(M)$, or equivalently that $n\text{length}_R(R) = \text{length}_R(R^n)$ for all $n \geq 1$. When $n = 1$, this is clearly true. Suppose that $(n-1)\text{length}_R(R) = \text{length}_R(R^{n-1})$. Then since there is a split short exact sequence

$$0 \to R \to R^n \to R^{n-1} \to 0$$

by Lemma 7.48.3 we have

$$\begin{aligned} \text{length}_R(R^n) &= \text{length}_R(R) + \text{length}_R(R^{n-1}) \\ &= \text{length}_R(R) + (n-1)\text{length}_R(R) \\ &= n\text{length}_R(R) \end{aligned}$$

as desired. $\square$

7.52. Graded rings

A *graded ring* will be for us a ring S endowed with a direct sum decomposition $S = \bigoplus_{d \geq 0} S_d$ such that $S_d \cdot S_e \subset S_{d+e}$. Note that we do not allow nonzero elements in negative degrees. The *irrelevant ideal* is the ideal $S_+ = \bigoplus_{d>0} S_d$. A *graded module* will be an S-module M endowed with a direct sum decomposition $M = \bigoplus_{n \in \mathbf{Z}} M_n$ such that $S_d \cdot M_e \subset M_{d+e}$. Note that for modules we do allow nonzero elements in negative degrees. We think of S as a graded S-module by setting $S_{-k} = (0)$ for $k > 0$. An element x (resp. f) of M (resp. S) is called *homogeneous* if $x \in M_d$ (resp. $f \in S_d$) for some d. A *map of graded S-modules* is a map of S-modules $\varphi : M \to M'$ such that $\varphi(M_d) \subset M'_d$. We do not allow maps to shift degrees. Let us denote $\text{GrHom}_0(M, N)$ the S_0-module of homomorphisms of graded modules from M to N.

At this point there are the notions of graded ideal, graded quotient ring, graded submodule, graded quotient module, graded tensor product, etc. We leave it to the reader to find the relevant definitions, and lemmas. For example: A short exact sequence of graded modules is short exact in every degree.

Given a graded ring S, a graded S-module M and $n \in \mathbf{Z}$ we denote $M(n)$ the graded S-module with $M(n)_d = M_{n+d}$. This is called the *twist of M by n*. In particular we get modules $S(n)$, $n \in \mathbf{Z}$ which will play an important role in the study of projective schemes. There are some obvious functorial isomorphisms such as $(M \oplus N)(n) = M(n) \oplus N(n)$, $(M \otimes_S N)(n) = M \otimes_S N(n) = M(n) \otimes_S N$. In addition we can define a graded S-module structure on the S_0-module

$$\mathrm{GrHom}(M, N) = \bigoplus\nolimits_{n \in \mathbf{Z}} \mathrm{GrHom}_n(M, N), \quad \mathrm{GrHom}_n(M, N) = \mathrm{GrHom}_0(M, N(n)).$$

We omit the definition of the multiplication.

Let S be a graded ring. Let $d \geq 1$ be an integer. We set $S^{(d)} = \bigoplus_{n \geq 0} S_{nd}$. We think of $S^{(d)}$ as a graded ring with degree n summand $(S^{(d)})_n = S_{nd}$. Given a graded S-module M we can similarly consider $M^{(d)} = \bigoplus_{n \in \mathbf{Z}} M_{nd}$ which is a graded $S^{(d)}$-module.

Lemma 7.52.1. *Let $R \to S$ be a homomorphism of graded rings. Let $S' \subset S$ be the integral closure of R in S. Then*

$$S' = \bigoplus\nolimits_{d \geq 0} S' \cap S_d,$$

i.e., S' is a graded R-subalgebra of S.

Proof. We have to show the following: If $s = s_n + s_{n+1} + \ldots + s_m \in S'$, then each homogeneous part $s_j \in S'$. We will prove this by induction on $m - n$ over all homomorphisms $R \to S$ of graded rings. First note that it is immediate that s_0 is integral over R_0 (hence over R) as there is a ring map $S \to S_0$ compatible with the ring map $R \to R_0$. Thus, after replacing s by $s - s_0$, we may assume $n > 0$. Consider the extension of graded rings $R[t, t^{-1}] \to S[t, t^{-1}]$ where t has degree 0. There is a commutative diagram

$$\begin{tikzcd}
S[t, t^{-1}] \arrow[r, "s \mapsto t^{\deg(s)} s"'] & S[t, t^{-1}] \\
R[t, t^{-1}] \arrow[u] \arrow[r, "r \mapsto r^{\deg(r)}"] & R[t, t^{-1}] \arrow[u]
\end{tikzcd}$$

where the horizontal maps are ring automorphisms. Hence the integral closure C of $S[t, t^{-1}]$ over $R[t, t^{-1}]$ maps into itself. Thus we see that

$$t^m(s_n + s_{n+1} + \ldots + s_m) - (t^n s_n + t^{n+1} s_{n+1} + \ldots + t^m s_m) \in C$$

which implies by induction hypothesis that each $(t^m - t^i)s_i \in C$ for $i = n, \ldots, m-1$. Note that for any ring A and $m > i \geq n > 0$ we have $A[t, t^{-1}]/(t^m - t^i - 1) \cong A[t]/(t^m - t^i - 1) \supset A$ because $t(t^{m-1} - t^{i-1}) = 1$ in $A[t]/(t^m - t^i - 1)$. Since $t^m - t^i$ maps to 1 we see the image of s_i in the ring $S[t]/(t^m - t^i - 1)$ is integral over $R[t]/(t^m - t^i - 1)$ for $i = n, \ldots, m-1$. Since $R \to R[t]/(t^m - t^i - 1)$ is finite we see that s_i is integral over R by transitivity, see Lemma 7.32.6. Finally, we also conclude that $s_m = s - \sum_{i=n,\ldots,m-1} s_i$ is integral over R. $\square$

7.53. Proj of a graded ring

Let S be a graded ring. A *homogeneous ideal* is simply an ideal $I \subset S$ which is also a graded submodule of S. Equivalently, it is an ideal generated by homogeneous elements. Equivalently, if $f \in I$ and

$$f = f_0 + f_1 + \ldots + f_n$$

is the decomposition of f into homogenous parts in S then $f_i \in I$ for each i. To check that a homogeneous ideal $\mathfrak{p}$ is prime it suffices to check that if $ab \in \mathfrak{p}$ with a, b homogeneous then either $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$.

Definition 7.53.1. Let S be a graded ring. We define Proj(S) to be the set of homogenous, prime ideals $\mathfrak{p}$ of S such that $S_+ \not\subset \mathfrak{p}$. As Proj($S$) is a subset of $Spec(S)$ and we endow it with the induced topology. The topological space Proj(S) is called the *homogeneous spectrum* of the graded ring S.

Note that by construction there is a continuous map

$$\mathrm{Proj}(S) \longrightarrow Spec(S_0)$$

Let $S = \oplus_{d \geq 0} S_d$ be a graded ring. Let $f \in S_d$ and assume that $d \geq 1$. We define $S_{(f)}$ to be the subring of S_f consisting of elements of the form r/f^n with r homogenous and $\deg(r) = nd$. If M is a graded S-module, then we define the $S_{(f)}$-module $M_{(f)}$ as the sub module of M_f consisting of elements of the form x/f^n with x homogeneous of degree nd.

Lemma 7.53.2. *Let S be a $\mathbf{Z}$-graded ring. Let $f \in S_d$, $d > 0$ and assume f is invertible in S. The set $G \subset Spec(S)$ of $\mathbf{Z}$-graded primes of S (with induced topology) maps homeomorphically to $Spec(S_0)$.*

Proof. First we show that the map is a bijection by constructing an inverse. Namely, if $\mathfrak{p}_0$ is a prime of S_0, then $\mathfrak{p}_0 S$ is a $\mathbf{Z}$-graded ideal of S such that $\mathfrak{p}_0 S \cap S_0 = \mathfrak{p}_0$. And if $ab \in \mathfrak{p}_0 S$ with a, b homogenenous, then $a^d b^d / f^{\deg(a)+\deg(b)} \in \mathfrak{p}_0$. Thus either $a^d/f^{\deg(a)} \in \mathfrak{p}_0$ or $b^d/f^{\deg(b)} \in \mathfrak{p}_0$, in other words either $a^d \in \mathfrak{p}_0 S$ or $b^d \in \mathfrak{p}_0 S$. It follows that $\sqrt{\mathfrak{p}_0 S}$ is a $\mathbf{Z}$-graded prime ideal of S whose intersection with S_0 is $\mathfrak{p}_0$.

To show that the map is a homeomorphism we show that the image of $G \cap D(g)$ is open. If $g = \sum g_i$ with $g_i \in S_i$, then by the above $G \cap D(g)$ maps onto the set $\bigcup D(g_i^d/f^i)$ which is open. □

For $f \in S$ homogenenous of degree > 0 we define

$$D_+(f) = \{\mathfrak{p} \in \mathrm{Proj}(S) \mid f \notin \mathfrak{p}\}.$$

Finally, for a homogenous ideal $I \subset S$ we define

$$V_+(I) = \{\mathfrak{p} \in \mathrm{Proj}(S) \mid I \subset \mathfrak{p}\}.$$

We will use more generally the notation $V_+(E)$ for any set E of homogeneous elements $E \subset S$.

Lemma 7.53.3. *(Topology on Proj(S).) Let $S = \oplus_{d \geq 0} S_d$ be a graded ring.*

(1) *The sets $D_+(f)$ are open in Proj(S).*
(2) *We have $D_+(ff') = D_+(f) \cap D_+(f')$.*
(3) *Let $g = g_0 + \ldots + g_m$ be an element of S with $g_i \in S_i$. Then*

$$D(g) \cap Proj(S) = (D(g_0) \cap Proj(S)) \cup \bigcup\nolimits_{i \geq 1} D_+(g_i).$$

(4) *Let $g_0 \in S_0$ be a homogenous element of degree 0. Then*

$$D(g_0) \cap Proj(S) = \bigcup\nolimits_{f \in S_d,\ d \geq 1} D_+(g_0 f).$$

(5) *The open sets $D_+(f)$ form a basis for the topology of Proj(S).*

(6) *Let $f \in S$ be homogeneous of positive degree. The ring S_f has a natural $\mathbf{Z}$-grading. The ring maps $S \to S_f \leftarrow S_{(f)}$ induce homeomorphisms*
$$D_+(f) \leftarrow \{\mathbf{Z}\text{-graded primes of } S_f\} \to Spec(S_{(f)}).$$
(7) *There exists an S such that $Proj(S)$ is not quasi-compact.*
(8) *The sets $V_+(I)$ are closed.*
(9) *Any closed subset $T \subset Proj(S)$ is of the form $V_+(I)$ for some homogeneous ideal $I \subset S$.*
(10) *For any graded ideal $I \subset S$ we have $V_+(I) = \emptyset$ if and only if $S_+ \subset \sqrt{I}$.*

Proof. Since $D_+(f) = \text{Proj}(S) \cap D(f)$, these sets are open. Similarly the sets $V_+(I) = \text{Proj}(S) \cap V(E)$ are closed.

Suppose that $T \subset \text{Proj}(S)$ is closed. Then we can write $T = \text{Proj}(S) \cap V(J)$ for some ideal $J \subset S$. By definition of a homogeneous ideal if $g \in J$, $g = g_0 + \ldots + g_m$ with $g_d \in S_d$ then $g_d \in \mathfrak{p}$ for all $\mathfrak{p} \in T$. Thus, letting $I \subset S$ be the ideal generated by the homogeneous parts of the elements of J we have $T = V_+(I)$.

The formula for $\text{Proj}(S) \cap D(g)$, with $g \in S$ is direct from the definitions. Consider the formula for $\text{Proj}(S) \cap D(g_0)$. The inclusion of the right hand side in the left hand side is obvious. For the other inclusion, suppose $g_0 \not\in \mathfrak{p}$ with $\mathfrak{p} \in \text{Proj}(S)$. If all $g_0 f \in \mathfrak{p}$ for all homogeneous f of positive degree, then we see that $S_+ \subset \mathfrak{p}$ which is a contradiction. This gives the other inclusion.

The collection of opens $D(g) \cap \text{Proj}(S)$ forms a basis for the topology since the standard opens $D(g) \subset Spec(S)$ form a basis for the topology on $Spec(S)$. By the formulas above we can express $D(g) \cap \text{proj}(S)$ as a union of opens $D_+(f)$. Hence the collection of opens $D_+(f)$ forms a basis for the topology also.

First we note that $D_+(f)$ may be identified with a subset (with induced topology) of $D(f) = Spec(S_f)$ via Lemma 7.16.6. Note that the ring S_f has a $\mathbf{Z}$-grading. The homogeneous elements are of the form r/f^n with $r \in S$ homogeneous and have degree $\deg(r/f^n) = \deg(r) - n\deg(f)$. The subset $D_+(f)$ corresponds exactly to those prime ideals $\mathfrak{p} \subset S_f$ which are $\mathbf{Z}$-graded ideals (i.e., generated by homogeneous elements). Hence we have to show that the set of $\mathbf{Z}$-graded prime ideals of S_f maps homeomorphically to $Spec(S_{(f)})$. This follows from Lemma 7.53.2.

Let $S = \mathbf{Z}[X_1, X_2, X_3, \ldots]$ with grading such that each X_i has degree 1. Then it is easy to see that
$$\text{Proj}(S) = \bigcup\nolimits_{i=1}^{\infty} D_+(X_i)$$
does not have a finite refinement.

Let $I \subset S$ be a graded ideal. If $\sqrt{I} \supset S_+$ then $V_+(I) = \emptyset$ since every prime $\mathfrak{p} \in \text{Proj}(S)$ does not contain S_+ by definition. Conversely, suppose that $S_+ \not\subset \sqrt{I}$. Then we can find an element $f \in S_+$ such that f is not nilpotent modulo I. Clearly this means that one of the homogeneous parts of f is not nilpotent modulo I, in other words we may (and do) assume that f is homogeneous. This implies that $IS_f \neq 0$, in other words that $(S/I)_f$ is not zero. Hence $(S/I)_{(f)} \neq 0$ since it is a ring which maps into $(S/I)_f$. Pick a prime $\mathfrak{q} \subset (S/I)_{(f)}$. This corresponds to a graded prime of S/I, not containing the irrelevant ideal $(S/I)_+$. And this in turn corresponds to a graded prime ideal $\mathfrak{p}$ of S, containing I but not containing S_+ as desired. $\square$

Example 7.53.4. Let R be a ring. If $S = R[X]$ with $\deg(X) = 1$, then the natural map $\text{Proj}(S) \to \mathit{Spec}(R)$ is a bijection and in fact a homeomorphism. Namely, suppose $\mathfrak{p} \in \text{Proj}(S)$. Since $S_+ \not\subset \mathfrak{p}$ we see that $X \notin \mathfrak{p}$. Thus if $aX^n \in \mathfrak{p}$ with $a \in R$ and $n > 0$, then $a \in \mathfrak{p}$. It follows that $\mathfrak{p} = \mathfrak{p}_0 S$ with $\mathfrak{p}_0 = \mathfrak{p} \cap R$.

If $\mathfrak{p} \in \text{Proj}(S)$, then we define $S_{(\mathfrak{p})}$ to be the ring whose elements are fractions r/f where $r, f \in S$ are homogeneous elements of the same degree such that $f \notin \mathfrak{p}$. As usual we say $r/f = r'/f'$ if and only if there exists some $f'' \in S$ homogeneous, $f'' \notin \mathfrak{p}$ such that $f''(rf' - r'f) = 0$. Given a graded S-module M we let $M_{(\mathfrak{p})}$ be the $S_{(\mathfrak{p})}$-module whose elements are fractions x/f with $x \in M$ and $f \in S$ homogeneous of the same degree such that $f \notin \mathfrak{p}$. We say $x/f = x'/f'$ if and only if there exists some $f'' \in S$ homogeneous, $f'' \notin \mathfrak{p}$ such that $f''(xf' - x'f) = 0$.

Lemma 7.53.5. *Let S be a graded ring. Let M be a graded S-module. Let $\mathfrak{p}$ be an element of $Proj(S)$. Let $f \in S$ be a homogeneous element of positive degree such that $f \notin \mathfrak{p}$, i.e., $\mathfrak{p} \in D_+(f)$. Let $\mathfrak{p}' \subset S_{(f)}$ be the element of $Spec(S_{(f)})$ corresponding to $\mathfrak{p}$ as in Lemma 7.53.3. Then $S_{(\mathfrak{p})} = (S_{(f)})_{\mathfrak{p}'}$ and compatibly $M_{(\mathfrak{p})} = (M_{(f)})_{\mathfrak{p}'}$.*

Proof. We define a map $\psi : M_{(\mathfrak{p})} \to (M_{(f)})_{\mathfrak{p}'}$. Let $x/g \in M_{(\mathfrak{p})}$. We set $\psi(x/g) = (x^{\deg(f)}/f^{\deg(x)})/(g^{\deg(f)}/f^{\deg(g)})$. This makes sense since $\deg(x) = \deg(g)$, and since $g^{\deg(f)}/f^{\deg(g)} \notin \mathfrak{p}'$. We omit the verification that ψ is well defined, a module map and an isomorphism. □

Here is a graded variant of Lemma 7.14.3.

Lemma 7.53.6. *Suppose S is a graded ring, $\mathfrak{p}_i$, $i = 1, \ldots, r$ homogeneous prime ideals and $I \subset S_+$ a graded ideal. Assume $I \not\subset \mathfrak{p}_i$ for all i. Then there exists a homogeneous element $x \in I$ of positive degree such that $x \notin \mathfrak{p}_i$ for all i.*

Proof. We may assume there are no inclusions among the $\mathfrak{p}_i$. The result is true for $r = 1$. Suppose the result holds for $r - 1$. Pick $x \in I$ homogeneous of positive degree such that $x \notin \mathfrak{p}_i$ for all $i = 1, \ldots, r-1$. If $x \notin \mathfrak{p}_r$ we are done. So assume $x \in \mathfrak{p}_r$. If $I\mathfrak{p}_1 \ldots \mathfrak{p}_{r-1} \subset \mathfrak{p}_r$ then $I \subset \mathfrak{p}_r$ a contradiction. Pick $y \in I\mathfrak{p}_1 \ldots \mathfrak{p}_{r-1}$ homogeneous and $y \notin \mathfrak{p}_r$. Then $x^{\deg(y)} + y^{\deg(x)}$ works. □

Lemma 7.53.7. *Let S be a graded ring. Let $\mathfrak{p} \subset S$ be a prime. Let $\mathfrak{q}$ be the homogeneous ideal of S generated by the homogeneous elements of $\mathfrak{p}$. Then $\mathfrak{q}$ is a prime ideal of S.*

Proof. Suppose $f, g \in S$ are such that $fg \in \mathfrak{q}$. Let f_d (resp. g_e) be the homogeneous part of f (resp. g) of degree d (resp. e). Assume d, e are maxima such that $f_d \neq 0$ and $g_e \neq 0$. By assumption we can write $fg = \sum a_i f_i$ with $f_i \in \mathfrak{p}$ homogeneous. Say $\deg(f_i) = d_i$. Then $f_d g_e = \sum a'_i f_i$ with a'_i to homogeneous par of degree $d+e-d_i$ of a_i (or 0 if $d+e-d_i < 0$). Hence $f_d \in \mathfrak{p}$ or $g_e \in \mathfrak{p}$. Hence $f_d \in \mathfrak{q}$ or $g_e \in \mathfrak{q}$. In the first case replace f by $f - f_d$, in the second case replace g by $g - g_e$. Then still $fg \in \mathfrak{q}$ but the discrete invariant $d + e$ has been decreased. Thus we may continue in this fashion until either f or g is zero. This clearly shows that $fg \in \mathfrak{q}$ implies either $f \in \mathfrak{q}$ or $g \in \mathfrak{q}$ as desired. □

Lemma 7.53.8. *Let S be a graded ring.*

(1) *Any minimal prime of S is a homogeneous ideal of S.*
(2) *Given a homogeneous ideal $I \subset S$ any minimal prime over I is homogeneous.*

Proof. The first assertion holds because the prime $\mathfrak{q}$ constructed in Lemma 7.53.7 satisfies $\mathfrak{q} \subset \mathfrak{p}$. The second because we may consider S/I and apply the first part. □

Lemma 7.53.9. *Let R be a ring. Let R' be a finite type R-algebra, and let M be a finite R'-module. There exists a graded R-algebra S, a graded S-module N and an element $f \in S$ homogeneous of degree 1 such that*

1. *$R' \cong S_{(f)}$ and $M \cong N_{(f)}$ (as modules),*
2. *$S_0 = R$ and S is generated by finitely many elements of degree 1 over R, and*
3. *N is a finite S-module.*

Proof. We may write $R' = R[x_1, \ldots, x_n]/I$ for some ideal I. For an element $g \in R[x_1, \ldots, x_n]$ denote $\tilde{g} \in R[x_0, \ldots, x_n]$ the element homogeneous of minimal degree such that $g = \tilde{g}(1, x_1, \ldots, x_n)$. Let $\tilde{I} \subset R[X_0, \ldots, X_n]$ generated by all elements $\tilde{g}$, $g \in I$. Set $S = R[X_0, \ldots, X_n]/\tilde{I}$ and denote f the image of X_0 in S. By construction we have an isomorphism

$$S_{(f)} \longrightarrow R', \quad X_i/X_0 \longmapsto x_i.$$

To do the same thing with the module M we choose a presentation

$$M = (R')^{\oplus r}/\sum\nolimits_{j \in J} R' k_j$$

with $k_j = (k_{1j}, \ldots, k_{rj})$. Let $d_{ij} = \deg(\tilde{k}_{ij})$. Set $d_j = \max\{d_{ij}\}$. Set $K_{ij} = X_0^{d_j - d_{ij}} \tilde{k}_{ij}$ which is homogeneous of degree d_j. With this notation we set

$$N = \mathrm{Coker}\Big(\bigoplus\nolimits_{j \in J} S(-d_j) \xrightarrow{(K_{ij})} S^{\oplus r} \Big)$$

which works. Some details omitted. □

7.54. Blow up algebras

In this section we make some elementary observations about blowing up.

Definition 7.54.1. Let R be a ring. Let $I \subset R$ be an ideal.

1. The *blowup algebra*, or the *Rees algebra*, associated to the pair (R, I) is the the graded R-algebra
$$\mathrm{Bl}_I(R) = \bigoplus\nolimits_{n \geq 0} I^n = R \oplus I \oplus I^2 \oplus \ldots$$
where the summand I^n is placed in degree n.
2. If $a \in I$ is an element, then the *affine blowup algebra* $R[\frac{I}{a}]$ is the algebra $(\mathrm{Bl}_I(R))_{(a)}$ constructed in Section 7.53.

In other words, an element of $R[\frac{I}{a}]$ is represented by an expression of the form x/a^n with $x \in I^n$. Two representatives x/a^n and y/a^m define the same element if and only if $a^k(a^m x - a^n y) = 0$ for some $k \geq 0$.

Lemma 7.54.2. *If R is a domain then every (affine) blowup algebra of R is a domain.*

Proof. Omitted. □

Lemma 7.54.3. *If R is reduced then every (affine) blowup algebra of R is reduced.*

Proof. Omitted. □

Lemma 7.54.4. *Let R be a ring. Let $I \subset R$ be an ideal. Let $a \in I$. If a is not contained in any minimal prime of R, then $Spec(R[\frac{I}{a}]) \to Spec(R)$ has dense image.*

Proof. If $a^k x = 0$ for $x \in R$, then x is contained in all the minimal primes of R and hence nilpotent, see Lemma 7.16.2. Thus the kernel of $R \to R[\frac{I}{a}]$ consists of nilpotent elements. Hence the result follows from Lemma 7.27.6. □

7.55. Noetherian graded rings

Lemma 7.55.1. *A graded ring S is Noetherian if and only if S_0 is Noetherian and S_+ is finitely generated. Furthermore, a set of homogenous elements $f_i \in S_+$ generates S as an algebra over S_0 if and only if they generate S_+ as an ideal.*

Proof. It is clear that if S is Noetherian then $S_0 = S/S_+$ is Noetherian and S_+ is finitely generated. It is also clear that if f_i generate S over S_0 then they generate S_+ as an ideal. Conversely, suppose that $S_+ = (f_1, \ldots, f_n)$ and S_0 Noetherian. By decomposing the f_i into homogenous pieces we may assume each f_i is homogeneous. Consider the map $\Psi : S_0[X_1, \ldots X_n] \to S$ which maps X_i to f_i. We claim this is surjective. Once we have seent his the result follows from Lemma 7.28.1. Namely, suppose that $f \in S_d$ for some d. By assumption we may write $f = \sum a_i f_i$. We may replace a_i by its piece of degree $\deg(f) - \deg(f_i)$ and still obtain a valid relation. Now each a_i is homogenous of strictly smaller degree than f_i, and hence by induction on the degree we may assume a_i is in the image of Ψ. Of course then f is in the image too. $\square$

Definition 7.55.2. Let A be an abelian group. We say that a function $f : n \mapsto f(n) \in A$ defined for all sufficient large integers n is a *numerical polynomial* if there exists $r \geq 0$, elements $a_0, \ldots, a_r \in A$ such that

$$f(n) = \sum\nolimits_{i=0}^{r} \binom{n}{i} a_i$$

for all $n \gg 0$.

The reason for using the binomial coefficients is the elementary fact that any polynomial $P \in \mathbf{Q}[T]$ all of whose values at integer points are integers, is equal to a sum $P(T) = \sum a_i \binom{T}{i}$ with $a_i \in \mathbf{Z}$. Note that in particular the expressions $\binom{T+1}{i+1}$ are of this form.

Lemma 7.55.3. *If $A \to A'$ is a homomorphism of abelian groups and if $f : n \mapsto f(n) \in A$ is a numerical polynomial, then so is the composition.*

Proof. This is immediate from the definitions. $\square$

Lemma 7.55.4. *Suppose that $f : n \mapsto f(n) \in A$ is defined for all n sufficiently large and suppose that $n \mapsto f(n) - f(n-1)$ is a numerical polynomial. Then f is a numerical polynomial.*

Proof. Let $f(n) - f(n-1) = \sum_{i=0}^{r} \binom{n}{i} a_i$ for all $n \gg 0$. Set $g(n) = f(n) - \sum_{i=0}^{r} \binom{n+1}{i+1} a_i$. Then $g(n) - g(n-1) = 0$ for all $n \gg 0$. Hence g is eventually constant, say equal to a_{-1}. We leave it to the reader to show that $a_{-1} + \sum_{i=0}^{r} \binom{n+1}{i+1} a_i$ has the required shape (see remark above the lemma). $\square$

Lemma 7.55.5. *If M is a finitely generated graded S-module, and if S is finitely generated over S_0, then each M_n is a finite S_0-module.*

Proof. Suppose the generators of M are m_i and the generators of S are f_i. By taking homogeneous components we may assume that the m_i and the f_i are homogeneous and we may assume $f_i \in S_+$. In this case it is clear that each M_n is generated over S_0 by the ``monomials'' $\prod f_i^{e_i} m_j$ whose degree is n. $\square$

Proposition 7.55.6. *Suppose that S is a Noetherian graded ring and M a finite graded S-module. Consider the function*

$$\mathbf{Z} \longrightarrow K'_0(S_0), \quad n \longmapsto [M_n]$$

see Lemma 7.55.5 above. If S_+ is generated by elements of degree 1, then this function is a numerical polynomial.

Proof. We prove this by induction on the minimal number of generators of S_1. If this number is 0, then $M_n = 0$ for all $n \gg 0$ and the result holds. To prove the induction step, let $x \in S_1$ be one of a minimal set of generators, such that the induction hypothesis applies to the graded ring $S/(x)$.

First we show the result holds if x is nilpotent on M. This we do by induction on the minimal integer r such that $x^r M = 0$. If $r = 1$, then M is a module over S/xS and the result holds (by the other induction hypothesis). If $r > 1$, then we can find a short exact sequence $0 \to M' \to M \to M'' \to 0$ such that the integers r', r'' are stricly smaller than r. Thus we know the result for M'' and M'. Hence we get the result for M because of the relation $[M_d] = [M'_d] + [M''_d]$ in $K'_0(R)$.

If x is not nilpotent on M, let $M' \subset M$ be the largest submodule on which x is nilpotent. Consider the exact sequence $0 \to M' \to M \to M/M' \to 0$ we see again it suffices to prove the result for M/M'. In other words we may assume that multiplication by x is injective.

Let $\overline{M} = M/xM$. Note that the map $x : M \to M$ is *not* a map of graded S-modules, since it does not map M_d into M_d. Namely, for each d we have the following short exact sequence

$$0 \to M_d \xrightarrow{x} M_{d+1} \to \overline{M}_{d+1} \to 0$$

This proves that $[M_{d+1}] - [M_d] = [\overline{M}_{d+1}]$. Hence we win by Lemma 7.55.4. $\square$

Remark 7.55.7. If S is still Noetherian but S is not generated in degree 1, then the function associated to a graded S-module is a periodic polynomial (i.e., it is a numerical polynomial on the congruence classes of integers modulo n for some n).

Example 7.55.8. Suppose that $S = k[X_1, \ldots, X_d]$. By Example 7.51.2 we may identify $K_0(k) = K'_0(k) = \mathbf{Z}$. Hence any finitely generated graded $k[X_1, \ldots, X_d]$-module gives rise to a numerical polynomial $n \mapsto \dim_k(M_n)$.

Lemma 7.55.9. *Let k be a field. Suppose that $I \subset k[X_1, \ldots, X_d]$ is a nonzero graded ideal. Let $M = k[X_1, \ldots, X_d]/I$. Then the numerical polynomial $n \mapsto \dim_k(M_n)$ (see Example 7.55.8 above) has degree $< d - 1$ (or is zero if $d = 1$).*

Proof. The numerical polynomial associated to the graded module $k[X_1, \ldots, X_n]$ is $n \mapsto \binom{n-1+d}{d-1}$. For any nonzero homogeneous $f \in I$ of degree e and any degree $n >> e$ we have $I_n \supset f \cdot k[X_1, \ldots, X_d]_{n-e}$ and hence $\dim_k(I_n) \geq \binom{n-e-1+d}{d-1}$. Hence $\dim_k(M_n) \leq \binom{n-1+d}{d-1} - \binom{n-e-1+d}{d-1}$. We win because the last expression has degree $< d - 1$ (or is zero if $d = 1$). $\square$

7.56. Noetherian local rings

In all of this section $(R, \mathfrak{m}, \kappa)$ is a Noetherian local ring. We develop some theory on Hilbert functions of modules in this section. Let M be a finite R-module. We define the *Hilbert function* of M to be the function

$$\varphi_M : n \longmapsto \text{length}_R(\mathfrak{m}^n M/\mathfrak{m}^{n+1}M).$$

Note that we have by Lemma 7.48.3 that

$$\text{length}_R(M/\mathfrak{m}^{n+1}M) = \sum\nolimits_{i=0}^{n} \varphi_M(i).$$

There is a variant of this construction which uses an ideal of definition.

Definition 7.56.1. Let $(R, \mathfrak{m})$ be a local Noetherian ring. An ideal $I \subset R$ such that $\sqrt{I} = \mathfrak{m}$ is called *an ideal of definition of R*.

Let $I \subset R$ be an ideal of definition. Because R is Noetherian this means that $\mathfrak{m}^r \subset I$ for some r, see Lemma 7.47.3. Hence any finite R-module annihilated by a power of I has a finite length, see Lemma 7.48.8. Thus in this case we may put

$$\varphi_{I,M} : n \longmapsto \text{length}_R(I^n M/I^{n+1}M).$$

Again we have that

$$\text{length}_R(M/I^{n+1}M) = \sum\nolimits_{i=0}^{n} \varphi_{I,M}(i).$$

Lemma 7.56.2. *Suppose that $M' \subset M$ are finite R-modules with finite length quotient. Then there exists a constants c_1, c_2 such that for all $n \gg c_2$ we have*

$$c_1 + \sum\nolimits_{i=0}^{n-c_2} \varphi_{I,M'}(i) \leq \sum\nolimits_{i=0}^{n} \varphi_{I,M}(i) \leq c_1 + \sum\nolimits_{i=0}^{n} \varphi_{I,M'}(i)$$

$\varphi_{I,M}(n) \geq \varphi_{I,M'}(n - c_1) - c_2$ *and* $\varphi_{I,M'}(n) \geq \varphi_{I,M}(n) - c_2$.

Proof. Let $c_1 = \text{length}_R(M/M')$. For $n \geq 1$ we have

$$\begin{aligned}
\sum\nolimits_{i=0}^{n} \varphi_{I,M}(i) &= \text{length}_R(M/I^{n+1}M) \\
&= c_1 + \text{length}_R(M'/I^{n+1}M) \\
&\leq c_1 + \text{length}_R(M'/I^{n+1}M') \\
&= c_1 + \sum\nolimits_{i=0}^{n} \varphi_{I,M'}(i)
\end{aligned}$$

On the other hand, let c_2 be such that $I^{c_2}M \subset M'$, so $I^n M \subset I^{n-c_2}M'$. We have

$$\begin{aligned}
\sum\nolimits_{i=0}^{n} \varphi_{I,M}(i) &= \text{length}_R(M/I^{n+1}M) \\
&= c_1 + \text{length}_R(M'/I^{n+1}M) \\
&\geq c_1 + \text{length}_R(M'/I^{n+1-c_2}M') \\
&= c_1 + \sum\nolimits_{i=0}^{n-c_2} \varphi_{I,M'}(i)
\end{aligned}$$

This works as soon as $n \geq c_2$. □

Lemma 7.56.3. *Suppose that $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of finite R-modules. Then there exists a submodule $N \subset M'$ with finite colength and an integer c such that $\varphi_{I,M}(n) = \varphi_{I,M''}(n) + \varphi_{I,N}(n - c)$ for all n large enough.*

Proof. Note that $M/I^nM \to M''/I^nM''$ is surjective with kernel $M'/M' \cap I^nM$. By the Artin-Rees Lemma 7.47.4 there exists a constant c such that $M' \cap I^nM = I^{n-c}(M' \cap I^cM)$. Denote $N = M' \cap I^cM$. Note that $I^cM' \subset N \subset M'$. Hence $\text{length}_R(M'/M' \cap I^nM) = \text{length}_R(M'/N) + \text{length}_R(N/I^{n-c}N)$. Then we obtain the equality

$$\sum_{i=0}^{n-1} \varphi_{I,M}(i) = \sum_{i=0}^{n-1} \varphi_{I,M''}(i) + \sum_{i=0}^{n-c-1} \varphi_{I,N}(i) + \text{length}_R(M'/N)$$

for n large enough. Thus we get $\varphi_{I,M}(n) = \varphi_{I,M''}(n) + \varphi_{I,N}(n-c)$ for n large enough. $\square$

Lemma 7.56.4. *Suppose that I, I' are two ideals of definition for the Noetherian local ring R. Let M be a finite R-module. There exists a constant a such that $\sum_{i=0}^{n} \varphi_{I,M}(i) \leq \sum_{i=0}^{an} \varphi_{I',M}(i)$.*

Proof. There exists an integer a such that $(I')^a \subset I$. Hence we get a surjection $M/(I')^{a(n+1)}M \to M/I^{n+1}M$. Whence the result (with $a+1$). $\square$

Proposition 7.56.5. *For every Noetherian local ring R, any $I \subset R$ such that $\sqrt{I} = \mathfrak{m}$ and every finite R-module M the Hilbert function $\varphi_{I,M}$ is a numerical polynomial.*

Proof. Consider the graded ring $S = R/I \oplus I/I^2 \oplus I^2/I^3 \oplus \ldots = \bigoplus_{d \geq 0} I^d/I^{d+1}$. Consider the graded S-module $N = M/IM \oplus IM/I^2M \oplus \ldots = \bigoplus_{d \geq 0} I^dM/I^{d+1}M$. This pair (S, N) satisfies the hypotheses of Proposition 7.55.6. Hence the result follows from that Proposition, and Lemma 7.51.1. $\square$

Lemma 7.56.6. *Suppose that M is a finite R-module. The degree of the numerical polynomial $\varphi_{I,M}$ is independent of the ideal of definition I.*

Proof. This follows immediately from Lemma 7.56.4. $\square$

Definition 7.56.7. If R is a local Noetherian ring and M a finite R-module. If $\mathfrak{m}^nM = 0$ for some $n \geq 0$ we set $d(M) = 0$. Otherwise we denote $d(M)$ the degree $+1$ of any of the numerical polynomials $\varphi_{I,M}$ above.

Thus $d(M)$ is the degree of the numerical polynomial $n \mapsto \text{length}_R(M/I^nM)$ for any ideal of definition I. We will denote this function

$$\chi_{I,M}(n) = \text{length}_R(M/I^{n+1}M).$$

We will frequently use that $\chi_{I,M}(n) = \sum_{i=0}^{n} \varphi_{I,M}(i)$ without further mention.

Lemma 7.56.8. *Suppose $M \subset M'$ with finite length quotient, but neither finite length. Then $\chi_{I,M} - \chi_{I,M'}$ is a polynomial of degree < degree of either polynomial.*

Proof. Immediate from Lemma 7.56.2 by elementary calculus. $\square$

Lemma 7.56.9. *Suppose that $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of finite R-modules. Then $\max\{\deg(\chi_{I,M'}), \deg(\chi_{I,M''})\} = \deg(\chi_{I,M})$. Suppose the length of M' is not finite. Then $\chi_{I,M} - \chi_{I,M''} - \chi_{I,M'}$ is a numerical polynomial of degree < the degree of $\chi_{I,M'}$.*

Proof. Immediate from Lemma 7.56.3, and 7.56.8 by elementary calculus. $\square$

7.57. Dimension

Definition 7.57.1. The *Krull dimension* of the ring R is the Krull dimension of the topological space $Spec(R)$, see Topology, Definition 5.7.1. In other words it is the supremum of the integers $n \geq 0$ such that there exists a chain of prime ideals of length n:

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_n, \quad \mathfrak{p}_i \neq \mathfrak{p}_{i+1}.$$

Definition 7.57.2. The *height* of a prime ideal $\mathfrak{p}$ of a ring R is the dimension of the local ring $R_{\mathfrak{p}}$.

Lemma 7.57.3. *The Krull dimension of R is the supremum of the heights of its (maximal) primes.*

Proof. This is so because we can always add a maximal ideal at the end of a chain of prime ideals. $\square$

Lemma 7.57.4. *A Noetherian ring of dimension* 0 *is Artinian. Conversely, any Artinian ring is Noetherian of dimension zero.*

Proof. By Lemma 7.28.5 the space $Spec(R)$ is Noetherian. By Topology, Lemma 5.6.2 we see that $Spec(R)$ has finitely many irreducible components, say $Spec(R) = Z_1 \cup \ldots Z_r$. According to Lemma 7.23.1, each $Z_i = V(\mathfrak{p}_i)$ with $\mathfrak{p}_i$ a minimal ideal. Since the dimension is 0 these $\mathfrak{p}_i$ are also maximal. Thus $Spec(R)$ is the discrete topological space with elements $\mathfrak{p}_i$. All elements f of the radical $I = \cap \mathfrak{p}_i$ are nilpotent since otherwise R_f would not be the zero ring and we would have another prime. Since I is finitely generated we conclude that I is nilpotent, Lemma 7.47.3. By Lemma 7.49.7 R is the product of its local rings. By Lemma 7.48.8 each of these has finite length over R. Hence we conclude that R is Artinian by Lemma 7.49.8.

If R is Artinian then by Lemma 7.49.8 it is Noetherian. All of its primes are maximal by a combination of Lemmas 7.49.3, 7.49.4 and 7.49.7. $\square$

In the following we will use the invariant $d(-)$ defined in Definition 7.56.7. Here is a warm up lemma.

Lemma 7.57.5. *Let R be a Noetherian local ring. Then* $\dim(R) = 0 \Leftrightarrow d(R) = 0$.

Proof. This is because $d(R) = 0$ if and only if R has finite length as an R-module. See Lemma 7.49.8. $\square$

Proposition 7.57.6. *Let R be a ring. The following are equivalent:*

(1) *R is Artinian,*
(2) *R is Noetherian and* $\dim(R) = 0$,
(3) *R has finite length as a module over itself,*
(4) *R is a finite product of Artinian local rings,*
(5) *R is Noetherian and $Spec(R)$ is a finite discrete topological space,*
(6) *R is a finite product of Noetherian local rings of dimension* 0,
(7) *R is a finite product of Noetherian local rings R_i with $d(R_i) = 0$,*
(8) *R is a finite product of Noetherian local rings R_i whose maximal ideals are nilpotent,*
(9) *R is Noetherian, has finitely many maximal ideals and its radical ideal is nilpotent, and*
(10) *R is Noetherian and there are no strict inclusions among its primes.*

Proof. This is a combination of Lemmas 7.49.7, 7.49.8, 7.57.4, and 7.57.5. ☐

Lemma 7.57.7. *Let R be a local Noetherian ring. The following are equivalent:*

1. $\dim(R) = 1$,
2. $d(R) = 1$,
3. *there exists an* $x \in \mathfrak{m}$, *x not nilpotent such that* $V(x) = \{\mathfrak{m}\}$,
4. *there exists an* $x \in \mathfrak{m}$, *x not nilpotent such that* $\mathfrak{m} = \sqrt{(x)}$, *and*
5. *there exists an ideal of definition generated by* 1 *element, and no ideal of definition is generated by* 0 *elements.*

Proof. First, assume that $\dim(R) = 1$. Let $\mathfrak{p}_i$ be the minimal primes of R. Because the dimension is 1 the only other prime of R is $\mathfrak{m}$. According to Lemma 7.28.6 there are finitely many. Hence we can find $x \in \mathfrak{m}$, $x \not\in \mathfrak{p}_i$, see Lemma 7.14.3. Thus the only prime containing x is $\mathfrak{m}$ and hence (3).

If (3) then $\mathfrak{m} = \sqrt{(x)}$ by Lemma 7.16.2, and hence (4). The converse is clear as well. The equivalence of (4) and (5) follows from directly the definitions.

Assume (5). Let $I = (x)$ be an ideal of definition. Note that I^n/I^{n+1} is a quotient of R/I via multiplication by x^n and hence $\text{length}_R(I^n/I^{n+1})$ is bounded. Thus $d(R) = 0$ or $d(R) = 1$, but $d(R) = 0$ is excluded by the assumption that 0 is not an ideal of definition.

Assume (2). To get a contradiction, assume there exist primes $\mathfrak{p} \subset \mathfrak{q} \subset \mathfrak{m}$, with both inclusions strict. Pick some ideal of definition $I \subset R$. We will repeatedly use Lemma 7.56.9. First of all it implies, via the exact sequence $0 \to \mathfrak{p} \to R \to R/\mathfrak{p} \to 0$, that $d(R/\mathfrak{p}) \leq 1$. But it clearly cannot be zero. Pick $x \in \mathfrak{q}$, $x \not\in \mathfrak{p}$. Consider the short exact sequence

$$0 \to R/\mathfrak{p} \to R/\mathfrak{p} \to R/(xR + \mathfrak{p}) \to 0.$$

This implies that $\chi_{I,R/\mathfrak{p}} - \chi_{I,R/\mathfrak{p}} - \chi_{I,R/(xR+\mathfrak{p})} = -\chi_{I,R/(xR+\mathfrak{p})}$ has degree < 1. In other words, $d(R/(xR + \mathfrak{p}) = 0$, and hence $\dim(R/(xR + \mathfrak{p})) = 0$, by Lemma 7.57.5. But $R/(xR + \mathfrak{p})$ has the distinct primes $\mathfrak{q}/(xR + \mathfrak{p})$ and $\mathfrak{m}/(xR + \mathfrak{p})$ which gives the desired contradiction. ☐

Proposition 7.57.8. *Let R be a local Noetherian ring. The following are equivalent:*

1. $\dim(R) = d$,
2. $d(R) = d$,
3. *there exists an ideal of definition generated by d elements, and no ideal of definition is generated by fewer than d elements.*

Proof. This proof is really just the same as the proof of Lemma 7.57.7. We will prove the proposition by induction on d. By Lemmas 7.57.5 and 7.57.7 we may assume that $d > 1$. Denote the minimal number of generators for an ideal of definition of R by $d'(R)$. We will prove that the inequalities $\dim(R) \geq d'(R) \geq d(R) \geq \dim(R)$, and hence they are all equal.

First, assume that $\dim(R) = d$. Let $\mathfrak{p}_i$ be the minimal primes of R. According to Lemma 7.28.6 there are finitely many. Hence we can find $x \in \mathfrak{m}$, $x \not\in \mathfrak{p}_i$, see Lemma 7.14.3. Note that every maximal chain of primes starts with some $\mathfrak{p}_i$, hence the dimension of R/xR is at most $d-1$. By induction there are $x_2, \ldots, x_d$ which generate an ideal of definition in R/xR. Hence R has an ideal of definition generated by (at most) d elements.

Assume $d'(R) = d$. Let $I = (x_1, \ldots, x_d)$ be an ideal of definition. Note that I^n/I^{n+1} is a quotient of a direct sum of $\binom{d+n-1}{d-1}$ copies R/I via multiplication by all degree n monomials in $x_1, \ldots, x_n$. Hence $\text{length}_R(I^n/I^{n+1})$ is bounded by a polynomial of degree $d-1$. Thus $d(R) \leq d$.

Assume $d(R) = d$. Consider a chain of primes $\mathfrak{p} \subset \mathfrak{q} \subset \mathfrak{q}_2 \subset \ldots \subset \mathfrak{p}_e = \mathfrak{m}$, with all inclusions strict, and $e \geq 2$. Pick some ideal of definition $I \subset R$. We will repeatedly use Lemma 7.56.9. First of all it implies, via the exact sequence $0 \to \mathfrak{p} \to R \to R/\mathfrak{p} \to 0$, that $d(R/\mathfrak{p}) \leq d$. But it clearly cannot be zero. Pick $x \in \mathfrak{q}$, $x \not\in \mathfrak{p}$. Consider the short exact sequence

$$0 \to R/\mathfrak{p} \to R/\mathfrak{p} \to R/(xR + \mathfrak{p}) \to 0.$$

This implies that $\chi_{I,R/\mathfrak{p}} - \chi_{I,R/\mathfrak{p}} - \chi_{I,R/(xR+\mathfrak{p})} = -\chi_{I,R/(xR+\mathfrak{p})}$ has degree $< d$. In other words, $d(R/(xR + \mathfrak{p})) \leq d - 1$, and hence $\dim(R/(xR + \mathfrak{p})) \leq d - 1$, by induction. Now $R/(xR + \mathfrak{p})$ has the chain of prime ideals $\mathfrak{q}/(xR + \mathfrak{p}) \subset \mathfrak{q}_2/(xR + \mathfrak{p}) \subset \ldots \subset \mathfrak{q}_e/(xR + \mathfrak{p})$ which gives $e - 1 \leq d - 1$. Since we started with an arbitrary chain of primes this proves that $\dim(R) \leq d(R)$.

Reading back the reader will see we proved the circular inequalities as desired. □

Let $(R, \mathfrak{m})$ be a Noetherian local ring. From the above it is clear that $\mathfrak{m}$ cannot be generated by fewer than $\dim(R)$ variables. By Nakayama's Lemma 7.14.5 the minimal number of generators of $\mathfrak{m}$ equals $\dim_{\kappa(\mathfrak{m})} \mathfrak{m}/\mathfrak{m}^2$. Hence we have the following fundamental inequality

$$\dim(R) \leq \dim_{\kappa(\mathfrak{m})} \mathfrak{m}/\mathfrak{m}^2.$$

It turns out that the rings where equality holds have a lot of good properties. They are called regular local rings.

Definition 7.57.9. Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d.

(1) A *system of parameters of* R is a sequence of elements $x_1, \ldots, x_d \in \mathfrak{m}$ which generates an ideal of definition of R,
(2) if there exist $x_1, \ldots, x_d \in \mathfrak{m}$ such that $\mathfrak{m} = (x_1, \ldots, x_d)$ then we call R a *regular local ring* and $x_1, \ldots, x_d$ a *regular system of parameters*.

The folllowing two lemmas are clear from the proofs of the lemmas and proposition above, but we spell them out so we have convenient references.

Lemma 7.57.10. *Let R be a Noetherian ring.*

(1) *Let $x \in R$, $\mathfrak{p}, \mathfrak{q} \in Spec(R)$. Suppose that $\mathfrak{p} \subset (\mathfrak{p}, x) \subset \mathfrak{q}$ and $\mathfrak{q}$ minimal over $(\mathfrak{p}, x)$. Then there is no prime strictly between $\mathfrak{p}$ and $\mathfrak{q}$.*
(2) *If $x \in R$ and $x \in \mathfrak{p}$ is minimal over (x) then the height of $\mathfrak{p}$ is* 0 *or* 1.

Proof. Consider the situation of the first assertion. The primes containing $\mathfrak{p}$ and contained in $\mathfrak{q}$ correspond to primes of $R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}}$, and the primes containing x correspond to the ones containing the image of x. Thus we may assume R is a Noetherian local domain, $\mathfrak{p} = (0)$ and $\mathfrak{q}$ maximal. Now since $\sqrt{(x)}$ is the intersection of the prime ideals containing it, and since $\mathfrak{q}$ is the only prime containing x by minimality, we see that $\sqrt{(x)} = \mathfrak{q}$. Hence Lemma 7.57.7 applies. The second assertion follows from the first. □

Lemma 7.57.11. *Suppose that R is a Noetherian local ring and $x \in \mathfrak{m}$ an element of its maximal ideal. Then* $\dim R \leq \dim R/xR + 1$. *If x is not contained in any of the minimal primes of R then equality holds. (For example if x is a nonzero divisor.)*

Proof. If $x_1, \ldots, x_{\dim R/xR} \in R$ map to elements of R/xR which generate an ideal of definition for R/xR, then $x, x_1, \ldots, x_{\dim R/xR}$ generate an ideal of definition for R. Hence the inequality by Proposition 7.57.8. On the other hand, if x is not contained in any minimal prime of R, then the chains of primes in R/xR all give rise to chains in R which are at least one step away from being maximal. $\square$

Lemma 7.57.12. *Let $(R, \mathfrak{m})$ be a Noetherian local ring. Suppose $x_1, \ldots, x_d \in \mathfrak{m}$ generate an ideal of definition and $d = \dim(R)$. Then $\dim(R/(x_1, \ldots, x_i)) = d - i$ for all $i = 1, \ldots, d$.*

Proof. Clear from the proof of Proposition 7.57.8, or use induction on d and Lemma 7.57.11 above. $\square$

7.58. Applications of dimension theory

We can use the results on dimension to prove certain rings are Jacobson.

Lemma 7.58.1. *Let R be a Noetherian local domain of dimension ≥ 2. A nonempty open subset $U \subset Spec(R)$ is infinite.*

Proof. To get a constradiction, assume that $U \subset Spec(R)$ is finite. In this case $(0) \in U$ and $\{(0)\}$ is an open subset of U (because the complement of $\{(0)\}$ is the union of the closures of the other points). Thus we may assume $U = \{(0)\}$. Let $\mathfrak{m} \subset R$ be the maximal ideal. We can find an $x \in \mathfrak{m}$, $x \neq 0$ such that $V(x) \cup U = Spec(R)$. In other words we see that $D(x) = \{(0)\}$. In particular we see that $\dim(R/xR) = \dim(R) - 1 \geq 1$, see Lemma 7.57.11. Let $\overline{y}_2, \ldots, \overline{y}_{\dim(R)} \in R/xR$ generate an ideal of definition of R/xR, see Proposition 7.57.8. Choose lifts $y_2, \ldots, y_{\dim(R)} \in R$, so that $x, y_2, \ldots, y_{\dim(R)}$ generate an ideal of definition in R. This implies that $\dim(R/(y_2)) = \dim(R) - 1$ and $\dim(R/(y_2, x)) = \dim(R) - 2$, see Lemma 7.57.12. Hence there exists a prime $\mathfrak{p}$ containing y_2 but not x. This constradicts the fact that $D(x) = \{(0)\}$. $\square$

Lemma 7.58.2. *(Noetherian Jacobson rings.)*

(1) *Any Noetherian domain R of dimension 1 with infinitely many primes is Jacobson.*
(2) *Any Noetherian ring such that every prime $\mathfrak{p}$ is either maximal or contained in infinitely many prime ideals is Jacobson.*

Proof. Part (1) is a reformulation of Lemma 7.31.6.

Let R be a Noetherian ring such that every non-maximal prime $\mathfrak{p}$ is contained in infinitely many prime ideals. Assume $Spec(R)$ is not Jacobson to get a contradiction. By Lemmas 7.23.1 and 7.28.5 we see that $Spec(R)$ is a sober, Noetherian topological space. By Topology, Lemma 5.13.3 we see that there exists a non-maximal ideal $\mathfrak{p} \subset R$ such that $\{\mathfrak{p}\}$ is a locally closed subset of $Spec(R)$. In other words, $\mathfrak{p}$ is not maximal and $\{\mathfrak{p}\}$ is an open subset of $V(\mathfrak{p})$. Consider a prime $\mathfrak{q} \subset R$ with $\mathfrak{p} \subset \mathfrak{q}$. Recall that the topology on the spectrum of $(R/\mathfrak{p})_{\mathfrak{q}} = R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}}$ is induced from that of $Spec(R)$, see Lemmas 7.16.5 and 7.16.7. Hence we see that $\{(0)\}$ is a locally closed subset of $Spec((R/\mathfrak{p})_{\mathfrak{q}})$. By Lemma 7.58.1 we conclude that $\dim((R/\mathfrak{p})_{\mathfrak{q}}) = 1$. Since this holds for every $\mathfrak{q} \supset \mathfrak{p}$ we conclude that $\dim(R/\mathfrak{p}) = 1$. At this point we use the assumption that $\mathfrak{p}$ is contained in infinitely many primes to see that $Spec(R/\mathfrak{p})$ is infinite. Hence by part (1) of the lemma we see that $V(\mathfrak{p}) \cong Spec(R/\mathfrak{p})$ is the closure of its closed points. This is the desired contradiction since it means that $\{\mathfrak{p}\} \subset V(\mathfrak{p})$ cannot be open. $\square$

7.59. Support and dimension of modules

Lemma 7.59.1. *Let R be a Noetherian ring, and let M be a finite R-module. There exists a filtration by R-submodules*

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$$

such that each quotient M_i/M_{i-1} is isomorphic to $R/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$ of R.

Proof. By Lemma 7.5.5 it suffices to do the case $M = R/I$ for some ideal I. Consider the set S of ideals J such that the lemma does not hold for the module R/J, and order it by inclusion. To arrive at a contradiction, assume that S is not empty. Because R is Noetherian, S has a maximal element J. By definition of S, the ideal J cannot be prime. Pick $a, b \in R$ such that $ab \in J$, but neither $a \in J$ nor $b \in J$. Consider the filtration $0 \subset aR/(J \cap aR) \subset R/J$. Note that $aR/(J \cap aR)$ is a quotient of $R/(J + bR)$ and the second quotient equals $R/(aR + J)$. Hence by maximality of J, each of these has a filtration as above and hence so does R/J. Contradiction. $\square$

Definition 7.59.2. Let R be a ring and let M be an R-module. The *support of M* is the set

$$\mathrm{Supp}(M) = \{\mathfrak{p} \in Spec(R) \mid M_{\mathfrak{p}} \neq 0\}$$

Lemma 7.59.3. *Let R be a ring. Let M be an R-module. Then*

$$M = (0) \Leftrightarrow Supp(M) = \emptyset.$$

Proof. Actually, Lemma 7.21.1 even shows that $\mathrm{Supp}(M)$ always contains a maximal ideal if M is not zero. $\square$

Lemma 7.59.4. *Let R be a ring and let M be an R-module. If M is finite, then $Supp(M)$ is closed. More precisely, let $I = \{f \in R \mid fM = 0\}$. Then $V(I) = Supp(M)$.*

Proof. We will show that $V(I) = \mathrm{Supp}(M)$.

Suppose $\mathfrak{p} \in \mathrm{Supp}(M)$. Then $M_{\mathfrak{p}} \neq 0$. Hence by Nakayam's Lemma 7.14.5 we have $M \otimes_R \kappa(\mathfrak{p}) \neq 0$. Hence $I \subset \mathfrak{p}$.

Conversely, suppose that $\mathfrak{p} \not\in \mathrm{Supp}(M)$. Then $M_{\mathfrak{p}} = 0$. Let $x_1, \ldots, x_r \in M$ be generators. By Lemma 7.9.9 there exists an $f \in R$, $f \not\in \mathfrak{p}$ such that $x_i/1 = 0$ in M_f. Hence $f^{n_i} x_i = 0$ for some $n_i \geq 1$. Hence $f^n M = 0$ for $n = \max\{n_i\}$ as desired. $\square$

Lemma 7.59.5. *Let R be a ring and let M be an R-module. If M is a finitely presented R-module, then $Supp(M)$ is a closed subset of $Spec(R)$ whose complement is quasi-compact.*

Proof. Choose a presentation

$$R^{\oplus m} \longrightarrow R^{\oplus n} \longrightarrow M \to 0$$

Let $A \in \mathrm{Mat}(n \times m, R)$ be the matrix of the first map. By Nakayama's Lemma 7.14.5 we see that

$$M_{\mathfrak{p}} \neq 0 \Leftrightarrow M \otimes \kappa(\mathfrak{p}) \neq 0 \Leftrightarrow \mathrm{rank}(A \bmod \mathfrak{p}) < n.$$

Hence, if I is the ideal of R generated by the $n \times n$ minors of A, then $\mathrm{Supp}(M) = V(I)$. Since I is finitely generated, say $I = (f_1, \ldots, f_t)$, we see that $Spec(R) \setminus V(I)$ is a finite union of the standard opens $D(f_i)$, hence quasi-compact. $\square$

Lemma 7.59.6. *Let R be a ring and let M be an R-module.*

1. *If M is finite then the support of M/IM is $Supp(M) \cap V(I)$.*
2. *If $N \subset M$, then $Supp(N) \subset Supp(M)$.*

(3) *If Q is a quotient module of M then Supp*$(Q) \subset$ *Supp*(M).
(4) *If* $0 \to N \to M \to Q \to 0$ *is a short exact sequence then Supp*$(M) =$ *Supp*$(Q) \cup$ *Supp*(N).

Proof. The functors $M \mapsto M_{\mathfrak p}$ are exact. This immediately implies all but the first assertion. For the first assertion we need to show that $M_{\mathfrak p}\neq 0$ and $I \subset \mathfrak p$ implies $(M/IM)_{\mathfrak p} = M_{\mathfrak p}/IM_{\mathfrak p}\neq 0$. This follows from Nakayama's Lemma 7.14.5. $\square$

Lemma 7.59.7. *Let R, M, M_i, $\mathfrak p_i$ as in Lemma 7.59.1. All of the primes $\mathfrak p_i$ are in the support of M.*

Proof. Since localization is exact, we see that $(R/\mathfrak p_i)_{\mathfrak p_i}$ is a subquotient of $M_{\mathfrak p_i}$. Hence $M_{\mathfrak p_i}$ is not zero. $\square$

Lemma 7.59.8. *Suppose that R is a Noetherian local ring with maximal ideal $\mathfrak m$. Let M be a finite R-module. Then Supp*$(M) = \{\mathfrak m\}$ *if and only if M has finite length over R.*

Proof. Assume that $\text{Supp}(M) = \{\mathfrak m\}$. It suffices to show that all the primes $\mathfrak p_i$ in the filtration of Lemma 7.59.1 are the maximal ideal. This is clear by Lemma 7.59.7.

Suppose that M has finite length over R. Then $\mathfrak m^n M = 0$ by Lemma 7.48.4. Since some element of $\mathfrak m$ maps to a unit in $R_{\mathfrak p}$ for any prime $\mathfrak p\neq\mathfrak m$ in R we see $M_{\mathfrak p} = 0$. $\square$

Lemma 7.59.9. *Let R be a Noetherian ring. Let $I \subset R$ be an ideal. Let M be a finite R-module. Then $I^n M = 0$ for some $n \geq 0$ if and only if Supp*$(M) \subset V(I)$.

Proof. It is clear that $I^n M = 0$ for some $n \geq 0$ implies $\text{Supp}(M) \subset V(I)$. Suppose that $\text{Supp}(M) \subset V(I)$. Choose a filtration $0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$ as in Lemma 7.59.1. Each of the primes $\mathfrak p_i$ is contained in $V(I)$ by Lemma 7.59.7. Hence $I \subset \mathfrak p_i$ and I annihilates M_i/M_{i-1}. Hence I^n annihilates M. $\square$

Lemma 7.59.10. *Let R, M, M_i, $\mathfrak p_i$ as in Lemma 7.59.1. The minimal elements of the set $\{\mathfrak p_i\}$ are the minimal elements of Supp*(M), *and the number of times a minimal prime $\mathfrak p$ occurs is*

$$\#\{i \mid \mathfrak p_i = \mathfrak p\} = \mathit{length}_{R_{\mathfrak p}} M_{\mathfrak p}.$$

Proof. We have already seen $\{\mathfrak p_i\} \subset \text{Supp}(M)$, in Lemma 7.59.7. Let $\mathfrak p \in \text{Supp}(M)$ be minimal. The support of $M_{\mathfrak p}$ is the set consisting of the maximal ideal $\mathfrak p R_{\mathfrak p}$. Hence by Lemma 7.59.8 the length of $M_{\mathfrak p}$ is finite and > 0. Next we note that $M_{\mathfrak p}$ has a filtration with subquotients $(R/\mathfrak p_i)_{\mathfrak p} = R_{\mathfrak p}/\mathfrak p_i R_{\mathfrak p}$ These are zero if $\mathfrak p_i\not\subset\mathfrak p$ and equal to $\kappa(\mathfrak p)$ if $\mathfrak p_i \subset \mathfrak p$ because by minimality of $\mathfrak p$ we have $\mathfrak p_i = \mathfrak p$ in this case. The result follows since $\kappa(\mathfrak p)$ has length 1. $\square$

Lemma 7.59.11. *Let R be a Noetherian local ring. Let M be a finite R-module. Then* $d(M) = \dim(\mathit{Supp}(M))$.

Proof. Let $M_i, \mathfrak p_i$ be as in Lemma 7.59.1. By Lemma 7.56.9 we obtain the equality $d(M) = \max\{d(R/\mathfrak p_i)\}$. By Proposition 7.57.8 we have $d(R/\mathfrak p_i) = \dim(R/\mathfrak p_i)$. Trivially $\dim(R/\mathfrak p_i) = \dim V(\mathfrak p_i)$. Since all minimal primes of $\text{Supp}(M)$ occur among the $\mathfrak p_i$ we win. $\square$

7.60. Associated primes

Here is the standard definition. For non-Noetherian rings and non-finite modules it may be more appropriate to use the definition in Section 7.63.

Definition 7.60.1. Let R be a ring. Let M be an R-module. A prime $\mathfrak{p}$ of R is *associated* to M if there exists an element $m \in M$ whose annihilator is $\mathfrak{p}$. The set of all such primes is denoted $\text{Ass}_R(M)$ or $\text{Ass}(M)$.

Lemma 7.60.2. *Let R be a ring. Let M be an R-module. Then $Ass(M) \subset Supp(M)$.*

Proof. If $m \in M$ has annihilator $\mathfrak{p}$, then in particular no element of $R \setminus \mathfrak{p}$ annihilates m. Hence m is a nonzero element of $M_{\mathfrak{p}}$, i.e., $\mathfrak{p} \in \text{Supp}(M)$. □

Lemma 7.60.3. *Let R be a ring. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of R-modules. Then $Ass(M') \subset Ass(M)$ and $Ass(M) \subset Ass(M') \cup Ass(M'')$.*

Proof. Omitted. □

Lemma 7.60.4. *Let R be a ring, and M an R-module. Suppose there exists a filtration by R-submodules*

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$$

such that each quotient M_i/M_{i-1} is isomorphic to $R/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$ of R. Then $Ass(M) \subset \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$.

Proof. By induction on the length n of the filtration $\{M_i\}$. Pick $m \in M$ whose annihilator is a prime $\mathfrak{p}$. If $m \in M_{n-1}$ we are done by induction. If not, then m maps to a nonzero element of $M/M_{n-1} \cong R/\mathfrak{p}_n$. Hence we have $\mathfrak{p} \subset \mathfrak{p}_n$. If equality does not hold, then we can find $f \in \mathfrak{p}_n$, $f \not\in \mathfrak{p}$. In this case the annihilator of fm is still $\mathfrak{p}$ and $fm \in M_{n-1}$. Thus we win by induction. □

Lemma 7.60.5. *Let R be a Noetherian ring. Let M be a finite R-module. Then $Ass(M)$ is finite.*

Proof. Immediate from Lemma 7.60.4 and Lemma 7.59.1. □

Proposition 7.60.6. *Let R be a Noetherian ring. Let M be a finite R-module. The following sets of primes are the same:*

1. *The minimal primes in the support of M.*
2. *The minimal primes in $Ass(M)$.*
3. *For any filtration $0 = M_0 \subset M_1 \subset \ldots \subset M_{n-1} \subset M_n = M$ with $M_i/M_{i-1} \cong R/\mathfrak{p}_i$ the minimal primes of the set $\{\mathfrak{p}_i\}$.*

Proof. Part of this we saw in Lemma 7.59.10. It suffices to prove that if $\mathfrak{p}$ is a minimal element of the set $\{\mathfrak{p}_i\}$ then it is the annihilator of an element of M. Let i be minimal such that $\mathfrak{p} = \mathfrak{p}_i$. Pick $m \in M_i$, $m \not\in M_{i-1}$. The annihilator of m is contained in $\mathfrak{p}_i = \mathfrak{p}$ and contains $\mathfrak{p}_1\mathfrak{p}_2 \ldots \mathfrak{p}_i$. By our choice of i we have $\mathfrak{p}_1\mathfrak{p}_2 \ldots \mathfrak{p}_{i-1} \not\subset \mathfrak{p}_i$. Pick $f \in \mathfrak{p}_1\mathfrak{p}_2 \ldots \mathfrak{p}_{i-1}$, $f \not\in \mathfrak{p}$. Then fm has annihilator $\mathfrak{p}$. □

Lemma 7.60.7. *Let R be a Noetherian ring. Let M be an R-module. Then*

$$M = (0) \Leftrightarrow Ass(M) = \emptyset.$$

Proof. If $M = (0)$, then $\text{Ass}(M) = \emptyset$ by definition. If $M \neq 0$, pick any nonzero finitely generated submodule $M' \subset M$, for example a submodule generated by a single nonzero element. By Lemma 7.59.3 we see that $\text{Supp}(M')$ is nonempty. By Proposition 7.60.6 this implies that $\text{Ass}(M')$ is nonempty. By Lemma 7.60.3 this implies $\text{Ass}(M) \neq \emptyset$. □

Lemma 7.60.8. *Let R be a Noetherian ring. Let M be an R-module. Any $\mathfrak{p} \in Supp(M)$ which is minimal among the elements of $Supp(M)$ is an element of $Ass(M)$.*

Proof. If M is a finite R-module, then this is a consequence of Proposition 7.60.6. In general write $M = \bigcup M_\lambda$ as the union of its finite submodules, and use that $\text{Supp}(M) = \bigcup \text{Supp}(M_\lambda)$ and $\text{Ass}(M) = \bigcup \text{Ass}(M_\lambda)$. □

Lemma 7.60.9. *Let R be a Noetherian ring. Let M be an R-module. The union $\bigcup_{\mathfrak{q} \in Ass(M)} \mathfrak{q}$ is the set of elements of R which are zero divisors on M.*

Proof. Any element in any associated prime clearly is a zero divisor on M. Conversely, suppose $x \in R$ is a zero divisor on M. Consider the submodule $N = \{m \in M \mid xm = 0\}$. Since N is not zero it has an associated prime $\mathfrak{q}$ by Lemma 7.60.7. Then $x \in \mathfrak{q}$ and $\mathfrak{q}$ is an associated prime of M by Lemma 7.60.3. □

Lemma 7.60.10. *Let $\varphi : R \to S$ be a ring map. Let M be an S-module. Then $Spec(\varphi)(Ass_S(M)) \subset Ass_R(M)$.*

Proof. If $\mathfrak{q} \in \text{Ass}_S(M)$, then there exists an m in M such that the annihilator of m in S is $\mathfrak{q}$. Then the annihilator of m in R is $\mathfrak{q} \cap R$. □

Remark 7.60.11. Let $\varphi : R \to S$ be a ring map. Let M be an S-module. Then it is not always the case that $Spec(\varphi)(\text{Ass}_S(M)) \supset \text{Ass}_R(M)$. For example, consider the ring map $R = k \to S = k[x_1, x_2, x_3, \ldots]/(x_i^2)$ and $M = S$. Then $\text{Ass}_R(M)$ is not empty, but $\text{Ass}_S(S)$ is empty.

Lemma 7.60.12. *Let $\varphi : R \to S$ be a ring map. Let M be an S-module. If S is Noetherian, then $Spec(\varphi)(Ass_S(M)) = Ass_R(M)$.*

Proof. We have already seen in Lemma 7.60.10 that $Spec(\varphi)(\text{Ass}_S(M)) \subset \text{Ass}_R(M)$. For the converse, choose a prime $\mathfrak{p} \in \text{Ass}_R(M)$. Let $m \in M$ be an element such that the annihilator of x in R is $\mathfrak{p}$. Let $I = \{g \in S \mid gm = 0\}$ be the annihilator of m in S. Then $R/\mathfrak{p} \subset S/I$ is injective, hence there exists a prime $\mathfrak{q} \subset S$ lying over $\mathfrak{p}$, see Lemma 7.27.5. By Proposition 7.60.6 we see that $\mathfrak{q}$ is an associated prime of S/I, hence an associated prime of M by Lemma 7.60.3 and we win. □

Lemma 7.60.13. *Let R be a ring. Let I be an ideal. Let M be an R/I-module. Via the canonical injection $Spec(R/I) \to Spec(R)$ we have $Ass_{R/I}(M) = Ass_R(M)$.*

Proof. Omitted. □

Lemma 7.60.14. *Let R be a ring. Let M be an R-module. Let $\mathfrak{p} \subset R$ be a prime.*

(1) *If $\mathfrak{p} \in Ass(M)$ then $\mathfrak{p}R_\mathfrak{p} \in Ass(M_\mathfrak{p})$.*
(2) *If $\mathfrak{p}$ is finitely generated then the converse holds as well.*

Proof. If $\mathfrak{p} \in \text{Ass}(M)$ there exists an element $m \in M$ whose annihilator is $\mathfrak{p}$. As localization is exact (Proposition 7.9.12) we see that the annihilator of $m/1$ in $M_\mathfrak{p}$ is $\mathfrak{p}R_\mathfrak{p}$ hence (1) holds. Assume $\mathfrak{p}R_\mathfrak{p} \in \text{Ass}(M_\mathfrak{p})$ and $\mathfrak{p} = (f_1, \ldots, f_n)$. Let m/g be an element of $M_\mathfrak{p}$ whose annihilator is $\mathfrak{p}R_\mathfrak{p}$. This implies that the annihilator of m is contained in $\mathfrak{p}$. As $f_i m/g = 0$ in $M_\mathfrak{p}$ we see there exists a $g_i \in R$, $g_i \notin \mathfrak{p}$ such that $g_i f_i m = 0$ in M. Combined we see the annihilator of $g_1 \ldots g_n m$ is $\mathfrak{p}$. Hence $\mathfrak{p} \in \text{Ass}(M)$. □

Lemma 7.60.15. *Let R be a ring. Let M be an R-module. Let $S \subset R$ be a multiplicative subset. Via the canonical injection $Spec(S^{-1}R) \to Spec(R)$ we have*

1. $Ass_R(S^{-1}M) = Ass_{S^{-1}R}(S^{-1}M)$,
2. $Ass_R(M) \cap Spec(S^{-1}R) \subset Ass_R(S^{-1}M)$, *and*
3. *if R is Noetherian this inclusion is an equality.*

Proof. The first equality follows, since if $m \in S^{-1}M$, then the annihilator of m in R is the intersection of the annihilator of m in $S^{-1}R$ with R. The displayed inclusion and equality in the Noetherian case follows from Lemma 7.60.14 since for $\mathfrak{p} \in R$, $S \cap \mathfrak{p} = \emptyset$ we have $M_{\mathfrak{p}} = (S^{-1}M)_{S^{-1}\mathfrak{p}}$. $\square$

Lemma 7.60.16. *Let R be a ring. Let M be an R-module. Let $S \subset R$ be a multiplicative subset. Assume that every $s \in S$ is a nonzero divisor on M. Then*

$$Ass_R(M) = Ass_R(S^{-1}M).$$

Proof. As $M \subset S^{-1}M$ by assumption we get the inclusion $\text{Ass}(M) = \text{Ass}(S^{-1}M)$ from Lemma 7.60.3. Conversely, suppose that $n/s \in S^{-1}M$ is an element whose annihilator is a prime ideal $\mathfrak{p}$. Then the annihilator of $n \in M$ is also $\mathfrak{p}$. $\square$

Lemma 7.60.17. *Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Let $I \subset \mathfrak{m}$ be an ideal. Let M be a finite R-module. The following are equivalent:*

1. *There exists an $x \in I$ which is not a zero divisor on M.*
2. *We have $I \not\subset \mathfrak{q}$ for all $\mathfrak{q} \in Ass(M)$.*

Proof. If there exists a nonzero divisor x in I, then x clearly cannot be in any associated prime of M. Conversely, suppose $I \not\subset \mathfrak{q}$ for all $\mathfrak{q} \in \text{Ass}(M)$. In this case we can choose $x \in I$, $x \notin \mathfrak{q}$ for all $\mathfrak{q} \in \text{Ass}(M)$ by Lemmas 7.60.5 and 7.14.3. By Lemma 7.60.9 the element x is not a zero divisor on M. $\square$

Lemma 7.60.18. *Let R be a ring. Let M be an R-module. If R is Noetherian the map*

$$M \longrightarrow \prod\nolimits_{\mathfrak{p} \in Ass(M)} M_{\mathfrak{p}}$$

is injective.

Proof. Let $x \in M$ be an element of the kernel of the map. Then if $\mathfrak{p}$ is an associated prime of $Rx \subset M$ we see on the one hand that $\mathfrak{p} \in \text{Ass}(M)$ (Lemma 7.60.3) and on the other hand that $(Rx)_{\mathfrak{p}} \subset M_{\mathfrak{p}}$ is not zero. This contradiction shows that $\text{Ass}(Rx) = \emptyset$. Hence $Rx = 0$ by Lemma 7.60.7. $\square$

7.61. Symbolic powers

We only make the following definition in the case of a Noetherian ring although the formula itself makes sense in general.

Definition 7.61.1. Let R be a Noetherian ring. Let $\mathfrak{p}$ be a prime ideal. Let $n \geq 1$. The nth *symbolic power* of $\mathfrak{p}$ is the ideal $\mathfrak{p}^{(n)} = R \cap \mathfrak{p}^n R_{\mathfrak{p}}$.

Note that $\mathfrak{p}^n \subset \mathfrak{p}^{(n)}$ but equality does not always hold.

Lemma 7.61.2. *Let R be a Noetherian ring. Let $\mathfrak{p}$ be a prime ideal. Let $n > 0$. Then $Ass(R/\mathfrak{p}^{(n)}) = \{\mathfrak{p}\}$.*

Proof. If $\mathfrak{q}$ is an associated prime of $R/\mathfrak{p}^{(n)}$ then clearly $\mathfrak{p} \subset \mathfrak{q}$. On the other hand, any element $x \in R$, $x \notin \mathfrak{p}$ is a nonzero divisor on $R/\mathfrak{p}^{(n)}$. Namely, if $y \in R$ and $xy \in \mathfrak{p}^{(n)} = R \cap \mathfrak{p}^n R_{\mathfrak{p}}$ then $y \in \mathfrak{p}^n R_{\mathfrak{p}}$, hence $y \in \mathfrak{p}^{(n)}$. Hence the lemma follows. $\square$

7.62. Relative assassin

Discussion of relative assassins. Let $R \to S$ be a ring map. Let N be an S-module. In this situation we can introduce the following sets of primes $\mathfrak{q}$ of S:

- A with $\mathfrak{p} = R \cap \mathfrak{q}$ we have that $\mathfrak{q} \in \text{Ass}_S(N \otimes_R \kappa(\mathfrak{p}))$,
- A' with $\mathfrak{p} = R \cap \mathfrak{q}$ we have that $\mathfrak{q}$ is in the image of $\text{Ass}_{S \otimes \kappa(\mathfrak{p})}(N \otimes_R \kappa(\mathfrak{p}))$ under the canonical map $Spec(S \otimes_R \kappa(\mathfrak{p})) \to Spec(S)$,
- A_{fin} with $\mathfrak{p} = R \cap \mathfrak{q}$ we have that $\mathfrak{q} \in \text{Ass}_S(N/\mathfrak{p}N)$,
- A'_{fin} for some prime $\mathfrak{p}' \subset R$ we have $\mathfrak{q} \in \text{Ass}_S(N/\mathfrak{p}'N)$,
- B for some R-module M we have $\mathfrak{q} \in \text{Ass}_S(N \otimes_R M)$, and
- B_{fin} for some finite R-module M we have $\mathfrak{q} \in \text{Ass}_S(N \otimes_R M)$.

Let us determine some of the relations between theses sets.

Lemma 7.62.1. *Let $R \to S$ be a ring map. Let N be an S-module. Let A, A', A_{fin}, B, and B_{fin} be the subsets of $Spec(S)$ introduced above.*

1. *We always have $A = A'$.*
2. *We always have $A_{fin} \subset A$, $B_{fin} \subset B$, $A_{fin} \subset A'_{fin} \subset B_{fin}$ and $A \subset B$.*
3. *If S is Noetherian, then $A = A_{fin}$ and $B = B_{fin}$.*
4. *If N is flat over R, then $A = A_{fin} = A'_{fin}$ and $B = B_{fin}$.*
5. *If R is Noetherian and N is flat over R, then all of the sets are equal, i.e., $A = A' = A_{fin} = A'_{fin} = B = B_{fin}$.*

Proof. Some of the arguments in the proof will be repeated in the proofs of later lemmas which are more precise than this one (because they deal with a given module M or a given prime $\mathfrak{p}$ and not with the collection of all of them).

Proof of (1). Let $\mathfrak{p}$ be a prime of R. Then we have

$$\text{Ass}_S(N \otimes_R \kappa(\mathfrak{p})) = \text{Ass}_{S/\mathfrak{p}S}(N \otimes_R \kappa(\mathfrak{p})) = \text{Ass}_{S \otimes_R \kappa(\mathfrak{p})}(N \otimes_R \kappa(\mathfrak{p}))$$

the first equality by Lemma 7.60.13 and the second by Lemma 7.60.15 part (1). This prove that $A = A'$. The inclusion $A_{fin} \subset A'_{fin}$ is clear.

Proof of (2). Each of the inclusions is immediate from the definitions except perhaps $A_{fin} \subset A$ which follows from Lemma 7.60.15 and the fact that we require $\mathfrak{p} = R \cap \mathfrak{q}$ in the formulation of A_{fin}.

Proof of (3). The equality $A = A_{fin}$ follows from Lemma 7.60.15 part (3) if S is Noetherian. Let $\mathfrak{q} = (g_1, \ldots, g_m)$ be a finitely generated prime ideal of S. Say $z \in N \otimes_R M$ is an element whose annihilator is $\mathfrak{q}$. We may pick a finite submodule $M' \subset M$ such that z is the image of $z' \in N \otimes_R M'$. Then $\text{Ann}_S(z') \subset \mathfrak{q} = \text{Ann}_S(z)$. Since $N \otimes_R -$ commutes with colimits and since M is the directed colimit of finite R-modules we can find $M' \subset M'' \subset M$ such that the image $z'' \in N \otimes_R M''$ is annihilated by $g_1, \ldots, g_m$. Hence $\text{Ann}_S(z'') = \mathfrak{q}$. This proves that $B = B_{fin}$ if S is Noetherian.

Proof of (4). If N is flat, then the functor $N \otimes_R -$ is exact. In particular, if $M' \subset M$, then $N \otimes_R M' \subset N \otimes_R M$. Hence if $z \in N \otimes_R M$ is an element whose annihilator $\mathfrak{q} = \text{Ann}_S(z)$ is a prime, then we can pick any finite R-submodule $M' \subset M$ such that $z \in N \otimes_R M'$ and we see that the annihilator of z as an element of $N \otimes_R M'$ is equal to $\mathfrak{q}$. Hence $B = B_{fin}$. Let $\mathfrak{p}'$ be a prime of R and let $\mathfrak{q}$ be a prime of S which is an associated prime of $N/\mathfrak{p}'N$. This implies that $\mathfrak{p}'S \subset \mathfrak{q}$. As N is flat over R we see that $N/\mathfrak{p}'N$ is flat over the integral domain $R/\mathfrak{p}'$. Hence every nonzero element of $R/\mathfrak{p}'$ is a nonzero divisor on $N/\mathfrak{p}'$. Hence none of these elements can map to an element of $\mathfrak{q}$ and we conclude that $\mathfrak{p}' = R \cap \mathfrak{q}$. Hence

$A_{fin} = A'_{fin}$. Finally, by Lemma 7.60.16 we see that $\text{Ass}_S(N/\mathfrak{p}'N) = \text{Ass}_S(N \otimes_R \kappa(\mathfrak{p}'))$, i.e., $A'_{fin} = A$.

Proof of (5). We only need to prove $A'_{fin} = B_{fin}$ as the other equalities have been proved in (4). To see this let M be a finite R-module. By Lemma 7.59.1 there exists a filtration by R-submodules

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$$

such that each quotient M_i/M_{i-1} is isomorphic to $R/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$ of R. Since N is flat we obtain a filtration by S-submodules

$$0 = N \otimes_R M_0 \subset N \otimes_R M_1 \subset \ldots \subset N \otimes_R M_n = N \otimes_R M$$

such that each subquotient is isomorphic to $N/\mathfrak{p}_i N$. By Lemma 7.60.3 we conclude that $\text{Ass}_S(N \otimes_R M) \subset \bigcup \text{Ass}_S(N/\mathfrak{p}_i N)$. Hence we see that $B_{fin} \subset A'_{fin}$. Since the other inclusion is part of (2) we win. $\square$

We define the relative assassin of N over S/R to be the set $A = A'$ above. As a motivation we point out that it depends only on the fibre modules $N \otimes_R \kappa(\mathfrak{p})$ over the fibre rings. As in the case of the assassin of a module we warn the reader that this notion makes most sense when the fibre rings $S \otimes_R \kappa(\mathfrak{p})$ are Noetherian, for example if $R \to S$ is of finite type.

Definition 7.62.2. Let $R \to S$ be a ring map. Let N be an S-module. The *relative assassin of N over S/R* is the set

$$\text{Ass}_{S/R}(N) = \{\mathfrak{q} \subset S \mid \mathfrak{q} \in \text{Ass}_S(N \otimes_R \kappa(\mathfrak{p})) \text{ with } \mathfrak{p} = R \cap \mathfrak{q}\}.$$

This is the set named A in Lemma 7.62.1.

The spirit of the next few results is that they are about the relative assassin, even though this may not apparant.

Lemma 7.62.3. *Let $R \to S$ be a ring map. Let M be an R-module, and let N be an S-module. If N is flat as R-module, then*

$$Ass_S(M \otimes_R N) \supset \bigcup\nolimits_{\mathfrak{p} \in Ass_R(M)} Ass_S(N/\mathfrak{p}N)$$

and if R is Noetherian then we have equality.

Proof. If $\mathfrak{p} \in \text{Ass}_R(M)$ then there exists an injection $R/\mathfrak{p} \to M$. As N is flat over R we obtain an injection $R/\mathfrak{p} \otimes_R N \to M \otimes_R M$. Since $R/\mathfrak{p} \otimes_R N = N/\mathfrak{p}N$ we conclude that $\text{Ass}_S(N/\mathfrak{p}N) \subset \text{Ass}_S(M \otimes_R N)$, see Lemma 7.60.3. Hence the right hand side is contained in the left hand side.

Write $M = \bigcup M_\lambda$ as the union of its finitely generated R-submodules. Then also $N \otimes_R M = \bigcup N \otimes_R M_\lambda$ (as N is R-flat). By definition of associated primes we see that $\text{Ass}_S(N \otimes_R M) = \bigcup \text{Ass}_S(N \otimes_R M_\lambda)$ and $\text{Ass}_R(M) = \bigcup \text{Ass}(M_\lambda)$. Hence we may assume M is finitely generated.

Let $\mathfrak{q} \in \text{Ass}_S(M \otimes_R N)$, and assume R is Noetherian and M is a finite R-module. To finish the proof we have to show that $\mathfrak{q}$ is an element of the right hand side. First we observe that $\mathfrak{q}S_\mathfrak{q} \in \text{Ass}_{S_\mathfrak{q}}((M \otimes_R N)_\mathfrak{q})$, see Lemma 7.60.14. Let $\mathfrak{p}$ be the corresponding prime of R. Note that

$$(M \otimes_R N)_\mathfrak{q} = M \otimes_R N_\mathfrak{q} = M_\mathfrak{p} \otimes_{R_\mathfrak{p}} N_\mathfrak{q}$$

If $\mathfrak{p}R_\mathfrak{p} \notin \text{Ass}_{R_\mathfrak{p}}(M_\mathfrak{p})$ then there exists an element $x \in \mathfrak{p}R_\mathfrak{p}$ which is a nonzero divisor in $M_\mathfrak{p}$ (see Lemma 7.60.17). Since $N_\mathfrak{q}$ is flat over $R_\mathfrak{p}$ we see that the image of x in $\mathfrak{q}S_\mathfrak{q}$ is a nonzero divisor on $(M \otimes_R N)_{\mathfrak q}$. This is a contradiction with the assumption that $\mathfrak q S_{\mathfrak q} \in \text{Ass}_S((M \otimes_R N)_{\mathfrak q})$. Hence we conclude that $\mathfrak p$ is one of the associated primes of M.

Continuing the argument we choose a filtration

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$$

such that each quotient M_i/M_{i-1} is isomorphic to $R/\mathfrak p_i$ for some prime ideal $\mathfrak p_i$ of R, see Lemma 7.59.1. (By Lemma 7.60.4 we have $\mathfrak p_i = \mathfrak p$ for at least one i.) This gives a filtration

$$0 = M_0 \otimes_R N \subset M_1 \otimes_R N \subset \ldots \subset M_n \otimes_R N = M \otimes_R N$$

with subquotients isomorphic to $N/\mathfrak p_i N$. If $\mathfrak p_i \neq \mathfrak p$ then $\mathfrak q$ cannot be associated to the module $N/\mathfrak p_i N$ by the result of the preceding paragraph (as $\text{Ass}_R(R/\mathfrak p_i) = \{\mathfrak p_i\}$). Hence we conclude that $\mathfrak q$ is associated to $N/\mathfrak p N$ as desired. □

Lemma 7.62.4. *Let $R \to S$ be a ring map. Let N be an S-module. Assume N is flat as an R-module and R is a domain with fraction field K. Then*

$$Ass_S(N) = Ass_S(N \otimes_R K) = Ass_{S \otimes_R K}(N \otimes_R K)$$

via the canonical inclusion $Spec(S \otimes_R K) \subset Spec(S)$.

Proof. Note that $S \otimes_R K = (R \setminus \{0\})^{-1} S$ and $N \otimes_R K = (R \setminus \{0\})^{-1} N$. For any nonzero $x \in R$ multiplication by x on N is injective as N is flat over R. Hence the lemma follows from Lemma 7.60.16 combined with Lemma 7.60.15 part (1). □

Lemma 7.62.5. *Let $R \to S$ be a ring map. Let M be an R-module, and let N be an S-module. Assume N is flat as R-module. Then*

$$Ass_S(M \otimes_R N) \supset \bigcup\nolimits_{\mathfrak p \in Ass_R(M)} Ass_{S \otimes_R \kappa(\mathfrak p)}(N \otimes_R \kappa(\mathfrak p))$$

where we use Remark 7.16.8 to think of the spectra of fibre rings as subsets of $Spec(S)$. If R is Noetherian then this inclusion is an equality.

Proof. This is equivalent to Lemma 7.62.3 by Lemmas 7.60.13, 7.35.6, and 7.62.4. □

Remark 7.62.6. Let $R \to S$ be a ring map. Let N be an S-module. Let $\mathfrak p$ be a prime of R. Then

$$\text{Ass}_S(N \otimes_R \kappa(\mathfrak p)) = \text{Ass}_{S/\mathfrak p S}(N \otimes_R \kappa(\mathfrak p)) = \text{Ass}_{S \otimes_R \kappa(\mathfrak p)}(N \otimes_R \kappa(\mathfrak p)).$$

The first equality by Lemma 7.60.13 and the second by Lemma 7.60.15 part (1).

7.63. Weakly associated primes

This is a variant on the notion of an associated prime that is useful for non-Noetherian ring and non-finite modules.

Definition 7.63.1. Let R be a ring. Let M be an R-module. A prime $\mathfrak p$ of R is *weakly associated* to M if there exists an element $m \in M$ such that $\mathfrak p$ is minimal among the prime ideals containing the annihilator $\text{Ann}(m) = \{f \in R \mid fm = 0\}$. The set of all such primes is denoted $\text{WeakAss}_R(M)$ or $\text{WeakAss}(M)$.

Thus an associated prime is a weakly associated prime. Here is a characterization in terms of the localization at the prime.

Lemma 7.63.2. *Let R be a ring. Let M be an R-module. Let $\mathfrak p$ be a prime of R. The following are equivalent:*

(1) *$\mathfrak p$ is weakly associated to M,*

(2) $\mathfrak{p}R_\mathfrak{p}$ *is weakly associated to* $M_\mathfrak{p}$, *and*
(3) $M_\mathfrak{p}$ *contains an element whose annihilator has radical equal to* $\mathfrak{p}R_\mathfrak{p}$.

Proof. Assume (1). Then there exists an element $m \in M$ such that $\mathfrak{p}$ is minimal among the primes containing the annihilator $I = \{x \in R \mid xm = 0\}$ of m. As localization is exact, the annihilator of m in $M_\mathfrak{p}$ is $I_\mathfrak{p}$. Hence $\mathfrak{p}R_\mathfrak{p}$ is a minimal prime of $R_\mathfrak{p}$ containing the annihilator $I_\mathfrak{p}$ of m in $M_\mathfrak{p}$. This implies (2) holds, and also (3) as it implies that $\sqrt{I_\mathfrak{p}} = \mathfrak{p}R_\mathfrak{p}$.

Applying the implication (1) ⇒ (3) to $M_\mathfrak{p}$ over $R_\mathfrak{p}$ we see that (2) ⇒ (3).

Finally, assume (3). This means there exists an element $m/f \in M_\mathfrak{p}$ whose annihilator has radical equal to $\mathfrak{p}R_\mathfrak{p}$. Then the annihilator $I = \{x \in R \mid xm = 0\}$ of m in M is such that $\sqrt{I_\mathfrak{p}} = \mathfrak{p}R_\mathfrak{p}$. Clearly this means that $\mathfrak{p}$ contains I and is minimal among the primes containing I, i.e., (1) holds. ☐

Lemma 7.63.3. *Let R be a ring. Let* $0 \to M' \to M \to M'' \to 0$ *be a short exact sequence of R-modules. Then* $WeakAss(M') \subset WeakAss(M)$ *and* $WeakAss(M) \subset WeakAss(M') \cup WeakAss(M'')$.

Proof. We will use the characterization of weakly associated primes of Lemma 7.63.2. Let $\mathfrak{p}$ be a prime of R. As localization is exact we obtain the short exact sequence $0 \to M'_\mathfrak{p} \to M_\mathfrak{p} \to M''_\mathfrak{p} \to 0$. Suppose that $m \in M_\mathfrak{p}$ is an element whose annihilator has radical $\mathfrak{p}R_\mathfrak{p}$. Then either the image $\overline{m}$ of m in $M''_\mathfrak{p}$ is zero and $m \in M'_\mathfrak{p}$, or the annihilator of $\overline{m}$ is $\mathfrak{p}R_\mathfrak{p}$. This proves that WeakAss(M) ⊂ WeakAss(M') ∪ WeakAss(M''). The inclusion WeakAss(M') ⊂ WeakAss(M) is immediate from the definitions. ☐

Lemma 7.63.4. *Let R be a ring. Let M be an R-module. Then*

$$M = (0) \Leftrightarrow WeakAss(M) = \emptyset$$

Proof. If $M = (0)$ then WeakAss(M) = ∅ by definition. Conversely, suppose that $M \neq 0$. Pick a nonzero element $m \in M$. Write $I = \{x \in R \mid xm = 0\}$ the annihilator of m. Then $R/I \subset M$. Hence WeakAss(R/I) ⊂ WeakAss(M) by Lemma 7.63.3. But as $I \neq R$ we have $V(I) = Spec(R/I)$ contains a minimal prime, see Lemmas 7.16.2 and 7.16.7, and we win. ☐

Lemma 7.63.5. *Let R be a ring. Let M be an R-module. Then*

$$Ass(M) \subset WeakAss(M) \subset Supp(M).$$

Proof. The first inclusion is immediate from the definitions. If $\mathfrak{p} \in$ WeakAss(M), then by Lemma 7.63.2 we have $M_\mathfrak{p} \neq 0$, hence $\mathfrak{p} \in$ Supp(M). ☐

Lemma 7.63.6. *Let R be a ring. Let M be an R-module. The union* $\bigcup_{\mathfrak{q} \in WeakAss(M)} \mathfrak{q}$ *is the set elements of R which are zero divisors on M.*

Proof. Suppose $f \in \mathfrak{q} \in$ WeakAss(M). Then there exists an element $m \in M$ such that $\mathfrak{q}$ is minimal over $I = \{x \in R \mid xm = 0\}$. Hence there exists a $g \in R$, $g \notin \mathfrak{p}$ and $n > 0$ such that $f^n gm = 0$. Note that $gm \neq 0$ as $g \notin I$. If we take n minimal as above, then $f(f^{n-1}gm) = 0$ and $f^{n-1}gm \neq 0$, so f is a zero divisor on M. Conversely, suppose $f \in R$ is a zero divisor on M. Consider the submodule $N = \{m \in M \mid fm = 0\}$. Since N is not zero it has a weakly associated prime $\mathfrak{q}$ by Lemma 7.63.4. Clearly $f \in \mathfrak{q}$ and by Lemma 7.63.3 $\mathfrak{q}$ is an associated prime of M. ☐

Lemma 7.63.7. *Let R be a ring. Let M be an R-module. Any* $\mathfrak{p} \in Supp(M)$ *which is minimal among the elements of* $Supp(M)$ *is an element of* $WeakAss(M)$.

Proof. Note that $\text{Supp}(M_{\mathfrak p}) = \{\mathfrak p R_{\mathfrak p}\}$ in $Spec(R_{\mathfrak p})$. In particular $M_{\mathfrak p}$ is nonzero, and hence $\text{WeakAss}(M_{\mathfrak p}) \neq \emptyset$ by Lemma 7.63.4. Since $\text{WeakAss}(M_{\mathfrak p}) \subset \text{Supp}(M_{\mathfrak p})$ by Lemma 7.63.5 we conclude that $\text{WeakAss}(M_{\mathfrak p}) = \{\mathfrak p R_{\mathfrak p}\}$, whence $\mathfrak p \in \text{WeakAss}(M)$ by Lemma 7.63.2. □

Lemma 7.63.8. *Let R be a ring. Let M be an R-module. Let $\mathfrak p$ be a prime ideal of R which is finitely generated. Then*

$$\mathfrak p \in Ass(M) \Leftrightarrow \mathfrak p \in WeakAss(M).$$

In particular, if R is Noetherian, then $Ass(M) = WeakAss(M)$.

Proof. Write $\mathfrak p = (g_1, \ldots, g_n)$ for some $g_i \in R$. It is enough the prove the implication ``$\Leftarrow$" as the other implication holds in general, see Lemma 7.63.5. Assume $\mathfrak p \in \text{WeakAss}(M)$. By Lemma 7.63.2 there exists an element $m \in M_{\mathfrak p}$ such that $I = \{x \in R_{\mathfrak p} \mid xm = 0\}$ has radical $\mathfrak p R_{\mathfrak p}$. Hence for each i there exists a smallest $e_i > 0$ such that $g_i^{e_i} m = 0$ in $M_{\mathfrak p}$. If $e_i > 1$ for some i, then we can replace m by $g_i^{e_i - 1} m \neq 0$ and decrease $\sum e_i$. Hence we may asssume that the annihilator of $m \in M_{\mathfrak p}$ is $(g_1, \ldots, g_n)R_{\mathfrak p} = \mathfrak p R_{\mathfrak p}$. By Lemma 7.60.14 we see that $\mathfrak p \in \text{Ass}(M)$. □

Remark 7.63.9. Let $\varphi : R \to S$ be a ring map. Let M be an S-module. Then it is not always the case that $Spec(\varphi)(\text{WeakAss}_S(M)) \subset \text{WeakAss}_R(M)$ contrary to the case of associated primes (see Lemma 7.60.10). An example is to consider the ring map

$$R = k[x_1, x_2, x_3, \ldots] \to S = k[x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots]/(x_1y_1, x_2y_2, x_3y_3, \ldots)$$

and $M = S$. In this case $\mathfrak q = \sum x_i S$ is a minimal prime of S, hence a weakly associated prime of $M = S$ (see Lemma 7.63.7). But on the other hand, for any nonzero element of S the annihilator in R is finitely generated, and hence does not have radical equal to $R \cap \mathfrak q = (x_1, x_2, x_3, \ldots)$ (details omitted).

Lemma 7.63.10. *Let $\varphi : R \to S$ be a ring map. Let M be an S-module. Then we have $Spec(\varphi)(WeakAss_S(M)) \supset WeakAss_R(M)$.*

Proof. Let $\mathfrak p$ be an element of $\text{WeakAss}_R(M)$. Then there exists an $m \in M_{\mathfrak p}$ whose annihilator $I = \{x \in R_{\mathfrak p} \mid xm = 0\}$ has radical $\mathfrak p R_{\mathfrak p}$. Consider the radical $J = \{x \in S_{\mathfrak p} \mid xm = 0\}$ of m in $S_{\mathfrak p}$. As $IS_{\mathfrak p} \subset J$ we see that any minimal prime $\mathfrak q \subset S_{\mathfrak p}$ over J lies over $\mathfrak p$. Moreover such a $\mathfrak q$ corresponds to a weakly associated prime of M for example by Lemma 7.63.2. □

Remark 7.63.11. Let $\varphi : R \to S$ be a ring map. Let M be an S-module. Denote $f : Spec(S) \to Spec(R)$ the associated map on spectra. Then we have

$$f(\text{Ass}_S(M)) \subset \text{Ass}_R(M) \subset \text{WeakAss}_R(M) \subset f(\text{WeakAss}_S(M))$$

see Lemmas 7.60.10, 7.63.10, and 7.63.5. In general all of the inclusions may be strict, see Remarks 7.60.11 and 7.63.9. If S is Noetherian, then all the inclusions are equalities as the outer two are equal by Lemma 7.63.8.

Lemma 7.63.12. *Let $\varphi : R \to S$ be a ring map. Let M be an S-module. Denote $f : Spec(S) \to Spec(R)$ the associated map on spectra. If φ is a finite ring map, then*

$$WeakAss_R(M) = f(WeakAss_S(M)).$$

Proof. One of the inclusions has already been proved, see Remark 7.63.11. To prove the other assume $\mathfrak q \in \text{WeakAss}_S(M)$ and let $\mathfrak p$ be the corresponding prime of R. Let $m \in M$ be an element such that $\mathfrak q$ is a minimal prime over $J = \{g \in S \mid gm = 0\}$. Thus the radical of

$JS_{\mathfrak q}$ is $\mathfrak q S_{\mathfrak q}$. As $R \to S$ is finite there are finitely many primes $\mathfrak q = \mathfrak q_1, \mathfrak q_2, \ldots, \mathfrak q_l$ over $\mathfrak p$, see Lemma 7.32.19. Pick $x \in \mathfrak q$ with $x \not\in \mathfrak q_i$ for $i > 1$, see Lemma 7.14.3. By the above there exists an element $y \in S$, $y \not\in \mathfrak q$ and an integer $t > 0$ such that $yx^t m = 0$. Thus the element $ym \in M$ is annihilated by x^t, hence ym maps to zero in $M_{\mathfrak q_i}$, $i = 2, \ldots, l$. To be sure, ym does not map to zero in $S_{\mathfrak q}$.

The ring $S_{\mathfrak p}$ is semi-local with maximal ideals $\mathfrak q_i S_{\mathfrak p}$ by going up for finite ring maps, see Lemma 7.32.20. If $f \in \mathfrak p R_{\mathfrak p}$ then some power of f ends up in $JS_{\mathfrak q}$ hence for some $n > 0$ we see that $f^t ym$ maps to zero in $M_{\mathfrak q}$. As ym vanishes at the other maximal ideals of $S_{\mathfrak p}$ we conclude that $f^t ym$ is zero in $M_{\mathfrak p}$, see Lemma 7.21.1. In this way we see that $\mathfrak p$ is a minimal prime over the annihilator of ym in R and we win. □

Lemma 7.63.13. *Let R be a ring. Let I be an ideal. Let M be an R/I-module. Via the canonical injection $Spec(R/I) \to Spec(R)$ we have $WeakAss_{R/I}(M) = WeakAss_R(M)$.*

Proof. Omitted. □

Lemma 7.63.14. *Let R be a ring. Let M be an R-module. Let $S \subset R$ be a multiplicative subset. Via the canonical injection $Spec(S^{-1}R) \to Spec(R)$ we have $WeakAss_R(S^{-1}M) = WeakAss_{S^{-1}R}(S^{-1}M)$ and*

$$WeakAss(M) \cap Spec(S^{-1}R) = WeakAss(S^{-1}M).$$

Proof. Suppose that $m \in S^{-1}M$. Let $I = \{x \in R \mid xm = 0\}$ and $I' = \{x' \in S^{-1}R \mid x'm = 0\}$. Then $I' = S^{-1}I$ and $I \cap S = \emptyset$ unless $I = R$ (verifications omitted). Thus primes in $S^{-1}R$ minimal over I' correspond bijectively to primes in R minimal over I and avoiding S. This proves the equality $\text{WeakAss}_R(S^{-1}M) = \text{WeakAss}_{S^{-1}R}(S^{-1}M)$. The second equality follows from Lemma 7.60.14 since for $\mathfrak p \in R$, $S \cap \mathfrak p = \emptyset$ we have $M_{\mathfrak p} = (S^{-1}M)_{S^{-1}\mathfrak p}$. □

Lemma 7.63.15. *Let R be a ring. Let M be an R-module. Let $S \subset R$ be a multiplicative subset. Assume that every $s \in S$ is a nonzero divisor on M. Then*

$$WeakAss(M) = WeakAss(S^{-1}M).$$

Proof. As $M \subset S^{-1}M$ by assumption we obtain $\text{WeakAss}(M) \subset \text{WeakAss}(S^{-1}M)$ from Lemma 7.63.3. Conversely, suppose that $n/s \in S^{-1}M$ is an element with annihilator I and $\mathfrak p$ a prime which is minimal over I. Then the annihilator of $n \in M$ is I and $\mathfrak p$ is a prime minimal over I. □

Lemma 7.63.16. *Let R be a ring. Let M be an R-module. The map*

$$M \longrightarrow \prod\nolimits_{\mathfrak p \in WeakAss(M)} M_{\mathfrak p}$$

is injective.

Proof. Let $x \in M$ be an element of the kernel of the map. Set $N = Rx \subset M$. If $\mathfrak p$ is a weakly associated prime of N we see on the one hand that $\mathfrak p \in \text{WeakAss}(M)$ (Lemma 7.63.3) and on the other hand that $N_{\mathfrak p} \subset M_{\mathfrak p}$ is not zero. This contradiction shows that $\text{WeakAss}(N) = \emptyset$. Hence $N = 0$, i.e., $x = 0$ by Lemma 7.63.4. □

Lemma 7.63.17. *Let $R \to S$ be a ring map. Let N be an S-module. Assume N is flat as an R-module and R is a domain with fraction field K. Then*

$$WeakAss_S(N) = WeakAss_{S \otimes_R K}(N \otimes_R K)$$

via the canonical inclusion $Spec(S \otimes_R K) \subset Spec(S)$.

Proof. Note that $S \otimes_R K = (R \setminus \{0\})^{-1}S$ and $N \otimes_R K = (R \setminus \{0\})^{-1}N$. For any nonzero $x \in R$ multiplication by x on N is injective as N is flat over R. Hence the lemma follows from Lemma 7.63.15. $\square$

7.64. Embedded primes

Here is the definition.

Definition 7.64.1. Let R be a ring. Let M be an R-module.

(1) The associated primes of M which are not minimal among the associated primes of M are called the *embedded associated primes* of M.
(2) The *embedded primes of* R are the embedded associated primes of R as an R-module.

Here is a way to get rid of these.

Lemma 7.64.2. *Let R be a Noetherian ring. Let M be a finite R-module. Consider the set of R-submodules*

$$\{K \subset M \mid Supp(K) \text{ nowhere dense in } Supp(M)\}.$$

This set has a maximal element K and the quotient $M' = M/K$ has the following properties

(1) *$Supp(M) = Supp(M')$,*
(2) *M' has no embedded associated primes,*
(3) *for any $f \in R$ which is contained in all embedded associated primes of M we have $M_f \cong M'_f$.*

Proof. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ denote the minimal primes in the support of M. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ denote the embedded associated primes of M. Then $\text{Ass}(M) = \{\mathfrak{q}_j, \mathfrak{p}_i\}$. There are finitely many of these, see Lemma 7.60.5. Set $I = \prod_{i=1,\ldots,s} \mathfrak{p}_i$. Then $I \not\subset \mathfrak{q}_j$ for any j. Hence by Lemma 7.14.3 we can find an $f \in I$ such that $f \not\in \mathfrak{q}_j$ for all $j = 1, \ldots, t$. Set $M' = \text{Im}(M \to M_f)$. This implies that $M_f \cong M'_f$. Since $M' \subset M_f$ we see that $\text{Ass}(M') \subset \text{Ass}(M_f) = \{\mathfrak{q}_j\}$. Thus M' has no embedded associated primes.

Moreover, the support of $K = \text{Ker}(M \to M')$ is contained in $V(\mathfrak{p}_1) \cup \ldots \cup V(\mathfrak{p}_s)$, because $\text{Ass}(K) \subset \text{Ass}(M)$ (see Lemma 7.60.3) and $\text{Ass}(K)$ contains none of the $\mathfrak{q}_i$ by construction. Clearly, K is in fact the largest submodule of M whose support is contained in $V(\mathfrak{p}_1) \cup \ldots \cup V(\mathfrak{p}_t)$. This implies that K is the maximal element of the set displayed in the lemma. $\square$

Lemma 7.64.3. *Let R be a Noetherian ring. Let M be a finite R-module. For any $f \in R$ we have $(M')_f = (M_f)'$ where $M \to M'$ and $M_f \to (M_f)'$ are the quotients constructed in Lemma 7.64.2.*

Proof. Omitted. $\square$

Lemma 7.64.4. *Let R be a Noetherian ring. Let M be a finite R-module without embedded associated primes. Let $I = \{x \in R \mid xM = 0\}$. Then the ring R/I has no embedded primes.*

Proof. We may replace R by R/I. Hence we may assume every nonzero element of R acts nontrivially on M. By Lemma 7.59.4 this implies that $Spec(R)$ equals the support of M. Suppose that $\mathfrak{p}$ is an embedded prime of R. Let $x \in R$ be an element whose annihilator is $\mathfrak{p}$. Consider the nonzero module $N = xM \subset M$. It is annihilated by $\mathfrak{p}$. Hence any associated prime $\mathfrak{q}$ of N contains $\mathfrak{p}$ and is also an associated prime of M. Then $\mathfrak{q}$ would be an embedded associated prime of M which contradicts the assumption of the lemma. $\square$

7.65. Regular sequences and depth

There is a characterization of depth in terms of Ext-groups that we will discuss in Section 7.67. Here we develop some basic properties of regular sequences and we prove the inequality between depth and dimension.

Definition 7.65.1. Let R be a ring. Let M be an R-module. A sequence of elements $f_1, \ldots, f_r$ of R is called *M-regular* if the following conditions hold:

(1) f_i is a nonzero divisor on $M/(f_1, \ldots, f_{r-1})M$ for each $i = 1, \ldots, r$, and
(2) the module $M/(f_1, \ldots, f_r)M$ is not zero.

If I is an ideal of R and $f_1, \ldots, f_r \in I$ then we call $f_1, \ldots, f_r$ a *M-regular sequence in I*. If $M = R$, we call $f_1, \ldots, f_r$ simply a *regular sequence* (in I).

Please pay attention to the fact that the definition depends on the order of the elements $f_1, \ldots, f_r$. Here are two examples.

Example 7.65.2. Let k be a field. In the ring $k[x, y, z]$ the sequence $x, y(1 - x), z(1 - x)$ is regular but the sequence $y(1 - x), z(1 - x), x$ is not.

Example 7.65.3. Let k be a field. Consider the ring $k[x, y, w_0, w_1, w_2, \ldots]/I$ where I is generated by yw_i, $i = 0, 1, 2, \ldots$ and $w_i - xw_{i+1}$, $i = 0, 1, 2, \ldots$. The sequence x, y is regular, but y is a zero divisor. Moreover you can localize at the maximal ideal (x, y, w_i) and still get an example.

Definition 7.65.4. Let R be a ring, and $I \subset R$ an ideal. Let M be an R-module. The *I-depth* of M is the supremum of the lengths of M-regular sequences in I; we denote it $\text{depth}_I(M)$. If $(R, \mathfrak{m})$ is local we call $\text{depth}_\mathfrak{m}(M)$ simply the *depth* of M.

Example 7.65.2 shows depth does not behave well even if the ring is Noetherian, and Example 7.65.3 shows that it does not behave well if the ring is local but non Noetherian. We will see later depth behaves well if the ring is local Noetherian. The following two lemmas are an indication of this.

Lemma 7.65.5. *Let R be a local Noetherian ring. Let M be a finite R-module. Let $x_1, \ldots, x_c$ be an M-regular sequence. Then any permutation of the x_i is a regular sequence as well.*

Proof. First we do the case $c = 2$. Consider $K \subset M$ the kernel of $x_2 : M \to M$. For any $z \in K$ we know that $z = x_1 z'$ for some $z' \in M$ because x_2 is a nonzero divisor on M/x_1M. Because x_1 is a nonzero divsor on M we see that $x_2 z' = 0$ as well. Hence $x_1 : K \to K$ is surjective. Thus $K = 0$ by Nakayama's Lemma 7.14.5. Next, consider multiplication by x_1 on M/x_2M. If $z \in M$ maps to an element $\overline{z} \in M/x_2M$ in the kernel of this map, then $x_1 z = x_2 y$ for some $y \in M$. But then since x_1, x_2 is a regular sequence we see that $y = x_1 y'$ for some $y' \in M$. Hence $x_1(z - x_2 y') = 0$ and hence $z = x_2 y'$ and hence $\overline{z} = 0$ as desired.

For the general case, observe that any permutation is a composition of transpositions of adjacent indices. Hence it suffices to prove that

$$x_1, \ldots, x_{i-2}, x_i, x_{i-1}, x_{i+1}, \ldots, x_c$$

is an M-regular sequence. This follows from the case we just did applied to the module $M/(x_1, \ldots, x_{i-2})$ and the length 2 regular sequence x_{i-1}, x_i. □

Lemma 7.65.6. *Let R be a Noetherian local ring. Let M be a finite R-module. Then* $\dim(\mathit{Supp}(M)) \geq \mathit{depth}(M)$.

Proof. By Lemma 7.59.11 it suffices to prove that if $f \in \mathfrak{m}$ is a nonzero divisor on M, then $d(M/fM) \leq d(M) - 1$. The existence of f shows that M does not have finite length. Consider the exact sequence

$$0 \to M \xrightarrow{f} M \to M/fM \to 0$$

and apply Lemma 7.56.9. It shows that $d(M/fM) < d(M)$. □

Here are a few more results on depth.

Lemma 7.65.7. *Let R, S be local rings. Let $R \to S$ be a flat local ring homomorphism. Let $x_1, \ldots, x_r$ be a sequence in R. Let M be an R-module. The following are equivalent*

(1) *$x_1, \ldots, x_r$ is an M-regular sequence in R, and*
(2) *the images of $x_1, \ldots, x_r$ in S form a $M \otimes_R S$-regular sequence.*

Proof. This is so because $R \to S$ is faithfully flat by Lemma 7.35.16. □

Lemma 7.65.8. *Let R be a Noetherian ring. Let M be a finite R-module. Let $\mathfrak{p}$ be a prime. Let $x_1, \ldots, x_r$ be a sequence in R whose image in $R_{\mathfrak{p}}$ forms an $M_{\mathfrak{p}}$-regular sequence. Then there exists a $g \in R$, $g \notin \mathfrak{p}$ such that the image of $x_1, \ldots, x_r$ in R_g forms an M_g-regular sequence.*

Proof. Set

$$K_i = \operatorname{Ker}\left(x_i : M/(x_1, \ldots, x_{i-1})M \to M/(x_1, \ldots, x_{i-1})M\right).$$

This is a finite R-module whose localization at $\mathfrak{p}$ is zero by assumption. Hence there exists a $g \in R$, $g \notin \mathfrak{p}$ such that $(K_i)_g = 0$ for all $i = 1, \ldots, r$. This g works. □

Lemma 7.65.9. *Let A be a ring. Let I be an ideal generated by a regular sequence $f_1, \ldots, f_n$ in A. Let $g_1, \ldots, g_m \in A$ be elements whose images $\overline{g}_1, \ldots, \overline{g}_m$ form a regular sequence in A/I. Then $f_1, \ldots, f_n, g_1, \ldots, g_m$ is a regular sequence in A.*

Proof. This follows immediately from the definitions. □

Lemma 7.65.10. *Let R be a ring. Let M be an R-module. Let $f_1, \ldots, f_r \in R$ be M-regula. Then for $e_1, \ldots, e_r > 0$ the sequence $f_1^{e_1}, \ldots, f_r^{e_r}$ is$M$-regular too.*

Proof. We will show that $f_1^e, f_2, \ldots, f_r$ is an M-regular sequence by induction on e. The case $e = 1$ is trivial. Since f_1 is a nonzero divisor we have a short exact sequence

$$0 \to M/f_1M \xrightarrow{f_1^{e-1}} M/f_1^eM \to M/f_1^{e-1}M \to 0$$

By induction the elements $f_2, \ldots, f_r$ are M/f_1M and $M/f_1^{e-1}M$-regular sequences. It follows from the snake lemma that they are also M/f_1^eM-regular sequences. □

Lemma 7.65.11. *Let R be a ring. Let $f_1, \ldots, f_r \in R$ which do not generate the unit ideal. The following are equivalent:*

(1) *any permutation of $f_1, \ldots, f_r$ is a regular sequence,*
(2) *any subsequence of $f_1, \ldots, f_r$ (in the given order) is a regular sequence, and*
(3) *$f_1x_1, \ldots, f_rx_r$ is a regular sequence in the polynomial ring $R[x_1, \ldots, x_r]$.*

Proof. It is clear that (1) implies (2). We prove (2) implies (1) by induction on r. The case $r = 1$ is trivial. The case $r = 2$ says that if $a, b \in R$ are a regular sequence and b is a nonzero divisor, then b, a is a regular sequence. This is clear because the kernel of $a : R/(b) \to R/(b)$ is isomorphic to the kernel of $b : R/(a) \to R/(a)$ if both a and b are nonzero divisors. The case $r > 2$. Assume (2) holds and say we want to prove $f_{\sigma(1)}, \ldots, f_{\sigma(r)}$ is a regular sequence for some permutation σ. We already know that $f_{\sigma(1)}, \ldots, f_{\sigma(r-1)}$ is a regular sequence by induction. Hence it suffices to show that f_s where $s = \sigma(r)$ is a nonzero divisor modulo $f_1, \ldots, \hat{f}_s, \ldots, f_r$. If $s = r$ we are done. If $s < r$, then note that f_s and f_r are both nonzero divisors in the ring $R/(f_1, \ldots, \hat{f}_s, \ldots, f_{r-1})$ (by induction hypothesis again). Since we know f_s, f_r is a regular sequence in that ring we conclude by the case of sequence of length 2 that f_r, f_s is too.

Note that $R[x_1, \ldots, x_r]/(f_1x_1, \ldots, f_ix_i)$ as an R-module is a direct sum of the modules

$$R/I_E \cdot x_1^{e_1} \ldots x_r^{e_r}$$

indexed by multi-indices $E = (e_1, \ldots, e_r)$ where I_E is the ideal generated by $f_j^{e_j}$ for $1 \leq j \leq i$ with $e_j > 0$. Hence $f_{i+1}x_i$ is a nonzero divisor on this if and only if the maps f_{i+1} is a nonzero divisor on R/I_E for all E. Thus it is clear that (3) implies (2) by taking $e_1, \ldots, e_i \in \{0, 1\}$. Conversely, if (2) holds, then any sequence of the form $f_1^{e_1}, \ldots, f_i^{e_i}, f_{i+1}$ (but omitting those powers with zero exponent) is a regular sequence by Lemma 7.65.10, i.e., f_{i+1} is a nonzero divisor on R/I_E. □

7.66. Quasi-regular sequences

There is a notion of regular sequence which is slightly weaker than that of a regular sequence and easier to use. Let R be a ring and let $f_1, \ldots, f_c \in R$. Set $J = (f_1, \ldots, f_c)$. Let M be an R-module. Then there is a canonical map

$$M/JM \otimes_{R/J} R/J[X_1, \ldots, X_c] \longrightarrow \bigoplus\nolimits_{n \geq 0} J^n M/J^{n+1}M \tag{7.66.0.1}$$

of graded $R/J[X_1, \ldots, X_c]$-modules defined by the rule

$$\overline{m} \otimes X_1^{e_1} \ldots X_c^{e_c} \longmapsto f_1^{e_1} \ldots f_c^{e_c} m \bmod J^{e_1+\ldots+e_c+1}M.$$

Note that (7.66.0.1) is always surjective.

Definition 7.66.1. Let R be a ring. Let M be an R-module. A sequence of elements $f_1, \ldots, f_c$ of R is called *M-quasi-regular* if (7.66.0.1) is an isomorphism. If $M = R$, we call $f_1, \ldots, f_c$ simply a *quasi-regular sequence*.

So if $f_1, \ldots, f_c$ is a quasi-regular sequence, then

$$R/J[X_1, \ldots, X_c] = \bigoplus\nolimits_{n \geq 0} J^n/J^{n+1}$$

where $J = (f_1, \ldots, f_c)$. It is clear that being a quasi-regular sequence is independent of the order of $f_1, \ldots, f_c$.

Lemma 7.66.2. *Let R be a ring.*

1. *A regular sequence $f_1, \ldots, f_c$ of R is a quasi-regular sequence.*
2. *Suppose that M is an R-module and that $f_1, \ldots, f_c$ is an M-regular sequence. Then $f_1, \ldots, f_c$ is an M-quasi-regular sequence.*

Proof. Set $J = (f_1, \ldots, f_c)$. We prove the first assertion by induction on c. We have to show that given any relation $\sum_{|I|=n} a_I f^I \in J^{n+1}$ with $a_I \in R$ we actually have $a_I \in J$ for all multi-indices I. Since any element of J^{n+1} is of the form $\sum_{|I|=n} b_I f^I$ with $b_I \in J$ we may assume, after replacing a_I by $a_I - b_I$, the relation reads $\sum_{|I|=n} a_I f^I = 0$. We can rewrite this as

$$\sum\nolimits_{e=0}^{n} \left(\sum\nolimits_{|I'|=n-e} a_{I',e} f^{I'} \right) f_c^e = 0$$

Here and below the ``primed'' multi-indices I' are required to be of the form $I' = (i_1, \ldots, i_{d-1}, 0)$. We will show by descending induction on $l \in \{0, \ldots, n\}$ that if we have a relation

$$\sum\nolimits_{e=0}^{l} \left(\sum\nolimits_{|I'|=n-e} a_{I',e} f^{I'}\right) f_c^e = 0$$

then $a_{I',e} \in J$ for all I', e. Namely, set $J' = (f_1, \ldots, f_{c-1})$. We observe that $\sum_{|I'|=n-l} a_{I',l} f^{I'}$ is mapped into J' by f_c^l and hence (because f_c is not a zero divisor on R/J') it is in J'. By induction hypotheses (for the induction on c), we see that $a_{I',l} \in J'$. This allows us to rewrite the term $(\sum_{|I'|=n-l} a_{I',l} f^{I'}) f_c^l$ in the form $(\sum_{|I'|=n-l+1} f_c b_{I',l-1} f^{I'}) f_c^{l-1}$. This gives a new relation of the form

$$\sum\nolimits_{|I'|=n-l+1} (a_{I',l-1} + f_c b_{I',l-1}) f^{I'}) f_c^{l-1} + \sum\nolimits_{e=0}^{l-2} \left(\sum\nolimits_{|I'|=n-e} a_{I',e} f^{I'}\right) f_c^e = 0$$

Now by the induction hypothesis (on l this time) we see that all $a_{I',l-1} + f_c b_{I',l-1} \in J$ and all $a_{I',e} \in J$ for $e \leq l-2$. This, combined with $a_{I',l} \in J' \subset J$ seen above, finishes the proof of the induction step.

The second assertion means that given any formal expression $F = \sum_{|I|=n} m_I X^I$, $m_I \in M$ with $\sum m_I f^I \in J^{n+1}M$, then all the coefficients m_I are in J. This is proved in exactly the same way as we prove the corresponding result for the first assertion above. $\square$

Lemma 7.66.3. *Let $R \to R'$ be a flat ring map. Let M be an R-module. Suppose that $f_1, \ldots, f_r \in R$ form an M-quasi-regular sequence. Then the images of $f_1, \ldots, f_r$ in R' form a $M \otimes_R R'$-quasi-regular sequence.*

Proof. Set $J = (f_1, \ldots, f_r)$, $J' = JR'$ and $M' = M \otimes_R R'$. Because $R \to S$ is flat the sequences $0 \to J^{n+1}M \to J^n M \to J^n M/J^{n+1}M \to 0$ remain exact on tensoring with S. Hence $(J')^n M'/(J')^{n+1}M' = J^n M/J^{n+1}M \otimes_R R'$. Thus the isomorphism $J^n/J^{n+1} \otimes_R M \to J^n M/J^{n+1}M$ gives rise to the corresponding isomorphism for M'. $\square$

Lemma 7.66.4. *Let R be a Noetherian ring. Let M be a finite R-module. Let $\mathfrak p$ be a prime. Let $x_1, \ldots, x_c$ be a sequence in R whose image in $R_{\mathfrak p}$ forms an $M_{\mathfrak p}$-quasi-regular sequence. Then there exists a $g \in R$, $g \not\in \mathfrak p$ such that the image of $x_1, \ldots, x_c$ in R_g forms an M_g-quasi-regular sequence.*

Proof. Consider the kernel K of the map (7.66.0.1). As $M/JM \otimes_{R/J} R/J[X_1, \ldots, X_c]$ is a finite $R/J[X_1, \ldots, X_c]$-module and as $R/J[X_1, \ldots, X_c]$ is Noetherian, we see that K is also a finite $R/J[X_1, \ldots, X_c]$-module. Pick homogeneous generators $k_1, \ldots, k_t \in K$. By assumption for each $i = 1, \ldots, t$ there exists a $g_i \in R$, $g_i \not\in \mathfrak p$ such that $g_i k_i = 0$. Hence $g = g_1 \ldots g_t$ works. $\square$

Lemma 7.66.5. *Let R be a ring. Let M be an R-module. Let $f_1, \ldots, f_c \in R$ be an M-quasi-regular sequence. For any i the sequence $\overline{f}_{i+1}, \ldots, \overline{f}_c$ of $\overline{R} = R/(f_1, \ldots, f_i)$ is an $\overline{M} = M/(f_1, \ldots, f_i)M$-quasi-regular sequence.*

Proof. It suffices to prove this for $i = 1$. Set $\overline{J} = (\overline{f}_2, \ldots, \overline{f}_c) \subset \overline{R}$. Then

$$\begin{aligned} \overline{J}^n \overline{M}/\overline{J}^{n+1}\overline{M} &= (J^n M + f_1 M)/(J^{n+1}M + f_1 M) \\ &= J^n M/(J^{n+1}M + J^n M \cap f_1 M). \end{aligned}$$

Thus, in order to prove the lemma it suffices to show that $J^{n+1}M + J^n M \cap f_1 M = J^{n+1}M + f_1 J^{n-1}M$ because that will show that $\bigoplus_{n \geq 0} \overline{J}^n \overline{M}/\overline{J}^{n+1}\overline{M}$ is the quotient of $\bigoplus_{n \geq 0} J^n M/J^{n+1} \cong$

$M/JM[X_1, \ldots, X_c]$ by X_1. Actually, we have $J^nM \cap f_1M = f_1J^{n-1}M$. Namely, if $m \notin J^{n-1}M$, then $f_1m \notin J^nM$ because $\bigoplus J^nM/J^{n+1}M$ is the polynomial algebra $M/J[X_1, \ldots, X_c]$ by assumption. □

Lemma 7.66.6. *Let $(R, \mathfrak{m})$ be a local Noetherian ring. Let M be a nonzero finite R-module. Let $f_1, \ldots, f_c \in \mathfrak{m}$ be an M-quasi-regular sequence. Then $f_1, \ldots, f_c$ is an M-regular sequence.*

Proof. Set $J = (f_1, \ldots, f_c)$. Let us show that f_1 is a nonzero divisor on M. Suppose $x \in M$ is not zero. By the Artin-Rees lemma there exists an integer r such that $x \in J^rM$ but $x \notin J^{r+1}M$, see Lemma 7.47.6. Then $f_1x \in J^{r+1}M$ is an element whose class in $J^{r+1}M/J^{r+2}M$ is nonzero by the assumed structure of $\bigoplus J^nM/J^{n+1}M$. Whence $f_1x \neq 0$.

Now we can finish the proof by induction on c using Lemma 7.66.5. □

Remark 7.66.7 (Koszul regular sequences)**.** In the paper [**Kab71**] the author introduces two more regularity conditions for sequences $x_1, \ldots, x_r$ of elements of a ring R. Namely, we say the sequence is *Koszul-regular* if $H_i(K_\bullet(R, x_\bullet)) = 0$ for $i \geq 1$ where $K_\bullet(R, x_\bullet)$ is the Koszul complex. The sequence is called *H_1-regular* if $H_1(K_\bullet(R, x_\bullet)) = 0$. If R is a local ring (possibly nonnoetherian) and the sequence consists of elements of the maximal ideal, then one has the implications regular $\Rightarrow$ Koszul-regular $\Rightarrow$ H_1-regular $\Rightarrow$ quasi-regular. By examples the author shows that these implications cannot be reversed in general. We introduce these notions in more detail in More on Algebra, Section 12.22.

Remark 7.66.8. Let k be a field. Consider the ring

$$A = k[x, y, w, z_0, z_1, z_2, \ldots]/(y^2z_0 - wx, z_0 - yz_1, z_1 - yz_2, \ldots)$$

In this ring x is a nonzero divisor and the image of y in A/xA gives a quasi-regular sequence. But it is not true that x, y is a quasi-regular sequence in A because $(x, y)/(x, y)^2$ isn't free of rank two over $A/(x, y)$ due to the fact that $wx = 0$ in $(x, y)/(x, y)^2$ but w isn't zero in $A/(x, y)$. Hence the analogue of Lemma 7.65.9 does not hold for quasi-regular sequences.

Lemma 7.66.9. *Let R be a ring. Let $J = (f_1, \ldots, f_r)$ be an ideal of R. Let M be an R-module. Set $\overline{R} = R/\bigcap_{n\geq 0} J^n$, $\overline{M} = M/\bigcap_{n\geq 0} J^nM$, and denote $\overline{f}_i$ the image of f_i in $\overline{R}$. Then $f_1, \ldots, f_r$ is M-quasi-regular if and only if $\overline{f}_1, \ldots, \overline{f}_r$ is $\overline{M}$-quasi-regular.*

Proof. This is true because $J^nM/J^{n+1}M \cong \overline{J}^n\overline{M}/\overline{J}^{n+1}\overline{M}$. □

7.67. Ext groups and depth

In this section we do a tiny bit of homological algebra, in order to establish some fundamental properties of depth over Noetherian local rings.

Lemma 7.67.1. *Let R be a ring. Let M be an R-module.*

(1) *The exists an exact complex*
$$\ldots \to F_2 \to F_1 \to F_0 \to M \to 0.$$
with F_i free R-modules.

(2) *If R is Noetherian and M finite R, then we choose the complex such that each F_i is finite free. In other words, we may find an exact complex*
$$\ldots \to R^{n_2} \to R^{n_1} \to R^{n_0} \to M \to 0.$$

Proof. Let us explain only the Noetherian case. As a first step choose a surjection $R^{n_0} \to M$. Then having constructed an exact complex of length e we simply choose a surjection $R^{n_{e+1}} \to \text{Ker}(R^{n_e} \to R^{n_{e-1}})$ which is possible because R is Noetherian. □

Definition 7.67.2. Let R be a ring. Let M be an R-module.

(1) A (left) *resolution* $F_\bullet \to M$ of M is an exact complex
$$\ldots \to F_2 \to F_1 \to F_0 \to M \to 0$$
of R-modules.

(2) A *resolution of M by free R-modules* is a resolution $F_\bullet \to M$ where each F_i is a free R-module.

(3) A *resolution of M by finite free R-modules* is a resolution $F_\bullet \to M$ where each F_i is a finite free R-module.

We often use the notation $F_\bullet$ to denote a complex of R-modules
$$\ldots \to F_i \to F_{i-1} \to \ldots$$
In this case we often use d_i or $d_{F,i}$ to denote the map $F_i \to F_{i-1}$. In this section we are always going to assume that F_0 is the last nonzero term in the complex. The *ith homology group of the complex $F_\bullet$* is the group $H_i = \text{Ker}(d_{F,i})/\text{Im}(d_{F,i+1})$. A *map of complexes* $\alpha : F_\bullet \to G_\bullet$ is given by maps $\alpha_i : F_i \to G_i$ such that $\alpha_{i-1} \circ d_{F,i} = d_{G,i-1} \circ \alpha_i$. Such a map induces a map on homology $H_i(\alpha) : H_i(F_\bullet) \to H_i(G_\bullet)$. If $\alpha, \beta : F_\bullet \to G_\bullet$ are maps of complexes, then a *homotopy* between α and β is given by a collection of maps $h_i : F_i \to G_{i+1}$ such that $\alpha_i - \beta_i = d_{G,i+1} \circ h_i + h_{i-1} \circ d_{F,i}$.

We will use a very similar notation regarding complexes of the form $F^\bullet$ which look like
$$\ldots \to F^i \xrightarrow{d^i} F^{i+1} \to \ldots$$
There are maps of complexes, homotopies, etc. In this case we set $H^i(F^\bullet) = \text{Ker}(d^i)/\text{Im}(d^{i-1})$ and we call it the *ith cohomology group*.

Lemma 7.67.3. *Any two homotopic maps of complexes induce the same maps on (co)homology groups.*

Proof. Omitted. □

Lemma 7.67.4. *Let R be a ring. Let $M \to N$ be a map of R-modules. Let $F_\bullet \to M$ be a resolution by free R-modules and let $N_\bullet \to N$ be an arbitrary resolution. Then*

(1) *there exists a map of complexes $F_\bullet \to N_\bullet$ inducing the given map*
$$M = Coker(F_1 \to F_0) \to Coker(N_1 \to N_0) = N$$

(2) *two maps $\alpha, \beta : F_\bullet \to N_\bullet$ inducing the same map $M \to N$ are homotopic.*

Proof. Proof of (1). Because F_0 is free we can find a map $F_0 \to N_0$ lifting the map $F_0 \to M \to N$. We obtain an induced map $F_1 \to F_0 \to N_0$ wich ends up in the image of $N_1 \to N_0$. Since F_1 is free we may lift this to a map $F_1 \to N_1$. This in turn induces a map $F_2 \to F_1 \to N_1$ which maps to zero into N_0. Since $N_\bullet$ is exact we see that the image of this map is contained in the image of $N_2 \to N_1$. Hence we may lift to get a map $F_2 \to N_2$. Repeat.

Proof of (2). To show that α, β are homotopic it suffices to show the difference $\gamma = \alpha - \beta$ is homotopic to zero. Note that the image of $\gamma_0 : F_0 \to N_0$ is contained in the image of $N_1 \to N_0$. Hence we may lift γ_0 to a map $h_0 : F_0 \to N_1$. Consider the map $\gamma_1' =$

$\gamma_1 - h_0 \circ d_{F,1}$. By our choice of h_0 we see that the image of γ_1' is contained in the kernel of $N_1 \to N_0$. Since $N_\bullet$ is exact we may lift γ_1' to a map $h_1 : F_1 \to N_2$. At this point we have $\gamma_1 = h_0 \circ d_{F,1} + d_{N,2} \circ h_1$. Repeat. □

At this point we are ready to define the groups $\text{Ext}^i_R(M, N)$. Namely, choose a resolution $F_\bullet$ of M by free R-modules, see Lemma 7.67.1. Consider the (cohomological) complex

$$Hom_R(F_\bullet, N) : Hom_R(F_0, N) \to Hom_R(F_1, N) \to Hom_R(F_2, N) \to \dots$$

We define $\text{Ext}^i_R(M, N)$ to be the ith cohomology group of this complex.[3] The following lemma explains in what sense this is well defined.

Lemma 7.67.5. *Let R be a ring. Let M_1, M_2, N be R-modules. Suppose that $F_\bullet$ is a free resolution of the module M_1, and $G_\bullet$ is a free resolution of the module M_2. Let $\varphi : M_1 \to M_2$ be a module map. Let $\alpha : F_\bullet \to G_\bullet$ be a map of complexes inducing φ on $M_1 = Coker(d_{F,1}) \to M_2 = Coker(d_{G,1})$, see Lemma 7.67.4. Then the induced maps*

$$H^i(\alpha) : H^i(Hom_R(F_\bullet, N)) \longrightarrow H^i(Hom_R(G_\bullet, N))$$

are independent of the choice of α. If φ is an isomorphism, so are all the maps $H^i(\alpha)$. If $M_1 = M_2$, $F_\bullet = G_\bullet$, and φ is the identity, so are all the maps $H_i(\alpha)$.

Proof. Another map $\beta : F_\bullet \to G_\bullet$ inducing φ is homotopic to α by Lemma 7.67.4. Hence the maps $Hom_R(F_\bullet, N) \to Hom_R(G_\bullet, N)$ are homotopic. Hence the independence result follows from Lemma 7.67.3.

Suppose that φ is an isomorphism. Let $\psi : M_2 \to M_1$ be an inverse. Choose $\beta : G_\bullet \to F_\bullet$ be a map inducing $\psi : M_2 = \text{Coker}(d_{G,1}) \to M_1 = \text{Coker}(d_{F,1})$, see Lemma 7.67.4. Ok, and now consider the map $H^i(\alpha) \circ H^i(\beta) = H^i(\alpha \circ \beta)$. By the above the map $H^i(\alpha \circ \beta)$ is the *same* as the map $H^i(\text{id}_{G_\bullet}) = \text{id}$. Similarly for the composition $H^i(\beta) \circ H^i(\alpha)$. Hence $H^i(\alpha)$ and $H^i(\beta)$ are inverses of each other. □

Lemma 7.67.6. *Let R be a ring. Let M be an R-module. Let $0 \to N' \to N \to N'' \to 0$ be a short exact sequence. Then we get a long exact sequence*

$$\begin{aligned} 0 \to Hom_R(M, N') \to Hom_R(M, N) \to Hom_R(M, N'') \\ \to Ext^1_R(M, N') \to Ext^1_R(M, N) \to Ext^1_R(M, N'') \to \dots \end{aligned}$$

Proof. Pick a free resolution $F_\bullet \to M$. Since each of the F_i are free we see that we get a short exact sequence of complexes

$$0 \to Hom_R(F_\bullet, N') \to Hom_R(F_\bullet, N) \to Hom_R(F_\bullet, N'') \to 0$$

Thus we get the long exact sequence from the snake lemma applied to this. □

Lemma 7.67.7. *Let R be a ring. Let N be an R-module. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence. Then we get a long exact sequence*

$$\begin{aligned} 0 \to Hom_R(M'', N) \to Hom_R(M, N) \to Hom_R(M', N) \\ \to Ext^1_R(M'', N) \to Ext^1_R(M, N) \to Ext^1_R(M', N) \to \dots \end{aligned}$$

Proof. Pick sets of generators $\{m'_{i'}\}_{i' \in I'}$ and $\{m''_{i''}\}_{i'' \in I''}$ of M' and M''. For each $i'' \in I''$ choose a lift $\tilde{m}''_{i''} \in M$ of the element $m''_{i''} \in M''$. Set $F' = \bigoplus_{i' \in I'} R$, $F'' = \bigoplus_{i'' \in I''} R$ and $F = F' \oplus F''$. Mapping the generators of these free modules to the corresponding chosen

[3] At this point it would perhaps be more appropriate to say ``an'' in stead of ``the'' Ext-group.

generators gives surjective R-module maps $F' \to M'$, $F'' \to M''$, and $F \to M$. We obtain a map of short exact sequences

$$\begin{array}{ccccccccc} 0 & \to & M' & \to & M & \to & M'' & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & F' & \to & F & \to & F'' & \to & 0 \end{array}$$

By the snake lemma we see that the sequence of kernels $0 \to K' \to K \to K'' \to 0$ is short exact sequence of R-modules. Hence we can continue this process indefinitively. In other words we obtain a short exact sequence of resolutions fitting into the diagram

$$\begin{array}{ccccccccc} 0 & \to & M' & \to & M & \to & M'' & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & F'_\bullet & \to & F_\bullet & \to & F''_\bullet & \to & 0 \end{array}$$

Because each of the sequences $0 \to F'_n \to F_n \to F''_n \to 0$ is split exact (by construction) we obtain a short exact sequence of complexes

$$0 \to Hom_R(F''_\bullet, N) \to Hom_R(F_\bullet, N) \to Hom_R(F'_\bullet, N) \to 0$$

by applying the $Hom_R(-, N)$ functor. Thus we get the long exact sequence from the snake lemma applied to this. $\square$

Lemma 7.67.8. *Let R be a ring. Let M, N be R-modules. Any $x \in R$ such that either $xN = 0$, or $xM = 0$ annihilates each of the modules $Ext^i_R(M, N)$.*

Proof. Pick a free resolution $F_\bullet$ of M. Since $\text{Ext}^i_R(M, N)$ is defined as the cohomology of the complex $Hom_R(F_\bullet, N)$ the lemma is clear when $xN = 0$. If $xM = 0$, then we see that multiplication by x on $F_\bullet$ lifts the zero map on M. Hence by Lemma 7.67.5 we see that it induces the same map on Ext groups as the zero map. $\square$

Lemma 7.67.9. *Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Let M be a finite R-module. Then $depth_R(M)$ is equal to the smallest integer i such that $Ext^i_R(R/\mathfrak{m}, M)$ is nonzero.*

Proof. Let $\delta(M)$ denote the depth of M and let $i = i(M)$ denote the smallest integer such that $\text{Ext}^i_R(R/\mathfrak{m}, M)$ is nonzero. We will see in a moment that $i(M) < \infty$. By Lemma 7.60.17 we have $\delta(M) = 0$ if and only if $i(M) = 0$, because $\mathfrak{m} \in \text{Ass}(M)$ exactly means that $i(M) = 0$. Hence if $\delta(M)$ or $i(M)$ is > 0, then we may choose $x \in \mathfrak{m}$ such that (a) x is a nonzero divisor on M, and (b) $\text{depth}(M/xM) = \delta(M) - 1$. Consider the long exact sequence of Ext-groups associated to the short exact sequence $0 \to M \to M \to M/xM \to 0$ by Lemma 7.67.6:

$$\begin{array}{l} 0 \to Hom_R(\kappa, M) \to Hom_R(\kappa, M) \to Hom_R(\kappa, M/xM) \\ \quad \to \text{Ext}^1_R(\kappa, M) \to \text{Ext}^1_R(\kappa, M) \to \text{Ext}^1_R(\kappa, M/xM) \to \ldots \end{array}$$

Since $x \in \mathfrak{m}$ all the maps $\text{Ext}^i_R(\kappa, M) \to \text{Ext}^i_R(\kappa, M)$ are zero, see Lemma 7.67.8. Thus it is clear that $i(M/xM) = i(M) - 1$. Induction, e.g., on $\dim(\text{Support}(M))$, finishes the proof. $\square$

Lemma 7.67.10. *Let R be a local Noetherian ring. Let $0 \to N' \to N \to N'' \to 0$ be a short exact sequence of finite R-modules.*

1. *$depth(N'') \geq \min\{depth(N), depth(N') - 1\}$*
2. *$depth(N') \geq \min\{depth(N), depth(N'') + 1\}$*

Proof. This is easy using the results above. Hint: Use the characterization of depth using the Ext groups $\mathrm{Ext}^i(\kappa, N)$, see Lemma 7.67.9, and use the long exact cohomology sequence

$$\begin{aligned}0 \to Hom_R(\kappa, N') \to Hom_R(\kappa, N) \to Hom_R(\kappa, N'') \\ \to \mathrm{Ext}^1_R(\kappa, N') \to \mathrm{Ext}^1_R(\kappa, N) \to \mathrm{Ext}^1_R(\kappa, N'') \to \ldots\end{aligned}$$

from Lemma 7.67.6. □

7.68. An application of Ext groups

This section should briefly discuss the relationship between $\mathrm{Ext}^1_R(Q, N)$ and extensions (see Homology, Section 10.4). Omitted.

Lemma 7.68.1. *Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Let $N \to M$ be a homomorphism of finite R-modules. Suppose that there exists arbitrarily large n such that $N/\mathfrak{m}^n N \to M/\mathfrak{m}^n M$ is a split injection. Then $N \to M$ is a split injection.*

Proof. Assume $\varphi : N \to M$ satisfies the assumptions of the lemma. Note that this implies that $\mathrm{Ker}(\varphi) \subset \mathfrak{m}^n N$ for arbitrarily large n. Hence by Lemma 7.47.6 we see that φ is injection. Let $Q = M/N$ so that we have a short exact sequence

$$0 \to N \to M \to Q \to 0.$$

Let

$$F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to Q \to 0$$

be a finite free resolution of Q. We can choose a map $\alpha : F_0 \to M$ lifting the map $F_0 \to Q$. This induces a map $\beta : F_1 \to N$ such that $\beta \circ d_2 = 0$. The extension above is split if and only if there exists a map $\gamma : F_0 \to N$ such that $\beta = \gamma \circ d_1$. In other words, the class of β in $\mathrm{Ext}^1_R(Q, N)$ is the obstruction to splitting the short exact sequence above.

Suppose n is a large integer such that $N/\mathfrak{m}^n N \to M/\mathfrak{m}^n M$ is a split injection. This implies

$$0 \to N/\mathfrak{m}^n N \to M/\mathfrak{m}^n M \to Q/\mathfrak{m}^n Q \to 0.$$

is still short exact. Also, the sequence

$$F_1/\mathfrak{m}^n F_1 \xrightarrow{d_1} F_0/\mathfrak{m}^n F_0 \to Q/\mathfrak{m}^n Q \to 0$$

is still exact. Arguing as above we see that the map $\overline{\beta} : F_1/\mathfrak{m}^n F_1 \to N/\mathfrak{m}^n N$ induced by β is equal to $\gamma_n \circ d_1$ for some map $\overline{\gamma_n} : F_0/\mathfrak{m}^n F_0 \to N/\mathfrak{m}^n$. Since F_0 is free we can lift $\overline{\gamma_n}$ to a map $\gamma_n : F_0 \to N$ and then we see that $\beta - \gamma_n \circ d_1$ is a map from F_1 into $\mathfrak{m}^n N$. In other words we conclude that

$$\beta \in \mathrm{Im}\Big(Hom_R(F_0, N) \to Hom_R(F_1, N)\Big) + \mathfrak{m}^n \, Hom_R(F_1, N).$$

for this n.

Since we have this property for arbitrarily large n by assumption we conclude (by Lemma 7.47.4) that β is actually in the image of the map $Hom_R(F_0, N) \to Hom_R(F_1, N)$ as desired. □

7.69. Tor groups and flatness

In this section we use some of the homological algebra developed in the previous section to explain what Tor groups are. Namely, suppose that R is a ring and that M, N are two R-modules. Choose a resolution $F_\bullet$ of M by free R-modules. See Lemma 7.67.1. Consider the homological complex

$$F_\bullet \otimes_R N : \ldots \to F_2 \otimes_R N \to F_1 \otimes_R N \to F_0 \otimes_R N$$

We define $\text{Tor}_i^R(M, N)$ to be the ith homology group of this complex. The following lemma explains in what sense this is well defined.

Lemma 7.69.1. *Let R be a ring. Let M_1, M_2, N be R-modules. Suppose that $F_\bullet$ is a free resolution of the module M_1 and that $G_\bullet$ is a free resolution of the module M_2. Let $\varphi : M_1 \to M_2$ be a module map. Let $\alpha : F_\bullet \to G_\bullet$ be a map of complexes inducing φ on $M_1 = Coker(d_{F,1}) \to M_2 = Coker(d_{G,1})$, see Lemma 7.67.4. Then the induced maps*

$$H_i(\alpha) : H_i(F_\bullet \otimes_R N) \longrightarrow H_i(G_\bullet \otimes_R N)$$

are independent of the choice of α. If φ is an isomorphism, so are all the maps $H_i(\alpha)$. If $M_1 = M_2$, $F_\bullet = G_\bullet$, and φ is the identity, so are all the maps $H_i(\alpha)$.

Proof. The proof of this lemma is identical to the proof of Lemma 7.67.5. □

Not only does this lemma imply that the Tor modules are well defined, but it also provides for the functoriality of the constructions $(M, N) \mapsto \text{Tor}_i^R(M, N)$ in the first variable. Of course the functoriality in the second variable is evident. We leave it to the reader to see that each of the Tor_i^R is in fact a functor

$$\text{Mod}_R \times \text{Mod}_R \to \text{Mod}_R.$$

Here Mod_R denotes the category of R-modules, and for the definition of the product category see Categories, Definition 4.2.20. Namely, given morphisms of R-modules $M_1 \to M_2$ and $N_1 \to N_2$ we get a commutative diagram

$$\begin{tikzcd}
\text{Tor}_i^R(M_1, N_1) \arrow[r] \arrow[d] & \text{Tor}_i^R(M_1, N_2) \arrow[d] \\
\text{Tor}_i^R(M_2, N_1) \arrow[r] & \text{Tor}_i^R(M_2, N_2)
\end{tikzcd}$$

Lemma 7.69.2. *Let R be a ring and let M be an R-module. Suppose that $0 \to N' \to N \to N'' \to 0$ is a short exact sequence of R-modules. There exists a long exact sequence*

$$\begin{array}{c} M \otimes_R N' \to M \otimes_R N \to M \otimes_R N'' \to 0 \\ Tor_1^R(M, N') \to Tor_1^R(M, N) \to Tor_1^R(M, N'') \to \end{array}$$

Proof. The proof of this is the same as the proof of Lemma 7.67.6. □

Consider a homological double complex of R-modules

$$\begin{tikzcd}
\ldots \arrow[r, "d"] & A_{2,0} \arrow[r, "d"] & A_{1,0} \arrow[r, "d"] & A_{0,0} \\
\ldots \arrow[r, "d"] & A_{2,1} \arrow[r, "d"] \arrow[u, "\delta"] & A_{1,1} \arrow[r, "d"] \arrow[u, "\delta"] & A_{0,1} \arrow[u, "\delta"] \\
\ldots \arrow[r, "d"] & A_{2,2} \arrow[r, "d"] \arrow[u, "\delta"] & A_{1,2} \arrow[r, "d"] \arrow[u, "\delta"] & A_{0,2} \arrow[u, "\delta"] \\
& \ldots \arrow[u, "\delta"] & \ldots \arrow[u, "\delta"] & \ldots \arrow[u, "\delta"]
\end{tikzcd}$$

This means that $d_{i,j} : A_{i,j} \to A_{i-1,j}$ and $\delta_{i,j} : A_{i,j} \to A_{i,j-1}$ have the following properties

(1) Any composition of two $d_{i,j}$ is zero. In other words the rows of the double complex are complexes.
(2) Any composition of two $\delta_{i,j}$ is zero. In other words the columns of the double complex are complexes.
(3) For any pair (i, j) we have $\delta_{i-1,j} \circ d_{i,j} = d_{i,j-1} \circ \delta_{i,j}$. In other words, all the squares commute.

The correct thing to do is to associate a spectral sequence to any such double complex. However, for the moment we can get away with doing something slightly easier.

Namely, for the purposes of this section only, given a double complex $(A_{\bullet,\bullet}, d, \delta)$ set $R(A)_j = \mathrm{Coker}(A_{1,j} \to A_{0,j})$ and $U(A)_i = \mathrm{Coker}(A_{i,1} \to A_{i,0})$. (The letters R and U are meant to suggest Right and Up.) We endow $R(A)_\bullet$ with the structure of a complex using the maps δ. Similarly we endow $U(A)_\bullet$ with the structure of a complex using the maps d. In other words we obtain the following huge commutative diagram

$$\begin{tikzcd}
\ldots \arrow[r, "d"] & U(A)_2 \arrow[r, "d"] & U(A)_1 \arrow[r, "d"] & U(A)_0 & \\
\ldots \arrow[r, "d"] & A_{2,0} \arrow[r, "d"] \arrow[u] & A_{1,0} \arrow[r, "d"] \arrow[u] & A_{0,0} \arrow[r] \arrow[u] & R(A)_0 \\
\ldots \arrow[r, "d"] & A_{2,1} \arrow[r, "d"] \arrow[u, "\delta"] & A_{1,1} \arrow[r, "d"] \arrow[u, "\delta"] & A_{0,1} \arrow[r] \arrow[u, "\delta"] & R(A)_1 \arrow[u, "\delta"] \\
\ldots \arrow[r, "d"] & A_{2,2} \arrow[r, "d"] \arrow[u, "\delta"] & A_{1,2} \arrow[r, "d"] \arrow[u, "\delta"] & A_{0,2} \arrow[r] \arrow[u, "\delta"] & R(A)_2 \arrow[u, "\delta"] \\
& \ldots \arrow[u, "\delta"] & \ldots \arrow[u, "\delta"] & \ldots \arrow[u, "\delta"] & \ldots \arrow[u, "\delta"]
\end{tikzcd}$$

(This is no longer a double complex of course.) It is clear what a morphism $\Phi : (A_{\bullet,\bullet}, d, \delta) \to (B_{\bullet,\bullet}, d, \delta)$ of double complexes is, and it is clear that this induces morphisms of complexes $R(\Phi) : R(A)_\bullet \to R(B)_\bullet$ and $U(\Phi) : U(A)_\bullet \to U(B)_\bullet$.

Lemma 7.69.3. *Let $(A_{\bullet,\bullet}, d, \delta)$ be a double complex such that*

(1) *Each row $A_{\bullet,j}$ is a resolution of $R(A)_j$.*

(2) *Each column $A_{i,\bullet}$ is a resolution of $U(A)_i$.*

Then there are canonical isomorphisms

$$H_i(R(A)_\bullet) \cong H_i(U(A)_\bullet).$$

The isomorphisms are functorial with respect to morphisms of double complexes with the properties above.

Proof. We will show that $H_i(R(A)_\bullet))$ and $H_i(U(A)_\bullet)$ are canonically isomorphic to a third group. Namely

$$\mathbf{H}_i(A) := \frac{\{(a_{i,0}, a_{i-1,1}, \ldots, a_{0,i}) \mid d(a_{i,0}) = \delta(a_{i-1,1}), \ldots, d(a_{1,i-1}) = \delta(a_{0,i})\}}{\{d(a_{i+1,0}) - \delta(a_{i,1}), d(a_{i,1}) - \delta(a_{i-1,2}), \ldots, d(a_{1,i}) - \delta(a_{0,i+1})\}}$$

Here we use the notational convention that $a_{i,j}$ denotes an element of $A_{i,j}$. In other words, an element of $\mathbf{H}_i$ is represented by a zig-zag, represented as follows for $i = 2$

$$\begin{tikzcd}
a_{2,0} \arrow[r, mapsto] & d(a_{2,0}) = \delta(a_{1,1}) & \\
& a_{1,1} \arrow[u, mapsto] \arrow[r, mapsto] & d(a_{1,1}) = \delta(a_{0,2}) \\
& & a_{0,2} \arrow[u, mapsto]
\end{tikzcd}$$

Naturally, we divide out by ``trivial'' zig-zags, namely the submodule generated by elements of the form $(0, \ldots, 0, -\delta(a_{t+1,t-i}), d(a_{t+1,t-i}), 0, \ldots, 0)$. Note that there are canonical homomorphisms

$$\mathbf{H}_i(A) \to H_i(R(A)_\bullet), \quad (a_{i,0}, a_{i-1,1}, \ldots, a_{0,i}) \mapsto \text{class of image of } a_{0,i}$$

and

$$\mathbf{H}_i(A) \to H_i(U(A)_\bullet), \quad (a_{i,0}, a_{i-1,1}, \ldots, a_{0,i}) \mapsto \text{class of image of } a_{i,0}$$

First we show that these maps are surjective. Suppose that $\bar{r} \in H_i(R(A)_\bullet)$. Let $r \in R(A)_i$ be a cocycle representing the class of $\bar{r}$. Let $a_{0,i} \in A_{0,i}$ be an element which maps to r. Because $\delta(r) = 0$, we see that $\delta(a_{0,i})$ is in the image of d. Hence there exists an element $a_{1,i-1} \in A_{1,i-1}$ such that $d(a_{1,i-1}) = \delta(a_{0,i})$. This in turn implies that $\delta(a_{1,i-1})$ is in the kernel of d (because $d(\delta(a_{1,i-1})) = \delta(d(a_{1,i-1})) = \delta(\delta(a_{0,i})) = 0$. By exactness of the rows we find an element $a_{2,i-2}$ such that $d(a_{2,i-2}) = \delta(a_{1,i-1})$. And so on until a full zig-zag is found. Of course surjectivity of $\mathbf{H}_i \to H_i(U(A))$ is shown similarly.

To prove injectivity we argue in exactly the same way. Namely, suppose we are given a zig-zag $(a_{i,0}, a_{i-1,1}, \ldots, a_{0,i})$ which maps to zero in $H_i(R(A)_\bullet)$. This means that $a_{0,i}$ maps to an element of $\operatorname{Coker}(A_{i,1} \to A_{i,0})$ which is in the image of $\delta : \operatorname{Coker}(A_{i+1,1} \to A_{i+1,0}) \to \operatorname{Coker}(A_{i,1} \to A_{i,0})$. In other words, $a_{0,i}$ is in the image of $\delta \oplus d : A_{0,i+1} \oplus A_{1,i} \to A_{0,i}$. From the definition of trivial zig-zags we see that we may modify our zig-zag by a trivial one and assume that $a_{0,i} = 0$. This immediately implies that $d(a_{1,i-1}) = 0$. As the rows are exact this implies that $a_{1,i-1}$ is in the image of $d : A_{2,i-1} \to A_{1,i-1}$. Thus we may modify our zig-zag once again by a trivial zig-zag and assume that our zig-zag looks like $(a_{i,0}, a_{i-1,1}, \ldots, a_{2,i-2}, 0, 0)$. Continuing like this we obtain the desired injectivity.

If $\Phi : (A_{\bullet,\bullet}, d, \delta) \to (B_{\bullet,\bullet}, d, \delta)$ is a morphism of double complexes both of which satisfy the conditions of the lemma, then we clearly obtain a commutative diagram

$$\begin{tikzcd} H_i(U(A)_\bullet) \arrow[d] & \mathbf{H}_i(A) \arrow[l] \arrow[r] \arrow[d] & H_i(R(A)_\bullet) \arrow[d] \\ H_i(U(B)_\bullet) & \mathbf{H}_i(B) \arrow[l] \arrow[r] & H_i(R(B)_\bullet) \end{tikzcd}$$

This proves the functoriality. □

Remark 7.69.4. The isomorphism constructed above is the ``correct'' one only up to signs. A good part of homological algebra is concerned with choosing signs for various maps and showing commutativity of diagrams with intervention of suitable signs. For the moment we will simply use the isomorphism as given in the proof above, and worry about signs later.

Lemma 7.69.5. *Let R be a ring. For any $i \geq 0$ the functors $Mod_R \times Mod_R \to Mod_R$, $(M, N) \mapsto Tor_i^R(M, N)$ and $(M, N) \mapsto Tor_i^R(N, M)$ are canonically isomorphic.*

Proof. Let $F_\bullet$ be a free resolution of the module M and let $G_\bullet$ be a free resolution of the module N. Consider the double complex $(A_{i,j}, d, \delta)$ defined as follows:

(1) set $A_{i,j} = F_i \otimes_R G_j$,
(2) set $d_{i,j} : F_i \otimes_R G_j \to F_{i-1} \otimes G_j$ equal to $d_{F,i} \otimes \text{id}$, and
(3) set $\delta_{i,j} : F_i \otimes_R G_j \to F_i \otimes G_{j-1}$ equal to $\text{id} \otimes d_{G,j}$.

This double complex is usually simply denoted $F_\bullet \otimes_R G_\bullet$.

Since each G_j is free, and hence flat we see that each row of the double complex is exact except in homological degree 0. Since each F_i is free and hence flat we see that each column of the double complex is exact except in homological degree 0. Hence the double complex satisfies the conditions of Lemma 7.69.3.

To see what the lemma says we compute $R(A)_\bullet$ and $U(A)_\bullet$. Namely,

$$\begin{aligned} R(A)_i &= \text{Coker}(A_{1,i} \to A_{0,i}) \\ &= \text{Coker}(F_1 \otimes_R G_i \to F_0 \otimes_R G_i) \\ &= \text{Coker}(F_1 \to F_0) \otimes_R G_i \\ &= M \otimes_R G_i \end{aligned}$$

In fact these isomorphisms are compatible with the differentials δ and we see that $R(A)_\bullet = M \otimes_R G_\bullet$ as homological complexes. In exactly the same way we see that $U(A)_\bullet = F_\bullet \otimes_R N$. We get

$$\begin{aligned} \text{Tor}_i^R(M, N) &= H_i(F_\bullet \otimes_R N) \\ &= H_i(U(A)_\bullet) \\ &= H_i(R(A)_\bullet) \\ &= H_i(M \otimes_R G_\bullet) \\ &= H_i(G_\bullet \otimes_R M) \\ &= \text{Tor}_i^R(N, M) \end{aligned}$$

Here the third equality is Lemma 7.69.3, and the fifth equality uses the isomorphism $V \otimes W = W \otimes V$ of the tensor product.

Functoriality. Suppose that we have R-modules M_ν, N_ν, $\nu = 1, 2$. Let $\varphi : M_1 \to M_2$ and $\psi : N_1 \to N_2$ be morphisms of R-modules. Suppose that we have free resolutions $F_{\nu,\bullet}$ for M_ν and free resolutions $G_{\nu,\bullet}$ for N_ν. By Lemma 7.67.4 we may choose maps of complexes $\alpha : F_{1,\bullet} \to F_{2,\bullet}$ and $\beta : G_{1,\bullet} \to G_{2,\bullet}$ compatible with φ and ψ. We claim that the pair (α, β) induces a morphism of double complexes

$$\alpha \otimes \beta : F_{1,\bullet} \otimes_R G_{1,\bullet} \longrightarrow F_{2,\bullet} \otimes_R G_{2,\bullet}$$

This is really a very straightforward check using the rule that $F_{1,i} \otimes_R G_{1,j} \to F_{2,i} \otimes_R G_{2,j}$ is given by $\alpha_i \otimes \beta_j$ where α_i, resp. β_j is the degree i, resp. j component of α, resp. β. The reader also readily verifies that the induced maps $R(F_{1,\bullet} \otimes_R G_{1,\bullet})_\bullet \to R(F_{2,\bullet} \otimes_R G_{2,\bullet})_\bullet$ agrees with the map $M_1 \otimes_R G_{1,\bullet} \to M_2 \otimes_R G_{2,\bullet}$ induced by $\varphi \otimes \beta$. Similarly for the map induced on the $U(-)_\bullet$ complexes. Thus the statement on functoriality follows from the statement on functoriality in Lemma 7.69.3. □

Remark 7.69.6. An interesting case occurs when $M = N$ in the above. In this case we get a canonical map $\text{Tor}_i^R(M, M) \to \text{Tor}_i^R(M, M)$. Note that this map is not the identity, because even when $i = 0$ this map is not the identity! For example, if V is a vector space of dimension n over a field, then the switch map $V \otimes_k V \to V \otimes_k V$ has $(n^2 + n)/2$ eigenvalues $+1$ and $(n^2 - n)/2$ eigenvalues -1. In characteristic 2 it is not even diagonalizable. Note that even changing the sign of the map will not get rid of this.

Lemma 7.69.7. *Let R be a ring. Let M be an R-module. The following are equivalent:*

(1) *The module M is flat over R.*
(2) *For all $i > 0$ the functor $Tor_i^R(M, -)$ is zero.*
(3) *The functor $Tor_1^R(M, -)$ is zero.*
(4) *For all ideals $I \subset R$ we have $Tor_1^R(M, R/I) = 0$.*
(5) *For all finitely generated ideals $I \subset R$ we have $Tor_1^R(M, R/I) = 0$.*

Proof. Suppose M is flat. Let N be an R-module. Let $F_\bullet$ be a free resolution of N. Then $F_\bullet \otimes_R M$ is a resolution of $N \otimes_R M$, by flatness of M. Hence all higher Tor groups vanish.

It now suffices to show that the last condition implies that M is flat. Let $I \subset R$ be an ideal. Consider the short exact sequence $0 \to I \to R \to R/I \to 0$. Apply Lemma 7.69.2. We get an exact sequence

$$\text{Tor}_1^R(M, R/I) \to M \otimes_R I \to M \otimes_R R \to M \otimes_R R/I \to 0$$

Since obviously $M \otimes_R R = M$ we conclude that the last hypothesis implies that $M \otimes_R I \to M$ is injective for every finitely generated ideal I. Thus M is flat by Lemma 7.35.4. □

Remark 7.69.8. The proof of Lemma 7.69.7 actually shows that

$$\text{Tor}_1^R(M, R/I) = \text{Ker}(I \otimes_R M \to M).$$

7.70. Functorialities for Tor

In this section we briefly discuss the functoriality of Tor with respect to change of ring, etc. Here is a list of items to work out.

(1) Given a ring map $R \to R'$, an R-module M and an R'-module N' the R-modules $\text{Tor}_i^R(M, N')$ have a natural R'-module structure.
(2) Given a ring map $R \to R'$ and R-modules M, N there is a natural R-module map $\text{Tor}_i^R(M, N) \to \text{Tor}_i^{R'}(M \otimes_R R', N \otimes_R R')$.

(3) Given a ring map $R \to R'$ an R-module M and an R'-module N' there exists a natural R'-module map $\mathrm{Tor}_i^R(M, N') \to \mathrm{Tor}_i^{R'}(M \otimes_R R', N')$.

Lemma 7.70.1. *Given a flat ring map $R \to R'$ and R-modules M, N the natural R-module map $Tor_i^R(M, N) \otimes_R R' \to Tor_i^{R'}(M \otimes_R R', N \otimes_R R')$ is an isomorphism for all i.*

Proof. Omitted. This is true because a free resolution $F_\bullet$ of M over R stays exact when tensoring with R' over R and hence $(F_\bullet \otimes_R N) \otimes_R R'$ computes the Tor groups over R'. □

7.71. Projective modules

Some lemmas on projective modules.

Definition 7.71.1. Let R be a ring. An R-module P is *projective* if and only if the functor $Hom_R(P, -) : \mathrm{Mod}_R \to \mathrm{Mod}_R$ is an exact functor.

The functor $Hom_R(M, -)$ is left exact for any R-module M, see Lemma 7.10.1. Hence the condition for P to be projective really signifies that given a surjection of R-modules $N \to N'$ the map $Hom_R(P, N) \to Hom_R(P, N')$ is surjective.

Lemma 7.71.2. *Let R be a ring. Let P be an R-module. The following are equivalent*

(1) *P is projective,*
(2) *P is a direct summand of a free R-module, and*
(3) *$Ext^1_R(P, M) = 0$ for every R-module M.*

Proof. Assume P is projective. Choose a surjection $\pi : F \to P$ where F is a free R-module. As P is projective there exists a $i \in Hom_R(P, F)$ such that $i \circ \pi = \mathrm{id}_P$. In other words $F \cong \mathrm{Ker}(\pi) \oplus i(P)$ and we see that P is a direct summand of F.

Conversely, assume that $P \oplus Q = F$ is a free R-module. Note that the free module $F = \bigoplus_{i \in I} R$ is projective as $Hom_R(F, M) = \prod_{i \in I} M$ and the functor $M \mapsto \prod_{i \in I} M$ is exact. Then $Hom_R(F, -) = Hom_R(P, -) \times Hom_R(Q, -)$ as functors, hence both P and Q are projective.

Assume $P \oplus Q = F$ is a free R-module. Then we have a free resolution $F_\bullet$ of the form

$$\ldots F \xrightarrow{a} F \xrightarrow{b} F \to P \to 0$$

where the maps a, b alternate and are equal to the projector onto P and Q. Hence the complex $Hom_R(F_\bullet, M)$ is split exact in degrees ≥ 1, whence we see the vanishing in (3).

Assume $\mathrm{Ext}^1_R(P, M) = 0$ for every R-module M. Pick a free resolution $F_\bullet \to P$. Set $M = \mathrm{Im}(F_1 \to F_0) = \mathrm{Ker}(F_0 \to M)$. Consider the element $\xi \in \mathrm{Ext}^1_R(P, M)$ given by the class of the quotient map $\pi : F_1 \to M$. Since ξ is zero there exists a map $s : F_0 \to M$ such that $\pi = s \circ (F_1 \to F_0)$. Clearly, this means that

$$F_0 = \mathrm{Ker}(s) \oplus \mathrm{Ker}(F_0 \to P) = P \oplus \ker(F_0 \to P)$$

and we win. □

Lemma 7.71.3. *A direct sum of projective modules is projective.*

Proof. This is true by the characterization of projectives as direct summands of free modules in Lemma 7.71.2. □

Lemma 7.71.4. *Let R be a ring. Let $I \subset R$ be a nilpotent ideal. Let $\overline{P}$ be a projective R-module. Then there exists a projective R-module P such that $P/IP \cong \overline{P}$.*

Proof. We can choose a set A and a direct sum decomposition $\bigoplus_{\alpha \in A} R/I = \overline{P} \oplus \overline{K}$ for some R/I-module $\overline{K}$. Write $F = \bigoplus_{\alpha \in A} R$ for the free R-module on A. Choose a lift $p : F \to F$ of the projector $\overline{p}$ associated to the direct summand $\overline{P}$ of $\bigoplus_{\alpha \in A} R/I$. Note that $p^2 - p \in \text{End}_R(F)$ is a nilpotent endomorphism of F (as I is nilpotent and the matrix entries of $p^2 - p$ are in I; more precisely, if $I^n = 0$, then $(p^2 - p)^n = 0$). Hence by Lemma 7.49.6 we can modify our choice of p and assume that p is a projector. Set $P = \text{Im}(p)$. □

Lemma 7.71.5. *Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module. Assume*

(1) *I is nilpotent,*
(2) *M/IM is a projective R/I-module,*
(3) *M is a flat R-module.*

Then M is a projective R-module.

Proof. By Lemma 7.71.4 we can find a projective R-module P and an isomorphism $P/IP \to M/IM$. We are going to show that M is isomorphic to P which will finish the proof. Because P is projective we can lift the map $P \to P/IP \to M/IM$ to an R-module map $P \to M$ which is an isomorphism modulo I. By Nakayama's Lemma 7.14.5 the map $P \to M$ is surjective. It remains to show that $P \to M$ is injective. Since $I^n = 0$ for some n, we can use the filtrations

$$0 = I^n M \subset I^{n-1} M \subset \ldots \subset IM \subset M$$

$$0 = I^n P \subset I^{n-1} P \subset \ldots \subset IP \subset P$$

to see that it suffices to show that the induced maps $I^a P/I^{a+1} P \to I^a M/I^{a+1} M$ are injective. Since both P and M are flat R-modules we can identify this with the map

$$I^a/I^{a+1} \otimes_{R/I} P/IP \longrightarrow I^a/I^{a+1} \otimes_{R/I} M/IM$$

induced by $P \to M$. Since we chose $P \to M$ such that the induced map $P/IP \to M/IM$ is an isomorphism, we win. □

7.72. Finite projective modules

Definition 7.72.1. Let R be a ring and M an R-module. We say that M is *locally free* if we can cover $Spec(R)$ by standard opens $D(f_i)$, $i \in I$ such that M_{f_i} is a free R_{f_i}-module for all $i \in I$. We say that M is *finite locally free* if each M_{f_i} is finite free.

Note that a finite locally free R-module is automatically finitely presented by Lemma 7.21.2.

Lemma 7.72.2. *Let R be a ring and let M be an R-module. The following are equivalent*

(1) *M is finitely presented and R-flat,*
(2) *M is finite projective,*
(3) *M is a direct summand of a finite free R-module,*
(4) *M is finitely presented and for all $\mathfrak{p} \in Spec(R)$ the localization $M_{\mathfrak{p}}$ is free,*
(5) *M is finitely presented and for all maximal ideals $\mathfrak{m} \subset R$ the localization $M_{\mathfrak{m}}$ is free,*
(6) *M is finite and locally free,*
(7) *M is finite locally free, and*
(8) *M is finite, for every prime $\mathfrak{p}$ the module $M_{\mathfrak{p}}$ is free, and the function*

$$\rho_M : Spec(R) \to \mathbf{Z}, \quad \mathfrak{p} \longmapsto \dim_{\kappa(\mathfrak{p})} M \otimes_R \kappa(\mathfrak{p})$$

is locally constant in the Zariski topology.

Proof. First suppose M is finite projective, i.e., (2) holds. Take a surjection $R^n \to M$ and let K be the kernel. Since M is projective, $0 \to K \to R^n \to M \to 0$ splits. Hence (2) $\Rightarrow$ (3). The implication (3) $\Rightarrow$ (2) follows from the fact that a direct summand of a projective is projective, see Lemma 7.71.2.

Assume (3), so we can write $K \oplus M \cong R^{\oplus n}$. So K is a direct summand of R^n and thus finitely generated. This shows $M = R^{\oplus n}/K$ is finitely presented. In other words, (3) $\Rightarrow$ (1).

Assume M is finitely presented and flat, i.e., (1) holds. We will prove that (7) holds. Pick any prime $\mathfrak p$ and $x_1, \ldots, x_r \in M$ which map to a basis of $M \otimes_R \kappa(\mathfrak p)$. By Nakayama's Lemma 7.14.5 these elements generate M_g for some $g \in R$, $g \not\in \mathfrak p$. The corresponding surjection $\varphi : R_g^{\oplus r} \to M_g^{\oplus r}$ has the following two properties: (a) $\text{Ker}(\varphi)$ is a finite R_g-module (see Lemma 7.5.3) and (b) $\text{Ker}(\varphi) \otimes \kappa(\mathfrak p) = 0$ by flatness of M_g over R_g (see Lemma 7.35.11). Hence by Nakayama's lemma again there exists a $g' \in R_g$ such that $\text{Ker}(\varphi)_{g'} = 0$. In other words, $M_{gg'}$ is free.

A finite locally free module is a finite module, see Lemma 7.21.2, hence (7) $\Rightarrow$ (6). It is clear that (6) $\Rightarrow$ (7) and that (7) $\Rightarrow$ (8).

A finite locally free module is a finitely presented module, see Lemma 7.21.2, hence (7) $\Rightarrow$ (4). Of course (4) implies (5). Since we may check flatness locally (see Lemma 7.35.19) we conclude that (5) implies (1). At this point we have

$$\begin{tikzcd}
(2) \arrow[r, Leftrightarrow] & (3) \arrow[r, Rightarrow] & (1) \arrow[r, Rightarrow] & (7) \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & (6) \\
 & & (5) \arrow[u, Rightarrow] & (4) \arrow[l, Rightarrow] & (8)
\end{tikzcd}$$

Suppose that M satisfies (1), (4), (5), (6), and (7). We will prove that (3) holds. It suffices to show that M is projective. We have to show that $Hom_R(M, -)$ is exact. Let $0 \to N'' \to N \to N' \to 0$ be a short exact sequence of R-module. We have to show that $0 \to Hom_R(M, N'') \to Hom_R(M, N) \to Hom_R(M, N') \to 0$ is exact. As M is finite locally free there exist a covering $Spec(R) = \bigcup D(f_i)$ such that M_{f_i} is finite free. By Lemma 7.10.2 we see that

$$0 \to Hom_R(M, N'')_{f_i} \to Hom_R(M, N)_{f_i} \to Hom_R(M, N')_{f_i} \to 0$$

is equal to $0 \to Hom_{R_{f_i}}(M_{f_i}, N''_{f_i}) \to Hom_{R_{f_i}}(M_{f_i}, N_{f_i}) \to Hom_{R_{f_i}}(M_{f_i}, N'_{f_i}) \to 0$ which is exact as M_{f_i} is free and as the localization $0 \to N''_{f_i} \to N_{f_i} \to N'_{f_i} \to 0$ is exact (as localization is exact). Whence we see that $0 \to Hom_R(M, N'') \to Hom_R(M, N) \to Hom_R(M, N') \to 0$ is exact by Lemma 7.21.2.

Finally, assume that (8) holds. Pick a maximal ideal $\mathfrak m \subset R$. Pick $x_1, \ldots, x_r \in M$ which map to a $\kappa(\mathfrak m)$-basis of $M \otimes_R \kappa(\mathfrak m) = M/\mathfrak m M$. In particular $\rho_M(\mathfrak m) = r$. By Nakayama's Lemma 7.14.5 there exists an $f \in R$, $f \not\in \mathfrak m$ such that $x_1, \ldots, x_r$ generate M_f over R_f. By the assumption that ρ_M is locally constant there exists a $g \in R$, $g \not\in \mathfrak m$ such that ρ_M is constant equal to r on $D(g)$. We claim that

$$\Psi : R_{fg}^{\oplus r} \longrightarrow M_{fg}, \quad (a_1, \ldots, a_r) \longmapsto \sum a_i x_i$$

is an isomorphism. This claim will show that M is finite locally free, i.e., that (7) holds. To see the claim it suffices to show that the induced map on localizations $\Psi_{\mathfrak p} : R_{\mathfrak p}^{\oplus r} \to M_{\mathfrak p}$ is an isomorphism for all $\mathfrak p \in D(fg)$, see Lemma 7.21.1. By our choice of f the map

$\Psi_{\mathfrak p}$ is surjective. By assumption (8) we have $M_{\mathfrak p} \cong R_{\mathfrak p}^{\oplus \rho_M(\mathfrak p)}$ and by our choice of g we have $\rho_M(\mathfrak p) = r$. Hence $\Psi_{\mathfrak p}$ determines a surjection $R_{\mathfrak p}^{\oplus r} \to M_{\mathfrak p} \cong R_{\mathfrak p}^{\oplus r}$ whence is an isomorphism by Lemma 7.15.4. (Of course this last fact follows from a simple matrix argument also.) □

Remark 7.72.3. It is not true that a finite R-module which is R-flat is automatically projective. A counter example is where $R = \mathcal{C}^\infty(\mathbf{R})$ is the ring of infinitely differentiable functions on $\mathbf{R}$, and $M = R_{\mathfrak m} = R/I$ where $\mathfrak m = \{f \in R \mid f(0) = 0\}$ and $I = \{f \in R \mid \exists \epsilon, \epsilon > 0 : f(x) = 0\ \forall x, |x| < \epsilon\}$.

Lemma 7.72.4. *(Warning: see Remark 7.72.3 above.) Suppose R is a local ring, and M is a finite flat R-module. Then M is finite free.*

Proof. Follows from the equational criterion of flatness, see Lemma 7.35.10. Namely, suppose that $x_1, \ldots, x_r \in M$ map to a basis of $M/\mathfrak m M$. By Nakayama's Lemma 7.14.5 these elements generate M. We want to show there is no relation among the x_i. Instead, we will show by induction on n that if $x_1, \ldots, x_n \in M$ are linearly independent in the vector space $M/\mathfrak m M$ then they are independent over R.

The base case of the induction is where we have $x \in M$, $x \not\in \mathfrak m M$ and a relation $fx = 0$. By the equational criterion there exist $y_j \in M$ and $a_j \in R$ such that $x = \sum a_j y_j$ and $fa_j = 0$ for all j. Since $x \not\in \mathfrak m M$ we see that at least one a_j is a unit and hence $f = 0$.

Suppose that $\sum f_i x_i$ is a relation among $x_1, \ldots, x_n$. By our choice of x_i we have $f_i \in \mathfrak m$. According to the equational criterion of flatness there exist $a_{ij} \in R$ and $y_j \in M$ such that $x_i = \sum a_{ij} y_j$ and $\sum f_i a_{ij} = 0$. Since $x_n \not\in \mathfrak m M$ we see that $a_{nj} \not\in \mathfrak m$ for at least one j. Since $\sum f_i a_{ij} = 0$ we get $f_n = \sum_{i=1}^{n-1} (-a_{ij}/a_{nj}) f_i$. The relation $\sum f_i x_i = 0$ now can be rewritten as $\sum_{i=1}^{n-1} f_i (x_i + (-a_{ij}/a_{nj}) x_n) = 0$. Note that the elements $x_i + (-a_{ij}/a_{nj}) x_n$ map to $n-1$ linearly independent elements of $M/\mathfrak m M$. By induction assumption we get that all the f_i, $i \leq n-1$ have to be zero, and also $f_n = \sum_{i=1}^{n-1} (-a_{ij}/a_{nj}) f_i$. This proves the induction step. □

Lemma 7.72.5. *Let $R \to S$ be a flat local homomorphism of local rings. Let M be a finite R-module. Then M is finite projective over R if and only if $M \otimes_R S$ is finite projective over S.*

Proof. Suppose that $M \otimes_R S$ is finite projective over S. By Lemma 7.72.2 it is finite free. Pick $x_1, \ldots, x_r \in M$ whose residue classes generate $M/\mathfrak m_R M$. Clearly we see that $x_1 \otimes 1, \ldots, x_r \otimes 1$ are a basis for $M \otimes_R S$. This implies that the map $R^{\oplus r} \to M, (a_i) \mapsto \sum a_i x_i$ becomes an isomorphism after tensoring with S. By faithful flatness of $R \to S$, see Lemma 7.35.16 we see that it is an isomorphism. □

Lemma 7.72.6. *Let R be a semi-local ring. Let M be a finite locally free module. If M has constant rank, then M is free. In particular, if R has connected spectrum, then M is free.*

Proof. Omitted. Hints: First show that $M/\mathfrak m_i M$ has the same dimension d for all maximal ideal $\mathfrak m_1, \ldots, \mathfrak m_n$ of R using the the spectrum is connected. Next, show that there exist elements $x_1, \ldots, x_d \in M$ which form a basis for each $M/\mathfrak m_i M$ by the Chinese remainder theorem. Finally show that $x_1, \ldots, x_d$ is a basis for M. □

Here is a technical lemma that is used in the chapter on groupoids.

Lemma 7.72.7. *Let R be a local ring with maximal ideal $\mathfrak m$ and infinite residue field. Let $R \to S$ be a ring map. Let M be an S-module and let $N \subset M$ be an R-submodule. Assume*

(1) *S is semi-local and $\mathfrak{m}S$ is contained in the radical of S,*
(2) *M is a finite free S-module, and*
(3) *N generates M as an S-module.*

Then N contains an S-basis of M.

Proof. Assume M is free of rank n. Let $I = \mathrm{rad}(S)$. By Nakayama's Lemma 7.14.5 a sequence of elements $m_1, \ldots, m_n$ is a basis for M if and only if $\overline{m}_i \in M/IM$ generate M/IM. Hence we may replace M by M/IM, N by $N/(N \cap IM)$, R by $R/\mathfrak{m}$, and S by S/IS. In this case we see that S is a finite product of fields $S = k_1 \times \ldots \times k_r$ and $M = k_1^{\oplus n} \times \ldots \times k_r^{\oplus n}$. The fact that $N \subset M$ generates M as an S-module means that there exist $x_j \in N$ such that a linear combination $\sum a_j x_j$ with $a_j \in S$ has a nonzero component in each factor $k_i^{\oplus n}$. Because $R = k$ is an infinite field, this means that also some linear combination $y = \sum c_j x_j$ with $c_j \in k$ has a nonzero component in each factor. Hence $y \in N$ generates a free direct summand $Sy \subset M$. By induction on n the result holds for M/Sy and the submodule $\overline{N} = N/(N \cap Sy)$. In other words there exist $\overline{y}_2, \ldots, \overline{y}_n$ in $\overline{N}$ which (freely) generate M/Sy. Then $y, y_2, \ldots, y_n$ (freely) generate M and we win. □

7.73. Open loci defined by module maps

The set of primes where a given module map is surjective, or an isomorphism is sometimes open. In the case of finite projective modules we can look at the rank of the map.

Lemma 7.73.1. *Let R be a ring. Let $\varphi : M \to N$ be a map of R-modules with N a finite R-module. Then we have the equality*

$$\begin{aligned} U &= \{\mathfrak{p} \subset R \mid \varphi_{\mathfrak{p}} : M_{\mathfrak{p}} \to N_{\mathfrak{p}} \text{ is surjective}\} \\ &= \{\mathfrak{p} \subset R \mid \varphi \otimes \kappa(\mathfrak{p}) : M \otimes \kappa(\mathfrak{p}) \to N \otimes \kappa(\mathfrak{p}) \text{ is surjective}\} \end{aligned}$$

and U is an open subset of $Spec(R)$.

Proof. This is a reformulation of Nakayama's lemma, see Lemma 7.14.5. Details omitted. □

Lemma 7.73.2. *Let R be a ring. Let $\varphi : M \to N$ be a map of finitely presented R-modules. Then*

$$U = \{\mathfrak{p} \subset R \mid \varphi_{\mathfrak{p}} : M_{\mathfrak{p}} \to N_{\mathfrak{p}} \text{ is an isomorphism}\}$$

is an open subset of $Spec(R)$.

Proof. Let $\mathfrak{p} \in U$. Pick a presentation $N = R^{\oplus n} / \sum_{j=1,\ldots,m} Rk_j$. Denote e_i the image in N of the ith basis vector of $R^{\oplus n}$. For each $i \in \{1, \ldots, n\}$ choose an element $m_i \in M_{\mathfrak{p}}$ such that $\varphi(m_i) = f_i e_i$ for some $f_i \in R$, $f_i \notin \mathfrak{p}$. This is possible as $\varphi_{\mathfrak{p}}$ is an isomorphism. Set $f = f_1 \ldots f_n$ and let $\psi : R_f^{\oplus n} \to M$ be the map which maps the ith basis vector to m_i/f_i. Note that $\varphi_f \circ \psi$ is the localization at f of the given map $R^{\oplus n} \to N$. As $\varphi_{\mathfrak{p}}$ is an isomorphism we see that $\psi(k_j)$ is an element of M which maps to zero in $M_{\mathfrak{p}}$. Hence we see that there exist $g_j \in R$, $g_j \notin \mathfrak{p}$ such that $g_j \psi(k_j) = 0$. Setting $g = g_1 \ldots g_m$, we see that ψ_g factors through N_{fg} to give a map $N_{fg} \to M_{fg}$. By construction this map is inverse to φ_{fg}. Hence φ_{fg} is an isomorphism, which implies that $D(fg) \subset U$ as desired. □

Lemma 7.73.3. *Let R be a ring. Let $\varphi : P_1 \to P_2$ be a map of finite projective modules. Then*

(1) *The set U of primes $\mathfrak{p} \in Spec(R)$ such that $\varphi \otimes \kappa(\mathfrak{p})$ is injective is open.*

(2) *For any* $f \in R$ *such that* $D(f) \subset U$ *the module* $Coker(\varphi)_f$ *is finite projective over* R_f.
(3) *The set* V *of primes* $\mathfrak{p} \in Spec(R)$ *such that* $\varphi \otimes \kappa(\mathfrak{p})$ *is an isomorphism is open.*
(4) *For any* $f \in R$ *such that* $D(f) \subset V$ *the map* $\varphi : P_{1,f} \to P_{2,f}$ *is an isomorphism of modules over* R_f.

Proof. To prove the sets U and V are open we may work locally on $Spec(R)$. Thus we may replace R by a suitable localization and assume that $P_1 = R^{n_1}$ and $P_2 = R^{n_2}$, see Lemma 7.72.2. In this case injectivity of $\varphi \otimes \kappa(\mathfrak{p})$ is equivalent to some $n_1 \times n_1$ minor f of the matrix of φ being invertible in $\kappa(\mathfrak{p})$. Thus $D(f) \subset U$. Similarly for V, but in that case with the added assumption that $m = n$ (and hence f is just the determinant of the map).

Now suppose $D(f) \subset U$. By Lemma 7.72.2 it suffices to prove that $\text{Coker}(\varphi)$ is finite projective locally on $D(f)$. Thus, as we saw above, we may assume that $P_1 = R^{n_1}$ and $P_2 = R^{n_2}$ and that some minor of the matrix of φ is invertible in R. If the minor in question corresponds to the first n_1 basis vectors of R^{n_2}, then using the last $n_2 - n_1$ basis vectors we get a map $R^{n_2 - n_1} \to R^{n_2} \to \text{Coker}(\varphi)$ which is easily seen to be an isomorphism. If $D(f) \subset V$ the argument is even easier. □

7.74. Faithfully flat descent for projectivity of modules

In the next few sections we prove, following Raynaud and Gruson [**GR71**], that the projectivity of modules descends along faithfully flat ring maps. The idea of the proof is to use dévissage à la Kaplansky [**Kap58**] to reduce to the case of countably generated modules. Given a well-behaved filtration of a module M, dévissage allows us to express M as a direct sum of successive quotients of the filtering submodules (see Section 7.78). Using this technique, we prove that a projective module is a direct sum of countably generated modules (Theorem 7.78.5). To prove descent of projectivity for countably generated modules, we introduce a ``Mittag-Leffler'' condition on modules, prove that a countably generated module is projective if and only if it is flat and Mittag-Leffler (Theorem 7.87.3), and then show that the property of being a flat Mittag-Leffler module descends (Lemma 7.89.1). Finally, given an arbitrary module M whose base change by a faithfully flat ring map is projective, we filter M by submodules whose successive quotients are countably generated projective modules, and then by dévissage conclude M is a direct sum of projectives, hence projective itself (Theorem 7.89.5).

We note that there is an error in the proof of faithfully flat descent of projectivity in [**GR71**]. There, descent of projectivity along faithfully flat ring maps is deduced from descent of projectivity along a more general type of ring map ([**GR71**, Example 3.1.4(1) of Part II]). However, the proof of descent along this more general type of map is incorrect, as explained in [**Gru73**]. Patching this hole in the proof of faithfully flat descent of projectivity comes down to proving that the property of being a flat Mittag-Leffler module descends along faithfully flat ring maps. We do this in Lemma 7.89.1.

7.75. Characterizing flatness

In this section we discuss criteria for flatness. The main result in this section is Lazard's theorem (Theorem 7.75.4 below), which says that a flat module is the colimit of a directed system of free finite modules. We remind the reader of the ``equational criterion for flatness'', see Lemma 7.35.10. It turns out that this can be massaged into a seemingly much stronger property.

Lemma 7.75.1. *Let M be an R-module. The following are equivalent:*

(1) *M is flat.*
(2) *If $f : R^n \to M$ is a module map and $x \in \ker(f)$, then there are module maps $h : R^n \to R^m$ and $g : R^m \to M$ such that $f = g \circ h$ and $x \in \ker(h)$.*
(3) *Suppose $f : R^n \to M$ is a module map, $N \subset \ker(f)$ any submodule, and $h : R^n \to R^m$ a map such that $N \subset \ker(h)$ and f factors through h. Then given any $x \in \ker(f)$ we can find a map $h' : R^n \to R^{m'}$ such that $N + Rx \subset \ker(h')$ and f factors through h'.*
(4) *If $f : R^n \to M$ is a module map and $N \subset \ker(f)$ is a finitely generated submodule, then there are module maps $h : R^n \to R^m$ and $g : R^m \to M$ such that $f = g \circ h$ and $N \subset \ker(h)$.*

Proof. That (1) is equivalent to (2) is just a reformulation of the equational criterion for flatness. To show (2) implies (3), let $g : R^m \to M$ be the map such that f factors as $f = g \circ h$. By (2) find $h'' : R^m \to R^{m'}$ such that h'' kills $h(x)$ and $g : R^m \to M$ factors through h''. Then taking $h' = h'' \circ h$ works. (3) implies (4) by induction on the number of generators of $N \subset \ker(f)$ in (4). Clearly (4) implies (2). □

Lemma 7.75.2. *Let M be an R-module. Then M is flat if and only if the following condition holds: if P is a finitely presented R-module and $f : P \to M$ a module map, then there is a free finite R-module F and module maps $h : P \to F$ and $g : F \to M$ such that $f = g \circ h$.*

Proof. This is just a reformulation of condition (4) from Lemma 7.75.1. □

Lemma 7.75.3. *Let M be an R-module. Then M is flat if and only if the following condition holds: for every finitely presented R-module P, if $N \to M$ is a surjective R-module map, then the induced map $Hom_R(P, N) \to Hom_R(P, M)$ is surjective.*

Proof. First suppose M is flat. We must show that if P is finitely presented, then given a map $f : P \to M$, it factors through the map $N \to M$. By Lemma 7.75.2 the map f factors through a map $F \to M$ where F is free and finite. Since F is free, this map factors through $N \to M$. Thus f factors through $N \to M$.

Conversely, suppose the condition of the lemma holds. Let $f : P \to M$ be a map from a finitely presented module P. Choose a free module N with a surjection $N \to M$ onto M. Then f factors through $N \to M$, and since P is finitely generated, f factors through a free finite submodule of N. Thus M satisfies the condition of Lemma 7.75.2, hence is flat. □

Theorem 7.75.4 (Lazard's theorem)**.** *Let M be an R-module. Then M is flat if and only if it is the colimit of a directed system of free finite R-modules.*

Proof. A colimit of a directed system of flat modules is flat, as taking directed colimits is exact and commutes with tensor product. Hence if M is the colimit of a directed system of free finite modules then M is flat.

For the converse, first recall that any module M can be written as the colimit of a directed system of finitely presented modules, in the following way. Choose a surjection $f : R^I \to M$ for some set I, and let K be the kernel. Let E be the set of ordered pairs (J, N) where J is a finite subset of I and N is a finitely generated submodule of $R^J \cap K$. Then E is made into a directed partially ordered set by defining $(J, N) \leq (J', N')$ if and only if $J \subset J'$ and $N \subset N'$. Define $M_e = R^J/N$ for $e = (J, N)$, and define $f_{ee'} : M_e \to M_{e'}$ to be the natural map for $e \leq e'$. Then $(M_e, f_{ee'})$ is a directed system and the natural maps $f_e : M_e \to M$ induce an isomorphism $colim_{e \in E} M_e \xrightarrow{\cong} M$.

Now suppose M is flat. Let $I = M \times \mathbf{Z}$, write (x_i) for the canonical basis of R^I, and take in the above discussion $f : R^I \to M$ to be the map sending x_i to the projection of i onto M. To prove the theorem it suffices to show that the $e \in E$ such that M_e is free form a cofinal subset of E. So let $e = (J, N) \in E$ be arbitrary. By Lemma 7.75.2 there is a free finite module F and maps $h : R^J/N \to F$ and $g : F \to M$ such that the natural map $f_e : R^J/N \to M$ factors as $R^J/N \xrightarrow{h} F \xrightarrow{g} M$. We are going to realize F as $M_{e'}$ for some $e' \geq e$.

Let $\{b_1, \ldots, b_n\}$ be a finite basis of F. Choose n distinct elements $i_1, \ldots, i_n \in I$ such that $i_\ell \notin J$ for all ℓ, and such that the image of x_{i_ℓ} under $f : R^I \to M$ equals the image of b_ℓ under $g : F \to M$. This is possible by our choice of I. Now let $J' = J \cup \{i_1, \ldots, i_n\}$, and define $R^{J'} \to F$ by $x_i \mapsto h(x_i)$ for $i \in J$ and $x_{i_\ell} \mapsto b_\ell$ for $\ell = 1, \ldots, n$. Let $N' = \ker(R^{J'} \to F)$. Observe:

(1) $R^{J'} \to F$ factors $f : R^I \to M$, hence $N' \subset K = \ker(f)$;
(2) $R^{J'} \to F$ is a surjection onto a free finite module, hence it splits and so N' is finitely generated;
(3) $J \subset J'$ and $N \subset N'$.

By (1) and (2) $e' = (J', N')$ is in E, by (3) $e' \geq e$, and by construction $M_{e'} = R^{J'}/N' \cong F$ is free. □

7.76. Universally injective module maps

Next we discuss universally injective module maps, which are in a sense complementary to flat modules (see Lemma 7.76.5). We follow Lazard's thesis [**Laz67**]; also see [**Lam99**].

Definition 7.76.1. Let $f : M \to N$ be a map of R-modules. Then f is called *universally injective* if for every R-module Q, the map $f \otimes_R \mathrm{id}_Q : M \otimes_R Q \to N \otimes_R Q$ is injective. A sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ of R-modules is called *universally exact* if it is exact and $M_1 \to M_2$ is universally injective.

Example 7.76.2. Examples of universally exact maps.

(1) A split short exact sequence is universally exact since tensoring commutes with taking direct sums.
(2) The colimit of a directed system of universally exact sequences is universally exact. This follows from the fact that taking directed colimits is exact and that tensoring commutes with taking colimits. In particular the colimit of a directed system of split exact sequences is universally exact. We will see below that, conversely, any universally exact sequence arises in this way.

Next we give a list of criteria for a short exact sequence to be universally exact. They are analogues of criteria for flatness given above. Parts (3)-(6) below correspond, respectively, to the criteria for flatness given in Lemmas 7.35.10, 7.75.1, 7.75.3, and Theorem 7.75.4.

Theorem 7.76.3. *Let*

$$0 \to M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to 0$$

be an exact sequence of R-modules. The following are equivalent:

(1) *The sequence* $0 \to M_1 \to M_2 \to M_3 \to 0$ *is universally exact.*
(2) *For every finitely presented R-module Q, the sequence*

$$0 \to M_1 \otimes_R Q \to M_2 \otimes_R Q \to M_3 \otimes_R Q \to 0$$

is exact.

(3) *Given elements* $x_i \in M_1$ $(i = 1, \ldots, n)$, $y_j \in M_2$ $(j = 1, \ldots, m)$, *and* $a_{ij} \in R$ $(i = 1, \ldots, n, j = 1, \ldots, m)$ *such that for all i*
$$f_1(x_i) = \sum\nolimits_j a_{ij} y_j,$$
there exists $z_j \in M_1$ $(j = 1, \ldots, m)$ *such that for all i,*
$$x_i = \sum\nolimits_j a_{ij} z_j.$$

(4) *Given a commutative diagram of R-module maps*
$$\begin{tikzcd} R^n \arrow[r] \arrow[d] & R^m \arrow[d] \\ M_1 \arrow[r, "f_1"] & M_2 \end{tikzcd}$$
where m and n are integers, there exists a map $R^m \to M_1$ *making the top triangle commute.*

(5) *For every finitely presented R-module P, the R-module map* $Hom_R(P, M_2) \to Hom_R(P, M_3)$ *is surjective.*

(6) *The sequence* $0 \to M_1 \to M_2 \to M_3 \to 0$ *is the colimit of a directed system of split exact sequences of the form*
$$0 \to M_1 \to M_{2,i} \to M_{3,i} \to 0$$
where the $M_{3,i}$ *are finitely presented.*

Proof. Obviously (1) implies (2).

Next we show (2) implies (3). Let $f_1(x_i) = \sum_j a_{ij} y_j$ be relations as in (3). Let (f_j) be a basis for R^m, (e_i) a basis for R^n, and $R^m \to R^n$ the map given by $f_j \mapsto \sum_i a_{ij} e_i$. Let Q be the cokernel of $R^m \to R^n$. Then tensoring $R^m \to R^n \to Q \to 0$ by the map $f_1 : M_1 \to M_2$, we get a commutative diagram
$$\begin{tikzcd} M_1^{\oplus m} \arrow[r] \arrow[d] & M_1^{\oplus n} \arrow[r] \arrow[d] & M_1 \otimes_R Q \arrow[r] \arrow[d] & 0 \\ M_2^{\oplus m} \arrow[r] & M_2^{\oplus n} \arrow[r] & M_2 \otimes_R Q \arrow[r] & 0 \end{tikzcd}$$
where $M_1^{\oplus m} \to M_1^{\oplus n}$ is given by
$$(z_1, \ldots, z_m) \mapsto (\sum\nolimits_j a_{1j} z_j, \ldots, \sum\nolimits_j a_{nj} z_j),$$
and $M_2^{\oplus m} \to M_2^{\oplus n}$ is given similarly. We want to show $x = (x_1, \ldots, x_n) \in M_1^{\oplus n}$ is in the image of $M_1^{\oplus m} \to M_1^{\oplus n}$. By (2) the map $M_1 \otimes Q \to M_2 \otimes Q$ is injective, hence by exactness of the top row it is enough to show x maps to 0 in $M_2 \otimes Q$, and so by exactness of the bottom row it is enough to show the image of x in $M_2^{\oplus n}$ is in the image of $M_2^{\oplus m} \to M_2^{\oplus n}$. This is true by assumption.

Condition (4) is just a translation of (3) into diagram form.

Next we show (4) implies (5). Let $\varphi : P \to M_3$ be a map from a finitely presented R-module P. We must show that φ lifts to a map $P \to M_2$. Choose a presentation of P,
$$R^n \xrightarrow{g_1} R^m \xrightarrow{g_2} P \to 0.$$

Using freeness of R^n and R^m, we can construct $h_2 : R^m \to M_2$ and then $h_1 : R^n \to M_1$ such that the following diagram commutes

$$\begin{tikzcd}
& R^n \arrow[r, "g_1"] \arrow[d, "h_1"] & R^m \arrow[r, "g_2"] \arrow[d, "h_2"] & P \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & M_1 \arrow[r, "f_1"] & M_2 \arrow[r, "f_2"] & M_3 \arrow[r] & 0.
\end{tikzcd}$$

By (4) there is a map $k_1 : R^m \to M_1$ such that $k_1 \circ g_1 = h_1$. Now define $h_2' : R^m \to M_2$ by $h_2' = h_2 - f_1 \circ k_1$. Then

$$h_2' \circ g_1 = h_2 \circ g_1 - f_1 \circ k_1 \circ g_1 = h_2 \circ g_1 - f_1 \circ h_1 = 0.$$

Hence by passing to the quotient h_2' defines a map $\varphi' : P \to M_2$ such that $\varphi' \circ g_2 = h_2'$. In a diagram, we have

$$\begin{tikzcd}
R^m \arrow[r, "g_2"] \arrow[d, "h_2'"'] & P \arrow[d, "\varphi"] \arrow[dl, "\varphi'"'] \\
M_2 \arrow[r, "f_2"] & M_3.
\end{tikzcd}$$

where the top triangle commutes. We claim that φ' is the desired lift, i.e. that $f_2 \circ \varphi' = \varphi$. From the definitions we have

$$f_2 \circ \varphi' \circ g_2 = f_2 \circ h_2' = f_2 \circ h_2 - f_2 \circ f_1 \circ k_1 = f_2 \circ h_2 = \varphi \circ g_2.$$

Since g_2 is surjective, this finishes the proof.

Now we show (5) implies (6). Write M_3 as the colimit of a directed system of finitely presented modules $M_{3,i}$. Let $M_{2,i}$ be the fiber product of $M_{3,i}$ and M_2 over M_3---by definition this is the submodule of $M_2 \times M_{3,i}$ consisting of elements whose two projections onto M_3 are equal. Let $M_{1,i}$ be the kernel of the projection $M_{2,i} \to M_{3,i}$. Then we have a directed system of exact sequences

$$0 \to M_{1,i} \to M_{2,i} \to M_{3,i} \to 0,$$

and for each i a map of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & M_{1,i} \arrow[r] \arrow[d] & M_{2,i} \arrow[r] \arrow[d] & M_{3,i} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_1 \arrow[r] & M_2 \arrow[r] & M_3 \arrow[r] & 0
\end{tikzcd}$$

compatible with the directed system. From the definition of the fiber product $M_{2,i}$, it follows that the map $M_{1,i} \to M_1$ is an isomorphism. By (5) there is a map $M_{3,i} \to M_2$ lifting $M_{3,i} \to M_3$, and by the universal property of the fiber product this gives rise to a section of $M_{2,i} \to M_{3,i}$. Hence the sequences

$$0 \to M_{1,i} \to M_{2,i} \to M_{3,i} \to 0$$

split. Passing to the colimit, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & colim\, M_{1,i} \arrow[r] \arrow[d, "\cong"] & colim\, M_{2,i} \arrow[r] \arrow[d] & colim\, M_{3,i} \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & M_1 \arrow[r] & M_2 \arrow[r] & M_3 \arrow[r] & 0
\end{tikzcd}$$

with exact rows and outer vertical maps isomorphisms. Hence $colim\, M_{2,i} \to M_2$ is also an isomorphism and (6) holds.

Condition (6) implies (1) by Example 7.76.2 (2). □

The previous theorem shows that a universally exact sequence is always a colimit of split short exact sequences. If the cokernel of a universally injective map is finitely presented, then in fact the map itself splits:

Lemma 7.76.4. *Let*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be an exact sequence of R-modules. Suppose M_3 is of finite presentation. Then

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is universally exact if and only if it is split.

Proof. A split sequence is always universally exact. Conversely, if the sequence is universally exact, then by Theorem 7.76.3 (5) applied to $P = M_3$, the map $M_2 \to M_3$ admits a section. □

The following lemma shows how universally injective maps are complementary to flat modules.

Lemma 7.76.5. *Let M be an R-module. Then M is flat if and only if any exact sequence of R-modules*

$$0 \to M_1 \to M_2 \to M \to 0$$

is universally exact.

Proof. This follows from Lemma 7.75.3 and Theorem 7.76.3 (5). □

Example 7.76.6. Non-split and non-flat universally exact sequences.

(1) In spite of Lemma 7.76.4, it is possible to have a short exact sequence of R-modules

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

that is universally exact but non-split. For instance, take $R = \mathbf{Z}$, let $M_1 = \bigoplus_{n=1}^{\infty} \mathbf{Z}$, let $M_2 = \prod_{n=1}^{\infty} \mathbf{Z}$, and let M_3 be the cokernel of the inclusion $M_1 \to M_2$. Then M_1, M_2, M_3 are all flat since they are torsion-free, so by Lemma 7.76.5,

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is universally exact. However there can be no section $s : M_3 \to M_2$. In fact, if x is the image of $(2, 2^2, 2^3, \ldots) \in M_2$ in M_3, then any module map $s : M_3 \to M_2$ must kill x. This is because $x \in 2^n M_3$ for any $n \geq 1$, hence $s(x)$ is divisible by 2^n for all $n \geq 1$ and so must be 0.

(2) In spite of Lemma 7.76.5, it is possible to have a short exact sequence of R-modules

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

that is universally exact but with M_1, M_2, M_3 all non-flat. In fact if M is any non-flat module, just take the split exact sequence

$$0 \to M \to M \oplus M \to M \to 0.$$

For instance over $R = \mathbf{Z}$, take M to be any torsion module.

(3) Taking the direct sum of an exact sequence as in (1) with one as in (2), we get a short exact sequence of R-modules
$$0 \to M_1 \to M_2 \to M_3 \to 0$$
that is universally exact, non-split, and such that M_1, M_2, M_3 are all non-flat.

We end this section with a simple observation.

Lemma 7.76.7. *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a universally exact sequence of R-modules, and suppose M_2 is flat. Then M_1 and M_3 are flat.*

Proof. Let $0 \to N \to N' \to N'' \to 0$ be a short exact sequence of R-modules. Consider the commutative diagram

$$\begin{tikzcd}
M_1 \otimes_R N \arrow[r] \arrow[d] & M_2 \otimes_R N \arrow[r] \arrow[d] & M_3 \otimes_R N \arrow[d] \\
M_1 \otimes_R N' \arrow[r] \arrow[d] & M_2 \otimes_R N' \arrow[r] \arrow[d] & M_3 \otimes_R N' \arrow[d] \\
M_1 \otimes_R N'' \arrow[r] & M_2 \otimes_R N'' \arrow[r] & M_3 \otimes_R N''
\end{tikzcd}$$

(we have dropped the 0's on the boundary). By assumption the rows give short exact sequences and the arrow $M_2 \otimes N \to M_2 \otimes N'$ is injective. Clearly this implies that $M_1 \otimes N \to M_1 \otimes N'$ is injective and we see that M_1 is flat. In particular the left and middle columns give rise to short exact sequences. It follows from a diagram chase that the arrow $M_3 \otimes N \to M_3 \otimes N'$ is injective. Hence M_3 is flat. □

Lemma 7.76.8. *Let R be a ring. Let $M \to M'$ be a universally injective R-module map. Then for any R-module N the map $M \otimes_R N \to M' \otimes_R N$ is universally injective.*

Proof. Omitted. □

Lemma 7.76.9. *Let R be a ring. A composition of universally injective R-module maps is universally injective.*

Proof. Omitted. □

Lemma 7.76.10. *Let R be a ring. Let $M \to M'$ and $M \to M''$ be R-module maps. If $M \to M''$ is universally injective, then $M \to M'$ is universally injective.*

Proof. Omitted. □

Lemma 7.76.11. *Let $R \to S$ be a faithfully flat ring map. Then $R \to S$ is universally injective as a map of R-modules. In particular $R \cap IS = I$ for any ideal $I \subset R$.*

Proof. Let N be an R-module. We have to show that $N \to N \otimes_R S$ is injective. As S is faithfully flat as an R-module, it suffices to prove this after tensoring with S. Hence it suffices to show that $N \otimes_R S \to N \otimes_R S \otimes_R S$, $n \otimes s \mapsto N \otimes 1 \otimes s$ is injective. This is true because there is a section, namely, $n \otimes s \otimes s' \mapsto n \otimes ss'$. □

Lemma 7.76.12. *Let $R \to S$ be a ring map. Let $M \to M'$ be a map of S-modules. The following are equivalent*

(1) *$M \to M'$ is universally injective as a map of R-modules,*

(2) *for each prime* $\mathfrak{q}$ *of* S *the map* $M_\mathfrak{q} \to M'_\mathfrak{q}$ *is universally injective as a map of* R*-modules,*
(3) *for each maximal ideal* $\mathfrak{m}$ *of* S *the map* $M_\mathfrak{m} \to M'_\mathfrak{m}$ *is universally injective as a map of* R*-modules,*
(4) *for each prime* $\mathfrak{q}$ *of* S *the map* $M_\mathfrak{q} \to M'_\mathfrak{q}$ *is universally injective as a map of* $R_\mathfrak{p}$*-modules, where* $\mathfrak{p}$ *is the inverse image of* $\mathfrak{q}$ *in* R*, and*
(5) *for each maximal ideal* $\mathfrak{m}$ *of* S *the map* $M_\mathfrak{m} \to M'_\mathfrak{m}$ *is universally injective as a map of* $R_\mathfrak{p}$*-modules, where* $\mathfrak{p}$ *is the inverse image of* $\mathfrak{m}$ *in* R*.*

Proof. Let N be an R-module. Let $\mathfrak{q}$ be a prime of S lying over the prime $\mathfrak{p}$ of R. Then we have

$$(M \otimes_R N)_\mathfrak{q} = M_\mathfrak{q} \otimes_R N = M_\mathfrak{q} \otimes_{R_\mathfrak{p}} N_\mathfrak{p}.$$

Moreover, the same thing holds for M' and localization is exact. Also, if N is an $R_\mathfrak{p}$-module, then $N_\mathfrak{p} = N$. Using this the equivalences can be proved in a straightforward manner.

For example, suppose that (5) holds. Let $K = \mathrm{Ker}(M \otimes_R N \to M' \otimes_R N)$. By the remarks above we see that $K_\mathfrak{m} = 0$ for each maximal ideal $\mathfrak{m}$ of S. Hence $K = 0$ by Lemma 7.21.1. Thus (1) holds. Conversely, suppose that (1) holds. Take any $\mathfrak{q} \subset S$ lying over $\mathfrak{p} \subset R$. Take any module N over $R_\mathfrak{p}$. Then by assumption $\mathrm{Ker}(M \otimes_R N \to M' \otimes_R N) = 0$. Hence by the formulae above and the fact that $N = N_\mathfrak{p}$ we see that $\mathrm{Ker}(M_\mathfrak{q} \otimes_{R_\mathfrak{p}} N \to M'_\mathfrak{q} \otimes_{R_\mathfrak{p}} N) = 0$. In other words (4) holds. Of course (4) $\Rightarrow$ (5) is immediate. Hence (1), (4) and (5) are all equivalent. We omit the proof of the other equivalences. $\square$

Lemma 7.76.13. *Let* $\varphi : A \to B$ *be a ring map. Let* $S \subset A$ *and* $S' \subset B$ *be multiplicative subsets such that* $\varphi(S) \subset S'$*. Let* $M \to M'$ *be a map of* B*-modules.*

(1) *If* $M \to M'$ *is universally injective as a map of* A*-modules, then* $(S')^{-1}M \to (S')^{-1}M'$ *is universally injective as a map of* A*-modules and as a map of* $S^{-1}A$*-modules.*
(2) *If* M *and* M' *are* $(S')^{-1}B$*-modules, then* $M \to M'$ *is universally injective as a map of* A*-modules if and only if it is universally injective as a map of* $S^{-1}A$*-modules.*

Proof. You can prove this using Lemma 7.76.12 but you can also prove it directly as follows. Assume $M \to M'$ is A-universally injective. Let Q be an A-module. Then $Q \otimes_A M \to Q \otimes_A M'$ is injective. Since localization is exact we see that $(S')^{-1}(Q \otimes_A M) \to (S')^{-1}(Q \otimes_A M')$ is injective. As $(S')^{-1}(Q \otimes_A M) = Q \otimes_A (S')^{-1}M$ and similarly for M' we see that $Q \otimes_A (S')^{-1}M \to Q \otimes_A (S')^{-1}M'$ is injective, hence $(S')^{-1}M \to (S')^{-1}M'$ is universally injective as a map of A-modules. This proves the first part of (1). To see (2) we can use the following two facts: (a) if Q is an $S^{-1}A$-module, then $Q \otimes_A S^{-1}A = Q$, i.e., tensoring with Q over A is the same thing as tensoring with Q over $S^{-1}A$, (b) if M is any A-module on which the elements of S are invertible, then $M \otimes_A Q = M \otimes_{S^{-1}A} S^{-1}Q$. Part (2) follows from this immediately. $\square$

7.77. Descent for finite projective modules

In this section we give an elementary proof of the fact that the property of being a *finite* projective module descends along faithfully flat ring maps. The proof does not apply when we drop the finiteness condition. However, the method is indicative of the one we shall use to prove descent for the property of being a *countably generated* projective module---see the comments at the end of this section.

Lemma 7.77.1. *Let* M *be an* R*-module. Then* M *is finite projective if and only if* M *is finitely presented and flat.*

Proof. This is part of Lemma 7.72.2. However, at this point we can give a more elegant proof of the implication (1) $\Rightarrow$ (2) of that lemma as follows. If M is finitely presented and flat, then take a surjection $R^n \to M$. By Lemma 7.75.3 applied to $P = M$, the map $R^n \to M$ admits a section. So M is a direct summand of a free module and hence projective. $\square$

Here are some properties of modules that descend.

Lemma 7.77.2. *Let $R \to S$ be a faithfully flat ring map. Let M be an R-module. Then*

1. *if the S-module $M \otimes_R S$ is of finite type, then M is of finite type,*
2. *if the S-module $M \otimes_R S$ is of finite presentation, then M is of finite presentation,*
3. *if the S-module $M \otimes_R S$ is flat, then M is flat, and*
4. *add more here as needed.*

Proof. Assume $M \otimes_R S$ is of finite type. Let $y_1, \ldots, y_m$ be generators of $M \otimes_R S$ over S. Write $y_j = \sum x_i \otimes f_i$ for some $x_1, \ldots, x_n \in M$. Then we see that the map $\varphi : R^{\oplus n} \to M$ has the property that $\varphi \otimes \text{id}_S : S^{\oplus n} \to M \otimes_R S$ is surjective. Since $R \to S$ is faithfully flat we see that φ is surjective, and M is finitely generated.

Assume $M \otimes_R S$ is of finite presentation. By (1) we see that M is of finite type. Choose a surjection $R^{\oplus n} \to M$ and denote K the kernel. As $R \to S$ is flat we see that $K \otimes_R S$ is the kernel of the base change $S^{\oplus n} \to M \otimes_R S$. As $M \otimes_R S$ is of finite presentation we conclude that $K \otimes_R S$ is of finite type. Hence by (1) we see that K is of finite type and hence M is of finite presentation.

Part (3) is Lemma 7.35.7. $\square$

Proposition 7.77.3. *Let $R \to S$ be a faithfully flat ring map. Let M be an R-module. If the S-module $M \otimes_R S$ is finite projective, then M is finite projective.*

Proof. Follows from Lemmas 7.77.1 and 7.77.2. $\square$

The next few sections are about removing the finiteness assumption by using dévissage to reduce to the countably generated case. In the countably generated case, the strategy is to find a characterization of countably generated projective modules analogous to Lemma 7.77.1, and then to prove directly that this characterization descends. We do this by introducing the notion of a Mittag-Leffer module and proving that if a module M is countably generated, then it is projective if and only if it is flat and Mittag-Leffler (Theorem 7.87.3). When M is finitely generated, this statement reduces to Lemma 7.77.1 (since, according to Example 7.85.1 (1), a finitely generated module is Mittag-Leffler if and only if it is finitely presented).

7.78. Transfinite dévissage of modules

In this section we introduce a dévissage technique for decomposing a module into a direct sum. The main result is that a projective module is a direct sum of countably generated modules (Theorem 7.78.5 below). We follow [**Kol96**].

Definition 7.78.1. Let M be an R-module. A *direct sum dévissage* of M is a family of submodules $(M_\alpha)_{\alpha \in S}$, indexed by an ordinal S and increasing (with respect to inclusion), such that:

(0) $M_0 = 0$;
(1) $M = \bigcup_\alpha M_\alpha$;
(2) if $\alpha \in S$ is a limit ordinal, then $M_\alpha = \bigcup_{\beta < \alpha} M_\beta$;
(3) if $\alpha + 1 \in S$, then M_α is a direct summand of $M_{\alpha+1}$.

If moreover

(4) $M_{\alpha+1}/M_\alpha$ is countably generated for $\alpha + 1 \in S$,

then $(M_\alpha)_{\alpha \in S}$ is called a *Kaplansky dévissage* of M.

The terminology is justified by the following lemma.

Lemma 7.78.2. *Let M be an R-module. If $(M_\alpha)_{\alpha\in S}$ is a direct sum dévissage of M, then $M \cong \bigoplus_{\alpha+1\in S} M_{\alpha+1}/M_\alpha$.*

Proof. By property (3) of a direct sum dévissage, there is an inclusion $M_{\alpha+1}/M_\alpha \to M$ for each $\alpha \in S$. Consider the map

$$f : \bigoplus_{\alpha+1\in S} M_{\alpha+1}/M_\alpha \to M$$

given by the sum of these inclusions. Transfinite induction on S shows that the image contains M_α for every $\alpha \in S$: for $\alpha = 0$ this is true by (0); if $\alpha + 1$ is a successor ordinal then it is clearly true; and if α is a limit ordinal and it is true for $\beta < \alpha$, then it is true for α by (2). Hence f is surjective by (1).

Transfinite induction on S also shows that for every $\beta \in S$ the restriction

$$f_\beta : \bigoplus_{\alpha+1\leq\beta} M_{\alpha+1}/M_\alpha \longrightarrow M$$

of f is injective: For $\beta = 0$ it is true. If it is true for all $\beta' < \beta$, then let x be in the kernel and write $x = (x_{\alpha+1})_{\alpha+1\leq\beta}$ in terms of its components $x_{\alpha+1} \in M_{\alpha+1}/M_\alpha$. By property (3) both $(x_{\alpha+1})_{\alpha+1<\beta}$ and $x_{\beta+1}$ map to 0. Hence $x_{\beta+1} = 0$ and, by the assumption that the restriction $f_{\beta'}$ is injective for all $\beta' < \beta$, also $x_{\alpha+1} = 0$ for every $\alpha + 1 < \beta$. So $x = 0$ and f_β is injective, which finishes the induction. We conclude that f is injective since f_β is for each $\beta \in S$. □

Lemma 7.78.3. *Let M be an R-module. Then M is a direct sum of countably generated R-modules if and only if it admits a Kaplansky dévissage.*

Proof. The lemma takes care of the ``if'' direction. Conversely, suppose $M = \bigoplus_{i\in I} N_i$ where each N_i is a countably generated R-module. Well-order I so that we can think of it as an ordinal. Then setting $M_i = \bigoplus_{j<i} N_j$ gives a Kaplansky dévissage $(M_i)_{i\in I}$ of M. □

Theorem 7.78.4. *Suppose M is a direct sum of countably generated R-modules. If P is a direct summand of M, then P is also a direct sum of countably generated R-modules.*

Proof. Write $M = P \oplus Q$. We are going to construct a Kaplansky dévissage $(M_\alpha)_{\alpha\in S}$ of M which, in addition to the defining properties (0)-(4), satisfies:

(5) Each M_α is a direct summand of M;

(6) $M_\alpha = P_\alpha \oplus Q_\alpha$, where $P_\alpha = P \cap M_\alpha$ and $Q = Q \cap M_\alpha$.

(Note: if properties (0)-(2) hold, then in fact property (3) is equivalent to property (5).)

To see how this implies the theorem, it is enough to show that $(P_\alpha)_{\alpha\in S}$ forms a Kaplansky dévissage of P. Properties (0), (1), and (2) are clear. By (5) and (6) for (M_α), each P_α is a direct summand of M. Since $P_\alpha \subset P_{\alpha+1}$, this implies P_α is a direct summand of $P_{\alpha+1}$; hence (3) holds for (P_α). For (4), note that

$$M_{\alpha+1}/M_\alpha \cong P_{\alpha+1}/P_\alpha \oplus Q_{\alpha+1}/Q_\alpha,$$

so $P_{\alpha+1}/P_\alpha$ is countably generated because this is true of $M_{\alpha+1}/M_\alpha$.

It remains to construct the M_α. Write $M = \bigoplus_{i\in I} N_i$ where each N_i is a countably generated R-module. Choose a well-ordering of I. By transfinite induction we are going to define an increasing family of submodules M_α of M, one for each ordinal α, such that M_α is a direct sum of some subset of the N_i.

For $\alpha = 0$ let $M_0 = 0$. If α is a limit ordinal and M_β has been defined for all $\beta < \alpha$, then define $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$. Since each M_β for $\beta < \alpha$ is a direct sum of a subset of the N_i, the same will be true of M_α. If $\alpha+1$ is a successor ordinal and M_α has been defined, then define $M_{\alpha+1}$ as follows. If $M_\alpha = M$, then let $M_{\alpha+1} = M$. If not, choose the smallest $j \in I$ such that N_j is not contained in M_α. We will construct an infinite matrix (x_{mn}), $m, n = 1, 2, 3, \ldots$ such that:

(1) N_j is contained in the submodule of M generated by the entries x_{mn};
(2) if we write any entry $x_{k\ell}$ in terms of its P- and Q-components, $x_{k\ell} = y_{k\ell} + z_{k\ell}$, then the matrix (x_{mn}) contains a set of generators for each N_i for which $y_{k\ell}$ or $z_{k\ell}$ has nonzero component.

Then we define $M_{\alpha+1}$ to be the submodule of M generated by M_α and all x_{mn}; by property (2) of the matrix (x_{mn}), $M_{\alpha+1}$ will be a direct sum of some subset of the N_i. To construct the matrix (x_{mn}), let $x_{11}, x_{12}, x_{13}, \ldots$ be a countable set of generators for N_j. Then if $x_{11} = y_{11}+z_{11}$ is the decomposition into P- and Q-components, let $x_{21}, x_{22}, x_{23}, \ldots$ be a countable set of generators for the sum of the N_i for which y_{11} or z_{11} have nonzero component. Repeat this process on x_{12} to get elements $x_{31}, x_{32}, \ldots$, the third row of our matrix. Repeat on x_{21} to get the fourth row, on x_{13} to get the fifth, and so on, going down along successive anti-diagonals as indicated below:

$$\begin{pmatrix}
x_{11} & & x_{12} & & x_{13} & & x_{14} & & \cdots \\
& \swarrow & & \swarrow & & \swarrow & & & \\
x_{21} & & x_{22} & & x_{23} & & \cdots & & \\
& \swarrow & & \swarrow & & & & & \\
x_{31} & & x_{32} & & \cdots & & & & \\
& \swarrow & & & & & & & \\
x_{41} & & \cdots & & & & & & \\
\cdots & & & & & & & &
\end{pmatrix}.$$

Transfinite induction on I (using the fact that we constructed $M_{\alpha+1}$ to contain N_j for the smallest j such that N_j is not contained in M_α) shows that for each $i \in I$, N_i is contained in some M_α. Thus, there is some large enough ordinal S satisfying: for each $i \in I$ there is $\alpha \in S$ such that N_i is contained in M_α. This means $(M_\alpha)_{\alpha\in S}$ satisfies property (1) of a Kaplansky dévissage of M. The family $(M_\alpha)_{\alpha\in S}$ moreover satisfies the other defining properties, and also (5) and (6) above: properties (0), (2), (4), and (6) are clear by construction; property (5) is true because each M_α is by construction a direct sum of some N_i; and (3) is implied by (5) and the fact that $M_\alpha \subset M_{\alpha+1}$. □

As a corollary we get the result for projective modules stated at the beginning of the section.

Theorem 7.78.5. *If P is a projective R-module, then P is a direct sum of countably generated projective R-modules.*

Proof. A module is projective if and only if it is a direct summand of a free module, so this follows from Theorem 7.78.4. □

7.79. Projective modules over a local ring

In this section we prove a very cute result: a projective module M over a local ring is free (Theorem 7.79.4 below). Note that with the additional assumption that M is finite, this result is Lemma 7.72.4. In general we have:

Lemma 7.79.1. *Let R be a ring. Then every projective R-module is free if and only if every countably generated projective R-module is free.*

Proof. Follows immediately from Theorem 7.78.5. □

Here is a criterion for a countably generated module to be free.

Lemma 7.79.2. *Let M be a countably generated R-module. Suppose any direct summand N of M satisfies: any element of N is contained in a free direct summand of N. Then M is free.*

Proof. Let $x_1, x_2, \ldots$ be a countable set of generators for M. By the assumption on M, we can construct by induction free R-modules $F_1, F_2, \ldots$ such that for every positive integer n, $\bigoplus_{i=1}^n F_i$ is a direct summand of M and contains $x_1, \ldots, x_n$. Then $M = \bigoplus_{i=1}^\infty F_i$. □

Lemma 7.79.3. *Let P be a projective module over a local ring R. Then any element of P is contained in a free direct summand of P.*

Proof. Since P is projective it is a direct summand of some free R-module F, say $F = P \oplus Q$. Let $x \in P$ be the element that we wish to show is contained in a free direct summand of P. Let B be a basis of F such that the number of basis elements needed in the expression of x is minimal, say $x = \sum_{i=1}^n a_i e_i$ for some $e_i \in B$ and $a_i \in R$. Then no a_j can be expressed as a linear combination of the other a_i; for if $a_j = \sum_{i \neq j} a_i b_i$ for some $b_i \in R$, then replacing e_i by $e_i + b_i e_j$ for $i \neq j$ and leaving unchanged the other elements of B, we get a new basis for F in terms of which x has a shorter expression.

Let $e_i = y_i + z_i, y_i \in P, z_i \in Q$ be the decomposition of e_i into its P- and Q-components. Write $y_i = \sum_{j=1}^n b_{ij} e_j + t_i$, where t_i is a linear combination of elements in B other than $e_1, \ldots, e_n$. To finish the proof it suffices to show that the matrix (b_{ij}) is invertible. For then the map $F \to F$ sending $e_i \mapsto y_i$ for $i = 1, \ldots, n$ and fixing $B \setminus \{e_1, \ldots, e_n\}$ is an isomorphism, so that $y_1, \ldots, y_n$ together with $B \setminus \{e_1, \ldots, e_n\}$ form a basis for F. Then the submodule N spanned by $y_1, \ldots, y_n$ is a free submodule of P; N is a direct summand of P since $N \subset P$ and both N and P are direct summands of F; and $x \in N$ since $x \in P$ implies $x = \sum_{i=1}^n a_i e_i = \sum_{i=1}^n a_i y_i$.

Now we prove that (b_{ij}) is invertible. Plugging $y_i = \sum_{j=1}^n b_{ij} e_j + t_i$ into $\sum_{i=1}^n a_i e_i = \sum_{i=1}^n a_i y_i$ and equating the coefficients of e_j gives $a_j = \sum_{i=1}^n a_i b_{ij}$. But as noted above, our choice of B guarantees that no a_j can be written as a linear combination of the other a_i. Thus b_{ij} is a non-unit for $i \neq j$, and $1 - b_{ii}$ is a non-unit---so in particular b_{ii} is a unit---for all i. But a matrix over a local ring having units along the diagonal and non-units elsewhere is invertible, as its determinant is a unit. □

Theorem 7.79.4. *If P is a projective module over a local ring R, then P is free.*

Proof. Follows from Lemmas 7.79.1, 7.79.2, and 7.79.3. □

7.80. Mittag-Leffler systems

The purpose of this section is to define Mittag-Leffler systems and why it is a useful property.

In the following, I will be a directed partially ordered set, see Categories, Definition 4.19.2. Let $(A_i, \varphi_{ji} : A_j \to A_i)$ be an inverse system of sets or of modules indexed by I, see Categories, Definition 4.19.2. This is a directed inverse system as we assumed I directed. For each $i \in I$, the images $\varphi_{ji}(A_j) \subset A_i$ for $j \geq i$ form a decreasing family. Let $A'_i = \bigcap_{j \geq i} \varphi_{ji}(A_j)$. Then $\varphi_{ji}(A'_j) \subset A'_i$ for $j \geq i$, hence by restricting we get a directed inverse system $(A'_i, \varphi_{ji}|_{A'_j})$. From the construction of the limit of an inverse system in the category of sets or modules, we have $lim\, A_i = lim\, A'_i$. The Mittag-Leffler condition on (A_i, φ_{ji}) is that A'_i equals $\varphi_{ji}(A_j)$ for some $j \geq i$ (and hence equals $\varphi_{ki}(A_k)$ for all $k \geq j$):

Definition 7.80.1. Let (A_i, φ_{ji}) be a directed inverse system of sets over I. Then we say (A_i, φ_{ji}) is *Mittag-Leffler inverse system* if for each $i \in I$, the decreasing family $\varphi_{ji}(A_j) \subset A_i$ for $j \geq i$ stabilizes. Explicitly, this means that for each $i \in I$, there exists $j \geq i$ such that for $k \geq j$ we have $\varphi_{ki}(A_k) = \varphi_{ji}(A_j)$. If (A_i, φ_{ji}) is a directed inverse system of modules over a ring R, we say that it is Mittag-Leffler if the underlying inverse system of sets is Mittag-Leffler.

Example 7.80.2. If (A_i, φ_{ji}) is a directed inverse system of sets or of modules and the maps φ_{ji} are surjective, then clearly the system is Mittag-Leffler. Conversely, suppose (A_i, φ_{ji}) is Mittag-Leffler. Let $A'_i \subset A_i$ be the stable image of $\varphi_{ji}(A_j)$ for $j \geq i$. Then $\varphi_{ji}|_{A'_j} : A'_j \to A'_i$ is surjective for $j \geq i$ and $lim\, A_i = lim\, A'_i$. Hence the limit of the Mittag-Leffler system (A_i, φ_{ji}) can also be written as the limit of a directed inverse system over I with surjective maps.

Lemma 7.80.3. *Let (A_i, φ_{ji}) be a directed inverse system over I. Suppose I is countable. If (A_i, φ_{ji}) is Mittag-Leffler and the A_i are nonempty, then $lim\, A_i$ is nonempty.*

Proof. Let $i_1, i_2, i_3, \ldots$ be an enumeration of the elements of I. Define inductively a sequence of elements $j_n \in I$ for $n = 1, 2, 3, \ldots$ by the conditions: $j_1 = i_1$, and $j_n \geq i_n$ and $j_n \geq j_m$ for $m < n$. Then the sequence j_n is increasing and forms a cofinal subset of I. Hence we may assume $I = \{1, 2, 3, \ldots\}$. So by Example 7.80.2 we are reduced to showing that the limit of an inverse system of nonempty sets with surjective maps indexed by the positive integers is nonempty. This is obvious. $\square$

The Mittag-Leffler condition will be important for us because of the following exactness property.

Lemma 7.80.4. *Let*

$$0 \to A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i \to 0$$

be an exact sequence of directed inverse systems of abelian groups over I. Suppose I is countable. If (A_i) is Mittag-Leffler, then

$$0 \to lim\, A_i \to lim\, B_i \to lim\, C_i \to 0$$

is exact.

Proof. Taking limits of directed inverse systems is left exact, hence we only need to prove surjectivity of $lim\, B_i \to lim\, C_i$. So let $(c_i) \in lim\, C_i$. For each $i \in I$, let $E_i = g_i^{-1}(c_i)$, which is nonempty since $g_i : B_i \to C_i$ is surjective. The system of maps $\varphi_{ji} : B_j \to B_i$ for (B_i) restrict to maps $E_j \to E_i$ which make (E_i) into an inverse system of nonempty sets. It is enough to show that (E_i) is Mittag-Leffler. For then Lemma 7.80.3 would show $lim\, E_i$ is nonempty, and taking any element of $lim\, E_i$ would give an element of $lim\, B_i$ mapping to (c_i).

By the injection $f_i : A_i \to B_i$ we will regard A_i as a subset of B_i. Since (A_i) is Mittag-Leffler, if $i \in I$ then there exists $j \geq i$ such that $\varphi_{ki}(A_k) = \varphi_{ji}(A_j)$ for $k \geq j$. We claim that also $\varphi_{ki}(E_k) = \varphi_{ji}(E_j)$ for $k \geq j$. Always $\varphi_{ki}(E_k) \subset \varphi_{ji}(E_j)$ for $k \geq j$. For the reverse inclusion let $e_j \in E_j$, and we need to find $x_k \in E_k$ such that $\varphi_{ki}(x_k) = \varphi_{ji}(e_j)$. Let $e'_k \in E_k$ be any element, and set $e'_j = \varphi_{kj}(e'_k)$. Then $g_j(e_j - e'_j) = c_j - c_j = 0$, hence $e_j - e'_j = a_j \in A_j$. Since $\varphi_{ki}(A_k) = \varphi_{ji}(A_j)$, there exists $a_k \in A_k$ such that $\varphi_{ki}(a_k) = \varphi_{ji}(a_j)$. Hence

$$\varphi_{ki}(e'_k + a_k) = \varphi_{ji}(e'_j) + \varphi_{ji}(a_j) = \varphi_{ji}(e_j),$$

so we can take $x_k = e'_k + a_k$. □

7.81. Inverse systems

In many papers (and in this section) the term *inverse system* is used to indicate an inverse system over the partially ordered set $(\mathbf{N}, \geq)$. We briefly discuss such systems in this section. This material will be discussed more broadly in Homology, Section 10.23. Suppose we are given a ring R and a sequence of R-modules

$$M_1 \xleftarrow{\varphi_2} M_2 \xleftarrow{\varphi_3} M_3 \leftarrow \ldots$$

with maps as indicated. By composing succesive maps we obtain maps $\varphi_{ii'} : M_i \to M_{i'}$ whenever $i \geq i'$ such that moreover $\varphi_{ii''} = \varphi_{i'i''} \circ \varphi_{ii'}$ whenever $i \geq i' \geq i''$. Conversely, given the system of maps $\varphi_{ii'}$ we can set $\varphi_i = \varphi_{i(i-1)}$ and recover the maps displayed above. In this case

$$lim\, M_i = \{(x_i) \in \prod M_i \mid \varphi_i(x_i) = x_{i-1},\ i = 2, 3, \ldots\}$$

compare with Categories, Section 4.14. As explained in Homology, Section 10.23 this is actually a limit in the category of R-modules, as defined in Categories, Section 4.13.

Lemma 7.81.1. *Let R be a ring. Let $0 \to K_i \to L_i \to M_i \to 0$ be short exact sequences of R-modules, $i \geq 1$ which fit into maps of short exact sequences*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_i & \longrightarrow & L_i & \longrightarrow & M_i & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & K_{i+1} & \longrightarrow & L_{i+1} & \longrightarrow & M_{i+1} & \longrightarrow & 0 \end{array}$$

If for every i there exists a $c = c(i) \geq i$ such that $Im(K_c \to K_i) = Im(K_j \to K_i)$ for all $j \geq c$, then the sequence

$$0 \to lim\, K_i \to lim\, L_i \to lim\, M_i \to 0$$

is exact.

Proof. This is a special case of the more general Lemma 7.80.4. □

7.82. Mittag-Leffler modules

A Mittag-Leffler module is (very roughly) a module which can be written as a directed limit whose dual is a Mittag-Leffler system. To be able to give a precise definition we need to do a bit of work.

Definition 7.82.1. Let (M_i, f_{ij}) be a directed system of R-modules. We say that (M_i, f_{ij}) is a *Mittag-Leffler directed system of modules* if each M_i is an R-module of finite presentation and if for every R-module N, the inverse system

$$(Hom_R(M_i, N), Hom_R(f_{ij}, N))$$

is Mittag-Leffler.

We are going to characterize those R-modules that are colimits of Mittag-Leffler directed systems of modules.

Definition 7.82.2. Let $f : M \to N$ and $g : M \to M'$ be maps of R-modules. Then we say g *dominates* f if for any R-module Q, we have $\ker(f \otimes_R \mathrm{id}_Q) \subset \ker(g \otimes_R \mathrm{id}_Q)$.

It is enough to check this condition for finitely presented modules.

Lemma 7.82.3. *Let $f : M \to N$ and $g : M \to M'$ be maps of R-modules. Then g dominates f if and only if for any finitely presented R-module Q, we have* $\ker(f \otimes_R id_Q) \subset \ker(g \otimes_R id_Q)$.

Proof. Suppose $\ker(f \otimes_R \mathrm{id}_Q) \subset \ker(g \otimes_R \mathrm{id}_Q)$ for all finitely presented modules Q. If Q is an arbitrary module, write $Q = colim_{i \in I} Q_i$ as a colimit of a directed system of finitely presented modules Q_i. Then $\ker(f \otimes_R \mathrm{id}_{Q_i}) \subset \ker(g \otimes_R \mathrm{id}_{Q_i})$ for all i. Since taking directed colimits is exact and commutes with tensor product, it follows that $\ker(f \otimes_R \mathrm{id}_Q) \subset \ker(g \otimes_R \mathrm{id}_Q)$. $\square$

The above definition of domination is related to the usual notion of domination of maps as follows.

Lemma 7.82.4. *Let $f : M \to N$ and $g : M \to M'$ be maps of R-modules. Suppose $Coker(f)$ is of finite presentation. Then g dominates f if and only if g factors through f, i.e. there exists a module map $h : N \to M'$ such that $g = h \circ f$.*

Proof. Consider the pushout of f and g,

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "g"'] & N \arrow[d, "g'"] \\ M' \arrow[r, "f'"] & N' \end{tikzcd}$$

where N' is $M' \oplus N$ modulo the submodule consisting of elements $(g(x), -f(x))$ for $x \in M$. We are going to show that the two conditions we wish to prove equivalent are each equivalent to f' being universally injective.

From the definition of N' we have a short exact sequence

$$0 \to \ker(f) \cap \ker(g) \to \ker(f) \to \ker(f') \to 0.$$

Since tensoring commutes with taking pushouts, we have such a short exact sequence

$$0 \to \ker(f \otimes \mathrm{id}_Q) \cap \ker(g \otimes \mathrm{id}_Q) \to \ker(f \otimes \mathrm{id}_Q) \to \ker(f' \otimes \mathrm{id}_Q) \to 0$$

for every R-module Q. So f' is universally injective if and only if $\ker(f \otimes \mathrm{id}_Q) \subset \ker(g \otimes \mathrm{id}_Q)$ for every Q, if and only if g dominates f.

On the other hand, from the definition of the pushout it follows that $\mathrm{Coker}(f') = \mathrm{Coker}(f)$, so $\mathrm{Coker}(f')$ is of finite presentation. Then by Lemma 7.76.4, f' is universally injective if and only if

$$0 \to M' \xrightarrow{f'} N' \to \mathrm{Coker}(f') \to 0$$

splits. This is the case if and only if there is a map $h' : N' \to M'$ such that $h' \circ f' = \mathrm{id}_{M'}$. From the universal property of the pushout, the existence of such an h' is equivalent to g factoring through f. □

Proposition 7.82.5. *Let M be an R-module. Let (M_i, f_{ij}) be a directed system of finitely presented R-modules, indexed by I, such that $M = colim\, M_i$. Let $f_i : M_i \to M$ be the canonical map. The following are equivalent:*

(1) *For every finitely presented R-module P and module map $f : P \to M$, there exists a finitely presented R-module Q and a module map $g : P \to Q$ such that g and f dominate each other, i.e.,* $\ker(f \otimes_R id_N) = \ker(g \otimes_R id_N)$ *for every R-module N.*
(2) *For each $i \in I$, there exists $j \geq i$ such that $f_{ij} : M_i \to M_j$ dominates $f_i : M_i \to M$.*
(3) *For each $i \in I$, there exists $j \geq i$ such that $f_{ij} : M_i \to M_j$ factors through $f_{ik} : M_i \to M_k$ for all $k \geq i$.*
(4) *For every R-module N, the inverse system $(Hom_R(M_i, N), Hom_R(f_{ij}, N))$ is Mittag-Leffler.*
(5) *For $N = \prod_{s\in I} M_s$, the inverse system $(Hom_R(M_i, N), Hom_R(f_{ij}, N))$ is Mittag-Leffler.*

Proof. First we prove the equivalence of (1) and (2). Suppose (1) holds and let $i \in I$. Corresponding to the map $f_i : M_i \to M$, we can choose $g : M_i \to Q$ as in (1). Since M_i and Q are of finite presentation, so is Coker(g). Then by Lemma 7.82.4, $f_i : M_i \to M$ factors through $g : M_i \to Q$, say $f_i = h \circ g$ for some $h : Q \to M$. Then since Q is finitely presented, h factors through $M_j \to M$ for some $j \geq i$, say $h = f_j \circ h'$ for some $h' : Q \to M_j$. In total we have a commutative diagram

$$\begin{tikzcd} & M & \\ M_i \arrow[ur, "f_i"] \arrow[rr, "f_{ij}"] \arrow[dr, "g"'] & & M_j. \arrow[ul, "f_j"'] \\ & Q \arrow[ur, "h'"'] & \end{tikzcd}$$

Thus f_{ij} dominates g. But g dominates f_i, so f_{ij} dominates f_i.

Conversely, suppose (2) holds. Let P be of finite presentation and $f : P \to M$ a module map. Then f factors through $f_i : M_i \to M$ for some $i \in I$, say $f = f_i \circ g'$ for some $g' : P \to M_i$. Choose by (2) a $j \geq i$ such that f_{ij} dominates f_i. We have a commutative diagram

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "g'"'] & M \\ M_i \arrow[ur, "f_i"] \arrow[r, "f_{ij}"'] & M_j . \arrow[u, "f_j"'] \end{tikzcd}$$

From the diagram and the fact that f_{ij} dominates f_i, we find that f and $f_{ij} \circ g'$ dominate each other. Hence taking $g = f_{ij} \circ g' : P \to M_j$ works.

Next we prove (2) is equivalent to (3). Let $i \in I$. It is always true that f_i dominates f_{ik} for $k \geq i$, since f_i factors through f_{ik}. If (2) holds, choose $j \geq i$ such that f_{ij} dominates

f_i. Then since domination is a transitive relation, f_{ij} dominates f_{ik} for $k \geq i$. All M_i are of finite presentation, so $\mathrm{Coker}(f_{ik})$ is of finite presentation for $k \geq i$. By Lemma 7.82.4, f_{ij} factors through f_{ik} for all $k \geq i$. Thus (2) implies (3). On the other hand, if (3) holds then for any R-module N, $f_{ij} \otimes_R \mathrm{id}_N$ factors through $f_{ik} \otimes_R \mathrm{id}_N$ for $k \geq i$. So $\ker(f_{ik} \otimes_R \mathrm{id}_N) \subset \ker(f_{ij} \otimes_R \mathrm{id}_N)$ for $k \geq i$. But $\ker(f_i \otimes_R \mathrm{id}_N : M_i \otimes_R N \to M \otimes_R N)$ is the union of $\ker(f_{ik} \otimes_R \mathrm{id}_N)$ for $k \geq i$. Thus $\ker(f_i \otimes_R \mathrm{id}_N) \subset \ker(f_{ij} \otimes_R \mathrm{id}_N)$ for any R-module N, which by definition means f_{ij} dominates f_i.

It is trivial that (3) implies (4) implies (5). We show (5) implies (3). Let $N = \prod_{s \in I} M_s$. If (5) holds, then given $i \in I$ choose $j \geq i$ such that

$$\mathrm{Im}(Hom(M_j, N) \to Hom(M_i, N)) = \mathrm{Im}(Hom(M_k, N) \to Hom(M_i, N))$$

for all $k \geq j$. Passing the product over $s \in I$ outside of the Hom's and looking at the maps on each component of the product, this says

$$\mathrm{Im}(Hom(M_j, M_s) \to Hom(M_i, M_s)) = \mathrm{Im}(Hom(M_k, M_s) \to Hom(M_i, M_s))$$

for all $k \geq j$ and $s \in I$. Taking $s = j$ we have

$$\mathrm{Im}(Hom(M_j, M_j) \to Hom(M_i, M_j)) = \mathrm{Im}(Hom(M_k, M_j) \to Hom(M_i, M_j))$$

for all $k \geq j$. Since f_{ij} is the image of $\mathrm{id} \in Hom(M_j, M_j)$ under $Hom(M_j, M_j) \to Hom(M_i, M_j)$, this shows that for any $k \geq j$ there is $h \in Hom(M_k, M_j)$ such that $f_{ij} = h \circ f_{ik}$. If $j \geq k$ then we can take $h = f_{kj}$. Hence (3) holds. $\square$

Definition 7.82.6. Let M be an R-module. We say that M is *Mittag-Leffler* if the equivalent conditions of Proposition 7.82.5 hold.

In particular a finitely presented module is Mittag-Leffler.

Remark 7.82.7. Let M be a flat R-module. By Lazard's theorem (Theorem 7.75.4) we can write $M = colim\, M_i$ as the colimit of a directed system (M_i, f_{ij}) where the M_i are free finite R-modules. For M to be Mittag-Leffler, it is enough for the inverse system of duals $(Hom_R(M_i, R), Hom_R(f_{ij}, R))$ to be Mittag-Leffler. This follows from criterion (4) of Proposition 7.82.5 and the fact that for a free finite R-module F, there is a functorial isomorphism $Hom_R(F, R) \otimes_R N \cong Hom_R(F, N)$ for any R-module N.

Lemma 7.82.8. *If R is a ring and M, N are Mittag-Leffler modules over R, then $M \otimes_R N$ is a Mittag-Leffler module.*

Proof. Write $M = colim_{i \in I} M_i$ and $N = colim_{j \in J} N_j$ as directed colimits of finitely presented R-modules. Denote $f_{ii'} : M_i \to M_{i'}$ and $g_{jj'} : N_j \to N_{j'}$ the transition maps. Then $M_i \otimes_R N_j$ is a finitely presented R-module (see Lemma 7.11.14), and $M \otimes_R N = colim_{(i,j) \in I \times J} M_i \otimes_R M_j$. Pick $(i, j) \in I \times J$. By the definition of a Mittag-Leffler module we have Proposition 7.82.5 (3) for both systems. In other words there exist $i' \geq i$ and $j' \geq j$ such that for every choice of $i'' \geq i$ and $j'' \geq j$ there exist maps $a : M_{i''} \to M_{i'}$ and $b : M_{j''} \to M_{j'}$ such that $f_{ii'} = a \circ f_{ii''}$ and $g_{jj'} = b \circ g_{jj''}$. Then it is clear that $a \otimes b : M_{i''} \otimes_R N_{j''} \to M_{i'} \otimes_R N_{j'}$ serves the same purpose for the system $(M_i \otimes_R N_j, f_{ii'} \otimes g_{jj'})$. Thus by the characterization Proposition 7.82.5 (3) we conclude that $M \otimes_R N$ is Mittag-Leffler. $\square$

Lemma 7.82.9. *Let R be a ring and M an R-module. Then M is Mittag-Leffler if and only if for every finite free R-module F and module map $f : F \to M$, there exists a finitely presented R-module Q and a module map $g : F \to Q$ such that g and f dominate each other, i.e.,* $\ker(f \otimes_R id_N) = \ker(g \otimes_R id_N)$ *for every R-module N.*

Proof. Since the condition is clear weaker than condition (1) of Proposition 7.82.5 we see that a Mittag-Leffler module satisfies the condition. Conversely, suppose that M satisfies the condition and that $f : P \to M$ is an R-module map from a finitely presented R-module P into M. Choose a surjection $F \to P$ where F is a finite free R-module. By assumption we can find a map $F \to Q$ where Q is a finitely presented R-module such that $F \to Q$ and $F \to M$ dominate each other. In particular, the kernel of $F \to Q$ contains the kernel of $F \to P$, hence we obtain an R-module map $g : P \to Q$ such that $F \to Q$ is equal to the composition $F \to P \to Q$. Let N be any R-module and consider the commutative diagram

$$\begin{tikzcd} F \otimes_R N \arrow[r] \arrow[d] & Q \otimes_R N \\ P \otimes_R N \arrow[r] \arrow[ur] & M \otimes_R N \end{tikzcd}$$

By assumption the kernels of $F \otimes_R N \to Q \otimes_R N$ and $F \otimes_R N \to M \otimes_R N$ are equal. Hence, as $F \otimes_R N \to P \otimes_R N$ is surjective, also the kernels of $P \otimes_R N \to Q \otimes_R N$ and $P \otimes_R N \to M \otimes_R N$ are equal. □

Lemma 7.82.10. *Let $R \to S$ be a finite and finitely presented ring map. Let M be an S-module. If M is a Mittag-Leffler module over S then M is a Mittag-Leffler module over R.*

Proof. Assume M is a Mittag-Leffler module over S. Write $M = colim\, M_i$ as a directed colimit of finitely presented S-modules M_i. As M is Mittag-Leffler over S there exists for each i an index $j \geq i$ such that for all $k \geq j$ there is a factorization $f_{ij} = h \circ f_{ik}$ (where h depends on i, the choice of j and k). Note that by Lemma 7.7.4 the modules M_i are also finitely presented as R-modules. Moreover, all the maps f_{ij}, f_{ik}, h are maps of R-modules. Thus we see that the system (M_i, f_{ij}) satisfies the same condition when viewed as a system of R-modules. Thus M is Mittag-Leffler as an R-module. □

Lemma 7.82.11. *Let R be a ring. Let $S = R/I$ for some finitely generated ideal I. Let M be an S-module. Then M is a Mittag-Leffler module over R if and only if M is a Mittag-Leffler module over S.*

Proof. One implication follows from Lemma 7.82.10. To prove the other, assume M is Mittag-Leffler as an R-module. Write $M = colim\, M_i$ as a directed colimit of finitely presented S-modules. As I is finitely generated, the ring S is finite and finitely presented as an R-algebra, hence the modules M_i are finitely presented as R-modules, see Lemma 7.7.4. Next, let N be any S-module. Note that for each i we have $Hom_R(M_i, N) = Hom_S(M_i, N)$ as $R \to S$ is surjective. Hence the condition that the inverse system $(Hom_R(M_i, N))_i$ satisfies Mittag-Leffler, implies that the system $(Hom_S(M_i, N))_i$ satisfies Mittag-Leffler. Thus M is Mittag-Leffler over S by definition. □

Remark 7.82.12. Let $R \to S$ be a finite and finitely presented ring map. Let M be an S-module which is Mittag-Leffler as an R-module. Then it is in general not the case that if M is Mittag-Leffler as an S-module. For example suppose that S is the ring of dual numbers over R, i.e., $S = R \oplus R\epsilon$ with $\epsilon^2 = 0$. Then an S-module consists of an R-module M endowed with a square zero R-linear endomorphism $\epsilon : M \to M$. Now suppose that M_0 is an R-module which is not Mittag-Leffler. Choose a presentation $F_1 \xrightarrow{u} F_0 \to M_0 \to 0$ with F_1 and F_0 free R-modules. Set $M = F_1 \oplus F_0$ with

$$\epsilon = \begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix} : M \longrightarrow M.$$

Then $M/\epsilon M \cong F_1 \oplus M_0$ is not Mittag-Leffler over $R = S/\epsilon S$, hence not Mittag-Leffler over S (see Lemma 7.82.11). On the other hand, $M/\epsilon M = M \otimes_S S/\epsilon S$ which would be Mittag-Leffler over S if M was, see Lemma 7.82.8.

7.83. Interchanging direct products with tensor

Let M be an R-module and let $(Q_\alpha)_{\alpha \in A}$ be a family of R-modules. Then there is a canonical map $M \otimes_R \left(\prod_{\alpha \in A} Q_\alpha\right) \to \prod_{\alpha \in A}(M \otimes_R Q_\alpha)$ given on pure tensors by $x \otimes (q_\alpha) \mapsto (x \otimes q_\alpha)$. This map is not necessarily injective or surjective, as the following example shows.

Example 7.83.1. Take $R = \mathbf{Z}$, $M = \mathbf{Q}$, and consider the family $Q_n = \mathbf{Z}/n$ for $n \geq 1$. Then $\prod_n (M \otimes Q_n) = 0$. However there is an injection $\mathbf{Q} \to M \otimes (\prod_n Q_n)$ obtained by tensoring the injection $\mathbf{Z} \to \prod_n Q_n$ by M, so $M \otimes (\prod_n Q_n)$ is nonzero. Thus $M \otimes (\prod_n Q_n) \to \prod_n (M \otimes Q_n)$ is not injective.

On the other hand, take again $R = \mathbf{Z}$, $M = \mathbf{Q}$, and let $Q_n = \mathbf{Z}$ for $n \geq 1$. The image of $M \otimes (\prod_n Q_n) \to \prod_n (M \otimes Q_n) = \prod_n M$ consists precisely of sequences of the form $(a_n/m)_{n \geq 1}$ with $a_n \in \mathbf{Z}$ and m some nonzero integer. Hence the map is not surjective.

We determine below the precise conditions needed on M for the map $M \otimes_R \left(\prod_\alpha Q_\alpha\right) \to \prod_\alpha (M \otimes_R Q_\alpha)$ to be surjective, bijective, or injective for all choices of $(Q_\alpha)_{\alpha \in A}$. This is relevant because the modules for which it is injective turn out to be exactly Mittag-Leffler modules (Proposition 7.83.5). In what follows, if M is an R-module and A a set, we write M^A for the product $\prod_{\alpha \in A} M$.

Proposition 7.83.2. *Let M be an R-module. The following are equivalent:*

(1) *M is finitely generated.*
(2) *For every family $(Q_\alpha)_{\alpha \in A}$ of R-modules, the canonical map $M \otimes_R \left(\prod_\alpha Q_\alpha\right) \to \prod_\alpha (M \otimes_R Q_\alpha)$ is surjective.*
(3) *For every R-module Q and every set A, the canonical map $M \otimes_R Q^A \to (M \otimes_R Q)^A$ is surjective.*
(4) *For every set A, the canonical map $M \otimes_R R^A \to M^A$ is surjective.*

Proof. First we prove (1) implies (2). Choose a surjection $R^n \to M$ and consider the commutative diagram

$$\begin{tikzcd} R^n \otimes_R (\prod_\alpha Q_\alpha) \arrow[r, "\cong"] \arrow[d] & \prod_\alpha (R^n \otimes_R Q_\alpha) \arrow[d] \\ M \otimes_R (\prod_\alpha Q_\alpha) \arrow[r] & \prod_\alpha (M \otimes_R Q_\alpha). \end{tikzcd}$$

The top arrow is an isomorphism and the vertical arrows are surjections. We conclude that the bottom arrow is a surjection.

Obviously (2) implies (3) implies (4), so it remains to prove (4) implies (1). In fact for (1) to hold it suffices that the element $d = (x)_{x \in M}$ of M^M is in the image of the map $f : M \otimes_R R^M \to M^M$. In this case $d = \sum_{i=1}^n f(x_i \otimes a_i)$ for some $x_i \in M$ and $a_i \in R^M$. If for $x \in M$ we write $p_x : M^M \to M$ for the projection onto the x-th factor, then

$$x = p_x(d) = \sum_{i=1}^n p_x(f(x_i \otimes a_i)) = \sum_{i=1}^n p_x(a_i) x_i.$$

Thus $x_1, \ldots, x_n$ generate M. $\square$

Proposition 7.83.3. *Let M be an R-module. The following are equivalent:*

(1) *M is finitely presented.*
(2) *For every family $(Q_\alpha)_{\alpha \in A}$ of R-modules, the canonical map $M \otimes_R \left(\prod_\alpha Q_\alpha\right) \to \prod_\alpha (M \otimes_R Q_\alpha)$ is bijective.*
(3) *For every R-module Q and every set A, the canonical map $M \otimes_R Q^A \to (M \otimes_R Q)^A$ is bijective.*
(4) *For every set A, the canonical map $M \otimes_R R^A \to M^A$ is bijective.*

Proof. First we prove (1) implies (2). Choose a presentation $R^m \to R^n \to M$ and consider the commutative diagram

$$\begin{tikzcd} R^m \otimes_R (\prod_\alpha Q_\alpha) \arrow[r] \arrow[d, "\cong"] & R^m \otimes_R (\prod_\alpha Q_\alpha) \arrow[r] \arrow[d, "\cong"] & M \otimes_R (\prod_\alpha Q_\alpha) \arrow[r] \arrow[d] & 0 \\ \prod_\alpha (R^m \otimes_R Q_\alpha) \arrow[r] & \prod_\alpha (R^n \otimes_R Q_\alpha) \arrow[r] & \prod_\alpha (M \otimes_R Q_\alpha) \arrow[r] & 0. \end{tikzcd}$$

The first two vertical arrows are isomorphisms and the rows are exact. This implies that the map $M \otimes_R (\prod_\alpha Q_\alpha) \to \prod_\alpha (M \otimes_R Q_\alpha)$ is surjective and, by a diagram chase, also injective. Hence (2) holds.

Obviously (2) implies (3) implies (4), so it remains to prove (4) implies (1). From Proposition 7.83.2, if (4) holds we already know that M is finitely generated. So we can choose a surjection $F \to M$ where F is free and finite. Let K be the kernel. We must show K is finitely generated. For any set A, we have a commutative diagram

$$\begin{tikzcd} & K \otimes_R R^A \arrow[r] \arrow[d, "f_3"'] & F \otimes_R R^A \arrow[r] \arrow[d, "f_2"', "\cong"] & M \otimes_R R^A \arrow[r] \arrow[d, "f_1"', "\cong"] & 0 \\ 0 \arrow[r] & K^A \arrow[r] & F^A \arrow[r] & M^A \arrow[r] & 0. \end{tikzcd}$$

The map f_1 is an isomorphism by assumption, the map f_2 is a isomorphism since F is free and finite, and the rows are exact. A diagram chase shows that f_3 is surjective, hence by Proposition 7.83.2 we get that K is finitely generated. □

We need the following lemma for the next proposition.

Lemma 7.83.4. *Let M be an R-module, P a finitely presented R-module, and $f : P \to M$ a map. Let Q be an R-module and suppose $x \in \ker(P \otimes Q \to M \otimes Q)$. Then there exists a finitely presented R-module P' and a map $f' : P \to P'$ such that f factors through f' and $x \in \ker(P \otimes Q \to P' \otimes Q)$.*

Proof. Write M as a colimit $M = colim_{i \in I} M_i$ of a directed system of finitely presented modules M_i. Since P is finitely presented, the map $f : P \to M$ factors through $M_j \to M$ for some $j \in I$. Upon tensoring by Q we have a commutative diagram

$$\begin{tikzcd} & M_j \otimes Q \arrow[dr] & \\ P \otimes Q \arrow[ur] \arrow[rr] & & M \otimes Q. \end{tikzcd}$$

The image y of x in $M_j \otimes Q$ is in the kernel of $M_j \otimes Q \to M \otimes Q$. Since $M \otimes Q = colim_{i \in I}(M_i \otimes Q)$, this means y maps to 0 in $M_{j'} \otimes Q$ for some $j' \geq j$. Thus we may take $P' = M_{j'}$ and f' to be the composite $P \to M_j \to M_{j'}$. □

Proposition 7.83.5. *Let M be an R-module. The following are equivalent:*

(1) *M is Mittag-Leffler.*
(2) *For every family $(Q_\alpha)_{\alpha \in A}$ of R-modules, the canonical map $M \otimes_R \left(\prod_\alpha Q_\alpha\right) \to \prod_\alpha (M \otimes_R Q_\alpha)$ is injective.*

Proof. First we prove (1) implies (2). Suppose M is Mittag-Leffler and let x be in the kernel of $M \otimes_R (\prod_\alpha Q_\alpha) \to \prod_\alpha (M \otimes_R Q_\alpha)$. Write M as a colimit $M = colim_{i \in I} M_i$ of a directed system of finitely presented modules M_i. Then $M \otimes_R (\prod_\alpha Q_\alpha)$ is the colimit of $M_i \otimes_R (\prod_\alpha Q_\alpha)$. So x is the image of an element $x_i \in M_i \otimes_R (\prod_\alpha Q_\alpha)$. We must show that x_i maps to 0 in $M_j \otimes_R (\prod_\alpha Q_\alpha)$ for some $j \geq i$. Since M is Mittag-Leffler, we may choose $j \geq i$ such that $M_i \to M_j$ and $M_i \to M$ dominate each other. Then consider the commutative diagram

$$\begin{tikzcd}
M \otimes_R (\prod_\alpha Q_\alpha) \arrow[r] & \prod_\alpha (M \otimes_R Q_\alpha) \\
M_i \otimes_R (\prod_\alpha Q_\alpha) \arrow[r, "\cong"] \arrow[u] \arrow[d] & \prod_\alpha (M_i \otimes_R Q_\alpha) \arrow[u] \arrow[d] \\
M_j \otimes_R (\prod_\alpha Q_\alpha) \arrow[r, "\cong"] & \prod_\alpha (M_j \otimes_R Q_\alpha)
\end{tikzcd}$$

whose bottom two horizontal maps are isomorphisms, according to Proposition 7.83.3. Since x_i maps to 0 in $\prod_\alpha (M \otimes_R Q_\alpha)$, its image in $\prod_\alpha (M_i \otimes_R Q_\alpha)$ is in the kernel of the map $\prod_\alpha (M_i \otimes_R Q_\alpha) \to \prod_\alpha (M \otimes_R Q_\alpha)$. But this kernel equals the kernel of $\prod_\alpha (M_i \otimes_R Q_\alpha) \to \prod_\alpha (M_j \otimes_R Q_\alpha)$ according to the choice of j. Thus x_i maps to 0 in $\prod_\alpha (M_j \otimes_R Q_\alpha)$ and hence to 0 in $M_j \otimes_R (\prod_\alpha Q_\alpha)$.

Now suppose (2) holds. We prove M satisfies formulation (1) of being Mittag-Leffler from Proposition 7.82.5. Let $f : P \to M$ be a map from a finitely presented module P to M. Choose a set B of representatives of the isomorphism classes of finitely presented R-modules. Let A be the set of pairs (Q, x) where $Q \in B$ and $x \in \ker(P \otimes Q \to M \otimes Q)$. For $\alpha = (Q, x) \in A$, we write Q_α for Q and x_α for x. Consider the commutative diagram

$$\begin{tikzcd}
M \otimes_R (\prod_\alpha Q_\alpha) \arrow[r] & \prod_\alpha (M \otimes_R Q_\alpha) \\
P \otimes_R (\prod_\alpha Q_\alpha) \arrow[r, "\cong"] \arrow[u] & \prod_\alpha (P \otimes_R Q_\alpha) \arrow[u]
\end{tikzcd}$$

The top arrow is an injection by assumption, and the bottom arrow is an isomorphism by Proposition 7.83.3. Let $x \in P \otimes_R (\prod_\alpha Q_\alpha)$ be the element corresponding to $(x_\alpha) \in \prod_\alpha (P \otimes_R Q_\alpha)$ under this isomorphism. Then $x \in \ker(P \otimes_R (\prod_\alpha Q_\alpha) \to M \otimes_R (\prod_\alpha Q_\alpha))$ since the top arrow in the diagram is injective. By Lemma 7.83.4, we get a finitely presented module P' and a map $f' : P \to P'$ such that $f : P \to M$ factors through f' and $x \in \ker(P \otimes_R (\prod_\alpha Q_\alpha) \to P' \otimes_R (\prod_\alpha Q_\alpha))$. We have a commutative diagram

$$\begin{tikzcd}
P' \otimes_R (\prod_\alpha Q_\alpha) \arrow[r, "\cong"] & \prod_\alpha (P' \otimes_R Q_\alpha) \\
P \otimes_R (\prod_\alpha Q_\alpha) \arrow[r, "\cong"] \arrow[u] & \prod_\alpha (P \otimes_R Q_\alpha) \arrow[u]
\end{tikzcd}$$

where both the top and bottom arrows are isomorphisms by Proposition 7.83.3. Thus since x is in the kernel of the left vertical map, (x_α) is in the kernel of the right vertical map. This means $x_\alpha \in \ker(P \otimes_R Q_\alpha \to P' \otimes_R Q_\alpha)$ for every $\alpha \in A$. By the definition of A this means $\ker(P \otimes_R Q \to P' \otimes_R Q) \supset \ker(P \otimes_R Q \to M \otimes_R Q)$ for all finitely presented Q and, since $f : P \to M$ factors through $f' : P \to P'$, actually equality holds. By Lemma 7.82.3, f and f' dominate each other. □

Lemma 7.83.6. *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a universally exact sequence of R-modules. Then:*

(1) *If M_2 is Mittag-Leffler, then M_1 is Mittag-Leffler.*
(2) *If M_1 and M_3 are Mittag-Leffler, then M_2 is Mittag-Leffler.*

Proof. For any family $(Q_\alpha)_{\alpha \in A}$ of R-modules we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & M_1 \otimes_R (\prod_\alpha Q_\alpha) \arrow[r] \arrow[d] & M_2 \otimes_R (\prod_\alpha Q_\alpha) \arrow[r] \arrow[d] & M_3 \otimes_R (\prod_\alpha Q_\alpha) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \prod_\alpha (M_1 \otimes Q_\alpha) \arrow[r] & \prod_\alpha (M_2 \otimes Q_\alpha) \arrow[r] & \prod_\alpha (M_3 \otimes Q_\alpha) \arrow[r] & 0 \end{tikzcd}$$

with exact rows. Thus (1) and (2) follow from Proposition 7.83.5. □

Lemma 7.83.7. *If $M = \bigoplus_{i \in I} M_i$ is a direct sum of R-modules, then M is Mittag-Leffler if and only if each M_i is Mittag-Leffler.*

Proof. The ``only if'' direction follows from Lemma 7.83.6 (1) and the fact that a split short exact sequence is universally exact. For the converse, first note that if I is finite then this follows from Lemma 7.83.6 (2). For general I, if all M_i are Mittag-Leffler then we prove the same of M by verifying condition (1) of Proposition 7.82.5. Let $f : P \to M$ be a map from a finitely presented module P. Then f factors as $P \xrightarrow{f'} \bigoplus_{i' \in I'} M_{i'} \hookrightarrow \bigoplus_{i \in I} M_i$ for some finite subset I' of I. By the finite case $\bigoplus_{i' \in I'} M_{i'}$ is Mittag-Leffler and hence there exists a finitely presented module Q and a map $g : P \to Q$ such that g and f' dominate each other. Then also g and f dominate each other. □

Lemma 7.83.8. *Let $R \to S$ be a ring map. Let M be an S-module. If S is Mittag-Leffler as an R-module, and M is flat and Mittag-Leffler as an S-module, then M is Mittag-Leffler as an R-module.*

Proof. We deduce using the characterization of Proposition 7.83.5. Namely, suppose that Q_α is a family of R-modules. Consider the composition

$$\begin{tikzcd} M \otimes_R \prod_\alpha Q_\alpha = M \otimes_S S \otimes_R \prod_\alpha Q_\alpha \arrow[d] \\ M \otimes_S \prod_\alpha (S \otimes_R Q_\alpha) \arrow[d] \\ \prod_\alpha (M \otimes_S \otimes_R Q_\alpha) = \prod_\alpha (M \otimes_R Q_\alpha) \end{tikzcd}$$

The first arrows is injective as M is flat over S and S is Mittag-Leffler over R and the second arrow is injective as M is Mittag-Leffler over S. Hence M is Mittag-Leffler over R. □

7.84. Coherent rings

We use the discussion on interchanging $\prod$ and $\otimes$ to determine for which rings products of flat modules are flat. It turns out that these are the so-called coherent rings. You may be more familiar with the notion of a coherent $\mathcal{O}_X$-module on a ringed space, see Modules, Section 7.84.

Definition 7.84.1. Let R be a ring. Let M be an R-module.

(1) We say M is a *coherent module* if it is finitely generated and every finitely generated submodule of M is finitely presented over R.
(2) We say R is a *coherent ring* if it is coherent as a module over itself.

Thus a ring is coherent if and only if every finitely generated ideal is finitely presented as a module. The category of coherent modules is abelian.

Lemma 7.84.2. *Let R be a ring.*

(1) *A finite type submodule of a coherent module is coherent.*
(2) *Let $\varphi : N \to M$ be a homomorphism from a finite module to a coherent module. Then $\mathrm{Ker}(\varphi)$ is finite.*
(3) *Let $\varphi : N \to M$ be a homomorphism of coherent modules. Then $\mathrm{Ker}(\varphi)$ and $\mathrm{Coker}(\varphi)$ are coherent modules.*
(4) *The category of coherent modules is an abelian subcategory of Mod_R.*
(5) *Given a short exact sequence of R-modules $0 \to M_1 \to M_2 \to M_3 \to 0$ if two out of three are coherent so is the third.*

Proof. The first statement is immediate from the definition. During the rest of the proof we will use the results of Lemma 7.5.4 without further mention.

Let $\varphi : N \to M$ satisfy the assumptions of (2). Suppose that N is generated by $x_1, \ldots, x_n$. By Definition 7.84.1 the kernel K of the induced map $\bigoplus_{i=1}^n R \to M$, $e_i \mapsto \varphi(x_i)$ is of finite type. Hence $\mathrm{Ker}(\varphi)$ which is the image of the composition $K \to \bigoplus_{i=1}^n R \to N$ is of finite type. This proves (2).

Let $\varphi : N \to M$ satisfy the assumptions of (3). By (2) the kernel of φ is of finite type and hence by (1) it is coherent.

With the same hypotheses let us show that $\mathrm{Coker}(\varphi)$ is coherent. Since M is finite so is $\mathrm{Coker}(\varphi)$. Let $\overline{x}_i \in \mathrm{Coker}(\varphi)$. We have to show that the kernel of the associated morphism $\overline{\Psi} : \bigoplus_{i=1}^n R \to \mathrm{Coker}(\varphi)$ is finite. Choose $x_i \in M$ lifting $\overline{x}_i$. Thus $\overline{\Psi}$ lifts to $\Psi : \bigoplus_{i=1}^n R \to M$. Consider the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(\Psi) \arrow[r] \arrow[d] & \bigoplus_{i=1}^n R \arrow[r] \arrow[d, equal] & R \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Ker}(\overline{\Psi}) \arrow[r] & \bigoplus_{i=1}^n R \arrow[r] & \mathrm{Coker}(\varphi) \arrow[r] & 0
\end{tikzcd}$$

By the snake lemma we get a short exact sequence $0 \to \mathrm{Ker}(\Psi) \to \mathrm{Ker}(\overline{\Psi}) \to \mathrm{Im}(\varphi) \to 0$. Hence we see that $\mathrm{Ker}(\overline{\Psi})$ is finite.

Statement (4) follows from (3).

Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence of R-modules. It suffices to prove that if M_1 and M_3 are coherent so is M_2. By Lemma 7.5.4 we see that M_2 is finite. Let $x_1, \ldots, x_n$ be finitely many elements of M_2. We have to show that the module of relations K between them is finite. Consider the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] \arrow[d] & \bigoplus_{i=1}^n R \arrow[r] \arrow[d] & \bigoplus_{i=1}^n R \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M_1 \arrow[r] & M_2 \arrow[r] & M_3 \arrow[r] & 0 \end{tikzcd}$$

with obvious notation. By the snake lemma we get an exact sequence $0 \to K \to K_3 \to M_1$ where K_3 is the module of relations among the images of the x_i in M_3. Since M_3 is coherent we see that K_3 is a finite module. Since M_1 is coherent we see that the image I of $K_3 \to M_1$ is coherent. Hence K is the kernel of the map $K_3 \to I$ between a finite module and a coherent module and hence finite by (2). $\square$

Lemma 7.84.3. *Let R be a ring. If R is coherent, then a module is coherent if and only if it is finitely presented.*

Proof. It is clear that a coherent module is finitely presented (over any ring). Coversely, if R is coherent, then $R^{\oplus n}$ is coherent and so is the cokernel of any map $R^{\oplus m} \to R^{\oplus n}$, see Lemma 7.84.2. $\square$

Lemma 7.84.4. *A Noetherian ring is a coherent ring.*

Proof. By Lemma 7.28.4 any finite R-module is finitely presented. In particular any ideal of R is finitely presented. $\square$

Proposition 7.84.5. *Let R be a ring. The following are equivalent*

(1) *R is coherent,*
(2) *any product of flat R-modules is flat, and*
(3) *for every set A the module R^A is flat.*

Proof. Assume R coherent, and let Q_α, $\alpha \in A$ be a set of flat R-modules. We have to show that $I \otimes_R \prod_\alpha Q_\alpha \to \prod Q_\alpha$ is injective for every finitely generated ideal I of R, see Lemma 7.35.4. Since R is coherent I is an R-module of finite presentation. Hence $I \otimes_R \prod_\alpha Q_\alpha = \prod I \otimes_R Q_\alpha$ by Proposition 7.83.3. The desired injectivity follows as $I \otimes_R Q_\alpha \to Q_\alpha$ is injective by flatness of Q_α.

The implication (2) $\Rightarrow$ (3) is trivial.

Assume that the R-module R^A is flat for every set A. Let I be a finitely generated ideal in R. Then $I \otimes_R R^A \to R^A$ is injective by assumption. By Proposition 7.83.2 and the finiteness of I the image is equal to I^A. Hence $I \otimes_R R^A = I^A$ for every set A and we conclude that I is finitely presented by Proposition 7.83.3. $\square$

7.85. Examples and non-examples of Mittag-Leffler modules

We end this section with some examples and non-examples of Mittag-Leffler modules.

Example 7.85.1. Mittag-Leffler modules.

(1) Any finitely presented module is Mittag-Leffler. This follows, for instance, from Proposition 7.82.5 (1). In general, it is true that a finitely generated module is Mittag-Leffler if and only it is finitely presented. This follows from Propositions 7.83.2, 7.83.3, and 7.83.5.

(2) A free module is Mittag-Leffler since it satisfies condition (1) of Proposition 7.82.5.
(3) By the previous example together with Lemma 7.83.7, projective modules are Mittag-Leffler.

We also want to add to our list of examples power series rings over a Noetherian ring R. This will be a consequence the following lemma.

Lemma 7.85.2. *Let M be a flat R-module. Suppose the following condition holds: if F is a free finite R-module and $x \in F \otimes_R M$, then there exists a smallest submodule F' of F such that $x \in F' \otimes_R M$. Then M is Mittag-Leffler.*

Proof. By Theorem 7.75.4 we can write M as the colimit $M = colim_{i \in I} M_i$ of a directed system (M_i, f_{ij}) of free finite R-modules. By Remark 7.82.7, it suffices to show that the inverse system $(Hom_R(M_i, R), Hom_R(f_{ij}, R))$ is Mittag-Leffler. In other words, fix $i \in I$ and for $j \geq i$ let Q_j be the image of $Hom_R(M_j, R) \to Hom_R(M_i, R)$; we must show that the Q_j stabilize.

Since M_i is free and finite, we can make the identification $Hom_R(M_i, M_j) = Hom_R(M_i, R) \otimes_R M_j$ for all j. Using the fact that the M_j are free, it follows that for $j \geq i$, Q_j is the smallest submodule of $Hom_R(M_i, R)$ such that $f_{ij} \in Q_j \otimes_R M_j$. Under the identification $Hom_R(M_i, M) = Hom_R(M_i, R) \otimes_R M$, the canonical map $f_i : M_i \to M$ is in $Hom_R(M_i, R) \otimes_R M$. By the assumption on M, there exists a smallest submodule Q of $Hom_R(M_i, R)$ such that $f_i \in Q \otimes_R M$. We are going to show that the Q_j stabilize to Q.

For $j \geq i$ we have a commutative diagram

$$\begin{tikzcd} Q_j \otimes_R M_j \arrow[r] \arrow[d] & Hom_R(M_i, R) \otimes_R M_j \arrow[d] \\ Q_j \otimes_R M \arrow[r] & Hom_R(M_i, R) \otimes_R M. \end{tikzcd}$$

Since $f_{ij} \in Q_j \otimes_R M_j$ maps to $f_i \in Hom_R(M_i, R) \otimes_R M$, it follows that $f_i \in Q_j \otimes_R M$. Hence, by the choice of Q, we have $Q \subset Q_j$ for all $j \geq i$.

Since the Q_j are decreasing and $Q \subset Q_j$ for all $j \geq i$, to show that the Q_j stabilize to Q it suffices to find a $j \geq i$ such that $Q_j \subset Q$. As an element of

$$Hom_R(M_i, R) \otimes_R M = colim_{j \in J}(Hom_R(M_i, R) \otimes_R M_j),$$

f_i is the colimit of f_{ij} for $j \geq i$, and f_i also lies in the submodule

$$colim_{j \in J}(Q \otimes_R M_j) \subset colim_{j \in J}(Hom_R(M_i, R) \otimes_R M_j).$$

It follows that for some $j \geq i$, f_{ij} lies in $Q \otimes_R M_j$. Since Q_j is the smallest submodule of $Hom_R(M_i, R)$ with $f_{ij} \in Q_j \otimes_R M_j$, we conclude $Q_j \subset Q$. □

Lemma 7.85.3. *Let R be a Noetherian ring and A a set. Then $M = R^A$ is a flat and Mittag-Leffler R-module.*

Proof. Combining Lemma 7.84.4 and Proposition 7.84.5 we see that M is flat over R. We show that M satisfies the condition of Lemma 7.85.2. Let F be a free finite R-module. If F'

is any submodule of F then it is finitely presented since R is Noetherian. So by Proposition 7.83.3 we have a commutative diagram

$$\begin{tikzcd} F' \otimes_R M \arrow[r] \arrow[d, "\cong"] & F \otimes_R M \arrow[d, "\cong"] \\ (F')^A \arrow[r] & F^A \end{tikzcd}$$

by which we can identify the map $F' \otimes_R M \to F \otimes_R M$ with $(F')^A \to F^A$. Hence if $x \in F \otimes_R M$ corresponds to $(x_\alpha) \in F^A$, then the submodule of F' of F generated by the x_α is the smallest submodule of F such that $x \in F' \otimes_R M$. $\square$

Lemma 7.85.4. *Let R be a Noetherian ring and n a positive integer. Then the R-module $M = R[[t_1, \ldots, t_n]]$ is flat and Mittag-Leffler.*

Proof. As an R-module, we have $M = R^A$ for a (countable) set A. Hence this lemma is a special case of Lemma 7.85.3. $\square$

Example 7.85.5. Non Mittag-Leffler modules.

(1) By Example 7.83.1 and Proposition 7.83.5, $\mathbf{Q}$ is not a Mittag-Leffler $\mathbf{Z}$-module.
(2) We prove below (Theorem 7.87.3) that for a flat and countably generated module, projectivity is equivalent to being Mittag-Leffler. Thus any flat, countably generated, non-projective module M is an example of a non-Mittag-Leffler module. For such an example, see Remark 7.72.3.
(3) Let k be a field. Let $R = k[[x]]$. The R-module $M = \prod_{n \in \mathbf{N}} R/(x^n)$ is not Mittag-Leffler. Namely, consider the element $\xi = (\xi_1, \xi_2, \xi_3, \ldots)$ defined by $\xi_{2^m} = x^{2^{m-1}}$ and $\xi_n = 0$ else, so
$$\xi = (0, x, 0, x^2, 0, 0, 0, x^4, 0, 0, 0, 0, 0, 0, 0, x^8, \ldots)$$
Then the annihilator of ξ in $M/x^{2^m}M$ is generated $x^{2^{m-1}}$ for $m \gg 0$. But if M was Mittag-Leffler, then there would exist a finite R-module Q and an element $\xi' \in Q$ such that the annihilator of ξ' in Q/x^lQ agrees with the annihilator of ξ in M/x^lM for all $l \geq 1$, see Proposition 7.82.5 (1). Now you can prove there exists an integer $a \geq 0$ such that the annihilator of ξ' in Q/x^lQ is generated by either x^a or x^{l-a} for all $l \gg 0$ (depending on whether $\xi' \in Q$ is torsion or not). The combination of the above would give for all $l = 2^m >> 0$ the equality $a = l/2$ or $l - a = l/2$ which is nonsensical.
(4) The same argument shows that (x)-adic completion of $\bigoplus_{n \in \mathbf{N}} R/(x^n)$ is not Mittag-Leffler over $R = k[[x]]$ (hint: ξ is actually an element of this completion).
(5) Let $R = k[a, b]/(a^2, ab, b^2)$. Let S be the finitely presented R-algebra with presentation $S = R[t]/(at - b)$. Then as an R-module S is countably generated and indecomposable (details omitted). On the other hand, R is Artinian local, hence complete local, hence a henselian local ring, see Lemma 7.139.10. If S was Mittag-Leffler as an R-module, then it would be a direct sum of finite R-modules by Lemma 7.139.26. Thus we conclude that S is not Mittag-Leffler as an R-module.

7.86. Countably generated Mittag-Leffler modules

It turns out that countably generated Mittag-Leffler modules have a particularly simple structure.

Lemma 7.86.1. *Let M be an R-module. Write $M = colim_{i\in I} M_i$ where (M_i, f_{ij}) is a directed system of finitely presented R-modules. If M is Mittag-Leffler and countably generated, then there is a directed countable subset $I' \subset I$ such that $M \cong colim_{i\in I'} M_i$.*

Proof. Let $x_1, x_2, \ldots$ be a countable set of generators for M. For each x_n choose $i \in I$ such that x_n is in the image of the canonical map $f_i : M_i \to M$; let $I'_0 \subset I$ be the set of all these i. Now since M is Mittag-Leffler, for each $i \in I'_0$ we can choose $j \in I$ such that $j \geq i$ and $f_{ij} : M_i \to M_j$ factors through $f_{ik} : M_i \to M_k$ for all $k \geq i$ (condition (3) of Proposition 7.82.5); let I'_1 be the union of I'_0 with all of these j. Since I'_1 is a countable, we can enlarge it to a countable directed set $I'_2 \subset I$. Now we can apply the same procedure to I'_2 as we did to I'_0 to get a new countable set $I'_3 \subset I$. Then we enlarge I'_3 to a countable directed set I'_4. Continuing in this way---adding in a j as in Proposition 7.82.5 (3) for each $i \in I'_\ell$ if ℓ is odd and enlarging I'_ℓ to a directed set if ℓ is even---we get a sequence of subsets $I'_\ell \subset I$ for $\ell \geq 0$. The union $I' = \bigcup I'_\ell$ satisfies:

(1) I' is countable and directed;
(2) each x_n is in the image of $f_i : M_i \to M$ for some $i \in I'$;
(3) if $i \in I'$, then there is $j \in I'$ such that $j \geq i$ and $f_{ij} : M_i \to M_j$ factors through $f_{ik} : M_i \to M_k$ for all $k \in I$ with $k \geq i$. In particular $\ker(f_{ik}) \subset \ker(f_{ij})$ for $k \geq i$.

We claim that the canonical map $colim_{i\in I'} M_i \to colim_{i\in I} M_i = M$ is an isomorphism. By (2) it is surjective. For injectivity, suppose $x \in colim_{i\in I'} M_i$ maps to 0 in $colim_{i\in I} M_i$. Representing x by an element $\tilde{x} \in M_i$ for some $i \in I'$, this means that $f_{ik}(\tilde{x}) = 0$ for some $k \in I, k \geq i$. But then by (3) there is $j \in I', j \geq i$, such that $f_{ij}(\tilde{x}) = 0$. Hence $x = 0$ in $colim_{i\in I'} M_i$. □

Lemma 7.86.1 above implies that a countably generated Mittag-Leffler module M over R is the colimit of a system

$$M_1 \to M_2 \to M_3 \to M_4 \to \ldots$$

with each M_n a finitely presented R-module. To see this argue as in the proof of Lemma 7.80.3 to see that a countable directed partially ordered set has a cofinal subset isomorphic to $(\mathbf{N}, \geq)$. Suppose $R = k[x_1, x_2, x_3, \ldots]$ and $M = R/(x_i)$. Then M is finitely generated but not finitely presented, hence not Mittag-Leffler (see Example 7.85.1 part (1)). But of course you can write $M = colim_n M_n$ by taking $M_n = R/(x_1, \ldots, x_n)$, hence the condition that you can write M as such a limit does not imply that M is Mittag-Leffler.

Lemma 7.86.2. *Let R be a ring. Let M be an R-module. Assume M is Mittag-Leffler and countably generated. For any R-module map $f : P \to M$ with P finitely generated there exists an endomorphism $\alpha : M \to M$ such that*

(1) *$\alpha : M \to M$ factors through a finitely presented R-module, and*
(2) *$\alpha \circ f = f$.*

Proof. Write $M = colim_{i\in I} M_i$ as a directed colimit of finitely presented R-modules with I countable, see Lemma 7.86.1. The transition maps are denoted f_{ij} and we use $f_i : M_i \to M$ to note the canonical maps into M. Set $N = \prod_{s\in I} M_s$. Denote

$$M_i^* = Hom_R(M_i, N) = \prod_{s\in S} Hom_R(M_i, M_s)$$

so that (M_i^*) is an inverse system of R-modules over I. Note that $Hom_R(M, N) = lim\, M_i^*$. As M is Mittag-Leffler, we find for every $i \in I$ an index $k(i) \geq i$ such that

$$E_i := \bigcap\nolimits_{i' \geq i} \mathrm{Im}(M_{i'}^* \to M_i^*) = \mathrm{Im}(M_{k(i)}^* \to M_i^*)$$

Choose $i \in I$ such that $\mathrm{Im}(P \to M) \subset \mathrm{Im}(M_i \to M)$. This is possible as P is finitely generated. Set $k = k(i)$ for this i. Let $x = (0, \ldots, 0, \mathrm{id}_{M_k}, 0, \ldots, 0) \in M_k^*$ and note that this maps to $y = (0, \ldots, 0, f_{ik}, 0, \ldots, 0) \in M_i^*$. By our choice of k we see that $y \in E_i$. By Example 7.80.2 the transition maps $E_{i'} \to E_i$ are surjective for each $i' \geq i$ and $lim\, E_i = lim\, M_i^* = Hom_R(M, N)$. Hence Lemma 7.80.3 guarantees there exists an element $z \in Hom_R(M, N)$ which maps to y in $E_i \subset M_i^*$. Let z_k be the kth component of z. Then $z_k : M \to M_k$ is a homomorphism such that

$$\begin{tikzcd} M \arrow[r, "z_k"'] & M_k \\ M_i \arrow[u, "f_i"] \arrow[ur, "f_{ik}"'] & \end{tikzcd}$$

commutes. Let $\alpha : M \to M$ be the composition $f_k \circ z_k : M \to M_k \to M$. Then α factors through a finitely presented module by construction and $\alpha \circ f_i = f_i$. Since the image of f is contained in the image of f_i this also implies that $\alpha \circ f = f$. □

We will see later (see Lemma 7.139.26) that Lemma 7.86.2 means that a countably generated Mittag-Leffler module over a henselian local ring is a direct sum of finitely presented modules.

7.87. Characterizing projective modules

The goal of this section is to prove that a module is projective if and only if it is flat, Mittag-Leffler, and a direct sum of countably generated modules (Theorem 7.87.3 below).

Lemma 7.87.1. *Let M be an R-module. If M is flat, Mittag-Leffler, and countably generated, then M is projective.*

Proof. By Lazard's theorem (Theorem 7.75.4), we can write $M = colim_{i \in I} M_i$ for a directed system of finite free R-modules (M_i, f_{ij}) indexed by a set I. By Lemma 7.86.1, we may assume I is countable. Now let

$$0 \to N_1 \to N_2 \to N_3 \to 0$$

be an exact sequence of R-modules. We must show that applying $Hom_R(M, -)$ preserves exactness. Since M_i is finite free,

$$0 \to Hom_R(M_i, N_1) \to Hom_R(M_i, N_2) \to Hom_R(M_i, N_3) \to 0$$

is exact for each i. Since M is Mittag-Leffler, $(Hom_R(M_i, N_1))$ is a Mittag-Leffler inverse system. So by Lemma 7.80.4,

$$0 \to lim_{i \in I} Hom_R(M_i, N_1) \to lim_{i \in I} Hom_R(M_i, N_2) \to lim_{i \in I} Hom_R(M_i, N_3) \to 0$$

is exact. But for any R-module N there is a functorial isomorphism $Hom_R(M, N) \cong lim_{i \in I} Hom_R(M_i, N)$, so

$$0 \to Hom_R(M, N_1) \to Hom_R(M, N_2) \to Hom_R(M, N_3) \to 0$$

is exact. □

Remark 7.87.2. Lemma 7.87.1 does not hold without the countable generation assumption. For example, the $\mathbf{Z}$-module $M = \mathbf{Z}[[x]]$ is flat and Mittag-Leffler but not projective. It is Mittag-Leffler by Lemma 7.85.4. Subgroups of free abelian groups are free, hence a projective $\mathbf{Z}$-module is in fact free and so are its submodules. Thus to show M is not projective it suffices to produce a non-free submodule. Fix a prime p and consider the submodule N consisting of power series $f(x) = \sum a_i x^i$ such that for every integer $m \geq 1$, p^m divides a_i for all but finitely many i. Then $\sum a_i p^i x^i$ is in N for all $a_i \in \mathbf{Z}$, so N is uncountable. Thus if N were free it would have uncountable rank and the dimension of N/pN over $\mathbf{Z}/p$ would be uncountable. This is not true as the elements $x^i \in N/pN$ for $i \geq 0$ span N/pN.

Theorem 7.87.3. *Let M be an R-module. Then M is projective if and only it satisfies:*

1. *M is flat,*
2. *M is Mittag-Leffler,*
3. *M is a direct sum of countably generated R-modules.*

Proof. First suppose M is projective. Then M is a direct summand of a free module, so M is flat and Mittag-Leffler since these properties pass to direct summands. By Kaplansky's theorem (Theorem 7.78.5), M satisfies (3).

Conversely, suppose M satisfies (1)-(3). Since being flat and Mittag-Leffler passes to direct summands, M is a direct sum of flat, Mittag-Leffler, countably generated R-modules. Lemma 7.87.1 implies M is a direct sum of projective modules. Hence M is projective. □

Lemma 7.87.4. *Let $f : M \to N$ be universally injective map of R-modules. Suppose M is a direct sum of countably generated R-modules, and suppose N is flat and Mittag-Leffler. Then M is projective.*

Proof. By Lemmas 7.76.7 and 7.83.6, M is flat and Mittag-Leffler, so the conclusion follows from Theorem 7.87.3. □

Lemma 7.87.5. *Let R be a Noetherian ring and let M be a R-module. Suppose M is a direct sum of countably generated R-modules, and suppose there is a universally injective map $M \to R[[t_1, \ldots, t_n]]$ for some n. Then M is projective.*

Proof. Follows from Lemmas 7.87.4 and 7.85.4. □

7.88. Ascending properties of modules

All of the properties of a module in Theorem 7.87.3 ascend along arbitrary ring maps:

Lemma 7.88.1. *Let $R \to S$ be a ring map. Let M be an R-module. Then:*

1. *If M is flat, then the S-module $M \otimes_R S$ is flat.*
2. *If M is Mittag-Leffler, then the S-module $M \otimes_R S$ is Mittag-Leffler.*
3. *If M is a direct sum of countably generated R-modules, then the S-module $M \otimes_R S$ is a direct sum of countably generated S-modules.*
4. *If M is projective, then the S-module $M \otimes_R S$ is projective.*

Proof. All are obvious except (2). For this, use formulation (3) of being Mittag-Leffler from Proposition 7.82.5 and the fact that tensoring commutes with taking colimits. □

7.89. Descending properties of modules

We address the faithfully flat descent of the properties from Theorem 7.87.3 that characterize projectivity. In the presence of flatness, the property of being a Mittag-Leffler module descends:

Lemma 7.89.1. *Let $R \to S$ be a faithfully flat ring map. Let M be an R-module. If the S-module $M \otimes_R S$ is flat and Mittag-Leffler, then M is flat and Mittag-Leffler.*

Proof. By Lemma 7.77.2, flatness descends, so M is flat. Thus by Lazard's theorem (Theorem 7.75.4) we can write $M = colim_{i \in I} M_i$ where (M_i, f_{ij}) is a directed system of free finite R-modules. According to Remark 7.82.7, to prove M is Mittag-Leffler it is enough to show that $(Hom_R(M_i, R))$ is a Mittag-Leffler inverse system.

Since tensoring commutes with colimits, $M \otimes_R S = colim(M_i \otimes_R S)$. Since $M \otimes_R S$ is Mittag-Leffler this means $(Hom_S(M_i \otimes_R S, S))$ is a Mittag-Leffler inverse system. So for every $i \in I$, the family $\text{Im}(Hom_S(M_j \otimes_R S, S) \to Hom_S(M_i \otimes_R S, S))$ for $j \geq i$ stabilizes. Because M_i is free and finite there is a functorial isomorphism $Hom_S(M_i \otimes_R S, S) \cong Hom_R(M_i, R) \otimes_R S$, and because $R \to S$ is faithfully flat, tensoring by S commutes with taking the image of a module map. Thus we find that for every $i \in I$, the family $\text{Im}(Hom_R(M_j, R) \to Hom_R(M_i, R)) \otimes_R S$ for $j \geq i$ stabilizes. But if N is an R-module and $N' \subset N$ a submodule such that $N' \otimes_R S = N \otimes_R S$, then $N' = N$ by faithful flatness of S. We conclude that for every $i \in I$, the family $\text{Im}(Hom_R(M_j, R) \to Hom_R(M_i, R))$ for $j \geq i$ stabilizes. So M is Mittag-Leffler. □

At this point the faithfully flat descent of countably generated projective modules follows easily.

Lemma 7.89.2. *Let $R \to S$ be a faithfully flat ring map. Let M be an R-module. If the S-module $M \otimes_R S$ is countably generated and projective, then M is countably generated and projective.*

Proof. Follows from Lemma 7.89.1, the fact that countable generation descends, and Theorem 7.87.3. □

All that remains is to use dévissage to reduce descent of projectivity in the general case to the countably generated case. First, two simple lemmas.

Lemma 7.89.3. *Let $R \to S$ be a ring map, let M be an R-module, and let Q be a countably generated S-submodule of $M \otimes_R S$. Then there exists a countably generated R-submodule P of M such that $Im(P \otimes_R S \to M \otimes_R S)$ contains Q.*

Proof. Let $y_1, y_2, \ldots$ be generators for Q and write $y_j = \sum_k x_{jk} \otimes s_{jk}$ for some $x_{jk} \in M$ and $s_{jk} \in S$. Then take P be the submodule of M generated by the x_{jk}. □

Lemma 7.89.4. *Let $R \to S$ be a ring map, and let M be an R-module. Suppose $M \otimes_R S = \bigoplus_{i \in I} Q_i$ is a direct sum of countably generated S-modules Q_i. If N is a countably generated submodule of M, then there is a countably generated submodule N' of M such that $N' \supset N$ and $Im(N' \otimes_R S \to M \otimes_R S) = \bigoplus_{i \in I'} Q_i$ for some subset $I' \subset I$.*

Proof. Let $N'_0 = N$. We construct by induction an increasing sequence of countably generated submodules $N'_\ell \subset M$ for $\ell = 0, 1, 2, \ldots$ such that: if I'_ℓ is the set of $i \in I$ such that the projection of $\text{Im}(N'_\ell \otimes_R S \to M \otimes_R S)$ onto Q_i is nonzero, then $\text{Im}(N'_{\ell+1} \otimes_R S \to M \otimes_R S)$ contains Q_i for all $i \in I'_\ell$. To construct $N'_{\ell+1}$ from N'_ℓ, let Q be the sum of (the countably many) Q_i for $i \in I'_\ell$, choose P as in Lemma 7.89.3, and then let $N'_{\ell+1} = N'_\ell + P$. Having constructed the N'_ℓ, just take $N' = \bigcup_\ell N'_\ell$ and $I' = \bigcup_\ell I'_\ell$. □

Theorem 7.89.5. *Let $R \to S$ be a faithfully flat ring map. Let M be an R-module. If the S-module $M \otimes_R S$ is projective, then M is projective.*

Proof. We are going to construct a Kaplansky dévissage of M to show that it is a direct sum of projective modules and hence projective. By Theorem 7.78.5 we can write $M \otimes_R S = \bigoplus_{i\in I} Q_i$ as a direct sum of countably generated S-modules Q_i. Choose a well-ordering on M. By transfinite induction we are going to define an increasing family of submodules M_α of M, one for each ordinal α, such that $M_\alpha \otimes_R S$ is a direct sum of some subset of the Q_i.

For $\alpha = 0$ let $M_0 = 0$. If α is a limit ordinal and M_β has been defined for all $\beta < \alpha$, then define $M_\beta = \bigcup_{\beta<\alpha} M_\beta$. Since each $M_\beta \otimes_R S$ for $\beta < \alpha$ is a direct sum of a subset of the Q_i, the same will be true of $M_\alpha \otimes_R S$. If $\alpha + 1$ is a successor ordinal and M_α has been defined, then define $M_{\alpha+1}$ as follows. If $M_\alpha = M$, then let $M_{\alpha+1} = M$. Otherwise choose the smallest $x \in M$ (with respect to the fixed well-ordering) such that $x \notin M_\alpha$. Since S is flat over R, $(M/M_\alpha)\otimes_R S = M\otimes_R S/M_\alpha \otimes_R S$, so since $M_\alpha \otimes_R S$ is a direct sum of some Q_i, the same is true of $(M/M_\alpha)\otimes_R S$. By Lemma 7.89.4, we can find a countably generated R-submodule P of M/M_α containing the image of x in M/M_α and such that $P\otimes_R S$ (which equals $\mathrm{Im}(P\otimes_R S \to M\otimes_R S)$ since S is flat over R) is a direct sum of some Q_i. Since $M\otimes_R S = \bigoplus_{i\in I} Q_i$ is projective and projectivity passes to direct summands, $P\otimes_R S$ is also projective. Thus by Lemma 7.89.2, P is projective. Finally we define $M_{\alpha+1}$ to be the preimage of P in M, so that $M_{\alpha+1}/M_\alpha = P$ is countably generated and projective. In particular M_α is a direct summand of $M_{\alpha+1}$ since projectivity of $M_{\alpha+1}/M_\alpha$ implies the sequence $0 \to M_\alpha \to M_{\alpha+1} \to M_{\alpha+1}/M_\alpha \to 0$ splits.

Transfinite induction on M (using the fact that we constructed $M_{\alpha+1}$ to contain the smallest $x \in M$ not contained in M_α) shows that each $x \in M$ is contained in some M_α. Thus, there is some large enough ordinal S satisfying: for each $x \in M$ there is $\alpha \in S$ such that $x \in M_\alpha$. This means $(M_\alpha)_{\alpha\in S}$ satisfies property (1) of a Kaplansky dévissage of M. The other properties are clear by construction. We conclude $M = \bigoplus_{\alpha+1\in S} M_{\alpha+1}/M_\alpha$. Since each $M_{\alpha+1}/M_\alpha$ is projective by construction, M is projective. □

7.90. Completion

Suppose that R is a ring and I is an ideal. We define the *completion of R with respect to I* to be the limit

$$R^\wedge = lim_n\, R/I^n.$$

An element of $R^\wedge$ is simply given by a sequence of elements $f_n \in R/I^n$ such that $f_n \equiv f_{n+1} \bmod I^n$ for all n. Similarly, if M is an R-module then we define the *completion of M with respect to I* to be the limit

$$M^\wedge = lim_n\, M/I^nM.$$

An element of $M^\wedge$ is simply given by a sequence of elements $m_n \in M/I^nM$ such that $m_n \equiv m_{n+1} \bmod I^nM$ for all n. From this description it is clear that there are always canonical maps

$$M \longrightarrow M^\wedge, \quad \text{and} \quad M\otimes_R R^\wedge \longrightarrow M^\wedge.$$

Moreover, given a map $\varphi : M \to N$ of modules we get an induced map $\varphi^\wedge : M^\wedge \to N^\wedge$ on completions making the diagram

$$\begin{tikzcd} M \arrow[r] \arrow[d] & N \arrow[d] \\ M^\wedge \arrow[r] & N^\wedge \end{tikzcd}$$

commute. In general completion is not an exact functor, see Examples, Section 64.4. Here are some initial positive results.

Lemma 7.90.1. *Let R be a ring. Let $I \subset R$ be an ideal. Let $\varphi : M \to N$ be a map of R-modules.*

(1) *If $M/IM \to N/IN$ is surjective, then $M^\wedge \to N^\wedge$ is surjective.*
(2) *If $M \to N$ is surjective, then $M^\wedge \to N^\wedge$ is surjective.*
(3) *If $0 \to K \to M \to N \to 0$ is a short exact sequence of R-modules and N is flat, then $0 \to K^\wedge \to M^\wedge \to N^\wedge \to 0$ is a short exact sequence.*
(4) *The map $M \otimes_R R^\wedge \to M^\wedge$ is surjective for any finite R-module M.*

Proof. Assume $M/IM \to N/IN$ is surjective. Then the map $M/I^nM \to N/I^nN$ is surjective for each $n \geq 1$ by Nakayama's lemma (Lemma 7.14.5). Set $K_n = \{x \in M \mid \varphi(x) \in I^nN\}$. Thus we get short exact sequences

$$0 \to K_n/I^nM \to M/I^nM \to N/I^nN \to 0$$

We claim that the canonical map $K_{n+1}/I^{n+1}M \to K_n/I^nM$ is surjective. Namely, if $x \in K_n$ write $\varphi(x) = \sum z_jn_j$ with $z_j \in I^n$, $n_j \in N$. By assumption we can write $n_j = \varphi(m_j) + \sum z_{jk}n_{jk}$ with $m_j \in M$, $z_{jk} \in I$ and $n_{jk} \in N$. Hence

$$\varphi(x - \sum z_jm_j) = \sum z_jz_{jk}n_{jk}.$$

This means that $x' = x - \sum z_jm_j \in K_{n+1}$ maps to x which proves the claim. Now we may apply Lemma 7.81.1 to the inverse system of short exact sequences above to see (1). Part (2) is a special case of (1). If the assumptions of (3) hold, then for each n the sequence

$$0 \to K/I^nK \to M/I^nM \to N/I^nN \to 0$$

is short exact by Lemma 7.35.11. Hence we can directly apply Lemma 7.81.1 to conclude (3) is true. To see (4) choose generators $x_i \in M$, $i = 1, \ldots, n$. Then the map $R^{\oplus n} \to M$, $(a_1, \ldots, a_n) \mapsto \sum a_ix_i$ is surjective. Hence by (2) we see $(R^\wedge)^{\oplus n} \to M^\wedge$, $(a_1, \ldots, a_n) \mapsto \sum a_ix_i$ is surjective. Assertion (4) follows from this. $\square$

Lemma 7.90.2. *Suppose R is Noetherian.*

(1) *If $N \to M$ is an injective map of finite R-modules, then the map on completions $N^\wedge \to M^\wedge$ is injective.*
(2) *If M is a finite R-module, then $M^\wedge = M \otimes_R R^\wedge$.*

Proof. For the first statement, by the Artin-Rees Lemma 7.47.4, we have a constant c such that $I^nM \cap N$ equals $I^{n-c}(I^cM \cap N) \subset I^{n-c}N$. Thus if $(n_i) \in N^\wedge$ maps to zero in $M^\wedge$, then each n_i maps to zero in $N/I^{i-c}N$. And hence $n_{i-c} = 0$. Thus $N^\wedge \to M^\wedge$ is injective.

For the second statement let $0 \to K \to R^t \to M \to 0$ be a presentation of M, corresponding the the generators $x_1, \ldots, x_t$ of M. By Lemma 7.90.1 $(R^t)^\wedge \to M^\wedge$ is surjective, and for any finitely generated R-module the canonical map $M \otimes_R R^\wedge \to M^\wedge$ is surjective. Hence to prove the second statement it suffices to prove the kernel of $(R^t)^\wedge \to M^\wedge$ is exactly $K^\wedge$.

Let $(x_n) \in (R^t)^\wedge$ be in the kernel. Note that each x_n is in the image of the map $K/I^nK \to (R/I^n)^t$. Choose c such that $(I^n)^t \cap K \subset I^{n-c}K$, which is possible by Artin-Rees (Lemma 7.47.4). For each $n \geq 0$ choose $y_n \in K/I^{n+c}K$ mapping to x_{n+c}, and set $z_n = y_n \bmod I^nK$. The elements z_n satisfy $z_{n+1} - z_n \bmod I^nK = y_{n+1} - y_n \bmod I^nK$, and $y_{n+1} - y_n \in I^{n+c}R^t$ by construction. Hence $z_{n+1} = z_n \bmod I^nK$ by the choice of c above. In other words $(z_n) \in K^\wedge$ maps to (x_n) as desired. □

Lemma 7.90.3. *Let R be a Noetherian ring. Let $I \subset R$ be an ideal.*

(1) *The ring map $R \to R^\wedge$ is flat.*
(2) *The functor $M \mapsto M^\wedge$ is exact on the category of finitely generated R-modules.*

Proof. Consider $I \otimes_R R^\wedge \to R \otimes_R R^\wedge = R^\wedge$. According to Lemma 7.90.2 this is identified with $I^\wedge \to R^\wedge$ and $I^\wedge \to R^\wedge$ is injective. Part (1) follows from Lemma 7.35.4. Part (2) follows from part (1) and Lemma 7.90.2 part (2). □

Lemma 7.90.4. *Let R be a Noetherian local ring. Let $\mathfrak{m} \subset R$ be the maximal ideal. Let $I \subset \mathfrak{m}$ be an ideal. The ring map $R \to R^\wedge$ is faithfully flat. In particular the completion with respect to $\mathfrak{m}$, namely $lim_n R/\mathfrak{m}^n$ is faithfully flat.*

Proof. By Lemma 7.90.3 it is flat. The composition $R \to R^\wedge \to R/\mathfrak{m}$ where the last map is the projection map $R^\wedge \to R/I$ combined with $R/I \to R/\mathfrak{m}$ shows that $\mathfrak{m}$ is in the image of $Spec(R^\wedge) \to Spec(R)$. Hence the map is faithfully flat by Lemma 7.35.14. □

Definition 7.90.5. Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module. We say M is *I-adically complete* if the map

$$M \longrightarrow M^\wedge = lim_n M/I^nM$$

is an isomorphism[4]. We say R is *I-adically complete* if R is complete as an R-module.

It is not true that the completion of an R-module M with respect to I is I-adically complete. For an example see Examples, Section 64.2. Here is a lemma from an unpublished note of Lenstra and de Smit.

Lemma 7.90.6. *Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module. Denote $K_n = Ker(M^\wedge \to M/I^nM)$. Then $M^\wedge$ is I-adically complete if and only if K_n is equal to $I^nM^\wedge$ for all $n \geq 1$.*

Proof. The module $I^nM^\wedge$ is contained in K_n. Thus for each $n \geq 1$ there is a canonical exact sequence

$$0 \to K_n/I^nM^\wedge \to M^\wedge/I^nM^\wedge \to M/I^nM \to 0.$$

As $I^nM^\wedge$ maps onto $I^nM/I^{n+1}M$ we see that $K_{n+1} + I^nM^\wedge = K_n$. Thus the inverse system $\{K_n/I^nM^\wedge\}_{n \geq 1}$ has surjective transition maps. By Lemma 7.81.1 we see that there is a short exact sequence

$$0 \to lim_n K_n/I^nM^\wedge \to (M^\wedge)^\wedge \to M^\wedge \to 0$$

Hence $M^\wedge$ is complete if and only if $K_n/I^nM^\wedge = 0$ for all $n \geq 1$. □

Lemma 7.90.7. *Let R be a ring. Let I be a finitely generated ideal of R. For any R-module M the completion $M^\wedge$ is complete. In particular $R^\wedge$ is complete.*

[4] This includes the condition that $\bigcap I^nM = (0)$.

Proof. Let $K_n = \text{Ker}(M^\wedge \to M/I^nM)$. By Lemma 7.90.6 we have to show that $K_n = I^nM^\wedge$. Write $I = (f_1, \ldots, f_t)$. Let $z \in K_n$. Write $z = (\overline{z}_m)$ with $\overline{z}_m \in M/I^mM$. Choose $z_m \in M$ mapping to $\overline{z}_m$ in M/I^mM. Then $z_{m+1} = z_m \bmod I^m$. Write $z_{n+1} = z_n + \delta_n$, $z_{n+2} = z_{n+1} + \delta_{n+1}$, etc. Then $\delta_m \in I^mM$. Thus the infinite sum

$$z = z_n + \delta_n + \delta_{n+1} + \delta_{n+2} + \ldots$$

converges in $M^\wedge$. For $m \geq n$ we have $\delta_m \in I^mM$ hence we can write

$$\delta_m = \sum\nolimits_{j_1+\ldots+j_t=n} f_1^{j_1} \ldots f_t^{j_t} \alpha_{J,m}$$

with $\alpha_{J,m} \in I^{m-n}M$. Our assumption $z \in K_n$ means $z_n \in I^nM$ hence we can also write

$$z_n = \sum\nolimits_{j_1+\ldots+j_t=n} f_1^{j_1} \ldots f_t^{j_t} \alpha_J$$

with $\alpha_J \in M$. Then we can set

$$z_J = \alpha_J + \alpha_{J,n} + \alpha_{J,n+1} + \alpha_{J,n+2} + \ldots$$

as an element of $M^\wedge$. By construction $z = \sum_J f_1^{j_1} \ldots f_t^{j_t} z_J$. Hence z is an element of $I^nM^\wedge$ as desired. □

Lemma 7.90.8. *Let R be a Noetherian ring. Let I be an ideal of R. Let M be an R-module. Then the completion $M^\wedge$ of M with respect to I is I-adically complete.*

Proof. This is a special case of Lemma 7.90.7 because I is a finitely generated ideal. □

Lemma 7.90.9. *Let R be a ring. Let $I \subset R$ be an ideal. Assume*

(1) *R/I is a Noetherian ring,*
(2) *I is finitely generated.*

Then $R^\wedge$ is a Noetherian ring complete with respect to $IR^\wedge$.

Proof. By Lemma 7.90.7 we see that $R^\wedge$ is I-adically complete. Hence it is also $IR^\wedge$-adically complete. Since $R^\wedge/IR^\wedge = R/I$ is Noetherian we see that after replacing R by $R^\wedge$ we may in addition to assumptions (1) and (2) assume that also R is I-adically complete.

Let $f_1, \ldots, f_t$ be generators of I. Then there is a surjection of rings $R/I[T_1, \ldots, T_t] \to \bigoplus I^n/I^{n+1}$ mapping T_i to the element $\overline{f}_i \in I/I^2$. Hence $\bigoplus I^n/I^{n+1}$ is a Noetherian ring. Let $J \subset R$ be an ideal. Consider the ideal

$$\bigoplus J \cap I^n/J \cap I^{n+1} \subset \bigoplus I^n/I^{n+1}.$$

Let $\overline{g}_1, \ldots, \overline{g}_m$ be generators of this ideal. We may choose $\overline{g}_j$ to be a homogeneous element of degree d_j and we may pick $g_j \in J \cap I^{d_j}$ mapping to $\overline{g}_j \in J \cap I^{d_j}/J \cap I^{d_j+1}$. We claim that $g_1, \ldots, g_m$ generate J.

Let $x \in J \cap I^n$. There exist $a_j \in I^{\max(0,n-d_j)}$ such that $x - \sum a_jg_j \in J \cap I^{n+1}$. The reason is that $J \cap I^n/J \cap I^{n+1}$ is equal to $\sum \overline{g}_jI^{n-d_j}/I^{n-d_j+1}$ by our choice of $g_1, \ldots, g_m$. Hence starting with $x \in J$ we can find a sequence of vectors $(a_{1,n}, \ldots, a_{m,n})_{n \geq 0}$ with $a_{j,n} \in I^{\max(0,n-d_j)}$ such that

$$x = \sum\nolimits_{n=0,\ldots,N} \sum\nolimits_{j=1,\ldots,m} a_{j,n}g_j \bmod I^{N+1}$$

Setting $A_j = \sum_{n \geq 0} a_{j,n}$ we see that $x = \sum A_jg_j$ as R is complete. Hence J is finitely generated and we win. □

Lemma 7.90.10. *Let R be a Noetherian ring. Let I be an ideal of R. The completion $R^\wedge$ of R with respect to I is Noetherian.*

Proof. This is a direct consequence of Lemma 7.90.9. It can also be seen directly as follows. Choose generators $f_1, \ldots, f_n$ of I. Consider the map

$$R[[x_1, \ldots, x_n]] \longrightarrow R^\wedge, \quad x_i \longmapsto f_i.$$

This is a well defined and surjective ring map (details omitted). Since $R[[x_1, \ldots, x_n]]$ is Noetherian (see Lemma 7.28.2) we win. □

Lemma 7.90.11. *Let R be a ring. Let $I \subset R$ be an ideal. Then*

1. *any element of $1 + I$ maps to an invertible element of $R^\wedge$,*
2. *any element of $1 + IR^\wedge$ is invertible in $R^\wedge$, and*
3. *the ideal $IR^\wedge$ is contained in the radical of $R^\wedge$.*

Proof. Let $x \in IR^\wedge$. Let u_n be the image of $1 - x + x^2 - x^3 + \ldots + (-x)^n$ in R/I^{n+1}. Note that (u_n) defines an element u of $R^\wedge$. By construction the element $(1 + x)u_n$ in R/I^n is 1. Hence we see that u is the inverse of $1 + x$ in $R^\wedge$. This proves (2) which implies (1). Let $\mathfrak m \subset R^\wedge$ be a maximal ideal and set $M = R^\wedge/\mathfrak m$. If $IR^\wedge \not\subset \mathfrak m$ then $IM = M$. By Nakayama's Lemma 7.14.5 there exists an $f \in IR^\wedge$ such that $(1 + f)M = 0$. This is a contradiction with (2). □

Lemma 7.90.12. *Let R be a ring. Let I, J be ideals of R. Assume there exist integers $c, d > 0$ such that $I^c \subset J$ and $J^d \subset I$. Then completion with respect to I agrees with completion with respect to J for any R-module. In particular an R-module M is I-adically complete if and only if it is J-adically complete.*

Proof. Consider the system of maps $M/I^nM \to M/J^{\lfloor n/d \rfloor}M$ and the system of maps $M/J^mM \to M/I^{\lfloor m/c \rfloor}M$ to get mutually inverse maps between the completions. □

Lemma 7.90.13. *Let R be a ring. Let I be an ideal of R. Let M be an I-adically complete R-module, and let $K \subset M$ be an R-submodule. The following are equivalent*

1. *$K = \bigcap(K + I^nM)$ and*
2. *M/K is I-adically complete.*

Proof. Set $N = M/K$. By Lemma 7.90.1 the map $M = M^\wedge \to N^\wedge$ is surjective. Hence $N \to N^\wedge$ is surjective. It is easy to see that the kernel of $N \to N^\wedge$ is the module $\bigcap(K + I^nM)/K$. □

Lemma 7.90.14. *Let R be a ring. Let I be an ideal of R. Let M be an R-module. If (a) R is I-adically complete, (b) M is a finite R-module, and (c) $\bigcap I^nM = (0)$, then M is I-adically complete.*

Proof. By Lemma 7.90.1 the map $M = M \otimes_R R = M \otimes_R R^\wedge \to M^\wedge$ is surjective. The kernel of this map is $\bigcap I^nM$ hence zero by assumption. Hence $M \cong M^\wedge$ and M is complete. □

Lemma 7.90.15. *Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module. Assume*

1. *R is I-adically complete,*
2. *$\bigcap_{n \geq 1} I^nM = (0)$, and*
3. *M/IM is a finite R/I-module.*

Then M is a finite R-module.

Proof. Let $x_1, \ldots, x_n \in M$ be elements whose images in M/IM generate M/IM as a R/I-module. Denote $M' \subset M$ the R-submodule generated by $x_1, \ldots, x_n$. By Lemma 7.90.1 the map $(M')^\wedge \to M^\wedge$ is surjective. Since $\bigcap I^nM = 0$ we see in particular that

$\bigcap I^n M' = (0)$. Hence by Lemma 7.90.14 we see that M' is complete, and we conclude that $M' \to M^\wedge$ is surjective. Finally, the kernel of $M \to M^\wedge$ is zero since it is equal to $\bigcap I^n M = (0)$. Hence we conclude that $M \cong M' \cong M^\wedge$ is finitely generated. $\square$

Suppose $R \to S$ is a local homomorphism of local rings $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$. Let $S^\wedge$ be the completion of S with respect to $\mathfrak{n}$. In general $S^\wedge$ is not the $\mathfrak{m}$-adic completion of S. If $\mathfrak{n}^t \subset \mathfrak{m}S$ for some $t \geq 1$ then we do have $S^\wedge = lim\, S/\mathfrak{m}^n S$ by Lemma 7.90.12. In some cases this even implies that $S^\wedge$ is finite over $R^\wedge$.

Lemma 7.90.16. *Let $R \to S$ be a local homomorphism of local rings $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$. Let $R^\wedge$, resp. $S^\wedge$ be the completion of R, resp. S with respect to $\mathfrak{m}$, resp. $\mathfrak{n}$. If $\mathfrak{m}$ and $\mathfrak{n}$ are finitely generated and* $\dim_{\kappa(\mathfrak{m})} S/\mathfrak{m}S < \infty$, *then*

(1) *$S^\wedge$ is equal to the $\mathfrak{m}$-adic completion of S, and*
(2) *$S^\wedge$ is a finite $R^\wedge$-module.*

Proof. We have $\mathfrak{m}S \subset \mathfrak{n}$ because $R \to S$ is a local ring map. The assumption $\dim_{\kappa(\mathfrak{m})} S/\mathfrak{m}S < \infty$ implies that $S/\mathfrak{m}S$ is an Artinian ring, see Lemma 7.49.2. Hence has dimension 0, see Lemma 7.57.4, hence $\mathfrak{n} = \sqrt{\mathfrak{m}S}$. This and the fact that $\mathfrak{n}$ is finitely generated implies that $\mathfrak{n}^t \subset \mathfrak{m}S$ for some $t \geq 1$. By Lemma 7.90.12 we see that $S^\wedge$ can be identified with the $\mathfrak{m}$-adic completion of S. As $\mathfrak{m}$ is finitely generated we see from Lemma 7.90.7 that $S^\wedge$ and $R^\wedge$ are $\mathfrak{m}$-adically complete. At this point we may apply Lemma 7.90.15 to $S^\wedge$ as an $R^\wedge$-module to conclude. $\square$

Lemma 7.90.17. *Let R be a Noetherian ring. Let $R \to S$ be a finite ring map. Let $\mathfrak{p} \subset R$ be a prime and let $\mathfrak{q}_1, \ldots, \mathfrak{q}_m$ be the primes of S lying over $\mathfrak{p}$ (Lemma 7.32.19). Then*

$$R_\mathfrak{p}^\wedge \otimes_R S = S_{\mathfrak{q}_1}^\wedge \times \ldots \times S_{\mathfrak{q}_m}^\wedge$$

where the local rings $R_\mathfrak{p}$ and $S_{\mathfrak{q}_i}$ are completed with respect to their maximal ideals.

Proof. We may replace R by the localization $R_\mathfrak{p}$ and S by $S_\mathfrak{p} = S \otimes_R R_\mathfrak{p}$. Hence we may assume that R is a local Noetherian ring and that $\mathfrak{p} = \mathfrak{m}$ is its maximal ideal. The $\mathfrak{q}_i S_{\mathfrak{q}_i}$-adic completion $S_{\mathfrak{q}_i}^\wedge$ is equal to the $\mathfrak{m}$-adic completion by Lemma 7.90.16. For every $n \geq 1$ prime ideals of $S/\mathfrak{m}^n S$ are in 1-to-1 correspondence with the maximal ideals $\mathfrak{q}_1, \ldots, \mathfrak{q}_m$ of S (by going up for S over R, see Lemma 7.32.20). Hence $S/\mathfrak{m}^n S = \prod S_{\mathfrak{q}_i}/\mathfrak{m}^n S_{\mathfrak{q}_i}$ by Lemma 7.49.8 (using for example Proposition 7.57.6 to see that $S/\mathfrak{m}^n S$ is Artinian). Hence the $\mathfrak{m}$-adic completion $S^\wedge$ of S is equal to $\prod S_{\mathfrak{q}_i}^\wedge$. Finally, we have $R^\wedge \otimes_R S = S^\wedge$ by Lemma 7.90.2. $\square$

Lemma 7.90.18. *Let R be a ring. Let $I \subset R$ be an ideal. Let $0 \to K \to P \to M \to 0$ be a short exact sequence of R-modules. If M is flat over R and M/IM is a projective R/I-module, then the sequence of I-adic completions*

$$0 \to K^\wedge \to P^\wedge \to M^\wedge \to 0$$

is a split exact sequence.

Proof. As M is flat, each of the sequences

$$0 \to K/I^n K \to P/I^n P \to M/I^n M \to 0$$

is short exact, see Lemma 7.35.11 and the sequence $0 \to K^\wedge \to P^\wedge \to M^\wedge \to 0$ is a short exact sequence, see Lemma 7.90.1. It suffices to show that we can find spittings $s_n : M/I^n M \to P/I^n P$ such that $s_{n+1} \bmod I^n = s_n$. We will construct these s_n by induction on n. Pick any splitting s_1, which exists as M/IM is a projective R/I-module. Assume given s_n for some $n > 0$. Set $P_{n+1} = \{x \in P \mid x \bmod I^nP \in \mathrm{Im}(s_n)\}$. The map $\pi : P_{n+1}/I^{n+1}P_{n+1} \to M/I^{n+1}M$ is surjective (details omitted). As $M/I^{n+1}M$ is projective as a R/I^{n+1}-module by Lemma 7.71.5 we may choose a section $t : M/I^{n+1}M \to P_{n+1}/I^{n+1}P_{n+1}$ of π. Setting s_{n+1} equal to the composition of t with the canonical map $P_{n+1}/I^{n+1}P_{n+1} \to P/I^{n+1}P$ works. □

7.91. Criteria for flatness

In this section we prove some important technical lemmas in the Noetherian case. We will (partially) generalize these to the non-Noetherian case in Section 7.119.

Lemma 7.91.1. *Suppose that $R \to S$ is a local homomorphism of Noetherian local rings. Denote $\mathfrak{m}$ the maximal ideal of R. Let $u : M \to N$ be a map of finite S-modules. Assume N flat over R. If $\bar{u} : M/\mathfrak{m}M \to N/\mathfrak{m}N$ is injective then u is injective. In this case $N/u(M)$ is flat over R.*

Proof. First we claim that $u_n : M/\mathfrak{m}^nM \to N/\mathfrak{m}^nN$ is injective for all $n \geq 1$. We proceed by induction, the base case given by assumption. By our assumption that N is flat over R we have a short exact sequence $0 \to N \otimes_R \mathfrak{m}^n/\mathfrak{m}^{n+1} \to N/\mathfrak{m}^{n+1}N \to N/\mathfrak{m}^nN \to 0$. Also, $N \otimes_R \mathfrak{m}^n/\mathfrak{m}^{n+1} = N/\mathfrak{m}N \otimes_{R/\mathfrak{m}} \mathfrak{m}^n/\mathfrak{m}^{n+1}$. We have a similar exact sequence $M \otimes_R \mathfrak{m}^n/\mathfrak{m}^{n+1} \to M/\mathfrak{m}^{n+1}M \to M/\mathfrak{m}^nM \to 0$ for M except we do not have the zero on the left. We also have $M \otimes_R \mathfrak{m}^n/\mathfrak{m}^{n+1} = M/\mathfrak{m}M \otimes_{R/\mathfrak{m}} \mathfrak{m}^n/\mathfrak{m}^{n+1}$. Thus the map u_{n+1} is injective as both u_n and the map $\bar{u} \otimes \mathrm{id}_{\mathfrak{m}^n/\mathfrak{m}^{n+1}}$ are.

Note that $lim_n\ M/\mathfrak{m}^nM$ is the completion of the module M with respect to the ideal $I = \mathfrak{m}S$, and similarly for N. Since M and N are finite S-modules we have $M^\wedge = M \otimes S^\wedge$ and similarly for N, see Lemma 7.90.2. We conclude that $u \otimes 1 : M \otimes S^\wedge \to N \otimes S^\wedge$ is injective. Since $S^\wedge$ is faithfully flat over S, see Lemma 7.90.4, we conclude that u is injective, see Lemma 7.35.4.

Finally, we have to prove that $I \otimes_R N/u(M) \to N/u(M)$ is injective for every ideal $I \subset R$. Consider the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & M/IM & \to & N/IN & \to & N/(IN + u(M)) & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & M & \to & N & \to & N/u(M) & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & M \otimes_R I & \to & N \otimes_R I & \to & N/u(M) \otimes_R I & \to & 0
\end{array}$$

The arrow $N \otimes_R I \to N$ is injective. Chasing through the diagram we see that it suffices to prove that M/IM injects into N/IN. Note that M/IM and N/IN are modules over the Noetherian ring S/IS, N/IN is flat over R/I and $u \bmod I : M/IM \to N/IN$ is injective module $\mathfrak{m}$ we may apply the result above to $u \bmod I$, and we win. □

Lemma 7.91.2. *Suppose that $R \to S$ is a flat and local ring homomorphism of Noetherian local rings. Denote $\mathfrak{m}$ the maximal ideal of R. Suppose $f \in S$ is a nonzero divisor in $S/\mathfrak{m}S$. Then S/fS is flat over R, and f is a nonzero divisor in S.*

Proof. Follows directly from Lemma 7.91.1. □

Lemma 7.91.3. *Suppose that $R \to S$ is a flat and local ring homomorphism of Noetherian local rings. Denote $\mathfrak{m}$ the maximal ideal of R. Suppose $f_1, \ldots, f_c$ is a sequence of elements* *of S such that the images $\overline{f}_1, \ldots, \overline{f}_c$ form a regular sequence in $S/\mathfrak{m}S$. Then $f_1, \ldots, f_c$ is a regular sequence in S and each of the quotients $S/(f_1, \ldots, f_i)$ is flat over R.*

Proof. Induction and Lemma 7.91.2 above. □

Lemma 7.91.4. *Let $R \to S$ be a local homomorphism of Noetherian local rings. Let $\mathfrak{m}$ be the maximal ideal of R. Let M be a finite S-modules. Suppose that (a) $M/\mathfrak{m}M$ is a free $S/\mathfrak{m}S$-module, and (b) M is flat over R. Then M is free and S is flat over R.*

Proof. Let $\overline{x}_1, \ldots, \overline{x}_n$ be a basis for the free module $M/\mathfrak{m}M$. Choose $x_1, \ldots, x_n \in M$ with x_i mapping to $\overline{x}_i$. Let $u : S^{\oplus n} \to M$ be the map which maps the ith standard basis vector to x_i. By Lemma 7.91.1 we see that u is injective. On the other hand, by Nakayama's Lemma 7.14.5 the map is surjective. The lemma follows. □

Lemma 7.91.5. *Let $R \to S$ be a local homomorphism of local Noetherian rings. Let $\mathfrak{m}$ be the maximal ideal of R. Let $0 \to F_e \to F_{e-1} \to \ldots \to F_0$ be a finite complex of finite S-modules. Assume that each F_i is R-flat, and that the complex $0 \to F_e/\mathfrak{m}F_e \to F_{e-1}/\mathfrak{m}F_{e-1} \to \ldots \to F_0/\mathfrak{m}F_0$ is exact. Then $0 \to F_e \to F_{e-1} \to \ldots \to F_0$ is exact, and moreover the module Coker$(F_1 \to F_0)$ is R-flat.*

Proof. By induction on e. If $e = 1$, then this is exactly Lemma 7.91.1. If $e > 1$, we see by Lemma 7.91.1 that $F_e \to F_{e-1}$ is injective and that $C = \text{Coker}(F_e \to F_{e-1})$ is a finite S-module flat over R. Hence we can apply the induction hypothesis to the complex $0 \to C \to F_{e-2} \to \ldots \to F_0$. We deduce that $C \to F_{e-2}$ is injective and the exactness of the complex follows, as well as the flatness of the cokernel of $F_1 \to F_0$. □

In the rest of this section we prove two versions of what is called the ``*local criterion of flatness*''. Note also the interesting Lemma 7.119.1 below.

Lemma 7.91.6. *Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field $\kappa = R/\mathfrak{m}$. Let M be an R-module. If $Tor_1^R(\kappa, M) = 0$, then for every finite length R-module N we have $Tor_1^R(N, M) = 0$.*

Proof. By descending induction on the length of N. If the length of N is 1, then $N \cong \kappa$ and we are done. If the length of N is more than 1, then we can fit N into a short exact sequence $0 \to N' \to N \to N'' \to 0$ where N', N'' are finite length R-modules of smaller length. The vanishing of $\text{Tor}_1^R(N, M)$ follows from the vanishing of $\text{Tor}_1^R(N', M)$ and $\text{Tor}_1^R(N'', M)$ (induction hypothesis) and the long exact sequence of Tor groups, see Lemma 7.69.2. □

Lemma 7.91.7 (Local criterion for flatness)**.** *Let $R \to S$ be a local homomorphism of local Noetherian rings. Let $\mathfrak{m}$ be the maximal ideal of R, and let $\kappa = R/\mathfrak{m}$. Let M be a finite S-module. If $Tor_1^R(\kappa, M) = 0$, then M is flat over R.*

Proof. Let $I \subset R$ be an ideal. By Lemma 7.35.4 it suffices to show that $I \otimes_R M \to M$ is injective. By Remark 7.69.8 we see that this kernel is equal to $\text{Tor}_1^R(M, R/I)$. By Lemma 7.91.6 we see that $J \otimes_R M \to M$ is injective for all ideals of finite colength.

Choose $n >> 0$ and consider the following short exact sequence

$$0 \to I \cap \mathfrak{m}^n \to I \oplus \mathfrak{m}^n \to I + \mathfrak{m}^n \to 0$$

This is a sub sequence of the short exact sequence $0 \to R \to R^{\oplus 2} \to R \to 0$. Thus we get the diagram

$$\begin{tikzcd} (I \cap \mathfrak{m}^n) \otimes_R M \arrow[r] \arrow[d] & I \otimes_R M \oplus \mathfrak{m}^n \otimes_R M \arrow[r] \arrow[d] & (I + \mathfrak{m}^n) \otimes_R M \arrow[d] \\ M \arrow[r] & M \oplus M \arrow[r] & M \end{tikzcd}$$

Note that $I + \mathfrak{m}^n$ and $\mathfrak{m}^n$ are ideals of finite colength. Thus a diagram chase shows that $\mathrm{Ker}((I \cap \mathfrak{m}^n) \otimes_R M \to M) \to \mathrm{Ker}(I \otimes_R M \to M)$ is surjective. We conclude in particular that $K = \mathrm{Ker}(I \otimes_R M \to M)$ is contained in the image of $(I \cap \mathfrak{m}^n) \otimes_R M$ in $I \otimes_R M$. By Artin-Rees, Lemma 7.47.4 we see that K is contained in $\mathfrak{m}^{n-c}(I \otimes_R M)$ for some $c > 0$ and all $n >> 0$. Since $I \otimes_R M$ is a finite S-module (!) and since S is Noetherian, we see that this implies $K = 0$. Namely, the above implies K maps to zero in the $\mathfrak{m}S$-adic completion of $I \otimes_R M$. But the map from S to its $\mathfrak{m}S$-adic completion is faithfully flat by Lemma 7.90.4. Hence $K = 0$, as desired. □

In the following we often encounter the conditions ``M/IM is flat over R/I and $\mathrm{Tor}_1^R(R/I, M) = 0$''. The following lemma gives some consequences of these conditions (it is a generalization of Lemma 7.91.6).

Lemma 7.91.8. *Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module. If M/IM is flat over R/I and $Tor_1^R(R/I, M) = 0$ then*

(1) *M/I^nM is flat over R/I^n for all $n \geq 1$, and*
(2) *for any module N which is annihilated by I^m for some $m \geq 0$ we have $Tor_1^R(N, M) = 0$.*

In particular, if I is nilpotent, then M is flat over R.

Proof. Assume M/IM is flat over R/I and $\mathrm{Tor}_1^R(R/I, M) = 0$. Let N be an R/I-module. Choose a short exact sequence

$$0 \to K \to \bigoplus\nolimits_{i \in I} R/I \to N \to 0$$

By the long exact sequence of Tor and the vanishing of $\mathrm{Tor}_1^R(R/I, M)$ we get

$$0 \to \mathrm{Tor}_1^R(N, M) \to K \otimes_R M \to (\bigoplus\nolimits_{i \in I} R/I) \otimes_R M \to N \otimes_R M \to 0$$

But since K, $\bigoplus_{i \in I} R/I$, and N are all annihilated by I we see that

$$\begin{aligned} K \otimes_R M &= K \otimes_{R/I} M/IM, \\ (\bigoplus\nolimits_{i \in I} R/I) \otimes_R M &= (\bigoplus\nolimits_{i \in I} R/I) \otimes_{R/I} M/IM, \\ N \otimes_R M &= N \otimes_{R/I} M/IM. \end{aligned}$$

As M/IM is flat over R/I we conclude that

$$0 \to K \otimes_{R/I} M/IM \to (\bigoplus\nolimits_{i \in I} R/I) \otimes_{R/I} M/IM \to N \otimes_{R/} M/IM \to 0$$

is exact. Combining this with the above we conclude that $\mathrm{Tor}_1^R(N, M) = 0$ for any R-module N annihilated by I.

In particular, if we apply this to the module I/I^2, then we conclude that the sequence

$$0 \to I^2 \otimes_R M \to I \otimes_R M \to I/I^2 \otimes_R M \to 0$$

is short exact. This implies that $I^2 \otimes_R M \to M$ is injective and it implies that $I/I^2 \otimes_{R/I} M/IM = IM/I^2M$.

Let us prove that M/I^2M is flat over R/I^2. Let $I^2 \subset J$ be an ideal. We have to show that $J/I^2 \otimes_{R/I^2} M/I^2M \to M/I^2M$ is injective, see Lemma 7.35.4. As M/IM is flat over R/I we know that the map $(I+J)/I \otimes_{R/I} M/IM \to M/IM$ is injective. The sequence

$$(I \cap J)/I^2 \otimes_{R/I^2} M/I^2M \to J/I^2 \otimes_{R/I^2} M/I^2M \to (I+J)/I \otimes_{R/I} M/IM \to 0$$

is exact, as you get it by tensoring the exact sequence $0 \to (I \cap J) \to J \to (I+J)/I \to 0$ by M/I^2M. Hence suffices to prove the injectivity of the map $(I \cap J)/I^2 \otimes_{R/I} M/IM \to IM/I^2M$. However, the map $(I \cap J)/I^2 \to I/I^2$ is injective and as M/IM is flat over R/I the map $(I \cap J)/I^2 \otimes_{R/I} M/IM \to I/I^2 \otimes_{R/I} M/IM$ is injective. Since we have previously seen that $I/I^2 \otimes_{R/I} M/IM = IM/I^2M$ we obtain the desired injectivity.

Hence we have proven that the assumptions imply: (a) $\text{Tor}_1^R(N, M) = 0$ for all N annihilated by I, (b) $I^2 \otimes_R M \to M$ is injective, and (c) M/I^2M is flat over R/I^2. Thus we can continue by induction to get the same results for I^n for all $n \geq 1$. □

Lemma 7.91.9 (Variant of the local criterion)**.** *Let $R \to S$ be a local homomorphism of Noetherian local rings. Let $I \neq R$ be an ideal in R. Let M be a finite S-module. If $Tor_1^R(M, R/I) = 0$ and M/IM is flat over R/I, then M is flat over R.*

Proof. First proof: By Lemma 7.91.8 we see that $\text{Tor}_1^R(\kappa, M)$ is zero where κ is the residue field of R. Hence we see that M is flat over R by Lemma 7.91.7.

Second proof: Let $\mathfrak{m}$ be the maximal ideal of R. We will show that $\mathfrak{m} \otimes_R M \to M$ is injective, and then apply Lemma 7.91.7. Suppose that $\sum f_i \otimes x_i \in \mathfrak{m} \otimes_R M$ and that $\sum f_i x_i = 0$ in M. By the equational criterion for flatness Lemma 7.35.10 applied to M/IM over R/I we see there exist $\overline{a}_{ij} \in R/I$ and $\overline{y}_j \in M/IM$ such that $x_i \bmod IM = \sum_j \overline{a}_{ij}\overline{y}_j$ and $0 = \sum_i (f_i \bmod I)\overline{a}_{ij}$. Let $a_{ij} \in R$ be a lift of $\overline{a}_{ij}$ and similarly let $y_j \in M$ be a lift of $\overline{y}_j$. Then we see that

$$\begin{aligned} \sum f_i \otimes x_i &= \sum f_i \otimes x_i + \sum f_i a_{ij} \otimes y_j - \sum f_i \otimes a_{ij} y_j \\ &= \sum f_i \otimes (x_i - \sum a_{ij} y_j) + \sum (\sum f_i a_{ij}) \otimes y_j \end{aligned}$$

Since $x_i - \sum a_{ij} y_j \in IM$ and $\sum f_i a_{ij} \in I$ we see that there exists an element in $I \otimes_R M$ which maps to our given element $\sum f_i \otimes x_i$ in $\mathfrak{m} \otimes_R M$. But $I \otimes_R M \to M$ is injective by assumption (see Remark 7.69.8) and we win. □

In particular, in the situation of Lemma 7.91.9, suppose that $I = (x)$ is generated by a single element x which is a nonzero divisor in R. Then $\text{Tor}_1^R(M, R/(x)) = (0)$ if and only if x is a nonzero divisor on M.

Lemma 7.91.10. *Let $R \to S$ be a ring map. Let $I \subset R$ be an ideal. Let M be an S-module. Assume*

(1) *R is a Noetherian ring,*
(2) *S is a Noetherian ring,*
(3) *M is a finite S-module, and*
(4) *for each $n \geq 1$ the module M/I^nM is flat over R/I^n.*

Then for every $\mathfrak{q} \in V(IS)$ the localization $M_{\mathfrak{q}}$ is flat over R. In particular, if S is local and IS is contained in its maximal ideal, then M is flat over R.

Proof. We are going to use Lemma 7.91.9. By assumption M/IM is flat over R/I. Hence it suffices to check that $\text{Tor}_1^R(M, R/I)$ is zero on localization at $\mathfrak{q}$. By Remark 7.69.8 this Tor group is equal to $K = \text{Ker}(I \otimes_R M \to M)$. We know for each $n \geq 1$ that the kernel $\mathrm{Ker}(I/I^n \otimes_{R/I^n} M/I^nM \to M/I^nM)$ is zero. Since there is a module map $I/I^n \otimes_{R/I^n} M/I^nM \to (I \otimes_R M)/I^{n-1}(I \otimes_R M)$ we conclude that $K \subset I^{n-1}(I \otimes_R M)$ for each n. By the Artin-Rees lemma, and more precisely Lemma 7.47.7 we conclude that $K_{\mathfrak q} = 0$, as desired. □

Lemma 7.91.11. *Let $R \to R' \to R''$ be ring maps. Let M be an R-module. Suppose that $M \otimes_R R'$ is flat over R'. Then the natural map $Tor_1^R(M, R') \otimes_{R'} R'' \to Tor_1^R(M, R'')$ is onto.*

Proof. Let $F_\bullet$ be a free resolution of M over R. The complex $F_2 \otimes_R R' \to F_1 \otimes_R R' \to F_0 \otimes_R R'$ computes $\mathrm{Tor}_1^R(M, R')$. The complex $F_2 \otimes_R R'' \to F_1 \otimes_R R'' \to F_0 \otimes_R R''$ computes $\mathrm{Tor}_1^R(M, R'')$. Note that $F_i \otimes_R R' \otimes_{R'} R'' = F_i \otimes_R R''$. Let $K' = \mathrm{Ker}(F_1 \otimes_R R' \to F_0 \otimes_R R')$ and similarly $K'' = \mathrm{Ker}(F_1 \otimes_R R'' \to F_0 \otimes_R R'')$. Thus we have an exact sequence

$$0 \to K' \to F_1 \otimes_R R' \to F_0 \otimes_R R' \to M \otimes_R R' \to 0.$$

By the assumption that $M \otimes_R R'$ is flat over R', the sequence $0 \to K' \otimes_{R'} R'' \to F_1 \otimes_R R'' \to F_0 \otimes_R R'' \to M \otimes_R R'' \to 0$ is still exact. This means that $K'' = K' \otimes_{R'} R''$. Since $\mathrm{Tor}_1^R(M, R')$ is a quotient of K' and $\mathrm{Tor}_1^R(M, R'')$ is a quotient of K'' we win. □

Lemma 7.91.12. *Let $R \to R'$ be a ring map. Let $I \subset R$ be an ideal and $I' = IR'$. Let M be an R-module and set $M' = M \otimes_R R'$. The natural map $Tor_1^R(R'/I', M) \to Tor_1^{R'}(R'/I', M')$ is surjective.*

Proof. Let $F_2 \to F_1 \to F_0 \to M \to 0$ be a free resolution of M over R. Set $F_i' = F_i \otimes_R R'$. The sequence $F_2' \to F_1' \to F_0' \to M' \to 0$ may no longer be exact at F_1'. A free resolution of M' over R' therefore looks like

$$F_2' \oplus F_2'' \to F_1' \to F_0' \to M' \to 0$$

for a suitable free module F_2'' over R'. Next, note that $F_i \otimes_R R'/I' = F_i'/IF_i' = F_i'/I'F_i'$. So the complex $F_2'/I'F_2' \to F_1'/I'F_1' \to F_0'/I'F_0'$ computes $\mathrm{Tor}_1^R(M, R'/I')$. On the other hand $F_i' \otimes_{R'} R'/I' = F_i'/I'F_i'$ and similarly for F_2''. Thus the complex $F_2'/I'F_2' \oplus F_2''/I'F_2'' \to F_1'/I'F_1' \to F_0'/I'F_0'$ computes $\mathrm{Tor}_1^{R'}(M', R'/I')$. Since the vertical map on complexes

$$\begin{tikzcd}
F_2'/I'F_2' \arrow[r] \arrow[d] & F_1'/I'F_1' \arrow[r] \arrow[d] & F_0'/I'F_0' \arrow[d] \\
F_2'/I'F_2' \oplus F_2''/I'F_2'' \arrow[r] & F_1'/I'F_1' \arrow[r] & F_0'/I'F_0'
\end{tikzcd}$$

clearly induces a surjection on cohomology we win. □

Lemma 7.91.13. *Let*

$$\begin{tikzcd}
S \arrow[r] & S' \\
R \arrow[r] \arrow[u] & R' \arrow[u]
\end{tikzcd}$$

be a commutative diagram of local homomorphisms of local Noetherian rings. Let $I \subset R$ be an ideal. Let M be an S-module. Denote $I' = IR'$ and $M' = M \otimes_S S'$. Assume that

(1) *S' is a localization of the tensor product $S \otimes_R R'$,*
(2) *M/IM is flat over R/I,*
(3) *$Tor_1^R(M, R/I) \to Tor_1^{R'}(M', R'/I')$ is zero.*

Then M' is flat over R'.

Proof. Since S' is a localization of $S \otimes_R R'$ we see that M' is a localization of $M \otimes_R R'$. Note that by Lemma 7.35.6 the module $M/IM \otimes_{R/I} R'/I' = M \otimes_R R'/I'(M \otimes_R R')$ is flat over R'/I'. Hence also $M'/I'M'$ is flat over R'/I' as the localization of a flat module is flat. By Lemma 7.91.9 it suffices to show that $\text{Tor}_1^{R'}(M', R'/I')$ is zero. Since M' is a localization of $M \otimes_R R'$, the last assumption implies that it suffices to show that $\text{Tor}_1^R(M, R/I) \otimes_R R' \to \text{Tor}_1^{R'}(M \otimes_R R', R'/I')$ is surjective.

By Lemma 7.91.12 we see that $\text{Tor}_1^R(M, R'/I') \to \text{Tor}_1^{R'}(M \otimes_R R', R'/I')$ is surjective. So now it suffices to show that $\text{Tor}_1^R(M, R/I) \otimes_R R' \to \text{Tor}_1^R(M, R'/I')$ is surjective. This follows from Lemma 7.91.11 by looking at the ring maps $R \to R/I \to R'/I'$ and the module M. □

Please compare the lemma below to Lemma 7.93.8 (the case of a nilpotent ideal) and Lemma 7.119.8 (the case of finitely presented algebras).

Lemma 7.91.14 (Critère de platitude par fibres; Noetherian case)**.** *Let R, S, S' be Noetherian local rings and let $R \to S \to S'$ be local ring homomorphisms. Let $\mathfrak{m} \subset R$ be the maximal ideal. Let M be an S'-module. Assume*

1. *The module M is finite over S'.*
2. *The module M is not zero.*
3. *The module $M/\mathfrak{m}M$ is a flat $S/\mathfrak{m}S$-module.*
4. *The module M is a flat R-module.*

Then S is flat over R and M is a flat S-module.

Proof. Set $I = \mathfrak{m}S \subset S$. Then we see that M/IM is a flat S/I-module because of (3). Since $\mathfrak{m} \otimes_R S' \to I \otimes_S S'$ is surjective we see that also $\mathfrak{m} \otimes_R M \to I \otimes_S M$ is surjective. Consider

$$\mathfrak{m} \otimes_R M \to I \otimes_S M \to M.$$

As M is flat over R the composition is injective and so both arrows are injective. In particular $\text{Tor}_1^S(S/I, M) = 0$ see Remark 7.69.8. By Lemma 7.91.9 we conclude that M is flat over S. Note that since $M/\mathfrak{m}_{S'}M$ is not zero by Nakayama's Lemma 7.14.5 we see that actually M is faithfully flat over S by Lemma 7.35.14 (since it forces $M/\mathfrak{m}_S M \neq 0$).

Consider the exact sequence $0 \to \mathfrak{m} \to R \to \kappa \to 0$. This gives an exact sequence $0 \to \text{Tor}_1^R(\kappa, S) \to \mathfrak{m} \otimes_R S \to I \to 0$. Since M is flat over S this gives an exact sequence $0 \to \text{Tor}_1^R(\kappa, S) \otimes_S M \to \mathfrak{m} \otimes_R M \to I \otimes_S M \to 0$. By the above this implies that $\text{Tor}_1^R(\kappa, S) \otimes_S M = 0$. Since M is faithfully flat over S this implies that $\text{Tor}_1^R(\kappa, S) = 0$ and we conclude that S is flat over R by Lemma 7.91.7. □

7.92. Base change and flatness

Some lemmas which deal with what happens with flatness when doing a base change.

Lemma 7.92.1. *Let*

$$\begin{array}{ccc} S & \longrightarrow & S' \\ \uparrow & & \uparrow \\ R & \longrightarrow & R' \end{array}$$

be a commutative diagram of local homomorphisms of local rings. Assume that S' is a localization of the tensor product $S \otimes_R R'$. Let M be an S-module and set $M' = S' \otimes_S M$.

(1) *If M is flat over R then M' is flat over R'.*
(2) *If M' is flat over R' and $R \to R'$ is flat then M is flat over R.*

In particular we have

(3) *If S is flat over R then S' is flat over R'.*
(4) *If $R' \to S'$ and $R \to R'$ are flat then S is flat over R.*

Proof. Proof of (1). If M is flat over R, then $M \otimes_R R'$ is flat over R' by Lemma 7.35.6. If $W \subset S \otimes_R R'$ is the multiplicative subset such that $W^{-1}(S \otimes_R R') = S'$ then $M' = W^{-1}(M \otimes_R R')$. Hence M' is flat over R' as the localization of a flat module, see Lemma 7.35.19 part (5). This proves (1) and in particular, we see that (3) holds.

Proof of (2). Suppose that M' is flat over R' and $R \to R'$ is flat. By (3) applied to the diagram reflected in the northwest diagonal we see that $S \to S'$ is flat. Thus $S \to S'$ is faithfully flat by Lemma 7.35.16. We are going to use the criterion of Lemma 7.35.4 (3) to show that M is flat. Let $I \subset R$ be an ideal. If $I \otimes_R M \to M$ has a kernel, so does $(I \otimes_R M) \otimes_S S' \to M \otimes_S S' = M'$. Note that $I \otimes_R R' = IR'$ as $R \to R'$ is flat, and that

$$(I \otimes_R M) \otimes_S S' = (I \otimes_R R') \otimes_{R'} (M \otimes_S S') = IR' \otimes_{R'} M'.$$

From flatness of M' over R' we conclude that this maps injectively into M'. This concludes the proof of (2), and hence (4) is true as well. □

7.93. Flatness criteria over Artinian rings

We discuss some flatness criteria for modules over Artinian rings.

Lemma 7.93.1. *Let $(R, \mathfrak{m})$ be a local Artinian ring. Let M be a flat R-module. If A is a set and $x_\alpha \in M$, $\alpha \in A$ is a collection of elements of M, then the following are equivalent:*

(1) *$\{\overline{x}_\alpha\}_{\alpha \in A}$ forms a basis for the vector space $M/\mathfrak{m}M$ over $R/\mathfrak{m}$, and*
(2) *$\{x_\alpha\}_{\alpha \in A}$ forms a basis for M over R.*

Proof. The implication (2) ⇒ (1) is immediate. We will prove the other implication by using induction on n to show that $\{x_\alpha\}_{\alpha \in A}$ forms a basis for $M/\mathfrak{m}^n M$ over $R/\mathfrak{m}^n$. The case $n = 1$ holds by assumption (1). Assume the statement holds for some $n \geq 1$. By Nakayama's Lemma 7.14.5 the elements x_α generate M, in particular $M/\mathfrak{m}^{n+1}M$. The exact sequence $0 \to \mathfrak{m}^n/\mathfrak{m}^{n+1} \to R/\mathfrak{m}^{n+1} \to R/\mathfrak{m}^n \to 0$ gives on tensoring with M the exact sequence

$$0 \to \mathfrak{m}^n M/\mathfrak{m}^{n+1}M \to M/\mathfrak{m}^{n+1}M \to M/\mathfrak{m}^n M \to 0$$

Here we are using that M is flat. Moreover, we have $\mathfrak{m}^n M/\mathfrak{m}^{n+1}M = M/\mathfrak{m}M \otimes_{R/\mathfrak{m}} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ by flatness of M again. Now suppose that $\sum f_\alpha x_\alpha = 0$ in $M/\mathfrak{m}^{n+1}M$. Then by induction hypothesis $f_\alpha \in \mathfrak{m}^n$ for each α. By the short exact sequence above we then conclude that $\sum \overline{f}_\alpha \otimes \overline{x}_\alpha$ is zero in $\mathfrak{m}^n/\mathfrak{m}^{n+1} \otimes_{R/\mathfrak{m}} M/\mathfrak{m}M$. Since $\overline{x}_\alpha$ forms a basis we conclude that each of the congruence classes $\overline{f}_\alpha \in \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is zero and we win. □

Lemma 7.93.2. *Let R be an Artinian local ring. Let M be an R-module. The following are equivalent*

(1) *M is flat over R,*
(2) *M is a free R-module, and*
(3) *M is a projective R-module.*

Proof. Since any projective module is flat (as a direct summand of a free module) and every free module is projective, it suffices to prove that a flat module is free. Let M be a flat module. Let A be a set and let $x_\alpha \in M$, $\alpha \in A$ be elements such that $\overline{x_\alpha} \in M/\mathfrak{m}M$

forms a basis over the residue field of R. By Lemma 7.93.1 the x_α are a basis for M over R and we win. □

Lemma 7.93.3. *Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module. Let A be a set and let $x_\alpha \in M$, $\alpha \in A$ be a collection of elements of M. Assume*

1. *I is nilpotent,*
2. *$\{\overline{x}_\alpha\}_{\alpha \in A}$ forms a basis for M/IM over R/I, and*
3. *$Tor_1^R(R/I, M) = 0$.*

Then M is free on $\{x_\alpha\}_{\alpha \in A}$ over R.

Proof. Let R, I, M, $\{x_\alpha\}_{\alpha \in A}$ be as in the lemma and satisfy assumptions (1), (2), and (3). By Nakayama's Lemma 7.14.5 the elements x_α generate M over R. The assumption $\text{Tor}_1^R(R/I, M) = 0$ implies that we have a short exact sequence

$$0 \to I \otimes_R M \to M \to M/IM \to 0.$$

Let $\sum f_\alpha x_\alpha = 0$ be a relation in M. By choice of x_α we see that $f_\alpha \in I$. Hence we conclude that $\sum f_\alpha \otimes x_\alpha = 0$ in $I \otimes_R M$. The map $I \otimes_R M \to I/I^2 \otimes_{R/I} M/IM$ and the fact that $\{x_\alpha\}_{\alpha \in A}$ forms a basis for M/IM implies that $f_\alpha \in I^2$! Hence we conclude that there are no relations among the images of the x_α in M/I^2M. In other words, we see that M/I^2M is free with basis the images of the x_α. Using the map $I \otimes_R M \to I/I^3 \otimes_{R/I^2} M/I^2M$ we then conclude that $f_\alpha \in I^3$! And so on. Since $I^n = 0$ for some n by assumption (1) we win. □

Lemma 7.93.4. *Let $\varphi : R \to R'$ be a ring map. Let $I \subset R$ be an ideal. Let M be an R-module. Assume*

1. *M/IM is flat over R/I, and*
2. *$R' \otimes_R M$ is flat over R'.*

Set $I_2 = \varphi^{-1}(\varphi(I^2)R')$. Then M/I_2M is flat over R/I_2.

Proof. We may replace R, M, and R' by R/I_2, M/I_2M, and $R'/\varphi(I)^2R'$. Then $I^2 = 0$ and φ is injective. By Lemma 7.91.8 and the fact that $I^2 = 0$ it suffices to prove that $\text{Tor}_1^R(R/I, M) = K = \text{Ker}(I \otimes_R M \to M)$ is zero. Set $M' = M \otimes_R R'$ and $I' = IR'$. By assumption the map $I' \otimes_{R'} M' \to M'$ is injective. Hence K maps to zero in

$$I' \otimes_{R'} M' = I' \otimes_R M = I' \otimes_{R/I} M/IM.$$

Then $I \to I'$ is an injective map of R/I-modules. Since M/IM is flat over R/I the map

$$I \otimes_{R/I} M/IM \longrightarrow I' \otimes_{R/I} M/IM$$

is injective. This implies that K is zero in $I \otimes_R M = I \otimes_{R/I} M/IM$ as desired. □

Lemma 7.93.5. *Let $\varphi : R \to R'$ be a ring map. Let $I \subset R$ be an ideal. Let M be an R-module. Assume*

1. *I is nilpotent,*
2. *$R \to R'$ is injective,*
3. *M/IM is flat over R/I, and*
4. *$R' \otimes_R M$ is flat over R'.*

Then M is flat over R.

Proof. Define inductively $I_1 = I$ and $I_{n+1} = \varphi^{-1}(\varphi(I_n)^2R')$ for $n \geq 1$. Note that by Lemma 7.93.4 we find that M/I_nM is flat over R/I_n for each $n \geq 1$. It is clear that $\varphi(I_n) \subset \varphi(I)^{2^n}R'$. Since I is nilpotent we see that $\varphi(I_n) = 0$ for some n. As φ is injective we conclude that $I_n = 0$ for some n and we win. □

Here is the local Artinian version of the local criterion for flatness.

Lemma 7.93.6. *Let R be an Artinian local ring. Let M be an R-module. Let $I \subset R$ be a proper ideal. The following are equivalent*

(1) *M is flat over R, and*
(2) *M/IM is flat over R/I and $Tor_1^R(R/I, M) = 0$.*

Proof. The implication (1) ⇒ (2) follows immediately from the definitions. Assume M/IM is flat over R/I and $\text{Tor}_1^R(R/I, M) = 0$. By Lemma 7.93.2 this implies that M/IM is free over R/I. Pick a set A and elements $x_\alpha \in M$ such that the images in M/IM form a basis. By Lemma 7.93.3 we conclude that M is free and in particular flat. □

It turns out that flatness descends along injective homomorphism whose source is an Artinian ring.

Lemma 7.93.7. *Let $R \to S$ be a ring map. Let M be an R-module. Assume*

(1) *R is Artinian*
(2) *$R \to S$ is injective, and*
(3) *$M \otimes_R S$ is a flat S-module.*

Then M is a flat R-module.

Proof. First proof: Let $I \subset R$ be the radical of R. Then I is nilpotent and M/IM is flat over R/I as R/I is a product of fields, see Section 7.49. Hence M is flat by an application of Lemma 7.93.5.

Second proof: By Lemma 7.49.8 we may write $R = \prod R_i$ as a finite product of local Artinian rings. This induces similar product decompositions for both R and S. Hence we reduce to the case where R is local Artinian (details omitted).

Assume that $R \to S$, M are as in the lemma satisfying (1), (2), and (3) and in addition that R is local with maximal ideal $\mathfrak{m}$. Let A be a set and $x_\alpha \in A$ be elements such that $\overline{x}_\alpha$ forms a basis for $M/\mathfrak{m}M$ over $R/\mathfrak{m}$. By Nakayama's Lemma 7.14.5 we see that the elements x_α generate M as an R-module. Set $N = S \otimes_R M$ and $I = \mathfrak{m}S$. Then $\{1 \otimes x_\alpha\}_{\alpha \in A}$ is a family of elements of N which form a basis for N/IN. Moreover, since N is flat over S we have $\text{Tor}_1^S(S/I, N) = 0$. Thus we conclude from Lemma 7.93.3 that N is free on $\{1 \otimes x_\alpha\}_{\alpha \in A}$. The injectivity of $R \to S$ then guarantees that there cannot be a nontrivial relation among the x_α with coefficients in R. □

Please compare the lemma below to Lemma 7.91.14 (the case of Noetherian local rings) and Lemma 7.119.8 (the case of finitely presented algebras).

Lemma 7.93.8 (Critère de platitude par fibres: Nilpotent case)**.** *Let*

$$\begin{tikzcd} S \arrow[rr] & & S' \\ & R \arrow[ul] \arrow[ur] & \end{tikzcd}$$

be a commutative diagram in the category of rings. Let $I \subset R$ be a nilpotent ideal and M an S'-module. Assume

(1) *The module M/IM is a flat S/IS-module.*
(2) *The module M is a flat R-module.*

Then M is a flat S-module and $S_\mathfrak{q}$ is flat over R for every $\mathfrak{q} \subset S$ such that $M \otimes_S \kappa(\mathfrak{q})$ is nonzero.

Proof. As M is flat over R tensoring with the short exact sequence $0 \to I \to R \to R/I \to 0$ gives a short exact sequence

$$0 \to I \otimes_R M \to M \to M/IM \to 0.$$

Note that $I \otimes_R M \to IS \otimes_S M$ is surjective. Combined with the above this means both maps in

$$I \otimes_R M \to IS \otimes_S M \to M$$

are injective. Hence $\mathrm{Tor}_1^S(IS, M) = 0$ (see Remark 7.69.8) and we conclude that M is a flat S-module by Lemma 7.91.8. To finish we need to show that $S_\mathfrak{q}$ is flat over R for any prime $\mathfrak{q} \subset S$ such that $M \otimes_S \kappa(\mathfrak{q})$ is nonzero. This follows from Lemma 7.35.14 and 7.35.9. □

7.94. What makes a complex exact?

Some of this material can be found in the paper [**BE73**] by Buchsbaum and Eisenbud.

Situation 7.94.1. Here R is a ring, and we have a complex

$$0 \to R^{n_e} \xrightarrow{\varphi_e} R^{n_{e-1}} \xrightarrow{\varphi_{e-1}} \ldots \xrightarrow{\varphi_{i+1}} R^{n_i} \xrightarrow{\varphi_i} R^{n_{i-1}} \xrightarrow{\varphi_{i-1}} \ldots \xrightarrow{\varphi_1} R^{n_0}$$

In other words we require $\varphi_i \circ \varphi_{i+1} = 0$ for $i = 1, \ldots, e-1$.

Lemma 7.94.2. *In Situation 7.94.1. Suppose R is a local ring with maximal ideal $\mathfrak{m}$. Suppose that for some i, $e \le i \le 1$ some matrix coefficient of the map φ_i is invertible. Then the complex $0 \to R^{n_e} \to R^{n_{e-1}} \to \ldots \to R^{n_0}$ is isomorphic to the direct sum of a complex $0 \to R^{n_e} \to \ldots \to R^{n_i - 1} \to R^{n_{i-1}-1} \to \ldots \to R^{n_0}$ and the complex $0 \to 0 \to \ldots \to R \to R \to 0 \to \ldots \to 0$ where the map $R \to R$ is the identity map.*

Proof. The assumption means, after a change of basis of R^{n_i} and $R^{n_{i-1}}$ that the first basis vector of R^{n_i} is mapped via φ_i to the first basis vector of $R^{n_{i-1}}$. Let e_j denote the jth basis vector of R^{n_i} and f_k the kth basis vector of $R^{n_{i-1}}$. Write $\varphi_i(e_j) = \sum a_{jk} f_k$. So $a_{1k} = 0$ unless $k = 1$ and $a_{11} = 1$. Change basis on R^{n_i} again by setting $e'_j = e_j - a_{j1} e_1$ for $j > 1$. After this change of coordinates we have $a_{j1} = 0$ for $j > 1$. Note the image of $R^{n_{i+1}} \to R^{n_i}$ is contained in the subspace spanned by e_j, $j > 1$. Note also that $R^{n_{i-1}} \to R^{n_{i-2}}$ has to annihilate f_1 since it is in the image. These conditions and the shape of the matrix (a_{jk}) for φ_i imply the lemma. □

Let us say that an acyclic complex of the form $\ldots \to 0 \to R \to R \to 0 \to \ldots$ is *trivial*. The lemma above clearly says that any finite complex of finite free modules over a local ring is up to direct sums with trivial complexes the same as a complex all of whose maps have all matrix coefficients in the maximal ideal.

Lemma 7.94.3. *In Situation 7.94.1. Let R be a Artinian local ring. Suppose that $0 \to R^{n_e} \to R^{n_{e-1}} \to \ldots \to R^{n_0}$ is an exact complex. Then the complex is isomorphic to a direct sum of trivial complexes.*

Proof. By induction on the integer $\sum n_i$. Clearly $\mathrm{Ass}(R) = \{\mathfrak{m}\}$. Pick $x \in R$, $x \neq 0$, $\mathfrak{m}x = 0$. Pick a basis vector $e_i \in R^{n_e}$. Since xe_i is not be mapped to zero by exactness of the complex we deduce that some matrix coefficient of the map $R^{n_e} \to R^{n_{e-1}}$ is not in $\mathfrak{m}$. Lemma 7.94.2 then allows us to decrease $\sum n_i$. □

Below we define the rank of a map of finite free modules. This is just one possible definition of rank. It is just the definition that works in this section; there are others that may be more convenient in other settings.

Definition 7.94.4. Let R be a ring. Suppose that $\varphi : R^m \to R^n$ is a map of finite free modules.

(1) The *rank* of φ is the maximal r such that $\wedge^r\varphi : \wedge^r R^m \to \wedge^r R^n$ is nonzero.
(2) We let $I(\varphi) \subset R$ be the ideal generated by the $r \times r$ minors of the matrix of φ, where r is the rank as defined above.

Lemma 7.94.5. *In Situation 7.94.1, suppose the complex is isomorphic to a direct sum of trivial complexes. Then we have*

(1) *the maps φ_i have rank $r_i = n_i - n_{i+1} + \ldots + (-1)^{e-i-1}n_{e-1} + (-1)^{e-i}n_e$,*
(2) *for all i, $1 \leq i \leq e$ we have $rank(\varphi_{i+1}) + rank(\varphi_i) = n_i$,*
(3) *each $I(\varphi_i) = R$.*

Proof. We may assume the complex is the direct sum of trivial complexes. Then for each i we can split the standard basis elements of R^{n_i} into those that map to a basis element of $R^{n_{i-1}}$ and those that are mapped to zero (and these are mapped onto by basis elements of $R^{n_{i+1}}$). Using descending induction starting with $i = e$ it is easy to prove that there are r_{i+1}-basis elements of R^{n_i} which are mapped to zero and r_i which are mapped to basis elements of $R^{n_{i-1}}$. From this the result follows. $\square$

Lemma 7.94.6. *Let R be a local Noetherian ring. Suppose that $\varphi : R^m \to R^n$ is a map of finite free modules. The following are equivalent*

(1) *φ is injective.*
(2) *the rank of φ is m and either $I(\varphi) = R$ or it contains a non zero divisor.*

Proof. If any matrix coefficient of φ is not in $\mathfrak{m}$, then we apply Lemma 7.94.2 to write φ as the sum of $1 : R \to R$ and a map $\varphi' : R^{m-1} \to R^{n-1}$. It is easy to see that the lemma for φ' implies the lemma for φ. Thus we may assume from the outset that all the matrix coefficients of φ are in $\mathfrak{m}$.

Suppose φ is injective. We may assume $m > 0$. Let $\mathfrak{q} \in \text{Ass}(R)$. Let $x \in R$ be an element whose annihilator is $\mathfrak{q}$. Note that φ induces a injective map $xR^m \to xR^n$ which is isomorphic to the map $\varphi_{\mathfrak{q}} : (R/\mathfrak{q})^m \to (R/\mathfrak{q})^n$ induced by φ. Since $R/\mathfrak{q}$ is a domain we deduce immediately by localizing to its fraction field that the rank of $\varphi_{\mathfrak{q}}$ is m and that $I(\varphi_{\mathfrak{q}})$ is not the zero ideal. Hence we conclude by Lemma 7.60.17.

Conversely, assume that the rank of φ is m and that $I(\varphi)$ contains a non zero divisor x. The rank being m implies $n \geq m$. By Lemma 7.14.6 we can find a map $\psi : R^n \to R^m$ such that $\psi \circ \varphi = f\text{id}_{R^m}$. Thus φ is injective. $\square$

Lemma 7.94.7. *In Situation 7.94.1. Suppose R is a local Noetherian ring with maximal ideal $\mathfrak{m}$. Assume $\mathfrak{m} \in Ass(R)$, in other words R has depth 0. Suppose that the complex is exact. In this case the complex is isomorphic to a direct sum of trivial complexes.*

Proof. The proof is the same as in Lemma 7.94.3, except using Lemma 7.94.6 to guarantee that $I(\varphi_e) = R$, and hence some matrix coefficient of φ_e is not in $\mathfrak{m}$. $\square$

Lemma 7.94.8. *In Situation 7.94.1, suppose R is a local Noetherian ring, and suppose that the complex is exact. Let x be an element of the maximal ideal which is a nonzero divisor. The complex $0 \to (R/xR)^{n_e} \to \ldots \to (R/xR)^{n_1}$ is still exact.*

Proof. Follows easily from the snake lemma. $\square$

Lemma 7.94.9. *(Acyclicity lemma.) Let R be a local Noetherian ring. Let $0 \to M_e \to M_{e-1} \to \ldots \to M_0$ be a complex of finite R-modules. Assume depth$(M_i) \geq i$. Let i be the largest index such that the complex is not exact at M_i. If $i > 0$ then Ker$(M_i \to M_{i-1})$/Im$(M_{i+1} \to M_i)$ has depth ≥ 1.*

Proof. Let $H = \mathrm{Ker}(M_i \to M_{i-1})/\mathrm{Im}(M_{i+1} \to M_i)$ be the cohomology group in question. We may break the complex into short exact sequences $0 \to M_e \to M_{e-1} \to K_{e-2} \to 0$, $0 \to K_j \to M_j \to K_{j-1} \to 0$, for $i + 2 \leq j \leq e - 2$, $0 \to K_{i+1} \to M_{i+1} \to B_i \to 0$, $0 \to K_i \to M_i \to M_{i-1}$, and $0 \to B_i \to K_i \to H \to 0$. We proceed up through these complexes to prove the statements about depths, repeatedly using Lemma 7.67.10. First of all, since depth$(M_e) \geq e$, and depth$(M_{e-1}) \geq e-1$ we deduce that depth$(K_{e-2}) \geq e-1$. At this point the sequences $0 \to K_j \to M_j \to K_{j-1} \to 0$ for $i+2 \leq j \leq e-2$ imply similarly that depth$(K_{j-1}) \geq j$ for $i + 2 \leq j \leq e - 2$. The sequence $0 \to K_{i+1} \to M_{i+1} \to B_i \to 0$ then shows that depth$(B_i) \geq i + 1$. The sequence $0 \to K_i \to M_i \to M_{i-1}$ shows that depth$(K_i) \geq 1$ since M_i has depth $\geq i \geq 1$ by assumption. The sequence $0 \to B_i \to K_i \to H \to 0$ then implies the result. □

Proposition 7.94.10. *In Situation 7.94.1, suppose R is a local Noetherian ring. The complex is exact if and only if for all i, $1 \leq i \leq e$ the following two conditions are satisfied:*

(1) *we have rank(φ_{i+1}) + rank$(\varphi_i) = n_i$, and*
(2) *$I(\varphi_i) = R$, or $I(\varphi_i)$ contains a regular sequence of length i.*

Proof. This proof is very similar to the proof of Lemma 7.94.6. As in the proof of Lemma 7.94.6 we may assume that all matrix entries of each φ_i are elements of the maximal ideal. We may also assume that $e \geq 1$.

Assume the complex is exact. Let $q \in \mathrm{Ass}(R)$. (There is at least one such prime.) Note that the ring $R_{\mathfrak q}$ has depth 0. We apply Lemmas 7.94.7 and 7.94.5 to the localized complex over $R_{\mathfrak q}$. all of the ideals $I(\varphi_i)_{\mathfrak q}$, $e \geq i \geq 1$ are equal to $R_{\mathfrak q}$. Thus none of the ideals $I(\varphi_i)$ is contained in $\mathfrak q$. This implies that $I(\varphi_e)I(\varphi_{e-1}) \ldots I(\varphi_1)$ is not contained in any of the associated primes of R. By Lemma 7.14.3 we may choose $x \in I(\varphi_e)I(\varphi_{e-1}) \ldots I(\varphi_1)$, $x \not\in \mathfrak q$ for all $q \in \mathrm{Ass}(R)$. According to Lemma 7.94.8 the complex $0 \to (R/xR)^{n_e} \to \ldots \to (R/xR)^{n_1}$ is exact. By induction on e all the ideals $I(\varphi_i)/xR$ have a regular sequence of length $i - 1$. This proves that $I(\varphi_i)$ contains a regular sequence of length i.

Assume the two conditions on the ranks of φ_i and the ideals $I(\varphi_i)$ is satisfied. Note that $I(\varphi_i) \subset \mathfrak m$ for all i because of what was said in the first paragraph of the proof. Hence the assumption in particular implies that depth$(R) \geq e$. By induction on the dimension of R we may assume the complex is exact when localized at any nonmaximal prime of R. Thus $\mathrm{Ker}(\varphi_i)/\mathrm{Im}(\varphi_{i+1})$ has support $\{\mathfrak m\}$ and hence (if nonzero) depth 0. By Lemma 7.94.9 we see that the complex is exact. □

7.95. Cohen-Macaulay modules

Here we just do a minimal amount of work to show that Cohen-Macaulay modules have good properties. We postpone using Ext groups to esthablish the connection with duality and so on.

Definition 7.95.1. Let R be a Noetherian local ring. Let M be a finite R-module. We say M is *Cohen-Macaulay* if dim(Support(M)) = depth(M).

Let R be a local Noetherian ring. Let M be a Cohen-Macaulay module, and let $f_1, \ldots, f_d$ be an M-regular sequence with $d = \dim(\mathrm{Support}(M))$. We say that $g \in \mathfrak m$ is *good with respect* *to* $(M, f_1, \dots, f_d)$ if for all $i = 0, 1, \dots, d-1$ we have $\dim(\text{Support}(M) \cap V(g, f_1, \dots, f_i)) = d - i - 1$. This is equivalent to the condition that $\dim((\text{Support}(M/(f_1, \dots, f_i)M) \cap V(g) = d - i - 1$ for $i = 0, 1, \dots, d-1$.

Lemma 7.95.2. *Notation and assumptions as above. If g is good with respect to $(M, f_1, \dots, f_d)$, then (a) g is a nonzero-divisor on M, and (b) M/gM is Cohen-Macaulay with maximal regular sequence $f_1, \dots, f_{d-1}$.*

Proof. We prove the lemma by induction on d. If $d = 0$, then M is finite and there is no case to which the lemma applies. If $d = 1$, then we have to show that $g : M \to M$ is injective. The kernel K has support $\{\mathfrak{m}\}$ because by assumption $\dim \text{Supp}(M) \cap V(g) = 0$. Hence K has finite length. Hence $f_1 : K \to K$ injective implies the length of the image is the length of K, and hence $f_1 K = K$, which by Nakayama's Lemma 7.14.5 implies $K = 0$. Also, $\dim \text{Supp}(M/gM) = 0$ and so M/gM is Cohen-Macaulay of depth 0.

For $d > 1$ we essentially argue in the same way. Let $K \subset M$ be the kernel of multiplication by g. As above $f_1 : K \to K$ cannot be surjective if $K \neq 0$ Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & M & \xrightarrow{f_1} & M & \to & M/f_1M & \to & 0 \\ & & \downarrow g & & \downarrow g & & \downarrow g & & \\ 0 & \to & M & \xrightarrow{f_1} & M & \to & M/f_1M & \to & 0 \end{array}$$

This shows that the kernel K_1 of $g : M/f_1M \to M/f_1M$ cannot be zero if K is not zero. But g is good for $(M/f_1M, f_2, \dots, f_d)$, as is easy seen from the definition. We conclude that $K_1 = 0$, and so $K = 0$. From the snake lemma we see that $0 \to M/gM \to M/gM \to M/(f_1, g)M \to 0$ is exact. By induction, we have that $M/(g, f_1)M$ is Cohen-Macaulay with regular sequence $f_2, \dots, f_{d-1}$. Thus M/gM is Cohen-Macaulay with regular sequence $f_1, \dots, f_{d-1}$. □

Lemma 7.95.3. *Let R be a Noetherian local ring. Let M be a Cohen-Macaulay module over R. Suppose $g \in \mathfrak{m}$ is such that* $\dim(\mathit{Supp}(M) \cap V(g)) = \dim(\mathit{Supp}(M)) - 1$. *Then (a) g is a nonzero divisor on M, and (b) M/gM is Cohen-Macaulay of depth one less.*

Proof. Choose a M-regular sequence $f_1, \dots, f_d$ with $d = \dim(\text{Supp}(M))$. If g is is good with respect to $(M, f_1, \dots, f_d)$ we win by Lemma 7.95.2. In particular the lemma holds if $d = 1$. (The case $d = 0$ does not occur.) Assume $d > 1$. Choose an element $h \in R$ such that (a) h is good with respect to $(M, f_1, \dots, f_d)$, and (b) $\dim(\text{Supp}(M) \cap V(h, g) = d - 2$. To see h exists, let $\{\mathfrak{q}_i\}$ be the (finite) set of minimal primes of the closed sets $\text{Supp}(M)$, $\text{Supp}(M) \cap V(f_1, \dots, f_i)$, $i = 1, \dots, d-1$, and $\text{Supp}(M) \cap V(g)$. None of these $\mathfrak{q}_i$ is equal to $\mathfrak{m}$ and hence we may find $h \in \mathfrak{m}$, $h \not\in \mathfrak{q}_i$ by Lemma 7.14.3. It is clear that h satisfies (a) and (b). At this point we may apply Lemma 7.95.2 to conclude that M/hM is Cohen-Macaulay. By (b) we see that the pair $(M/hM, g)$ satisfies the induction hypothesis. Hence $M/(h, g)M$ is Cohen-Macaulay, and $g : M/hM \to M/hM$ is injective. From this it follows easily that $g : M \to M$ is injective, by a snake lemma argument. This in its turn implies that $h : M/gM \to M/gM$ is injective. Combined with the fact that $M/(g, h)M$ is Cohen-Macaulay this finishes the proof. □

Proposition 7.95.4. *Let R be a Noetherian local ring, with maximal ideal $\mathfrak{m}$. Let M be a Cohen-Macaulay module over R whose support has dimension d. Suppose that $g_1, \dots, g_c$ are elements of $\mathfrak{m}$ such that* $\dim(\mathit{Supp}(M/(g_1, \dots, g_c)M)) = d - c$. *Then $g_1, \dots, g_c$ is an M-regular sequence, and can be extended to a maximal M-regular sequence.*

Proof. Let $Z = \mathrm{Supp}(M) \subset Spec(R)$. By Lemma 7.57.11 in the chain $Z \supset Z \cap V(g_1) \supset \ldots \supset Z \cap V(g_1, \ldots, g_c)$ each step decreases the dimension at most by 1. Hence by assumption each step decreases the dimension by exactly 1 each time. Thus we may succesively apply Lemma 7.95.3 above to the modules $M/(g_1, \ldots, g_i)$ and the element g_{i+1}.

To extend $g_1, \ldots, g_c$ by one element if $c < d$ we simply choose an element $g_{c+1} \in \mathfrak{m}$ which is not in any of the finitely many minimal primes of $Z \cap V(g_1, \ldots, g_c)$, using Lemma 7.14.3. □

7.96. Cohen-Macaulay rings

Definition 7.96.1. A Noetherian local ring R is called *Cohen-Macaulay* if it is Cohen-Macaulay as a module over itself.

Note that this is equivalent to requiring the existence of a R-regular sequence $x_1, \ldots, x_d$ of the maximal ideal such that $R/(x_1, \ldots, x_d)$ has dimension 0. We will usually just say ``regular sequence'' and not ``R-regular sequence''.

Lemma 7.96.2. *Let R be a Noetherian local Cohen-Macaulay ring with maximal ideal $\mathfrak{m}$. Let $x_1, \ldots, x_c \in \mathfrak{m}$ be elements. Then*

$$x_1, \ldots, x_c \textit{ is a regular sequence } \Leftrightarrow \dim(R/(x_1, \ldots, x_c)) = \dim(R) - c$$

If so $x_1, \ldots, x_c$ can be extended to a regular sequence of length $\dim(R)$ *and each quotient $R/(x_1, \ldots, x_i)$ is a Cohen-Macaulay ring of dimension* $\dim(R) - i$.

Proof. This is a reformulation of Proposition 7.95.4 in the case where the module is equal to R. □

Lemma 7.96.3. *Let R be Noetherian local. Suppose R is Cohen-Macaulay of dimension d. Any maximal chain of ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_n$ has length $n = d$.*

Proof. Choose an element $x \in \mathfrak{p}_1$, with x not in any of the minimal primes of R, and in particular $x \not\in \mathfrak{p}_0$. (See Lemma 7.14.3.) Then $\dim(R/xR) < \dim(R)$ and R/xR is Cohen-Macaulay by Proposition 7.95.4. By induction the chain $\mathfrak{p}_1/xR \subset \ldots \subset \mathfrak{p}_n/xR$ has length $d - 1$. □

Lemma 7.96.4. *Suppose R is a Noetherian local Cohen-Macaulay ring of dimension d. For any prime $\mathfrak{p} \subset R$ we have*

$$\dim(R) = \dim(R_{\mathfrak{p}}) + \dim(R/\mathfrak{p}).$$

Proof. This is immediate from the result on maximal sequences above, by looking at maximal sequences which have $\mathfrak{p}$ in them. □

Lemma 7.96.5. *Suppose R is a Cohen-Macaulay local ring. For any prime $\mathfrak{p} \subset R$ the ring $R_{\mathfrak{p}}$ is Cohen-Macaulay as well.*

Proof. Suppose that $\dim(R) = d$ and that $\dim(R/\mathfrak{p}) = d - c$. We may choose $f_1, \ldots, f_c \in \mathfrak{p}$ such that $\dim V(f_1, \ldots, f_i) = d - i$, using Lemma 7.14.3 at each step to avoid minimal primes of $V(f_1, \ldots, f_{i-1})$. Then $\mathfrak{p}$ is minimal over $(f_1, \ldots, f_c)$ and hence the support of $R_{\mathfrak{p}}/(f_1, \ldots, f_c)R_{\mathfrak{p}}$ consists of the maximal ideal of $R_{\mathfrak{p}}$. In other words $R_{\mathfrak{p}}$ has an ideal of definition generated by c elements, and has dimension c by Lemma 7.96.4. □

Definition 7.96.6. A Noetherian ring R is called *Cohen-Macaulay* if all its local rings are Cohen-Macaulay.

Lemma 7.96.7. *Suppose R is a Cohen-Macaulay ring. Any polynomial algebra over R is Cohen-Macaulay.*

Proof. By induction on the number of variables it suffices to prove that $R[x]$ is Cohen-Macaulay if R is. Let $\mathfrak q \subset R[x]$ be a prime, and let $\mathfrak p$ be its image. Let $f_1, \ldots, f_d$ be a regular sequence in the maximal ideal of $R_{\mathfrak p}$ of length $d = \dim(R_{\mathfrak p})$. Note that since $R[x]$ is flat over R the localization $R[x]_{\mathfrak q}$ is flat over $R_{\mathfrak p}$. Hence, by Lemma 7.65.7, the sequence $f_1, \ldots, f_d$ is a regular sequence of length d in $R[x]_{\mathfrak q}$. The quotient $R[x]_{\mathfrak q}/(f_1, \ldots, f_d)$ is a localization of $(R_{\mathfrak p}/(f_1, \ldots, f_d))[x]$ at a prime $\overline{\mathfrak q}$. It is clear that either $\overline{\mathfrak q}$ contains a monic polynomial f in $(R_{\mathfrak p}/(f_1, \ldots, f_d))[x]$, or $\overline{\mathfrak q}$ equals the kernel of $(R_{\mathfrak p}/(f_1, \ldots, f_d))[x] \to \kappa(\mathfrak p)[x]$. In the first case the monic polynomial f is a nonzero divisor in $(R_{\mathfrak p}/(f_1, \ldots, f_d))[x]$ and hence in $R[x]_{\mathfrak q}/(f_1, \ldots, f_d)$, and $x_1, \ldots, x_d, f$ is a regular sequence in $R[x]_{\mathfrak q}$ such that $\dim(R[x]_{\mathfrak q}/(x_1, \ldots, x_d, f)) = 0$. In the second case it is already the case that $\dim R[x]_{\mathfrak q}/(f_1, \ldots, f_d) = 0$. □

Lemma 7.96.8. *Suppose that R is a Noetherian local Cohen-Macaulay ring of dimension d. Suppose that M is a finite R-module, and suppose that $0 \to K \to R^n \to M \to 0$ is an exact sequence of R-modules. Then either depth$(K) >$ depth(M), or depth$(K) =$ depth$(M) = d$.*

Proof. If depth$(M) = 0$ the lemma is clear. Let $x \in \mathfrak m$ be a nonzero divisor on M and on R. Then x is a nonzero divisor on M and on K and it follows by an easy diagram chase that $0 \to K/xK \to (R/xR)^n \to M/xM \to 0$ is exact. Thus the result follows from the result for K/xK over R/xR which has smaller dimension. □

Definition 7.96.9. Let R be a Noetherian local Cohen-Macaulay ring. A finite module M over R is called a *maximal Cohen-Macaulay* module if depth$(M) = \dim(R)$.

Lemma 7.96.10. *Let R be a local Noetherian Cohen-Macaulay ring of dimension d Let M be a finite R module of depth e. There exists an exact complex*

$$0 \to K \to F_{d-e-1} \to \ldots \to F_0 \to M \to 0$$

with each F_i finite free and K maximal Cohen-Macaulay.

Proof. Immediate from the definition and Lemma 7.96.8. □

Lemma 7.96.11. *Let $\varphi : A \to B$ be a map of local rings. Assume that B is Noetherian and Cohen-Macaulay and that $\mathfrak m_B = \sqrt{\varphi(\mathfrak m_A)B}$. Then there exists a sequence of elements $f_1, \ldots, f_{\dim(B)}$ in A such that $\varphi(f_1), \ldots, \varphi(f_{\dim(B)})$ is a regular sequence in B.*

Proof. By induction on $\dim(B)$ it suffices to prove: If $\dim(B) \geq 1$, then we can find an element f of A which maps to a nonzero divisor in B. By Lemma 7.96.2 it suffices to find $f \in A$ whose image in B is not contained in any of the finitely many minimal primes $\mathfrak q_1, \ldots, \mathfrak q_r$ of B. By the assumption that $\mathfrak m_B = \sqrt{\varphi(\mathfrak m_A)B}$ we see that $\mathfrak m_A \not\subset \varphi^{-}(\mathfrak q_i)$. Hence we can find f by Lemma 7.14.3. □

7.97. Catenary rings

Definition 7.97.1. A ring R is said to be *catenary* if for any pair of prime ideals $\mathfrak p \subset \mathfrak q$, all maximal chains of primes $\mathfrak p = \mathfrak p_0 \subset \mathfrak p_1 \subset \ldots \subset \mathfrak p_e = \mathfrak q$ have the same (finite) length.

Lemma 7.97.2. *A ring R is catenary if and only if the topological space Spec(R) is catenary (see Topology, Definition 5.8.1).*

Proof. Immediate from the definition and the characterization of irreducible closed subsets. □

Lemma 7.97.3. *Any localization of a catenary ring is catenary.*

Proof. Omitted. □

Lemma 7.97.4. *Any quotient of a catenary ring is catenary.*

Proof. Omitted. □

In general it is not the case that a finitely generated R-algebra is catenary if R is. Thus we make the following definition.

Definition 7.97.5. A ring R is said to be *universally catenary* if R is Noetherian and every R algebra of finite type is catenary.

By Lemma 7.97.4 this just means that R is Noetherian and that each polynomial algebra $R[x_1, \ldots, x_n]$ is catenary.

Lemma 7.97.6. *A Cohen-Macaulay ring is universally catenary.*

Proof. Since a polynomial algebra over R is Cohen-Macaulay, by Lemma 7.96.7, it suffices to show that a Cohen-Macaulay ring is catenary. Let R be Cohen-Macaulay and $\mathfrak{p} \subset \mathfrak{q}$ primes of R. By definition $R_{\mathfrak{q}}$ and $R_{\mathfrak{p}}$ are Cohen-Macaulay. Take a maximal chain of primes $\mathfrak{p} = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_n = \mathfrak{q}$. Next choose a maximal chain of primes $\mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \ldots \subset \mathfrak{q}_m = \mathfrak{p}$. By Lemma 7.96.3 we have $n + m = \dim(R_{\mathfrak{q}})$. And we have $m = \dim(R_{\mathfrak{p}})$ by the same lemma. Hence $n = \dim(R_{\mathfrak{q}}) - \dim(R_{\mathfrak{p}})$ is independent of choices. □

7.98. Regular local rings

It is not that easy to show that all prime localizations of a regular local ring are regular. In fact, quite a bit of the material developped sofar is geared towards a proof of this fact. See Proposition 7.102.5, and trace back the references.

Lemma 7.98.1. *Let R be a regular local ring with maximal ideal $\mathfrak{m}$. The graded ring $\bigoplus \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is isomorphic to the graded polynomial algebra $\kappa(\mathfrak{m})[X_1, \ldots, X_d]$.*

Proof. Let $x_1, \ldots, x_d$ be a minimal set of generators for the maximal ideal $\mathfrak{m}$. Write $\kappa = \kappa(\mathfrak{m})$. There is a surjection $\kappa[X_1, \ldots, X_d] \to \bigoplus \mathfrak{m}^n/\mathfrak{m}^{n+1}$, which maps the class of x_i in $\mathfrak{m}/\mathfrak{m}^2$ to X_i. Since $d(R) = d$ we know that the numerical polynomial $n \mapsto \dim_\kappa \mathfrak{m}^n/\mathfrak{m}^{n+1}$ has degree d. By Lemma 7.55.9 we conclude that the surjection $\kappa[X_1, \ldots, X_d] \to \bigoplus \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is an isomorphism. □

Lemma 7.98.2. *Any regular local ring is a domain.*

Proof. We will use that $\bigcap \mathfrak{m}^n = 0$ by Lemma 7.47.6. Let $f, g \in R$ such that $fg = 0$. Suppose that $f \in \mathfrak{m}^a$ and $g \in \mathfrak{m}^b$, with a, b maximal. Since $fg = 0 \in \mathfrak{m}^{a+b+1}$ we see from the result of Lemma 7.98.1 that either $f \in \mathfrak{m}^{a+1}$ or $g \in \mathfrak{m}^{b+1}$. Contradiction. □

Lemma 7.98.3. *Let R be a regular local ring and let $x_1, \ldots, x_d$ be a minimal set of generators for the maximal ideal $\mathfrak{m}$. Then $x_1, \ldots, x_d$ is a regular sequence, and each $R/(x_1, \ldots, x_c)$ is a regular local ring of dimension $d - c$. In particular R is Cohen-Macaulay.*

Proof. Note that R/x_1R is a Noetherian local ring of dimension $\geq d - 1$ by Lemma 7.57.11 with $x_2, \ldots, x_d$ generating the maximal ideal. Hence it is a regular local ring by definition. Since R is a domain by Lemma 7.98.2 x_1 is a nonzero divisor. □

Lemma 7.98.4. *Let R be a regular local ring. Let $I \subset R$ be an ideal such that R/I is a regular local ring as well. Then there exists a minimal set of generators $x_1, \ldots, x_d$ for the maximal $\mathfrak{m}$ of R such that $I = (x_1, \ldots, x_c)$ for some $0 \leq c \leq d$.*

Proof. Say $\dim(R) = d$ and $\dim(R/I) = d - c$. Denote $\overline{\mathfrak{m}} = \mathfrak{m}/I$ the maximal ideal of R/I. Let $\kappa = R/\mathfrak{m}$. We have $\dim_\kappa(I/\mathfrak{m}^2) = \dim_\kappa(\mathfrak{m}/\mathfrak{m}^2) - \dim(\overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2) = d - (d - c) = c$ by the definition of a regular local ring. Hence we can choose $x_1, \ldots, x_c \in I$ whose images in $\mathfrak{m}/\mathfrak{m}^2$ are linearly independent, and supplement with $x_{c+1}, \ldots, x_d$ to get a minimal system of generators. □

Lemma 7.98.5. *Let R be a Noetherian local ring. Let $x \in \mathfrak{m}$. Let M be a finite R-module such that x is a nonzero divisor on M and M/xM is free over R/xR. Then M is free over R.*

Proof. Let $m_1, \ldots, m_r$ be elements of M which map to a R/xR-basis of M/xM. By Nakayama's Lemma 7.14.5 $m_1, \ldots, m_r$ generate M. If $\sum a_i m_i = 0$ is a relation, then $a_i \in xR$ for all i. Hence $a_i = b_i x$ for some $b_i \in R$. Hence the kernel K of $R^r \to M$ satisfies $xK = K$ and hence is zero by Nakayama's lemma. □

Lemma 7.98.6. *Let R be a regular local ring. Any maximal Cohen-Macaulay module over R is free.*

Proof. Let M be a maximal Cohen-Macaulay module over R. Let $x \in \mathfrak{m}$ be part of a regular sequence generating $\mathfrak{m}$. Then x is a nonzero divisor on M by Proposition 7.95.4, and M/xM is a maximal Cohen-Macaulay module over R/xR. By induction on $\dim(R)$ we see that M/xM is free. We win by Lemma 7.98.5. □

Lemma 7.98.7. *Suppose R is a Noetherian local ring. Let $x \in \mathfrak{m}$ be a nonzero divisor such that R/xR is a regular local ring. Then R is a regular local ring. More generally, if $x_1, \ldots, x_r$ is a regular sequence in R such that $R/(x_1, \ldots, x_r)$ is a regular local ring, then R is a regular local ring.*

Proof. This is true because x together with the lifts of a system of minimal generators of the maximal ideal of R/xR will give $\dim(R)$ generators of $\mathfrak{m}$. Use Lemma 7.57.11. The last statement follows from the first and induction. □

Lemma 7.98.8. *Let $(R_i, \varphi_{ii'})$ be a directed system of local rings whose transition maps are local ring maps. If each R_i is a regular local ring and $R = colim\, R_i$ is Noetherian, then R is a regular local ring.*

Proof. Let $\mathfrak{m} \subset R$ be the maximal ideal; it is the colimit of the maximal ideal $\mathfrak{m}_i \subset R_i$. We prove the lemma by induction on $d = \dim \mathfrak{m}/\mathfrak{m}^2$. If $d = 0$, then $R = R/\mathfrak{m}$ is a field and R is a regular local ring. If $d > 0$ pick an $x \in \mathfrak{m}$, $x \notin \mathfrak{m}^2$. For some i we can find an $x_i \in \mathfrak{m}_i$ mapping to x. Note that $R/xR = colim_{i' \geq i} R_{i'}/x_i R_{i'}$ is a Noetherian local ring. By Lemma 7.98.3 we see that $R_{i'}/x_i R_{i'}$ is a regular local ring. Hence by induction we see that R/xR is a regular local ring. Since each R_i is a domain (Lemma 7.98.1) we see that R is a domain. Hence x is a nonzero divisor and we conclude that R is a regular local ring by Lemma 7.98.7. □

7.99. Epimorphisms of rings

In any category there is a notion of an *epimorphism*. Some of this material is taken from [**Laz69**] and [**Maz68**].

Lemma 7.99.1. *Let $R \to S$ be a ring map. The following are equivalent*

(1) *$R \to S$ is an epimorphism,*
(2) *the two ring maps $S \to S \otimes_R S$ are equal,*
(3) *either of the ring maps $S \to S \otimes_R S$ is an isomorphism, and*
(4) *the ring map $S \otimes_R S \to S$ is an isomorphism.*

Proof. Omitted. ☐

Lemma 7.99.2. *The composition of two epimorphisms of rings is an epimorphism.*

Proof. Omitted. Hint: This is true in any category. ☐

Lemma 7.99.3. *If $R \to S$ is an epimorphism of rings and $R \to R'$ is any ring map, then $R' \to R' \otimes_R S$ is an epimorphism.*

Proof. Omitted. Hint: True in any category with pushouts. ☐

Lemma 7.99.4. *If $A \to B \to C$ are ring maps and $A \to C$ is an epimorphism, so is $B \to C$.*

Proof. Omitted. Hint: This is true in any category. ☐

This means in particular, that if $R \to S$ is an epimorphism with image $\overline{R} \subset S$, then $\overline{R} \to S$ is an epimorphism. Hence while proving results for epimorphisms we may often assume the map is injective.

Lemma 7.99.5. *Let $R \to S$ be a ring map. The following are equivalent:*

(1) *$R \to S$ is an epimorphism, and*
(2) *$R_{\mathfrak p} \to S_{\mathfrak p}$ is an epimorphism for each prime $\mathfrak p$ of R.*

Proof. Since $S_{\mathfrak p} = R_{\mathfrak p} \otimes_R S$ (see Lemma 7.11.15) we see that (1) implies (2) by Lemma 7.99.3. Conversely, assume that (2) holds. Let $a, b : S \to A$ be two ring maps from S to a ring A equalizing the maop $R \to S$. By assumption we see that for evey prime $\mathfrak p$ of R the induced maps $a_{\mathfrak p}, b_{\mathfrak p} : S_{\mathfrak p} \to A_{\mathfrak p}$ are the same. Hence $a = b$ as $A \subset \prod_{\mathfrak p} A_{\mathfrak p}$, see Lemma 7.21.1. ☐

Lemma 7.99.6. *Let $R \to S$ be a ring map. The following are equivalent*

(1) *$R \to S$ is an epimorphism and finite, and*
(2) *$R \to S$ is surjective.*

Proof. (This lemma seems to have been reproved many times in the literature, and has many different proofs.) It is clear that a surjective ring map is an epimorphism. Suppose that $R \to S$ is a finite ring map such that $S \otimes_R S \to S$ is an isomorphism. Our goal is to show that $R \to S$ is surjective. Assume S/R is not zero. The exact sequence $R \to S \to S/R \to 0$ leads to an exact sequence

$$R \otimes_R S \to S \otimes_R S \to S/R \otimes_R S \to 0.$$

Our assumption implies that the first arrow is an isomorphism, hence we conclude that $S/R \otimes_R S = 0$. Hence also $S/R \otimes_R S/R = 0$. By Lemma 7.5.5 there exists a surjection of R-modules $S/R \to R/I$ for some proper ideal $I \subset R$. Hence there exists a surjection $S/R \otimes_R S/R \to R/I \otimes_R R/I = R/I \neq 0$, contradiction. ☐

Lemma 7.99.7. *A faithfully flat epimorphism is an isomorphism.*

Proof. This is clear from Lemma 7.99.1 part (3) as the map $S \to S \otimes_R S$ is the map $R \to S$ tensored with S. ☐

Lemma 7.99.8. *If $k \to S$ is an epimorphism and k is a field, then $S = k$ or $S = 0$.*

Proof. This is clear from the result of Lemma 7.99.7 (as any nonzero algebra over k is faithfully flat), or by argueing directly that $R \to R \otimes_k R$ cannot be surjective unless $\dim_k(R) \leq 1$. □

Lemma 7.99.9. *Let $R \to S$ be an epimorphism of rings. Then $Spec(S) \to Spec(R)$ is injective.*

Proof. Let $\mathfrak{p}$ be a prime of R. The fibre of the map is the spectrum of the fibre ring $S \otimes_R \kappa(\mathfrak{p})$. By Lemma 7.99.3 the map $\kappa(\mathfrak{p}) \to S \otimes_R \kappa(\mathfrak{p})$ is an epimorphism, and hence by Lemma 7.99.8 there is either one point or no points lying over $\mathfrak{p}$. □

Lemma 7.99.10. *Let R be a ring. Let M, N be R-modules. Let $\{x_i\}_{i \in I}$ be a set of generators of M. Let $\{y_j\}_{j \in J}$ be a set of generators of N. Let $\{m_j\}_{j \in J}$ be a family of elements of M with $m_j = 0$ for all but finitely many j. Then*

$$\sum\nolimits_{j \in J} m_j \otimes y_j = 0 \text{ in } M \otimes_R N$$

is equivalent to the following: There exist $a_{i,j} \in R$ with $a_{i,j} = 0$ for all but finitely many pairs (i, j) such that

$$m_j = \sum\nolimits_{i \in I} a_{i,j} x_i \quad \text{for all } j \in J,$$
$$0 = \sum\nolimits_{j \in J} a_{i,j} y_j \quad \text{for all } i \in I.$$

Proof. The sufficiency is immediate. Suppose that $\sum_{j \in J} m_j \otimes y_j = 0$. Consider the short exact sequence

$$0 \to K \to \bigoplus\nolimits_{j \in J} R \to N \to 0$$

where the jth basis vector of $\bigoplus_{j \in J} R$ maps to y_j. Tensor this with M to get the exact sequence

$$K \otimes_R M \to \bigoplus\nolimits_{j \in J} M \to N \otimes_R M \to 0.$$

The assumption implies that there exist elements $k_i \in K$ such that $\sum k_i \otimes x_i$ maps to the element $(m_j)_{j \in J}$ of the middle. Writing $k_i = (a_{i,j})_{j \in J}$ and we obtain what we want. □

Lemma 7.99.11. *Let $\varphi : R \to S$ be a ring map. Let $g \in S$. The following are equivalent:*

1. *$g \otimes 1 = 1 \otimes g$ in $S \otimes_R S$, and*
2. *there exist $n \geq 0$ and elements $y_i, z_j \in S$ and $x_{i,j} \in R$ for $1 \leq i, j \leq n$ such that*
   (a) *$g = \sum_{i,j \leq n} x_{i,j} y_i z_j$,*
   (b) *for each j we have $\sum x_{i,j} y_i \in \varphi(R)$, and*
   (c) *for each i we have $\sum x_{i,j} z_j \in \varphi(R)$.*

Proof. It is clear that (2) implies (1). Conversely, suppose that $g \otimes 1 = 1 \otimes g$. Choose generators $\{s_i\}_{i \in I}$ of S as an R-module with $0, 1 \in I$ and $s_0 = 1$ and $s_1 = g$. Apply Lemma 7.99.10 to the relation $g \otimes s_0 + (-1) \otimes s_1 = 0$. We see that there exist $a_{i,j} \in R$ such that $g = \sum_i a_{i,0} s_i$, $-1 = \sum_i a_{i,1} s_i$, and for $j \neq 0, 1$ we have $0 = \sum_i a_{i,j} s_i$, and moreover for all i we have $\sum_j a_{i,j} s_j = 0$. Then we have

$$\sum\nolimits_{i,j \neq 0} a_{i,j} s_i s_j = -g + a_{0,0}$$

and for each $j \neq 0$ we have $\sum_{i \neq 0} a_{i,j} s_i \in R$. This proves that $-g + a_{0,0}$ can be written as in (2). It follows that g can be written as in (2). Details omitted. Hint: Show that the set of elements of S which have an expression as in (2) form an R-subalgebra of S. □

Remark 7.99.12. Let $R \to S$ be a ring map. Sometimes the set of elements $g \in S$ such that $g \otimes 1 = 1 \otimes g$ is called the *epicenter* of S. It is an R-algebra. By the construction of Lemma 7.99.11 we get for each g in the epicenter a matrix factorization

$$(g) = YXZ$$

with $X \in \mathrm{Mat}(n \times n, R)$, $Y \in \mathrm{Mat}(1 \times n, S)$, and $Z \in \mathrm{Mat}(n \times 1, S)$. Namely, let $x_{i,j}, y_i, z_j$ be as in part (2) of the lemma. Set $X = (x_{i,j})$, let y be the row vector whose entries are the y_i and let z be the column vector whose entries are the z_j. With this notation conditions (b) and (c) of Lemma 7.99.11 mean exactly that $YX \in \mathrm{Mat}(1 \times n, R)$, $XZ \in \mathrm{Mat}(n \times 1, R)$. It turns out to be very convenient to consider the triple of matrices (X, YX, XZ). Given $n \in \mathbf{N}$ and a triple (P, U, V) we say that (P, U, V) is a *n-triple associated to g* if there exists a matrix factorization as above such that $P = X$, $U = YX$ and $V = XZ$.

Lemma 7.99.13. *Let $R \to S$ be an epimorphism of rings. Then the cardinality of S is at most the cardinality of R. In a formula: $|S| \leq |R|$.*

Proof. The condition that $R \to S$ is an epimorphism means that each $g \in S$ satisfies $g \otimes 1 = 1 \otimes g$, see Lemma 7.99.1. We are going to use the notation introduced in Remark 7.99.12. Suppose that $g, g' \in S$ and suppose that (P, U, V) is an n-triple which is associated to both g and g'. Then we claim that $g = g'$. Namely, write $(P, U, V) = (X, YX, XZ)$ for a matrix factorization $(g) = YXZ$ of g and write $(P, U, V) = (X', Y'X', X'Z')$ for a matrix factorization $(g') = Y'X'Z'$ of g'. Then we see that

$$(g) = YXZ = UZ = Y'X'Z = Y'PZ = Y'XZ = Y'V = Y'X'Z' = (g')$$

and hence $g = g'$. This implies that the cardinality of S is bounded by the number of possible triples, which has cardinality at most $\sup_{n \in \mathbf{N}} |R|^n$. If R is infinite then this is at most $|R|$, see [**Kun83**, Ch. I, 10.13].

If R is a finite ring then the argument above only proves that S is at worst countable. In fact in this case R is Artinian and the map $R \to S$ is surjective. We omit the proof of this case. □

7.100. Pure ideals

The material in this section is discussed in many papers, see for example [**Laz67**], [**Bko70**], and [**DM83**].

Definition 7.100.1. Let R be a ring. We say that $I \subset R$ is *pure* if the quotient ring R/I is flat over R.

Lemma 7.100.2. *Let R be a ring. Let $I \subset R$ be an ideal. The following are equivalent:*

(1) *I is pure,*
(2) *for every ideal $J \subset R$ we have $J \cap I = IJ$,*
(3) *for every finitely generated ideal $J \subset R$ we have $J \cap I = JI$,*
(4) *for every $x \in R$ we have $(x) \cap I = xI$,*
(5) *for every $x \in I$ we have $x = yx$ for some $y \in I$,*
(6) *for every $x_1, \ldots, x_n \in I$ there exists a $y \in I$ such that $x_i = yx_i$ for all $i = 1, \ldots, n$,*
(7) *for every prime $\mathfrak{p}$ of R we have $IR_{\mathfrak{p}} = 0$ or $IR_{\mathfrak{p}} = R_{\mathfrak{p}}$,*
(8) *$Supp(I) = Spec(R) \setminus V(I)$,*
(9) *I is the kernel of the map $R \to (1 + I)^{-1}R$,*
(10) *$R/I \cong S^{-1}R$ as R-algebras for some multiplicative subset S of R, and*
(11) *$R/I \cong (1 + I)^{-1}R$ as R-algebras.*

Proof. For any ideal J of R we have the short exact sequence $0 \to J \to R \to R/J \to 0$. Tensoring with R/I we get an exact sequence $J \otimes_R R/I \to R/I \to R/I + J \to 0$ and $J \otimes_R R/I = R/JI$. Thus the equivalence of (1), (2), and (3) follows from Lemma 7.35.4. Moreover, these imply (4).

The implication (4) $\Rightarrow$ (5) is trivial. Assume (5) and let $x_1, \ldots, x_n \in I$. Choose $y_i \in I$ such that $x_i = y_i x_i$. Let $y \in I$ be the element such that $1 - y = \prod_{i=1,\ldots,n}(1 - y_i)$. Then $x_i = yx_i$ for all $i = 1, \ldots, n$. Hence (6) holds, and it follows that (5) $\Leftrightarrow$ (6).

Assume (5). Let $x \in I$. Then $x = yx$ for some $y \in I$. Hence $x(1 - y) = 0$, which shows that x maps to zero in $(1 + I)^{-1}R$. Of course the kernel of the map $R \to (1 + I)^{-1}R$ is always contained in I. Hence we see that (5) implies (9). Assume (9). Then for any $x \in I$ we see that $x(1 - y) = 0$ for some $y \in I$. In other words, $x = yx$. We conclude that (5) is equivalent to (9).

Assume (5). Let $\mathfrak p$ be a prime of R. If $\mathfrak p \not\in V(I)$, then $IR_{\mathfrak p} = R_{\mathfrak p}$. If $\mathfrak p \in V(I)$, in other words, if $I \subset \mathfrak p$, then $x \in I$ implies $x(1 - y) = 0$ for some $y \in I$, implies x maps to zero in $R_{\mathfrak p}$, i.e., $IR_{\mathfrak p} = 0$. Thus we see that (7) holds.

Assume (7). Then $(R/I)_{\mathfrak p}$ is either 0 or $R_{\mathfrak p}$ for any prime $\mathfrak p$ of R. Hence by Lemma 7.35.19 we see that (1) holds. At this point we see that all of (1) -- (7) and (9) are equivalent.

As $IR_{\mathfrak p} = I_{\mathfrak p}$ we see that (7) implies (8). Finally, if (8) holds, then this means exactly that $I_{\mathfrak p}$ is the zero module if and only if $\mathfrak p \in V(I)$, which is clearly saying that (7) holds. Now (1) -- (9) are equivalent.

Assume (1) -- (9) hold. Then $R/I \subset (1 + I)^{-1}R$ by (9) and the map $R/I \to (1 + I)^{-1}R$ is also surjective by the description of localizations at primes afforded by (7). Hence (11) holds.

The implication (11) $\Rightarrow$ (10) is trivial. And (10) implies that (1) holds because a localization of R is flat over R, see Lemma 7.35.19. $\square$

Lemma 7.100.3. *Let R be a ring. If $I, J \subset R$ are pure ideals, then $V(I) = V(J)$ implies $I = J$.*

Proof. For example, by property (7) of Lemma 7.100.2 we see that $I = \ker(R \to \prod_{\mathfrak p \in V(I)} R_{\mathfrak p})$ can be recovered from the closed subset associated to it. $\square$

Lemma 7.100.4. *Let R be a ring. The rule $I \mapsto V(I)$ determines a bijection*

$$\{I \subset R \textit{ pure}\} \leftrightarrow \{Z \subset \textit{Spec}(R) \textit{ closed and closed under generalizations}\}$$

Proof. Let I be a pure ideal. Then since $R \to R/I$ is flat, by going up generalizations lift along the map $Spec(R/I) \to Spec(R)$. Hence $V(I)$ is closed under generalizations. This shows that the map is well defined. By Lemma 7.100.3 the map is injective. Suppose that $Z \subset Spec(R)$ is closed and closed under generalizations. Let $J \subset R$ be the radical ideal such that $Z = V(J)$. Let $I = \{x \in R : x \in xJ\}$. Note that I is an ideal. We claim that I is pure and that $V(I) = V(J)$. If the claim is true then the map of the lemma is surjective and the lemma holds.

Note that $I \subset J$, so that $V(J) \subset V(I)$. Let $I \subset \mathfrak p$ be a prime. Consider the multiplicative subset $S = (R \setminus \mathfrak p)(1 + J)$. By definition of I and $I \subset \mathfrak p$ we see that $0 \not\in S$. Hence we can find a prime $\mathfrak q$ of R which is disjoint from S, see Lemmas 7.9.4 and 7.16.5. Hence $\mathfrak q \subset \mathfrak p$ and $\mathfrak q \cap (1 + J) = \emptyset$. This implies that $\mathfrak q + J$ is a proper ideal of R. Let $\mathfrak m$ be a maximal ideal containing $\mathfrak q + J$. Then we get $\mathfrak m \in V(J)$ and hence $\mathfrak q \in V(J) = Z$ as Z was assumed to be closed under generalization. This in turn implies $\mathfrak{p} \in V(J)$ as $\mathfrak{q} \subset \mathfrak{p}$. Thus we see that $V(I) = V(J)$.

Finally, since $V(I) = V(J)$ (and J radical) we see that $J = \sqrt{I}$. Pick $x \in I$, so that $x = xy$ for some $y \in J$ by definition. Then $x = xy = xy^2 = \ldots = xy^n$. Since $y^n \in I$ for some $n > 0$ we conclude that property (5) of Lemma 7.100.2 holds and we see that I is indeed pure. ☐

Lemma 7.100.5. *Let R be a ring. Let $I \subset R$ be an ideal. The following are equivalent*

(1) *I is pure and finitely generated,*
(2) *I is generated by an idempotent,*
(3) *I is pure and $V(I)$ is open, and*
(4) *R/I is a projective R-module.*

Proof. If (1) holds, then $I = I \cap I = I^2$ by Lemma 7.100.2. Hence I is generated by an idempotent by Lemma 7.18.5. Thus (1) ⇒ (2). If (2) holds, then $I = (e)$ and $R = (1-e) \bigoplus (e)$ as an R-module hence R/I is flat and I is pure and $V(I) = D(1 - e)$ is open. Thus (2) ⇒ (1) + (3). Finally, assume (3). Then $V(I)$ is open and closed, hence $V(I) = D(1 - e)$ for some idempotent e of R, see Lemma 7.18.3. The ideal $J = (e)$ is a pure ideal such that $V(J) = V(I)$ hence $I = J$ by Lemma 7.100.3. In this way we see that (3) ⇒ (2). By Lemma 7.72.2 we see that (4) is equivalent to the assertion that I is pure and R/I finitely presented. Moreover, R/I is finitely presented if and only if I is finitely generated, see Lemma 7.5.3. Hence (4) is equivalent to (1). ☐

We can use the above to characterize those rings for which every finite flat module is finitely presented.

Lemma 7.100.6. *Let R be a ring. The following are equivalent:*

(1) *every $Z \subset Spec(R)$ which is closed and closed under generalizations is also open, and*
(2) *any finite flat R-module is finite locally free.*

Proof. If any finite flat R-module is finite locally free then the support of R/I where I is a pure ideal is open. Hence the implication (2) ⇒ (1) follows from Lemma 7.100.3.

For the converse assume that R satisfies (1). Let M be a finite flat R-module. The support $Z = \text{Supp}(M)$ of M is closed, see Lemma 7.59.4. On the other hand, if $\mathfrak{p} \subset \mathfrak{p}'$, then by Lemma 7.72.4 the module $M_{\mathfrak{p}'}$ is free, and $M_{\mathfrak{p}} = M_{\mathfrak{p}'} \otimes_{R_{\mathfrak{p}'}} R_{\mathfrak{p}}$ Hence $\mathfrak{p}' \in \text{Supp}(M) \Rightarrow \mathfrak{p} \in \text{Supp}(M)$, in other words, the support is closed under generalization. As R satisfies (1) we see that the support of M is open and closed. Suppose that M is generated by r elements $m_1, \ldots, m_r$. The modules $\wedge^i(M)$, $i = 1, \ldots, r$ are finite flat R-modules also, because $\wedge^i(M)_{\mathfrak{p}} = \wedge^i(M_{\mathfrak{p}})$ is free over $R_{\mathfrak{p}}$. Note that $\text{Supp}(\wedge^{i+1}(M)) \subset \text{Supp}(\wedge^i(M))$. Thus we see that there exists a decomposition

$$Spec(R) = U_0 \coprod U_1 \coprod \ldots \coprod U_r$$

by open and closed subsets such that the support of $\wedge^i(M)$ is $U_r \cup \ldots \cup U_i$ for all $i = 0, \ldots, r$. Let $\mathfrak{p}$ be a prime of R, and say $\mathfrak{p} \in U_i$. Note that $\wedge^i(M) \otimes_R \kappa(\mathfrak{p}) = \wedge^i(M \otimes_R \kappa(\mathfrak{p}))$. Hence, after possibly renumbering $m_1, \ldots, m_r$ we may assume that $m_1, \ldots, m_i$ generate $M \otimes_R \kappa(\mathfrak{p})$. By Nakayama's Lemma 7.14.5 we get a surjection

$$R_f^{\oplus i} \longrightarrow M_f, \quad (a_1, \ldots, a_i) \longmapsto \sum a_i m_i$$

for some $f \in R$, $f \notin \mathfrak{p}$. We may also assume that $D(f) \subset U_i$. This means that $\wedge^i(M_f) = \wedge^i(M)_f$ is a flat R_f module whose support is all of $Spec(R_f)$. By the above it is generated by a single element, namely $m_1 \wedge \ldots \wedge m_i$. Hence $\wedge^i(M)_f \cong R_f/J$ for some pure ideal $J \subset R_f$ with $V(J) = Spec(R_f)$. Clearly this means that $J = (0)$, see Lemma 7.100.3. Thus $m_1 \wedge \ldots \wedge m_i$ is a basis for $\wedge^i(M_f)$ and it follows that the displayed map is injective as well as surjective. This proves that M is finite locally free as desired. □

7.101. Rings of finite global dimension

The following lemma is often used to compare different projective resolutions of a given module.

Lemma 7.101.1. *(Schanuel's lemma.) Let R be a ring. Let M be an R-module. Suppose that $0 \to K \to P_1 \to M \to 0$ and $0 \to L \to P_2 \to M \to 0$ are two short exact sequences, with P_i projective. Then $K \oplus P_2 \cong L \oplus P_1$.*

Proof. Consider the module N defined by the short exaxt sequence $0 \to N \to P_1 \oplus P_2 \to M \to 0$, where the last map is the sum of the two maps $P_i \to M$. It is easy to see that the projection $N \to P_1$ is surjective with kernel L, and that $N \to P_2$ is surjective with kernel K. Since P_i are projective we have $N \cong K \oplus P_2 \cong L \oplus P_1$. □

Definition 7.101.2. Let R be a ring. Let M be an R-module. We say M has *finite projective dimension* if it has a finite length resolution by projective R-modules. The minimal length of such a resolution is called the *projective dimension* of M.

It is clear that the projective dimension of M is 0 if and only if M is a projective module. The following lemma explains to what extend the projective dimension is independent of the choice of a projective resolution.

Lemma 7.101.3. *Let R be a ring. Suppose that M is an R-module of projective dimension d. Suppose that $F_e \to F_{e-1} \to \ldots \to F_0 \to M \to 0$ is exact with F_i projective and $e \geq d-1$. Then the kernel of $F_e \to F_{e-1}$ is projective (or the kernel of $F_0 \to M$ is projective in case $e = 0$).*

Proof. We prove this by induction on d. If $d = 0$, then M is projective. In this case there is a splitting $F_0 = \text{Ker}(F_0 \to M) \oplus M$, and hence $\text{Ker}(F_0 \to M)$ is projective. This finishes the proof if $e = 0$, and if $e > 0$, then replacing M by $\text{Ker}(F_0 \to M)$ we decrease e.

Next assume $d > 0$. Let $0 \to P_d \to P_{d-1} \to \ldots \to P_0 \to M \to 0$ be a minimal length finite resolution with P_i projective. According to Schanuel's Lemma 7.101.1 we have $P_0 \oplus \text{Ker}(F_0 \to M) \cong F_0 \oplus \text{Ker}(P_0 \to M)$. This proves the case $d = 1$, $e = 0$, because then the right hand side is $F_0 \oplus P_1$ which is projective. Hence now we may assume $e > 0$. The module $F_0 \oplus \text{Ker}(P_0 \to M)$ has the finite projective resolution $0 \to P_d \oplus F_0 \to P_{d-1} \oplus F_0 \to \ldots \to P_1 \oplus F_0 \to \text{Ker}(P_0 \to M) \oplus F_0 \to 0$ of length $d-1$. By induction on d we see that the kernel of $F_e \oplus P_0 \to F_{e-1} \oplus P_0$ is projective. This implies the lemma. □

Lemma 7.101.4. *Let R be a ring. Let M be an R-module. Let $n \geq 0$. The following are equivalent*

(1) *M has projective dimension $\leq n$,*
(2) *$Ext^i_R(M, N) = 0$ for all R-modules N and all $i \geq n+1$, and*
(3) *$Ext^{n+1}_R(M, N) = 0$ for all R-modules N.*

Proof. Assume (1). Choose a free resolution $F_\bullet \to M$ of M. Denote $d_e : F_e \to F_{e-1}$. By Lemma 7.101.3 we see that $P_e = \text{Ker}(d_e)$ is projective for $e \geq n - 1$. This implies that $F_e \cong P_e \oplus P_{e-1}$ for $e \geq n$ where d_e maps the summand P_{e-1} isomorphically to P_{e-1} in F_{e-1}. Hence, for any R-module N the complex $Hom_R(F_\bullet, N)$ is split exact in degrees $\geq n + 1$. Whence (2) holds. The implication (2) $\Rightarrow$ (3) is trivial.

Assume (3) holds. If $n = 0$ then M is projective by Lemma 7.71.2 and we see that (1) holds. If $n > 0$ choose a free R-module F and a surjection $F \to M$ with kernel K. By Lemma 7.67.7 and the vanishing of $\text{Ext}^i_R(F, N)$ for all $i > 0$ by part (1) we see that $\text{Ext}^n_R(K, N) = 0$ for all R-modules N. Hence by induction we see that N has projective dimension $\leq n - 1$. Then M has projective dimension $\leq n$ as any finite projective resolution of N gives a projective resolution of length one more for M by adding F to the front. $\square$

Lemma 7.101.5. *Let R be a ring. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of R-modules.*

1. *If M has projective dimension $\leq n$ and M'' has projective dimension $\leq n + 1$, then M' has projective dimension $\leq n$.*
2. *If M' and M'' have projective dimension $\leq n$ then M has projective dimension $\leq n$.*
3. *If M' has projective dimension $\leq n$ and M has projective dimension $\leq n+1$ then M'' has projective dimension $\leq n + 1$.*

Proof. Combine the characterization of projective dimension in Lemma 7.101.4 with the long exact sequence of ext groups in Lemma 7.67.7. $\square$

Definition 7.101.6. Let R be a ring. The ring R is said to have *finite global dimension* if there exists an integer n such that every R-module has a resolution by projective R-modules of length at most n. The minimal such n is then called the *global dimension* of R.

The argument in the proof of the following lemma can be found in the paper [**Aus55**] by Auslander.

Lemma 7.101.7. *Let R be a ring. The following are equivalent*

1. *R has finite global dimension $\leq n$,*
2. *every finite R-module has projective dimension $\leq n$, and*
3. *every cyclic R-module R/I has projective dimension $\leq n$.*

Proof. It is clear that (1) $\Rightarrow$ (3). Assume (3). Since every finite R-module has a finite filtration by cyclic modules, see Lemma 7.5.5 we see that (2) follows by Lemma 7.101.5.

Assume (2). Let M be an arbitrary R-module. Choose a set $E \subset M$ of generators of M. Choose a well ordering on E. For $e \in E$ denote M_e the submodule of M generated by the elements $e' \in E$ with $e' \leq e$. Then $M = \bigcup_{e \in E} M_e$. Note that for each $e \in E$ the quotient

$$M_e / \bigcup\nolimits_{e' < e} M_{e'}$$

is either zero or generated by one element, hence has projective dimension $\leq n$. To finish the proof we claim that any time we have a well ordered set E and a module $M = \bigcup_{e \in E} M_e$ such that the quotients $M_e / \bigcup_{e' < e} M_{e'}$ have projective dimesion $\leq n$, then M has projective dimension $\leq n$.

We may prove this statement by induction on n. If $n = 0$, then we will show, by transfinite induction that M is projective. Namely, for each $e \in E$ we may choose a splitting $M_e =$

$\bigcup_{e'<e} M_{e'} \oplus P_e$ because $P_e = M_e / \bigcup_{e'<e} M_{e'}$ is projective. Hence it follows that $M = \bigoplus_{e \in E} P_e$ and we conclude that M is projective, see Lemma 7.71.3.

If $n > 0$, then for $e \in E$ we denote F_e the free R-module on the set of elements of M_e. Then we have a system of short exact sequences

$$0 \to K_e \to F_e \to M_e \to 0$$

over the well ordered set E. Note that the transition maps $F_{e'} \to F_e$ and $K_{e'} \to K_e$ are injective too. Set $F = \bigcup F_e$ and $K = \bigcup K_e$. Then

$$0 \to K_e / \bigcup\nolimits_{e'<e} K_{e'} \to F_e / \bigcup\nolimits_{e'<e} F_{e'} \to M_e / \bigcup\nolimits_{e'<e} M_{e'} \to 0$$

is a short exact sequence of R-modules too and $F_e / \bigcup_{e'<e} F_{e'}$ is the free R-module on the set of elements in M_e which are not contained in $\bigcup_{e'<e} M_{e'}$. Hence by Lemma 7.101.5 we see that the projective dimension of $K_e / \bigcup_{e'<e} K_{e'}$ is at most $n-1$. By induction we conclude that K has projective dimension at most $n-1$. Whence M has projective dimension at most n and we win. □

Lemma 7.101.8. *Let R be a ring. Let M be an R-module. Let $S \subset R$ be a multiplicative subset.*

1. *If M has projective dimension $\leq n$, then $S^{-1}M$ has projective dimension $\leq n$ over $S^{-1}R$.*
2. *If R has finite global dimension $\leq n$, then $S^{-1}R$ has finite global dimension $\leq n$.*

Proof. Let $0 \to P_n \to P_{n-1} \to \ldots \to P_0 \to M \to 0$ be a projective resolution. As localization is exact, see Proposition 7.9.12, and as each $S^{-1}P_i$ is a projective $S^{-1}R$-module, see Lemma 7.88.1, we see that $0 \to S^{-1}P_n \to \ldots \to S^{-1}P_0 \to S^{-1}M \to 0$ is a projective resolution of $S^{-1}M$. This proves (1). Let M' be an $S^{-1}R$-module. Note that $M' = S^{-1}M'$. Hence we see that (2) follows from (1). □

7.102. Regular rings and global dimension

We can use the material on rings of finite global dimension to give another characterization of regular local rings.

Proposition 7.102.1. *Let R be a regular local ring of dimension d. Every finite R-module M of depth e has a finite free resolution*

$$0 \to F_{d-e} \to \ldots \to F_0 \to M \to 0.$$

In particular a regular local ring has global dimension $\leq d$.

Proof. This is clear in view of Lemma 7.98.6 and Lemma 7.96.10. □

Lemma 7.102.2. *Let R be a Noetherian ring. Then R has finite global dimension if and only if there exists an integer n such that for all maximal ideals $\mathfrak{m}$ of R the ring $R_{\mathfrak{m}}$ has global dimension $\leq n$.*

Proof. We saw, Lemma 7.101.8 that if R has finite global dimension n, then all the localizations $R_{\mathfrak{m}}$ have finite global dimension at most n. Conversely, suppose that all the $R_{\mathfrak{m}}$ have global dimension n. Let M be a finite R-module. Let $0 \to K_n \to F_{n-1} \to \ldots \to F_0 \to M \to 0.$ be a resolution with F_i finite free. Then K_n is a finite R-module. According to Lemma 7.101.3 and the assumption all the modules $K_n \otimes_R R_{\mathfrak{m}}$ are projective. Hence by Lemma 7.72.2 the module K_n is finite projective. □

Lemma 7.102.3. *Suppose that R is a Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field κ. In this case the projective dimension of κ is $\geq \dim_\kappa \mathfrak{m}/\mathfrak{m}^2$.*

Proof. Let $x_1, \ldots x_n$ be elements of $\mathfrak{m}$ whose images in $\mathfrak{m}/\mathfrak{m}^2$ form a basis. Consider the *Koszul complex* on $x_1, \ldots, x_n$. This is the complex

$$0 \to \wedge^n R^n \to \wedge^{n-1} R^n \to \wedge^{n-2} R^n \to \ldots \to \wedge^i R^n \to \ldots \to R^n \to R$$

with maps given by

$$e_{j_1} \wedge \ldots \wedge e_{j_i} \longmapsto \sum_{a=1}^{i} (-1)^{i+1} x_{j_a} e_{j_1} \wedge \ldots \wedge \hat{e}_{j_a} \wedge \ldots \wedge e_{j_i}$$

It is easy to see that this is a complex $K_\bullet(R, x_\bullet)$. Note that the cokernel of the last map of $K_\bullet(R, x_\bullet)$ is clearly κ.

Now, let $F_\bullet \to \kappa$ by any finite resolution by finite free R-modules. By Lemma 7.94.2 we may assume all the maps in the complex $F_\bullet$ have to property that $\mathrm{Im}(F_i \to F_{i-1}) \subset \mathfrak{m}F_{i-1}$, because removing a trivial summand from the resolution can at worst shorten the resolution. By Lemma 7.67.4 we can find a map of complexes $\alpha : K_\bullet(R, x_\bullet) \to F_\bullet$ inducing the identity on κ. We will prove by induction that the maps $\alpha_i : \wedge^i R^n = K_i(R, x_\bullet) \to F_i$ have the property that $\alpha_i \otimes \kappa : \wedge^i \kappa^n \to F_i \otimes \kappa$ are injective. This will prove the lemma since it clearly shows that $F_n \neq 0$.

The result is clear for $i = 0$ because the composition $R \xrightarrow{\alpha_0} F_0 \to \kappa$ is nonzero. Note that F_0 must have rank 1 since otherwise the map $F_1 \to F_0$ whose cokernel is a single copy of κ cannot have image contained in $\mathfrak{m}F_0$. For α_1 we use that $x_1, \ldots, x_n$ is a minimal set of generators for $\mathfrak{m}$. Namely, we saw above that $F_0 = R$ and $F_1 \to F_0 = R$ has image $\mathfrak{m}$. We have a commutative diagram

$$\begin{array}{ccccccc} R^n & = & K_1(R, x_\bullet) & \to & K_0(R, x_\bullet) & = & R \\ & & \downarrow & & \downarrow & & \downarrow \\ & & F_1 & \to & F_0 & = & R \end{array}$$

where the rightmost vertical arrow is given by multiplication by a unit. Hence we see that the image of the composition $R^n \to F_1 \to F_0 = R$ is also equal to $\mathfrak{m}$. Thus the map $R^n \otimes \kappa \to F_1 \otimes \kappa$ has to be injective since $\dim_\kappa(\mathfrak{m}/\mathfrak{m}^2) = n$.

Suppose the injectivity of $\alpha_j \otimes \kappa$ has been proved for all $j \leq i-1$. Consider the commutative diagram

$$\begin{array}{ccccccc} \wedge^i R^n & = & K_i(R, x_\bullet) & \to & K_{i-1}(R, x_\bullet) & = & \wedge^{i-1} R^n \\ & & \downarrow & & \downarrow & & \\ & & F_i & \to & F_{i-1} & & \end{array}$$

We know that $\wedge^{i-1}\kappa^n \to F_{i-1} \otimes \kappa$ is injective. This proves that $\wedge^{i-1}\kappa^n \otimes_\kappa \mathfrak{m}/\mathfrak{m}^2 \to F_{i-1} \otimes \mathfrak{m}/\mathfrak{m}^2$ is injective. Also, by our choice of the complex, F_i maps into $\mathfrak{m}F_{i-1}$, and similarly for the Koszul complex. Hence we get a commutative diagram

$$\begin{array}{ccc} \wedge^i \kappa^n & \to & \wedge^{i-1}\kappa^n \otimes \mathfrak{m}/\mathfrak{m}^n \\ \downarrow & & \downarrow \\ F_i \otimes \kappa & \to & F_{i-1} \otimes \mathfrak{m}/\mathfrak{m}^2 \end{array}$$

At this point it suffices to verify the map $\wedge^i \kappa^n \to \wedge^{i-1}\kappa^n \otimes \mathfrak{m}/\mathfrak{m}^n$ is injective, which can be done by hand. $\square$

Lemma 7.102.4. *Let R be a Noetherian local ring. Suppose that the residue field κ has finite projective dimension n over R. In this case* $\dim(R) \geq n$.

Proof. Let $F_\bullet$ be a finite resolution of κ by finite free R-modules. By Lemma 7.94.2 we may assume all the maps in the complex $F_\bullet$ have to property that $\mathrm{Im}(F_i \to F_{i-1}) \subset \mathfrak{m}F_{i-1}$, because removing a trivial summand from the resolution can at worst shorten the resolution. Say $F_n \neq 0$ and $F_i = 0$ for $i > n$, so that the projective dimension of κ is n. By Proposition 7.94.10 we see that $\mathrm{depth}(I(\varphi_n)) \geq n$ since $I(\varphi_n)$ cannot equal R by our choice of the complex. Thus by Lemma 7.65.6 also $\dim(R) \geq n$. □

Proposition 7.102.5. *A Noetherian local ring whose residue field has finite projective dimension is a regular local ring. In particular a Noetherian local ring of finite global dimension is a regular local ring.*

Proof. By Lemmas 7.102.3 and 7.102.4 we see that $\dim(R) \geq \dim_\kappa(\mathfrak{m}/\mathfrak{m}^2)$. Thus the result follows immediately from Definition 7.57.9. □

By Propositions 7.102.5 and 7.102.1 we see that a Noetherian local ring is a regular local ring if and only if it has finite global dimension. Furthermore, any localization $R_\mathfrak{p}$ has finite global dimension, see Lemma 7.101.8, and hence is a regular local ring. Thus it now makes sense to make the following definition, because it does not conclict with the earlier definition of a regular local ring.

Definition 7.102.6. A Noetherian ring R is said to be *regular* if all the localizations $R_\mathfrak{p}$ are regular local rings.

Note that this is not the same as asking R to have finite global dimension, even assuming R is Noetherian. This is because there is an example of a regular Noetherian ring which does not have finite global dimension, namely because it does not have finite dimension.

Lemma 7.102.7. *Let R be a Noetherian ring. The following are equivalent:*

(1) *R has finite global dimension n,*
(2) *there exists an integer n such that all the localizations $R_\mathfrak{m}$ at maximal ideals are regular of dimension $\leq n$ with equality for at least one $\mathfrak{m}$, and*
(3) *there exists an integer n such that all the localizations $R_\mathfrak{p}$ at prime ideals are regular of dimension $\leq n$ with equality for at least one $\mathfrak{p}$.*

Proof. This is a reformulation of Lemma 7.102.2 in view of the discussion surrouding Definition 7.102.6. See especially Propositions 7.102.1 and 7.102.5. □

Lemma 7.102.8. *Let $R \to S$ be a local homomorphism of local Noetherian rings. Assume that $R \to S$ is flat and that S is regular. Then R is regular.*

Proof. Let $\mathfrak{m} \subset R$ be the maximal ideal and let $\kappa = R/\mathfrak{m}$ be the residue field. Let $d = \dim S$. Choose any resolution $F_\bullet \to \kappa$ with each F_i a finite free R-module. Set $K_d = \mathrm{Ker}(F_{d-1} \to F_{d-2})$. By flatness of $R \to S$ the complex $0 \to K_d \otimes_R S \to F_{d-1} \otimes_R S \to \ldots \to F_0 \otimes_R S \to \kappa \otimes_R S \to 0$ is still exact. Because the global dimension of S is d, see Proposition 7.102.1, we see that $K_d \otimes_R S$ is a finite free S-module (see also Lemma 7.101.3). By Lemma 7.72.5 we see that K_d is a finite free R-module. Hence κ has finite projective dimension and R is regular by Proposition 7.102.5. □

7.103. Homomorphisms and dimension

This section contains a collection of easy results relating dimensions of rings when there are maps between them.

Lemma 7.103.1. *Suppose $R \to S$ is a ring map satisfying either going up, see Definition 7.36.1, or going down see Definition 7.36.1. Assume in addition that Spec$(S) \to$ Spec(R) is surjective. Then* $\dim(R) \leq \dim(S)$.

Proof. Assume going up. Take any chain $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_e$ of prime ideals in R. By surjectivity we may choose a prime $\mathfrak{q}_0$ mapping to $\mathfrak{p}_0$. By going up we may extend this to a chain of length e of primes $\mathfrak{q}_i$ lying over $\mathfrak{p}_i$. Thus $\dim(S) \geq \dim(R)$. The case of going down is exactly the same. See also Topology, Lemma 5.14.8 for a purely topological version. □

Lemma 7.103.2. *Suppose that $R \to S$ is a ring map with the going up property, see Definition 7.36.1. If $\mathfrak{q} \subset S$ is a maximal ideal. Then the inverse image of $\mathfrak{q}$ in R is a maximal ideal too.*

Proof. Trivial. □

Lemma 7.103.3. *Suppose that $R \to S$ is a ring map such that S is integral over R. Then* $\dim(R) \geq \dim(S)$, *and every closed point of Spec(S) maps to a closed point of Spec(R).*

Proof. Immediate from Lemmas 7.32.18 and 7.103.2 and the definitions. □

Lemma 7.103.4. *Suppose $R \subset S$ and S integral over R. Then* $\dim(R) = \dim(S)$.

Proof. This is a combination of Lemmas 7.32.20, 7.32.15, 7.103.1, and 7.103.3. □

Definition 7.103.5. Suppose that $R \to S$ is a ring map. Let $\mathfrak{q} \subset S$ be a prime lying over the prime $\mathfrak{p}$ of R. The *local ring of the fibre at* $\mathfrak{q}$ is the local ring

$$S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}} = (S/\mathfrak{p}S)_{\mathfrak{q}} = (S \otimes_R \kappa(\mathfrak{p}))_{\mathfrak{q}}$$

Lemma 7.103.6. *Let $R \to S$ be a homomorphism of Noetherian rings. Let $\mathfrak{q} \subset S$ be a prime lying over the prime $\mathfrak{p}$. Then*

$$\dim(S_{\mathfrak{q}}) \leq \dim(R_{\mathfrak{p}}) + \dim(S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}}).$$

Proof. We use the characterization of dimension of Proposition 7.57.8. Let $x_1, \ldots, x_d$ be elements of $\mathfrak{p}$ generating an ideal of definition of $R_{\mathfrak{p}}$ with $d = \dim(R_{\mathfrak{p}})$. Let $y_1, \ldots, y_e$ be elements of $\mathfrak{q}$ generating an ideal of definition of $S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}}$ with $e = \dim(S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}})$. It is clear that $S_{\mathfrak{q}}/(x_1, \ldots, x_d, y_1, \ldots, y_e)$ has a nilpotent maximal ideal. Hence $x_1, \ldots, x_d, y_1, \ldots, y_e$ generate an ideal of definition if $S_{\mathfrak{q}}$. □

Lemma 7.103.7. *Let $R \to S$ be a homomorphism of Noetherian rings. Let $\mathfrak{q} \subset S$ be a prime lying over the prime $\mathfrak{p}$. Assume the going down property holds for $R \to S$ (for example if $R \to S$ is flat, see Lemma 7.35.17). Then*

$$\dim(S_{\mathfrak{q}}) = \dim(R_{\mathfrak{p}}) + \dim(S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}}).$$

Proof. By Lemma 7.103.6 we have an inequality $\dim(S_{\mathfrak{q}}) \leq \dim(R_{\mathfrak{p}}) + \dim(S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}})$. To get equality, choose a chain of primes $\mathfrak{p}S \subset \mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \ldots \subset \mathfrak{q}_d = \mathfrak{q}$ with $d = \dim(S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}})$. On the other hand, choose a chain of primes $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_e = \mathfrak{p}$ with $e = \dim(R_{\mathfrak{p}})$. By the going down theorem we may choose $\mathfrak{q}_{-1} \subset \mathfrak{q}_0$ lying over $\mathfrak{p}_{e-1}$. And then we may choose $\mathfrak{q}_{-2} \subset \mathfrak{q}_{e-1}$ lying over $\mathfrak{p}_{e-2}$. Inductively we keep going until we get a chain $\mathfrak{q}_{-e} \subset \ldots \subset \mathfrak{q}_d$ of length $e + d$. □

Lemma 7.103.8. *Let $R \to S$ be a local homomorphism of local Noetherian rings. Assume*

(1) *R is regular,*
(2) *$S/\mathfrak{m}_R S$ is regular, and*
(3) *$R \to S$ is flat.*

Then S is regular.

Proof. By Lemma 7.103.7 we have $\dim(S) = \dim(R) + \dim(S/\mathfrak{m}_R S)$. Pick generators $x_1, \ldots, x_d \in \mathfrak{m}_R$ with $d = \dim(R)$, and pick $y_1, \ldots, y_e \in \mathfrak{m}_S$ which generate the maximal ideal of $S/\mathfrak{m}_R S$ with $e = \dim(S/\mathfrak{m}_R S)$. Then we see that $x_1, \ldots, x_d, y_1, \ldots, y_e$ are elements which generate the maximal ideal of S and $e + d = \dim(S)$. $\square$

The lemma below will later be used to show that rings of finite type over a field are Cohen-Macaulay if and only if they are quasi-finite flat over a polynomial ring. It is a partial converse to Lemma 7.119.1.

Lemma 7.103.9. *Let $R \to S$ be a local homomorphism of Noetherian local rings. Assume R Cohen-Macaulay. If S is finite flat over R, or if S is flat over R and* $\dim(S) \leq \dim(R)$, *then S is Cohen-Macaulay and* $\dim(R) = \dim(S)$.

Proof. Let $x_1, \ldots, x_d \in \mathfrak{m}_R$ be a regular sequence of length $d = \dim(R)$. By Lemma 7.65.7 this maps to a regular sequence in S. Hence S is Cohen-Macaulay if $\dim(S) \leq d$. This is true if S is finite flat over R by Lemma 7.103.4. And in the second case we assumed it. $\square$

7.104. The dimension formula

Recall the definitions of catenary (Definition 7.97.1) and universally catenary (Definition 7.97.5).

Lemma 7.104.1. *Let $R \to S$ be a ring map. Let $\mathfrak{q}$ be a prime of S lying over the prime $\mathfrak{p}$ of R. Assume that*

(1) *R is Noetherian,*
(2) *$R \to S$ is of finite type,*
(3) *R, S are domains, and*
(4) *$R \subset S$.*

Then we have

$$height(\mathfrak{q}) \leq height(\mathfrak{p}) + trdeg_R(S) - trdeg_{\kappa(\mathfrak{p})}\kappa(\mathfrak{q})$$

with equality if R is universally catenary.

Proof. Suppose that $R \subset S' \subset S$ is a finitely generated R-subalgebra of S. In this case set $\mathfrak{q}' = S' \cap \mathfrak{q}$. The lemma for the ring maps $R \to S'$ and $S' \to S$ implies the lemma for $R \to S$ by additivity of transcendence degree in towers of fields. Hence we can use induction on the number of generators of S over R and reduce to the case where S is generated by one element over R.

Case I: $S = R[x]$ is a polynomial algebra over R. In this case we have $\text{trdeg}_R(S) = 1$. Also $R \to S$ is flat and hence

$$\dim(S_{\mathfrak{q}}) = \dim(R_{\mathfrak{p}}) + \dim(S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}})$$

see Lemma 7.103.7. Let $\mathfrak{r} = \mathfrak{m}S$. Then $\text{trdeg}_{\kappa(\mathfrak{p})}\kappa(\mathfrak{q}) = 1$ is equivalent to $\mathfrak{q} = \mathfrak{r}$, and implies that $\dim(S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}}) = 0$. In the same vein $\text{trdeg}_{\kappa(\mathfrak{p})}\kappa(\mathfrak{q}) = 0$ is equivalent to having a strict inclusion $\mathfrak{q} \subset \mathfrak{r}$, which implies that $\dim(S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}}) = 1$. Thus we are done with case I with equality in every instance.

Case II: $S = R[x]/\mathfrak{n}$ with $\mathfrak{n} \neq 0$. In this case we have $\text{trdeg}_R(S) = 0$. Denote $\mathfrak{q}' \subset R[x]$ the prime corresponding to $\mathfrak{q}$. Thus we have

$$S_{\mathfrak{q}} = (R[x])_{\mathfrak{q}'}/\mathfrak{n}(R[x])_{\mathfrak{q}'}$$

By the previous case we have $\dim((R[x])_{\mathfrak{q}'}) = \dim(R_{\mathfrak{p}}) + 1 - \text{trdeg}_{\kappa(\mathfrak{p})}\kappa(\mathfrak{q})$. Since $\mathfrak{n} \neq 0$ we see that the dimension of $S_{\mathfrak{q}}$ decreases by at least one, see Lemma 7.57.11, which proves the inequality of the lemma. To see the equality in case R is universally catenary note that $\mathfrak{n} \subset R[x]$ is a height one prime as it corresponds to a nonzero prime in $f.f.(R)[x]$. Hence any maximal chain of primes in $R[x]_{\mathfrak{q}'}/\mathfrak{n}$ corresponds to a maximal chain of primes with length 1 greater between $\mathfrak{q}'$ and (0) in $R[x]$. If R is universally catenary these all have the same length equal to the height of $\mathfrak{q}'$. This proves that $\dim(R[x]_{\mathfrak{q}'}/\mathfrak{n}) = \dim(R[x]_{\mathfrak{q}'}) - 1$ as desired. □

The following lemma says that generically finite maps tend to be quasi-finite in codimension 1.

Lemma 7.104.2. *Let $A \to B$ be a ring map. Assume*

(1) *$A \subset B$ is an extension of domains.*
(2) *A is Noetherian,*
(3) *$A \to B$ is of finite type, and*
(4) *the extension $f.f.(A) \subset f.f.(B)$ is finite.*

Let $\mathfrak{p} \subset A$ be a prime such that $\dim(A_{\mathfrak{p}}) = 1$. *Then there are at most finitely many primes of B lying over $\mathfrak{p}$.*

Proof. By the dimension formula (Lemma 7.104.1) for any prime $\mathfrak{q}$ lying over $\mathfrak{p}$ we have

$$\dim(B_{\mathfrak{q}}) \leq \dim(A_{\mathfrak{p}}) - \text{trdeg}_{\kappa(\mathfrak{p})}\kappa(\mathfrak{q}).$$

As the domain $B_{\mathfrak{q}}$ has at least 2 prime ideals we see that $\dim(B_{\mathfrak{q}}) \geq 1$. We conclude that $\dim(B_{\mathfrak{q}}) = 1$ and that the extension $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q})$ is algebraic. Hence $\mathfrak{q}$ defines a closed point of its fibre $Spec(B \otimes_A \kappa(\mathfrak{p}))$, see Lemma 7.31.9. Since $B \otimes_A \kappa(\mathfrak{p})$ is a Noetherian ring the fibre $Spec(B \otimes_A \kappa(\mathfrak{p}))$ is a Noetherian topological space, see Lemma 7.28.5. A Noetherian topological space consisting of closed points is finite, see for example Topology, Lemma 5.6.2. □

7.105. Dimension of finite type algebras over fields

In this section we compute the dimension of a polynomial ring over a field. We also prove that the dimension of a finite type domain over a field is the dimension of its local rings at maximal ideals. We will esthablish the connection with the transcendence degree over the ground field in Section 7.107.

Lemma 7.105.1. *Let $\mathfrak{m}$ be a maximal ideal in $k[x_1, \ldots, x_n]$. The ideal $\mathfrak{m}$ is generated by n elements. The dimension of $k[x_1, \ldots, x_n]_{\mathfrak{m}}$ is n. Hence $k[x_1, \ldots, x_n]_{\mathfrak{m}}$ is a regular local ring of dimension n.*

Proof. By the Hilbert Nullstellensatz (Theorem 7.30.1) we know the residue field $\kappa = \kappa(\mathfrak{m})$ is a finite extension of k. Denote $\alpha_i \in \kappa$ the image of x_i. Denote $\kappa_i = k(\alpha_1, \ldots, \alpha_i) \subset \kappa$, $i = 1, \ldots, n$ and $\kappa_0 = k$. Note that $\kappa_i = k[\alpha_1, \ldots, \alpha_i]$ by field theory. Define inductively elements $f_i \in \mathfrak{m} \cap k[x_1, \ldots, x_i]$ as follows: Let $P_i(T) \in \kappa_{i-1}[T]$ be the monic minimal polynomial of α_i over κ_{i-1}. Let $Q_i(T) \in k[x_1, \ldots, x_{i-1}][T]$ be a monic lift of $P_i(T)$ (of the same degree). Set $f_i = Q_i(x_i)$. Note that if $d_i = \deg_T(P_i) = \deg_T(Q_i) = \deg_{x_i}(f_i)$ then $d_1 d_2 \ldots d_i = [\kappa_i : k]$ by elementary field theory.

We claim that for all $i = 0, 1, \ldots, n$ there is an isomorphism

$$\psi_i : k[x_1, \ldots, x_i]/(f_1, \ldots, f_i) \cong \kappa_i.$$

By construction the composition $k[x_1, \ldots, x_i] \to k[x_1, \ldots, x_n] \to \kappa$ is surjective onto κ_i and $f_1, \ldots, f_i$ are in the kernel. This gives a surjective homomorphism. We prove ψ_i is injective by induction. It is clear for $i = 0$. Given the statement for i we prove it for $i + 1$. The ring extension $k[x_1, \ldots, x_i]/(f_1, \ldots, f_i) \to k[x_1, \ldots, x_{i+1}]/(f_1, \ldots, f_{i+1})$ is generated by 1 element over a field and one irreducible equation. By elementrary field theory $k[x_1, \ldots, x_{i+1}]/(f_1, \ldots, f_{i+1})$ is a field, and hence ψ_i is injective.

This implies that $\mathfrak{m} = (f_1, \ldots, f_n)$. Moreover, we also conclude that

$$k[x_1, \ldots, x_n]/(f_1, \ldots, f_i) \cong \kappa_i[x_{i+1}, \ldots, x_n].$$

Hence $(f_1, \ldots, f_i)$ is a prime ideal. Thus

$$(0) \subset (f_1) \subset (f_1, f_2) \subset \ldots \subset (f_1, \ldots, f_n) = \mathfrak{m}$$

is a chain of primes of length n. The lemma follows. □

Proposition 7.105.2. *A polynomial algebra in n variables over a field is a regular ring. It has global dimension n. All localizations at maximal ideals are regular local rings of dimension n.*

Proof. By Lemma 7.105.1 all localizations $k[x_1, \ldots, x_n]_{\mathfrak{m}}$ at maximal ideals are regular local rings of dimension n. Hence we conclude by Lemma 7.102.7. □

Lemma 7.105.3. *Let k be a field. Let $\mathfrak{p} \subset \mathfrak{q} \subset k[x_1, \ldots, x_n]$ be a pair of primes. Any maximal chain of primes between $\mathfrak{p}$ and $\mathfrak{q}$ has length height$(\mathfrak{q})$ − height$(\mathfrak{p})$.*

Proof. By Proposition 7.105.2 any local ring of $k[x_1, \ldots, x_n]$ is regular. Hence all local rings are Cohen-Macaulay, see Lemma 7.98.3. The local rings at maximal ideals have dimension n hence every maximal chain of primes in $k[x_1, \ldots, x_n]$ has length n, see Lemma 7.96.3. Hence every maximal chain of primes between (0) and $\mathfrak{p}$ has length height($\mathfrak{p}$), see Lemma 7.96.4 for example. Putting these together leads to the assertion of the lemma. □

Lemma 7.105.4. *Let k be a field. Let S be a finite type k-algebra which is an integral domain. Then* $\dim(S) = \dim(S_{\mathfrak{m}})$ *for any maximal ideal $\mathfrak{m}$ of S. In words: every maximal chain of primes has length equal to the dimension of S.*

Proof. Write $S = k[x_1, \ldots, x_n]/\mathfrak{p}$. By Proposition 7.105.2 and Lemma 7.105.3 above all the maximal chains of primes in S (which necessarily end with a maximal ideal) have length $n - \text{height}(\mathfrak{p})$. Thus this number is the dimension of S and of $S_{\mathfrak{m}}$ for any maximal ideal $\mathfrak{m}$ of S. □

Recall that we defined the dimension $\dim_x(X)$ of a topological space X at a point x in Topology, Definition 5.7.1.

Lemma 7.105.5. *Let k be a field. Let S be a finite type k-algebra. Let $X = Spec(S)$. Let $\mathfrak{p} \subset S$ be a prime ideal and let $x \in X$ be the associated point. The following numbers are equal*

1. $\dim_x(X)$,
2. $\max \dim(Z)$ *where the maximum is over those irreducible components Z of X passing through x, and*
3. $\min \dim(S_{\mathfrak{m}})$ *where the minimum is over maximal ideals $\mathfrak{m}$ with $\mathfrak{p} \subset \mathfrak{m}$.*

Proof. Let $X = \bigcup_{i \in I} Z_i$ be the decomposition of X into its irreducible components. There are finitely many of them (see Lemmas 7.28.3 and 7.28.5). Let $I' = \{i \mid x \in Z_i\}$, and let $T = \bigcup_{x \notin I'} Z_i$. Then $U = X \setminus T$ is an open subset of X containing the point x. The number (2) is $\max_{i \in I'} \dim(Z_i)$. For any open $W \subset U$, with $x \in W$ the irreducible components of W are the irreducible sets $W_i = Z_i \cap W$ for $i \in I'$. Note that each W_i, $i \in I'$ contains a closed point because X is Jacobson, see Section 7.31. By Lemma 7.105.4 we see that $\dim(W_i) = \dim(Z_i)$ for any $i \in I'$. Hence $\dim(W)$ is equal to the number (2). This proves that (1) = (2).

Let $\mathfrak{m} \supset \mathfrak{p}$ be any maximal ideal containing $\mathfrak{p}$. Let $x_0 \in X$ be the corresponding point. First of all, x_0 is contained in all the irreducible components Z_i, $i \in I'$. Let $\mathfrak{q}_i$ denote the minimal primes of S corresponding to the irreducible componenents Z_i. For each i such that $x_0 \in Z_i$ (which is equivalent to $\mathfrak{m} \supset \mathfrak{q}_i$) we have a surjection

$$S_{\mathfrak{m}} \longrightarrow S_{\mathfrak{m}}/\mathfrak{q}_i S_{\mathfrak{m}} = (S/\mathfrak{q}_i)_{\mathfrak{m}}$$

Moreover, the primes $\mathfrak{q}_i S_{\mathfrak{m}}$ so obtained exhaust the minimal primes of the Noetherian local ring $S_{\mathfrak{m}}$, see Lemma 7.23.2. We conclude, using Lemma 7.105.4, that the dimension of $S_{\mathfrak{m}}$ is the maximum of the dimensions of the Z_i passing through x_0. To finish the proof of the lemma it suffices to show that we can choose x_0 such that $x_0 \in Z_i \Rightarrow i \in I'$. Because S is Jacobson (as we saw above) it is enough to show that $V(\mathfrak{p}) \setminus T$ (with T as above) is nonempty. And this is clear since it contains the point x (i.e. $\mathfrak{p}$). $\square$

Lemma 7.105.6. *Let k be a field. Let S be a finite type k-algebra. Let $X = Spec(S)$. Let $\mathfrak{m} \subset S$ be a maximal ideal and let $x \in X$ be the associated closed point. Then* $\dim_x(X) = \dim(S_{\mathfrak{m}})$.

Proof. This is a special case of Lemma 7.105.5 above. $\square$

Lemma 7.105.7. *Let k be a field. Let S be a finite type k algebra. Assume that S is Cohen-Macaulay. Then $Spec(S) = \coprod T_d$ is a finite disjoint union of open and closed subsets T_d with T_d equidimensional (see Topology, Definition 5.7.4) of dimension d. Equivalently, S is a product of rings S_d, $d = 0, \ldots, \dim(S)$ such that every maximal ideal $\mathfrak{m}$ of S_d has height d.*

Proof. The equivalence of the two statements follows from Lemma 7.20.3. Let $\mathfrak{m} \subset S$ be a maximal ideal. Every maximal chain of primes in $S_{\mathfrak{m}}$ has the same length equal to $\dim(S_{\mathfrak{m}})$, see Lemma 7.96.3. Hence, the dimension of the irreducible components passing through the point corresponding to $\mathfrak{m}$ all have dimension equal to $\dim(S_{\mathfrak{m}})$, see Lemma 7.105.4. Since $Spec(S)$ is a Jacobson topological space the intersection of any two irreducible components of it contains a closed point if nonempty, see Lemmas 7.31.2 and 7.31.4. Thus we have shown that any two irreducible components that meet have the same dimension. The lemma follows easily from this, and the fact that $Spec(S)$ has a finite number of irreducible components (see Lemmas 7.28.3 and 7.28.5). $\square$

7.106. Noether normalization

In this section we prove variants of the Noether normalization lemma. The key ingredient we will use is contained in the following two lemmas.

Lemma 7.106.1. *Let $n \in \mathbf{N}$. Let N be a finite nonempty set of multi-indices $\nu = (\nu_1, \ldots, \nu_n)$. Given $e = (e_1, \ldots, e_n)$ we set $e \cdot \nu = \sum e_i \nu_i$. Then for $e_1 \gg e_2 \gg \ldots \gg e_{n-1} \gg e_n$ we have: If $\nu, \nu' \in N$ then*

$$(e \cdot \nu = e \cdot \nu') \Leftrightarrow (\nu = \nu')$$

Proof. Let $A_i = \max_i \nu_i - \min_i \nu_i$. If for each i we have $e_{i-1} > A_i e_i + A_{i+1}e_{i+1} + \ldots + A_n e_n$ then the lemma holds. Details omitted. $\square$

Lemma 7.106.2. *Let R be a ring. Let $g \in R[x_1, \ldots, x_n]$ be an element which is nonconstant, i.e., $g \not\in R$. For $e_1 \gg e_2 \gg \ldots \gg e_{n-1} \gg e_n = 1$ the polynomial*

$$g(x_1 + x_n^{e_1}, x_2 + x_n^{e_2}, \ldots, x_{n-1} + x_n^{e_{n-1}}, x_n) = ax_n^d + \text{lower order terms in } x_n$$

where $d > 0$ and $a \in R$ is one of the nonzero coefficients of g.

Proof. Write $g = \sum_{\nu \in N} a_\nu x^\nu$ with $a_\nu \in R$ not zero. Here N is a finite set of multi-indices as in Lemma 7.106.1 and $x^\nu = x_1^{\nu_1} \ldots x_n^{\nu_n}$. Note that the leading term in

$$(x_1 + x_n^{e_1})^{\nu_1} \ldots (x_{n-1} + x_n^{e_{n-1}})^{\nu_{n-1}} x_n^{\nu_n} \quad \text{is} \quad x_n^{e_1\nu_1 + \ldots + e_{n-1}\nu_{n-1} + \nu_n}.$$

Hence the lemma follows from Lemma 7.106.1 which garantees that there is exactly one nonzero term $a_\nu x^\nu$ of g which gives rise to the leading term of $g(x_1 + x_n^{e_1}, x_2 + x_n^{e_2}, \ldots, x_{n-1} + x_n^{e_{n-1}}, x_n)$, i.e., $a = a_\nu$ for the unique $\nu \in N$ such that $e \cdot \nu$ is maximal. $\square$

Lemma 7.106.3. *Let k be a field. Let $S = k[x_1, \ldots, x_n]/I$ for some ideal I. If $I \neq 0$, then there exist $y_1, \ldots, y_{n-1} \in k[x_1, \ldots, x_n]$ such that S is finite over $k[y_1, \ldots, y_{n-1}]$. Moreover we may choose y_i to be in the $\mathbf{Z}$-subalgebra of $k[x_1, \ldots, x_n]$ generated by $x_1, \ldots, x_n$.*

Proof. Pick $f \in I$, $f \neq 0$. It suffices to show the lemma for $k[x_1, \ldots, x_n]/(f)$ since S is a quotient of that ring. We will take $y_i = x_i - x_n^{e_i}$, $i = 1, \ldots, n-1$ for suitable integers e_i. When does this work? It suffices to show that $\overline{x_n} \in k[x_1, \ldots, x_n]/(f)$ is integral over the ring $k[y_1, \ldots, y_{n-1}]$. The equation for $\overline{x_n}$ over this ring is

$$f(y_1 + x_n^{e_1}, \ldots, y_{n-1} + x_n^{e_{n-1}}, x_n) = 0.$$

Hence we are done if we can show there exists integers e_i such that the leading coefficient w.r.t. x_n of the equation above is a nonzero element of k. This can be achieved for example by choosing $e_1 \gg e_2 \gg \ldots \gg e_{n-1}$, see Lemma 7.106.2. $\square$

Lemma 7.106.4. *Let k be a field. Let $S = k[x_1, \ldots, x_n]/I$ for some ideal I. There exist $r \geq 0$, and $y_1, \ldots, y_r \in k[x_1, \ldots, x_n]$ such that (a) the map $k[y_1, \ldots, y_r] \to S$ is injective, and (b) the map $k[y_1, \ldots, y_r] \to S$ is finite. In this case the integer r is the dimension of S. Moreover we may choose y_i to be in the $\mathbf{Z}$-subalgebra of $k[x_1, \ldots, x_n]$ generated by $x_1, \ldots, x_n$.*

Proof. By induction on n, with $n = 0$ being trivial. If $I = 0$, then take $r = n$ and $y_i = x_i$. If $I \neq 0$, then choose $y_1, \ldots, y_{n-1}$ as in Lemma 7.106.3. Let $S' \subset S$ be the subring generated by the images of the y_i. By induction we can choose r and $z_1, \ldots, z_r \in k[y_1, \ldots, y_{n-1}]$ such that (a), (b) hold for $k[z_1, \ldots, z_r] \to S'$. Since $S' \to S$ is injective and finite we see (a), (b) hold for $k[z_1, \ldots, z_r] \to S$. The last assertion follows from Lemma 7.103.4. $\square$

Lemma 7.106.5. *Let k be a field. Let S be a finite type k algebra and denote $X = Spec(S)$. Let $\mathfrak{q}$ be a prime of S, and let $x \in X$ be the corresponding point. There exists a $g \in S$, $g \not\in \mathfrak{q}$ such that $\dim(S_g) = \dim_x(X) =: d$ and such that there exists a finite injective map $k[y_1, \ldots, y_d] \to S_g$.*

Proof. Note that by definition $\dim_x(X)$ is the minimum of the dimensions of S_g for $g \in S$, $g \not\in \mathfrak{q}$, i.e., the minimum is attained. Thus the lemma follows from Lemma 7.106.4. $\square$

Lemma 7.106.6. *Let k be a field. Let $\mathfrak q \subset k[x_1, \ldots, x_n]$ be a prime ideal. Set $r = trdeg_k\ \kappa(\mathfrak q)$. Then there exists a finite ring map $\varphi : k[y_1, \ldots, y_n] \to k[x_1, \ldots, x_n]$ such that $\varphi^{-1}(\mathfrak q) = (y_{r+1}, \ldots, y_n)$.*

Proof. By induction on n. The case $n = 0$ is clear. Assume $n > 0$. If $r = n$, then $\mathfrak q = (0)$ and the result is clear. Choose a nonzero $f \in \mathfrak q$. Of course f is nonconstant. After applying an automorphism of the form

$$k[x_1, \ldots, x_n] \longrightarrow k[x_1, \ldots, x_n], \quad x_n \mapsto x_n, \quad x_i \mapsto x_i + x_n^{e_i}\ (i < n)$$

we may assume that f is monic in x_n over $k[x_1, \ldots, x_n]$, see Lemma 7.106.2. Hence the ring map

$$k[y_1, \ldots, y_n] \longrightarrow k[x_1, \ldots, x_n], \quad y_n \mapsto f, \quad y_i \mapsto x_i\ (i < n)$$

is finite. Moreover $y_n \in \mathfrak q \cap k[y_1, \ldots, y_n]$ by construction. Thus $\mathfrak q \cap k[y_1, \ldots, y_n] = \mathfrak p k[y_1, \ldots, y_n] + (y_n)$ where $\mathfrak p \subset k[y_1, \ldots, y_{n-1}]$ is a prime ideal. Note that $\kappa(\mathfrak p) \subset \kappa(\mathfrak q)$ is finite, and hence $r = \text{trdeg}_k\ \kappa(\mathfrak p)$. Apply the induction hypothesis to the pair $(k[y_1, \ldots, y_{n-1}], \mathfrak p)$ and we obtain a finite ring map $k[z_1, \ldots, z_{n-1}] \to k[y_1, \ldots, y_{n-1}]$ such that $\mathfrak p \cap k[z_1, \ldots, z_{n-1}] = (z_{r+1}, \ldots, z_{n-1})$. We extend the ring map $k[z_1, \ldots, z_{n-1}] \to k[y_1, \ldots, y_{n-1}]$ to a ring map $k[z_1, \ldots, z_n] \to k[y_1, \ldots, y_n]$ by mapping z_n to y_n. The composition of the ring maps

$$k[z_1, \ldots, z_n] \to k[y_1, \ldots, y_n] \to k[x_1, \ldots, x_n]$$

solves the problem. □

Lemma 7.106.7. *Let $R \to S$ be an injective finite type map of domains. Then there exists an integer d and factorization*

$$R \to R[y_1, \ldots, y_d] \to S' \to S$$

by injective maps such that S' is finite over $R[y_1, \ldots, y_d]$ and such that $S'_f \cong S_f$ for some nonzero $f \in R$.

Proof. Pick $x_1, \ldots, x_n \in S$ which generate S over R. Let $K = f.f.(R)$ and $S_K = S \otimes_R K$. By Lemma 7.106.4 we can find $y_1, \ldots, y_d \in S$ such that $K[y_1, \ldots, y_d] \to S_K$ is a finite injective map. Note that $y_i \in S$ because we may pick the y_j in the $\mathbf{Z}$-algebra generated by $x_1, \ldots, x_n$. As a finite ring map is integral (see Lemma 7.32.3) we can find monic $P_i \in K[y_1, \ldots, y_d][T]$ such that $P_i(x_i) = 0$ in S_K. Let $f \in R$ be a nonzero element such that $fP_i \in R[y_1, \ldots, y_d][T]$ for all i. Set $x_i' = fx_i$ and let $S' \subset S$ be the subalgebra generated by $y_1, \ldots, y_d$ and $x_1', \ldots, x_n'$. Note that x_i' is integral over $R[y_1, \ldots, y_d]$ as we have $Q_i(x_i') = 0$ where $Q_i = f^{-\deg_T(P_i)} P_i(fT)$ which is a monic polynomial in T with coefficients in $R[y_1, \ldots, y_d]$ by our choice of f. Hence $R[y_1, \ldots, y_n] \subset S'$ is finite by Lemma 7.32.5. By construction $S'_f \cong S_f$ and we win. □

7.107. Dimension of finite type algebras over fields, reprise

This section is a continuation of Section 7.105. In this section we esthablish the connection between dimension and transcendence degree over the ground field for finite type domains over a field.

Lemma 7.107.1. *Let k be a field. Let S be a finite type k algebra which is an integral domain. Let $K = f.f.(S)$ be the field of fractions of S. Let $r = trdeg(K/k)$ be the transcendence degree of K over k. Then $\dim(S) = r$. Moreover, the local ring of S at every maximal ideal has dimension r.*

Proof. We may write $S = k[x_1, \ldots, x_n]/\mathfrak{p}$. By Lemma 7.105.3 all local rings of S at maximal ideals have the same dimension. Apply Lemma 7.106.4 above. We get a finite injective ring map

$$k[y_1, \ldots, y_d] \to S$$

with $d = \dim(S)$. Clearly, $k(y_1, \ldots, y_d) \subset K$ is a finite extension and we win. □

Lemma 7.107.2. *Let k be a field. Let S be a finite type k-algebra. Let $\mathfrak{q} \subset \mathfrak{q}' \subset S$ be distinct prime ideals. Then $trdeg_k\ \kappa(\mathfrak{q}') < trdeg_k\ \kappa(\mathfrak{q})$.*

Proof. By Lemma 7.107.1 we have $\dim V(\mathfrak{q}) = \text{trdeg}_k\ \kappa(\mathfrak{q})$ and similarly for $\mathfrak{q}'$. Hence the result follows as the strict inclusion $V(\mathfrak{q}') \subset V(\mathfrak{q})$ implies a strict inequality of dimensions. □

The following lemma generalizes Lemma 7.105.6.

Lemma 7.107.3. *Let k be a field. Let S be a finite type k algebra. Let $X = Spec(S)$. Let $\mathfrak{p} \subset S$ be a prime ideal, and let $x \in X$ be the corresponding point. Then we have*

$$\dim_x(X) = \dim(S_{\mathfrak{p}}) + trdeg_k\ \kappa(\mathfrak{p}).$$

Proof. By Lemma 7.107.1 above we know that $r = \text{trdeg}_k\ \kappa(\mathfrak{p})$ is equal to the dimension of $V(\mathfrak{p})$. Pick any maximal chain of primes $\mathfrak{p} \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_r$ starting with $\mathfrak{p}$ in S. This has length r by Lemma 7.105.4. Let $\mathfrak{q}_i$, $i \in I'$ be the minimal primes of S which are contained in $\mathfrak{p}$. These correspond $1-1$ to minimal primes in $S_{\mathfrak{p}}$ via the rule $\mathfrak{q}_i \mapsto \mathfrak{q}_i S_{\mathfrak{p}}$. By Lemma 7.105.5 we know that $\dim_x(X)$ is equal to the maximum of the dimensions of the rings $S/\mathfrak{q}_i$. For each i pick a maximal chain of primes $\mathfrak{q}_i \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_{s(i)} = \mathfrak{p}$. Then $\dim(S_{\mathfrak{p}}) = \max_{i \in I'} s(i)$. Now, each chain

$$\mathfrak{q}_i \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_{s(i)} = \mathfrak{p} \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_r$$

is a maximal chain in $S/\mathfrak{q}_i$, and by what was said before we have $\dim_x(X) = \max_{i \in I'} r + s(i)$. The lemma follows. □

The following lemma says that the codimension of one finite type Spec in another is the difference of heights.

Lemma 7.107.4. *Let k be a field. Let $S' \to S$ be a surjection of finite type k algebras. Let $\mathfrak{p} \subset S$ be a prime ideal, and let $\mathfrak{p}'$ be the corresponding prime ideal of S'. Let $X = Spec(S)$, resp. $X' = Spec(S')$, and let $x \in X$, resp. $x' \in X'$ be the point corrseponding to $\mathfrak{p}$, resp. $\mathfrak{p}'$. Then*

$$\dim_{x'} X' - \dim_x X = height(\mathfrak{p}') - height(\mathfrak{p}).$$

Proof. Immediate from Lemma 7.107.3 above. □

Lemma 7.107.5. *Let k be a field. Let S be a finite type k-algebra. Let $k \subset K$ be a field extension. Then* $\dim(S) = \dim(K \otimes_k S)$.

Proof. By Lemma 7.106.4 there exists a finite injective map $k[y_1, \ldots, y_d] \to S$ with $d = \dim(S)$. Since K is flat over k we also get a finite injective map $K[y_1, \ldots, y_d] \to K \otimes_k S$. The result follows from Lemma 7.103.4. □

Lemma 7.107.6. *Let k be a field. Let S be a finite type k-algebra. Set $X = Spec(S)$. Let $k \subset K$ be a field extension. Set $S_K = K \otimes_k S$, and $X_K = Spec(S_K)$. Let $\mathfrak{q} \subset S$ be a prime corresponding to $x \in X$ and let $\mathfrak{q}_K \subset S_K$ be a prime corresponding to $x_K \in X_K$ lying over $\mathfrak{q}$. Then* $\dim_x X = \dim_{x_K} X_K$.

Proof. Choose a presentation $S = k[x_1, \ldots, x_n]/I$. This gives a presentation $K \otimes_k S = K[x_1, \ldots, x_n]/(K \otimes_k I)$. Let $\mathfrak{q}'_K \subset K[x_1, \ldots, x_n]$, resp. $\mathfrak{q}' \subset k[x_1, \ldots, x_n]$ be the corresponding primes. Consider the following commutative diagram of Noetherian local rings

$$\begin{tikzcd} K[x_1, \ldots, x_n]_{\mathfrak{q}'_K} \arrow[r] & (K \otimes_k S)_{\mathfrak{q}_K} \\ k[x_1, \ldots, x_n]_{\mathfrak{q}'} \arrow[r] \arrow[u] & S_{\mathfrak{q}} \arrow[u] \end{tikzcd}$$

Both vertical arrows are flat because they are localizations of the flat ring maps $S \to S_K$ and $k[x_1, \ldots, x_n] \to K[x_1, \ldots, x_n]$. Moreover, the vertical arrows have the same fibre rings. Hence, we see from Lemma 7.103.7 that $\text{height}(\mathfrak{q}') - \text{height}(\mathfrak{q}) = \text{height}(\mathfrak{q}'_K) - \text{height}(\mathfrak{q}_K)$. Denote $x' \in X' = Spec(k[x_1, \ldots, x_n])$ and $x'_K \in X'_K = Spec(K[x_1, \ldots, x_n])$ the points corresponding to $\mathfrak{q}'$ and $\mathfrak{q}'_K$. By Lemma 7.107.4 and what we showed above we have

$$\begin{aligned} n - \dim_x X &= \dim_{x'} X' - \dim_x X \\ &= \text{height}(\mathfrak{q}') - \text{height}(\mathfrak{q}) \\ &= \text{height}(\mathfrak{q}'_K) - \text{height}(\mathfrak{q}_K) \\ &= \dim_{x'_K} X'_K - \dim_{x_K} X_K \\ &= n - \dim_{x_K} X_K \end{aligned}$$

and the lemma follows. □

7.108. Dimension of graded algebras over a field

Here is a basic result.

Lemma 7.108.1. *Let k be a field. Let S be a finitely generated graded algebra over k. Assume $S_0 = k$. Let $P(T) \in \mathbf{Q}[T]$ be the polynomial such that $\dim(S_d) = P(d)$ for all $d \gg 0$. See Proposition 7.55.6. Then*

1. *The irrelevant ideal S_+ is a maximal ideal $\mathfrak{m}$.*
2. *Any minimal prime of S is a homogeneous ideal and is contained in $S_+ = \mathfrak{m}$.*
3. *We have $\dim(S) = \deg(P)+1 = \dim_x Spec(S)$ (with the convention that $\deg(0) = -1$) where x is the point corresponding to the maximal ideal $S_+ = \mathfrak{m}$.*
4. *The Hilbert function of the local ring $R = S_{\mathfrak{m}}$ is equal to the Hilbert function of S.*

Proof. The first statement is obvious. The second follows from Lemma 7.53.8. The equality $\dim(S) = \dim_x Spec(S)$ follows from the fact that every irreducible component passes through x according to (2). Hence we may compute this dimension as the dimension of the local ring $R = S_{\mathfrak{m}}$ with $\mathfrak{m} = S_+$ by Lemma 7.105.6. Since $\mathfrak{m}^d/\mathfrak{m}^{d+1} \cong \mathfrak{m}^d R/\mathfrak{m}^{d+1} R$ we see that the Hilbert function of the local ring R is equal to the Hilbert function of S, which is (4). We conclude the last equality of (3) by Proposition 7.57.8. □

7.109. Generic flatness

Basically this says that a finite type algebra over a domain becomes flat after inverting a single element of the domain. There are several versions of this result (in increasing order of strength).

Lemma 7.109.1. *Let $R \to S$ be a ring map. Let M be an S-module. Assume*

1. *R is Noetherian,*

(2) *R is a domain,*
(3) *$R \to S$ is of finite type, and*
(4) *M is a finite type S-module.*

Then there exists a nonzero $f \in R$ such that M_f is a free R_f-module.

Proof. Let K be the fraction field of R. Set $S_K = K \otimes_R S$. This is an algebra of finite type over K. We will argue by induction on $d = \dim(S_K)$ (which is finite for example by Noether normalization, see Section 7.106). Fix $d \geq 0$. Assume we know that the lemma holds in all cases where $\dim(S_K) < d$.

Suppose given $R \to S$ and M as in the lemma with $\dim(S_K) = d$. By Lemma 7.59.1 there exists a filtration $0 \subset M_1 \subset M_2 \subset \ldots \subset M_n = M$ so that M_i/M_{i-1} is isomorphic to $S/\mathfrak{q}$ for some prime $\mathfrak{q}$ of S. Note that $\dim((S/\mathfrak{q})_K) \leq \dim(S_K)$. Also, note that an extension of free modules is free (see basic notion 50). Thus we may assume $M = S$ and that S is a domain of finite type over R.

If $R \to S$ has a nontrivial kernel, then take a nonzero $f \in R$ in this kernel. In this case $S_f = 0$ and the lemma holds. (This is really the case $d = -1$ and the start of the induction.) Hence we may assume that $R \to S$ is a finite type extension of Noetherian domains.

Apply Lemma 7.106.7 and replace R by R_f (with f as in the lemma) to get a factorization

$$R \subset R[y_1, \ldots, y_d] \subset S$$

where the second extension is finite. Note that $f.f.(R[y_1, \ldots, y_d]) \subset f.f.(S)$ is a finite extension of fields. Choose $z_1, \ldots, z_r \in S$ which form a basis for $f.f.(S)$ over $f.f.(R[y_1, \ldots, y_d])$. This gives a short exact sequence

$$0 \to R[y_1, \ldots, y_d]^{\oplus r} \xrightarrow{(z_1, \ldots, z_r)} S \to N \to 0$$

By construction N is a finite $R[y_1, \ldots, y_d]$-module whose support does not contain the generic point (0) of $Spec(R[y_1, \ldots, y_d])$. By Lemma 7.59.4 there exists a nonzero $g \in R[y_1, \ldots, y_d]$ such that g annihilates N, so we may view N as a finite module over $S' = R[y_1, \ldots, y_d]/(g)$. Since $\dim(S'_K) < d$ by induction there exists a nonzero $f \in R$ such that N_f is a free R_f-module. Since $(R[y_1, \ldots, y_d])_f \cong R_f[y_1, \ldots, y_d]$ is free also we conclude by the already mentioned fact that an extension of free modules is free. $\square$

Lemma 7.109.2. *Let $R \to S$ be a ring map. Let M be an S-module. Assume*
(1) *R is a domain,*
(2) *$R \to S$ is of finite presentation, and*
(3) *M is an S-module of finite presentation.*

Then there exists a nonzero $f \in R$ such that M_f is a free R_f-module.

Proof. Write $S = R[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$. For $g \in R[x_1, \ldots, x_n]$ denote $\overline{g}$ its image in S. We may write $M = S^{\oplus t}/\sum Sn_i$ for some $n_i \in S^{\oplus t}$. Write $n_i = (\overline{g}_{i1}, \ldots, \overline{g}_{it})$ for some $g_{ij} \in R[x_1, \ldots, x_n]$. Let $R_0 \subset R$ be the subring generated by all the coefficients of all the elements $g_i, g_{ij} \in R[x_1, \ldots, x_n]$. Define $S_0 = R_0[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$. Define $M_0 = S_0^{\oplus t}/\sum S_0 n_i$. Then R_0 is a domain of finite type over $\mathbf{Z}$ and hence Noetherian (see Lemma 7.28.1). Moreover via the injection $R_0 \to R$ we have $S \cong R \otimes_{R_0} S_0$ and $M \cong R \otimes_{R_0} M_0$. Applying Lemma 7.109.1 we obtain a nonzero $f \in R_0$ such that $(M_0)_f$ is a free $(R_0)_f$-module. Hence $M_f = R_f \otimes_{(R_0)_f} (M_0)_f$ is a free R_f-module. $\square$

Lemma 7.109.3. *Let $R \to S$ be a ring map. Let M be an S-module. Assume*
(1) *R is a domain,*

(2) *$R \to S$ is of finite type, and*
(3) *M is a finite type S-module.*

Then there exists a nonzero $f \in R$ such that

(a) *M_f and S_f are free as R_f-modules, and*
(b) *S_f is a finitely presented R_f-algebra and M_f is a finitely presented S_f-module.*

Proof. We first prove the lemma for $S = R[x_1, \ldots, x_n]$, and then we deduce the result in general.

Assume $S = R[x_1, \ldots, x_n]$. Choose elements $m_1, \ldots, m_t$ which generate M. This gives a short exact sequence

$$0 \to N \to S^{\oplus t} \xrightarrow{(m_1,\ldots,m_t)} M \to 0.$$

Denote K the fraction field of R. Denote $S_K = K \otimes_R S = K[x_1, \ldots, x_n]$, and similarly $N_K = K \otimes_R N$, $M_K = K \otimes_R M$. As $R \to K$ is flat the sequence remains flat after tensoring with K. As $S_K = K[x_1, \ldots, x_n]$ is a Noetherian ring (see Lemma 7.28.1) we can find finitely many elements $n'_1, \ldots, n'_s \in N_K$ which generate it. Choose $n_1, \ldots, n_r \in N$ such that $n'_i = \sum a_{ij} n_j$ for some $a_{ij} \in K$. Set

$$M' = S^{\oplus t} / \sum\nolimits_{i=1,\ldots,r} S n_i$$

By construction M' is a finitely presented S-module, and there is a surjection $M' \to M$ which induces an isomorphism $M'_K \cong M_K$. We may apply Lemma 7.109.2 to $R \to S$ and M' and we find an $f \in R$ such that M'_f is a free R_f-module. Thus $M'_f \to M_f$ is a surjection of modules over the domain R_f where the source is a free module and which becomes an isomorphism upon tensoring with K. Thus it is injective as $M'_f \subset M'_K$ as it is free over the domain R_f. Hence $M'_f \to M_f$ is an isomorphism and the result is proved.

For the general case, choose a surjection $R[x_1, \ldots, x_n] \to S$. Think of both S and M as finite modules over $R[x_1, \ldots, x_n]$. By the special case proved above there exists a nonzero $f \in R$ such that both S_f and M_f are free as R_f-modules and finitely presented as $R_f[x_1, \ldots, x_n]$-modules. Clearly this implies that S_f is a finitely presented R_f-algebra and that M_f is a finitely presented S_f-module. □

Let $R \to S$ be a ring map. Let M be an S-module. Consider the following condition on an element $f \in R$:

$$(7.109.3.1) \qquad \begin{cases} S_f & \text{is of finite presentation over } R_f \\ M_f & \text{is of finite presentation as } S_f\text{-module} \\ S_f, M_f & \text{are free as } R_f\text{-modules} \end{cases}$$

We define

$$(7.109.3.2) \qquad U(R \to S, M) = \bigcup\nolimits_{f \in R \text{ with } (7.109.3.1)} D(f)$$

which is an open subset of $Spec(R)$.

Lemma 7.109.4. *Let $R \to S$ be a ring map. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence of S-modules. Then*

$$U(R \to S, M_1) \cap U(R \to S, M_3) \subset U(R \to S, M_2).$$

Proof. Let $u \in U(R \to S, M_1) \cap U(R \to S, M_3)$. Choose $f_1, f_3 \in R$ such that $u \in D(f_1)$, $u \in D(f_3)$ and such that (7.109.3.1) holds for f_1 and M_1 and for f_3 and M_3. Then set $f = f_1 f_3$. Then $u \in D(f)$ and (7.109.3.1) holds for f and both M_1 and M_3. An extension of free modules is free, and an extension of finitely presented modules is finitely presented (Lemma 7.5.4). Hence we see that (7.109.3.1) holds for f and M_2. Thus $u \in U(R \to S, M_2)$ and we win. □

Lemma 7.109.5. *Let $R \to S$ be a ring map. Let M be an S-module. Let $f \in R$. Using the identification $Spec(R_f) = D(f)$ we have $U(R_f \to S_f, M_f) = D(f) \cap U(R \to S, M)$.*

Proof. Suppose that $u \in U(R_f \to S_f, M_f)$. Then there exists an element $g \in R_f$ such that $u \in D(g)$ and such that (7.109.3.1) holds for the pair $((R_f)_g \to (S_f)_g, (M_f)_g)$. Write $g = a/f^n$ for some $a \in R$. Set $h = af$. Then $R_h = (R_f)_g$, $S_h = (S_f)_g$, and $M_h = (M_f)_g$. Moreover $u \in D(h)$. Hence $u \in U(R \to S, M)$. Conversely, suppose that $u \in D(f) \cap U(R \to S, M)$. Then there exists an element $g \in R$ such that $u \in D(g)$ and such that (7.109.3.1) holds for the pair $(R_g \to S_g, M_g)$. Then it is clear that (7.109.3.1) also holds for the pair $(R_{fg} \to S_{fg}, M_{fg}) = ((R_f)_g \to (S_f)_g, (M_f)_g)$. Hence $u \in U(R_f \to S_f, M_f)$ and we win. □

Lemma 7.109.6. *Let $R \to S$ be a ring map. Let M be an S-module. Let $U \subset Spec(R)$ be a dense open. Assume there is a covering $U = \bigcup_{i \in I} D(f_i)$ of opens such that $U(R_{f_i} \to S_{f_i}, M_{f_i})$ is dense in $D(f_i)$ for each $i \in I$. Then $U(R \to S, M)$ is dense in $Spec(R)$.*

Proof. In view of Lemma 7.109.5 this is a purely topological statement. Namely, by that lemma we see that $U(R \to S, M) \cap D(f_i)$ is dense in $D(f_i)$ for each $i \in I$. By Topology, Lemma 5.17.4 we see that $U(R \to S, M) \cap U$ is dense in U. Since U is dense in $Spec(R)$ we conclude that $U(R \to S, M)$ is dense in $Spec(R)$. □

Lemma 7.109.7. *Let $R \to S$ be a ring map. Let M be an S-module. Assume*
  (1) *$R \to S$ is of finite type,*
  (2) *M is a finite S-module, and*
  (3) *R is reduced.*

Then there exists a subset $U \subset Spec(R)$ such that
  (1) *U is open and dense in $Spec(R)$,*
  (2) *for every $u \in U$ there exists an $f \in R$ such that $u \in D(f) \subset U$ and such that we have*
    (a) *M_f and S_f are free over R_f,*
    (b) *S_f is a finitely presented R_f-algebra, and*
    (c) *M_f is a finitely presented S_f-module.*

Proof. Note that the lemma is equivalent to the statement that the open $U(R \to S, M)$, see Equation (7.109.3.2), is dense in $Spec(R)$. We first prove the lemma for $S = R[x_1, \ldots, x_n]$, and then we deduce the result in general.

Proof of the case $S = R[x_1, \ldots, x_n]$ and M any finite module over S. Note that in this case $S_f = R_f[x_1, \ldots, x_n]$ is free and of finite presentation over R_f, so we do not have to worry about the conditions regarding S, only those that concern M. We will use induction on n.

There exists a finite filtration

$$0 \subset M_1 \subset M_2 \subset \ldots \subset M_t = M$$

such that $M_i/M_{i-1} \cong S/J_i$ for some ideal $J_i \subset S$, see Lemma 7.5.5. Since a finite intersection of dense opens is dense open, we see from Lemma 7.109.4 that it suffices to prove the lemma for each of the modules R/J_i. Hence we may assume that $M = S/J$ for some ideal J of $S = R[x_1, \ldots, x_n]$.

Let $I \subset R$ be the ideal generated by the coefficients of elements of J. Let $U_1 = Spec(R) \setminus V(I)$ and let

$$U_2 = Spec(R) \setminus \overline{U_1}.$$

Then it is clear that $U = U_1 \cup U_2$ is dense in $Spec(R)$. Let $f \in R$ be an element such that either (a) $D(f) \subset U_1$ or (b) $D(f) \subset U_2$. If for any such f the lemma holds for the pair $(R_f \to R_f[x_1, \ldots, x_n], M_f)$ then by Lemma 7.109.6 we see that $U(R \to S, M)$ is dense in $Spec(R)$. Hence we may assume either (a) $I = R$, or (b) $V(I) = Spec(R)$.

In case (b) we actually have $I = 0$ as R is reduced! Hence $J = 0$ and $M = S$ and the lemma holds in this case.

In case (a) we have to do a little bit more work. Note that every element of I is actually the coefficient of a monomial of an element of J, because the set of coefficients of elements of J forms an ideal (details omitted). Hence we find an element

$$g = \sum\nolimits_{K \in E} a_K x^K \in J$$

where E is a finite set of multi-indices $K = (k_1, \ldots, k_n)$ with at least one coefficient a_{K_0} a unit in R. Actually we can find one which has a coefficient equal to 1 as $1 \in I$ in case (a). Let $m = \#\{K \in E \mid a_K \text{ is not a unit}\}$. Note that $0 \leq m \leq \#E - 1$. We will argue by induction on m.

The case $m = 0$. In this case all the coefficients a_K, $K \in E$ of g are units and $E \neq \emptyset$. If $E = \{K_0\}$ is a singleton and $K_0 = (0, \ldots, 0)$, then g is a unit and $J = S$ so the result holds for sure. (This happens in particular when $n = 0$ and it provides the base case of the induction on n.) If not $E = \{(0, \ldots, 0)\}$, then at least one K is not equal to $(0, \ldots, 0)$, i.e., $g \notin R$. At this point we employ the usual trick of Noether normalization. Namely, we consider

$$G(y_1, \ldots, y_n) = g(y_1 + y_n^{e_1}, y_2 + y_n^{e_2}, \ldots, y_{n-1} + y_n^{e_{n-1}}, y_n)$$

with $0 \ll e_{n-1} \ll e_{n-2} \ll \ldots \ll e_1$. By Lemma 7.106.2 it follows that $G(y_1, \ldots, y_n)$ as a polynomial in y_n looks like

$$a_K y_n^{k_n + \sum_{i=1,\ldots,n-1} e_i k_i} + \text{lower order terms in } y_n$$

As a_K is a unit we conclude that $M = R[x_1, \ldots, x_n]/J$ is finite over $R[y_1, \ldots, y_{n-1}]$. Hence $U(R \to R[x_1, \ldots, x_n], M) = U(R \to R[y_1, \ldots, y_{n-1}], M)$ and we win by induction on n.

The case $m > 0$. Pick a multi-index $K \in E$ such that a_K is not a unit. As before set $U_1 = Spec(R_{a_K}) = Spec(R) \setminus V(a_K)$ and set

$$U_2 = Spec(R) \setminus \overline{U_1}.$$

Then it is clear that $U = U_1 \cup U_2$ is dense in $Spec(R)$. Let $f \in R$ be an element such that either (a) $D(f) \subset U_1$ or (b) $D(f) \subset U_2$. If for any such f the lemma holds for the pair $(R_f \to R_f[x_1, \ldots, x_n], M_f)$ then by Lemma 7.109.6 we see that $U(R \to S, M)$ is dense in $Spec(R)$. Hence we may assume either (a) $a_K R = R$, or (b) $V(a_K) = Spec(R)$. In case (a) the number m drops, as a_K has turned into a unit. In case (b), since R is reduced, we conclude that $a_K = 0$. Hence the set E decreases so the number m drops as well. In both cases we win by induction on m.

At this point we have proven the lemma in case $S = R[x_1, \ldots, x_n]$. Assume that $(R \to S, M)$ is an arbitrary pair satisfying the conditions of the lemma. Choose a surjection $R[x_1, \ldots, x_n] \to S$. Observe that, with the notation introduced in (7.109.3.2), we have

$$U(R \to S, M) = U(R \to R[x_1, \ldots, x_n], S) \cap U(R \to R[x_1, \ldots, x_n], S)$$

Hence as we've just finished proving the right two opens are dense also the open on the left is dense. □

7.110. Around Krull-Akizuki

One application of Krull-Akizuki is to show that there are plenty of discrete valuation rings. More generally in this section we show how to construct discrete valution rings dominating Noetherian local rings.

First we show how to dominate a Noetherian local domain by a 1-dimensional Noetherian local domain by blowing up the maximal ideal.

Lemma 7.110.1. *Let R be a local Noetherian domain with fraction field K. Assume R is not a field. Then there exist $R \subset R' \subset K$ with*

1. *R' local Noetherian of dimension 1,*
2. *$R \to R'$ a local ring map, i.e., R' dominates R, and*
3. *$R \to R'$ essentially of finite type.*

Proof. Choose any valuation ring $A \subset K$ dominating R (which exist by Lemma 7.46.2). Denote v the corresponding valuation. Let $x_1, \ldots, x_r$ be a minimal set of generators of the maximal ideal $\mathfrak{m}$ of R. We may and do assume that $v(x_r) = \min\{v(x_1), \ldots, v(x_r)\}$. Consider the ring

$$S = R[x_1/x_r, x_2/x_r, \ldots, x_{r-1}/x_r] \subset K.$$

Note that $\mathfrak{m}S = x_rS$ is a principal ideal. Note that $S \subset A$ and that $v(x_r) > 0$, hence we see that $x_rS \neq S$. Choose a minimal prime $\mathfrak{q}$ over x_rS. Then height$(\mathfrak{q}) = 1$ by Lemma 7.57.10 and $\mathfrak{q}$ lies over $\mathfrak{m}$. Hence we see that $R' = S_{\mathfrak{q}}$ is a solution. □

The spectrum of the ring R' in the following lemma is really the blow up of $Spec(R)$ in the maximal ideal of R (at least if case R is reduced).

Lemma 7.110.2. *Let R be a local ring with maximal ideal $\mathfrak{m}$. Assume R is Noetherian, dimension 1 and that $\dim(\mathfrak{m}/\mathfrak{m}^2) > 1$. Then there exists a ring map $R \to R'$ such that*

1. *$R \to R'$ is finite,*
2. *$R \to R'$ is not an isomorphism, and*
3. *for every $f \in \mathfrak{m}$ the map $R_f \to R'_f$ is an isomorphism.*

Proof. If $\mathfrak{m}$ is an associated prime of R then we can take $R' = R/I$ with $I = \{x \in R \mid \mathfrak{m}x = 0\}$. Hence we may assume that depth$(R) = 1$. In other words, we may assume R is Cohen-Macaulay.

Denote $\kappa = R/\mathfrak{m}$ the residue field of R. Consider the graded κ-algebra $S = \bigoplus_{d \geq 0} \mathfrak{m}^d/\mathfrak{m}^{d+1}$. This is a Noetherian ring, and hence has finitely many minimal primes $\mathfrak{q}_j$. Since the dimension of R is 1 we know the Hilbert function of R is eventually constant, see Proposition 7.57.8. Hence there exists an integer $d_0 \geq 0$ and an integer $r > 0$ such that $\dim_\kappa(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = r$ for all $d \geq d_0$. By Lemma 7.108.1 we have $\dim(S) = 1$ and each $\mathfrak{q}_i$ is a homogeneous prime ideal. Note that $\dim(S) = 1$ implies none of the $\mathfrak{q}_i$ is equal to S_+. Hence by Lemma 7.53.6 we may choose $f \in S_+$ homogeneous not contained in any $\mathfrak{q}_j$. Then $\dim(S/fS) = 0$. This implies that $\dim_\kappa(S/fS) < \infty$, see for example Lemma 7.106.4. Hence we see that $fS_n = S_{n+\deg(f)}$ for all $n \gg 0$.

Set $d = \deg(f)$. Choose $x \in \mathfrak{m}^d$ which maps to f. Note that since $fS_n = S_{n+d}$ for $n \gg 0$ we have $x\mathfrak{m}^n = \mathfrak{m}^{n+d}$ by Nakayama's Lemma 7.14.5 for $n \gg 0$. Hence $\mathfrak{m} = \sqrt{(x)}$. Since

R is Cohen-Macaulay this implies that x is a nonzero divisor. Choose generators $x_1, \ldots, x_t$ of $\mathfrak{m}^d$ as an ideal of R. Set

$$R' = R[x_1/x, x_2/x, \ldots, x_t/x] \subset R[1/x].$$

Note that since x is a nonzero divisor we have $R \subset R'$. Since $\mathfrak{m} = \sqrt{(x)}$ we see that x is invertible in R_g for any $g \in \mathfrak{m}$, whence (3).

We claim that R' is finite over R. Namely, choose n such that $x\mathfrak{m}^{nd} = \mathfrak{m}^{(n+1)d}$. Then we can write $x_i^{n+1} = xf_i(x_1, \ldots, x_t)$ with $f_i \in R[X_1, \ldots, X_t]$ homogeneous of degree n. Hence we see that

$$(x_i/x)^{n+1} = f_i(x_1/x, \ldots, x_t/x)$$

in R' with the right hand side of degree $\leq n$. Hence any element in R' can be expressed as a sum of $(x_1/x)^{i_1} \ldots (x_t/x)^{i_t}$ with $i_j \leq n$. This proves that R' is finite over R.

Finally we show that $R \neq R'$. We argue by contradiction. Suppose $R' = R$. This means that $x_1/x, \ldots, x_t/x \in R$ for all i. In other words this means that $\mathfrak{m}^d = (x)$. Choose $y_1, \ldots, y_s \in \mathfrak{m}$ a minimal generating set. The assumption of the lemma implies $s \geq 2$. For some $i_1, \ldots, i_s \geq 0$, $\sum i_j = d$ we have $x = uy_1^{i_1} \ldots y_s^{i_s}$ for some unit u in R. We may assume $i_1 > 0$. Then

$$y_1^{i_1-1} y_2^{i_2+1} y_3^{i_3} \ldots y_s^{i_s} \in \mathfrak{m}^d$$

is a multiple of x hence a multiple of $y_1^{i_1} \ldots y_s^{i_s}$. Hence we see that $y_2/y_1 \in R$. This is a contradiction with the minimality of $y_1, \ldots, y_s$. □

Example 7.110.3. Consider the Noetherian local ring

$$R = k[[x, y]]/(y^2)$$

It has dimension 1 and it is Cohen-Macaulay. The result of applying the procedure of Lemma 7.110.2 to R is the extension

$$k[[x, y]]/(y^2) \subset k[[x, z]]/(z^2), \quad y \mapsto xz$$

in other words it is gotten by adjoining y/x to R. The effect of repeating the construction $n > 1$ times is to adjoin the element y/x^n.

Example 7.110.4. Let k be a field of characteristic $p > 0$ such that k has infinite degree over its subfield k^p of pth powers. For example $k = \mathbf{F}_p(t_1, t_2, t_3, \ldots)$. Consider the ring

$$A = \left\{ \sum a_i x^i \in k[[x]] \text{ such that } [k^p(a_0, a_1, a_2, \ldots) : k^p] < \infty \right\}$$

Then A is a discrete valuation ring and its completion is $A^\wedge = k[[x]]$. Note that the field extension $f.f.(A) \subset f.f.(k[[x]])$ is infinite purely inseparable. Choose any $f \in k[[x]]$, $f \notin A$. Let $R = A[f] \subset k[[x]]$. Then R is a Noetherian local domain of dimension 1 whose completion $R^\wedge$ is nonreduced (think!).

Remark 7.110.5. Suppose that R is a 1-dimensional semi-local Noetherian domain. If there is a maximal ideal $\mathfrak{m} \subset R$ such that $R_\mathfrak{m}$ is not regular, then we may apply the procedure of (the proof of) Lemma 7.110.2 to $(R, \mathfrak{m})$ to get a finite ring extension $R \subset R_1$. (Note that $Spec(R_1) \to Spec(R)$ is the blow up of $Spec(R)$ in the ideal $\mathfrak{m}$.) Of course R_1 is a 1-dimensional semi-local Noetherian domain with the same fraction field as R. If R_1 is not a regular semi-liocal ring, then we may repeat the construction to get $R_1 \subset R_2$. Thus we get a sequence

$$R \subset R_1 \subset R_2 \subset R_3 \subset \ldots$$

of finite ring extensions which may stop if R_n is regular for some n. Resolution of singularities would be the claim that eventually R_n is indeed regular. In reality this is not the case. Namely, there exists a characteristic 0 Noetherian local domain A of dimension 1 whose completion is nonreduced, see [**FR70**, Proposition 3.1] or our Examples, Section 64.8. For an example in characteristic $p > 0$ see Example 7.110.4. Since the construction of blowing up commutes with completion it is easy to see the sequence never stabilizes. See [**Ben73**] for a discussion (mostly in positive characteristic). On the other hand, if the completion of R in all of its maximal ideals is reduced, then the procedure stops (insert future reference here).

Lemma 7.110.6. *Let A be a ring. The following are equivalent.*

(1) *The ring A is a discrete valuation ring.*
(2) *The ring A is a valuation ring and Noetherian.*
(3) *The ring A is a regular local ring of dimension* 1.
(4) *The ring A is a Noetherian local domain with maximal ideal $\mathfrak{m}$ generated by a single nonzero element.*
(5) *The ring A is a Noetherian local normal domain of dimension* 1.

In this case if π is a generator of the maximal ideal of A, then every element of A can be uniquely written as $u\pi^n$, where $u \in A$ is a unit.

Proof. The equivalence of (1) and (2) is Lemma 7.46.12. Moreover, in the proof of Lemma 7.46.12 we saw that if A is a dscrete valuation ring, then A is a PID, hence (3). Note that a regular local ring is a domain (see Lemma 7.98.2). Using this the equivalence of (3) and (4) follows from dimension theory, see Section 7.57.

Assume (3) and let π be a generator of the maximal ideal $\mathfrak{m}$. For all $n \geq 0$ we have $\dim_{A/\mathfrak{m}} \mathfrak{m}^n/\mathfrak{m}^{n+1} = 1$ because it is generated by π^n (and it cannot be zero). In particular $\mathfrak{m}^n = (\pi^n)$ and the graded ring $\bigoplus \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is isomorphic to the polynomial ring $A/\mathfrak{m}[T]$. For $x \in A \setminus \{0\}$ define $v(x) = \max\{n \mid x \in \mathfrak{m}^n\}$. In other words $x = u\pi^{v(x)}$ with $u \in A^*$. By the remarks above we have $v(xy) = v(x) + v(y)$ for all $x, y \in A \setminus \{0\}$. We extend this to the field of fractions K of A by setting $v(a/b) = v(a) - v(b)$ (well defined by multiplicativity shown above). Then it is clear that A is the set of elements of K which have valuation ≥ 0. Hence we see that A is a valuation ring by Lemma 7.46.10.

A valuation ring is a normal domain by Lemma 7.46.6. Hence we see that the equivalent conditions (1) -- (3) imply (5). Assume (5). Suppose that $\mathfrak{m}$ cannot be generated by 1 element to get a contradiction. Then Lemma 7.110.2 implies there is a finite ring map $A \to A'$ which is an isomorphism after inverting any nonzero element of $\mathfrak{m}$ but not an isomorphism. In particular $A' \subset f.f.(A)$. Since $A \to A'$ is finite it is integral (see Lemma 7.32.3). Since A is normal we get $A = A'$ a contradiction. $\square$

Lemma 7.110.7. *Let R be a domain with fraction field K. Let M be an R-submodule of $K^{\oplus r}$. Assume R is local Noetherian of dimension* 1. *For any nonzero $x \in R$ we have $length_R(R/xR) < \infty$ and*

$$length_R(M/xM) \leq r \cdot length_R(R/xR).$$

Proof. If x is a unit then the result is true. Hence we may assume $x \in \mathfrak{m}$ the maximal ideal of R. Since x is not zero and R is a domain we have $\dim(R/xR) = 0$, and hence R/xR has finite length. Consider $M \subset K^{\oplus r}$ as in the lemma. We may assume that the elements of M generate $K^{\oplus r}$ as a K-vector space after replacing $K^{\oplus r}$ by a smaller subspace if necessary.

Suppose first that M is a finite R-module. In that case we can clear denominators and assume $M \subset R^{\oplus r}$. Since M generates $K^{\oplus r}$ as a vectors space we see that $R^{\oplus r}/M$ has finite length. In particular there exists an integer $c \geq 0$ such that $x^c R^{\oplus r} \subset M$. Note that $M \supset xM \supset x^2 M \supset \ldots$ is a sequence of modules with succesive quotients each isomorphic to M/xM. Hence we see that

$$n\text{length}_R(M/xM) = \text{length}_R(M/x^n M).$$

The same argument for $M = R^{\oplus r}$ shows that

$$n\text{length}_R(R^{\oplus r}/xR^{\oplus r}) = \text{length}_R(R^{\oplus r}/x^n R^{\oplus r}).$$

By our choice of c above we see that $x^n M$ is sandwiched between $x^n R^{\oplus r}$ and $x^{n+c} R^{\oplus r}$. This easily gives that

$$r(n+c)\text{length}_R(R/xR) \geq n\text{length}_R(M/xM) \geq r(n-c)\text{length}_R(R/xR)$$

Hence in the finite case we actually get the result of the lemma with equality.

Suppose now that M is not finite. Suppose that the length of M/xM is $\geq k$ for some natural number k. Then we can find

$$0 \subset N_0 \subset N_1 \subset N_2 \subset \ldots N_k \subset M/xM$$

with $N_i \neq N_{i+1}$ for $i = 0, \ldots k-1$. Choose an element $m_i \in M$ whose congruence class mod xM falls into N_i but not into N_{i-1} for $i = 1, \ldots, k$. Consider the finite R-module $M' = Rm_1 + \ldots + Rm_k \subset M$. Let $N_i' \subset M'/xM'$ be the inverse image of N_i. It is clear that $N_i' \neq N_{i+1}'$ by our choice of m_i. Hence we see that $\text{length}_R(M'/xM') \geq k$. By the finite case we conclude $k \leq r\text{length}_R(R/xR)$ as desired. □

Here is a first application.

Lemma 7.110.8. *Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local homomorphism of local domains with fraction fields $K \subset L$. If R is Noetherian of dimension 1, and $[L : K] < \infty$ then $[\kappa(\mathfrak{n}) : \kappa(\mathfrak{m})] < \infty$.*

Proof. This is clear on applying Lemma 7.110.7 to the submodule $S \subset L \cong K^{\oplus n}$ where $n = [L : K]$. Namely, this shows that for any nonzero $x \in \mathfrak{m}$ the ring S/xS has finite length over R, which shows that $\kappa(\mathfrak{n})$ has finite length over R, which implies that $\dim_{\kappa(\mathfrak{m})} \kappa(\mathfrak{n})$ is finite (Lemma 7.48.6). □

Lemma 7.110.9. *Let R be a domain with fraction field K. Let M be an R-submodule of $K^{\oplus r}$. Assume R is Noetherian of dimension 1. For any nonzero $x \in R$ we have $\text{length}_R(M/xM) < \infty$.*

Proof. Since R has dimension 1 we see that x is contained in finitely many primes $\mathfrak{m}_i$, $i = 1, \ldots, n$, each maximal. Since R is Noetherian we see that R/xR is Artinian, see Proposition 7.57.6. Hence R/xR is a quotient of $\prod R/\mathfrak{m}_i^{e_i}$ for certain e_i because that $\mathfrak{m}_1^{e_1} \ldots \mathfrak{m}_n^{e_n} \subset (x)$ for suitably large e_i as R/xR is Artinian (see Section 7.49). Hence M/xM similarly decomposes as a product $\prod (M/xM)_{\mathfrak{m}_i} = \prod M/(\mathfrak{m}_i^{e_i}, x)M$ of its localizations at the $\mathfrak{m}_i$. By Lemma 7.110.7 applied to $M_{\mathfrak{m}_i}$ over $R_{\mathfrak{m}_i}$ we see each $M_{\mathfrak{m}_i}/xM_{\mathfrak{m}_i} = (M/xM)_{\mathfrak{m}_i}$ has finite length over $R_{\mathfrak{m}_i}$. It easily follows that M/xM has finite length over R. □

Lemma 7.110.10. *(Krull-Akizuki) Let R be a domain with fraction field K. Let $K \subset L$ be a finite extension of fields. Assume R is Noetherian and* $\dim(R) = 1$. *In this case any ring A with $R \subset A \subset L$ is Noetherian.*

Proof. To begin we may assume that L is the fraction field of A by replacing L by the fraction field of A if necessary. Let $I \subset A$ be an ideal. Clearly I generates L as a K-vectors space. Hence we see that $I \cap R \neq (0)$. Pick any nonzero $x \in I \cap R$. Then we get $I/xA \subset A/xA$. By Lemma 7.110.9 the R-module A/xA has finite length as an R-module. Hence I/xA has finite length as an R-module. Hence I is finitely generated as an ideal in A. □

Lemma 7.110.11. *Let R be a Noetherian local domain with fraction field K. Assume that R is not a field. Let $K \subset L$ be a finitely generated field extension. Then there exists discrete valuation ring A with fraction field L which dominates R.*

Proof. If L is not finite over K choose a transcendence basis $x_1, \dots, x_r$ of L over K and replace R by $R[x_1, \dots, x_r]$ localized at the maximal ideal generated by $\mathfrak{m}_R$ and $x_1, \dots, x_r$. Thus we may assume $K \subset L$ finite.

By Lemma 7.110.1 we may assume $\dim(R) = 1$.

Let $A \subset L$ be the integral closure of R in L. By Lemma 7.110.10 this is Noetherian. By Lemma 7.32.15 there is a prime ideal $\mathfrak{q} \subset A$ lying over the maximal ideal of R. By Lemma 7.110.6 the ring $A_{\mathfrak{q}}$ is a discrete valuation ring dominating R as desired. □

7.111. Factorization

Here are some notions and relations between them that are typically taught in a first year course on algebra at the undergraduate level.

Definition 7.111.1. Let R be a domain.

(1) Elements $x, y \in R$ are called *associates* if there exists a unit $u \in R^*$ such that $x = uy$.
(2) An element $x \in R$ is called *irreducible* if it is nonzero, not a unit and whenever $x = yz$, $y, z \in R$, then y is either a unit or an associate of x.
(3) An element $x \in R$ is called *prime* if the ideal generated by x is a prime ideal.

Lemma 7.111.2. *Let R be a domain. Let $x, y \in R$. Then x, y are associates if and only if $(x) = (y)$.*

Proof. Omitted. □

Lemma 7.111.3. *Let R be a domain. Consider the following conditions:*

(1) *The ring R satisfies the ascending chain condition for principal ideals.*
(2) *Every nonzero, nonunit element $a \in R$ has a factorization $a = b_1 \dots b_k$ with each b_i an irreducible element of R.*

Then (1) implies (2).

Proof. Omitted. □

Definition 7.111.4. A *unique factorization domain*, abbreviated *UFD*, is a domain R such that if $x \in R$ is a nonzero, nonunit, then x has a factorization into irreducibles, and if

$$x = a_1 \dots a_m = b_1 \dots b_n$$

are factorizations into irreducibles then $n = m$ and there exists a permutation $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ such that a_i and $b_{\sigma(i)}$ are associates.

Lemma 7.111.5. *Let R be a domain. Assume every nonzero, nonunit factors into irreducibles. Then R is a UFD if and only if every irreducible element is prime.*

Proof. Omitted. □

Definition 7.111.6. A *principal ideal domain*, abbreviated *PID*, is a domain R such that every ideal is a principal ideal.

Lemma 7.111.7. *A principal ideal domain is a unique factorization domain.*

Proof. Omitted. □

Definition 7.111.8. A *Dedekind domain* is a domain R such that every nonzero ideal $I \subset R$ can be written

$$I = \mathfrak{p}_1 \ldots \mathfrak{p}_r$$

uniquely up to permutation of the prime ideals $\mathfrak{p}_i$.

Lemma 7.111.9. *Let R be a ring. The following are equivalent*

(1) *R is a Dedeking domain,*
(2) *R is a Noetherian domain, and for every maximal ideal $\mathfrak{m}$ the local ring $R_\mathfrak{m}$ is a discrete valuation ring, and*
(3) *R is a Noetherian, normal domain, and* $\dim(R) \leq 1$.

Proof. Omitted. □

7.112. Orders of vanishing

Lemma 7.112.1. *Let R be a semi-local Noetherian ring of dimension* 1. *If $a, b \in R$ are not zero divisors then*

$$length_R(R/(ab)) = length_R(R/(a)) + length_R(R/(b))$$

and these lengths are finite.

Proof. We saw the finiteness in Lemma 7.110.9. Additivity holds since there is a short exact sequence $0 \to R/(a) \to R/(ab) \to R/(b) \to 0$ where the first map is given by multiplication by b. (Use length is additive, see Lemma 7.48.3.) □

Definition 7.112.2. Suppose that K is a field, and $R \subset K$ is a local[5] Noetherian subring of dimension 1 with fraction field K. In this case we define the *order of vanishing along R*

$$\mathrm{ord}_R : K^* \longrightarrow \mathbf{Z}$$

by the rule

$$\mathrm{ord}_R(x) = \mathrm{length}_R(R/(x))$$

if $x \in R$ and we set $\mathrm{ord}_R(x/y) = \mathrm{ord}_R(x) - \mathrm{ord}_R(y)$ for $x, y \in R$ both nonzero.

We can use the order of vanishing to compare lattices in a vector space. Here is the definition.

Definition 7.112.3. Let R be a Noetherian local domain of dimension 1 with fraction field K. Let V be a finite dimensional K-vector space. A *lattice in V* is a finite R-submodule $M \subset V$ such that $V = K \otimes_R M$.

The condition $V = K \otimes_R M$ signifies that M contains a basis for the vector space K. We remark that in many places in the literature the notion of a lattice may be defined only in case the ring R is a discrete valuation ring. If R is a discrete valuation ring then any lattice is a free R-module, and this may not be the case in general.

Lemma 7.112.4. *Let R be a Noetherian local domain of dimension* 1 *with fraction field K. Let V be a finite dimensional K-vector space.*

[5]We could also define this when R is only semi-local but this is probably never really what you want!

(1) *If M is a lattice in V and $M \subset M' \subset V$ is an R-submodule of V containing M then the following are equivalent*
   (a) *M' is a lattice,*
   (b) *$length_R(M'/M)$ is finite, and*
   (c) *M' is finitely generated.*
(2) *If M is a lattice in V and $M' \subset M$ is an R-submodule of M then M' is a lattice if and only if $length_R(M/M')$ is finite.*
(3) *If M, M' are lattices in V, then so are $M \cap M'$ and $M + M'$.*
(4) *If $M \subset M' \subset M'' \subset V$ are lattices in V then*
$$length_R(M''/M) = length_R(M'/M) + length_R(M''/M').$$
(5) *If M, M', N, N' are lattices in V and $N \subset M \cap M'$, $M + M' \subset N'$, then we have*
$$\begin{aligned} & length_R(M/M \cap M') - length_R(M'/M \cap M') \\ = \quad & length_R(M/N) - length_R(M'/N) \\ = \quad & length_R(M + M'/M') - length_R(M + M'/M) \\ = \quad & length_R(N'/M') - length_R(N'/M) \end{aligned}$$

Proof. Proof of (1). Assume (1)(a). Say $y_1, \ldots, y_m$ generate M'. Then each $y_i = x_i/f_i$ for some $x_i \in M$ and nonzero $f_i \in R$. Hence we see that $f_1 \ldots f_m M' \subset M$. Since R is Noetherian local of dimension 1 we see that $\mathfrak{m}^n \subset (f_1 \ldots f_m)$ for some n (for example combine Lemmas 7.57.11 and Proposition 7.57.6 or combine Lemmas 7.110.7 and 7.48.4). In other words $\mathfrak{m}^n M' \subset M$ for some n Hence length$(M'/M) < \infty$ by Lemma 7.48.8, in other words (1)(b) holds. Assume (1)(b). Then M'/M is a finite R-module (see Lemma 7.48.2). Hence M' is a finite R-module as an extension of finite R-modules. Hence (1)(c). The implication (1)(c) $\Rightarrow$ (1)(a) follows from the remark following Definition 7.112.3.

Proof of (2). Suppose M is a lattice in V and $M' \subset M$ is an R-submodule. We have seen in (1) that if M' is a lattice, then length$_R(M/M') < \infty$. Conversely, assume that length$_R(M/M') < \infty$. Then M' is finitely generated as R is Noetherian and for some n we have $\mathfrak{m}^n M \subset M'$ (Lemma 7.48.4). Hence it follows that M' contains a basis for V, and M' is a lattice.

Proof of (3). Assume M, M' are lattices in V. Since R is Noetherian the submodule $M \cap M'$ of M is finite. As M is a lattice we can find $x_1, \ldots, x_n \in M$ which form a K-basis for V. Because M' is a lattice we can write $x_i = y_i/f_i$ with $y_i \in M'$ and $f_i \in R$. Hence $f_i x_i \in M \cap M'$. Hence $M \cap M'$ is a lattice also. The fact that $M + M'$ is a lattice follows from part (1).

Part (4) follows from additivity of lengths (Lemma 7.48.3) and the exact sequence
$$0 \to M'/M \to M''/M \to M''/M' \to 0$$
Part (5) follows from repeatedly applying part (4). □

Definition 7.112.5. Let R be a Noetherian local domain of dimension 1 with fraction field K. Let V be a finite dimensional K-vector space. Let M, M' be two lattices in V. The *distance between M and M'* is the integer
$$d(M, M') = \text{length}_R(M/M \cap M') - \text{length}_R(M'/M \cap M')$$
of Lemma 7.112.4 part (5).

In particular, if $M' \subset M$, then $d(M, M') = \text{length}_R(M/M')$.

Lemma 7.112.6. *Let R be a Noetherian local domain of dimension 1 with fraction field K. Let V be a finite dimensional K-vector space. This distance function has the property that*

$$d(M, M'') = d(M, M') + d(M', M'')$$

whenever given three lattices M, M', M'' of V. In particular we have $d(M, M') = -d(M', M)$.

Proof. Omitted. □

Lemma 7.112.7. *Let R be a Noetherian local domain of dimension 1 with fraction field K. Let V be a finite dimensional K-vector space. Let $\varphi : V \to V$ be a K-linear isomorphism. For any lattice $M \subset V$ we have*

$$d(M, \varphi(M)) = ord_R(\det(\varphi))$$

Proof. We can see that the integer $d(M, \varphi(M))$ does not depend on the lattice M as follows. Suppose that M' is a second such lattice. Then we see that

$$\begin{aligned} d(M, \varphi(M)) &= d(M, M') + d(M', \varphi(M)) \\ &= d(M, M') + d(\varphi(M'), \varphi(M)) + d(M', \varphi(M')) \end{aligned}$$

Since φ is an isomorphism we see that $d(\varphi(M'), \varphi(M)) = d(M', M) = -d(M, M')$, and hence $d(M, \varphi(M)) = d(M', \varphi(M'))$. Moreover, both sides of the equation (of the lemma) are additive in φ, i.e.,

$$\mathrm{ord}_R(\det(\varphi \circ \psi)) = \mathrm{ord}_R(\det(\varphi)) + \mathrm{ord}_R(\det(\psi))$$

and also

$$\begin{aligned} d(M, \varphi(\psi((M))) &= d(M, \psi(M)) + d(\psi(M), \varphi(\psi(M))) \\ &= d(M, \psi(M)) + d(M, \varphi(M)) \end{aligned}$$

by the independence shown above. Hence it suffices to prove the lemma for generators of $\mathrm{GL}(V)$. Choose an isomorphism $K^{\oplus n} \cong V$. Then $\mathrm{GL}(V) = \mathrm{GL}_n(K)$ is generated by elementary matrices E. The result is clear for E equal to the identity matrix. If $E = E_{ij}(\lambda)$ with $i \neq j$, $\lambda \in K$, $\lambda \neq 0$, for example

$$E_{12}(\lambda) = \begin{pmatrix} 1 & \lambda & \ldots \\ 0 & 1 & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix}$$

then with respect to a different basis we get $E_{12}(1)$. The result is clear for $E = E_{12}(1)$ by taking as lattice $R^{\oplus n} \subset K^{\oplus n}$. Finally, if $E = E_i(a)$, with $a \in K^*$ for example

$$E_1(a) = \begin{pmatrix} a & 0 & \ldots \\ 0 & 1 & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix}$$

then $E_1(a)(R^{\oplus b}) = aR \oplus R^{\oplus n-1}$ and it is clear that $d(R^{\oplus n}, aR \oplus R^{\oplus n-1}) = \mathrm{ord}_R(a)$ as desired. □

Lemma 7.112.8. *Let $A \to B$ be a ring map. Assume*

(1) *A is a Noetherian local domain of dimension 1,*
(2) *$A \subset B$ is a finite extension of domains.*

Let $K = f.f.(A)$ and $L = f.f.(B)$ so that L is a finite field extension of K. Let $y \in L^*$ and $x = Nm_{L/K}(y)$. In this situation B is semi-local. Let $\mathfrak{m}_i$, $i = 1, \ldots, n$ be the maximal ideals of B. Then

$$ord_A(x) = \sum\nolimits_i [\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m}_A)] ord_{B_{\mathfrak{m}_i}}(y)$$

where ord is defined as in Definition 7.112.2.

Proof. The ring B is semi-local by Lemma 7.104.2. Write $y = b/b'$ for some $b, b' \in B$. By the additivity of ord and multiplicativity of Nm it suffices to prove the lemma for $y = b$ or $y = b'$. In other words we may assume $y \in B$. In this case the left hand side of the formula is

$$\sum [\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m}_A)] \text{length}_{B_{\mathfrak{m}_i}}((B/yB)_{\mathfrak{m}_i})$$

By Lemma 7.48.12 this is equal to $\text{length}_A(B/yB)$. By Lemma 7.112.7 we have

$$\text{length}_A(B/yB) = d(B, yB) = \text{ord}_A(\det_K(L \xrightarrow{y} L)).$$

Since $x = \text{Nm}_{L/K}(y) = \det_K(L \xrightarrow{y} L)$ by definition the lemma is proved. □

We can extend some of the results above to reduced 1-dimensional Noetherian local rings which are not domains by the following lemma.

Lemma 7.112.9. *Let $(R, \mathfrak{m})$ be a reduced Noetherian local ring of dimension 1 and let $x \in \mathfrak{m}$ be a nonzero divisor. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ be the minimal primes of R. Then*

$$length_R(R/(x)) = \sum\nolimits_i ord_{R/\mathfrak{q}_i}(x)$$

Proof. Note that $R_i = R/\mathfrak{q}_i$ is a Noetherian 1-dimensional local domain. Denote $K_i = f.f.(R_i)$. If x is a unit in R, then both sides are zero. Hence we may assume $x \in \mathfrak{m}$. Consider the map $\Psi : R \to \prod R_i$. As R is reduced this map is injective, see Lemma 7.16.2. By Lemma 7.22.2 we have $Q(R) = \prod K_i$. Hence the finite R-module $\text{Coker}(\Psi)$ is annihilated by a nonzero divisor $y \in R$, hence has support $\{\mathfrak{m}\}$, is annihilated by some power of x and has finite length over R, see Lemma 7.59.8. Consider the short exact sequence

$$0 \to R \to \prod R_i \to \text{Coker}(\Psi) \to 0$$

Applying multiplication by x^n to this for $n \gg 0$ we obtain from the snake lemma

$$0 \to \text{Coker}(\Psi) \to R/x^nR \to \prod R_i/x^nR_i \to \text{Coker}(\Psi) \to 0$$

Thus we see that

$$\text{length}_R(R/x^nR) = \text{length}_R(\prod R_i/x^nR_i) = \sum \text{length}_R(R_i/x^nR_i)$$

by Lemma 7.48.3. By Lemma 7.48.5 we have $\text{length}_R(R_i/x^nR_i) = \text{length}_{R_i}(R_i/x^nR_i)$. Now the result follows from the additivity of Lemma 7.112.1 and the definition of the order of vanishing along R_i. □

7.113. Quasi-finite maps

Consider a ring map $R \to S$ of finite type. A map $Spec(S) \to Spec(R)$ is quasi-finite at a point if that point is isolated in its fibre. This means that the fibre is zero dimensional at that point. In this section we study the basic properties of this important but technical notion. More advanced material can be found in the next section.

Lemma 7.113.1. *Let k be a field. Let S be a finite type k algebra. Let $\mathfrak{q}$ be a prime of S. The following are equivalent:*

(1) *$\mathfrak{q}$ is an isolated point of $Spec(S)$,*

(2) *$S_{\mathfrak q}$ is finite over k,*
(3) *there exists a $g \in S$, $g \not\in \mathfrak q$ such that $D(g) = \{\mathfrak q\}$,*
(4) $\dim_{\mathfrak q} Spec(S) = 0$,
(5) *$\mathfrak q$ is a closed point of $Spec(S)$ and $\dim(S_{\mathfrak q}) = 0$, and*
(6) *the field extension $k \subset \kappa(\mathfrak q)$ is finite and $\dim(S_{\mathfrak q}) = 0$.*

In this case $S = S_{\mathfrak q} \times S'$ for some finite type k-algebra S'. Also, the element g as in (3) has the property $S_{\mathfrak q} = S_g$.

Proof. Suppose $\mathfrak q$ is an isolated point of $Spec(S)$, i.e., $\{\mathfrak q\}$ is open in $Spec(S)$. Because $Spec(S)$ is a Jacobson space (see Lemmas 7.31.2 and 7.31.4) we see that $\mathfrak q$ is a closed point. Hence $\{\mathfrak q\}$ is open and closed in $Spec(S)$. By Lemmas 7.18.3 and 7.20.3 we may write $S = S_1 \times S_2$ with $\mathfrak q$ corresponding to the only point $Spec(S_1)$. Hence $S_1 = S_{\mathfrak q}$ is a zero dimensional ring of finite type over k. Hence it is finite over k for example by Lemma 7.106.4. We have proved (1) implies (2).

Suppose $S_{\mathfrak q}$ is finite over k. Then $S_{\mathfrak q}$ is Artinian local, see Lemma 7.49.2. So $Spec(S_{\mathfrak q}) = \{\mathfrak q S_{\mathfrak q}\}$ by Lemma 7.49.8. Consider the exact sequence $0 \to K \to S \to S_{\mathfrak q} \to Q \to 0$. It is clear that $K_{\mathfrak q} = Q_{\mathfrak q} = 0$. Also, K is a finite S-module as S is Noetherian and Q is a finite S-modules since $S_{\mathfrak q}$ is finite over k. Hence there exists $g \in S$, $g \not\in \mathfrak q$ such that $K_g = Q_g = 0$. Thus $S_{\mathfrak q} = S_g$ and $D(g) = \{\mathfrak q\}$. We have proved that (2) implies (3).

Suppose $D(g) = \{\mathfrak q\}$. Since $D(g)$ is open by construction of the topology on $Spec(S)$ we see that $\mathfrak q$ is an isolated point of $Spec(S)$. We have proved that (3) implies (1). In other words (1), (2) and (3) are equivalent.

Assume $\dim_{\mathfrak q} Spec(S) = 0$. This means that there is some open neighbourhood of $\mathfrak q$ in $Spec(S)$ which has dimension zero. Then there is an open neighbourhood of the form $D(g)$ which has dimension zero. Since S_g is Noetherian we conclude that S_g is Artinian and $D(g) = Spec(S_g)$ is a finite discrete set, see Proposition 7.57.6. Thus $\mathfrak q$ is an isolated point of $D(g)$ and, by the equivalence of (1) and (2) above applied to $\mathfrak q S_g \subset S_g$, we see that $S_{\mathfrak q} = (S_g)_{\mathfrak q S_g}$ is finite over k. Hence (4) implies (2). It is clear that (1) implies (4). Thus (1) -- (4) are all equivalent.

Lemma 7.105.6 gives the implication (5) $\Rightarrow$ (4). The implication (4) $\Rightarrow$ (6) follows from Lemma 7.107.3. The implication (6) $\Rightarrow$ (5) follows from Lemma 7.31.9. At this point we know (1) -- (6) are equivalent.

The two statements at the end of the lemma we saw during the course of the proof of the equivalence of (1), (2) and (3) above. $\square$

Lemma 7.113.2. *Let $R \to S$ be a ring map of finite type. Let $\mathfrak q \subset S$ be a prime lying over $\mathfrak p \subset R$. Let $F = Spec(S \otimes_R \kappa(\mathfrak p))$ be the fibre of $Spec(S) \to Spec(R)$, see Remark 7.16.8. Denote $\overline{\mathfrak q} \in F$ the point corresponding to $\mathfrak q$. The following are equivalent*

(1) *$\overline{\mathfrak q}$ is an isolated point of F,*
(2) *$S_{\mathfrak q}/\mathfrak p S_{\mathfrak q}$ is finite over $\kappa(\mathfrak p)$,*
(3) *there exists a $g \in S$, $g \not\in \mathfrak q$ such that the only prime of $D(g)$ mapping to $\mathfrak p$ is $\mathfrak q$,*
(4) $\dim_{\overline{\mathfrak q}}(F) = 0$,
(5) *$\overline{\mathfrak q}$ is a closed point of F and $\dim(S_{\mathfrak q}/\mathfrak p S_{\mathfrak q}) = 0$, and*
(6) *the field extension $\kappa(\mathfrak p) \subset \kappa(\mathfrak q)$ is finite and $\dim(S_{\mathfrak q}/\mathfrak p S_{\mathfrak q}) = 0$.*

Proof. Note that $S_{\mathfrak q}/\mathfrak p S_{\mathfrak q} = (S \otimes_R \kappa(\mathfrak p))_{\overline{\mathfrak q}}$. Moreover $S \otimes_R \kappa(\mathfrak p)$ is of finite type over $\kappa(\mathfrak p)$. The conditions correspong exactly to the conditions of Lemma 7.113.1 for the $\kappa(\mathfrak p)$-algebra $S \otimes_R \kappa(\mathfrak p)$ and the prime $\overline{\mathfrak q}$, hence they are equivalent. $\square$

Definition 7.113.3. Let $R \to S$ be a finite type ring map. Let $\mathfrak q \subset S$ be a prime.

(1) If the equivalent conditions of Lemma 7.113.2 are satisfied then we say $R \to S$ is *quasi-finite at* $\mathfrak q$.
(2) We say a ring map $A \to B$ is *quasi-finite* if it is of finite type and quasi-finite at all primes of B.

Lemma 7.113.4. *Let $R \to S$ be a finite type ring map. Then $R \to S$ is quasi-finite if and only if for all primes $\mathfrak p \subset R$ the fibre $S \otimes_R \kappa(\mathfrak p)$ is finite over $\kappa(\mathfrak p)$.*

Proof. If the fibres are finite then the map is clearly quasi-finite. For the converse, note that $S \otimes_R \kappa(\mathfrak p)$ is a $\kappa(\mathfrak p)$-algebra of finite type over k of dimension 0. Hence it is finite over k for example by Lemma 7.106.4. $\square$

Lemma 7.113.5. *Let $R \to S$ be a finite type ring map. Let $\mathfrak q \subset S$ be a prime lying over $\mathfrak p \subset R$. Let $f \in R$, $f \not\in \mathfrak p$ and $g \in S$, $g \not\in \mathfrak q$. Then $R \to S$ is quasi-finite at $\mathfrak q$ if and only if $R_f \to S_{fg}$ is quasi-finite at $\mathfrak q S_{fg}$.*

Proof. The fibre of $\text{Spec}(S_{fg}) \to \text{Spec}(R_f)$ is homeomorphic to an open subset of the fibre of $\text{Spec}(S) \to \text{Spec}(R)$. Hence the lemma follows from part (1) of the equivalent conditions of Lemma 7.113.2. $\square$

Lemma 7.113.6. *Let*

$$\begin{tikzcd} S \arrow[r] & S' \\ R \arrow[u] \arrow[r] & R' \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} \mathfrak q \arrow[r, dash] & \mathfrak q' \\ \mathfrak p \arrow[u, dash] \arrow[r, dash] & \mathfrak p' \arrow[u, dash] \end{tikzcd}$$

be a commutative diagram of rings with primes as indicated. Assume $R \to S$ of finite type, and $S \otimes_R R' \to S'$ surjective. If $R \to S$ is quasi-finite at $\mathfrak q$, then $R' \to S'$ is quasi-finite at $\mathfrak q'$.

Proof. Write $S \otimes_R \kappa(\mathfrak p) = S_1 \times S_2$ with S_1 finite over $\kappa(\mathfrak p)$ and such that $\mathfrak q$ corresponds to a point of S_1 as in Lemma 7.113.1. Because $S \otimes_R R' \to S'$ surjective the canonical map $(S \otimes_R \kappa(\mathfrak p)) \otimes_{\kappa(\mathfrak p)} \kappa(\mathfrak p') \to S' \otimes_{R'} \kappa(\mathfrak p')$ is surjective. Let S_i' be the image of $S_i \otimes_{\kappa(\mathfrak p)} \kappa(\mathfrak p')$ in $S' \otimes_{R'} \kappa(\mathfrak p')$. Then $S' \otimes_{R'} \kappa(\mathfrak p') = S_1' \times S_2'$ and S_1' is finite over $\kappa(\mathfrak p')$. The map $S' \otimes_{R'} \kappa(\mathfrak p') \to \kappa(\mathfrak q')$ factors through S_1' (i.e. it annihilates the factor S_2') because the map $S \otimes_R \kappa(\mathfrak p) \to \kappa(\mathfrak q)$ factors through S_1 (i.e. it annihilates the factor S_2). Thus $\mathfrak q'$ corresponds to a point of $Spec(S_1')$ in the disjoint union decomposition of the fibre: $Spec(S' \otimes_{R'} \kappa(\mathfrak p')) = Spec(S_1') \amalg Spec(S_1')$. (See Lemma 7.18.2.) Since S_1' is finite over a field, it is Artinian ring, and hence $Spec(S_1')$ is a finite discrete set. (See Proposition 7.57.6.) We conclude $\mathfrak q'$ is isolated in its fibre as desired. $\square$

Lemma 7.113.7. *A composition of quasi-finite ring maps is quasi-finite.*

Proof. Suppose $A \to B$ and $B \to C$ are quasi-finite ring maps. By Lemma 7.6.2 we see that $A \to C$ is of finite type. Let $\mathfrak r \subset C$ be a prime of C lying over $\mathfrak q \subset B$ and $\mathfrak p \subset A$. Since $A \to B$ and $B \to C$ are quasi-finite at $\mathfrak q$ and $\mathfrak r$ respectively, then there exist $b \in B$ and $c \in C$ such that $\mathfrak q$ is the only prime of $D(b)$ which maps to $\mathfrak p$ and similarly $\mathfrak r$ is the only prime of

$D(c)$ which maps to $\mathfrak{q}$. If $c' \in C$ is the image of $b \in B$, then $\mathfrak{r}$ is the only prime of $D(cc')$ which maps to $\mathfrak{p}$. Therefore $A \to C$ is quasi-finite at $\mathfrak{r}$. □

Lemma 7.113.8. *Let $R \to S$ be a ring map of finite type. Let $R \to R'$ be any ring map. Set $S' = R' \otimes_R S$.*

(1) *The set $\{\mathfrak{q}' \mid R' \to S' \text{ quasi-finite at } \mathfrak{q}'\}$ is the inverse image of the corresponding set of $Spec(S)$ under the canonical map $Spec(S') \to Spec(S)$.*
(2) *If $Spec(R') \to Spec(R)$ is surjective, then $R \to S$ is quasi-finite if and only if $R' \to S'$ is quasi-finite.*
(3) *Any base change of a quasi-finite ring map is quasi-finite.*

Proof. Let $\mathfrak{p}' \subset R'$ be a prime lying over $\mathfrak{p} \subset R$. Then the fibre ring $S' \otimes_{R'} \kappa(\mathfrak{p}')$ is the base change of the fibre ring $S \otimes_R \kappa(\mathfrak{p})$ by the field extension $\kappa(\mathfrak{p}) \to \kappa(\mathfrak{p}')$. Hence the first assertion follows from the invariance of dimension under field extension (Lemma 7.107.6) and Lemma 7.113.1. The stability of quasi-finite maps under base change follows from this and the stability of finite type property under base change. The second assertion follows since the assumption implies that given a prime $\mathfrak{q} \subset S$ we can find a prime $\mathfrak{q}' \subset S'$ lying over it. □

The following lemma is not quite about quasi-finite ring maps, but it does not seem to fit anywhere else so well.

Lemma 7.113.9. *Let $R \to S$ be a ring map of finite type. Let $\mathfrak{p} \subset R$ be a minimal prime. Assume that there are at most finitely many primes of S lying over $\mathfrak{p}$. Then there exists a $g \in R$, $g \not\in \mathfrak{p}$ such that the ring map $R_g \to S_g$ is finite.*

Proof. Let $x_1, \ldots, x_n$ be generators of S over R. Since $\mathfrak{p}$ is a minimal prime we have that $\mathfrak{p}R_{\mathfrak{p}}$ is a locally nilpotent ideal, see Lemma 7.23.3. Hence $\mathfrak{p}S_{\mathfrak{p}}$ is a locally nilpotent ideal, see Lemma 7.14.1. By assumption the finite type $\kappa(\mathfrak{p})$-algebra $S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}}$ has finitely many primes. Hence (for example by Lemma 7.106.4) $\kappa(\mathfrak{p}) \to S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}}$ is a finite ring map. Thus we may find monic polynomials $P_i \in R_{\mathfrak{p}}[X]$ such that $P_i(x_i)$ maps to zero in $S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}}$. By what we said above there exist $e_i \geq 1$ such that $P(x_i)^{e_i} = 0$ in $S_{\mathfrak{p}}$. Let $g_1 \in R$, $g_1 \not\in \mathfrak{p}$ be an element such that $P_i \in R[1/g_1]$ for all i. Next, let $g_2 \in R$, $g_2 \not\in \mathfrak{p}$ be an element such that $P(x_i) = 0$ in $S_{g_1 g_2}$. Setting $g = g_1 g_2$ we win. □

7.114. Zariski's Main Theorem

In this section our aim is to prove the algebraic version of Zariski's Main theorem. This theorem will be the basis of many further developments in the theory of schemes and morphisms of schemes later in the project.

Let $R \to S$ be a ring map of finite type. Our goal in this section is to show that the set of points of $Spec(S)$ where the map is quasi-finite is *open* (Theorem 7.114.13). In fact, it will turn out that there exists a finite ring map $R \to S'$ such that in some sense the quasi-finite locus of S/R is open in $Spec(S')$ (but we will not prove this in the algebra chapter since we do not develop the language of schemes here -- for the case where $R \to S$ is quasi-finite see Lemma 7.114.15). These statements are somewhat tricky to prove and we do it by a long list of lemmas concering integral and finite extensions of rings. This material may be found in [**Ray70**], and [**Pes66**]. We also found notes by Thierry Coquand helpful.

Lemma 7.114.1. *Let $\varphi : R \to S$ be a ring map. Suppose $t \in S$ satisfies the relation $\varphi(a_0) + \varphi(a_1)t + \ldots + \varphi(a_n)t^n = 0$. Then $\varphi(a_n)t$ is integral over R.*

Proof. Namely, multiply the equation $\varphi(a_0) + \varphi(a_1)t + \ldots + \varphi(a_n)t^n = 0$ with $\varphi(a_n)^{n-1}$ and write it as $\varphi(a_0 a_n^{n-1}) + \varphi(a_1 a_n^{n-2})(\varphi(a_n)t) + \ldots + (\varphi(a_n)t)^n = 0$. $\square$

The following lemma is in some sense the key lemma in this section.

Lemma 7.114.2. *Let R be a ring. Let $\varphi : R[x] \to S$ be a ring map. Let $t \in S$. Assume that (a) t is integral over $R[x]$, and (b) there exists a monic $p \in R[x]$ such that $t\varphi(p) \in Im(\varphi)$. Then there exists a $q \in R[x]$ such that $t - \varphi(q)$ is integral over R.*

Proof. Write $t\varphi(p) = \varphi(r)$ for some $r \in R[x]$. Using euclidean division, write $r = qp + r'$ with $q, r' \in R[x]$ and $\deg(r') < \deg(p)$. We may replace t by $t - \varphi(q)$ which is still integral over $R[x]$, so that we obtain $t\varphi(p) = \varphi(r')$. In the ring S_t we may write this as $\varphi(p) - (1/t)\varphi(r') = 0$. This implies that $\varphi(x)$ gives an element of the localization S_t which is integral over $\varphi(R)[1/t] \subset S_t$. On the other hand, t is integral over the subring $\varphi(R)[\varphi(x)] \subset S$. Combined we conclude that t is integral over the subring $\varphi(R)[1/t] \subset S_t$, see Lemma 7.32.6. In other words there exists an equation of the form $t^d + \sum_{i<d} (\varphi(r_i)/t^{n_i}) t^i = 0$ in S_t with $r_i \in R$. This means that $t^{d+N} + \sum_{i<d} \varphi(r_i) t^{i+N-n_i} = 0$ in S for some N large enough. In other words t is integral over R. $\square$

Lemma 7.114.3. *Let R be a ring and let $\varphi : R[x] \to S$ be a ring map. Let $t \in S$. If t is integral over $R[x]$, then there exists an $\ell \geq 0$ such that for every $a \in R$ the element $\varphi(a)^\ell t$ is integral over $\varphi_a : R[y] \to S$, defined by $y \mapsto \varphi(ax)$ and $r \mapsto \varphi(r)$ for $r \in R$.*

Proof. Say $t^d + \sum_{i<d} \varphi(f_i) t^i = 0$ with $f_i \in R[x]$. Let ℓ be the maximum degree in x of all the f_i. Multiply the equation by $\varphi(a)^\ell$ to get $\varphi(a)^\ell t^d + \sum_{i<d} \varphi(a^\ell f_i) t^i = 0$. Note that each $\varphi(a^\ell f_i)$ is in the image of φ_a. The result follows from Lemma 7.114.1. $\square$

Lemma 7.114.4. *Let R be a ring. Let $\varphi : R[x] \to S$ be a ring map. Let $t \in S$. Assume t is integral over $R[x]$. Let $p \in R[x]$, $p = a_0 + a_1 x + \ldots + a_k x^k$ such that $t\varphi(p) \in Im(\varphi)$. Then there exists a $q \in R[x]$ and $n \geq 0$ such that $\varphi(a_k)^n t - \varphi(q)$ is integral over R.*

Proof. By Lemma 7.114.3 there exists an $\ell \geq 0$ such that the element $\varphi(a_k)^\ell t$ is integral over the map $\varphi' : R[y] \to S$, $\varphi'(y) = \varphi(a_k x)$ and $\varphi'(r) = \varphi(r)$, for $r \in R$. The polynomial $p' = a_k^{k-1} a_0 + a_k^{k-2} a_1 y + \ldots + y^k$ is monic and $t\varphi'(p') = \varphi(a_k^{k-1}) t\varphi(p) \in \mathrm{Im}(\varphi)$. By definition of φ' this implies there exists a $n \geq k-1$ such that $\varphi(a_k^n) t\varphi'(p') \in \mathrm{Im}(\varphi')$. If also $n \geq \ell$, then $\varphi(a_k)^n t$ is still integral over $R[y]$. By Lemma 7.114.2 we see that $\varphi(a_k)^n t - \varphi'(q)$ is integral over R for some $q \in R[y]$. Again by the simple relationship between φ' and φ this implies the lemma. $\square$

Situation 7.114.5. Let R be a ring. Let $\varphi : R[x] \to S$ be finite. Let

$$J = \{g \in S \mid gS \subset \mathrm{Im}(\varphi)\}$$

be the ``conductor ideal" of φ. Assume $\varphi(R) \subset S$ integrally closed in S.

Lemma 7.114.6. *In Situation 7.114.5. Suppose $u \in S$, $a_0, \ldots, a_k \in R$, $u\varphi(a_0 + a_1 x + \ldots + a_k x^k) \in J$. Then there exists an $m \geq 0$ such that $u\varphi(a_k)^m \in J$.*

Proof. Assume that S is generated by $t_1, \ldots, t_n$ as an $R[x]$-module. In this case $J = \{g \in S \mid gt_i \in \mathrm{Im}(\varphi) \text{ for all } i\}$. Note that each element ut_i is integral over $R[x]$, see Lemma 7.32.3. We have $\varphi(a_0 + a_1 x + \ldots + a_k x^k) ut_i \in \mathrm{Im}(\varphi)$. By Lemma 7.114.4, for each i there exists an integer n_i and an element $q_i \in R[x]$ such that $\varphi(a_k^{n_i}) ut_i - \varphi(q_i)$ is integral over R. By assumption this element is in $\varphi(R)$ and hence $\varphi(a_k^{n_i}) ut_i \in \mathrm{Im}(\varphi)$. It follows that $m = \max\{n_1, \ldots, n_n\}$ works. $\square$

Lemma 7.114.7. *In Situation 7.114.5. Suppose $u \in S$, $a_0, \ldots, a_k \in R$, $u\varphi(a_0 + a_1x + \ldots + a_kx^k) \in \sqrt{J}$. Then $u\varphi(a_i) \in \sqrt{J}$ for all i.*

Proof. Under the assumptions of the lemma we have $u^n\varphi(a_0 + a_1x + \ldots + a_kx^k)^n \in J$ for some $n \geq 1$. By Lemma 7.114.6 we deduce $u^n\varphi(a_k^{nm}) \in J$ for some $m \geq 1$. Thus $u\varphi(a_k) \in \sqrt{J}$, and so $u\varphi(a_0 + a_1x + \ldots + a_kx^k) - u\varphi(a_k) = u\varphi(a_0 + a_1x + \ldots + a_{k-1}x^{k-1}) \in \sqrt{J}$. We win by induction on k. $\square$

This lemma suggests the following definition.

Definition 7.114.8. Given an inclusion of rings $R \subset S$ and an element $x \in S$ we say that x is *strongly transcendental over R* if whenever $u(a_0 + a_1x + \ldots + a_kx^k) = 0$ with $u \in S$ and $a_i \in R$, then we have $ua_i = 0$ for all i.

Note that if S is a domain then this is the same as saying that x as an element of the fraction field of S is transcendental over the fraction field of R.

Lemma 7.114.9. *Suppose $R \subset S$ is an inclusion of reduced rings and suppose that $x \in S$ is strongly transcendental over R. Let $\mathfrak{q} \subset S$ be a minimal prime and let $\mathfrak{p} = R \cap \mathfrak{q}$. Then the image of x in $S/\mathfrak{q}$ is strongly transcendental over the subring $R/\mathfrak{p}$.*

Proof. Suppose $u(a_0 + a_1x + \ldots + a_kx^k) \in \mathfrak{q}$. By Lemma 7.23.3 the local ring $S_{\mathfrak{q}}$ is a field, and hence $u(a_0 + a_1x + \ldots + a_kx^k)$ is zero in $S_{\mathfrak{q}}$. Thus $uu'(a_0 + a_1x + \ldots + a_kx^k) = 0$ for some $u' \in S$, $u' \notin \mathfrak{q}$. Since x is strongly transcendental over R we get $uu'a_i = 0$ for all i. This in turn implies that $ua_i \in \mathfrak{q}$. $\square$

Lemma 7.114.10. *Suppose $R \subset S$ is an inclusion of domains and let $x \in S$. Assume x is (strongly) transcendental over R and that S is finite over $R[x]$. Then $R \to S$ is not quasi-finite at any prime of S.*

Proof. As a first case, assume that R is normal, see Definition 7.33.10. By Lemma 7.33.13 we see that $R[x]$ is normal. Take a prime $\mathfrak{q} \subset S$, and set $\mathfrak{p} = R \cap \mathfrak{q}$. Assume that the extension $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q})$ is finite. This would be the case if $R \to S$ is quasi-finite at $\mathfrak{q}$. Let $\mathfrak{r} = R[x] \cap \mathfrak{q}$. Then since $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{r}) \subset \kappa(\mathfrak{q})$ we see that the extension $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{r})$ is finite too. Thus the inclusion $\mathfrak{r} \supset \mathfrak{p}R[x]$ is strict. By going down for $R[x] \subset S$, see Proposition 7.34.7, we find a prime $\mathfrak{q}' \subset \mathfrak{q}$, lying over the prime $\mathfrak{p}R[x]$. Hence the fibre $Spec(S \otimes_R \kappa(\mathfrak{p}))$ contains a point not equal to $\mathfrak{q}$, namely $\mathfrak{q}'$, whose closure contains $\mathfrak{q}$ and hence $\mathfrak{q}$ is not isolated in its fibre.

If R is not normal, let $R \subset R' \subset K$ be the integral closure R' of R in its field of fractions K. Let $S \subset S' \subset L$ be the subring S' of the field of fractions L of S generated by R' and S. Note that by construction the map $S \otimes_R R' \to S'$ is surjective. This implies that $R'[x] \subset S'$ is finite. Also, the map $S \subset S'$ induces a surjection on $Spec$, see Lemma 7.32.15. We conclude by Lemma 7.113.6 and the normal case we just discussed. $\square$

Lemma 7.114.11. *Suppose $R \subset S$ is an inclusion of reduced rings. Assume $x \in S$ be strongly transcendental over R, and S finite over $R[x]$. Then $R \to S$ is not quasi-finite at any prime of S.*

Proof. Let $\mathfrak{q} \subset S$ be any prime. Choose a minimal prime $\mathfrak{q}' \subset \mathfrak{q}$. According to Lemmas 7.114.9 and 7.114.10 the extension $R/(R \cap \mathfrak{q}') \subset S/\mathfrak{q}'$ is not quasi-finite at the prime corresponding to $\mathfrak{q}$. By Lemma 7.113.6 the extension $R \to S$ is not quasi-finite at $\mathfrak{q}$. $\square$

Lemma 7.114.12. *Let R be a ring. Let $S = R[x]/I$. Let $\mathfrak q \subset S$ be a prime. Assume $R \to S$ is quasi-finite at $\mathfrak q$. Let $S' \subset S$ be the integral closure of R in S. Then there exists an element $g \in S'$, $g \not\in \mathfrak q$ such that $S'_g \cong S_g$.*

Proof. Let $\mathfrak p$ be the image of $\mathfrak q$ in $Spec(R)$. The assumption that $R \to S$ is quasi-finite at $\mathfrak q$ implies there exists an $f \in I$, $f = a_n x^n + \ldots + a_0$ such that some $a_i \not\in \mathfrak p$. In particular there exists a relation $b_m x^m + \ldots + b_0 = 0$ with $b_j \in S'$, $j = 0, \ldots, m$ and $b_j \not\in \mathfrak q \cap S'$ for some j. We prove the lemma by induction on m.

The case $b_m \in \mathfrak q$. In this case we have $b_m x \in S'$ by Lemma 7.114.1. Set $b'_{m-1} = b_m x + b_{m-1}$. Then

$$b'_{m-1} x^{m-1} + b_{m-2} x^{m-2} + \ldots + b_0 = 0$$

Since b'_{m-1} is congruent to b_{m-1} modulo $S' \cap \mathfrak q$ we see that it is still the case that one of $b'_{m-1}, b_{m-2}, \ldots, b_0$ is not in $S' \cap \mathfrak q$. Thus we win by induction on m.

The case $b_m \not\in \mathfrak q$. In this case x is intregral over S'_{b_m}, in fact $b_m x \in S'$ by Lemma 7.114.1. Hence the injective map $S'_{b_m} \to S_{b_m}$ is also surjective, i.e., an isomorphism as desired. $\square$

Theorem 7.114.13. *(Zariski's Main Theorem.) Let R be a ring. Let $R \to S$ be a finite type R-algebra. Let $S' \subset S$ be the integral closure of R in S. Let $\mathfrak q \subset S$ be a prime of S. If $R \to S$ is quasi-finite at $\mathfrak q$ then there exists a $g \in S'$, $g \not\in \mathfrak q$ such that $S'_g \cong S_g$.*

Proof. There exist finitely many elements $x_1, \ldots, x_n \in S$ such that S is finite over the R-sub algebra generated by $x_1, \ldots, x_n$. (For example generators of S over R.) We prove the proposition by induction on the minimal such number n.

The case $n = 0$ is trivial, because in this case $S' = S$, see Lemma 7.32.3.

The case $n = 1$. We may and do replace R by its integral closure in S, in particular this means that $R \subset S$. Consider the map $\varphi : R[x] \to S$, $x \mapsto x_1$. (We will see that φ is not injective below.) By assumption φ is finite. Hence we are in Situation 7.114.5. Let $J \subset S$ be the ``conductor ideal" defined in Situation 7.114.5. Consider the diagram

$$\begin{tikzcd}
R[x] \arrow[r] & S \arrow[r] & S/\sqrt{J} & R/(R \cap \sqrt{J})[x] \arrow[l] \\
& R \arrow[ul] \arrow[u] \arrow[r] & R/(R \cap \sqrt{J}) \arrow[u] \arrow[ur] &
\end{tikzcd}$$

According to Lemma 7.114.7 the image of x in the quotient $S/\sqrt{J}$ is strongly transcendental over $R/(R \cap \sqrt{J})$. Hence by Lemma 7.114.11 the ring map $R/(R \cap \sqrt{J}) \to S/\sqrt{J}$ is not quasi-finite at any prime of $S/\sqrt{J}$. By Lemma 7.113.6 we deduce that $\mathfrak q$ does not lie in $V(J) \subset Spec(S)$. Thus there exists an element $s \in J$, $s \not\in \mathfrak q$. By definition of J we may write $s = \varphi(f)$ for some polynomial $f \in R[x]$. Now let $I = \text{Ker}(R[x] \to S)$. Since $\varphi(f) \in J$ we get $(R[x]/I)_f \cong S_{\varphi(f)}$. Also $s \not\in \mathfrak q$ means that $f \not\in \varphi^{-1}(\mathfrak q)$. Thus $\varphi^{-1}(\mathfrak q)$ is a prime of $R[x]/I$ at which $R \to R[x]/I$ is quasi-finite, see Lemma 7.113.5. Let $C \subset R[x]/I$ be the integral closure of R. By Lemma 7.114.12 there exists an element $h \in C$, $h \not\in \varphi^{-1}(\mathfrak q)$ such that $C_h \cong (R[x]/I)_h$. We conclude that $(R[x]/I)_{fh} = S_{\varphi(fh)}$ is isomorphic to a principal localization $C_{h'}$ of C for some $h' \in C$, $h' \not\in \varphi^{-1}(\mathfrak q)$. Since $\varphi(C) \subset S'$ we get $g = \varphi(h') \in S'$, $g \not\in \mathfrak q$ and moreover the injective map $S'_g \to S_g$ is also surjective because by our choice of h' the map $C_{h'} \to S_g$ is surjective.

The case $n > 1$. Consider the subring $R' \subset S$ which is the integral closure of $R[x_1, \ldots, x_{n-1}]$ in S. By Lemma 7.113.6 the extension S/R' is quasi-finite at $\mathfrak q$. Also, note that S is finite over $R'[x_n]$. By the case $n = 1$ above, there exists a $g' \in R'$, $g' \not\in \mathfrak q$ such that $(R')_{g'} \cong S_{g'}$. At this point we cannot apply induction to $R \to R'$ since R' may not be finite type over R. Since S is finitely generated over R we deduce in particular that $(R')_{g'}$ is finitely generated over R. Say the elements g', and $y_1/(g')^{n_1}, \ldots, y_N/(g')^{n_N}$ with $y_i \in R'$ generate $(R')_{g'}$ over R. Let R'' be the R-sub algebra of R' generated by $x_1, \ldots, x_{n-1}, y_1, \ldots, y_N, g'$. This has the property $(R'')_{g'} \cong S_{g'}$. Surjectivity because of how we chose y_i, injectivity because $R'' \subset R'$, and localization is exact. Note that R'' is finite over $R[x_1, \ldots, x_{n-1}]$ because of our choice of R', see Lemma 7.32.4. Let $\mathfrak q'' = R'' \cap \mathfrak q$. Since $(R'')_{\mathfrak q''} = S_{\mathfrak q}$ we see that $R \to R''$ is quasi-finite at $\mathfrak q''$, see Lemma 7.113.2. We apply our induction hypothesis to $R \to R''$, $\mathfrak q''$ and $x_1, \ldots, x_{n-1} \in R''$ and we find a subring $R''' \subset R''$ which is integral over R and an element $g'' \in R'''$, $g'' \not\in \mathfrak q''$ such that $(R''')_{g''} \cong (R'')_{g''}$. Write the image of g' in $(R'')_{g''}$ as $g'''/(g'')^n$ for some $g''' \in R'''$. Set $g = g''g''' \in R'''$. Then it is clear that $g \not\in \mathfrak q$ and $(R''')_g \cong S_g$. Since by construction we have $R''' \subset S'$ we also have $S'_g \cong S_g$ as desired. □

Lemma 7.114.14. *Let $R \to S$ be a finite type ring map. The set of points $\mathfrak q$ of $Spec(S)$ at which S/R is quasi-finite is open in $Spec(S)$.*

Proof. Let $\mathfrak q \subset S$ be a point at which the ring map is quasi-finite. By Theorem 7.114.13 there exists an integral ring extension $R \to S'$, $S' \subset S$ and an element $g \in S'$, $g \not\in \mathfrak q$ such that $S'_g \cong S_g$. Since S and hence S_g are of finite type over R we may find finitely many elements $y_1, \ldots, y_N$ of S' such that $S''_g \cong S$ where $S'' \subset S'$ is the sub R-algebra generated by $g, y_1, \ldots, y_N$. Since S'' is finite over R (see Lemma 7.32.4) we see that S'' is quasi-finite over R (see Lemma 7.113.4). It is easy to see that this implies that S''_g is quasi-finite over R, for example because the property of being quasi-finite at a prime depends only on the local ring at the prime. Thus we see that S_g is quasi-finite over R. By the same token this implies that $R \to S$ is quasi-finite at every prime of S which lies in $D(g)$. □

Lemma 7.114.15. *Let $R \to S$ be a finite type ring map. Suppose that S is quasi-finite over R. Let $S' \subset S$ be the integral closure of R in S. Then*

(1) *$Spec(S) \to Spec(S')$ is a homeomorphism onto an open subset,*
(2) *if $g \in S'$ and $D(g)$ is contained in the image of the map, then $S'_g \cong S_g$, and*
(3) *there exists a finite R-algebra $S'' \subset S'$ such that (1) and (2) hold for the ring map $S'' \to S$.*

Proof. Because S/R is quasi-finite we may apply Theorem 7.114.13 to each point $\mathfrak q$ of $Spec(S)$. Since $Spec(S)$ is quasi-compact, see Lemma 7.16.10, we may choose a finite number of $g_i \in S'$, $i = 1, \ldots, n$ such that $S'_{g_i} = S_{g_i}$, and such that $g_1, \ldots, g_n$ generate the unit ideal in S (in other words the standard opens of $Spec(S)$ associated to $g_1, \ldots, g_n$ cover all of $Spec(S)$).

Suppose that $D(g) \subset Spec(S')$ is contained in the image. Then $D(g) \subset \bigcup D(g_i)$. In other words, $g_1, \ldots, g_n$ generate the unit ideal of S'_g. Note that $S'_{gg_i} \cong S_{gg_i}$ by our choice of g_i. Hence $S'_g \cong S_g$ by Lemma 7.21.2.

We construct a finite algebra $S'' \subset S'$ as in (3). To do this note that each $S'_{g_i} \cong S_{g_i}$ is a finite type R-algebra. For each i pick some elements $y_{ij} \in S'$ such that each S'_{g_i} is generated as R-algebra by $1/g_i$ and the elements y_{ij}. Then set S'' equal to the sub R-algebra of S' generated by all g_i and all the y_{ij}. Details omitted. □

7.115. Applications of Zariski's Main Theorem

Here is an immediate application characterizing the finite maps of 1-dimensional semi-local rings among the quasi-finite ones as those where equality always holds in the formula of Lemma 7.112.8.

Lemma 7.115.1. *Let $A \subset B$ be an extension of domains. Assume*

1. *A is a local Noetherian ring of dimension 1,*
2. *$A \to B$ is of finite type, and*
3. *the extension $K = f.f.(A) \subset L = f.f.(B)$ is a finite field extension.*

Then B is semi-local. Let $x \in \mathfrak{m}_A$, $x \neq 0$. Let $\mathfrak{m}_i$, $i = 1, \ldots, n$ be the maximal ideals of B. Then

$$[L : K]ord_A(x) \geq \sum\nolimits_i [\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m}_A)]ord_{B_{\mathfrak{m}_i}}(x)$$

where ord is defined as in Definition 7.112.2. We have equality if and only if $A \to B$ is finite.

Proof. The ring B is semi-local by Lemma 7.104.2. Let B' be the integral closure of A in B. By Lemma 7.114.15 we can find a finite A-subalgebra $C \subset B'$ such that on setting $\mathfrak{n}_i = C \cap \mathfrak{m}_i$ we have $C_{\mathfrak{n}_i} \cong B_{\mathfrak{m}_i}$ and the primes $\mathfrak{n}_1, \ldots, \mathfrak{n}_n$ are pairwise distinct. The ring C is semi-local by Lemma 7.104.2. Let $\mathfrak{p}_j$, $j = 1, \ldots, m$ be the other maximal ideals of C (the ``missing points"). By Lemma 7.112.8 we have

$$\mathrm{ord}_A(x^{[L:K]}) = \sum\nolimits_i [\kappa(\mathfrak{n}_i) : \kappa(\mathfrak{m}_A)]\mathrm{ord}_{C_{\mathfrak{n}_i}}(x) + \sum\nolimits_j [\kappa(\mathfrak{p}_j) : \kappa(\mathfrak{m}_A)]\mathrm{ord}_{C_{\mathfrak{p}_j}}(x)$$

hence the inequality follows. In case of equality we conclude that $m = 0$ (no ``missing points"). Hence $C \subset B$ is an inclusion of semi-local rings inducing a bijection on maximal ideals and an isomorphism on all localizations at maximal ideals. So if $b \in B$, then $I = \{x \in C \mid xb \in C\}$ is an ideal of C which is not contained in any of the maximal ideals of C, and hence $I = C$, hence $b \in C$. Thus $B = C$ and B is finite over A. □

Here is a more standard application of Zariski's main theorem to the structure of local homomorphisms of local rings.

Lemma 7.115.2. *Let $(R, \mathfrak{m}_R) \to (S, \mathfrak{m}_S)$ be a local homomorphism of local rings. Assume*

1. *$R \to S$ is essentially of finite type,*
2. *$\kappa(\mathfrak{m}_R) \subset \kappa(\mathfrak{m}_S)$ is finite, and*
3. *$\dim(S/\mathfrak{m}_R S) = 0$.*

Then S is the localization of a finite R-algebra.

Proof. Let S' be a finite type R-algebra such that $S = S'_{\mathfrak{q}'}$ for some prime $\mathfrak{q}'$ of S'. By Definition 7.113.3 we see that $R \to S'$ is quasi-finite at $\mathfrak{q}'$. After replacing S' by $S'_{g'}$ for some $g' \in S'$, $g' \notin \mathfrak{q}'$ we may assume that $R \to S'$ is quasi-finite, see Lemma 7.114.14. Then by Lemma 7.114.15 there exists a finite R-algebra S'' and elements $g' \in S'$, $g' \notin \mathfrak{q}'$ and $g'' \in S''$ such that $S'_{g'} \cong S''_{g''}$ as R-algebras. This proves the lemma. □

Lemma 7.115.3. *Let $R \to S$ be a ring map, $\mathfrak{q}$ a prime of S lying over $\mathfrak{p}$ in R. If*

1. *R is Noetherian,*
2. *$R \to S$ is of finite type, and*
3. *$R \to S$ is quasi-finite at $\mathfrak{q}$,*

then $R_{\mathfrak{p}}^\wedge \otimes_R S = S_{\mathfrak{q}}^\wedge \times B$ for some $R_{\mathfrak{p}}^\wedge$-algebra B.

Proof. There exists a finite R-algebra $S' \subset S$ and an element $g \in S'$, $g \not\in \mathfrak{q}' = S' \cap \mathfrak{q}$ such that $S'_g = S_g$ and in particular $S'_{\mathfrak{q}'} = S_{\mathfrak{q}}$, see Lemma 7.114.15. We have

$$R_{\mathfrak{p}}^\wedge \otimes_R S' = (S'_{\mathfrak{q}'})^\wedge \times B'$$

by Lemma 7.90.17. Note that we have a commutative diagram

$$\begin{tikzcd}
R_{\mathfrak{p}}^\wedge \otimes_R S \arrow[r] & S_{\mathfrak{q}}^\wedge \\
R_{\mathfrak{p}}^\wedge \otimes_R S' \arrow[r] \arrow[u] & (S'_{\mathfrak{q}'})^\wedge \arrow[u]
\end{tikzcd}$$

where the right vertical is an isomorphism and the lower horizontal arrow is the projection map of the product decomposition above. The lemma follows. □

7.116. Dimension of fibres

We study the behaviour of dimensions of fibres, using Zariski's main theorem. Recall that we defined the dimension $\dim_x(X)$ of a topological space X at a point x in Topology, Definition 5.7.1.

Definition 7.116.1. Suppose that $R \to S$ is of finite type, and let $\mathfrak{q} \subset S$ be a prime lying over a prime $\mathfrak{p}$ of R. We define the *relative dimension of S/R at* $\mathfrak{q}$, denoted $\dim_{\mathfrak{q}}(S/R)$, to be the dimension of $Spec(S \otimes_R \kappa(\mathfrak{p}))$ at the point corresponding to $\mathfrak{q}$. We let $\dim(S/R)$ be the supremum of $\dim_{\mathfrak{q}}(S/R)$ over all $\mathfrak{q}$. This is called the *relative dimension of S/R*.

In particular, $R \to S$ is quasi-finite at $\mathfrak{q}$ if and only if $\dim_{\mathfrak{q}}(S/R) = 0$. The following lemma is more or less a reformulation of Zariski's Main Theorem.

Lemma 7.116.2. *Let $R \to S$ be a finite type ring map. Let $\mathfrak{q} \subset S$ be a prime. Suppose that $\dim_{\mathfrak{q}}(S/R) = n$. There exists a $g \in S$, $g \not\in \mathfrak{q}$ such that S_g is quasi-finite over a polynomial algebra $R[t_1, \ldots, t_n]$.*

Proof. The ring $\overline{S} = S \otimes_R \kappa(\mathfrak{p})$ is of finite type over $\kappa(\mathfrak{p})$. Let $\overline{\mathfrak{q}}$ be the prime of $\overline{S}$ corresponding to $\mathfrak{q}$. By definition of the dimension of a topological space at a point there exists an open $U \subset Spec(\overline{S})$ with $\overline{q} \in U$ and $\dim(U) = n$. Since the topology on $Spec(\overline{S})$ is induced from the topology on $Spec(S)$ (see Remark 7.16.8), we can find a $g \in S$, $g \not\in \mathfrak{q}$ with image $\overline{g} \in \overline{S}$ such that $D(\overline{g}) \subset U$. Thus after replacing S by S_g we see that $\dim(\overline{S}) = n$.

Next, choose generators $x_1, \ldots, x_N$ for S as an R-algebra. By Lemma 7.106.4 there exist elements $y_1, \ldots, y_n$ in the $\mathbf{Z}$-subalgebra of S generated by $x_1, \ldots, x_N$ such that the map $R[t_1, \ldots, t_n] \to S$, $t_i \mapsto y_i$ has the property that $\kappa(\mathfrak{p})[t_1 \ldots, t_n] \to \overline{S}$ is finite. In particular, S is quasi-finite over $R[t_1, \ldots, t_n]$ at $\mathfrak{q}$. Hence, by Lemma 7.114.14 we may replace S by S_g for some $g \in S$, $g \not\in \mathfrak{q}$ such that $R[t_1, \ldots, t_n] \to S$ is quasi-finite. □

Lemma 7.116.3. *Let $R \to S$ be a ring map. Let $\mathfrak{q} \subset S$ be a prime lying over the prime $\mathfrak{p}$ of R. Assume*

(1) *$R \to S$ is of finite type,*
(2) *$\dim_{\mathfrak{q}}(S/R) = n$, and*
(3) *$trdeg_{\kappa(\mathfrak{p})}\kappa(\mathfrak{q}) = r$.*

Then there exist $f \in R$, $f \not\in \mathfrak{p}$, $g \in S$, $g \not\in \mathfrak{q}$ *and a quasi-finite ring map*

$$\varphi : R_f[x_1, \ldots, x_n] \longrightarrow S_g$$

such that $\varphi^{-1}(\mathfrak{q}S_g) = (\mathfrak{p}, x_{r+1}, \ldots, x_n)R_f[x_{r+1}, \ldots, x_n]$

Proof. After replacing S by a principal localization we may assume there exists a quasi-finite ring map $\varphi : R[t_1, \ldots, t_n] \to S$, see Lemma 7.116.2. Set $\mathfrak{q}' = \varphi^{-1}(\mathfrak{q})$. Let $\overline{\mathfrak{q}}' \subset \kappa(\mathfrak{p})[t_1, \ldots, t_n]$ be the prime corresponding to $\mathfrak{q}'$. By Lemma 7.106.6 there exists a finite ring map $\kappa(\mathfrak{p})[x_1, \ldots, x_n] \to \kappa(\mathfrak{p})[t_1, \ldots, t_n]$ such that the inverse image of $\overline{\mathfrak{q}}'$ is $(x_{r+1}, \ldots, x_n)$. Let $\overline{h}_i \in \kappa(\mathfrak{p})[t_1, \ldots, t_n]$ be the image of x_i. We can find an element $f \in R$, $f \not\in \mathfrak{p}$ and $h_i \in R_f[t_1, \ldots, t_n]$ which map to $\overline{h}_i$ in $\kappa(\mathfrak{p})[t_1, \ldots, t_n]$. Then the ring map

$$R_f[x_1, \ldots, x_n] \longrightarrow R_f[t_1, \ldots, t_n]$$

becomes finite after tensoring with $\kappa(\mathfrak{p})$. In particular, $R_f[t_1, \ldots, t_n]$ is quasi-finite over $R_f[x_1, \ldots, x_n]$ at the prime $\mathfrak{q}'R_f[t_1, \ldots, t_n]$. Hence, by Lemma 7.114.14 there exists a $g \in R_f[t_1, \ldots, t_n]$, $g \not\in \mathfrak{q}'R_f[t_1, \ldots, t_n]$ such that $R_f[x_1, \ldots, x_n] \to R_f[t_1, \ldots, t_n, 1/g]$ is quasi-finite. Thus we see that the composition

$$R_f[x_1, \ldots, x_n] \longrightarrow R_f[t_1, \ldots, t_n, 1/g] \longrightarrow S_{\varphi(g)}$$

is quasi-finite and we win. □

Lemma 7.116.4. *Let* $R \to S$ *be a finite type ring map. Let* $\mathfrak{q} \subset S$ *be a prime lying over* $\mathfrak{p} \subset R$*. If* $R \to S$ *is quasi-finite at* $\mathfrak{q}$*, then* $\dim(S_{\mathfrak{q}}) \leq \dim(R_{\mathfrak{p}})$.

Proof. If $R_{\mathfrak{p}}$ is Noetherian (and hence $S_{\mathfrak{q}}$ Noetherian since it is essentially of finite type over $R_{\mathfrak{p}}$) then this follows immediately from Lemma 7.103.6 and the definitions. In the general case we can use Zariski's Main Theorem 7.114.13 to write $S_{\mathfrak{q}} = S'_{\mathfrak{q}'}$ for some ring S' integral over $R_{\mathfrak{p}}$. Thus the result follows from Lemma 7.103.3. □

Lemma 7.116.5. *Let* k *be a field. Let* S *be a finite type* k*-algebra. Suppose there is a quasi-finite* k*-algebra map* $k[t_1, \ldots, t_n] \subset S$*. Then* $\dim(S) \leq n$.

Proof. By Lemma 7.105.1 the dimension of any local ring of $k[t_1, \ldots, t_n]$ is at most n. Thus the result follows from Lemma 7.116.4 above. □

Lemma 7.116.6. *Let* $R \to S$ *be a finite type ring map. Let* $\mathfrak{q} \subset S$ *be a prime. Suppose that* $\dim_{\mathfrak{q}}(S/R) = n$*. There exists an open neighbourhood* V *of* $\mathfrak{q}$ *in* $Spec(S)$ *such that* $\dim_{\mathfrak{q}'}(S/R) \leq n$ *for all* $\mathfrak{q}' \in V$.

Proof. By Lemma 7.116.2 we see that we may assume that S is quasi-finite over a polynomial algebra $R[t_1, \ldots, t_n]$. Considering the fibres, we reduce to Lemma 7.116.5. □

In other words, the lemma says that the set of points where the fibre has dimension $\leq n$ is open in $Spec(S)$. The next lemma says that formation of this open commutes with base change. If the ring map is of finite presentation then this set is quasi-compact open (see below).

Lemma 7.116.7. *Let* $R \to S$ *be a finite type ring map. Let* $R \to R'$ *be any ring map. Set* $S' = R' \otimes_R S$ *and denote* $f : Spec(S') \to Spec(S)$ *the associated map on spectra. Let* $n \geq 0$*. The inverse image* $f^{-1}(\{\mathfrak{q} \in Spec(S) \mid \dim_{\mathfrak{q}}(S/R) \leq n\})$ *is equal to* $\{\mathfrak{q}' \in Spec(S') \mid \dim_{\mathfrak{q}'}(S'/R') \leq n\}$.

Proof. The condition is formulated in terms of dimensions of fibre rings which are of finite type over a field. Combined with Lemma 7.107.6 this yields the lemma. □

Lemma 7.116.8. *Let $R \to S$ be a ring homomorphism of finite presentation. Let $n \geq 0$. The set*
$$V_n = \{\mathfrak{q} \in Spec(S) \mid \dim_{\mathfrak{q}}(S/R) \leq n\}$$
is a quasi-compact open subset of $Spec(S)$.

Proof. It is open by Lemma 7.116.6 above. Let $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ be a presentation of S. Let R_0 be the $\mathbf{Z}$-subalgebra of R generated by the coefficients of the polynomials f_i. Let $S_0 = R_0[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. Then $S = R \otimes_{R_0} S_0$. By Lemma 7.116.7 V_n is the inverse image of an open $V_{0,n}$ under the quasi-compact continuous map $Spec(S) \to Spec(S_0)$. Since S_0 is Noetherian we see that $V_{0,n}$ is quasi-compact. ☐

Lemma 7.116.9. *Let R be a valuation ring with residue field k and field of fractions K. Let S be a domain containing R such that S is of finite type over R. If $S \otimes_R k$ is not the zero ring then*
$$\dim(S \otimes_R k) = \dim(S \otimes_R K)$$
In fact, $Spec(S \otimes_R k)$ is equidimensional.

Proof. It suffices to show that $\dim_{\mathfrak{q}}(S/k)$ is equal to $\dim(S \otimes_R K)$ for every prime $\mathfrak{q}$ of S containing $\mathfrak{m}_R S$. Pick such a prime. By Lemma 7.116.6 the inequality $\dim_{\mathfrak{q}}(S/k) \geq \dim(S \otimes_R K)$ holds. Set $n = \dim_{\mathfrak{q}}(S/k)$. By Lemma 7.116.2 after replacing S by S_g for some $g \in S, g \not\in \mathfrak{q}$ there exists a quasi-finite ring map $R[t_1, \ldots, t_n] \to S$. If $\dim(S \otimes_R K) < n$, then $K[t_1, \ldots, t_n] \to S \otimes_R K$ has a nonzero kernel. Say $f = \sum a_I t_1^{i_1} \ldots t_n^{i_n}$. After dividing f by a nonzero coefficient of f with minimal valuation, we may assume $f \in R[t_1, \ldots, t_n]$ and some a_I does not map to zero in k. Hence the ring map $k[t_1, \ldots, t_n] \to S \otimes_R k$ has a nonzero kernel which implies that $\dim(S \otimes_R k) < n$. Contradiction. ☐

7.117. Algebras and modules of finite presentation

In this section we dicuss some standard results where the key feature is that the assumption involves a finite type or finite presentation assumption.

Lemma 7.117.1. *Let $R \to S$ be a ring map. Let $R \to R'$ be a faithfully flat ring map. Set $S' = R' \otimes_R S$. Then $R \to S$ is of finite type if and only if $R' \to S'$ is of finite type.*

Proof. It is clear that if $R \to S$ is of finite type then $R' \to S'$ is of finite type. Assume that $R' \to S'$ is of finite type. Say $y_1, \ldots, y_m$ generate S' over R'. Write $y_j = \sum_i a_{ij} \otimes x_{ji}$ for some $a_{ij} \in R'$ and $x_{ji} \in S$. Let $A \subset S$ be the R-subalgebra generated by the x_{ij}. By flatness we have $A' := R' \otimes_R A \subset S'$, and by construction $y_j \in A'$. Hence $A' = S'$. By faithful flatness $A = S$. ☐

Lemma 7.117.2. *Let $R \to S$ be a ring map. Let $R \to R'$ be a faithfully flat ring map. Set $S' = R' \otimes_R S$. Then $R \to S$ is of finite presentation if and only if $R' \to S'$ is of finite presentation.*

Proof. It is clear that if $R \to S$ is of finite presentation then $R' \to S'$ is of finite presentation. Assume that $R' \to S'$ is of finite presentation. By Lemma 7.117.1 above we see that $R \to S$ is of finite type. Write $S = R[x_1, \ldots, x_n]/I$. By flatness $S' = R'[x_1, \ldots, x_n]/R' \otimes I$. Say $g_1, \ldots, g_m$ generate $R' \otimes I$ over $R'[x_1, \ldots, x_n]$. Write $g_j = \sum_i a_{ij} \otimes f_{ji}$ for some $a_{ij} \in R'$ and $f_{ji} \in I$. Let $J \subset I$ be the ideal generated by the f_{ij}. By flatness we have $R' \otimes_R J \subset R' \otimes_R I$, and both are ideals over $R'[x_1, \ldots, x_n]$. By construction $g_j \in R' \otimes_R J$. Hence $R' \otimes_R J = R' \otimes_R I$. By faithful flatness $J = I$. ☐

Lemma 7.117.3. *Let R be a ring. Let $I \subset R$ be an ideal. Let $S \subset R$ be a multiplicative subset. Set $R' = S^{-1}(R/I) = S^{-1}R/S^{-1}I$.*

(1) *For any finite R'-module M' there exists a finite R-module M such that $S^{-1}(M/IM) \cong M'$.*
(2) *For any finitely presented R'-module M' there exists a finitely presented R-module M such that $S^{-1}(M/IM) \cong M'$.*

Proof. Proof of (1). Choose a short exact sequence $0 \to K' \to (R')^{\oplus n} \to M' \to 0$. Let $K \subset R^{\oplus n}$ be the inverse image of K' under the map $R^{\oplus n} \to (R')^{\oplus n}$. Then $M = R^{\oplus n}/K$ works.

Proof of (2). Choose a presentation $(R')^{\oplus m} \to (R')^{\oplus n} \to M' \to 0$. Suppose that the first map is given by the matrix $A' = (a'_{ij})$ and the second map is determined by generators $x'_i \in M'$, $i = 1, \ldots, n$. As $R' = S^{-1}(R/I)$ we can choose $s \in S$ and a matrix $A = (a_{ij})$ with coefficients in R such that $a'_{ij} = a_{ij}/s \bmod S^{-1}I$. Let M be the finitely presented R-module with presentation $R^{\oplus m} \to R^{\oplus n} \to M \to 0$ where the first map is given by the matrix A and the second map is determined by generators $x_i \in M$, $i = 1, \ldots, n$. Then the map $M \to M'$, $x_i \mapsto x'_i$ induces an isomorphism $S^{-1}(M/IM) \cong M'$. □

Lemma 7.117.4. *Let R be a ring. Let $S \subset R$ be a multiplicative subset. Let M be an R-module.*

(1) *If $S^{-1}M$ is a finite $S^{-1}R$-module then there exists a finite R-module M' and a map $M' \to M$ which induces an isomorphism $S^{-1}M' \to S^{-1}M$.*
(2) *If $S^{-1}M$ is a finitely presented $S^{-1}R$-module then there exists an R-module M' of finite presentation and a map $M' \to M$ which induces an isomorphism $S^{-1}M' \to S^{-1}M$.*

Proof. Proof of (1). Let $x_1, \ldots, x_n \in M$ be elements which generate $S^{-1}M$ as an $S^{-1}R$-module. Let M' be the R-submodule of M generated by $x_1, \ldots, x_n$.

Proof of (2). Let $x_1, \ldots, x_n \in M$ be elements which generate $S^{-1}M$ as an $S^{-1}R$-module. Let $K = \mathrm{Ker}(R^{\oplus n} \to M)$ where the map is given by the rule $(a_1, \ldots, a_n) \mapsto \sum a_i x_i$. By Lemma 7.5.3 we see that $S^{-1}K$ is a finite $S^{-1}R$-module. By (1) we can find a finite type submodule $K' \subset K$ with $S^{-1}K' = S^{-1}K$. Take $M' = \mathrm{Coker}(K' \to R^{\oplus n})$. □

Lemma 7.117.5. *Let R be a ring. Let $\mathfrak{p} \subset R$ be a prime ideal. Let M be an R-module.*

(1) *If $M_{\mathfrak{p}}$ is a finite $R_{\mathfrak{p}}$-module then there exists a finite R-module M' and a map $M' \to M$ which induces an isomorphism $M'_{\mathfrak{p}} \to M_{\mathfrak{p}}$.*
(2) *If $M_{\mathfrak{p}}$ is a finitely presented $R_{\mathfrak{p}}$-module then there exists an R-module M' of finite presentation and a map $M' \to M$ which induces an isomorphism $M'_{\mathfrak{p}} \to M_{\mathfrak{p}}$.*

Proof. This is a special case of Lemma 7.117.4 □

Lemma 7.117.6. *Let R be a ring. Let $\varphi : S' \to S$ be a homomorphism of R-algebras. Assume*

(1) *S' is of finite type over R,*
(2) *S is of finite presentation over R, and*
(3) *$\mathfrak{q}' \subset S'$ and $\mathfrak{q} \subset S$ are primes such that φ induces an isomorphism $S'_{\mathfrak{q}'} \cong S_{\mathfrak{q}}$.*

Then there exist $g \in S'$, $g \notin \mathfrak{q}'$ and such that φ induces an isomorphism $S'_g \cong S_{\varphi(g)}$.

Proof. By Lemma 7.6.2 the map $S' \to S$ is of finite presentation. Write $S = S'[y_1, \ldots, y_a]/(g_1, \ldots, g_b)$. We may, after replacing S' by S'_g and S by $S_{\varphi(g)}$ for a suitable g, assume that the elements y_j are in the image of φ. This implies that $S' \to S$ is surjective. Say $x_j \in S'$ maps to y_j. After further replacing S' by S'_g and S by $S_{\varphi(g)}$ for a suitable g we may assume the expressions $g_i(x_1, \ldots, x_a)$ are zero in S'. This means that $S' \to S$ is an isomorphism. □

Lemma 7.117.7. *Let R be a ring. Let S, S' be of finite presentation over R. Let $\mathfrak{q} \subset S$ and $\mathfrak{q}' \subset S'$ be primes. If $S_{\mathfrak{q}} \cong S_{\mathfrak{q}'}$ as R-algebras, then there exist $g \in S$, $g \not\in \mathfrak{q}$ and $g' \in S'$, $g' \not\in \mathfrak{q}'$ such that $S_g \cong S'_{g'}$ as R-algebras.*

Proof. Let $\psi : S_{\mathfrak{q}} \to S_{\mathfrak{q}'}$ be the isomorphism of the hypothesis of the lemma. Write $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$ and $S' = R[y_1, \ldots, y_m]/J$. For each $i = 1, \ldots, n$ choose a fraction h_i/g_i with $h_i, g_i \in R[y_1, \ldots, y_m]$ and $g_i \bmod J$ not in $\mathfrak{q}'$ which represents the image of x_i under ψ. After replacing S' by $S'_{g_1 \ldots g_n}$ and $R[y_1, \ldots, y_m, y_{m+1}]$ (mapping y_{m+1} to $1/(g_1 \ldots g_n)$) we may assume that $\psi(x_i)$ is the image of some $h_i \in R[y_1, \ldots, y_m]$. Consider the elements $f_j(h_1, \ldots, h_n) \in R[y_1, \ldots, y_m]$. Since ψ kills each f_j we see that there exists a $g \in R[y_1, \ldots, y_m]$, $g \bmod J \not\in \mathfrak{q}'$ such that $g f_j(h_1, \ldots, h_n) \in J$ for each $j = 1, \ldots, r$. After replacing S' by S'_g and $R[y_1, \ldots, y_m, y_{m+1}]$ as before we may assume that $f_j(h_1, \ldots, h_n) \in J$. Thus we obtain a ring map $S \to S'$, $x_i \mapsto h_i$ which induces ψ on local rings. We win by Lemma 7.117.6 above. □

7.118. Colimits and maps of finite presentation

In this section we prove some preliminary lemmas which will eventually help us prove result using absolute Noetherian reduction. we begin discussing how we will think about colimits in this section.

Let $(\Lambda, \geq)$ a partially ordered set. A system of rings over Λ is given by a ring R_λ for every $\lambda \in \Lambda$, and a morphism $R_\lambda \to R_\mu$ whenever $\lambda \leq \mu$. These morphisms have to satisfy the rule that $R_\lambda \to R_\mu \to R_\nu$ is equal to the map $R_\lambda \to R_\nu$ for all $\lambda \leq \mu \leq \nu$. See Categories, Section 4.19. We will often assume that $(I, \leq)$ is *directed*, which means that Λ is nonempty and given $\lambda, \mu \in \Lambda$ there exists a $\nu \in \Lambda$ with $\lambda \leq \nu$ and $\mu \leq \nu$. Recall that the colimit $colim_\lambda R_\lambda$ is sometimes called a ``direct limit" in this case (but we will not use this terminology).

Lemma 7.118.1. *Let $R \to A$ be a ring map. There exists a directed system A_λ of R-algebras of finite presentation such that $A = colim_\lambda A_\lambda$. If A is of finite type over R we may arrange it so that all the transition maps are surjective.*

Proof. Compare with the proof of Lemma 7.8.13. Consider any finite subset $S \subset A$, and any finite collection of polynomial relations E among the elements of S. So each $s \in S$ corresponds to $x_s \in A$ and each $e \in E$ consists of a polynomial $f_e \in R[X_s; s \in S]$ such that $f_e(x_s) = 0$. Let $A_{S,E} = R[X_s; s \in S]/(f_e; e \in E)$ which is a finitely presented R-algebra. There are canonical maps $A_{S,E} \to A$. If $S \subset S'$ and if the elements of E correspond, via the map $R[X_s; s \in S] \to R[X_s; s \in S']$, to a subset of E', then there is an obvious map $A_{S,E} \to A_{S',E'}$ commuting with the maps to A. Thus, setting Λ equal the set of pairs (S, E) with ordering by inclusion as above, we get a directed partially ordered set. It is clear that the colimit of this directed system is A.

For the last statement, suppose $A = R[x_1, \ldots, x_n]/I$. In this case, consider the subset $\Lambda' \subset \Lambda$ consisting of those systems (S, E) above with $S = \{x_1, \ldots, x_n\}$. It is easy to see that still $A = colim_{\lambda' \in \Lambda'} A_{\lambda'}$. Moreover, the transition maps are clearly surjective. □

It turns out that we can characterize ring maps of finite presentation as follows. This in some sense says that the algebras of finite presentation are the ``compact'' objects in the category of R-algebras.

Lemma 7.118.2. *Let $\varphi : R \to S$ be a ring map. Then φ is of finite presentation if and only if for every directed system A_λ of R-algebras we have*

$$colim_\lambda\, Hom_R(S, A_\lambda) = Hom_R(S, colim_\lambda\, A_\lambda)$$

Proof. Suppose $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. If $\chi : S \to colim\, A_\lambda$ is a map, then each x_i maps to some element in the image of some A_{λ_i}. We may pick $\mu \geq \lambda_i$, $i = 1, \ldots, n$ and assume $\chi(x_i)$ is the image of $y_i \in A_\mu$ for $i = 1, \ldots, n$. Consider $z_j = f_j(y_1, \ldots, y_n) \in A_\mu$. Since χ is a homomorphism the image of z_j in $colim_\lambda\, A_\lambda$ is zero. Hence there exists a $\mu_j \geq \mu$ such that z_j maps to zero in A_{μ_j}. Pick $\nu \geq \mu_j$, $j = 1, \ldots, m$. Then the images of $z_1, \ldots, z_m$ are zero in A_ν. This exactly means that the y_i map to elements $y'_i \in A_\nu$ which satsify the relations $f_j(y'_1, \ldots, y'_n) = 0$. Thus we obtain a ring map $S \to A_\nu$ as desired.

Conversely, suppose the displayed formula holds always. By Lemma 7.118.1 we may write $S = colim_\lambda\, S_\lambda$ with S_λ of finite presentation over R. Then the identity map factors as

$$S \to S_\lambda \to S$$

for some λ. Hence we see that S is finitely generated over R (because S_λ is). Thus we may choose the system such that all transition maps are surjective. In this case a factorization of the identity as above can only exist if $S = S_\lambda$. □

But more is true. Namely, given $R = colim_\lambda\, R_\lambda$ we see that the category of finitely presented R-modules is equivalent to the limit of the category of finitely presented R_λ-modules. Similarly for the categories of finitely presented R-algebras.

Lemma 7.118.3. *Let A be a ring and let M, N be A-modules. Suppose that $R = colim_{i\in I}\, R_i$ is a directed colimit of A-algebras.*

1. *If M is a finite A-module, and $u, u' : M \to N$ are A-module maps such that $u \otimes 1 = u' \otimes 1 : M \otimes_A R \to N \otimes_A R$ then for some i we have $u \otimes 1 = u' \otimes 1 : M \otimes_A R_i \to N \otimes_A R_i$.*
2. *If N is a finite A-module and $u : M \to N$ is an A-module map such that $u \otimes 1 : M \otimes_A R \to N \otimes_A R$ is surjective, then for some i the map $u \otimes 1 : M \otimes_A R_i \to N \otimes_A R_i$ is surjective.*
3. *If N is a finitely presented A-module, and $v : N \otimes_A R \to M \otimes_A R$ is an R-module map, then there exists an i and an R_i-module map $v_i : N \otimes_A R_i \to M \otimes_A R_i$ such that $v = v_i \otimes 1$.*
4. *If M is a finite A-module, N is a finitely presented A-module, and $u : M \to N$ is an R-module map such that $u \otimes 1 : M \otimes_A R \to N \otimes_A R$ is an isomorphism, then for some i the map $u \otimes 1 : M \otimes_A R_i \to N \otimes_A R_i$ is an isomorphism.*

Proof. To prove (1) assume u is as in (1) and let $x_1, \ldots, x_m \in M$ be generators. Since $N \otimes_A R = colim_i\, N \otimes_A R_i$ we may pick an $i \in I$ such that $u(x_j) \otimes 1 = u'(x_j) \otimes 1$ in $M \otimes_A R_i$, $j = 1, \ldots, m$. For such an i we have $u \otimes 1 = u' \otimes 1 : M \otimes_A R_i \to N \otimes_A R_i$.

To prove (2) assume $u \otimes 1$ surjective and let $y_1, \ldots, y_m \in N$ be generators. Since $N \otimes_A R = colim_i\, N \otimes_A R_i$ we may pick an $i \in I$ and $z_j \in M \otimes_A R_i$, $j = 1, \ldots, m$ whose images in $N \otimes_A R$ equal $y_j \otimes 1$. For such an i the map $u \otimes 1 : M \otimes_A R_i \to N \otimes_A R_i$ is surjective.

To prove (3) let $y_1, \ldots, y_m \in N$ be generators. Let $K = \text{Ker}(A^{\oplus m} \to N)$ where the map is given by the rule $(a_1, \ldots, a_m) \mapsto \sum a_j x_j$. Let $k_1, \ldots, k_t$ be generators for K. Say $k_s =$

$(k_{s1}, \ldots, k_{sm})$. Since $M \otimes_A R = colim_i\, M \otimes_A R_i$ we may pick an $i \in I$ and $z_j \in M \otimes_A R_i$, $j = 1, \ldots, m$ whose images in $M \otimes_A R$ equal $v(y_j \otimes 1)$. We want to use the z_j to define the map $v_i : N \otimes_A R_i \to M \otimes_A R_i$. Since $K \otimes_A R_i \to R_i^{\oplus m} \to N \otimes_A R_i \to 0$ is a presentation, it suffices to check that $\xi_s = \sum_j k_{sj} z_j$ is zero in $M \otimes_A R_i$ for each $s = 1, \ldots, t$. This may not be the case, but since the image of ξ_s in $M \otimes_A R$ is zero we see that it will be the case after increasing i a bit.

To prove (4) assume $u \otimes 1$ is an isomorphism, that M is finite, and that N is finitely presented. Let $v : N \otimes_A R \to M \otimes_A R$ be an inverse to $u \otimes 1$. Apply part (3) to get a map $v_i : N \otimes_A R_i \to M \otimes_A R_i$ for some i. Apply part (1) to see that, after increasing i we have $v_i \circ (u \otimes 1) = \mathrm{id}_{M \otimes_R R_i}$ and $(u \otimes 1) \circ v_i = \mathrm{id}_{N \otimes_R R_i}$. □

Lemma 7.118.4. *Suppose that $R = colim_{i \in I} R_i$ is a directed colimit of rings. Then the category of finitely presented R-modules is the colimit of the categories of finitely presented R_λ-modules. More precisely*

1. *Given a finitely presented R-module M there exists a $\lambda \in \Lambda$ and a finitely presented R_λ-module M_λ such that $M \cong M_\lambda \otimes_{R_\lambda} R$.*
2. *Given a $\lambda \in \Lambda$, finitely presented R_λ-modules M_λ, N_λ, and an R-module map $\varphi : M_\lambda \otimes_{R_\lambda} R \to N_\lambda \otimes_{R_\lambda} R$, then there exists a $\mu \geq \lambda$ and an R_μ-module map $\varphi_\mu : M_\lambda \otimes_{R_\lambda} R_\mu \to N_\lambda \otimes_{R_\lambda} R_\mu$ such that $\varphi = \varphi_\mu \otimes 1_R$.*
3. *Given a $\lambda \in \Lambda$, finitely presented R_λ-modules M_λ, N_λ, and R-module maps $\varphi_\lambda, \psi_\lambda : M_\lambda \to N_\lambda$ such that $\varphi \otimes 1_R = \psi \otimes 1_R$, then $\varphi \otimes 1_{R_\mu} = \psi \otimes 1_{R_\mu}$ for some $\mu \geq \lambda$.*

Proof. To prove (1) choose a presentation $R^{\oplus m} \to R^{\oplus n} \to M \to 0$. Suppose that the first map is given by the matrix $A = (a_{ij})$. We can choose a $\lambda \in \Lambda$ and a matrix $A_\lambda = (a_{\lambda, ij})$ with coefficients in R_λ which maps to A in R. Then we simply let M_λ be the R_λ-module with presentation $R_\lambda^{\oplus m} \to R_\lambda^{\oplus n} \to M_\lambda \to 0$ where the first arrow is given by A_λ.

Parts (3) and (4) follow from Lemma 7.118.3. □

Lemma 7.118.5. *Let A be a ring and let B, C be A-algebras. Suppose that $R = colim_{i \in I} R_i$ is a directed colimit of A-algebras.*

1. *If B is a finite type A-algebra, and $u, u' : B \to C$ are A-algebra maps such that $u \otimes 1 = u' \otimes 1 : B \otimes_A R \to C \otimes_A R$ then for some i we have $u \otimes 1 = u' \otimes 1 : B \otimes_A R_i \to C \otimes_A R_i$.*
2. *If C is a finite type A-algebra and $u : B \to C$ is an A-algebra map such that $u \otimes 1 : B \otimes_A R \to C \otimes_A R$ is surjective, then for some i the map $u \otimes 1 : B \otimes_A R_i \to C \otimes_A R_i$ is surjective.*
3. *If C is of finite presentation over A and $v : C \otimes_A R \to B \otimes_A R$ is an R-algebra map, then there exists an i and an R_i-algebra map $v_i : C \otimes_A R_i \to B \otimes_A R)i$ such that $v = v_i \otimes 1$.*
4. *If B is a finite type A-algebra, C is a finitely presented A-algebra, and $u \otimes 1 : B \otimes_A R \to C \otimes_A R$ is an isomorphism, then for some i the map $u \otimes 1 : B \otimes_A R_i \to C \otimes_A R_i$ is an isomorphism.*

Proof. To prove (1) assume u is as in (1) and let $x_1, \ldots, x_m \in B$ be generators. Since $B \otimes_A R = colim_i\, B \otimes_A R_i$ we may pick an $i \in I$ such that $u(x_j) \otimes 1 = u'(x_j) \otimes 1$ in $B \otimes_A R_i$, $j = 1, \ldots, m$. For such an i we have $u \otimes 1 = u' \otimes 1 : B \otimes_A R_i \to C \otimes_A R_i$.

To prove (2) assume $u \otimes 1$ surjective and let $y_1, \ldots, y_m \in C$ be generators. Since $B \otimes_A R = colim_i\, B \otimes_A R_i$ we may pick an $i \in I$ and $z_j \in B \otimes_A R_i$, $j = 1, \ldots, m$ whose images in $C \otimes_A R$ equal $y_j \otimes 1$. For such an i the map $u \otimes 1 : B \otimes_A R_i \to C \otimes_A R_i$ is surjective.

To prove (3) let $c_1, \ldots, c_m \in C$ be generators. Let $K = \mathrm{Ker}(A[x_1, \ldots, x_m] \to N)$ where the map is given by the rule $x_j \mapsto \sum c_j$. Let $f_1, \ldots, f_t$ be generators for K as an ideal in $A[x_1, \ldots, x_m]$. We think of $f_j = f_j(x_1, \ldots, x_m)$ as a polynomial. Since $B \otimes_A R = colim_i\, B \otimes_A R_i$ we may pick an $i \in I$ and $z_j \in B \otimes_A R_i$, $j = 1, \ldots, m$ whose images in $B \otimes_A R$ equal $v(c_j \otimes 1)$. We want to use the z_j to define a map $v_i : C \otimes_A R_i \to B \otimes_A R_i$. Since $K \otimes_A R_i \to R_i[x_1, \ldots, x_m] \to C \otimes_A R_i \to 0$ is a presentation, it suffices to check that $\xi_s = f_j(z_1, \ldots, z_m)$ is zero in $B \otimes_A R_i$ for each $s = 1, \ldots, t$. This may not be the case, but since the image of ξ_s in $B \otimes_A R$ is zero we see that it will be the case after increasing i a bit.

To prove (4) assume $u \otimes 1$ is an isomorphism, that B is a finite type A-algebra, and that C is a finitely presented A-algebra. Let $v : B \otimes_A R \to C \otimes_A R$ be an inverse to $u \otimes 1$. let $v_i : C \otimes_A R_i \to B \otimes_A R_i$ be as in part (3). Apply part (1) to see that, after increasing i we have $v_i \circ (u \otimes 1) = \mathrm{id}_{B \otimes_R R_i}$ and $(u \otimes 1) \circ v_i = \mathrm{id}_{C \otimes_R R_i}$. □

Lemma 7.118.6. *Suppose that $R = colim_{i \in I} R_i$ is a directed colimit of rings. Then the category of finitely presented R-algebras is the colimit of the categories of finitely presented R_λ-algebras. More precisely*

1. *Given a finitely presented R-algebra A there exists a $\lambda \in \Lambda$ and a finitely presented R_λ-algebra A_λ such that $A \cong A_\lambda \otimes_{R_\lambda} R$.*
2. *Given a $\lambda \in \Lambda$, finitely presented R_λ-algebras A_λ, B_λ, and an R-algebra map $\varphi : A_\lambda \otimes_{R_\lambda} R \to B_\lambda \otimes_{R_\lambda} R$, then there exists a $\mu \geq \lambda$ and an R_μ-algebra map $\varphi_\mu : A_\lambda \otimes_{R_\lambda} R_\mu \to B_\lambda \otimes_{R_\lambda} R_\mu$ such that $\varphi = \varphi_\mu \otimes 1_R$.*
3. *Given a $\lambda \in \Lambda$, finitely presented R_λ-algebras A_λ, B_λ, and R-algebra maps $\varphi_\lambda, \psi_\lambda : A_\lambda \to B_\lambda$ such that $\varphi \otimes 1_R = \psi \otimes 1_R$, then $\varphi \otimes 1_{R_\mu} = \psi \otimes 1_{R_\mu}$ for some $\mu \geq \lambda$.*

Proof. To prove (1) choose a presentation $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. We can choose a $\lambda \in \Lambda$ and elements $f_{\lambda,j} \in R_\lambda[x_1, \ldots, x_n]$ mapping to $f_j \in R[x_1, \ldots, x_n]$. Then we simply let $A_\lambda = R_\lambda[x_1, \ldots, x_n]/(f_{\lambda,1}, \ldots, f_{\lambda,m})$.

Parts (3) and (4) follow from Lemma 7.118.5. □

Lemma 7.118.7. *Suppose $R \to S$ is a local homomorphism of local rings. There exists a directed set $(\Lambda, \leq)$, and a system of local homomorphisms $R_\lambda \to S_\lambda$ of local rings such that*

1. *The colimit of the system $R_\lambda \to S_\lambda$ is equal to $R \to S$.*
2. *Each R_λ is essentially of finite type over $\mathbf{Z}$.*
3. *Each S_λ is essentially of finite type over R_λ.*

Proof. Denote $\varphi : R \to S$ the ring map. Let $\mathfrak{m} \subset R$ be the maximal ideal of R and let $\mathfrak{n} \subset S$ be the maximal ideal of S. Let

$$\Lambda = \{(A, B) \mid A \subset R, B \subset S, \#A < \infty, \#B < \infty, \varphi(A) \subset B\}.$$

As partial ordering we take the inclusion relation. For each $\lambda = (A, B) \in \Lambda$ we let R'_λ be the sub $\mathbf{Z}$-algebra generated by $a \in A$, and we let S'_λ be the sub $\mathbf{Z}$-algebra generated by

b, $b \in B$. Let R_λ be the localization of R'_λ at the prime ideal $R'_\lambda \cap \mathfrak{m}$ and let S_λ be the localization of S'_λ at the prime ideal $S'_\lambda \cap \mathfrak{n}$. In a picture

$$\begin{tikzcd} B \arrow[r] & S'_\lambda \arrow[r] & S_\lambda \arrow[r] & S \\ A \arrow[r] \arrow[u] & R'_\lambda \arrow[r] \arrow[u] & R_\lambda \arrow[r] \arrow[u] & R \arrow[u] \end{tikzcd}$$

The transition maps are clear. We leave the proofs of the other assertions to the reader. □

Lemma 7.118.8. *Suppose $R \to S$ is a local homomorphism of local rings. Assume that S is essentially of finite type over R. Then there exists a directed set $(\Lambda, \leq)$, and a system of local homomorphisms $R_\lambda \to S_\lambda$ of local rings such that*

1. *The colimit of the system $R_\lambda \to S_\lambda$ is equal to $R \to S$.*
2. *Each R_λ is essentially of finite type over $\mathbf{Z}$.*
3. *Each S_λ is essentially of finite type over R_λ.*
4. *For each $\lambda \leq \mu$ the map $S_\lambda \otimes_{R_\lambda} R_\mu \to S_\mu$ presents S_μ as the localization of a quotient of $S_\lambda \otimes_{R_\lambda} R_\mu$.*

Proof. Denote $\varphi : R \to S$ the ring map. Let $\mathfrak{m} \subset R$ be the maximal ideal of R and let $\mathfrak{n} \subset S$ be the maximal ideal of S. Let $x_1, \ldots, x_n \in S$ be elements such that S is a localization of the sub R-algebra of S generated by $x_1, \ldots, x_n$. In other words, S is a quotient of a localization of the polynomial ring $R[x_1, \ldots, x_n]$.

Let $\Lambda = \{A \subset R \mid \#A < \infty\}$ be the set of finite subsets of R. As partial ordering we take the inclusion relation. For each $\lambda = A \in \Lambda$ we let R'_λ be the sub $\mathbf{Z}$-algebra generated by $a \in A$, and we let S'_λ be the sub $\mathbf{Z}$-algebra generated by $\varphi(a)$, $a \in A$ and the elements $x_1, \ldots, x_n$. Let R_λ be the localization of R'_λ at the prime ideal $R'_\lambda \cap \mathfrak{m}$ and let S_λ be the localization of S'_λ at the prime ideal $S'_\lambda \cap \mathfrak{n}$. In a picture

$$\begin{tikzcd} \varphi(A) \coprod \{x_i\} \arrow[r] & S'_\lambda \arrow[r] & S_\lambda \arrow[r] & S \\ A \arrow[r] \arrow[u] & R'_\lambda \arrow[r] \arrow[u] & R_\lambda \arrow[r] \arrow[u] & R \arrow[u] \end{tikzcd}$$

It is clear that if $A \subset B$ corresponds to $\lambda \leq \mu$ in Λ, then there are canonical maps $R_\lambda \to R_\mu$, and $S_\lambda \to S_\mu$ and we obtain a system over the directed set Λ.

The assertion that $R = colim\, R_\lambda$ is clear because all the maps $R_\lambda \to R$ are injective and any element of R eventually is in the image. The same argument works for $S = colim\, S_\lambda$. Assertions (2), (3) are true by construction. The final assertion holds because clearly the maps $S'_\lambda \otimes_{R'_\lambda} R'_\mu \to S'_\mu$ are surjective. □

Lemma 7.118.9. *Suppose $R \to S$ is a local homomorphism of local rings. Assume that S is essentially of finite presentation over R. Then there exists a directed set $(\Lambda, \leq)$, and a system of local homomorphism $R_\lambda \to S_\lambda$ of local rings such that*

1. *The colimit of the system $R_\lambda \to S_\lambda$ is equal to $R \to S$.*
2. *Each R_λ is essentially of finite type over $\mathbf{Z}$.*
3. *Each S_λ is essentially of finite type over R_λ.*
4. *For each $\lambda \leq \mu$ the map $S_\lambda \otimes_{R_\lambda} R_\mu \to S_\mu$ presents S_μ as the localization of $S_\lambda \otimes_{R_\lambda} R_\mu$ at a prime ideal.*

Proof. By assumption we may choose an isomorphism $\Phi : (R[x_1, \ldots, x_n]/I)_{\mathfrak q} \to S$ where $I \subset R[x_1, \ldots, x_n]$ is a finitely generated ideal, and $\mathfrak q \subset R[x_1, \ldots, x_n]/I$ is a prime. (Note that the pull back of $\mathfrak q$ to R is equal to the maximal ideal $\mathfrak m$ of R.) We also choose generators $f_1, \ldots, f_m \in I$ for the ideal I. Write R in any way as a colimit $R = colim\, R_\lambda$ over a directed set $(\Lambda, \leq)$, with each R_λ local and essentially of finite type over $\mathbf{Z}$. There exists some $\lambda_0 \in \Lambda$ such that f_j is the image of some $f_{j,\lambda_0} \in R_{\lambda_0}[x_1, \ldots, x_n]$. For all $\lambda \geq \lambda_0$ denote $f_{j,\lambda} \in R_\lambda[x_1, \ldots, x_n]$ the image of f_{j,λ_0}. Thus we obtain a system of ring maps

$$R_\lambda[x_1, \ldots, x_n]/(f_{1,\lambda}, \ldots, f_{n,\lambda}) \to R[x_1, \ldots, x_n]/(f_1, \ldots, f_n) \to S$$

Set $\mathfrak q_\lambda$ the inverse image of $\mathfrak q$. Set $S_\lambda = (R_\lambda[x_1, \ldots, x_n]/(f_{1,\lambda}, \ldots, f_{n,\lambda}))_{\mathfrak q_\lambda}$. We leave it to the reader to see that this works. ☐

Remark 7.118.10. Suppose that $R \to S$ is a local homomorphism of local rings, which is essentially of finite presentation. Take any system $(\Lambda, \leq)$, $R_\lambda \to S_\lambda$ with the properties listed in Lemma 7.118.8. What may happen is that this is the ``wrong'' system, namely, it may happen that property (4) of Lemma 7.118.9 is not satisfied. Here is an example. Let k be a field. Consider the ring

$$R = k[[z, y_1, y_2, \ldots]]/(y_i^2 - zy_{i+1}).$$

Set $S = R/zR$. As system take $\Lambda = \mathbf{N}$ and $R_n = k[[z, y_1, \ldots, y_n]]/(\{y_i^2 - zy_{i+1}\}_{i\leq n-1})$ and $S_n = R_n/(z, y_n^2)$. All the maps $S_n \otimes_{R_n} R_{n+1} \to S_{n+1}$ are not localizations (i.e., isomorphisms in this case) since $1 \otimes y_{n+1}^2$ maps to zero. If we take instead $S'_n = R_n/zR_n$ then the maps $S'_n \otimes_{R_n} R_{n+1} \to S'_{n+1}$ are isomorphisms. The moral of this remark is that we do have to be a little careful in choosing the systems.

Lemma 7.118.11. *Suppose $R \to S$ is a local homomorphism of local rings. Assume that S is essentially of finite presentation over R. Let M be a finitely presented S-module. Then there exists a directed set $(\Lambda, \leq)$, and a system of local homomorphisms $R_\lambda \to S_\lambda$ of local rings together with S_λ-modules M_λ, such that*

1. *The colimit of the system $R_\lambda \to S_\lambda$ is equal to $R \to S$. The colimit of the system M_λ is M.*
2. *Each R_λ is essentially of finite type over $\mathbf{Z}$.*
3. *Each S_λ is essentially of finite type over R_λ.*
4. *Each M_λ is finite over S_λ.*
5. *For each $\lambda \leq \mu$ the map $S_\lambda \otimes_{R_\lambda} R_\mu \to S_\mu$ presents S_μ as the localization of $S_\lambda \otimes_{R_\lambda} R_\mu$ at a prime ideal.*
6. *For each $\lambda \leq \mu$ the map $M_\lambda \otimes_{S_\lambda} S_\mu \to M_\mu$ is an isomorphism.*

Proof. As in the proof of Lemma 7.118.9 we may first write $R = colim\, R_\lambda$ as a directed colimit of local $\mathbf{Z}$-algebras which are essentially of finite type. Next, we may assume that for some $\lambda_1 \in \Lambda$ there exist $f_{j,\lambda_1} \in R_{\lambda_1}[x_1, \ldots, x_n]$ such that

$$S = colim_{\lambda \geq \lambda_1} S_\lambda, \text{ with } S_\lambda = (R_\lambda[x_1, \ldots, x_n]/(f_{1,\lambda}, \ldots, f_{m,\lambda}))_{\mathfrak q_\lambda}$$

Choose a presentation

$$S^{\oplus s} \to S^{\oplus t} \to M \to 0$$

of M over S. Let $A \in \mathrm{Mat}(t \times s, S)$ be the matrix of the presentation. For some $\lambda_2 \in \Lambda$, $\lambda_2 \geq \lambda_1$ we can find a matrix $A_{\lambda_2} \in \mathrm{Mat}(t \times s, S_{\lambda_2})$ which maps to A. For all $\lambda \geq \lambda_2$ we let $M_\lambda = \mathrm{Coker}(S_\lambda^{\oplus s} \xrightarrow{A_\lambda} S_\lambda^{\oplus t})$. We leave it to the reader to see that this works. ☐

Lemma 7.118.12. *Suppose $R \to S$ is a ring map. Then there exists a directed set $(\Lambda, \leq)$, and a system of ring maps $R_\lambda \to S_\lambda$ such that*

1. *The colimit of the system $R_\lambda \to S_\lambda$ is equal to $R \to S$.*
2. *Each R_λ is of finite type over $\mathbf{Z}$.*
3. *Each S_λ is of finite type over R_λ.*

Proof. This is the non-local version of Lemma 7.118.7. Proof is similar and left to the reader. □

Lemma 7.118.13. *Suppose $R \to S$ is a ring map. Assume that S is of finite type over R. Then there exists a directed set $(\Lambda, \leq)$, and a system of ring maps $R_\lambda \to S_\lambda$ such that*

1. *The colimit of the system $R_\lambda \to S_\lambda$ is equal to $R \to S$.*
2. *Each R_λ is of finite type over $\mathbf{Z}$.*
3. *Each S_λ is of finite type over R_λ.*
4. *For each $\lambda \leq \mu$ the map $S_\lambda \otimes_{R_\lambda} R_\mu \to S_\mu$ presents S_μ as a quotient of $S_\lambda \otimes_{R_\lambda} R_\mu$.*

Proof. This is the non-local version of Lemma 7.118.8. Proof is similar and left to the reader. □

Lemma 7.118.14. *Suppose $R \to S$ is a ring map. Assume that S is of finite presentation over R. Then there exists a directed set $(\Lambda, \leq)$, and a system of ring maps $R_\lambda \to S_\lambda$ such that*

1. *The colimit of the system $R_\lambda \to S_\lambda$ is equal to $R \to S$.*
2. *Each R_λ is of finite type over $\mathbf{Z}$.*
3. *Each S_λ is of finite type over R_λ.*
4. *For each $\lambda \leq \mu$ the map $S_\lambda \otimes_{R_\lambda} R_\mu \to S_\mu$ is an isomorphism.*

Proof. This is the non-local version of Lemma 7.118.9. Proof is similar and left to the reader. □

Lemma 7.118.15. *Suppose $R \to S$ is a ring map. Assume that S is of finite presentation over R. Let M be a finitely presented S-module. Then there exists a directed set $(\Lambda, \leq)$, and a system of ring maps $R_\lambda \to S_\lambda$ together with S_λ-modules M_λ, such that*

1. *The colimit of the system $R_\lambda \to S_\lambda$ is equal to $R \to S$. The colimit of the system M_λ is M.*
2. *Each R_λ is of finite type over $\mathbf{Z}$.*
3. *Each S_λ is of finite type over R_λ.*
4. *Each M_λ is finite over S_λ.*
5. *For each $\lambda \leq \mu$ the map $S_\lambda \otimes_{R_\lambda} R_\mu \to S_\mu$ is an isomorphism.*
6. *For each $\lambda \leq \mu$ the map $M_\lambda \otimes_{S_\lambda} S_\mu \to S_\mu$ is an isomorphism.*

In particular, for every $\lambda \in \Lambda$ we have

$$M = M_\lambda \otimes_{S_\lambda} S = M_\lambda \otimes_{R_\lambda} R.$$

Proof. This is the non-local version of Lemma 7.118.11. Proof is similar and left to the reader. □

7.119. More flatness criteria

The following lemma is often used in algebraic geometry to show that a finite morphism from a normal surface to a smooth surface is flat. It is a partial converse to Lemma 7.103.9 because a finite local ring map certainly satisfies condition (3).

Lemma 7.119.1. *Let $R \to S$ be a local homomorphism of Noetherian local rings. Assume*

1. *R is regular,*
2. *S Cohen-Macaulay,*
3. *$\mathfrak{m}_S = \sqrt{\mathfrak{m}_R S}$, and*
4. $\dim(R) = \dim(S)$.

Then $R \to S$ is flat.

Proof. By induction on $\dim(R)$. The case $\dim(R) = 0$ is trivial, because then R is a field. Assume $\dim(R) > 0$. By (4) this implies that $\dim(S) > 0$. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ be the minimal primes of S. Note that $\mathfrak{q}_i \neq \mathfrak{m}_S$, hence $\mathfrak{p}_i = R \cap \mathfrak{q}_i$ is not equal to $\mathfrak{m}_R$ by (3). Pick $x \in \mathfrak{m}$, $x \notin \mathfrak{m}^2$, and $x \notin \mathfrak{p}_i$, see Lemma 7.14.3. Hence we see that x is not contained in any of the minimal primes of S. Hence x is a nonzero divisor on S by (2), see Lemma 7.96.2 and S/xS is Cohen-Macaulay with $\dim(S/xS) = \dim(S) - 1$. By (1) and Lemma 7.98.3 the ring R/xR is regular with $\dim(R/xR) = \dim(R) - 1$. By induction we see that $R/xR \to S/xS$ is flat. Hence we conclude by Lemma 7.91.9 (see also the remarks following it). □

Lemma 7.119.2. *Let $R \to S$ be a homomorphism of Noetherian local rings. Assume that R is a regular local ring and that a regular system of parameters maps to a regular sequence in S. Then $R \to S$ is flat.*

Proof. Suppose that $x_1, \ldots, x_d$ are a system of parameters of R which map to a regular sequence in S. Note that $S/(x_1, \ldots, x_d)S$ is flat over $R/(x_1, \ldots, x_d)$ as the latter is a field. Then x_d is a nonzero divisor in $S/(x_1, \ldots, x_{d-1})S$ hence $S/(x_1, \ldots, x_{d-1})S$ is flat over $R/(x_1, \ldots, x_{d-1})$ by the local criterion of flatness (see Lemma 7.91.9 and remarks following). Then x_{d-1} is a nonzero divisor in $S/(x_1, \ldots, x_{d-2})S$ hence $S/(x_1, \ldots, x_{d-2})S$ is flat over $R/(x_1, \ldots, x_{d-2})$ by the local criterion of flatness (see Lemma 7.91.9 and remarks following). Continue till one reaches the conclusion that S is flat over R. □

The following lemma is the key to proving that results for finitely presented modules over finitely presented rings over a base ring follow from the corresponding results for finite modules in the Noetherian case.

Lemma 7.119.3. *Let $R \to S$, M, Λ, $R_\lambda \to S_\lambda$, M_λ be as in Lemma 7.118.11. Assume that M is flat over R. Then for some $\lambda \in \Lambda$ the module M_λ is flat over R_λ.*

Proof. Pick some $\lambda \in \Lambda$ and consider

$$\mathrm{Tor}_1^{R_\lambda}(M_\lambda, R_\lambda/\mathfrak{m}_\lambda) = \mathrm{Ker}(\mathfrak{m}_\lambda \otimes_{R_\lambda} M_\lambda \to M_\lambda).$$

See Remark 7.69.8. The right hand side shows that this is a finitely generated S_λ-module (because S_λ is Noetherian and the modules in question are finite). Let $\xi_1, \ldots, \xi_n$ be generators. Because M is flat over R we have that $0 = \mathrm{Ker}(\mathfrak{m}_\lambda R \otimes_R M \to M)$. Since $\otimes$ commutes with colimits we see there exists a $\lambda' \geq \lambda$ such that each ξ_i maps to zero in $\mathfrak{m}_\lambda R_{\lambda'} \otimes_{R_{\lambda'}} M_{\lambda'}$. Hence we see that

$$\mathrm{Tor}_1^{R_\lambda}(M_\lambda, R_\lambda/\mathfrak{m}_\lambda) \longrightarrow \mathrm{Tor}_1^{R_{\lambda'}}(M_\lambda, R_{\lambda'}/\mathfrak{m}_\lambda R_{\lambda'})$$

is zero. Note that $M_\lambda \otimes_{R_\lambda} R_\lambda/\mathfrak{m}_\lambda$ is flat over $R_\lambda/\mathfrak{m}_\lambda$ because this last ring is a field. Hence we may apply Lemma 7.91.13 to get that $M_{\lambda'}$ is flat over $R_{\lambda'}$. □

Using the lemma above we can start to reprove the results of Section 7.91 in the non-Noetherian case.

Lemma 7.119.4. *Suppose that $R \to S$ is a local homomorphism of local rings. Denote $\mathfrak{m}$ the maximal ideal of R. Let $u : M \to N$ be a map of S-modules. Assume*

(1) *S is essentially of finite presentation over R,*
(2) *M, N are finitely presented over S,*
(3) *N is flat over R, and*
(4) *$\bar{u} : M/\mathfrak{m}M \to N/\mathfrak{m}N$ is injective.*

Then u is injective, and $N/u(M)$ is flat over R.

Proof. By Lemma 7.118.11 and its proof we can find a system $R_\lambda \to S_\lambda$ of local ring maps together with maps of S_λ-modules $u_\lambda : M_\lambda \to N_\lambda$ satisfying the conclusions (1) -- (6) for both N and M of that lemma and such that the colimit of the maps u_λ is u. By Lemma 7.119.3 we may assume that N_λ is flat over R_λ for all sufficiently large λ. Denote $\mathfrak{m}_\lambda \subset R_\lambda$ the maximal ideal and $\kappa_\lambda = R_\lambda/\mathfrak{m}_\lambda$, resp. $\kappa = R/\mathfrak{m}$ the residue fields.

Consider the map

$$\Psi_\lambda : M_\lambda/\mathfrak{m}_\lambda M_\lambda \otimes_{\kappa_\lambda} \kappa \longrightarrow M/\mathfrak{m}M.$$

Since $S_\lambda/\mathfrak{m}_\lambda S_\lambda$ is essentially of finite type over the field κ_λ we see that the tensor product $S_\lambda/\mathfrak{m}_\lambda S_\lambda \otimes_{\kappa_\lambda} \kappa$ is essentially of finite type over κ. Hence it is a Noetherian ring and we conclude the kernel of Ψ_λ is finitely generated. Since $M/\mathfrak{m}M$ is the colimit of the system $M_\lambda/\mathfrak{m}_\lambda M_\lambda$ and κ is the colimit of the fields κ_λ there exists a $\lambda' > \lambda$ such that the kernel of Ψ_λ is generated by the kernel of

$$\Psi_{\lambda,\lambda'} : M_\lambda/\mathfrak{m}_\lambda M_\lambda \otimes_{\kappa_\lambda} \kappa_{\lambda'} \longrightarrow M_{\lambda'}/\mathfrak{m}_{\lambda'} M_{\lambda'}.$$

By construction there exists a multiplicative subset $W \subset S_\lambda \otimes_{R_\lambda} R_{\lambda'}$ such that $S_{\lambda'} = W^{-1}(S_\lambda \otimes_{R_\lambda} R_{\lambda'})$ and

$$W^{-1}(M_\lambda/\mathfrak{m}_\lambda M_\lambda \otimes_{\kappa_\lambda} \kappa_{\lambda'}) = M_{\lambda'}/\mathfrak{m}_{\lambda'} M_{\lambda'}.$$

Now suppose that x is an element of the kernel of

$$\Psi_{\lambda'} : M_{\lambda'}/\mathfrak{m}_{\lambda'} M_{\lambda'} \otimes_{\kappa_{\lambda'}} \kappa \longrightarrow M/\mathfrak{m}M.$$

Then for some $w \in W$ we have $wx \in M_\lambda/\mathfrak{m}_\lambda M_\lambda \otimes \kappa$. Hence $wx \in \mathrm{Ker}(\Psi_\lambda)$. Hence wx is a linear combination of elements in the kernel of $\Psi_{\lambda,\lambda'}$. Hence $wx = 0$ in $M_{\lambda'}/\mathfrak{m}_{\lambda'} M_{\lambda'} \otimes_{\kappa_{\lambda'}} \kappa$, hence $x = 0$ because w is invertible in $S_{\lambda'}$. We conclude that the kernel of $\Psi_{\lambda'}$ is zero for all sufficiently large λ'!

By the result of the preceding paragraph we may assume that the kernel of Ψ_λ is zero for all λ sufficiently large, which implies that the map $M_\lambda/\mathfrak{m}_\lambda M_\lambda \to M/\mathfrak{m}M$ is injective. Combined with $\bar{u}$ being injective this formally implies that also $\overline{u_\lambda} : M_\lambda/\mathfrak{m}_\lambda M_\lambda \to N_\lambda/\mathfrak{m}_\lambda N_\lambda$ is injective. By Lemma 7.91.1 we conclude that (for all sufficiently large λ) the map u_λ is injective and that $N_\lambda/u_\lambda(M_\lambda)$ is flat over R_λ. The lemma follows. □

Lemma 7.119.5. *Suppose that $R \to S$ is a local ring homomorphism of local rings. Denote $\mathfrak{m}$ the maximal ideal of R. Suppose*

(1) *S is essentially of finite presentation over R,*
(2) *S is flat over R, and*
(3) *$f \in S$ is a nonzero divisor in $S/\mathfrak{m}S$.*

Then S/fS is flat over R, and f is a nonzero divisor in S.

Proof. Follows directly from Lemma 7.119.4. □

Lemma 7.119.6. *Suppose that $R \to S$ is a local ring homomorphism of local rings. Denote $\mathfrak{m}$ the maximal ideal of R. Suppose*

(1) *$R \to S$ is essentially of finite presentation,*

(2) *$R \to S$ is flat, and*
(3) *$f_1, \ldots, f_c$ is a sequence of elements of S such that the images $\overline{f}_1, \ldots, \overline{f}_c$ form a regular sequence in $S/\mathfrak{m}S$.*

Then $f_1, \ldots, f_c$ is a regular sequence in S and each of the quotients $S/(f_1, \ldots, f_i)$ is flat over R.

Proof. Induction and Lemma 7.119.5 above. □

Here is the version of the local criterion of flatness for the case of local ring maps which are locally of finite presentation.

Lemma 7.119.7. *Let $R \to S$ be a local homomorphism of local rings. Let $I \neq R$ be an ideal in R. Let M be an S-module. Assume*

(1) *S is essentially of finite presentation over R,*
(2) *M is of finite presentation over S,*
(3) *$Tor_1^R(M, R/I) = 0$, and*
(4) *M/IM is flat over R/I.*

Then M is flat over R.

Proof. Let Λ, $R_\lambda \to S_\lambda$, M_λ be as in Lemma 7.118.11. Denote $I_\lambda \subset R_\lambda$ the inverse image of I. In this case the system $R/I \to S/IS$, M/IM, $R_\lambda \to S_\lambda/I_\lambda S_\lambda$, and $M_\lambda/I_\lambda M_\lambda$ satisfies the conclusions of Lemma 7.118.11 as well. Hence by Lemma 7.119.3 we may assume (after shrinking the index set Λ) that $M_\lambda/I_\lambda M_\lambda$ is flat for all λ. Pick some λ and consider

$$\mathrm{Tor}_1^{R_\lambda}(M_\lambda, R_\lambda/I_\lambda) = \mathrm{Ker}(I_\lambda \otimes_{R_\lambda} M_\lambda \to M_\lambda).$$

See Remark 7.69.8. The right hand side shows that this is a finitely generated S_λ-module (because S_λ is Noetherian and the modules in question are finite). Let $\xi_1, \ldots, \xi_n$ be generators. Because $\mathrm{Tor}^1_R(M, R/I) = 0$ and since $\otimes$ commutes with colimits we see there exists a $\lambda' \geq \lambda$ such that each ξ_i maps to zero in $\mathrm{Tor}_1^{R_{\lambda'}}(M_{\lambda'}, R_{\lambda'}/I_{\lambda'})$. The composition of the maps

$$\begin{array}{c}
R_{\lambda'} \otimes_{R_\lambda} \mathrm{Tor}_1^{R_\lambda}(M_\lambda, R_\lambda/I_\lambda) \\
\Big\downarrow \text{surjective by Lemma 7.91.11} \\
\mathrm{Tor}_1^{R_\lambda}(M_\lambda, R_{\lambda'}/I_\lambda R_{\lambda'}) \\
\Big\downarrow \text{surjective up to localization by Lemma 7.91.12} \\
\mathrm{Tor}_1^{R_{\lambda'}}(M_{\lambda'}, R_{\lambda'}/I_\lambda R_{\lambda'}) \\
\Big\downarrow \text{surjective by Lemma 7.91.11} \\
\mathrm{Tor}_1^{R_{\lambda'}}(M_{\lambda'}, R_{\lambda'}/I_{\lambda'}).
\end{array}$$

is surjective up to a localization by the reasons indicated. The localization is necessary since $M_{\lambda'}$ is not equal to $M_\lambda \otimes_{R_\lambda} R_{\lambda'}$. Namely, it is equal to $M_\lambda \otimes_{S_\lambda} S_{\lambda'}$ and $S_{\lambda'}$ is the localization of $S_\lambda \otimes_{R_\lambda} R_{\lambda'}$ whence the statement up to a localization (or tensoring with $S_{\lambda'}$). Note that Lemma 7.91.11 applies to the first and third arrows because $M_\lambda/I_\lambda M_\lambda$ is flat over R_λ/I_λ and because $M_{\lambda'}/I_\lambda M_{\lambda'}$ is flat over $R_{\lambda'}/I_\lambda R_{\lambda'}$ as it is a base change of the flat module $M_\lambda/I_\lambda M_\lambda$. The composition maps the generators ξ_i to zero as we explained above. We finally conclude that $\text{Tor}_1^{R_{\lambda'}}(M_{\lambda'}, R_{\lambda'}/I_{\lambda'})$ is zero. This implies that $M_{\lambda'}$ is flat over $R_{\lambda'}$ by Lemma 7.91.9. ☐

Please compare the lemma below to Lemma 7.91.14 (the case of Noetherian local rings) and Lemma 7.93.8 (the case of a nilpotent ideal in the base).

Lemma 7.119.8 (Critère de platitude par fibres)**.** *Let R, S, S' be local rings and let $R \to S \to S'$ be local ring homomorphisms. Let M be an S'-module. Let $\mathfrak{m} \subset R$ be the maximal ideal. Assume*

1. *The ring maps $R \to S$ and $R \to S'$ are essentially of finite presentation.*
2. *The module M is of finite presentation over S'.*
3. *The module M is not zero.*
4. *The module $M/\mathfrak{m}M$ is a flat $S/\mathfrak{m}S$-module.*
5. *The module M is a flat R-module.*

Then S is flat over R and M is a flat S-module.

Proof. As in the proof of Lemma 7.118.9 we may first write $R = colim\, R_\lambda$ as a directed colimit of local $\mathbf{Z}$-algebras which are essentially of finite type. Denote $\mathfrak{p}_\lambda$ the maximal ideal of R. Next, we may assume that for some $\lambda_1 \in \Lambda$ there exist $f_{j,\lambda_1} \in R_{\lambda_1}[x_1, \ldots, x_n]$ such that

$$S = colim_{\lambda \geq \lambda_1} S_\lambda, \text{ with } S_\lambda = (R_\lambda[x_1, \ldots, x_n]/(f_{1,\lambda}, \ldots, f_{u,\lambda}))_{\mathfrak{q}_\lambda}$$

For some $\lambda_2 \in \Lambda$, $\lambda_2 \geq \lambda_1$ there exist $g_{j,\lambda_2} \in R_{\lambda_2}[x_1, \ldots, x_n, y_1, \ldots, y_m]$ with images $\overline{g}_{j,\lambda_2} \in S_{\lambda_2}[y_1, \ldots, y_m]$ such that

$$S' = colim_{\lambda \geq \lambda_2} S'_\lambda, \text{ with } S'_\lambda = (S_\lambda[y_1, \ldots, y_m]/(\overline{g}_{1,\lambda}, \ldots, \overline{g}_{v,\lambda}))_{\overline{\mathfrak{q}}'_\lambda}$$

Note that this also implies that

$$S'_\lambda = (R_\lambda[x_1, \ldots, x_n, y_1, \ldots, y_m]/(g_{1,\lambda}, \ldots, g_{v,\lambda}))_{\mathfrak{q}'_\lambda}$$

Choose a presentation

$$(S')^{\oplus s} \to (S')^{\oplus t} \to M \to 0$$

of M over S'. Let $A \in \text{Mat}(t \times s, S')$ be the matrix of the presentation. For some $\lambda_3 \in \Lambda$, $\lambda_3 \geq \lambda_2$ we can find a matrix $A_{\lambda_3} \in \text{Mat}(t \times s, S_{\lambda_3})$ which maps to A. For all $\lambda \geq \lambda_3$ we let $M_\lambda = \text{Coker}((S'_\lambda)^{\oplus s} \xrightarrow{A_\lambda} (S'_\lambda)^{\oplus t})$.

With these choices, we have for each $\lambda_3 \leq \lambda \leq \mu$ that $S_\lambda \otimes_{R_\lambda} R_\mu \to S_\mu$ is a localization, $S'_\lambda \otimes_{S_\lambda} S_\mu \to S'_\mu$ is a localization, and the map $M_\lambda \otimes_{S'_\lambda} S_\mu \to M_\mu$ is an isomorphism. This also implies that $S'_\lambda \otimes_{R_\lambda} R_\mu \to S'_\mu$ is a localization. Thus, since M is flat over R we see by Lemma 7.119.3 that for all λ big enough the module M_λ is flat over R_λ. Moreover, note that $\mathfrak{m} = colim\, \mathfrak{p}_\lambda$, $S/\mathfrak{m}S = colim\, S_\lambda/\mathfrak{p}_\lambda S_\lambda$, $S'/\mathfrak{m}S' = colim\, S'_\lambda/\mathfrak{p}_\lambda S'_\lambda$, and $M/\mathfrak{m}M = colim\, M_\lambda/\mathfrak{p}_\lambda M_\lambda$. Also, for each $\lambda_3 \leq \lambda \leq \mu$ we see (from the properties listed above) that

$$S'_\lambda/\mathfrak{p}_\lambda S'_\lambda \otimes_{S_\lambda/\mathfrak{p}_\lambda S_\lambda} S_\mu/\mathfrak{p}_\mu S_\mu \longrightarrow S'_\mu/\mathfrak{p}_\mu S'_\mu$$

is a localization, and the map

$$M_\lambda/\mathfrak{p}_\lambda M_\lambda \otimes_{S'_\lambda/\mathfrak{p}_\lambda S'_\lambda} S_\mu/\mathfrak{p}_\mu S'_\mu \longrightarrow M_\mu/\mathfrak{p}_\mu M_\mu$$

is an isomorphism. Hence the system $(S_\lambda/\mathfrak{p}_\lambda S_\lambda \to S'_\lambda/\mathfrak{p}_\lambda S'_\lambda, M_\lambda/\mathfrak{p}_\lambda M_\lambda)$ is a system as in Lemma 7.118.11 as well. We may apply Lemma 7.119.3 again because $M/\mathfrak{m}M$ is assumed flat over $S/\mathfrak{m}S$ and we see that $M_\lambda/\mathfrak{p}_\lambda M_\lambda$ is flat over $S_\lambda/\mathfrak{p}_\lambda S_\lambda$ for all λ big enough. Thus for λ big enough the data $R_\lambda \to S_\lambda \to S'_\lambda, M_\lambda$ satisfies the hypotheses of Lemma 7.91.14.

Pick such a λ. Then $S = S_\lambda \otimes_{R_\lambda} R$ is flat over R, and $M = M_\lambda \otimes_{S_\lambda} S'_\lambda$ is flat over S (since the base change of a flat module is flat). $\square$

The following is an easy consequence of the usual version of the ``critére de platitude par fibres". For more results of this kind see More on Flatness, Section 34.1.

Lemma 7.119.9. *(Critère de platitude par fibres; the case of essentially finite presentation over essentially finite type.) Let R, S, S' be local rings and let $R \to S \to S'$ be local ring homomorphisms. Let M be an S'-module. Let $\mathfrak{m} \subset R$ be the maximal ideal. Assume*

(1) *$R \to S'$ is essentially of finite presentation,*
(2) *$R \to S$ is essentially of finite type,*
(3) *M is of finite presentation over S',*
(4) *M is not zero,*
(5) *$M/\mathfrak{m}M$ is a flat $S/\mathfrak{m}S$-module, and*
(6) *M is a flat R-module.*

Then S is essentially of finite presentation and flat over R and M is a flat S-module.

Proof. As S is essentially of finite presentation over R we can write $S = C_{\overline{\mathfrak{q}}}$ for some finite type R-algebra C. Write $C = R[x_1, \ldots, x_n]/I$. Denote $\mathfrak{q} \subset R[x_1, \ldots, x_n]$ be the prime ideal corresponding to $\overline{\mathfrak{q}}$. Then we see that $S = B/J$ where $B = R[x_1, \ldots, x_n]_{\mathfrak{q}}$ is essentially of finite presentation over R and $J = IB$. We can find $f_1, \ldots, f_k \in J$ such that the images $\overline{f}_i \in B/\mathfrak{m}B$ generate the image $\overline{J}$ of J in the Noetherian ring $B/\mathfrak{m}B$. Hence there exist finitely generated ideals $J' \subset J$ such that $B/J' \to B/J$ induces an isomorphism

$$(B/J') \otimes_R R/\mathfrak{m} \longrightarrow B/J \otimes_R R/\mathfrak{m} = S/\mathfrak{m}S.$$

For any J' as above we see that Lemma 7.119.8 applies to the ring maps

$$R \longrightarrow B/J' \longrightarrow S'$$

and the module M. Hence we conclude that B/J' is flat over R for any choice J' as above. Now, if $J' \subset J' \subset J$ are two finitely generated ideals as above, then we conclude that $B/J' \to B/J''$ is a surjective map between flat R-algebras which are essentially of finite presentation which is an isomorphism modulo $\mathfrak{m}$. Hence Lemma 7.119.4 implies that $B/J' = B/J''$, i.e., $J' = J''$. Clearly this means that J is finitely generated, i.e., S is essentially of finite presentation over R. Thus we may apply Lemma 7.119.8 to $R \to S \to S'$ and we win. $\square$

7.120. Openness of the flat locus

Lemma 7.120.1. *Let k be a field. Let S be a finite type k-algebra. Let $f_1, \ldots, f_i$ be elements of S. Assume that S is Cohen-Macaulay and equidimensional of dimension d, and that* $\dim V(f_1, \ldots, f_i) \leq d - i$. *Then equality holds and $f_1, \ldots, f_i$ forms a regular sequence in $S_\mathfrak{q}$ for every prime $\mathfrak{q}$ of $V(f_1, \ldots, f_i)$.*

Proof. If S is Cohen-Macaulay and equidimensional of dimension d, then we have $\dim(S_\mathfrak{m}) = d$ for all maximal ideals $\mathfrak{m}$ of S, see Lemma 7.105.7. By Proposition 7.95.4 we see that for all maximal ideals $\mathfrak{m} \in V(f_1, \ldots, f_i)$ the sequence is a regular sequence in $S_\mathfrak{m}$ and the local ring $S_\mathfrak{m}/(f_1, \ldots, f_i)$ is Cohen-Macaulay of dimension $d - i$. This actually means that $S/(f_1, \ldots, f_i)$ is Cohen-Macaulay and equidimensional of dimension $d - i$. $\square$

Lemma 7.120.2. *Suppose that $R \to S$ is a ring map which is finite type, flat. Let d be an integer such that all fibres $S \otimes_R \kappa(\mathfrak{p})$ are Cohen-Macaulay and equidimensional of dimension d. Let $f_1, \ldots, f_i$ be elements of S. The set*

$$\{\mathfrak{q} \in V(f_1, \ldots, f_i) \mid f_1, \ldots, f_i \text{ are a regular sequence in } S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q} \text{ where } \mathfrak{p} = R \cap \mathfrak{q}\}$$

is open in $V(f_1, \ldots, f_i)$.

Proof. Write $\overline{S} = S/(f_1, \ldots, f_i)$. Suppose $\mathfrak{q}$ is an element of the set defined in the lemma, and $\mathfrak{p}$ is the corresponding prime of R. We will use relative dimension as defined in Definition 7.116.1. First, note that $d = \dim_\mathfrak{q}(S/R) = \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) + \text{trdeg}_{\kappa(\mathfrak{p})}\ \kappa(\mathfrak{q})$ by Lemma 7.107.3. Since $f_1, \ldots, f_i$ form a regular sequence in the Noetherian local ring $S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}$ general dimension theory tells us that $\dim(\overline{S}_\mathfrak{q}/\mathfrak{p}\overline{S}_\mathfrak{q}) = \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) - i$. By the same Lemma 7.107.3 we then conclude that $\dim_\mathfrak{q}(\overline{S}/R) = \dim(\overline{S}_\mathfrak{q}/\mathfrak{p}\overline{S}_\mathfrak{q}) + \text{trdeg}_{\kappa(\mathfrak{p})}\ \kappa(\mathfrak{q}) = d - i$. By Lemma 7.116.6 we have $\dim_{\mathfrak{q}'}(\overline{S}/R) \le d - i$ for all $\mathfrak{q}' \in V(f_1, \ldots, f_i) = Spec(\overline{S})$ in a neighbourhood of $\mathfrak{q}$. Thus after replacing S by S_g for some $g \in S$, $g \notin \mathfrak{q}$ we may assume that the inequality holds for all $\mathfrak{q}'$. The result follows from Lemma 7.120.1. $\square$

Lemma 7.120.3. *Let $R \to S$ is a ring map. Consider a finite homological complex of finite free S-modules:*

$$F_\bullet : 0 \to S^{n_e} \xrightarrow{\varphi_e} S^{n_{e-1}} \xrightarrow{\varphi_{e-1}} \ldots \xrightarrow{\varphi_{i+1}} S^{n_i} \xrightarrow{\varphi_i} S^{n_{i-1}} \xrightarrow{\varphi_{i-1}} \ldots \xrightarrow{\varphi_1} S^{n_0}$$

For every prime $\mathfrak{q}$ of S consider the complex $\overline{F}_{\bullet,\mathfrak{q}} = F_{\bullet,\mathfrak{q}} \otimes_R \kappa(\mathfrak{p})$ where $\mathfrak{p}$ is inverse image of $\mathfrak{q}$ in R. Assume there exists an integer d such that $R \to S$ is finite type, flat with fibres $S \otimes_R \kappa(\mathfrak{p})$ Cohen-Macaulay of dimension d. The set

$$\{\mathfrak{q} \in Spec(S) \mid \overline{F}_{\bullet,\mathfrak{q}} \text{ is exact}\}$$

is open in $Spec(S)$.

Proof. Let $\mathfrak{q}$ be an element of the set defined in the lemma. We are going to use Proposition 7.94.10 to show there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $D(g)$ is contained in the set defined in the lemma. In other words, we are going to show that after replacing S by S_g, the set of the lemma is all of $Spec(S)$. Thus during the proof we will, finitely often, replace S by such a localization. Recall that Proposition 7.94.10 characterizes exactness of complexes in terms of ranks of the maps φ_i and the ideals $I(\varphi_i)$, in case the ring is local. We first address the rank condition. Set $r_i = n_i - n_{i+1} + \ldots + (-1)^{e-i} n_e$. Note that $r_i + r_{i+1} = n_i$ and note that r_i is the expected rank of φ_i (in the exact case).

By Lemma 7.91.5 we see that if $\overline{F}_{\bullet,\mathfrak{q}}$ is exact, then the localization $F_{\bullet,\mathfrak{q}}$ is exact. In particular the complex $F_\bullet$ becomes exact after localizing by an element $g \in S$, $g \notin \mathfrak{q}$. In this case Proposition 7.94.10 applied to all localizations of S at prime ideals implies that all $(r_i + 1) \times (r_i + 1)$-minors of φ_i are zero. Thus we see that the rank of of φ_i is at most r_i.

Let $I_i \subset S$ denote the ideal generated by the $r_i \times r_i$-minors of the matrix of φ_i. By Proposition 7.94.10 the complex $\overline{F}_{\bullet,\mathfrak{q}}$ is exact if and only if for every $1 \le i \le e$ we have either $(I_i)_\mathfrak{q} = S_\mathfrak{q}$ or $(I_i)_\mathfrak{q}$ contains a $S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}$-regular sequence of length i. Namely, by our choice of r_i above and by the bound on the ranks of the φ_i this is the only way the conditions of Proposition 7.94.10 can be satisfied.

If $(I_i)_\mathfrak{q} = S_\mathfrak{q}$, then after localizing S at some element $g \notin \mathfrak{q}$ we may assume that $I_i = S$. Clearly, this is an open condition.

If $(I_i)_\mathfrak{q} \neq S_\mathfrak{q}$, then we have a sequence $f_1, \ldots, f_i \in (I_i)_\mathfrak{q}$ which form a regular sequence in $S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}$. Note that for any prime $\mathfrak{q}' \subset S$ such that $(f_1, \ldots, f_i) \not\subset \mathfrak{q}'$ we have $(I_i)_{\mathfrak{q}'} = S_{\mathfrak{q}'}$. Thus the result follows from Lemma 7.120.2. □

Theorem 7.120.4. *Let R be a ring. Let $R \to S$ be a ring map of finite presentation. Let M be a finitely presented S-module. The set*

$$\{\mathfrak{q} \in Spec(S) \mid M_\mathfrak{q} \text{ is flat over } R\}$$

is open in $Spec(S)$.

Proof. Let $\mathfrak{q} \in Spec(S)$ be a prime. Let $\mathfrak{p} \subset R$ be the inverse image of $\mathfrak{q}$ in R. Note that $M_\mathfrak{q}$ is flat over R if and only if it is flat over $R_\mathfrak{p}$. Let us assume that $M_\mathfrak{q}$ is flat over R. We claim that there exists a $g \in S$, $g \notin \mathfrak{q}$ such that M_g is flat over R.

We first reduce to the case where R and S are of finite type over $\mathbf{Z}$. Choose a directed partially ordered set Λ and a system $(R_\lambda \to S_\Lambda, M_\lambda)$ as in Lemma 7.118.15. Set $\mathfrak{p}_\lambda$ equal to the inverse image of $\mathfrak{p}$ in R_λ. Set $\mathfrak{q}_\lambda$ equal to the inverse image of $\mathfrak{q}$ in S_λ. Then the system

$$((R_\lambda)_{\mathfrak{p}_\lambda}, (S_\lambda)_{\mathfrak{q}_\lambda}, (M_\lambda)_{\mathfrak{q}_\lambda})$$

is a system as in Lemma 7.118.11. Hence by Lemma 7.119.3 we see that for some λ the module M_λ is flat over R_λ at the prime $\mathfrak{q}_\lambda$. Suppose we can prove our claim for the system $(R_\lambda \to S_\lambda, M_\lambda, \mathfrak{q}_\lambda)$. In other words, suppose that we can find a $g \in S_\lambda$, $g \notin \mathfrak{q}_\lambda$ such that $(M_\lambda)_g$ is flat over R_λ. By Lemma 7.118.15 we have $M = M_\lambda \otimes_{R_\lambda} R$ and hence also $M_g = (M_\lambda)_g \otimes_{R_\lambda} R$. Thus by Lemma 7.35.6 we deduce the claim for the system $(R \to S, M, \mathfrak{q})$.

At this point we may assume that R and S are of finite type over $\mathbf{Z}$. We may write S as a quotient of a polynomial ring $R[x_1, \ldots, x_n]$. Of course, we may replace S by $R[x_1, \ldots, x_n]$ and assume that S is a polynomial ring over R. In particular we see that $R \to S$ is flat and all fibres rings $S \otimes_R \kappa(\mathfrak{p})$ have global dimension n.

Choose a resolution $F_\bullet$ of M over S with each F_i finite free, see Lemma 7.67.1. Let $K_n = \text{Ker}(F_{n-1} \to F_{n-2})$. Note that $(K_n)_\mathfrak{q}$ is flat over R, since each F_i is flat over R and by assumption on M, see Lemma 7.35.12. In addition, the sequence

$$0 \to K_n/\mathfrak{p}K_n \to F_{n-1}/\mathfrak{p}F_{n-1} \to \ldots \to F_0/\mathfrak{p}F_0 \to M/\mathfrak{p}M \to 0$$

is exact upon localizing at $\mathfrak{q}$, because of vanishing of $\text{Tor}_i^{R_\mathfrak{p}}(\kappa(\mathfrak{p}), M_\mathfrak{q})$. Since the global dimension of $S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}$ is n we conclude that $K_n/\mathfrak{p}K_n$ localized at $\mathfrak{q}$ is a finite free module over $S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}$. By Lemma 7.91.4 $(K_n)_\mathfrak{q}$ is free over $S_\mathfrak{q}$. In particular, there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $(K_n)_g$ is finite free over S_g.

By Lemma 7.120.3 there exists a further localization S_g such that the complex

$$0 \to K_n \to F_{n-1} \to \ldots \to F_0$$

is exact on *all fibres* of $R \to S$. By Lemma 7.91.5 this implies that the cokernel of $F_1 \to F_0$ is flat. This proves the theorem in the Noetherian case. □

Here is a technical application of the openness of flatness. It says that we can approximate flat modules by flat modules which is sometimes useful. Please skip on a first reading.

Lemma 7.120.5. *Let $R \to S$ be a ring map. Let M be an S-module. Assume that*

(1) *$R \to S$ is of finite presentation,*
(2) *M is a finitely presented S-module, and*
(3) *M is flat over R.*

In this case we have the following:

1. *There exists a finite type* $\mathbf{Z}$*-algebra* R_0 *and a finite type ring map* $R_0 \to S_0$ *and a finite* S_0*-module* M_0 *such that* M_0 *is flat over* R_0*, together with a ring maps* $R_0 \to R$ *and* $S_0 \to S$ *and an* S_0*-module map* $M_0 \to M$ *such that* $S \cong R \otimes_{R_0} S_0$ *and* $M = S \otimes_{S_0} M_0$*.*
2. *If* $R = colim_{\lambda \in \Lambda} R_\lambda$ *is written as a directed colimit, then there exists a* λ *and a ring map* $R_\lambda \to S_\lambda$ *of finite presentation, and an* S_λ*-module* M_λ *of finite presentation such that* M_λ *is flat over* R_λ *and such that* $S = R \otimes_{R_\lambda} S_\lambda$ *and* $M = S \otimes_{S_\lambda} M_\lambda$*.*
3. *If*
$$(R \to S, M) = colim_{\lambda \in \Lambda}(R_\lambda \to S_\lambda, M_\lambda)$$
is written as a directed colimit such that for $\mu \geq \lambda$ *the maps* $R_\mu \otimes_{R_\lambda} S_\lambda \to S_\mu$ *and* $S_\mu \otimes_{S_\lambda} M_\lambda \to M_\mu$ *are isomorphisms then for all sufficiently large* λ *the module* M_λ *is flat over* R_λ*.*

Proof. We first write $(R \to S, M)$ as the directed colimit of a system $(R_\lambda \to S_\lambda, M_\lambda)$ as in as in Lemma 7.118.15. Let $\mathfrak{q} \subset S$ be a prime. Let $\mathfrak{p} \subset R$, $\mathfrak{q}_\lambda \subset S_\lambda$, and $\mathfrak{p}_\lambda \subset R_\lambda$ the corresponding primes. As seen in the proof of Theorem 7.120.4

$$((R_\lambda)_{\mathfrak{p}_\lambda}, (S_\lambda)_{\mathfrak{q}_\lambda}, (M_\lambda)_{\mathfrak{q}_\lambda})$$

is a system as in Lemma 7.118.11, and hence by Lemma 7.119.3 we see that for some $\lambda_{\mathfrak{q}} \in \Lambda$ for all $\lambda \geq \lambda_{\mathfrak{q}}$ the module M_λ is flat over R_λ at the prime $\mathfrak{q}_\lambda$.

By Theorem 7.120.4 we get an open subset $U_\lambda \subset Spec(S_\lambda)$ such that M_λ flat over R_λ at all the primes of U_λ. Denote $V_\lambda \subset Spec(S)$ the inverse image of U_λ under the map $Spec(S) \to Spec(S_\lambda)$. The argument above shows that for every $\mathfrak{q} \in Spec(S)$ there exists a $\lambda_{\mathfrak{q}}$ such that $\mathfrak{q} \in V_\lambda$ for all $\lambda \geq \lambda_{\mathfrak{q}}$. Since $Spec(S)$ is quasi-compact we see this implies there exists a single $\lambda_0 \in \Lambda$ such that $V_{\lambda_0} = Spec(S)$.

The complement $Spec(S_{\lambda_0}) \setminus U_{\lambda_0}$ is $V(I)$ for some ideal $I \subset S_{\lambda_0}$. As $V_{\lambda_0} = Spec(S)$ we see that $IS = S$. Choose $f_1, \ldots, f_r \in I$ and $s_1, \ldots, s_n \in S$ such that $\sum f_i s_i = 1$. Since $colim\, S_\lambda = S$, after increasing λ_0 we may assume there exist $s_{i,\lambda_0} \in S_{\lambda_0}$ such that $\sum f_i s_{i,\lambda_0} = 1$. Hence for this λ_0 we have $U_{\lambda_0} = Spec(S_{\lambda_0})$. This proves (1).

It turns out that (2) and (3) follow in a mechanical way from part (1). Namely, let $(R_0 \to S_0, M_0)$ be as in (1) and suppose that $R = colim\, R_\lambda$. Since R_0 is a finite type $\mathbf{Z}$ algebra, there exists a lambda and a map $R_0 \to R_\lambda$ such that $R_0 \to R_\lambda \to R$ is the given map $R_0 \to R$. Then, part (2) follows by taking $S_\lambda = R_\lambda \otimes_{R_0} S_0$ and $M_\lambda = S_\lambda \otimes_{S_0} M_0$.

Finally, we come to the proof of (3). We strongly suggest not reading this proof ever! Assume the directed system $(R_\lambda \to S_\lambda, M_\lambda)$ satisfies the hypotheses of (3) and that $(R_0 \to S_0, M_0)$, $R_0 \to R$ satisfies the conclusion of (1). As above, there exits a λ_0 and a ring map $R_0 \to R_{\lambda_0}$ such that $R_0 \to R_{\lambda_0} \to R$ is the given map $R_0 \to R$. Then for all $\lambda \geq \lambda_0$ we get an induced ring map $R_0 \to R_\lambda$ with the same property. Then, since S_0 is of finite type over $\mathbf{Z}$ too, and as $S = colim\, S_\lambda$ we see that for some $\lambda_1 \geq \lambda_0$ we also get an R_0-algebra map $S_0 \to S_{\lambda_1}$ such that the composition $S_0 \to S_{\lambda_1} \to S$ is the given map $S_0 \to S$. For all $\lambda \geq \lambda_1$ this gives maps

$$\Psi_\lambda : R_\lambda \otimes_{R_0} S_0 \longrightarrow R_\lambda \otimes_{R_{\lambda_1}} S_{\lambda_1} \cong S_\lambda$$

the last isomorphism by assumption. Let us argue that this map is an isomorphism for all λ large enough. First, pick generators $x_1, \ldots, x_n$ of the finitely presented algebra S_{λ_1} over R_{λ_1}. Since $S = R \otimes_{R_0} S_0$ and since $R = colim\, R_\lambda$ hence $S = colim\, R_\lambda \otimes_{R_0} S_0$ there exists a $\lambda_2 \geq \lambda_1$ such that $x_i \in S_{\lambda_2}$ is in the image of Ψ_{λ_2}. Hence for $\lambda \geq \lambda_2$ the map Ψ_{λ_2} is surjective. Write $S_0 = R_0[y_1, \ldots, y_m]/I_0$. Write

$$S_{\lambda_2} = R_{\lambda_2}[y_1, \ldots, y_m]/(g_1, \ldots, g_k).$$

We know that $g_1, \ldots, g_k \in I_0R[y_1, \ldots, y_m]$ as $S = R[y_1, \ldots, y_m]/(g_1, \ldots, g_m)$ and also $S = R[y_1, \ldots, y_m]/I_0R[y_1, \ldots, y_m]$ by all our assumptions. Hence for some $\lambda_3 \geq \lambda_2$ we see that $g_1, \ldots, g_m \in I_0R_{\lambda_3}[y_1, \ldots, y_m]$. Then it is clear that Ψ_λ is an isomorphism for all $\lambda \geq \lambda_3$. Phew!

In the same vein, there exists a λ_4 and an S_0-module map $M_0 \to M_{\lambda_4}$ such that $M_0 \to M_{\lambda_4} \to M$ is the given map $M_0 \to M$. We claim that for all λ large enough the induced maps $S_\lambda \otimes_{S_0} M_0 \to M_\lambda$ are isomorphisms. This is proved in exactly the same way as above and we omit it. Of course this implies (3), because M_0 is flat over R_0. □

Lemma 7.120.6. *Let $R \to A \to B$ be ring maps. Assume $A \to B$ faithfully flat of finite presentation. Then there exists a commutative diagram*

$$\begin{tikzcd} R \arrow[r] \arrow[d, equal] & A_0 \arrow[r] \arrow[d] & B_0 \arrow[d] \\ R \arrow[r] & A \arrow[r] & B \end{tikzcd}$$

with $R \to A_0$ of finite presentation, $A_0 \to B_0$ faithfully flat of finite presentation and $B = A \otimes_{A_0} B_0$.

Proof. We first prove the lemma with R replaced $\mathbf{Z}$. By Lemma 7.120.5 there exists a diagram

$$\begin{tikzcd} A_0 \arrow[r] & A \\ B_0 \arrow[u] \arrow[r] & B \arrow[u] \end{tikzcd}$$

where A_0 is of finite type over $\mathbf{Z}$, B_0 is flat of finite presentation over A_0 such that $B = A \otimes_{A_0} B_0$. As $A_0 \to B_0$ is flat of finite presentation we see that the image of $Spec(B_0) \to Spec(A_0)$ is open, see Proposition 7.36.8. Hence the complement of the image is $V(I_0)$ for some ideal $I_0 \subset A_0$. As $A \to B$ is faithfully flat the map $Spec(B) \to Spec(A)$ is surjective, see Lemma 7.35.15. Now we use that the base change of the image is the image of the base change. Hence $I_0A = A$. Pick a relation $\sum f_ir_i = 1$, with $r_i \in A$, $f_i \in I_0$. Then after enlarging A_0 to contain the elements r_i (and correspondingly enlarging B_0) we see that $A_0 \to B_0$ is surjective on spectra also, i.e., faithfully flat.

Thus the lemma holds in case $R = \mathbf{Z}$. In the general case, take the solution $A_0' \to B_0'$ just obtained and set $A_0 = A_0' \otimes_{\mathbf{Z}} R$, $B_0 = B_0' \otimes_{\mathbf{Z}} R$. □

7.121. Openness of Cohen-Macaulay loci

In this section we characterize the Cohen-Macaulay property of finite type algebras in terms of flatness. We then use this to prove the set of points where such an algebra is Cohen-Macaulay is open.

Lemma 7.121.1. *Let S be a finite type algebra over a field k. Let $\varphi : k[y_1, \ldots, y_d] \to S$ be a finite ring map. As subsets of $Spec(S)$ we have*

$$\{\mathfrak{q} \mid S_\mathfrak{q} \text{ flat over } k[y_1, \ldots, y_d]\} = \{\mathfrak{q} \mid S_\mathfrak{q} \text{ CM and } \dim_\mathfrak{q}(S/k) = d\}$$

For notation see Definition 7.116.1.

Proof. Let $\mathfrak{q} \subset S$ be a prime. Denote $\mathfrak{p} = k[y_1, \ldots, y_d] \cap \mathfrak{q}$. Note that always $\dim(S_\mathfrak{q}) \leq \dim(k[y_1, \ldots, y_d]_\mathfrak{p})$ by Lemma 7.116.4 for example. Moreover, the field extension $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q})$ is finite and hence $\text{trdeg}_k(\kappa(\mathfrak{p})) = \text{trdeg}_k(\kappa(\mathfrak{q}))$.

Let $\mathfrak{q}$ be an element of the left hand side. Then Lemma 7.103.9 applies and we conclude that $S_\mathfrak{q}$ is Cohen-Macaulay and $\dim(S_\mathfrak{q}) = \dim(k[y_1, \ldots, y_d]_\mathfrak{p})$. Combined with the equality of transcendence degrees above and Lemma 7.107.3 this implies that $\dim_\mathfrak{q}(S/k) = d$. Hence $\mathfrak{q}$ is an element of the right hand side.

Let $\mathfrak{q}$ be an element of the right hand side. By the equality of transcendence degrees above, the assumption that $\dim_\mathfrak{q}(S/k) = d$ and Lemma 7.107.3 we conclude that $\dim(S_\mathfrak{q}) = \dim(k[y_1, \ldots, y_d]_\mathfrak{p})$. Hence Lemma 7.119.1 applies and we see that $\mathfrak{q}$ is an element of the left hand side. □

Lemma 7.121.2. *Let S be a finite type algebra over a field k. The set of primes $\mathfrak{q}$ such that $S_\mathfrak{q}$ is Cohen-Macaulay is open in S.*

This lemma is a special case of Lemma 7.121.4 below, so you can skip straight to the proof of that lemma if you like.

Proof. Let $\mathfrak{q} \subset S$ be a prime such that $S_\mathfrak{q}$ is Cohen-Macaulay. We have to show there exists a $g \in S$, $g \notin \mathfrak{q}$ such that the ring S_g is Cohen-Macaulay. For any $g \in S$, $g \notin \mathfrak{q}$ we may replace S by S_g and $\mathfrak{q}$ by $\mathfrak{q}S_g$. Combining this with Lemmas 7.106.5 and 7.107.3 we may assume that there exists a finite injective ring map $k[y_1, \ldots, y_d] \to S$ with $d = \dim(S_\mathfrak{q}) + \text{trdeg}_k(\kappa(\mathfrak{q}))$. Set $\mathfrak{p} = k[y_1, \ldots, y_d] \cap \mathfrak{q}$. By construction we see that $\mathfrak{q}$ is an element of the right hand side of the displayed equality of Lemma 7.121.1. Hence it is also an element of the left hand side.

By Theorem 7.120.4 we see that for some $g \in S$, $g \notin \mathfrak{q}$ the ring S_g is flat over $k[y_1, \ldots, y_d]$. Hence by the equality of Lemma 7.121.1 again we conclude that all local rings of S_g are Cohen-Macaulay as desired. □

Lemma 7.121.3. *Let k be a field. Let S be a finite type k algebra. The set of Cohen-Macaulay primes forms a dense open $U \subset Spec(S)$.*

Proof. The set is open by Lemma 7.121.2 above. It contains all minimal primes $\mathfrak{q} \subset S$ since the local ring at a minimal prime $S_\mathfrak{q}$ has dimension zero and hence is Cohen-Macaulay. □

Lemma 7.121.4. *Let R be a ring. Let $R \to S$ be of finite presentation and flat. For any $d \geq 0$ the set*

$$\left\{ \begin{matrix} \mathfrak{q} \in Spec(S) \text{ such that setting } \mathfrak{p} = R \cap \mathfrak{q} \text{ the fibre ring} \\ S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q} \text{ is Cohen-Macaulay and } \dim_\mathfrak{q}(S/R) = d \end{matrix} \right\}$$

is open in $Spec(S)$.

Proof. Let $\mathfrak{q}$ be an element of the set indicated, with $\mathfrak{p}$ the corresponding prime of R. We have to find a $g \in S$, $g \notin \mathfrak{q}$ such that all fibre rings of $R \to S_g$ are Cohen-Macaulay. During the course of the proof we may (finitely many times) replace S by S_g for a $g \in S$, $g \notin \mathfrak{q}$. Thus by Lemma 7.116.2 we may assume there is a quasi-finite ring map $R[t_1, \ldots, t_d] \to S$ with $d = \dim_\mathfrak{q}(S/R)$. Let $\mathfrak{q}' = R[t_1, \ldots, t_d] \cap \mathfrak{q}$. By Lemma 7.121.1 we see that the ring map

$$R[t_1, \ldots, t_d]_{\mathfrak{q}'}/\mathfrak{p}R[t_1, \ldots, t_d]_{\mathfrak{q}'} \longrightarrow S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}$$

is flat. Hence by the critère de platitude par fibres Lemma 7.119.8 we see that $R[t_1, \ldots, t_d]_{\mathfrak q'} \to S_{\mathfrak q}$ is flat. Hence by Theorem 7.120.4 we see that for some $g \in S$, $g \not\in \mathfrak q$ the ring map $R[t_1, \ldots, t_d] \to S_g$ is flat. Replacing S by S_g we see that for every prime $\mathfrak r \subset S$, setting $\mathfrak r' = R[t_1, \ldots, t_d] \cap \mathfrak r$ and $\mathfrak p' = R \cap \mathfrak r$ the local ring map $R[t_1, \ldots, t_d]_{\mathfrak r'} \to S_{\mathfrak r}$ is flat. Hence also the base change

$$R[t_1, \ldots, t_d]_{\mathfrak r'}/\mathfrak p' R[t_1, \ldots, t_d]_{\mathfrak r'} \longrightarrow S_{\mathfrak r}/\mathfrak p' S_{\mathfrak r}$$

is flat. Hence by Lemma 7.121.1 applied with $k = \kappa(\mathfrak p')$ we see $\mathfrak r$ is in the set of the lemma as desired. □

Lemma 7.121.5. *Let R be a ring. Let $R \to S$ be flat of finite presentation. The set of primes $\mathfrak q$ such that the fibre ring $S_{\mathfrak q} \otimes_R \kappa(\mathfrak p)$, with $\mathfrak p = R \cap \mathfrak q$ is Cohen-Macaulay is open and dense in every fibre of $Spec(S) \to Spec(R)$.*

Proof. The set, call it W, is open by Lemma 7.121.4 above. It is dense in the fibres because the intersection of W with a fibre is the corresponding set of the fibre to which Lemma 7.121.3 applies. □

Lemma 7.121.6. *Let k be a field. Let S be a finite type k-algebra. Let $k \subset K$ be a field extension, and set $S_K = K \otimes_k S$. Let $\mathfrak q \subset S$ be a prime of S. Let $\mathfrak q_K \subset S_K$ be a prime of S_K lying over $\mathfrak q$. Then $S_{\mathfrak q}$ is Cohen-Macaulay if and only if $(S_K)_{\mathfrak q_K}$ is Cohen-Macaulay.*

Proof. During the course of the proof we may (finitely many times) replace S by S_g for any $g \in S$, $g \not\in \mathfrak q$. Hence using Lemma 7.106.5 we may assume that $\dim(S) = \dim_{\mathfrak q}(S/k) =: d$ and find a finite injective map $k[x_1, \ldots, x_d] \to S$. Note that this also induces a finite injective map $K[x_1, \ldots, x_d] \to S_K$ by base change. By Lemma 7.107.6 we have $\dim_{\mathfrak q_K}(S_K/K) = d$. Set $\mathfrak p = k[x_1, \ldots, x_d] \cap \mathfrak q$ and $\mathfrak p_K = K[x_1, \ldots, x_d] \cap \mathfrak q_K$. Consider the following commutative diagram of Noetherian local rings

$$\begin{tikzcd} S_{\mathfrak q} \arrow[r] & (S_K)_{\mathfrak q_K} \\ k[x_1, \ldots, x_d]_{\mathfrak p} \arrow[u] \arrow[r] & K[x_1, \ldots, x_d]_{\mathfrak p_K} \arrow[u] \end{tikzcd}$$

By Lemma 7.121.1 above we have to show that the left vertical arrow is flat if and only if the right vertical arrow is flat. Because the bottom arrow is flat this equivalence holds by Lemma 7.92.1. □

Lemma 7.121.7. *Let R be a ring. Let $R \to S$ be of finite type. Let $R \to R'$ be any ring map. Set $S' = R' \otimes_R S$. Denote $f : Spec(S') \to Spec(S)$ the map associated to the ring map $S \to S'$. Set W equal to the set of primes $\mathfrak q$ such that the fibre ring $S_{\mathfrak q} \otimes_R \kappa(\mathfrak p)$, $\mathfrak p = R \cap \mathfrak q$ is Cohen-Macaulay, and let W' denote the analogue for S'/R'. Then $W' = f^{-1}(W)$.*

Proof. Trivial from Lemma 7.121.6 and the definitions. □

Lemma 7.121.8. *Let R be a ring. Let $R \to S$ be a ring map which is (a) flat, (b) of finite presentation, (c) has Cohen-Macaulay fibres. Then $S = S_0 \times \ldots \times S_n$ is a product of rings S_d such that each S_d satisfies (a), (b), (c) and has all fibres equidimensional of dimension d.*

Proof. For each integer d denote $W_d \subset Spec(S)$ the set defined in Lemma 7.121.4. Clearly we have $Spec(S) = \coprod W_d$, and each W_d is open by the lemma we just quoted. Hence the result follows from Lemma 7.20.3. □

Lemma 7.121.9. *Let $R \to S$ be a faithfully flat ring map of finite presentation. Then there exists a commutative diagram*

$$\begin{tikzcd} S \arrow[rr] & & S' \\ & R \arrow[ul] \arrow[ur] & \end{tikzcd}$$

where $R \to S'$ is quasi-finite, faithfully flat and of finite presentation.

Proof. As a first step we reduce this lemma to the case where R is of finite type over $\mathbf{Z}$. By Lemma 7.120.6 there exists a diagram

$$\begin{tikzcd} S_0 \arrow[r] & S \\ R_0 \arrow[u] \arrow[r] & R \arrow[u] \end{tikzcd}$$

where R_0 is of finite type over $\mathbf{Z}$, and S_0 is faithfully flat of finite presentation over R_0 such that $S = R \otimes_{R_0} S_0$. If we prove the lemma for the ring map $R_0 \to S_0$, then the lemma follows for $R \to S$ by base change, as the base change of a quasi-finite ring map is quasi-finite, see Lemma 7.113.8. (Of course we also use that base changes of flat maps are flat and base changes of maps of finite presentation are of finite presentation.)

Assume $R \to S$ is a faithfully flat ring map of finite presentation and that R is Noetherian (which we may assume by the preceding paragraph). Let $W \subset Spec(S)$ be the open set of Lemma 7.121.4. As $R \to S$ is faithfully flat the map $Spec(S) \to Spec(R)$ is surjective, see Lemma 7.35.15. By Lemma 7.121.5 the map $W \to Spec(R)$ is also surjective. Hence by replacing S with a product $S_{g_1} \times \ldots \times S_{g_m}$ we may assume $W = Spec(S)$; here we use that $Spec(R)$ is quasi-compact (Lemma 7.16.10), and that the map $Spec(S) \to Spec(R)$ is open (Proposition 7.36.8). Suppose that $\mathfrak{p} \subset R$ is a prime. Choose a prime $\mathfrak{q} \subset S$ lying over $\mathfrak{p}$ which corresponds to a maximal ideal of the fibre ring $S \otimes_R \kappa(\mathfrak{p})$. The Noetherian local ring $\overline{S}_{\mathfrak{q}} = S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}}$ is Cohen-Macaulay, say of dimension d. We may choose $f_1, \ldots, f_d$ in the maximal ideal of $S_{\mathfrak{q}}$ which map to a regular sequence in $\overline{S}_{\mathfrak{q}}$. Choose a common denominator $g \in S$, $g \notin \mathfrak{q}$ of $f_1, \ldots, f_d$, and consider the R-algebra

$$S' = S_g/(f_1, \ldots, f_d).$$

By construction there is a prime ideal $\mathfrak{q}' \subset S'$ lying over $\mathfrak{p}$ and corresponding to $\mathfrak{q}$ (via $S_g \to S'_g$). Also by construction the ring map $R \to S'$ is quasi-finite at $\mathfrak{q}$ as the local ring

$$S'_{\mathfrak{q}'}/\mathfrak{p}S'_{\mathfrak{q}'} = S_{\mathfrak{q}}/(f_1, \ldots, f_d) + \mathfrak{p}S_{\mathfrak{q}} = \overline{S}_{\mathfrak{q}}/(\overline{f}_1, \ldots, \overline{f}_d)$$

has dimension zero, see Lemma 7.113.2. Also by construction $R \to S'$ is of finite presentation. Finally, by Lemma 7.91.3 the local ring map $R_{\mathfrak{p}} \to S'_{\mathfrak{q}'}$ is flat (this is where we use that R is Noetherian). Hence, by openness of flatness (Theorem 7.120.4), and openness of quasi-finiteness (Lemma 7.114.14) we may after replacing g by gg' for a suitable $g' \in S$, $g' \notin \mathfrak{q}$ assume that $R \to S'$ is flat and quasi-finite. The image $Spec(S') \to Spec(R)$ is open and contains $\mathfrak{p}$. In other words we have shown a ring S' as in the statement of the lemma exists (except possibly the faithfulness part) whose image contains any given prime. Using one more time the quasi-compactness of $Spec(R)$ we see that a finite product of such rings does the job. $\square$

7.122. Differentials

Definition 7.122.1. Let $\varphi : R \to S$ be a ring map and let M be an S-module. A *derivation*, or more precisely an *R-derivation* into M is a map $D : S \to M$ which is additive, annihilates elements of $\varphi(R)$, and satisfies the *Leibniz rule*: $D(ab) = aD(b) + D(a)b$.

Note that $D(ra) = rD(a)$ if $r \in R$ and $a \in S$. The set of all R-derivations forms an S-module: Given two R-derivations D, D' the sum $D + D' : S \to M$, $a \mapsto D(a) + D'(a)$ is an R-derivation, and given an R-derivation D and an element $c \in S$ the scalar multiple $cD : S \to M$, $a \mapsto cD(a)$ is an R-derivation. We denote this S-module

$$\mathrm{Der}_R(S, M).$$

Also, if $\alpha : M \to N$ is an S-module map, then the composition $\alpha \circ D$ is an R-derivation into N. In this way the assignment $M \mapsto \mathrm{Der}_R(S, M)$ is a covariant functor.

Consider the following map of free S-modules

$$\bigoplus\nolimits_{(a,b)\in S^2} S[(a,b)] \oplus \bigoplus\nolimits_{(f,g)\in S^2} S[(f,g)] \oplus \bigoplus\nolimits_{r\in R} S[r] \longrightarrow \bigoplus\nolimits_{a\in S} S[a]$$

defined by the rules

$$[(a,b)] \longmapsto [a+b] - [a] - [b], \quad [(f,g)] \longmapsto [fg] - f[g] - g[f], \quad [r] \longmapsto [\varphi(r)]$$

with obvious notation. Let $\Omega_{S/R}$ be the cokernel of this map. There is a map $\mathrm{d} : S \to \Omega_{S/R}$ which maps a to the class $\mathrm{d}a$ of $[a]$ in the cokernel. This is an R-derivation by the relations imposed on $\Omega_{S/R}$, in other words

$$\mathrm{d}(a+b) = \mathrm{d}a + \mathrm{d}b, \quad \mathrm{d}(fg) = f\mathrm{d}g + g\mathrm{d}f, \quad \mathrm{d}r = 0$$

where $a, b, f, g \in S$ and $r \in R$.

Definition 7.122.2. The pair $(\Omega_{S/R}, \mathrm{d})$ is called the *module of Kähler diffentials* or the *module of differentials* of S over R.

Lemma 7.122.3. *The module of differentials of S over R has the following universal property. The map*

$$Hom_S(\Omega_{S/R}, M) \longrightarrow Der_R(S, M), \quad \alpha \longmapsto \alpha \circ d$$

is an isomorphism of functors.

Proof. By definition an R-derivation is a rule which associates to each $a \in S$ an element $D(a) \in M$. Thus D gives rise to a map $[D] : \bigoplus S[a] \to M$. However, the conditions of being an R-derivation exactly mean that $[D]$ annihilates the image of the map in the displayed presentation of $\Omega_{S/R}$ above. □

Lemma 7.122.4. *Let I be a directed partially ordered set. Let $(R_i \to S_i, \varphi_{ii'})$ be a system of ring maps over I, see Categories, Section 4.19. Then we have*

$$\Omega_{S/R} = colim_i \, \Omega_{S_i/R_i}.$$

Proof. This is clear from the presentation of $\Omega_{S/R}$ given above. □

Lemma 7.122.5. *Suppose that $R \to S$ is surjective. Then $\Omega_{S/R} = 0$.*

Proof. You can see this either because all R-derivations clearly have to be zero, or because the map in the presentation of $\Omega_{S/R}$ is surjective. □

Suppose that

$$\begin{tikzcd} S \arrow[r, "\varphi"'] & S' \\ R \arrow[u, "\alpha"] \arrow[r, "\psi"] & R' \arrow[u, "\beta"'] \end{tikzcd} \tag{7.122.5.1}$$

is a commutative diagram of rings. In this case there is a natural map of modules of differentials fitting into the commutative diagram

$$\begin{tikzcd} \Omega_{S/R} \arrow[r] & \Omega_{S'/R'} \\ S \arrow[u, "\mathrm{d}"] \arrow[r] & S' \arrow[u, "\mathrm{d}"'] \end{tikzcd}$$

To construct the map just use the obvious map between the presentations for $\Omega_{S/R}$ and $\Omega_{S'/R'}$. Namely,

$$\begin{tikzcd} \bigoplus S'[(a',b')] \oplus \bigoplus S'[(f',g')] \oplus \bigoplus S'[r'] \arrow[r] & \bigoplus S'[a'] \\ \bigoplus S[(a,b)] \oplus \bigoplus S[(f,g)] \oplus \bigoplus S[r] \arrow[u, "{\begin{array}{c}[(a,b)] \mapsto [(\varphi(a),\varphi(b))] \\ {[(f,g)]} \mapsto [(\varphi(f),\varphi(g))] \\ {[r]} \mapsto [\psi(r)]\end{array}}"] \arrow[r] & \bigoplus S[a] \arrow[u, "{[a]\mapsto[\varphi(a)]}"'] \end{tikzcd}$$

The result is simply that $f\mathrm{d}g \in \Omega_{S/R}$ is mapped to $\varphi(f)\mathrm{d}\varphi(g)$.

Lemma 7.122.6. *In diagram (7.122.5.1), suppose that $S \to S'$ is surjective with kernel $I \subset S$. Then $\Omega_{S/R} \to \Omega_{S'/R'}$ is surjective with kernel generated as an S-module by the elements the elements da, where $a \in S$ is such that $\varphi(a) \in \beta(R')$. (This includes in particular the elements $d(i)$, $i \in I$.)*

Proof. Consider the map of presentations above. Clearly the right vertical map of free modules is surjective. Thus the map is surjective. A diagram chase shows that the following elements generate the kernel as an S-module for sure: $i\mathrm{d}a$, $i \in I$, $a \in S$, and $\mathrm{d}a$, with $a \in S$ such that $\varphi(a) = \beta(r')$ for some $r' \in R'$. Note that $\varphi(i) = \varphi(ia) = 0 = \beta(0)$, and that $\mathrm{d}(ia) = i\mathrm{d}a + a\mathrm{d}i$. Hence $i\mathrm{d}a = \mathrm{d}(ia) - a\mathrm{d}i$ is an S-linear combination of elements of the second kind. □

Lemma 7.122.7. *Let $A \to B \to C$ be ring maps. Then there is a canonical exact sequence*

$$C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

of C-modules.

Proof. We get a diagram (7.122.5.1) by putting $R = A$, $S = C$, $R' = B$, and $S' = C$. By Lemma 7.122.6 the map $\Omega_{C/A} \to \Omega_{C/B}$ is surjective, and the kernel is generated by the elements $\mathrm{d}(c)$, where $c \in C$ is in the image of $B \to C$. The lemma follows. □

Lemma 7.122.8. *Let $\varphi : A \to B$ be a ring map.*

(1) *If $S \subset A$ is a multiplicative subset mapping to invertible elements of B, then $\Omega_{B/A} = \Omega_{B/S^{-1}A}$.*
(2) *If $S \subset B$ is a multiplicative subset then $S^{-1}\Omega_{B/A} = \Omega_{S^{-1}B/A}$.*

Proof. To show the equality of (1) it is enough to show that any A-derivation $D : B \to M$ annihilates the elements $\varphi(s)^{-1}$. This is clear from the Leibniz rule applied to $1 = \varphi(s)\varphi(s)^{-1}$. To show (2) note that there is an obvious map $S^{-1}\Omega_{B/A} \to \Omega_{S^{-1}B/A}$. To show it is an isomorphism it is enough to show that there is a A-derivation d' of $S^{-1}B$ into $S^{-1}\Omega_{B/A}$. To define it we simply set $\mathrm{d}'(b/s) = (1/s)\mathrm{d}b - (1/s^2)b\mathrm{d}s$. Details omitted. □

Lemma 7.122.9. *In diagram (7.122.5.1), suppose that $S \to S'$ is surjective with kernel $I \subset S$, and assume that $R' = R$. Then there is a canonical exact sequence of S'-modules*

$$I/I^2 \longrightarrow \Omega_{S/R} \otimes_S S' \longrightarrow \Omega_{S'/R} \longrightarrow 0$$

The leftmost map is characterized by the rule that $f \in I$ maps to $\mathrm{d}f \otimes 1$.

Proof. The middle term is $\Omega_{S/R} \otimes_S S/I$. For $f \in I$ denote $\overline{f}$ the image of f in I/I^2. To show that the map $\overline{f} \mapsto \mathrm{d}f \otimes 1$ is well defined we just have to check that $\mathrm{d}f_1f_2 \otimes 1 = 0$ if $f_1, f_2 \in I$. And this is clear from the Leibniz rule $\mathrm{d}f_1f_2 \otimes 1 = (f_1\mathrm{d}f_2 + f_2\mathrm{d}f_1) \otimes 1 = \mathrm{d}f_2 \otimes f_1 + \mathrm{d}f_2 \otimes f_1 = 0$. A similar computation show this map is $S' = S/I$-linear.

The map $\Omega_{S/R} \otimes_S S' \to \Omega_{S'/R}$ is the canonical S'-linear map associated to the S-linear map $\Omega_{S/R} \to \Omega_{S'/R}$. It is surjective because $\Omega_{S/R} \to \Omega_{S'/R}$ is surjective by Lemma 7.122.6.

The composite of the two maps is zero because $\mathrm{d}f$ maps to zero in $\Omega_{S'/R}$ for $f \in I$. Note that exactness just says that the kernel of $\Omega_{S/R} \to \Omega_{S'/R}$ is generated as an S-submodule by the submodule $I\Omega_{S/R}$ together with the elements $\mathrm{d}f$, with $f \in I$. We know by Lemma 7.122.6 that this kernel is generated by the elements $\mathrm{d}(a)$ where $\varphi(a) = \beta(r)$ for some $r \in R$. But then $a = \alpha(r) + a - \alpha(r)$, so $\mathrm{d}(a) = \mathrm{d}(a - \alpha(r))$. And $a - \alpha(r) \in I$ since $\varphi(a - \alpha(r)) = \varphi(a) - \varphi(\alpha(r)) = \beta(r) - \beta(r) = 0$. We conclude the elements $\mathrm{d}f$ with $f \in I$ already generate the kernel as an S-module, as desired. □

Lemma 7.122.10. *In diagram (7.122.5.1), suppose that $S \to S'$ is surjective with kernel $I \subset S$, and assume that $R' = R$. Moreover, assume that there exists an R-algebra map $S' \to S$ which is a right inverse to $S' \to S$. Then the exact sequence of S'-modules of Lemma 7.122.9 turns into a short exact sequence*

$$0 \longrightarrow I/I^2 \longrightarrow \Omega_{S/R} \otimes_S S' \longrightarrow \Omega_{S'/R} \longrightarrow 0$$

which is even a split short exact sequence.

Proof. Let $\beta : S' \to S$ be the right inverse to the surjection $\alpha : S \to S'$, so $S = I \oplus \beta(S')$. Clearly we can use $\beta : \Omega_{S'/R} \to \Omega_{S/R}$, to get a right inverse to the map $\Omega_{S/R} \otimes_S S' \to \Omega_{S'/R}$. On the other hand, consider the map

$$D : S \longrightarrow I/I^2, \quad x \longmapsto x - \beta(\alpha(x))$$

It is easy to show that D is an R-derivation (omitted). Moreover $xD(s) = 0$ if $x \in I, s \in S$. Hence, by the universal property D induces a map $\tau : \Omega_{S/R} \otimes_R S' \to I/I^2$. We omit the verification that it is a left inverse to $\mathrm{d} : I/I^2 \to \Omega_{S/R} \otimes_S S'$. Hence we win. □

Lemma 7.122.11. *Let $R \to S$ be a ring map. Let $I \subset S$ be an ideal. Let $n \geq 1$ be an integer. Set $S' = S/I^{n+1}$. The map $\Omega_{S/R} \to \Omega_{S'/R}$ induces an isomorphism*

$$\Omega_{S/R} \otimes_S S/I^n \longrightarrow \Omega_{S'/R} \otimes_{S'} S/I^n.$$

Proof. This follows from Lemma 7.122.9 and the fact that $\mathrm{d}(I^{n+1}) \subset I^n\Omega_{S/R}$ by the Leibniz rule for d. □

Lemma 7.122.12. *Suppose that we have ring maps $R \to R'$ and $R \to S$. Set $S' = S \otimes_R R'$, so that we obtain a diagram (7.122.5.1). Then the canonical map defined above induces an isomorphism $\Omega_{S/R} \otimes_R R' = \Omega_{S'/R'}$.*

Proof. Let $\mathrm{d}' : S' = S \otimes_R R' \to \Omega_{S/R} \otimes_R R'$ denote the map $\mathrm{d}'(\sum a_i \otimes x_i) = \mathrm{d}(a_i) \otimes x_i$. It exists because the map $S \times R' \to \Omega_{S/R} \otimes_R R'$, $(a, x) \mapsto \mathrm{d}a \otimes_R x$ is R-bilinear. This is an R'-derivation, as can be verified by a simple computation. We will show that $(\Omega_{S/R} \otimes_R R', \mathrm{d}')$ satisfies the universal property. Let $D : S' \to M'$ be an R' derivation into an S'-module. The composition $S \to S' \to M'$ is an R-derivation, hence we get an S-linear map $\varphi_D : \Omega_{S/R} \to M'$. We may tensor this with R' and get the map $\varphi'_D : \Omega_{S/R} \otimes_R R' \to M'$, $\varphi'_D(\eta \otimes x) = x\varphi_D(\eta)$. It is clear that $D = \varphi'_D \circ \mathrm{d}'$. □

The multiplication map $S \otimes_R S \to S$ is the R-algebra map which maps $a \otimes b$ to ab in S. It is also an S-algebra map, if we think of $S \otimes_R S$ as an S-algebra via either of the maps $S \to S \otimes_R S$.

Lemma 7.122.13. *Let $R \to S$ be a ring map. Let $J = Ker(S \otimes_R S \to S)$ be the kernel of the multiplication map. There is a canonical isomorphism of S-modules $\Omega_{S/R} \to J/J^2$, $adb \mapsto a \otimes b - ab \otimes 1$.*

Proof. First we show that the rule $a\mathrm{d}b \mapsto a \otimes b - ab \otimes 1$ is well defined. In order to do this we have to show that $\mathrm{d}r$ and $a\mathrm{d}b + b\mathrm{d}a - d(ab)$ map to zero. The first because $r \otimes 1 - 1 \otimes r = 0$ by definition of the tensor product. The second because $a \otimes b - ab \otimes 1 + b \otimes a - ba \otimes 1 = (a \otimes 1 - 1 \otimes a)(1 \otimes b - b \otimes 1)$ is in J^2.

We construct a map in the other direction. We may think of $S \to S \otimes_R S$, $a \mapsto a \otimes 1$ as the base change of $R \to S$. Hence we have $\Omega_{S \otimes_R S/S} = \Omega_{S/R} \otimes_S (S \otimes_R S)$, by Lemma 7.122.12. At this point the sequence of Lemma 7.122.9 gives a map

$$J/J^2 \to \Omega_{S \otimes_R S/S} \otimes_{S \otimes_R S} S = (\Omega_{S/R} \otimes_S (S \otimes_R S)) \otimes_{S \otimes_R S} S = \Omega_{S/R}.$$

We leave it to the reader to see it is the inverse of the map above. □

Lemma 7.122.14. *If $S = R[x_1, \ldots, x_n]$, then $\Omega_{S/R}$ is a finite free S-module with basis $dx_1, \ldots, dx_n$.*

Proof. We first show that $\mathrm{d}x_1, \ldots, \mathrm{d}x_n$ generate $\Omega_{S/R}$ as an S-module. To prove this we show that $\mathrm{d}g$ can be expressed as a sum $\sum g_i \mathrm{d}x_i$ for any $g \in R[x_1, \ldots, x_n]$. We do this by induction on the (total) degree of g. It is clear if the degree of g is 0, because then $\mathrm{d}g = 0$. If the degree of g is > 0, then we may write g as $c + \sum g_i x_i$ with $c \in R$ and $\deg(g_i) < \deg(g)$. By the Leibnize rule we have $\mathrm{d}g = \sum g_i \mathrm{d}x_i + \sum x_i \mathrm{d}g_i$, and hence we win by induction.

Consider the R-derivation $\partial/\partial x_i : R[x_1, \ldots, x_n] \to R[x_1, \ldots, x_n]$. (We leave it to the reader to define this; the defining property being that $\partial/\partial x_i(x_j) = \delta_{ij}$.) By the universal property this corresponds to an S-module map $l_i : \Omega_{S/R} \to R[x_1, \ldots, x_n]$ which maps $\mathrm{d}x_i$ to 1 and $\mathrm{d}x_j$ to 0 for $j \neq i$. Thus it is clear that there are no S-linear relations among the elements $\mathrm{d}x_1, \ldots, \mathrm{d}x_n$. □

Lemma 7.122.15. *Suppose $R \to S$ is of finite presentation. Then $\Omega_{S/R}$ is a finitely presented S-module.*

Proof. Write $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. Write $I = (f_1, \ldots, f_m)$. According to Lemma 7.122.9 there is an exact sequence of S-modules

$$I/I^2 \to \Omega_{R[x_1,\ldots,x_n]/R} \otimes_{R[x_1,\ldots,x_n]} S \to \Omega_{S/R} \to 0$$

The result follows from the fact that I/I^2 is a finite S-module (generated by the images of the f_i), and that the middle term is finite free by Lemma 7.122.14. □

Lemma 7.122.16. *Suppose $R \to S$ is of finite type. Then $\Omega_{S/R}$ is finitely generated S-module.*

Proof. This is very similar to, but easier than the proof of Lemma 7.122.15. □

7.123. The naive cotangent complex

Let $R \to S$ be a ring map. Denote $R[S]$ the polynomial ring whose variables are the elements $s \in S$. Let's denote $[s] \in R[S]$ the variable corresponding to $s \in S$. Thus $R[S]$ is a free R-module on the basis elements $[s_1] \ldots [s_n]$ where $s_1, \ldots, s_n$ is an unordered sequence of elements of S. There is a canonical surjection

$$R[S] \longrightarrow S, \quad [s] \longmapsto s \tag{7.123.0.1}$$

whose kernel we denote $I \subset R[S]$. It is a simple observation that I is generated by the elements $[s][s'] - [ss']$ and $r[s] - [rs]$. According to Lemma 7.122.9 there is a canonical map

$$I/I^2 \longrightarrow \Omega_{R[S]/S} \otimes_R S \tag{7.123.0.2}$$

whose cokernel is canonically isomorphic to $\Omega_{S/R}$. Observe that the S-module $\Omega_{R[S]/R} \otimes_R S$ is free on the generators $\text{d}[s]$.

Definition 7.123.1. Let $R \to S$ be a ring map. The *naive cotangent complex $NL_{S/R}$* is the chain complex (7.123.0.2)

$$NL_{S/R} = \left(I/I^2 \longrightarrow \Omega_{R[S]/S} \otimes_R S\right)$$

with I/I^2 placed in (homological) degree 1 and $\Omega_{R[S]/S} \otimes_R S$ placed in degree 0. We will denote $H_1(L_{S/R}) = H_1(NL_{S/R})$[6] the homology in degree 1.

Before we continue let us say a few words about the actual cotangent complex (insert future reference here). Given a ring map $R \to S$ there exists a canonical augmented cosimplicial R-algebra $P_{S/R,\bullet}$ whose terms are polynomial algebras and which comes equipped with a canonical homotopy equivalence

$$P_{S/R,\bullet} \longrightarrow S$$

The cotangent complex $L_{S/R}$ of S over R is defined as the chain complex associated to the cosimplicial module

$$\Omega_{P_{S/R,\bullet}/R} \otimes_{P_{S/R,\bullet}} S$$

The naive cotangent complex as defined above is canonically isomorphic to the truncation $\tau_{\leq 1} L_{S/R}$ (see Homology, Section 10.11). In particular, it is indeed the case that $H_1(NL_{S/R}) = H_1(L_{S/R})$ so our definition is compatible with the one using the cotangent complex. Moreover, $H_0(L_{S/R}) = H_0(NL_{S/R}) = \Omega_{S/R}$ as we've seen above.

Let $R \to S$ be a ring map. A *presentation of S over R* is a surjection $\alpha : P \to S$ of R-algebras where P is a polynomial algebra (on a set of variables). Often, when S is of finite type over R we will indicate this by saying: ``Let $R[x_1, \ldots, x_n] \to S$ be a presentation of $S/R$", or ``Let $0 \to I \to R[x_1, \ldots, x_n] \to S \to 0$ be a presentation of S/R" if we want to indicate that I is the kernel of the presentation. Note that the map $R[S] \to S$ used to define the naive cotangent complex is an example of a presentation.

[6] This module is sometimes denoted $\Gamma_{S/R}$ in the literature.

Note that for every presentation α we obtain a two term chain complex of S-modules

$$NL(\alpha) : I/I^2 \longrightarrow \Omega_{P/R} \otimes_P S.$$

Here the term I/I^2 is placed in degree 1 and the term $\Omega_{P/R} \otimes S$ is placed in degree 0. The class of $f \in I$ in I/I^2 is mapped to $\mathrm{d}f \otimes 1$ in $\Omega_{P/R} \otimes S$. The cokernel of this complex is canonically $\Omega_{S/R}$, see Lemma 7.122.9. We call the complex $NL(\alpha)$ the *naive cotangent complex associated to the presentation $\alpha : P \to S$ of S/R*. Note that if $P = R[S]$ with its canonical surjection onto S, then we recover $NL_{S/R}$. If $P = R[x_1, \ldots, x_n]$ then will sometimes use the notation $I/I^2 \to \bigoplus_{i=1,\ldots,n} S\mathrm{d}x_i$ to denote this complex.

Suppose we are given a commutative diagram

$$\begin{tikzcd} S \arrow[r, "\phi"] & S' \\ R \arrow[u] \arrow[r] & R' \arrow[u] \end{tikzcd} \tag{7.123.1.1}$$

of rings. Let $\alpha : P \to S$ be a presentation of S over R and let $\alpha : P' \to S'$ be a presentation of S' over R'. A *morphism of presentations from $\alpha : P \to S$ to $\alpha' : P' \to S'$* is defined to be an R-algebra map

$$\varphi : P \to P'$$

such that $\phi \circ \alpha = \alpha' \circ \varphi$. Note that in this case $\varphi(I) \subset I'$, where $I = \ker(\alpha)$ and $I' = \ker(\alpha')$. Thus φ induces a map of S-modules $I/I^2 \to I'/(I')^2$ and by functoriality of differentials also an S-module map $\Omega_{P/R} \otimes S \to \Omega_{P'/R'} \otimes S'$. These maps are compatible with the differentials of $NL(\alpha)$ and $NL(\alpha')$ and we obtain a map of naive cotangent complexes

$$NL(\alpha) \longrightarrow NL(\alpha').$$

It is often convenient to consider the induced map $NL(\alpha) \otimes_S S' \to NL(\alpha')$.

In the special case that $P = R[S]$ and $P' = R'[S']$ the map $\phi : S \to S'$ induces a canonical ring map $\varphi : P \to P'$ by the rule $[s] \mapsto [\phi(s)]$. Hence the construction above determines a canonical(!) maps of chain complexes

$$NL_{S/R} \longrightarrow NL_{S'/R'}, \quad \text{and} \quad NL_{S/R} \otimes_S S' \longrightarrow NL_{S'/R'}$$

associated to the diagram (7.123.1.1). Note that this construction is compatible with composition: given a commutative diagram

$$\begin{tikzcd} S \arrow[r, "\phi"] & S' \arrow[r, "\phi'"] & S'' \\ R \arrow[u] \arrow[r] & R' \arrow[u] \arrow[r] & R'' \arrow[u] \end{tikzcd}$$

we see that the composition of

$$NL_{S/R} \longrightarrow NL_{S'/R'} \longrightarrow NL_{S'/R'}$$

is the map $NL_{S/R} \to NL_{S''/R''}$ given by the outer square.

It turns out that $NL(\alpha)$ is homotopy equivalent to $NL_{S/R}$ and that the maps constructed above are well defined up to homotopy (homotopies of maps of complexes are discussed in Homology, Section 10.10 but we also spell out the exact meaning of the statements in the lemma below in its proof).

Lemma 7.123.2. *Suppose given a diagram (7.123.1.1). Let $\alpha : P \to S$ and $\alpha' : P' \to S'$ be presentations.*

(1) *There exist a morphism of presentations from α to α'.*
(2) *Any two morphisms of presentations induce homotopic morphisms of complexes $NL(\alpha) \to NL(\alpha')$.*
(3) *The construction is compatible with compositions of morphisms of presentations (see proof for exact statement).*
(4) *If $R \to R'$ and $S \to S'$ are isomorphisms, then for any map φ of presentations from α to α' the induced map $NL(\alpha) \to NL(\alpha')$ is a homotopy equivalence and a quasi-isomorphism.*

In particular, comparing α to the canonical presentation (7.123.0.1) we conclude there is a quasi-isomorphism $NL(\alpha) \to NL_{S/R}$ well defined up to homotopy and compatible with all functorialities (up to homotopy).

Proof. To construct a morphism φ from α to α', write $P = R[x_a, a \in A]$. Choose for every a an element $f_a \in P'$ such that $\alpha'(f_a) = \phi(\alpha(x_i))$. Let $\varphi : P = R[x_a, a \in A] \to P'$ be the unique R-algebra map such that $\varphi(x_a) = f_a$. This gives the morphism.

Let φ and φ' morphisms of presentations from α to α'. Let $I = \mathrm{Ker}(\alpha)$ and $I' = \mathrm{Ker}(\alpha')$. We have to construct the diagonal map h in the diagram

$$\begin{tikzcd} I/I^2 \arrow[r, "\mathrm{d}"] \arrow[d, shift right, "\varphi_1"'] \arrow[d, shift left, "\varphi_1'"] & \Omega_{P/R} \otimes_P S \arrow[d, shift right, "\varphi_0"'] \arrow[d, shift left, "\varphi_0'"] \arrow[dl, "h"'] \\ J/J^2 \arrow[r, "\mathrm{d}"] & \Omega_{P'/R'} \otimes_{P'} S' \end{tikzcd}$$

where the vertical maps are induced by φ, φ' such that

$$\varphi_1 - \varphi_1' = h \circ d \quad \text{and} \quad \varphi_0 - \varphi_0' = d \circ h$$

Consider the map $D = \varphi - \varphi' : P \to P'$. Since both φ and φ are compatible with α and α' we conclude that $\varphi - \varphi' : P \to I'$. Also $\varphi - \varphi'$ is R-linear and

$$(\varphi - \varphi')(fg) = \varphi(f)(\varphi - \varphi')(g) + (\varphi - \varphi')(f)\varphi'(g)$$

Hence the induced map $D : P \to I'/(I')^2$ is a R-derivation. Thus we obtain a canonical map $h : \Omega_{P/S} \otimes_P S \to I'/(I')^2$ such that $D = h \circ \mathrm{d}$. A calculation (omitted) shows that h is the desired homotopy.

Suppose that we have a commutative diagram

$$\begin{tikzcd} S \arrow[r, "\phi"'] & S' \arrow[r, "\phi'"'] & S'' \\ R \arrow[r] \arrow[u] & R' \arrow[r] \arrow[u] & R'' \arrow[u] \end{tikzcd}$$

with finite type vertical arrows. Suppose that

(1) $\alpha : P \to S$,
(2) $\alpha' : P' \to S'$, and
(3) $\alpha'' : P'' \to S''$

are presentations. Suppose that

(1) $\varphi : P \to P$ is a morphism of presentations from α to α' and
(2) $\varphi' : P' \to P''$ is a morphism of presentations from α' to α''.

Then it is immediate that $\varphi' \circ \varphi : P \to P''$ is a morphism of presentations from α to α'' and that the induced map $NL(\alpha) \to NL(\alpha'')$ of naive cotangent complexes is the composition of the maps $NL(\alpha) \to NL(\alpha')$ and $NL(\alpha) \to NL(\alpha')$ induced by φ and φ'.

In the simple case of complexes with 2 terms a quasi-isomorphism is just a map that induces an isomorphism on both the cokernel and the kernel of the maps between the terms. Note that homotopic maps of 2 term complexes (as explained above) define the same maps on kernel and cokernel. Hence if φ is a map from a presentation α of S over R to itself, then the induced map $NL(\alpha) \to NL(\alpha)$ is a quasi-isomorphism being homotopic to the identity by part (2). To prove (4) in full generality, consider a morphism φ' from α' to α which exists by (1). The compositions $NL(\alpha) \to NL(\alpha') \to NL(\alpha)$ and $NL(\alpha') \to NL(\alpha) \to NL(\alpha')$ are homotopic to the identity maps by (3), hence these maps are homotopy equivalences by definition. It follows formally that both maps $NL(\alpha) \to NL(\alpha')$ and $NL(\alpha') \to NL(\alpha)$ are quasi-isomorphisms. Some details omitted. $\square$

The following lemma is part of the motivation for introducing the naive cotangent complex. The cotangent complex extends this to a genuine long exact cohomology sequence. If $B \to C$ is a local complete intersection, then one can extend the sequence with a zero on the left, see More on Algebra, Lemma 12.24.6.

Lemma 7.123.3 (Jacobi-Zariski sequence)**.** *Let $A \to B \to C$ be ring maps. Choose a presentation $\alpha : A[x_s, s \in S] \to B$ with kernel I. Choose a presentation $\beta : B[y_t, t \in T] \to C$ with kernel J. Let $\gamma : A[x_s, y_t] \to C$ be the induced presentation of C with kernel K. Then we get a canonical commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & \Omega_{A[x_s]/A} \otimes C \arrow[r] & \Omega_{A[x_s,y_t]/A} \otimes C \arrow[r] & \Omega_{B[y_t]/B} \otimes C \arrow[r] & 0 \\ & I/I^2 \otimes C \arrow[r] \arrow[u] & K/K^2 \arrow[r] \arrow[u] & J/J^2 \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

with exact rows. We get the following exact sequence of homology groups

$$H_1(NL_{B/A} \otimes_B C) \to H_1(L_{C/A}) \to H_1(L_{C/B}) \to C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

of C-modules extending the sequence of Lemma 7.122.7. If $Tor_1^B(\Omega_{B/A}, C) = 0$, then $H_1(NL_{B/A} \otimes_B C) = H_1(L_{B/A}) \otimes_B C$.

Proof. The precise definition of the maps is omitted. The exactness of the top row follows as the $\mathrm{d}x_s$, $\mathrm{d}y_t$ form a basis for the middle module. The map γ factors

$$A[x_s, y_t] \to B[y_t] \to C$$

with surjective first arrow and second arrow equal to β. Thus we see that $K \to J$ is surjective. Moreover, the kernel of the first displayed arrow is $IA[x_s, y_t]$. Hence $I/I^2 \otimes C$ surjects onto the kernel of $J/J^2 \to K/K^2$. Finally, we can use Lemma 7.123.2 to identify the terms as homology groups of the naive cotangent complexes. The final assertion follows as the degree 0 term of the complex $NL_{B/A}$ is a free B-module. $\square$

Lemma 7.123.4. *Let $A \to B$ be a surjective ring map with kernel I. Then $NL_{B/A}$ is homotopy equivalent to the chain complex $(I/I^2 \to 0)$ with I/I^2 in degree 1. In particular $H_1(L_{B/A}) = I/I^2$.*

Proof. Follows from Lemma 7.123.2 and the fact that $A \to B$ is a presentation of B over A. $\square$

Lemma 7.123.5. *Let $A \to B \to C$ be ring maps. Assume $A \to C$ is surjective (so also $B \to C$ is). Denote $I = Ker(A \to C)$ and $J = Ker(B \to C)$. Then the sequence*

$$I/I^2 \to J/J^2 \to \Omega_{B/A} \otimes_B B/J \to 0$$

is exact.

Proof. Follows from Lemma 7.123.3 and the description of the naive cotangent complexes $NL_{C/B}$ and $NL_{C/A}$ in Lemma 7.123.4. □

Lemma 7.123.6 (Flat base change)**.** *Let $R \to S$ be a ring map. Let $\alpha : P \to S$ be a presentation. Let $R \to R'$ be a flat ring map. Let $\alpha' : P' \otimes_R R' \to S' = S \otimes_R R'$ be the induced presentation. Then $NL(\alpha) \otimes_R R' = NL(\alpha) \otimes_S S' = NL(\alpha')$. In particular, the canonical map*

$$NL_{S/R} \otimes_R R' \longrightarrow NL_{S \otimes_R R'/R'}$$

is a homotopy equivalence if $R \to R'$ is flat.

Proof. This is true because $\mathrm{Ker}(\alpha') = R' \otimes_R \mathrm{Ker}(\alpha)$ since $R \to R'$ is flat. □

Lemma 7.123.7. *Let $R_i \to S_i$ be a system of ring maps over the directed partially ordered set I. Set $R = colim\, R_i$ and $S = colim\, S_i$. Then $NL_{S/R} = colim\, NL_{S_i/R_i}$.*

Proof. Recall that $NL_{S/R}$ is the complex $I/I^2 \to \bigoplus_{s \in S} S\mathrm{d}[s]$ where $I \subset R[S]$ is the kernel of the canonical presentation $R[S] \to S$. Now it is clear that $R[S] = colim\, R_i[S_i]$ and similarly that $I = colim\, I_i$ where $I_i = \mathrm{Ker}(R_i[S_i] \to S_i)$. Hence the lemma is clear. □

Lemma 7.123.8. *If $S \subset A$ is a multiplicative subset of A, then $NL_{S^{-1}A/A}$ is homotopy equivalent to the zero complex.*

Proof. Since $A \to S^{-1}A$ is flat we see that $NL_{S^{-1}A/A} \otimes_A S^{-1}A \to NL_{S^{-1}A/S^{-1}A}$ is a homotopy equivalence. Since the source of the arrow is isomorphic to $NL_{S^{-1}A/A}$ and the target of the arrow is zero (by Lemma 7.123.4) we win. □

Lemma 7.123.9. *Let $S \subset A$ is a multiplicative subset of A. Let $S^{-1}A \to B$ be a ring map. Then $NL_{B/A} \to NL_{B/S^{-1}A}$ is an homotopy equivalence.*

Proof. Follows from Lemmas 7.123.3 and 7.123.8. □

Lemma 7.123.10. *Let $A \to B$ be a ring map. Let $S \subset B$ be a multiplicative subset. Then there exists a quasi-isomorphism $NL_{B/A} \otimes_B S^{-1}B \to NL_{S^{-1}B/A}$.*

Proof. Note that $S^{-1}B = colim_{g \in S}\, B_g$ where we think of S as a directed partially ordered set (ordering by divisibility). Hence by Lemma 7.123.7 it suffices to prove $NL_{B/A} \otimes_B B_g \to NL_{B_g/A}$. Suppose $\alpha : P \to B$ is a presentation with kernel I. Then a presentation of B_g over A is

$$\beta : P[x] \longrightarrow B_g$$

mapping x to $1/g$ in B_g. Then kernel J of β is generated by I and the element $fx - 1$. Here $f \in P$ is a polynomial that maps to g in B. In this case

$$J/J^2 \cong (I/I^2)_g \oplus B_g \cdot (fx - 1).$$

Moreover, the term of degree zero of the naive cotangent complex for the presentation of B_g has one more summand, namely $B_g\mathrm{d}x$. Thus we see that there is a short exact sequence of complexes

$$0 \to NL(\alpha) \otimes_B B_g \to NL(\beta) \to (B_g \xrightarrow{g} B_g) \to 0$$

which proves that $NL(\beta)$ is quasi-isomorphic to $NL(\alpha) \otimes_S S_g$. By Lemma 7.123.2 this implies $NL_{B/A} \otimes_B B_g \to NL_{B_g/A}$ is a quasi-isomorphism as desired. $\square$

Lemma 7.123.11. *Let R be a ring. Let $A_1 \to A_0$, and $B_1 \to B_0$ be two two term complexes. Suppose that there exist morphisms of complexes $\varphi : A_\bullet \to B_\bullet$ and $\psi : B_\bullet \to A_\bullet$ such that $\varphi \circ \psi$ and $\psi \circ \varphi$ are homotopic to the identity maps. Then $A_1 \oplus B_0 \cong B_1 \oplus A_0$ as R-modules.*

Proof. Choose a map $h : A_0 \to B_1$ such that

$$\mathrm{id}_{A_1} - \psi_1 \circ \varphi_1 = h \circ d_A \text{ and } \mathrm{id}_{A_0} - \psi_0 \circ \varphi_0 = d_A \circ h.$$

Similarly, choose a map $h' : B_0 \to A_1$ such that

$$\mathrm{id}_{B_1} - \varphi_1 \circ \psi_1 = h \circ d_B \text{ and } \mathrm{id}_{B_0} - \varphi_0 \circ \psi_0 = d_B \circ h.$$

A trivial computation shows that

$$\begin{pmatrix} \mathrm{id}_{A_1} & -h' \circ \psi_1 + h \circ \psi_0 \\ 0 & \mathrm{id}_{B_0} \end{pmatrix} = \begin{pmatrix} \psi_1 & h \\ -d_B & \varphi_0 \end{pmatrix} \begin{pmatrix} \varphi_1 & -h' \\ d_A & \psi_0 \end{pmatrix}$$

This shows that both matrices on the right hand side are invertible and proves the lemma. $\square$

Lemma 7.123.12. *Let $R \to S$ be a ring map of finite type. For any presentations $\alpha : R[x_1, \ldots, x_n] \to S$, and $\beta : R[y_1, \ldots, y_m] \to S$ we have*

$$I/I^2 \oplus S^{\oplus m} \cong J/J^2 \oplus S^{\oplus n}$$

as S-modules where $I = Ker(\alpha)$ and $J = ker(\beta)$.

Proof. See Lemmas 7.123.2 and 7.123.11. $\square$

Lemma 7.123.13. *Let $R \to S$ be a ring map of finite type. Let $g \in S$. For any presentations $\alpha : R[x_1, \ldots, x_n] \to S$, and $\beta : R[y_1, \ldots, y_m] \to S_g$ we have*

$$(I/I^2)_g \oplus S_g^{\oplus m} \cong J/J^2 \oplus S_g^{\oplus n}$$

as S_g-modules where $I = Ker(\alpha)$ and $J = ker(\beta)$.

Proof. By Lemma 7.123.12 above, we see that it suffices to prove this for a single choice of α and β. Thus take β to be the presentation

$$\beta : R[x_1, \ldots, x_n, x_{n+1}] \longrightarrow S_g$$

which maps x_i to $\alpha(x_i)$ and x_{n+1} to $1/g$. Clearly $J = Ik[x_1, \ldots, x_n, x_{n+1}] + (x_{n+1}g - 1)$. Hence $J/J^2 \cong (I/I^2)_g \oplus S_g$ and we win. $\square$

7.124. Local complete intersections

The property of being a local complete intersection is somehow an intrinsic property of a Noetherian local ring. However, for the moment we just define this property for finite type algebras over a field.

Definition 7.124.1. Let k be a field. Let S be a finite type k-algebra.

(1) We say that S is a *global complete intersection over k* if there exists a presentation $S = k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ such that $\dim(S) = n - c$.
(2) We say that S is a *local complete intersection over k* if there exists a covering $Spec(S) = \bigcup D(g_i)$ such that each of the rings S_{g_i} is a global complete intersection over k.

We will also use the convention that the zero ring is a global complete intersection over k.

Suppose S is a global complete intersection $S = k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ as in the definition. Recall that $\dim(S) = n - c$ means that all irreducible components of $Spec(S)$ have dimension $\leq n - c$. Since all maximal ideals of the polynomial ring have local rings of dimension n we conclude that all irreducible components of $Spec(S)$ have dimension $\geq n - c$. See Section 7.57. In other words, $Spec(S)$ is equidimensional of dimension $n - c$.

Lemma 7.124.2. *Let k be a field. Let S be a finite type k-algebra. Let $g \in S$.*

1. *If S is a global complete intersection so is S_g.*
2. *If S is a local complete intersection so is S_g.*

Proof. The second statement follows immediately from the first. For the first, say that $S = k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ with $n - c = \dim(S)$. By the remarks above S is equidimensional of dimension $n - c$, so $\dim(S_g) = n - c$ as well (or it is the zero ring in which case the lemma is true by convention). Let $g' \in k[x_1, \ldots, x_n]$ be an element whose residue class corresponds to g. Then $S_g = k[x_1, \ldots, x_n, x_{n+1}]/(f_1, \ldots, f_c, x_{n+1}g' - 1)$ as desired. $\square$

Lemma 7.124.3. *Let k be a field. Let S be a finite type k-algebra. If S is a local complete intersection, then S is a Cohen-Macaulay ring.*

Proof. Choose a maximal prime $\mathfrak{m}$ of S. We have to show that $S_{\mathfrak{m}}$ is Cohen-Macaulay. By assumption we may assume $S = k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ with $\dim(S) = n - c$. Let $\mathfrak{m}' \subset k[x_1, \ldots, x_n]$ be the maximal ideal corresponding to $\mathfrak{m}$. According to Proposition 7.105.2 the local ring $k[x_1, \ldots, x_n]_{\mathfrak{m}'}$ is regular local of dimension n. Hence, by dimension theory (see Section 7.57) the ring $S_{\mathfrak{m}} = k[x_1, \ldots, x_n]_{\mathfrak{m}'}/(f_1, \ldots, f_c)$ has dimension $\geq n - c$. By assumption $\dim(S_{\mathfrak{m}}) \leq n - c$. Thus we get equality. This implies that $f_1, \ldots, f_c$ is a regular sequence in $k[x_1, \ldots, x_n]_{\mathfrak{m}'}$ and that $S_{\mathfrak{m}}$ is Cohen-Macaulay, see Proposition 7.95.4. $\square$

The following is the technical key to the rest of the material in this section. An important feature of this lemma is that we may choose any presentation for the ring S, but that condition (1) does not depend on this choice.

Lemma 7.124.4. *Let k be a field. Let S be a finite type k-algebra. Let $\mathfrak{q}$ be a prime of S. Choose any presentation $S = k[x_1, \ldots, x_n]/I$. Let $\mathfrak{q}'$ be the prime of $k[x_1, \ldots, x_n]$ corresponding to $\mathfrak{q}$. Set $c = height(\mathfrak{q}') - height(\mathfrak{q})$, in other words $\dim_{\mathfrak{q}}(S) = n - c$ (see Lemma 7.107.4). The following are equivalent*

1. *There exists a $g \in S$, $g \notin \mathfrak{q}$ such that S_g is a global complete intersection over k.*
2. *The ideal $I_{\mathfrak{q}'} \subset k[x_1, \ldots, x_n]_{\mathfrak{q}'}$ can be generated by c elements.*
3. *The conormal module $(I/I^2)_{\mathfrak{q}}$ can be generated by c elements over $S_{\mathfrak{q}}$.*
4. *The conormal module $(I/I^2)_{\mathfrak{q}}$ is a free $S_{\mathfrak{q}}$-module of rank c.*
5. *The ideal $I_{\mathfrak{q}'}$ can be generated by a regular sequence in the regular local ring $k[x_1, \ldots, x_n]_{\mathfrak{q}'}$.*

In this case any c elements of $I_{\mathfrak{q}'}$ which generate $I_{\mathfrak{q}'}/\mathfrak{q}'I_{\mathfrak{q}'}$ form a regular sequence in the local ring $k[x_1, \ldots, x_n]_{\mathfrak{q}'}$.

Proof. Set $R = k[x_1, \ldots, x_n]_{\mathfrak{q}'}$. This is a regular local ring of dimension height$(\mathfrak{q}')$. Moreover, $\overline{R} = R/IR = R/I_{\mathfrak{q}'} = S_{\mathfrak{q}}$ is a quotient of dimension height$(\mathfrak{q})$. Let $f_1, \ldots, f_c \in I_{\mathfrak{q}'}$ be elements which generate $(I/I^2)_{\mathfrak{q}}$. By Lemma 7.14.5 we see that $f_1, \ldots, f_c$ generate $I_{\mathfrak{q}'}$. Since the dimensions work out, we conclude by Proposition 7.95.4 that $f_1, \ldots, f_c$ is a regular sequence in R. By Lemma 7.66.2 we see that $(I/I^2)_{\mathfrak{q}}$ is free. These arguments show that (2), (3), (4) are equivalent and that they imply the last statement of the lemma, and therefore they imply (5).

If (5) holds, say $I_{\mathfrak{q}'}$ is generated by a regular sequence of length e, then $\text{height}(\mathfrak{q}) = \dim(S_{\mathfrak{q}}) = \dim(k[x_1, \ldots, x_n]_{\mathfrak{q}'}) - e = \text{height}(\mathfrak{q}') - e$ by dimension theory, see Section 7.57. We conclude that $e = c$. Thus (5) implies (2).

We continue with the notation introduced in the first paragraph. For each f_i we may find $d_i \in k[x_1, \ldots, x_n]$, $d_i \not\in \mathfrak{q}'$ such that $f_i' = d_i f_i \in k[x_1, \ldots, x_n]$. Then it is still true that $I_{\mathfrak{q}'} = (f_1', \ldots, f_c')R$. Hence there exists a $g' \in k[x_1, \ldots, x_n]$, $g' \not\in \mathfrak{q}'$ such that $I_{g'} = (f_1', \ldots, f_c')$. Moreover, pick $g'' \in k[x_1, \ldots, x_n]$, $g'' \not\in \mathfrak{q}'$ such that $\dim(S_{g''}) = \dim_{\mathfrak{q}} Spec(S)$. By Lemma 7.107.4 this dimension is equal to $n - c$. Finally, set g equal to the image of $g'g''$ in S. Then we see that

$$S_g \cong k[x_1, \ldots, x_n, x_{n+1}]/(f_1', \ldots, f_c', x_{n+1}g'g'' - 1)$$

and by our choice of g'' this ring has dimension $n - c$. Therefore it is a global complete intersection. Thus each of (2), (3), and (4) implies (1).

Assume (1). Let $S_g \cong k[y_1, \ldots, y_m]/(f_1, \ldots, f_t)$ be a presentation of S_g as a global complete intersection. Write $J = (f_1, \ldots, f_t)$. Let $\mathfrak{q}'' \subset k[y_1, \ldots, y_m]$ be the prime corresponding to $\mathfrak{q}S_g$. Note that $t = m - \dim(S_g) = \text{height}(\mathfrak{q}) - \text{height}(\mathfrak{q}'')$, see Lemma 7.107.4 for the last equality. As seen in the proof of Lemma 7.124.3 (and also above) the elements $f_1, \ldots, f_t$ form a regular sequence in the local ring $k[y_1, \ldots, y_m]_{\mathfrak{q}''}$. By Lemma 7.66.2 we see that $(J/J^2)_{\mathfrak{q}}$ is free of rank t. By Lemma 7.123.13 we have

$$J/J^2 \oplus S_g^n \cong (I/I^2)_g \oplus S_g^m$$

Thus $(I/I^2)_{\mathfrak{q}}$ is free of rank $t + n - m = m - \dim(S_g) + n - m = n - \dim(S_g) = \text{height}(\mathfrak{q}) - \text{height}(\mathfrak{q}') = c$. Thus we obtain (4). □

The result of Lemma 7.124.4 suggests the following definition.

Definition 7.124.5. Let k be a field. Let S be a local k-algebra essentially of finite type over k. We say S is a *complete intersection (over k)* if there exists a local k-algebra R and elements $f_1, \ldots, f_c \in \mathfrak{m}_R$ such that

(1) R is essentially of finite type over k,
(2) R is a regular local ring,
(3) $f_1, \ldots, f_c$ form a regular sequence in R, and
(4) $S \cong R/(f_1, \ldots, f_c)$ as k-algebras.

By the Cohen structure theorem (see Theorem 7.143.8) any complete Noetherian local ring may be written as the quotient of some regular complete local ring. Hence we may use the definition above to define the notion of a complete intersection ring for any complete Noetherian local ring. We will discuss this later, see (insert future reference here). In the meantime the following lemma shows that such a definition makes sense.

Lemma 7.124.6. *Let $A \to B \to C$ be surjective local ring homomorphisms. Assume A and B are regular local rings. The following are equivalent*

(1) *$Ker(A \to C)$ is generated by a regular sequence,*
(2) *$Ker(A \to C)$ is generated by* $\dim(A) - \dim(C)$ *elements,*
(3) *$Ker(B \to C)$ is generated by a regular sequence, and*
(4) *$Ker(B \to C)$ is generated by* $\dim(B) - \dim(C)$ *elements.*

Proof. A regular local ring is Cohen-Macaulay, see Lemma 7.98.3. Hence the equivalences (1) $\Leftrightarrow$ (2) and (3) $\Leftrightarrow$ (4), see Proposition 7.95.4. By Lemma 7.98.4 the ideal $\mathrm{Ker}(A \to B)$ can be generated by $\dim(A) - \dim(B)$ elements. Hence we see that (4) implies (2).

It remains to show that (1) implies (4). We do this by induction on $\dim(A) - \dim(B)$. The case $\dim(A) - \dim(B) = 0$ is trivial. Assume $\dim(A) > \dim(B)$. Write $I = \mathrm{Ker}(A \to C)$ and $J = \mathrm{Ker}(A \to B)$. Note that $J \subset I$. Our assumption is that the minimal number of generators of I is $\dim(A) - \dim(C)$. Let $\mathfrak{m} \subset A$ be the maximal ideal. Consider the maps

$$J/\mathfrak{m}J \to I/\mathfrak{m}I \to \mathfrak{m}/\mathfrak{m}^2$$

By Lemma 7.98.4 and its proof the composition is injective. Take any element $x \in J$ which is not zero in $J/\mathfrak{m}J$. By the above and Nakayama's lemma x is an element of a minimal set of generators of I. Hence we may replace A by A/xA and I by I/xA which decreases both $\dim(A)$ and the minimal number of generators of I by 1. Thus we win. □

Lemma 7.124.7. *Let k be a field. Let S be a local k-algebra essentially of finite type over k. The following are equivalent:*

1. *S is a complete intersection over k,*
2. *for any surjection $R \to S$ with R a regular local ring essentially of finite presentation over k the ideal $\mathrm{Ker}(R \to S)$ can be generated by a regular sequence,*
3. *for some surjection $R \to S$ with R a regular local ring essentially of finite presentation over k the ideal $\mathrm{Ker}(R \to S)$ can be generated by* $\dim(R) - \dim(S)$ *elements,*
4. *there exists a global complete intersection A over k and a prime $\mathfrak{a}$ of A such that $S \cong A_{\mathfrak{a}}$, and*
5. *there exists a local complete intersection A over k and a prime $\mathfrak{a}$ of A such that $S \cong A_{\mathfrak{a}}$.*

Proof. It is clear that (2) implies (1) and (1) implies (3). It is also clear that (4) implies (5). Let us show that (3) implies (4). Thus we assume there exists a surjection $R \to S$ with R a regular local ring essentially of finite presentation over k such that the ideal $\mathrm{Ker}(R \to S)$ can be generated by $\dim(R) - \dim(S)$ elements. We may write $R = (k[x_1, \ldots, x_n]/J)_{\mathfrak{q}}$ for some $J \subset k[x_1, \ldots, x_n]$ and some prime $\mathfrak{q} \subset k[x_1, \ldots, x_n]$ with $J \subset \mathfrak{q}$. Let $I \subset k[x_1, \ldots, x_n]$ be the kernel of the map $k[x_1, \ldots, x_n] \to S$ so that $S \cong (k[x_1, \ldots, x_n]/I)_{\mathfrak{q}}$. By assumption $(I/J)_{\mathfrak{q}}$ is generated by $\dim(R) - \dim(S)$ elements. We conclude that $I_{\mathfrak{q}}$ can be generated by $\dim(k[x_1, \ldots, x_n]_{\mathfrak{q}}) - \dim(S)$ elements by Lemma 7.124.6. From Lemma 7.124.4 we see that for some $g \in k[x_1, \ldots, x_n]$, $g \notin \mathfrak{q}$ the algebra $(k[x_1, \ldots, x_n]/I)_g$ is a global complete intersection and S is isomorphic to a local ring of it.

To finish the proof of the lemma we have to show that (5) implies (2). Assume (5) and let $\pi : R \to S$ be a surjection with R a regular local k-algebra essentially of finite type over k. By assumption we have $S = A_{\mathfrak{a}}$ for some local complete intersection A over k. Choose a presentation $R = (k[y_1, \ldots, y_m]/J)_{\mathfrak{q}}$ with $J \subset \mathfrak{q} \subset k[y_1, \ldots, y_m]$. We may and do assume that J is the kernel of the map $k[y_1, \ldots, y_m] \to R$. Let $I \subset k[y_1, \ldots, y_m]$ be the kernel of the map $k[y_1, \ldots, y_m] \to S = A_{\mathfrak{a}}$. Then $J \subset I$ and $(I/J)_{\mathfrak{q}}$ is the kernel of the surjection $\pi : R \to S$. So $S = (k[y_1, \ldots, y_m]/I)_{\mathfrak{q}}$.

By Lemma 7.117.7 we see that there exist $g \in A$, $g \notin \mathfrak{a}$ and $g' \in k[y_1, \ldots, y_m]$, $g' \notin \mathfrak{q}$ such that $A_g \cong (k[y_1, \ldots, y_m]/I)_{g'}$. After replacing A by A_g and $k[y_1, \ldots, y_m]$ by $k[y_1, \ldots, y_{m+1}]$ we may assume that $A \cong k[y_1, \ldots, y_m]/I$. Consider the surjective maps of local rings

$$k[y_1, \ldots, y_m]_{\mathfrak{q}} \to R \to S.$$

We have to show that the kernel of $R \to S$ is generated by a regular sequence. By Lemma 7.124.4 we know that $k[y_1, \ldots, y_m]_{\mathfrak q} \to A_{\mathfrak a} = S$ has this property (as A is a local complete intersection over k). We win by Lemma 7.124.6. $\square$

Lemma 7.124.8. *Let k be a field. Let S be a finite type k-algebra. Let $\mathfrak q$ be a prime of S. The following are equivalent:*

1. *The local ring $S_{\mathfrak q}$ is a complete intersection ring (Definition 7.124.5).*
2. *There exists a $g \in S$, $g \not\in \mathfrak q$ such that S_g is a local complete intersection over k.*
3. *There exists a $g \in S$, $g \not\in \mathfrak q$ such that S_g is a global complete intersection over k.*
4. *For any presentation $S = k[x_1, \ldots, x_n]/I$ with $\mathfrak q' \subset k[x_1, \ldots, x_n]$ corresponding to $\mathfrak q$ any of the equivalent conditions (1) -- (5) of Lemma 7.124.4 hold.*

Proof. This is a combination of Lemmas 7.124.4 and 7.124.7 and the definitions. $\square$

Lemma 7.124.9. *Let k be a field. Let S be a finite type k-algebra. The following are equivalent:*

1. *The ring S is a local complete intersection over k.*
2. *All local rings of S are complete intersection rings over k.*
3. *All localizations of S at maximal ideals are complete intersection rings over k.*

Proof. This follows from Lemma 7.124.8, the fact that $Spec(S)$ is quasi-compact and the definitions. $\square$

The following lemma says that being a complete intersection is preserved under change of base field (in a strong sense).

Lemma 7.124.10. *Let $k \subset K$ be a field extension. Let S be a finite type algebra over k. Let $\mathfrak q_K$ be a prime of $S_K = K \otimes_k S$ and let $\mathfrak q$ be the corresponding prime of S. Then $S_{\mathfrak q}$ is a complete intersection over k (Definition 7.124.5) if and only if $(S_K)_{\mathfrak q_K}$ is a complete intersection over K.*

Proof. Choose a presentation $S = k[x_1, \ldots, x_n]/I$. This gives a presentation $S_K = K[x_1, \ldots, x_n]/I_K$ where $I_K = K \otimes_k I$. Let $\mathfrak q'_K \subset K[x_1, \ldots, x_n]$, resp. $\mathfrak q' \subset k[x_1, \ldots, x_n]$ be the corresponding prime. We will show that the equivalent conditions of Lemma 7.124.4 hold for the pair $(S = k[x_1, \ldots, x_n]/I, \mathfrak q)$ if and only if they hold for the pair $(S_K = K[x_1, \ldots, x_n]/I_K, \mathfrak q_K)$. The lemma will follow from this (see Lemma 7.124.8).

By Lemma 7.107.6 we have $\dim_{\mathfrak q} S = \dim_{\mathfrak q_K} S_K$. Hence the integer c occuring in Lemma 7.124.4 is the same for the pair $(S = k[x_1, \ldots, x_n]/I, \mathfrak q)$ as for the pair $(S_K = K[x_1, \ldots, x_n]/I_K, \mathfrak q_K)$. On the other hand we have

$$\begin{aligned} I \otimes_{k[x_1,\ldots,x_n]} \kappa(\mathfrak q') \otimes_{\kappa(\mathfrak q')} \kappa(\mathfrak q'_K) &= I \otimes_{k[x_1,\ldots,x_n]} \kappa(\mathfrak q'_K) \\ &= I \otimes_{k[x_1,\ldots,x_n]} K[x_1, \ldots, x_n] \otimes_{K[x_1,\ldots,x_n]} \kappa(\mathfrak q'_K) \\ &= (K \otimes_k I) \otimes_{K[x_1,\ldots,x_n]} \kappa(\mathfrak q'_K) \\ &= I_K \otimes_{K[x_1,\ldots,x_n]} \kappa(\mathfrak q'_K). \end{aligned}$$

Therefore, $\dim_{\kappa(\mathfrak q')} I \otimes_{k[x_1,\ldots,x_n]} \kappa(\mathfrak q') = \dim_{\kappa(\mathfrak q'_K)} I_K \otimes_{K[x_1,\ldots,x_n]} \kappa(\mathfrak q'_K)$. Thus it follows from Nakayama's Lemma 7.14.5 that the minimal number of generators of $I_{\mathfrak q'}$ is the same as the minimal number of generators of $(I_K)_{\mathfrak q'_K}$. Thus the lemma follows from characterization (2) of Lemma 7.124.4. $\square$

Lemma 7.124.11. *Let $k \to K$ be a field extension. Let S be a finite type k-algebra. Then S is a local complete intersection over k if and only if $S \otimes_k K$ is a local complete intersection over K.*

Proof. This follows from a combination of Lemmas 7.124.9 and 7.124.10. But we also give a different proof here (based on the same principles).

Set $S' = S \otimes_k K$. Let $\alpha : k[x_1, \ldots, x_n] \to S$ be a presentation with kernel I. Let $\alpha' : K[x_1, \ldots, x_n] \to S'$ be the induced presentation with kernel I'.

Suppose that S is a local complete intersection. Pick a prime $\mathfrak{q} \subset S'$. Denote $\mathfrak{q}'$ the corresponding prime of $K[x_1, \ldots, x_n]$, $\mathfrak{p}$ the corresponding prime of S, and $\mathfrak{p}'$ the corresponding prime of $k[x_1, \ldots, x_n]$. Consider the following diagram of Noetherian local rings

$$\begin{tikzcd} S'_{\mathfrak{q}} & K[x_1, \ldots, x_n]_{\mathfrak{q}'} \arrow[l] \\ S_{\mathfrak{p}} \arrow[u] & k[x_1, \ldots, x_n]_{\mathfrak{p}'} \arrow[l] \arrow[u] \end{tikzcd}$$

By Lemma 7.124.4 we know that $S_{\mathfrak{p}}$ is cut out by some regular sequence $f_1, \ldots, f_e$ in $k[x_1, \ldots, x_n]_{\mathfrak{p}'}$. Since the right vertical arrow is flat we see that the images of $f_1, \ldots, f_c$ form a regular sequence in $K[x_1, \ldots, x_n]_{\mathfrak{q}'}$. Because tensoring with K over k is an exact functor we have $S'_{\mathfrak{q}} = K[x_1, \ldots, x_n]_{\mathfrak{q}'}/(f_1, \ldots, f_e)$. Hence by Lemma 7.124.4 again we see that S' is a local complete intersection in a neighbourhood of $\mathfrak{q}$. Since $\mathfrak{q}$ was arbitrary we see that S' is a local complete intersection over K.

Suppose that S' is a local complete intersection. Pick a maximal ideal $\mathfrak{m}$ of S. Let $\mathfrak{m}'$ denote the corresponding maximal ideal of $k[x_1, \ldots, x_n]$. Denote $\kappa = \kappa(\mathfrak{m})$ the residue field. By Remark 7.16.8 the primes of S' lying over $\mathfrak{m}$ correspond to primes in $K \otimes_k \kappa$. By the Hilbert-Nullstellensatz Theorem 7.30.1 we have $[\kappa : k] < \infty$. Hence $K \otimes_k \kappa$ is finite nonzero over K. Hence $K \otimes_k \kappa$ has a finite number > 0 of primes which are all maximal, each of which has a residue field finite over K (see Section 7.49). Hence there are finitely many > 0 prime ideals $\mathfrak{n} \subset S'$ lying over $\mathfrak{m}$, each of which is maximal and has a residue field which is finite over K. Pick one, say $\mathfrak{n} \subset S'$, and let $\mathfrak{n}' \subset K[x_1, \ldots, x_n]$ denote the corresponding prime ideal of $K[x_1, \ldots, x_n]$. Note that since $V(\mathfrak{m}S')$ is finite, we see that $\mathfrak{n}$ is an isolated closed point of it, and we deduce that $\mathfrak{m}S'_{\mathfrak{n}}$ is an ideal of definition of $S'_{\mathfrak{n}}$. This implies that $\dim(S_{\mathfrak{m}}) \geq \dim(S'_{\mathfrak{n}})$, for example by Lemma 7.103.6 or by the characterization of dimension in terms of minimal number of generators of ideal of definition, see Section 7.57. (In reality the dimensions are equal but we do not need this.) Consider the corresponding diagram of Noetherian local rings

$$\begin{tikzcd} S'_{\mathfrak{n}} & K[x_1, \ldots, x_n]_{\mathfrak{n}'} \arrow[l] \\ S_{\mathfrak{m}} \arrow[u] & k[x_1, \ldots, x_n]_{\mathfrak{m}'} \arrow[l] \arrow[u] \end{tikzcd}$$

According to Lemma 7.123.6 we have $NL(\alpha) \otimes_S S' = NL(\alpha')$, in particular $I'/(I')^2 = I/I^2 \otimes_S S'$. Thus $(I/I^2)_{\mathfrak{m}} \otimes_{S_{\mathfrak{m}}} \kappa$ and $(I'/(I')^2)_{\mathfrak{n}} \otimes_{S'_{\mathfrak{n}}} \kappa(\mathfrak{n})$ have the same dimension. Since $(I'/(I')^2)_{\mathfrak{n}}$ is free of rank $n - \dim S'_{\mathfrak{n}}$ we deduce that $(I/I^2)_{\mathfrak{m}}$ can be generated by $n - \dim S'_{\mathfrak{n}} \leq n - \dim S_{\mathfrak{m}}$ elements. By Lemma 7.124.4 we see that S is a local complete intersection in a neighbourhood of $\mathfrak{m}$. Since $\mathfrak{m}$ was any maximal ideal we conclude that S is a local complete intersection. □

We end with a lemma which we will later use to prove that given ring maps $T \to A \to B$ where B is syntomic over T, and B is syntomic over A, then A is syntomic over T.

Lemma 7.124.12. *Let*

$$\begin{tikzcd} B & S \arrow[l] \\ A \arrow[u] & R \arrow[l] \arrow[u] \end{tikzcd}$$

be a commutative square of local rings. Assume

(1) *R and $\overline{S} = S/\mathfrak{m}_R S$ are regular local rings,*
(2) *$A = R/I$ and $B = S/J$ for some ideals I, J,*
(3) *$J \subset S$ and $\overline{J} = J/\mathfrak{m}_R \cap J \subset \overline{S}$ are generated by regular sequences, and*
(4) *$A \to B$ and $R \to S$ are flat.*

Then I is generated by a regular sequence.

Proof. Set $\overline{B} = B/\mathfrak{m}_R B = B/\mathfrak{m}_A B$ so that $\overline{B} = \overline{S}/\overline{J}$. Let $f_1, \ldots, f_{\overline{c}} \in J$ be elements such that $\overline{f}_1, \ldots, \overline{f}_{\overline{c}} \in \overline{J}$ form a regular sequence generating $\overline{J}$. Note that $\overline{c} = \dim(\overline{S}) - \dim(\overline{B})$, see Lemma 7.124.6. By Lemma 7.91.3 the ring $S/(f_1, \ldots, f_{\overline{c}})$ is flat over R. Hence $S/(f_1, \ldots, f_{\overline{c}}) + IS$ is flat over A. The map $S/(f_1, \ldots, f_{\overline{c}}) + IS \to B$ is therefore a surjection of finite S/IS-modules flat over A which is an isomorphism modulo $\mathfrak{m}_A$, and hence an isomorphism by Lemma 7.91.1. In other words, $J = (f_1, \ldots, f_{\overline{c}}) + IS$.

By Lemma 7.124.6 again the ideal J is generated by a regular sequence of $c = \dim(S) - \dim(B)$ elements. Hence $J/\mathfrak{m}_S J$ is a vector space of dimension c. By the description of J above there exist $g_1, \ldots, g_{c-\overline{c}} \in I$ such that J is generated by $f_1, \ldots, f_{\overline{c}}, g_1, \ldots, g_{c-\overline{c}}$ (use Nakayama's Lemma 7.14.5). Consider the ring $A' = R/(g_1, \ldots, g_{c-\overline{c}})$ and the surjection $A' \to A$. We see from the above that $B = S/(f_1, \ldots, f_{\overline{c}}, g_1, \ldots, g_{c-\overline{c}})$ is flat over A' (as $S/(f_1, \ldots, f_{\overline{c}})$ is flat over R). Hence $A' \to B$ is injective (as it is faithfully flat, see Lemma 7.35.16). Since this map factors through A we get $A' = A$. Note that $\dim(B) = \dim(A) + \dim(\overline{B})$, and $\dim(S) = \dim(R) + \dim(\overline{S})$, see Lemma 7.103.7. Hence $c - \overline{c} = \dim(R) - \dim(A)$ by elementary algebra. Thus $I = (g_1, \ldots, g_{c-\overline{c}})$ is generated by a regular sequence according to Lemma 7.124.6. □

7.125. Syntomic morphisms

Definition 7.125.1. A ring map $R \to S$ is called *syntomic*, or we say S is a *flat local complete intersection over* R if it is flat, of finite presentation, and if all of its fibre rings $S \otimes_R \kappa(\mathfrak{p})$ are local complete intersections, see Definition 7.124.1.

Clearly, an algebra over a field is syntomic over the field if and only if it is a local complete intersection. Here is a pleasing feature of this definition.

Lemma 7.125.2. *Let $R \to S$ be a ring map. Let $R \to R'$ be a faithfully flat ring map. Set $S' = R' \otimes_R S$. Then $R \to S$ is syntomic if and only if $R' \to S'$ is syntomic.*

Proof. By Lemma 7.117.2 and Lemma 7.35.7 this holds for the property of being flat and for the property of being of finite presentation. The map $Spec(R') \to Spec(R)$ is surjective, see Lemma 7.35.15. Thus it suffices to show given primes $\mathfrak{p}' \subset R'$ lying over $\mathfrak{p} \subset R$ that $S \otimes_R \kappa(\mathfrak{p})$ is a local complete intersection if and only if $S' \otimes_{R'} \kappa(\mathfrak{p}')$ is a local complete intersection. Note that $S' \otimes_{R'} \kappa(\mathfrak{p}') = S \otimes_R \kappa(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p}')$. Thus Lemma 7.124.11 applies. □

Lemma 7.125.3. *Any base change of a syntomic map is syntomic.*

Proof. This is true for being flat, for being of finite presentation, and for having local complete intersections as fibres by Lemmas 7.35.6, 7.6.2 and 7.124.11. $\square$

Lemma 7.125.4. *Let $R \to S$ be a ring map. Suppose we have $g_1, \ldots g_m \in S$ which generate the unit ideal such that each $R \to S_{g_i}$ is syntomic. Then $R \to S$ is syntomic.*

Proof. This is true for being flat and for being of finite presentation by Lemmas 7.35.19 and 7.21.3. The property of having fibre rings which are local complete intersections is local on S by its very definition, see Definition 7.124.1. $\square$

Definition 7.125.5. Let $R \to S$ be a ring map. We say that $R \to S$ is a *relative global complete intersection* if we are given a presentation $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ such that every nonempty fibre has dimension $n - c$.

The following lemma is occasionally useful to find global presentations.

Lemma 7.125.6. *Let S be a finitely presented R-algebra which has a presentation $S = R[x_1, \ldots, x_n]/I$ such that I/I^2 is free over S. Then S has a presentation $S = R[y_1, \ldots, y_m]/(f_1, \ldots, f_c)$ such that $(f_1, \ldots, f_c)/(f_1, \ldots, f_c)^2$ is free with basis given by the classes of $f_1, \ldots, f_c$.*

Proof. Note that I is a finitely generated ideal by Lemma 7.6.3. Let $f_1, \ldots, f_c \in I$ be elements which map to a basis of I/I^2. By Nakayama's lemma (Lemma 7.14.5) there exists a $g \in 1 + I$ such that

$$g \cdot I \subset (f_1, \ldots, f_c)$$

Hence we see that

$$S \cong R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)[1/g] \cong R[x_1, \ldots, x_n, x_{n+1}]/(f_1, \ldots, f_c, gx_{n+1} - 1)$$

as desired. $\square$

Example 7.125.7. Let $n, m \geq 1$ be integers. Consider the ring map

$$\begin{array}{rcl} R = \mathbf{Z}[a_1, \ldots, a_{n+m}] & \longrightarrow & S = \mathbf{Z}[b_1, \ldots, b_n, c_1, \ldots, c_m] \\ a_1 & \longmapsto & b_1 + c_1 \\ a_2 & \longmapsto & b_2 + b_1c_1 + c_2 \\ \ldots & \ldots & \ldots \\ a_{n+m} & \longmapsto & b_nc_m \end{array}$$

In other words, this is the unique ring map of polynomial rings as indicated such that the polynomial factorization

$$x^n + a_1x^{n-1} + \ldots + a_{n+m} = (x^n + b_1x^{n-1} + \ldots + b_n)(x^m + c_1x^{m-1} + \ldots + c_m)$$

holds. Note that S is generated by $n + m$ elements over R (namely, b_i, c_j) and that there are $n + m$ equations (namely $a_k = a_k(b_i, c_j)$). In order to show that S is a relative global complete intersection over R it suffices to prove that all fibres have dimension 0.

To prove this, let $R \to k$ be a ring map into a field k. Say a_i maps to $\alpha_i \in k$. Consider the fibre ring $S_k = k \otimes_R S$. Let $k \to K$ be a field extension. A k-algebra map of $S_k \to K$ is the same thing as finding $\beta_1, \ldots, \beta_n, \gamma_1, \ldots, \gamma_m \in K$ such that

$$x^n + \alpha_1x^{n-1} + \ldots + \alpha_{n+m} = (x^n + \beta_1x^{n-1} + \ldots + \beta_n)(x^m + \gamma_1x^{m-1} + \ldots + \gamma_m).$$

Hence we see there are at most finitely many choices of such $n + m$-tuples in K. This proves that all fibres have finitely many closed points (use Hilbert's Nullstellensatz to see they all correspond to solutions in $\overline{k}$ for example) and hence that $R \to S$ is a relative global complete intersection.

Another way to argue this is to show $\mathbf{Z}[a_1, \ldots, a_{n+m}] \to \mathbf{Z}[b_1, \ldots, b_n, c_1, \ldots, c_m]$ is actually also a *finite* ring map. Namely, by Lemma 7.34.5 each of b_i, c_j is integral over R, and hence $R \to S$ is finite by Lemma 7.32.4.

Example 7.125.8. Consider the ring map

$$\begin{array}{ccc} R = \mathbf{Z}[a_1, \ldots, a_n] & \longrightarrow & S = \mathbf{Z}[\alpha_1, \ldots, \alpha_n] \\ a_1 & \longmapsto & \alpha_1 + \ldots + \alpha_n \\ \ldots & \ldots & \ldots \\ a_n & \longmapsto & \alpha_1 \ldots \alpha_n \end{array}$$

In other words this is the unique ring map of polynomial rings as indicated such that

$$x^n + a_1 x^{n-1} + \ldots + a_n = \prod\nolimits_{i=1}^n (x + \alpha_i)$$

holds in $\mathbf{Z}[\alpha_i, x]$. Another way to say this is that a_i maps to the ith elementary symmetric function in $\alpha_1, \ldots, \alpha_n$. Note that S is generated by n elements over R subject to n equations. Hence to show that S is a global relative complete intersection over R we have to show that the fibre rings $S \otimes_R \kappa(\mathfrak{p})$ have dimension 0. This follows as in Example 7.125.7 above because the ring map $\mathbf{Z}[a_1, \ldots, a_n] \to \mathbf{Z}[\alpha_1, \ldots, \alpha_n]$ is actually *finite* since each $\alpha_i \in S$ satisfies the monic equation $x^n - a_1 x^{n-1} + \ldots + (-1)^n a_n$ over R.

Lemma 7.125.9. *Suppose that A is a ring, and $P(x) = x^n + b_1 x^{n-1} + \ldots + b_n \in A[x]$ is a monic polynomial over A. Then there exists a syntomic, finite locally free, faithfully flat ring extension $A \subset A'$ such that $P(x) = \prod_{i=1,\ldots,n}(x - \beta_i)$ for certain $\beta_i \in A'$.*

Proof. Take $A' = A \otimes_R S$, where R and S are as in Example 7.125.8 above, where $R \to A$ maps a_i to b_i, and let $\beta_i = -1 \otimes \alpha_i$. □

Lemma 7.125.10. *Let $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ be a relative global complete intersection over R.*

(1) *For any $R \to R'$ the base change $R' \otimes_R S = R'[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a relative global complete intersection.*
(2) *For any $g \in S$ which is the image of $h \in R[x_1, \ldots, x_n]$ the ring $S_g = R[x_1, \ldots, x_n, x_{n+1}]/(f_1, \ldots, f_c, hx_{n+1} - 1)$ is a relative global complete intersection.*
(3) *If $R \to S$ factors as $R \to R_f \to S$ for some $f \in R$. Then the ring $S = R_f[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a relative global complete intersection over R_f.*

Proof. By Lemma 7.107.5 the fibres of a base change have the same dimension as the fibres of the original map. Moreover $R' \otimes_R R[x_1, \ldots, x_n]/(f_1, \ldots, f_c) = R'[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$. Thus (1) follows. The proof of (2) is that the localization at one element can be described as $S_g \cong S[x_{n+1}]/(gx_{n+1} - 1)$. Assertion (3) follows from (1) since under the assumptions of (3) we have $R_f \otimes_R S \cong S$. □

Lemma 7.125.11. *Let R be a ring. Let $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$. Let $\mathfrak{q} \subset S$ be a prime lying over $\mathfrak{p} \subset R$. If $\dim_{\mathfrak{q}}(S/R) = n - c$, then there exists a $h \in R[x_1, \ldots, x_n]$ which maps to $g \in S$, $g \notin \mathfrak{q}$ such that*

$$S_g = R[x_1, \ldots, x_n, x_{n+1}]/(f_1, \ldots, f_c, hx_{n+1} - 1)$$

is a relative global complete intersection over R.

Proof. By Lemma 7.116.6 there exists a $g \in S$, $g \notin \mathfrak q$ such that all nonempty fibres of $R \to S_g$ have dimension $\leq n-c$. Let $h \in R[x_1, \ldots, x_n]$ be an element that maps to g. Then $S_g \cong R[x_1, \ldots, x_n, x_{n+1}]/(f_1, \ldots, f_c, f_{c+1})$ with $f_{c+1} = hx_{n+1} - 1$. Thus S_g is a relative global complete intersection. □

The following lemma ``almost'' states that we can do absolute Noetherian approximation for relative complete intersections. It is a bit awkward because we have not yet developed enough theory to deal with the condition on the dimensions of fibres. We will state and prove the correct version of this lemma in More on Morphisms, Lemma 33.24.10.

Lemma 7.125.12. *Let R be a ring. Let S be a relative global complete intersection with presentation $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$. Let $\mathfrak q \subset S$ be a prime. There exist*

1. *a finite type $\mathbf{Z}$-subalgebra $R_0 \subset R$ such that $f_i \in R_0[x_1, \ldots, x_n]$, and*
2. *an element $h \in R_0[x_1, \ldots, x_n]$*

such that with $f_{c+1} = hx_{n+1} - 1$ we have

1. *h maps to an element g of S which is not in $\mathfrak q$, and*
2. *$R_0[x_1, \ldots, x_n, x_{n+1}]/(f_1, \ldots, f_c, f_{c+1})$ is a relative global complete intersection over R_0.*

In particular S_g is isomorphic to the base change of a relative global complete intersection over R_0.

Proof. Let $R_0 \subset R$ be the $\mathbf{Z}$-algebra of R generated by all the coefficients of the polynomials $f_1, \ldots, f_c$. Let $S_0 = R_0[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$. Clearly, $S = R \otimes_{R_0} S_0$. Suppose our $\mathfrak q \subset S$ lies over the prime $\mathfrak p \subset R$, and let $\mathfrak q_0 \subset S_0$ be the corresponding prime lying over the prime $\mathfrak p_0$ of R_0. Because $\dim(S \otimes_R \kappa(\mathfrak p)) = n - c$ we also have $\dim(S_0 \otimes_{R_0} \kappa(\mathfrak p_0)) = n-c$, see Lemma 7.107.5 for example. By Lemma 7.125.11 we conclude that there exists a $g \in S_0$, $g \notin \mathfrak q_0$ such that $R_0 \to (S_0)_g$ is a relative global complete intersection. Let $h \in R_0[x_1, \ldots, x_n]$ be any element mapping to g. □

Lemma 7.125.13. *Let R be a ring. Let $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ be a relative global complete intersection. For every prime $\mathfrak q$ of S, let $\mathfrak q'$ denote the corresponding prime of $R[x_1, \ldots, x_n]$. Then*

1. *$f_1, \ldots, f_c$ is a regular sequence in the local ring $R[x_1, \ldots, x_n]_{\mathfrak q'}$,*
2. *each of the rings $R[x_1, \ldots, x_n]_{\mathfrak q'}/(f_1, \ldots, f_i)$ is flat over R, and*
3. *the S-module $(f_1, \ldots, f_c)/(f_1, \ldots, f_c)^2$ is free with basis given by the elements $f_i \bmod (f_1, \ldots, f_c)^2$.*

Proof. First, by Lemma 7.66.2, the last statement follows from the first.

We first reduce (1) and (2) to the Noetherian case. Namely, assume the lemma holds in the Noetherian case. For every $\mathfrak q$ we may choose R_0, h, $f_{c+1} = hx_{n+1} - 1$ and g as in Lemma 7.125.12. Denote $\mathfrak q'' \subset R[x_1, \ldots, x_{n+1}]$ the unique prime containing $\mathfrak q$ and f_{c+1}, i.e., the one that corresponds to the prime $\mathfrak q S_g$ via the isomorphism $S_g = R[x_1, \ldots, x_n, x_{n+1}]/(f_1, \ldots, f_c, f_{c+1})$. Denote $\mathfrak q_0'' \subset R_0[x_1, \ldots, x_n, x_{n+1}]$ the prime corresponding to q''. Because $S_0 = R_0[x_1, \ldots, x_n, x_{n+1}]/(f_1, \ldots, f_c, f_{c+1})$ is a relative global complete intersection over a Noetherian ring we see that $f_{c+1}, f_1, \ldots, f_c$ is a regular sequence in $R_0[x_1, \ldots, x_{n+1}]_{\mathfrak q_0''}$. Since after all we may reorder the elements at will without destroying the property of being a relative global complete intersection. Similarly each ring $R_0[x_1, \ldots, x_{n+1}]_{\mathfrak q_0''}/(f_{c+1}, f_1, \ldots, f_i)$ is flat over R_0. Thus each short exact sequence

$$0 \to \frac{(P_0)_{\mathfrak q_0''}}{(f_{c+1}, f_1, \ldots, f_{i-1})} \to \frac{(P_0)_{\mathfrak q_0''}}{(f_{c+1}, f_1, \ldots, f_{i-1})} \to \frac{(P_0)_{\mathfrak q_0''}}{(f_{c+1}, f_1, \ldots, f_i)} \to 0$$

and the short exact sequence

$$0 \to (P_0)_{\mathfrak q_0''} \to (P_0)_{\mathfrak q_0''} \to (P_0)_{\mathfrak q_0''}/(f_{c+1}) \to 0$$

with $P_0 = R_0[x_1, \ldots, x_n, x_{n+1}]$ remain exact upon tensoring over R_0 with R, see Lemma 7.35.11. Since $R[x_1, \ldots, x_n, x_{n+1}]_{\mathfrak q''}$ is a localization of $R \otimes_{R_0} P_0$ we conclude that $f_{c+1}, f_1, \ldots, f_c$ form a regular sequence in the local ring $R[x_1, \ldots, x_n, x_{n+1}]_{\mathfrak q''}$. Finally we use the obvious isomorphism $R[x_1, \ldots, x_{n+1}]_{\mathfrak q''}/(f_{c+1}) \cong R[x_1, \ldots, x_n]_{\mathfrak q'}$, to conclude that $f_1, \ldots, f_c$ form a regular sequence in the local ring $R[x_1, \ldots, x_n]_{\mathfrak q'}$. Similarly the quotients

$$R[x_1, \ldots, x_n, x_{n+1}]_{\mathfrak q''}/(f_{c+1}, f_1, \ldots, f_i) \cong R[x_1, \ldots, x_n]_{\mathfrak q'}/(f_1, \ldots, f_i)$$

are flat over R as desired.

It remains to show (1) and (2) in case R is Noetherian. By Lemma 7.124.4 for example we see that $f_1, \ldots, f_c$ form a regular sequence in the local ring $R[x_1, \ldots, x_n]_{\mathfrak q'} \otimes_R \kappa(\mathfrak p)$. Moreover, the local ring $R[x_1, \ldots, x_n]_{\mathfrak q'}$ is flat over $R_{\mathfrak p}$. Since R, and hence $R[x_1, \ldots, x_n]_{\mathfrak q'}$ is Noetherian we may apply Lemma 7.91.3 to conclude. ☐

Lemma 7.125.14. *A relative global complete intersection is syntomic.*

Proof. Let $R \to S$ be a relative global complete intersection. The fibres are global complete intersections, and S is of finite presentation over R. Thus the only thing to prove is that $R \to S$ is flat. This is true by (2) of Lemma 7.125.13 above. ☐

The following technical lemma says that you can lift any sequence of relations from a fibre to the whole space of a ring map which is essentially of finite type, in a suitable sense.

Lemma 7.125.15. *Let $R \to S$ be a ring map. Let $\mathfrak p \subset R$ be a prime. Let $\mathfrak q \subset S$ be a prime lying over $\mathfrak p$. Assume $S_{\mathfrak q}$ is essentially of finite type over $R_{\mathfrak p}$. Assume given*

(1) *an integer $n \geq 0$,*
(2) *a prime $\mathfrak a \subset \kappa(\mathfrak p)[x_1, \ldots, x_n]$,*
(3) *a surjective $\kappa(\mathfrak p)$-homomorphism*

$$\psi : (\kappa(\mathfrak p)[x_1, \ldots, x_n])_{\mathfrak a} \longrightarrow S_{\mathfrak q}/\mathfrak p S_{\mathfrak q},$$

and

(4) *elements $\overline{f}_1, \ldots, \overline{f}_e$ in Ker(ψ).*

Then there exist

(1) *an integer $m \geq 0$,*
(2) *and element $g \in S$, $g \notin \mathfrak q$,*
(3) *a map*

$$\Psi : R[x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}] \longrightarrow S_g,$$

and

(4) *elements $f_1, \ldots, f_e, f_{e+1}, \ldots, f_{e+m}$ of Ker(Ψ)*

such that

(1) *the following diagram commutes*

$$\begin{tikzcd} R[x_1, \ldots, x_{n+m}] \arrow[r, "x_{n+j}\mapsto 0"'] \arrow[d, "\Psi"'] & (\kappa(\mathfrak p)[x_1, \ldots, x_n])_{\mathfrak a} \arrow[d, "\psi"] \\ S_g \arrow[r] & S_{\mathfrak q}/\mathfrak p S_{\mathfrak q} \end{tikzcd},$$

(2) *the element f_i, $i \leq n$ maps to a unit times $\overline{f}_i$ in the local ring*

$$(\kappa(\mathfrak p)[x_1, \ldots, x_{n+m}])_{(\mathfrak a, x_{n+1}, \ldots, x_{n+m})},$$

(3) *the element f_{e+j} maps to a unit times x_{n+j} in the same local ring, and*

(4) *the induced map $R[x_1, \ldots, x_{n+m}]_{\mathfrak b} \to S_{\mathfrak q}$ is surjective, where $\mathfrak b = \Psi^{-1}(\mathfrak q S_g)$.*

Proof. We claim that it suffices to prove the lemma in case R and S are local with maximal ideals $\mathfrak p$ and $\mathfrak q$. Namely, suppose we have constructed

$$\Psi' : R_{\mathfrak p}[x_1, \ldots, x_{n+m}] \longrightarrow S_{\mathfrak q}$$

and $f'_1, \ldots, f'_{e+m} \in R_{\mathfrak p}[x_1, \ldots, x_{n+m}]$ with all the required properties. Then there exists an element $f \in R$, $f \not\in \mathfrak p$ such that each ff'_k comes from an element $f_k \in R[x_1, \ldots, x_{n+m}]$. Moreover, for a suitable $g \in S$, $g \not\in \mathfrak q$ the elements $\Psi'(x_i)$ are the image of elements $y_i \in S_g$. Let Ψ be the R-algebra map defined by the rule $\Psi(x_i) = y_i$. Since $\Psi(f_i)$ is zero in the localization $S_{\mathfrak q}$ we may after possibly replacing g assume that $\Psi(f_i) = 0$. This proves the claim.

Thus we may assume R and S are local with maximal ideals $\mathfrak p$ and $\mathfrak q$. Pick $y_1, \ldots, y_n \in S$ such that $y_i \bmod \mathfrak p S = \psi(x_i)$. Let $y_{n+1}, \ldots, y_{n+m} \in S$ be elements which generate an R-subalgebra of which S is the localization. These exist by the assumption that S is essentially of finite type over R. Since ψ is surjective we may write $y_{n+j} \bmod \mathfrak p S = \psi(h_j)$ for some $h_j \in \kappa(\mathfrak p)[x_1, \ldots, x_n]_{\mathfrak a}$. Write $h_j = g_j/d$, $g_j \in \kappa(\mathfrak p)[x_1, \ldots, x_n]$ for some common denominator $d \in \kappa(\mathfrak p)[x_1, \ldots, x_n]$, $d \not\in \mathfrak a$. Choose lifts $G_j, D \in R[x_1, \ldots, x_n]$ of g_j and d. Set $y'_{n+j} = D(y_1, \ldots, y_n)y_{n+j} - G_j(y_1, \ldots, y_n)$. By construction $y'_{n+j} \in \mathfrak p S$. It is clear that $y_1, \ldots, y_n, y'_n, \ldots, y'_{n+m}$ generate an R-subalgebra of S whose localization is S. We define

$$\Psi : R[x_1, \ldots, x_{n+m}] \to S$$

to be the map that sends x_i to y_i for $i = 1, \ldots, n$ and x_{n+j} to y'_{n+j} for $j = 1, \ldots, m$. Properties (1) and (4) are clear by construction. Moreover the ideal $\mathfrak b$ maps onto the ideal $(\mathfrak a, x_{n+1}, \ldots, x_{n+m})$ in the polynomial ring $\kappa(\mathfrak p)[x_1, \ldots, x_{n+m}]$.

Denote $J = \text{Ker}(\Psi)$. We have a short exact sequence

$$0 \to J_{\mathfrak b} \to R[x_1, \ldots, x_{n+m}]_{\mathfrak b} \to S_{\mathfrak q} \to 0.$$

The surjectivity comes from our choice of $y_1, \ldots, y_n, y'_n, \ldots, y'_{n+m}$ above. This implies that

$$J_{\mathfrak b}/\mathfrak p J_{\mathfrak b} \to \kappa(\mathfrak p)[x_1, \ldots, x_{n+m}]_{(\mathfrak a, x_{n+1}, \ldots, x_{n+m})} \to S_{\mathfrak q}/\mathfrak p S_{\mathfrak q} \to 0$$

is exact. By construction x_i maps to $\psi(x_i)$ and x_{n+j} maps to zero under the last map. Thus it is easy to choose f_i as in (2) and (3) of the lemma. □

Lemma 7.125.16. *Let $R \to S$ be a ring map. Let $\mathfrak q \subset S$ be a prime lying over the prime $\mathfrak p$ of R. The following are equivalent:*

(1) *There exists an element $g \in S$, $g \not\in \mathfrak q$ such that $R \to S_g$ is syntomic.*

(2) *There exists an element $g \in S$, $g \not\in \mathfrak q$ such that S_g is a relative global complete intersection over R.*
(3) *There exists an element $g \in S$, $g \not\in \mathfrak q$, such that $R \to S_g$ is of finite presentation, the local ring map $R_{\mathfrak p} \to S_{\mathfrak q}$ is flat, and the local ring $S_{\mathfrak q}/\mathfrak pS_{\mathfrak q}$ is a complete intersection ring over $\kappa(\mathfrak p)$ (see Definition 7.124.5).*

Proof. The implication (1) ⇒ (3) is clear (see Lemma 7.124.8). The implication (2) ⇒ (1) follows from Lemma 7.125.14. Assume (3). After replacing S by S_g for some $g \in S$, $g \not\in \mathfrak q$ we may assume S is finitely presented over R.

We use this to reduce to the case where R is Noetherian. Namely, write $R \to S$ as a directed colimit of map $R_\lambda \to S_\lambda$ as in Lemma 7.118.14. Denote $\mathfrak q_\lambda \subset S_\lambda$ and $\mathfrak p_\lambda \subset R_\lambda$ the corresponding prime ideals. Note that

$$S_\lambda \otimes_{R_\lambda} \kappa(\mathfrak p_\lambda) \otimes_{\kappa(\mathfrak p_\lambda)} \kappa(\mathfrak p) \cong S \otimes_R \kappa(\mathfrak p).$$

Our assumption implies that $S \otimes_R \kappa(\mathfrak p)$ satisfies any of the conditions (1) - (5) of Lemma 7.124.4 at the prime corresponding to $\mathfrak q$, see Lemma 7.124.8. By Lemma 7.124.10 we see that the same holds for $S_\lambda \otimes_{R_\lambda} \kappa(\mathfrak p_\lambda)$ at the prime corresponding to $\mathfrak q_\lambda$. Moreover, for some sufficiently large λ the local ring map $(R_\lambda)_{\mathfrak p_\lambda} \to (S_\lambda)_{\mathfrak q_\lambda}$ is flat, by Lemma 7.119.3. In other words, all the conditions of (3) hold for $(R_\lambda \to S_\lambda, \mathfrak q_\lambda)$. If we can show that (2) holds for $(R_\lambda \to S_\lambda, \mathfrak q_\lambda)$ then it follows for $(R \to S, \mathfrak q)$. Thus we have reduced to the case where R is Noetherian.

By the last statement of Lemma 7.124.4 we may find a surjective $\kappa(\mathfrak p)$-algebra map $\overline{\psi} : \kappa(\mathfrak p)[x_1, \ldots, x_n]_{\mathfrak a} \to S_{\mathfrak q}/\mathfrak pS_{\mathfrak q}$ whose kernel is generated by a regular sequence $\overline{f}_1, \ldots, \overline{f}_c$ of $\kappa(\mathfrak p)[x_1, \ldots, x_n]_{\mathfrak a}$. We apply Lemma 7.125.15. Thus we find a $g \in S$, $g \not\in \mathfrak q$, a map $\Psi : R[x_1, \ldots, x_{n+m}] \to S_g$ and elements $f_1, \ldots, f_{c+m}$ in the kernel of Ψ which (up to units) give the elements $\overline{f}_1, \ldots, \overline{f}_c, x_{n+1}, \ldots, x_{n+m}$ in the local ring $\kappa(\mathfrak p)[x_1, \ldots, x_{n+m}]_{(\mathfrak a, x_{n+1}, \ldots, x_{n+m})}$. Moreover, the referenced lemma shows the induced map $R[x_1, \ldots, x_{n+m}]_{\mathfrak b} \to S_{\mathfrak q}$ is surjective, where $\mathfrak b \subset R[x_1, \ldots, x_{n+m}]$ is a suitable prime ideal. Consider the induced map

$$\overline{\Psi} : S' := R[x_1, \ldots, x_{n+m}]/(f_1, \ldots, f_{c+m}) \longrightarrow S_g.$$

We now know it has the following properties:

(1) it induces an surjection between the localizations at the primes $\mathfrak q' = \mathfrak b/(f_i)$ and $\mathfrak qS_g$
(2) it induces an isomorphism $S'_{\mathfrak q'}/\mathfrak pS'_{\mathfrak q'} \to S_{\mathfrak q}/\mathfrak pS_{\mathfrak q}$,
(3) the local ring $S_{\mathfrak q}$ is flat over $R_{\mathfrak p}$.

Denoting $J = \text{Ker}(\overline{\Psi})$, we see that $0 \to J_{\mathfrak q'} \to S'_{\mathfrak q'} \to S_{\mathfrak q} \to 0$ is exact. By flatness of $S_{\mathfrak q}$ over $R_{\mathfrak p}$ we see that $0 \to J_{\mathfrak q'}/\mathfrak pJ_{\mathfrak q'} \to S'_{\mathfrak q'}/\mathfrak pS_{\mathfrak q} \to S_{\mathfrak q}/\mathfrak pS_{\mathfrak q} \to 0$ is exact. By the second property above we conclude that $J_{\mathfrak q'}/\mathfrak pJ_{\mathfrak q'} = 0$. Because R and hence S' is Noetherian, we conclude that $J_{\mathfrak q'} = 0$, in other words $S'_{\mathfrak q'} \cong S_{\mathfrak q}$. By Lemma 7.117.6, we conclude there exists a $g' \in S'$, $g' \not\in \mathfrak q'$ such that $S'_{g'} \cong S_{g\overline{\Psi}(g')}$. By Lemma 7.125.11 applied to S' and the prime $\mathfrak q'$ (note that $\dim_{\mathfrak q'}(S'/R) = n + m - (c + m)$ by the explicit description of the sequence f_j) there exists a further element $g'' \in S'$, $g'' \not\in \mathfrak q'$ such that $S'_{g''}$ is a relative global complete intersection over R. By Lemma 7.125.10 we conclude that $S'_{g'g''} \cong S_{g\overline{\Psi}(g'g'')}$ is a relative global complete intersection over R, as desired. □

Lemma 7.125.17. *Let R be a ring. Let $S = R[x_1, \ldots, x_n]/I$ for some finitely generated ideal I. If $g \in S$ is such that S_g is syntomic over R, then $(I/I^2)_g$ is a finite projective S_g-module.*

Proof. By Lemma 7.125.16 there exist finitely many elements $g_1, \ldots, g_m \in S$ which generate the unit ideal in S_g such that each S_{gg_j} is a relative global complete intersection over R. Since it suffices to prove that $(I/I^2)_{gg_j}$ is finite projective, see Lemma 7.72.2, we may asssume that S_g is a relative global complete intersection. In this case the result follows from Lemmas 7.123.13 and 7.125.13. □

Lemma 7.125.18. *Let $R \to S$, $S \to S'$ be ring maps.*

(1) *If $R \to S$ and $S \to S'$ are syntomic, then $R \to S'$ is syntomic.*
(2) *If $R \to S$ and $S \to S'$ are relative global complete intersections so is $R \to S'$.*

Proof. Assume $R \to S$ and $S \to S'$ are syntomic. This implies that $R \to S'$ is flat by Lemma 7.35.3. It also implies that $R \to S'$ is of finite presentation by Lemma 7.6.2. Thus it suffices to show that the fibres of $R \to S'$ are local complete intersections. Choose a prime $\mathfrak{p} \subset R$. We have a factorization

$$\kappa(\mathfrak{p}) \to S \otimes_R \kappa(\mathfrak{p}) \to S' \otimes_R \kappa(\mathfrak{p}).$$

By assumption $S \otimes_R \kappa(\mathfrak{p})$ is a local complete intersection, and by Lemma 7.125.3 we see that $S \otimes_R \kappa(\mathfrak{p})$ is syntomic over $S \otimes_R \kappa(\mathfrak{p})$. After replacing S by $S \otimes_R \kappa(\mathfrak{p})$ and S' by $S' \otimes_R \kappa(\mathfrak{p})$ we may assume that R is a field. Say $R = k$.

Choose a prime $\mathfrak{q}' \subset S'$ lying over the prime $\mathfrak{q}$ of S. Our goal is to find a $g' \in S'$, $g' \not\in \mathfrak{q}'$ such that $S'_{g'}$ is a global complete intersection over k. Choose a $g \in S$, $g \not\in \mathfrak{q}$ such that $S_g = k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a global complete intersection over k. Since $S_g \to S'_g$ is still syntomic also, and $g \not\in \mathfrak{q}'$ we may replace S by S_g and S' by S'_g and assume that $S = k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a global complete intersection over k. Next we choose a $g' \in S'$, $g' \not\in \mathfrak{q}'$ such that $S' = S[y_1, \ldots, y_m]/(h_1, \ldots, h_d)$ is a relative global complete intersection over S. Hence we have reduced to part (2) of the lemma.

Suppose that $R \to S$ and $S \to S'$ are relative global complete intersections. Say $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ and $S' = S[y_1, \ldots, y_m]/(h_1, \ldots, h_d)$. Then

$$S' \cong R[x_1, \ldots, x_n, y_1, \ldots, y_m]/(f_1, \ldots, f_c, h'_1, \ldots, h'_d)$$

for some lifts $h'_j \in R[x_1, \ldots, x_n, y_1, \ldots, y_m]$ of the h_j. Hence it suffices to bound the dimensions of the fibres. Thus we may yet again assume $R = k$ is a field. In this case we see that we have a ring, namely S, which is of finite type over k and equidimensional of dimension $n - c$, and a finite type ring map $S \to S'$ all of whose nonempty fibre rings are equidimensional of dimension $m - d$. Then, by Lemma 7.103.6 for example applied to localizations at maximal ideals of S', we see that $\dim(S') \leq n - c + m - d$ as desired. □

Lemma 7.125.19. *Let R be a ring and let $I \subset R$ be an ideal. Let $R/I \to \overline{S}$ be a syntomic map. Then there exists elements $\overline{g}_i \in \overline{S}$ which generate the unit ideal of $\overline{S}$ such that each $\overline{S}_{g_i} \cong S_i/IS_i$ for some relative global complete intersection S_i over R.*

Proof. By Lemma 7.125.16 we find a collection of elements $\overline{g}_i \in \overline{S}$ which generate the unit ideal of $\overline{S}$ such that each $\overline{S}_{g_i}$ is a relative global complete intersection over R/I. Hence we may assume that $\overline{S}$ is a relative global complete intersection. Write $\overline{S} = (R/I)[x_1, \ldots, x_n]/(\overline{f}_1, \ldots, \overline{f}_c)$ as in Definition 7.125.5. Choose $f_1, \ldots, f_c \in R[x_1, \ldots, x_n]$ lifting $\overline{f}_1, \ldots, \overline{f}_c$. Set $S =$

$R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$. Note that $S/IS \cong \overline{S}$. Choose a prime $\overline{\mathfrak q}$ of $\overline{S}$, and let $\mathfrak q \subset S$ be the corresponding prime of S. By Lemma 7.125.11 there exists an element $g \in S$, $g \notin \mathfrak q$ such that S_g is a relative global complete intersection over R. And $\overline{S}_{\overline{g}} \cong S_g/IS_g$. This proves the lemma. □

7.126. Smooth ring maps

Let us motivate the definition of a smooth ring map by an example. Suppose R is a ring and $S = R[x, y]/(f)$ for some nonzero $f \in R[x, y]$. In this case there is an exact sequence

$$S \to S\mathrm{d}x \oplus S\mathrm{d}y \to \Omega_{S/R} \to 0$$

where the first arrow maps 1 to $\frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y$ see Section 7.123. We conclude that $\Omega_{S/R}$ is locally free of rank 1 if the partial derivatives of f generate the unit ideal in S. In this case S is smooth of relative dimension 1 over R. But it can happen that $\Omega_{S/R}$ is locally free of rank 2 namely if both partial derivatives of f are zero. For example if for a prime p we have $p = 0$ in R and $f = x^p + y^p$ then this happens. Here $R \to S$ is a relative global complete intersection of relative dimension 1 which is not smooth. Hence, in order to check that a ring map is smooth it is not sufficient to check whether the module of differentials is free. The correct condition is the following.

Definition 7.126.1. A ring map $R \to S$ is *smooth* if it is of finite presentation and the naive cotangent complex $NL_{S/R}$ is quasi-isomorphic to a finite projective S-module placed in degree 0.

In particular, if $R \to S$ is smooth then the module $\Omega_{S/R}$ is a finite projective S-module. Moreover, by Lemma 7.126.2 the naive cotangent complex of any presentation has the same structure. Thus, for a surjection $\alpha : R[x_1, \ldots, x_n] \to S$ with kernel I the map

$$I/I^2 \longrightarrow \Omega_{R[x_1,\ldots,x_n]/R} \otimes_{R[x_1,\ldots,x_n]} S$$

is a split injection. In other words $\bigoplus_{i=1}^n S\mathrm{d}x_i \cong I/I^2 \oplus \Omega_{S/R}$ as S-modules. This implies that I/I^2 is a finite projective S-module too!

Lemma 7.126.2. *Let $R \to S$ be a ring map of finite presentation. If for some presentation α of S over R the naive cotangent complex $NL(\alpha)$ is quasi-isomorphic to a finite projective S-module placed in degree 0, then this holds for any presentation.*

Proof. Immediate from Lemma 7.123.2. □

Lemma 7.126.3. *Let $R \to S$ be a smooth ring map. Any localization S_g is smooth over R. If $f \in R$ maps to an invertible element of S, then $R_f \to S$ is smooth.*

Proof. By Lemma 7.123.10 we see that the naive cotangent complex for S_g over R is the base change of the naive cotangent complex of S over R. The assumption is that the naive cotangent complex of S/R is $\Omega_{S/R}$ and that this is a finite projective S-module. Hence so is its base change. Thus S_g is smooth over R.

For the last assertion: A presentation of S over R_f is $R_f[x_1, \ldots, x_n]/I_f$. Since $I_f/I_f^2 = (I/I^2)_f = I/I^2$ we see that this presentation has isomorphic naive cotangent complex to the presentation of S over R. The result follows. □

Lemma 7.126.4. *Let $R \to S$ be a smooth ring map. Let $R \to R'$ be any ring map. Then the base change $R' \to S' = R' \otimes_R S$ is smooth.*

Proof. Let $\alpha : R[x_1, \ldots, x_n] \to S$ be a presentation with kernel I. Let $\alpha' : R'[x_1, \ldots, x_n] \to R' \otimes_R S$ be the induced presentation. Let $I' = \mathrm{Ker}(\alpha')$. Since $0 \to I \to R[x_1, \ldots, x_n] \to S \to 0$ is exact, the sequence $R' \otimes_R I \to R'[x_1, \ldots, x_n] \to R' \otimes_r S \to 0$ is exact. Thus $R' \otimes_R I \to I'$ is surjective. By Definition 7.126.1 there is a short exact sequence

$$0 \to I/I^2 \to \Omega_{R[x_1,\ldots,x_n]/R} \otimes_{R[x_1,\ldots,x_n]} S \to \Omega_{S/R} \to 0$$

and the S-module $\Omega_{S/R}$ is finite projective. In particular I/I^2 is a direct summand of $\Omega_{R[x_1,\ldots,x_n]/R} \otimes_{R[x_1,\ldots,x_n]} S$. Consider the commutative diagram

$$\begin{tikzcd}
R' \otimes_R (I/I^2) \arrow[r] \arrow[d] & R' \otimes_R (\Omega_{R[x_1,\ldots,x_n]/R} \otimes_{R[x_1,\ldots,x_n]} S) \arrow[d] \\
I'/(I')^2 \arrow[r] & \Omega_{R'[x_1,\ldots,x_n]/R'} \otimes_{R'[x_1,\ldots,x_n]} (R' \otimes_R S)
\end{tikzcd}$$

Since the right vertical map is an isomorphism we see that the left vertical map is injective and surjective by what was said above. Thus we conclude that $NL(\alpha')$ is quasi-isomorphic to $\Omega_{S'/R'} \cong S' \otimes_S \Omega_{S/R}$. And this is finite projective since it is the base change of a finite projective module. □

Lemma 7.126.5. *Let k be a field. Let S be a smooth k-algebra. Then S is a local complete intersection.*

Proof. By Lemmas 7.126.4 and 7.124.11 it suffices to prove this when k is algebraically closed. Choose a presentation $\alpha : k[x_1, \ldots, x_n] \to S$ with kernel I. Let $\mathfrak{m}$ be a maximal ideal of S, and let $\mathfrak{m}' \subset I$ be the corresponding maximal ideal of $k[x_1, \ldots, x_n]$. We will show that condition (5) of Lemma 7.124.4 holds (with $\mathfrak{m}$ instead of $\mathfrak{q}$). We may write $\mathfrak{m}' = (x_1 - a_1, \ldots, x_n - a_n)$ for some $a_i \in k$, because k is algebraically closed, see Theorem 7.30.1. By our assumption that $k \to S$ is smooth the S-module map $\mathrm{d} : I/I^2 \to \bigoplus_{i=1}^n S\mathrm{d}x_i$ is a split injection. Hence the corresponding map $I/\mathfrak{m}'I \to \bigoplus \kappa(\mathfrak{m}')\mathrm{d}x_i$ is injective. Say $\dim_{\kappa(\mathfrak{m}')}(I/\mathfrak{m}'I) = c$ and pick $f_1, \ldots, f_c \in I$ which map to a $\kappa(\mathfrak{m}')$-basis of $I/\mathfrak{m}'I$. By Nakayama's Lemma 7.14.5 we see that $f_1, \ldots, f_c$ generate $I_{\mathfrak{m}'}$ over $k[x_1, \ldots, x_n]_{\mathfrak{m}'}$. Consider the commutative diagram

$$\begin{tikzcd}
I \arrow[r] \arrow[d] & I/I^2 \arrow[r] \arrow[d] & I/\mathfrak{m}'I \arrow[d] \\
\Omega_{k[x_1,\ldots,x_n]/k} \arrow[r] & \bigoplus S\mathrm{d}x_i \arrow[r, "\mathrm{d}x_i \mapsto x_i - a_i"] & \mathfrak{m}'/(\mathfrak{m}')^2
\end{tikzcd}$$

(proof commutativity omitted). The middle vertical map is the one defining the naive cotangent complex of α. Note that the right lower horizontal arrow induces an isomorphism $\bigoplus \kappa(\mathfrak{m}')\mathrm{d}x_i \to \mathfrak{m}'/(\mathfrak{m}')^2$. Hence our generators $f_1, \ldots, f_c$ of $I_{\mathfrak{m}'}$ map to a collection of elements in $k[x_1, \ldots, x_n]_{\mathfrak{m}'}$ whose classes in $\mathfrak{m}'/(\mathfrak{m}')^2$ are linearly independent over $\kappa(\mathfrak{m}')$. Therefore they form a regular sequence in the ring $k[x_1, \ldots, x_n]_{\mathfrak{m}'}$ by Lemma 7.98.3. This verifies condition (5) of Lemma 7.124.4 hence S_g is a global complete intersection over k for some $g \in S$, $g \notin \mathfrak{m}$. As this works for any maximal ideal of S we conclude that S is a local complete intersection over k. □

Definition 7.126.6. Let R be a ring. A *standard smooth algebra over R* is a an algebra S with a (given) presentation

$$S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$$

such that the polynomial

$$g = \det \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_2/\partial x_1 & \ldots & \partial f_c/\partial x_1 \\ \partial f_1/\partial x_2 & \partial f_2/\partial x_2 & \ldots & \partial f_c/\partial x_2 \\ \ldots & \ldots & \ldots & \ldots \\ \partial f_1/\partial x_c & \partial f_2/\partial x_c & \ldots & \partial f_c/\partial x_c \end{pmatrix}$$

maps to an invertible element in S.

Lemma 7.126.7. *Let $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c) = R[x_1, \ldots, x_n]/I$ be a standard smooth algebra. Then*

(1) *the ring map $R \to S$ is smooth,*
(2) *the S-module $\Omega_{S/R}$ is free on $dx_{c+1}, \ldots, dx_n$,*
(3) *the S-module I/I^2 is free on the classes of $f_1, \ldots, f_c$,*
(4) *for any $g \in S$ the ring map $R \to S_g$ is standard smooth,*
(5) *for any ring map $R \to R'$ the base change $R' \to R' \otimes_R S$ is standard smooth,*
(6) *if $f \in R$ maps to an invertible element in S, then $R_f \to S$ is standard smooth, and*
(7) *the ring S is a relative global complete intersection over R.*

Proof. Consider the naive cotangent complex of the given presentation

$$(f_1, \ldots, f_c)/(f_1, \ldots, f_c)^2 \longrightarrow \bigoplus\nolimits_{i=1}^{n} S\mathrm{d}x_i$$

Let us compose this map with the projection onto the first c direct summands of the direct sum. According to the definition of a standard smooth algebra the classes f_i mod $(f_1, \ldots, f_c)^2$ map to a basis of $\bigoplus_{i=1}^{c} S\mathrm{d}x_i$. We conclude that $(f_1, \ldots, f_c)/(f_1, \ldots, f_c)^2$ is free of rank c with a basis given by the elements f_i mod $(f_1, \ldots, f_c)^2$, and that the homology in degree 0, i.e., $\Omega_{S/R}$, of the naive cotangent complex is a free S-module with basis the images of $\mathrm{d}x_{c+j}$, $j = 1, \ldots, n-c$. In particular, this proves $R \to S$ is smooth.

The proofs of (4) and (6) are omitted. But see the example below and the proof of Lemma 7.125.10.

Let $\varphi : R \to R'$ be any ring map. Denote $S' = R'[x_1, \ldots, x_n]/(f_1^{\varphi}, \ldots, f_c^{\varphi})$ where f^{φ} is the polynomial obtained from $f \in R[x_1, \ldots, x_n]$ by applying φ to all the coefficients. Then $S' \cong R' \otimes_R S$. Moreover, the determinant of Definition 7.126.6 for S'/R' is equal to g^{φ}. Its image in S' is therefore the image of g via $R[x_1, \ldots, x_n] \to S \to S'$ and hence invertible. This proves (5).

To prove (7) it suffices to show that $S \otimes_R \kappa(\mathfrak{p})$ has dimension $n-c$. By (5) it suffices to prove that any standard smooth algebra $k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ over a field k has dimension $n-c$. We already know that $k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a local complete intersection by Lemma 7.126.5 above. Hence, since I/I^2 is free of rank c we see that it dimension $n-c$, by Lemma 7.124.4 for example. □

Example 7.126.8. Let R be a ring. Let $f_1, \ldots, f_c \in R[x_1, \ldots, x_n]$. Let

$$h = \det \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_2/\partial x_1 & \ldots & \partial f_c/\partial x_1 \\ \partial f_1/\partial x_2 & \partial f_2/\partial x_2 & \ldots & \partial f_c/\partial x_2 \\ \ldots & \ldots & \ldots & \ldots \\ \partial f_1/\partial x_c & \partial f_2/\partial x_c & \ldots & \partial f_c/\partial x_c \end{pmatrix}.$$

Set $S = R[x_1, \ldots , x_{n+1}]/(f_1, \ldots , f_c, x_{n+1}h - 1)$. This is an example of a standard smooth algebra, except that the presentation is wrong and the variables should be in the following order: $x_1, \ldots , x_c, x_{n+1}, x_{c+1}, \ldots , x_n$.

Lemma 7.126.9. *A composition of standard smooth ring maps is standard smooth.*

Proof. Suppose that $R \to S$ and $S \to S'$ are standard smooth. We choose presentations $S = R[x_1, \ldots , x_n]/(f_1, \ldots , f_c)$ and $S' = S[y_1, \ldots , y_m]/(g_1, \ldots , g_d)$. Choose elements $g'_j \in R[x_1, \ldots , x_n, y_1, \ldots , y_m]$ mapping to the g_j. In this way we see $S' = R[x_1, \ldots , x_n, y_1, \ldots , y_m]/(f_1, \ldots , f_c, g'_1, \ldots , g'_d)$. To show that S' is standard smooth it suffices to verify that the determinant

$$\det \begin{pmatrix} \partial f_1/\partial x_1 & \ldots & \partial f_c/\partial x_1 & \partial g_1/\partial x_1 & \ldots & \partial g_d/\partial x_1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \partial f_1/\partial x_c & \ldots & \partial f_c/\partial x_c & \partial g_1/\partial x_c & \ldots & \partial g_d/\partial x_c \\ 0 & \ldots & 0 & \partial g_1/\partial y_1 & \ldots & \partial g_d/\partial y_1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & \partial g_1/\partial y_d & \ldots & \partial g_d/\partial y_d \end{pmatrix}$$

is invertible in S'. This is clear since it is the product of the two determinants which were assumed to be invertible by hypothesis. $\square$

Lemma 7.126.10. *Let $R \to S$ be a smooth ring map. There exists an open covering of $Spec(S)$ by standard opens $D(g)$ such that each S_g is standard smooth over R. In particular $R \to S$ is syntomic.*

Proof. Choose a presentation $\alpha : R[x_1, \ldots , x_n] \to S$ with kernel $I = (f_1, \ldots , f_m)$. For every subset $E \subset \{1, \ldots , m\}$ consider the open subset U_E where the classes $f_e, e \in E$ freely generate the finite projective S-module I/I^2, see Lemma 7.73.3. We may cover $Spec(S)$ by standard opens $D(g)$ each completely contained in one of the opens U_E. For such a g we look at the presentation

$$S_g = R[x_1, \ldots , x_n, x_{n+1}]/(f_1, \ldots , f_n, f_{n+1}) = R[x_1, \ldots , x_n, x_{n+1}]/J$$

with $f_{n+1} = gx_{n+1} - 1$. Since $J/J^2 = (I/I^2)_g \oplus S_g \cdot f_{n+1}$ we see that J/J^2 is freely generated by $f_e, e \in E$ and f_{n+1}. This reduces us to the case where S has a presentation $S = R[x_1, \ldots , x_n]/I$ with $I = (f_1, \ldots , f_m)$ and with I/I^2 free on the classes of $f_1, \ldots , f_c$.

Next, we more or less repeat this argument with the basis elements $\mathrm{d}x_1, \ldots , \mathrm{d}x_n$ of $\Omega_{R[x_1,\ldots,x_n]/R} \otimes_R S$. Namely, for any subset $E \subset \{1, \ldots , n\}$ we may consider the open subset U_E of $Spec(S)$, where the differential of $NL(\alpha)$ composed with the projection

$$S^{\oplus c} \cong I/I^2 \longrightarrow \Omega_{R[x_1,\ldots,x_n]/R} \otimes_R S \longrightarrow \bigoplus\nolimits_{e\in E} S\mathrm{d}x_i$$

is an isomorphism. Again we may find a covering of $Spec(S)$ by (finitely many) standard opens $D(g)$ such that each $D(g)$ is completely contained in one of the opens U_E. For a g with $D(g) \subset U_E$ we look at the presentation

$$S_g = R[x_1, \ldots , x_n, x_{n+1}]/(f_1, \ldots , f_n, f_{n+1}) = R[x_1, \ldots , x_n, x_{n+1}]/J$$

with $f_{n+1} = gx_{n+1} - 1$. Ok, and now we have $J/J^2 = (I/I^2)_g \oplus S_g \cdot f_{n+1}$, and $\Omega_{R[x_1,\ldots,x_{n+1}]/R} \otimes_R S_g = \bigoplus_{i=1}^{n+1} S_g \mathrm{d}x_i$, and $\mathrm{d}(f_{n+1}) = g\mathrm{d}x_{n+1} + x_{n+1}\mathrm{d}g$. From this we see that $J/J^2 \to (\bigoplus_{e\in E} S_g \mathrm{d}x_e) \oplus S_g \mathrm{d}x_{n+1}$ is an isomorphism. This reduces us to the case where S has a presentation $S = R[x_1, \ldots , x_n]/I$ with $I = (f_1, \ldots , f_m)$ and with I/I^2 free on the classes of $f_1, \ldots , f_c$, and furthermore the composition

$$I/I^2 \longrightarrow \Omega_{R[x_1,\ldots,x_n]/R} \otimes_R S \longrightarrow \bigoplus\nolimits_{i=1}^{c} S\mathrm{d}x_i$$

is an isomorphism.

At this point we consider the surjective map of R-algebras

$$S' := R[x_1, \ldots, x_n]/(f_1, \ldots, f_c) \longrightarrow S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m).$$

This is surjective. Let J be the kernel. Note that J is finitely generated (by the images of $f_{c+1}, \ldots, f_m$ in S'). Since $(f_1, \ldots, f_m)/(f_1, \ldots, f_m)^2$ is freely generated by $f_1, \ldots, f_c$ we see that $J/J^2 = 0$. By Lemma 7.18.5 we see that $Spec(S')$ contains $Spec(S)$ as an open and closed subset, and moreover that S is a localization $S = S'_{g'}$ for some element $g' \in S'$. Note that the determinant

$$g = \det \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_2/\partial x_1 & \ldots & \partial f_c/\partial x_1 \\ \partial f_1/\partial x_2 & \partial f_2/\partial x_2 & \ldots & \partial f_c/\partial x_2 \\ \ldots & \ldots & \ldots & \ldots \\ \partial f_1/\partial x_c & \partial f_2/\partial x_c & \ldots & \partial f_c/\partial x_c \end{pmatrix}$$

maps to an invertible element in S (by the conclusion of the previous paragraph). Hence we actually have $S \cong S_{gg'}$. Since S'_g is standard smooth (see Example 7.126.8), we win because a principal localization of a standard smooth algebra is standard smooth, see Lemma 7.126.7. $\square$

Definition 7.126.11. Let $R \to S$ be a ring map. Let $\mathfrak{q}$ be a prime of S. We say $R \to S$ is *smooth at* $\mathfrak{q}$ if there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is smooth.

For ring maps of finite presentation we can characterize this as follows.

Lemma 7.126.12. *Let $R \to S$ be of finite presentation. Let $\mathfrak{q}$ be a prime of S. The following are equivalent*

(1) *$R \to S$ is smooth at $\mathfrak{q}$,*
(2) *$H_1(L_{S/R})_{\mathfrak{q}} = 0$ and $\Omega_{S/R,\mathfrak{q}}$ is a projective $S_{\mathfrak{q}}$-module, and*
(3) *$H_1(L_{S/R})_{\mathfrak{q}} = 0$ and $\Omega_{S/R,\mathfrak{q}}$ is a flat $S_{\mathfrak{q}}$-module.*

Proof. We will use without further mention that formation of the naive cotangent complex commutes with localization, see Section 7.123, especially Lemma 7.123.10. It is clear that (1) implies (2) implies (3). Assume (3) holds. Note that $\Omega_{S/R}$ is a finitely presented S-module, see Lemma 7.122.15. Hence $\Omega_{S/R,\mathfrak{q}}$ is a finite free module by Lemma 7.72.4. Writing $S_{\mathfrak{q}}$ as the colimit of principal localizations we see from Lemma 7.118.4 that we can find a $g \in S$, $g \notin \mathfrak{q}$ such that $(\Omega_{S/R})_g$ is finite free. Choose a presentation $\alpha : R[x_1, \ldots, x_n] \to S$ with kernel I. We may work with $NL(\alpha)$ instead of $NL_{S/R}$, see Lemma 7.123.2. The surjection

$$\Omega_{R[x_1,\ldots,x_n]/R} \otimes_R S \to \Omega_{S/R} \to 0$$

has a right inverse after inverting g because $(\Omega_{S/R})_g$ is projective. Hence the image of $\text{d} : (I/I^2)_g \to \Omega_{R[x_1,\ldots,x_n]/R} \otimes_R S_g$ is a direct summand and this map has a right inverse too. We conclude that $H_1(L_{S/R})_g$ is a quotient of $(I/I^2)_g$. In particular $H_1(L_{S/R})_g$ is a finite S_g-module. Thus the vanishing of $H_1(L_{S/R})_{\mathfrak{q}}$ implies the vanishing of $H_1(L_{S/R})_{gg'}$ for some $g' \in S$, $g' \notin \mathfrak{q}$. Then $R \to S_{gg'}$ is smooth by definition. $\square$

Lemma 7.126.13. *Let $R \to S$ be a ring map. Then $R \to S$ is smooth if and only if $R \to S$ is smooth at every prime $\mathfrak{q}$ of S.*

Proof. The direct implication is trivial. Suppose that $R \to S$ is smooth at every prime $\mathfrak{q}$ of S. Since $Spec(S)$ is quasi-compact, see Lemma 7.16.10, there exists a finite covering $Spec(S) = \bigcup D(g_i)$ such that each S_{g_i} is smooth. By Lemma 7.21.3 this implies that S is of finite presentation over R. According to Lemma 7.123.10 we see that $NL_{S/R} \otimes_S S_{g_i}$ is quasi-isomorphic to a finite projective S_{g_i}-module. By Lemma 7.72.2 this implies that $NL_{S/R}$ is quasi-isomorphic to a finite projective S-module. □

Lemma 7.126.14. *A composition of smooth ring maps is smooth.*

Proof. This follows from a combination of Lemmas 7.126.10, 7.126.9 and 7.126.13. (You can also prove this in many different ways; including easier ones.) □

Lemma 7.126.15. *Let R be a ring. Let $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ be a relative global complete intersection. Let $\mathfrak{q} \subset S$ be a prime. Then $R \to S$ is smooth at $\mathfrak{q}$ if and only if there exists a subset $I \subset \{1, \ldots, n\}$ of cardinality c such that the polynomial*

$$g_I = \det(\partial f_j/\partial x_i)_{j=1,\ldots,c,\ i\in I}.$$

does not map to an element of $\mathfrak{q}$.

Proof. By Lemma 7.125.13 we see that the naive cotangent complex associated to the given presentation of S is the complex

$$\bigoplus\nolimits_{j=1}^{c} S \cdot f_j \longrightarrow \bigoplus\nolimits_{i=1}^{n} S \cdot \mathrm{d}x_i, \quad f_j \longmapsto \sum \frac{\partial f_j}{\partial x_i} \mathrm{d}x_i.$$

The maximal minors of the matrix giving the map are exactly the polynomials g_I.

Assume g_I maps to $g \in S$, with $g \notin \mathfrak{q}$. Then the algebra S_g is smooth over R. Namely, its naive cotangent complex is quasi-isomorphic to the complex above localized at g, see Lemma 7.123.10. And by construction it is quasi-isomorphic to a free rank $n - c$ module in degree 0.

Conversely, suppose that all g_I end up in $\mathfrak{q}$. In this case the complex above tensored with $\kappa(\mathfrak{q})$ does not have maximal rank, and hence there is no localization by an element $g \in S$, $g \notin \mathfrak{q}$ where this map becomes a split injection. By Lemma 7.123.10 again there is no such localization which is smooth over R. □

Lemma 7.126.16. *Let $R \to S$ be a ring map. Let $\mathfrak{q} \subset S$ be a prime lying over the prime $\mathfrak{p}$ of R. Assume*

(1) *there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is of finite presentation,*
(2) *the local ring homomorphism $R_{\mathfrak{p}} \to S_{\mathfrak{q}}$ is flat,*
(3) *the fibre $S \otimes_R \kappa(\mathfrak{p})$ is smooth over $\kappa(\mathfrak{p})$ at the prime corresponding to $\mathfrak{q}$.*

Then $R \to S$ is smooth at $\mathfrak{q}$.

Proof. By Lemmas 7.125.16 and 7.126.5 we see that there exists a $g \in S$ such that S_g is a relative global complete intersection. Replacing S by S_g we may assume $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a relative global complete intersection. For any subset $I \subset \{1, \ldots, n\}$ of cardinality c consider the polynomial $g_I = \det(\partial f_j/\partial x_i)_{j=1,\ldots,c,i\in I}$ of Lemma 7.126.15. Note that the image $\overline{g}_I$ of g_I in the polynomial ring $\kappa(\mathfrak{p})[x_1, \ldots, x_n]$ is the determinant of the partial derivatives of the images $\overline{f}_j$ of the f_j in the ring $\kappa(\mathfrak{p})[x_1, \ldots, x_n]$. Thus the lemma follows by applying Lemma 7.126.15 both to $R \to S$ and to $\kappa(\mathfrak{p}) \to S \otimes_R \kappa(\mathfrak{p})$. □

Note that the sets U, V in the following lemma are open by definition.

Lemma 7.126.17. *Let $R \to S$ be a ring map of finite presentation. Let $R \to R'$ be a flat ring map. Denote $S' = R' \otimes_R S$ the base change. Let $U \subset Spec(S)$ be the set of primes at which $R \to S$ is smooth. Let $V \subset Spec(S')$ the set of primes at which $R' \to S'$ is smooth. Then V is the inverse image of U under the map $f : Spec(S') \to Spec(S)$.*

Proof. By Lemma 7.123.6 we see that $NL_{S/R} \otimes_S S'$ is homotopy equivalent to $NL_{S'/R'}$. This already implies that $f^{-1}(U) \subset V$.

Let $\mathfrak{q}' \subset S'$ be a prime lying over $\mathfrak{q} \subset S$. Assume $\mathfrak{q}' \in V$. We have to show that $\mathfrak{q} \in U$. Since $S \to S'$ is flat, we see that $S_{\mathfrak{q}} \to S'_{\mathfrak{q}'}$ is faithfully flat (Lemma 7.35.16). Thus the vanishing of $H_1(L_{S'/R'})_{\mathfrak{q}'}$ implies the vanishing of $H_1(L_{S/R})_{\mathfrak{q}}$. By Lemma 7.72.5 applied to the $S_{\mathfrak{q}}$-module $(\Omega_{S/R})_{\mathfrak{q}}$ and the map $S_{\mathfrak{q}} \to S'_{\mathfrak{q}'}$ we see that $(\Omega_{S/R})_{\mathfrak{q}}$ is projective. Hence $R \to S$ is smooth at $\mathfrak{q}$ by Lemma 7.126.12. ☐

Lemma 7.126.18. *Let $k \subset K$ be a field extension. Let S be a finite type algebra over k. Let $\mathfrak{q}_K$ be a prime of $S_K = K \otimes_k S$ and let $\mathfrak{q}$ be the corresponding prime of S. Then S is smooth over k at $\mathfrak{q}$ if and only if S_K is smooth at $\mathfrak{q}_K$ over K.*

Proof. This is a special case of Lemma 7.126.17 above. ☐

Lemma 7.126.19. *Let R be a ring and let $I \subset R$ be an ideal. Let $R/I \to \overline{S}$ be a smooth ring map. Then there exists elements $\overline{g}_i \in \overline{S}$ which generate the unit ideal of $\overline{S}$ such that each $\overline{S}_{g_i} \cong S_i/IS_i$ for some (standard) smooth ring S_i over R.*

Proof. By Lemma 7.126.10 we find a collection of elements $\overline{g}_i \in \overline{S}$ which generate the unit ideal of $\overline{S}$ such that each $\overline{S}_{g_i}$ is standard smooth over R/I. Hence we may assume that $\overline{S}$ is standard smooth over R/I. Write $\overline{S} = (R/I)[x_1, \ldots, x_n]/(\overline{f}_1, \ldots, \overline{f}_c)$ as in Definition 7.126.6. Choose $f_1, \ldots, f_c \in R[x_1, \ldots, x_n]$ lifting $\overline{f}_1, \ldots, \overline{f}_c$. Set $S = R[x_1, \ldots, x_n, x_{n+1}]/(f_1, \ldots, f_c, x_{n+1}\Delta - 1)$ where $\Delta = \det(\frac{\partial f_i}{\partial x_j})_{i,j=1,\ldots,c}$ as in Example 7.126.8. This proves the lemma. ☐

7.127. Formally smooth maps

In this section we define formally smooth ring maps. It will turn out that a ring map of finite presentation is formally smooth if and only if it is smooth, see Proposition 7.127.13.

Definition 7.127.1. Let $R \to S$ be a ring map. We say S is *formally smooth over R* if for every commutative solid diagram

$$\begin{tikzcd} S \arrow[r] \arrow[rd, dashed] & A/I \\ R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

where $I \subset A$ is an ideal of square zero, a dotted arrow exists which makes the diagram commute.

Lemma 7.127.2. *Let $R \to S$ be a formally smooth ring map. Let $R \to R'$ be any ring map. Then the base change $S' = R' \otimes_R S$ is formally smooth over R'.*

Proof. Let a solid diagram

$$\begin{tikzcd} S \arrow[r] \arrow[rrd, dashed] & R' \otimes_R S \arrow[r] \arrow[rd, dashed] & A/I \\ R \arrow[u] \arrow[r] & R' \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

as in Definition 7.127.1 be given. By assumption the longer dotted arrow exists. By the universal property of tensor product we obtain the shorter dotted arrow. ☐

Lemma 7.127.3. *A composition of formally smooth ring maps is formally smooth.*

Proof. Omitted. (Hint: This is completely formal, and follows from considering a suitable diagram.) ☐

Lemma 7.127.4. *A polynomial ring over R is formally smooth over R.*

Proof. Suppose we have a diagram as in Definition 7.127.1 with $S = R[x_j; j \in J]$. Then there exists a dotted arrow simply by choosing lifts $a_j \in A$ of the elements in A/I to which the elements x_j map to under the top horizontal arrow. ☐

Lemma 7.127.5. *Let $R \to S$ be a ring map. Let $P \to S$ be a surjective R-algebra map from a polynomial ring P onto S. Denote $J \subset P$ the kernel. Then $R \to S$ is formally smooth if and only if there exists an R-algebra map $\sigma : S \to P/J^2$ which is a right inverse to the surjection $P/J^2 \to S$.*

Proof. Assume $R \to S$ is formally smooth. Consider the commutative diagram

$$\begin{tikzcd} S \arrow[r] \arrow[rd, dashed] & P/J \\ R \arrow[u] \arrow[r] & P/J^2 \arrow[u] \end{tikzcd}$$

By assumption the dotted arrow exists. This proves that σ exists.

Conversely, suppose we have a σ as in the lemma. Let a solid diagram

$$\begin{tikzcd} S \arrow[r] \arrow[rd, dashed] & A/I \\ R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

as in Definition 7.127.1 be given. Because P is formally smooth by Lemma 7.127.4, there exists an R-algebra homomorphism $\psi : P \to A$ which lifts the map $P \to S \to A/I$. Clearly $\psi(J) \subset I$ and since $I^2 = 0$ we conclude that $\psi(J^2) = 0$. Hence ψ factors as $\overline{\psi} : P/J^2 \to A$. The desired dotted arrow is the composition $\overline{\psi} \circ \sigma : S \to A$. ☐

Remark 7.127.6. Lemma 7.127.5 above holds more generally whenever P is formally smooth over R.

Lemma 7.127.7. *Let $R \to S$ be a ring map. Let $P \to S$ be a surjective R-algebra map from a polynomial ring P onto S. Denote $J \subset P$ the kernel. Then $R \to S$ is formally smooth if and only if the sequence*

$$0 \to J/J^2 \to \Omega_{P/R} \otimes_R S \to \Omega_{S/R} \to 0$$

of Lemma 7.122.9 is a split exact sequence.

Proof. Assume S is formally smooth over R. By Lemma 7.127.5 this means there exists an R-algebra map $S \to P/J^2$ which is a left inverse to the canonical map $P/J^2 \to S$. This means that

$$P/J^2 \cong S \oplus J/J^2$$

as R-algebras. Note that the middle term of the exact sequence is $\Omega_{P/R} \otimes_P S \cong \Omega_{(P/J^2)/R} \otimes_R S$ by Lemma 7.122.11. A direct computation shows that

$$\Omega_{(S \oplus J/J^2)/R} \otimes_{(S \oplus J/J^2)} S = \Omega_{S/R} \oplus J/J^2$$

as desired.

Assume the exact sequence of the lemma is split exact. Choose a splitting $\sigma : \Omega_{S/R} \to \Omega_{P/R} \otimes_R S$. For each $\lambda \in S$ choose $x_\lambda \in P$ which maps to λ. Next, for each $\lambda \in S$ choose $f_\lambda \in J$ such that

$$\mathrm{d}f_\lambda = \mathrm{d}x_\lambda - \sigma(\mathrm{d}\lambda)$$

in the middle term of the exact sequence. We claim that $s : \lambda \mapsto x_\lambda - f_\lambda \mod J^2$ is an R-algebra homomorphism $s : S \to P/J^2$. To prove this we will repeatedly use that if $h \in J$ and $\mathrm{d}h = 0$ in $\Omega_{P/R} \otimes_R S$, then $h \in J^2$. Let $\lambda, \mu \in S$. Then $\sigma(\mathrm{d}\lambda + \mathrm{d}\mu - \mathrm{d}(\lambda + \mu)) = 0$. This implies

$$\mathrm{d}(x_\lambda + x_\mu - x_{\lambda+\mu} - f_\lambda - f_\mu + f_{\lambda+\mu}) = 0$$

which means that $x_\lambda + x_\mu - x_{\lambda+\mu} - f_\lambda - f_\mu + f_{\lambda+\mu} \in J^2$, which in turn means that $s(\lambda) + s(\mu) = s(\lambda + \mu)$. Similarly, we have $\sigma(\lambda \mathrm{d}\mu + \mu \mathrm{d}\lambda - \mathrm{d}\lambda\mu) = 0$ which implies that

$$\mu(\mathrm{d}x_\lambda - \mathrm{d}f_\lambda) + \lambda(\mathrm{d}x_\mu - \mathrm{d}f_\mu) - \mathrm{d}x_{\lambda\mu} - \mathrm{d}f_{\lambda\mu} = 0$$

in the middle term of the exact sequence. Moreover we have

$$\mathrm{d}(x_\lambda x_\mu) = x_\lambda \mathrm{d}x_\mu + x_\mu \mathrm{d}x_\lambda = \lambda \mathrm{d}x_\mu + \mu \mathrm{d}x_\lambda$$

in the middle term again. Combined these equations mean that $x_\lambda x_\mu - x_{\lambda\mu} - \mu f_\lambda - \lambda f_\mu + f_{\lambda\mu} \in J^2$ which means that $s(\lambda)s(\mu) = s(\lambda\mu)$. If $\lambda \in R$, then $\mathrm{d}\lambda = 0$ and we see that $\mathrm{d}f_\lambda = \mathrm{d}x_\lambda$, hence $\lambda - x_\lambda + f_\lambda \in J^2$ and hence $s(\lambda) = \lambda$ as desired. At this point we can apply Lemma 7.127.5 to conclude that S/R is formally smooth. □

Proposition 7.127.8. *Let $R \to S$ be a ring map. Consider a formally smooth R-algebra P and a surjection $P \to S$ with kernel J. The following are equivalent*

(1) *S is formally smooth over R,*
(2) *for some $P \to S$ as above there exists a section to $P/J^2 \to S$,*
(3) *for all $P \to S$ as above there exists a section to $P/J^2 \to S$,*
(4) *for some $P \to S$ as above the sequence $0 \to J/J^2 \to \Omega_{P/R} \otimes S \to \Omega_{S/R} \to 0$ is split exact,*
(5) *for all $P \to S$ as above the sequence $0 \to J/J^2 \to \Omega_{P/R} \otimes S \to \Omega_{S/R} \to 0$ is split exact, and*
(6) *the naive cotangent complex $NL_{S/R}$ is quasi-isomorphic to a projective S-module placed in degree 0.*

Proof. It is clear that (1) implies (3) implies (2), see first part of the proof of Lemma 7.127.5. It is also true that (3) implies (5) implies (4) and that (2) implies (4), see first part of the proof of Lemma 7.127.7. Finally, Lemma 7.127.7 applied to the canonical surjection $R[S] \to S$ (7.123.0.1) shows that (1) implies (6).

Assume (4) and let's prove (6). Consider the sequence of Lemma 7.123.3 associated to the ring maps $R \to P \to S$. By the implication (1) ⇒ (6) proved above we see that $NL_{P/R} \otimes_R S$ is quasi-isomorphic to $\Omega_{P/R} \otimes_P S$ placed in degree 0. Hence $H_1(NL_{P/R} \otimes_P S) = 0$. Since $P \to S$ is surjective we see that $NL_{S/P}$ is homotopy equivalent to J/J^2 placed in degee 1 (Lemma 7.123.4). Thus we obtain the exact sequence $0 \to H_1(L_{S/R}) \to J/J^2 \to$

$\Omega_{P/R} \otimes_P S \to \Omega_{S/R} \to 0$. By assumption we see that $H_1(L_{S/R}) = 0$ and that $\Omega_{S/R}$ is a projective S-module. Thus (6) follows.

Finally, let's prove that (6) implies (1). The assumption means that the complex $J/J^2 \to \Omega_{P/R} \otimes S$ where $P = R[S]$ and $P \to S$ is the canonical surjection (7.123.0.1). Hence Lemma 7.127.7 shows that S is formally smooth over R. □

Lemma 7.127.9. *Let $A \to B \to C$ be ring maps. Assume $B \to C$ is formally smooth. Then the sequence*

$$0 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

of Lemma 7.122.7 is a split short exact sequence.

Proof. Follows from Proposition 7.127.8 and Lemma 7.123.3. □

Lemma 7.127.10. *Let $A \to B \to C$ be ring maps with $A \to C$ formally smooth and $B \to C$ surjective with kernel $J \subset B$. Then the exact sequence*

$$0 \to J/J^2 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to 0$$

of Lemma 7.122.9 is split exact.

Proof. Follows from Proposition 7.127.8, Lemma 7.123.3, and Lemma 7.122.9. □

Lemma 7.127.11. *Let $A \to B \to C$ be ring maps. Assume $A \to C$ is surjective (so also $B \to C$ is) and $A \to B$ formally smooth. Denote $I = Ker(A \to C)$ and $J = Ker(B \to C)$. Then the sequence*

$$0 \to I/I^2 \to J/J^2 \to \Omega_{B/A} \otimes_B B/J \to 0$$

of Lemma 7.123.5 is split exact.

Proof. Since $A \to B$ is formally smooth there exists a ring map $\sigma : B \to A/I^2$ whose composition with $A \to B$ equals the quotient map $A \to A/I^2$. Then σ induces a map $J/J^2 \to I/I^2$ which is inverse to the map $I/I^2 \to J/J^2$. □

Lemma 7.127.12. *Let $R \to S$ be a ring map. Let $I \subset R$ be an ideal. Assume*

(1) *$I^2 = 0$,*
(2) *$R \to S$ is flat, and*
(3) *$R/I \to S/IS$ is formally smooth.*

Then $R \to S$ is formally smooth.

Proof. Assume (1), (2) and (3). Let $P = R[\{x_t\}_{t\in T}] \to S$ be a surjection of R-algebras with kernel J. Thus $0 \to J \to P \to S \to 0$ is a short exact sequence of flat R-modules. This implies that $I \otimes_R S = IS$, $I \otimes_R P = IP$ and $I \otimes_R J = IJ$ as well as $J \cap IP = IJ$. We will use throughout the proof that

$$\Omega_{(S/IS)/(R/I)} = \Omega_{S/R} \otimes_S (S/IS) = \Omega_{S/R} \otimes_R R/I = \Omega_{S/R}/I\Omega_{S/R}$$

and similarly for P (see Lemma 7.122.12). By Lemma 7.127.7 the sequence

$$(7.127.12.1) \qquad 0 \to J/(IJ + J^2) \to \Omega_{P/R} \otimes_P S/IS \to \Omega_{S/R} \otimes_S S/IS \to 0$$

is split exact. Of course the middle term is $\bigoplus_{t\in T} S/IS\mathrm{d}x_t$. Choose a splitting $\sigma : \Omega_{P/R} \otimes_P S/IS \to J/(IJ + J^2)$. For each $t \in T$ choose an element $f_t \in J$ which maps to $\sigma(\mathrm{d}x_t)$ in $J/(IJ + J^2)$. This determines a unique S-module map

$$\tilde{\sigma} : \Omega_{P/R} \otimes_R S = \bigoplus S\mathrm{d}x_t \longrightarrow J/J^2$$

with the property that $\tilde{\sigma}(\mathrm{d}x_t) = f_t$. As σ is a section to d the difference

$$\Delta = \mathrm{id}_{J/J^2} - \tilde{\sigma} \circ \mathrm{d}$$

is a self map $J/J^2 \to J/J^2$ whose image is contained in $(IJ + J^2)/J^2$. In particular $\Delta((IJ + J^2)/J^2) = 0$ because $I^2 = 0$. This means that Δ factors as

$$J/J^2 \to J/(IJ + J^2) \xrightarrow{\overline{\Delta}} (IJ + J^2)/J^2 \to J/J^2$$

where $\overline{\Delta}$ is a S/IS-module map. Using again that the sequence (7.127.12.1) is split, we can find a S/IS-module map $\overline{\delta} : \Omega_{P/R} \otimes_P S/IS \to (IJ + J^2)/J^2$ such that $\overline{\delta} \circ d$ is equal to $\overline{\Delta}$. In the same manner as above the map $\overline{\delta}$ determines an S-module map $\delta : \Omega_{P/R} \otimes_P S \to J/J^2$. After replacing $\tilde{\sigma}$ by $\tilde{\sigma} + \delta$ a simple computation shows that $\Delta = 0$. In other words $\tilde{\sigma}$ is a section of $J/J^2 \to \Omega_{P/R} \otimes_P S$. By Lemma 7.127.7 we conclude that $R \to S$ is formally smooth. ☐

Proposition 7.127.13. *Let $R \to S$ be a ring map. The following are equivalent*

1. *$R \to S$ is of finite presentation and formally smooth,*
2. *$R \to S$ is smooth.*

Proof. Follows from Proposition 7.127.8 and Definition 7.126.1. (Note that $\Omega_{S/R}$ is a finitely presented S-module if $R \to S$ is of finite presentation, see Lemma 7.122.15.) ☐

Lemma 7.127.14. *Let $R \to S$ be a smooth ring map. Then there exists a subring $R_0 \subset R$ of finite type over $\mathbf{Z}$ and a smooth ring map $R_0 \to S_0$ such that $S \cong R \otimes_{R_0} S_0$.*

Proof. We are going to use that smooth is equivalent to finite presentation and formally smooth, see Proposition 7.127.13. Write $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ and denote $I = (f_1, \ldots, f_m)$. Choose a right inverse $\sigma : S \to R[x_1, \ldots, x_n]/I^2$ to the projection to S as in Lemma 7.127.5. Choose $h_i \in R[x_1, \ldots, x_n]$ such that $\sigma(x_i \bmod I) = h_i \bmod I^2$. The fact that σ is an R-algebra homomorphism $R[x_1, \ldots, x_n]/I \to R[x_1, \ldots, x_n]/I^2$ is equivalent to the condition that

$$f_j(h_1, \ldots, h_n) = \sum\nolimits_{j_1 j_2} a_{j_1 j_2} f_{j_1} f_{j_2}$$

for certain $a_{kl} \in R[x_1, \ldots, x_n]$. Let $R_0 \subset R$ be the subring generated over $\mathbf{Z}$ by all the coefficients of the polynomials f_j, h_i, a_{kl}. Set $S_0 = R_0[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$, with $I_0 = (f_1, \ldots, f_m)$. Let $\sigma_0 : S_0 \to R_0[x_1, \ldots, x_n]/I_0^2$ defined by the rule $x_i \mapsto h_i \bmod I_0^2$; this works since the a_{lk} are defined over R_0 and satisfy the same relations. Thus by Lemma 7.127.5 the ring S_0 is formally smooth over R_0. ☐

Lemma 7.127.15. *Let $R \to S$ be a ring map. Let $R \to R'$ be a faithfully flat ring map. Set $S' = S \otimes_R R'$. Then $R \to S$ is formally smooth if and only if $R' \to S'$ is formally smooth.*

Proof. If $R \to S$ is formally smooth, then $R' \to S'$ is formally smooth by Lemma 7.127.2. To prove the converse, assume $R' \to S'$ is formally smooth. Note that $N \otimes_R R' = N \otimes_S S'$ for any S-module N. In particular $S \to S'$ is faithfully flat also. Choose a polynomial ring $P = R[\{x_i\}_{i \in I}]$ and a surjection of R-algebras $P \to S$ with kernel J. Note that $P' = P \otimes_R R'$ is a polynomial algebra over R'. Since $R \to R'$ is flat the kernel J' of the surjection $P' \to S'$ is $J \otimes_R R'$. Hence the split exact sequence (see Lemma 7.127.7)

$$0 \to J'/(J')^2 \to \Omega_{P'/R'} \otimes_{P'} S' \to \Omega_{S'/R'} \to 0$$

is the base change via $S \to S'$ of the corresponding sequence

$$J/J^2 \to \Omega_{P/R} \otimes_P S \to \Omega_{S/R} \to 0$$

see Lemma 7.122.9. As $S \to S'$ is faithfully flat we conclude two things: (1) this sequence (without $'$) is exact too, and (2) $\Omega_{S/R}$ is a projective S-module. Namely, $\Omega_{S'/R'}$ is projective as a direct sum of the free module $\Omega_{P'/R'} \otimes_{P'} S'$ and $\Omega_{S/R} \otimes_S S' = \Omega_{S'/R'}$ by what we said above. Thus (2) follows by descent of projectivity through faithfully flat ring maps, see Theorem 7.89.5. Hence the sequence $0 \to J/J^2 \to \Omega_{P/R} \otimes_P S \to \Omega_{S/R} \to 0$ is exact also and we win by applying Lemma 7.127.7 once more. $\square$

It turns out that smooth ring maps satisfy the following strong lifting property.

Lemma 7.127.16. *Let $R \to S$ be a smooth ring map. Given a commutative solid diagram*

$$\begin{tikzcd} S \arrow[r] \arrow[rd, dashed] & A/I \\ R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

where $I \subset A$ is a locally nilpotent ideal, a dotted arrow exists which makes the diagram commute.

Proof. By Lemma 7.127.14 we can extend the diagram to a commutative diagram

$$\begin{tikzcd} S_0 \arrow[r] & S \arrow[r] \arrow[rd, dashed] & A/I \\ R_0 \arrow[u] \arrow[r] & R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

with $R_0 \to S_0$ smooth, R_0 of finite type over $\mathbf{Z}$, and $S = S_0 \otimes_{R_0} R$. Let $x_1, \ldots, x_n \in S_0$ be generators of S_0 over R_0. Let $a_1, \ldots, a_n$ be elements of A which map to the same elements in A/I as the elements $x_1, \ldots, x_n$. Denote $A_0 \subset A$ the subring generated by the image of R_0 and the elements $a_1, \ldots, a_n$. Set $I_0 = A_0 \cap I$. Then $A_0/I_0 \subset A/I$ and $S_0 \to A/I$ maps into A_0/I_0. Thus it suffices to find the dotted arrow in the diagram

$$\begin{tikzcd} S_0 \arrow[r] \arrow[rd, dashed] & A_0/I_0 \\ R_0 \arrow[u] \arrow[r] & A_0 \arrow[u] \end{tikzcd}$$

The ring A_0 is of finite type over $\mathbf{Z}$ by construction. Hence A_0 is Noetherian, whence I_0 is nilpotent, see Lemma 7.47.3. Say $I_0^n = 0$. By Proposition 7.127.13 we can successively lift the R_0-algebra map $S_0 \to A_0/I_0$ to $S_0 \to A_0/I_0^2$, $S_0 \to A_0/I_0^3$, ..., and finally $S_0 \to A_0/I_0^n = A_0$. $\square$

7.128. Smoothness and differentials

Some results on differentials and smooth ring maps.

Lemma 7.128.1. *Given ring maps $A \to B \to C$ with $B \to C$ smooth, then the sequence*

$$0 \to C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

of Lemma 7.122.7 is exact.

Proof. This follows from the more general Lemma 7.127.9 because a smooth ring map is formally smooth, see Proposition 7.127.13. But it also follows directly from Lemma 7.123.3 since $H_1(L_{C/B}) = 0$ is part of the definition of smoothness of $B \to C$. $\square$

Lemma 7.128.2. *Let $A \to B \to C$ be ring maps with $A \to C$ smooth and $B \to C$ surjective with kernel $J \subset B$. Then the exact sequence*

$$0 \to J/J^2 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to 0$$

of Lemma 7.122.9 is split exact.

Proof. This follows from the more general Lemma 7.127.10 because a smooth ring map is formally smooth, see Proposition 7.127.13. □

Lemma 7.128.3. *Let $A \to B \to C$ be ring maps. Assume $A \to C$ is surjective (so also $B \to C$ is) and $A \to B$ smooth. Denote $I = Ker(A \to C)$ and $J = Ker(B \to C)$. Then the sequence*

$$0 \to I/I^2 \to J/J^2 \to \Omega_{B/A} \otimes_B B/J \to 0$$

of Lemma 7.123.5 is exact.

Proof. This follows from the more general Lemma 7.127.11 because a smooth ring map is formally smooth, see Proposition 7.127.13. □

Lemma 7.128.4. *Let $\varphi : R \to S$ be a smooth ring map. Let $\sigma : S \to R$ be a left inverse to φ. Set $I = Ker(\sigma)$. Then*

(1) *I/I^2 is a finite locally free R-module, and*
(2) *if I/I^2 is free, then $S^\wedge \cong R[[t_1, \ldots, t_d]]$ as R-algebras, where $S^\wedge$ is the I-adic completion of S.*

Proof. By Lemma 7.122.10 applied to $R \to S \to R$ we see that $I/I^2 = \Omega_{S/R} \otimes_{S,\sigma} R$. Since by definition of a smooth morphism the module $\Omega_{S/R}$ is finite locally free over S we deduce that (1) holds. If I/I^2 is free, then choose $f_1, \ldots, f_d \in I$ whose images in I/I^2 form an R-basis. Consider the R-algebra map defined by

$$\Psi : R[[x_1, \ldots, x_d]] \longrightarrow S^\wedge, \quad x_i \longmapsto f_i.$$

Denote $P = R[[x_1, \ldots, x_d]]$ and $J = (x_1, \ldots, x_d) \subset P$. We write $\Psi_n : P/J^n \to S/I^n$ for the induced map of quotient rings. Note that $S/I^2 = \varphi(R) \oplus I/I^2$. Thus Ψ_2 is an isomorphism. Denote $\sigma_2 : S/I^2 \to P/J^2$ the inverse of Ψ_2. We will prove by induction on n that for all $n > 2$ there exists an inverse $\sigma_n : S/I^n \to P/J^n$ of Ψ_n. Namely, as S is formally smooth over R (by Proposition 7.127.13) we see that in the solid diagram

$$\begin{tikzcd} S \arrow[r, dotted] \arrow[dr, "\sigma_{n-1}"'] & P/J^n \arrow[d] \\ & P/J^{n-1} \end{tikzcd}$$

of R-algebras we can fill in the dotted arrow by some R-algebra map $\tau : S \to P/J^n$ making the diagram commute. This induces an R-algebra map $\overline{\tau} : S/I^n \to P/J^n$ which is equal to σ_{n-1} modulo J^n. By construction the map Ψ_n is surjective and now $\overline{\tau} \circ \Psi_n$ is an R-algebra endomorphism of P/J^n which maps x_i to $x_i + \delta_{i,n}$ with $\delta_{i,n} \in J^{n-1}/J^n$. It follows that Ψ_n is an isomorphism and hence it has an inverse σ_n. This proves the lemma. □

7.129. Smooth algebras over fields

Warning: The following two lemmas do not hold over nonperfect fields in general.

Lemma 7.129.1. *Let k be an algebraically closed field. Let S be a finite type k-algebra. Let $\mathfrak{m} \subset S$ be a maximal ideal. Then*

$$\dim_{\kappa(\mathfrak{m})} \Omega_{S/k} \otimes_S \kappa(\mathfrak{m}) = \dim_{\kappa(\mathfrak{m})} \mathfrak{m}/\mathfrak{m}^2.$$

Proof. Since k is algebraically closed we have $\kappa(\mathfrak{m}) = k$. We may choose a presentation $0 \to I \to k[x_1, \ldots, x_n] \to S \to 0$ such that all x_i end up in $\mathfrak{m}$. Write $I = (f_1, \ldots, f_m)$. Note that each f_i is contained in $(x_1, \ldots, x_n)$, i.e., each f_i has zero constant term. Hence we may write

$$f_j = \sum a_{ij} x_i + \text{h.o.t.}$$

By Lemma 7.122.9 there is an exact sequence

$$\bigoplus S \cdot f_j \to \bigoplus S \cdot \mathrm{d}x_i \to \Omega_{S/k} \to 0.$$

Tensoring with $\kappa(\mathfrak{m}) = k$ we get an exact sequence

$$\bigoplus k \cdot f_j \to \bigoplus k \cdot \mathrm{d}x_i \to \Omega_{S/k} \otimes \kappa(\mathfrak{m}) \to 0.$$

The matrix of the map is given by the partial derivatives of the f_j evaluated at 0. In other words by the matrix (a_{ij}). Similarly there is a short exact sequence

$$(f_1, \ldots, f_m)/(x_1 f_1, \ldots, x_n f_m) \to (x_1, \ldots, x_n)/(x_1, \ldots, x_n)^2 \to \mathfrak{m}/\mathfrak{m}^2 \to 0.$$

Note that the first map is given by expanding the f_j in terms of the x_i, i.e., by the same matrix (a_{ij}). Hence the two numbers are the same. $\square$

Lemma 7.129.2. *Let k be an algebraically closed field. Let S be a finite type k-algebra. Let $\mathfrak{m} \subset S$ be a maximal ideal. The following are equivalent:*

1. *The ring $S_{\mathfrak{m}}$ is a regular local ring.*
2. *We have* $\dim_{\kappa(\mathfrak{m})} \Omega_{S/k} \otimes_S \kappa(\mathfrak{m}) \leq \dim(S_{\mathfrak{m}})$.
3. *We have* $\dim_{\kappa(\mathfrak{m})} \Omega_{S/k} \otimes_S \kappa(\mathfrak{m}) = \dim(S_{\mathfrak{m}})$.
4. *There exists a $g \in S$, $g \notin \mathfrak{m}$ such that S_g is smooth over k. In other words S/k is smooth at $\mathfrak{m}$.*

Proof. Note that (1), (2) and (3) are equivalent by Lemma 7.129.1 and Definition 7.102.6.

Assume that S is smooth at $\mathfrak{q}$. By Lemma 7.126.10 we see that S_g is standard smooth over k for a suitable $g \in S$, $g \notin \mathfrak{m}$. Hence by Lemma 7.126.7 we see that $\Omega_{S_g/k}$ is free of rank $\dim(S_g)$. Hence by Lemma 7.129.1 we see that $\dim(S_m) = \dim(\mathfrak{m}/\mathfrak{m}^2)$ in other words $S_{\mathfrak{m}}$ is regular.

Conversely, suppose that $S_{\mathfrak{m}}$ is regular. Let $d = \dim(S_{\mathfrak{m}}) = \dim \mathfrak{m}/\mathfrak{m}^2$. Choose a presentation $S = k[x_1, \ldots, x_n]/I$ such that x_i maps to an element of $\mathfrak{m}$ for all i. In other words, $\mathfrak{m}'' = (x_1, \ldots, x_n)$ is the corresponding maximal ideal of $k[x_1, \ldots, x_n]$. Note that we have a short exact sequence

$$I/\mathfrak{m}'' I \to \mathfrak{m}''/(\mathfrak{m}'')^2 \to \mathfrak{m}/(\mathfrak{m})^2 \to 0$$

Pick $c = n - d$ elements $f_1, \ldots, f_d \in I$ such that their images in $\mathfrak{m}''/(\mathfrak{m}'')^2$ span the kernel of the map to $\mathfrak{m}/(\mathfrak{m})^2$. This is clearly possible. Denote $J = (f_1, \ldots, f_c)$. So $J \subset I$.

Denote $S' = k[x_1, \ldots, x_n]/J$ so there is a surjection $S' \to S$. Denote $\mathfrak m' = \mathfrak m'' S'$ the corresponding maximal ideal of S'. Hence we have

$$\begin{tikzcd} k[x_1, \ldots, x_n] \arrow[r] & S' \arrow[r] & S \\ \mathfrak m'' \arrow[r] \arrow[u] & \mathfrak m' \arrow[r] \arrow[u] & \mathfrak m \arrow[u] \end{tikzcd}$$

By our choice of J the exact sequence

$$J/\mathfrak m'' J \to \mathfrak m''/(\mathfrak m'')^2 \to \mathfrak m'/(\mathfrak m')^2 \to 0$$

shows that $\dim(\mathfrak m'/(\mathfrak m')^2) = d$. Since $S'_{\mathfrak m'}$ surjects onto $S_{\mathfrak m}$ we see that $\dim(S_{\mathfrak m'}) \geq d$. Hence by the discussion preceding Definition 7.57.9 we conclude that $S'_{\mathfrak m'}$ is regular of dimension d as well. Because S' was cut out by $c = n - d$ equations we conclude that there exists a $g' \in S'$, $g' \not\in \mathfrak m'$ such that $S'_{g'}$ is a global complete intersection over k, see Lemma 7.124.4. Also the map $S'_{\mathfrak m'} \to S_{\mathfrak m}$ is a surjection of Noetherian local domains of the same dimension and hence an isomorphism. By Lemma 7.117.6 we see that $S'_{g''} \cong S_{g''}$ for some $g'' \in S'$, $g'' \not\in \mathfrak m'$. All in all we conclude that after replacing S by a principal localization we may assume that S is a global complete intersection.

At this point we may write $S = k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ with $\dim S = n - c$. Recall that the naive cotangent complex of this algebra is given by

$$\bigoplus S \cdot f_j \to \bigoplus S \cdot \mathrm{d}x_i$$

see Lemma 7.126.15. By this same lemma in order to show that S is smooth at $\mathfrak m$ we have to show that one of the $c \times c$ minors g_I of the matrix ``A'' giving the map above does not vanish at $\mathfrak m$. By Lemma 7.129.1 the matrix A mod $\mathfrak m$ has rank c. Thus we win. ☐

Lemma 7.129.3. *Let k be any field. Let S be a finite type k-algebra. Let $X = Spec(S)$. Let $\mathfrak q \subset S$ be a prime corresponding to $x \in X$. The following are equivalent:*

1. *The k-algebra S is smooth at $\mathfrak q$ over k.*
2. *We have $\dim_{\kappa(\mathfrak q)} \Omega_{S/k} \otimes_S \kappa(\mathfrak q) \leq \dim_x X$.*
3. *We have $\dim_{\kappa(\mathfrak q)} \Omega_{S/k} \otimes_S \kappa(\mathfrak q) = \dim_x X$.*

Moreover, in this case the local ring $S_{\mathfrak q}$ is regular.

Proof. If S is smooth at $\mathfrak q$ over k, then there exists a $g \in S$, $g \not\in \mathfrak q$ such that S_g is standard smooth over k, see Lemma 7.126.10. A standard smooth algebra over k has a module of differentials which is free of rank equal to the dimension, see Lemma 7.126.7. Thus we see that (1) implies (3). To finish the proof of the lemma it suffices to show that (2) implies (1) and that it implies that $S_{\mathfrak q}$ is regular.

Assume (2). By Nakayama's Lemma 7.14.5 we see that $\Omega_{S/k, \mathfrak q}$ can be generated by $\leq \dim_x X$ elements. We may replace S by S_g for some $g \in S$, $g \not\in \mathfrak q$ such that $\Omega_{S/k}$ is generated by at most $\dim_x X$ elements. Let $K \supset k$ be an algebraically closed field extension such that there exists a k-algebra map $\psi : \kappa(\mathfrak q) \to K$. Consider $S_K = K \otimes_k S$. Let $\mathfrak m \subset S_K$ be the maximal ideal corresponding to the surjection

$$S_K = K \otimes_k S \longrightarrow K \otimes_k \kappa(\mathfrak q) \xrightarrow{\mathrm{id}_K \otimes \psi} K.$$

Note that $\mathfrak m \cap S = \mathfrak q$, in other words $\mathfrak m$ lies over $\mathfrak q$. By Lemma 7.107.6 the dimension of $X_K = Spec(S_K)$ at the point corresponding to $\mathfrak m$ is $\dim_x X$. By Lemma 7.105.6 this is equal to $\dim((S_K)_\mathfrak{m})$. By Lemma 7.122.12 the module of differentials of S_K over K is the base change of $\Omega_{S/k}$, hence also generated by at most $\dim_x X = \dim((S_K)_\mathfrak{m})$ elements. By Lemma 7.129.2 we see that S_K is smooth at $\mathfrak{m}$ over K. By Lemma 7.126.17 this implies that S is smooth at $\mathfrak{q}$ over k. This proves (1). Moreover, we know by Lemma 7.129.2 that the local ring $(S_K)_\mathfrak{m}$ is regular. Since $S_\mathfrak{q} \to (S_K)_\mathfrak{m}$ is flat we conclude from Lemma 7.102.8 that $S_\mathfrak{q}$ is regular. □

The following lemma can be significantly generalized (in several different ways).

Lemma 7.129.4. *Let k be a field. Let R be a Noetherian local ring containing k. Assume that the residue field $\kappa = R/\mathfrak{m}$ is a finitely generated separable extension of k. Then the map*
$$d : \mathfrak{m}/\mathfrak{m}^2 \longrightarrow \Omega_{R/k} \otimes_R \kappa(\mathfrak{m})$$
is injective.

Proof. We may replace R by $R/\mathfrak{m}^2$. Hence we may assume that $\mathfrak{m}^2 = 0$. By assumption we may write $\kappa = k(\overline{x}_1, \ldots, \overline{x}_r, \overline{y})$ where $\overline{x}_1, \ldots, \overline{x}_r$ is a transcendence basis of κ over k and $\overline{y}$ is separable algebraic over $k(\overline{x}_1, \ldots, \overline{x}_r)$. Say its minimal equation is $P(\overline{y}) = 0$ with $P(T) = T^d + \sum_{i<d} a_i T^i$, with $a_i \in k(\overline{x}_1, \ldots, \overline{x}_r)$ and $P'(\overline{y}) \neq 0$. Choose any lifts $x_i \in R$ of the elements $\overline{x}_i \in \kappa$. This gives a commutative diagram

$$\begin{tikzcd} R \arrow[r] & \kappa \\ & k(\overline{x}_1, \ldots, \overline{x}_r) \arrow[u] \arrow[ul, "\varphi"] \end{tikzcd}$$

of k-algebras. We want to extend the left upwards arrow φ to a k-algebra map from κ to R. To do this choose any $y \in R$ lifting $\overline{y}$. To see that it defines a k-algebra map defined on $\kappa \cong k(\overline{x}_1, \ldots, \overline{x}_r)[T]/(P)$ all we have to show is that we may choose y such that $P^\varphi(y) = 0$. If not then we compute for $\delta \in \mathfrak{m}$ that
$$P(y + \delta) = P(y) + P'(y)\delta$$
because $\mathfrak{m}^2 = 0$. Since $P'(y)\delta = P'(\overline{y})\delta$ we see that we can adjust our choice as desired. This shows that $R \cong \kappa \oplus \mathfrak{m}$ as k-algebras! From a direct computation of $\Omega_{\kappa \oplus \mathfrak{m}/k}$ the lemma follows. □

Lemma 7.129.5. *Let k be a field. Let S be a finite type k-algebra. Let $\mathfrak{q} \subset S$ be a prime. Assume $\kappa(\mathfrak{q})$ is separable over k. The following are equivalent:*

(1) *The algebra S is smooth at $\mathfrak{q}$ over k.*
(2) *The ring $S_\mathfrak{q}$ is regular.*

Proof. Denote $R = S_\mathfrak{q}$ and denote its maximal by $\mathfrak{m}$ and its residue field κ. By Lemma 7.129.4 and 7.122.9 we see that there is a short exact sequence
$$0 \to \mathfrak{m}/\mathfrak{m}^2 \to \Omega_{R/k} \otimes_R \kappa \to \Omega_{\kappa/k} \to 0$$
Note that $\Omega_{R/k} = \Omega_{S/k,\mathfrak{q}}$, see Lemma 7.122.8. Moreover, since κ is separable over k we have $\dim_\kappa \Omega_{\kappa/k} = \text{trdeg}_k(\kappa)$. Hence we get
$$\dim_\kappa \Omega_{R/k} \otimes_R \kappa = \dim_\kappa \mathfrak{m}/\mathfrak{m}^2 + \text{trdeg}_k(\kappa) \geq \dim R + \text{trdeg}_k(\kappa) = \dim_\mathfrak{q} S$$
(see Lemma 7.107.3 for the last equality) with equality if and only if R is regular. Thus we win by applying Lemma 7.129.3. □

Lemma 7.129.6. *Let $R \to S$ be a $\mathbf{Q}$-algebra map. Let $f \in S$ be such that $\Omega_{S/R} = S\mathrm{d}f \oplus C$ for some S-submodule C. Then*

(1) *f is not nilpotent, and*
(2) *if S is a Noetherian local ring, then f is a nonzero divisor in S.*

Proof. For $a \in S$ write $\mathrm{d}(a) = \theta(a)\mathrm{d}f + c(a)$ for some $\theta(a) \in S$ and $c(a) \in C$. Consider the R-derivation $S \to S$, $a \mapsto \theta(a)$. Note that $\theta(f) = 1$.

If $f^n = 0$ with $n > 1$ minimal, then $0 = \theta(f^n) = nf^{n-1}$ contradicting the minimality of n. We conclude that f is not nilpotent.

Suppose $fa = 0$. If f is a unit then $a = 0$ and we win. Assume f is not a unit. Then $0 = \theta(fa) = f\theta(a) + a$ by the Leibniz rule and hence $a \in (f)$. By induction suppose we have shown $fa = 0 \Rightarrow a \in (f^n)$. Then writing $a = f^n b$ we get $0 = \theta(f^{n+1}b) = (n+1)f^n b + f^{n+1}\theta(b)$. Hence $a = f^n b = -f^{n+1}\theta(b)/(n+1) \in (f^{n+1})$. Since in the Noetherian local ring S we have $\bigcap (f^n) = 0$, see Lemma 7.47.6 we win. $\square$

The following is probably quite useless in applications.

Lemma 7.129.7. *Let k be a field of characteristic 0. Let S be a finite type k-algebra. Let $\mathfrak{q} \subset S$ be a prime. The following are equivalent:*

(1) *The algebra S is smooth at $\mathfrak{q}$ over k.*
(2) *The $S_{\mathfrak{q}}$-module $\Omega_{S/k,\mathfrak{q}}$ is (finite) free.*
(3) *The ring $S_{\mathfrak{q}}$ is regular.*

Proof. In characteristic zero any field extension is separable and hence the equivalence of (1) and (3) follows from Lemma 7.129.5. Also (1) implies (2) by definition of smooth algebras. Assume that $\Omega_{S/k,\mathfrak{q}}$ is free over $S_{\mathfrak{q}}$. We are going to use the notation and observations made in the proof of Lemma 7.129.5. So $R = S_{\mathfrak{q}}$ with maximal ideal $\mathfrak{m}$ and residue field κ. Our goal is to prove R is regular.

If $\mathfrak{m}/\mathfrak{m}^2 = 0$, then $\mathfrak{m} = 0$ and $R \cong \kappa$. Hence R is regular and we win.

If $\mathfrak{m}/\mathfrak{m}^2 \neq 0$, then choose any $f \in \mathfrak{m}$ whose image in $\mathfrak{m}/\mathfrak{m}^2$ is not zero. By Lemma 7.129.4 we see that $\mathrm{d}f$ has nonzero image in $\Omega_{R/k}/\mathfrak{m}\Omega_{R/k}$. By assumption $\Omega_{R/k} = \Omega_{S/k,\mathfrak{q}}$ is finite free and hence by Nakayama's Lemma 7.14.5 we see that $\mathrm{d}f$ generates a direct summand. We apply Lemma 7.129.6 to deduce that f is a nonzero divisor in R. Furthermore, by Lemma 7.122.9 we get an exact sequence

$$(f)/(f^2) \to \Omega_{R/k} \otimes_R R/fR \to \Omega_{(R/fR)/k} \to 0$$

This implies that $\Omega_{(R/fR)/k}$ is finite free as well. Hence by induction we see that R/fR is a regular local ring. Since $f \in \mathfrak{m}$ was a nonzero divisor we conclude that R is regular, see Lemma 7.98.7. $\square$

Example 7.129.8. Lemma 7.129.7 does not hold in characteristic $p > 0$. The standard examples are the ring maps

$$\mathbf{F}_p \longrightarrow \mathbf{F}_p[x]/(x^p)$$

whose module of differentials is free but is clearly not smooth, and the ring map ($p > 2$)

$$\mathbf{F}_p(t) \to \mathbf{F}_p(t)[x, y]/(x^p + y^2 + \alpha)$$

which is not smooth at the prime $\mathfrak{q} = (y, x^p - \alpha)$ but is regular.

Using the material above we can characterize smoothness at the generic point in terms of field extensions.

Lemma 7.129.9. *Let $R \to S$ be an injective finite type ring map with R and S domains. Then $R \to S$ is smooth at $\mathfrak{q} = (0)$ if and only if $f.f.(R) \subset f.f.(S)$ is a separable extension of fields.*

Proof. Assume $R \to S$ is smooth at (0). We may replace S by S_g for some nonzero $g \in S$ and assume that $R \to S$ is smooth. Set $K = f.f.(R)$. Then $K \to S \otimes_R K$ is smooth (Lemma 7.126.4). Moreover, for any field extension $K \subset K'$ the ring map $K' \to S \otimes_R K'$ is smooth as well. Hence $S \otimes_R K'$ is a regular ring by Lemma 7.129.3, in particular reduced. It follows that $S \otimes_R K$ is a geometrically reduced over K. Hence $f.f.(S)$ is geometricaly reduced over K, see Lemma 7.40.3. Hence $f.f.(S)/K$ is separable by Lemma 7.41.1.

Conversely, assume that $f.f.(R) \subset f.f.(S)$ is separable. We may assume $R \to S$ is of finite presentation, see Lemma 7.27.1. It suffices to prove that $K \to S \otimes_R K$ is smooth at (0), see Lemma 7.126.17. This follows from Lemma 7.129.5, the fact that a field is a regular ring, and the assumption that $f.f.(R) \to f.f.(S)$ is separable. $\square$

7.130. Smooth ring maps in the Noetherian case

Definition 7.130.1. Let $\varphi : B' \to B$ be a ring map. We say φ is a *small extension* if B' and B are local Artinian rings, φ is surjective and $I = \mathrm{Ker}(\varphi)$ has length 1 as a B'-module.

Clearly this means that $I^2 = 0$ and that $I = (x)$ for some $x \in B'$ such that $\mathfrak{m}'x = 0$ where $\mathfrak{m}' \subset B'$ is the maximal ideal.

Lemma 7.130.2. *Let $R \to S$ be a ring map. Let $\mathfrak{q}$ be a prime ideal of S lying over $\mathfrak{p} \subset R$. Assume R is Noetherian and $R \to S$ of finite type. The following are equivalent:*

(1) *$R \to S$ is smooth at $\mathfrak{q}$,*
(2) *for every surjection of local R-algebras $(B', \mathfrak{m}') \to (B, \mathfrak{m})$ with Ker$(B' \to B)$ having square zero and every solid commutative diagram*

$$\begin{tikzcd} S \arrow[r] \arrow[dr, dashed] & B \\ R \arrow[u] \arrow[r] & B' \arrow[u] \end{tikzcd}$$

such that $\mathfrak{q} = S \cap \mathfrak{m}$ there exists a dotted arrow making the diagram commute,
(3) *same as in (2) but with $B' \to B$ ranging over small extensions, and*
(4) *same as in (2) but with $B' \to B$ ranging over small extensions such that in addition $S \to R$ induces an isomorphism $\kappa(\mathfrak{q}) \cong \kappa(\mathfrak{m})$.*

Proof. Assume (1). This means there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is smooth. By Proposition 7.127.13 we know that $R \to S_g$ is formally smooth. Note that given any diagram as in (2) the map $S \to B$ factors automatically through $S_{\mathfrak{q}}$ and a fortiori through S_g. The formal smoothness of S_g over R gives us a morphism $S_g \to B'$ fitting into a similar diagram with S_g at the upper left corner. Composing with $S \to S_g$ gives the desired arrow. In other words, we have shown that (1) implies (2).

Clearly (2) implies (3) and (3) implies (4).

Assume (4). We are going to show that (1) holds, thereby finishing the proof of the lemma. Choose a presentation $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. This is possible as S is of finite type over R and therefore of finite presentation (see Lemma 7.28.4). Set $I = (f_1, \ldots, f_m)$. Consider the naive cotangent complex

$$\mathrm{d} : I/I^2 \longrightarrow \bigoplus\nolimits_{j=1}^{m} S\mathrm{d}x_j$$

of this presentation (see Section 7.123). It suffices to show that when we localize this complex at $\mathfrak q$ then the map becomes a split injection, see Lemma 7.126.12. Denote $S' = R[x_1, \ldots, x_n]/I^2$. By Lemma 7.122.11 we have

$$S \otimes_{S'} \Omega_{S'/R} = S \otimes_{R[x_1, \ldots, x_n]} \Omega_{R[x_1, \ldots, x_n]/R} = \bigoplus\nolimits_{j=1}^m S\mathrm{d}x_j.$$

Thus the map

$$\mathrm{d} : I/I^2 \longrightarrow S \otimes_{S'} \Omega_{S'/R}$$

is the same as the map in the naive cotangent complex above. In particular the truth of the assertion we are trying to prove depends only on the three rings $R \to S' \to S$. Let $\mathfrak q' \subset R[x_1, \ldots, x_n]$ be the prime ideal corresponding to $\mathfrak q$. Since localization commutes with taking modules of differentials (Lemma 7.122.8) we see that it suffices to show that the map

$$\mathrm{d} : I_{\mathfrak q'}/I^2_{\mathfrak q'} \longrightarrow S_{\mathfrak q} \otimes_{S'_{\mathfrak q'}} \Omega_{S'_{\mathfrak q'}/R} \tag{7.130.2.1}$$

coming from $R \to S'_{\mathfrak q'} \to S_{\mathfrak q}$ is a split injection.

Let $N \in \mathbf{N}$ be an integer. Consider the ring

$$B'_N = S'_{\mathfrak q'}/(\mathfrak q')^N S'_{\mathfrak q'} = (S'/(\mathfrak q')^N S')_{\mathfrak q'}$$

and its quotient $B_N = B'_N/IB'_N$. Note that $B_N \cong S_{\mathfrak q}/\mathfrak q^N S_{\mathfrak q}$. Observe that B'_N is an Artinian local ring since it is the quotient of a local Noetherian ring by a power of its maximal ideal. Consider a filtration of the kernel I_N of $B'_N \to B_N$ by B'_N-submodules

$$0 \subset J_{N,1} \subset J_{N,2} \subset \ldots \subset J_{N,n(N)} = I_N$$

such that each successive quotient $J_{N,i}/J_{N,i-1}$ has length 1. (As B'_N is Artinian such a filtration exists.) This gives a sequence of small extensions

$$B'_N \to B'_N/J_{N,1} \to B'_N/J_{N,2} \to \ldots \to B'_N/J_{N,n(N)} = B'_N/I_N = B_N = S_{\mathfrak q}/\mathfrak q^N S_{\mathfrak q}$$

Applying condition (4) successively to these small extensions starting with the map $S \to B_N$ we see there exists a commutative diagram

$$\begin{tikzcd} S \arrow[r] \arrow[dr] & B_N \\ R \arrow[u] \arrow[r] & B'_N \arrow[u] \end{tikzcd}$$

Clearly the ring map $S \to B'_N$ factors as $S \to S_{\mathfrak q} \to B'_N$ where $S_{\mathfrak q} \to B'_N$ is a local homomorphism of local rings. Moreover, since the maximal ideal of B'_N to the Nth power is zero we conclude that $S_{\mathfrak q} \to B'_N$ factors through $S_{\mathfrak q}/(\mathfrak q)^N S_{\mathfrak q} = B_N$. In other words we have shown that for all $N \in \mathbf{N}$ the surjection of R-algebras $B'_N \to B_N$ has a splitting.

Consider the presentation

$$I_N \to B_N \otimes_{B'_N} \Omega_{B'_N/R} \to \Omega_{B_N/R} \to 0$$

coming from the surjection $B'_N \to B_N$ with kernel I_N (see Lemma 7.122.9). By the above the R-algebra map $B'_N \to B_N$ has a right inverse. Hence by Lemma 7.122.10 we see that the sequence above is split exact! Thus for every N the map

$$I_N \longrightarrow B_N \otimes_{B'_N} \Omega_{B'_N/R}$$

is a split injection. The rest of the proof is gotten by unwinding what this means exactly. Note that

$$I_N = I_{\mathfrak{q}'}/(I_{\mathfrak{q}'}^2 + (\mathfrak{q}')^N \cap I_{\mathfrak{q}'})$$

By Artin-Rees (Lemma 7.47.4) we find a $c \geq 0$ such that

$$S_\mathfrak{q}/\mathfrak{q}^{N-c}S_\mathfrak{q} \otimes_{S_\mathfrak{q}} I_N = S_\mathfrak{q}/\mathfrak{q}^{N-c}S_\mathfrak{q} \otimes_{S_\mathfrak{q}} I_{\mathfrak{q}'}/I_{\mathfrak{q}'}^2$$

for all $N \geq c$ (these tensor product are just a fancy way of dividing by $\mathfrak{q}^{N-c}$). We may of course assume $c \geq 1$. By Lemma 7.122.11 we see that

$$S'_{\mathfrak{q}'}/(\mathfrak{q}')^{N-c}S'_{\mathfrak{q}'} \otimes_{S'_{\mathfrak{q}'}} \Omega_{B'_N/R} = S'_{\mathfrak{q}'}/(\mathfrak{q}')^{N-c}S'_{\mathfrak{q}'} \otimes_{S'_{\mathfrak{q}'}} \Omega_{S'_{\mathfrak{q}'}/R}$$

we can further tensor this by $B_N = S_\mathfrak{q}/\mathfrak{q}^N$ to see that

$$S_\mathfrak{q}/\mathfrak{q}^{N-c}S_\mathfrak{q} \otimes_{S'_{\mathfrak{q}'}} \Omega_{B'_N/R} = S_\mathfrak{q}/\mathfrak{q}^{N-c}S_\mathfrak{q} \otimes_{S'_{\mathfrak{q}'}} \Omega_{S'_{\mathfrak{q}'}/R}.$$

Since a split injection remains a split injection after tensoring with anything we see that

$$S_\mathfrak{q}/\mathfrak{q}^{N-c}S_\mathfrak{q} \otimes_{S_\mathfrak{q}} (7.130.2.1) = S_\mathfrak{q}/\mathfrak{q}^{N-c}S_\mathfrak{q} \otimes_{S_\mathfrak{q}} (I_N \longrightarrow B_N \otimes_{B'_N} \Omega_{B'_N/R})$$

is a split injection for all $N \geq c$. By Lemma 7.68.1 we see that (7.130.2.1) is a split injection. This finishes the proof. □

7.131. Overview of results on smooth ring maps

Here is a list of results on smooth ring maps that we proved in the preceding sections. For more precise statements and definitions please consult the references given.

(1) A ring map $R \to S$ is smooth if it is of finite presentation and the naive cotangent complex of S/R is quasi-isomorphic to a finite projective S-module in degree 0, see Definition 7.126.1.
(2) If S is smooth over R, then $\Omega_{S/R}$ is a finite projective S-module, see discussion following Definition 7.126.1.
(3) The property of being smooth is local on S, see Lemma 7.126.13.
(4) The property of being smooth is stable under base change, see Lemma 7.126.4.
(5) The property of being smooth is stable under composition, see Lemma 7.126.14.
(6) A smooth ring map is syntomic, in particular flat, see Lemma 7.126.10.
(7) A finitely presented, flat ring map with smooth fibre rings is smooth, see Lemma 7.126.16.
(8) A finitely presented ring map $R \to S$ is smooth if and only if it is formally smooth, see Proposition 7.127.13.
(9) If $R \to S$ is a finite type ring map with R Noetherian then to check that $R \to S$ is smooth it suffices to check the lifting property of formal smoothness along small extensions of Artinian local rings, see Lemma 7.130.2.
(10) A smooth ring map $R \to S$ is the base change of a smooth ring map $R_0 \to S_0$ with R_0 of finite type over $\mathbf{Z}$, see Lemma 7.127.14.
(11) Formation of the set of points where a ring map is smooth commutes with flat base change, see Lemma 7.126.17.
(12) If S is of finite type over an algebraically closed field k, and $\mathfrak{m} \subset S$ a maximal ideal, then the following are equivalent
   (a) S is smooth over k in a neighbourhood of $\mathfrak{m}$,
   (b) $S_\mathfrak{m}$ is a regular local ring,
   (c) $\dim(S_\mathfrak{m}) = \dim_{\kappa(m)} \Omega_{S/k} \otimes_S \kappa(\mathfrak{m})$.

see Lemma 7.129.2.

(13) If S is of finite type over a field k, and $\mathfrak{q} \subset S$ a prime ideal, then the following are equivalent
  (a) S is smooth over k in a neighbourhood of $\mathfrak{q}$,
  (b) $\dim_{\mathfrak{q}}(S/k) = \dim_{\kappa(\mathfrak{q})} \Omega_{S/k} \otimes_S \kappa(\mathfrak{q})$.
  see Lemma 7.129.3.

(14) If S is smooth over a field, then all its local rings are regular, see Lemma 7.129.3.

(15) If S is of finite type over a field k, $\mathfrak{q} \subset S$ a prime ideal, the field extension $k \subset \kappa(\mathfrak{q})$ is separable and $S_{\mathfrak{q}}$ is regular, then S is smooth over k at $\mathfrak{q}$, see Lemma 7.129.5.

(16) If S is of finite type over a field k, if k has characteristic 0, if $\mathfrak{q} \subset S$ a prime ideal, and if $\Omega_{S/k,\mathfrak{q}}$ is free, then S is smooth over k at $\mathfrak{q}$, see Lemma 7.129.7.

Some of these results were proved using the notion of a standard smooth ring map, see Definition 7.126.6. This is the analogue of what a relative global complete intersection map is for the case of syntomic morphisms. It is also the easiest way to make examples.

7.132. Étale ring maps

An étale ring map is a smooth ring map whose relative dimension is equal to zero. This is the same as the following slightly more direct definition.

Definition 7.132.1. Let $R \to S$ be a ring map. We say $R \to S$ is *étale* if it is of finite presentation and the naive cotangent complex $NL_{S/R}$ is quasi-isomorphic to zero. Given a prime $\mathfrak{q}$ of S we say that $R \to S$ is *étale at* $\mathfrak{q}$ if there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is étale.

In particular we see that $\Omega_{S/R} = 0$ if S is étale over R. If $R \to S$ is smooth, then $R \to S$ is étale if and only if $\Omega_{S/R} = 0$. From our results on smooth ring maps we automatically get a whole host of results for étale maps. We summarize these in Lemma 7.132.3 below. But before we do so we prove that *any* étale ring map is standard smooth.

Lemma 7.132.2. *Any étale ring map is standard smooth. More precisely, if $R \to S$ is étale, then there exists a presentation $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ such that the image of* $\det(\partial f_j/\partial x_i)$ *is invertible in S.*

Proof. Let $R \to S$ be étale. Choose a presentation $S = R[x_1, \ldots, x_n]/I$. As $R \to S$ is étale we know that

$$\mathrm{d} : I/I^2 \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} S\mathrm{d}x_i$$

is an isomorphism, in particular I/I^2 is a free S-module. Thus by Lemma 7.125.6 we may assume (after possibly changing the presentation), that $I = (f_1, \ldots, f_c)$ such that the classes $f_i \bmod I^2$ form a basis of I/I^2. It follows immediately from the fact that the displayed map above is an isomorphism that $c = n$ and that $\det(\partial f_j/\partial x_i)$ is invertible in S. □

Lemma 7.132.3. *Results on étale ring maps.*

1. *If $R \to R_f$ is étale for any ring R and any $f \in R$.*
2. *Compositions of étale ring maps are étale.*
3. *A base change of an étale ring map is étale.*
4. *The property of being étale is local: Given a ring map $R \to S$ and elements $g_1, \ldots, g_m \in S$ which generate the unit ideal such that $R \to S_{g_j}$ is étale for $j = 1, \ldots, m$ then $R \to S$ is étale.*

(5) *Given $R \to S$ of finite presentation, and a flat ring map $R \to R'$, set $S' = R' \otimes_R S$. The set of primes where $R \to S'$ is étale is the inverse image via $Spec(S') \to Spec(S)$ of the set of primes where $R \to S$ is étale.*
(6) *An étale ring map is syntomic, in particular flat.*
(7) *If S is finite type over a field k, then S is étale over k if and only if $\Omega_{S/k} = 0$.*
(8) *Any étale ring map $R \to S$ is the base change of an étale ring map $R_0 \to S_0$ with R_0 of finite type over $\mathbf{Z}$.*
(9) *Let $A = colim\, A_i$ be a filtered colimit of rings. Let $A \to B$ be an étale ring map. Then there exists an étale ring map $A_i \to B_i$ for some i such that $B \cong A \otimes_{A_i} B_i$.*
(10) *Let A be a ring. Let S be a multiplicative subset of A. Let $S^{-1}A \to B'$ be étale. Then there exists an étale ring map $A \to B$ such that $B' \cong S^{-1}B$.*

Proof. In each case we use the corresponding result for smooth ring maps with a small argument added to show that $\Omega_{S/R}$ is zero.

Proof of (1). The ring map $R \to R_f$ is smooth and $\Omega_{R_f/R} = 0$.

Proof of (2). The composition $A \to C$ of smooth maps $A \to B$ and $B \to C$ is smooth, see Lemma 7.126.14. By Lemma 7.122.7 we see that $\Omega_{C/A}$ is zero as both $\Omega_{C/B}$ and $\Omega_{B/A}$ are zero.

Proof of (3). Let $R \to S$ be étale and $R \to R'$ be arbitrary. Then $R' \to S' = R' \otimes_R S$ is smooth, see Lemma 7.126.4. Since $\Omega_{S'/R'} = S' \otimes_S \Omega_{S/R}$ by Lemma 7.122.12 we conclude that $\Omega_{S'/R'} = 0$. Hence $R' \to S'$ is étale.

Proof of (4). Assume the hypotheses of (4). By Lemma 7.126.13 we see that $R \to S$ is smooth. We are also given that $\Omega_{S_{g_i}/R} = (\Omega_{S/R})_{g_i} = 0$ for all i. Then $\Omega_{S/R} = 0$, see Lemma 7.21.2.

Proof of (5). The result for smooth maps is Lemma 7.126.17. In the proof of that lemma we used that $NL_{S/R} \otimes_S S'$ is homotopy equivalent to $NL_{S'/R'}$. This reduces us to showing that if M is a finitely presented S-module the set of primes $\mathfrak{q}'$ of S' such that $(M \otimes_S S')_{\mathfrak{q}'} = 0$ is the inverse image of the set of primes $\mathfrak{q}$ of S such that $M_{\mathfrak{q}} = 0$. This is true (proof omitted).

Proof of (6). Follows directly from the corresponding result for smooth ring maps (Lemma 7.126.10).

Proof of (7). Follows from Lemma 7.129.3 and the definitions.

Proof of (8). Lemma 7.127.14 gives the result for smooth ring maps. The resulting smooth ring map $R_0 \to S_0$ satisfies the hypotheses of Lemma 7.121.8, and hence we may replace S_0 by the factor of relative dimension 0 over R_0.

Proof of (9). Follows from (8) since $R_0 \to A$ will factor through A_i for some i.

Proof of (10). Follows from (9), (1), and (2) since $S^{-1}A$ is a filtered colimit of principal localizations of A. □

Next we work out in more detail what it means to be étale over a field.

Lemma 7.132.4. *Let k be a field. A ring map $k \to S$ is étale if and only if S is isomorphic as a k-algebra to a finite product of finite separable extensions of k.*

Proof. If $k \to k'$ is a finite separable field extension then we can write $k' = k(\alpha) \cong k[x]/(f)$. Here f is the minimal polynomial of the element α. Since k' is separable over k we have $\gcd(f, f') = 1$. This implies that $\mathrm{d} : k' \cdot f \to k' \cdot \mathrm{d}x$ is an isomorphism. Hence $k \to k'$ is étale.

Conversely, suppose that $k \to S$ is étale. Let $\overline{k}$ be an algebraic closure of k. Then $S \otimes_k \overline{k}$ is étale over $\overline{k}$. Suppose we have the result over $\overline{k}$. Then $S \otimes_k \overline{k}$ is reduced and hence S is reduced. Also, $S \otimes_k \overline{k}$ is finite over $\overline{k}$ and hence S is finite over k. Hence S is a finite product $S = \prod k_i$ of fields, see Lemma 7.49.2 and Proposition 7.57.6. The result over $\overline{k}$ means $S \otimes_k \overline{k}$ is isomorphic to a finite product of copies of $\overline{k}$, which implies that each $k \subset k_i$ is finite separable, see for example Lemmas 7.41.1 and 7.41.3. Thus we have reduced to the case $k = \overline{k}$. In this case Lemma 7.129.2 (combined with $\Omega_{S/k} = 0$) we see that $S_\mathfrak{m} \cong k$ for all maximal ideals $\mathfrak{m} \subset S$. This implies the result because S is the product of the localizations at its maximal ideals by Lemma 7.49.2 and Proposition 7.57.6 again. □

Lemma 7.132.5. *Let $R \to S$ be a ring map. Let $\mathfrak{q} \subset S$ be a prime lying over $\mathfrak{p}$ in R. If S/R is étale at $\mathfrak{q}$ then*

(1) *we have $\mathfrak{p}S_\mathfrak{q} = \mathfrak{q}S_\mathfrak{q}$ is the maximal ideal of the local ring $S_\mathfrak{q}$, and*
(2) *the field extension $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q})$ is finite separable.*

Proof. First we may replace S by S_g for some $g \in S$, $g \notin \mathfrak{q}$ and assume that $R \to S$ is étale. Then the lemma follows from Lemma 7.132.4 by unwinding the fact that $S \otimes_R \kappa(\mathfrak{p})$ is étale over $\kappa(\mathfrak{p})$. □

Lemma 7.132.6. *An étale ring map is quasi-finite.*

Proof. Let $R \to S$ be an étale ring map. By definition $R \to S$ is of finite type. For any prime $\mathfrak{p} \subset R$ the fibre ring $S \otimes_R \kappa(\mathfrak{p})$ is étale over $\kappa(\mathfrak{p})$ and hence a finite products of fields finite separable over $\kappa(\mathfrak{p})$, in particular finite over $\kappa(\mathfrak{p})$. Thus $R \to S$ is quasi-finite by Lemma 7.113.4. □

Lemma 7.132.7. *Let $R \to S$ be a ring map. Let $\mathfrak{q}$ be a prime of S lying over a prime $\mathfrak{p}$ of R. If*

(1) *$R \to S$ is of finite presentation,*
(2) *$R_\mathfrak{p} \to S_\mathfrak{q}$ is flat*
(3) *$\mathfrak{p}S_\mathfrak{q}$ is the maximal ideal of the local ring $S_\mathfrak{q}$, and*
(4) *the field extension $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q})$ is finite separable,*

then $R \to S$ is étale at $\mathfrak{q}$.

Proof. Apply Lemma 7.113.2 to find a $g \in S$, $g \notin \mathfrak{q}$ such that $\mathfrak{q}$ is the only prime of S_g lying over $\mathfrak{p}$. We may and do replace S by S_g. Then $S \otimes_R \kappa(\mathfrak{p})$ has a unique prime, hence is a local ring, hence is equal to $S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q} \cong \kappa(\mathfrak{q})$. By Lemma 7.126.16 there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is smooth. Replace S by S_g again we may assume that $R \to S$ is smooth. By Lemma 7.126.10 we may even assume that $R \to S$ is standard smooth, say $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$. Since $S \otimes_R \kappa(\mathfrak{p}) = \kappa(\mathfrak{q})$ has dimension 0 we conclude that $n = c$, i.e., if $R \to S$ is étale. □

Here is a completely new phenomenon.

Lemma 7.132.8. *Let $R \to S$ and $R \to S'$ be étale. Then any R-algebra map $S' \to S$ is étale.*

Proof. First of all we note that $S' \to S$ is of finite presentation by Lemma 7.6.2. Let $\mathfrak{q} \subset S$ be a prime ideal lying over the primes $\mathfrak{q}' \subset S'$ and $\mathfrak{p} \subset R$. By Lemma 7.132.5 the ring map $S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q} \to S'_{\mathfrak{q}'}/\mathfrak{p}S'_{\mathfrak{q}'}$ is a map finite separable extensions of $\kappa(\mathfrak{p})$. In particular it is flat. Hence by Lemma 7.119.8 we see that $S'_{\mathfrak{q}'} \to S_\mathfrak{q}$ is flat. Thus $S' \to S$ is flat. Moreover, the above also shows that $\mathfrak{q}'S_\mathfrak{q}$ is the maximal ideal of $S_\mathfrak{q}$ and that the residue field extension of $S'_{\mathfrak{q}'} \to S_\mathfrak{q}$ is finite separable. Hence from Lemma 7.132.7 above we conclude that $S' \to S$ is étale at $\mathfrak{q}$. Since being étale is local (see Lemma 7.132.3) we win. $\square$

Lemma 7.132.9. *Let $\varphi : R \to S$ be a ring map. If $R \to S$ is surjective, flat and finitely presented then there exist an idempotent $e \in R$ such that $S = R_e$.*

Proof. Since $Spec(S) \to Spec(R)$ is a homeomorphism onto a closed subset (see Lemma 7.16.7) and is open (see Proposition 7.36.8) we see that the image is $D(e)$ for some idempotent $e \in R$ (see Lemma 7.18.3). Thus $R_e \to S$ induces a bijection on spectra. Now this map induces an isomorphism on all local rings for example by Lemmas 7.72.4 and 7.14.5. Then it follows that $R_e \to S$ is also injective, for example see Lemma 7.21.1. $\square$

Lemma 7.132.10. *Let R be a ring and let $I \subset R$ be an ideal. Let $R/I \to \overline{S}$ be an étale ring map. Then there exists an étale ring map $R \to S$ such that $\overline{S} \cong S/IS$ as R/I-algebras.*

Proof. By Lemma 7.132.2 we can write $\overline{S} = (R/I)[x_1, \ldots, x_n]/(\overline{f}_1, \ldots, \overline{f}_n)$ as in Definition 7.126.6 with $\overline{\Delta} = \det(\frac{\partial \overline{f}_i}{\partial x_j})_{i,j=1,\ldots,n}$ invertible in $\overline{S}$. Just take some lifts f_i and set $S = R[x_1, \ldots, x_n, x_{n+1}]/(f_1, \ldots, f_c, x_{n+1}\Delta - 1)$ where $\Delta = \det(\frac{\partial f_i}{\partial x_j})_{i,j=1,\ldots,c}$ as in Example 7.126.8. This proves the lemma. $\square$

Lemma 7.132.11. *Consider a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & J \arrow[r] & B' \arrow[r] & B \arrow[r] & 0 \\
0 \arrow[r] & I \arrow[r] \arrow[u] & A' \arrow[r] \arrow[u] & A \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

with exact rows where $B' \to B$ and $A' \to A$ are surjective ring maps whose kernels are ideals of square zero. If $A \to B$ is étale, and $J = I \otimes_A B$, then $A' \to B'$ is étale.

Proof. By Lemma 7.132.10 there exists an étale ring map $A' \to C$ such that $C/IC = B$. Then $A' \to C$ is formally smooth (by Proposition 7.127.13) hence we get an A'-algebra map $\varphi : C \to B'$. Since $A' \to C$ is flat we have $I \otimes_A B = I \otimes_A C/IC = IC$. Hence the assumption that $J = I \otimes_A B$ implies that φ induces an isomorphism $IC \to J$ and an isomorphism $C/IC \to B'/IB'$, whence φ is an isomorphism. $\square$

Example 7.132.12. Let $n, m \geq 1$ be integers. Consider the ring map

$$\begin{array}{rcl}
R = \mathbf{Z}[a_1, \ldots, a_{n+m}] & \longrightarrow & S = \mathbf{Z}[b_1, \ldots, b_n, c_1, \ldots, c_m] \\
a_1 & \longmapsto & b_1 + c_1 \\
a_2 & \longmapsto & b_2 + b_1 c_1 + c_2 \\
\ldots & \ldots & \ldots \\
a_{n+m} & \longmapsto & b_n c_m
\end{array}$$

of Example 7.125.7. Write symbolically

$$S = R[b_1, \ldots, c_m]/(\{a_k(b_i, c_j) - a_k\}_{k=1,\ldots,n+m})$$

where for example $a_1(b_i, c_j) = b_1 + c_1$. The matrix of partial derivatives is

$$\begin{pmatrix} 1 & c_1 & \ldots & c_m & 0 & \ldots & 0 \\ 0 & 1 & c_1 & \ldots & c_m & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & 1 & c_1 & \ldots & c_m \\ 1 & b_1 & \ldots & b_n & 0 & \ldots & 0 \\ 0 & 1 & b_1 & \ldots & b_n & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & 1 & b_1 & \ldots & b_n \end{pmatrix}$$

The determinant Δ of this matrix is better known as the *resultant* of the polynomials $g = x^n + b_1 x^{n-1} + \ldots + b_n$ and $h = x^m + c_1 x^{m-1} + \ldots + c_m$, and the matrix above is known as the *Sylvester matrix* associated to g, h. In a formula $\Delta = \text{Res}_x(g, h)$. The Sylvester matrix is the tranpose of the matrix of the linear map

$$\begin{array}{ccc} S[x]_{<m} \oplus S[x]_{<n} & \longrightarrow & S[x]_{<n+m} \\ a \oplus b & \longmapsto & ag + bh \end{array}$$

Let $\mathfrak{q} \subset S$ be any prime. By the above the following are equivalent:

(1) $R \to S$ is étale at $\mathfrak{q}$,
(2) $\Delta = \text{Res}_x(g, h) \not\in \mathfrak{q}$,
(3) the images $\overline{g}, \overline{h} \in \kappa(\mathfrak{q})[x]$ of the polynomials g, h are relatively prime in $\kappa(\mathfrak{q})[x]$.

The equivalence of (2) and (3) holds because the image of the Sylvester matrix in $\text{Mat}(n + m, \kappa(\mathfrak{q}))$ has a kernel if and only if the polynomials $\overline{g}, \overline{h}$ have a factor in common. We conclude that the ring map

$$R \longrightarrow S[\frac{1}{\Delta}] = S[\frac{1}{\text{Res}_x(g, h)}]$$

is étale.

Lemma 7.132.2 tells us that it does not really make sense to define a standard étale morphism to be a standard smooth morphism of relative dimension 0. As a model for an étale morphism we take the example given by a finite separable extension $k \subset k'$ of fields. Namely, we can always find an element $\alpha \in k'$ such that $k' = k(\alpha)$ and such that the minimal polynomial $f(x) \in k[x]$ of α has derivative f' which is relatively prime to f.

Definition 7.132.13. Let R be a ring. Let $g, f \in R[x]$. Assume that f is monic and the derivative f' is invertible in the localization $R[x]_g$. In this case the ring map $R \to R[x]_g/(f)$ is said to be *standard étale*.

Lemma 7.132.14. *Let $R \to R[x]_g/(f)$ be standard étale.*

(1) *The ring map $R \to R[x]_g/(f)$ is étale.*
(2) *For any ring map $R \to R'$ the base change $R' \to R'[x]_g/(f)$ of the standard étale ring map $R \to R[x]_g/(f)$ is standard étale.*
(3) *Any principal localization of $R[x]_g/(f)$ is standard étale over R.*
(4) *A composition of standard étale maps is* **not** *standard étale in general.*

Proof. Omitted. Here is an example for (4). The ring map $\mathbf{F}_2 \to \mathbf{F}_{2^2}$ is standard étale. The ring map $\mathbf{F}_{2^2} \to \mathbf{F}_{2^2} \times \mathbf{F}_{2^2} \times \mathbf{F}_{2^2} \times \mathbf{F}_{2^2}$ is standard étale. But the ring map $\mathbf{F}_2 \to \mathbf{F}_{2^2} \times \mathbf{F}_{2^2} \times \mathbf{F}_{2^2} \times \mathbf{F}_{2^2}$ is not standard étale. □

Standard étale morphisms are a convenient way to produce étale maps. Here is an example.

Lemma 7.132.15. *Let R be a ring. Let $\mathfrak{p}$ be a prime of R. Let $\kappa(\mathfrak{p}) \subset L$ be a finite separable field extension. There exists an étale ring map $R \to R'$ together with a prime $\mathfrak{p}'$ lying over $\mathfrak{p}$ such that the field extension $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{p}')$ is isomorphic to $\kappa(\mathfrak{p}) \subset L$.*

Proof. By the theorem of the primitive element we may write $L = \kappa(\mathfrak{p})[\alpha]$. Let $\overline{f} \in \kappa(\mathfrak{p})[x]$ denote the minimal polynomial for α (in particular this is monic). After replacing α by $c\alpha$ for some $c \in R$, $c \not\in \mathfrak{p}$ we may assume all the coefficients of $\overline{f}$ are in the image of $R \to \kappa(\mathfrak{p})$ (verification omitted). Thus we can find a monic polynomial $f \in R[x]$ which maps to $\overline{f}$ in $\kappa(\mathfrak{p})[x]$. Since $\kappa(\mathfrak{p}) \subset L$ is separable, we see that $\gcd(\overline{f}, \overline{f}') = 1$. Hence there is an element $\gamma \in L$ such that $\overline{f}'(\alpha)\gamma = 1$. Thus we get a R-algebra map

$$\begin{array}{rcl} R[x, 1/f']/(f) & \longrightarrow & L \\ x & \longmapsto & \alpha \\ 1/f' & \longmapsto & \gamma \end{array}$$

The left hand side is a standard étale algebra R' over R and the kernel of the ring map gives the desired prime. □

Proposition 7.132.16. *Let $R \to S$ be a ring map. Let $\mathfrak{q} \subset S$ be a prime. If $R \to S$ is étale at $\mathfrak{q}$, then there exists a $g \in S$, $g \not\in \mathfrak{q}$ such that $R \to S_g$ is standard étale.*

Proof. The following proof is a little roundabout and there may be ways to shorten it.

Step 1. By Definition 7.132.1 there exists a $g \in S$, $g \not\in \mathfrak{q}$ such that $R \to S_g$ is étale. Thus we may assume that S is étale over R.

Step 2. By Lemma 7.132.3 there exists an étale ring map $R_0 \to S_0$ with R_0 of finite type over $\mathbf{Z}$, and a ring map $R_0 \to R$ such that $R = R \otimes_{R_0} S_0$. Denote $\mathfrak{q}_0$ the prime of S_0 corresponding to $\mathfrak{q}$. If we show the result for $(R_0 \to S_0, \mathfrak{q}_0)$ then the result follows for $(R \to S, \mathfrak{q})$ by base change. Hence we may assume that R is Noetherian.

Step 3. Note that $R \to S$ is quasi-finite by Lemma 7.132.6. By Lemma 7.114.15 there exists a finite ring map $R \to S'$, an R-algebra map $S' \to S$, an element $g' \in S'$ such that $g' \not\in \mathfrak{q}$ such that $S' \to S$ induces an isomorphism $S'_{g'} \cong S_{g'}$. (Note that of course S' is not étale over R in general.) Thus we may assume that (a) R is Noetherian, (b) $R \to S$ is finite and (c) $R \to S$ is étale at $\mathfrak{q}$ (but no longer necessarily étale at all primes).

Step 4. Let $\mathfrak{p} \subset R$ be the prime corresponding to $\mathfrak{q}$. Consider the fibre ring $S \otimes_R \kappa(\mathfrak{p})$. This is a finite algebra over $\kappa(\mathfrak{p})$. Hence it is Artinian (see Lemma 7.49.2) and so a finite product of local rings

$$S \otimes_R \kappa(\mathfrak{p}) = \prod\nolimits_{i=1}^{n} A_i$$

see Proposition 7.57.6. One of the factors, say A_1, is the local ring $S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}}$ which is isomorphic to $\kappa(\mathfrak{q})$, see Lemma 7.132.5. The other factors correspond to the other primes, say $\mathfrak{q}_2, \ldots, \mathfrak{q}_n$ of S lying over $\mathfrak{p}$.

Step 5. We may choose a nonzero element $\alpha \in \kappa(\mathfrak{q})$ which generates the finite separable field extension $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q})$ (so even if the field extension is trivial we do not allow $\alpha = 0$). Note that for any $\lambda \in \kappa(\mathfrak{p})^*$ the element $\lambda\alpha$ also generates $\kappa(\mathfrak{q})$ over $\kappa(\mathfrak{p})$. Consider the element

$$\bar{t} = (\alpha, 0, \ldots, 0) \in \prod\nolimits_{i=1}^{n} A_i = S \otimes_R \kappa(\mathfrak{p}).$$

After possibly replacing α by $\lambda\alpha$ as above we may assume that $\bar{t}$ is the image of $t \in S$. Let $I \subset R[x]$ be the kernel of the R-algebra map $R[x] \to S$ which maps x to t. Set $S' = R[x]/I$, so $S' \subset S$. Here is a diagram

$$\begin{tikzcd} R[x] \arrow[r] & S' \arrow[r] & S \\ R \arrow[u] \arrow[ur] \arrow[urr] & & \end{tikzcd}$$

By construction the primes $\mathfrak{q}_j$, $j \geq 2$ of S all lie over the prime $(\mathfrak{p}, x)$ of $R[x]$, whereas the prime $\mathfrak{q}$ lies over a different prime of $R[x]$ because $\alpha \neq 0$.

Step 6. Denote $\mathfrak{q}' \subset S'$ the prime of S' corresponding to $\mathfrak{q}$. By the above $\mathfrak{q}$ is the only prime of S lying over $\mathfrak{q}'$. Thus we see that $S_\mathfrak{q} = S_{\mathfrak{q}'}$, see Lemma 7.36.11 (we have going up for $S' \to S$ by Lemma 7.32.20 since $S' \to S$ is finite as $R \to S$ is finite). It follows that $S'_{\mathfrak{q}'} \to S_\mathfrak{q}$ is finite and injective as the localization of the finite injective ring map $S' \to S$. Consider the maps of local rings

$$R_\mathfrak{p} \to S'_{\mathfrak{q}'} \to S_\mathfrak{q}$$

The second map is finite and injective. We have $S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q} = \kappa(\mathfrak{q})$, see Lemma 7.132.5. Hence a fortiori $S_\mathfrak{q}/\mathfrak{q}'S_\mathfrak{q} = \kappa(\mathfrak{q})$. Since

$$\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q}') \subset \kappa(\mathfrak{q})$$

and since α is in the image of $\kappa(\mathfrak{q}')$ in $\kappa(\mathfrak{q})$ we conclude that $\kappa(\mathfrak{q}') = \kappa(\mathfrak{q})$. Hence by Nakayama's Lemma 7.14.5 applied to the $S'_{\mathfrak{q}'}$-module map $S'_{\mathfrak{q}'} \to S_\mathfrak{q}$, the map $S'_{\mathfrak{q}'} \to S_\mathfrak{q}$ is surjective. In other words, $S'_{\mathfrak{q}'} \cong S_\mathfrak{q}$.

Step 7. By Lemma 7.117.6 there exists a $g' \in S'$, $g' \notin \mathfrak{q}'$ such that $S'_{g'} \cong S_{g'}$. As R is Noetherian the ring S' is finite over R as it is an R-submodule of the finite R-module S. Hence after replacing S by S' we may assume that (a) R is Noetherian, (b) S finite over R, (c) S is étale over R at $\mathfrak{q}$, and (d) $S = R[x]/I$.

Step 8. Consider the ring $S \otimes_R \kappa(\mathfrak{p}) = \kappa(\mathfrak{p})[x]/\overline{I}$ where $\overline{I} = I \cdot \kappa(\mathfrak{p})[x]$ is the ideal generated by I in $\kappa(\mathfrak{p})[x]$. As $\kappa(\mathfrak{p})[x]$ is a PID we know that $\overline{I} = (\overline{h})$ for some monic $\overline{h} \in \kappa(\mathfrak{p})$. After replacing $\overline{h}$ by $\lambda \cdot \overline{h}$ for some $\lambda \in \kappa(\mathfrak{p})$ we may assume that $\overline{h}$ is the image of some $h \in R[x]$. (The problem is that we do not know if we may choose h monic.) Also, as in Step 4 we know that $S \otimes_R \kappa(\mathfrak{p}) = A_1 \times \ldots \times A_n$ with $A_1 = \kappa(\mathfrak{q})$ a finite separable extension of $\kappa(\mathfrak{p})$ and $A_2, \ldots, A_n$ local. This implies that

$$\overline{h} = \overline{h}_1 \overline{h}_2^{e_2} \ldots \overline{h}_n^{e_n}$$

for certain pairwise coprime irreducible monic polynomials $\overline{h}_i \in \kappa(\mathfrak{p})[x]$ and certain $e_2, \ldots, e_n \geq 1$. Here the numbering is chosen so that $A_i = \kappa(\mathfrak{p})[x]/(\overline{h}_i^{e_i})$ as $\kappa(\mathfrak{p})[x]$-algebras. Note that $\overline{h}_1$ is the minimal polynomial of $\alpha \in \kappa(\mathfrak{q})$ and hence is a separable polynomial (its derivative is prime to itself).

Step 9. Let $m \in I$ be a monic element; such an element exists because the ring extension $R \to R[x]/I$ is finite hence integral. Denote $\overline{m}$ the image in $\kappa(\mathfrak{p})[x]$. We may factor

$$\overline{m} = \overline{k}\overline{h}_1^{d_1} \overline{h}_2^{d_2} \ldots \overline{h}_n^{d_n}$$

for some $d_1 \geq 1$, $d_j \geq e_j$, $j = 2, \ldots, n$ and $\overline{k} \in \kappa(\mathfrak{p})[x]$ prime to all the $\overline{h}_i$. Set $f = m^l + h$ where $l\deg(m) > \deg(h)$, and $l \geq 2$. Then f is monic as a polynomial over R. Also, the image $\overline{f}$ of f in $\kappa(\mathfrak{p})[x]$ factors as

$$\overline{f} = \overline{h}_1\overline{h}_2^{e_2} \ldots \overline{h}_n^{e_n} + \overline{k}^l\overline{h}_1^{ld_1}\overline{h}_2^{ld_2} \ldots \overline{h}_n^{ld_n} = \overline{h}_1(\overline{h}_2^{e_2} \ldots \overline{h}_n^{e_n} + \overline{k}^l\overline{h}_1^{ld_1-1}\overline{h}_2^{ld_2} \ldots \overline{h}_n^{ld_n}) = \overline{h}_1\overline{w}$$

with $\overline{w}$ a polynomial relatively prime to $\overline{h}_1$. Set $g = f'$ (the derivative with respect to x).

Step 10. The ring map $R[x] \to S = R[x]/I$ has the properties: (1) it maps f to zero, and (2) it maps g to an element of $S \setminus \mathfrak{q}$. The first assertion is clear since f is an element of I. For the second assertion we just have to show that g does not map to zero in $\kappa(\mathfrak{q}) = \kappa(\mathfrak{p})[x]/(\overline{h}_1)$. The image of g in $\kappa(\mathfrak{p})[x]$ is the derivative of $\overline{f}$. Thus (2) is clear because

$$\overline{g} = \frac{\mathrm{d}\overline{f}}{\mathrm{d}x} = \overline{w}\frac{\mathrm{d}\overline{h}_1}{\mathrm{d}x} + \overline{h}_1\frac{\mathrm{d}\overline{w}}{\mathrm{d}x},$$

$\overline{w}$ is prime to $\overline{h}_1$ and $\overline{h}_1$ is separable.

Step 11. We conclude that $\varphi : R[x]/(f) \to S$ is a surjective ring map, $R[x]_g/(f)$ is étale over R (because it is standard étale, see Lemma 7.132.14) and $\varphi(g) \not\in \mathfrak{q}$. Pick an element $g' \in R[x]/(f)$ such that also $\varphi(g') \not\in \mathfrak{q}$ and $S_{\varphi(g')}$ is étale over R (which exists since S is étale over R at $\mathfrak{q}$). Then the ring map $R[x]_{gg'}/(f) \to S_{\varphi(g)}$ is a surjective map of étale algebras over R. Hence it is étale by Lemma 7.132.8. Hence it is a localization by Lemma 7.132.9. Thus a localization of S at an element not in $\mathfrak{q}$ is isomorphic to a localization of a standard étale algebra over R which is what we wanted to show. □

The following two lemmas say that the étale topology is coarser than the topology generated by Zariski coverings and finite flat morphisms. They should be skipped on a first reading.

Lemma 7.132.17. *Let $R \to S$ be a standard étale morphism. There exists a ring map $R \to S'$ with the following properties*

1. *$R \to S'$ is finite, finitely presented, and flat (in other words S' is finite projective as an R-module),*
2. *$Spec(S') \to Spec(R)$ is surjective,*
3. *for every prime $\mathfrak{q} \subset S$, lying over $\mathfrak{p} \subset R$ and every prime $\mathfrak{q}' \subset S'$ lying over $\mathfrak{p}$ there exists a $g' \in S'$, $g' \not\in \mathfrak{q}'$ such that the ring map $R \to S'_{g'}$ factors through a map $\varphi : S \to S'_{g'}$ with $\varphi^{-1}(\mathfrak{q}'S'_{g'}) = \mathfrak{q}$.*

Proof. Let $S = R[x]_g/(f)$ be a presentation of S as in Definition 7.132.13. Write $f = x^n + a_1x^{n-1} + \ldots + a_n$ with $a_i \in R$. By Lemma 7.125.9 there exists a finite locally free and faithfully flat ring map $R \to S'$ such that $f = \prod(x - \alpha_i)$ for certain $\alpha_i \in S'$. Hence $R \to S'$ satisfies conditions (1), (2). Let $\mathfrak{q} \subset R[x]/(f)$ be a prime ideal with $g \not\in \mathfrak{q}$ (i.e., it corresponds to a prime of S). Let $\mathfrak{p} = R \cap \mathfrak{q}$ and let $\mathfrak{q}' \subset S'$ be a prime lying over $\mathfrak{p}$. Note that there are n maps of R-algebras

$$\begin{array}{rccc} \varphi_i : & R[x]/(f) & \longrightarrow & S' \\ & x & \longmapsto & \alpha_i \end{array}$$

To finish the proof we have to show that for some i we have (a) the image of $\varphi_i(g)$ in $\kappa(\mathfrak{q}')$ is not zero, and (b) $\varphi_i^{-1}(\mathfrak{q}') = \mathfrak{q}$. Because then we can just take $g' = \varphi_i(g)$, and $\varphi = \varphi_i$ for that i.

Let $\overline{f}$ denote the image of f in $\kappa(\mathfrak{p})[x]$. Note that as a point of $Spec(\kappa(\mathfrak{p})[x]/(\overline{f}))$ the prime $\mathfrak{q}$ corresponds to an irreducible factor f_1 of $\overline{f}$. Moreover, $g \not\in \mathfrak{q}$ means that f_1 does not divide the image $\overline{g}$ of g in $\kappa(\mathfrak{p})[x]$. Denote $\overline{\alpha}_1, \ldots, \overline{\alpha}_n$ the images of $\alpha_1, \ldots, \alpha_n$ in $\kappa(\mathfrak{q}')$. Note that the polynomial $\overline{f}$ splits completely in $\kappa(\mathfrak{q}')[x]$, namely

$$\overline{f} = \prod\nolimits_i (x - \overline{\alpha}_i)$$

Moreover $\varphi_i(g)$ reduces to $\overline{g}(\overline{\alpha}_i)$. It follows we may pick i such that $f_1(\overline{\alpha}_i) = 0$ and $\overline{g}(\overline{\alpha}_i) \neq 0$. For this i properties (a) and (b) hold. Some details omitted. □

Lemma 7.132.18. *Let $R \to S$ be a ring map. Assume that*

1. *$R \to S$ is étale, and*
2. *$Spec(S) \to Spec(R)$ is surjective.*

Then there exists a ring map $R \to S'$ such that

1. *$R \to S'$ is finite, finitely presented, and flat (in other words it is finite projective as an R-module),*
2. *$Spec(S') \to Spec(R)$ is surjective,*
3. *for every prime $\mathfrak{q}' \subset S'$ there exists a $g' \in S'$, $g' \notin \mathfrak{q}'$ such that the ring map $R \to S'_{g'}$ factors as $R \to S \to S'_{g'}$.*

Proof. By Proposition 7.132.16 and the quasi-compactness of $Spec(S)$ (see Lemma 7.16.10) we can find $g_1, \ldots, g_n \in S$ generating the unit ideal of S such that each $R \to S_{g_i}$ is standard étale. If we prove the lemma for the ring map $R \to \prod_{i=1,\ldots,n} S_{g_i}$ then the lemma follows for the ring map $R \to S$. Hence we may assume that $S = \prod_{i=1,\ldots,n} S_i$ is a finite product of standard étale morphisms.

For each i choose a ring map $R \to S'_i$ as in Lemma 7.132.17 adapted to the standard étale morphism $R \to S_i$. Set $S' = S'_1 \otimes_R \ldots \otimes_R S'_n$; we will use the R-algebra maps $S'_i \to S'$ without further mention below. We claim this works. Properties (1) and (2) are immediate. For property (3) suppose that $\mathfrak{q}' \subset S'$ is a prime. Denote $\mathfrak{p}$ its image in $Spec(R)$. Choose $i \in \{1, \ldots, n\}$ such that $\mathfrak{p}$ is in the image of $Spec(S_i) \to Spec(R)$; this is possible by assumption. Set $\mathfrak{q}'_i \subset S'_i$ the image of $\mathfrak{q}'$ in the spectrum of S'_i. By construction of S'_i there exists a $g'_i \in S'_i$ such that $R \to (S'_i)_{g'_i}$ factors as $R \to S_i \to (S'_i)_{g'_i}$. Hence also $R \to S'_{g'_i}$ factors as

$$R \to S_i \to (S'_i)_{g'_i} \to S'_{g'_i}$$

as desired. □

Lemma 7.132.19. *Let R be a ring. Let $f \in R[x]$ be a monic polynomial. Let $\mathfrak{p}$ be a prime of R. Let f mod $\mathfrak{p} = \overline{g}\overline{h}$ be a factorization of the image of f in $\kappa(\mathfrak{p})[x]$. If $\gcd(\overline{g}, \overline{h}) = 1$, then there exist*

1. *an étale ring map $R \to R'$,*
2. *a prime $\mathfrak{p}' \subset R'$ lying over $\mathfrak{p}$, and*
3. *a factorization $f = gh$ in $R'[x]$*

such that

1. *$\kappa(\mathfrak{p}) = \kappa(\mathfrak{p}')$,*
2. *$\overline{g} = g$ mod $\mathfrak{p}'$, $\overline{h} = h$ mod $\mathfrak{p}'$, and*
3. *the polynomials g, h generate the unit ideal in $R'[x]$.*

Proof. Suppose $\overline{g} = \overline{b}_0 x^n + \overline{b}_1 x^{n-1} + \ldots + \overline{b}_n$, and $\overline{h} = \overline{c}_0 x^m + \overline{c}_1 x^{m-1} + \ldots + \overline{c}_m$ with $\overline{b}_0, \overline{c}_0 \in \kappa(\mathfrak{p})$ nonzero. After localizing R at some element of R not contained in $\mathfrak{p}$ we may assume $\overline{b}_0$ is the image of an invertible element $b_0 \in R$. Replacing $\overline{g}$ by $\overline{g}/b_0$ and $\overline{h}$ by $b_0\overline{h}$ we reduce to the case where $\overline{g}, \overline{h}$ are monic (verification omitted). Say $\overline{g} = x^n + \overline{b}_1 x^{n-1} + \ldots + \overline{b}_n$, and $\overline{h} = x^m + \overline{c}_1 x^{m-1} + \ldots + \overline{c}_m$. Write $f = x^{n+m} + a_1 x^{n-1} + \ldots + a_{n+m}$. Consider the fibre product

$$R' = R \otimes_{\mathbf{Z}[a_1,\ldots,a_{n+m}]} \mathbf{Z}[b_1, \ldots, b_n, c_1, \ldots, c_m]$$

where the map $\mathbf{Z}[a_k] \to \mathbf{Z}[b_i, c_j]$ is as in Examples 7.125.7 and 7.132.12. By construction there is an R-algebra map

$$R' = R \otimes_{\mathbf{Z}[a_1,\ldots,a_{n+m}]} \mathbf{Z}[b_1, \ldots, b_n, c_1, \ldots, c_m] \longrightarrow \kappa(\mathfrak{p})$$

which maps b_i to $\overline{b}_i$ and c_j to $\overline{c}_j$. Denote $\mathfrak{p}' \subset R'$ the kernel of this map. Since by assumption the polynomials $\overline{g}, \overline{h}$ are relatively prime we see that the element $\Delta = \text{Res}_x(g, h) \in \mathbf{Z}[b_i, c_j]$ (see Example 7.132.12) does not map to zero in $\kappa(\mathfrak{p})$ under the displayed map. We conclude that $R \to R'$ is étale at $\mathfrak{p}'$. In fact a solution to the problem posed in the lemma is the ring map $R \to R'[1/\Delta]$ and the prime $\mathfrak{p}' R'[1/\Delta]$. Because $\text{Res}_x(f, g)$ is invertible in this ring the Sylvester matrix is invertible over R' and hence $1 = ag + bh$ for some $a, b \in R'[x]$ see Example 7.132.12. □

The following lemmas say roughly that after an étale extension a quasi-finite ring map becomes finite. To help interpret the results recall that the locus where a finite type ring map is quasi-finite is open (see Lemma 7.114.14) and that formation of this locus commutes with arbitrary base change (see Lemma 7.113.8).

Lemma 7.132.20. *Let $R \to S' \to S$ be ring maps. Let $\mathfrak{p} \subset R$ be a prime. Let $g \in S'$ be an element. Assume*

1. *$R \to S'$ is integral,*
2. *$R \to S$ is finite type,*
3. *$S'_g \cong S_g$, and*
4. *g invertible in $S' \otimes_R \kappa(\mathfrak{p})$.*

Then there exists a $f \in R$, $f \notin \mathfrak{p}$ such that $R_f \to S_f$ is finite.

Proof. By assumption the image T of $V(g) \subset Spec(S')$ under the morphism $Spec(S') \to Spec(R)$ does not contain $\mathfrak{p}$. By Section 7.36 especially, Lemma 7.36.6 we see T is closed. Pick $f \in R$, $f \notin \mathfrak{p}$ such that $T \cap V(f) = \emptyset$. Then we see that g becomes invertible in S'_f. Hence $S'_f \cong S_f$. Thus S_f is both of finite type and integral over R_f, hence finite. □

Lemma 7.132.21. *Let $R \to S$ be a ring map. Let $\mathfrak{q} \subset S$ be a prime lying over the prime $\mathfrak{p} \subset R$. Assume $R \to S$ finite type and quasi-finite at $\mathfrak{q}$. Then there exists*

1. *an étale ring map $R \to R'$,*
2. *a prime $\mathfrak{p}' \subset R'$ lying over $\mathfrak{p}$,*
3. *a product decomposition*

$$R' \otimes_R S = A \times B$$

with the following properties

1. *$\kappa(\mathfrak{p}) = \kappa(\mathfrak{p}')$,*
2. *$R' \to A$ is finite,*
3. *A has exactly one prime $\mathfrak{r}$ lying over $\mathfrak{p}'$, and*
4. *$\mathfrak{r}$ lies over $\mathfrak{q}$.*

Proof. Let $S' \subset S$ be the integral closure of R in S. Let $\mathfrak{q}' = S' \cap \mathfrak{q}$. By Zariski's Main Theorem 7.114.13 there exists a $g \in S'$, $g \notin \mathfrak{q}'$ such that $S'_g \cong S_g$. Consider the fibre rings $F = S \otimes_R \kappa(\mathfrak{p})$ and $F' = S' \otimes_R \kappa(\mathfrak{p})$. Denote $\overline{\mathfrak{q}}'$ the prime of F' corresponding to $\mathfrak{q}'$. Since

F' is integral over $\kappa(\mathfrak{p})$ we see that $\overline{\mathfrak{q}}'$ is a closed point of $Spec(F')$, see Lemma 7.32.17. Note that $\mathfrak{q}$ defines an isolated closed point $\overline{\mathfrak{q}}$ of $Spec(F)$ (see Definition 7.113.3). Since $S'_g \cong S_g$ we have $F'_g \cong F_g$, so $\overline{\mathfrak{q}}$ and $\overline{\mathfrak{q}}'$ have isomorphic open neighbourhoods in $Spec(F)$ and $Spec(F')$. We conclude the set $\{\overline{\mathfrak{q}}'\} \subset Spec(F')$ is open. Combined with closedness shown above we conclude that $\overline{\mathfrak{q}}'$ defines an isolated closed point of $Spec(F')$ as well.

An additional small remark is that under the map $Spec(F) \to Spec(F')$ the point $\overline{\mathfrak{q}}$ is the only point mapping to $\overline{\mathfrak{q}}'$. This follows from the discussion above.

By Lemma 7.20.3 we may write $F' = F'_1 \times F'_2$ with $Spec(F'_1) = \{\overline{\mathfrak{q}}'\}$. Since $F' = S' \otimes_R \kappa(\mathfrak{p})$, there exists an $s' \in S'$ which maps to the element $(r, 0) \in F'_1 \times F'_2 = F'$ for some $r \in R$, $r \not\in \mathfrak{p}$. In fact, what we will use about s' is that it is an element of S', not contained in $\mathfrak{q}'$, and contained in any other prime lying over $\mathfrak{p}$.

Let $f(x) \in R[x]$ be a monic polynomial such that $f(s') = 0$. Denote $\overline{f} \in \kappa(\mathfrak{p})[x]$ the image. We can factor it as $\overline{f} = x^e \overline{h}$ where $\overline{h}(0) \neq 0$. By Lemma 7.132.19 we can find an étale ring extension $R \to R'$, a prime $\mathfrak{p}'$ lying over $\mathfrak{p}$, and a factorization $f = hi$ in $R'[x]$ such that $\kappa(\mathfrak{p}) = \kappa(\mathfrak{p}')$, $x^e = h \bmod \mathfrak{p}'$, $\overline{i} = i \bmod \mathfrak{p}'$, and we can write $ah + bi = 1$ in $R'[x]$ (for suitable a, b).

Consider the elements $h(s'), i(s') \in R' \otimes_R S'$. By construction we have $h(s')i(s') = f(s') = 0$. On the other hand they generate the unit ideal since $a(s')h(s') + b(s')i(s') = 1$. Thus we see that $R' \otimes_R S'$ is the product of the localizations at these elements:

$$R' \otimes_R S' = (R' \otimes_R S')_{h(s')} \times (R' \otimes_R S')_{i(s')} = S'_1 \times S'_2$$

Moreover this product decomposition is compatible with the product decomposition we found for the fibre ring F'; this comes from our choice of s', h which garantee that $\overline{\mathfrak{q}}'$ is the only prime of F' which does not contain the image of $h(s')$ in F'. Here we use that the fibre ring of $R' \otimes_R S'$ over R' at $\mathfrak{p}'$ is the same as F' due to the fact that $\kappa(\mathfrak{p}) = \kappa(\mathfrak{p}')$. It follows that S'_1 has exactly one prime, say $\mathfrak{r}'$, lying over $\mathfrak{p}'$ and that this prime lies over $\mathfrak{q}$. Hence the element $g \in S'$ maps to an element of S'_1 not contained in $\mathfrak{r}'$.

The base change $R' \otimes_R S$ inherits a similar product decomposition

$$R' \otimes_R S = (R' \otimes_R S)_{h(s')} \times (R' \otimes_R S)_{i(s')} = S_1 \times S_2$$

It follows from the above that S_1 has exactly one prime, say $\mathfrak{r}$, lying over $\mathfrak{p}'$ (consider the fibre ring as above), and that this prime lies over $\mathfrak{q}$.

Now we may apply Lemma 7.132.20 to the ring maps $R' \to S'_1 \to S_1$, the prime $\mathfrak{p}'$ and the element g to see that after replacing R' by a principal localization we can assume that S_1 is finite over R' as desired. ☐

Lemma 7.132.22. *Let $R \to S$ be a ring map. Let $\mathfrak{p} \subset R$ be a prime. Assume $R \to S$ finite type. Then there exists*

1. *an étale ring map $R \to R'$,*
2. *a prime $\mathfrak{p}' \subset R'$ lying over $\mathfrak{p}$,*
3. *a product decomposition*

$$R' \otimes_R S = A_1 \times \ldots \times A_n \times B$$

with the following properties

1. *we have $\kappa(\mathfrak{p}) = \kappa(\mathfrak{p}')$,*
2. *each A_i is finite over R',*
3. *each A_i has exactly one prime $\mathfrak{r}_i$ lying over $\mathfrak{p}'$, and*

(4) *$R' \to B$ not quasi-finite at any prime lying over $\mathfrak{p}'$.*

Proof. Denote $F = S \otimes_R \kappa(\mathfrak{p})$ the fibre ring of S/R at the prime $\mathfrak{p}$. As F is of finite type over $\kappa(\mathfrak{p})$ it is Noetherian and hence $Spec(F)$ has finitely many isolated closed points. If there are no isolated closed points, i.e., no primes $\mathfrak{q}$ of S over $\mathfrak{p}$ such that S/R is quasi-finite at $\mathfrak{q}$, then the lemma holds. If there exists at least one such prime $\mathfrak{q}$, then we may apply Lemma 7.132.21 above. This gives a diagram

$$\begin{tikzcd} S \arrow[r] & R' \otimes_R S \arrow[r, equal] & A_1 \times B' \\ R \arrow[u] \arrow[r] & R' \arrow[u] \arrow[ur] & \end{tikzcd}$$

as in said lemma. Since the residue fields at $\mathfrak{p}$ and $\mathfrak{p}'$ are the same, the fibre rings of S/R and $(A \times B)/R'$ are the same. Hence, by induction on the number of isolated closed points of the fibre we may assume that the lemma holds for $R' \to B$ and $\mathfrak{p}'$. Thus we get an étale ring map $R' \to R''$, a prime $\mathfrak{p}'' \subset R''$ and a decomposition

$$R'' \otimes_{R'} B' = A_2 \times \ldots \times A_n \times B$$

We omit the verification that the ring map $R \to R''$, the prime $\mathfrak{p}''$ and the resulting decomposition

$$R'' \otimes_R S = (R'' \otimes_{R'} A_1) \times A_2 \times \ldots \times A_n \times B$$

is a solution to the problem posed in the lemma. □

Lemma 7.132.23. *Let $R \to S$ be a ring map. Let $\mathfrak{p} \subset R$ be a prime. Assume $R \to S$ finite type. Then there exists*

(1) *an étale ring map $R \to R'$,*
(2) *a prime $\mathfrak{p}' \subset R'$ lying over $\mathfrak{p}$,*
(3) *a product decomposition*

$$R' \otimes_R S = A_1 \times \ldots \times A_n \times B$$

with the following properties

(1) *each A_i is finite over R',*
(2) *each A_i has exactly one prime $\mathfrak{r}_i$ lying over $\mathfrak{p}'$,*
(3) *the finite field extensions $\kappa(\mathfrak{p}') \subset \kappa(\mathfrak{r}_i)$ are purely inseparable, and*
(4) *$R' \to B$ not quasi-finite at any prime lying over $\mathfrak{p}'$.*

Proof. The strategy of the proof is to make two étale ring extensions: first we control the residue fields, then we apply Lemma 7.132.22 above.

Denote $F = S \otimes_R \kappa(\mathfrak{p})$ the fibre ring of S/R at the prime $\mathfrak{p}$. As in the proof of Lemma 7.132.22 there are finitely may primes, say $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ of S lying over R at which the ring map $R \to S$ is quasi-finite. Let $\kappa(\mathfrak{p}) \subset L_i \subset \kappa(\mathfrak{q}_i)$ be the subfield such that $\kappa(\mathfrak{p}) \subset L_i$ is separable, and the field extension $L_i \subset \kappa(\mathfrak{q}_i)$ is purely inseparable. Let $\kappa(\mathfrak{p}) \subset L$ be a finite Galois extension into which L_i embeds for $i = 1, \ldots, n$. By Lemma 7.132.15 we can find an étale ring extension $R \to R'$ together with a prime $\mathfrak{p}'$ lying over $\mathfrak{p}$ such that the field extension $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{p}')$ is isomorphic to $\kappa(\mathfrak{p}) \subset L$. Thus the fibre ring of $R' \otimes_R S$ at $\mathfrak{p}'$ is isomorphic to $F \otimes_{\kappa(\mathfrak{p})} L$. The primes lying over $\mathfrak{q}_i$ correspond to primes of $\kappa(\mathfrak{q}_i) \otimes_{\kappa(\mathfrak{p})} L$ which is a product of fields purely inseparable over L by our choice of L and elementary field theory. These are also the only primes over $\mathfrak{p}'$ at which $R' \to R' \otimes_R S$ is quasi-finite, by Lemma 7.113.8. Hence after replacing R by R', $\mathfrak{p}$ by $\mathfrak{p}'$, and S by $R' \otimes_R S$ we may assume that for all primes $\mathfrak{q}$ lying over $\mathfrak{p}$ for which S/R is quasi-finite the field extensions $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q})$ are purely inseparable.

Next apply Lemma 7.132.22. The result is what we want since the field extensions do not change under this étale ring extension. □

7.133. Local homomorphisms

Lemma 7.133.1. *Let $(R, \mathfrak{m}_R) \to (S, \mathfrak{m}_S)$ be a local homomorphism of local rings. Assume S is the localization of an étale ring extension of R. Then there exists a finite, finitely presented, faithfully flat ring map $R \to S'$ such that for every maximal ideal $\mathfrak{m}'$ of S' there is a factorization*

$$R \to S \to S'_{\mathfrak{m}'}.$$

of the ring map $R \to S'_{\mathfrak{m}'}$.

Proof. Write $S = T_{\mathfrak{q}}$ for some étale R-algebra T. By Proposition 7.132.16 we may assume T is standard étale. Apply Lemma 7.132.17 to the ring map $R \to T$ to get $R \to S'$. Then in particular for every maximal ideal $\mathfrak{m}'$ of S' we get a factorization $\varphi : T \to S'_{g'}$ for some $g' \not\in \mathfrak{m}'$ such that $\mathfrak{q} = \varphi^{-1}(\mathfrak{m}' S'_{g'})$. Thus φ induces the desired local ring map $S \to S'_{\mathfrak{m}'}$. □

7.134. Integral closure and smooth base change

Lemma 7.134.1. *Let R be a ring. Let $f \in R[x]$ be a monic polynomial. Let $R \to B$ be a ring map. If $h \in B[x]/(f)$ is integral over R, then the element $f'h$ can be written as $f'h = \sum_i b_i x^i$ with $b_i \in B$ integral over R.*

Proof. Say $h^e + r_1 h^{e-1} + \ldots + r_e = 0$ in the ring $B[x]/(f)$ with $r_i \in R$. There exists a finite free ring extension $B \subset B'$ such that $f = (x - \alpha_1) \ldots (x - \alpha_d)$ for some $\alpha_i \in B'$, see Lemma 7.125.9. Note that each α_i is integral over R. We may represent $h = h_0 + h_1 x + \ldots + h_{d-1} x^{d-1}$ with $h_i \in B$. Then it is a universal fact that

$$f'h \equiv \sum\nolimits_{i=1,\ldots,d} h(\alpha_i)(x - \alpha_1) \ldots \widehat{(x - \alpha_i)} \ldots (x - \alpha_d)$$

as elements of $B[x]/(f)$. You prove this by evaluating both sides at the points α_i over the ring $B_{univ} = \mathbf{Z}[\alpha_i, h_j]$ (some details omitted). By our assumption that h satisfies $h^e + r_1 h^{e-1} + \ldots + r_e = 0$ in the ring $B[x]/(f)$ we see that

$$h(\alpha_i)^e + r_1 h(\alpha_i)^{e-1} + \ldots + r_e = 0$$

in B'. Hence $h(\alpha_i)$ is integral over R. Using the formula above we see that $f'h \equiv \sum_{j=0,\ldots,d-1} b'_j x^j$ in $B'[x]/(f)$ with $b'_j \in B'$ integral over R. However, since $f'h \in B[x]/(f)$ and since $1, x, \ldots, x^{d-1}$ is a B'-basis for $B'[x]/(f)$ we see that $b'_j \in B$ as desired. □

Lemma 7.134.2. *Let $R \to S$ be an étale ring map. Let $R \to B$ be any ring map. Let $A \subset B$ be the integral closure of R in B. Let $A' \subset S \otimes_R B$ be the integral closure of S in $S \otimes_R B$. Then the canonical map $S \otimes_R A \to A'$ is an isomorphism.*

Proof. The map $S \otimes_R A \to A'$ is injective because $A \subset B$ and $R \to S$ is flat. We are going to use repeatedly that taking integral closure commutes with localization, see Lemma 7.32.9. Hence we may localize on S, by Lemma 7.21.2 (the criterion for checking whether an S-module map is an isomorphism). Thus we may assume that $S = R[x]_g/(f) = (R[x]/(f))_g$ is standard étale over R, see Proposition 7.132.16. Applying localization one more time we see that A' is $(A'')_g$ where A'' is the integral closure of $R[x]/(f)$ in $B[x]/(f)$. Suppose that $a \in A''$. It suffices to show that a is in $S \otimes_R A$. By Lemma 7.134.1 we see that $f' a = \sum a_i x^i$ with $a_i \in A$. Since f' is invertible in $B[x]_g/(f)$ (by definition of a standard étale ring map) we conclude that $a \in S \otimes_R A$ as desired. □

Example 7.134.3. Let p be a prime number. For every $n > 0$ the ring extension

$$R = \mathbf{Z}[1/p] \subset R' = \mathbf{Z}[1/p][x]/(x^{p^n} - 1)$$

has the following property: For $d < p^n$ there exist elements $\alpha_0, \ldots, \alpha_{d-1} \in R'$ such that

$$\prod\nolimits_{0 \leq i < j < d} (\alpha_i - \alpha_j)$$

is a unit in R'. Namely, take α_i equal to the class of x^i in R'. Then we have

$$T^{p^n} - 1 = \prod\nolimits_{i = 0, \ldots, p^n - 1} (T - \alpha_i)$$

(for example because this is clear over $\mathbf{Q}$) and hence by taking derivatives on both sides

$$p^n \alpha_i^{p^n - 1} = (\alpha_i - \alpha_1) \ldots \widehat{(\alpha_i - \alpha_i)} \ldots (\alpha_i - \alpha_1)$$

and we see this is invertible in R'.

Lemma 7.134.4. *Let $R \to S$ be a smooth ring map. Let $R \to B$ be any ring map. Let $A \subset B$ be the integral closure of R in B. Let $A' \subset S \otimes_R B$ be the integral closure of S in $S \otimes_R B$. Then the canonical map $S \otimes_R A \to A'$ is an isomorphism.*

Proof. Arguing as in the proof of Lemma 7.134.2 we may localize on S. Hence we may assume that $R \to S$ is a standard smooth ring map, see Lemma 7.126.10. By definition of a standard smooth ring map we see that S is étale over a polynomial ring $R[x_1, \ldots, x_n]$. Since we have seen the result in the case of an étale ring extension (Lemma 7.134.2) this reduces us to the case where $S = R[x]$. Thus we have to show

$$f = \sum b_i x^i \text{ integral over } R[x] \Leftrightarrow \text{each } b_i \text{ integral over } R.$$

The implication from right to left holds because the set of elements in $B[x]$ integral over $R[x]$ is a ring (Lemma 7.32.7) and contains x.

Suppose that $f \in B[x]$ is integral over $R[x]$, and assume that $f = \sum_{i<d} b_i x^i$ has degree $< d$. Since integral closure and localization commute, it suffices to show that each b_i is integral over $R[1/2]$ and over $R[1/3]$. Hence, we can find a finite free ring extension $R \subset R'$ such that R' contains $\alpha_1, \ldots, \alpha_d$ with the property that $\prod_{i<j}(\alpha_i - \alpha_j)$ is a unit in R', see Example 7.134.3. In this case we have the universal equality

$$f = \sum_i f(\alpha_i) \frac{(x - \alpha_1) \ldots \widehat{(x - \alpha_i)} \ldots (x - \alpha_d)}{(\alpha_i - \alpha_1) \ldots \widehat{(\alpha_i - \alpha_i)} \ldots (\alpha_i - \alpha_d)}.$$

OK, and the elements $f(\alpha_i)$ are integral over R' since $(R' \otimes_R B)[x] \to R' \otimes_R B$, $h \mapsto h(\alpha_i)$ is a ring map. Hence we see that the coefficients of f in $(R' \otimes_R B)[x]$ are integral over over R'. Since R' is finite over R (hence integral over R) we see that they are integral over R also, as desired. □

7.135. Formally unramified maps

It turns out to be logically more efficient to define the notion of a formally unramified map before introducing the notion of a formally étale one.

Definition 7.135.1. Let $R \to S$ be a ring map. We say S is *formally unramified over R* if for every commutative solid diagram

$$\begin{tikzcd} S \arrow[r] \arrow[rd, dashed] & A/I \\ R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

where $I \subset A$ is an ideal of square zero, there exists at most one dotted arrow making the diagram commute.

Lemma 7.135.2. *Let $R \to S$ be a ring map. The following are equivalent:*

(1) *$R \to S$ is formally unramified,*
(2) *the module of differentials $\Omega_{S/R}$ is zero.*

Proof. Let $J = \mathrm{Ker}(S \otimes_R S \to S)$ be the kernel of the multiplication map. Let $A_{univ} = S \otimes_R S/J^2$. Recall that $I_{univ} = J/J^2$ is isomorphic to $\Omega_{S/R}$, see Lemma 7.122.13. Moreover, the two R-algebra maps $\sigma_1, \sigma_2 : S \to A_{univ}$, $\sigma_1(s) = s \otimes 1 \bmod J^2$, and $\sigma_2(s) = 1 \otimes s \bmod J^2$ differ by the universal derivation $\mathrm{d} : S \to \Omega_{S/R} = I_{univ}$.

Assume $R \to S$ formally unramified. Then we see that $\sigma_1 = \sigma_2$. Hence $\mathrm{d}(s) = 0$ for all $s \in S$. Hence $\Omega_{S/R} = 0$.

Assume that $\Omega_{S/R} = 0$. Let $A, I, R \to A, S \to A/I$ be a solid diagram as in Definition 7.135.1. Let $\tau_1, \tau_2 : S \to A$ be two dotted arrows making the diagram commute. Consider the R-algebra map $A_{univ} \to A$ defined by the rule $s_1 \otimes s_2 \mapsto \tau_1(s_1)\tau_2(s_2)$. We omit the verification that this is well defined. Since $A_{univ} \cong S$ as $I_{univ} = \Omega_{S/R} = 0$ we conclude that $\tau_1 = \tau_2$. □

Lemma 7.135.3. *Let $R \to S$ be a ring map. The following are equivalent:*

(1) *$R \to S$ is formally unramified,*
(2) *$R \to S_{\mathfrak q}$ is formally unramified for all primes $\mathfrak q$ of S, and*
(3) *$R_{\mathfrak p} \to S_{\mathfrak q}$ is formally unramified for all primes $\mathfrak q$ of S with $\mathfrak p = R \cap \mathfrak q$.*

Proof. We have seen in Lemma 7.135.2 that (1) is equivalent to $\Omega_{S/R} = 0$. Similarly, by Lemma 7.122.8 we see that (2) and (3) are equivalent to $(\Omega_{S/R})_{\mathfrak q} = 0$ for all $\mathfrak q$. Hence the equivalence follows from Lemma 7.21.1. □

Lemma 7.135.4. *Let $A \to B$ be a formally unramified ring map.*

(1) *For $S \subset A$ a multiplicative subset, $S^{-1}A \to S^{-1}B$ is formally unramified.*
(2) *For $S \subset B$ a multiplicative subset, $A \to S^{-1}B$ is formally unramified.*

Proof. Follows from Lemma 7.135.3. (You can also deduce it from Lemma 7.135.2 combined with Lemma 7.122.8.) □

7.136. Conormal modules and universal thickenings

It turns out that one can define the first infinitesimal neighbourhood not just for a closed immersion of schemes, but already for any formally unramified morphism. This is based on the following algebraic fact.

Lemma 7.136.1. *Let $R \to S$ be a formally unramified ring map. There exists a surjection of R-algebras $S' \to S$ whose kernel is an ideal of square zero with the following universal property: Given any commutative diagram*

$$\begin{tikzcd} S \arrow[r, "a"'] & A/I \\ R \arrow[u] \arrow[r, "b"] & A \arrow[u] \end{tikzcd}$$

where $I \subset A$ is an ideal of square zero, there is a unique R-algebra map $a' : S' \to A$ such that $S' \to A \to A/I$ is equal to $S' \to S \to A$.

Proof. Choose a set of generators $z_i \in S, i \in I$ for S as an R-algebra. Let $P = R[\{x_i\_i \in I]$ denote the polynomial ring on generators x_i, $i \in I$. Consider the R-algebra map $P \to S$ which maps x_i to z_i. Let $J = \text{Ker}(P \to S)$. Consider the map

$$\mathrm{d} : J/J^2 \longrightarrow \Omega_{P/R} \otimes_P S$$

see Lemma 7.122.9. This is surjective since $\Omega_{S/R} = 0$ by assumption, see Lemma 7.135.2. Note that $\Omega_{P/R}$ is free on $\mathrm{d}x_i$, and hence the module $\Omega_{P/R} \otimes_P S$ is free over S. Thus we may choose a splitting of the surjection above and write

$$J/J^2 = K \oplus \Omega_{P/R} \otimes_P S$$

Let $J^2 \subset J' \subset J$ be the ideal of P such that J'/J^2 is the second summand in the decomposition above. Set $S' = P/J'$. We obtain a short exact sequence

$$0 \to J/J' \to S' \to S \to 0$$

and we see that $J/J' \cong K$ is a square zero ideal in S'. Hence

$$\begin{tikzcd} S \arrow[r, "1"'] & S \\ R \arrow[u] \arrow[r] & S' \arrow[u] \end{tikzcd}$$

is a diagram as above. In fact we claim that this is an initial object in the category of diagrams. Namely, let $(I \subset A, a, b)$ be an arbitrary diagram. We may choose an R-algebra map $\beta : P \to A$ such that

$$\begin{tikzcd} S \arrow[r, "1"'] & S \arrow[r, "a"'] & A/I \\ R \arrow[u] \arrow[r] \arrow[rr, bend right, "b"'] & P \arrow[u] \arrow[r, "\beta"] & A \arrow[u] \end{tikzcd}$$

is commutative. Now it may not be the case that $\beta(J') = 0$, in other words it may not be true that β factors through $S' = P/J'$. But what is clear is that $\beta(J') \subset I$ and since $\beta(J) \subset I$ and $I^2 = 0$ we have $\beta(J^2) = 0$. Thus the ``obstruction" to finding a morphism from $(J/J' \subset S', 1, R \to S')$ to $(I \subset A, a, b)$ is the corresponding S-linear map $\overline{\beta} : J'/J^2 \to I$. The choice in picking β lies in the choice of $\beta(x_i)$. A different choice of β, say β', is gotten by taking $\beta'(x_i) = \beta(x_i) + \delta_i$ with $\delta_i \in I$. In this case, for $g \in J'$, we obtain

$$\beta'(g) = \beta(g) + \sum\nolimits_i \delta_i \frac{\partial g}{\partial x_i}.$$

Since the map $\mathrm{d}|_{J'/J^2} : J'/J^2 \to \Omega_{P/R} \otimes_P S$ given by $g \mapsto \frac{\partial g}{\partial x_i}\mathrm{d}x_i$ is an isomorphism by construction, we see that there is a unique choice of $\delta_i \in I$ such that $\beta'(g) = 0$ for all $g \in J'$.

(Namely, δ_i is $-\overline{\beta}(g)$ where $g \in J'/J^2$ is the unique element with $\frac{\partial g}{\partial x_j} = 1$ if $i = j$ and 0 else.) The uniqueness of the solution implies the uniqueness required in the lemma. □

In the situation of Lemma 7.136.1 the R-algebra map $S' \to S$ is unique up to unique isomorphism.

Definition 7.136.2. Let $R \to S$ be a formally unramified ring map.

(1) The *universal first order thickening* of S over R is the surjection of R-algebras $S' \to S$ of Lemma 7.136.1.
(2) The *conormal module* of $R \to S$ is the kernel I of the universal first order thickening $S' \to S$, seen as a S-module.

We often denote the conormal module $C_{S/R}$ in this situation.

Lemma 7.136.3. *Let $I \subset R$ be an ideal of a ring. The universal first order thickening of R/I over R is the surjection $R/I^2 \to R/I$. The conormal module of R/I over R is $C_{(R/I)/R} = I/I^2$.*

Proof. Omitted. □

Lemma 7.136.4. *Let $A \to B$ be a formally unramified ring map. Let $\varphi : B' \to B$ be the universal first order thickening of B over A.*

(1) *Let $S \subset A$ be a multiplicative subset. Then $S^{-1}B' \to S^{-1}B$ is the universal first order thickening of $S^{-1}B$ over $S^{-1}A$. In particular $S^{-1}C_{B/A} = C_{S^{-1}B/S^{-1}A}$.*
(2) *Let $S \subset B$ be a multiplicative subset. Then $S' = \varphi^{-1}(S)$ is a multiplicative subset in B' and $(S')^{-1}B' \to S^{-1}B$ is the universal first order thickening of $S^{-1}B$ over A. In particular $S^{-1}C_{B/A} = C_{S^{-1}B/A}$.*

Note that the lemma makes sense by Lemma 7.135.4.

Proof. With notation and assumptions as in (1). Let $(S^{-1}B)' \to S^{-1}B$ be the universal first order thickening of $S^{-1}B$ over $S^{-1}A$. Note that $S^{-1}B' \to S^{-1}B$ is a surjection of $S^{-1}A$-algebras whose kernel has square zero. Hence by definition we obtain a map $(S^{-1}B)' \to S^{-1}B'$ compatible with the maps towards $S^{-1}B$. Consider any commutative diagram

$$\begin{tikzcd} B \arrow[r] & S^{-1}B \arrow[r] & D/I \\ A \arrow[r] \arrow[u] & S^{-1}A \arrow[r] \arrow[u] & D \arrow[u] \end{tikzcd}$$

where $I \subset D$ is an ideal of square zero. Since B' is the universal first order thickening of B over A we obtain an A-algebra map $B' \to D$. But it is clear that the image of S in D is mapped to invertible elements of D, and hence we obtain a compatible map $S^{-1}B' \to D$. Applying this to $D = (S^{-1}B)'$ we see that we get a map $S^{-1}B' \to (S^{-1}B)'$. We omit the verification that this map is inverse to the map described above.

With notation and assumptions as in (2). Let $(S^{-1}B)' \to S^{-1}B$ be the universal first order thickening of $S^{-1}B$ over A. Note that $(S')^{-1}B' \to S^{-1}B$ is a surjection of A-algebras whose kernel has square zero. Hence by definition we obtain a map $(S^{-1}B)' \to (S')^{-1}B'$

compatible with the maps towards $S^{-1}B$. Consider any commutative diagram

$$\begin{tikzcd} B \arrow[r] & S^{-1}B \arrow[r] & D/I \\ A \arrow[u] \arrow[r] & A \arrow[u] \arrow[r] & D \arrow[u] \end{tikzcd}$$

where $I \subset D$ is an ideal of square zero. Since B' is the universal first order thickening of B over A we obtain an A-algebra map $B' \to D$. But it is clear that the image of S' in D is mapped to invertible elements of D, and hence we obtain a compatible map $(S')^{-1}B' \to D$. Applying this to $D = (S^{-1}B)'$ we see that we get a map $(S')^{-1}B' \to (S^{-1}B)'$. We omit the verification that this map is inverse to the map described above. $\square$

Lemma 7.136.5. *Let $R \to A \to B$ be ring maps. Assume $A \to B$ formally unramified. Let $B' \to B$ be the universal first order thickening of B over A. Then B' is formally unramified over A, and the canonical map $\Omega_{A/R} \otimes_A B \to \Omega_{B'/R} \otimes_{B'} B$ is an isomorphism.*

Proof. We are going to use the construction of B' from the proof of Lemma 7.136.1 allthough in principle it should be possible to deduce these results formally from the definition. Namely, we choose a presentation $B = P/J$, where $P = A[x_i]$ is a polynomial ring over A. Next, we choose elements $f_i \in J$ such that $\mathrm{d}f_i = \mathrm{d}x_i \otimes 1$ in $\Omega_{P/A} \otimes_P B$. Having made these choices we have $B' = P/J'$ with $J' = (f_i) + J^2$, see proof of Lemma 7.136.1.

Consider the canonical exact sequence

$$J'/(J')^2 \to \Omega_{P/A} \otimes_P B' \to \Omega_{B'/A} \to 0$$

see Lemma 7.122.9. By construction the classes of the $f_i \in J'$ map to elements of the module $\Omega_{P/A} \otimes_P B'$ which generate it modulo J'/J^2 by construction. Since J'/J^2 is a nilpotent ideal, we see that these elements generate the module alltogether (by Nakayama's Lemma 7.14.5). This proves that $\Omega_{B'/A} = 0$ and hence that B' is formally unramified over A, see Lemma 7.135.2.

Since P is a polynomial ring over A we have $\Omega_{P/R} = \Omega_{A/R} \otimes_A P \oplus \bigoplus P\mathrm{d}x_i$. We are going to use this decomposition. Consider the following exact sequence

$$J'/(J')^2 \to \Omega_{P/R} \otimes_P B' \to \Omega_{B'/R} \to 0$$

see Lemma 7.122.9. We may tensor this with B and obtain the exact sequence

$$J'/(J')^2 \otimes_{B'} B \to \Omega_{P/R} \otimes_P B \to \Omega_{B'/R} \otimes_{B'} B \to 0$$

If we remember that $J' = (f_i) + J^2$ then we see that the first arrow annihilates the submodule $J^2/(J')^2$. In terms of the direct sum decomposition $\Omega_{P/R} \otimes_P B = \Omega_{A/R} \otimes_A B \oplus \bigoplus B\mathrm{d}x_i$ given we see that the submodule $(f_i)/(J')^2 \otimes_{B'} B$ maps isomorphically onto the summand $\bigoplus B\mathrm{d}x_i$. Hence what is left of this exact sequence is an isomorphism $\Omega_{A/R} \otimes_A B \to \Omega_{B'/R} \otimes_{B'} B$ as desired. $\square$

7.137. Formally étale maps

Definition 7.137.1. Let $R \to S$ be a ring map. We say S is *formally étale over R* if for every commutative solid diagram

$$\begin{tikzcd} S \arrow[r] \arrow[dr, dashed] & A/I \\ R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

where $I \subset A$ is an ideal of square zero, there exists a unique dotted arrow making the diagram commute.

Clearly a ring map is formally étale if and only if it is booth formally smooth and formally unramified.

Lemma 7.137.2. *Let $R \to S$ be a ring map of finite presentation. The following are equivalent:*

(1) *$R \to S$ is formally étale,*
(2) *$R \to S$ is étale.*

Proof. Assume that $R \to S$ is formally étale. Then $R \to S$ is smooth by Proposition 7.127.13. By Lemma 7.135.2 we have $\Omega_{S/R} = 0$. Hence $R \to S$ is étale by definition.

Assume that $R \to S$ is étale. Then $R \to S$ is formally smooth by Proposition 7.127.13. By Lemma 7.135.2 it is formally unramified. Hence $R \to S$ is formally étale. □

Lemma 7.137.3. *Let R be a ring. Let I be a directed partially ordered set. Let $(S_i, \varphi_{ii'})$ be a system of R-algebras over I. If each $R \to S_i$ is formally étale, then $S = colim_{i\in I} S_i$ is formally étale over R*

Proof. Consider a diagram as in Definition 7.137.1. By assumption we get unique R-algebra maps $S_i \to A$ lifting the compositions $S_i \to S \to A/I$. Hence these are compatible with the transition maps $\varphi_{ii'}$ and define a lift $S \to A$. This proves existence. The uniqueness is clear by restricting to each S_i. □

Lemma 7.137.4. *Let R be a ring. Let $S \subset R$ be any multiplicative subset. Then the ring map $R \to S^{-1}R$ is formally étale.*

Proof. Let $I \subset A$ be an ideal of square zero. What we are saying here is that given a ring map $\varphi : R \to A$ such that $\varphi(f) \mod I$ is invertible for all $f \in S$ we have also that $\varphi(f)$ is invertible in A for all $f \in S$. This is true because A^* is the inverse image of $(A/I)^*$ under the canonical map $A \to A/I$. □

7.138. Unramified ring maps

The definition of a G-unramified ring map is the one from EGA. The definition of an unramified ring map is the one from [**Ray70**].

Definition 7.138.1. Let $R \to S$ be a ring map.

(1) We say $R \to S$ is *unramified* if $R \to S$ is of finite type and $\Omega_{S/R} = 0$.
(2) We say $R \to S$ is *G-unramified* if $R \to S$ is of finite presentation and $\Omega_{S/R} = 0$.
(3) Given a prime $\mathfrak{q}$ of S we say that S is *unramified at* $\mathfrak{q}$ if there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is unramified.
(4) Given a prime $\mathfrak{q}$ of S we say that S is *G-unramified at* $\mathfrak{q}$ if there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is G-unramified.

Of course a G-unramified map is unramified.

Lemma 7.138.2. *Let $R \to S$ be a ring map. The following are equivalent*

(1) *$R \to S$ is formally unramified and of finite type, and*
(2) *$R \to S$ is unramified.*

Moreover, also the following are equivalent

(1) *$R \to S$ is formally unramified and of finite presentation, and*

(2) *$R \to S$ is G-unramified.*

Proof. Follows from Lemma 7.135.2 and the definitions. $\square$

Lemma 7.138.3. *Properties of unramified and G-unramified ring maps.*

1. *The base change of an unramified ring map is unramified. The base change of a G-unramified ring map is G-unramified.*
2. *The composition of unramified ring maps is unramified. The composition of G-unramified ring maps is G-unramified.*
3. *Any principal localization $R \to R_f$ is G-unramified and unramified.*
4. *If $I \subset R$ is an ideal, then $R \to R/I$ is unramified. If $I \subset R$ is a finitely generated ideal, then $R \to R/I$ is G-unramified.*
5. *An étale ring map is G-unramified and unramified.*
6. *If $R \to S$ is of finite type (resp. finite presentation), $\mathfrak{q} \subset S$ is a prime and $(\Omega_{S/R})_{\mathfrak{q}} = 0$, then $R \to S$ is unramified (resp. G-unramified) at $\mathfrak{q}$.*
7. *If $R \to S$ is of finite type (resp. finite presentation), $\mathfrak{q} \subset S$ is a prime and $\Omega_{S/R} \otimes_S \kappa(\mathfrak{q}) = 0$, then $R \to S$ is unramified (resp. G-unramified) at $\mathfrak{q}$.*
8. *If $R \to S$ is of finite type (resp. finite presentation), $\mathfrak{q} \subset S$ is a prime lying over $\mathfrak{p} \subset R$ and $(\Omega_{S \otimes_R \kappa(\mathfrak{p})/\kappa(\mathfrak{p})})_{\mathfrak{q}} = 0$, then $R \to S$ is unramified (resp. G-unramified) at $\mathfrak{q}$.*
9. *If $R \to S$ is of finite type (resp. presentation), $\mathfrak{q} \subset S$ is a prime lying over $\mathfrak{p} \subset R$ and $(\Omega_{S \otimes_R \kappa(\mathfrak{p})/\kappa(\mathfrak{p})}) \otimes_{S \otimes_R \kappa(\mathfrak{p})} \kappa(\mathfrak{q}) = 0$, then $R \to S$ is unramified (resp. G-unramified) at $\mathfrak{q}$.*
10. *If $R \to S$ is a ring map, $g_1, \ldots, g_m \in S$ generate the unit ideal and $R \to S_{g_j}$ is unramified (resp. G-unramified) for $j = 1, \ldots, m$, then $R \to S$ is unramified (resp. G-unramified).*
11. *If $R \to S$ is a ring map which is unramified (resp. G-unramified) at every prime of S, then $R \to S$ is unramified (resp. G-unramified).*
12. *If $R \to S$ is G-unramified, then there exists a finite type $\mathbf{Z}$-algebra R_0 and a G-unramified ring map $R_0 \to S_0$ and a ring map $R_0 \to R$ such that $S = R \otimes_{R_0} S_0$.*
13. *If $R \to S$ is unramified, then there exists a finite type $\mathbf{Z}$-algebra R_0 and an unramified ring map $R_0 \to S_0$ and a ring map $R_0 \to R$ such that S is a quotient of $R \otimes_{R_0} S_0$.*

Proof. We prove each point, in order.

Ad (1). Follows from Lemmas 7.122.12 and 7.13.2.

Ad (2). Follows from Lemmas 7.122.7 and 7.13.2.

Ad (3). Follows by direct computation of $\Omega_{R_f/R}$ which we omit.

Ad (4). We have $\Omega_{(R/I)/R} = 0$, see Lemma 7.122.5, and the ring map $R \to R/I$ is of finite type. If I is a finitely generated ideal then $R \to R/I$ is of finite presentation.

Ad (5). See discussion following Definition 7.132.1.

Ad (6). In this case $\Omega_{S/R}$ is a finite S-module (see Lemma 7.122.16) and hence there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $(\Omega_{S/R})_g = 0$. By Lemma 7.122.8 this means that $\Omega_{S_g/R} = 0$ and hence $R \to S_g$ is unramified as desired.

Ad (7). Use Nakayama's lemma (Lemma 7.14.5) to see that the condition is equivalent to the condition of (6).

Ad (8) & (9). These are equivalent in the same manner that (6) and (7) are equivalent. Moreover $\Omega_{S\otimes_R\kappa(\mathfrak p)/\kappa(\mathfrak p)} = \Omega_{S/R} \otimes_S (S \otimes_R \kappa(\mathfrak p))$ by Lemma 7.122.12. Hence we see that (9) is equivalent to (7) since the $\kappa(\mathfrak q)$ vector spaces in both are canonically isomorphic.

Ad (10). Follows from from Lemmas 7.21.2 and 7.122.8.

Ad (11). Follows from (6) and (7) and the fact that the spectrum of S is quasi-compact.

Ad (12). Write $S = R[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$. As $\Omega_{S/R} = 0$ we can write

$$\mathrm{d}x_i = \sum h_{ij}\mathrm{d}g_j + \sum a_{ijk}g_j\mathrm{d}x_k$$

in $\Omega_{R[x_1,\ldots,x_n]/R}$ for some $h_{ij}, a_{ijk} \in R[x_1, \ldots, x_n]$. Choose a finitely generated $\mathbf{Z}$-subalgebra $R_0 \subset R$ containing all the coefficients of the polynomials g_i, h_{ij}, a_{ijk}. Set $S_0 = R_0[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$. This works.

Ad (13). Write $S = R[x_1, \ldots, x_n]/I$. As $\Omega_{S/R} = 0$ we can write

$$\mathrm{d}x_i = \sum h_{ij}\mathrm{d}g_{ij} + \sum g'_{ik}\mathrm{d}x_k$$

in $\Omega_{R[x_1,\ldots,x_n]/R}$ for some $h_{ij} \in R[x_1, \ldots, x_n]$ and $g_{ij}, g'_{ik} \in I$. Choose a finitely generated $\mathbf{Z}$-subalgebra $R_0 \subset R$ containing all the coefficients of the polynomials g_{ij}, h_{ij}, g'_{ik}. Set $S_0 = R_0[x_1, \ldots, x_n]/(g_{ij}, g'_{ik})$. This works. □

Lemma 7.138.4. *Let $R \to S$ be a ring map. If $R \to S$ is unramified, then there exists an idempotent $e \in S \otimes_R S$ such that $S \otimes_R S \to S$ is isomorphic to $S \otimes_R S \to (S \otimes_R S)_e$.*

Proof. Let $J = \mathrm{Ker}(S \otimes_R S \to S)$. By assumption $J/J^2 = 0$, see Lemma 7.122.13. Since S is of finite type over R we see that J is finitely generated, namely by $x_i \otimes 1 - 1 \otimes x_i$, where x_i generate S over R. We win by Lemma 7.18.5. □

Lemma 7.138.5. *Let $R \to S$ be a ring map. Let $\mathfrak q \subset S$ be a prime lying over $\mathfrak p$ in R. If S/R is unramified at $\mathfrak q$ then*

(1) *we have $\mathfrak pS_{\mathfrak q} = \mathfrak qS_{\mathfrak q}$ is the maximal ideal of the local ring $S_{\mathfrak q}$, and*
(2) *the field extension $\kappa(\mathfrak p) \subset \kappa(\mathfrak q)$ is finite separable.*

Proof. We may first replace S by S_g for some $g \in S$, $g \notin \mathfrak q$ and assume that $R \to S$ is unramified. The base change $S \otimes_R \kappa(\mathfrak p)$ is unramified over $\kappa(\mathfrak p)$ by Lemma 7.138.3. By Lemma 7.129.3 it is smooth hence étale over $\kappa(\mathfrak p)$. Hence we see that $S \otimes_R \kappa(\mathfrak p) = (R \setminus \mathfrak p)^{-1}S/\mathfrak pS$ is a product of finite separable field extensions of $\kappa(\mathfrak p)$ by Lemma 7.132.4. This implies the lemma. □

Lemma 7.138.6. *Let $R \to S$ be a finite type ring map. Let $\mathfrak q$ be a prime of S. If $R \to S$ is unramified at $\mathfrak q$ then $R \to S$ is quasi-finite at $\mathfrak q$. In particular, an unramified ring map is quasi-finite.*

Proof. An unramified ring map is of finite type. Thus it is clear that the second statement follows from the first. To see the first statement apply the characterization of Lemma 7.113.2 part (2) using Lemma 7.138.5. □

Lemma 7.138.7. *Let $R \to S$ be a ring map. Let $\mathfrak q$ be a prime of S lying over a prime $\mathfrak p$ of R. If*

(1) *$R \to S$ is of finite type,*
(2) *$\mathfrak pS_{\mathfrak q}$ is the maximal ideal of the local ring $S_{\mathfrak q}$, and*
(3) *the field extension $\kappa(\mathfrak p) \subset \kappa(\mathfrak q)$ is finite separable,*

then $R \to S$ is unramified at $\mathfrak q$.

Proof. By Lemma 7.138.3 (8) it suffices to show that $\Omega_{S \otimes_R \kappa(\mathfrak{p})/\kappa(\mathfrak{p})}$ is zero when localized at $\mathfrak{q}$. Hence we may replace S by $S \otimes_R \kappa(\mathfrak{p})$ and R by $\kappa(\mathfrak{p})$. In other words, we may assume that $R = k$ is a field and S is a finite type k-algebra. In this case the hypotheses imply that $S_{\mathfrak{q}} \cong \kappa(\mathfrak{q})$ and hence $S = \kappa(\mathfrak{q}) \times S'$ (see Lemma 7.113.1). Hence $(\Omega_{S/k})_{\mathfrak{q}} = \Omega_{\kappa(\mathfrak{q})/k}$ which is zero as desired. □

Proposition 7.138.8. *Let $R \to S$ be a ring map. Let $\mathfrak{q} \subset S$ be a prime. If $R \to S$ is unramified at $\mathfrak{q}$, then there exist*

(1) *a $g \in S$, $g \notin \mathfrak{q}$,*
(2) *a standard étale ring map $R \to S'$, and*
(3) *a surjective R-algebra map $S' \to S_g$.*

Proof. This proof is the ``same'' as the proof of Proposition 7.132.16. The proof is a little roundabout and there may be ways to shorten it.

Step 1. By Definition 7.138.1 there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is unramified. Thus we may assume that S is unramified over R.

Step 2. By Lemma 7.138.3 there exists an unramified ring map $R_0 \to S_0$ with R_0 of finite type over $\mathbf{Z}$, and a ring map $R_0 \to R$ such that S is a quotient of $R \otimes_{R_0} S_0$. Denote $\mathfrak{q}_0$ the prime of S_0 corresponding to $\mathfrak{q}$. If we show the result for $(R_0 \to S_0, \mathfrak{q}_0)$ then the result follows for $(R \to S, \mathfrak{q})$ by base change. Hence we may assume that R is Noetherian.

Step 3. Note that $R \to S$ is quasi-finite by Lemma 7.138.6. By Lemma 7.114.15 there exists a finite ring map $R \to S'$, an R-algebra map $S' \to S$, an element $g' \in S'$ such that $g' \notin \mathfrak{q}$ such that $S' \to S$ induces an isomorphism $S'_{g'} \cong S_{g'}$. (Note that S' may not unramified over R.) Thus we may assume that (a) R is Noetherian, (b) $R \to S$ is finite and (c) $R \to S$ is unramified at $\mathfrak{q}$ (but no longer necessarily unramified at all primes).

Step 4. Let $\mathfrak{p} \subset R$ be the prime corresponding to $\mathfrak{q}$. Consider the fibre ring $S \otimes_R \kappa(\mathfrak{p})$. This is a finite algebra over $\kappa(\mathfrak{p})$. Hence it is Artinian (see Lemma 7.49.2) and so a finite product of local rings

$$S \otimes_R \kappa(\mathfrak{p}) = \prod\nolimits_{i=1}^n A_i$$

see Proposition 7.57.6. One of the factors, say A_1, is the local ring $S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}}$ which is isomorphic to $\kappa(\mathfrak{q})$, see Lemma 7.138.5. The other factors correspond to the other primes, say $\mathfrak{q}_2, \ldots, \mathfrak{q}_n$ of S lying over $\mathfrak{p}$.

Step 5. We may choose a nonzero element $\alpha \in \kappa(\mathfrak{q})$ which generates the finite separable field extension $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q})$ (so even if the field extension is trivial we do not allow $\alpha = 0$). Note that for any $\lambda \in \kappa(\mathfrak{p})^*$ the element $\lambda\alpha$ also generates $\kappa(\mathfrak{q})$ over $\kappa(\mathfrak{p})$. Consider the element

$$\bar{t} = (\alpha, 0, \ldots, 0) \in \prod\nolimits_{i=1}^n A_i = S \otimes_R \kappa(\mathfrak{p}).$$

After possibly replacing α by $\lambda\alpha$ as above we may assume that $\bar{t}$ is the image of $t \in S$. Let $I \subset R[x]$ be the kernel of the R-algebra map $R[x] \to S$ which maps x to t. Set $S' = R[x]/I$, so $S' \subset S$. Here is a diagram

$$\begin{tikzcd} R[x] \arrow[r] & S' \arrow[r] & S \\ R \arrow[u] \arrow[ur] \arrow[urr] & & \end{tikzcd}$$

By construction the primes $\mathfrak{q}_j$, $j \geq 2$ of S all lie over the prime $(\mathfrak{p}, x)$ of $R[x]$, whereas the prime $\mathfrak{q}$ lies over a different prime of $R[x]$ because $\alpha \neq 0$.

Step 6. Denote $\mathfrak{q}' \subset S'$ the prime of S' corresponding to $\mathfrak{q}$. By the above $\mathfrak{q}$ is the only prime of S lying over $\mathfrak{q}'$. Thus we see that $S_\mathfrak{q} = S_{\mathfrak{q}'}$, see Lemma 7.36.11 (we have going up for $S' \to S$ by Lemma 7.32.20 since $S' \to S$ is finite as $R \to S$ is finite). It follows that $S'_{\mathfrak{q}'} \to S_\mathfrak{q}$ is finite and injective as the localization of the finite injective ring map $S' \to S$. Consider the maps of local rings

$$R_\mathfrak{p} \to S'_{\mathfrak{q}'} \to S_\mathfrak{q}$$

The second map is finite and injective. We have $S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q} = \kappa(\mathfrak{q})$, see Lemma 7.138.5. Hence a fortiori $S_\mathfrak{q}/\mathfrak{q}'S_\mathfrak{q} = \kappa(\mathfrak{q})$. Since

$$\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q}') \subset \kappa(\mathfrak{q})$$

and since α is in the image of $\kappa(\mathfrak{q}')$ in $\kappa(\mathfrak{q})$ we conclude that $\kappa(\mathfrak{q}') = \kappa(\mathfrak{q})$. Hence by Nakayama's Lemma 7.14.5 applied to the $S'_{\mathfrak{q}'}$-module map $S'_{\mathfrak{q}'} \to S_\mathfrak{q}$, the map $S'_{\mathfrak{q}'} \to S_\mathfrak{q}$ is surjective. In other words, $S'_{\mathfrak{q}'} \cong S_\mathfrak{q}$.

Step 7. By Lemma 7.117.6 there exists a $g' \in S'$, $g' \notin \mathfrak{q}'$ such that $S'_{g'} \cong S_{g'}$. As R is Noetherian the ring S' is finite over R as it is an R-submodule of the finite R-module S. Hence after replacing S by S' we may assume that (a) R is Noetherian, (b) S finite over R, (c) S is unramified over R at $\mathfrak{q}$, and (d) $S = R[x]/I$.

Step 8. Consider the ring $S \otimes_R \kappa(\mathfrak{p}) = \kappa(\mathfrak{p})[x]/\overline{I}$ where $\overline{I} = I \cdot \kappa(\mathfrak{p})[x]$ is the ideal generated by I in $\kappa(\mathfrak{p})[x]$. As $\kappa(\mathfrak{p})[x]$ is a PID we know that $\overline{I} = (\overline{h})$ for some monic $\overline{h} \in \kappa(\mathfrak{p})$. After replacing $\overline{h}$ by $\lambda \cdot \overline{h}$ for some $\lambda \in \kappa(\mathfrak{p})$ we may assume that $\overline{h}$ is the image of some $h \in R[x]$. (The problem is that we do not know if we may choose h monic.) Also, as in Step 4 we know that $S \otimes_R \kappa(\mathfrak{p}) = A_1 \times \ldots \times A_n$ with $A_1 = \kappa(\mathfrak{q})$ a finite separable extension of $\kappa(\mathfrak{p})$ and $A_2, \ldots, A_n$ local. This implies that

$$\overline{h} = \overline{h}_1 \overline{h}_2^{e_2} \ldots \overline{h}_n^{e_n}$$

for certain pairwise coprime irreducible monic polynomials $\overline{h}_i \in \kappa(\mathfrak{p})[x]$ and certain $e_2, \ldots, e_n \geq 1$. Here the numbering is chosen so that $A_i = \kappa(\mathfrak{p})[x]/(\overline{h}_i^{e_i})$ as $\kappa(\mathfrak{p})[x]$-algebras. Note that $\overline{h}_1$ is the minimal polynomial of $\alpha \in \kappa(\mathfrak{q})$ and hence is a separable polynomial (its derivative is prime to itself).

Step 9. Let $m \in I$ be a monic element; such an element exists because the ring extension $R \to R[x]/I$ is finite hence integral. Denote $\overline{m}$ the image in $\kappa(\mathfrak{p})[x]$. We may factor

$$\overline{m} = \overline{k} \overline{h}_1^{d_1} \overline{h}_2^{d_2} \ldots \overline{h}_n^{d_n}$$

for some $d_1 \geq 1$, $d_j \geq e_j$, $j = 2, \ldots, n$ and $\overline{k} \in \kappa(\mathfrak{p})[x]$ prime to all the $\overline{h}_i$. Set $f = m^l + h$ where $l \deg(m) > \deg(h)$, and $l \geq 2$. Then f is monic as a polynomial over R. Also, the image $\overline{f}$ of f in $\kappa(\mathfrak{p})[x]$ factors as

$$\overline{f} = \overline{h}_1 \overline{h}_2^{e_2} \ldots \overline{h}_n^{e_n} + \overline{k}^l \overline{h}_1^{ld_1} \overline{h}_2^{ld_2} \ldots \overline{h}_n^{ld_n} = \overline{h}_1 (\overline{h}_2^{e_2} \ldots \overline{h}_n^{e_n} + \overline{k}^l \overline{h}_1^{ld_1 - 1} \overline{h}_2^{ld_2} \ldots \overline{h}_n^{ld_n}) = \overline{h}_1 \overline{w}$$

with $\overline{w}$ a polynomial relatively prime to $\overline{h}_1$. Set $g = f'$ (the derivative with respect to x).

Step 10. The ring map $R[x] \to S = R[x]/I$ has the properties: (1) it maps f to zero, and (2) it maps g to an element of $S \setminus \mathfrak{q}$. The first assertion is clear since f is an element of I. For the second assertion we just have to show that g does not map to zero in $\kappa(\mathfrak{q}) = \kappa(\mathfrak{p})[x]/(\overline{h}_1)$. The image of g in $\kappa(\mathfrak{p})[x]$ is the derivative of $\overline{f}$. Thus (2) is clear because

$$\overline{g} = \frac{\mathrm{d}\overline{f}}{\mathrm{d}x} = \overline{w}\frac{\mathrm{d}\overline{h}_1}{\mathrm{d}x} + \overline{h}_1 \frac{\mathrm{d}\overline{w}}{\mathrm{d}x},$$

$\overline{w}$ is prime to $\overline{h}_1$ and $\overline{h}_1$ is separable.

Step 11. We conclude that $\varphi : R[x]/(f) \to S$ is a surjective ring map, $R[x]_g/(f)$ is étale over R (because it is standard étale, see Lemma 7.132.14) and $\varphi(g) \not\in \mathfrak{q}$. Thus the map $(R[x]/(f))_g \to S_{\varphi(g)}$ is the desired surjection. □

Lemma 7.138.9. *Let $R \to S$ be a ring map. Let $\mathfrak{q}$ be a prime of S lying over $\mathfrak{p} \subset R$. Assume that $R \to S$ is of finite type and unramified at $\mathfrak{q}$. Then there exist*

(1) *an étale ring map $R \to R'$,*
(2) *a prime $\mathfrak{p}' \subset R'$ lying over $\mathfrak{p}$.*
(3) *a product decomposition*

$$R' \otimes_R S = A \times B$$

with the following properties

(1) *$R' \to A$ is surjective, and*
(2) *$\mathfrak{p}'A$ is a prime of A lying over $\mathfrak{p}'$ and over $\mathfrak{q}$.*

Proof. We may replace $(R \to S, \mathfrak{p}, \mathfrak{q})$ with any base change $(R' \to R' \otimes_R S, \mathfrak{p}', \mathfrak{q}')$ by a étale ring map $R \to R'$ with a prime $\mathfrak{p}'$ lying over $\mathfrak{p}$, and a choice of $\mathfrak{q}'$ lying over both $\mathfrak{q}$ and $\mathfrak{p}'$. Note also that given $R \to R'$ and $\mathfrak{p}'$ a suitable $\mathfrak{q}'$ can always be found.

The assumption that $R \to S$ is of finite type means that we may apply Lemma 7.132.23. Thus we may assume that $S = A_1 \times \ldots \times A_n \times B$, that each $R \to A_i$ is finite with exactly one prime $\mathfrak{r}_i$ lying over $\mathfrak{p}$ such that $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{r}_i)$ is purely inseparable and that $R \to B$ is not quasi-finite at any prime lying over $\mathfrak{p}$. Then clearly $\mathfrak{q} = \mathfrak{r}_i$ for some i, since an unramified morphism is quasi-finite (see Lemma 7.138.6). Say $\mathfrak{q} = \mathfrak{r}_1$. By Lemma 7.138.5 we see that $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{r}_1)$ is separable hence the trivial field extension, and that $\mathfrak{p}(A_1)_{\mathfrak{r}_1}$ is the maximal ideal. Also, by Lemma 7.36.11 (which applies to $R \to A_1$ because a finite ring map satisfies going up by Lemma 7.32.20) we have $(A_1)_{\mathfrak{r}_1} = (A_1)_{\mathfrak{p}}$. It follows from Nakayama's Lemma 7.14.5 that the map of local rings $R_{\mathfrak{p}} \to (A_1)_{\mathfrak{p}} = (A_1)_{\mathfrak{r}_1}$ is surjective. Since A_1 is finite over R we see that there exists a $f \in R$, $f \not\in \mathfrak{p}$ such that $R_f \to (A_1)_f$ is surjective. After replacing R by R_f we win. □

Lemma 7.138.10. *Let $R \to S$ be a ring map. Let $\mathfrak{p}$ be a prime of R. If $R \to S$ is unramified then there exist*

(1) *an étale ring map $R \to R'$,*
(2) *a prime $\mathfrak{p}' \subset R'$ lying over $\mathfrak{p}$.*
(3) *a product decomposition*

$$R' \otimes_R S = A_1 \times \ldots \times A_n \times B$$

with the following properties

(1) *$R' \to A_i$ is surjective,*
(2) *$\mathfrak{p}'A_i$ is a prime of A_i lying over $\mathfrak{p}'$, and*
(3) *there is no prime of B lying over $\mathfrak{p}'$.*

Proof. We may apply Lemma 7.132.23. Thus, after an étale base change, we may assume that $S = A_1 \times \ldots \times A_n \times B$, that each $R \to A_i$ is finite with exactly one prime $\mathfrak{r}_i$ lying over $\mathfrak{p}$ such that $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{r}_i)$ is purely inseparable, and that $R \to B$ is not quasi-finite at any prime lying over $\mathfrak{p}$. Since $R \to S$ is quasi-finite (see Lemma 7.138.6) we see there is no prime of B lying over $\mathfrak{p}$. By Lemma 7.138.5 we see that $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{r}_i)$ is separable hence the trivial field extension, and that $\mathfrak{p}(A_i)_{\mathfrak{r}_i}$ is the maximal ideal. Also, by Lemma 7.36.11 (which applies to $R \to A_i$ because a finite ring map satisfies going up by Lemma 7.32.20) we have $(A_i)_{\mathfrak{r}_i} = (A_i)_{\mathfrak{p}}$. It follows from Nakayama's Lemma 7.14.5 that the map of local rings $R_{\mathfrak{p}} \to (A_i)_{\mathfrak{p}} = (A_i)_{\mathfrak{r}_i}$ is surjective. Since A_i is finite over R we see that there exists a $f \in R$, $f \notin \mathfrak{p}$ such that $R_f \to (A_i)_f$ is surjective. After replacing R by R_f we win. □

7.139. Henselian local rings

In this section we discuss a bit the notion of a henselian local ring. Let $(R, \mathfrak{m}, \kappa)$ be a local ring. For $a \in R$ we denote $\overline{a}$ the image of a in κ. For a polynomial $f \in R[T]$ we often denote $\overline{f}$ the image of f in $\kappa[T]$. Given a polynomial $f \in R[T]$ we denote f' the derivative of f with respect to T. Note that $\overline{f}' = \overline{f'}$.

Definition 7.139.1. Let $(R, \mathfrak{m}, \kappa)$ be a local ring.

1. We say R is *henselian* if for every monic $f \in R[T]$ and every root $a_0 \in \kappa$ of $\overline{f}$ such that $\overline{f'}(a_0) \neq 0$ there exists an $a \in R$ such that $f(a) = 0$ and $a_0 = \overline{a}$.
2. We say R is *strictly henselian* if R is henselian and its residue field is separably algebraically closed.

Note that the condition $\overline{f'}(a_0) \neq 0$ is equivalent to the condition that a_0 is a simple root of the polynomial $\overline{f}$. In fact, it implies that the lift $a \in R$, if it exists, is unique.

Lemma 7.139.2. *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Let $f \in R[T]$. Let $a, b \in R$ such that $f(a) = f(b) = 0$, $a = b \bmod \mathfrak{m}$, and $f'(a) \notin \mathfrak{m}$. Then $a = b$.*

Proof. Write $f(x+y) - f(x) = f'(x)y + g(x, y)y^2$ in $R[x, y]$ (this is possible as one sees by expaning $f(x+y)$; details omitted). Then we see that $0 = f(b) - f(a) = f(a+(b-a)) - f(a) = f'(a)(b-a) + c(b-a)^2$ for some $c \in R$. By assumption $f'(a)$ is a unit in R. Hence $(b-a)(1 + f'(a)^{-1}c(b-a)) = 0$. By assumption $b - a \in \mathfrak{m}$, hence $1 + f'(a)^{-1}c(b-a)$ is a unit in R. Hence $b - a = 0$ in R. □

Here is the characterization of henselian local rings.

Lemma 7.139.3. *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. The following are equivalent*

1. *R is henselian,*
2. *for every $f \in R[T]$ and every root $a_0 \in \kappa$ of $\overline{f}$ such that $\overline{f'}(a_0) \neq 0$ there exists an $a \in R$ such that $f(a) = 0$ and $a_0 = \overline{a}$,*
3. *for any monic $f \in R[T]$ and any factorization $\overline{f} = g_0 h_0$ with $\gcd(g_0, h_0) = 1$ there exists a factorization $f = gh$ in $R[T]$ such that $g_0 = \overline{g}$ and $h_0 = \overline{h}$,*
4. *for any monic $f \in R[T]$ and any factorization $\overline{f} = g_0 h_0$ with $\gcd(g_0, h_0) = 1$ there exists a factorization $f = gh$ in $R[T]$ such that $g_0 = \overline{g}$ and $h_0 = \overline{h}$ and moreover $\deg_T(g) = \deg_T(g_0)$,*
5. *for any $f \in R[T]$ and any factorization $\overline{f} = g_0 h_0$ with $\gcd(g_0, h_0) = 1$ there exists a factorization $f = gh$ in $R[T]$ such that $g_0 = \overline{g}$ and $h_0 = \overline{h}$,*

(6) *for any* $f \in R[T]$ *and any factorization* $\overline{f} = g_0 h_0$ *with* $\gcd(g_0, h_0) = 1$ *there exists a factorization* $f = gh$ *in* $R[T]$ *such that* $g_0 = \overline{g}$ *and* $h_0 = \overline{h}$ *and moreover* $\deg_T(g) = \deg_T(g_0)$*,*
(7) *for any étale ring map* $R \to S$ *and prime* $\mathfrak{q}$ *of* S *lying over* $\mathfrak{m}$ *with* $\kappa = \kappa(\mathfrak{q})$ *there exists a section* $\tau : S \to R$ *of* $R \to S$*,*
(8) *for any étale ring map* $R \to S$ *and prime* $\mathfrak{q}$ *of* S *lying over* $\mathfrak{m}$ *with* $\kappa = \kappa(\mathfrak{q})$ *there exists a section* $\tau : S \to R$ *of* $R \to S$ *with* $\mathfrak{q} = \tau^{-1}(\mathfrak{m})$*,*
(9) *any finite* R*-algebra is a product of local rings,*
(10) *any finite* R*-algebra is a finite product of local rings,*
(11) *any finite type* R*-algebra* S *can be written as* $A \times B$ *with* $R \to A$ *finite and* $R \to B$ *not quasi-finite at any prime lying over* $\mathfrak{m}$*,*
(12) *any finite type* R*-algebra* S *can be written as* $A \times B$ *with* $R \to A$ *finite such that each irreducible component of* $\mathrm{Spec}(B \otimes_R \kappa)$ *has dimension* ≥ 1*, and*
(13) *any quasi-finite* R*-algebra* S *can be written as* $S = A \times B$ *with* $R \to A$ *finite such that* $B \otimes_R \kappa = 0$*.*

Proof. Here is a list of the easier implications:

2⇒1 because in (2) we consider all polynomials and in (1) only monic ones,
5⇒3 because in (5) we consider all polynomials and in (3) only monic ones,
6⇒4 because in (6) we consider all polynomials and in (4) only monic ones,
4⇒3 is obvious,
6⇒5 is obvious,
8⇒7 is obvious,
10⇒9 is obvious,
11⇔12 by definition of being quasi-finite at a prime,
11⇒13 by definition of being quasi-finite,

Proof of 1⇒8. Assume (1). Let $R \to S$ be étale, and let $\mathfrak{q} \subset S$ be a prime ideal such that $\kappa(\mathfrak{q}) \cong \kappa$. By Proposition 7.132.16 we can find a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is standard étale. After replacing S by S_g we may assume that $S = R[t]_g/(f)$ is standard étale. Since the prime $\mathfrak{q}$ has residue field κ it corresponds to a root a_0 of $\overline{f}$ which is not a root of $\overline{g}$. By definition of a standard étale algebra this also means that $\overline{f'}(a_0) \neq 0$. Since also f is monic by definition of a standard étale algebra again we may use that R is henselian to conclude that there exists an $a \in R$ with $a_0 = \overline{a}$ such that $f(a) = 0$. This implies that $g(a)$ is a unit of R and we obtain the desired map $\tau : S = R[t]_g/(f) \to R$ by the rule $t \mapsto a$. By construction $\tau^{-1}(\mathfrak{q}) = \mathfrak{m}$. This proves (8) holds.

Proof of 7⇒8. (This is really unimportant and should be skipped.) Assume (7) holds and assume $R \to S$ is étale. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ be the other primes of S lying over $\mathfrak{m}$. Then we can find a $g \in S$, $g \notin \mathfrak{q}$ and $g \in \mathfrak{q}_i$ for $i = 1, \ldots, r$, see Lemma 7.14.3. Apply (7) to the étale ring map $R \to S_g$ and the prime $\mathfrak{q}S_g$. This gives a section $\tau_g : S_g \to R$ such that the composition $\tau : S \to S_g \to R$ has the property $\tau^{-1}(\mathfrak{q}) = \mathfrak{m}$. Minor details omitted.

Proof of 8⇒11. Assume (8) and let $R \to S$ be a finite type ring map. Apply Lemma 7.132.22. We find an étale ring map $R \to R'$ and a prime $\mathfrak{m}' \subset R'$ lying over $\mathfrak{m}$ with $\kappa = \kappa(\mathfrak{m}')$ such that $R' \otimes_R S = A' \times B'$ with A' finite over R' and B' not quasi-finite over R' at any prime lying over $\mathfrak{m}'$. Apply (8) to get a section $\tau : R' \to R$ with $\mathfrak{m} = \tau^{-1}(\mathfrak{m}')$. Then use that

$$S = (S \otimes_R R') \otimes_{R',\tau} R = (A' \times B') \otimes_{R',\tau} R = (A' \otimes_{R',\tau} R) \times (B' \otimes_{R',\tau} R)$$

which gives a decomposition as in (11).

Proof of 8⇒10. Assume (8) and let $R \to S$ be a finite ring map. Apply Lemma 7.132.22. We find an étale ring map $R \to R'$ and a prime $\mathfrak{m}' \subset R'$ lying over $\mathfrak{m}$ with $\kappa = \kappa(\mathfrak{m}')$ such that $R' \otimes_R S = A'_1 \times \ldots \times A'_n \times B'$ with A'_i finite over R' having exactly one prime over $\mathfrak{m}'$ and B' not quasi-finite over R' at any prime lying over $\mathfrak{m}'$. Apply (8) to get a section $\tau : R' \to R$ with $\mathfrak{m} = \tau^{-1}(\mathfrak{m}')$. Then we obtain

$$\begin{aligned} S &= (S \otimes_R R') \otimes_{R',\tau} R \\ &= (A'_1 \times \ldots \times A'_n \times B') \otimes_{R',\tau} R \\ &= (A'_1 \otimes_{R',\tau} R) \times \ldots \times (A'_1 \otimes_{R',\tau} R) \times (B' \otimes_{R',\tau} R) \\ &= A_1 \times \ldots \times A_n \times B \end{aligned}$$

The factor B is finite over R but $R \to B$ is not quasi-finite at any prime lying over $\mathfrak{m}$. Hence $B = 0$. The factors A_i are finite R-algebras having exactly one prime lying over $\mathfrak{m}$, hence they are local rings. This proves that S is a finite product of local rings.

Proof of 9⇒10. This holds because if S is finite over the local ring R, then it has at most finitely many maximal ideals. Namely, by going up for $R \to S$ the maximal ideals of S all lie over $\mathfrak{m}$, and $S/\mathfrak{m}S$ is Artinian hence has finitely many primes.

Proof of 10⇒1. Assume (10). Let $f \in R[T]$ be a monic polynomial and $a_0 \in \kappa$ a simple root of $\overline{f}$. Then $S = R[T]/(f)$ is a finite R-algebra. Applying (10) we get $S = A_1 \times \ldots \times A_r$ is a finite product of local R-algebras. In particular we see that $S/\mathfrak{m}S = \prod A_i/\mathfrak{m}A_i$ is the decomposition of $\kappa[T]/(\overline{f})$ as a product of local rings. This means that one of the factors, say $A_1/\mathfrak{m}A_1$ is the quotient $\kappa[T]/(\overline{f}) \to \kappa[T]/(T - a_0)$. Since A_1 is a summand of the finite free R-module S it is a finite free R-module itself. As $A_1/\mathfrak{m}A_1$ is a κ-vector space of dimension 1 we see that $A_1 \cong R$ as an R-module. Clearly this means that $R \to A_1$ is an isomorphism. Let $a \in R$ be the image of T under the map $R[T] \to S \to A_1 \to R$. Then $f(a) = 0$ and $\overline{a} = a_0$ as desired.

Proof of 13⇒1. Assume (13). Let $f \in R[T]$ be a monic polynomial and $a_0 \in \kappa$ a simple root of $\overline{f}$. Then $S_1 = R[T]/(f)$ is a finite R-algebra. Let $g \in R[T]$ be any element such that $\overline{g} = \overline{f}/(T - a_0)$. Then $S = (S_1)_g$ is a quasi-finite R-algebra such that $S \otimes_R \kappa \cong \kappa[T]_{\overline{g}}/(\overline{f}) \cong \kappa[T]/(T - a_0) \cong \kappa$. Applying (13) to S we get $S = A \times B$ with A finite over R and $B \otimes_R \kappa = 0$. In particular we see that $\kappa \cong S/\mathfrak{m}S = A/\mathfrak{m}A$. Since A is a summand of the flat R-algebra S we see that it is finite flat, hence free over R. As $A/\mathfrak{m}A$ is a κ-vector space of dimension 1 we see that $A \cong R$ as an R-module. Clearly this means that $R \to A$ is an isomorphism. Let $a \in R$ be the image of T under the map $R[T] \to S \to A \to R$. Then $f(a) = 0$ and $\overline{a} = a_0$ as desired.

Proof of 8⇒2. Assume (8). Let $f \in R[T]$ be any polynomial and let $a_0 \in \kappa$ be a simple root. Then the algebra $S = R[T]_{f'}/(f)$ is étale over R. Let $\mathfrak{q} \subset S$ be the prime generated by $\mathfrak{m}$ and $T - b$ where $b \in R$ is any element such that $\overline{b} = a_0$. Apply (8) to S and $\mathfrak{q}$ to get $\tau : S \to R$. Then the image $\tau(T) = a \in R$ works in (2).

At this point we see that (1), (2), (7), (8), (9), (10), (11), (12), (13) are all equivalent. The weakest assertion of (3), (4), (5) and (6) is (3) and the strongest is (6). Hence we still have to prove that (3) implies (1) and (1) implies (6).

Proof of 3⇒1. Assume (3). Let $f \in R[T]$ be monic and let $a_0 \in \kappa$ be a simple root of $\overline{f}$. This gives a factorization $\overline{f} = (T - a_0)h_0$ with $h_0(a_0) \neq 0$, so $\gcd(T - a_0, h_0) = 1$.

Apply (3) to get a factorization $f = gh$ with $\overline{g} = T - a_0$ and $\overline{h} = h_0$. Set $S = R[T]/(f)$ which is a finite free R-algebra. We will write g, h also for the images of g and h in S. Then $gS + hS = S$ by Nakayama's Lemma 7.14.5 as the equality holds modulo $\mathfrak{m}$. Since $gh = f = 0$ in S this also implies that $gS \cap hS = 0$. Hence by the Chinese Remainder theorem we obtain $S = S/(g) \times S/(h)$. This implies that $A = S/(g)$ is a summand of a finite free R-module, hence finite free. Moreover, the rank of A is 1 as $A/\mathfrak{m}A = \kappa[T]/(T - a_0)$. Thus the map $R \to A$ is an isomorphism. Setting $a \in R$ equal to the image of T under the maps $R[T] \to S \to A \to R$ gives an element of R with $f(a) = 0$ and $\overline{a} = a_0$.

Proof of 1⇒6. Assume (1) or equivalently all of (1), (2), (7), (8), (9), (10), (11), (12), (13). Let $f \in R[T]$ be a polynomial. Suppose that $\overline{f} = g_0 h_0$ is a factorization with $\gcd(g_0, h_0) = 1$. We may and do assume that g_0 is monic. Consider $S = R[T]/(f)$. Because we have the factorization we see that the coefficients of f generate the unit ideal in R. This implies that S has finite fibres over R, hence is quasi-finite over R. It also implies that S is flat over R by Lemma 7.91.2. Combining (13) and (10) we may write $S = A_1 \times \ldots \times A_n \times B$ where each A_i is local and finite over R, and $B \otimes_R \kappa = 0$. After reordering the factors $A_1, \ldots, A_n$ we may assume that

$$\kappa[T]/(g_0) = A_1/\mathfrak{m}A_1 \times \ldots \times A_r/\mathfrak{m}A_r,\ \kappa[T]/(h_0) = A_{r+1}/\mathfrak{m}A_{r+1} \times \ldots \times A_n/\mathfrak{m}A_n$$

as quotients of $\kappa[T]$. The finite flat R-algebra $A = A_1 \times \ldots \times A_r$ is free as an R-module, see Lemma 7.72.4. Its rank is $\deg_T(g_0)$. Let $g \in R[T]$ be the characteristic polynomial of the R-linear operator $T : A \to A$. Then g is a monic polynomial of degree $\deg_T(g) = \deg_T(g_0)$ and moreover $\overline{g} = g_0$. By Cayley-Hamilton (Lemma 7.15.1) we see that $g(T_A) = 0$ where T_A indicates the image of T in A. Hence we obtain a well defined surjective map $R[T]/(g) \to A$ which is an isomorphism by Nakayama's Lemma 7.14.5. The map $R[T] \to A$ factors through $R[T]/(f)$ by construction hence we may write $f = gh$ for some h. This finishes the proof. □

Lemma 7.139.4. *Let $(R, \mathfrak{m}, \kappa)$ be a henselian local ring.*

1. *If $R \subset S$ is a finite ring extension then S is a finite product of henselian local rings.*
2. *If $R \subset S$ is a finite local homomorphism of local rings, then S is a henselian local ring.*
3. *If $R \to S$ is a finite type ring map, and $\mathfrak{q}$ is a prime of S lying over $\mathfrak{m}$ at which $R \to S$ is quasi-finite, then $S_{\mathfrak{q}}$ is henselian.*
4. *If $R \to S$ is quasi-finite then $S_{\mathfrak{q}}$ is henselian for every prime $\mathfrak{q}$ lying over $\mathfrak{m}$.*

Proof. Part (2) implies part (1) since S as in part (1) is a finite product of its localizations at the primes lying over $\mathfrak{m}$. Part (2) follows from Lemma 7.139.3 part (10) since any finite S-algebra is also a finite R-algebra. If $R \to S$ and $\mathfrak{q}$ are as in (3), then $S_{\mathfrak{q}}$ is a local ring of a finite R-algebra by Lemma 7.139.3 part (11). Hence (3) follows from (1). Part (4) follows from part (3). □

Lemma 7.139.5. *A filtered colimit of henselian local rings along local homorphisms is henselian.*

Proof. Categories, Lemma 4.19.3 says that this is really just a question about a colimit of henselian local rings over a directed partially ordered set. Let $(R_i, \varphi_{ii'})$ be such a system with each $\varphi_{ii'}$ local. Then $R = colim_i\, R_i$ is local, and its residue field κ is $colim\, \kappa_i$ (argument omitted). Suppose that $f \in R[T]$ is monic and that $a_0 \in \kappa$ is a simple root of $\overline{f}$. Then for some large enough i there exists an $f_i \in R_i[T]$ mapping to f and an $a_{0,i} \in \kappa_i$ mapping to a_0. Since $\overline{f_i}(a_{0,i}) \in \kappa_i$, resp. $\overline{f_i'}(a_{0,i}) \in \kappa_i$ maps to $0 = \overline{f}(a_0) \in \kappa$, resp. $0 \neq \overline{f'}(a_0) \in \kappa$ we conclude that $a_{0,i}$ is a simple root of $\overline{f_i}$. As R_i is henselian we can find $a_i \in R_i$ such that $f_i(a_i) = 0$ and $a_{0,i} = \overline{a_i}$. Then the image $a \in R$ of a_i is the desired solution. Thus R is henselian. □

Lemma 7.139.6. *Let $(R, \mathfrak{m}, \kappa)$ be a henselian local ring. Any finite type R-algebra S can be written as $S = A_1 \times \ldots \times A_n \times B$ with A_i local and finite over R and $R \to B$ not quasi-finite at any prime of B lying over $\mathfrak{m}$.*

Proof. This is a combination of parts (11) and (10) of Lemma 7.139.3. □

Lemma 7.139.7. *Let $(R, \mathfrak{m}, \kappa)$ be a strictly henselian local ring. Any finite type R-algebra S can be written as $S = A_1 \times \ldots \times A_n \times B$ with A_i local and finite over R and $\kappa \subset \kappa(\mathfrak{m}_{A_i})$ finite purely inseparable and $R \to B$ not quasi-finite at any prime of B lying over $\mathfrak{m}$.*

Proof. First write $S = A_1 \times \ldots \times A_n \times B$ as in Lemma 7.139.6. The field extension $\kappa \subset \kappa(\mathfrak{m}_{A_i})$ is finite and κ is separably algebraically closed, hence it is finite purely inseparable. □

Lemma 7.139.8. *Let $(R, \mathfrak{m}, \kappa)$ be a henselian local ring. The category of finite étale ring extensions $R \to S$ is equivalent to the category of finite étale algebras $\kappa \to \overline{S}$ via the functor $S \mapsto S/\mathfrak{m}S$.*

Proof. Denote $\mathcal{C} \to \mathcal{D}$ the functor of categories of the statement. Suppose that $R \to S$ is finite étale. Then we may write

$$S = A_1 \times \ldots \times A_n$$

with A_i local and finite étale over S, use either Lemma 7.139.6 or Lemma 7.139.3 part (10). In particular $A_i/\mathfrak{m}A_i$ is a finite separable field extension of κ, see Lemma 7.132.5. Thus we see that every object of $\mathcal{C}$ and $\mathcal{D}$ decomposes canonically into irreducible pieces which correspond via the given functor. Next, suppose that S_1, S_2 are finite étale over R such that $\kappa_1 = S_1/\mathfrak{m}S_1$ and $\kappa_2 = S_2/\mathfrak{m}S_2$ are fields (finite separable over κ). Then $S_1 \otimes_R S_2$ is finite étale over R and we may write

$$S_1 \otimes_R S_2 = A_1 \times \ldots \times A_n$$

as before. Then we see that $Hom_R(S_1, S_2)$ is identified with the set of indices $i \in \{1, \ldots, n\}$ such that $S_2 \to A_i$ is an isomorphism. To see this use that given any R-algebra map $\varphi : S_1 \to S_2$ the map $\varphi \times 1 : S_1 \otimes_R S_2 \to S_2$ is surjective, and hence is equal to projection onto one of the factors A_i. But in exectly the same way we see that $Hom_\kappa(\kappa_1, \kappa_2)$ is identified with the set of indices $i \in \{1, \ldots, n\}$ such that $\kappa_2 \to A_i/\mathfrak{m}A_i$ is an isomorphism. By the discussion above these sets of indices match, and we conclude that our functor is fully faithful. Finally, let $\kappa \subset \kappa'$ be a finite separable field extension. By Lemma 7.132.15 there exists an étale ring map $R \to S$ and a prime $\mathfrak{q}$ of S lying over $\mathfrak{m}$ such that $\kappa \subset \kappa(\mathfrak{q})$ is isomorphic to the given extension. By part (1) we may write $S = A_1 \times \ldots \times A_n \times B$. Since $R \to S$ is quasi-finite we see that there exists no prime of B over $\mathfrak{m}$. Hence $S_\mathfrak{q}$ is equal to A_i for some i. Hence $R \to A_i$ is finite étale and produces the given residue field extension. Thus the functor is essentially surjective and we win. □

Lemma 7.139.9. *Let $(R, \mathfrak{m}, \kappa)$ be a strictly henselian local ring. Let $R \to S$ be an unramified ring map. Then*

$$S = A_1 \times \ldots \times A_n \times B$$

with each $R \to A_i$ surjective and no prime of B lying over $\mathfrak{m}$.

Proof. First write $S = A_1 \times \ldots \times A_n \times B$ as in Lemma 7.139.6. Now we see that $R \to A_i$ is finite unramified and A_i local. Hence the maximal ideal of A_i is $\mathfrak{m}A_i$ and its residue field $A_i/\mathfrak{m}A_i$ is a finite separable extension of κ, see Lemma 7.138.5. However, the condition that R is strictly henselian means that κ is separably algebraically closed, so $\kappa = A_i/\mathfrak{m}A_i$. By Nakayama's Lemma 7.14.5 we conclude that $R \to A_i$ is surjective as desired. □

Lemma 7.139.10. *Let $(R, \mathfrak{m}, \kappa)$ be a complete local ring, see Definition 7.143.1. Then R is henselian.*

Proof. Let $f \in R[T]$ be monic. Denote $f_n \in R/\mathfrak{m}^{n+1}[T]$ the image. Denote f'_n the derivative of f_n with respect to T. Let $a_0 \in \kappa$ be a simple root of f_0. We lift this to a solution of f over R inductively as follows: Suppose given $a_n \in R/\mathfrak{m}^{n+1}$ such that $a_n \bmod \mathfrak{m} = a_0$ and $f_n(a_n) = 0$. Pick any element $b \in R/\mathfrak{m}^{n+2}$ such that $a_n = b \bmod \mathfrak{m}^{n+1}$. Then $f_{n+1}(b) \in \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2}$. Set

$$a_{n+1} = b - f_{n+1}(b)/f'_{n+1}(b)$$

(Newton's method). This makes sense as $f'_{n+1}(b) \in R_{n+1}$ is invertible by the condition on a_0. Then we compute $f_{n+1}(a_{n+1}) = f_{n+1}(b) - f_{n+1}(b) = 0$ in $R/\mathfrak{m}^{n+2}$. Since the system of elements $a_n \in R/\mathfrak{m}^{n+1}$ so constructed is compatible we get an element $a \in lim\, R/\mathfrak{m} = R$ (here we use that R is complete). Moreover, $f(a) = 0$ since it maps to zero in each $R/\mathfrak{m}^n$. Finally $\overline{a} = a_0$ and we win. □

Lemma 7.139.11. *Let $(R, \mathfrak{m})$ be a local ring of dimension* 0*. Then R is henselian.*

Proof. Let $R \to S$ be a finite ring map. By Lemma 7.139.3 it suffices to show that S is a product of local rings. By Lemma 7.32.19 S has finitely many primes $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ which all lie over $\mathfrak{m}$. There are no inclusions among these primes, see Lemma 7.32.18, hence they are all maximal. Every element of $\mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_r$ is nilpotent by Lemma 7.16.2. It follows S is the product of the localizations of S at the primes $\mathfrak{m}_i$ by Lemma 7.49.7. □

Lemma 7.139.12. *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. There exists a canonical local ring map $R \to R^h$ with the following properties*

(1) *R^h is henselian,*
(2) *R^h is a filtered colimit of étale R-algebras,*
(3) *$\mathfrak{m}R^h$ is the maximal ideal of R^h, and*
(4) *$\kappa = R^h/\mathfrak{m}R^h$.*

Proof. Consider the cateory of pairs $(S, \mathfrak{q})$ where $R \to S$ is an étale ring map, and $\mathfrak{q}$ is a prime of S lying over $\mathfrak{m}$ with $\kappa = \kappa(\mathfrak{q})$. A morphism of pairs $(S, \mathfrak{q}) \to (S', \mathfrak{q}')$ is given by an R-algebra map $\varphi : S \to S'$ such that $\varphi^{-1}(\mathfrak{q}') = \mathfrak{q}$. We set

$$R^h = colim_{(S,\mathfrak{q})}\, S.$$

This clearly implies that R^h is canonical, since no choices were made in this construction. Moreover, property (2) is clear.

Let us show that the category of pairs is filtered, see Categories, Definition 4.17.1. The category contains the pair $(R, \mathfrak{m})$ and hence is not empty, which proves part (1) of Categories, Definition 4.17.1. Note that for any pair $(S, \mathfrak{q})$ the prime ideal $\mathfrak{q}$ is maximal, for example since $\kappa \to S/\mathfrak{q} \subset \kappa(\mathfrak{q})$ are isomorphisms. Suppose that $(S, \mathfrak{q})$ and $(S', \mathfrak{q}')$ are two objects. Set $S'' = S \otimes_R S'$ and $\mathfrak{q}'' = \mathfrak{q}S'' + \mathfrak{q}'S''$. Then $S''/\mathfrak{q}'' = S/\mathfrak{q} \otimes_R S'/\mathfrak{q}' = \kappa$ by what we said above. Moreover, $R \to S''$ is étale by Lemma 7.132.3. This proves part (2) of Categories, Definition 4.17.1. Next, suppose that $\varphi, \psi : (S, \mathfrak{q}) \to (S', \mathfrak{q}')$ are two morphisms of pairs. Consider

$$S'' = (S' \otimes_{\varphi, S, \psi} S') \otimes_{S' \otimes_R S'} S'$$

with prime ideal

$$\mathfrak{q}'' = (\mathfrak{q}' \otimes S' + S' \otimes \mathfrak{q}') \otimes S' + (S' \otimes_{\varphi, S, \psi} S') \otimes \mathfrak{q}'$$

Arguing as above (base change of étale maps is étale, composition of étale maps is étale) we see that S'' is étale over R. Moreover, the canonical map $S' \to S''$ (using the right most factor for example) equalizes φ and ψ. This proves part (3) of Categories, Definition 4.17.1. Hence we conclude that R^h consists of triples $(S, \mathfrak{q}, f)$ with $f \in S$, and two such triples $(S, \mathfrak{q}, f)$, $(S', \mathfrak{q}', f')$ define the same element of R^h if and only if there exists a pair $(S'', \mathfrak{q}'')$ and morphisms of pairs $\varphi : (S, \mathfrak{q}) \to (S'', \mathfrak{q}'')$ and $\varphi' : (S', \mathfrak{q}') \to (S'', \mathfrak{q}'')$ such that $\varphi(f) = \varphi'(f')$.

Suppose that $x \in R^h$. Represent x by a triple $(S, \mathfrak{q}, f)$. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ be the other primes of S lying over $\mathfrak{m}$. Then we can find a $g \in S$, $g \notin \mathfrak{q}$ and $g \in \mathfrak{q}_i$ for $i = 1, \ldots, r$, see Lemma 7.14.3. Consider the morphism of pairs $(S, \mathfrak{q}) \to (S_g, \mathfrak{q}S_g)$. In this way we see that we may always assume that x is given by a triple $(S, \mathfrak{q}, f)$ where $\mathfrak{q}$ is the only prime of S lying over $\mathfrak{m}$, i.e., $\sqrt{\mathfrak{m}S} = \mathfrak{q}$. But since $R \to S$ is étale, we have $\mathfrak{m}S_\mathfrak{q} = \mathfrak{q}S_\mathfrak{q}$, see Lemma 7.132.5. Hence we actually get that $\mathfrak{m}S = \mathfrak{q}$.

Suppose that $x \notin \mathfrak{m}R^h$. Represent x by a triple $(S, \mathfrak{q}, f)$ with $\mathfrak{m}S = \mathfrak{q}$. Then $f \notin \mathfrak{m}S$, i.e., $f \notin \mathfrak{q}$. Hence $(S, \mathfrak{q}) \to (S_f, \mathfrak{q}S_f)$ is a morphism of pairs such that the image of f becomes invertible. Hence x is invertible with inverse represented by the triple $(S_f, \mathfrak{q}S_f, 1/f)$. We conclude that R^h is a local ring with maximal ideal $\mathfrak{m}R^h$. The residue field is κ since we can define $R^h/\mathfrak{m}R^h \to \kappa$ by mapping a triple $(S, \mathfrak{q}, f)$ to the residue class of f module $\mathfrak{q}$.

We still have to show that R^h is henselian. Namely, suppose that $P \in R^h[T]$ is a monic polynomial and $a_0 \in \kappa$ is a simple root of the reduction $\overline{P} \in \kappa[T]$. Then we can find a pair $(S, \mathfrak{q})$ such that P is the image of a monic polynomial $Q \in S[T]$. Since $S \to R^h$ induces an isomorphism of residue fields we see that $S' = S[T]/(Q)$ has a prime ideal $\mathfrak{q}' = (\mathfrak{q}, T - a_0)$ at which $S \to S'$ is standard étale. Moreover, $\kappa = \kappa(\mathfrak{q}')$. Pick $g \in S'$, $g \notin \mathfrak{q}'$ such that $S'' = S'_g$ is étale over S. Then $(S, \mathfrak{q}) \to (S'', \mathfrak{q}'S'')$ is a morphism of pairs. Now that triple $(S'', \mathfrak{q}'S'', \text{class of } T)$ determines an element $a \in R^h$ with the properties $P(a) = 0$, and $\overline{a} = a_0$ as desired. $\square$

Lemma 7.139.13. *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Let $\kappa \subset \kappa^{sep}$ be a separable algebraic closure. There exists a canonical commutative diagram*

$$\begin{array}{ccccc} \kappa & \longrightarrow & \kappa & \longrightarrow & \kappa^{sep} \\ \uparrow & & \uparrow & & \uparrow \\ R & \longrightarrow & R^h & \longrightarrow & R^{sh} \end{array}$$

with the following properties

1. *the map $R^h \to R^{sh}$ is local*
2. *R^{sh} is strictly henselian,*
3. *R^{sh} is a filtered colimit of étale R-algebras,*
4. *$\mathfrak{m}R^{sh}$ is the maximal ideal of R^{sh}, and*
5. *$\kappa^{sep} = R^{sh}/\mathfrak{m}R^{sh}$.*

Proof. This can be proved using the method followed in the proof of Lemma 7.139.12. The only difference is that, instead of pairs, one uses triples $(S, \mathfrak{q}, \alpha)$ where $R \to S$ étale, $\mathfrak{q}$ is a prime of S lying over $\mathfrak{m}$, and $\alpha : \kappa(\mathfrak{q}) \to \kappa^{sep}$ is an embedding of extensions of κ.

But we can also deduce the result directly from the result of Lemma 7.139.12. Namely, for any finite separable field sub extension $\kappa \subset \kappa' \subset \kappa^{sep}$ there exists a unique (up to unique isomorphism) finite étale local ring extension $R^h \subset R^h(\kappa')$ whose residue field extension extension reproduces the given extension, see Lemma 7.139.8. Hence we can set

$$R^{sh} = \bigcup\nolimits_{\kappa \subset \kappa' \subset \kappa^{sep}} R^h(\kappa')$$

The arrows in this system, compatible with the arrows on the level of residue fields, exist by Lemma 7.139.8. This will produce a henselian local ring by Lemma 7.139.5 since each of the rings $R^h(\kappa')$ is henselian by Lemma 7.139.4. By construction the residue field extension induced by $R^h \to R^{sh}$ is the field extension $\kappa \subset \kappa^{sep}$. We omit the proof that R^{sh} is a colimit of étale R-algebras. □

Definition 7.139.14. Let $(R, \mathfrak{m}, \kappa)$ be a local ring.

(1) The local ring map $R \to R^h$ constructed in Lemma 7.139.12 is called the *henselization* of R.
(2) Given a separable algebraic closure $\kappa \subset \kappa^{sep}$ the local ring map $R \to R^{sh}$ constructed in Lemma 7.139.13 is called the *strict henselization of R with respect to* $\kappa \subset \kappa^{sep}$.
(3) A local ring map $R \to R^{sh}$ is called a *strict henselization* of R if it is isomorphic to one of the local ring maps constructed in Lemma 7.139.13

Note that $R \to R^h \to R^{sh}$ are flat local ring homomorphisms. The first by the construction of R^h in Lemma 7.139.12 as a directed colimit of étale hence flat R-algebras and the second by the construction of R^{sh} in the proof of Lemma 7.139.13 as a directed colimit of finite étale hence flat R^h-algebras (see Lemma 7.35.2). In the following lemmas we discuss functoriality properties of the (strict) henselizations. This should make it clear exactly how canonical these constructions really are.

Lemma 7.139.15. *Let $R \to S$ be a local map of local rings. Let $S \to S^h$ be the henselization. Let $R \to A$ be an étale ring map and let $\mathfrak{q}$ be a prime of A lying over $\mathfrak{m}_R$ such that $R/\mathfrak{m}_R \cong \kappa(\mathfrak{q})$. Then there exists a unique morphism of rings $f : A \to S^h$ fitting into the commutative diagram*

$$\begin{tikzcd} A \arrow[r, "f"] & S^h \\ R \arrow[u] \arrow[r] & S \arrow[u] \end{tikzcd}$$

such that $f^{-1}(\mathfrak{m}_{S^h}) = \mathfrak{q}$.

Proof. Consider $A \otimes_R S^h$. This is an étale algebra over S^h, see Lemma 7.132.3. Moreover $\mathfrak{q}' = \mathfrak{q} \otimes S^h + A \otimes \mathfrak{m}_{S^h}$ is a maximal ideal lying over $\mathfrak{m}_{S^h}$ with residue field equal to the residue field of S^h. Hence by Lemma 7.139.3 there exists a (unique) splitting $\tau : A \otimes_R S^h \to S^h$ with $\tau^{-1}(\mathfrak{m}_{S^h}) = \mathfrak{q}'$. Set f equal to the composition $A \to A \otimes_R S^h \to S^h$. Minor details omitted. □

Lemma 7.139.16. *Let $R \to S$ be a local map of local rings. Let $R \to R^h$ and $S \to S^h$ be the henselizations. There exists a unique local ring map $R^h \to S^h$ fitting into the* *commutative diagram*

$$\begin{tikzcd} R^h \arrow[r, "f"'] & S^h \\ R \arrow[u] \arrow[r] & S \arrow[u] \end{tikzcd}$$

Proof. Write $R^h = colim\, R_i$ as a filtered colimit of étale R-algebras R_i. Note that $\mathfrak{m}_i = \mathfrak{m}_{R^h} \cap R_i$ is a maximal ideal of R_i with residue field κ. Hence by Lemma 7.139.15 we obtain canonical ring maps $f_i : R_i \to S^h$. Setting $f = colim\, f_i$ we win. $\square$

Lemma 7.139.17. *Let $R \to S$ be a ring map. Let $\mathfrak{q}$ be a prime of S lying over $\mathfrak{p}$ in R. Assume $R \to S$ is quasi-finite at $\mathfrak{q}$. The commutative diagram*

$$\begin{tikzcd} R_{\mathfrak{p}}^h \arrow[r] & S_{\mathfrak{q}}^h \\ R_{\mathfrak{p}} \arrow[u] \arrow[r] & S_{\mathfrak{q}} \arrow[u] \end{tikzcd}$$

of Lemma 7.139.16 identifies $S_{\mathfrak{q}}^h$ with the localization of $R_{\mathfrak{p}}^h \otimes_{R_{\mathfrak{p}}} S_{\mathfrak{q}}$ at the prime generated by $\mathfrak{q}$.

Proof. Note that $R_{\mathfrak{p}}^h \otimes_R S$ is quasi-finite over $R_{\mathfrak{p}}^h$ at the prime ideal corresponding to $\mathfrak{q}$, see Lemma 7.113.6. Hence the localization S' of $R_{\mathfrak{p}}^h \otimes_{R_{\mathfrak{p}}} S_{\mathfrak{q}}$ is henselian, see Lemma 7.139.4. On the other hand, since $R_{\mathfrak{p}}^h$ is a filtered colimit of étale R-algebras, also $R_{\mathfrak{p}}^h \otimes_{R_{\mathfrak{p}}} S_{\mathfrak{q}}$ is a filtered colimit of étale $S_{\mathfrak{q}}$-algebras, and hence S' is a filtered colimit of étale $S_{\mathfrak{q}}$-algebras. Moreover, each of those étale algebras occurs in the limit defining $S_{\mathfrak{q}}^h$. Hence the map $S' \to S_{\mathfrak{q}}^h$ exihibits $S_{\mathfrak{q}}^h$ as a colimit of étale algebras over S' with trivial residue field extensions at the primes lying over $\mathfrak{q}$. Since S' is henselian, this map is an isomorphism. $\square$

Lemma 7.139.18. *Let R be a local ring with henselization R^h. Let $I \subset \mathfrak{m}_R$. Then R^h/IR^h is the henselization of R/I.*

Proof. This is a special case of Lemma 7.139.17. $\square$

Lemma 7.139.19. *Let $\varphi : R \to S$ be a local map of local rings. Let $S/\mathfrak{m}_S \subset \kappa^{sep}$ be a separable algebraic closure. Let $S \to S^{sh}$ be the strict henselization of S with respect to $S/\mathfrak{m}_S \subset \kappa^{sep}$. Let $R \to A$ be an étale ring map and let $\mathfrak{q}$ be a prime of A lying over $\mathfrak{m}_R$. Given any commutative diagram*

$$\begin{tikzcd} \kappa(\mathfrak{q}) \arrow[r, "\phi"'] & \kappa^{sep} \\ R/\mathfrak{m}_R \arrow[u] \arrow[r, "\varphi"] & S/\mathfrak{m}_S \arrow[u] \end{tikzcd}$$

there exists a unique morphism of rings $f : A \to S^{sh}$ fitting into the commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"'] & S^{sh} \\ R \arrow[u] \arrow[r, "\varphi"] & S \arrow[u] \end{tikzcd}$$

such that $f^{-1}(\mathfrak{m}_{S^h}) = \mathfrak{q}$ and the induced map $\kappa(\mathfrak{q}) \to \kappa^{sep}$ is the given one.

Proof. Consider $A \otimes_R S^h$. This is an étale algebra over S^h, see Lemma 7.132.3. Moreover $I = \mathfrak{q} \otimes S^h + A \otimes \mathfrak{m}_{S^h}$ is an ideal lying over $\mathfrak{m}_{S^h}$ and our map ϕ induces a surjection

$$A \otimes_R S^h/I = \kappa(\mathfrak{q}) \otimes_{R/\mathfrak{m}_R} \kappa^{sep} \xrightarrow{\phi\otimes 1} \kappa^{sep}$$

The kernel of this surjection determines a maximal ideal $\mathfrak{q}'$ of $A \otimes_R S^h$. Hence by Lemma 7.139.3 there exists a (unique) splitting $\tau : A \otimes_R S^h \to S^h$ with $\tau^{-1}(\mathfrak{m}_{S^h}) = \mathfrak{q}'$. Set f equal to the composition $A \to A \otimes_R S^h \to S^h$. Minor details omitted. □

Lemma 7.139.20. *Let $R \to S$ be a local map of local rings. Choose separable algebraic closures $R/\mathfrak{m}_R \subset \kappa_1^{sep}$ and $S/\mathfrak{m}_S \subset \kappa_2^{sep}$. Let $R \to R^{sh}$ and $S \to S^{sh}$ be the corresponding strict henselizations. Given any commutative diagram*

$$\begin{tikzcd} \kappa_1^{sep} \arrow[r, "\phi"'] & \kappa_2^{sep} \\ R/\mathfrak{m}_R \arrow[u] \arrow[r, "\varphi"] & S/\mathfrak{m}_S \arrow[u] \end{tikzcd}$$

There exists a unique local ring map $R^{sh} \to S^{sh}$ fitting into the commutative diagram

$$\begin{tikzcd} R^{sh} \arrow[r, "f"'] & S^{sh} \\ R \arrow[u] \arrow[r] & S \arrow[u] \end{tikzcd}$$

and inducing ϕ on the residue fields of R^{sh} and S^{sh}.

Proof. Write $R^{sh} = colim\, R_i$ as a filtered colimit of étale R-algebras R_i. Note that $\mathfrak{m}_i = \mathfrak{m}_{R^h} \cap R_i$ is a maximal ideal of R_i with residue field contained in κ_1^{sep}. Hence by Lemma 7.139.19 we obtain canonical ring maps $f_i : R_i \to S^{sh}$. Setting $f = colim\, f_i$ we win. □

Lemma 7.139.21. *Let $R \to S$ be a ring map. Let $\mathfrak{q}$ be a prime of S lying over $\mathfrak{p}$ in R. Let $\kappa(\mathfrak{q}) \subset \kappa^{sep}$ be a separable algebraic closure. Assume $R \to S$ is quasi-finite at $\mathfrak{q}$. The commutative diagram*

$$\begin{tikzcd} R_{\mathfrak{p}}^{sh} \arrow[r] & S_{\mathfrak{q}}^{sh} \\ R_{\mathfrak{p}} \arrow[u] \arrow[r] & S_{\mathfrak{q}} \arrow[u] \end{tikzcd}$$

of Lemma 7.139.20 identifies $S_{\mathfrak{q}}^{sh}$ with a localization of $R_{\mathfrak{p}}^{sh} \otimes_{R_{\mathfrak{p}}} S_{\mathfrak{q}}$.

Proof. The residue field of $R_{\mathfrak{p}}^{sh}$ is the separable algebraic closure of $\kappa(\mathfrak{p})$ in κ^{sep}. Note that $R_{\mathfrak{p}}^{sh} \otimes_R S$ is quasi-finite over $R_{\mathfrak{p}}^{sh}$ at the prime ideal corresponding to $\mathfrak{q}$, see Lemma 7.113.6. Hence the localization S' of $R_{\mathfrak{p}}^{sh} \otimes_{R_{\mathfrak{p}}} S_{\mathfrak{q}}$ is henselian, see Lemma 7.139.4. Note that the residue field of S' is κ^{sep} since it contains both the separable algebraic closure of $\kappa(\mathfrak{p})$ and $\kappa(\mathfrak{q})$. Since $R_{\mathfrak{p}}^{sh}$ is a filtered colimit of étale R-algebras, also $R_{\mathfrak{p}}^{sh} \otimes_{R_{\mathfrak{p}}} S_{\mathfrak{q}}$ is a filtered colimit of étale $S_{\mathfrak{q}}$-algebras, and hence S' is a filtered colimit of étale $S_{\mathfrak{q}}$-algebras. Moreover, each of those étale algebras occurs in the limit defining $S_{\mathfrak{q}}^{sh}$. Hence the map $S' \to S_{\mathfrak{q}}^{sh}$ exihibits

$S_{\mathfrak q}^{sh}$ as a colimit of étale algebras over S'! Since S' is strictly henselian, this map is an isomorphism. □

Lemma 7.139.22. *Let R be a local ring with strict henselization R^{sh}. Let $I \subset \mathfrak m_R$. Then R^{sh}/IR^{sh} is a strict henselization of R/I.*

Proof. This is a special case of Lemma 7.139.21. □

Here is a slightly different construction of the henselization.

Lemma 7.139.23. *Let R be a ring. Let $\mathfrak p \subset R$ be a prime ideal. Consider the category of pairs $(S, \mathfrak q)$ where $R \to S$ is étale and $\mathfrak q$ is a prime lying over $\mathfrak p$ such that $\kappa(\mathfrak p) = \kappa(\mathfrak q)$. This category is filtered and*

$$(R_{\mathfrak p})^h = colim_{(S,\mathfrak q)}\, S = colim_{(S,\mathfrak q)}\, S_{\mathfrak q}$$

canonically.

Proof. A morphism of pairs $(S, \mathfrak q) \to (S', \mathfrak q')$ is given by an R-algebra map $\varphi : S \to S'$ such that $\varphi^{-1}(\mathfrak q') = \mathfrak q$. Let us show that the category of pairs is filtered, see Categories, Definition 4.17.1. The category contains the pair $(R, \mathfrak p)$ and hence is not empty, which proves part (1) of Categories, Definition 4.17.1. Suppose that $(S, \mathfrak q)$ and $(S', \mathfrak q')$ are two pairs. Note that $\mathfrak q$, resp. $\mathfrak q'$ correspond to primes of the fibre rings $S \otimes \kappa(\mathfrak p)$, resp. $S' \otimes \kappa(\mathfrak p)$ with residue fields $\kappa(\mathfrak p)$, hence they correspond to maximal ideals of $S \otimes \kappa(\mathfrak p)$, resp. $S' \otimes \kappa(\mathfrak p)$. Set $S'' = S \otimes_R S'$. By the above there exists a unique prime $\mathfrak q'' \subset S''$ lying over $\mathfrak q$ and over $\mathfrak q'$ whose residue field is $\kappa(\mathfrak p)$. The ring map $R \to S''$ is étale by Lemma 7.132.3. This proves part (2) of Categories, Definition 4.17.1. Next, suppose that $\varphi, \psi : (S, \mathfrak q) \to (S', \mathfrak q')$ are two morphisms of pairs. Consider

$$S'' = (S' \otimes_{\varphi, S, \psi} S') \otimes_{S' \otimes_R S'} S'$$

Arguing as above (base change of étale maps is étale, composition of étale maps is étale) we see that S'' is étale over R. The fibre ring of S'' over $\mathfrak p$ is

$$F'' = (F' \otimes_{\varphi, F, \psi} F') \otimes_{F' \otimes_{\kappa(\mathfrak p)} F'} F'$$

where F', F are the fibre rings of S' and S. Since φ and ψ are morphisms of pairs the map $F' \to \kappa(\mathfrak p)$ corresponding to $\mathfrak p'$ extends to a map $F'' \to \kappa(\mathfrak p)$ and in turn corresponds to a prime ideal $\mathfrak q'' \subset S''$ whose residue field is $\kappa(\mathfrak p)$. The canonical map $S' \to S''$ (using the right most factor for example) is a morphism of pairs $(S', \mathfrak q') \to (S'', \mathfrak q'')$ which equalizes φ and ψ. This proves part (3) of Categories, Definition 4.17.1. Hence we conclude that the category is filtered.

Recall that in the proof of Lemma 7.139.12 we constructed $(R_{\mathfrak p})^h$ as the corresponding colimit but starting with $R_{\mathfrak p}$ and its maximal ideal $\mathfrak pR_{\mathfrak p}$. Now, given any pair $(S, \mathfrak q)$ for $(R, \mathfrak p)$ we obtain a pair $(S_{\mathfrak p}, \mathfrak qS_{\mathfrak p})$ for $(R_{\mathfrak p}, \mathfrak pR_{\mathfrak p})$. Moreover, in this situation

$$S_{\mathfrak p} = colim_{f \in R, f \not\in \mathfrak p}\, S_f.$$

Hence in order to show the equalities of the lemma, it suffices to show that any pair $(S_{loc}, \mathfrak q_{loc})$ for $(R_{\mathfrak p}, \mathfrak pR_{\mathfrak p})$ is of the form $(S_{\mathfrak p}, \mathfrak qS_{\mathfrak p})$ for some pair $(S, \mathfrak q)$ over $(R, \mathfrak p)$ (some details omitted). This follows from Lemma 7.132.3. □

Lemma 7.139.24. *Let R be a ring. Let $\mathfrak p \subset R$ be a prime ideal. Let $\kappa(\mathfrak p) \subset \kappa^{sep}$ be a separable algebraic closure. Consider the category of triples $(S, \mathfrak q, \phi)$ where $R \to S$ is* *étale, $\mathfrak{q}$ is a prime lying over $\mathfrak{p}$, and $\phi : \kappa(\mathfrak{q}) \to \kappa^{sep}$ is a $\kappa(\mathfrak{p})$-algebra map. This category is filtered and*

$$(R_{\mathfrak{p}})^{sh} = colim_{(S,\mathfrak{q},\phi)} S = colim_{(S,\mathfrak{q},\phi)} S_{\mathfrak{q}}$$

canonically.

Proof. A morphism of triples $(S, \mathfrak{q}, \phi) \to (S', \mathfrak{q}', \phi')$ is given by an R-algebra map $\varphi : S \to S'$ such that $\varphi^{-1}(\mathfrak{q}') = \mathfrak{q}$ and such that $\phi' \circ \varphi = \phi$. Let us show that the category of pairs is filtered, see Categories, Definition 4.17.1. The category contains the triple $(R, \mathfrak{p}, \kappa(\mathfrak{p}) \subset \kappa^{sep})$ and hence is not empty, which proves part (1) of Categories, Definition 4.17.1. Suppose that $(S, \mathfrak{q}, \phi)$ and $(S', \mathfrak{q}', \phi')$ are two triples. Note that $\mathfrak{q}$, resp. $\mathfrak{q}'$ correspond to primes of the fibre rings $S \otimes \kappa(\mathfrak{p})$, resp. $S' \otimes \kappa(\mathfrak{p})$ with residue fields finite separable over $\kappa(\mathfrak{p})$ and ϕ, resp. ϕ' correspond to maps into κ^{sep}. Hence this data corresponds to $\kappa(\mathfrak{p})$-algebra maps

$$\phi : S \otimes_R \kappa(\mathfrak{p}) \longrightarrow \kappa^{sep}, \quad \phi' : S' \otimes_R \kappa(\mathfrak{p}) \longrightarrow \kappa^{sep}.$$

Set $S'' = S \otimes_R S'$. Combining the maps the above we get a unique $\kappa(\mathfrak{p})$-algerba map

$$\phi'' = \phi \otimes \phi' : S'' \otimes_R \kappa(\mathfrak{p}) \longrightarrow \kappa^{sep}$$

whose kernel corresponds to a prime $\mathfrak{q}'' \subset S''$ lying over $\mathfrak{q}$ and over $\mathfrak{q}'$, and whose residue field maps via ϕ'' to the compositum of $\phi(\kappa(\mathfrak{q}))$ and $\phi'(\kappa(\mathfrak{q}'))$ in κ^{sep}. The ring map $R \to S''$ is étale by Lemma 7.132.3. Hence $(S'', \mathfrak{q}'', \phi'')$ is a triple dominating both $(S, \mathfrak{q}, \phi)$ and $(S', \mathfrak{q}', \phi')$. This proves part (2) of Categories, Definition 4.17.1. Next, suppose that $\varphi, \psi : (S, \mathfrak{q}, \phi) \to (S', \mathfrak{q}', \phi')$ are two morphisms of pairs. Consider

$$S'' = (S' \otimes_{\varphi,S,\psi} S') \otimes_{S' \otimes_R S'} S'$$

Arguing as above (base change of étale maps is étale, composition of étale maps is étale) we see that S'' is étale over R. The fibre ring of S'' over $\mathfrak{p}$ is

$$F'' = (F' \otimes_{\varphi,F,\psi} F') \otimes_{F' \otimes_{\kappa(\mathfrak{p})} F'} F'$$

where F', F are the fibre rings of S' and S. Since φ and ψ are morphisms of triples the map $\phi' : F' \to \kappa^{sep}$ extends to a map $\phi'' : F'' \to \kappa^{sep}$ which in turn corresponds to a prime ideal $\mathfrak{q}'' \subset S''$. The canonical map $S' \to S''$ (using the right most factor for example) is a morphism of triples $(S', \mathfrak{q}', \phi') \to (S'', \mathfrak{q}'', \phi'')$ which equalizes φ and ψ. This proves part (3) of Categories, Definition 4.17.1. Hence we conclude that the category is filtered.

We still have to show that the colimit R_{colim} of the system is equal to the strict henselization of $R_{\mathfrak{p}}$ with respect to κ^{sep}. To see this note that the system of triples $(S, \mathfrak{q}, \phi)$ contains as a subsystem the pairs $(S, \mathfrak{q})$ of Lemma 7.139.23. Hence R_{colim} contains $R_{\mathfrak{p}}^h$ by the result of that lemma. Moreover, it is clear that $R_{\mathfrak{p}}^h \subset R_{colim}$ is a directed colimit of étale ring extensions. It follows that R_{colim} is henselian by Lemmas 7.139.4 and 7.139.5. Finally, by Lemma 7.132.15 we see that the residue field of R_{colim} is equal to κ^{sep}. Hence we conclude that R_{colim} is strictly henselian and hence equals the strict henselization of $R_{\mathfrak{p}}$ as desired. Some details omitted. □

Lemma 7.139.25. *Let $\varphi : R \to S$ be a local homomorphism of strictly henselian local rings. Let $P_1, \ldots, P_n \in R[x_1, \ldots, x_n]$ be polynomials such that $R[x_1, \ldots, x_n]/(P_1, \ldots, P_n)$ is étale over R. Then the map*

$$R^n \longrightarrow S^n, \quad (h_1, \ldots, h_n) \longmapsto (\varphi(h_1), \ldots, \varphi(h_n))$$

induces a bijection between

$$\{(r_1, \ldots, r_n) \in R^n \mid P_i(r_1, \ldots, r_n) = 0,\ i = 1, \ldots, n\}$$

and

$$\{(s_1, \ldots, s_n) \in S^n \mid P_i'(s_1, \ldots, s_n) = 0,\ i = 1, \ldots, n\}$$

where $P_i' \in S[x_1, \ldots, x_n]$ *are the images of the* P_i *under* φ.

Proof. The first solution set is canonically isomorphic to the set

$$Hom_R(R[x_1, \ldots, x_n]/(P_1, \ldots, P_n), R).$$

As R is henselian the map $R \to R/\mathfrak{m}_R$ induces a bijection between this set and the set of solutions in the residue field $R/\mathfrak{m}_R$, see Lemma 7.139.3. The same is true for S. Now since $R[x_1, \ldots, x_n]/(P_1, \ldots, P_n)$ is étale over R and $R/\mathfrak{m}_R$ is separably algebraically closed we see that $R/\mathfrak{m}_R[x_1, \ldots, x_n]/(\overline{P_1}, \ldots, \overline{P_n})$ is a finite product of copies of $R/\mathfrak{m}_R$. Hence the tensor product

$$R/\mathfrak{m}_R[x_1, \ldots, x_n]/(\overline{P_1}, \ldots, \overline{P_n}) \otimes_{R/\mathfrak{m}_R} S/\mathfrak{m}_S = S/\mathfrak{m}_S[x_1, \ldots, x_n]/(\overline{P_1'}, \ldots, \overline{P_n'})$$

is also a finite product of copies of $S/\mathfrak{m}_S$ with the same index set. This proves the lemma. □

Lemma 7.139.26. *Let R be a henselian local ring. Any countably generated Mittag-Leffler module over R is a direct sum of finitely presented R-modules.*

Proof. Let M be a countably generated and Mittag-Leffler R-module. We claim that for any element $x \in M$ there exists a direct sum decomposition $M = N \oplus K$ with $x \in N$, the module N finitely presented, and K Mittag-Leffler.

Suppose the claim is true. Choose generators $x_1, x_2, x_3, \ldots$ of M. By the claim we can inductively find direct sum decompositions

$$M = N_1 \oplus N_2 \oplus \ldots \oplus N_n \oplus K_n$$

with N_i finitely presented, $x_1, \ldots, x_n \in N_1 \oplus \ldots \oplus N_n$, and K_n Mittag-Leffler. Repeating ad infinitum we see that $M = \bigoplus N_i$.

We still have to prove the claim. Let $x \in M$. By Lemma 7.86.2 there exists an endomorphism $\alpha : M \to M$ such that α factors through a finitely presented module, and $\alpha(x) = x$. Say α factors as

$$M \xrightarrow{\pi} P \xrightarrow{i} M$$

Set $a = \pi \circ \alpha \circ i : P \to P$, so $i \circ a \circ \pi = \alpha^3$. By Lemma 7.15.2 there exists a monic polynomial $P \in R[T]$ such that $P(a) = 0$. Note that this implies formally that $\alpha^2 P(\alpha) = 0$. Hence we may think of M as a module over $R[T]/(T^2 P)$. Assume that $x \neq 0$. Then $\alpha(x) = x$ implies that $0 = \alpha^2 P(\alpha) x = P(1) x$ hence $P(1) = 0$ in R/I where $I = \{r \in R \mid rx = 0\}$ is the annihilator of x. As $x \neq 0$ we see $I \subset \mathfrak{m}_R$, hence 1 is a root of $\overline{P} = P \bmod \mathfrak{m}_R \in R/\mathfrak{m}_R[T]$. As R is henselian we can find a factorization

$$T^2 P = (T^2 Q_1) Q_2$$

for some $Q_1, Q_2 \in R[T]$ with $Q_2 = (T-1)^e \bmod \mathfrak{m}_R R[T]$ and $Q_1(1) \neq 0 \bmod \mathfrak{m}_R$, see Lemma 7.139.3. Let $N = \mathrm{Im}(\alpha^2 Q_1(\alpha) : M \to M)$ and $K = \mathrm{Im}(Q_2(\alpha) : M \to M)$. As $T^2 Q_1$ and Q_2 generate the unit ideal of $R[T]$ we get a direct sum decomposition $M = N \oplus K$. Moreover, Q_2 acts as zero on N and $T^2 Q_1$ acts as zero on K. Note that N is a quotient of P hence is finitely generated. Also $x \in N$ because $\alpha^2 Q_1(\alpha) x = Q_1(1) x$ and $Q_1(1)$ is a unit in R. By Lemma 7.83.7 the modules N and K are Mittag-Leffler. Finally, the finitely generated module N is finitely presented as a finitely generated Mittag-Leffler module is finitely presented, see Example 7.85.1 part (1). $\square$

7.140. Serre's criterion for normality

We introduce the following properties of Noetherian rings.

Definition 7.140.1. Let R be a Noetherian ring. Let $k \geq 0$ be an integer.

(1) We say R has property (R_k) if for every prime $\mathfrak{p}$ of height $\leq k$ the local ring $R_{\mathfrak{p}}$ is regular. We also say that R is *regular in codimension* $\leq k$.
(2) We say R has property (S_k) if for every prime $\mathfrak{p}$ the local ring $R_{\mathfrak{p}}$ has depth at least $\min\{k, \dim(R_{\mathfrak{p}})\}$.
(3) Let M be a finite R-module. We say M has property (S_k) if for every prime $\mathfrak{p}$ the module $M_{\mathfrak{p}}$ has depth at least $\min\{k, \dim(M_{\mathfrak{p}})\}$.

Any Noetherian ring has property (S_0) (and so does any finite module over it).

Lemma 7.140.2. *Let R be a Noetherian ring. Let M be a finite R-module. The following are equivalent:*

(1) *M has no embedded associated prime, and*
(2) *M has property (S_1).*

Proof. Let $\mathfrak{p}$ be an embedded associated prime of M. Then there exists another associated prime $\mathfrak{q}$ of M such that $\mathfrak{p} \subset \mathfrak{q}$. In particular this implies that $\dim(M_{\mathfrak{p}}) \geq 1$ (since $\mathfrak{q}$ is in the support as well). On the other hand $\mathfrak{p}R_{\mathfrak{p}}$ is associated to $M_{\mathfrak{p}}$ (Lemma 7.60.14) and hence $\text{depth}(M_{\mathfrak{p}}) = 0$ (see Lemma 7.60.17). In other words (S_1) does not hold. Conversely, if (S_1) does not then there exists a prime $\mathfrak{p}$ such that $\dim(M_{\mathfrak{p}}) \geq 1$ and $\text{depth}(M_{\mathfrak{p}}) = 0$. Then we see (arguing backwards using the lemmas cited above) that $\mathfrak{p}$ is an embedded associated prime. $\square$

Lemma 7.140.3. *Let R be a Noetherian ring. The following are equivalent:*

(1) *R is reduced, and*
(2) *R has properties (R_0) and (S_1).*

Proof. Suppose that R is reduced. Then $R_{\mathfrak{p}}$ is a field for every minimal prime $\mathfrak{p}$ of R, according to Lemma 7.23.3. Hence we have (R_0). Let $\mathfrak{p}$ be a prime of height ≥ 1. Then $A = R_{\mathfrak{p}}$ is a reduced local ring of dimension ≥ 1. Hence its maximal ideal $\mathfrak{m}$ is not an associated prime since this would mean there exists a $x \in \mathfrak{m}$ with annihilator $\mathfrak{m}$ so $x^2 = 0$. Hence the depth of $A = R_{\mathfrak{p}}$ is at least one, by Lemma 7.60.9. This shows that (S_1) holds.

Conversely, assume that R satisfies (R_0) and (S_1). If $\mathfrak{p}$ is a minimal prime of R, then $R_{\mathfrak{p}}$ is a field by (R_0), and hence is reduced. If $\mathfrak{p}$ is not minimal, then we see that $R_{\mathfrak{p}}$ has depth ≥ 1 by (S_1) and we conclude there exists an element $t \in \mathfrak{p}R_{\mathfrak{p}}$ such that $R_{\mathfrak{p}} \to R_{\mathfrak{p}}[1/t]$ is injective. This implies that $R_{\mathfrak{p}}$ is a subring of localizations of R at primes of smaller height. Thus by induction on the height we conclude that R is reduced. $\square$

Lemma 7.140.4. *(Serre's criterion for normality) Let R be a Noetherian ring. The following are equivalent:*

(1) *R is a normal ring, and*
(2) *R has properties (R_1) and (S_2).*

Proof. Suppose that R is normal. This means by definition that R is reduced and all localizations $R_{\mathfrak p}$ are normal domains. In particular we see that R has (R_0) and (S_1) by Lemma 7.140.3. Hence it suffices to show that a local Noetherian normal domain R of dimension d has depth $\geq \min(2, d)$ and is regular if $d = 1$. The assertion if $d = 1$ follows from Lemma 7.110.6.

Let R be a local Noetherian normal domain with maximal ideal $\mathfrak m$ and dimension $d \geq 2$. Choose $x \in \mathfrak m$, $x \not\in \mathfrak m^2$. If $\text{depth}(R/xR) \geq 1$ then $\text{depth}(R) \geq 2$ and we win. Assume $\text{depth}(R/xR) = 0$ to get a contradiction. This means that $\mathfrak m/(x)$ is an associated prime of R/xR. In other words, there exists an element $z \in R$ such that $\mathfrak m z \subset (x)$, but $z \not\in (x)$. Consider the element z/x of the fraction field of R. Let $c \in \mathfrak m$ be an arbitrary nonzero element. We claim that $cz^n/x^n \in R$. Namely, $cz^n/x^n = (cz/x)z^{n-1}/x^{n-1}$. By choice of z we have $cz = c'x$ for some $c' \in R$. Note that $c' \in \mathfrak m$ since $x \not\in \mathfrak m^2$. Hence $cz^n/x^n = c'z^{n-1}/x^{n-1}$ which is an element of R by induction on n. In other words, this shows that z/x is almost integral over R, see Definition 7.33.3. By Lemma 7.33.4 we see that z/x is integral over R. As R is normal we see that $z/x \in R$ which is the desired contradiction.

Suppose that R satisfies (R_1) and (S_2). By Lemma 7.140.3 we conclude that R is reduced. Hence it suffices to show that if R is a reduced local Noetherian ring of dimension d satisfying (S_2) and (R_1) then R is a normal domain. If $d = 0$, the result is clear. If $d = 1$, then the result follows from Lemma 7.110.6.

Let R be a reduced local Noetherian ring with maximal ideal $\mathfrak m$ and dimension d which satisfies (R_1) and (S_2). By Lemma 7.33.14 it suffices to show that R is integrally closed in its total ring of fractions. Pick $x = f/g$, with $f, g \in R$ and g a nonzero divisor which satisfies a monic equation

$$(f/g)^n + \sum\nolimits_{i=1}^n a_i (f/g)^{n-i} = 0$$

with $a_i \in R$. Our goal is to show that $f \in (g) = gR$. We will prove this by induction on d. By the remarks in the previous paragraph we know this is the case when $d = 0$, and when $d = 1$, which starts the induction. Assume $d \geq 2$. Consider the short exact sequence

$$0 \to R \to R \to R/(g) \to 0.$$

By Lemma 7.67.10 this implies $\text{depth}(R/(g)) \geq 1$. Hence there exists an element $t \in \mathfrak m$ which is a nonzero divisor on $R/(g)$. Hence if f has a nonzero image in $R/(g)$ then it has a nonzero image in $(R/(g))[1/t] \cong R_t/gR_t$. But by induction on the dimension the image of f is zero in R_t/gR_t (for example by localizing at all the primes of $D(t) \subset Spec(R)$). Hence we win. □

Lemma 7.140.5. *A regular ring is normal.*

Proof. Let R be a regular ring. By Lemma 7.140.4 it suffices to prove that R is (R_1) and (S_2). As a regular local ring is Cohen-Macaulay, see Lemma 7.98.3, it is clear that R is (S_2). Property (R_1) is immediate. □

Lemma 7.140.6. *Let R be a Noetherian normal domain with fraction field K. Then*

(1) *for any nonzero $a \in R$ the quotient R/aR has no embedded primes, and all its associated primes have height* 1

(2) $$R = \bigcap\nolimits_{height(\mathfrak p)=1} R_{\mathfrak p}$$

(3) *For any nonzero $x \in K$ the quotient $R/(R \cap xR)$ has no embedded primes, and all its associates primes have height* 1.

Proof. By Lemma 7.140.4 we see that R has (S_2). Hence for any nonzero element $a \in R$ we see that R/aR has (S_1) (use Lemma 7.67.10 for example) Hence R/aR has no embedded primes (Lemma 7.140.2). We conclude the associated primes of R/aR are exactly the minimal primes $\mathfrak p$ over (a), which have height 1 as a is not zero (Lemma 7.57.10). This proves (1).

Thus, given $b \in R$ we have $b \in aR$ if and only if $b \in aR_{\mathfrak p}$ for every minimal prime $\mathfrak p$ over (a) (see Lemma 7.60.18). These primes all have height 1 as seen above so $b/a \in R$ if and only if $b/a \in R_{\mathfrak p}$ for all height 1 primes. Hence (2) holds.

For (3) write $x = a/b$. Let $\mathfrak p_1, \ldots, \mathfrak p_r$ be the minimal primes over (ab). These all have height 1 by the above. Then we see that $R \cap xR = \bigcap_{i=1,\ldots,r}(R \cap xR_{\mathfrak p_i})$ by part (2) of the lemma. Hence $R/(R \cap xR)$ is a submodule of $\bigoplus R/(R \cap xR_{\mathfrak p_i})$. As $R_{\mathfrak p_i}$ is a discrete valuation ring (by property (R_1) for the Noetherian normal domain R, see Lemma 7.140.4) we have $xR_{\mathfrak p_i} = \mathfrak p_i^{e_i}R_{\mathfrak p_i}$ for some $e_i \in \mathbf Z$. Hence the direct sum is equal to $\bigoplus_{e_i>0} R/\mathfrak p_i^{(e_i)}$, see Definition 7.61.1. By Lemma 7.61.2 the only associated prime of the module $R/\mathfrak p^{(n)}$ is $\mathfrak p$. Hence the set of associate primes of $R/(R \cap xR)$ is a subset of $\{\mathfrak p_i\}$ and there are no inclusion relations among them. This proves (3). □

7.141. Formal smoothness of fields

In this section we show that field extensions are formally smooth if and only if they are separable.

Lemma 7.141.1. *Let K be a field of characteristic $p > 0$. Let $a \in K$. Then $da = 0$ in $\Omega_{K/\mathbf F_p}$ if and only if a is a pth power.*

Proof. By Lemma 7.122.4 we see that there exists a subfield $\mathbf F_p \subset L \subset K$ such that $\mathbf F_p \subset L$ is a finitely generated field extension and such that $\mathrm da$ is zero in $\Omega_{L/\mathbf F_p}$. Hence we may assume that K is a finitely generated field extension of $\mathbf F_p$.

Choose a transcendence basis $x_1, \ldots, x_r \in K$ such that K is finite separable over $\mathbf F_p(x_1, \ldots, x_r)$. We remark that the result holds for the purely transcendental subfield $\mathbf F_p(x_1, \ldots, x_r) \subset K$. Namely,

$$\Omega_{\mathbf F_p(x_1,\ldots,x_r)/\mathbf F_p} = \bigoplus\nolimits_{i=1}^r \mathbf F_p(x_1, \ldots, x_r)\mathrm dx_i$$

and any rational function all of whose partial derivatives are zero is a pth power. Moreover, we also have

$$\Omega_{K/\mathbf F_p} = \bigoplus\nolimits_{i=1}^r K\mathrm dx_i$$

since $\mathbf F_p(x_1, \ldots, x_r) \subset K$ is finite separable (computation omitted). Suppose $a \in K$ is an element such that $\mathrm da = 0$ in the module of differentials. By our choice of x_i we see that the minimal polynomial $P(T) \in k(x_1, \ldots, x_r)[T]$ of a is separable. Write

$$P(T) = T^d + \sum\nolimits_{i=1}^d a_iT^{d-i}$$

and hence

$$0 = \mathrm dP(a) = \sum\nolimits_{i=1}^d a^{d-i}\mathrm da_i$$

in $\Omega_{K/\mathbf F_p}$. By the description of $\Omega_{K/\mathbf F_p}$ above and the fact that P was the minimal polynomial of a, we see that this implies $\mathrm da_i = 0$. Hence $a_i = b_i^p$ for each i. Therefore by Lemma 7.38.8 we see that a is a pth power. □

Lemma 7.141.2. *Let k be a field of characteristic $p > 0$. Let $a_1, \ldots, a_n \in k$ be elements such that $da_1, \ldots, da_n$ are linearly independent in $\Omega_{k/\mathbf{F}_p}$. Then the field extension $k(a_1^{1/p}, \ldots, a_n^{1/p})$ has degree p^n over k.*

Proof. By induction on n. If $n = 1$ the result is Lemma 7.141.1. For the induction step, suppose that $k(a_1^{1/p}, \ldots, a_{n-1}^{1/p})$ has degree p^{n-1} over k. We have to show that a_n does not map to a pth power in $k(a_1^{1/p}, \ldots, a_{n-1}^{1/p})$. If it does then we can write

$$\begin{aligned} a_n &= \left(\sum\nolimits_{I=(i_1,\ldots,i_{n-1}),\ 0\le i_j\le p-1} \lambda_I a_1^{i_1/p} \ldots a_{n-1}^{i_{n-1}/p}\right)^p \\ &= \sum\nolimits_{I=(i_1,\ldots,i_{n-1}),\ 0\le i_j\le p-1} \lambda_I^p a_1^{i_1} \ldots a_{n-1}^{i_{n-1}} \end{aligned}$$

Applying d we see that da_n is linearly dependent on da_i, $i < n$. This is a contradiction. $\square$

Lemma 7.141.3. *Let k be a field of characteristic $p > 0$. The following are equivalent:*

(1) *the field extension K/k is separable (see Definition 7.39.1), and*
(2) *the map $K \otimes_k \Omega_{k/\mathbf{F}_p} \to \Omega_{K/\mathbf{F}_p}$ is injective.*

Proof. Write K as a directed colimit $K = colim_i K_i$ of finitely generated field extensions $k \subset K_i$. By definition K is separable if and only if each K_i is separable over k, and by Lemma 7.122.4 we see that $K \otimes_k \Omega_{k/\mathbf{F}_p} \to \Omega_{K/\mathbf{F}_p}$ is injective if and only if each $K_i \otimes_k \Omega_{k/\mathbf{F}_p} \to \Omega_{K_i/\mathbf{F}_p}$ is injective. Hence we may assume that K/k is a finitely generated field extension.

Assume $k \subset K$ is a finitely generated field extension which is separable. Choose $x_1, \ldots, x_{r+1} \in K$ as in Lemma 7.39.3. In this case there exists an irreducible polynomial $G(X_1, \ldots, X_{r+1}) \in k[X_1, \ldots, X_{r+1}]$ such that $G(x_1, \ldots, x_{r+1}) = 0$ and such that $\partial G/\partial X_{r+1}$ is not identically zero. Moreover K is the field of fractions of the domain. $S = K[X_1, \ldots, X_{r+1}]/(G)$. Write

$$G = \sum a_I X^I, \quad X^I = X_1^{i_1} \ldots X_{r+1}^{i_{r+1}}.$$

Using the presentation of S above we see that

$$\Omega_{S/\mathbf{F}_p} = \frac{S \otimes_k \Omega_k \oplus \bigoplus_{i=1,\ldots,r+1} S\mathrm{d}X_i}{\langle \sum X^I \mathrm{d}a_I + \sum \partial G/\partial X_i \mathrm{d}X_i \rangle}$$

Since $\Omega_{K/\mathbf{F}_p}$ is the localization of the S-module $\Omega_{S/\mathbf{F}_p}$ (see Lemma 7.122.8) we conclude that

$$\Omega_{K/\mathbf{F}_p} = \frac{K \otimes_k \Omega_k \oplus \bigoplus_{i=1,\ldots,r+1} K\mathrm{d}X_i}{\langle \sum X^I \mathrm{d}a_I + \sum \partial G/\partial X_i \mathrm{d}X_i \rangle}$$

Now, since the polynomial $\partial G/\partial X_{r+1}$ is not identically zero we conclude that the map $K \otimes_k \Omega_{k/\mathbf{F}_p} \to \Omega_{S/\mathbf{F}_p}$ is injective as desired.

Assume $k \subset K$ is a finitely generated field extension and that $K \otimes_k \Omega_{k/\mathbf{F}_p} \to \Omega_{K/\mathbf{F}_p}$ is injective. (This part of the proof is the same as the argument proving Lemma 7.41.1.) Let $x_1, \ldots, x_r$ be a transcendence basis of K over k such that the degree of inseparability of the finite extension $k(x_1, \ldots, x_r) \subset K$ is minimal. If K is separable over $k(x_1, \ldots, x_r)$ then we win. Assume this is not the case to get a contradiction. Then there exists an element $\alpha \in K$ which is not separable over $k(x_1, \ldots, x_r)$. Let $P(T) \in k(x_1, \ldots, x_r)[T]$ be its minimal polynomial. Because α is not separable actually P is a polynomial in T^p. Clear denominators to get an irreducible polynomial

$$G(X_1, \ldots, X_r, T) = \sum a_{I,i} X^I T^i \in k[X_1, \ldots, X_r, T]$$

such that $G(x_1, \ldots, x_r, \alpha) = 0$ in L. Note that this means $k[X_1, \ldots, X_r, T]/(G) \subset L$. We may assume that for some pair (I_0, i_0) the coefficient $a_{I_0,i_0} = 1$. We claim that dG/dX_i is not identically zero for at least one i. Namely, if this is not the case, then G is actually a polynomial in $X_1^p, \ldots, X_r^p, T^p$. Then this means that

$$\sum\nolimits_{(I,i)\neq(I_0,i_0)} x^I\alpha^i \mathrm{d}a_{I,i}$$

is zero in $\Omega_{K/\mathbf{F}_p}$. Note that there is no k-linear relation among the elements

$$\{x^I\alpha^i \mid a_{I,i}\neq 0 \text{ and } (I,i)\neq(I_0,i_0)\}$$

of K. Hence the assumption that $K\otimes_k \Omega_{k/\mathbf{F}_p} \to \Omega_{K/\mathbf{F}_p}$ is injective this implies that $\mathrm{d}a_{I,i} = 0$ in $\Omega_{k/\mathbf{F}_p}$ for all (I, i). By Lemma 7.141.1 we see that each $a_{I,i}$ is a pth power, which implies that G is a pth power contradicting the irreducibility of G. Thus, after renumbering, we may assume that dG/dX_1 is not zero. Then we see that x_1 is separably algebraic over $k(x_2, \ldots, x_r, \alpha)$, and that $x_2, \ldots, x_r, \alpha$ is a transcendence basis of L over k. This means that the degree of inseparability of the finite extension $k(x_2, \ldots, x_r, \alpha) \subset L$ is less than the degree of inseparability of the finite extension $k(x_1, \ldots, x_r) \subset L$, which is a contradiction. □

Lemma 7.141.4. *Let $k \subset K$ be an extension of fields. If K is formally smooth over k, then K is a separable extension of k.*

Proof. Assume K is formally smooth over k. By Lemma 7.127.9 we see that $K\otimes_k \Omega_{k/\mathbf{Z}} \to \Omega_{K/\mathbf{Z}}$ is injective. Hence K is separable over k by Lemma 7.141.3 above. □

Lemma 7.141.5. *Let $k \subset K$ be an extension of fields. Then K is formally smooth over k if and only if $H_1(L_{K/k}) = 0$.*

Proof. This follows from Proposition 7.127.8 and the fact that a vector spaces is free (hence projective). □

Lemma 7.141.6. *Let $k \subset K$ be an extension of fields.*

(1) *If K is purely transcendental over k, then K is formally smooth over k.*
(2) *If K is separable algebraic over k, then K is formally smooth over k.*
(3) *If K is separable over k, then K is formally smooth over k.*

Proof. For (1) write $K = k(x_j; j \in J)$. Suppose that A is a k-algebra, and $I \subset A$ is an ideal of square zero. Let $\varphi : K \to A/I$ be a k-algebra map. Let $a_j \in A$ be an element such that $a_j \mod I = \varphi(x_j)$. Then it is easy to see that there is a unique k-algebra map $K \to A$ which maps x_j to a_j and which reduces to φ mod I. Hence $k \subset K$ is formally smooth.

In case (2) we see that $k \subset K$ is a colimit of étale ring extensions. An étale ring map is formally étale (Lemma 7.137.2). Hence this case follows from Lemma 7.137.3 and the trivial observation that a formally étale ring map is formally smooth.

In case (3), write $K = colim\, K_i$ as the filtered colimit of its finitely generated sub k-extensions. By Definition 7.39.1 each K_i is separable algebraic over a purely transcendental extension of k. Hence K_i/k is formally smooth by cases (1) and (2) and Lemma 7.127.3. Thus $H_1(L_{K_i/k}) = 0$ by Lemma 7.141.5. Hence $H_1(L_{K/k}) = 0$ by Lemma 7.123.7. Hence K/k is formally smooth by Lemma 7.141.5 again. □

Lemma 7.141.7. *Let k be a field.*

(1) *If the characteristic of k is zero, then any extension field of k is formally smooth over k.*

(2) *If the characteristic of k is $p > 0$, then $k \subset K$ is formally smooth if and only if it is a separable field extension.*

Proof. Combine Lemmas 7.141.4 and 7.141.6. □

Here we put together all the different characterizations of separable field extensions.

Proposition 7.141.8. *Let $k \subset K$ be a field extension. If the characteristic of k is zero then*

(1) *K is separable over k,*
(2) *K is geometrically reduced over k,*
(3) *K is formally smooth over k,*
(4) *$H_1(L_{K/k}) = 0$, and*
(5) *the map $K \otimes_k \Omega_{k/\mathbf{Z}} \to \Omega_{K/\mathbf{Z}}$ is injective.*

If the characteristic of k is $p > 0$, then the following are equivalent:

(1) *K is separable over k,*
(2) *the ring $K \otimes_k k^{1/p}$ is reduced,*
(3) *K is geometrically reduced over k,*
(4) *the map $K \otimes_k \Omega_{k/\mathbf{F}_p} \to \Omega_{K/\mathbf{F}_p}$ is injective,*
(5) *$H_1(L_{K/k}) = 0$, and*
(6) *K is formally smooth over k.*

Proof. This is a combination of Lemmas 7.41.1, 7.141.7 7.141.4, and 7.141.3. □

Here is yet another characterization of finitely generated separable field extensions.

Lemma 7.141.9. *Let $k \subset K$ be a finitely generated field extension. Then K is separable over k if and only if K is the localization of a smooth k-algebra.*

Proof. Choose a finite type k-algebra R which is a domain whose fraction field is K. Lemma 7.129.9 says that $k \to R$ is smooth at (0) if and only if K/k is separable. This proves the lemma. □

Lemma 7.141.10. *Let $k \subset K$ be a field extension. Then K is a filtered colimit of global complete intersection algebras over k. If K/k is separable, then K is a filtered colimit of smooth algebras over k.*

Proof. Suppose that $E \subset K$ is a finite subset. It suffices to show that there exists a k subalgebra $A \subset K$ which contains E and which is a global complete intersection (resp. smooth) over k. The separable/smooth case follows from Lemma 7.141.9. In general let $L \subset K$ be the subfield generated by E. Pick a transcendence basis $x_1, \ldots, x_d \in L$ over k. The extension $k(x_1, \ldots, x_d) \subset L$ is finite. Say $L = k(x_1, \ldots, x_d)[y_1, \ldots, y_r]$. Pick inductively polynomials $P_i \in k(x_1, \ldots, x_d)[Y_1, \ldots, Y_r]$ such that $P_i = P_i(Y_1, \ldots, Y_i)$ is monic in Y_i over $k(x_1, \ldots, x_d)[Y_1, \ldots, Y_{i-1}]$ and maps to the minimum polynomial of y_i in $k(x_1, \ldots, x_d)[y_1, \ldots, y_{i-1}][Y_i]$. Then it is clear that $P_1, \ldots, P_r$ is a regular sequence in $k(x_1, \ldots, x_r)[Y_1, \ldots, Y_r]$ and that $L = k(x_1, \ldots, x_r)[Y_1, \ldots, Y_r]/(P_1, \ldots, P_r)$. If $h \in k[x_1, \ldots, x_d]$ is a polynomial such that $P_i \in k[x_1, \ldots, x_d, 1/h, Y_1, \ldots, Y_r]$, then we see that $P_1, \ldots, P_r$ is a regular sequence in $k[x_1, \ldots, x_d, 1/h, Y_1, \ldots, Y_r]$ and $A = k[x_1, \ldots, x_d, 1/h, Y_1, \ldots, Y_r]/(P_1, \ldots, P_r)$ is a global complete intersection. After adjusting our choice of h we may assume $E \subset A$ and we win. □

7.142. Constructing flat ring maps

The following lemma is occasionally useful.

Lemma 7.142.1. *Let $(R, \mathfrak m, k)$ be a local ring. Let $k \subset K$ be a field extension. There exists a local ring R', a flat local ring map $R \to R'$ such that $\mathfrak m' = \mathfrak m R'$ and the residue field extension $k = R/\mathfrak m \subset R'/\mathfrak m'$ is isomorphic to $k \subset K$.*

Proof. Suppose that $k \subset k' = k(\alpha)$ is a monogenic extension of fields. Then k' is the residue field of a flat local extension $R \subset R'$ as in the lemma. Namely, if α is transcendental over k, then we let R' be the localization of $R[x]$ at the prime $\mathfrak m R[x]$. If α is algebraic with minimal polynomial $T^d + \sum \overline{\lambda}_i T^{d-i}$, then we let $R' = R[T]/(T^d + \sum \lambda_i T^{d-i})$.

Consider the collection of triples $(k', R \to R', \phi)$, where $k \subset k' \subset K$ is a subfield, $R \to R'$ is a local ring map as in the lemma, and $\phi : R' \to k'$ induces an isomorphism $R'/\mathfrak m R' \cong k'$ of k-extensions. These form a ``big'' category $\mathcal C$ with morphisms $(k_1, R_1, \phi_1) \to (k_2, R_2, \phi_2)$ given by ring maps $\psi : R_1 \to R_2$ such that

$$\begin{tikzcd} R_1 \arrow[r, "\phi_1"'] \arrow[d, "\psi"'] & k_1 \arrow[r] & K \arrow[d, equal] \\ R_2 \arrow[r, "\phi_2"] & k_2 \arrow[r] & K \end{tikzcd}$$

commutes. This implies that $k_1 \subset k_2$.

Suppose that I is a directed partially ordered set, and $((R_i, k_i, \phi_i), \psi_{ii'})$ is a system over I, see Categories, Section 4.19. In this case we can consider

$$R' = colim_{i \in I} R_i$$

This is a local ring with maximal ideal $\mathfrak m R'$, and residue field $k' = \bigcup_{i \in I} k_i$. Moreover, the ring map $R \to R'$ is flat as it is a colimit of flat maps (and tensor products commute with directed colimits). Hence we see that (R', k', ϕ') is an ``upper bound'' for the system.

An almost trivial application of Zorn's Lemma would finish the proof if $\mathcal C$ was a set, but it isn't. (Actually, you can make this work by finding a reasonable bound on the cardinals of the local rings occuring.) To get around this problem we choose a total ordering on K. For $x \in K$ we let $K(x)$ be the subfield of K generated by all elements of K which are $\leq x$. By transfinite induction on $x \in K$ we will produce ring maps $R \subset R(x)$ as in the lemma with residue field extension $k \subset K(x)$. Moreover, by construction we will have that $R(x)$ will contain $R(y)$ for all $y \leq x$. Namely, if x has a predecessor x', then $K(x) = K(x')[x]$ and hence we can let $R(x') \subset R(x)$ be the local ring extension constructed in the first paragraph of the proof. If x does not have a predecessor, then we first set $R'(x) = colim_{x' < x} R(x')$ as in the third paragraph of the proof. The residue field of $R'(x)$ is $K'(x) = \bigcup_{x' < x} K(x')$. Since $K(x) = K'(x)[x]$ we see that we can use the construction of the first paragraph of the proof to produce $R'(x) \subset R(x)$. This finishes the proof of the lemma. $\square$

Lemma 7.142.2. *Let R be a ring. Let $\mathfrak p \subset R$ be a prime and let $\kappa(\mathfrak p) \subset L$ be a finite extension of fields. Then there exists a finite free ring map $R \to S$ such that $\mathfrak q = \mathfrak p S$ is prime and $\kappa(\mathfrak p) \subset \kappa(\mathfrak q)$ is isomorphic to the given extension $\kappa(\mathfrak p) \subset L$.*

Proof. By induction of the degree of $\kappa(\mathfrak p) \subset L$. If the degree is 1, then we take $R = S$. In general, if there exists a sub extension $\kappa(\mathfrak p) \subset L' \subset L$ then we win by induction on the degree (by first constructing $R \subset S'$ corresponding to $L'/\kappa(\mathfrak p)$ and then construction $S' \subset S$ corresponding to L/L'). Thus we may assume that $L \supset \kappa(\mathfrak p)$ is generated by a single element $\alpha \in L$. Let $X^d + \sum_{i<d} a_i X^i$ be the minimal polynomial of α over $\kappa(\mathfrak p)$, so $a_i \in \kappa(\mathfrak p)$. We may write a_i as the image of f_i/g for some $f_i, g \in R$ and $g \not\in \mathfrak p$. After replacing α by $g\alpha$ (and correspondingly replacing a_i by $g^{d-i}a_i$) we may assume that a_i is the image of some $f_i \in R$. Then we simply take $S = R[x]/(x^d + \sum f_i x^i)$. $\square$

7.143. The Cohen structure theorem

Here is a fundamental notion in commutative algebra.

Definition 7.143.1. Let $(R, \mathfrak m)$ be a local ring. We say R is a *complete local ring* if the canonical map

$$R \longrightarrow lim_n R/\mathfrak m^n$$

to the completion of R with respect to $\mathfrak m$ is an isomorphism[7].

Note that an Artinian local ring R is a complete local ring because $\mathfrak m_R^n = 0$ for some $n > 0$. In this section we mostly focus on Noetherian complete local rings.

Lemma 7.143.2. *Let R be a Noetherian complete local ring. Any quotient of R is also a Noetherian complete local ring. Given a finite ring map $R \to S$, then S is a product of Noetherian complete local rings.*

Proof. The ring S is Noetherian by Lemma 7.28.1. As an R-module S is complete by Lemma 7.90.2. Hence S is the product of the completions at its maximal ideals by Lemma 7.90.17. $\square$

Lemma 7.143.3. *Let $(R, \mathfrak m)$ be a complete local ring. If $\mathfrak m$ is a finitely generated ideal then R is Noetherian.*

Proof. See Lemma 7.90.9. $\square$

Definition 7.143.4. Let $(R, \mathfrak m)$ be a complete local ring. A subring $\Lambda \subset R$ is called a *coefficient ring* if the following conditions hold:

1. Λ is a complete local ring with maximal ideal $\Lambda \cap \mathfrak m$,
2. the residue field of Λ maps isomorphically to the residue field of R, and
3. $\Lambda \cap \mathfrak m = p\Lambda$, where p is the characteristic of the residue field of R.

Let us make some remarks on this definition. We split the discussion into the following cases:

1. The local ring R contains a field. This happens if either $\mathbf Q \subset R$, or $pR = 0$ where p is the characteristic of $R/\mathfrak m$. In this case a coefficient ring Λ is a field contained in R which maps isomorphically to $R/\mathfrak m$.
2. The characteristic of $R/\mathfrak m$ is $p > 0$ but no power of p is zero in R. In this case Λ is a complete discrete valuation ring with uniformizer p and residue field $R/\mathfrak m$.
3. The characteristic of $R/\mathfrak m$ is $p > 0$, and for some $n > 1$ we have $p^{n-1} \neq 0$, $p^n = 0$ in R. In this case Λ is an Artinian local ring whose maximal ideal is generated by p and which has residue field $R/\mathfrak m$.

The complete discrete valuation rings with uniformizer p above play a special role and we baptize them as follows.

[7] This includes the condition that $\bigcap \mathfrak m^n = (0)$; in some texts this may be indicated by saying that R is complete and separated. Warning: It can happen that the completion $lim_n R/\mathfrak m^n$ of a local ring is non-complete, see Examples, Lemma 64.2.1. This does not happen when $\mathfrak m$ is finitely generated, see Lemma 7.90.7 in which case the completion is Noetherian, see Lemma 7.90.9.

Definition 7.143.5. A *Cohen ring* is a complete discrete valuation ring with uniformizer p a prime number.

Lemma 7.143.6. *Let p be a prime number. Let k be a field of characteristic p. There exists a Cohen ring Λ with $\Lambda/p\Lambda \cong k$.*

Proof. First note that the p-adic integers $\mathbf{Z}_p$ form a Cohen ring for $\mathbf{F}_p$. Let k be an arbitrary field of characteristic p. Let $\mathbf{Z}_p \to R$ be a flat local ring map such that $\mathfrak{m}_R = pR$ and $R/pR = k$, see Lemma 7.142.1. Then clearly R is a discrete valuation ring. Hence its completion is a Cohen ring for k. $\square$

Lemma 7.143.7. *Let $p > 0$ be a prime. Let Λ be a Cohen ring with residue field of characteristic p. For every $n \geq 1$ the ring map*

$$\mathbf{Z}/p^n\mathbf{Z} \to \Lambda/p^n\Lambda$$

is formally smooth.

Proof. If $n = 1$, this follows from Proposition 7.141.8. For general n we argue by induction on n. Namely, if $\mathbf{Z}/p^n\mathbf{Z} \to \Lambda/p^n\Lambda$ is formally smooth, then we can apply Lemma 7.127.12 to the ring map $\mathbf{Z}/p^{n+1}\mathbf{Z} \to \Lambda/p^{n+1}\Lambda$ and the ideal $I = (p^n) \subset \mathbf{Z}/p^{n+1}\mathbf{Z}$. $\square$

Theorem 7.143.8. *(Cohen structure theorem) Let $(R, \mathfrak{m})$ be a complete local ring.*

(1) *R has a coefficient ring (see Definition 7.143.4),*
(2) *if $\mathfrak{m}$ is a finitely generated ideal, then R is isomorphic to a quotient*

$$\Lambda[[x_1, \ldots, x_n]]/I$$

where Λ is either a field or a Cohen ring.

Proof. Let us prove a coefficient ring exists. First we prove this in case the characteristic of the residue field κ is zero. Namely, in this case we will prove by induction on $n > 0$ that there exists a section

$$\varphi_n : \kappa \longrightarrow R/\mathfrak{m}^n$$

to the canonical map $R/\mathfrak{m}^n \to \kappa = R/\mathfrak{m}$. This is trivial for $n = 1$. If $n > 1$, let φ_{n-1} be given. The field extension $\mathbf{Q} \subset \kappa$ is formally smooth by Proposition 7.141.8. Hence we can find the dotted arrow in the following diagram

$$\begin{tikzcd} R/\mathfrak{m}^{n-1} & R/\mathfrak{m}^n \arrow[l] \\ \kappa \arrow[u, "\varphi_{n-1}"] \arrow[ur, dotted] & \mathbf{Q} \arrow[l] \arrow[u] \end{tikzcd}$$

This proves the induction step. Putting these maps together

$$lim_n\ \varphi_n : \kappa \longrightarrow R = lim_n\ R/\mathfrak{m}^n$$

gives a map whose image is the desired coefficient ring.

Next, we prove the existence of a coefficient ring in the case where the characteristic of the residue field κ is $p > 0$. Namely, choose a Cohen ring Λ with $\kappa = \Lambda/p\Lambda$, see Lemma 7.143.6. In this case we will prove by induction on $n > 0$ that there exists a map

$$\varphi_n : \Lambda/p^n\Lambda \longrightarrow R/\mathfrak{m}^n$$

whose composition with the reduction map $R/\mathfrak{m}^n \to \kappa$ produces the given isomorphism $\Lambda/p\Lambda = \kappa$. This is trivial for $n = 1$. If $n > 1$, let φ_{n-1} be given. The ring map $\mathbf{Z}/p^n\mathbf{Z} \to$

$\Lambda/p^n\Lambda$ is formally smooth by Lemma 7.143.7. Hence we can find the dotted arrow in the following diagram

$$\begin{tikzcd} R/\mathfrak{m}^{n-1} & R/\mathfrak{m}^n \arrow[l] \\ \Lambda/p^n\Lambda \arrow[u, "\varphi_{n-1}"] \arrow[ur, dotted] & \mathbf{Z}/p^n\mathbf{Z} \arrow[l] \arrow[u] \end{tikzcd}$$

This proves the induction step. Putting these maps together

$$lim_n\ \varphi_n : \Lambda = lim_n\ \Lambda/p^n\Lambda \longrightarrow R = lim_n\ R/\mathfrak{m}^n$$

gives a map whose image is the desired coefficient ring.

The final statement of the theorem is now clear. Namely, if $y_1, \ldots, y_n$ are generators of the ideal $\mathfrak{m}$, then we can use the map $\Lambda \to R$ just constructed to get a map

$$\Lambda[[x_1, \ldots, x_n]] \longrightarrow R, \quad x_i \longmapsto y_i.$$

This map is surjective on each $R/\mathfrak{m}^n$ and hence is surjective as R is complete. Some details omitted. $\square$

Remark 7.143.9. If k is a field then the power series ring $k[[X_1, \ldots, X_d]]$ is a Noetherian complete local regular ring of dimension d. If Λ is a Cohen ring then $\Lambda[[X_1, \ldots, X_d]]$ is a complete local Noetherian regular ring of dimension $d + 1$. Hence the Cohen structure theorem implies that any Noetherian complete local ring is a quotient of a regular local ring. In particular we see that a Noetherian complete local ring is universally catenary, see Lemma 7.97.6 and Lemma 7.98.3.

Lemma 7.143.10. *Let $(R, \mathfrak{m})$ be a Noetherian complete local domain. Then there exists a $R_0 \subset R$ with the following properties*

1. *R_0 is a regular complete local ring,*
2. *$R_0 \subset R$ is finite and induces an isomorphism on residue fields,*
3. *R_0 is either isomorphic to $k[[X_1, \ldots, X_d]]$ where k is a field or $\Lambda[[X_1, \ldots, X_d]]$ where Λ is a Cohen ring.*

Proof. Let Λ be a coefficient ring of R. Since R is a domain we see that either Λ is a field or Λ is a Cohen ring.

Case I: $\Lambda = k$ is a field. Let $d = \dim(R)$. Choose $x_1, \ldots, x_d \in \mathfrak{m}$ which generate an ideal of definition $I \subset R$. (See Section 7.57.) By Lemma 7.90.12 we see that R is I-adically complete as well. Consider the map $R_0 = k[[X_1, \ldots, X_d]] \to R$ which maps X_i to x_i. Note that R_0 is complete with respect to the ideal $I_0 = (X_1, \ldots, X_d)$, and that $R/I_0R \cong R/IR$ is finite over $k = R_0/I_0$ (because $\dim(R/I) = 0$, see Section 7.57.) Hence we conclude that $R_0 \to R$ is finite by Lemma 7.90.15. Since $\dim(R) = \dim(R_0)$ this implies that $R_0 \to R$ is injective (see Lemma 7.103.3), and the lemma is proved.

Case II: Λ is a Cohen ring. Let $d+1 = \dim(R)$. Let $p > 0$ be the characteristic of the residue field k. As R is a domain we see that p is a nonzero divisor in R. Hence $\dim(R/pR) = d$, see Lemma 7.57.11. Choose $x_1, \ldots, x_d \in R$ which generate an ideal of definition in R/pR. Then $I = (p, x_1, \ldots, x_d)$ is an ideal of definition of R. By Lemma 7.90.12 we see that R is I-adically complete as well. Consider the map $R_0 = \Lambda[[X_1, \ldots, X_d]] \to R$ which maps X_i to x_i. Note that R_0 is complete with respect to the ideal $I_0 = (p, X_1, \ldots, X_d)$, and that $R/I_0R \cong R/IR$ is finite over $k = R_0/I_0$ (because $\dim(R/I) = 0$, see Section 7.57.) Hence we conclude that $R_0 \to R$ is finite by Lemma 7.90.15. Since $\dim(R) = \dim(R_0)$ this implies that $R_0 \to R$ is injective (see Lemma 7.103.3), and the lemma is proved. $\square$

7.144. Nagata and Japanese rings

In this section we discuss finiteness of integral closure. It turns out that this is closely related to the relationship between a local ring and its completion.

Definition 7.144.1. Let R be a a domain with field of fractions K.

(1) We say R is *N-1* if the integral closure of R in K is a finite R-module.
(2) We say R is *N-2*, or *Japanese* if for any finite extension $K \subset L$ of fields the integral closure of R in L is finite over R.

The main interest in these notions is for Noetherian rings, but here is a non-Noetherian example.

Example 7.144.2. Let k be a field. The domain $R = k[x_1, x_2, x_3, \ldots]$ is Japanese, but not Noetherian. The reason is the following. Suppose that $R \subset L$ and the field L is a finite extension of the fraction field of R. Then there exists an integer n such that L comes from a finite extension $k(x_1, \ldots, x_n) \subset L_0$ by adjoining the (transcendental) elements x_{n+1}, x_{n+2}, etc. Let S_0 be the integral closure of $k[x_1, \ldots, x_n]$ in L_0. By Proposition 7.144.31 below it is true that S_0 is finite over $k[x_1, \ldots, x_n]$. Moreover, the integral closure of R in L is $S = S_0[x_{n+1}, x_{n+2}, \ldots]$ (use Lemma 7.33.8) and hence finite over R. The same argument works for $R = \mathbf{Z}[x_1, x_2, x_3, \ldots]$.

Lemma 7.144.3. *Let R be a domain. If R is N-1 then so is any localization of R. Same for N-2.*

Proof. These statements hold because taking integral closure commutes with localization, see Lemma 7.32.9. □

Lemma 7.144.4. *Let R be a domain. Let $f_1, \ldots, f_n \in R$ generate the unit ideal. If each domain R_{f_i} is N-1 then so is R. Same for N-2.*

Proof. Assume R_{f_i} is N-2 (or N-1). Let L be a finite extension of the fraction field of R (equal to the fraction field in the N-1 case). Let S be the integral closure of R in L. By Lemma 7.32.9 we see that S_{f_i} is the integral closure of R_{f_i} in L. Hence S_{f_i} is finite over R_{f_i} by assumption. Thus S is finite over R by Lemma 7.21.2. □

Lemma 7.144.5. *Let R be a domain. Let $R \subset S$ be a quasi-finite extension of domains (for example finite). Assume R is N-2 and Noetherian. Then S is N-2.*

Proof. Let $K = f.f.(R) \subset L = f.f.(S)$. Note that this is a finite field extension (for example by Lemma 7.113.2 (2) applied to the fibre $S \otimes_R K$, and the definition of a quasi-finite ring map). Let S' be the integral closure of R in S. Then S' is contained in the integral closure of R in L which is finite over R by assumption. As R is Noetherian this implies S' is finite over R. By Lemma 7.114.15 there exist elements $g_1, \ldots, g_n \in S'$ such that $S'_{g_i} \cong S_{g_i}$ and such that $g_1, \ldots, g_n$ generate the unit ideal in S. Hence it suffices to show that S' is N-2 by Lemmas 7.144.3 and 7.144.4. Thus we have reduced to the case where S is finite over R.

Assume $R \subset S$ with hypotheses as in the lemma and moreover that S is finite over R. Let M be a finite field extension of the fraction field of S. Then M is also a finite field extension of $f.f(R)$ and we conclude that the integral closure T of R in M is finite over R. By Lemma 7.32.14 we see that T is also the integral closure of S in M and we win by Lemma 7.32.13. □

Lemma 7.144.6. *Let R be a Noetherian domain. If $R[z, z^{-1}]$ is N-1, then so is R.*

Proof. Let R' be the integral closure of R in its field of fractions K. Let S' be the integral closure of $R[z, z^{-1}]$ in its field of fractions. Clearly $R' \subset S'$. Since $K[z, z^{-1}]$ is a normal domain we see that $S' \subset K[z, z^{-1}]$. Suppose that $f_1, \ldots, f_n \in S'$ generate S' as $R[z, z^{-1}]$-module. Say $f_i = \sum a_{ij} z^j$ (finite sum), with $a_{ij} \in K$. For any $x \in R'$ we can write

$$x = \sum h_i f_i$$

with $h_i \in R[z, z^{-1}]$. Thus we see that R' is contained in the finite R-submodule $\sum R a_{ij} \subset K$. Since R is Noetherian we conclude that R' is a finite R-module. □

Lemma 7.144.7. *Let R be a Noetherian domain, and let $R \subset S$ be a finite extension of domains. If S is N-1, then so is R. If S is N-2, then so is R.*

Proof. Omitted. (Hint: Integral closures of R in extension fields are contained in integral closures of S in extension fields.) □

Lemma 7.144.8. *Let R be a Noetherian normal domain with fraction field K. Let $K \subset L$ be a finite separable field extension. Then the integral closure of R in L is finite over R.*

Proof. Consider the trace pairing

$$L \times L \longrightarrow K, \quad (x, y) \longmapsto \langle x, y \rangle := \mathrm{Tr}_{L/K}(xy).$$

Since L/K is separable this is nondegenerate (exercise in Galois theory). Moreover, if $x \in L$ is integral over R, then $\mathrm{Tr}_{L/K}(x)$ is integral over R also, and since R is normal we see $\mathrm{Tr}_{L/K}(x) \in R$. Pick $x_1, \ldots, x_n \in L$ which are integral over R and which form a K-basis of L. Then the integral closure $S \subset L$ is contained in the R-module

$$M = \{ y \in L \mid \langle x_i, y \rangle \in R,\ i = 1, \ldots, n \}$$

By linear algebra we see that $M \cong R^{\oplus n}$ as an R-module. Hence $S \subset R^{\oplus n}$ is a finitely generated R-module as R is Noetherian. □

Example 7.144.9. Lemma 7.144.8 does not work if the ring is not Noetherian. For example consider the action of $G = \{+1, -1\}$ on $A = \mathbf{C}[x_1, x_2, x_3, \ldots]$ where -1 acts by mapping x_i to $-x_i$. The invariant ring $R = A^G$ is the $\mathbf{C}$-algebra generated by all $x_i x_j$. Hence $R \subset A$ is not finite. But R is a normal domain with fraction field $K = L^G$ the G-invariants in the fraction field L of A. And clearly A is the integral closure of R in L.

Lemma 7.144.10. *A Noetherian domain of characteristic zero is N-1 if and only if it is N-2 (i.e., Japanese).*

Proof. This is clear from Lemma 7.144.8 since every field extension in characteristic zero is separable. □

Lemma 7.144.11. *Let R be a Noetherian domain with fraction field K of characteristic $p > 0$. Then R is Japanese if and only if for every finite purely inseparable extension $K \subset L$ the integral closure of R in L is finite over R.*

Proof. Assume the integral closure of R in every finite purely inseparable field extension of K is finite. Let $K \subset L$ be any finite extension. We have to show the integral closure of R in L is finite over R. Choose a finite normal field extension $K \subset M$ containing L. As R is Noetherian it suffices to show that the integral closure of R in M is finite over R. By Lemma 7.38.4 there exists a subextension $K \subset M_{insep} \subset M$ such that M_{insep}/K is purely inseparable, and M/M_{insep} is separable. By assumption the integral closure R' of R in M_{insep} is finite over R. By Lemma 7.144.8 the integral closure R'' of R' in M is finite over

R'. Then R'' is finite over R by Lemma 7.7.3. Since R'' is also the integral closure of R in M (see Lemma 7.32.14) we win. □

Lemma 7.144.12. *Let R be a Noetherian domain. If R is N-1 then $R[x]$ is N-1. If R is N-2 then $R[x]$ is N-2.*

Proof. Assume R is N-1. Let R' be the integral closure of R which is finite over R. Hence also $R'[x]$ is finite over $R[x]$. The ring $R'[x]$ is normal (see Lemma 7.33.8), hence N-1. This proves the first assertion.

For the second assertion, by Lemma 7.144.7 it suffices to show that $R'[x]$ is N-2. In other words we may and do assume that R is a normal N-2 domain. In characteristic zero we are done by Lemma 7.144.10. In characteristic $p > 0$ we have to show that the integral closure of $R[x]$ is finite in any finite purely inseparable extension of $f.f.(R[x]) = K(x) \subset L$ with $K = f.f.(R)$. Clearly there exists a finite purely inseparable field extension $K \subset L'$ and $q = p^e$ such that $L \subset L'(x^{1/q})$. As $R[x]$ is Noetherian it suffices to show that the integral closure of $R[x]$ in $L'(x^{1/q})$ is finite over $R[x]$. And this integral closure is equal to $R'[x^{1/q}]$ with $R \subset R' \subset L'$ the integral closure of R in L'. Since R is N-2 we see that R' is finite over R and hence $R'[x^{1/q}]$ is finite over $R[x]$. □

Lemma 7.144.13 (Tate)**.** *Let R be a ring. Let $x \in R$. Assume*

(1) *R is a normal Noetherian domain,*
(2) *R/xR is a Japanese domain,*
(3) *$R \cong lim_n R/x^n R$ is complete with respect to x.*

Then R is Japanese.

Proof. We may assume $x \neq 0$ since otherwise the lemma is trivial. Let K be the fraction field of R. If the characteristic of K is zero the lemma follows from (1), see Lemma 7.144.10. Hence we may assume that the characteristic of K is $p > 0$, and we may apply Lemma 7.144.11. Thus given $K \subset L$ be a finite purely inseparable field extension we have to show that the integral closure S of R in L is finite over R.

Let q be a power of p such that $L^q \subset K$. By enlarging L if necessary we may assume there exists an element $y \in L$ such that $y^q = x$. Since $R \to S$ induces a homeomorphism of spectra (see Lemma 7.43.2) there is a unique prime ideal $\mathfrak{q} \subset S$ lying over the prime ideal $\mathfrak{p} = xR$. It is clear that

$$\mathfrak{q} = \{f \in S \mid f^q \in \mathfrak{p}\} = yS$$

since $y^q = x$. Hence $R_{\mathfrak{p}}$ and $S_{\mathfrak{q}}$ are discrete valuation rings, see Lemma 7.110.6. By Lemma 7.110.8 we see that $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q})$ is a finite field extension. Hence the integral closure $S' \subset \kappa(\mathfrak{q})$ of R/xR is finite over R/xR by assumption (2). Since $S/yS \subset S'$ this implies that S/yS is finite over R. Note that S/y^nS has a finite filtration whose subquotients are the modules $y^iS/y^{i+1}S \cong S/yS$. Hence we see that each S/y^nS is finite over R. In particular S/xS is finite over R. Also, it is clear that $\bigcap x^nS = (0)$ since an element in the intersection has qth power contained in $\bigcap x^nR = (0)$ (Lemma 7.47.6). Thus we may apply Lemma 7.90.15 to conclude that S is finite over R, and we win. □

Lemma 7.144.14. *Let R be a ring. If R is Noetherian, a domain, and N-2, then so is $R[[x]]$.*

Proof. Apply Lemma 7.144.13 to the element $x \in R[[x]]$. □

Definition 7.144.15. Let R be a ring.

(1) We say R is *universally Japanese* if for any finite type ring map $R \to S$ with S a domain we have that S is Japanese (i.e., N-2).

(2) We say that R is a *Nagata ring* if R is Noetherian and for every prime ideal $\mathfrak{p}$ the ring $R/\mathfrak{p}$ is Japanese.

It is clear that a Noetherian universally Japanese ring is a Nagata ring. It is our goal to show that a Nagata ring is universally Japanese. This is not obvious at all, and requires some work. But first, here is a useful lemma.

Lemma 7.144.16. *Let R be a Nagata ring. Let $R \to S$ be essentially of finite type with S reduced. Then the integral closure of R in S is finite over R.*

Proof. As S is essentially of finite type over R it is Noetherian and has finitely many minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_m$, see Lemma 7.28.6. Since S is reduced we have $S \subset \prod S_{\mathfrak{q}_i}$ and each $S_{\mathfrak{q}_i} = K_i$ is a field, see Lemmas 7.22.2 and 7.23.3. It suffices to show that the integral closure A_i' of R in each K_i is finite over R. This is true because R is Noetherian and $A \subset \prod A_i'$. Let $\mathfrak{p}_i \subset R$ be the prime of R corresponding to $\mathfrak{q}_i$. As S is essentially of finite type over R we see that $K_i = S_{\mathfrak{q}_i} = \kappa(\mathfrak{q}_i)$ is a finitely generated field extension of $\kappa(\mathfrak{p}_i)$. Hence the algebraic closure L_i of $\kappa(\mathfrak{p}_i)$ in $\subset K_i$ is finite over $\kappa(\mathfrak{p}_i)$, see Lemma 7.38.7. It is clear that A_i' is the integral closure of $R/\mathfrak{p}_i$ in L_i, and hence we win by definition of a Nagata ring. ☐

Lemma 7.144.17. *Let R be a ring. To check that R is universally Japanese it suffices to show: If $R \to S$ is of finite type, and S a domain then S is N-1.*

Proof. Namely, assume the condition of the lemma. Let $R \to S$ be a finite type ring map with S a domain. Let $f.f.(S) \subset L$ be a finite extension of its fraction field. Then there exists a finite ring extension $S \subset S' \subset L$ with $f.f.(S') = L$. By assumption S' is N-1, and hence the integral closure S'' of S' in L is finite over S'. Thus S'' is finite over S (Lemma 7.7.3) and S'' is the integral closure of S in L (Lemma 7.32.14). We conclude that R is universally Japanese. ☐

Lemma 7.144.18. *If R is universally Japanese then any algebra essentially of finite type over R is universally Japanese.*

Proof. The case of an algebra of finite type over R is immediate from the definition. The general case follows on applying Lemma 7.144.3. ☐

Lemma 7.144.19. *Let R be a Nagata ring. If $R \to S$ is a quasi-finite ring map (for example finite) then S is a Nagata ring also.*

Proof. First note that S is Noetherian as R is Noetherian and a quasi-finite ring map is of finite type. Let $\mathfrak{q} \subset S$ be a prime ideal, and set $\mathfrak{p} = R \cap \mathfrak{q}$. Then $R/\mathfrak{p} \subset S/\mathfrak{q}$ is quasi-finite and hence we conclude that $S/\mathfrak{q}$ is N-2 by Lemma 7.144.5 as desired. ☐

Lemma 7.144.20. *A localization of a Nagata ring is a Nagata ring.*

Proof. Clear from Lemma 7.144.3. ☐

Lemma 7.144.21. *Let R be a ring. Let $f_1, \ldots, f_n \in R$ generate the unit ideal.*

(1) *If each R_{f_i} is universally Japanese then so is R.*
(2) *If each R_{f_i} is Nagata then so is R.*

Proof. Let $\varphi : R \to S$ be a finite type ring map so that S is a domain. Then $\varphi(f_1), \ldots, \varphi(f_n)$ generate the unit ideal in S. Hence if each $S_{f_i} = S_{\varphi(f_i)}$ is N-1 then so is S, see Lemma 7.144.4. This proves (1).

If each R_{f_i} is Nagata, then each R_{f_i} is Noetherian and hence R is Noetherian, see Lemma 7.21.2. And if $\mathfrak{p} \subset R$ is a prime, then we see each $R_{f_i}/\mathfrak{p}R_{f_i} = (R/\mathfrak{p})_{f_i}$ is Japanese and hence we conclude $R/\mathfrak{p}$ is Japanese by Lemma 7.144.4. This proves (2). □

Lemma 7.144.22. *A Noetherian complete local ring is a Nagata ring.*

Proof. Let R be a complete local Noetherian ring. Let $\mathfrak{p} \subset R$ be a prime. Then $R/\mathfrak{p}$ is also a complete local Noetherian ring, see Lemma 7.143.2. Hence it suffices to show that a Noetherian complete local domain R is N-2. By Lemmas 7.144.5 and 7.143.10 we reduce to the case $R = k[[X_1, \ldots, X_d]]$ where k is a field or $R = \Lambda[[X_1, \ldots, X_d]]$ where Λ is a Cohen ring.

In the case $k[[X_1, \ldots, X_d]]$ we reduce to the statement that a field is N-2 by Lemma 7.144.14. This is clear. In the case $\Lambda[[X_1, \ldots, X_d]]$ we reduce to the statement that a Cohen ring Λ is N-2. Applying Lemma 7.144.13 once more with $x = p \in \Lambda$ we reduce yet again to the case of a field. Thus we win. □

Definition 7.144.23. Let $(R, \mathfrak{m})$ be a Noetherian local ring. We say R is *analytically unramified* if its completion $R^\wedge = lim_n R/\mathfrak{m}^n$ is reduced. A prime ideal $\mathfrak{p} \subset R$ is said to be *analytically unramified* if $R/\mathfrak{p}$ is analytically unramified.

At this point we know the following are true for any Noetherian local ring R: The map $R \to R^\wedge$ is a faithfully flat local ring homomorphism (Lemma 7.90.4). The completion $R^\wedge$ is Noetherian (Lemma 7.90.9) and complete (Lemma 7.90.8). Hence the completion $R^\wedge$ is a Nagata ring (Lemma 7.144.22). Moreover, we have seen in Section 7.143 that $R^\wedge$ is a quotient of a regular local ring (Theorem 7.143.8), and hence universally catenary (Remark 7.143.9).

Lemma 7.144.24. *Let $(R, \mathfrak{m})$ be a Noetherian local ring.*

(1) *If R is analytically unramified, then R is reduced.*
(2) *If R is analytically unramified, then each minimal prime of R is analytically unramified.*
(3) *If R is reduced with minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$, and each $\mathfrak{q}_i$ is analytically unramified, then R is analytically unramified.*
(4) *If R is analytically unramified, then the integral closure of R in its total ring of fractions $Q(R)$ is finite over R.*
(5) *If R is a domain and analytically unramified, then R is N-1.*

Proof. In this proof we will use the remarks immediately following Definition 7.144.23. As $R \to R^\wedge$ is a faithfully flat local ring homomorphism it is injective and (1) follows.

Let $\mathfrak{q}$ be a minimal prime of R, and assume R is analytically unramified. Then $\mathfrak{q}$ is an associated prime of R (see Proposition 7.60.6). Hence there exists an $f \in R$ such that $\{x \in R \mid fx = 0\} = \mathfrak{q}$. Note that $(R/\mathfrak{q})^\wedge = R^\wedge/\mathfrak{q}^\wedge$, and that $\{x \in R^\wedge \mid fx = 0\} = \mathfrak{q}^\wedge$, because completion is exact (Lemma 7.90.3). If $x \in R^\wedge$ is such that $x^2 \in \mathfrak{q}^\wedge$, then $fx^2 = 0$ hence $(fx)^2 = 0$ hence $fx = 0$ hence $x \in \mathfrak{q}^\wedge$. Thus $\mathfrak{q}$ is analytically unramified and (2) holds.

Assume R is reduced with minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$, and each $\mathfrak{q}_i$ is analytically unramified. Then $R \to R/\mathfrak{q}_1 \times \ldots \times R/\mathfrak{q}_t$ is injective. Since completion is exact (see Lemma 7.90.3) we see that $R^\wedge \subset (R/\mathfrak{q}_1)^\wedge \times \ldots \times (R/\mathfrak{q}_t)^\wedge$. Hence (3) is clear.

Assume R is analytically unramified. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ be the minimal primes of $R^\wedge$. Then we see that

$$Q(R^\wedge) = R_{\mathfrak{p}_1} \times \ldots \times R_{\mathfrak{p}_s}$$

with each $R_{\mathfrak{p}_i}$ a field as $R^\wedge$ is reduced (see Lemma 7.22.2). Hence the integral closure S of $R^\wedge$ in $Q(R^\wedge)$ is equal to $S = S_1 \times \ldots \times S_s$ with S_i the integral closure of $R/\mathfrak{p}_i$ in its fraction field. In particular S is finite over $R^\wedge$. Denote R' the integral closure of R in $Q(R)$. As $R \to R^\wedge$ is flat we see that $R' \otimes_R R^\wedge \subset Q(R) \otimes_R R^\wedge \subset Q(R^\wedge)$. Moreover $R' \otimes_R R^\wedge$ is integral over $R^\wedge$ (Lemma 7.32.11). Hence $R' \otimes_R R^\wedge \subset S$ is a $R^\wedge$-submodule. As $R^\wedge$ is Noetherian it is a finite $R^\wedge$-module. Thus we may find $f_1, \ldots, f_n \in R'$ such that $R' \otimes_R R^\wedge$ is generated by the elements $f_i \otimes 1$ as a $R^\wedge$-module. By faithful flatness we see that R' is generated by $f_1, \ldots, f_n$ as an R-module. This proves (4).

Part (5) is a special case of part (4). □

Lemma 7.144.25. *Let R be a Noetherian local ring. Let $\mathfrak{p} \subset R$ be a prime. Assume*

1. *$R_\mathfrak{p}$ is a discrete valuation ring, and*
2. *$\mathfrak{p}$ is analytically unramified.*

Then for any associated prime $\mathfrak{q}$ of $R^\wedge/\mathfrak{p}R^\wedge$ the local ring $(R^\wedge)_\mathfrak{q}$ is a discrete valuation ring.

Proof. Assumption (2) says that $R^\wedge/\mathfrak{p}R^\wedge$ is a reduced ring. Hence an associated prime $\mathfrak{q} \subset R^\wedge$ of $R^\wedge/\mathfrak{p}R^\wedge$ is the same thing as a minimal prime over $\mathfrak{p}R^\wedge$. In particular we see that the maximal ideal of $(R^\wedge)_\mathfrak{q}$ is $\mathfrak{p}(R^\wedge)_\mathfrak{q}$. Choose $x \in R$ such that $xR_\mathfrak{p} = \mathfrak{p}R_\mathfrak{p}$. By the above we see that $x \in (R^\wedge)_\mathfrak{q}$ generates the maximal ideal. As $R \to R^\wedge$ is faithfully flat we see that x is a nonzero divisor in $(R^\wedge)_\mathfrak{q}$. Hence we win. □

Lemma 7.144.26. *Let $(R, \mathfrak{m})$ be a Noetherian local domain. Let $x \in \mathfrak{m}$. Assume*

1. *$x \neq 0$,*
2. *R/xR has no embedded primes, and*
3. *for each associated prime $\mathfrak{p} \subset R$ of R/xR we have*
   (a) *the local ring $R_\mathfrak{p}$ is regular, and*
   (b) *$\mathfrak{p}$ is analytically unramified.*

Then R is analytically unramified.

Proof. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ be the associated primes of the R-module R/xR. Since R/xR has no embedded primes we see that each $\mathfrak{p}_i$ has height 1, and is a minimal prime over (x). For each i, let $\mathfrak{q}_{i1}, \ldots, \mathfrak{q}_{is_i}$ be the associated primes of the $R^\wedge$-module $R^\wedge/\mathfrak{p}_iR^\wedge$. By Lemma 7.144.25. we see that $(R^\wedge)_{\mathfrak{q}_{ij}}$ is regular. By Lemma 7.62.3 we see that

$$\text{Ass}_{R^\wedge}(R^\wedge/xR^\wedge) = \bigcup\nolimits_{\mathfrak{p} \in \text{Ass}_R(R/xR)} \text{Ass}_{R^\wedge}(R^\wedge/\mathfrak{p}R^\wedge) = \{\mathfrak{q}_{ij}\}.$$

Let $y \in R^\wedge$ with $y^2 = 0$. As $(R^\wedge)_{\mathfrak{q}_{ij}}$ is regular, and hence a domain (Lemma 7.98.2) we see that y maps to zero in $(R^\wedge)_{\mathfrak{q}_{ij}}$. Hence y maps to zero in $R^\wedge/xR^\wedge$ by Lemma 7.60.18. Hence $y = xy'$. Since x is a nonzero divisor (as $R \to R^\wedge$ is flat) we see that $(y')^2 = 0$. Hence we conclude that $y \in \bigcap x^nR^\wedge = (0)$ (Lemma 7.47.6). □

Lemma 7.144.27. *Let $(R, \mathfrak{m})$ be a local ring. If R is Noetherian, a domain, and Nagata, then R is analytically unramified.*

Proof. By induction on $\dim(R)$. The case $\dim(R) = 0$ is trivial. Hence we assume $\dim(R) = d$ and that the lemma holds for all Noetherian Nagata domains of dimension $< d$.

Let $R \subset S$ be the integral closure of R in the field of fractions of R. By assumption S is a finite R-module. By Lemma 7.144.19 we see that S is Nagata. By Lemma 7.103.4 we see $\dim(R) = \dim(S)$. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_t$ be the maximal ideals of S. Each of these lies over the maximal ideal $\mathfrak{m}$ of R. Moreover

$$(\mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_t)^n \subset \mathfrak{m}S$$

for sufficiently large n as $S/\mathfrak{m}S$ is Artinian. By Lemma 7.90.3 $R^\wedge \to S^\wedge$ is an injective map, and by the Chinese Remainder Lemma 7.14.4 combined with Lemma 7.90.12 we have $S^\wedge = \prod S_i^\wedge$ where $S_i^\wedge$ is the completion of S with respect to the maximal ideal $\mathfrak{m}_i$. Hence it suffices to show that $S_{\mathfrak{m}_i}$ is analytically unramified. In other words, we have reduced to the case where R is a Noetherian normal Nagata domain.

Assume R is a Noetherian, normal, local Nagata domain. Pick a nonzero $x \in \mathfrak{m}$ in the maximal ideal. We are going to apply Lemma 7.144.26. We have to check properties (1), (2), (3)(a) and (3)(b). Property (1) is clear. We have that R/xR has no embedded primes by Lemma 7.140.6. Thus property (2) holds. The same lemma also tells us each associated prime $\mathfrak{p}$ of R/xR has height 1. Hence $R_\mathfrak{p}$ is a 1-dimensional normal domain hence regular (Lemma 7.110.6). Thus (3)(a) holds. Finally (3)(b) holds by induction hypothesis, since $R/\mathfrak{p}$ is Nagata (by Lemma 7.144.19 or directly from the definition). Thus we conclude R is analytically unramified. □

Lemma 7.144.28. *Let R be a Noetherian domain. If there exists an $f \in R$ such that R_f is normal then*

$$U = \{\mathfrak{p} \in Spec(R) \mid R_\mathfrak{p} \text{ is normal}\}$$

is open in $Spec(R)$.

Proof. It is clear that the standard open $D(f)$ is contained in U. By Serre's criterion Lemma 7.140.4 we see that $\mathfrak{p} \notin U$ implies that for some $\mathfrak{q} \subset \mathfrak{p}$ we have either

(1) Case I: $\text{depth}(R_\mathfrak{q}) < 2$ and $\dim(R_\mathfrak{q}) \geq 2$, and
(2) Case II: $R_\mathfrak{q}$ is not regular and $\dim(R_\mathfrak{q}) = 1$.

This in particular also means that $R_\mathfrak{q}$ is not normal, and hence $f \in \mathfrak{q}$. In case I we see that $\text{depth}(R_\mathfrak{q}) = \text{depth}(R_\mathfrak{q}/fR_\mathfrak{q}) + 1$. Hence such a prime $\mathfrak{q}$ is the same thing as an embedded associated prime of R/fR. In case II $\mathfrak{q}$ is an associated prime of R/fR of height 1. Thus there is a finite set E of such primes $\mathfrak{q}$ (see Lemma 7.60.5) and

$$Spec(R) \setminus U = \bigcup\nolimits_{\mathfrak{q} \in E} V(\mathfrak{q})$$

as desired. □

Lemma 7.144.29. *Let R be a Noetherian domain. Assume*

(1) *there exists a nonzero $f \in R$ such that R_f is normal, and*
(2) *for every maximal ideal $\mathfrak{m} \subset R$ the local ring $R_\mathfrak{m}$ is N-1.*

Then R is N-1.

Proof. Set $K = f.f.(R)$. Suppose that $R \subset R' \subset K$ is a finite extension of R contained in K. Note that $R_f = R'_f$ since R_f is already normal. Hence by Lemma 7.144.28 the set of primes $\mathfrak{p}' \in Spec(R')$ with $R'_{\mathfrak{p}'}$ non-normal is closed in $Spec(R')$. Since $Spec(R') \to$

$Spec(R)$ is closed the image of this set is closed in $Spec(R)$. For such a ring R' denote $Z_{R'} \subset Spec(R)$ this image.

Pick a maximal ideal $\mathfrak{m} \subset R$. Let $R_\mathfrak{m} \subset R'_\mathfrak{m}$ be the integral closure of the local ring in K. By assumption this is a finite ring extension. By Lemma 7.32.9 we can find finitely many elements $r_1, \ldots, r_n \in K$ integral over R such that $R'_\mathfrak{m}$ is generated by $r_1, \ldots, r_n$ over $R_\mathfrak{m}$. Let $R' = R[x_1, \ldots, x_n] \subset K$. With this choice it is clear that $\mathfrak{m} \notin Z_{R'}$.

As $Spec(R)$ is quasi-compact, the above shows that we can find a finite collection $R \subset R'_i \subset K$ such that $\bigcap Z_{R'_i} = \emptyset$. Let R' be the subring of K generated by all of these. It is finite over R. Also $Z_{R'} = \emptyset$. Namely, every prime $\mathfrak{p}'$ lies over a prime $\mathfrak{p}'_i$ such that $(R'_i)_{\mathfrak{p}'_i}$ is normal. This implies that $R'_{\mathfrak{p}'} = (R'_i)_{\mathfrak{p}'_i}$ is normal too. Hence R' is normal, in other words R' is the integral closure of R in K. □

The following proposition says in particular that an algebra of finite type over a Nagata ring is a Nagata ring.

Proposition 7.144.30 (Nagata). *Let R be a ring. The following are equivalent:*

(1) *R is a Nagata ring,*
(2) *any finite type R-algebra is Nagata, and*
(3) *R is universally Japanese and Noetherian.*

Proof. It is clear that a Noetherian universally Japanese ring is universally Nagata (i.e., condition (2) holds). Let R be a Nagata ring. We will show that any finitely generated R-algebra S is Nagata. This will prove the proposition.

Step 1. There exists a sequence of ring maps $R = R_0 \to R_1 \to R_2 \to \ldots \to R_n = S$ such that each $R_i \to R_{i+1}$ is generated by a single element. Hence by induction it suffices to prove S is Nagata if $S \cong R[x]/I$.

Step 2. Let $\mathfrak{q} \subset S$ be a prime of S, and let $\mathfrak{p} \subset R$ be the corresponding prime of R. We have to show that $S/\mathfrak{q}$ is N-2. Hence we have reduced to the proving the following: (*) Given a Nagata domain R and a monogenic extension $R \subset S$ of domains then S is N-2.

Step 3. Let R be a Nagata domain and $R \subset S$ a monogenic extension of domains. Let $R \subset R'$ be the integral closure of R in its fraction field. Let S' be the subring of $f.f.(S)$ generated by R' and S. As R' is finite over R (by the Nagata property) also S' is finite over S. Since S is Noetherian it suffices to prove that S' is N-2 (Lemma 7.144.7). Hence we have reduced to proving the following: (**) Given a normal Nagata domain R and a monogenic extension $R \subset S$ of domains then S is N-2.

Step 4: Let R be a normal Nagata domain and let $R \subset S$ be a monogenic extension of domains. Suppose the extension of fraction fields $f.f.(R) \subset f.f.(S)$ is purely transcendental. In this case $S = R[x]$. By Lemma 7.144.12 we see that S is N-2. Hence we have reduced to proving the following: (**) Given a normal Nagata domain R and a monogenic extension $R \subset S$ of domains inducing a finite extension of fraction fields then S is N-2.

Step 5. Let R be a normal Nagata domain and let $R \subset S$ be a monogenic extension of domains inducing a finite extension of fraction fields $K = f.f.(R) \subset f.f.(S) = L$. Choose an element $x \in S$ which generates S as an R-algebra. Let $L \subset M$ be a finite extension of fields. Let R' be the integral closure of R in M. Then the integral closure S' of S in M is equal to the integral closure of $R'[x]$ in M. Also $f.f.(R') = M$, and $R \subset R'$ is finite (by the Nagata property of R). This implies that R' is a Nagata ring (Lemma 7.144.19). To show that S' is finite over S is the same as showing that S' is finite over $R'[x]$. Replace

R by R' and S by S' to reduce to the following statement: (***) Given a normal Nagata domain R with fraction field K, and $x \in K$, the ring $S \subset K$ generated by R and x is N-1.

Step 6. Let R be a normal Nagata domain with fraction field K. Let $x = b/a \in K$. We have to show that the ring $S \subset K$ generated by R and x is N-1. Note that $S_a \cong R_a$ is normal. Hence by Lemma 7.144.29 it suffices to show that $S_{\mathfrak m}$ is N-1 for every maximal ideal $\mathfrak m$ of S.

With assumptions as in the preceding paragraph, pick such a maximal ideal and set $\mathfrak n = R \cap \mathfrak m$. The residue field extension $\kappa(\mathfrak n) \subset \kappa(\mathfrak m)$ is finite (Theorem 7.30.1) and generated by the image of x. Hence there exists a monic polynomial $f(X) = X^d + \sum_{i=1,\ldots,d} a_i X^{d-i}$ with $f(x) \in \mathfrak m$. Let $K \subset K''$ be a finite extension of fields such that $f(X)$ splits completely in $K''[X]$. Let R' be the integral closure of R in K''. Let $S' \subset K'$ be the subring generated by R' and x. As R is Nagata we see R' is finite over R and Nagata (Lemma 7.144.19). Moreover, S' is finite over S. If for every maximal ideal $\mathfrak m'$ of S' the local ring $S'_{\mathfrak m'}$ is N-1, then $S'_{\mathfrak m}$ is N-1 by Lemma 7.144.29, which in turn implies that $S_{\mathfrak m}$ is N-1 by Lemma 7.144.7. After replacing R by R' and S by S', and $\mathfrak m$ by any of the maximal ideals $\mathfrak m'$ lying over $\mathfrak m$ we reach the situation where the polynomial f above split completely: $f(X) = \prod_{i=1,\ldots,d}(X - a_i)$ with $a_i \in R$. Since $f(x) \in \mathfrak m$ we see that $x - a_i \in \mathfrak m$ for some i. Finally, after replacing x by $x - a_i$ we may assume that $x \in \mathfrak m$.

To recapitulate: R is a normal Nagata domain with fraction field K, $x \in K$ and S is the subring of K generated by x and R, finally $\mathfrak m \subset S$ is a maximal ideal with $x \in \mathfrak m$. We have to show $S_{\mathfrak m}$ is N-1.

We will show that Lemma 7.144.26 applies to the local ring $S_{\mathfrak m}$ and the element x. This will imply that $S_{\mathfrak m}$ is analytically unramified, whereupon we see that it is N-1 by Lemma 7.144.24.

We have to check properties (1), (2), (3)(a) and (3)(b). Property (1) is trivial. Let $I = \mathrm{Ker}(R[X] \to S)$ where $X \mapsto x$. We claim that I is generated by all linear forms $aX + b$ such that $ax = b$ in K. Clearly all these linear forms are in I. If $g = a_d X^d + \ldots a_1 X + a_0 \in I$, then we see that $a_d x$ is integral over R (Lemma 7.114.1) and hence $b := a_d x \in R$ as R is normal. Then $g - (a_d X - b)X^{d-1} \in I$ and we win by induction on the degree. As a consequence we see that

$$S/xS = R[X]/(X, I) = R/J$$

where

$$J = \{b \in R \mid ax = b \text{ for some } a \in R\} = xR \cap R$$

By Lemma 7.140.6 we see that $S/xS = R/J$ has no embedded primes as an R-module, hence as an R/J-module, hence as an S/xS-module, hence as an S-module. This proves property (2). Take such an associated prime $\mathfrak q \subset S$ with the property $\mathfrak q \subset \mathfrak m$ (so that it is an associated prime of $S_{\mathfrak m}/xS_{\mathfrak m}$ -- it does not matter for the arguments). Then $\mathfrak q$ is minimal over xS and hence has height 1. By the sequence of equalities above we see that $\mathfrak p = R \cap \mathfrak q$ is an associated prime of R/J, and so has height 1 (see Lemma 7.140.6). Thus $R_{\mathfrak p}$ is a discrete valuation ring and therefore $R_{\mathfrak p} \subset S_{\mathfrak q}$ is an equality. This shows that $S_{\mathfrak q}$ is regular. This proves property (3)(a). Finally, $(S/\mathfrak q)_{\mathfrak m}$ is a localization of $S/\mathfrak q$, which is a quotient of $S/xS = R/J$. Hence $(S/\mathfrak q)_{\mathfrak m}$ is a localization of a quotient of the Nagata ring R, hence Nagata (Lemmas 7.144.19 and 7.144.20) and hence analytically unramified (Lemma 7.144.27). This shows (3)(b) holds and we are done. □

Proposition 7.144.31. *The following types of rings are Nagata and in particular universally Japanese:*

(1) *fields,*
(2) *Noetherian complete local rings,*
(3) $\mathbf{Z}$*,*
(4) *Dedekind domains with fraction field of characteristic zero,*
(5) *finite type ring extensions of any of the above.*

Proof. The Noetherian complete local ring case is Lemma 7.144.22. In the other cases you just check if $R/\mathfrak{p}$ is N-2 for every prime ideal $\mathfrak{p}$ of the ring. This is clear whenever $R/\mathfrak{p}$ is a field, i.e., $\mathfrak{p}$ is maximal. Hence for the Dedeking ring case we only need to check it when $\mathfrak{p} = (0)$. But since we assume the fraction field has characteristic zero Lemma 7.144.10 kicks in. $\square$

7.145. Ascending properties

In this section we start proving some algebraic facts concerning the ``ascent'' of properties of rings.

Lemma 7.145.1. *Suppose that $R \to S$ is a flat and local ring homomorphism of Noetherian local rings. Then*
$$depth_S(S) = depth_R(R) + depth_S(S/\mathfrak{m}_R S).$$

Proof. Denote n the right hand side of the equality of the lemma. First assume that n is zero. Then $\text{depth}(R) = 0$ (resp. $\text{depth}(S/\mathfrak{m}_R S) = 0$) which means there is a $z \in R$ (resp. $\overline{y} \in S/\mathfrak{m}_R S$) whose annihilator is $\mathfrak{m}_R$ (resp. $\mathfrak{m}_S/\mathfrak{m}_R$). Let $y \in S$ be a lift of $\overline{y}$. It follows from the fact that $R \to S$ is flat that the annihilator of z in S is $\mathfrak{m}_R S$, and hence the annihilator of zy is $\mathfrak{m}_S$. Thus $\text{depth}_S(S) = 0$ as well.

Assume $n > 0$. If $\text{depth}(S/\mathfrak{m}_R S) > 0$, then choose an $f \in \mathfrak{m}_S$ which maps to a nonzero divisor in $S/\mathfrak{m}_R S$. According to Lemma 7.91.2 the element $f \in S$ is a nonzero divisor and S/fS is flat over R. Hence by induction on n we have
$$\text{depth}(S/fS) = \text{depth}(R) + \text{depth}(S/(f, \mathfrak{m}_R)).$$
Since f and $\overline{f}$ are nonzero divisors we see that $\text{depth}(S) = \text{depth}(S/fS)+1$ and $\text{depth}(S/\mathfrak{m}_R S) = \text{depth}(S/(f, \mathfrak{m}_R))+1$, see Lemma 7.67.10. Hence we see that the equality holds for $R \to S$ as well.

If $n > 0$, but $\text{depth}(S/\mathfrak{m}_R S) = 0$, then choose $f \in \mathfrak{m}_R$ a nonzero divisor. As $R \to S$ is flat it is also the case that f is a nonzero divisor on S. By induction on n again we have
$$\text{depth}(S/fS) = \text{depth}(R/fR) + \text{depth}(S/\mathfrak{m}_R).$$
By a similar argument as above we conclude that the equality holds for $R \to S$ as well. $\square$

Here is a more general statement involving modules, see [**DG67**, IV, Proposition 6.3.1].

Lemma 7.145.2. *We have*
$$depth_S(M \otimes_R N) = depth_R(M) + depth_{S/\mathfrak{m}_R S}(N/\mathfrak{m}_R N)$$
where $R \to S$ is a local homomorphism of local Noetherian rings, M is a finite R-module, and N is a finite S-module flat over R.

Proof. Omitted, but similar to the proof of Lemma 7.145.1. $\square$

Lemma 7.145.3. *Let $R \to S$ be a local homomorphism of local Noetherian rings. Assume*
(1) *$S/\mathfrak{m}_R S$ is Cohen-Macaulay, and*
(2) *$R \to S$ is flat.*

Then S is Cohen-Macaulay if and only if R is Cohen-Macaulay.

Proof. This follows from the definitions combined with Lemmas 7.145.1 and 7.103.7. □

Lemma 7.145.4. *Let $\varphi : R \to S$ be a ring map. Assume*

1. *R is Noetherian,*
2. *S is Noetherian,*
3. *φ is flat,*
4. *the fibre rings $S \otimes_R \kappa(\mathfrak{p})$ are Cohen-Macaulay, and*
5. *R has property (S_k).*

Then S has property (S_k).

Proof. Let $\mathfrak{q}$ be a prime of S lying over a prime $\mathfrak{p}$ of R. By Lemma 7.145.1 we have

$$\text{depth}(S_\mathfrak{q}) = \text{depth}(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) + \text{depth}(R_\mathfrak{p}).$$

On the other hand, we have

$$\dim(S_\mathfrak{q}) \leq \dim(R_\mathfrak{p}) + \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}).$$

by Lemma 7.103.6. (Actually equality holds, by Lemma 7.103.7 but strictly speaking we do not need this.) Finally, as the fibre rings of the map are assumed Cohen-Macaulay we see that $\text{depth}(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) = \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q})$. Thus the lemma follows by the following string of inequalities

$$\begin{array}{rcl} \text{depth}(S_\mathfrak{q}) & = & \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) + \text{depth}(R_\mathfrak{p}) \\ & \geq & \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) + \min(k, \dim(R_\mathfrak{p})) \\ & = & \min(\dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) + k, \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) + \dim(R_\mathfrak{p})) \\ & \geq & \min(k, \dim(S_\mathfrak{q})) \end{array}$$

as desired. □

Lemma 7.145.5. *Let $\varphi : R \to S$ be a ring map. Assume*

1. *R is Noetherian,*
2. *S is Noetherian*
3. *φ is flat,*
4. *the fibre rings $S \otimes_R \kappa(\mathfrak{p})$ are regular, and*
5. *R has property (R_k).*

Then S has property (R_k).

Proof. Let $\mathfrak{q}$ be a prime of S lying over a prime $\mathfrak{p}$ of R. Assume that $\dim(S_\mathfrak{q}) \leq k$. Since $\dim(S_\mathfrak{q}) = \dim(R_\mathfrak{p}) + \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q})$ by Lemma 7.103.7 we see that $\dim(R_\mathfrak{p}) \leq k$. Hence $R_\mathfrak{p}$ is regular by assumption. It follows that $S_\mathfrak{q}$ is regular by Lemma 7.103.8. □

Lemma 7.145.6. *Let $\varphi : R \to S$ be a ring map. Assume*

1. *φ is smooth,*
2. *R is reduced.*

Then S is reduced.

Proof. First assume R is Noetherian. In this case being reduced is the same as having properties (S_1) and (R_0), see Lemma 7.140.3. Note that S is noetherian, and $R \to S$ is flat with regular fibres (see the list of results on smooth ring maps in Section 7.131). Hence we may apply Lemmas 7.145.4 and 7.145.5 and we see that S is (S_1) and (R_0), in other words reduced by Lemma 7.140.3 again.

In the general case we may find a finitely generated $\mathbf{Z}$-subalgebra $R_0 \subset R$ and a smooth ring map $R_0 \to S_0$ such that $S \cong R \otimes_{R_0} S_0$, see remark (10) in Section 7.131. Now, if $x \in S$ is an element with $x^2 = 0$, then we can enlarge R_0 and assume that x comes from an element $x_0 \in S_0$. After enlarging R_0 once more we may assume that $x_0^2 = 0$ in S_0. However, since $R_0 \subset R$ is reduced we see that S_0 is reduced and hence $x_0 = 0$ as desired. $\square$

Lemma 7.145.7. *Let $\varphi : R \to S$ be a ring map. Assume*

(1) *φ is smooth,*
(2) *R is normal.*

Then S is normal.

Proof. First assume R is Noetherian. In this case being reduced is the same as having properties (S_2) and (R_1), see Lemma 7.140.4. Note that S is noetherian, and $R \to S$ is flat with regular fibres (see the list of results on smooth ring maps in Section 7.131). Hence we may apply Lemmas 7.145.4 and 7.145.5 and we see that S is (S_2) and (R_1), in other words reduced by Lemma 7.140.4 again.

The general case. First note that R is reduced and hence S is reduced by Lemma 7.145.6. Let $\mathfrak{q}$ be a prime of S and let $\mathfrak{p}$ be the corresponding prime of R. Note that $R_{\mathfrak{p}}$ is a normal domain. We have to show that $S_{\mathfrak{q}}$ is a normal domain. To do this we may replace R by $R_{\mathfrak{p}}$ and S by $S_{\mathfrak{p}}$. Hence we may assume that R is a normal domain.

Assume $R \to S$ smooth, and R a normal domain. We may find a finitely generated $\mathbf{Z}$-subalgebra $R_0 \subset R$ and a smooth ring map $R_0 \to S_0$ such that $S \cong R \otimes_{R_0} S_0$, see remark (10) in Section 7.131. As R_0 is a Nagata domain (see Proposition 7.144.31) we see that its integral closure R_0' is finite over R_0. Moreover, as R is a normal domain it is clear that $R_0' \subset R$. Hence we may replace R_0 by R_0' and S_0 by $R_0' \otimes_{R_0} S_0$ and assume that R_0 is a normal Noetherian domain. By the first paragraph of the proof we conclude that S_0 is a normal ring (it need not be a domain of course). In this way we see that $R = \bigcup R_\lambda$ is the union of normal Noetherian domains and correspondingly $S = colim\, R_\lambda \otimes_{R_0} S_0$ is the colimit of normal rings. This implies that S is a normal ring. Some details omitted. $\square$

Lemma 7.145.8. *Let $\varphi : R \to S$ be a ring map. Assume*

(1) *φ is smooth,*
(2) *R is a regular ring.*

Then S is regular.

Proof. This follows from Lemma 7.145.5 applied for all (R_k) using Lemma 7.129.3 to see that the hypotheses are satisfied. $\square$

7.146. Descending properties

In this section we start proving some algebraic facts concerning the ``descent" of properties of rings. It turns out that it is often ``easier" to descend properties than it is to ascend them. In other words, the assumption on the ring map $R \to S$ are often weaker than the assumptions in the corresponding lemma of the preceding section. However, we warn the reader that the results on descent are often useless unless the corresponding ascent can also be shown! Here is a typical result which illustrates this phenomenon.

Lemma 7.146.1. *Let $R \to S$ be a ring map. Assume that*

(1) *$R \to S$ is faithfully flat, and*
(2) *S is Noetherian.*

Then R is Noetherian.

Proof. Let $I_0 \subset I_1 \subset I_2 \subset \ldots$ be a growing sequence of ideals of R. By assumption we have $I_n S = I_{n+1} S = I_{n+2} S = \ldots$ for some n. Since $R \to S$ is flat we have $I_k S = I_k \otimes_R S$. Hence, as $R \to S$ is faithfully flat we see that $I_n S = I_{n+1} S = I_{n+2} S = \ldots$ implies that $I_n = I_{n+1} = I_{n+2} = \ldots$ as desired. $\square$

Lemma 7.146.2. *Let $R \to S$ be a ring map. Assume that*

(1) *$R \to S$ is faithfully flat, and*
(2) *S is reduced.*

Then R is reduced.

Proof. This is clear as $R \to S$ is injective. $\square$

Lemma 7.146.3. *Let $R \to S$ be a ring map. Assume that*

(1) *$R \to S$ is faithfully flat, and*
(2) *S is a normal ring.*

Then R is a normal ring.

Proof. Since S is reduced it follows that R is reduced. Let $\mathfrak{p}$ be a prime of R. We have to show that $R_\mathfrak{p}$ is a normal domain. Since $S_\mathfrak{p}$ is faithfully over $R_\mathfrak{p}$ too we may assume that R is local with maximal ideal $\mathfrak{m}$. Let $\mathfrak{q}$ be a prime of S lying over $\mathfrak{m}$. Then we see that $R \to S_\mathfrak{q}$ is faithfully flat (Lemma 7.35.16). Hence we may assume S is local as well. In particular S is a normal domain. Since $R \to S$ is faithfully flat and S is a normal domain we see that R is a domain. Next, suppose that a/b is integral over R with $a, b \in R$. Then $a/b \in S$ as S is normal. Hence $a \in bS$. This means that $a : R \to R/bR$ becomes the zero map after base change to S. By faithful flatness we see that $a \in bR$, so $a/b \in R$. Hence R is normal. $\square$

Lemma 7.146.4. *Let $R \to S$ be a ring map. Assume that*

(1) *$R \to S$ is faithfully flat, and*
(2) *S is a regular ring.*

Then R is a regular ring.

Proof. We see that R is Noetherian by Lemma 7.146.1. Let $\mathfrak{p} \subset R$ be a prime. Choose a prime $\mathfrak{q} \subset S$ lying over $\mathfrak{p}$. Then Lemma 7.102.8 applies to $R_\mathfrak{p} \to S_\mathfrak{q}$ and we conclude that $R_\mathfrak{p}$ is regular. Since $\mathfrak{p}$ was arbitrary we see R is regular. $\square$

Lemma 7.146.5. *Let $R \to S$ be a ring map. Assume that*

(1) *$R \to S$ is faithfully flat of finite presentation, and*
(2) *S is Noetherian and has property (S_k).*

Then R is Noetherian and has property (S_k).

Proof. We have already seen that (1) and (2) imply that R is Noetherian, see Lemma 7.146.1. Let $\mathfrak{p} \subset R$ be a prime ideal. Choose a prime $\mathfrak{q} \subset S$ lying over $\mathfrak{p}$ which corresponds to a minimal prime of the fibre ring $S \otimes_R \kappa(\mathfrak{p})$. Then $A = R_\mathfrak{p} \to S_\mathfrak{q} = B$ is a flat local ring homomorphism of Noetherian local rings with $\mathfrak{m}_A B$ an ideal of definition of B. Hence $\dim(A) = \dim(B)$ (Lemma 7.103.7) and $\text{depth}(A) = \text{depth}(B)$ (Lemma 7.145.1). Hence since B has (S_k) we see that A has (S_k). $\square$

Lemma 7.146.6. *Let $R \to S$ be a ring map. Assume that*

(1) *$R \to S$ is faithfully flat and of finite presentation, and*

(2) *S is Noetherian and has property (R_k).*

Then R is Noetherian and has property (R_k).

Proof. We have already seen that (1) and (2) imply that R is Noetherian, see Lemma 7.146.1. Let $\mathfrak{p} \subset R$ be a prime ideal and assume $\dim(R_\mathfrak{p}) \leq k$. Choose a prime $\mathfrak{q} \subset S$ lying over $\mathfrak{p}$ which corresponds to a minimal prime of the fibre ring $S \otimes_R \kappa(\mathfrak{p})$. Then $A = R_\mathfrak{p} \to S_\mathfrak{q} = B$ is a flat local ring homomorphism of Noetherian local rings with $\mathfrak{m}_A B$ an ideal of definition of B. Hence $\dim(A) = \dim(B)$ (Lemma 7.103.7). As S has (R_k) we conclude that B is a regular local ring. By Lemma 7.102.8 we conclude that A is regular. □

Lemma 7.146.7. *Let $R \to S$ be a ring map. Assume that*

(1) *$R \to S$ is smooth and surjective on spectra, and*
(2) *S is a Nagata ring.*

Then R is a Nagata ring.

Proof. Recall that a Nagata ring is the same thing as a Noetherian universally Japanese ring (Proposition 7.144.30). We have already seen that R is Noetherian in Lemma 7.146.1. Let $R \to A$ be a finite type ring map into a domain. According to Lemma 7.144.17 it suffices to check that A is N-1. It is clear that $B = A \otimes_R S$ is a finite type S-algebra and hence Nagata (Proposition 7.144.30). Since $A \to B$ is smooth (Lemma 7.126.4) we see that B is reduced (Lemma 7.145.6). Since B is Noetherian it has only a finite number of minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ (see Lemma 7.28.6). As $A \to B$ is flat each of these lies over $(0) \subset A$ (by going down, see Lemma 7.35.17) The total ring of fractions $Q(B)$ is the product of the $L_i = f.f.(B/\mathfrak{q}_i)$ (Lemmas 7.22.2 and 7.23.3). Moreover, the integral closure B' of B in $Q(B)$ is the product of the integral closures B'_i of the $B/\mathfrak{q}_i$ in the factors L_i (compare with Lemma 7.33.14). Since B is universally Japanese the ring extensions $B/\mathfrak{q}_i \subset B'_i$ are finite and we conclude that $B' = \prod B'_i$ is finite over B. Since $A \to B$ is flat we see that any nonzero divisor on A maps to a nonzero divisor on B. The corresponding map

$$Q(A) \otimes_A B = (A \setminus \{0\})^{-1} A \otimes_A B = (A \setminus \{0\})^{-1} B \to Q(B)$$

is injective (we used Lemma 7.11.15). Via this map A' maps into B'. This induces a map

$$A' \otimes_A B \longrightarrow B'$$

which is injective (by the above and the flatness of $A \to B$). Since B' is a finite B-module and B is Noetherian we see that $A' \otimes_A B$ is a finite B-module. Hence there exist finitely many elements $x_i \in A'$ such that the elements $x_i \otimes 1$ generate $A' \otimes_A B$ as a B-module. Finally, by faithful flatness of $A \to B$ we conclude that the x_i also generated A' as an A-module, and we win. □

Remark 7.146.8. The property of being ``universally catenary" does not descend; not even along étale ring maps. In Examples, Section 64.9 there is a construction of a finite ring map $A \to B$ with A local Noetherian and not universally catenary, B semi-local with two maximal ideals $\mathfrak{m}$, $\mathfrak{n}$ with $B_\mathfrak{m}$ and $B_\mathfrak{n}$ regular of dimension 2 and 1 respectively, and the same residue fields as that of A. Moreover, $\mathfrak{m}_A$ generates the maximal ideal in both $B_\mathfrak{m}$ and $B_\mathfrak{n}$ (so $A \to B$ is unramified as well as finite). By Lemma 7.138.10 there exists a local étale ring map $A \to A'$ such that $B \otimes_A A' = B_1 \times B_2$ decomposes with $A' \to B_i$ surjective. This shows that A' has two minimal primes $\mathfrak{q}_i$ with $A'/\mathfrak{q}_i \cong B_i$. Since B_i is regular local (since it is étale over either $B_\mathfrak{m}$ or $B_\mathfrak{n}$) we conclude that A' is universally catenary.

7.147. Geometrically normal algebras

In this section we put some applications of ascent and descent of properties of rings.

Lemma 7.147.1. *Let k be a field. Let A be a k-algebra. The following properties of A are equivalent:*

(1) *$k' \otimes_k A$ is a normal ring for every field extension $k \subset k'$,*
(2) *$k' \otimes_k A$ is a normal ring for every finitely generated field extension $k \subset k'$, and*
(3) *$k' \otimes_k A$ is a normal ring for every finite purely inseparable extension $k \subset k'$.*

where normal ring is as defined in Definition 7.33.10.

Proof. It is clear that (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

Assume (2) and let $k \subset k'$ be any field extension. Then we can write $k' = colim_i\, k_i$ as a directed colimit of finitely generated field extensions. Hence we see that $k' \otimes_k A = colim_i\, k_i \otimes_k A$ is a directed colimit of normal rings. Thus we see that $k' \otimes_k A$ is a normal ring by Lemma 7.33.15. Hence (1) holds.

Assume (3) and let $k \subset K$ be a finitely generated field extension. By Lemma 7.42.3 we can find a diagram

$$\begin{tikzcd} K \arrow[r] & K' \\ k \arrow[u] \arrow[r] & k' \arrow[u] \end{tikzcd}$$

where $k \subset k'$, $K \subset K'$ are finite purely inseparable field extensions such that $k' \subset K'$ is separable. By Lemma 7.141.9 there exists a smooth k'-algebra B such that K' is the fraction field of B. Now we can argue as follows: Step 1: $k' \otimes_k A$ is a normal ring because we assumed (3). Step 2: $B \otimes_{k'} k' \otimes_k A$ is a normal ring as $k' \otimes_k A \to B \otimes_{k'} k' \otimes_k A$ is smooth (Lemma 7.126.4) and ascent of normality along smooth maps (Lemma 7.145.7). Step 3. $K' \otimes_{k'} k' \otimes_k A = K' \otimes_k A$ is a normal ring as it is a localization of a normal ring (Lemma 7.33.12). Step 4. Finally $K \otimes_k A$ is a normal ring by descent of normality along the faithfully flat ring map $K \otimes_k A \to K' \otimes_k A$ (Lemma 7.146.3). This proves the lemma. $\square$

Definition 7.147.2. Let k be a field. A k-algebra R is called *geometrically normal* over k if the equivalent conditions of Lemma 7.147.1 hold.

Lemma 7.147.3. *Let k be a field. A localization of a geometrically normal k-algebra is geometrically normal.*

Proof. This is clear as being a normal ring is checked at the localizations at prime ideals. $\square$

Lemma 7.147.4. *Let k be a field. Let A, B be k-algebras. Assume A is geometrically normal over k and B is a normal ring. Then $A \otimes_k B$ is a normal ring.*

Proof. Let $\mathfrak{r}$ be a prime ideal of $A \otimes_k B$. Denote $\mathfrak{p}$, resp. $\mathfrak{q}$ the corresponding prime of A, resp. B. Then $(A \otimes_k B)_{\mathfrak{r}}$ is a localization of $A_{\mathfrak{p}} \otimes_k B_{\mathfrak{q}}$. Hence it suffices to prove the result for the ring $A_{\mathfrak{p}} \otimes_k B_{\mathfrak{q}}$, see Lemma 7.33.12 and Lemma 7.147.3. Thus we may assume A and B are domains.

Assume that A and B are domains with fractions fields K and L. Note that B is the filtered colimit of its finite type normal k-sub algebras (as k is a Nagata ring, see Proposition 7.144.31, and hence the integral closure of a finite type k-sub algebra is still a finite type

k-sub algebra by Proposition 7.144.30). By Lemma 7.33.15 we reduce to the case that B is of finite type over k.

Assume that A and B are domains with fractions fields K and L and B of finite type over k. In this case the ring $K \otimes_k B$ is of finite type over K, hence Noetherian (Lemma 7.28.1). In particular $K \otimes_k B$ has finitely many minimal primes (Lemma 7.28.6). Since $A \to A \otimes_k B$ is flat, this implies that $A \otimes_k B$ has finitely many minimal primes (by going down for flat ring maps -- Lemma 7.35.17 -- these primes all lie over $(0) \subset A$). Thus it suffices to prove that $A \otimes_k B$ is integrally closed in its total ring of fractions (Lemma 7.33.14).

We claim that $K \otimes_k B$ and $A \otimes_k L$ are both normal rings. If this is true then any element x of $Q(A \otimes_k B)$ which is integral over $A \otimes_k B$ is (by Lemma 7.33.11) contained in $K \otimes_k B \cap A \otimes_k L = A \otimes_k B$ and we're done. Since $A \otimes_K L$ is a normal ring by assumption, it suffices to prove that $K \otimes_k B$ is normal.

As A is geometrically normal over k we see K is geometrically normal over k (Lemma 7.147.3) hence K is geometrically reduced over k. Hence $K = \bigcup K_i$ is the union of finitely generated field extensions of k which are geometrically reduced (Lemma 7.40.2). Each K_i is the localization of a smooth k-algebra (Lemma 7.141.9). So $K_i \otimes_k B$ is the localization of a smooth B-algebra hence normal (Lemma 7.145.7). Thus $K \otimes_k B$ is a normal ring (Lemma 7.33.15) and we win. □

7.148. Geometrically regular algebras

Let k be a field. Let A be a Noetherian k-algebra. Let $k \subset K$ be a finitely generated field extension. Then the ring $K \otimes_k A$ is Noetherian as well, see Lemma 7.28.7. Thus the following lemma makes sense.

Lemma 7.148.1. *Let k be a field. Let A be a k-algebra. Assume A is Noetherian. The following properties of A are equivalent:*

(1) *$k' \otimes_k A$ is a regular ring for every finitely generated field extension $k \subset k'$, and*
(2) *$k' \otimes_k A$ is a regular ring for every finite purely inseparable extension $k \subset k'$.*

where regular ring is as defined in Definition 7.102.6.

Proof. The lemma makes sense by the remarks preceding the lemma. It is clear that (1) $\Rightarrow$ (2).

Assume (2) and let $k \subset K$ be a finitely generated field extension. By Lemma 7.42.3 we can find a diagram

$$\begin{tikzcd} K \arrow[r] & K' \\ k \arrow[u] \arrow[r] & k' \arrow[u] \end{tikzcd}$$

where $k \subset k'$, $K \subset K'$ are finite purely inseparable field extensions such that $k' \subset K'$ is separable. By Lemma 7.141.9 there exists a smooth k'-algebra B such that K' is the fraction field of B. Now we can argue as follows: Step 1: $k' \otimes_k A$ is a regular ring because we assumed (2). Step 2: $B \otimes_{k'} k' \otimes_k A$ is a regular ring as $k' \otimes_k A \to B \otimes_{k'} k' \otimes_k A$ is smooth (Lemma 7.126.4) and ascent of regularity along smooth maps (Lemma 7.145.8). Step 3. $K' \otimes_{k'} k' \otimes_k A = K' \otimes_k A$ is a regular ring as it is a localization of a regular ring (immediate from the definition). Step 4. Finally $K \otimes_k A$ is a regular ring by descent of regularity along the faithfully flat ring map $K \otimes_k A \to K' \otimes_k A$ (Lemma 7.146.4). This proves the lemma. □

Definition 7.148.2. Let k be a field. Let R be a Noetherian k-algebra. The k-algebra R is called *geometrically regular* over k if the equivalent conditions of Lemma 7.148.1 hold.

We will see later (More on Algebra, Proposition 12.26.1) that it suffices to check $R \otimes_k k'$ is regular whenever $k \subset k' \subset k^{1/p}$ (finite).

Lemma 7.148.3. *Let k be a field. Let $A \to B$ be a faithfully flat k-algebra map. If B is geometrically regular over k, so is A.*

Proof. Assume B is geometrically regular over k. Let $k \subset k'$ be a finite, purely inseparable extension. Then $A \otimes_k k' \to B \otimes_k k'$ is faithfully flat as a base change of $A \to B$ (by Lemmas 7.27.3 and 7.35.6) and $B \otimes_k k'$ is regular by our assumption on B over k. Then $A \otimes_k k'$ is regular by Lemma 7.146.4. □

7.149. Geometrically Cohen-Macaulay algebras

This section is a bit of a misnomer, since Cohen-Macaulay algebras are automatically geometrically Cohen-Macaulay. Namely, see Lemma 7.121.6 and Lemma 7.149.2 below.

Lemma 7.149.1. *Let k be a field and let $k \subset K$ and $k \subset L$ be two field extensions such that one of them is a field extension of finite type. Then $K \otimes_k L$ is a Noetherian Cohen-Macaulay ring.*

Proof. The ring $K \otimes_k L$ is Noetherian by Lemma 7.28.7. Say K is a finite extension of the purely transcendental extension $k(t_1, \ldots, t_r)$. Then $k(t_1, \ldots, t_r) \otimes_k L \to K \otimes_k L$ is a finite free ring map. By Lemma 7.103.9 it suffices to show that $k(t_1, \ldots, t_r) \otimes_k L$ is Cohen-Macaulay. This is clear because it is a localization of the polynomial ring $L[t_1, \ldots, t_r]$. (See for example Lemma 7.96.7 for the fact that a polynomial ring is Cohen-Macaulay.) □

Lemma 7.149.2. *Let k be a field. Let S be a Noetherian k-algebra. Let $k \subset K$ be a finitely generated field extension, and set $S_K = K \otimes_k S$. Let $\mathfrak{q} \subset S$ be a prime of S. Let $\mathfrak{q}_K \subset S_K$ be a prime of S_K lying over $\mathfrak{q}$. Then $S_{\mathfrak{q}}$ is Cohen-Macaulay if and only if $(S_K)_{\mathfrak{q}_K}$ is Cohen-Macaulay.*

Proof. By Lemma 7.28.7 the ring S_K is Noetherian. Hence $S_{\mathfrak{q}} \to (S_K)_{\mathfrak{q}_K}$ is a flat local homomorphism of Noetherian local rings. Note that the fibre

$$(S_K)_{\mathfrak{q}_K}/\mathfrak{q}(S_K)_{\mathfrak{q}_K} \cong (\kappa(\mathfrak{q}) \otimes_k K)_{\mathfrak{q}'}$$

is the localization of the Cohen-Macaulay (Lemma 7.149.1) ring $\kappa(\mathfrak{q}) \otimes_k K$ at a suitable prime ideal $\mathfrak{q}'$. Hence the lemma follows from Lemma 7.145.3. □

7.150. Other chapters

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CHAPTER 8

Brauer groups

8.1. Introduction

A reference are the lectures by Serre in the Seminaire Cartan, see [**Ser55a**]. Serre in turn refers to [**Deu68**] and [**ANT44**]. We changed some of the proofs, in particular we used a fun argument of Rieffel to prove Wedderburn's theorem. Very likely this change is not an improvement and we strongly encourage the reader to read the original exposition by Serre.

8.2. Noncommutative algebras

Let k be a field. In this chapter an *algebra* A over k is a possibly noncommutative ring A together with a ring map $k \to A$ such that k maps into the center of A and such that 1 maps to an identity element of A. An A-*module* is a right A-module such that the identity of A acts as the identity.

Definition 8.2.1. Let A be a k-algebra. We say A is *finite* if $\dim_k(A) < \infty$. In this case we write $[A : k] = \dim_k(A)$.

Definition 8.2.2. A *skew field* is a possibly noncommutative ring with an identity element 1, with $1 \neq 0$, such that in which every nonzero element has a multiplicative inverse.

A skew field is a k-algebra for some k (e.g., for the prime field contained in it). We will use below that any module over a skew field is free because a maximal linearly independent set of vectors forms a basis and exists by Zorn's lemma.

Definition 8.2.3. Let A be a k-algebra. We say an A-module M is *simple* if it is nonzero and the only A-submodules are 0 and M. We say A is *simple* if the only two-sided ideals of A are 0 and A.

Definition 8.2.4. A k-algebra A is *central* if the center of A is the image of $k \to A$.

Definition 8.2.5. Given a k-algebra A we denote A^{op} the k-algebra we get by reversing the order of multiplication in A. This is called the *opposite algebra*.

8.3. Wedderburn's theorem

The following cute argument can be found in a paper of Rieffel, see [**Rie65**]. The proof could not be simpler (quote from Carl Faith's review).

Lemma 8.3.1. *Let A be a possibly noncommutative ring with* 1 *which contains no nontrivial two-sided ideal. Let M be a nonzero right ideal in A, and view M as a right A-module. Then A coincides with the bicommutant of M.*

Proof. Let $A' = \text{End}_A(M)$, and let $A'' = \text{End}_{A'}(M)$ (the bicommutant of M). Let $R : A \to A''$ be the natural homomorphism $R(a)(m) = ma$. Then R is injective, since $R(1) = \text{id}_M$ and A contains no nontrivial two-sided ideal. We claim that $R(M)$ is a right ideal in

A''. Namely, $R(m)a'' = R(ma'')$ for $a'' \in A''$ and m in M, because *left* multiplication of M by any element n of M represents an element of A', and so $(nm)a'' = n(ma'')$, that is, $(R(m)a'')(n) = R(ma'')(n)$ for all n in M. Finally, the product ideal AM is a two-sided ideal, and so $A = AM$. Thus $R(A) = R(A)R(M)$, so that $R(A)$ is a right ideal in A''. But $R(A)$ contains the identity element of A'', and so $R(A) = A''$. □

Lemma 8.3.2. *Let A be a k-algebra. If A is finite, then*

(1) *A has a simple module,*
(2) *any nonzero module contains a simple submodule,*
(3) *a simple module over A has finite dimension over k, and*
(4) *if M is a simple A-module, then $End_A(M)$ is a skew field.*

Proof. Of course (1) follows from (2) since A is a nonzero A-module. For (2), any sub-module of minimal (finite) dimension as a k-vector space will be simple. There exists a finite dimensional one because a cyclic submodule is one. If M is simple, then $mA \subset M$ is a sub-module, hence we see (3). Any nonzero element of $\text{End}_A(M)$ is an isomorphism, hence (4) holds. □

Theorem 8.3.3. *Let A be a simple finite k-algebra. Then A is a matrix algebra over a finite k-algebra K which is a skew field.*

Proof. We may choose a simple submodule $M \subset A$ and then the k-algebra $K = \text{End}_A(M)$ is a skew field, see Lemma 8.3.2. By Lemma 8.3.1 we see that $A = \text{End}_K(M)$. Since K is a skew field and M is finitely generated (since $\dim_k(M) < \infty$) we see that M is finite free as a left K-module. It follows immediately that $A \cong \text{Mat}(n \times n, K^{op})$. □

8.4. Lemmas on algebras

Let A be a k-algebra. Let $B \subset A$ be a subalgebra. The *centralizer of B in A* is the subalgebra

$$C = \{y \in A \mid xy = yx \text{ for all } x \in B\}.$$

It is a k-algebra.

Lemma 8.4.1. *Let A, A' be k-algebras. Let $B \subset A$, $B' \subset A'$ be subalgebras with centralizers C, C'. Then the centralizer of $B \otimes_k B'$ in $A \otimes_k A'$ is $C \otimes_k C'$.*

Proof. Denote $C'' \subset A \otimes_k A'$ the centralizer of $B \otimes_k B'$. It is clear that $C \otimes_k C' \subset C''$. Conversely, every element of C'' commutes with $B \otimes 1$ hence is contained in $C \otimes_k A'$. Similarly $C'' \subset A \otimes_k C'$. Thus $C'' \subset C \otimes_k A' \cap A \otimes_k C' = C \otimes_k C'$. □

Lemma 8.4.2. *Let A be a finite simple k-algebra. Then the center k' of A is a finite field extension of k.*

Proof. Write $A = \text{Mat}(n \times n, K)$ for some skew field K finite over k, see Theorem 8.3.3. By Lemma 8.4.1 the center of A is $k \otimes_k k'$ where $k' \subset K$ is the center of K. Since the center of a skew field is a field, we win. □

Lemma 8.4.3. *Let V be a k vector space. Let K be a central k-algebra which is a skew field. Let $W \subset V \otimes_k K$ be a two-sided K-sub vector space. Then W is generated as a left K-vector space by $W \cap (V \otimes 1)$.*

Proof. Let $V' \subset V$ be the k-sub vector space generated by $v \in V$ such that $v \otimes 1 \in W$. Then $V' \otimes_k K \subset W$ and we have

$$W/V' \otimes_k K \subset V/V' \otimes_k K.$$

If $\overline{v} \in V/V'$ is a nonzero vector such that $\overline{v} \otimes 1$ is contained in $W/V' \otimes_k K$, then we see that $v \otimes 1 \in W$ where $v \in V$ lifts $\overline{v}$. This contradicts our construction of V'. Hence we may replace V by V/V' and W by $W/V' \otimes_k K$ and it suffices to prove that $W \cap (V \otimes 1)$ is nonzero if W is nonzero.

To see this let $w \in W$ be a nonzero element which can be written as $W = \sum_{i=1,\ldots,n} v_i \otimes k_i$ with n minimal. We may right multiply with k_1^{-1} and assume that $k_1 = 1$. If $n = 1$, then we win because $v_1 \otimes 1 \in W$. If $n > 1$, then we see that for any $c \in K$

$$cv - vc = \sum\nolimits_{i=2,\ldots,n} v_i \otimes (ck_i - k_i c) \in W$$

and hence $ck_i - k_i c = 0$ by minimality of n. This implies that k_i is in the center of K which is k by assumption. Hence $v = (v_1 + \sum k_i v_i) \otimes 1$ contradicting the minimality of n. $\square$

Lemma 8.4.4. *Let A be a k-algebra. Let K be a central k-algebra which is a skew field. Then any two-sided ideal $I \subset A \otimes_k K$ is of the form $J \otimes_k K$ for some two-sided ideal $J \subset A$. In particular, if A is simple, then so is $A \otimes_k K$.*

Proof. Set $J = \{a \in A \mid a \otimes 1 \in I\}$. This is a two-sided ideal of A. And $I = J \otimes_k K$ by Lemma 8.4.3. $\square$

Lemma 8.4.5. *Let R be a possibly noncommutative ring. Let $n \geq 1$ be an integer. Let $R_n = Mat(n \times n, R)$.*

(1) *The functors $M \mapsto M^{\oplus n}$ and $N \mapsto Ne_{11}$ define quasi-inverse equivalences of categories $Mod_R \leftrightarrow Mod_{R_n}$.*
(2) *A two-sided ideal of R_n is of the form IR_n for some two-sided ideal I of R.*
(3) *The center of R_n is equal to the center of R.*

Proof. Part (1) proves itself. If $J \subset R_n$ is a two-sided ideal, then $J = \bigoplus e_{ii} J e_{jj}$ and all of the summands $e_{ii} J e_{jj}$ are equal to each other and are a two-sided ideal I of R. This proves (2). Part (3) is clear. $\square$

Lemma 8.4.6. *Let A be a finite simple k-algebra.*

(1) *There exists exactly one simple A-module M up to isomorphism.*
(2) *Any finite A-module is a direct sum of copies of a simple module.*
(3) *Two finite A-modules are isomorphic if and only if they have the same dimension over k.*
(4) *If $A = Mat(n \times n, K)$ with K a finite skew field extension of k, then $M = K^{\oplus n}$ is a simple A-module and $End_A(M) = K^{op}$.*
(5) *If M is a simple A-module, then $L = End_A(M)$ is a skew field finite over k acting on the left on M, we have $A = End_L(M)$, and the centers of A and L agree. Also $[A : k][L : k] = \dim_k(M)^2$.*
(6) *For a finite A-module N the algebra $B = End_A(N)$ is a matrix algebra over the skew field L of (3). Moreover $End_B(N) = A$.*

Proof. By Theorem 8.3.3 we can write $A = \text{Mat}(n \times n, K)$ for some finite skew field extension K of k. By Lemma 8.4.5 the category of modules over A is equivalent to the category of modules over K. Thus (1), (2), and (3) hold because every module over K is free. Part (4) holds because the equivalence transforms the K-module K to $M = K^{\oplus n}$. Using $M = K^{\oplus n}$ in (5) we see that $L = K^{op}$. The statement about the center of $L = K^{op}$ follows from Lemma 8.4.5. The statement about $\text{End}_L(M)$ follows from the explicit form of M. The formula of dimensions is clear. Part (6) follows as N is isomorphic to a direct sum of copies of a simple module. $\square$

Lemma 8.4.7. *Let A, A' be two simple k-algebras one of which is finite and central over k. Then $A \otimes_k A'$ is simple.*

Proof. Suppose that A' is finite and central over k. Write $A' = \text{Mat}(n \times n, K')$, see Theorem 8.3.3. Then the center of K' is k and we conclude that $A \otimes_k K'$ is simple by Lemma 8.4.4. Hence $A \otimes_k A' = \text{Mat}(n \times n, A \otimes_k K')$ is simple by Lemma 8.4.5. □

Lemma 8.4.8. *The tensor product of finite central simple algebras over k is finite, central, and simple.*

Proof. Combine Lemmas 8.4.1 and 8.4.7. □

Lemma 8.4.9. *Let A be a finite central simple algebra over k. Let $k \subset k'$ be a field extension. Then $A' = A \otimes_k k'$ is a finite central simple algebra over k'.*

Proof. Combine Lemmas 8.4.1 and 8.4.7. □

Lemma 8.4.10. *Let A be a finite central simple algebra over k. Then $A \otimes_k A^{op} \cong Mat(n \times n, k)$ where $n = [A : k]$.*

Proof. By Lemma 8.4.8 the algebra $A \otimes_k A^{op}$ is simple. Hence the map

$$A \otimes_k A^{op} \longrightarrow \text{End}_k(A), \quad a \otimes a' \longmapsto (x \mapsto axa')$$

is injective. Since both sides of the arrow have the same dimension we win. □

8.5. The Brauer group of a field

Let k be a field. Consider two finite central simple algebras A and B over k. We say A and B are *similar* if there exist $n, m > 0$ such that $\text{Mat}(n \times n, A) \cong \text{Mat}(m \times m, B)$ as k-algebras.

Lemma 8.5.1. *Similarity.*

(1) *Similarity defines an equivalence relation on the set of isomorphism classes of finite central simple algebras over k.*
(2) *Every similarity class contains a unique (up to isomorphism) finite central skew field extension of k.*
(3) *If $A = Mat(n \times n, K)$ and $B = Mat(m \times m, K')$ for some finite central skew fields K, K' over k then A and B are similar if and only if $K \cong K'$ as k-algebras.*

Proof. Note that by Wedderburn's theorem (Theorem 8.3.3) we can always write a finite central simple algebra as a matrix algebra over a finite central skew field. Hence it suffices to prove the third assertion. To see this it suffices to show that if $A = \text{Mat}(n \times n, K) \cong \text{Mat}(m \times m, K') = B$ then $K \cong K'$. To see this note that for a simple module M of A we have $\text{End}_A(M) = K^{op}$, see Lemma 8.4.6. Hence $A \cong B$ implies $K^{op} \cong (K')^{op}$ and we win. □

Given two finite central simple k-algebras A, B the tensor product $A \otimes_k B$ is another, see Lemma 8.4.8. Moreover if A is similar to A', then $A \otimes_k B$ is similar to $A' \otimes_k B$ because tensor products and taking matrix algebras commute. Hence tensor product defines an operation on equivalence classes of finite central simple algebras which is clearly associative and commutative. Finally, Lemma 8.4.10 shows that $A \otimes_k A^{op}$ is isomorphic to a matrix algebra, i.e., that $A \otimes_k A^{op}$ is in the similarity class of k. Thus we obtain an abelian group.

Definition 8.5.2. Let k be a field. The *Brauer group* of k is the abelian group of similarity classes of finite central simple k-algebras defined above. Notation $\text{Br}(k)$.

For any map of fields $k \to k'$ we obtain a group homomorphism

$$\mathrm{Br}(k) \longrightarrow \mathrm{Br}(k'), \quad A \longmapsto A \otimes_k k'$$

see Lemma 8.4.9. In other words, $\mathrm{Br}(-)$ is a functor from the category of fields to the category of abelian groups. Observe that the Brauer group of a field is zero if and only if every finite central skew field extension $k \subset K$ is trivial.

Lemma 8.5.3. *The Brauer group of an algebraically closed field is zero.*

Proof. Let $k \subset K$ be a finite central skew field extension. For any element $x \in K$ the subring $k[x] \subset K$ is a commutative finite integral k-sub algebra, hence a field, see Algebra, Lemma 7.32.17. Since k is algebraically closed we conclude that $k[x] = k$. Since x was arbitrary we conclude $k = K$. □

Lemma 8.5.4. *Let A be a finite central simple algebra over a field k. Then $[A : k]$ is a square.*

Proof. This is true because $A \otimes_k \overline{k}$ is a matrix algebra over $\overline{k}$ by Lemma 8.5.3. □

8.6. Skolem-Noether

Theorem 8.6.1. *Let A be a finite central simple k-algebra. Let B be a simple k-algebra. Let $f, g : B \to A$ be two k-algebra homomorphisms. Then there exists an invertible element $x \in A$ such that $f(b) = xg(b)x^{-1}$ for all $b \in B$.*

Proof. Choose a simple A-module M. Set $L = \mathrm{End}_A(M)$. Then L is a skew field with center k which acts on the left on M, see Lemmas 8.3.2 and 8.4.6. Then M has two $B \otimes_k L^{op}$-module structures defined by $m \cdot_1 (b \otimes l) = lmf(b)$ and $m \cdot_2 (b \otimes l) = lmg(b)$. Since $B \otimes_k L^{op}$ is a finite simple k-algebra by Lemma 8.4.7 we see that these module structures are isomorphic by Lemma 8.4.6. Hence we find $\varphi : M \to M$ intertwining these operations. In particular φ is in the commutant of L which implies that φ is multiplication by some $x \in A$, see Lemma 8.4.6. Working out the definitions we see that x is a solution to our problem. □

Lemma 8.6.2. *Let A be a finite simple k-algebra. Any automorphism of A is inner. In particular, any automorphism of $Mat(n \times n, k)$ is inner.*

Proof. Note that A is a finite central simple algebra over the center of A which is a finite field extension of k, see Lemma 8.4.2. Hence the Skolem-Noether theorem (Theorem 8.6.1) applies. □

8.7. The centralizer theorem

Theorem 8.7.1. *Let A be a finite central simple algebra over k, and let B be a simple subalgebra of A. Then*

(1) *the centralizer C of B in A is simple,*
(2) *$[A : k] = [B : k][C : k]$, and*
(3) *the centralizer of C in A is B.*

Proof. Throughout this proof we use the results of Lemma 8.4.6 freely. Choose a simple A-module M. Set $L = \mathrm{End}_A(M)$. Then L is a skew field with center k which acts on the left on M and $A = \mathrm{End}_L(M)$. Then M is a right $B \otimes_k L^{op}$-module and $C = \mathrm{End}_{B \otimes_k L^{op}}(M)$. Since the algebra $B \otimes_k L^{op}$ is simple by Lemma 8.4.7 we see that C is simple (by Lemma 8.4.6 again).

Write $B \otimes_k L^{op} = \mathrm{Mat}(m \times m, K)$ for some skew field K finite over k. Then $C = \mathrm{Mat}(n \times n, K^{op})$ if M is isomorphic to a direct sum of n copies of the simple $B \otimes_k L^{op}$-module $K^{\oplus m}$ (the lemma again). Thus we have $\dim_k(M) = nm[K : k]$, $[B : k][L : k] = m^2[K : k]$, $[C : k] = n^2[K : k]$, and $[A : k][L : k] = \dim_k(M)^2$ (by the lemma again). We conclude that (2) holds.

Part (3) follows because of (2) applied to $C \subset A$ shows that $[B : k] = [C' : k]$ where C' is the centralizer of C in A (and the obvious fact that $B \subset C'$). □

Lemma 8.7.2. *Let A be a finite central simple algebra over k, and let B be a simple subalgebra of A. If B is a central k-algebra, then $A = B \otimes_k C$ where C is the (central simple) centralizer of B in A.*

Proof. We have $\dim_k(A) = \dim_k(B \otimes_k C)$ by Theorem 8.7.1. By Lemma 8.4.7 the tensor product is simple. Hence the natural map $B \otimes_k C \to A$ is injective hence an isomorphism. □

Lemma 8.7.3. *Let A be a finite central simple algebra over k. If $K \subset A$ is a subfield, then the following are equivalent*

(1) $[A : k] = [K : k]^2$,
(2) *K is its own centralizer, and*
(3) *K is a maximal commutative subring.*

Proof. Theorem 8.7.1 shows that (1) and (2) are equivalent. It is clear that (3) and (2) are equivalent. □

Lemma 8.7.4. *Let A be a finite central skew field over k. Then every maximal subfield $K \subset A$ satisfies $[A : k] = [K : k]^2$.*

Proof. Special case of Lemma 8.7.3. □

8.8. Splitting fields

Definition 8.8.1. Let A be a finite central simple k-algebra. We say a field extension $k \subset k'$ *splits* A, or k' is a *splitting field* for A if $A \otimes_k k'$ is a matrix algebra over k'.

Another way to say this is that the class of A maps to zero under the map $\mathrm{Br}(k) \to \mathrm{Br}(k')$.

Theorem 8.8.2. *Let A be a finite central simple k-algebra. Let $k \subset k'$ be a finite field extension. The following are equivalent*

(1) *k' splits A, and*
(2) *there exists a finite central simple algebra B similar to A such that $k' \subset B$ and $[B : k] = [k' : k]^2$.*

Proof. Assume (2). It suffices to show that $B \otimes_k k'$ is a matrix algebra. We know that $B \otimes_k B^{op} \cong \mathrm{End}_k(B)$. Since k' is the centralizer of k' in B^{op} by Lemma 8.7.3 we see that $B \otimes_k k'$ is the centralizer of $k \otimes k'$ in $B \otimes_k B^{op} = \mathrm{End}_k(B)$. Of course this centralizer is just $\mathrm{End}_{k'}(B)$ where we view B as a k' vector space via the embedding $k' \to B$. Thus the result.

Assume (1). This means that we have an isomorphism $A \otimes_k k' \cong \mathrm{End}_{k'}(V)$ for some k'-vector space V. Let B be the commutant of A in $\mathrm{End}_k(V)$. Note that k' sits in B. By Lemma 8.7.2 the classes of A and B add up to zero in $\mathrm{Br}(k)$. From the dimension formula in Theorem 8.7.1 we see that

$$[B : k][A : k] = \dim_k(V)^2 = [k' : k]^2 \dim_{k'}(V)^2 = [k' : k]^2[A : k].$$

Hence $[B : k] = [k' : k]^2$. Thus we have proved the result for the opposite to the Brauer class of A. However, k' splits the Brauer class of A if and only if it splits the Brauer class of the opposite algebra, so we win anyway. □

Lemma 8.8.3. *A maximal subfield of a finite central skew field K over k is a splitting field for K.*

Proof. Combine Lemma 8.7.4 with Theorem 8.8.2. □

Lemma 8.8.4. *Consider a finite central skew field K over k. Let $d^2 = [K : k]$. For any finite splitting field k' for K the degree $[k' : k]$ is divisible by d.*

Proof. By Theorem 8.8.2 there exists a finite central simple algebra B in the Brauer class of K such that $[B : k] = [k' : k]^2$. By Lemma 8.5.1 we see that $B = \text{Mat}(n \times n, K)$ for some n. Then $[k' : k]^2 = n^2 d^2$ whence the result. □

Proposition 8.8.5. *Consider a finite central skew field K over k. There exists a maximal subfield $k \subset k' \subset K$ which is separable over k. In particular, every Brauer class has a finite separable spitting field.*

Proof. Since every Brauer class is represented by a finite central skew field over k, we see that the second statement follows from the first by Lemma 8.8.3.

To prove the first statement, suppose that we are given a separable subfield $k' \subset K$. Then the centralizer K' of k' in K has center k', and the problem reduces to finding a maximal subfield of K' separable over k'. Thus it suffices to prove, if $k \neq K$, that we can find an element $x \in K$, $x \notin K$ which is separable over k. This statement is clear in characteristic zero. Hence we may assume that k has characteristic $p > 0$. If the ground field k is finite then, the result is clear as well (because extensions of finite fields are always separable). Thus we may assume that k is an infinite field of positive characteristic.

To get a contradiction assume no element of K is separable over k. By the discussion in Algebra, Section 7.38 this means the minimal polynomial of any $x \in K$ is of the form $T^q - a$ where q is a power of p and $a \in k$. Since it is clear that every element of K has a minimal polynomial of degree $\leq \dim_k(K)$ we conclude that there exists a fixed p-power q such that $x^q \in k$ for all $x \in K$.

Consider the map

$$(-)^q : K \longrightarrow K$$

and write it out in terms of a k-basis $\{a_1, \ldots, a_n\}$ of K with $a_1 = 1$. So

$$(\sum x_i a_i)^q = \sum f_i(x_1, \ldots, x_n) a_i.$$

Since multiplication on A is k-bilinear we see that each f_i is a polynomial in $x_1, \ldots, x_n$ (details omitted). The choice of q above and the fact that k is infinite shows that f_i is identically zero for $i \geq 2$. Hence we see that it remains zero on extending k to its algebraic closure $\overline{k}$. But the algebra $A \otimes_k \overline{k}$ is a matrix algebra, which implies there are some elements whose qth power is not central (e.g., e_{11}). This is the desired contradiction. □

The results above allow us to characterize finite central simple algebras as follows.

Lemma 8.8.6. *Let k be a field. For a k-algebra A the following are equivalent*

(1) *A is finite central simple k-algebra,*
(2) *A is a finite dimensional k-vector space, k is the center of A, and A has no non-trivial two-sided ideal,*

(3) *there exists* $d \geq 1$ *such that* $A \otimes_k \bar{k} \cong Mat(d \times d, \bar{k})$*,*
(4) *there exists* $d \geq 1$ *such that* $A \otimes_k k^{sep} \cong Mat(d \times d, k^{sep})$*,*
(5) *there exist* $d \geq 1$ *and a finite Galois extension* $k \subset k'$ *such that* $A \otimes_{k'} k' \cong Mat(d \times d, k')$*,*
(6) *there exist* $n \geq 1$ *and a finite central skew field* K *over* k *such that* $A \cong Mat(n \times n, K)$*.*

The integer d *is called the degree of* A*.*

Proof. The equivalence of (1) and (2) is a consequence of the definitions, see Section 8.2. Assume (1). By Proposition 8.8.5 there exists a separable splitting field $k \subset k'$ for A. Of course, then a Galois closure of k'/k is a splitting field also. Thus we see that (1) implies (5). It is clear that (5) $\Rightarrow$ (4) $\Rightarrow$ (3). Assume (3). Then $A \otimes_k \overline{k}$ is a finite central simple $\overline{k}$-algebra for example by Lemma 8.4.5. This trivially implies that A is a finite central simple k-algebra. Finally, the equivalence of (1) and (6) is Wedderburn's theorem, see Theorem 8.3.3. □

8.9. Other chapters

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CHAPTER 9

Sites and Sheaves

9.1. Introduction

The notion of a site was introduced by Grothendieck to be able to study sheaves in the étale topology of schemes. The basic reference for this notion is perhaps [**MA71**]. Our notion of a site differs from that in [**MA71**]; what we call a site is called a category endowed with a pretopology in [**MA71**, Exposé II, Définition 1.3]. The reason we do this is that in algebraic geometry it is often convenient to work with a given class of coverings, for example when defining when a property of schemes is local in a given topology, see Descent, Section 31.11. Our exposition will closely follow [**Art62**]. We will not use universes.

9.2. Presheaves

Let $\mathcal{C}$ be a category. A *presheaf of sets* is a contravariant functor $\mathcal{F}$ from $\mathcal{C}$ to *Sets* (see Categories, Remark 4.2.11). So for every object U of $\mathcal{C}$ we have a set $\mathcal{F}(U)$. The elements of this set are called the *sections* of $\mathcal{F}$ over U. For every morphism $f : V \to U$ the map $\mathcal{F}(f) : \mathcal{F}(U) \to \mathcal{F}(V)$ is called the *restricton map* and is often denoted $f^* : \mathcal{F}(U) \to \mathcal{F}(V)$. Another way of expressing this is to say that $f^*(s)$ is the *pullback* of s via f. Functoriality means that $g^* f^*(s) = (f \circ g)^*(s)$. Sometimes we use the notation $s|_V := f^*(s)$. This notation is consistent with the notion of restriction of functions from topology because if $W \to V \to U$ are morphisms in $\mathcal{C}$ and s is a section of $\mathcal{F}$ over U then $s|_W = (s|_V)|_W$ by the functorial nature of $\mathcal{F}$. Of course we have to be careful since it may very well happen that there is more than one morphism $V \to U$ and it is certainly not going to be the case that the corresponding pull back maps are equal.

Definition 9.2.1. A *presheaf of sets* on $\mathcal{C}$ is a contravariant functor from $\mathcal{C}$ to *Sets*. *Morphisms of presheaves* are transformations of functors. The category of presheaves of sets is denoted $PSh(\mathcal{C})$.

Note that for any object U of $\mathcal{C}$ the functor of points h_U, see Categories, Example 4.3.4 is a presheaf. These are called the *representable presheaves*. These presheaves have the pleasing property that for any presheaf $\mathcal{F}$ we have

$$Mor_{PSh(\mathcal{C})}(h_U, \mathcal{F}) = \mathcal{F}(U).$$

This is similar to the Yoneda lemma (Categories, Lemma 4.3.5) and left as a good exercise to the reader.

Similarly, we can define the notion of a presheaf of abelian groups, rings, etc. More generally we may define a presheaf with values in a category.

Definition 9.2.2. Let $\mathcal{C}$, $\mathcal{A}$ be categories. A *presheaf* $\mathcal{F}$ on $\mathcal{C}$ with values in $\mathcal{A}$ is a contravariant functor from $\mathcal{C}$ to $\mathcal{A}$, i.e., $\mathcal{F} : \mathcal{C}^{o} \to \mathcal{A}$. A *morphism* of presheaves $\mathcal{F} \to \mathcal{G}$ on $\mathcal{C}$ with values in $\mathcal{A}$ is a transformation of functors from $\mathcal{F}$ to $\mathcal{G}$.

These form the objects and morphisms of the category of presheaves on $\mathcal{C}$ with values in $\mathcal{A}$.

Remark 9.2.3. As already pointed out we may consider the category presheaves with values in any of the ``big'' categories listed in Categories, Remark 4.2.2. These will be ``big'' categories as well and they will be listed in the above mentioned remark as we go along.

9.3. Injective and surjective maps of presheaves

Definition 9.3.1. Let $\mathcal{C}$ be a category, and let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of presheaves of sets.

(1) We say that φ is *injective* if for every object U of $\mathcal{C}$ we have $\alpha : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective.
(2) We say that φ is *surjective* if for every object U of $\mathcal{C}$ we have $\alpha : \mathcal{F}(U) \to \mathcal{G}(U)$ is surjective.

Lemma 9.3.2. *The injective (resp. surjective) maps defined above are exactly the monomorphisms (resp. epimorphisms) of $PSh(\mathcal{C})$. A map is an isomorphism if and only if it is both injective and surjective.*

Proof. Omitted. □

Definition 9.3.3. We say $\mathcal{F}$ is a *subpresheaf* of $\mathcal{G}$ if for every object $U \in Ob(\mathcal{C})$ the set $\mathcal{F}(U)$ is a subset of $\mathcal{G}(U)$, compatibly with the restriction mappings.

In other words, the inclusion maps $\mathcal{F}(U) \to \mathcal{G}(U)$ glue together to give an (injective) morphism of presheaves $\mathcal{F} \to \mathcal{G}$.

Lemma 9.3.4. *Let $\mathcal{C}$ be a category. Suppose that $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves of setson $\mathcal{C}$. There exists a unique subpresheaf $\mathcal{G}' \subset \mathcal{G}$ such that φ factors as $\mathcal{F} \to \mathcal{G}' \to \mathcal{G}$ and such that the first map is surjective.*

Proof. Omitted. □

Definition 9.3.5. Notation as in Lemma 9.3.4. We say that $\mathcal{G}'$ is the *image of φ*.

9.4. Limits and colimits of presheaves

Let $\mathcal{C}$ be a category. Limits and colimits exist in the category $PSh(\mathcal{C})$. In addition, for any $U \in \mathrm{ob}(\mathcal{C})$ the functor

$$PSh(\mathcal{C}) \longrightarrow Sets, \quad \mathcal{F} \longmapsto \mathcal{F}(U)$$

commutes with limits and colimits. Perhaps the easiest way to prove these statement is the following. Given a diagram $\mathcal{F} : \mathcal{I} \to PSh(\mathcal{C})$ define presheaves

$$\mathcal{F}_{lim} : U \longmapsto lim_{i\in\mathcal{I}}\mathcal{F}_i(U) \text{ and } \mathcal{F}_{colim} : U \longmapsto colim_{i\in\mathcal{I}}\mathcal{F}_i(U)$$

There are clearly projection maps $\mathcal{F}_{lim} \to \mathcal{F}_i$ and canonical maps $\mathcal{F}_i \to \mathcal{F}_{colim}$. These maps satisfy the requirements of the maps of a limit (reps. colimit) of Categories, Definition 4.13.1 (resp. Categories, Definition 4.13.2). Finally, if $(\mathcal{G}, q_i : \mathcal{G} \to \mathcal{F}_i)$ is another system (as in the definition of a limit), then we get for every U a system of maps $\mathcal{G}(U) \to \mathcal{F}_i(U)$ with suitable functoriality requirements. And thus a unique map $\mathcal{G}(U) \to \mathcal{F}_{lim}(U)$. It is easy to verify these are compatible as we vary U and arise from the desired map $\mathcal{G} \to \mathcal{F}_{lim}$. A similar argument works in the case of the colimit.

9.5. Functoriality of categories of presheaves

Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between categories. In this case we denote

$$u^p : PSh(\mathcal{D}) \longrightarrow PSh(\mathcal{C})$$

the functor that associates to $\mathcal{G}$ on $\mathcal{D}$ the presheaf $u^p\mathcal{G} = \mathcal{G} \circ u$. Note that by the previous section this functor commutes with all limits.

For $V \in \text{ob}(\mathcal{D})$ let $\mathcal{I}_V^u$ denote the category with

$$(9.5.0.1) \qquad \begin{array}{rcl} Ob(\mathcal{I}_V^u) & = & \{(U, \phi) \mid U \in Ob(\mathcal{C}), \phi : V \to u(U)\} \\ Mor_{\mathcal{I}_V^u}((U, \phi), (U', \phi')) & = & \{f : U \to U' \text{ in } \mathcal{C} \mid u(f) \circ \phi = \phi'\} \end{array}$$

We sometimes drop the subscript ^{u} from the notation and we simply write $\mathcal{I}_V$. We will use these categories to define a left adjoint to the functor u^p. Before we do so we prove a few technical lemmas.

Lemma 9.5.1. *Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between categories. Suppose that $\mathcal{C}$ has fibre products and equalizers, and that u commutes with them. Then the categories $(\mathcal{I}_V)^{opp}$ satisfy the hypotheses of Categories, Lemma 4.17.3.*

Proof. There are two conditions to check.

First, suppose we are given three objects $\phi : V \to u(U)$, $\phi' : V \to u(U')$, and $\phi'' : V \to u(U'')$ and morphisms $a : U' \to U$, $b : U'' \to U$ such that $u(a) \circ \phi = \phi'$ and $u(b) \circ \phi = \phi''$. We have to show there exists another object $\phi''' : V \to u(U''')$ and morphisms $c : U''' \to U'$ and $d : U''' \to U''$ such that $u(c) \circ \phi = \phi'''$, $u(d) \circ \phi = \phi'''$ and $a \circ c = b \circ d$. We take $U''' = U' \times_U U''$ with c and d the projection morphisms. This works as u commutes with fibre products; we omit the verification.

Second, suppose we are given two objects $\phi : V \to u(U)$ and $\phi' : V \to u(U')$ and morphisms $a, b : (U, \phi) \to (U', \phi')$. We have to find a morphism $c : (U'', \phi'') \to (U, \phi)$ which equalizes a and b. Let $c : U'' \to U$ be the equalizer of a and b in the category $\mathcal{C}$. As u commutes with equalizers and since $u(a) \circ \phi = u(b) \circ \phi = \phi'$ we obtain a morphism $\phi'' : V \to u(U'')$. $\square$

Lemma 9.5.2. *Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between categories. Assume*

1. *the category $\mathcal{C}$ has a final object X and $u(X)$ is a final object of $\mathcal{D}$, and*
2. *the category $\mathcal{C}$ has fibre products and u commutes with them.*

Then the index categories $(\mathcal{I}_V^u)^{opp}$ of are filtered (see Categories, Definition 4.17.1).

Proof. The assumptions imply that the assumptions of Lemma 9.5.1 are satisfied (see the discussion in Categories, Section 4.16). By Categories, Lemma 4.17.3 we see that $\mathcal{I}_V$ is a (possibly empty) disjoint union of directed categories. Hence it suffices to show that $\mathcal{I}_V$ is nonempty and connected.

First, we show that $\mathcal{I}_V$ is nonempty. Namely, let X be the final object of $\mathcal{C}$, which exists by assumption. Let $V \to u(X)$ be the morphism coming from the fact that $u(X)$ is final in $\mathcal{D}$ by assumption. This gives an object of $\mathcal{I}_V$.

Second, we show that $\mathcal{I}_V$ is connected. Let $\phi_1 : V \to u(U_1)$ and $\phi_2 : V \to u(U_2)$ be in $Ob(\mathcal{I}_V)$. By assumption $U_1 \times U_2$ exists and $u(U_1 \times U_2) = u(U_1) \times u(U_2)$. Consider the morphism $\phi : V \to u(U_1 \times U_2)$ corresponding to (ϕ_1, ϕ_2) by the universal property of products. Clearly the object $\phi : V \to u(U_1 \times U_2)$ maps to both $\phi_1 : V \to u(U_1)$ and $\phi_2 : V \to u(U_2)$. $\square$

Given $g : V' \to V$ in $\mathcal{D}$ we get a functor $\overline{g} : \mathcal{I}_V \to \mathcal{I}_{V'}$ by setting $\overline{g}(U, \phi) = (U, \phi \circ g)$ on objects. Given a presheaf $\mathcal{F}$ on $\mathcal{C}$ we obtain a functor

$$\mathcal{F}_V : \mathcal{I}_V^{opp} \longrightarrow Sets, \quad (U, \phi) \longmapsto \mathcal{F}(U).$$

In other words, $\mathcal{F}_V$ is a presheaf of sets on $\mathcal{I}_V$. Note that we have $\mathcal{F}_{V'} \circ \overline{g} = \mathcal{F}_V$. We define

$$u_p\mathcal{F}(V) := colim_{\mathcal{I}_V^{opp}} \mathcal{F}_V$$

As a colimit we obtain for each $(U, \phi) \in Ob(\mathcal{I}_V)$ a canonical map $\mathcal{F}(U) \xrightarrow{c(\phi)} u_p\mathcal{F}(V)$. For $g : V' \to V$ as above there is a canonical restriction map $g^* : u_p\mathcal{F}(V) \to u_p\mathcal{F}(V')$ compatible with $\mathcal{F}_{V'} \circ \overline{g} = \mathcal{F}_V$ by Categories, Lemma 4.13.7. It is the unique map so that for all $(U, \phi) \in Ob(\mathcal{I}_V)$ the diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "c(\phi)"] \arrow[d, "\mathrm{id}"'] & u_p\mathcal{F}(V) \arrow[d, "g^*"] \\ \mathcal{F}(U) \arrow[r, "c(\phi \circ g)"] & u_p\mathcal{F}(V') \end{tikzcd}$$

commutes. The uniquess of these maps implies that we obtain a presheaf. This presheaf will be denoted $u_p\mathcal{F}$.

Lemma 9.5.3. *There is a canonical map $\mathcal{F}(U) \to u_p\mathcal{F}(u(U))$, which is compatible with restriction maps (on $\mathcal{F}$ and on $u_p\mathcal{F}$).*

Proof. This is just the map $c(\mathrm{id}_{u(U)})$ introduced above. □

Note that any map of presheaves $\mathcal{F} \to \mathcal{F}'$ gives rise to compatible systems of maps between functors $\mathcal{F}_Y \to \mathcal{F}'_Y$, and hence to a map of presheaves $u_p\mathcal{F} \to u_p\mathcal{F}'$. In other words, we have defined a functor

$$u_p : PSh(\mathcal{C}) \longrightarrow PSh(\mathcal{D})$$

Lemma 9.5.4. *The functor u_p is a left adjoint to the functor u^p. In other words the formula*

$$Mor_{PSh(\mathcal{C})}(\mathcal{F}, u^p\mathcal{G}) = Mor_{PSh(\mathcal{D})}(u_p\mathcal{F}, \mathcal{G})$$

holds bifunctorially in $\mathcal{F}$ and $\mathcal{G}$.

Proof. Let $\mathcal{G}$ be a presheaf on $\mathcal{D}$ and let $\mathcal{F}$ be a presheaf on $\mathcal{C}$. We will show that the displayed formula holds by constructing maps either way. We will leave it to the reader to verify they are each others inverse.

Given a map $\alpha : u_p\mathcal{F} \to \mathcal{G}$ we get $u^p\alpha : u^p u_p\mathcal{F} \to u^p\mathcal{G}$. Lemma 9.5.3 says that there is a map $\mathcal{F} \to u^p u_p\mathcal{F}$. The composition of the two gives the desired map. (The good thing about this construction is that it is clearly functorial in everything in sight.)

Conversely, given a map $\beta : \mathcal{F} \to u^p\mathcal{G}$ we get a map $u_p\beta : u_p\mathcal{F} \to u_p u^p\mathcal{G}$. We claim that the functor $u^p\mathcal{G}_Y$ on $\mathcal{I}_Y$ has a canonical map to the constant functor with value $\mathcal{G}(Y)$. Namely, for every object (X, ϕ) of $\mathcal{I}_Y$, the value of $u^p\mathcal{G}_Y$ on this object is $\mathcal{G}(u(X))$ which maps to $\mathcal{G}(Y)$ by $\mathcal{G}(\phi) = \phi^*$. This is a transformation of functors because $\mathcal{G}$ is a functor itself. This leads to a map $u_p u^p\mathcal{G}(Y) \to \mathcal{G}(Y)$. Another trivial verification shows that this is functorial in Y leading to a map of presheaves $u_p u^p\mathcal{G} \to \mathcal{G}$. The composition $u_p\mathcal{F} \to u_p u^p\mathcal{G} \to \mathcal{G}$ is the desired map. □

Remark 9.5.5. Suppose that $\mathcal{A}$ is a category such that any diagram $\mathcal{I}_Y \to \mathcal{A}$ has a colimit in $\mathcal{A}$. In this case it is clear that there are functors u^p and u_p, defined in exactly the same way as above, on the categories of presheaves with values in $\mathcal{A}$. Moreover, the adjointness of the pair u^p and u_p continues to hold in this setting.

Lemma 9.5.6. *Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between categories. For any object U of $\mathcal{C}$ we have $u_p h_U = h_{u(U)}$.*

Proof. By adjointness of u_p and u^p we have

$$Mor_{PSh(\mathcal{D})}(u_p h_U, \mathcal{G}) = Mor_{PSh(\mathcal{C})}(h_U, u^p\mathcal{G}) = u^p\mathcal{G}(U) = \mathcal{G}(u(U))$$

and hence by Yoneda's lemma we see that $u_p h_U = h_{u(U)}$ as presheaves. □

9.6. Sites

Our notion of a site uses the following type of structures.

Definition 9.6.1. Let $\mathcal{C}$ be a category, see Conventions, Section 2.3. A *family of morphisms with fixed target* in $\mathcal{C}$ is given by an object $U \in Ob(\mathcal{C})$, a set I and for each $i \in I$ a morphism $U_i \to U$ of $\mathcal{C}$ with target U. We use the notation $\{U_i \to U\}_{i\in I}$ to indicate this.

It can happen that the set I is empty! This notation is meant to suggest an open covering as in topology.

Definition 9.6.2. A *site*[1] is given by a category $\mathcal{C}$ and a set $\mathrm{Cov}(\mathcal{C})$ of families of morphisms with fixed target $\{U_i \to U\}_{i\in I}$, called *coverings of $\mathcal{C}$*, satisfying the following axioms

(1) If $V \to U$ is an isomorphism then $\{V \to U\} \in \mathrm{Cov}(\mathcal{C})$.
(2) If $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ and for each i we have $\{V_{ij} \to U_i\}_{j\in J_i} \in \mathrm{Cov}(\mathcal{C})$, then $\{V_{ij} \to U\}_{i\in I, j\in J_i} \in \mathrm{Cov}(\mathcal{C})$.
(3) If $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ and $V \to U$ is a morphism of $\mathcal{C}$ then $U_i \times_U V$ exists for all i and $\{U_i \times_U V \to V\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$.

Remark 9.6.3. (On set theoretic issues -- skip on a first reading.) The main reason for introducing sites is to study the category of sheaves on a site, because it is the generalization of the category of sheaves on a topological space that has been so important in algebraic geometry. In order to avoid thinking about things like ``classes of classes'' and so on, we will not allow sites to be ``big'' categories, in contrast to what we do for categories and 2-categories.

Suppose that $\mathcal{C}$ is a category and that $\mathrm{Cov}(\mathcal{C})$ is a proper class of coverings satisfying (1), (2) and (3) above. We will not allow this as a site either, mainly because we are going to take limits over coverings. However, there are several natural ways to replace $\mathrm{Cov}(\mathcal{C})$ by a set of coverings or a slightly different structure that give rise to the same category of sheaves. For example:

(1) In Sets, Section 3.11 we show how to pick a suitable set of coverings that gives the same category of sheaves.
(2) Another thing we can do is to take the associated topology (see Definition 9.41.2). The resulting topology on $\mathcal{C}$ has the same category of sheaves. Two topologies have the same categories of sheaves if and only if they are equal, see Theorem 9.43.2. A topology on a category is given by a choice of sieves on objects. The collection of all possible sieves and even all possible topologies on $\mathcal{C}$ is a set.

[1] This notation differs from that of [**MA71**], as explained in the introduction.

(3) We could also slightly modify the notion of a site, see Remark 9.41.4 below, and end up with a canonical set of coverings which is contained in the powerset of the set of arrows of $\mathcal{C}$.

Each of these solutions has some minor drawback. For the first, one has to check that constructions later on do not depend on the choice of the set of coverings. For the second, one has to learn about topologies and redo many of the arguments for sites. For the third, see the last sentence of Remark 9.41.4.

Our approach will be to work with sites as in Definition 9.6.2 above. Given a category $\mathcal{C}$ with a proper class of coverings as above, we will replace this by a set of coverings producing a site using Sets, Lemma 3.11.1. It is shown in Lemma 9.8.6 below that the resulting category of sheaves (the topos) is independent of this choice. We leave it to the reader to use one of the other two strategies to deal with these issues if he/she so desires.

Example 9.6.4. Let X be a topological space. Let $\mathcal{T}_X$ be the category whose objects consist of all the open sets U in X and whose morphisms are just the inclusion maps. That is, there is at most one morphism between any two objects in $\mathcal{T}_X$. Now define $\{U_i \to U\}_{i\in I} \in \text{Cov}(\mathcal{T}_X)$ if and only if $\bigcup U_i = U$. Conditions (1) and (2) above are clear, and (3) is also clear once we realize that in $\mathcal{T}_X$ we have $U\times V = U\cap V$. Note that in particular the empty set has to be an element of $\mathcal{T}_X$ since otherwise this would not work in general. Furthermore, it is equally important, as we will see later, to allow the *empty covering of the empty set as a covering*! We turn $\mathcal{T}_X$ into a site by choosing a suitable set of coverings $\text{Cov}(\mathcal{T}_X)_{\kappa,\alpha}$ as in Sets, Lemma 3.11.1. Presheaves and sheaves (as defined below) on the site $\mathcal{T}_X$ will agree exactly with the usual notion of a presheaves and sheaves on a topological space, as defined in Sheaves, Section 6.1.

Example 9.6.5. Let G be a group. Consider the category G-*Sets* whose objects are sets X with a left G-action, with G-equivariant maps as the morphisms. An important example is ${}_GG$ which is the G-set whose underlying set is G and action given by left multiplication. This category has fiber products, see Categories, Section 4.7. We declare $\{\varphi_i : U_i \to U\}_{i\in I}$ to be a covering if $\bigcup_{i\in I} \varphi_i(U_i) = U$. This gives a class of coverings on G-*Sets* which is easily see to satisfy conditions (1), (2), and (3) of Definition 9.6.2. The result is not a site since both the collection of objects of the underlying category and the collection of coverings form a proper class. We first replace by G-*Sets* by a full subcategory G-*Sets*$_\alpha$ as in Sets, Lemma 3.10.1. After this the site $(G\text{-}Sets_\alpha, \text{Cov}_{\kappa,\alpha'}(G\text{-}Sets_\alpha))$ gotten by suitably restricting the collection of coverings as in Sets, Lemma 3.11.1 will be denoted $\mathcal{T}_G$.

Example 9.6.6. Let $\mathcal{C}$ be a category. There is a canonical way to turn this into a site where $\{\text{id}_U : U \to U\}$ are the coverings. Sheaves on this site are the presheaves on $\mathcal{C}$. This corresponding topology is called the *chaotic* or *indiscrete topology*.

9.7. Sheaves

Let $\mathcal{C}$ be a site. Before we introduce the notion of a sheaf with values in a category we explain what it means for a presheaf of sets to be a sheaf. Let $\mathcal{F}$ be a presheaf of sets on $\mathcal{C}$ and let $\{U_i \to U\}_{i\in I}$ be an element of $\text{Cov}(\mathcal{C})$. By assumption all the fibre products $U_i \times_U U_j$ exist in $\mathcal{C}$. There are two natural maps

$$\prod\nolimits_{i\in I} \mathcal{F}(U_i) \underset{\text{pr}_1^*}{\overset{\text{pr}_0^*}{\rightrightarrows}} \prod\nolimits_{(i_0,i_1)\in I\times I} \mathcal{F}(U_{i_0} \times_U U_{i_1})$$

which we will denote pr_i^*, $i = 0, 1$ as indicated in the displayed equation. Namely, an element of the left hand side corresponds to a family $(s_i)_{i\in I}$, where each s_i is a section of $\mathcal{F}$ over U_i. For each pair $(i_0, i_1) \in I \times I$ we have the projection morphisms

$$\mathrm{pr}^{(i_0,i_1)}_{i_0} : U_{i_0} \times_U U_{i_1} \longrightarrow U_{i_0} \text{ and } \mathrm{pr}^{(i_0,i_1)}_{i_1} : U_{i_0} \times_U U_{i_1} \longrightarrow U_{i_1}.$$

Thus we may pull back either the section s_{i_0} via the first of these maps or the section s_{i_1} via the second. Explicitly the maps we refered to above are

$$\mathrm{pr}_0^* : (s_i)_{i\in I} \longmapsto \left(\mathrm{pr}^{(i_0,i_1),*}_{i_0}(s_{i_0})\right)_{(i_0,i_1)\in I\times I}$$

and

$$\mathrm{pr}_1^* : (s_i)_{i\in I} \longmapsto \left(\mathrm{pr}^{(i_0,i_1),*}_{i_1}(s_{i_1})\right)_{(i_0,i_1)\in I\times I}.$$

Finally consider the natural map

$$\mathcal{F}(U) \longrightarrow \prod\nolimits_{i\in I} \mathcal{F}(U_i), \quad s \longmapsto (s|_{U_i})_{i\in I}$$

where we have used the notation $s|_{U_i}$ to indicate the pullback of s via the map $U_i \to U$. It is clear from the functorial natural of $\mathcal{F}$ and the commutativity of the fibre product diagrams that $\mathrm{pr}_0^*((s|_{U_i})_{i\in I}) = \mathrm{pr}_1^*((s|_{U_i})_{i\in I})$.

Definition 9.7.1. Let $\mathcal{C}$ be a site, and let $\mathcal{F}$ be a presheaf of sets on $\mathcal{C}$. We say $\mathcal{F}$ is a *sheaf* if for every covering $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ the diagram

$$\mathcal{F}(U) \longrightarrow \prod\nolimits_{i\in I} \mathcal{F}(U_i) \underset{\mathrm{pr}_1^*}{\overset{\mathrm{pr}_0^*}{\rightrightarrows}} \prod\nolimits_{(i_0,i_1)\in I\times I} \mathcal{F}(U_{i_0} \times_U U_{i_1}) \tag{9.7.1.1}$$

represents the first arrow as the equalizer of pr_0^* and pr_1^*.

Loosely speaking this means that given sections $s_i \in \mathcal{F}(U_i)$ such that

$$s_i|_{U_i\times_U U_j} = s_j|_{U_i\times_U U_j}$$

in $\mathcal{F}(U_i \times_U U_j)$ for all pairs $(i, j) \in I \times I$ then there exists a unique $s \in \mathcal{F}(U)$ such that $s_i = s|_{U_i}$.

Remark 9.7.2. If the covering $\{U_i \to U\}_{i\in I}$ is the empty family (this means that $I = \emptyset$), then the sheaf condition signifies that $\mathcal{F}(U) = \{*\}$ is a singleton set. This is true because in (9.7.1.1) the second and third sets are empty products in the category of sets, which are final objects in the category of sets, hence singletons.

Example 9.7.3. Let X be a topological space. Let $\mathcal{T}_X$ be the site constructed in Example 9.6.4. The notion of a sheaf on $\mathcal{T}_X$ coincides with the notion of a sheaf on X introduced in Sheaves, Definition 6.7.1.

Example 9.7.4. Let X be a topological space. Let us consider the site $\mathcal{T}'_X$ which is the same as the site $\mathcal{T}_X$ of Example 9.6.4 except that we disallow the empty covering of the empty set. In other words, we do allow the covering $\{\emptyset \to \emptyset\}$ but we do not allow the covering whose index set is empty. It is easy to show that this still defines a site. However, we claim that the sheaves on $\mathcal{T}'_X$ are different from the sheaves on $\mathcal{T}_X$. For example, as an extreme case consider the situation where $X = \{p\}$ is a singleton. Then the objects of $\mathcal{T}'_X$ are $\emptyset, X$ and the coverings are $\{\{\emptyset \to \emptyset\}, \{X \to X\}\}$. Clearly, a sheaf on this is given by any choice of a set $\mathcal{F}(\emptyset)$ and any choice of a set $\mathcal{F}(X)$, together with any restricion map $\mathcal{F}(X) \to \mathcal{F}(\emptyset)$. Thus sheaves on $\mathcal{T}'_X$ are the same as usual sheaves on the two point space

$\{\eta, p\}$ with open sets $\{\emptyset, \{\eta\}, \{p, \eta\}\}$. In general sheaves on $\mathcal{T}'_X$ are the same as sheaves on the space $X \amalg \{\eta\}$, with opens given by the empty set and any set of the form $U \cup \{\eta\}$ for $U \subset X$ open.

Definition 9.7.5. The category $Sh(\mathcal{C})$ of sheaves of sets is the full subcategory of the category $PSh(\mathcal{C})$ whose objects are the sheaves of sets.

Let $\mathcal{A}$ be a category. If products indexed by I, and $I \times I$ exist in $\mathcal{A}$ for any I that occurs as an index set for covering families then Definition 9.7.1 above makes sense, and defines a notion of a sheaf on $\mathcal{C}$ with values in $\mathcal{A}$. Note that the diagram in $\mathcal{A}$

$$\mathcal{F}(U) \longrightarrow \prod\nolimits_{i\in I} \mathcal{F}(U_i) \overset{\mathrm{pr}_0^*}{\underset{\mathrm{pr}_1^*}{\rightrightarrows}} \prod\nolimits_{(i_0,i_1)\in I\times I} \mathcal{F}(U_{i_0} \times_U U_{i_1})$$

is an equalizer diagram if and only if for every object X of $\mathcal{A}$ the diagram of sets

$$Mor_{\mathcal{A}}(X, \mathcal{F}(U)) \longrightarrow \prod Mor_{\mathcal{A}}(X, \mathcal{F}(U_i)) \overset{\mathrm{pr}_0^*}{\underset{\mathrm{pr}_1^*}{\rightrightarrows}} \prod Mor_{\mathcal{A}}(X, \mathcal{F}(U_{i_0} \times_U U_{i_1}))$$

is an equalizer diagram.

Suppose $\mathcal{A}$ is arbitrary. Let $\mathcal{F}$ be a presheaf with values in $\mathcal{A}$. Choose any object $X \in Ob(\mathcal{A})$. Then we get a presheaf of sets $\mathcal{F}_X$ defined by the rule

$$\mathcal{F}_X(U) = Mor_{\mathcal{A}}(X, \mathcal{F}(U)).$$

From the above it follows that a good definition is obtained by requiring all the presheaves $\mathcal{F}_X$ to be sheaves of sets.

Definition 9.7.6. Let $\mathcal{C}$ be a site, let $\mathcal{A}$ be a category and let $\mathcal{F}$ be a presheaf on $\mathcal{C}$ with values in $\mathcal{A}$. We say that $\mathcal{F}$ is a *sheaf* if for all objects X of $\mathcal{A}$ the presheaf of sets $\mathcal{F}_X$ (defined above) is a sheaf.

9.8. Families of morphisms with fixed target

This section is meant to introduce some notions regarding families of morphisms with the same target.

Definition 9.8.1. Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{U_i \to U\}_{i\in I}$ be a family of morphisms of $\mathcal{C}$ with fixed target. Let $\mathcal{V} = \{V_j \to V\}_{j\in J}$ be another.

1. A *morphism of families of maps with fixed target of $\mathcal{C}$ from $\mathcal{U}$ to $\mathcal{V}$*, or simply a *morphism from $\mathcal{U}$ to $\mathcal{V}$* is given by a morphism $U \to V$, a map of sets $\alpha : I \to J$ and for each $i \in I$ a morphism $U_i \to V_{\alpha(i)}$ such that the diagram
$$\begin{tikzcd} U_i \arrow[r] \arrow[d] & V_{\alpha(i)} \arrow[d] \\ U \arrow[r] & V \end{tikzcd}$$
is commutative.
2. In the special case that $U = V$ and $U \to V$ is the identity we call $\mathcal{U}$ a *refinement* of the family $\mathcal{V}$.

A trivial but important remark is that if $\mathcal{U} = \{U_i \to U\}_{i\in I}$ is the *empty family of maps*, i.e., if $I = \emptyset$, then no family $\mathcal{V} = \{V_j \to V\}_{j\in J}$ with $J \neq \emptyset$ can refine $\mathcal{U}$!

Definition 9.8.2. Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{\varphi_i : U_i \to U\}_{i \in I}$, and $\mathcal{V} = \{\psi_j : V_j \to U\}_{j \in J}$ be two families of morphisms with fixed target.

(1) We say $\mathcal{U}$ and $\mathcal{V}$ are *combinatorially equivalent* if there exist maps $\alpha : I \to J$ and $\beta : J \to I$ such that $\varphi_i = \psi_{\alpha(i)}$ and $\psi_j = \varphi_{\beta(j)}$.
(2) We say $\mathcal{U}$ and $\mathcal{V}$ are *tautologically equivalent* if there exist maps $\alpha : I \to J$ and $\beta : J \to I$ and for all $i \in I$ and $j \in J$ commutative diagrams

$$\begin{tikzcd} U_i \arrow[rr] \arrow[dr] & & V_{\alpha(i)} \arrow[dl] \\ & U & \end{tikzcd} \qquad \begin{tikzcd} V_j \arrow[rr] \arrow[dr] & & U_{\beta(j)} \arrow[dl] \\ & U & \end{tikzcd}$$

with isomorphisms as horizontal arrows.

Lemma 9.8.3. *Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{\varphi_i : U_i \to U\}_{i \in I}$, and $\mathcal{V} = \{\psi_j : V_j \to U\}_{j \in J}$ be two families of morphisms with the same fixed target.*

(1) *If $\mathcal{U}$ and $\mathcal{V}$ are combinatorially equivalent then they are tautologically equivalent.*
(2) *If $\mathcal{U}$ and $\mathcal{V}$ are tautologically equivalent then $\mathcal{U}$ is a refinement of $\mathcal{V}$ and $\mathcal{V}$ is a refinement of $\mathcal{U}$.*
(3) *The relation ``being combinatorially equivalent'' is an equivalence relation on all families of morphisms with fixed target.*
(4) *The relation ``being tautologically equivalent'' is an equivalence relation on all families of morphisms with fixed target.*
(5) *The relation ``$\mathcal{U}$ refines $\mathcal{V}$ and $\mathcal{V}$ refines $\mathcal{U}$'' is an equivalence relation on all families of morphisms with fixed target.*

Proof. Omitted. $\square$

In the following lemma, given a category $\mathcal{C}$, a presheaf $\mathcal{F}$ on $\mathcal{C}$, a family $\mathcal{U} = \{U_i \to U\}_{i \in I}$ such that all fibre products $U_i \times_U U_{i'}$ exist, we say that *the sheaf condition for $\mathcal{F}$ with respect to $\mathcal{U}$* holds if the diagram (9.7.1.1) is an equalizer diagram.

Lemma 9.8.4. *Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{\varphi_i : U_i \to U\}_{i \in I}$, and $\mathcal{V} = \{\psi_j : V_j \to U\}_{j \in J}$ be two families of morphisms with the same fixed target. Assume that the fibre products $U_i \times_U U_{i'}$ and $V_j \times_U V_{j'}$ exist. If $\mathcal{U}$ and $\mathcal{V}$ are tautologically equivalent, then for any presheaf $\mathcal{F}$ on $\mathcal{C}$ the sheaf condition for $\mathcal{F}$ with respect to $\mathcal{U}$ is equivalent to the sheaf condition for $\mathcal{F}$ with respect to $\mathcal{V}$.*

Proof. First, note that if $\varphi : A \to B$ is an isomorphism in the category $\mathcal{C}$, then $\varphi^* : \mathcal{F}(B) \to \mathcal{F}(A)$ is an isomorphism. Let $\beta : J \to I$ be a map and let $\psi_j : V_j \to U_{\beta(j)}$ be isomorphisms over U which are assumed to exist by hypothesis. Let us show that the sheaf condition for $\mathcal{V}$ implies the sheaf condition for $\mathcal{U}$. Suppose given sections $s_i \in \mathcal{F}(U_i)$ such that

$$s_i|_{U_i \times_U U_{i'}} = s_{i'}|_{U_i \times_U U_{i'}}$$

in $\mathcal{F}(U_i \times_U U_{i'})$ for all pairs $(i, i') \in I \times I$. Then we can define $s_j = \psi_j^* s_{\beta(j)}$. For any pair $(j, j') \in J \times J'$ the morphism $\psi_j \times_{\mathrm{id}_U} \psi_{j'} : V_j \times_U V_{j'} \to U_{\beta(j)} \times_U U_{\beta(j')}$ is an isomorphism as well. Hence by transport of structure we see that

$$s_j|_{V_j \times_U V_{j'}} = s_{j'}|_{V_j \times_U V_{j'}}$$

as well. The sheaf condition w.r.t. $\mathcal{V}$ implies there exists a unique s such that $s|_{V_j} = s_j$ for all $j \in J$. By the first remark of the proof this implies that $s|_{U_i} = s_i$ for all $i \in \mathrm{Im}(\beta)$ as well.

Suppose that $i \in I$, $i \notin \mathrm{Im}(\beta)$. For such an i we have isomorphisms $U_i \to V_{\alpha(i)} \to U_{\beta(\alpha(i))}$ over U. This gives a morphism $U_i \to U_i \times_U U_{\beta(\alpha(i))}$ which is a section of the projection. Because s_i and $s_{\beta(\alpha(i))}$ restrict to the same element on the fibre product we conclude that $s_{\beta(\alpha(i))}$ pulls back to s_i via $U_i \to U_{\beta(\alpha(i))}$. Thus we see that also $s_i = s|_{U_i}$ as desired. $\square$

Lemma 9.8.5. *Let $\mathcal{C}$ be a category. Let Cov_i, $i = 1, 2$ be two sets of families of morphisms with fixed target which each define the structure of a site on $\mathcal{C}$.*

(1) *If every $\mathcal{U} \in Cov_1$ is tautologically equivalent to some $\mathcal{V} \in Cov_2$, then $Sh(\mathcal{C}, Cov_2) \subset Sh(\mathcal{C}, Cov_1)$. If also, every $\mathcal{U} \in Cov_2$ is tautologically quivalent to some $\mathcal{V} \in Cov_1$ then the category of sheaves are equal.*
(2) *Suppose that for each $\mathcal{U} \in Cov_1$ there exists a $\mathcal{V} \in Cov_2$ such that $\mathcal{V}$ refines $\mathcal{U}$. In this case $Sh(\mathcal{C}, Cov_2) \subset Sh(\mathcal{C}, Cov_1)$. If also for every $\mathcal{U} \in Cov_2$ there exists a $\mathcal{V} \in Cov_1$ such that $\mathcal{V}$ refines $\mathcal{U}$, then the categories of sheaves are equal.*

Proof. Part (1) follows directly from Lemma 9.8.4 and the definitions.

We advise the reader to **skip the proof of (2)** on a first reading. Let $\mathcal{F}$ be a sheaf of sets for the site $(\mathcal{C}, \mathrm{Cov}_2)$. Let $\mathcal{U} \in \mathrm{Cov}_1$, say $\mathcal{U} = \{U_i \to U\}_{i \in I}$. Choose a refinement $\mathcal{V} \in \mathrm{Cov}_2$ of $\mathcal{U}$, say $\mathcal{V} = \{V_j \to U\}_{j \in J}$ and refinement given by $\alpha : J \to I$ and $f_j : V_j \to U_{\alpha(j)}$.

First let $s, s' \in \mathcal{F}(U)$. If for all $i \in I$ we have $s|_{U_i} = s'|_{U_i}$, then we also have $s|_{V_j} = s'|_{V_j}$ for all $j \in J$. This implies that $s = s'$ by the sheaf condition for $\mathcal{F}$ with respect to Cov_2. Hence we see that the unicity in the sheaf condition for $\mathcal{F}$ and the site $(\mathcal{C}, \mathrm{Cov}_1)$ holds.

Next, suppose given $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_i \times_U U_{i'}} = s_{i'}|_{U_i \times_U U_{i'}}$ for all $i, i' \in I$. Set $s_j = f_j^*(s_{\alpha(j)}) \in \mathcal{F}(V_j)$. Since the morphisms f_j are morphisms over U we obtain induced morphisms $f_{jj'} : V_j \times_U V_{j'} \to U_{\alpha(i)} \times_U U_{\alpha(i')}$ compatible with the $f_j, f_{j'}$ via the projection maps. It follows that

$$s_j|_{V_j \times_U V_{j'}} = f_{jj'}^*(s_{\alpha(j)}|_{U_{\alpha(j)} \times_U U_{\alpha(j')}}) = f_{jj'}^*(s_{\alpha(j')}|_{U_{\alpha(j)} \times_U U_{\alpha(j')}}) = s_{j'}|_{V_j \times_U V_{j'}}$$

for all $j, j' \in J$. Hence, by the sheaf condition for $\mathcal{F}$ with respect to Cov_2, we get a section $s \in \mathcal{F}(U)$ which restricts to s_j on each V_j. We are done if we show s restricts to s_{i_0} on U_{i_0} for any $i_0 \in I$. For each $i_0 \in I$ the family $\mathcal{U}' = \{U_i \times_U U_{i_0} \to U_{i_0}\}_{i \in I}$ is an element of Cov_1 by the axioms of a site. Also, the family $\mathcal{V}' = \{V_j \times_U U_{i_0} \to U_{i_0}\}_{j \in J}$ is an element of Cov_2. Then $\mathcal{V}'$ refines $\mathcal{U}'$ via $\alpha : J \to I$ and the maps $f'_j = f_j \times \mathrm{id}_{U_{i_0}}$. The element s_{i_0} restricts to $s_i|_{U_i \times_U U_{i_0}}$ on the members of the covering $\mathcal{U}'$ and hence via $(f'_j)^*$ to the elements $s_j|_{V_j \times_U U_{i_0}}$ on the members of the covering $\mathcal{V}'$. By construction of s this is the same as the family of restrictions of $s|_{U_{i_0}}$ to the members of the covering $\mathcal{V}'$. Hence by the sheaf condition for $\mathcal{F}$ with respect to Cov_2 we see that $s|_{U_{i_0}} = s_{i_0}$ as desired. $\square$

Lemma 9.8.6. *Let $\mathcal{C}$ be a category. Let $Cov(\mathcal{C})$ be a proper class of coverings satisfying conditions (1), (2) and (3) of Definition 9.6.2. Let $Cov_1, Cov_2 \subset Cov(\mathcal{C})$ be two subsets of $Cov(\mathcal{C})$ which endow $\mathcal{C}$ with the structure of a site. If every covering $\mathcal{U} \in Cov(\mathcal{C})$ is combinatorially equivalent to a covering in Cov_1 and combinatorially equivalent to a covering in Cov_2, then $Sh(\mathcal{C}, Cov_1) = Sh(\mathcal{C}, Cov_2)$.*

Proof. This is clear from Lemmas 9.8.5 and 9.8.3 above as the hypothesis implies that every covering $\mathcal{U} \in \mathrm{Cov}_1 \subset \mathrm{Cov}(\mathcal{C})$ is combinatorially equivalent to an element of Cov_2, and similarly with the roles of Cov_1 and Cov_2 reversed. $\square$

9.9. The example of G-sets

As an example, consider the site $\mathcal{T}_G$ of Example 9.6.5. We will describe the category of sheaves on $\mathcal{T}_G$. The answer will turn out to be independent of the choices made in defining $\mathcal{T}_G$. In fact, during the proof we will need only the following properties of the site $\mathcal{T}_G$:

(a) $\mathcal{T}_G$ is a full subcategory of G-$Sets$,
(b) $\mathcal{T}_G$ contains the G-set ${}_GG$,
(c) $\mathcal{T}_G$ has fibre products and they are the same as in G-$Sets$,
(d) given $U \in Ob(\mathcal{T}_G)$ and a G-invariant subset $O \subset U$, there exists an object of $\mathcal{T}_G$ isomorphic to O, and
(e) any surjective family of maps $\{U_i \to U\}_{i\in I}$, with $U, U_i \in Ob(\mathcal{T}_G)$ is combinatorially equivalent to a covering of $\mathcal{T}_G$.

These properties hold by Sets, Lemmas 3.10.2 and 3.11.1.

Remark that the map

$$Hom_G({}_GG, {}_GG) \longrightarrow G^{opp}, \varphi \longmapsto \varphi(1)$$

is an isomorphism of groups. The inverse map sends $g \in G$ to the map $R_g : s \mapsto sg$ (i.e. right multiplication). Note that $R_{g_1g_2} = R_{g_2} \circ R_{g_1}$ so the opposite is necessary.

This implies that for every presheaf $\mathcal{F}$ on $\mathcal{T}_G$ the value $\mathcal{F}({}_GG)$ inherets the structure of a G-set as follows: $g \cdot s$ for $g \in G$ and $s \in \mathcal{F}({}_GG)$ defined by $\mathcal{F}(R_g)(s)$. This is a left action because

$$(g_1g_2) \cdot s = \mathcal{F}(R_{g_1g_2})(s) = \mathcal{F}(R_{g_2} \circ R_{g_1})(s) = \mathcal{F}(R_{g_1})(\mathcal{F}(R_{g_2})(s)) = g_1 \cdot (g_2 \cdot s).$$

Here we've used that $\mathcal{F}$ is contravariant. Note that if $\mathcal{F} \to \mathcal{G}$ is a morphism of presheaves of sets on $\mathcal{T}_G$ then we get a map $\mathcal{F}({}_GG) \to \mathcal{G}({}_GG)$ which is compatible with the G-actions we have just defined. All in all we have constructed a functor

$$PSh(\mathcal{T}_G) \longrightarrow G\text{-}Sets, \quad \mathcal{F} \longmapsto \mathcal{F}({}_GG).$$

We leave it to the reader to verify that this construction has the pleasing property that the representable presheaf h_U is mapped to something canonically isomorphic to U. In a formula $h_U({}_GG) = Hom_G({}_GG, U) \cong U$.

Suppose that S is a G-set. We define a presheaf $\mathcal{F}_S$ by the formula[2]

$$\mathcal{F}_S(U) = Mor_{G\text{-}Sets}(U, S).$$

This is clearly a presheaf. On the other hand, suppose that $\{U_i \to U\}_{i\in I}$ is a covering in $\mathcal{T}_G$. This implies that $\coprod_i U_i \to U$ is surjective. Thus it is clear that the map

$$\mathcal{F}_S(U) = Mor_{G\text{-}Sets}(U, S) \longrightarrow \prod \mathcal{F}_S(U_i) = \prod Mor_{G\text{-}Sets}(U_i, S)$$

is injective. And, given a family of G-equivariant maps $s_i : U_i \to S$, such that all the diagrams

$$\begin{tikzcd} U_i \times_U U_j \arrow[r] \arrow[d] & U_j \arrow[d, "s_j"] \\ U_i \arrow[r, "s_i"] & S \end{tikzcd}$$

commute, there is a unique G-equivariant map $s : U \to S$ such that s_i is the composition $U_i \to U \to S$. Namely, we just define $s(u) = s_i(u_i)$ where $i \in I$ is any index such that

[2] It may appear this is the representable presheaf defined by S. This may not be the case because S may not be an object of $\mathcal{T}_G$ which was chosen to be a sufficiently large set of G-sets.

there exists some $u_i \in U_i$ mapping to u under the map $U_i \to U$. The commutativity of the diagrams above implies exactly that this construction is well defined. All in all we have constructed a functor

$$G\text{-}Sets \longrightarrow Sh(\mathcal{T}_G), \quad S \longmapsto \mathcal{F}_S.$$

We now have the following diagram of categories and functors

$$\begin{tikzcd}
PSh(\mathcal{T}_G) \arrow[rr, "\mathcal{F}\mapsto\mathcal{F}({}_GG)"] & & G\text{-}Sets \arrow[dl, "S\mapsto\mathcal{F}_S"] \\
& Sh(\mathcal{T}_G) \arrow[ul] &
\end{tikzcd}$$

It is immediate from the definitions that $\mathcal{F}_S({}_GG) = Mor_G({}_GG, S) = S$, the last equality by evaluation at 1. This almost proves the following.

Proposition 9.9.1. *The functors $\mathcal{F} \mapsto \mathcal{F}({}_GG)$ and $S \mapsto \mathcal{F}_S$ define quasi-inverse equivalences between $Sh(\mathcal{T}_G)$ and G-Sets.*

Proof. We have already seen that composing the functors one way around is isomorphic to the identity functor. In the other direction, for any sheaf $\mathcal{H}$ there is a natural map of sheaves

$$can : \mathcal{H} \longrightarrow \mathcal{F}_{\mathcal{H}({}_GG)}.$$

Namely, for any object U of $\mathcal{T}_G$ we let can_U be the map

$$\begin{array}{ccc}
\mathcal{H}(U) & \longrightarrow & \mathcal{F}_{\mathcal{H}({}_GG)}(U) = Mor_G(U, \mathcal{H}({}_GG)) \\
s & \longmapsto & (u \mapsto \alpha_u^* s).
\end{array}$$

Here $\alpha_u : {}_GG \to U$ is the map $\alpha_u(g) = gu$ and $\alpha_u^* : \mathcal{H}(U) \to \mathcal{H}({}_GG)$ is the pullback map. A trivial but confusing verification shows that this is indeed a map of presheaves. We have to show that can is an isomorphism. We do this by showing can_U is an isomorphism for all $U \in \mathrm{ob}(\mathcal{T}_G)$. We leave the (important but easy) case that $U = {}_GG$ to the reader. A general object U of $\mathcal{T}_G$ is a disjoint union of G-orbits: $U = \coprod_{i\in I} O_i$. The family of maps $\{O_i \to U\}_{i\in I}$ is tautologically equivalent to a covering in $\mathcal{T}_G$ (by the properties of $\mathcal{T}_G$ listed at the beginning of this section). Hence by Lemma 9.8.4 the sheaf $\mathcal{H}$ satisfies the sheaf property with respect to $\{O_i \to U\}_{i\in I}$. The sheaf property for this covering implies $\mathcal{H}(U) = \prod_i \mathcal{H}(O_i)$. Hence it suffices to show that can_U is an isomorphism when U consists of a single G-orbit. Let $u \in U$ and let $H \subset G$ be its stabilizer. Clearly, $Mor_G(U, \mathcal{H}({}_GG)) = \mathcal{H}({}_GG)^H$ equals the subset of H-invariant elements. On the other hand consider the covering $\{{}_GG \to U\}$ given by $g \mapsto gu$ (again it is just combinatorially equivalent to some covering of $\mathcal{T}_G$, and again this doesn't matter). Note that the fibre product $({}_GG) \times_U ({}_GG)$ is equal to $\{(g, gh), g \in G, h \in H\} \cong \coprod_{h\in H} {}_GG$. Hence the sheaf property for this covering reads as

$$\begin{tikzcd}
\mathcal{H}(U) \arrow[r] & \mathcal{H}({}_GG) \arrow[r, shift left, "\mathrm{pr}_0^*"] \arrow[r, shift right, "\mathrm{pr}_1^*"'] & \prod_{h\in H} \mathcal{H}({}_GG).
\end{tikzcd}$$

The two maps pr_i^* into the factor $\mathcal{H}({}_GG)$ differ by multiplication by h. Now the result follows from this and the fact that can is an isomorphism for $U = {}_GG$. $\square$

9.10. Sheafification

In order to define the sheafification we study the zeroth Cech cohomology group of a covering and its functoriality properties.

Let $\mathcal{F}$ be a presheaf of sets on $\mathcal{C}$, and let $\mathcal{U} = \{U_i \to U\}_{i\in I}$ be a covering of $\mathcal{C}$. Let us use the notation $\mathcal{F}(\mathcal{U})$ to indicate the equalizer

$$H^0(\mathcal{U}, \mathcal{F}) = \{(s_i)_{i\in I} \in \prod\nolimits_i \mathcal{F}(U_i) \mid s_i|_{U_i\times_U U_j} = s_j|_{U_i\times_U U_j}\ \forall i,j\in I\}.$$

As we will see later, this is the zeroth Cech cohomology of $\mathcal{F}$ over U with respect to the covering $\mathcal{U}$. A small remark is that we can define $H^0(\mathcal{U}, \mathcal{F})$ as soon as all the morphisms $U_i \to U$ are representable, i.e., $\mathcal{U}$ need not be a covering of the site. There is a canonical map $\mathcal{F}(U) \to H^0(\mathcal{U}, \mathcal{F})$. It is clear that a morphism of coverings $\mathcal{U} \to \mathcal{V}$ induces commutative diagrams

$$\begin{tikzcd}
 & U_i \arrow[rr] & & V_{\alpha(i)} \\
U_i \times_U U_j \arrow[ur] \arrow[r] \arrow[dr] & & V_{\alpha(i)} \times_V V_{\alpha(j)} \arrow[ur] \arrow[dr] & \\
 & U_j \arrow[rr] & & V_{\alpha(j)}
\end{tikzcd}.$$

This in turn produces a map $H^0(\mathcal{V}, \mathcal{F}) \to H^0(\mathcal{U}, \mathcal{F})$, compatible with the map $\mathcal{F}(V) \to \mathcal{F}(U)$.

By construction, a presheaf $\mathcal{F}$ is a sheaf if and only if for every covering $\mathcal{U}$ of $\mathcal{C}$ the natural map $\mathcal{F}(U) \to H^0(\mathcal{U}, \mathcal{F})$ is bijective. We will use this notion to prove the following simple lemma about limits of sheaves.

Lemma 9.10.1. *Let $\mathcal{F} : \mathcal{I} \to Sh(\mathcal{C})$ be a diagram. Then $lim_{\mathcal{I}} \mathcal{F}$ exists and is equal to the limit in the category of presheaves.*

Proof. Let $lim_i \mathcal{F}_i$ be the limit as a presheaf. We will show that this is a sheaf and then it will trivially follow that it is a limit in the category of sheaves. To prove the sheaf property, let $\mathcal{V} = \{V_j \to V\}_{j\in J}$ be a covering. Let $(s_j)_{j\in J}$ be an element of $H^0(\mathcal{V}, lim_i \mathcal{F}_i)$. Using the projection maps we get elements $(s_{j,i})_{j\in J}$ in $H^0(\mathcal{V}, \mathcal{F}_i)$. By the sheaf poperty for $\mathcal{F}_i$ we see that there is a unique $s_i \in \mathcal{F}_i(V)$ such that $s_{j,i} = s_i|_{V_j}$. Let $\phi : i \to i'$ be a morphism of the index category. We would like to show that $\mathcal{F}(\phi) : \mathcal{F}_i \to \mathcal{F}_{i'}$ maps s_i to $s_{i'}$. We know this is true for the sections $s_{i,j}$ and $s_{i',j}$ for all j and hence by the sheaf property for $\mathcal{F}_{i'}$ this is true. At this point we have an element $s = (s_i)_{i\in Ob(\mathcal{I})}$ of $(lim_i \mathcal{F}_i)(V)$. We leave it to the reader to see this element has the required property that $s_j = s|_{V_j}$. □

Example 9.10.2. A particular example is the limit over the empty diagram. This gives the final object in the category of (pre)sheaves. It is the sheaf that associates to each object U of $\mathcal{C}$ a singleton set, with unique restriction mappings. We often denote this sheaf by $*$.

Let $\mathcal{J}_U$ be the category of all coverings of U. In other words, the objects of $\mathcal{J}_U$ are the coverings of U in $\mathcal{C}$, and the morphisms are the refinements. By our conventions on sites this is indeed a category, i.e., the collection of objects and morphisms forms a set. Note that $Ob(\mathcal{J}_U)$ is not empty since $\{\mathrm{id}_U\}$ is an object of it. According to the remarks above the construction $\mathcal{U} \mapsto H^0(\mathcal{U}, \mathcal{F})$ is a contravariant functor on $\mathcal{J}_U$. We define

$$\mathcal{F}^+(U) = colim_{\mathcal{J}_U^{opp}} H^0(\mathcal{U}, \mathcal{F})$$

See Categories, Section 4.13 for a discussion of limits and colimits. We point out that later we will see that $\mathcal{F}^+(U)$ is the zeroth Cech cohomology of $\mathcal{F}$ over U.

Before we say more about the structure of the colimit, we turn the collection of sets $\mathcal{F}^+(U)$, $U \in Ob(\mathcal{C})$ into a presheaf. Namely, let $V \to U$ be a morphism of $\mathcal{C}$. By the axioms of a site there is a functor[3]

$$\mathcal{J}_U \longrightarrow \mathcal{J}_V, \quad \{U_i \to U\} \longmapsto \{U_i \times_U V \to V\}.$$

Note that the projection maps furnish a functorial morphism of coverings $\{U_i \times_U V \to V\} \to \{U_i \to U\}$ and hence, by the construction above, a functorial map of sets $H^0(\{U_i \to U\}, \mathcal{F}) \to H^0(\{U_i \times_U V \to V\}, \mathcal{F})$. In other words, there is a transformation of functors from $H^0(-, \mathcal{F}) : \mathcal{J}_U \to Sets$ to the compostion $\mathcal{J}_U \to \mathcal{J}_V \xrightarrow{H^0(-,\mathcal{F})} Sets$. Hence by generalities of colimits we obtain a canonical map $\mathcal{F}^+(U) \to \mathcal{F}^+(V)$. In terms of the description of the set $\mathcal{F}^+(U)$ above, it just takes the element associated with $s = (s_i) \in H^0(\{U_i \to U\}, \mathcal{F})$ to the element associated with $(s_i|_{V\times_U U_i}) \in H^0(\{U_i \times_U V \to V\}, \mathcal{F})$.

Lemma 9.10.3. *The constructions above define a presheaf $\mathcal{F}^+$ together with a canonical map of presheaves $\mathcal{F} \to \mathcal{F}^+$.*

Proof. All we have to do is to show that given morphisms $W \to V \to U$ the composition $\mathcal{F}^+(U) \to \mathcal{F}^+(V) \to \mathcal{F}^+(W)$ equals the map $\mathcal{F}^+(U) \to \mathcal{F}^+(W)$. This can be shown directly by verifying that, given a covering $\{U_i \to U\}$ and $s = (s_i) \in H^0(\{U_i \to U\}, \mathcal{F})$, we have canonically $W \times_U U_i \cong W \times_V (V \times_U U_i)$, and $s_i|_{W\times_U U_i}$ corresponds to $(s_i|_{V\times_U U_i})|_{W\times_V(V\times_U U_i)}$ via this isomorphism. □

More indirectly, the result of Lemma 9.10.6 shows that we may pullback an element s as above via any morphism from any covering of W to $\{U_i \to U\}$ and we will always end up with the same element in $\mathcal{F}^+(W)$.

Lemma 9.10.4. *The association $\mathcal{F} \mapsto (\mathcal{F} \to \mathcal{F}^+)$ is a functor.*

Proof. Instead of proving this we state exactly what needs to be proven. Let $\mathcal{F} \to \mathcal{G}$ be a map of presehaves. Prove the commutativity of:

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F}^+ \arrow[d] \\ \mathcal{G} \arrow[r] & \mathcal{G}^+ \end{tikzcd}$$

□

The next two lemmas imply that the colimits above are colimits over a directed partially ordered set.

Lemma 9.10.5. *Given a pair of coverings $\{U_i \to U\}$ and $\{V_j \to U\}$ of a given object U of the site $\mathcal{C}$, there exists a covering which is a common refinement.*

[3]This construction actually involves a choice of the fibre products $U_i \times_U V$ and hence the axiom of choice. The resulting map does not depend on the choices made, see below.

Proof. Since $\mathcal{C}$ is a site we have that for every i the family $\{V_j \times_U U_i \to U_i\}_j$ is a covering. And, then another axiom implies that $\{V_j \times_U U_i \to U\}_{i,j}$ is a covering of U. Clearly this covering refines both given coverings. $\square$

Lemma 9.10.6. *Any two morphisms $f, g : \mathcal{U} \to \mathcal{V}$ of coverings inducing the same morphism $U \to V$ induce the same map $H^0(\mathcal{V}, \mathcal{F}) \to H^0(\mathcal{U}, \mathcal{F})$.*

Proof. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ and $\mathcal{V} = \{V_j \to V\}_{j \in J}$. The morphism f consists of a map $U \to V$, a map $\alpha : I \to J$ and maps $f_i : U_i \to V_{\alpha(i)}$. Likewise, g determines a map $\beta : I \to J$ and maps $g_i : U_i \to V_{\beta(i)}$. As f and g induce the same map $U \to V$, the diagram

$$\begin{tikzcd} & V_{\alpha(i)} \arrow[dr] & \\ U_i \arrow[ur, "f_i"] \arrow[dr, "g_i"'] & & V \\ & V_{\beta(i)} \arrow[ur] & \end{tikzcd}$$

is commutative for every $i \in I$. Hence f and g factor through the fibre product

$$\begin{tikzcd} & V_{\alpha(i)} \\ U_i \arrow[ur, "f_i"] \arrow[r, "\varphi"] \arrow[dr, "g_i"'] & V_{\alpha(i)} \times_V V_{\beta(i)} \arrow[u, "\mathrm{pr}_1"'] \arrow[d, "\mathrm{pr}_2"] \\ & V_{\beta(i)}. \end{tikzcd}$$

Now let $s = (s_j)_j \in H^0(\mathcal{V}, \mathcal{F})$. Then for all $i \in I$:

$$(f^* s)_i = f_i^*(s_{\alpha(i)}) = \varphi^* \mathrm{pr}_1^*(s_{\alpha(i)}) = \varphi^* \mathrm{pr}_2^*(s_{\beta(i)}) = g_i^*(s_{\beta(i)}) = (g^* s)_i,$$

where the middle equality is given by the definition of $H^0(\mathcal{V}, \mathcal{F})$. This shows that the maps $H^0(\mathcal{V}, \mathcal{F}) \to H^0(\mathcal{U}, \mathcal{F})$ induced by f and g are equal. $\square$

Remark 9.10.7. In particular this lemma shows that if $\mathcal{U}$ is a refinement of $\mathcal{V}$, and if $\mathcal{V}$ is a refinement of $\mathcal{U}$, then there is a canonical identification $H^0(\mathcal{U}, \mathcal{F}) = H^0(\mathcal{V}, \mathcal{F})$.

From these two lemmas, and the fact that $\mathcal{J}_U$ is nonempty, it follows that the diagram $H^0(-, \mathcal{F}) : \mathcal{J}_U^{opp} \to Sets$ is filtered, see Categories, Definition 4.17.1. Hence, by Categories, Section 4.17 the colimit $\mathcal{F}^+(U)$ may be described in the following straightforward manner. Namely, every element in the set $\mathcal{F}^+(U)$ arises from an element $s \in H^0(\mathcal{U}, \mathcal{F})$ for some covering $\mathcal{U}$ of U. Given a second element $s' \in H^0(\mathcal{U}', \mathcal{F})$ then s and s' determine the same element of the colimit if and only if there exists a covering $\mathcal{V}$ of U and refinements $f : \mathcal{V} \to \mathcal{U}$ and $f' : \mathcal{V} \to \mathcal{U}'$ such that $f^* s = (f')^* s'$ in $H^0(\mathcal{V}, \mathcal{F})$. Since the trivial covering $\{\mathrm{id}_U\}$ is an object of $\mathcal{J}_U$ we get a canonical map $\mathcal{F}(U) \to \mathcal{F}^+(U)$.

Lemma 9.10.8. *The map $\theta : \mathcal{F} \to \mathcal{F}^+$ has the following property: For every object U of $\mathcal{C}$ and every section $s \in \mathcal{F}^+(U)$ there exists a covering $\{U_i \to U\}$ such that $s|_{U_i}$ is in the image of $\theta : \mathcal{F}(U_i) \to \mathcal{F}^+(U_i)$.*

Proof. Namely, let $\{U_i \to U\}$ be a covering such that s arises from the element $(s_i) \in H^0(\{U_i \to U\}, \mathcal{F})$. According to Lemma 9.10.6 we may consider the covering $\{U_i \to U_i\}$ and the (obvious) morphism of coverings $\{U_i \to U_i\} \to \{U_i \to U\}$ to compute the pullback of s to an element of $\mathcal{F}^+(U_i)$. And indeed, using this covering we get exactly $\theta(s_i)$ for the restriction of s to U_i. □

Definition 9.10.9. We say that a presheaf of sets $\mathcal{F}$ on a site $\mathcal{C}$ is *separated* if, for all coverings of $\{U_i \to U\}$, the map $\mathcal{F}(U) \to \prod \mathcal{F}(U_i)$ is injective.

Theorem 9.10.10. *With $\mathcal{F}$ as above*

(1) *The presheaf $\mathcal{F}^+$ is separated.*
(2) *If $\mathcal{F}$ is separated, then $\mathcal{F}^+$ is a sheaf and the map of presheaves $\mathcal{F} \to \mathcal{F}^+$ is injective.*
(3) *If $\mathcal{F}$ is a sheaf, then $\mathcal{F} \to \mathcal{F}^+$ is an isomorphism.*
(4) *The presheaf $\mathcal{F}^{++}$ is always a sheaf.*

Proof. Proof of (1). Suppose that $s, s' \in \mathcal{F}^+(U)$ and suppose that there exists some covering $\{U_i \to U\}$ such that $s|_{U_i} = s'|_{U_i}$ for all i. We now have three coverings of U: the covering $\{U_i \to U\}$ above, a covering $\mathcal{U}$ for s as in Lemma 9.10.8, and a similar covering $\mathcal{U}'$ for s'. By Lemma 9.10.5, we can find a common refinement, say $\{W_j \to U\}$. This means we have $s_j, s'_j \in \mathcal{F}(W_j)$ such that $s|_{W_j} = \theta(s_j)$, similarly for $s'|_{W_j}$, and such that $\theta(s_j) = \theta(s'_j)$. This last equality means that there exists some covering $\{W_{jk} \to W_j\}$ such that $s_j|_{W_{jk}} = s'_j|_{W_{jk}}$. Then since $\{W_{jk} \to U\}$ is a covering we see that s, s' map to the same element of $H^0(\{W_{jk} \to U\}, \mathcal{F})$ as desired.

Proof of (2). It is clear that $\mathcal{F} \to \mathcal{F}^+$ is injective because all the maps $\mathcal{F}(U) \to H^0(\mathcal{U}, \mathcal{F})$ are injective. It is also clear that, if $\mathcal{U} \to \mathcal{U}'$ is a refinement, then $H^0(\mathcal{U}', \mathcal{F}) \to H^0(\mathcal{U}, \mathcal{F})$ is injective. Now, suppose that $\{U_i \to U\}$ is a covering, and let (s_i) be a family of elements of $\mathcal{F}^+(U_i)$ satisfying the sheaf condition $s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j}$ for all $i, j \in I$. Choose coverings (as in Lemma 9.10.8) $\{U_{ij} \to U_i\}$ such that $s_i|_{U_{ij}}$ is the image of the (unique) element $s_{ij} \in \mathcal{F}(U_{ij})$. The sheaf condition implies that s_{ij} and $s_{i'j'}$ agree over $U_{ij} \times_U U_{i'j'}$ because it maps to $U_i \times_U U_{i'}$ and we have the equality there. Hence $(s_{ij}) \in H^0(\{U_{ij} \to U\}, \mathcal{F})$ gives rise to an element $s \in \mathcal{F}^+(U)$. We leave it to the reader to verify that $s|_{U_i} = s_i$.

Proof of (3). This is immediate from the definitions because the sheaf property says exactly that every map $\mathcal{F} \to H^0(\mathcal{U}, \mathcal{F})$ is bijective (for every covering $\mathcal{U}$ of U).

Statement (4) is now obvious. □

Definition 9.10.11. Let $\mathcal{C}$ be a site and let $\mathcal{F}$ be a presheaf of sets on $\mathcal{C}$. The sheaf $\mathcal{F}^\# := \mathcal{F}^{++}$ together with the canonical map $\mathcal{F} \to \mathcal{F}^\#$ is called the *sheaf associated to $\mathcal{F}$*.

Proposition 9.10.12. *The canonical map $\mathcal{F} \to \mathcal{F}^\#$ has the following universal property: For any map $\mathcal{F} \to \mathcal{G}$, where $\mathcal{G}$ is a sheaf of sets, there is a unique map $\mathcal{F}^\# \to \mathcal{G}$ such that $\mathcal{F} \to \mathcal{F}^\# \to \mathcal{G}$ equals the given map.*

Proof. By Lemma 9.10.4 we get a commutative diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F}^+ \arrow[r] \arrow[d] & \mathcal{F}^{++} \arrow[d] \\ \mathcal{G} \arrow[r] & \mathcal{G}^+ \arrow[r] & \mathcal{G}^{++} \end{tikzcd}$$

and by Theorem 9.10.10 the lower horizontal maps are isomorphisms. The uniqueness follows from Lemma 9.10.8 which says that every section of $\mathcal{F}^\#$ locally comes from sections of $\mathcal{F}$. □

It is clear from this result that the functor $\mathcal{F} \mapsto (\mathcal{F} \to \mathcal{F}^\#)$ is unique up to unique isomorphism of functors. Actually, let us temporarily denote $i : Sh(\mathcal{C}) \to PSh(\mathcal{C})$ the functor of inclusion. The result above actually says that

$$Mor_{PSh(\mathcal{C})}(\mathcal{F}, i(\mathcal{G})) = Mor_{Sh(\mathcal{C})}(\mathcal{F}^\#, \mathcal{G}).$$

In other words, the functor of sheafification is the left adjoint to the inclusion functor i. We finish this section with a couple of lemmas.

Lemma 9.10.13. *Let $\mathcal{F} : \mathcal{I} \to Sh(\mathcal{C})$ be a diagram. Then $colim_\mathcal{I} \mathcal{F}$ exists and is the sheafification of the colimit in the category of presheaves.*

Proof. Since the sheafification functor is a left adjoint it commutes with all colimits, see Categories, Lemma 4.22.2. Hence, since $PSh(\mathcal{C})$ has colimits, we deduce that $Sh(\mathcal{C})$ has colimits (which are the sheafifications of the colimits in presheaves). □

Lemma 9.10.14. *The functor $PSh(\mathcal{C}) \to Sh(\mathcal{C})$, $\mathcal{F} \mapsto \mathcal{F}^\#$ is exact.*

Proof. Since it is a left adjoint it is right exact, see Categories, Lemma 4.22.3. On the other hand, by Lemmas 9.10.5 and Lemma 9.10.6 the colimits in the construction of $\mathcal{F}^+$ are really over the directed partially ordered set $Ob(\mathcal{J}_U)$ where $\mathcal{U} \geq \mathcal{U}'$ if and only if $\mathcal{U}$ is a refinement of $\mathcal{U}'$. Hence by Categories, Lemma 4.17.2 we see that $\mathcal{F} \to \mathcal{F}^+$ commutes with finite limits (as a functor from presheaves to presheaves). Then we conclude using Lemma 9.10.1. □

Lemma 9.10.15. *Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be a presheaf of sets on $\mathcal{C}$. Denote $\theta^2 : \mathcal{F} \to \mathcal{F}^\#$ the canonical map of $\mathcal{F}$ into its sheafification. Let U be an object of $\mathcal{C}$. Let $s \in \mathcal{F}^\#(U)$. There exists a covering $\{U_i \to U\}$ and sections $s_i \in \mathcal{F}(U_i)$ such that*

1. *$s|_{U_i} = \theta^2(s_i)$, and*
2. *for every i, j there exists a covering $\{U_{ijk} \to U_i \times_U U_j\}$ of $\mathcal{C}$ such that the pullback of s_i and s_j to each U_{ijk} agree.*

Conversely, given any covering $\{U_i \to U\}$, elements $s_i \in \mathcal{F}(U_i)$ such that (2) holds, then there exists a unique section $s \in \mathcal{F}^\#(U)$ such that (1) holds.

Proof. Omitted. □

9.11. Injective and surjective maps of sheaves

Definition 9.11.1. Let $\mathcal{C}$ be a site, and let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of sheaves of sets.

1. We say that φ is *injective* if for every object U of $\mathcal{C}$ the map $\varphi : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective.
2. We say that φ is *surjective* if for every object U of $\mathcal{C}$ and every section $s \in \mathcal{F}(U)$ there exists a covering $\{U_i \to U\}$ such that for all i the restriction $s|_{U_i}$ is in the image of $\varphi : \mathcal{F}(U_i) \to \mathcal{G}(U_i)$.

Lemma 9.11.2. *The injective (resp. surjective) maps defined above are exactly the monomorphisms (resp. epimorphisms) of the category $Sh(\mathcal{C})$. A map of sheaves is an isomorphism if and only if it is both injective and surjective.*

Proof. Omitted. □

9.12. Representable sheaves

Let $\mathcal{C}$ be a category. The canonical topology is the finest topology such that all representable presheaves are sheaves (it is formally defined in Definition 9.40.12 but we will not need this). This topology is not always the topology associated to the structure of a site on $\mathcal{C}$. We will give a collection of coverings that generates this topology in case $\mathcal{C}$ has fibered products. First we give the following general definition.

Definition 9.12.1. Let $\mathcal{C}$ be a category. We say that a family $\{U_i \to U\}$ is an *effective epimorphism* if all the morphisms $U_i \to U$ are representable (see Categories, Definition 4.6.3), and for any $X \in Ob(\mathcal{C})$ the sequence

$$Mor_{\mathcal{C}}(U, X) \longrightarrow Mor_{\mathcal{C}}(U_i, X) \rightrightarrows Mor_{\mathcal{C}}(U_i \times_U U_j, X)$$

is an equalizer diagram. We say that a family $\{U_i \to U\}$ is a *universal effective epimorphism* if for any morphism $V \to U$ the base change $\{U_i \times_U V \to V\}$ is an effective epimorphism.

The class of families which are universal effective epimorphisms satisfies the axioms of Definition 9.6.2. If $\mathcal{C}$ has fibre products, then the associated topology is the canonical topology. (In this case, to get a site argue as in Sets, Lemma 3.11.1.)

Conversely, suppose that $\mathcal{C}$ is a site such that all representable presheaves are sheaves. Then clearly, all coverings are universal effective epimorphisms. Thus the following definition is the ``correct'' one in the setting of sites.

Definition 9.12.2. We say that the topology on a site $\mathcal{C}$ is *weaker than the canonical topology*, or that the topology is *subcanonical* if all the coverings of $\mathcal{C}$ are universal effective epimorphisms.

A representable sheaf is a representable presheaf which is also a sheaf. Since it is perhap better to avoid this terminology when the topology is not subcanonical, we only define it formally in that case.

Definition 9.12.3. Suppose that the topology on the site $\mathcal{C}$ is weaker than the canonical topology. The Yoneda embedding h (see Categories, Section 4.3) actually presents $\mathcal{C}$ as a full subcategory of the category of sheaves of $\mathcal{C}$. In this case we sometimes write $\underline{U} = h_U$ or simply U for the *representable sheaf* associated to the object U of $\mathcal{C}$.

Note that we have in the situation of the definition

$$Mor_{Sh(\mathcal{C})}(\underline{U}, \mathcal{F}) = \mathcal{F}(U)$$

for every sheaf $\mathcal{F}$, since after all the same thing was true for presheaves. In general (but only rarely) the presheaves h_U are not sheaves and to get a sheaf you have to sheafifiy them. In this case it will still be true that $Mor_{Sh(\mathcal{C})}(h_U^\#, \mathcal{F}) = \mathcal{F}(U)$ for every sheaf $\mathcal{F}$ by the adjointness property of #.

The next lemma says that, if the topology is weaker than the canonical topology, every sheaf is made up out of representable sheaves in a way.

Lemma 9.12.4. *Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be a sheaf of sets. There exists a diagram of sheaves of sets*

$$\mathcal{F}_1 \rightrightarrows \mathcal{F}_0 \longrightarrow \mathcal{F}$$

which represents $\mathcal{F}$ as a coequalizer, such that $\mathcal{F}_i$, $i = 0, 1$ are coproducts of sheaves of the form $h_U^\#$.

Proof. First we show there is an epimorphism $\mathcal{F}_0 \to \mathcal{F}$ of the desired type. Namely, just take

$$\mathcal{F}_0 = \coprod\nolimits_{U \in Ob(\mathcal{C}), s \in \mathcal{F}(U)} (h_U)^\# \longrightarrow \mathcal{F}$$

Here the arrow restricted to the component corresponding to (U, s) maps the element $\mathrm{id}_U \in h_U^\#(U)$ to the section $s \in \mathcal{F}(U)$. This is an epimorphism according to Lemma 9.11.2 above. To construct $\mathcal{F}_1$ first set $\mathcal{G} = \mathcal{F}_0 \times_{\mathcal{F}} \mathcal{F}_0$ and then construct an epimorphism $\mathcal{F}_1 \to \mathcal{G}$ as above. □

Lemma 9.12.5. *Let $\mathcal{C}$ be a site. If $\{U_i \to U\}_{i \in I}$ is a covering of the site $\mathcal{C}$, then the morphism of presheaves of sets*

$$\coprod\nolimits_{i \in I} h_{U_i} \to h_U$$

becomes surjective after sheafification.

Proof. By Lemma 9.11.2 above we have to show that $\coprod_{i \in I} h_{U_i}^\# \to h_U^\#$ is an epimorphism. Let $\mathcal{F}$ be a sheaf of sets. A morphism $h_U^\# \to \mathcal{F}$ corresponds to a section $s \in \mathcal{F}(U)$. Hence the injectivity of $Mor(h_U^\#, \mathcal{F}) \to \prod_i Mor(h_{U_i}^\#, \mathcal{F})$ follows directly from the sheaf property of $\mathcal{F}$. □

9.13. Continuous functors

Definition 9.13.1. Let $\mathcal{C}$ and $\mathcal{D}$ be sites. A functor $u : \mathcal{C} \to \mathcal{D}$ is called *continuous* if for every $\{V_i \to V\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})$ we have the following

(1) $\{u(V_i) \to u(V)\}_{i \in I}$ is in $\mathrm{Cov}(\mathcal{D})$, and
(2) for any morphism $T \to V$ in $\mathcal{C}$ the morphism $u(T \times_V V_i) \to u(T) \times_{u(V)} u(V_i)$ is an isomorphism.

Recall that given a functor u as above, and a presheaf of sets $\mathcal{F}$ on $\mathcal{D}$ we have defined $u^p\mathcal{F}$ to be simply the presheaf $\mathcal{F} \circ u$, in other words

$$u^p\mathcal{F}(V) = \mathcal{F}(u(V))$$

for every object V of $\mathcal{C}$.

Lemma 9.13.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous functor. If $\mathcal{F}$ is a sheaf on $\mathcal{D}$ then $u^p\mathcal{F}$ is a sheaf as well.*

Proof. Let $\{V_i \to V\}$ be a covering. By assumption $\{u(V_i) \to u(V)\}$ is a covering in $\mathcal{D}$ and $u(V_i \times_V V_j) = u(V_i) \times_{u(V)} u(V_j)$. Hence the sheaf condition for $u^p\mathcal{F}$ and the covering $\{V_i \to V\}$ is precisely the same as the sheaf condition for $\mathcal{F}$ and the covering $\{u(V_i) \to u(V)\}$. □

In order to avoid confusion we sometimes denote

$$u^s : Sh(\mathcal{D}) \longrightarrow Sh(\mathcal{C})$$

the functor u^p restricted to the subcategory of sheaves of sets.

Lemma 9.13.3. *In the situation of Lemma 9.13.2. The functor $u_s : \mathcal{G} \mapsto (u_p\mathcal{G})^\#$ is a left adjoint to u^s.*

Proof. Follows directly from Lemma 9.5.4 and Proposition 9.10.12. □

Here is a technical lemma.

Lemma 9.13.4. *In the situation of Lemma 9.13.2. For any presheaf $\mathcal{G}$ on $\mathcal{C}$ we have $(u_p\mathcal{G})^\# = (u_p(\mathcal{G}^\#))^\#$.*

Proof. For any sheaf $\mathcal{F}$ on $\mathcal{D}$ we have

$$\begin{aligned} Mor_{Sh(\mathcal{D})}(u_s(\mathcal{G}^\#), \mathcal{F}) &= Mor_{Sh(\mathcal{C})}(\mathcal{G}^\#, u^s\mathcal{F}) \\ &= Mor_{PSh(\mathcal{C})}(\mathcal{G}^\#, u^p\mathcal{F}) \\ &= Mor_{PSh(\mathcal{C})}(\mathcal{G}, u^p\mathcal{F}) \\ &= Mor_{PSh(\mathcal{D})}(u_p\mathcal{G}, \mathcal{F}) \\ &= Mor_{Sh(\mathcal{D})}((u_p\mathcal{G})^\#, \mathcal{F}) \end{aligned}$$

and the result follows from the Yoneda lemma. □

Lemma 9.13.5. *Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous functor between sites. For any object U of $\mathcal{C}$ we have $u_s h_U^\# = h_{u(U)}^\#$.*

Proof. Follows from Lemmas 9.5.6 and 9.13.4. □

Remark 9.13.6. (Skip on first reading.) Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let us use the definition of tautologically equivalent families of maps, see Definition 9.8.2 to (slightly) weaken the conditions defining continuity. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Let us call u *quasi-continuous* if for every $\mathcal{V} = \{V_i \to V\}_{i\in I} \in \text{Cov}(\mathcal{C})$ we have the following

(1') the family of maps $\{u(V_i) \to u(V)\}_{i\in I}$ is tautologically equivalent to an element of $\text{Cov}(\mathcal{D})$, and
(2) for any morphism $T \to V$ in $\mathcal{C}$ the morphism $u(T \times_V V_i) \to u(T) \times_{u(V)} u(V_i)$ is an isomorphism.

We are going to see that Lemmas 9.13.2 and 9.13.3 hold in case u is quasi-continuous as well.

We first remark that the morphisms $u(V_i) \to u(V)$ are representable, since they are isomorphic to representable morphisms (by the first condition). In particular, the family $u(\mathcal{V}) = \{u(V_i) \to u(V)\}_{i\in I}$ gives rise to a zeroth Cech cohomology group $H^0(u(\mathcal{V}), \mathcal{F})$ for any presheaf $\mathcal{F}$ on $\mathcal{D}$. Let $\mathcal{U} = \{U_j \to u(V)\}_{j\in J}$ be an element of $\text{Cov}(\mathcal{D})$ tautologically equivalent to $\{u(V_i) \to u(V)\}_{i\in I}$. Note that $u(\mathcal{V})$ is a refinement of $\mathcal{U}$ and vice versa. Hence by Remark 9.10.7 we see that $H^0(u(\mathcal{V}), \mathcal{F}) = H^0(\mathcal{U}, \mathcal{F})$. In particular, if $\mathcal{F}$ is a sheaf, then $\mathcal{F}(u(V)) = H^0(u(\mathcal{V}), \mathcal{F})$ because of the sheaf property expressed in terms of zeroth Cech cohomology groups. We conclude that $u^p\mathcal{F}$ is a sheaf if $\mathcal{F}$ is a sheaf, since $H^0(\mathcal{V}, u^p\mathcal{F}) = H^0(u(\mathcal{V}), \mathcal{F})$ which we just observed is equal to $\mathcal{F}(u(V)) = u^p\mathcal{F}(V)$. Thus Lemma 9.13.2 holds. Lemma 9.13.3 follows immediately.

9.14. Morphisms of sites

Definition 9.14.1. Let $\mathcal{C}$ and $\mathcal{D}$ be sites. A *morphism of sites* $f : \mathcal{D} \to \mathcal{C}$ is given by a continuous functor $u : \mathcal{C} \to \mathcal{D}$ such that the functor u_s is exact.

Notice how the functor u goes in the direction *opposite* the morphism f. If $f \leftrightarrow u$ is a morphism of sites then we use the notation $f^{-1} = u_s$ and $f_* = u^s$. The functor f^{-1} is called the *pullback functor* and the functor f_* is called the *push forward functor*. As in topology we have the following adjointness property

$$Mor_{Sh(\mathcal{D})}(f^{-1}\mathcal{G}, \mathcal{F}) = Mor_{Sh(\mathcal{C})}(\mathcal{G}, f_*\mathcal{F})$$

The motivation for this definition comes from the following example.

Example 9.14.2. Let $f : X \to Y$ be a continuous map of topological spaces. Recall that we have sites $\mathcal{T}_X$ and $\mathcal{T}_Y$, see Example 9.6.4. Consider the functor $u : \mathcal{T}_Y \to \mathcal{T}_X$, $V \mapsto f^{-1}(V)$. This functor is clearly continuous because inverse images of open coverings are open coverings. (Actually, this depends on how you chose sets of coverings for $\mathcal{T}_X$ and $\mathcal{T}_Y$. But in any case the functor is quasi-continuous, see Remark 9.13.6.) It is easy to verify that the functor u^s equals the usual pushforward functor f_* from topology. Hence, since u_s is an adjoint and since the usual topological pullback functor f^{-1} is an adjoint as well, we get a canonical isomorphism $f^{-1} \cong u_s$. Since f^{-1} is exact we deduce that u_s is exact. Hence u defines a morphism of sites $f : \mathcal{T}_X \to \mathcal{T}_Y$, which we may denote f as well since we've already seen the functors u_s, u^s agree with their usual notions anyway.

Lemma 9.14.3. *Let $\mathcal{C}_i$, $i = 1, 2, 3$ be sites. Let $u : \mathcal{C}_2 \to \mathcal{C}_1$ and $v : \mathcal{C}_3 \to \mathcal{C}_2$ be continuous functors which induce morphisms of sites. Then the functor $u \circ v : \mathcal{C}_3 \to \mathcal{C}_1$ is continuous and defines a morphism of sites $\mathcal{C}_1 \to \mathcal{C}_3$.*

Proof. It is immediate from the definitions that $u \circ v$ is a continuous functor. In addition, we clearly have $(u \circ v)^p = v^p \circ u^p$, and hence $(u \circ v)^s = v^s \circ u^s$. Hence functors $(u \circ v)_s$ and $u_s \circ v_s$ are both left adjoints of $(u \circ v)^s$. Therefore $(u \circ v)_s \cong u_s \circ v_s$ and we conclude that $(u \circ v)_s$ is exact as a composition of exact functors. $\square$

Definition 9.14.4. Let $\mathcal{C}_i$, $i = 1, 2, 3$ be sites. Let $f : \mathcal{C}_1 \to \mathcal{C}_2$ and $g : \mathcal{C}_2 \to \mathcal{C}_3$ be morphisms of sites given by continuous functors $u : \mathcal{C}_2 \to \mathcal{C}_1$ and $v : \mathcal{C}_3 \to \mathcal{C}_2$. The *composition* $g \circ f$ is the morphism of sites corresponding to the functor $u \circ v$.

In this situation we have $(g \circ f)_* = g_* \circ f_*$ and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ (see proof of Lemma 9.14.3).

Lemma 9.14.5. *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be continuous. Assume all the categories $(\mathcal{I}_V^u)^{opp}$ of Section 9.5 are filtered. Then u defines a morphism of sites $\mathcal{D} \to \mathcal{C}$, in other words u_s is exact.*

Proof. Since u_s is the left adjoint of u^s we see that u_s is right exact, see Categories, Lemma 4.22.3. Hence it suffices to show that u_s is left exact. In other words we have to show that u_s commutes with finite limits. Because the categories $\mathcal{I}_Y^{opp}$ are filtered we see that u_p commutes with finite limits, see Categories, Lemma 4.17.2 (this also uses the description of limits in *PSh*, see Section 9.4). And since sheafification commutes with finite limits as well (Lemma 9.10.14) we conclude because $u_s = \# \circ u_p$. $\square$

Proposition 9.14.6. *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be continuous. Assume furthermore the following:*

(1) *the category $\mathcal{C}$ has a final object X and $u(X)$ is a final object of $\mathcal{D}$, and*
(2) *the category $\mathcal{C}$ has fibre products and u commutes with them.*

Then u defines a morphism of sites $\mathcal{D} \to \mathcal{C}$, in other words u_s is exact.

Proof. This follows from Lemmas 9.5.2 and 9.14.5. $\square$

Remark 9.14.7. The conditions of Proposition 9.14.6 above are equivalent to saying that u is left exact, i.e., commutes with finite limits. See Categories, Lemmas 4.16.4 and 4.21.2. It seems more natural to phrase it in terms of final objects and fibre products since this seems to have more geometric meaning in the examples.

Remark 9.14.8. (Skip on first reading.) Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Analogously to Definition 9.14.1 we say that a *quasi-morphism of sites* $f : \mathcal{D} \to \mathcal{C}$ is given by a quasi-continuous functor $u : \mathcal{C} \to \mathcal{D}$ (see Remark 9.13.6) such that u_s is exact. The analogue of Proposition 9.14.6 in this setting is obtained by replacing the word ``continuous" by the word ``quasi-continuous", and replacing the word ``morphism" by ``quasi-morphism". The proof is literally the same.

9.15. Topoi

Here is a definition of a topos which is suitable for our purposes. Namely, a topos is the category of sheaves on a site. In order to specify a topos you just specify the site. The real difference between a topos and a site lies in the definition of morphisms. Namely, it turns out that there are lots of morphisms of topoi which do not come from morphisms of the underlying sites.

Definition 9.15.1. Topoi.

(1) A *topos* is the category $Sh(\mathcal{C})$ of sheaves of sets on a site $\mathcal{C}$.
(2) Let $\mathcal{C}$, $\mathcal{D}$ be sites. A *morphism of topoi* f from $Sh(\mathcal{D})$ to $Sh(\mathcal{C})$ is given by a pair of functors $f_* : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ and $f^{-1} : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ such that
  (a) we have
  $$Mor_{Sh(\mathcal{D})}(f^{-1}\mathcal{G}, \mathcal{F}) = Mor_{Sh(\mathcal{C})}(\mathcal{G}, f_*\mathcal{F})$$
  bifunctorially, and
  (b) the functor f^{-1} commutes with finite limits, i.e., is left exact.
(3) Let $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ be sites. Given morphisms of topoi $f : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ and $g : Sh(\mathcal{E}) \to Sh(\mathcal{D})$ the *composition* $f \circ g$ is the morphism of topoi defined by the functors $(f \circ g)_* = f_* \circ g_*$ and $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

Note that, being an adjoint pair, the functor f_* commutes with all limits and that f^{-1} commutes with all colimits, see Categories, Lemma 4.22.2. In particular, f^{-1} is exact.

Suppose that $\alpha : \mathcal{S}_1 \to \mathcal{S}_2$ is an equivalence of (possibly ``big") categories. If $\mathcal{S}_1$, $\mathcal{S}_2$ are topoi, then setting $f_* = \alpha$ and f^{-1} equal to the quasi-inverse of α gives a morphism $f : \mathcal{S}_1 \to \mathcal{S}_2$ of topoi. Moreover this morphism is an equivalence in the 2-category of topoi (see Section 9.32). Thus it makes sense to say ``$\mathcal{S}$ is a topos" if $\mathcal{S}$ is equivalent to the category of sheaves on a site (and not necessarily equal to the the category of sheaves on a site). We will occasionally use this abuse of notation.

Remark 9.15.2. (Set theoretical issues related to morphisms of topoi. Skip on a first reading.) A morphism of topoi as defined above is not a set but a class. In other words it is given by a mathematical formula rather than a mathematical object. Allthough we may contemplate the collection of all morphisms between two given topoi, it is not a good idea to introduce it as a mathematical object. On the other hand, suppose $\mathcal{C}$ and $\mathcal{D}$ are given sites. Consider a functor $\Phi : \mathcal{C} \to Sh(\mathcal{D})$. Such a thing is a set, in other words, it is a mathematical object. We may, in succession, ask the following questions on Φ.

(1) Is it true, given a sheaf $\mathcal{F}$ on $\mathcal{D}$, that the rule $U \mapsto Mor_{Sh(\mathcal{D})}(\Phi(U), \mathcal{F})$ defines a sheaf on $\mathcal{C}$? If so, this defines a functor $\Phi_* : Sh(\mathcal{D}) \to Sh(\mathcal{C})$.
(2) Is it true that Φ_* has a left adjoint? If so, write Φ^{-1} for this left adjoint.
(3) Is it true that Φ^{-1} is exact?

If the last question still has the answer ``yes", then we obtain a morphism of topoi (Φ_*, Φ^{-1}). Moreover, given any morphism of topoi (f_*, f^{-1}) we may set $\Phi(U) = f^{-1}(h_U^\#)$ and obtain a functor Φ as above with $f_* \cong \Phi_*$ and $f^{-1} \cong \Phi^{-1}$ (compatible with adjoint property). The upshot is that by working with the collection of Φ instead of morphisms of topoi, we (a) replaced the notion of a morphism of topoi by a mathematical object, and (b) the collection of Φ forms a class (and not a collection of classes). Of course, more can be said, for example one can work out more precisely the significance of condition (2) above for example; we do this in the case of points of topoi in Section 9.28.

Most geometrically interesting morphisms of topoi come about via Lemma 9.19.1 and the following lemma.

Lemma 9.15.3. *Given a morphism of sites $f : \mathcal{D} \to \mathcal{C}$ corresponding to the functor $u : \mathcal{C} \to \mathcal{D}$ the pair of functors $(f^{-1} = u_s, f_* = u^s)$ is a morphism of topoi.*

Proof. This is obvious from Definition 9.14.1. □

The simplest example of a site is perhaps the site whose category has exactly one object pt and one morphism id_{pt} and whose only covering is the covering $\{\mathrm{id}_{pt}\}$. We will simply write pt for this site. It is clear that the category of sheaves = the category of presheaves = the category of sets. In a formula $Sh(pt) = Sets$.

Remark 9.15.4. There are many sites that give rise to the topos $Sh(pt)$. A useful example is the following. Suppose that S is a set (of sets) which contains at least one nonempty element. Let $\mathcal{S}$ be the category whose objects are elements of S and whose morphisms are arbitrary set maps. Assume that $\mathcal{S}$ has fibre products. For example this will be the case if $S = \mathcal{P}(\text{infinite set})$ is the power set of any infinite set (exercise in set theory). Make $\mathcal{S}$ into a site by declaring surjective families of maps to be coverings (and choose a suitable sufficiently large set of covering families as in Sets, Section 3.11). We claim that $Sh(\mathcal{S})$ is equivalent to the category of sets.

We first prove this in case S contains $e \in S$ which is a singleton. In this case, there is an equivalence of topoi $i : Sh(pt) \to Sh(\mathcal{S})$ given by the functors

$$i^{-1}\mathcal{F} = \mathcal{F}(e), \quad i_*E = (U \mapsto Mor_{Sets}(U, E)) \tag{9.15.4.1}$$

Namely, suppose that $\mathcal{F}$ is a sheaf on $\mathcal{S}$. For any $U \in Ob(\mathcal{S}) = S$ we can find a covering $\{\varphi_u : e \to U\}_{u\in U}$, where φ_u maps the unique element of e to $u \in U$. The sheaf condition implies in this case that $\mathcal{F}(U) = \prod_{u\in U} \mathcal{F}(e)$. In other words $\mathcal{F}(U) = Mor_{Sets}(U, \mathcal{F}(e))$. Moreover, this rule is compatible with restriction mappings. Hence the functor

$$i_* : Sets = Sh(pt) \longrightarrow Sh(\mathcal{S}), \quad E \longmapsto (U \mapsto Mor_{Sets}(U, E))$$

is an equivalence of categories, and its inverse is the functor i^{-1} given above.

If $\mathcal{S}$ does not contain a singleton, then the functor i_* as defined above still makes sense. To show that it is still an equivalence in this case, choose any nonempty $\tilde{e} \in S$ and a map $\varphi : \tilde{e} \to \tilde{e}$ whose image is a singleton. For any sheaf $\mathcal{F}$ set

$$\mathcal{F}(e) := \mathrm{Im}(\mathcal{F}(\varphi) : \mathcal{F}(\tilde{e}) \longrightarrow \mathcal{F}(\tilde{e}))$$

and show that this is a quasi-inverse to i_*. Details omitted.

Remark 9.15.5. (Skip on first reading.) Let $\mathcal{C}$ and $\mathcal{D}$ be sites. A quasi-morphism of sites $f : \mathcal{D} \to \mathcal{C}$ (see Remark 9.14.8) gives rise to a morphis of topoi f from $Sh(\mathcal{D})$ to $Sh(\mathcal{C})$ exactly as in Lemma 9.15.3.

9.16. G-sets and morphisms

Let $\varphi : G \to H$ be a homomorphism of groups. Choose (suitable) sites $\mathcal{T}_G$ and $\mathcal{T}_H$ as in Example 9.6.5 and Section 9.9. Let $u : \mathcal{T}_H \to \mathcal{T}_G$ be the functor which assigns to a H-set U the G-set U_φ which has the same underlying set but G action defined by $g \cdot u = \varphi(g)u$. It is clear that u commutes with finite limits and is continuous[4]. Applying Proposition 9.14.6 and Lemma 9.15.3 we obtain a morphism of topoi

$$f : Sh(\mathcal{T}_G) \longrightarrow Sh(\mathcal{T}_H)$$

associated with φ. Using Proposition 9.9.1 we see that we get a pair of adjoint functors

$$f_* : G\textit{-Sets} \longrightarrow H\textit{-Sets}, \quad f^{-1} : H\textit{-Sets} \longrightarrow G\textit{-Sets}.$$

Let's work out what are these functors in this case.

We first work out a formula for f_*. Recall that given a G-set S the corresponding sheaf $\mathcal{F}_S$ on $\mathcal{T}_G$ is given by the rule $\mathcal{F}_S(U) = Mor_G(U, S)$. And on the other hand, given a sheaf $\mathcal{G}$ on $\mathcal{T}_H$ the corresponding H-set is given by the rule $\mathcal{G}({}_HH)$. Hence we see that

$$f_*S = Mor_{G\textit{-Sets}}(({}_HH)_\varphi, S).$$

If we work this out a little bit more then we get

$$f_*S = \{a : H \to S \mid a(gh) = ga(h)\}$$

with left H-action given by $(h \cdot a)(h') = a(h'h)$ for any element $a \in f_*S$.

Next, we explicitly compute f^{-1}. Note that since the topology on $\mathcal{T}_G$ and $\mathcal{T}_H$ is subcanonical, all representable presheaves are sheaves. Moreover, given an object V of $\mathcal{T}_H$ we see that $f^{-1}h_V$ is equal to $h_{u(V)}$ (see Lemma 9.13.5). Hence we see that $f^{-1}S = S_\varphi$ for representable sheaves. Since every sheaf on $\mathcal{T}_H$ is a coproduct of representable sheaves we conclude that this is true in general. Hence we see that for any H-set T we have

$$f^{-1}T = T_\varphi.$$

The adjunction between f^{-1} and f_* is evidenced by the formula

$$Mor_{G\textit{-Sets}}(T_\varphi, S) = Mor_{H\textit{-Sets}}(T, f_*S)$$

with f_*S as above. This can be proved directly. Moreover, it is then clear that (f^{-1}, f_*) form an adjoint pair and that f^{-1} is exact. So alternatively to the above the morphism of topoi $f : G\textit{-Sets} \to H\textit{-Sets}$ can be defined directly in this manner.

9.17. More functoriality of presheaves

In this section we revisit the material of Section 9.5. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between categories. Recall that

$$u^p : PSh(\mathcal{D}) \longrightarrow PSh(\mathcal{C})$$

is the functor that associates to $\mathcal{G}$ on $\mathcal{D}$ the presheaf $u^p\mathcal{G} = \mathcal{G} \circ u$. It turns out that this functor not only has a left adjoint (namely u_p) but also a right adjoint.

Namely, for any $V \in Ob(\mathcal{D})$ we define a category ${}_V\mathcal{I} = {}^u_V\mathcal{I}$. Its objects are pairs $(U, \psi : u(U) \to V)$. Note that the arrow is in the opposite direction from the arrow we used in defining the category $\mathcal{I}_V^u$ in Section 9.5. A morphism $(U, \psi) \to (U', \psi')$ is given by a morphism $\alpha : U \to U'$ such that $\psi = \psi' \circ u(\alpha)$. In addition, given any presheaf of

[4]Set theoretical remark: First choose $\mathcal{T}_H$. Then choose $\mathcal{T}_G$ to contain $u(\mathcal{T}_H)$ and such that every covering in $\mathcal{T}_H$ corresponds to a covering in $\mathcal{T}_G$. This is possible by Sets, Lemmas 3.10.1, 3.10.2 and 3.11.1.

sets $\mathcal{F}$ on $\mathcal{C}$ we introduce the functor ${}_V\mathcal{F} : {}_V\mathcal{I}^{opp} \to Sets$, which is defined by the rule ${}_V\mathcal{F}(U, \psi) = \mathcal{F}(U)$. We define

$$ {}_pu(\mathcal{F})(V) := lim_{{}_V\mathcal{I}^{opp}}\, {}_V\mathcal{F} $$

As a limit there are projection maps $c(\psi) : {}_pu(\mathcal{F})(V) \to \mathcal{F}(U)$ for every object (U, ψ) of ${}_V\mathcal{I}$. In fact,

$$ {}_pu(\mathcal{F})(V) = \left\{ \begin{matrix} \text{collections } s_{(U,\psi)} \in \mathcal{F}(U) \\ \forall \beta : (U_1, \psi_1) \to (U_2, \psi_2) \text{ in } {}_V\mathcal{I} \\ \text{we have } \beta^* s_{(U_2,\psi_2)} = s_{(U_1,\psi_1)} \end{matrix} \right\} $$

where the correspondence is given by $s \mapsto s_{(U,\psi)} = c(\psi)(s)$. We leave it to the reader to define the restriction mappings ${}_pu(\mathcal{F})(V) \to {}_pu(\mathcal{F})(V')$ associated to any morphism $V' \to V$ of $\mathcal{D}$. The resulting presheaf will be denoted ${}_pu\mathcal{F}$.

Lemma 9.17.1. *There is a canonical map ${}_pu\mathcal{F}(u(U)) \to \mathcal{F}(U)$, which is compatible with restriction maps.*

Proof. This is just the projection map $c(\mathrm{id}_{u(U)})$ above. $\square$

Note that any map of presheaves $\mathcal{F} \to \mathcal{F}'$ gives rise to compatible systems of maps between functors ${}_V\mathcal{F} \to {}_V\mathcal{F}'$, and hence to a map of presheaves ${}_pu\mathcal{F} \to {}_pu\mathcal{F}'$. In other words, we have defined a functor

$$ {}_pu : PSh(\mathcal{C}) \longrightarrow PSh(\mathcal{D}) $$

Lemma 9.17.2. *The functor ${}_pu$ is a right adjoint to the functor u^p. In other words the formula*

$$ Mor_{PSh(\mathcal{C})}(u^p\mathcal{G}, \mathcal{F}) = Mor_{PSh(\mathcal{D})}(\mathcal{G}, {}_pu\mathcal{F}) $$

holds bifunctorially in $\mathcal{F}$ and $\mathcal{G}$.

Proof. This is proved in exactly the same way as the proof of Lemma 9.5.4. We note that the map $u^p\,{}_pu\mathcal{F} \to \mathcal{F}$ from Lemma 9.17.1 is the map that is used to go from the right to the left.

Alternately, think of a presheaf of sets $\mathcal{F}$ on $\mathcal{C}$ as a presheaf $\mathcal{F}'$ on $\mathcal{C}^{opp}$ with values in $Sets^{opp}$, and similarly on $\mathcal{D}$. Check that $({}_pu\mathcal{F})' = u_p(\mathcal{F}')$, and that $(u^p\mathcal{G})' = u^p(\mathcal{G}')$. By Remark 9.5.5 we have the adjointness of u_p and u^p for presheaves with values in $Sets^{opp}$. The result then follows formally from this. $\square$

9.18. Cocontinuous functors

There is another way to construct morphisms of topoi. This involves using cocontinuous functors defined as follows.

Definition 9.18.1. Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. The functor u is called *cocontinuous* if for every $U \in Ob(\mathcal{C})$ and every covering $\{V_j \to u(U)\}_{j\in J}$ of $\mathcal{D}$ there exists a covering $\{U_i \to U\}_{i\in I}$ of $\mathcal{C}$ such that the family of maps $\{u(U_i) \to u(U)\}_{i\in I}$ refines the covering $\{V_j \to u(U)\}_{j\in J}$.

Note that $\{u(U_i) \to u(U)\}_{i\in I}$ is in general *not* a covering of the site $\mathcal{D}$.

Lemma 9.18.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be cocontinuous. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$. Then ${}_pu\mathcal{F}$ is a sheaf on $\mathcal{D}$, which we will denote ${}_su\mathcal{F}$.*

Proof. Let $\{V_j \to V\}_{j\in J}$ be a covering of the site $\mathcal{D}$. We have to show that

$$ {}_pu\mathcal{F}(V) \longrightarrow \prod {}_pu\mathcal{F}(V_j) \rightrightarrows \prod {}_pu\mathcal{F}(V_j \times_V V_{j'}) $$

is an equalizer diagram. Since ${}_pu$ is right adjoint to u^p we have

$$ {}_pu\mathcal{F}(V) = \mathrm{Mor}_{PSh(\mathcal{D})}(h_V, {}_pu\mathcal{F}) = \mathrm{Mor}_{PSh(\mathcal{C})}(u^ph_V, \mathcal{F}) = \mathrm{Mor}_{Sh(\mathcal{C})}((u^ph_V)^\#, \mathcal{F}) $$

Hence it suffices to show that

$$ \coprod u^ph_{V_j\times_V V_{j'}} \rightrightarrows \coprod u^ph_{V_j} \longrightarrow u^ph_V \tag{9.18.2.1} $$

becomes a coequalizer diagram after sheafification. (Recall that a coproduct in the category of sheaves is the sheafification of the coproduct in the category of presheaves, see Lemma 9.10.13.)

We first show that the second arrow of (9.18.2.1) becomes surjective after sheafification. To do this we use Lemma 9.11.2. Thus it suffices to show a section s of u^ph_V over U lifts to a section of $\coprod u^ph_{V_j}$ on the members of a covering of U. Note that s is a morphism $s : u(U) \to V$. Then $\{V_j\times_{V,s}u(U) \to u(U)\}$ is a covering of $\mathcal{D}$. Hence, as u is cocontinuous, there is a covering $\{U_i \to U\}$ such that $\{u(U_i) \to u(U)\}$ refines $\{V_j\times_{V,s}u(U) \to u(U)\}$. This means that each restriction $s|_{U_i} : u(U_i) \to V$ factors through a morphism $s_i : u(U_i) \to V_j$ for some j, i.e., $s|_{U_i}$ is in the image of $u^ph_{V_j}(U_i) \to u^ph_V(U_i)$ as desired.

Let $s, s' \in (\coprod u^ph_{V_j})^\#(U)$ map to the same element of $(u^ph_V)^\#(U)$. To finish the proof of the lemma we show that after replacing U by the members of a covering that s, s' are the image of the same section of $\coprod u^ph_{V_j\times_V V_{j'}}$ by the two maps of (9.18.2.1). We may first replace U by the members of a covering and assume that $s \in u^ph_{V_j}(U)$ and $s' \in u^ph_{V_{j'}}(U)$. A second such replacement guarantees that s and s' have the same image in $u^ph_V(U)$ instead of in the sheafification. Hence $s : u(U) \to V_j$ and $s' : u(U) \to V_{j'}$ are morphisms of $\mathcal{D}$ such that

$$ \begin{tikzcd} u(U) \arrow[r, "s'"'] \arrow[d, "s"'] & V_{j'} \arrow[d] \\ V_j \arrow[r] & V \end{tikzcd} $$

is commutative. Thus we obtain $t = (s, s') : u(U) \to V_j \times_V V_{j'}$, i.e., a section $t \in u^ph_{V_j\times_V V_{j'}}(U)$ which maps to s, s' as desired. □

Lemma 9.18.3. *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be cocontinuous. The functor $Sh(\mathcal{D}) \to Sh(\mathcal{C})$, $\mathcal{G} \mapsto (u^p\mathcal{G})^\#$ is a left adjoint to the functor ${}_su$ introduced in Lemma 9.18.2 above. Moreover, it is exact.*

Proof. Let us prove the adjointness property as follows

$$ \begin{aligned} Mor_{Sh(\mathcal{C})}((u^p\mathcal{G})^\#, \mathcal{F}) &= Mor_{PSh(\mathcal{C})}(u^p\mathcal{G}, \mathcal{F}) \\ &= Mor_{PSh(\mathcal{D})}(\mathcal{G}, {}_pu\mathcal{F}) \\ &= Mor_{Sh(\mathcal{D})}(\mathcal{G}, {}_su\mathcal{F}). \end{aligned} $$

Thus it is a left adjoint and hence right exact, see Categories, Lemma 4.22.3. We have seen that sheafification is left exact, see Lemma 9.10.14. Moreover, the inclusion $i : Sh(\mathcal{D}) \to PSh(\mathcal{D})$ is left exact by Lemma 9.10.1. Finally, the functor u^p is left exact because it is a right adjoint (namely to u_p). Thus the functor is the composition $^\# \circ u^p \circ i$ of left exact functors, hence left exact. □

We finish this section with a technical lemma.

Lemma 9.18.4. *In the situation of Lemma 9.18.3. For any presheaf $\mathcal{G}$ on $\mathcal{D}$ we have $(u^p\mathcal{G})^\# = (u^p(\mathcal{G}^\#))^\#$.*

Proof. For any sheaf $\mathcal{F}$ on $\mathcal{C}$ we have

$$\begin{aligned} Mor_{Sh(\mathcal{C})}((u^p(\mathcal{G}^\#))^\#, \mathcal{F}) &= Mor_{Sh(\mathcal{D})}(\mathcal{G}^\#, {}_su\mathcal{F}) \\ &= Mor_{PSh(\mathcal{D})}(\mathcal{G}^\#, {}_pu\mathcal{F}) \\ &= Mor_{PSh(\mathcal{D})}(\mathcal{G}, {}_pu\mathcal{F}) \\ &= Mor_{PSh(\mathcal{C})}(u^p\mathcal{G}, \mathcal{F}) \\ &= Mor_{Sh(\mathcal{C})}((u^p\mathcal{G})^\#, \mathcal{F}) \end{aligned}$$

and the result follows from the Yoneda lemma. □

9.19. Cocontinuous functors and morphisms of topoi

It is clear from the above that a cocontinuous functor u gives a morphism of topoi in the same direction as u. Thus this is in the opposite direction from the morphism of topoi associated (under certain conditions) to a continuous u as in Definition 9.14.1, Proposition 9.14.6, and Lemma 9.15.3.

Lemma 9.19.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be cocontinuous. The functors $g_* = {}_su$ and $g^{-1} = (u^p\ )^\#$ define a morphism of topoi g from $Sh(\mathcal{C})$ to $Sh(\mathcal{D})$.*

Proof. This is exactly the content of Lemma 9.18.3. □

Lemma 9.19.2. *Let $u : \mathcal{C} \to \mathcal{D}$, and $v : \mathcal{D} \to \mathcal{E}$ be cocontinuous functors. Then $v \circ u$ is cocontinuous and we have $h = g \circ f$ where $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$, resp. $g : Sh(\mathcal{D}) \to Sh(\mathcal{E})$, resp. $h : Sh(\mathcal{C}) \to Sh(\mathcal{E})$ is the morphism of topoi associated to u, resp. v, resp. $v \circ u$.*

Proof. Let $U \in Ob(\mathcal{C})$. Let $\{E_i \to v(u(U))\}$ be a covering of U in $\mathcal{E}$. By assumption there exists a covering $\{D_j \to u(U)\}$ in $\mathcal{D}$ such that $\{v(D_j) \to v(u(U))\}$ refines $\{E_i \to v(u(U))\}$. Also by assumption there exists a covering $\{C_l \to U\}$ in $\mathcal{C}$ such that $\{u(C_l) \to u(U)\}$ refines $\{D_j \to u(U)\}$. Then it is true that $\{v(u(C_l)) \to v(u(U))\}$ refines the covering $\{E_i \to v(u(U))\}$. This proves that $v \circ u$ is cocontinuous. To prove the last assertion it suffices to show that ${}_sv \circ {}_su = {}_s(v \circ u)$. It suffices to prove that ${}_pv \circ {}_pu = {}_p(v \circ u)$, see Lemma 9.18.2. Since ${}_pu$, resp. ${}_pv$, resp. ${}_p(v \circ u)$ is right adjoint to u^p, resp. v^p, resp. $(v \circ u)^p$ it suffices to prove that $u^p \circ v^p = (v \circ u)^p$. And this is direct from the definitions. □

Example 9.19.3. Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subspace. Recall that we have sites $\mathcal{T}_X$ and $\mathcal{T}_U$, see Example 9.6.4. Recall that we have the functor $u : \mathcal{T}_X \to \mathcal{T}_U$ associated to j which is continuous and gives rise to a morphism of sites $\mathcal{T}_U \to \mathcal{T}_X$, see Example 9.14.2. This also gives a morphism of topoi (j_*, j^{-1}). Next, consider the functor $v : \mathcal{T}_U \to \mathcal{T}_X$, $V \mapsto v(V) = V$ (just the same open but now thought of as an object of $\mathcal{T}_X$). This functor is cocontinuous. Namely, if $v(V) = \bigcup_{j\in J} W_j$ is an open covering in X, then each W_j must be a subset of U and hence is of the form $v(V_j)$, and trivially $V = \bigcup_{j\in J} V_j$ is an open covering in U. We conclude by Lemma 9.19.1 above that there is a morphism of topoi associated to v

$$Sh(U) \longrightarrow Sh(X)$$

given by ${}_sv$ and $(v^p\ )^\#$. We claim that actually $(v^p\ )^\# = j^{-1}$ and that ${}_sv = j_*$, in other words, that this is the same morphism of topoi as the one given above. Perhaps the easiest way to see this is to realize that for any sheaf $\mathcal{G}$ on X we have $v^p\mathcal{G}(V) = \mathcal{G}(V)$ which according to Sheaves, Lemma 6.31.1 is a description of $j^{-1}\mathcal{G}$ (and hence sheafification is superfluous in this case). The equality of ${}_sv$ and j_* follows by uniqueness of adjoint functors (but may also be computed directly).

Example 9.19.4. This example is a slight generalization of Example 9.19.3. Let $f : X \to Y$ be a continuous map of topological spaces. Assume that f is open. Recall that we have sites $\mathcal{T}_X$ and $\mathcal{T}_Y$, see Example 9.6.4. Recall that we have the functor $u : \mathcal{T}_Y \to \mathcal{T}_X$ associated to f which is continuous and gives rise to a morphism of sites $\mathcal{T}_X \to \mathcal{T}_Y$, see Example 9.14.2. This also gives a morphism of topoi (f_*, f^{-1}). Next, consider the functor $v : \mathcal{T}_X \to \mathcal{T}_Y$, $U \mapsto v(U) = f(U)$. This functor is cocontinuous. Namely, if $f(U) = \bigcup_{j\in J} V_j$ is an open covering in Y, then setting $U_j = f^{-1}(V_j) \cap U$ we get an open covering $U = \bigcup U_j$ such that $f(U) = \bigcup f(U_j)$ is a refinement of $f(U) = \bigcup V_j$. We conclude by Lemma 9.19.1 above that there is a morphism of topoi associated to v

$$Sh(X) \longrightarrow Sh(Y)$$

given by ${}_sv$ and $(v^p)^\#$. We claim that actually $(v^p)^\# = f^{-1}$ and that ${}_sv = f_*$, in other words, that this is the same morphism of topoi as the one given above. For any sheaf $\mathcal{G}$ on Y we have $v^p\mathcal{G}(U) = \mathcal{G}(f(U))$. On the other hand, we may compute $u_p\mathcal{G}(U) = colim_{f(U)\subset V}\mathcal{G}(V) = \mathcal{G}(f(U))$ because clearly $(f(U), U \subset f^{-1}(f(U)))$ is an initial object of the category $\mathcal{I}^u_U$ of Section 9.5. Hence $u_p = v^p$ and we conclude $f^{-1} = u_s = (v^p)^\#$. The equality of ${}_sv$ and f_* follows by uniqueness of adjoint functors (but may also be computed directly).

In the first Example 9.19.3 the functor v is also continuous. But in the second Example 9.19.4 it is generally not continuous because condition (2) of Definition 9.13.1 may fail. Hence the following lemma applies to the first example, but not to the second.

Lemma 9.19.5. *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that*

(a) *u is cocontinuous, and*
(b) *u is continuous.*

Let $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be the associated morphism of topoi. Then

(1) *sheafification in the formula $g^{-1} = (u^p)^\#$ is unnecessary, in other words $g^{-1}(\mathcal{G})(U) = \mathcal{G}(u(U))$,*
(2) *g^{-1} has a left adjoint $g_! = (u_p)^\#$, and*
(3) *g^{-1} commutes with arbitrary limits and colimits.*

Proof. By Lemma 9.13.2 for any sheaf $\mathcal{G}$ on $\mathcal{D}$ the presheaf $u^p\mathcal{G}$ is a sheaf on $\mathcal{C}$. And then we see the adjointness by the following string of equalities

$$\begin{aligned} Mor_{Sh(\mathcal{C})}(\mathcal{F}, g^{-1}\mathcal{G}) &= Mor_{PSh(\mathcal{C})}(\mathcal{F}, u^p\mathcal{G}) \\ &= Mor_{PSh(\mathcal{D})}(u_p\mathcal{F}, \mathcal{G}) \\ &= Mor_{Sh(\mathcal{D})}(g_!\mathcal{F}, \mathcal{G}) \end{aligned}$$

The statement on limits and colimits follows from the discussion in Categories, Section 4.22. □

In the situation of Lemma 9.19.5 above we see that we have a sequence of adjoint functors

$$g_!,\ g^{-1},\ g_*.$$

The functor $g_!$ is *not* exact in general, because it does not transform a final object of $Sh(\mathcal{C})$ into a final object of $Sh(\mathcal{D})$ in general. See Sheaves, Remark 6.31.13. On the other hand, in the topological setting of Example 9.19.3 the functor $j_!$ is exact on abelian sheaves, see Modules, Lemma 15.3.5. The following lemma gives the generalization to the case of sites.

Lemma 9.19.6. *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that*

(a) *u is cocontinuous,*
(b) *u is continuous, and*
(c) *fibre products and equalizers exist in $\mathcal{C}$ and u commutes with them.*

In this case the functor $g_!$ above commutes with fibre products and equalizers (and more generally with any finite, nonempty connected limits).

Proof. Assume (a), (b), and (c). We have $g_! = (u_p\ )^\#$. Recall (Lemma 9.10.1) that limits of sheaves are equal to the corresponding limits as presheaves. And sheafification commutes with finite limits (Lemma 9.10.14). Thus it suffices to show that u_p commutes with fibre products and equalizers. To do this it suffices that colimits over the categories $(\mathcal{I}_V^u)^{opp}$ of Section 9.5 commute with fibre products and equalizers. This follows from Lemma 9.5.1 and Categories, Lemma 4.17.4. □

The following lemma deals with a case that is even more like the morphism associated to an open immersion of topological spaces.

Lemma 9.19.7. *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that*

(a) *u is cocontinuous,*
(b) *u is continuous, and*
(c) *u is fully faithful.*

For $g_!, g^{-1}, g_*$ as above the canonical maps $\mathcal{F} \to g^{-1}g_!\mathcal{F}$ and $g^{-1}g_*\mathcal{F} \to \mathcal{F}$ are isomorphisms for all sheaves $\mathcal{F}$ on $\mathcal{C}$.

Proof. Let X be an object of $\mathcal{C}$. In Lemmas 9.18.2 and 9.19.5 we have seen that sheafification is not necessary for the functors $g^{-1} = (u^p\ )^\#$ and $g_* = ({}_pu\ )^\#$. We may compute $(g^{-1}g_*\mathcal{F})(X) = g_*\mathcal{F}(u(X)) = lim\,\mathcal{F}(Y)$. Here the limit is over the category of pairs $(Y, u(Y) \to u(X))$ where the morphisms $u(Y) \to u(X)$ are not required to be of the form $u(\alpha)$ with α a morphism of $\mathcal{C}$. By assumption (c) we see that they automatically come from morphisms of $\mathcal{C}$ and we deduce that the limit is the value on $(X, u(\mathrm{id}_X))$, i.e., $\mathcal{F}(X)$. This proves that $g^{-1}g_*\mathcal{F} = \mathcal{F}$.

On the other hand, $(g^{-1}g_!\mathcal{F})(X) = g_!\mathcal{F}(u(X)) = (u_p\mathcal{F})^\#(u(X))$, and $u_p\mathcal{F}(u(X)) = colim\,\mathcal{F}(Y)$. Here the colimit is over the category of pairs $(Y, u(X) \to u(Y))$ where the morphisms $u(X) \to u(Y)$ are not required to be of the form $u(\alpha)$ with α a morphism of $\mathcal{C}$. By assumption (c) we see that they automatically come from morphisms of $\mathcal{C}$ and we deduce that the colimit is the value on $(X, u(\mathrm{id}_X))$, i.e., $\mathcal{F}(X)$. Thus for every $X \in Ob(\mathcal{C})$ we have $u_p\mathcal{F}(u(X)) = \mathcal{F}(X)$. Since u is cocontinuous and continuous any covering of $u(X)$ in $\mathcal{D}$ can be refined by a covering (!) $\{u(X_i) \to u(X)\}$ of $\mathcal{D}$ where $\{X_i \to X\}$ is a covering in $\mathcal{C}$. This implies that $(u_p\mathcal{F})^+(u(X)) = \mathcal{F}(X)$ also, since in the colimit defining the value of $(u_p\mathcal{F})^+$ on $u(X)$ we may restrict to the cofinal system of coverings $\{u(X_i) \to u(X)\}$ as above. Hence we see that $(u_p\mathcal{F})^+(u(X)) = \mathcal{F}(X)$ for all objects X of $\mathcal{C}$ as well. Repeating this argument one more time gives the equality $(u_p\mathcal{F})^\#(u(X)) = \mathcal{F}(X)$ for all objects X of $\mathcal{C}$. This produces the desired equality $g^{-1}g_!\mathcal{F} = \mathcal{F}$. □

Finally, here is a case that does not have any corresponding topological example. Namely, this lemma explains what happens when we enlarge a ``partial universe'' of schemes keeping the same topology.

Lemma 9.19.8. *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that*

(a) *u is cocontinuous,*
(b) *u is continuous,*
(c) *u is fully faithful,*
(d) *fibre products exist in $\mathcal{C}$ and u commutes with them, and*
(e) *there exist final objects $e_{\mathcal{C}} \in Ob(\mathcal{C})$, $e_{\mathcal{D}} \in Ob(\mathcal{D})$ such that $u(e_{\mathcal{C}}) = e_{\mathcal{D}}$.*

Let $g_!, g^{-1}, g_*$ be as above. Then, u defines a morphism of sites $f : \mathcal{D} \to \mathcal{C}$ with $f_* = g^{-1}$, $f^{-1} = g_!$. The composition

$$Sh(\mathcal{C}) \xrightarrow{g} Sh(\mathcal{D}) \xrightarrow{f} Sh(\mathcal{C})$$

is isomorphic to the identity morphism of the topos $Sh(\mathcal{C})$. Moreover, the functor f^{-1} is fully faithful.

Proof. By assumption the functor u satisfies the hypotheses of Proposition 9.14.6. Hence u defines a morphism of sites and hence a morphism of topoi f as in Lemma 9.15.3. The formulas $f_* = g^{-1}$ and $f^{-1} = g_!$ are clear from the lemma cited and Lemma 9.19.5. We have $f_* \circ g_* = g^{-1} \circ g_* \cong \mathrm{id}$, and $g^{-1} \circ f^{-1} = g^{-1} \circ g_! \cong \mathrm{id}$ by Lemma 9.19.7.

We still have to show that f^{-1} is fully faithful. Let $\mathcal{F}, \mathcal{G} \in Ob(Sh(\mathcal{C}))$. We have to show that the map

$$Mor_{Sh(\mathcal{C})}(\mathcal{F}, \mathcal{G}) \longrightarrow Mor_{Sh(\mathcal{D})}(f^{-1}\mathcal{F}, f^{-1}\mathcal{G})$$

is bijective. But the right hand side is equal to

$$\begin{aligned} Mor_{Sh(\mathcal{D})}(f^{-1}\mathcal{F}, f^{-1}\mathcal{G}) &= Mor_{Sh(\mathcal{C})}(f_*f^{-1}\mathcal{F}, \mathcal{G}) \\ &= Mor_{Sh(\mathcal{C})}(g^{-1}f^{-1}\mathcal{F}, \mathcal{G}) \\ &= Mor_{Sh(\mathcal{C})}(\mathcal{F}, \mathcal{G}) \end{aligned}$$

(the first equality by adjunction) which proves what we want. □

Example 9.19.9. Let X be a topological space. Let $i : Z \to X$ be the inclusion of a subset (with induced topology). Consider the functor $u : \mathcal{T}_X \to \mathcal{T}_Z$, $U \mapsto u(U) = Z \cap U$. At first glance it may appear that this functor is cocontinuous as well. After all, since Z has the induced topology, shouldn't any covering of $U \cap Z$ it come from a covering of U in X? Not so! Namely, what if $U \cap Z = \emptyset$? In that case, the empty covering is a covering of $U \cap Z$, and the empty covering can only be refined by the empty covering. Thus we conclude that u cocontinuous $\Rightarrow$ every nonempty open U of X has nonempty intersection with Z. But this is not sufficient. For example, if $X = \mathbf{R}$ the real number line with the usual topology, and $Z = \mathbf{R} \setminus \{0\}$, then there is an open covering of Z, namely $Z = \{x < 0\} \cup \bigcup_n \{1/n < x\}$ which cannot be refined by the restriction of any open covering of X.

9.20. Cocontinuous functors which have a right adjoint

It may happen that a cocontinuous functor u has a right adjoint v. In this case it is often the case that v is continuous, and if so, then it defines a morphism of topoi (which is the same as the one defined by u).

Lemma 9.20.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$, and $v : \mathcal{D} \to \mathcal{C}$ be functors. Assume that u is cocontinuous, and that v is a right adjoint to u. Let $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be the morphism of topoi associated to u, see Lemma 9.19.1. Then $g_*\mathcal{F}$ is equal to the presheaf $v^p\mathcal{F}$, in other words, $(g_*\mathcal{F})(V) = \mathcal{F}(v(V))$.*

Proof. Let V be an object of $\mathcal{D}$. We have $u^p h_V = h_{v(V)}$ because $u^p h_V(U) = Mor_{\mathcal{D}}(u(U), V) = Mor_{\mathcal{C}}(U, v(V))$ by assumption. By Lemma 9.18.4 this implies that $g^{-1}(h_V^\#) = (u^p h_V^\#)^\# = (u^p h_V)^\# = h_{v(V)}^\#$. Hence for any sheaf $\mathcal{F}$ on $\mathcal{C}$ we have

$$\begin{aligned} (g_*\mathcal{F})(V) &= Mor_{Sh(\mathcal{D})}(h_V^\#, g_*\mathcal{F}) \\ &= Mor_{Sh(\mathcal{C})}(g^{-1}(h_V^\#), \mathcal{F}) \\ &= Mor_{Sh(\mathcal{C})}(h_{v(V)}^\#, \mathcal{F}) \\ &= \mathcal{F}(v(V)) \end{aligned}$$

which proves the lemma. □

In the situation of the lemma we see that v^p transforms sheaves into sheaves. Hence we can define $v^s = v^p$ restricted to sheaves. Just as in Lemma 9.13.3 we see that $v_s : \mathcal{G} \mapsto (v_p\mathcal{G})^\#$ is a left adjoint to v^s. On the other hand, we have $v^s = g_*$ and g^{-1} is a left adjoint of g_* as well. We conclude that $g^{-1} = v_s$ is exact.

Lemma 9.20.2. *In the situation of Lemma 9.20.1. We have $g_* = v^s = v^p$ and $g^{-1} = v_s = (v_p)^\#$. If v is continuous then v defines a morphism of sites f from $\mathcal{C}$ to $\mathcal{D}$ whose associated morphism of topoi is equal to the morphism g associated to the cocontinuous functor u.*

Proof. Clear from the discussion above the lemma and Definitions 9.14.1 and Lemma 9.15.3. □

9.21. Localization

Let $\mathcal{C}$ be a site. Let $U \in Ob(\mathcal{C})$. See Categories, Example 4.2.13 for the definition of the category $\mathcal{C}/U$ of objects over U. We turn $\mathcal{C}/U$ into a site by declaring a family of morphisms $\{V_j \to V\}$ of objects over U to be a covering of $\mathcal{C}/U$ if and only if it is a covering in $\mathcal{C}$. Consider the forgetful functor

$$j_U : \mathcal{C}/U \longrightarrow \mathcal{C}.$$

This is clearly cocontinuous and continuous. Hence by the results of the previous sections we obtain a morphism of topoi

$$j_U : Sh(\mathcal{C}/U) \longrightarrow Sh(\mathcal{C})$$

given by j_U^{-1} and j_{U*}, as well as a functor $j_{U!}$.

Definition 9.21.1. Let $\mathcal{C}$ be a site. Let $U \in Ob(\mathcal{C})$.

(1) The site $\mathcal{C}/U$ is called the *localization of the site $\mathcal{C}$ at the object U*.
(2) The morphism of topoi $j_U : Sh(\mathcal{C}/U) \to Sh(\mathcal{C})$ is called the *localization morphism*.
(3) The functor j_{U*} is called the *direct image functor*.
(4) For a sheaf $\mathcal{F}$ on $\mathcal{C}$ the sheaf $j_U^{-1}\mathcal{F}$ is called the *restriction of $\mathcal{F}$ to $\mathcal{C}/U$*.
(5) For a sheaf $\mathcal{G}$ on $\mathcal{C}/U$ the sheaf $j_{U!}\mathcal{G}$ is called the *extension of $\mathcal{G}$ by the empty set*.

The restriction $j_U^{-1}\mathcal{F}$ is the sheaf defined by the rule $j_U^{-1}\mathcal{F}(X/U) = \mathcal{F}(X)$ as expected. The extension by the empty set also has a very easy description in this case; here it is.

Lemma 9.21.2. *Let $\mathcal{C}$ be a site. Let $U \in Ob(\mathcal{C})$. Let $\mathcal{G}$ be a presheaf on $\mathcal{C}/U$. Then $j_{U!}(\mathcal{G}^\#)$ is the sheaf associated to the presheaf*

$$V \longmapsto \coprod\nolimits_{\varphi \in Mor_{\mathcal{C}}(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U)$$

with obvious restriction mappings.

Proof. By Lemma 9.19.5 we have $j_{U!}(\mathcal{G}^\#) = ((j_U)_p\mathcal{G}^\#)^\#$. By Lemma 9.13.4 this is equal to $((j_U)_p\mathcal{G})^\#$. Hence it suffices to prove that $(j_U)_p$ is given by the formula above for any presheaf $\mathcal{G}$ on $\mathcal{C}/U$. OK, and by the definition in Section 9.5 we have

$$(j_U)_p\mathcal{G}(V) = colim_{(W/U, V\to W)}\,\mathcal{G}(W)$$

Now it is clear that the category of pairs $(W/U, V \to W)$ has an object $O_\varphi = (\varphi : V \to U, \mathrm{id} : V \to V)$ for every $\varphi : V \to U$, and moreover for any object there is a unique morphism from one of the O_φ into it. The result follows. $\square$

Lemma 9.21.3. *Let $\mathcal{C}$ be a site. Let $U \in Ob(\mathcal{C})$. Let X/U be an object of $\mathcal{C}/U$. Then we have $j_{U!}(h^\#_{X/U}) = h^\#_X$.*

Proof. Denote $p : X \to U$ the structure morphism of X. By Lemma 9.21.2 we see $j_{U!}(h^\#_{X/U})$ is the sheaf associated to the presheaf

$$V \longmapsto \coprod\nolimits_{\varphi\in Mor_{\mathcal{C}}(V,U)} \{\psi : V \to X \mid p \circ \psi = \varphi\}$$

This is clearly the same thing as $Mor_{\mathcal{C}}(V, X)$. Hence the lemma follows. $\square$

We have $j_{U!}(*) = h^\#_U$ by either of the two lemmas above. Hence for every sheaf $\mathcal{G}$ over $\mathcal{C}/U$ there is a canonical map of sheaves $j_{U!}\mathcal{G} \to h^\#_U$. This characterizes sheaves in the essential image of $j_{U!}$.

Lemma 9.21.4. *Let $\mathcal{C}$ be a site. Let $U \in Ob(\mathcal{C})$. The functor $j_{U!}$ gives an equivalence of categories*

$$Sh(\mathcal{C}/U) \longrightarrow Sh(\mathcal{C})/h^\#_U$$

Proof. We explain how to get a functor from $Sh(\mathcal{C})/h^\#_U$ to $Sh(\mathcal{C}/U)$. Suppose that $\varphi : \mathcal{F} \to h^\#_U$ is given. For any object $a : X \to U$ of $\mathcal{C}/U$ we consider the set $\mathcal{F}_\varphi(X \to U)$ of elements $s \in \mathcal{F}(X)$ which under φ map to the image of $a \in Mor_{\mathcal{C}}(X, U) = h_U(X)$ in $h^\#_U(X)$. It is easy to see that $(X \to U) \mapsto \mathcal{F}_\varphi(X \to U)$ is a sheaf on $\mathcal{C}/U$. The verification that $(\mathcal{F}, \varphi) \mapsto \mathcal{F}_\varphi$ is an inverse to the functor $j_{U!}$ is omitted. $\square$

The lemma says the functor $j_{U!}$ is the composition

$$Sh(\mathcal{C}/U) \to Sh(\mathcal{C})/h^\#_U \to Sh(\mathcal{C})$$

where the first arrow is an equivalence.

Lemma 9.21.5. *Let $\mathcal{C}$ be a site. Let $U \in Ob(\mathcal{C})$. The functor $j_{U!}$ commutes with with fibre products and equalizers (and more generally finite, nonempty, connected limits). In particular, if $\mathcal{F} \subset \mathcal{F}'$ in $Sh(\mathcal{C}/U)$, then $j_{U!}\mathcal{F} \subset j_{U!}\mathcal{F}'$.*

Proof. This follows from the fact that an isomorphism of categories commutes with all limits and the functor $Sh(\mathcal{C})/h^\#_U \to Sh(\mathcal{C})$ commutes with fibre products and equalizers. Alternatively, one can prove this directly using the description of $j_{U!}$ in Lemma 9.21.2 using that sheafification is exact. (Also, in case $\mathcal{C}$ has fibre products and equalizers, the result follows from Lemma 9.19.6.) $\square$

Lemma 9.21.6. *Let $\mathcal{C}$ be a site. Let $U \in Ob(\mathcal{C})$. For any sheaf $\mathcal{F}$ on $\mathcal{C}$ we have $j_{U!}j_U^{-1}\mathcal{F} = \mathcal{F} \times h^\#_U$.*

Proof. This is clear from the description of $j_{U!}$ in Lemma 9.21.2. $\square$

Lemma 9.21.7. *Let $\mathcal{C}$ be a site. Let $f : V \to U$ be a morphism of $\mathcal{C}$. Then there exists a commutative diagram*

$$\begin{tikzcd} \mathcal{C}/V \arrow[rr, "j"] \arrow[dr, "j_V"'] & & \mathcal{C}/U \arrow[dl, "j_U"] \\ & \mathcal{C} & \end{tikzcd}$$

of cocontinuous functors. Here $j : \mathcal{C}/V \to \mathcal{C}/U$, $(a : W \to V) \mapsto (f \circ a : W \to U)$ is identified with the functor $j_{V/U} : (\mathcal{C}/U)/(V/U) \to \mathcal{C}/U$ via the identification $(\mathcal{C}/U)/(V/U) = \mathcal{C}/V$. Moreover we have $j_{V!} = j_{U!} \circ j_!$, $j_V^{-1} = j^{-1} \circ j_U^{-1}$, and $j_{V*} = j_{U*} \circ j_*$.

Proof. The commutativity of the diagram is immediate. The agreement of j with $j_{V/U}$ follows from the definitions. By Lemma 9.19.2 we see that the following diagram of morphisms of topoi

(9.21.7.1)
$$\begin{tikzcd} Sh(\mathcal{C}/V) \arrow[rr, "j"] \arrow[dr, "j_V"'] & & Sh(\mathcal{C}/U) \arrow[dl, "j_U"] \\ & Sh(\mathcal{C}) & \end{tikzcd}$$

is commutative. This proves that $j_V^{-1} = j^{-1} \circ j_U^{-1}$ and $j_{V*} = j_{U*} \circ j_*$. The equality $j_{V!} = j_{U!} \circ j_!$ follows formally from adjointness properties. □

Lemma 9.21.8. *Notation $\mathcal{C}$, $f : V \to U$, j_U, j_V, and j as in Lemma 9.21.7. Via the identifications $Sh(\mathcal{C}/V) = Sh(\mathcal{C})/h_V^\#$ and $Sh(\mathcal{C}/U) = Sh(\mathcal{C})/h_U^\#$ of Lemma 9.21.4 the functor j^{-1} has the following description*

$$j^{-1}(\mathcal{H} \xrightarrow{\varphi} h_U^\#) = (\mathcal{H} \times_{\varphi, h_U^\#, f} h_V^\# \to h_V^\#).$$

Proof. Suppose that $\varphi : \mathcal{H} \to h_U^\#$ is an object of $Sh(\mathcal{C})/h_U^\#$. By the proof of Lemma 9.21.4 this corresponds to the sheaf $\mathcal{H}_\varphi$ on $\mathcal{C}/U$ defined by the rule

$$(a : W \to U) \longmapsto \{s \in \mathcal{H}(W) \mid \varphi(s) = a\}$$

on $\mathcal{C}/U$. The pullback $j^{-1}\mathcal{H}_\varphi$ to $\mathcal{C}/V$ is given by the rule

$$(a : W \to V) \longmapsto \{s \in \mathcal{H}(W) \mid \varphi(s) = f \circ a\}$$

by the description of $j^{-1} = j_{U/V}^{-1}$ as the restriction of $\mathcal{H}_\varphi$ to $\mathcal{C}/V$. On the other hand, applying the rule to the object

$$\mathcal{H}' = \mathcal{H} \times_{\varphi, h_U^\#, f} h_V^\# \xrightarrow{\varphi'} h_V^\#$$

of $Sh(\mathcal{C})/h_V^\#$ we get $\mathcal{H}'_{\varphi'}$ given by

$$\begin{aligned} (a : W \to V) \longmapsto & \{s' \in \mathcal{H}'(W) \mid \varphi'(s') = a\} \\ = & \{(s, a') \in \mathcal{H}(W) \times h_V^\#(W) \mid a' = a \text{ and } \varphi(s) = f \circ a'\} \end{aligned}$$

which is exactly the same rule as the one describing $j^{-1}\mathcal{H}_\varphi$ above. □

Remark 9.21.9. Localization and presheaves. Let $\mathcal{C}$ be a category. Let U be an object of $\mathcal{C}$. Strictly speaking the functors j_U^{-1}, j_{U*} and $j_{U!}$ have not been defined for presheaves. But of course, we can think of a presheaf as a sheaf for the chaotic topology on $\mathcal{C}$ (see Example 9.6.6). Hence we also obtain a functor

$$j_U^{-1} : PSh(\mathcal{C}) \longrightarrow PSh(\mathcal{C}/U)$$

and functors

$$j_{U*}, j_{U!} : PSh(\mathcal{C}/U) \longrightarrow PSh(\mathcal{C})$$

which are right, left adjoint to j_U^{-1}. By Lemma 9.21.2 we see that $j_{U!}\mathcal{G}$ is the presheaf

$$V \longmapsto \coprod\nolimits_{\varphi \in Mor_{\mathcal{C}}(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U)$$

In addition the functor $j_{U!}$ commutes with fibre products and equalizers.

9.22. Glueing sheaves

This section is the analogue of Sheaves, Section 6.33.

Lemma 9.22.1. *Let $\mathcal{C}$ be a site. Let $\{U_i \to U\}$ be a covering of $\mathcal{C}$. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves on $\mathcal{C}$. Given a collection*

$$\varphi_i : \mathcal{F}|_{\mathcal{C}/U_i} \longrightarrow \mathcal{G}|_{\mathcal{C}/U_i}$$

of maps of sheaves such that for all $i, j \in I$ the maps φ_i, φ_j restrict to the same map $\mathcal{F}|_{\mathcal{C}/U_i \times_U U_j} \to \mathcal{G}|_{\mathcal{C}/U_i \times_U U_j}$ then there exists a unique map of sheaves

$$\varphi : \mathcal{F}|_{\mathcal{C}/U} \longrightarrow \mathcal{G}|_{\mathcal{C}/U}$$

whose restriction to each $\mathcal{C}/U_i$ agrees with φ_i.

Proof. Omitted. Note that the restrictions are always those of Lemma 9.21.7. ☐

The previous lemma implies that given two sheaves $\mathcal{F}$, $\mathcal{G}$ on a site $\mathcal{C}$ the rule

$$U \longmapsto Mor_{Sh(\mathcal{C}/U)}(\mathcal{F}|_{\mathcal{C}/U}, \mathcal{G}|_{\mathcal{C}/U})$$

defines a sheaf. This is a kind of *internal hom sheaf*. It is seldom used in the setting of sheaves of sets, and more usually in the setting of sheaves of modules, see Modules on Sites, Section 16.25.

Let $\mathcal{C}$ be a site. Let $\{U_i \to U\}_{i \in I}$ be a covering of $\mathcal{C}$. For each $i \in I$ let $\mathcal{F}_i$ be a sheaf of sets on $\mathcal{C}/U_i$. For each pair $i, j \in I$, let

$$\varphi_{ij} : \mathcal{F}_i|_{\mathcal{C}/U_i \times_U U_j} \longrightarrow \mathcal{F}_j|_{\mathcal{C}/U_i \times_U U_j}$$

be an isomorphism of sheaves of sets. Assume in addition that for every triple of indices $i, j, k \in I$ the following diagram is commutative

$$\begin{tikzcd}
\mathcal{F}_i|_{\mathcal{C}/U_i \times_U U_j \times_U U_k} \arrow[rr, "\varphi_{ik}"] \arrow[dr, "\varphi_{ij}"'] & & \mathcal{F}_k|_{\mathcal{C}/U_i \times_U U_j \times_U U_k} \\
& \mathcal{F}_j|_{\mathcal{C}/U_i \times_U U_j \times_U U_k} \arrow[ur, "\varphi_{jk}"'] &
\end{tikzcd}$$

We will call such a collection of data $(\mathcal{F}_i, \varphi_{ij})$ a *glueing data for sheaves of sets with respect to the covering $\{U_i \to U\}_{i \in I}$*.

Lemma 9.22.2. *Let $\mathcal{C}$ be a site. Let $\{U_i \to U\}_{i \in I}$ be a covering of $\mathcal{C}$. Given any glueing data $(\mathcal{F}_i, \varphi_{ij})$ for sheaves of sets with respect to the covering $\{U_i \to U\}_{i \in I}$ there exists a sheaf of sets $\mathcal{F}$ on $\mathcal{C}/U$ together with isomorphisms*

$$\varphi_i : \mathcal{F}|_{\mathcal{C}/U_i} \to \mathcal{F}_i$$

such that the diagrams

$$\begin{tikzcd}
\mathcal{F}|_{\mathcal{C}/U_i\times_U U_j} \arrow[r, "\varphi_i"'] \arrow[d, "id"'] & \mathcal{F}_i|_{\mathcal{C}/U_i\times_U U_j} \arrow[d, "\varphi_{ij}"] \\
\mathcal{F}|_{\mathcal{C}/U_i\times_U U_j} \arrow[r, "\varphi_j"] & \mathcal{F}_j|_{\mathcal{C}/U_i\times_U U_j}
\end{tikzcd}$$

are commutative.

Proof. Let us describe how to construct the sheaf $\mathcal{F}$ on $\mathcal{C}/U$. Let $a : V \to U$ be an object of $\mathcal{C}/U$. Then

$$\mathcal{F}(V/U) = \{(s_i)_{i\in I} \in \prod_{i\in I} \mathcal{F}_i(U_i \times_U V/U_i) \mid \varphi_{ij}(s_i|_{U_i\times_U U_j\times_U V}) = s_j|_{U_i\times_U U_j\times_U V}\}$$

We omit the construction of the restriction mappings. We omit the verification that this is a sheaf. We omit the construction of the isomorphisms φ_i, and we omit proving the commutativity of the diagrams of the lemma. □

Let $\mathcal{C}$ be a site. Let $\{U_i \to U\}_{i\in I}$ be a covering of $\mathcal{C}$. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}/U$. Associated to $\mathcal{F}$ we have its *canonical glueing data* given by the restrictions $\mathcal{F}|_{\mathcal{C}/U_i}$ and the canonical isomorphisms

$$\left(\mathcal{F}|_{\mathcal{C}/U_i}\right)|_{\mathcal{C}/U_i\times_U U_j} = \left(\mathcal{F}|_{\mathcal{C}/U_j}\right)|_{\mathcal{C}/U_i\times_U U_j}$$

coming from the fact that the composition of the functors $\mathcal{C}/U_i \times_U U_j \to \mathcal{C}/U_i \to \mathcal{C}/U$ and $\mathcal{C}/U_i \times_U U_j \to \mathcal{C}/U_j \to \mathcal{C}/U$ are equal.

Lemma 9.22.3. *Let $\mathcal{C}$ be a site. Let $\{U_i \to U\}_{i\in I}$ be a covering of $\mathcal{C}$. The category $Sh(\mathcal{C}/U)$ is equivalent to the category of glueing data via the functor that associates to $\mathcal{F}$ on $\mathcal{C}/U$ the canonical glueing data.*

Proof. In Lemma 9.22.1 we saw that the functor is fully faithful, and in Lemma 9.22.2 we proved that it is essentially surjective (by explicitly constructing a quasi-inverse functor). □

9.23. More localization

In this section we prove a few lemmas on localization where we impose some additional hypotheses on the site on or the object we are localizing at.

Lemma 9.23.1. *Let $\mathcal{C}$ be a site. Let $U \in Ob(\mathcal{C})$. If the topology on $\mathcal{C}$ is subcanonical, see Definition 9.12.2, and if $\mathcal{G}$ is a sheaf on $\mathcal{C}/U$, then*

$$j_{U!}(\mathcal{G})(V) = \coprod\nolimits_{\varphi\in Mor_{\mathcal{C}}(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U),$$

in other words sheafification is not necessary in Lemma 9.21.2.

Proof. Let $\mathcal{V} = \{V_i \to V\}_{i\in I}$ be a covering of V in the site $\mathcal{C}$. We are going to check the sheaf condition for the presheaf $\mathcal{H}$ of Lemma 9.21.2 directly. Let $(s_i, \varphi_i)_{i\in I} \in \prod_i \mathcal{H}(V_i)$, This means $\varphi_i : V_i \to U$ is a morphism in $\mathcal{C}$, and $s_i \in \mathcal{G}(V_i \xrightarrow{\varphi_i} U)$. The restriction of the pair (s_i, φ_i) to $V_i \times_V V_j$ is the pair $(s_i|_{V_i\times_V V_j/U}, \mathrm{pr}_1 \circ \varphi_i)$, and likewise the restriction of the pair (s_j, φ_j) to $V_i \times_V V_j$ is the pair $(s_j|_{V_i\times_V V_j/U}, \mathrm{pr}_2 \circ \varphi_j)$. Hence, if the family (s_i, φ_i) lies in $\check{H}^0(\mathcal{V}, \mathcal{H})$, then we see that $\mathrm{pr}_1 \circ \varphi_i = \mathrm{pr}_2 \circ \varphi_j$. The condition that the topology on $\mathcal{C}$ is weaker than the canonical topology then implies that there exists a unique morphism $\varphi : V \to U$ such that φ_i is the composition of $V_i \to V$ with φ. At this point the sheaf condition for $\mathcal{G}$ garantees that the sections s_i glue to a unique section $s \in \mathcal{G}(V \xrightarrow{\varphi} U)$. Hence $(s, \varphi) \in \mathcal{H}(V)$ as desired. $\square$

Lemma 9.23.2. *Let $\mathcal{C}$ be a site. Let $U \in Ob(\mathcal{C})$. Assume $\mathcal{C}$ has products of pairs of objects. Then*

1. *the functor j_U has a continuous right adjoint, namely the functor $v(X) = X \times U/U$,*
2. *the functor v defines a morphism of sites $\mathcal{C}/U \to \mathcal{C}$ whose associated morphism of topoi equals $j_U : Sh(\mathcal{C}/U) \to Sh(\mathcal{C})$, and*
3. *we have $j_{U*}\mathcal{F}(X) = \mathcal{F}(X \times U/U)$.*

Proof. The functor v being right adjoint to j_U means that given Y/U and X we have

$$Mor_{\mathcal{C}}(Y, X) = Mor_{\mathcal{C}/U}(Y/U, X \times U/U)$$

which is clear. To check that v is continous let $\{X_i \to X\}$ be a convering of $\mathcal{C}$. By the third axiom of a site (Definition 9.6.2) we see that

$$\{X_i \times_X (X \times U) \to X \times_X (X \times U)\} = \{X_i \times U \to X \times U\}$$

is a covering of $\mathcal{C}$ also. Hence v is continuous. The other statements of the lemma follow from Lemmas 9.20.1 and 9.20.2. $\square$

A fundamental property of an open immersion is that the restriction of the pushforward and the restriction of the extension by the empty set produces back the original sheaf. This is not always true for the functors associated to j_U above. It is true when U is a ``subobject of the final object''.

Lemma 9.23.3. *Let $\mathcal{C}$ be a site. Let $U \in Ob(\mathcal{C})$. Assume that every X in $\mathcal{C}$ has at most one morphism to U. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}/U$. The canonical maps $\mathcal{F} \to j_U^{-1} j_{U!}\mathcal{F}$ and $j_U^{-1} j_{U*}\mathcal{F} \to \mathcal{F}$ are isomorphisms.*

Proof. If $\mathcal{C}$ has fibre products, then this is a special case of Lemma 9.19.7. In general we have the following direct proof.

Let X/U be an object over U. In Lemmas 9.18.2 and 9.19.5 we have seen that sheafification is not necessary for the functors $j_U^{-1} = (u^p\ )^\#$ and $j_{U*} = ({}_pu)^\#$. We may compute $(j_U^{-1} j_{U*}\mathcal{F})(X/U) = j_{U*}\mathcal{F}(X) = lim\, \mathcal{F}(Y/U)$. Here the limit is over the category of pairs $(Y/U, Y \to X)$ where the morphisms $Y \to X$ are not required to be over U. By our assumption however we see that they are automatically morphisms over U and we deduce that the limit is the value on id_X, i.e., $\mathcal{F}(X/U)$. This proves that $j_U^{-1} j_{U*}\mathcal{F} = \mathcal{F}$.

On the other hand, $(j_U^{-1} j_{U!}\mathcal{F})(X/U) = j_{U!}\mathcal{F}(X) = (u_p\mathcal{F})^\#(X)$, and $u_p\mathcal{F}(X) = colim\, \mathcal{F}(Y/U)$. Here the colimit is over the category of pairs $(Y/U, X \to Y)$ where the morphisms $X \to Y$ are not required to be over U. By our assumption however we see that they are automatically morphisms over U and we deduce that the colimit is the value on id_X, i.e., $\mathcal{F}(X/U)$. This shows that the sheafification is not necessary (since any object over X is automatically in a unique way an object over U) and the result follows. $\square$

9.24. Localization and morphisms

The following lemma is important in order to understand relation between localization and morphisms of sites and topoi.

Lemma 9.24.1. *Let $f : \mathcal{C} \to \mathcal{D}$ be a morphism of sites corresponding to the continuous functor $u : \mathcal{D} \to \mathcal{C}$. Let $V \in Ob(\mathcal{D})$ and set $U = u(V)$. Then the functor $u' : \mathcal{D}/V \to \mathcal{C}/U$, $V'/V \mapsto u(V')/U$ determines a morphism of sites $f' : \mathcal{C}/U \to \mathcal{D}/V$. The morphism f' fits into a commutative diagram of topoi*

$$\begin{tikzcd} Sh(\mathcal{C}/U) \arrow[r, "j_U"'] \arrow[d, "f'"'] & Sh(\mathcal{C}) \arrow[d, "f"] \\ Sh(\mathcal{D}/V) \arrow[r, "j_V"] & Sh(\mathcal{D}). \end{tikzcd}$$

Using the identifications $Sh(\mathcal{C}/U) = Sh(\mathcal{C})/h_U^\#$ and $Sh(\mathcal{D}/V) = Sh(\mathcal{D})/h_V^\#$ of Lemma 9.21.4 the functor $(f')^{-1}$ is described by the rule

$$(f')^{-1}(\mathcal{H} \xrightarrow{\varphi} h_V^\#) = (f^{-1}\mathcal{H} \xrightarrow{f^{-1}\varphi} h_U^\#).$$

Finally, we have $f'_* j_U^{-1} = j_V^{-1} f_*$.

Proof. It is clear that u' is continuous, and hence we get functors $f'_* = (u')^s = (u')^p$ (see Sections 9.5 and 9.13) and an adjoint $(f')^{-1} = (u')_s = ((u')_p\ )^\#$. The assertion $f'_* j_U^{-1} = j_V^{-1} f_*$ follows as

$$(j_V^{-1} f_*\mathcal{F})(V'/V) = f_*\mathcal{F}(V') = \mathcal{F}(u(V')) = (j_U^{-1}\mathcal{F})(u(V')/U) = (f'_* j_U^{-1}\mathcal{F})(V'/V)$$

which holds even for presheaves. What isn't clear a priori is that $(f')^{-1}$ is exact, that the diagram commutes, and that the description of $(f')^{-1}$ holds.

Let $\mathcal{H}$ be a sheaf on $\mathcal{D}/V$. Let us compute $j_{U!}(f')^{-1}\mathcal{H}$. We have

$$\begin{aligned} j_{U!}(f')^{-1}\mathcal{H} &= ((j_U)_p(u'_p\mathcal{H})^\#)^\# \\ &= ((j_U)_p u'_p\mathcal{H})^\# \\ &= (u_p(j_V)_p\mathcal{H})^\# \\ &= f^{-1} j_{V!}\mathcal{H} \end{aligned}$$

The first equality by unwinding the definitions. The second equality by Lemma 9.13.4. The third equality because $u \circ j_V = j_U \circ u'$. The fourth equality by Lemma 9.13.4 again. All of the equalities above are isomorphisms of functors, and hence we may interpret this as saying that the following diagram of categories and functors is commutative

$$\begin{tikzcd} Sh(\mathcal{C}/U) \arrow[r, "j_{U!}"'] & Sh(\mathcal{C})/h_U^\# \arrow[r] & Sh(\mathcal{C}) \\ Sh(\mathcal{D}/V) \arrow[r, "j_{V!}"] \arrow[u, "(f')^{-1}"] & Sh(\mathcal{D})/h_V^\# \arrow[r] \arrow[u, "f^{-1}"] & Sh(\mathcal{D}) \arrow[u, "f^{-1}"] \end{tikzcd}$$

The middle arrow makes sense as $f^{-1}h_V^\# = (h_{u(V)})^\# = h_U^\#$, see Lemma 9.13.5. In particular this proves the description of $(f')^{-1}$ given in the statement of the lemma. Since by Lemma 9.21.4 the left horizontal arrows are equivalences and since f^{-1} is exact by assumption we conclude that $(f')^{-1} = u'_s$ is exact. Namely, because it is a left adjoint it is already right exact (Categories, Lemma 4.22.2). Hence we only need to show that it transforms a final object into a final object and commutes with fibre products (Categories, Lemma 4.21.2). Both are clear for the induced functor $f^{-1} : Sh(\mathcal{D})/h_V^\# \to Sh(\mathcal{C})/h_U^\#$. This proves that f' is a morphism of sites.

We still have to verify that $(f')^{-1}j_V^{-1} = j_U^{-1}f^{-1}$. To see this use the formula above and the description in Lemma 9.21.6. Namely, combined these give, for any sheaf $\mathcal{G}$ on $\mathcal{D}$, that

$$j_{U!}(f')^{-1}j_V^{-1}\mathcal{G} = f^{-1}j_{V!}j_V^{-1}\mathcal{G} = f^{-1}(\mathcal{G}\times h_V^\#) = f^{-1}\mathcal{G}\times h_U^\# = j_{U!}j_U^{-1}f^{-1}\mathcal{G}.$$

Since the functor $j_{U!}$ induces an equivalence $Sh(\mathcal{C}/U) \to Sh(\mathcal{C})/h_U^\#$ we conclude. □

The following lemma is a special case of the more general Lemma 9.24.1 above.

Lemma 9.24.2. *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{D} \to \mathcal{C}$ be a functor. Let $V \in Ob(\mathcal{D})$. Set $U = u(V)$. Assume that*

1. *$\mathcal{C}$ and $\mathcal{D}$ have all finite limits,*
2. *u is continuous, and*
3. *u commutes with finite limits.*

There exists a commutative diagram of morphisms of sites

$$\begin{tikzcd} \mathcal{C}/U \arrow[r, "j_U"'] \arrow[d, "f'"'] & \mathcal{C} \arrow[d, "f"] \\ \mathcal{D}/V \arrow[r, "j_V"] & \mathcal{D} \end{tikzcd}$$

where the right vertical arrow corresponds to u, the left vertical arrow corresponds to the functor $u' : \mathcal{D}/V \to \mathcal{C}/U$, $V'/V \mapsto u(V')/u(V)$ and the horizontal arrows correspond to the functors $\mathcal{C} \to \mathcal{C}/U$, $X \mapsto X \times U$ and $\mathcal{D} \to \mathcal{D}/V$, $Y \mapsto Y \times V$ as in Lemma 9.23.2. Moreover, the associated diagram of morphisms of topoi is equal to the diagram of Lemma 9.24.1. In particular we have $f'_* j_U^{-1} = j_V^{-1} f_*$.

Proof. Note that u satisfies the assumptions of Proposition 9.14.6 and hence induces a morphism of sites $f : \mathcal{C} \to \mathcal{D}$ by that proposition. It is clear that u induces a functor u' as indicated. It is clear that this functor also satisfies the assumptions of Proposition 9.14.6. Hence we get a morphism of sites $f' : \mathcal{C}/U \to \mathcal{D}/V$. The diagram commutes by our definition of composition of morphisms of sites (see Definition 9.14.4) and because

$$u(Y \times V) = u(Y) \times u(V) = u(Y) \times U$$

which shows that the diagram of categories and functors opposite to the diagram of the lemma commutes. □

At this point we can localize a site, we know how to relocalize, and we can localize a morphism of sites at an object of the site downstairs. If we combine these then we get the following kind of diagram.

Lemma 9.24.3. *Let $f : \mathcal{C} \to \mathcal{D}$ be a morphism of sites corresponding to the continuous functor $u : \mathcal{C} \to \mathcal{D}$. Let $V \in Ob(\mathcal{D})$, $U \in Ob(\mathcal{C})$ and $c : U \to u(V)$ a morphism of $\mathcal{C}$. There exists a commutative diagram of topoi*

$$\begin{tikzcd} Sh(\mathcal{C}/U) \arrow[r, "j_U"'] \arrow[d, "f_c"'] & Sh(\mathcal{C}) \arrow[d, "f"] \\ Sh(\mathcal{D}/V) \arrow[r, "j_V"] & Sh(\mathcal{D}). \end{tikzcd}$$

We have $f_c = f' \circ j_{U/u(V)}$ where $f' : Sh(\mathcal{C}/u(V)) \to Sh(\mathcal{D}/V)$ is as in Lemma 9.24.1 and $j_{U/u(V)} : Sh(\mathcal{C}/U) \to Sh(\mathcal{C}/u(V))$ is as in Lemma 9.21.7. Using the identifications

$Sh(\mathcal{C}/U) = Sh(\mathcal{C})/h_U^\#$ and $Sh(\mathcal{D}/V) = Sh(\mathcal{D})/h_V^\#$ of Lemma 9.21.4 the functor $(f_c)^{-1}$ is described by the rule

$$(f_c)^{-1}(\mathcal{H} \xrightarrow{\varphi} h_V^\#) = (f^{-1}\mathcal{H} \times_{f^{-1}\varphi, h^\#_{u(V)}, c} h_U^\# \to h_U^\#).$$

Finally, given any morphisms $b : V' \to V$, $a : U' \to U$ and $c' : U' \to u(V')$ such that

$$\begin{tikzcd} U' \arrow[r, "c'"'] \arrow[d, "a"'] & u(V') \arrow[d, "u(b)"] \\ U \arrow[r, "c"] & u(V) \end{tikzcd}$$

commutes, then the diagram

$$\begin{tikzcd} Sh(\mathcal{C}/U') \arrow[r, "j_{U'/U}"'] \arrow[d, "f_{c'}"'] & Sh(\mathcal{C}/U) \arrow[d, "f_c"] \\ Sh(\mathcal{D}/V') \arrow[r, "j_{V'/V}"] & Sh(\mathcal{D}/V). \end{tikzcd}$$

commutes.

Proof. This lemma proves itself, and is more a collection of things we know at this stage of the development of theory. For example the commutativity of the first square follows from the commutativity of Diagram (9.21.7.1) and the commutativity of the diagram in Lemma 9.24.1. The description of f_c^{-1} follows on combining Lemma 9.21.8 with Lemma 9.24.1. The commutativity of the last square then follows from the equality

$$f^{-1}\mathcal{H} \times_{h^\#_{u(V)}, c} h_U^\# \times_{h_U^\#} h_{U'}^\# = f^{-1}(\mathcal{H} \times_{h_V^\#} h_{V'}^\#) \times_{h^\#_{u(V')}, c'} h_{U'}^\#$$

which is formal using that $f^{-1}h_V^\# = h^\#_{u(V)}$ and $f^{-1}h^\#_{V'} = h^\#_{u(V')}$, see Lemma 9.13.5. □

In the following lemma we find another kind of functoriality of localization, in case the morphism of topoi comes from a cocontinuous functor. This is a kind of diagram which is different from the diagram in Lemma 9.24.1, and in particular, in general the equality $f'_* j_U^{-1} = j_V^{-1} f_*$ seen in Lemma 9.24.1 does not hold in the situation of the following lemma.

Lemma 9.24.4. *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a cocontinuous functor. Let U be an object of $\mathcal{C}$, and set $V = u(U)$. We have a commutative diagram*

$$\begin{tikzcd} \mathcal{C}/U \arrow[r, "j_U"'] \arrow[d, "u'"'] & \mathcal{C} \arrow[d, "u"] \\ \mathcal{D}/V \arrow[r, "j_V"] & \mathcal{D} \end{tikzcd}$$

where the left vertical arrow is $u' : \mathcal{C}/U \to \mathcal{D}/V$, $U'/U \mapsto V'/V$. Then u' is cocontinuous also and we get a commutative diagram of topoi

$$\begin{tikzcd} Sh(\mathcal{C}/U) \arrow[r, "j_U"'] \arrow[d, "f'"'] & Sh(\mathcal{C}) \arrow[d, "f"] \\ Sh(\mathcal{D}/V) \arrow[r, "j_V"] & Sh(\mathcal{D}) \end{tikzcd}$$

where f (resp. f') corresponds to u (resp. u').

Proof. The commutativity of the first diagram is clear. It implies the commutativity of the second diagram provided we show that u' is cocontinuous.

Let U'/U be an object of $\mathcal{C}/U$. Let $\{V_j/V \to u(U')/V\}_{j\in J}$ be a covering of $u(U')/V$ in $\mathcal{D}/V$. Since u is cocontinuous there exists a covering $\{U'_i \to U'\}_{i\in I}$ such that the family $\{u(U'_i) \to u(U')\}$ refines the covering $\{V_j \to u(U')\}$ in $\mathcal{D}$. In other words, there exists a map of index sets $\alpha : I \to J$ and morphisms $\phi_i : u(U'_i) \to V_{\alpha(i)}$ over U'. Think of U'_i as an object over U via the composition $U'_i \to U' \to U$. Then $\{U'_i/U \to U'/U\}$ is a covering of $\mathcal{C}/U$ such that $\{u(U'_i)/V \to u(U')/V\}$ refines $\{V_j/V \to u(U')/V\}$ (use the same α and the same maps ϕ_i). Hence $u' : \mathcal{C}/U \to \mathcal{D}/V$ is cocontinuous. □

9.25. Morphisms of topoi

In this section we show that any morphism of topoi is equivalent to a morphism of topoi which comes from a morphism of sites.

Lemma 9.25.1. *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that*

(1) *u is cocontinuous,*
(2) *u is continuous,*
(3) *given $a, b : U' \to U$ in $\mathcal{C}$ such that $u(a) = u(b)$, then there exists a covering $\{f_i : U'_i \to U'\}$ in $\mathcal{C}$ such that $a \circ f_i = b \circ f_i$,*
(4) *given $U', U \in Ob(\mathcal{C})$ and a morphism $c : u(U') \to u(U)$ in $\mathcal{D}$ there exists a covering $\{f_i : U'_i \to U'\}$ in $\mathcal{C}$ and morphisms $c_i : U'_i \to U$ such that $u(c_i) = c \circ u(f_i)$, and*
(5) *given $V \in Ob(\mathcal{D})$ there exists a covering of V in $\mathcal{D}$ of the form $\{u(U_i) \to V\}_{i\in I}$.*

Then the morphism of topoi

$$g : Sh(\mathcal{C}) \longrightarrow Sh(\mathcal{D})$$

associated to the cocontinuous functor u by Lemma 9.19.1 is an equivalence.

Proof. Assume u satisfies properties (1) -- (5). We will show that the adjunction mappings

$$\mathcal{G} \longrightarrow g_*g^{-1}\mathcal{G} \quad \text{and} \quad g^{-1}g_*\mathcal{F} \longrightarrow \mathcal{F}$$

are isomorphisms.

Note that Lemma 9.19.5 applies and we have $g^{-1}\mathcal{G}(U) = \mathcal{G}(u(U))$ for any sheaf $\mathcal{G}$ on $\mathcal{D}$. Next, let $\mathcal{F}$ be a sheaf on $\mathcal{C}$, and let V be an object of $\mathcal{D}$. By definition we have $g_*\mathcal{F}(V) = lim_{u(U)\to V}\mathcal{F}(U)$. Hence

$$g^{-1}g_*\mathcal{F}(U) = lim_{U', u(U')\to u(U)}\mathcal{F}(U')$$

where the morphisms $\psi : u(U') \to u(U)$ need not be of the form $u(\alpha)$. The category of such pairs (U', ψ) has a final object, namely (U, id), which gives rise to the map from the limit into $\mathcal{F}(U)$. Let $(s_{(U',\psi)})$ be an element of the limit. We want to show that $s_{(U',\psi)}$ is uniquely determined by the value $s_{(U,\text{id})} \in \mathcal{F}(U)$. By property (4) given any (U', ψ) there exists a covering $\{U'_i \to U'\}$ such that the compositions $u(U'_i) \to u(U') \to u(U)$ are of the form $u(c_i)$ for some $c_i : U'_i \to U$ in $\mathcal{C}$. Hence

$$s_{(U',\psi)}|_{U'_i} = c_i^*(s_{U,\text{id}}).$$

Since $\mathcal{F}$ is a sheaf it follows that indeed $s_{(U',\psi)}$ is determined by $s_{(U,\text{id})}$. This proves uniqueness. For existence, assume given any $s \in \mathcal{F}(U)$, $\psi : u(U') \to u(U)$, $\{f_i : U'_i \to U'\}$ and

$c_i : U'_i \to U$ such that $\psi \circ u(f_i) = u(c_i)$ as above. We claim there exists a (unique) element $s_{(U',\psi)} \in \mathcal{F}(U')$ such that

$$s_{(U',\psi)}|_{U'_i} = c_i^*(s).$$

Namely, a priori it is not clear the elements $c_i^*(s)|_{U'_i \times_{U'} U'_j}$ and $c_j^*(s)|_{U'_i \times_{U'} U'_j}$ agree, since the diagram

$$\begin{tikzcd} U'_i \times_{U'} U'_j \arrow[r, "\mathrm{pr}_2"'] \arrow[d, "\mathrm{pr}_1"'] & U'_j \arrow[d, "c_j"] \\ U'_i \arrow[r, "c_i"] & U \end{tikzcd}$$

need not commute. But condition (3) of the lemma garantees that there exist coverings $\{f_{ijk} : U'_{ijk} \to U'_i \times_{U'} U'_j\}_{k \in K_{ij}}$ such that $c_i \circ \mathrm{pr}_1 \circ f_{ijk} = c_j \circ \mathrm{pr}_2 \circ f_{ijk}$. Hence

$$f_{ijk}^*\left(c_i^* s|_{U'_i \times_{U'} U'_j}\right) = f_{ijk}^*\left(c_j^* s|_{U'_i \times_{U'} U'_j}\right)$$

Hence $c_i^*(s)|_{U'_i \times_{U'} U'_j} = c_j^*(s)|_{U'_i \times_{U'} U'_j}$ by the sheaf condition for $\mathcal{F}$ and hence the existence of $s_{U',\psi}$ also by the sheaf condition for $\mathcal{F}$. The uniqueness garantees that the collection $(s_{U',\psi})$ so obtained is an element of the limit with $s_{(U,\psi)} = s$. This proves that $g^{-1}g_*\mathcal{F} \to \mathcal{F}$ is an isomorphism.

Let $\mathcal{G}$ be a sheaf on $\mathcal{D}$. Let V be an object of $\mathcal{D}$. Then we see that

$$g_*g^{-1}\mathcal{G}(V) = lim_{U,\psi : u(U) \to V}\mathcal{G}(u(U))$$

By the preceding paragraph we see that the value of the sheaf $g_*g^{-1}\mathcal{G}$ on an object V of the form $V = u(U)$ is equal to $\mathcal{G}(u(U))$. (Formally, this holds because we have $g^{-1}g_*g^{-1} \cong g^{-1}$, and the description of g^{-1} given at the beginning of the proof; informally just by comparing limits here and above.) Hence the adjunction mapping $\mathcal{G} \to g_*g^{-1}\mathcal{G}$ has the property that it is a bijection on sections over any object of the form $u(U)$. Since by axiom (5) there exists a covering of V by objects of the form $u(U)$ we see easily that the adjunction map is an isomorphism. $\square$

It will be convenient to give cocontinuous functors as in Lemma 9.25.1 a name.

Definition 9.25.2. Let $\mathcal{C}$, $\mathcal{D}$ be sites. A *special cocontinuous functor u from $\mathcal{C}$ to $\mathcal{D}$* is a cocontinuous functor $u : \mathcal{C} \to \mathcal{D}$ satisfying the assumptions and conclusions of Lemma 9.25.1.

Lemma 9.25.3. *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a special cocontinuous functor. For every object U of $\mathcal{C}$ we have a commutative diagram*

$$\begin{tikzcd} \mathcal{C}/U \arrow[r, "j_U"'] \arrow[d] & \mathcal{C} \arrow[d, "u"] \\ \mathcal{D}/u(U) \arrow[r, "j_{u(U)}"] & \mathcal{D} \end{tikzcd}$$

as in Lemma 9.24.4. The left vertical arrow is a special cocontinuous functor. Hence in the commutative diagram of topoi

$$\begin{tikzcd} Sh(\mathcal{C}/U) \arrow[r, "j_U"'] \arrow[d] & Sh(\mathcal{C}) \arrow[d, "u"] \\ Sh(\mathcal{D}/u(U)) \arrow[r, "j_{u(U)}"] & Sh(\mathcal{D}) \end{tikzcd}$$

the vertical arrows are equivalences.

Proof. We have seen the existence and commutativity of the diagrams in Lemma 9.24.4. We have to check hypotheses (1) -- (5) of Lemma 9.25.1 for the induced functor $u : \mathcal{C}/U \to \mathcal{D}/u(U)$. This is completely mechanical.

Property (1). This is Lemma 9.24.4.

Property (2). Let $\{U'_i/U' \to U'/U\}_{i \in I}$ be a covering of U'/U in $\mathcal{C}/U$. Because u is continuous we see that $\{u(U'_i)/u(U') \to u(U')/u(U)\}_{i \in I}$ is a covering of $u(U')/u(U)$ in $\mathcal{D}/u(U)$. Hence (2) holds for $u : \mathcal{C}/U \to \mathcal{D}/u(U)$.

Property (3). Let $a, b : U''/U \to U'/U$ in $\mathcal{C}/U$ be morphisms such that $u(a) = u(b)$ in $\mathcal{D}/u(U)$. Because u satisfies (3) we see there exists a covering $\{f_i : U''_i \to U''\}$ in $\mathcal{C}$ such that $a \circ f_i = b \circ f_i$. This gives a covering $\{f_i : U''_i/U \to U''/U\}$ in $\mathcal{C}/U$ such that $a \circ f_i = b \circ f_i$. Hence (3) holds for $u : \mathcal{C}/U \to \mathcal{D}/u(U)$.

Property (4). Let $U''/U, U'/U \in Ob(\mathcal{C}/U)$ and a morphism $c : u(U'')/u(U) \to u(U')/u(U)$ in $\mathcal{D}/u(U)$ be given. Because u satisfies property (4) there exists a covering $\{f_i : U''_i \to U''\}$ in $\mathcal{C}$ and morphisms $c_i : U''_i \to U'$ such that $u(c_i) = c \circ u(f_i)$. We think of U''_i as an object over U via the composition $U''_i \to U'' \to U$. It may not be true that c_i is a morphism over U! But since $u(c_i)$ is a morphism over $u(U)$ we may apply property (3) for u and find coverings $\{f_{ik} : U''_{ik} \to U''_i\}$ such that $c_{ik} = c_i \circ f_{ik} : U''_{ik} \to U'$ are morphisms over U. Hence $\{f_i \circ f_{ik} : U''_{ik}/U \to U''/U\}$ is a covering in $\mathcal{C}/U$ such that $u(c_{ik}) = c \circ u(f_{ik})$. Hence (4) holds for $u : \mathcal{C}/U \to \mathcal{D}/u(U)$.

Property (5). Let $h : V \to u(U)$ be an object of $\mathcal{D}/u(U)$. Because u saitisfies property (5) there exists a covering $\{c_i : u(U_i) \to V\}$ in $\mathcal{D}$. By property (3) we can find coverings $\{f_{ij} : U_{ij} \to U_i\}$ and morphisms $c_{ij} : U_{ij} \to U$ such that $u(c_{ij}) = h \circ c_i \circ u(f_{ij})$. Hence $\{u(U_{ij})/u(U) \to V/u(U)\}$ is a covering in $\mathcal{D}/u(U)$ of the desired shape and we conclude that (5) holds for $u : \mathcal{C}/U \to \mathcal{D}/u(U)$. □

Lemma 9.25.4. *Let $\mathcal{C}$ be a site. Let $\mathcal{C}' \subset Sh(\mathcal{C})$ be a full subcategory (with a set of objects) such that*

(1) $h_U^\# \in Ob(\mathcal{C}')$ *for all* $U \in Ob(\mathcal{C})$, *and*
(2) $\mathcal{C}'$ *is preserved under fibre products in* $Sh(\mathcal{C})$.

Declare a covering of $\mathcal{C}'$ to be any family $\{\mathcal{F}_i \to \mathcal{F}\}_{i \in I}$ of maps such that $\coprod_{i \in I} \mathcal{F}_i \to \mathcal{F}$ is a surjective map of sheaves. Then

(1) $\mathcal{C}'$ *is a site (after choosing a set of coverings, see Sets, Lemma 3.11.1),*
(2) *representable presheaves on $\mathcal{C}'$ are sheaves (i.e., the topology on $\mathcal{C}'$ is subcanonical, see Definition 9.12.2),*
(3) *the functor $v : \mathcal{C} \to \mathcal{C}'$, $U \mapsto h_U^\#$ is a special cocontinuous functor, hence induces an equivalence $g : Sh(\mathcal{C}) \to Sh(\mathcal{C}')$,*
(4) *for any $\mathcal{F} \in Ob(\mathcal{C}')$ we have $g^{-1}h_{\mathcal{F}} = \mathcal{F}$, and*
(5) *for any $U \in Ob(\mathcal{C})$ we have $g_*h_U^\# = h_{v(U)} = h_{h_U^\#}$.*

Proof. Warning: Some of the statements above may look be a bit confusing at first; this is because objects of $\mathcal{C}'$ can also be viewed as sheaves on $\mathcal{C}$! We omit the proof that the coverings of $\mathcal{C}'$ as described in the lemma satisfy the conditions of Definition 9.6.2.

Suppose that $\{\mathcal{F}_i \to \mathcal{F}\}$ is a surjective family of morphisms of sheaves. Let $\mathcal{G}$ be another sheaf. Part (2) of the lemma says that the equalizer of

$$Mor_{Sh(\mathcal{C})}(\coprod\nolimits_{i\in I} \mathcal{F}_i, \mathcal{G}) \rightrightarrows Mor_{Sh(\mathcal{C})}(\coprod\nolimits_{(i_0,i_1)\in I\times I} \mathcal{F}_{i_0} \times_{\mathcal{F}} \mathcal{F}_{i_1}, \mathcal{G})$$

is $Mor_{Sh(\mathcal{C})}(\mathcal{F}, \mathcal{G})$. This is clear (for example use Lemma 9.11.2).

To prove (3) we have to check conditions (1) -- (5) of Lemma 9.25.1. The fact that v is cocontinuous is equivalent to the description of surjective maps of sheaves in Lemma 9.11.2. The functor v is continuous because $U \mapsto h_U^\#$ commutes with fibre products, and transforms coverings into coverings (see Lemma 9.10.14, and Lemma 9.12.5). Properties (3), (4) of Lemma 9.25.1 are statements about morphisms $f : h_U^\# \to h_{U'}^\#$. Such a morphism is the same thing as an element of $h_{U'}^\#(U)$. Hence (3) and (4) are immediate from the construction of the sheafification. Property (5) of Lemma 9.25.1 is Lemma 9.12.4. Denote $g : Sh(\mathcal{C}) \to Sh(\mathcal{C}')$ the equivalence of topoi associated with v by Lemma 9.25.1.

Let $\mathcal{F}$ be as in part (4) of the lemma. For any $U \in Ob(\mathcal{C})$ we have

$$g^{-1}h_{\mathcal{F}}(U) = h_{\mathcal{F}}(v(U)) = Mor_{Sh(\mathcal{C})}(h_U^\#, \mathcal{F}) = \mathcal{F}(U)$$

The first equality by Lemma 9.19.5. Thus part (4) holds.

Let $\mathcal{F} \in Ob(\mathcal{C}')$. Let $U \in Ob(\mathcal{C})$. Then

$$\begin{aligned} g_*h_U^\#(\mathcal{F}) &= Mor_{Sh(\mathcal{C}')}(h_{\mathcal{F}}, g_*h_U^\#) \\ &= Mor_{Sh(\mathcal{C})}(g^{-1}h_{\mathcal{F}}, h_U^\#) \\ &= Mor_{Sh(\mathcal{C})}(\mathcal{F}, h_U^\#) \\ &= Mor_{\mathcal{C}'}(\mathcal{F}, h_U^\#) \end{aligned}$$

as desired (where the third equality was shown above). $\square$

Using this we can massage any topos to live over a site having all finite limits.

Lemma 9.25.5. *Let $Sh(\mathcal{C})$ be a topos. There exists an equivalence of topoi $g : Sh(\mathcal{C}) \to Sh(\mathcal{C}')$ induced by a special cocontinuous functor $u : \mathcal{C} \to \mathcal{C}'$ such that $\mathcal{C}'$ is a site with a subcanonical topology having fibre products and a final object (in other words, $\mathcal{C}'$ has all finite limits). Moreover, given a set of sheaves $\{\mathcal{F}_i\}_{i\in I}$ we may choose $\mathcal{C}'$ such that each $g_*\mathcal{F}_i$ is a representable sheaf.*

Proof. Consider the full subcategory $\mathcal{C}_1 \subset Sh(\mathcal{C})$ consisting of all $h_U^\#$ for all $U \in Ob(\mathcal{C})$, the given sheaves $\mathcal{F}_i$ and the final sheaf $*$ (see Example 9.10.2). Let $\mathcal{C}_{n+1}$ be a full subcategory consisting of all fibre products of objects of $\mathcal{C}_n$. Set $\mathcal{C}' = \bigcup_{n\geq 1} \mathcal{C}_n$. A covering in $\mathcal{C}'$ is any family $\{\mathcal{F}_i \to \mathcal{F}\}_{i\in I}$ such that $\coprod_{i\in I} \mathcal{F}_i \to \mathcal{F}$ is surjective as a map of sheaves on $\mathcal{C}$. The functor $v : \mathcal{C} \to \mathcal{C}'$ is given by $U \mapsto h_U^\#$. Apply Lemma 9.25.4. $\square$

Here is the goal of the current section.

Lemma 9.25.6. *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. Then there exists a site $\mathcal{C}'$ and a diagram of functors*

$$\mathcal{C} \xrightarrow{v} \mathcal{C}' \xleftarrow{u} \mathcal{D}$$

such that

(1) *the functor v is a special cocontinuous functor,*

(2) *the functor u commutes with fibre products, is continuous and defines a morphism of sites $\mathcal{C}' \to \mathcal{D}'$, and*
(3) *the morphism of topoi f agrees with the composition of morphisms of topoi*

$$Sh(\mathcal{C}) \longrightarrow Sh(\mathcal{C}') \longrightarrow Sh(\mathcal{D})$$

where the first arrow comes from v via Lemma 9.25.1 and the second arrow from u via Lemma 9.15.3.

Proof. Consider the full subcategory $\mathcal{C}_1 \subset Sh(\mathcal{C})$ consisting of all $h_U^\#$ and all $f^{-1}h_V^\#$ for all $U \in Ob(\mathcal{C})$ and all $V \in Ob(\mathcal{D})$. Let $\mathcal{C}_{n+1}$ be a full subcategory consisting of all fibre products of objects of $\mathcal{C}_n$. Set $\mathcal{C}' = \bigcup_{n \geq 1} \mathcal{C}_n$. A covering in $\mathcal{C}'$ is any family $\{\mathcal{F}_i \to \mathcal{F}\}_{i \in I}$ such that $\coprod_{i \in I} \mathcal{F}_i \to \mathcal{F}$ is surjective as a map of sheaves on $\mathcal{C}$. The functor $v : \mathcal{C} \to \mathcal{C}'$ is given by $U \mapsto h_U^\#$. The functor $u : \mathcal{D} \to \mathcal{C}'$ is given by $V \mapsto f^{-1}h_V^\#$.

Part (1) follows from Lemma 9.25.4.

Proof of (2) and (3) of the lemma. The functor u commutes with fibre products as both $V \mapsto h_V^\#$ and f^{-1} do. Moreover, since f^{-1} is exact and commutes with arbitrary colimits we see that it transforms a covering into a surjective family of morphisms of sheaves. Hence u is continuous. To see that it defines a morphism of sites we still have to see that u_s is exact. In order to do this we will show that $g^{-1} \circ u_s = f^{-1}$. Namely, then since g^{-1} is an equivalence and f^{-1} is exact we will conclude. Because g^{-1} is adjoint to g_*, and u_s is adjoint to u^s, and f^{-1} is adjoint to f_* it also suffices to prove that $u^s \circ g_* = f_*$. Let U be an object of $\mathcal{C}$ and let V be an object of $\mathcal{D}$. Then

$$\begin{aligned}
(u^s g_* h_U^\#)(V) &= g_* h_U^\#(f^{-1}h_V^\#) \\
&= Mor_{Sh(\mathcal{C})}(f^{-1}h_V^\#, h_U^\#) \\
&= Mor_{Sh(\mathcal{D})}(h_V^\#, f_* h_U^\#) \\
&= f_* h_U^\#(V)
\end{aligned}$$

The first equality because $u^s = u^p$. The second equality by Lemma 9.25.4 (5). The third equality by adjointness of f_* and f^{-1} and the final equality by properties of sheafification and the Yoneda lemma. We omit the verification that these identities are functorial in U and V. Hence we see that we have $u^s \circ g_* = f_*$ for sheaves of the form $h_U^\#$. This implies that $u^s \circ g_* = f_*$ and we win (some details omitted). □

Remark 9.25.7. Notation and assumptions as in Lemma 9.25.6. If the site $\mathcal{D}$ has a final object and fibre products then the functor $u : \mathcal{D} \to \mathcal{C}'$ satisfies all the assumptions of Proposition 9.14.6. Namely, in addition to the properties mentioned in the lemma u also transforms the final object of $\mathcal{D}$ into the final object of $\mathcal{C}'$. This is clear from the construction of u. Hence, if we first apply Lemmas 9.25.5 to $\mathcal{D}$ and then Lemma 9.25.6 to the resulting morphism of topoi $Sh(\mathcal{C}) \to Sh(\mathcal{D}')$ we obtain the following statement: Any morphism of topoi $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ fits into a commutative diagram

$$\begin{tikzcd}
Sh(\mathcal{C}) \arrow[r, "f"'] \arrow[d, "g"'] & Sh(\mathcal{D}) \arrow[d, "e"] \\
Sh(\mathcal{C}') \arrow[r, "f'"] & Sh(\mathcal{D}')
\end{tikzcd}$$

where the following properties hold:

(1) the morphisms e and g are equivalences given by special cocontinuous functors $\mathcal{C} \to \mathcal{C}'$ and $\mathcal{D} \to \mathcal{D}'$,
(2) the sites $\mathcal{C}'$ and $\mathcal{D}'$ have fibre products, final objects and have subcanonical topologies,
(3) the morphism $f' : \mathcal{C}' \to \mathcal{D}'$ comes from a morphism of sites corresponding to a functor $u : \mathcal{D}' \to \mathcal{C}'$ to which Proposition 9.14.6 applies, and
(4) given any set of sheaves $\mathcal{F}_i$ (resp. $\mathcal{G}_j$) on $\mathcal{C}$ (resp. $\mathcal{D}$) we may assume each of these is a representable sheaf on $\mathcal{C}'$ (resp. $\mathcal{D}'$).

It is often useful to replace $\mathcal{C}$ and $\mathcal{D}$ by $\mathcal{C}'$ and $\mathcal{D}'$.

Remark 9.25.8. Notation and assumptions as in Lemma 9.25.6. Suppose that in addition the original morphism of topoi $Sh(\mathcal{C}) \to Sh(\mathcal{D})$ is an equivalence. Then the construction in the proof of Lemma 9.25.6 gives two functors

$$\mathcal{C} \to \mathcal{C}' \leftarrow \mathcal{D}$$

which are both special continuous functors. Hence in this case we can actually factor the morphism of topoi as a composition

$$Sh(\mathcal{C}) \to Sh(\mathcal{C}') = Sh(\mathcal{D}') \leftarrow Sh(\mathcal{D})$$

as in Remark 9.25.7, but with the middle morphism an identity.

9.26. Localization of topoi

We repeat some of the material on localization to the apparantly more general case of topoi. In reality this is not more general since we may always enlarge the underlying sites to assume that we are localizing at objects of the site.

Lemma 9.26.1. *Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$. Then the category $Sh(\mathcal{C})/\mathcal{F}$ is a topos. There is a canonical morphism of topoi*

$$j_{\mathcal{F}} : Sh(\mathcal{C})/\mathcal{F} \longrightarrow Sh(\mathcal{C})$$

which is a localization as in Section 9.21 such that

(1) *the functor $j_{\mathcal{F}}^{-1}$ is the functor $\mathcal{H} \mapsto \mathcal{H} \times \mathcal{F}/\mathcal{F}$, and*
(2) *the functor $j_{\mathcal{F}!}$ is the forgetful functor $\mathcal{G}/\mathcal{F} \mapsto \mathcal{G}$.*

Proof. Apply Lemma 9.25.5. This means we may assume $\mathcal{C}$ is a site with subcanonical topology, and $\mathcal{F} = h_U = h_U^\#$ for some $U \in Ob(\mathcal{C})$. Hence the material of Section 9.21 applies. In particular, there is an equivalence $Sh(\mathcal{C}/U) = Sh(\mathcal{C})/h_U^\#$ such that the composition

$$Sh(\mathcal{C}/U) \to Sh(\mathcal{C})/h_U^\# \to Sh(\mathcal{C})$$

is equal to $j_{U!}$, see Lemma 9.21.4. Denote $a : Sh(\mathcal{C})/h_U^\# \to Sh(\mathcal{C}/U)$ the inverse functor, so $j_{\mathcal{F}!} = j_{U!} \circ a$, $j_{\mathcal{F}}^{-1} = j_U^{-1} \circ a$ and $j_{\mathcal{F},*} = j_{U,*} \circ a$. The description of $j_{\mathcal{F}!}$ follows from the above. The description of $j_{\mathcal{F}}^{-1}$ follows from Lemma 9.21.6. □

Remark 9.26.2. In the situation of Lemma 9.26.1 we can also describe the functor $j_{\mathcal{F},*}$. It is the functor which associates to $\varphi : \mathcal{G} \to \mathcal{F}$ the sheaf

$$U \longmapsto \{\alpha : \mathcal{F}|_U \to \mathcal{G}|_U \text{ such that } \alpha \text{ is a right inverse to } \varphi|_U\}$$

In order to prove that this works the introduction of $\mathcal{H}om$-sheaves is desirable, hence we postpone this to a later time.

Lemma 9.26.3. *Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$. Let $\mathcal{C}/\mathcal{F}$ be the category of pairs (U, s) where $U \in Ob(\mathcal{C})$ and $s \in \mathcal{F}(U)$. Let a covering in $\mathcal{C}/\mathcal{F}$ be a family $\{(U_i, s_i) \to (U, s)\}$ such that $\{U_i \to U\}$ is a covering of $\mathcal{C}$. Then $j : \mathcal{C}/\mathcal{F} \to \mathcal{C}$ is a continuous and cocontinuous functor of sites which induces a morphism of topi $j : Sh(\mathcal{C}/\mathcal{F}) \to Sh(\mathcal{C})$. In fact, there is an equivalence $Sh(\mathcal{C}/\mathcal{F}) = Sh(\mathcal{C})/\mathcal{F}$ which turns j into $j_{\mathcal{F}}$.*

Proof. We omit the verification that $\mathcal{C}/\mathcal{F}$ is a site and that j is continuous and cocontinuous. By Lemma 9.19.5 there exists a morphism of topoi j as indicated, with $j^{-1}\mathcal{G}(U, s) = \mathcal{G}(U)$, and there is a left adjoint $j_!$ to j^{-1}. A morphism $\varphi : * \to g^{-1}\mathcal{G}$ on $\mathcal{C}/\mathcal{F}$ is the same thing as a rule which assigns to every pair (U, s) a section $\varphi(s) \in \mathcal{G}(U)$ compatible with restriction maps. Hence this is the same thing as a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ over $\mathcal{C}$. We conclude that $j_!* = \mathcal{F}$. In particular, for every $\mathcal{H} \in Sh(\mathcal{C}/\mathcal{F})$ there is a canonical map

$$j_!\mathcal{H} \to j_!* = \mathcal{F}$$

i.e., we obtain a functor $j'_! : Sh(\mathcal{C}/\mathcal{F}) \to Sh(\mathcal{C})/\mathcal{F}$. An inverse to this functor is the rule which assigns to an object $\varphi : \mathcal{G} \to \mathcal{F}$ of $Sh(\mathcal{C})/\mathcal{F}$ the sheaf

$$a(\mathcal{G}/\mathcal{F}) : (U, s) \longmapsto \{t \in \mathcal{G}(U) \mid \varphi(t) = s\}$$

We omit the verification that $a(\mathcal{G}/\mathcal{F})$ is a sheaf and that a is inverse to $j'_!$. □

Definition 9.26.4. Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$.

(1) The topos $Sh(\mathcal{C})/\mathcal{F}$ is called the *localization of the topos $Sh(\mathcal{C})$ at $\mathcal{F}$.*
(2) The morphism of topoi $j_{\mathcal{F}} : Sh(\mathcal{C})/\mathcal{F} \to Sh(\mathcal{C})$ of Lemma 9.26.1 is called the *localization morphism*.

We are going to show that whenever the sheaf $\mathcal{F}$ is equal to $h_U^\#$ for some object U of the site, then the localization of the topos is equal to the category of sheaves on the localization of the site at U. Moreover, we are going to check that any functorialities are compatible with this identification.

Lemma 9.26.5. *Let $\mathcal{C}$ be a site. Let $\mathcal{F} = h_U^\#$ for some object U of $\mathcal{C}$. Then $j_{\mathcal{F}} : Sh(\mathcal{C})/\mathcal{F} \to Sh(\mathcal{C})$ constructed in Lemma 9.26.1 agrees with the morphism of topoi $j_U : Sh(\mathcal{C}/U) \to Sh(\mathcal{C})$ constructed in Section 9.21 via the identification $Sh(\mathcal{C}/U) = Sh(\mathcal{C})/h_U^\#$ of Lemma 9.21.4.*

Proof. We have seen in Lemma 9.21.4 that the composition $Sh(\mathcal{C}/U) \to Sh(\mathcal{C})/h_U^\# \to Sh(\mathcal{C})$ is $j_{U!}$. The functor $Sh(\mathcal{C})/h_U^\# \to Sh(\mathcal{C})$ is $j_{\mathcal{F}!}$ by Lemma 9.26.1. Hence $j_{\mathcal{F}!} = j_{U!}$ via the identification. So $j_{\mathcal{F}}^{-1} = j_U^{-1}$ (by adjointness) and so $j_{\mathcal{F},*} = j_{U,*}$ (by adjointness again). □

Lemma 9.26.6. *Let $\mathcal{C}$ be a site. If $s : \mathcal{G} \to \mathcal{F}$ is a morphism of sheaves on $\mathcal{C}$ then there exists a natural commutative diagram of morphisms of topoi*

$$\begin{tikzcd} Sh(\mathcal{C})/\mathcal{G} \arrow[rr, "j"'] \arrow[dr, "j_{\mathcal{G}}"'] & & Sh(\mathcal{C})/\mathcal{F} \arrow[dl, "j_{\mathcal{F}}"] \\ & Sh(\mathcal{C}) & \end{tikzcd}$$

where $j = j_{\mathcal{G}/\mathcal{F}}$ is the localization of the topos $Sh(\mathcal{C})/\mathcal{F}$ at the object $\mathcal{G}/\mathcal{F}$. In particular we have

$$j^{-1}(\mathcal{H} \to \mathcal{F}) = (\mathcal{H} \times_{\mathcal{F}} \mathcal{G} \to \mathcal{G}).$$

and

$$j_!(\mathcal{E} \xrightarrow{e} \mathcal{F}) = (\mathcal{E} \xrightarrow{s \circ e} \mathcal{G}).$$

Proof. The description of j^{-1} and $j_!$ comes from the description of those functors in Lemma 9.26.1. The equality of functors $j_{\mathcal{G}!} = j_{\mathcal{F}!} \circ j_!$ is clear from the description of these functors (as forgetful functors). By adjointness we also obtain the equalities $j_{\mathcal{G}}^{-1} = j^{-1} \circ j_{\mathcal{F}}^{-1}$, and $j_{\mathcal{G},*} = j_{\mathcal{F},*} \circ j_*$. □

Lemma 9.26.7. *Assume $\mathcal{C}$ and $s : \mathcal{G} \to \mathcal{F}$ are as in Lemma 9.26.6. If $\mathcal{G} = h_V^\#$ and $\mathcal{F} = h_U^\#$ and $s : \mathcal{G} \to \mathcal{F}$ comes from a morphism $V \to U$ of $\mathcal{C}$ then the diagram in Lemma 9.26.6 is identified with diagram (9.21.7.1) via the identifications $Sh(\mathcal{C}/V) = Sh(\mathcal{C})/h_V^\#$ and $Sh(\mathcal{C}/U) = Sh(\mathcal{C})/h_U^\#$ of Lemma 9.21.4.*

Proof. This is true because the descriptions of j^{-1} agree. See Lemma 9.21.8 and Lemma 9.26.6. □

9.27. Localization and morphisms of topoi

This section is the analogue of Section 9.24 for morphisms of topoi.

Lemma 9.27.1. *Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. Let $\mathcal{G}$ be a sheaf on $\mathcal{D}$. Set $\mathcal{F} = f^{-1}\mathcal{G}$. Then there exists a commutative diagram of topoi*

$$\begin{array}{ccc} Sh(\mathcal{C})/\mathcal{F} & \xrightarrow{j_{\mathcal{F}}} & Sh(\mathcal{C}) \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Sh(\mathcal{D})/\mathcal{G} & \xrightarrow{j_{\mathcal{G}}} & Sh(\mathcal{D}). \end{array}$$

The morphism f' is characterized by the property that

$$(f')^{-1}(\mathcal{H} \xrightarrow{\varphi} \mathcal{G}) = (f^{-1}\mathcal{H} \xrightarrow{f^{-1}\varphi} \mathcal{F})$$

and we have $f'_* j_{\mathcal{F}}^{-1} = j_{\mathcal{G}}^{-1} f_*$.

Proof. Since the statement is about topoi and does not refer to the underlying sites we may change sites at will. Hence by the discussion in Remark 9.25.7 we may assume that f is given by a continuous functor $u : \mathcal{D} \to \mathcal{C}$ satisfying the assumptions of Proposition 9.14.6 between sites having all finite limits and subcanonical topologies, and such that $\mathcal{G} = h_V$ for some object V of $\mathcal{D}$. Then $\mathcal{F} = f^{-1}h_V = h_{u(V)}$ by Lemma 9.13.5. By Lemma 9.24.1 we obtain a commutative diagram of morphisms of topoi

$$\begin{array}{ccc} Sh(\mathcal{C}/U) & \xrightarrow{j_U} & Sh(\mathcal{C}) \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Sh(\mathcal{D}/V) & \xrightarrow{j_V} & Sh(\mathcal{D}), \end{array}$$

and we have $f'_* j_U^{-1} = j_V^{-1} f_*$. By Lemma 9.26.5 we may identify $j_{\mathcal{F}}$ and j_U and $j_{\mathcal{G}}$ and j_V. The description of $(f')^{-1}$ is given in Lemma 9.24.1. □

Lemma 9.27.2. *Let $f : \mathcal{C} \to \mathcal{D}$ be a morphism of sites given by the continuous functor $u : \mathcal{D} \to \mathcal{C}$. Let V be an object of $\mathcal{D}$. Set $U = u(V)$. Set $\mathcal{G} = h_V^\#$, and $\mathcal{F} = h_U^\# = f^{-1}h_V^\#$ (see Lemma 9.13.5). Then the diagram of morphisms of topoi of Lemma 9.27.1 agrees with the diagram of morphisms of topoi of Lemma 9.24.1 via the identifications $j_{\mathcal{F}} = j_U$ and $j_{\mathcal{G}} = j_V$ of Lemma 9.26.5.*

Proof. This is not a complete triviality as the choice of morphism of sites giving rise to f made in the proof of Lemma 9.27.1 may be different from the morphisms of sites given to us in the lemma. But in both cases the functor $(f')^{-1}$ is described by the same rule. Hence they agree and the associated morphism of topoi is the same. Some details omitted. $\square$

Lemma 9.27.3. *Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. Let $\mathcal{G} \in Sh(\mathcal{D})$, $\mathcal{F} \in Sh(\mathcal{C})$ and $s : \mathcal{F} \to f^{-1}\mathcal{G}$ a morphism of sheaves. There exists a commutative diagram of topoi*

$$\begin{tikzcd} Sh(\mathcal{C})/\mathcal{F} \arrow[r, "j_{\mathcal{F}}"'] \arrow[d, "f_s"'] & Sh(\mathcal{C}) \arrow[d, "f"] \\ Sh(\mathcal{D})/\mathcal{G} \arrow[r, "j_{\mathcal{G}}"] & Sh(\mathcal{D}). \end{tikzcd}$$

We have $f_s = f' \circ j_{\mathcal{F}/f^{-1}\mathcal{G}}$ where $f' : Sh(\mathcal{C})/f^{-1}\mathcal{G} \to Sh(\mathcal{D})/\mathcal{F}$ is as in Lemma 9.27.1 and $j_{\mathcal{F}/f^{-1}\mathcal{G}} : Sh(\mathcal{C})/\mathcal{F} \to Sh(\mathcal{C})/f^{-1}\mathcal{G}$ is as in Lemma 9.26.6. The functor $(f_s)^{-1}$ is described by the rule

$$(f_s)^{-1}(\mathcal{H} \xrightarrow{\varphi} \mathcal{G}) = (f^{-1}\mathcal{H} \times_{f^{-1}\varphi, f^{-1}\mathcal{G}, s} \mathcal{F} \to \mathcal{F}).$$

Finally, given any morphisms $b : \mathcal{G}' \to \mathcal{G}$, $a : \mathcal{F}' \to \mathcal{F}$ and $s' : \mathcal{F}' \to f^{-1}\mathcal{G}'$ such that

$$\begin{tikzcd} \mathcal{F}' \arrow[r, "s'"'] \arrow[d, "a"'] & f^{-1}\mathcal{G}' \arrow[d, "f^{-1}b"] \\ \mathcal{F} \arrow[r, "s"] & f^{-1}\mathcal{G} \end{tikzcd}$$

commutes, then the diagram

$$\begin{tikzcd} Sh(\mathcal{C})/\mathcal{F}' \arrow[r, "j_{\mathcal{F}'/\mathcal{F}}"'] \arrow[d, "f_{s'}"'] & Sh(\mathcal{C})/\mathcal{F} \arrow[d, "f_s"] \\ Sh(\mathcal{D})/\mathcal{G}' \arrow[r, "j_{\mathcal{G}'/\mathcal{G}}"] & Sh(\mathcal{D})/\mathcal{G}. \end{tikzcd}$$

commutes.

Proof. The commutativity of the first square follows from the commutativity of the diagram in Lemma 9.26.6 and the commutativity of the diagram in Lemma 9.27.1. The description of f_s^{-1} follows on combining the descriptions of $(f')^{-1}$ in Lemma 9.27.1 with the description of $(j_{\mathcal{F}/f^{-1}\mathcal{G}})^{-1}$ in Lemma 9.26.6. The commutativity of the last square then follows from the equality

$$f^{-1}\mathcal{H} \times_{f^{-1}\mathcal{G}, s} \mathcal{F} \times_{\mathcal{F}} \mathcal{F}' = f^{-1}(\mathcal{H} \times_{\mathcal{G}} \mathcal{G}') \times_{\mathcal{G}', s'} \mathcal{F}'$$

which is formal. $\square$

Lemma 9.27.4. *Let $f : \mathcal{C} \to \mathcal{D}$ be a morphism of sites given by the continuous functor $u : \mathcal{D} \to \mathcal{C}$. Let V be an object of $\mathcal{D}$. Let $c : U \to u(V)$ be a morphism. Set $\mathcal{G} = h_V^\#$ and $\mathcal{F} = h_U^\# = f^{-1}h_V^\#$. Let $s : \mathcal{F} \to f^{-1}\mathcal{G}$ be the map induced by c. Then the diagram of morphisms of topoi of Lemma 9.24.3 agrees with the diagram of morphisms of topoi of Lemma 9.27.3 via the identifications $j_{\mathcal{F}} = j_U$ and $j_{\mathcal{G}} = j_V$ of Lemma 9.26.5.*

Proof. This follows on combining Lemmas 9.26.7 and 9.27.2. $\square$

9.28. Points

Definition 9.28.1. Let $\mathcal{C}$ be a site. A *point of the topos* $Sh(\mathcal{C})$ is a morphism of topoi p from $Sh(pt)$ to $Sh(\mathcal{C})$.

We will define a point of a site in terms of a functor $u : \mathcal{C} \to Sets$. It will turn out later that u will define a morphism of sites which gives rise to a point of the topos associated to $\mathcal{C}$, see Lemma 9.28.8.

Let $\mathcal{C}$ be a site. Let $p = u$ be a functor $u : \mathcal{C} \to Sets$. This curious language is introduced because it seems funny to talk about neighbourhoods of functors; so we think of a ``point'' p as a geometric thing which is given by a categorical datum, namely the functor u. The fact that p is actually equal to u does not matter. A *neighbourhood* of p is a pair (U, x) with $U \in Ob(\mathcal{C})$ and $x \in u(U)$. A *morphism of neighbourhoods* $(V, y) \to (U, x)$ is given by a morphism $\alpha : V \to U$ of $\mathcal{C}$ such that $u(\alpha)(y) = x$. Note that the category of neighbourhoods isn't a ``big'' category.

We define the *stalk* of a presheaf $\mathcal{F}$ at p as

$$\mathcal{F}_p = colim_{\{(U,x)\}^{opp}} \mathcal{F}(U). \tag{9.28.1.1}$$

The colimit is over the opposite of the category of neighbourhoods of p. In other words, an element of $\mathcal{F}_p$ is given by a triple (U, x, s), where (U, x) is a neighbourhood of p and $s \in \mathcal{F}(U)$. Equality of triples is the equivalence relation generated by $(U, x, s) \sim (V, y, \alpha^* s)$ when α is as above.

Note that if $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves of sets, then we get a canonical map of stalks $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$. Thus we obtain a *stalk functor*

$$PSh(\mathcal{C}) \longrightarrow Sets, \quad \mathcal{F} \longmapsto \mathcal{F}_p.$$

We have defined the stalk functor using any functor $p = u : \mathcal{C} \to Sets$. No conditions are necessary for the definition to work[5]. On the other hand, it is probably better not to use this notion unless p actually is a point (see definition below), since in general the stalk functor does not have good properties.

Definition 9.28.2. Let $\mathcal{C}$ be a site. A *point p of the site $\mathcal{C}$* is given by a functor $u : \mathcal{C} \to Sets$ such that

(1) For every covering $\{U_i \to U\}$ of $\mathcal{C}$ the map $\coprod u(U_i) \to u(U)$ is surjective.
(2) For every covering $\{U_i \to U\}$ of $\mathcal{C}$ and every morphism $V \to U$ the maps $u(U_i \times_U V) \to u(U_i) \times_{u(U)} u(V)$ are bijective.
(3) The stalk functor $Sh(\mathcal{C}) \to Sets$, $\mathcal{F} \to \mathcal{F}_p$ is left exact.

The conditions should be familiar since they are modeled after those of Definitions 9.13.1 and 9.14.1. Note that (3) implies that $*_p = \{*\}$, see Example 9.10.2. Hence $u(U) \neq \emptyset$ for at least some U (because the empty colimit produces the empty set). We will show below (Lemma 9.28.7) that this does give rise to a point of the topos $Sh(\mathcal{C})$. Before we do so, we prove some lemmas for general functors u.

Lemma 9.28.3. *Let $\mathcal{C}$ be a site. Let $p = u : \mathcal{C} \to Sets$ be a functor. There are functorial isomorphisms $(h_U)_p = u(U)$ for $U \in Ob(\mathcal{C})$.*

[5] One should try to avoid the case where $u(U) = \emptyset$ for all U.

Proof. An element of $(h_U)_p$ is given by a triple (V, y, f), where $V \in Ob(\mathcal{C})$, $y \in u(V)$ and $f \in h_U(V) = Mor_{\mathcal{C}}(V, U)$. Two such (V, y, f), (V', y', f') determine the same object if there exists a morphism $\phi : V \to V'$ such that $u(\phi)(x) = x'$ and $f' \circ \phi = f$, and in general you have to take chains of identities like this to get the correct equivalence relation. In any case, every (V, y, f) is equivalent to the element $(U, u(f)(y), \mathrm{id}_U)$. If ϕ exists as above, then the triples (V, y, f), (V', y', f') determine the same triple $(U, u(f)(y), \mathrm{id}_U) = (U, u(f')(y'), \mathrm{id}_U)$. This proves that the map $u(U) \to (h_U)_p$, $x \mapsto$ class of (U, x, id_U) is bijective. □

Let $\mathcal{C}$ be a site. Let $p = u : \mathcal{C} \to Sets$ be a functor. In analogy with the constructions in Section 9.5 given a set E we define a presheaf $u^p E$ by the rule

(9.28.3.1) $$U \longmapsto u^p E(U) = Mor_{Sets}(u(U), E) = \mathrm{Map}(u(U), E).$$

This defines a functor $u^p : Sets \to PSh(\mathcal{C})$, $E \mapsto u^p E$.

Lemma 9.28.4. *For any functor $u : \mathcal{C} \to Sets$. The functor u^p is a right adjoint to the stalk functor on presheaves.*

Proof. Let $\mathcal{F}$ be a presheaf on $\mathcal{C}$. Let E be a set. A morphism $\mathcal{F} \to u^p E$ is given by a compatible system of maps $\mathcal{F}(U) \to \mathrm{Map}(u(U), E)$, i.e., a compatible system of maps $\mathcal{F}(U) \times u(U) \to E$. And by definition of $\mathcal{F}_p$ a map $\mathcal{F}_p \to E$ is given by a rule associating with each triple (U, x, σ) an element in E such that equivalent triples map to the same element, see discussion surrounding Equation (9.28.1.1). This also means a compatible system of maps $\mathcal{F}(U) \times u(U) \to E$. □

In analogy with Section 9.13 we have the following lemma.

Lemma 9.28.5. *Let $\mathcal{C}$ be a site. Let $p = u : \mathcal{C} \to Sets$ be a functor. Suppose that for every covering $\{U_i \to U\}$ of $\mathcal{C}$*

(1) *the map $\coprod u(U_i) \to u(U)$ is surjective and*
(2) *the maps $u(U_i \times_U U_j) \to u(U_i) \times_{u(U)} u(U_j)$ are surjective.*

Then we have

(1) *the presheaf $u^p E$ is a sheaf for all sets E, denote it $u^s E$,*
(2) *the stalk functor $Sh(\mathcal{C}) \to Sets$ and the functor $u^s : Sets \to Sh(\mathcal{C})$ are adjoint, and*
(3) *we have $\mathcal{F}_p = \mathcal{F}^\#_p$ for every presheaf of sets $\mathcal{F}$.*

Proof. The first assertion is immediate from the definition of a sheaf, assumptions (1) and (2), and the definition of $u^p E$. The second is a restatement of the adjointness of u^p and the stalk functor (but now restricted to sheaves). The third assertion follows as, for any set E, we have

$$\mathrm{Map}(\mathcal{F}_p, E) = Mor_{PSh(\mathcal{C})}(\mathcal{F}, u^p E) = Mor_{Sh(\mathcal{C})}(\mathcal{F}^\#, u^s E) = \mathrm{Map}(\mathcal{F}^\#_p, E)$$

by the adjointness property of sheafification. □

In particular Lemma 9.28.5 holds when $p = u$ is a point. In this case we think of the sheaf $u^s E$ as the ``skyscraper'' sheaf with value E at p.

Definition 9.28.6. Let p be a point of the site $\mathcal{C}$ given by the functor u. For a set E we define $p_* E = u^s E$ the sheaf described in Lemma 9.28.5 above. We sometimes call this a *skyscraper sheaf.*

In particular we have the following adjointness property of skyscraper sheaves and stalks:

$$Mor_{Sh(\mathcal{C})}(\mathcal{F}, p_*E) = \mathrm{Map}(\mathcal{F}_p, E)$$

This motivates the notation $p^{-1}\mathcal{F} = \mathcal{F}_p$ which we will sometimes use.

Lemma 9.28.7. *Let $\mathcal{C}$ be a site.*

1. *Let p be a point of the site $\mathcal{C}$. Then the pair of functors (p_*, p^{-1}) introduced above define a morphism of topoi $Sh(pt) \to Sh(\mathcal{C})$.*
2. *Let $p = (p_*, p^{-1})$ be a point of the topos $Sh(\mathcal{C})$. Then the functor $u : U \mapsto p^{-1}(h_U^\#)$ gives rise to a point p' of the site $\mathcal{C}$ whose associated morphism of topoi $(p'_*, (p')^{-1})$ is equal to p.*

Proof. Proof of (1). By the above the functors p_* and p^{-1} are adjoint. The functor p^{-1} is required to be exact by Definition 9.28.2. Hence the conditions imposed in Definition 9.15.1 are all satisfied and we see that (1) holds.

Proof of (2). Let $\{U_i \to U\}$ be a covering of $\mathcal{C}$. Then $\coprod (h_{U_i})^\# \to h_U^\#$ is surjective, see Lemma 9.12.5. Since p^{-1} is exact (by definition of a morphism of topoi) we conclude that $\coprod u(U_i) \to u(U)$ is surjective. This proves part (1) of Definition 9.28.2. Sheafification is exact, see Lemma 9.10.14. Hence if $U \times_V W$ exists in $\mathcal{C}$, then

$$h_{U\times_V W}^\# = h_U^\# \times_{h_V^\#} h_W^\#$$

and we see that $u(U \times_V W) = u(U) \times_{u(V)} u(W)$ since p^{-1} is exact. This proves part (2) of Definition 9.28.2. Let $p' = u$, and let $\mathcal{F}_{p'}$ be the stalk functor defined by Equation (9.28.1.1) using u. There is a canonical comparison map $c : \mathcal{F}_{p'} \to \mathcal{F}_p = p^{-1}\mathcal{F}$. Namely, given a triple (U, x, σ) representing an element ξ of $\mathcal{F}_{p'}$ we think of σ as a map $\sigma : h_U^\# \to \mathcal{F}$ and we can set $c(\xi) = p^{-1}(\sigma)(x)$ since $x \in u(U) = p^{-1}(h_U^\#)$. By Lemma 9.28.3 we see that $(h_U)_{p'} = u(U)$. Since conditions (1) and (2) of Definition 9.28.2 hold for p' we also have $(h_U^\#)_{p'} = (h_U)_{p'}$ by Lemma 9.28.5. Hence we have

$$(h_U^\#)_{p'} = (h_U)_{p'} = u(U) = p^{-1}(h_U^\#)$$

We claim this bijection equals the comparison map $c : (h_U^\#)_{p'} \to p^{-1}(h_U^\#)$ (verification omitted). Any sheaf on $\mathcal{C}$ is a coequalizer of maps of coproducts of sheaves of the form $h_U^\#$, see Lemma 9.12.4. The stalk functor $\mathcal{F} \mapsto \mathcal{F}_{p'}$ and the functor p^{-1} commute with arbitrary colimits (as they are both left adjoints). We conclude c is an isomorphism for every sheaf $\mathcal{F}$. Thus the stalk functor $\mathcal{F} \mapsto \mathcal{F}_{p'}$ is isomorphic to p^{-1} and we in particular see that it is exact. This proves condition (3) of Definition 9.28.2 holds and p' is a point. The final assertion has already been shown above, since we saw that $p^{-1} = (p')^{-1}$. □

Actually a point always corresponds to a morphism of sites as we show in the following lemma.

Lemma 9.28.8. *Let $\mathcal{C}$ be a site. Let p be a point of $\mathcal{C}$ given by $u : \mathcal{C} \to Sets$. Let S_0 be an infinite set such that $u(U) \subset S_0$ for all $U \in Ob(\mathcal{C})$. Let $\mathcal{S}$ be the site constructed out of the powerset $S = \mathcal{P}(S_0)$ in Remark 9.15.4. Then*

1. *there is an equivalence $i : Sh(pt) \to Sh(\mathcal{S})$,*
2. *the functor $u : \mathcal{C} \to \mathcal{S}$ induces a morphism of sites $f : \mathcal{S} \to \mathcal{C}$, and*

(3) *the composition*

$$Sh(pt) \to Sh(\mathcal{S}) \to Sh(\mathcal{C})$$

is the morphism of topoi (p_*, p^{-1}) *of Lemma 9.28.7.*

Proof. Part (1) we saw in Remark 9.15.4. Moreover, recall that the equivalence associates to the set E the sheaf i_*E on $\mathcal{S}$ defined by the rule $V \mapsto Mor_{Sets}(V, E)$. Part (2) is clear from the definition of a point of $\mathcal{C}$ (Definition 9.28.2) and the definition of a morphism of sites (Definition 9.14.1). Finally, consider f_*i_*E. By construction we have

$$f_*i_*E(U) = i_*E(u(U)) = Mor_{Sets}(u(U), E)$$

which is equal to $p_*E(U)$, see Equation (9.28.3.1). This proves (3). □

Contrary to what happens in the topological case it is not always true that the stalk of the skyscraper sheaf with value E is E. Here is what is true in general.

Lemma 9.28.9. *Let $\mathcal{C}$ be a site. Let $p : Sh(pt) \to Sh(\mathcal{C})$ be a point of the topos associated to $\mathcal{C}$. For any set E there are canonical maps*

$$E \longrightarrow (p_*E)_p \longrightarrow E$$

whose composition is id_E.

Proof. There is always an adjunction map $(p_*E)_p = p^{-1}p_*E \to E$. This map is an isomorphism when $E = \{*\}$ because p_* and p^{-1} are both left exact, hence transform the final object into the final object. Hence given $e \in E$ we can consider the map $i_e : \{*\} \to E$ which gives

$$\begin{tikzcd} p^{-1}p_*\{*\} \arrow[r, "p^{-1}p_*i_e"'] \arrow[d, "\cong"'] & p^{-1}p_*E \arrow[d] \\ \{*\} \arrow[r, "i_e"] & E \end{tikzcd}$$

whence the map $E \to (p_*E)_p = p^{-1}p_*E$ as desired. □

Lemma 9.28.10. *Let $\mathcal{C}$ be a site. Let $p : Sh(pt) \to Sh(\mathcal{C})$ be a point of the topos associated to $\mathcal{C}$. The functor $p_* : Sets \to Sh(\mathcal{C})$ has the following properties: It commutes with arbitrary limits, it is left exact, it is faithful, it transforms surjections into surjections, it commutes with coequalizers, it reflects injections, it reflects surjections, and it reflects isomorpisms.*

Proof. Because p_* is a right adjoint it commutes with arbitrary limits and it is left exact. The fact that $p^{-1}p_*E \to E$ is a canonically split surjection implies that p_* is faithful, reflects injections, reflects surjections, and reflects isomorphisms. By Lemma 9.28.7 we may assume that p comes from a point $u : \mathcal{C} \to Sets$ of the underlying site $\mathcal{C}$. In this case the sheaf p_*E is given by

$$p_*E(U) = Mor_{Sets}(u(U), E)$$

see Equation (9.28.3.1) and Definition 9.28.6. It follows immediately from this formula that p_* transforms surjections into surjections and coequalizers into coequalizers. □

9.29. Constructing points

In this section we give criteria for when a functor from a site to the category of sets defines a point of that site.

Lemma 9.29.1. *Let $\mathcal{C}$ be a site. Assume that $\mathcal{C}$ has a final object X and fibred products. Let $p = u : \mathcal{C} \to Sets$ be a functor such that*

(1) *$u(X)$ is a singleton set, and*
(2) *for every pair of morphisms $U \to W$ and $V \to W$ with the same target the map $u(U \times_W V) \to u(U) \times_{u(W)} u(V)$ is bijective.*

Then the opposite of the category of neighbourhoods of p is filtered. Moreover, the stalk functor $Sh(\mathcal{C}) \to Sets$, $\mathcal{F} \to \mathcal{F}_p$ commutes with finite limits.

Proof. This is analogous to the proof of Lemma 9.5.2 above. The assumptions on $\mathcal{C}$ imply that $\mathcal{C}$ has finite limits. See Categories, Lemma 4.16.4. Assumption (1) implies that the category of neighbourhoods is nonempty. Suppose (U, x) and (V, y) are neighbourhoods. Then $u(U \times V) = u(U \times_X V) = u(U) \times_{u(X)} u(V) = u(U) \times u(V)$ by (2). Hence there exists a neighbourhood $(U \times_X V, z)$ mapping to both (U, x) and (V, y). Let $a, b : (V, y) \to (U, x)$ be two morphisms in the category of neighbourhoods. Let W be the equalizer of $a, b : V \to U$. As in the proof of Categories, Lemma 4.16.4 we may write W in terms of fibre products:

$$W = (V \times_{a,U,b} V) \times_{(pr_1, pr_2), V \times V, \Delta} V$$

The bijectivity in (2) garantees there exists an element $z \in u(W)$ which maps to $((y, y), y)$. Then $(W, z) \to (V, y)$ equalizes a, b as desired.

Let $\mathcal{I} \to Sh(\mathcal{C})$, $i \mapsto \mathcal{F}_i$ be a finite diagram of sheaves. We have to show that the stalk of the limit of this system agrees with the limit of the stalks. Let $\mathcal{F}$ be the limit of the system as a *presheaf*. According to Lemma 9.10.1 this is a sheaf and it is the limit in the category of sheaves. Hence we have to show that $\mathcal{F}_p = lim_{\mathcal{I}} \mathcal{F}_{i,p}$. Recall also that $\mathcal{F}$ has a simple description, see Section 9.4. Thus we have to show that

$$lim_i \; colim_{\{(U,x)\}^{opp}} \mathcal{F}_i(U) = colim_{\{(U,x)\}^{opp}} \; lim_i \mathcal{F}_i(U).$$

This holds, by Categories, Lemma 4.17.2, because we just showed the opposite of the category of neighbourhoods is filtered. □

Proposition 9.29.2. *Let $\mathcal{C}$ be a site. Assume that finite limits exist in $\mathcal{C}$. (I.e., $\mathcal{C}$ has fibre products, and a final object.) A point p of such a site $\mathcal{C}$ is given by a functor $u : \mathcal{C} \to Sets$ such that*

(1) *u commutes with finite limits, and*
(2) *if $\{U_i \to U\}$ is a covering, then $\coprod_i u(U_i) \to u(U)$ is surjective.*

Proof. Suppose first that p is a point (Definition 9.28.2) given by a functor u. Condition (2) is satisfied directly from the definition of a point. By Lemma 9.28.3 we have $(h_U)_p = u(U)$. By Lemma 9.28.5 we have $(h_U^\#)_p = (h_U)_p$. Thus we see that u is equal to the composition of functors

$$\mathcal{C} \xrightarrow{h} PSh(\mathcal{C}) \xrightarrow{\#} Sh(\mathcal{C}) \xrightarrow{()_p} Sets$$

Each of these functors is left exact, and hence we see u satisfies (1).

Conversely, suppose that u satisfies (1) and (2). In this case we immediately see that u satisfies the first two conditions of Definition 9.28.2. And its stalk functor is exact, because it is a left adjoint by Lemma 9.28.5 and it commutes with finite limits by Lemma 9.29.1. □

Remark 9.29.3. In fact, let $\mathcal{C}$ be a site. Assume $\mathcal{C}$ has a final object X and fibre products. Let $p = u : \mathcal{C} \to Sets$ be a functor such that

(1) $u(X) = \{*\}$ a singleton, and
(2) for every pair of morphisms $U \to W$ and $V \to W$ with the same target the map $u(U \times_W V) \to u(U) \times_{u(W)} u(V)$ is surjective.
(3) for every covering $\{U_i \to U\}$ the map $\coprod u(U_i) \to u(U)$ is surjective.

Then, in general, p is **not** a point of $\mathcal{C}$. An example is the category $\mathcal{C}$ with two objects $\{U, X\}$ and exactly one non-identity arrow, namely $U \to X$. We endow $\mathcal{C}$ with the trivial topology, i.e., the only coverings are $\{U \to U\}$ and $\{X \to X\}$. A sheaf $\mathcal{F}$ is the same thing as a presheaf and consists of a triple $(A, B, A \to B)$: namely $A = \mathcal{F}(X)$, $B = \mathcal{F}(U)$ and $A \to B$ is the restriction mapping corresponding to $U \to X$. Note that $U \times_X U = U$ so fibre products exist. Consider the functor $u = p$ with $u(X) = \{*\}$ and $u(U) = \{*_1, *_2\}$. This satisfies (1), (2), and (3), but the corresponding stalk functor (9.28.1.1) is the functor

$$(A, B, A \to B) \longmapsto B \coprod\nolimits_A B$$

which isn't exact. Namely, consider $(\emptyset, \{1\}, \emptyset \to \{1\}) \to (\{1\}, \{1\}, \{1\} \to \{1\})$ which is an injective map of sheaves, but is transformed into the noninjective map of sets

$$\{1\} \coprod \{1\} \longrightarrow \{1\} \coprod\nolimits_{\{1\}} \{1\}$$

by the stalk functor.

Example 9.29.4. Let X be a topological space. Let $\mathcal{T}_X$ be the site of Example 9.6.4. Let $x \in X$ be a point. Consider the functor

$$u : \mathcal{T}_X \longrightarrow Sets, \quad U \mapsto \begin{cases} \emptyset & \text{if} \quad x \notin U \\ \{*\} & \text{if} \quad x \in U \end{cases}$$

This functor commutes with product and fibred products, and turns coverings into surjective families of maps. Hence we obtain a point p of the site $\mathcal{T}_X$. It is immediately verified that the stalk functor agrees with the stalk at x defined in Sheaves, Section 6.11.

Example 9.29.5. Let X be a topological space. What are the points of the topos $Sh(X)$? To see this, let $\mathcal{T}_X$ be the site of Example 9.6.4. By Lemma 9.28.7 a point of $Sh(X)$ corresponds to a point of this site. Let p be a point of the site $\mathcal{T}_X$ given by the functor $u : \mathcal{T}_X \to Sets$. We are going to use the characterization of such a u in Proposition 9.29.2. This implies immediately that $u(\emptyset) = \emptyset$ and $u(U \cap V) = u(U) \times u(V)$. In particular we have $u(U) = u(U) \times u(U)$ via the diagonal map which implies that $u(U)$ is either a singleton or empty. Moreover, if $U = \bigcup U_i$ is an open covering then

$$u(U) = \emptyset \Rightarrow \forall i,\ u(U_i) = \emptyset \quad \text{and} \quad u(U) \neq \emptyset \Rightarrow \exists i,\ u(U_i) \neq \emptyset.$$

We conclude that there is a unique largest open $W \subset X$ with $u(W) = \emptyset$, namely the union of all the opens U with $u(U) = \emptyset$. Let $Z = X \setminus W$. If $Z = Z_1 \cup Z_2$ with $Z_i \subset Z$ closed, then $W = (X \setminus Z_1) \cap (X \setminus Z_2)$ so $\emptyset = u(W) = u(X \setminus Z_1) \times u(X \setminus Z_2)$ and we conclude that $u(X \setminus Z_1) = \emptyset$ or that $u(X \setminus Z_2) = \emptyset$. This means that $X \setminus Z_1 = W$ or that $X \setminus Z_2 = W$. In other words, Z is irreducible. Now we see that u is described by the rule

$$u : \mathcal{T}_X \longrightarrow Sets, \quad U \mapsto \begin{cases} \emptyset & \text{if} \quad Z \cap U = \emptyset \\ \{*\} & \text{if} \quad Z \cap U \neq \emptyset \end{cases}$$

Note that for any irreducible closed $Z \subset X$ this functor satisfies assumptions (1), (2) of Proposition 9.29.2 and hence defines a point. In other words we see that points of the site $\mathcal{T}_X$ are in one-to-one correspondence with irreducible closed subsets of X. In particular, if X is a sober topological space, then points of $\mathcal{T}_X$ and points of X are in one to one correspondence, see Example 9.29.4.

Example 9.29.6. Consider the site $\mathcal{T}_G$ described in Example 9.6.5 and Section 9.9. The forgetful functor $u : \mathcal{T}_G \to Sets$ commutes with products and fibred products and turns coverings into surjective families. Hence it defines a point of $\mathcal{T}_G$. We identify $Sh(\mathcal{T}_G)$ and G-$Sets$. The stalk functor

$$p^{-1} : Sh(\mathcal{T}_G) = G\text{-}Sets \longrightarrow Sets$$

is the forgetful functor. The pushforward p_* is the functor

$$Sets \longrightarrow Sh(\mathcal{T}_G) = G\text{-}Sets$$

which maps a set S to the G-set $\mathrm{Map}(G, S)$ with action $g \cdot \psi = \psi \circ R_g$ where R_g is right multiplication. In particular we have $p^{-1}p_*S = \mathrm{Map}(G, S)$ as a set and the maps $S \to \mathrm{Map}(G, S) \to S$ of Lemma 9.28.9 are the obvious ones.

9.30. Points and and morphisms of topoi

In this section we make a few remarks about points and morphisms of topoi.

Lemma 9.30.1. *Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites given by a continuous functor $u : \mathcal{C} \to \mathcal{D}$. Let p be a point of $\mathcal{D}$ given by the functor $v : \mathcal{D} \to Sets$, see Definition 9.28.2. Then the functor $v \circ u : \mathcal{C} \to Sets$ defines a point q of $\mathcal{C}$ and moreover there is a canonical identification*

$$(f^{-1}\mathcal{F})_p = \mathcal{F}_q$$

for any sheaf $\mathcal{F}$ on $\mathcal{C}$.

First proof Lemma 9.30.1. Note that since u is continuous and since v defines a point, it is immediate that $v \circ u$ satisfies conditions (1) and (2) of Definition 9.28.2. Let us prove the displayed equality. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$. Then

$$\mathcal{F}_q = colim_{(U,x)}\, \mathcal{F}(U)$$

where the colimit is over objects U in $\mathcal{C}$ and elements $x \in v(u(U))$. Similarly, we have

$$\begin{aligned}
(f^{-1}\mathcal{F})_p &= (u_p\mathcal{F})_p \\
&= colim_{(V,x)}\, colim_{U,\phi : V \to u(U)}\, \mathcal{F}(U) \\
&= colim_{(V,x,U,\phi : V \to u(U))}\, \mathcal{F}(U) \\
&= colim_{(U,x)}\, \mathcal{F}(U) \\
&= \mathcal{F}_q
\end{aligned}$$

Explanation: The first equality holds because $f^{-1}\mathcal{F} = (u_p\mathcal{F})^\#$ and because $\mathcal{G}_p = \mathcal{G}_p^\#$ for any presheaf $\mathcal{G}$, see Lemma 9.28.5. The second equality holds by the definition of u_p. In the third equality we simply combine colimits. To see the fourth equality we apply Categories, Lemma 4.15.5 to the functor F of diagram categories defined by the rule $F((V, x, U, \phi : V \to u(U))) = (U, v(\phi)(x))$. The lemma applies, because F has a right inverse, namely $(U, x) \mapsto (u(U), x, U, \mathrm{id} : u(U) \to u(U))$ and because there is always a morphism

$$(V, x, U, \phi : V \to u(U)) \longrightarrow (u(U), v(\phi)(x), U, \mathrm{id} : u(U) \to u(U))$$

in the fibre category over (U, x) which shows the fibre categories are nonempty and connected. The fifth equality is clear. Hence now we see that q also satisfies condition (3) of Definition 9.28.2 because it is a composition of exact functors. This finishes the proof. $\square$

Second proof Lemma 9.30.1. By Lemma 9.28.8 we may factor (p_*, p^{-1}) as

$$Sh(pt) \xrightarrow{i} Sh(\mathcal{S}) \xrightarrow{h} Sh(\mathcal{D})$$

where the second morphism of topoi comes from a morphism of sites $h : \mathcal{S} \to \mathcal{D}$ induced by the functor $v : \mathcal{D} \to \mathcal{S}$ (which makes sense as $\mathcal{S} \subset Sets$ is a full subcategory containing every object in the image of v). By Lemma 9.14.3 the composition $v \circ u : \mathcal{C} \to \mathcal{S}$ defines a morphism of sites $g : \mathcal{S} \to \mathcal{C}$. In particular, the functor $v \circ u : \mathcal{C} \to \mathcal{S}$ is continuous which by the definition of the coverings in $\mathcal{S}$, see Remark 9.15.4, means that $v \circ u$ satisfies conditions (1) and (2) of Definition 9.28.2. On the other hand, we see that

$$g_* i_* E(U) = i_* E(v(u(U)) = Mor_{Sets}(v(u(U)), E)$$

by the construction of i in Remark 9.15.4. Note that this is the same as the formula for which is equal to $(v \circ u)^p E$, see Equation (9.28.3.1). By Lemma 9.28.5 the functor $g_* i_* = (v \circ u)^p = (v \circ u)^s$ is right adjoint to the the stalk functor $\mathcal{F} \mapsto \mathcal{F}_q$. Hence we see that the stalk functor q^{-1} is canonically isomorphic to $i^{-1} \circ g^{-1}$. Hence it is exact and we conclude that q is a point. Finally, as we have $g = f \circ h$ by construction we see that $q^{-1} = i^{-1} \circ h^{-1} \circ f^{-1} = p^{-1} \circ f^{-1}$, i.e., we have the displayed formula of the lemma. □

Lemma 9.30.2. *Let $f : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ be a morphism of topoi. Let $p : Sh(pt) \to Sh(\mathcal{D})$ be a point. Then $q = f \circ p$ is a point of the topos $Sh(\mathcal{C})$ and we have a canonical identification*

$$(f^{-1}\mathcal{F})_p = \mathcal{F}_q$$

for any sheaf $\mathcal{F}$ on $\mathcal{C}$.

Proof. This is immediate from the definitions and the fact that we can compose morphisms of topoi. □

9.31. Localization and points

In this section we show that points of a localization $\mathcal{C}/U$ are constructed in a simple manner from the points of $\mathcal{C}$.

Lemma 9.31.1. *Let $\mathcal{C}$ be a site. Let p be a point of $\mathcal{C}$ given by $u : \mathcal{C} \to Sets$. Let U be an object of $\mathcal{C}$ and let $x \in u(U)$. The functor*

$$v : \mathcal{C}/U \longrightarrow Sets, \quad (\varphi : V \to U) \longmapsto \{y \in u(V) \mid u(\varphi)(y) = x\}$$

defines a point q of the site $\mathcal{C}/U$ such that the diagram

$$\begin{tikzcd} & Sh(pt) \arrow[dl, "q"'] \arrow[d, "p"] \\ Sh(\mathcal{C}/U) \arrow[r, "j_U"] & Sh(\mathcal{C}) \end{tikzcd}$$

commutes. In other words $\mathcal{F}_p = (j_U^{-1}\mathcal{F})_q$ for any sheaf on $\mathcal{C}$.

Proof. Choose S and $\mathcal{S}$ as in Lemma 9.28.8. We may identify $Sh(pt) = Sh(\mathcal{S})$ as in that lemma, and we may write $p = f : Sh(\mathcal{S}) \to Sh(\mathcal{C})$ for the morphism of topoi induced by u. By Lemma 9.24.1 we get a commutative diagram of topoi

$$\begin{tikzcd} Sh(\mathcal{S}/u(U)) \arrow[r, "j_{u(U)}"'] \arrow[d, "p'"'] & Sh(\mathcal{S}) \arrow[d, "p"] \\ Sh(\mathcal{C}/U) \arrow[r, "j_U"] & Sh(\mathcal{C}), \end{tikzcd}$$

where p' is given by the functor $u' : \mathcal{C}/U \to \mathcal{S}/u(U)$, $V/U \mapsto u(V)/u(U)$. Consider the functor $j_x : \mathcal{S} \cong \mathcal{S}/x$ obtained by assigning to a set E the set E endowed with the constant map $E \to u(U)$ with value x. Then j_x is a fully faithful cocontinuous functor which has a continuous right adjoint $v_x : (\psi : E \to u(U)) \mapsto \psi^{-1}(\{x\})$. Note that $j_U \circ j_x = \mathrm{id}_{\mathcal{S}}$, and $v_x \circ u' = v$. These observations imply that we have the following commutative diagram of topoi

$$\begin{tikzcd} Sh(\mathcal{S}) \arrow[dr, "a"] \arrow[ddr, "q"'] \arrow[rrr, bend left, "p"] & & & \\ & Sh(\mathcal{S}/u(U)) \arrow[r, "j_{u(U)}"'] \arrow[d, "p'"] & Sh(\mathcal{S}) \arrow[d, "p"] & \\ & Sh(\mathcal{C}/U) \arrow[r, "j_U"] & Sh(\mathcal{C}) & \end{tikzcd}$$

Namely:

(1) The morphism $a : Sh(\mathcal{S}) \to Sh(\mathcal{S}/u(U))$ is the morphism of topoi assoicated to the cocontinuous functor j_x, which equals the morphism associated to the continuous functor v_x, see Lemma 9.19.1 and Section 9.20.
(2) The composition $p \circ j_{u(U)} \circ a = p$ since $j_{u(U)} \circ j_x = \mathrm{id}_{\mathcal{S}}$.
(3) The compostion $p' \circ a$ gives a morphism of topoi. Moreover, it is the morphism of topoi associated to the continuous functor $v_x \circ u' = v$. Hence v does indeed define a point q of $\mathcal{C}/U$ which fits into the diagram above by construction.

This ends the proof of the lemma. □

Lemma 9.31.2. *Let $\mathcal{C}$, p, u, U be as in Lemma 9.31.1. The construction of Lemma 9.31.1 gives a one to one correspondence between points q of $\mathcal{C}/U$ lying over p and elements x of $u(U)$.*

Proof. Let q be a point of $\mathcal{C}/U$ given by the functor $v : \mathcal{C}/U \to Sets$ such that $j_U \circ q = p$ as morphisms of topoi. Recall that $u(V) = p^{-1}(h_V^\#)$ for any object V of $\mathcal{C}$, see Lemma 9.28.7. Similarly $v(V/U) = q^{-1}(h_{V/U}^\#)$ for any object V/U of $\mathcal{C}/U$. Consider the following two diagrams

$$\begin{tikzcd} Mor_{\mathcal{C}/U}(W/U, V/U) \arrow[r] \arrow[d] & Mor_{\mathcal{C}}(W, V) \arrow[d] \\ Mor_{\mathcal{C}/U}(W/U, U/U) \arrow[r] & Mor_{\mathcal{C}}(W, U) \end{tikzcd} \qquad \begin{tikzcd} h_{V/U}^\# \arrow[r] \arrow[d] & j_U^{-1}(h_V^\#) \arrow[d] \\ h_{U/U}^\# \arrow[r] & j_U^{-1}(h_U^\#) \end{tikzcd}$$

The right hand diagram is the sheafification of the diagram of presheaves on $\mathcal{C}/U$ which maps W/U to the left hand diagram of sets. (There is a small technical point to make here, namely, that we have $(j_U^{-1}h_V)^\# = j_U^{-1}(h_V^\#)$ and similarly for h_U, see Lemma 9.18.4.) Note that the left hand diagram of sets is cartesian. Since sheafification is exact (Lemma 9.10.14) we conclude that the right hand diagram is cartesian.

Apply the exact functor q^{-1} to the right hand diagram to get a cartesian diagram

$$\begin{tikzcd} v(V/U) \arrow[r] \arrow[d] & u(V) \arrow[d] \\ v(U/U) \arrow[r] & u(U) \end{tikzcd}$$

of sets. Here we have used that $q^{-1} \circ j^{-1} = p^{-1}$. Since U/U is a final object of $\mathcal{C}/U$ we see that $v(U/U)$ is a singleton. Hence the image of $v(U/U)$ in $u(U)$ is an element x, and the top horizontal map gives a bijection $v(V/U) \to \{y \in u(V) \mid y \mapsto x \text{ in } u(U)\}$ as desired. $\square$

Lemma 9.31.3. *Let $\mathcal{C}$ be a site. Let p be a point of $\mathcal{C}$ given by $u : \mathcal{C} \to Sets$. Let U be an object of $\mathcal{C}$. For any sheaf $\mathcal{G}$ on $\mathcal{C}/U$ we have*

$$(j_{U!}\mathcal{G})_p = \coprod\nolimits_q \mathcal{G}_q$$

where the coproduct is over the points q of $\mathcal{C}/U$ associated to elements $x \in u(U)$ as in Lemma 9.31.1.

Proof. We use the description of $j_{U!}\mathcal{G}$ as the sheaf associated to the presheaf $V \mapsto \coprod_{\varphi \in Mor_{\mathcal{C}}(V,U)} \mathcal{G}(V/_\varphi U)$ of Lemma 9.21.2. Also, the stalk of $j_{U!}\mathcal{G}$ at p is equal to the stalk of this presheaf, see Lemma 9.28.5. Hence we see that

$$(j_{U!}\mathcal{G})_p = colim_{(V,y)} \coprod\nolimits_{\varphi : V \to U} \mathcal{G}(V/_\varphi U)$$

To each element (V, y, φ, s) of this colimit, we can assign $x = u(\varphi)(y) \in u(U)$. Hence we obtain

$$(j_{U!}\mathcal{G})_p = \coprod\nolimits_{x \in u(U)} colim_{(\varphi : V \to U, y),\ u(\varphi)(y) = x} \mathcal{G}(V/_\varphi U).$$

This is equal to the expression of the lemma by our construction of the points q. $\square$

Remark 9.31.4. Warning: The result of Lemma 9.31.3 has no analogue for $j_{U,*}$.

9.32. 2-morphisms of topoi

This is a brief section concerning the notion of a 2-morphism of topoi.

Definition 9.32.1. Let $f, g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be two morphisms of topoi. A *2-morphism from f to g* is given by a transformation of functors $t : f_* \to g_*$.

Pictorially we sometimes represent t as follows:

$$\begin{tikzcd} Sh(\mathcal{C}) \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] & Sh(\mathcal{D}) \arrow[from=U, to=D, phantom, "\Downarrow t"] \end{tikzcd}$$

Note that since f^{-1} is adjoint to f_* and g^{-1} is adjoint to g_* we see that t induces also a transformation of functors $t : g^{-1} \to f^{-1}$ (usually denoted by the same symbol) uniquely characterized by the condition that the diagram

$$\begin{tikzcd} Mor_{Sh(\mathcal{C})}(\mathcal{G}, f_*\mathcal{F}) \arrow[r, equal] \arrow[d, "t \circ -"'] & Mor_{Sh(\mathcal{C})}(f^{-1}\mathcal{G}, \mathcal{F}) \arrow[d, "- \circ t"] \\ Mor_{Sh(\mathcal{C})}(\mathcal{G}, g_*\mathcal{F}) \arrow[r, equal] & Mor_{Sh(\mathcal{C})}(g^{-1}\mathcal{G}, \mathcal{F}) \end{tikzcd}$$

commutes. Because of set theoretic difficulties (see Remark 9.15.2) we do not obtain a 2-category of topoi. But we can still define horizontal and vertical composition and show that the axioms of a strict 2-category listed in Categories, Section 4.26 hold. Namely, vertical composition of 2-morphisms is clear (just compose transformations of functors), composition of 1-morphisms has been defined in Definition 9.15.1, and horizontal composition of

$$\begin{tikzcd} Sh(\mathcal{C}) \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow t"] & Sh(\mathcal{D}) \arrow[r, bend left, "f'"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow s"] & Sh(\mathcal{E}) \end{tikzcd}$$

is defined by the transformation of functors $s \star t$ introduced in Categories, Definition 4.25.1. Explicitly, $s \star t$ is given by

$$f'_* f_* \mathcal{F} \xrightarrow{f'_* t} f'_* g_* \mathcal{F} \xrightarrow{s} g'_* g_* \mathcal{F} \quad \text{or} \quad f'_* f_* \mathcal{F} \xrightarrow{s} g'_* f_* \mathcal{F} \xrightarrow{g'_* t} g'_* g_* \mathcal{F}$$

(these maps are equal). Since these definitions agree with the ones in Categories, Section 4.25 it follows from Categories, Lemma 4.25.2 that the axioms of a strict 2-category hold with these definitions.

9.33. Morphisms between points

Lemma 9.33.1. *Let $\mathcal{C}$ be a site. Let $u, u' : \mathcal{C} \to Sets$ be two functors, and let $t : u' \to u$ be a transformation of functors. Then we obtain a canonical transformation of stalk functors $t_{stalk} : \mathcal{F}_{p'} \to \mathcal{F}_p$ which agrees with t via the identifications of Lemma 9.28.3.*

Proof. Omitted. □

Definition 9.33.2. Let $\mathcal{C}$ be a site. Let p, p' be points of $\mathcal{C}$ given by functors $u, u' : \mathcal{C} \to Sets$. A *morphism* $f : p \to p'$ is given by a transformation of functors

$$f_u : u' \to u.$$

Note how the transformation of functors goes the other way. This makes sense, as we will see later, by thinking of the morphism f as a kind of 2-arrow pictorially as follows:

$$\begin{tikzcd} Sets = Sh(pt) \arrow[r, bend left, "p"] \arrow[r, bend right, "p'"'] \arrow[r, phantom, "\Downarrow f"] & Sh(\mathcal{C}) \end{tikzcd}$$

Namely, we will see later that f_u induces a canonical transformation of functors $p_* \to p'_*$ between the skyscraper sheaf constructions.

This is a fairly important notion, and deserves a more complete treatment here. List of desiderata

1. Describe the automorphisms of the point of $\mathcal{T}_G$ described in Example 9.29.6.
2. Describe $\boldsymbol{Mor}(p, p')$ in terms of $\boldsymbol{Mor}(p_*, p'_*)$.
3. Specialization of points in topological spaces. Show that if $x' \in \overline{\{x\}}$ in the topological space X, then there is a morphism $p \to p'$, where p (resp. p') is the point of $\mathcal{T}_X$ associated to x (resp. x').

9.34. Sites with enough points

Definition 9.34.1. Let $\mathcal{C}$ be a site.

1. A family of points $\{p_i\}_{i \in I}$ is called *conservative* if for every map of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ which is an isomorphism on all the fibres $\mathcal{F}_{p_i} \to \mathcal{G}_{p_i}$ is an isomorphism.
2. We say that $\mathcal{C}$ *has enough points* if there exists a conservative family of points.

It turns out that you can then check ``exactness'' at the stalks.

Lemma 9.34.2. *Let $\mathcal{C}$ be a site and let $\{p_i\}_{i \in I}$ be a conservative family of points. Then*

(1) *Given any map of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ we have $\forall i, \varphi_{p_i}$ injective implies φ injective.*
(2) *Given any map of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ we have $\forall i, \varphi_{p_i}$ surjective implies φ surjective.*
(3) *Given any pair of maps of sheaves $\varphi_1, \varphi_2 : \mathcal{F} \to \mathcal{G}$ we have $\forall i, \varphi_{1,p_i} = \varphi_{2,p_i}$ implies $\varphi_1 = \varphi_2$.*
(4) *Given a finite diagram $\mathcal{G} : \mathcal{J} \to Sh(\mathcal{C})$, a sheaf $\mathcal{F}$ and morphisms $q_j : \mathcal{F} \to \mathcal{G}_j$ then $(\mathcal{F}, q_j)$ is a limit of the diagram if and only if for each i the stalk $(\mathcal{F}_{p_i}, (q_j)_{p_i})$ is one.*
(5) *Given a finite diagram $\mathcal{F} : \mathcal{J} \to Sh(\mathcal{C})$, a sheaf $\mathcal{G}$ and morphisms $e_j : \mathcal{F}_j \to \mathcal{G}$ then $(\mathcal{G}, e_j)$ is a colimit of the diagram if and only if for each i the stalk $(\mathcal{G}_{p_i}, (e_j)_{p_i})$ is one.*

Proof. We will use over and over again that all the stalk functors commute with any finite limits and colimits and hence with products, fibred products, etc. We will also use that injective maps are the monomorphisms and the surjective maps are the epimorphisms. A map of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ is injective if and only if $\mathcal{F} \to \mathcal{F} \times_{\mathcal{G}} \mathcal{F}$ is an isomorphism. Hence (1). Similarly, $\varphi : \mathcal{F} \to \mathcal{G}$ is surjective if and only if $\mathcal{G} \amalg_{\mathcal{F}} \mathcal{G} \to \mathcal{G}$ is an isomorphism. Hence (2). The maps $a, b : \mathcal{F} \to \mathcal{G}$ are equal if and only if $\mathcal{F} \times_{a,\mathcal{G},b} \mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is an isomorphism. Hence (3). The assertions (4) and (5) follow immediately from the definitions and the remarks at the start of this proof. $\square$

Lemma 9.34.3. *Let $\mathcal{C}$ be a site and let $\{(p_i, u_i)\}_{i \in I}$ be a family of points. The family is conservative if and only if for every sheaf $\mathcal{F}$ and every $U \in Ob(\mathcal{C})$ and every pair of distinct sections $s, s' \in \mathcal{F}(U)$, $s \neq s'$ there exists an i and $x \in u_i(U)$ such that the triples (U, x, s) and (U, x, s') define distinct elements of $\mathcal{F}_{p_i}$.*

Proof. Suppose that the family is conservative and that $\mathcal{F}$, U, and s, s' are as in the lemma. The sections s, s' define maps $a, a' : (h_U)^\# \to \mathcal{F}$ which are distinct. Hence, by Lemma 9.34.2 there is an i such that $a_{p_i} \neq a'_{p_i}$. Recall that $(h_U)^\#_{p_i} = u_i(U)$, by Lemmas 9.28.3 and 9.28.5. Hence there exists an $x \in u_i(U)$ such that $a_{p_i}(x) \neq a'_{p_i}(x)$ in $\mathcal{F}_{p_i}$. Unwinding the definitions you see that (U, x, s) and (U, x, s') are as in the statement of the lemma.

To prove the converse, assume the condition on the existence of points of the lemma. Let $\phi : \mathcal{F} \to \mathcal{G}$ be a map of sheaves which is an isomorphism at all the stalks. We have to show that ϕ is both injective and surjective, see Lemma 9.11.2. Injectivity is an immediate consequence of the assumption. Let $*$ denote the final object of the category of sheaves, see Example 9.10.2. Consider the sheaf $\mathcal{H} = \mathcal{G} \coprod_{\mathcal{F}} *$. The map $\mathcal{F} \to \mathcal{G}$ is surjective if and only if the map $* \to \mathcal{H}$ is an isomorphism. By construction all the maps on stalks $*_{p_i} = \{*\} \to \mathcal{H}_{p_i}$ are bijective. If ϕ is not surjective, then there exists a U and a section $s \in \mathcal{H}(U)$ which is not equal to the section $*$. By assumption we see there exists an index i and $x \in u_i(U)$ such that (U, x, s) and $(U, x, *)$ define distinct points of $\mathcal{H}_{p_i}$. This is a contradiction. $\square$

In the following lemma the points $q_{i,x}$ are exactly all the points of $\mathcal{C}/U$ lying over the point p_i according to Lemma 9.31.2.

Lemma 9.34.4. *Let $\mathcal{C}$ be a site. Let U be an object of $\mathcal{C}$. let $\{(p_i, u_i)\}_{i \in I}$ be a family of points of $\mathcal{C}$. For $x \in u_i(U)$ let $q_{i,x}$ be the point of $\mathcal{C}/U$ constructed in Lemma 9.31.1. If $\{p_i\}$ is a conservative family of points, then $\{q_{i,x}\}_{i \in I, x \in u_i(U)}$ is a conservative family of points of $\mathcal{C}/U$. In particular, if $\mathcal{C}$ has enough points, then so does every localization $\mathcal{C}/U$.*

Proof. We know that $j_{U!}$ induces an equivalence $j_{U!} : Sh(\mathcal{C}/U) \to Sh(\mathcal{C})/h_U^\#$, see Lemma 9.21.4. Moreover, we know that $(j_{U!}\mathcal{G})_{p_i} = \coprod_x \mathcal{G}_{q_{i,x}}$, see Lemma 9.31.3. Hence the result follows formally. $\square$

The following lemma tells us we can check the existence of points locally on the site.

Lemma 9.34.5. *Let $\mathcal{C}$ be a site. Let $\{U_i\}_{i\in I}$ be a family of objects of $\mathcal{C}$. Assume*

(1) *$\coprod h_{U_i}^\# \to *$ is a surjective map of sheaves, and*
(2) *each localization $\mathcal{C}/U_i$ has enough points.*

Then $\mathcal{C}$ has enough points.

Proof. For each $i \in I$ let $\{p_j\}_{j\in J_i}$ be a conservative family of points of $\mathcal{C}/U_i$. For $j \in J_i$ denote $q_j : Sh(pt) \to Sh(\mathcal{C})$ the composition of p_j with the localization morphism $Sh(\mathcal{C}/U_i) \to Sh(\mathcal{C})$. Then q_j is a point, see Lemma 9.30.2. We claim that the family of points $\{q_j\}_{j\in\coprod J_i}$ is conservative. Namely, let $\mathcal{F} \to \mathcal{G}$ be a map of sheaves on $\mathcal{C}$ such that $\mathcal{F}_{q_j} \to \mathcal{G}_{q_j}$ is an isomorphism for all $j \in \coprod J_i$. Let W be an object of $\mathcal{C}$. By assumption (1) there exists a covering $\{W_a \to W\}$ and morphisms $W_a \to U_{i(a)}$. Since $(\mathcal{F}|_{\mathcal{C}/U_{i(a)}})_{p_j} = \mathcal{F}_{q_j}$ and $(\mathcal{G}|_{\mathcal{C}/U_{i(a)}})_{p_j} = \mathcal{G}_{q_j}$ by Lemma 9.30.2 we see that $\mathcal{F}|_{U_{i(a)}} \to \mathcal{G}|_{U_{i(a)}}$ is an isomorphism since the family of points $\{p_j\}_{j\in J_{i(a)}}$ is conservative. Hence $\mathcal{F}(W_a) \to \mathcal{G}(W_a)$ is bijective for each a. Similarly $\mathcal{F}(W_a\times_W W_b) \to \mathcal{G}(W_a\times_W W_b)$ is bijective for each a, b. By the sheaf condition this shows that $\mathcal{F}(W) \to \mathcal{G}(W)$ is bijective, i.e., $\mathcal{F} \to \mathcal{G}$ is an isomorphism. $\square$

9.35. Criterion for existence of points

This section corresponds to Deligne's appendix to [**MA71**, Exposé VI]. In fact it is almost literally the same.

Let $\mathcal{C}$ be a site. Suppose that $(I, \geq)$ is a directed partially ordered set, and that $(U_i, f_{ii'})$ is an inverse system over I, see Categories, Definition 4.19.1. Given the data $(I, \geq, U_i, f_{ii'})$ we define

$$u : \mathcal{C} \longrightarrow Sets, \quad u(V) = colim_i\ Mor_\mathcal{C}(U_i, V)$$

Let $\mathcal{F} \mapsto \mathcal{F}_p$ be the stalk functor associated to u as in Section 9.28. It is direct from the definition that actually

$$\mathcal{F}_p = colim_i\ \mathcal{F}(U_i)$$

in this special case. Note that u commutes with all finite limits (I mean those that are representable in $\mathcal{C}$) because each of the functors $V \mapsto Mor_\mathcal{C}(U_i, V)$ do, see Categories, Lemma 4.17.2.

We say that a system $(I, \geq, U_i, f_{ii'})$ is a *refinement* of $(J, \geq, V_j, g_{jj'})$ if $J \subset I$, the ordening on J induced from that of I and $V_j = U_j$, $g_{jj'} = f_{jj'}$ (in words, the inverse system over J is induced by that over I). Let u be the functor associated to $(I, \geq, U_i, f_{ii'})$ and let u' be the functor associated to $(J, \geq, V_j, g_{jj'})$. This induces a transformation of functors

$$u' \longrightarrow u$$

simply because the colimits for u' are over a subsystem of the systems in the colimits for u. In particular we get an associated transformation of stalk functors $\mathcal{F}_{p'} \to \mathcal{F}_p$, see Lemma 9.33.1.

Lemma 9.35.1. *Let $\mathcal{C}$ be a site. Let $(J, \geq, V_j, g_{jj'})$ be a system as above with associated pair of functors (u', p'). Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$. Let $s, s' \in \mathcal{F}_{p'}$ be distinct elements. Let $\{W_k \to W\}$ be a finite covering of $\mathcal{C}$. Let $f \in u(W)$. There exists a refinement $(I, \geq, U_i, f_{ii'})$*

of $(J, \geq, V_j, g_{jj'})$ such that s, s' map to distinct elements of $\mathcal{F}_p$ and that the image of f in $u'(W)$ is in the image of one of the $u'(W_k)$.

Proof. There exists a $j_0 \in J$ such that f is defined by $f' : V_{j_0} \to W$. For $j \geq j_0$ we set $V_{j,k} = V_j \times_{f' \circ f_{jj_0}, W} W_k$. Then $\{V_{j,k} \to V_j\}$ is a finite covering in the site $\mathcal{C}$. Hence $\mathcal{F}(V_j) \subset \prod_k \mathcal{F}(V_{j,k})$. By Categories, Lemma 4.17.2 once again we see that

$$\mathcal{F}_{p'} = colim_j\, \mathcal{F}(V_j) \longrightarrow \prod\nolimits_k colim_j\, \mathcal{F}(V_{j,k})$$

is injective. Hence there exists a k such that s and s' have distinct image in $colim_j\, \mathcal{F}(V_{j,k})$. Let $J_0 = \{j \in J, j \geq j_0\}$ and $I = J \amalg J_0$. We order I so that no element of the second summand is smaller than any element of the first, but otherwise using the ordering on J. If $j \in I$ is in the first summand then we use V_j and if $j \in I$ is in the second summand then we use $V_{j,k}$. We omit the definition of the transition maps of the inverse system. By the above it follows that s, s' have distinct image in $\mathcal{F}_p$. Moreover, the restriction of f' to $V_{j,k}$ factors through W_k by construction. □

Lemma 9.35.2. *Let $\mathcal{C}$ be a site. Let $(J, \geq, V_j, g_{jj'})$ be a system as above with associated pair of functors (u', p'). Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$. Let $s, s' \in \mathcal{F}_{p'}$ be distinct elements. There exists a refinement $(I, \geq, U_i, f_{ii'})$ of $(J, \geq, V_j, g_{jj'})$ such that s, s' map to distinct elements of $\mathcal{F}_p$ and such that for every finite covering $\{W_k \to W\}$ of the site $\mathcal{C}$, and any $f \in u'(W)$ the image of f in $u(W)$ is in the image of one of the $u(W_k)$.*

Proof. Let E be the set of pairs $(\{W_k \to W\}, f \in u'(W))$. Consider pairs $(E' \subset E, (I, \geq, U_i, f_{ii'}))$ such that

1. $(I, \geq, U_i, g_{ii'})$ is a refinement of $(J, \geq, V_j, g_{jj'})$,
2. s, s' map to distinct elements of $\mathcal{F}_p$, and
3. for every pair $(\{W_k \to W\}, f \in u'(W)) \in E'$ we have that the image of f in $u(W)$ is in the image of one of the $u(W_k)$.

We order such pairs by inclusion in the first factor and by refinement in the second. Denote $\mathcal{S}$ the class of all pairs $(E' \subset E, (I, \geq, U_i, f_{ii'}))$ as above. We claim that the hypothesis of Zorn's lemma holds for $\mathcal{S}$. Namely, suppose that $(E'_a, (I_a, \geq, U_i, f_{ii'}))_{a \in A}$ is a totally ordered subset of $\mathcal{S}$. Then we can define $E' = \bigcup_{a \in A} E'_a$ and we can set $I = \bigcup_{a \in A} I_a$. We claim that the corresponding pair $(E', (I, \geq, U_i, f_{ii'}))$ is an element of $\mathcal{S}$. Conditions (1) and (3) are clear. For condition (2) you note that

$$u = colim_{a \in A}\, u_a \text{ and correspondingly } \mathcal{F}_p = colim_{a \in A}\, \mathcal{F}_{p_a}$$

The distinctness of the images of s, s' in this stalk follows from the description of a directed colimit of sets, see Categories, Section 4.17. We will simply write $(E', (I, \ldots)) = \bigcup_{a \in A}(E'_a, (I_a, \ldots))$ in this situation.

OK, so Zorn's Lemma would apply if $\mathcal{S}$ was a set, and this would, combined with Lemma 9.35.1 above easily prove the lemma. It doesn't since $\mathcal{S}$ is a class. In order to circumvent this we choose a well ordering on E. For $e \in E$ set $E'_e = \{e' \in E \mid e' \leq e\}$. By transfinite induction we construct pairs $(E'_e, (I_e, \ldots)) \in \mathcal{S}$ such that $e_1 \leq e_2 \Rightarrow (E'_{e_1}, (I_{e_1}, \ldots)) \leq (E'_{e_2}, (I_{e_2}, \ldots))$. Let $e \in E$, say $e = (\{W_k \to W\}, f \in u'(W))$. If e has a predecessor $e - 1$, then we let $(I_e, \ldots)$ be a refinement of $(I_{e-1}, \ldots)$ as in Lemma 9.35.1 with respect to the system $e = (\{W_k \to W\}, f \in u'(W))$. If e does not have a predecessor, then we let $(I_e, \ldots)$ be a refinement of $\bigcup_{e' < e}(I_{e'}, \ldots)$ with respect to the system $e = (\{W_k \to W\}, f \in u'(W))$. Finally, the union $\bigcup_{e \in E} I_e$ will be a solution to the problem posed in the lemma. □

Proposition 9.35.3. *Let $\mathcal{C}$ be a site. Assume that*

(1) *finite limits exist in $\mathcal{C}$, and*
(2) *every covering $\{U_i \to U\}_{i \in I}$ has a refinement by a finite covering of $\mathcal{C}$.*

Then $\mathcal{C}$ has enough points.

Proof. We have to show that given any sheaf $\mathcal{F}$ on $\mathcal{C}$, any $U \in Ob(\mathcal{C})$, and any distinct sections $s, s' \in \mathcal{F}(U)$, there exists a point p such that s, s' have distinct image in $\mathcal{F}_p$. See Lemma 9.34.3. Consider the system $(J, \geq, V_j, g_{jj'})$ with $J = \{1\}$, $V_1 = U$, $g_{11} = \mathrm{id}_U$. Apply Lemma 9.35.2. By the result of that lemma we get a system $(I, \geq, U_i, f_{ii'})$ refining our system such that $s_p \neq s'_p$ and such that moreover for every finite covering $\{W_k \to W\}$ of the site $\mathcal{C}$ the map $\coprod_k u(W_k) \to u(W)$ is surjective. Since every covering of $\mathcal{C}$ can be refined by a finite covering we conclude that $\coprod_k u(W_k) \to u(W)$ is surjective for *any* covering $\{W_k \to W\}$ of the site $\mathcal{C}$. This implies that $u = p$ is a point, see Proposition 9.29.2 (and the discussion at the beginning of this section which garantees that u commutes with finite limits). $\square$

9.36. Exactness properties of pushforward

Let f be a morphism of topoi. The functor f_* in general is only left exact. There are many additional conditions one can impose on this functor to single out particular classes of morphisms of topoi. We collect them here and note some of the logical dependencies. Some parts of the following lemma are purely category theoretical (i.e., they do not depend on having a morphism of topoi, just having a pair of adjoint functors is enough).

Lemma 9.36.1. *Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. Consider the following properties (on sheaves of sets):*

(1) *f_* is faithful,*
(2) *f_* is fully faithful,*
(3) *$f^{-1}f_*\mathcal{F} \to \mathcal{F}$ is surjective for all $\mathcal{F}$ in $Sh(\mathcal{C})$,*
(4) *f_* transforms surjections into surjections,*
(5) *f_* commutes with coequalizers,*
(6) *f_* commutes with pushouts,*
(7) *$f^{-1}f_*\mathcal{F} \to \mathcal{F}$ is an isomorphism for all $\mathcal{F}$ in $Sh(\mathcal{C})$,*
(8) *f_* reflects injections,*
(9) *f_* reflects surjections,*
(10) *f_* reflects bijections, and*
(11) *for any surjection $\mathcal{F} \to f^{-1}\mathcal{G}$ there exists a surjection $\mathcal{G}' \to \mathcal{G}$ such that $f^{-1}\mathcal{G}' \to f^{-1}\mathcal{G}$ factors through $\mathcal{F} \to f^{-1}\mathcal{G}$.*

Then we have the following implications

(a) *(2) $\Rightarrow$ (1),*
(b) *(3) $\Rightarrow$ (1),*
(c) *(7) $\Rightarrow$ (1), (2), (3), (8), (9), (10).*
(d) *(3) $\Leftrightarrow$ (9),*
(e) *(6) $\Rightarrow$ (4),*
(f) *(4) $\Leftrightarrow$ (11), and*
(g) *(8) + (9) $\Rightarrow$ (10).*

Proof. Proof of (a): This is immediate from the definitions.

Proof of (b). Suppose that $a, b : \mathcal{F} \to \mathcal{F}'$ are maps of sheaves on $\mathcal{C}$. If $f_*a = f_*b$, then $f^{-1}f_*a = f^{-1}f_*b$. Consider the commutative diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, shift left] \arrow[r, shift right] & \mathcal{F}' \\ f^{-1}f_*\mathcal{F} \arrow[u] \arrow[r, shift left] \arrow[r, shift right] & f^{-1}f_*\mathcal{F}' \arrow[u] \end{tikzcd}$$

If the bottom two arrows are equal and the vertical arrows are surjective then the top two arrows are equal. Hence (b) follows.

Proof of (c). Suppose that $a : \mathcal{F} \to \mathcal{F}'$ is a map of sheaves on $\mathcal{C}$. Consider the commutative diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r] & \mathcal{F}' \\ f^{-1}f_*\mathcal{F} \arrow[u] \arrow[r] & f^{-1}f_*\mathcal{F}' \arrow[u] \end{tikzcd}$$

If (7) holds, then the vertical arrows are isomorphisms. Hence if f_*a is injective (resp. surjective, resp. bijective) then the bottom arrow is injective (resp. surjective, resp. bijective) and hence the top arrow is injective (resp. surjective, resp. bijective). Thus we see that (7) implies (8), (9), (10). It is clear that (7) implies (3) and hence (1). Finally, if $\beta : f_*\mathcal{F} \to f_*\mathcal{F}'$ is a map of sheaves, then $\alpha = f^{-1}\beta : \mathcal{F} = f^{-1}f_*\mathcal{F} \to f^{-1}f_*\mathcal{F}' = \mathcal{F}'$ is a map of sheaves on $\mathcal{C}$. Chasing diagrams we see that the following diagram

$$\begin{tikzcd} f_*\mathcal{F} \arrow[r, "f_*\alpha"] & f_*\mathcal{F}' \\ f_*f^{-1}f_*\mathcal{F} \arrow[u] \arrow[r, "f_*f^{-1}\beta"] & f_*f^{-1}f_*\mathcal{F}' \arrow[u] \\ f_*\mathcal{F} \arrow[u] \arrow[r, "\beta"] & f_*\mathcal{F}' \arrow[u] \end{tikzcd}$$

is commutative, in other words $f_*\alpha = \beta$. Hence we see that (2) holds.

Proof of (d). Assume (3). Suppose that $a : \mathcal{F} \to \mathcal{F}'$ is a map of sheaves on $\mathcal{C}$ such that f_*a is surjective. As f^{-1} is exact this implies that $f^{-1}f_*a : f^{-1}f_*\mathcal{F} \to f^{-1}f_*\mathcal{F}'$ is surjective. Combined with (3) this implies that a is surjective. This means that (9) holds. Assume (9). Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$. We have to show that the map $f^{-1}f_*\mathcal{F} \to \mathcal{F}$ is surjective. It suffices to show that $f_*f^{-1}f_*\mathcal{F} \to f_*\mathcal{F}$ is surjective. And this is true because there is a canonical map $f_*\mathcal{F} \to f_*f^{-1}f_*\mathcal{F}$ which is a one-sided inverse.

Proof of (e). If $\mathcal{F} \to \mathcal{F}'$ is surjective then the map $\mathcal{F}' \amalg_{\mathcal{F}} \mathcal{F}' \to \mathcal{F}'$ is injective. Hence (6) implies that $f_*\mathcal{F}' \amalg_{f_*\mathcal{F}} f_*\mathcal{F}' \to f_*\mathcal{F}'$ is injective also. And this in turn implies that $f_*\mathcal{F} \to f_*\mathcal{F}'$ is surjective. Hence we see that (6) implies (4).

Proof of (f). Assume (4). Let $\mathcal{F} \to f^{-1}\mathcal{G}$ be a surjective map of sheaves on $\mathcal{C}$. By (4) we see that $f_*\mathcal{F} \to f_*f^{-1}\mathcal{G}$ is surjective. Let $\mathcal{G}'$ be the fibre product

$$\begin{tikzcd} f_*\mathcal{F} \arrow[r] & f_*f^{-1}\mathcal{G} \\ \mathcal{G}' \arrow[u] \arrow[r] & \mathcal{G} \arrow[u] \end{tikzcd}$$

so that $\mathcal{G}' \to \mathcal{G}$ is surjective also. Consider the commutative diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r] & f^{-1}\mathcal{G} \\ f^{-1}f_*\mathcal{F} \arrow[u] \arrow[r] & f^{-1}f_*f^{-1}\mathcal{G} \arrow[u] \\ f^{-1}\mathcal{G}' \arrow[u] \arrow[r] & f^{-1}\mathcal{G} \arrow[u] \end{tikzcd}$$

and we see the required result. Conversely, assume (11). Let $a : \mathcal{F} \to \mathcal{F}'$ be surjective map of sheaves on $\mathcal{C}$. Consider the fibre product diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r] & \mathcal{F}' \\ \mathcal{F}'' \arrow[u] \arrow[r] & f^{-1}f_*\mathcal{F}' \arrow[u] \end{tikzcd}$$

Because the lower horizontal arrow is surjective and by (11) we can find a surjection $\gamma : \mathcal{G}' \to f_*\mathcal{F}'$ such that $f^{-1}\gamma$ factors through $\mathcal{F}'' \to f^{-1}f_*\mathcal{F}'$:

$$\begin{tikzcd} & \mathcal{F} \arrow[r] & \mathcal{F}' \\ f^{-1}\mathcal{G}' \arrow[r] & \mathcal{F}'' \arrow[u] \arrow[r] & f^{-1}f_*\mathcal{F}' \arrow[u] \end{tikzcd}$$

Pushing this down using f_* we get a commutative diagram

$$\begin{tikzcd} & f_*\mathcal{F} \arrow[r] & f_*\mathcal{F}' \\ f_*f^{-1}\mathcal{G}' \arrow[r] & f_*\mathcal{F}'' \arrow[u] \arrow[r] & f_*f^{-1}f_*\mathcal{F}' \arrow[u] \\ \mathcal{G}' \arrow[u] \arrow[rr] & & f_*\mathcal{F}' \arrow[u] \end{tikzcd}$$

which proves that (4) holds.

Proof of (g). This is immediate from the definitions. □

Here is a condition on a morphism of sites which garantees that the functor f_* transforms surjective maps into surjective maps.

Lemma 9.36.2. *Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites associated to the continuous functor $u : \mathcal{C} \to \mathcal{D}$. Assume that for any object U of $\mathcal{C}$ and any covering $\{V_j \to u(U)\}$ in $\mathcal{D}$ there exists a covering $\{U_i \to U\}$ in $\mathcal{C}$ such that the map of sheaves*

$$\coprod h^\#_{u(U_i)} \to h^\#_{u(U)}$$

factors through the map of sheaves

$$\coprod h^\#_{V_j} \to h^\#_{u(U)}.$$

Then f_* transforms surjective maps of sheaves into surjective maps of sheaves.

Proof. Let $a : \mathcal{F} \to \mathcal{G}$ be a surjective map of sheaves on $\mathcal{D}$. Let U be an object of $\mathcal{C}$ and let $s \in f_*\mathcal{G}(U) = \mathcal{G}(u(U))$. By assumption there exists a covering $\{V_j \to u(U)\}$ and sections $s_j \in \mathcal{F}(V_j)$ with $a(s_j) = s|_{V_j}$. Now we may think of the sections s, s_j and a as giving a commutative diagram of maps of sheaves

$$\begin{tikzcd} \coprod h^\#_{V_j} \arrow[r, "\coprod s_j"'] \arrow[d] & \mathcal{F} \arrow[d, "a"] \\ h^\#_{u(U)} \arrow[r, "s"] & \mathcal{G} \end{tikzcd}$$

By assumption there exists a covering $\{U_i \to U\}$ such that we can enlarge the commutative diagram above as follows

$$\begin{tikzcd} & \coprod h^\#_{V_j} \arrow[r, "\coprod s_j"'] \arrow[d] & \mathcal{F} \arrow[d, "a"] \\ \coprod h^\#_{u(U_i)} \arrow[r] \arrow[ur] & h^\#_{u(U)} \arrow[r, "s"] & \mathcal{G} \end{tikzcd}$$

Because $\mathcal{F}$ is a sheaf the map from the left lower corner to the right upper corner corresponds to a family of sections $s_i \in \mathcal{F}(u(U_i))$, i.e., sections $s_i \in f_*\mathcal{F}(U_i)$. The commutativity of the diagram implies that $a(s_i)$ is equal to the restriction of s to U_i. In other words we have shown that f_*a is a surjective map of sheaves. $\square$

Example 9.36.3. Assume $f : \mathcal{D} \to \mathcal{C}$ satisfies the assumptions of Lemma 9.36.2. Then it is in general not the case that f_* commutes with coequalizers or pushouts. Namely, suppose that f is the morphism of sites associated to the morphism of topological spaces $X = \{1, 2\} \to Y = \{*\}$ (see Example 9.14.2), where Y is a singleton space, and $X = \{1, 2\}$ is a discrete space with two points. A sheaf $\mathcal{F}$ on X is given by a pair (A_1, A_2) of sets. Then $f_*\mathcal{F}$ corresponds to the set $A_1 \times A_2$. Hence if $a = (a_1, a_2), b = (b_1, b_2) : (A_1, A_2) \to (B_1, B_2)$ are maps of sheaves on X, then the coequalizer of a, b is (C_1, C_2) where C_i is the coequalizer of a_i, b_i, and the coequalizer of f_*a, f_*b is the coequalizer of

$$a_1 \times a_2, b_1 \times b_2 : A_1 \times A_2 \longrightarrow B_1 \times B_2$$

which is in general different from $C_1 \times C_2$. Namely, if $A_2 = \emptyset$ then $A_1 \times A_2 = \emptyset$, and hence the coequalizer of the displayed arrows is $B_1 \times B_2$, but in general $C_1 \neq B_1$. A similar example works for pushouts.

The following lemma gives a criterion for when a morphism of sites has a functor f_* which reflects injections and surjections. Note that this also implies that f_* is faithful and that the map $f^{-1}f_*\mathcal{F} \to \mathcal{F}$ is always surjective.

Lemma 9.36.4. *Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites given by the functor $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that for every object V of $\mathcal{D}$ there exist objects U_i of $\mathcal{C}$ and morphisms $u(U_i) \to V$ such that $\{u(U_i) \to V\}$ is a covering of $\mathcal{D}$. In this case the functor $f_* : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ reflects injections and surjections.*

Proof. Let $\alpha : \mathcal{F} \to \mathcal{G}$ be maps of sheaves on $\mathcal{D}$. By assumption for every object V of $\mathcal{D}$ we get $\mathcal{F}(V) \subset \prod \mathcal{F}(u(U_i)) = \prod (u^s\mathcal{F})(u(U_i))$ by the sheaf condition for some U_i objects of $\mathcal{C}$ and similarly for $\mathcal{G}$. Hence it is clear that if $f_*\alpha$ is injective, then α is injective. In other words f_* reflects injections.

Suppose that $f_*\alpha$ is surjective. Then for $V, U_i, u(U_i) \to V$ as above and a section $s \in \mathcal{G}(V)$, there exist coverings $\{U_{ij} \to U_i\}$ such that $s|_{u(U_{ij})}$ is in the image of $\mathcal{F}(u(U_{ij}))$. Since $\{u(U_{ij}) \to V\}$ is a covering (as u is continuous and by the axioms of a site) we conclude that s is locally in the image. Thus α is surjective. In other words f_* reflects surjections. □

9.37. Almost cocontinuous functors

Let $\mathcal{C}$ be a site. The category $PSh(\mathcal{C})$ has an initial object, namely the presheaf which assigns the empty set to each object of $\mathcal{C}$. Let us denote this presheaf by $\emptyset$. It follows from the properties of sheafification that the sheafification $\emptyset^\#$ of $\emptyset$ is an initial object of the category $Sh(\mathcal{C})$ of sheaves on $\mathcal{C}$.

Definition 9.37.1. Let $\mathcal{C}$ be a site. We say an object U of $\mathcal{C}$ is *sheaf theoretically empty* if $\emptyset^\# \to h_U^\#$ is an isomorphism of sheaves.

The following lemma makes this notion more explicit.

Lemma 9.37.2. *Let $\mathcal{C}$ be a site. Let U be an object of $\mathcal{C}$. The following are equivalent:*

(1) *U is sheaf theoretically empty,*
(2) *$\mathcal{F}(U)$ is a singleton for each sheaf $\mathcal{F}$,*
(3) *$\emptyset^\#(U)$ is a singleton,*
(4) *$\emptyset^\#(U)$ is nonempty, and*
(5) *the empty family is a covering of U in $\mathcal{C}$.*

Moreover, if U is sheaf theoretically empty, then for any morphism $U' \to U$ of $\mathcal{C}$ the object U' is sheaf theoretically empty.

Proof. For any sheaf $\mathcal{F}$ we have $\mathcal{F}(U) = Mor_{Sh(\mathcal{C})}(h_U^\#, \mathcal{F})$. Hence, we see that (1) and (2) are equivalent. It is clear that (2) implies (3) implies (4). If every covering of U is given by a nonempty family, then $\emptyset^+(U)$ is empty by definition of the plus construction. Note that $\emptyset^+ = \emptyset^\#$ as $\emptyset$ is a separated presheaf, see Theorem 9.10.10. Thus we see that (4) implies (5). If (5) holds, then $\mathcal{F}(U)$ is a singleton for every sheaf $\mathcal{F}$ by the sheaf condition for $\mathcal{F}$, see Remark 9.7.2. Thus (5) implies (2) and (1) -- (5) are equivalent. The final assertion of the lemma follows from Axiom (3) of Definition 9.6.2 applied the the empty covering of U. □

Definition 9.37.3. Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. We say u is *almost cocontinuous* if for every object U of $\mathcal{C}$ and every covering $\{V_j \to u(U)\}_{j\in J}$ there exists a covering $\{U_i \to U\}_{i\in I}$ in $\mathcal{C}$ such that for each i in I we have at least one of the following two conditions

(1) $u(U_i)$ is sheaf theoretically empty, or
(2) the morphism $u(U_i) \to u(U)$ factors through V_j for some $j \in J$.

The motivation for this definition comes from a closed immersion $i : Z \to X$ of topological spaces. As discussed in Example 9.19.9 the continuous functor $\mathcal{T}_X \to \mathcal{T}_Z$, $U \mapsto Z \cap U$ is not cocontinuous. But it is almost continuous in the sense defined above. We know that i_* while not exact on sheaves of sets, is exact on sheaves of abelian groups, see Sheaves, Remark 6.32.5. And this holds in general for continuous and almost continuous functors.

Lemma 9.37.4. *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that u is continuous and almost continuous. Let $\mathcal{G}$ be a presheaf on $\mathcal{D}$ such that $\mathcal{G}(V)$ is a singleton whenever V is sheaf theoretically empty. Then $(u^p\mathcal{G})^\# = u^p(\mathcal{G}^\#)$.*

Proof. Let $U \in Ob(\mathcal{C})$. We have to show that $(u^p\mathcal{G})^\#(U) = u^p(\mathcal{G}^\#)(U)$. It suffices to show that $(u^p\mathcal{G})^+(U) = u^p(\mathcal{G}^+)(U)$ since $\mathcal{G}^+$ is another presheaf for which the assumption of the lemma holds. We have

$$u^p(\mathcal{G}^+)(U) = \mathcal{G}^+(u(U)) = colim_{\mathcal{V}}\, \check{H}^0(\mathcal{V}, \mathcal{G})$$

where the colimit is over the coverings $\mathcal{V}$ of $u(U)$ in $\mathcal{D}$. On the other hand, we see that

$$u^p(\mathcal{G})^+(U) = colim_{\mathcal{U}}\, \check{H}^0(u(\mathcal{U}), \mathcal{G})$$

where the colimit is over the category of coverings $\mathcal{U} = \{U_i \to U\}_{i\in I}$ of U in $\mathcal{C}$ and $u(\mathcal{U}) = \{u(U_i) \to u(U)\}_{i\in I}$. The condition that u is continuous means that each $u(\mathcal{U})$ is a covering. Write $I = I_1 \amalg I_2$, where

$$I_2 = \{i \in I \mid u(U_i) \text{ is sheaf theoretically empty}\}$$

Then $u(\mathcal{U})' = \{u(U_i) \to u(U)\}_{i\in I_1}$ is still a covering of because each of the other pieces can be covered by the empty family and hence can be dropped by Axiom (2) of Definition 9.6.2. Moreover, $\check{H}^0(u(\mathcal{U}), \mathcal{G}) = \check{H}^0(u(\mathcal{U})', \mathcal{G})$ by our assumption on $\mathcal{G}$. Finally, the condition that u is almost cocontinuous implies that for every covering $\mathcal{V}$ of $u(U)$ there exists a covering $\mathcal{U}$ of U such that $u(\mathcal{U})'$ refines $\mathcal{V}$. It follows that the two colimits displayed above have the same value as desired. □

Lemma 9.37.5. *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that u is continuous and almost cocontinuous. Then $u^s = u^p : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ commutes with pushouts and coequalizers (and more generally finite, nonempty, connected colimits).*

Proof. Let $\mathcal{I}$ be a finite, nonempty, connected index category. Let $\mathcal{I} \to Sh(\mathcal{D})$, $i \mapsto \mathcal{G}_i$ by a diagram. We know that the colimit of this diagram is the sheafification of the colimit in the category of presheaves, see Lemma 9.10.13. Denote $colim^{Psh}$ the colimit in the category of presheaves. Since $\mathcal{I}$ is finite, nonempty and connected we see that $colim_i^{Psh}\, \mathcal{G}_i$ is a presheaf satisfying the assumptions of Lemma 9.37.4 (because a finite nonempty connected colimit of singleton sets is a singleton). Hence that lemma gives

$$\begin{aligned} u^s(colim_i\, \mathcal{G}_i) &= u^s((colim_i^{Psh}\, \mathcal{G}_i)^\#) \\ &= (u^p(colim_i^{Psh}\, \mathcal{G}_i))^\# \\ &= (colim_i^{PSh}\, u^p(\mathcal{G}_i))^\# \\ &= colim_i\, u^s(\mathcal{G}_i) \end{aligned}$$

as desired. □

Lemma 9.37.6. *Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites associated to the continuous functor $u : \mathcal{C} \to \mathcal{D}$. If u is almost cocontinuous then f_* commutes with pushouts and coequalizers (and more generally finite, nonempty, connected colimits).*

Proof. This is a special case of Lemma 9.37.5. □

9.38. Sheaves of algebraic structures

In Sheaves, Section 6.15 we introduced a type of algebraic struture to be a pair $(\mathcal{A}, s)$, where $\mathcal{A}$ is a category, and $s : \mathcal{A} \to Sets$ is a functor such that

(1) s is faithful,
(2) $\mathcal{A}$ has limits and s commutes with limits,
(3) $\mathcal{A}$ has filtered colimits and s commutes with them, and
(4) s reflects isomorphisms.

For such a type of algebraic structure we saw that a presheaf $\mathcal{F}$ with values in $\mathcal{A}$ on a space X is a sheaf if and only if the associated presheaf of sets is a sheaf. Moreover, we worked out the notion of stalk, and given a continuous map $f : X \to Y$ we defined adjoint functors pushforward and pullback on sheaves of algebraic structures which agrees with pushforward and pullback on the underlying sheaves of sets. In addition extending a sheaf of algebraic structures from a basis to all opens of a space, works as expected.

Part of this material still works in the setting of sites and sheaves. Let $(\mathcal{A}, s)$ be a type of algebraic structure. Let $\mathcal{C}$ be a site. Let us denote $PSh(\mathcal{C}, \mathcal{A})$, resp. $Sh(\mathcal{C}, \mathcal{A})$ the category of presheaves, resp. sheaves with values in $\mathcal{A}$ on $\mathcal{C}$.

(α) A presheaf with values in $\mathcal{A}$ is a sheaf if and only if its underlying presheaf of sets is a sheaf. See the proof of Sheaves, Lemma 6.9.2.
(β) Given a presheaf $\mathcal{F}$ with values in $\mathcal{A}$ the presheaf $\mathcal{F}^\# = (\mathcal{F}^+)^+$ is a sheaf. This is true since the colimits in the sheafification process are filtered, and even colimits over directed partially ordered sets (see Section 9.10, especially the proof of Lemma 9.10.14) and since s commutes with filtered colimits.
(γ) We get the following commutative diagram

$$\begin{tikzcd} Sh(\mathcal{C}, \mathcal{A}) \arrow[r, shift left] \arrow[d, "s"'] & PSh(\mathcal{C}, \mathcal{A}) \arrow[l, shift left, "\#"] \arrow[d, "s"] \\ Sh(\mathcal{C}) \arrow[r, shift left] & PSh(\mathcal{C}) \arrow[l, shift left] \end{tikzcd}$$

(δ) We have $\mathcal{F} = \mathcal{F}^\#$ if and only if $\mathcal{F}$ is a sheaf of algebraic structures.
(ϵ) The functor $^\#$ is adjoint to the inclusion functor:

$$Mor_{PSh(\mathcal{C},\mathcal{A})}(\mathcal{G}, \mathcal{F}) = Mor_{Sh(\mathcal{C},\mathcal{A})}(\mathcal{G}^\#, \mathcal{F})$$

The proof is the same as the proof of Proposition 9.10.12.
(ζ) The functor $\mathcal{F} \mapsto \mathcal{F}^\#$ is left exact. The proof is the same as the proof of Lemma 9.10.14.

Definition 9.38.1. Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites given by a functor $u : \mathcal{C} \to \mathcal{D}$. We define the *pushforward* functor for presheaves of algebraic structures by the rule $u^p\mathcal{F}(U) = \mathcal{F}(uU)$, and for sheaves of algebraic structures by the same rule, namely $f_*\mathcal{F}(U) = \mathcal{F}(uU)$.

The problem comes with trying the define the pullback. The reason is that the colimits defining the functor u_p in Section 9.5 may not be filtered. Thus the axioms above are not enough in general to define the pullback of a (pre)sheaf of algebraic structures. Nonetheless, in almost all cases the following lemma is sufficient to define pushfoward, and pullback of (pre)sheaves of algebraic structures.

Lemma 9.38.2. *Suppose the functor $u : \mathcal{C} \to \mathcal{D}$ satisfies the hypotheses of Proposition 9.14.6, and hence gives rise to a morphism of sites $f : \mathcal{D} \to \mathcal{C}$. In this case the pullback* *functor f^{-1} (resp. u^p) and the pushforward functor f_* (resp. u_p) extend to an adjoint pair of functors on the categories of sheaves (resp. presheaves) of algebraic structures. Moreover, these functors commute with taking the underlying sheaf (resp. presheaf) of sets.*

Proof. We have defined $f_* = u^p$ above. In the course of the proof of Proposition 9.14.6 we saw that all the colimits used to define u_p are filtered under the assumptions of the proposition. Hence we conclude from the definition of a type of algebraic structure that we may define u_p by exactly the same colimits as a functor on presheaves of algebraic structures. Adjointness of u_p and u^p is proved in exactly the same way as the proof of Lemma 9.5.4. The discussion of sheafification of presheaves of algebraic structures above then implies that we may define $f^{-1}(\mathcal{F}) = (u_p\mathcal{F})^\#$. □

We briefly discuss a method for dealing with pullback and pushforward for a general morphism of sites, and more generally for any morphism of topoi.

Let $\mathcal{C}$ be a site. In the case $\mathcal{A} = Ab$, we may think of an abelian (pre)sheaf on $\mathcal{C}$ as a quadruple $(\mathcal{F}, +, 0, i)$. Here the data are

(D1) $\mathcal{F}$ is a sheaf of sets,
(D2) $+ : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ is a morphism of sheaves of sets,
(D3) $0 : * \to \mathcal{F}$ is a morphism from the singleton sheaf (see Example 9.10.2) to $\mathcal{F}$, and
(D4) $i : \mathcal{F} \to \mathcal{F}$ is a morphism of sheaves of sets.

These data have to satisfy the following axioms

(A1) $+$ is associative and commutative,
(A2) 0 is a unit for $+$, and
(A3) $+ \circ (1, i) = 0 \circ (\mathcal{F} \to *)$.

Compare Sheaves, Lemma 6.4.3. Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites. Note that since f^{-1} is exact we have $f^{-1}* = *$ and $f^{-1}(\mathcal{F} \times \mathcal{F}) = f^{-1}\mathcal{F} \times f^{-1}\mathcal{F}$. Thus we can define $f^{-1}\mathcal{F}$ simply as the quadruple $(f^{-1}\mathcal{F}, f^{-1}+, f^{-1}0, f^{-1}i)$. The axioms are going to be preserved because f^{-1} is a functor which commutes with finite limits. Finally it is not hard to check that f_* and f^{-1} are adjoint as usual.

In [**MA71**] this method is used. They introduce something called an ``*espèce the structure algébrique ≪définie par limites projectives finie≫*". For such an espèce you can use the method described above to define a pair of adjoint functors f^{-1} and f_* as above. This clearly works for most algebraic structures that one encounters in practice. Instead of formalizing this construction we simply list those algebraic structures for which this method works (to be verified case by case). In fact, this method works for any morphism of topoi.

Proposition 9.38.3. *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $f = (f^{-1}, f_*)$ be a morphism of topoi from $Sh(\mathcal{D}) \to Sh(\mathcal{C})$. The method introduced above gives rise to an adjoint pair of functors (f^{-1}, f_*) on sheaves of algebraic structures compatible with taking the underlying sheaves of sets for the following types of algebraic structures:*

(1) *pointed sets,*
(2) *abelian groups,*
(3) *groups,*
(4) *monoids,*
(5) *rings,*
(6) *modules over a fixed ring, and*
(7) *lie algebras over a fixed field.*

Moreover, in each of these cases the results above labeled (α), (β), (γ), (δ), (ϵ), and (ζ) hold.

Proof. The final statement of the proposition holds simply since each of the listed categories, endowed with the obvious forgetful functor, is indeed a type of algebraic structure in the sense explained at the beginning of this section. See Sheaves, Lemma 6.15.2.

Proof of (2). We think of a sheaf of abelian groups as a quadruple $(\mathcal{F}, +, 0, i)$ as explained in the discussion preceding the proposition. If $(\mathcal{F}, +, 0, i)$ lives on $\mathcal{C}$, then its pullback is defined as $(f^{-1}\mathcal{F}, f^{-1}+, f^{-1}0, f^{-1}i)$. If $(\mathcal{G}, +, 0, i)$ lives on $\mathcal{D}$, then its pushforward is defined as $(f_*\mathcal{G}, f_*+, f_*0, f_*i)$. This works because $f_*(\mathcal{G}\times\mathcal{G}) = f_*\mathcal{G}\times f_*\mathcal{G}$. Adjointness follows from adjointness of the set based functors, since

$$Mor_{Ab(\mathcal{C})}((\mathcal{F}_1, +, 0, i), (\mathcal{F}_2, +, 0, i)) = \left\{ \begin{matrix} \varphi \in Mor_{Sh(\mathcal{C})}(\mathcal{F}_1, \mathcal{F}_2) \\ \varphi \text{ is compatible with } +, 0, i \end{matrix} \right\}$$

Details left to the reader.

This method also works for sheaves of rings by thinking of a sheaf of rings (with unit) as a sixtuple $(\mathcal{O}, +, 0, i, \cdot, 1)$ satisfying a list of axioms that you can find in any elementary algebra book.

A sheaf of pointed sets is a pair $(\mathcal{F}, p)$, where $\mathcal{F}$ is a sheaf of sets, and $p : * \to \mathcal{F}$ is a map of sheaves of sets.

A sheaf of groups is given by a quadruple $(\mathcal{F}, \cdot, 1, i)$ with suitable axioms.

A sheaf of monoids is given by a pair $(\mathcal{F}, \cdot)$ with suitable axiom.

Let R be a ring. An sheaf of R-modules is given by a quintuple $(\mathcal{F}, +, 0, i, \{\lambda_r\}_{r\in R})$, where the quadruple $(\mathcal{F}, +, 0, i)$ is a sheaf of abelian groups as above, and $\lambda_r : \mathcal{F} \to \mathcal{F}$ is a family of morphisms of sheaves of sets such that $\lambda_r \circ 0 = 0$, $\lambda_r \circ + = + \circ (\lambda_r, \lambda_r)$, $\lambda_{r+r'} = + \circ \lambda_r \times \lambda_{r'} \circ (\text{id}, \text{id})$, $\lambda_{rr'} = \lambda_r \circ \lambda_{r'}$, $\lambda_1 = \text{id}$, $\lambda_0 = 0 \circ (\mathcal{F} \to *)$. $\square$

We will discuss the category of sheaves of modules over a sheaf of rings in Modules on Sites, Section 16.10.

Remark 9.38.4. Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{D} \to \mathcal{C}$ be a continuous functor which gives rise to a morphism of sites $\mathcal{C} \to \mathcal{D}$. Note that even in the case of abelian groups we have not defined a pullback functor for presheaves of abelian groups. Since all colimits are representable in the category of abelian groups, we certainly may define a functor u_p^{ab} on abelian presheaves by the same colimits as we have used to define u_p on presheaves of sets. It will also be the case that u_p^{ab} is adjoint to u^p on the categories of abelian presheaves. However, it will not always be the case that u_p^{ab} agrees with u_p on the underlying presheaves of sets.

9.39. Pullback maps

It sometimes happens that a site $\mathcal{C}$ does not have a final object. In this case we define the global section functor as follows.

Definition 9.39.1. The *global sections* of a presheaf of sets $\mathcal{F}$ over a site $\mathcal{C}$ is the set

$$\Gamma(\mathcal{C}, \mathcal{F}) = Mor_{PSh(\mathcal{C})}(*, \mathcal{F})$$

where $*$ is the final object in the category of presheaves on $\mathcal{C}$, i.e., the presheaf which associates to every object a singleton.

Of course the same definition applies to sheaves as well. Here is one way to compute global sections.

Lemma 9.39.2. *Let $\mathcal{C}$ be a site. Let $a, b : V \to U$ be objects of $\mathcal{C}$ such that*

$$h_V^\# \rightrightarrows h_U^\# \longrightarrow *$$

is a coequalizer in $Sh(\mathcal{C})$. Then $\Gamma(\mathcal{C}, \mathcal{F})$ is the equalizer of $a^*, b^* : \mathcal{F}(U) \to \mathcal{F}(V)$.

Proof. Since $\mathrm{Mor}_{Sh(\mathcal{C})}(h_U^\#, \mathcal{F}) = \mathcal{F}(U)$ this is clear from the definitions. □

Now, let $f : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ be a morphism of topoi. Then for any sheaf $\mathcal{G}$ on $\mathcal{D}$ there is a pullback map

$$f^{-1} : \Gamma(\mathcal{D}, \mathcal{F}) \longrightarrow \Gamma(\mathcal{C}, f^{-1}\mathcal{F})$$

Namely, as f^{-1} is exact it transforms $*$ into $*$. We can generalize this a bit by considering a pair of sheaves $\mathcal{F}, \mathcal{G}$ on $\mathcal{C}, \mathcal{D}$ together with a map $f^{-1}\mathcal{F} \to \mathcal{G}$. Then we compose to get a map

$$\Gamma(\mathcal{D}, \mathcal{F}) \longrightarrow \Gamma(\mathcal{C}, \mathcal{G})$$

A slightly more general construction which occurs frequently in nature is the following. Suppose that we have a commutative diagram of morphisms of topoi

$$\begin{tikzcd} Sh(\mathcal{D}) \arrow[rr, "f"'] \arrow[dr, "h"'] & & Sh(\mathcal{C}) \arrow[dl, "g"] \\ & Sh(\mathcal{B}) & \end{tikzcd}$$

Next, suppose that we have a sheaf $\mathcal{F}$ on $\mathcal{D}$. Then there is a *pullback map*

$$f^{-1} : g_*\mathcal{F} \longrightarrow h_*f^{-1}\mathcal{F}$$

Namely, it is just the map coming from the identification $h_*f^{-1}\mathcal{F} = g_*f_*f^{-1}\mathcal{F}$ together with the canonical map $\mathcal{F} \to f_*f^{-1}\mathcal{F}$ pushed down to $\mathcal{B}$. Again, if we have a pair of sheaves $\mathcal{F}, \mathcal{G}$ on $\mathcal{C}, \mathcal{D}$ together with a map $f^{-1}\mathcal{F} \to \mathcal{G}$, then we compose to get a map

$$g_*\mathcal{F} \longrightarrow h_*\mathcal{G}$$

Restricting to sections over an object of $\mathcal{B}$ one recovers the pullback map on global sections in many cases, see (insert future reference here). A seemingly more general situation is where we have a commutative diagram of topoi

$$\begin{tikzcd} Sh(\mathcal{D}) \arrow[r, "f"'] \arrow[d, "h"'] & Sh(\mathcal{C}) \arrow[d, "g"] \\ Sh(\mathcal{B}) \arrow[r, "e"] & Sh(\mathcal{A}) \end{tikzcd}$$

and a sheaf $\mathcal{G}$ on $\mathcal{C}$. Then there is a map $e^{-1}g_*\mathcal{G} \to h_*f^{-1}\mathcal{G}$. Namely, this map is adjoint to a map $g_*\mathcal{G} \to e_*h_*f^{-1}\mathcal{G} = (e \circ h)_*f^{-1}\mathcal{G}$ which is the pullback map just described.

9.40. Topologies

In this section we define what a topology on a category is. However, the case of most interest for algebraic geometry is the topology defined by a site on its underlying category. We strongly suggest the first time reader **skip this section and all other sections of this chapter**!

Definition 9.40.1. Let $\mathcal{C}$ be a category. Let $U \in Ob(\mathcal{C})$. A *sieve S on U* is a subpresheaf $S \subset h_U$.

In other words, a sieve on U picks out for each object $T \in Ob(\mathcal{C})$ a subset $S(T)$ of the set of all morphisms $T \to U$. In fact, the only condition on the collection of subsets $S(T) \subset h_U(T) = Mor_\mathcal{C}(T, U)$ is the following rule

$$\left.\begin{matrix} (\alpha : T \to U) \in S(T) \\ g : T' \to T \end{matrix}\right\} \Rightarrow (\alpha \circ g : T' \to U) \in S(T') \tag{9.40.1.1}$$

A good mental picture to keep in mind is to think of the map $S \to h_U$ as a ``morphism from S to U".

Lemma 9.40.2. *Let $\mathcal{C}$ be a category. Let $U \in Ob(\mathcal{C})$.*

1. *The collection of sieves on U is a set.*
2. *Inclusion defines a partial ordering on this set.*
3. *Unions and intersections of sieves are sieves.*
4. *Given a family of morphisms $\{U_i \to U\}_{i\in I}$ of $\mathcal{C}$ with target U there exists a unique smallest sieve S on U such that each $U_i \to U$ belongs to $S(U_i)$.*
5. *The sieve $S = h_U$ is the maximal sieve.*
6. *The empty subpresheaf is the minimal sieve.*

Proof. By our definition of subpresheaf, the collection of all subpresheaves of a presheaf $\mathcal{F}$ is a subset of $\prod_{U\in Ob(\mathcal{C})} \mathcal{P}(\mathcal{F}(U))$. And this is a set. (Here $\mathcal{P}(A)$ denotes the powerset of A.) Hence the collection of sieves on U is a set.

The partial ordering is defined by: $S \leq S'$ if and only if $S(T) \subset S'(T)$ for all $T \to U$. Notation: $S \subset S'$.

Given a collection of sieves S_i, $i \in I$ on U we can define $\bigcup S_i$ as the sieve with values $(\bigcup S_i)(T) = \bigcup S_i(T)$ for all $T \in Ob(\mathcal{C})$. We define the intersection $\bigcap S_i$ in the same way.

Given $\{U_i \to U\}_{i\in I}$ as in the statement, consider the morphisms of presheaves $h_{U_i} \to h_U$. We simply define S as the union of the images (Definition 9.3.5) of these maps of presheaves.

The last two statements of the lemma are obvious. $\square$

Definition 9.40.3. Let $\mathcal{C}$ be a category. Given a family of morphisms $\{f_i : U_i \to U\}_{i\in I}$ of $\mathcal{C}$ with target U we say the sieve S on U described in Lemma 9.40.2 part (4) is the *sieve on U generated by the morphisms f_i*.

Definition 9.40.4. Let $\mathcal{C}$ be a category. Let $f : V \to U$ be a morphism of $\mathcal{C}$. Let $S \subset h_U$ be a sieve. We define the *pullback of S by f* to be the sieve $S \times_U V$ of V defined by the rule

$$(\alpha : T \to V) \in (S \times_U V)(T) \Leftrightarrow (f \circ \alpha : T \to U) \in S(T)$$

We leave it to the reader to see that this is indeed a sieve (hint: use Equation 9.40.1.1). We also sometimes call $S \times_U V$ the *base change* of S by $f : V \to U$.

Lemma 9.40.5. *Let $\mathcal{C}$ be a category. Let $U \in Ob(\mathcal{C})$. Let S be a sieve on U. If $f : V \to U$ is in S, then $S \times_U V = h_V$ is maximal.*

Proof. Trivial from the definitions. □

Definition 9.40.6. Let $\mathcal{C}$ be a category. A *topology on $\mathcal{C}$* is given by the following datum:

For every $U \in Ob(\mathcal{C})$ a subset $J(U)$ of the set of all sieves on U.

These sets $J(U)$ have to satisfy the following conditions

(1) For every morphism $f : V \to U$ in $\mathcal{C}$, and every element $S \in J(U)$ the pullback $S \times_U V$ is an element of $J(V)$.
(2) If S and S' are sieves on $U \in Ob(\mathcal{C})$, if $S \in J(U)$, and if for all $f \in S(V)$ the pullback $S' \times_U V$ belongs to $J(V)$, then S' belongs to $J(U)$.
(3) For every $U \in Ob(\mathcal{C})$ the maximal sieve $S = h_U$ belongs to $J(U)$.

In this case, the sieves belonging to $J(U)$ are called the *covering sieves*.

Lemma 9.40.7. *Let $\mathcal{C}$ be a category. Let J be a topology on $\mathcal{C}$. Let $U \in Ob(\mathcal{C})$.*

(1) *Finite intersections of elements of $J(U)$ are in $J(U)$.*
(2) *If $S \in J(U)$ and $S' \supset S$, then $S' \in J(U)$.*

Proof. Let $S, S' \in J(U)$. Consider $S'' = S \cap S'$. For every $V \to U$ in $S(U)$ we have

$$S' \times_U V = S'' \times_U V$$

simply because $V \to U$ already is in S. Hence by the second axiom of the definition we see that $S'' \in J(U)$.

Let $S \in J(U)$ and $S' \supset S$. For every $V \to U$ in $S(U)$ we have $S' \times_U V = h_V$ by Lemma 9.40.5. Thus $S' \times_U V \in J(V)$ by the third axiom. Hence $S' \in J(U)$ by the second axiom. □

Definition 9.40.8. Let $\mathcal{C}$ be a category. Let J, J' be two topologies on $\mathcal{C}$. We say that J is *finer* than J' if and only if for every object U of $\mathcal{C}$ we have $J'(U) \subset J(U)$.

In other words, any covering sieve of J' is a covering sieve of J. There exists a finest topology on $\mathcal{C}$, namely that topology where any sieve is a covering sieve. This is called the *discrete topology* of $\mathcal{C}$. There also exists a coarsest topology. Namely, the topology where $J(U) = \{h_U\}$ for all objects U. This is called the *chaotic* or *indiscrete topology*.

Lemma 9.40.9. *Let $\mathcal{C}$ be a category. Let $\{J_i\}_{i \in I}$ be a set of topologies.*

(1) *The rule $J(U) = \bigcap J_i(U)$ defines a topology on $\mathcal{C}$.*
(2) *There is a coarsest topology finer than all of the topologies J_i.*

Proof. The first part is direct from the definitions. The second follows by taking the intersection of all topologies finer than all of the J_i. □

At this point we can define without any motivation what a sheaf is.

Definition 9.40.10. Let $\mathcal{C}$ be a category endowed with a topology J. Let $\mathcal{F}$ be a presheaf of sets on $\mathcal{C}$. We say that $\mathcal{F}$ is a *sheaf* on $\mathcal{C}$ if for every $U \in Ob(\mathcal{C})$ and for every covering sieve S of U the canonical map

$$Mor_{PSh(\mathcal{C})}(h_U, \mathcal{F}) \longrightarrow Mor_{PSh(\mathcal{C})}(S, \mathcal{F})$$

is bijective.

Recall that the left hand side of the displayed formula equals $\mathcal{F}(U)$. In other words, $\mathcal{F}$ is a sheaf if and only if a section of $\mathcal{F}$ over U is the same thing as a compatible collection of sections $s_{T,\alpha} \in \mathcal{F}(T)$ parametrized by $(\alpha : T \to U) \in S(T)$, and this for every covering sieve S on U.

Lemma 9.40.11. *Let $\mathcal{C}$ be a category. Let $\{\mathcal{F}_i\}_{i\in I}$ be a collection of presheaves of sets on $\mathcal{C}$. For each $U \in Ob(\mathcal{C})$ denote $J(U)$ the set of sieves S with the following property: For every morphism $V \to U$, the maps*

$$Mor_{PSh(\mathcal{C})}(h_V, \mathcal{F}_i) \longrightarrow Mor_{PSh(\mathcal{C})}(S \times_U V, \mathcal{F}_i)$$

are bijective for all $i \in I$. Then J defines a topology on $\mathcal{C}$. This topology is the finest topology in which all of the $\mathcal{F}_i$ are sheaves.

Proof. If we show that J is a topology, then the last statement of the lemma immediately follows. The first and second axioms of a topology are immediately verified. Thus, assume that we have an object U, and sieves S, S' of U such that $S \in J(U)$, and for all $V \to U$ in $S(V)$ we have $S' \times_U V \in J(V)$. We have to show that $S' \in J(U)$. In other words, we have to show that for any $f : W \to U$, the maps

$$\mathcal{F}_i(W) = Mor_{PSh(\mathcal{C})}(h_W, \mathcal{F}_i) \longrightarrow Mor_{PSh(\mathcal{C})}(S' \times_U W, \mathcal{F}_i)$$

are bijective for all $i \in I$. Pick an element $i \in I$ and pick an element $\varphi \in Mor_{PSh(\mathcal{C})}(S' \times_U W, \mathcal{F}_i)$. We will construct a section $s \in \mathcal{F}_i(W)$ mapping to φ.

Suppose $\alpha : V \to W$ is an element of $S \times_U W$. According to the definition of pullbacks we see that the composition $f \circ \alpha : V \to W \to U$ is in S. Hence $S' \times_U V$ is in $J(W)$ by assumption on the pair of sieves S, S'. Now we have a commutative diagram of presheaves

$$\begin{tikzcd} S' \times_U V \arrow[r] \arrow[d] & h_V \arrow[d] \\ S' \times_U W \arrow[r] & h_W \end{tikzcd}$$

The restriction of φ to $S' \times_U V$ corresponds to an element $s_{V,\alpha} \in \mathcal{F}_i(V)$. This we see from the definition of J, and because $S' \times_U V$ is in $J(W)$. We leave it to the reader to check that the rule $(V, \alpha) \mapsto s_{V,\alpha}$ defines an element $\psi \in Mor_{PSh(\mathcal{C})}(S \times_U W, \mathcal{F}_i)$. Since $S \in J(U)$ we see immediately from the definition of J that ψ corresponds to an element s of $\mathcal{F}_i(W)$.

We leave it to the reader to verify that the construction $\varphi \mapsto s$ is inverse to the natural map displayed above. □

Definition 9.40.12. Let $\mathcal{C}$ be a category. The finest topology on $\mathcal{C}$ such that all representable presheaves are sheaves, see Lemma 9.40.11, is called the *canonical topology* of $\mathcal{C}$.

9.41. The topology defined by a site

Suppose that $\mathcal{C}$ is a category, and suppose that $\text{Cov}_1(\mathcal{C})$ and $\text{Cov}_2(\mathcal{C})$ are sets of coverings that define the structure of a site on $\mathcal{C}$. In this situation it can happen that the categories of sheaves (of sets) for $\text{Cov}_1(\mathcal{C})$ and $\text{Cov}_2(\mathcal{C})$ are the same, see for example Lemma 9.8.5.

It turns out that the category of sheaves on $\mathcal{C}$ with respect to some topology J determines and is determined by the topology J. This is a nontrivial statement which we will address later, see Theorem 9.43.2.

Accepting this for the moment it makes sense to study the topology determined by a site.

Lemma 9.41.1. *Let $\mathscr{C}$ be a site with coverings $Cov(\mathscr{C})$. For every object U of $\mathscr{C}$, let $J(U)$ denote the set of sieves S on U with the following property: there exists a covering $\{f_i : U_i \to U\}_{i\in I} \in Cov(\mathscr{C})$ so that the sieve S' generated by the f_i (see Definition 9.40.3) is contained in S.*

(1) *This J is a topology on $\mathscr{C}$.*
(2) *A presheaf $\mathscr{F}$ is a sheaf for this topology (see Definition 9.40.10) if and only if it is a sheaf on the site (see Definition 9.7.1).*

Proof. To prove the first assertion we just note that axioms (1), (2) and (3) of the definition of a site (Definition 9.6.2) directly imply the axioms (3), (2) and (1) of the definition of a topology (Definition 9.40.6). As an example we prove J has property (2). Namely, let U be an object of $\mathscr{C}$, let S, S' be sieves on U such that $S \in J(U)$, and such that for every $V \to U$ in $S(V)$ we have $S' \times_U V \in J(V)$. By definition of $J(U)$ we can find a covering $\{f_i : U_i \to U\}$ of the site such that S the image of $h_{U_i} \to h_U$ is contained in S. Since each $S' \times_U U_i$ is in $J(U_i)$ we see that there are coverings $\{U_{ij} \to U_i\}$ of the site such that $h_{U_{ij}} \to h_{U_i}$ is contained in $S' \times_U U_i$. By definition of the base change this means that $h_{U_{ij}} \to h_U$ is contained in the subpresheaf $S' \subset h_U$. By axiom (2) for sites we see that $\{U_{ij} \to U\}$ is a covering of U and we conclude that $S' \in J(U)$ by definition of J.

Let $\mathscr{F}$ be a presheaf. Suppose that $\mathscr{F}$ is a sheaf in the topology J. We will show that $\mathscr{F}$ is a sheaf on the site as well. Let $\{f_i : U_i \to U\}_{i\in I}$ be a covering of the site. Let $s_i \in \mathscr{F}(U_i)$ be a family of sections such that $s_i|_{U_i\times_U U_j} = s_j|_{U_i\times_U U_j}$ for all i, j. We have to show that there exists a unique section $s \in \mathscr{F}(U)$ restricting back to the s_i on the U_i. Let $S \subset h_U$ be the sieve generated by the f_i. Note that $S \in J(U)$ by definition. In stead of constructing s, by the sheaf condition in the topology, it suffices to construct an element

$$\varphi \in Mor_{PSh(\mathscr{C})}(S, \mathscr{F}).$$

Take $\alpha \in S(T)$ for some object $T \in \mathscr{U}$. This means exactly that $\alpha : T \to U$ is a morphism which factors through f_i for some $i \in I$ (and maybe more than 1). Pick such an index i and a factorization $\alpha = f_i \circ \alpha_i$. Define $\varphi(\alpha) = \alpha_i^* s_i$. If i', $\alpha = f_i \circ \alpha'_{i'}$ is a second choice, then $\alpha_i^* s_i = (\alpha'_{i'})^* s_{i'}$ exactly because of our condition $s_i|_{U_i\times_U U_j} = s_j|_{U_i\times_U U_j}$ for all i, j. Thus $\varphi(\alpha)$ is well defined. We leave it to the reader to verify that φ, which in turn determines s is correct in the sense that s restricts back to s_i.

Let $\mathscr{F}$ be a presheaf. Suppose that $\mathscr{F}$ is a sheaf on the site $(\mathscr{C}, \mathrm{Cov}(\mathscr{C}))$. We will show that $\mathscr{F}$ is a sheaf for the topology J as well. Let U be an object of $\mathscr{C}$. Let S be a covering sieve on U with respect to the topology J. Let

$$\varphi \in Mor_{PSh(\mathscr{C})}(S, \mathscr{F}).$$

We have to show there is a unique element in $\mathscr{F}(U) = Mor_{PSh(\mathscr{C})}(h_U, \mathscr{F})$ which restricts back to φ. By definition there exists a covering $\{f_i : U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathscr{C})$ such that $f_i : U_i \in U$ belongs to $S(U_i)$. Hence we can set $s_i = \varphi(f_i) \in \mathscr{F}(U_i)$. Then it is a pleasant exercise to see that $s_i|_{U_i\times_U U_j} = s_j|_{U_i\times_U U_j}$ for all i, j. Thus we obtain the desired section s by the sheaf condition for $\mathscr{F}$ on the site $(\mathscr{C}, \mathrm{Cov}(\mathscr{C}))$. Details left to the reader. $\square$

Definition 9.41.2. Let $\mathscr{C}$ be a site with coverings $\mathrm{Cov}(\mathscr{C})$. The *topology associated to* $\mathscr{C}$ is the topology J contructed in Lemma 9.41.1 above.

Let $\mathscr{C}$ be a category. Let $\mathrm{Cov}_1(\mathscr{C})$ and $\mathrm{Cov}_2(\mathscr{C})$ be two coverings defining the structure of a site on $\mathscr{C}$. It may very well happen that the topologies defined by these are the same. If this happens then we say $\mathrm{Cov}_1(\mathcal{C})$ and $\mathrm{Cov}_2(\mathcal{C})$ *define the same topology* on $\mathcal{C}$. And if this happens then the categories of sheaves are the same, by Lemma 9.41.1.

It is usually the case that we only care about the topology defined by a collection of coverings, and we view the possibility of choosing different sets of coverings as a tool to study the topology.

Remark 9.41.3. Enlarging the class of coverings. Clearly, if $\mathrm{Cov}(\mathcal{C})$ defines the structure of a site on $\mathcal{C}$ then we may add to $\mathcal{C}$ any set of families of morphisms with fixed target tautologically equivalent (see Definition 9.8.2) to elements of $\mathrm{Cov}(\mathcal{C})$ without changing the topology.

Remark 9.41.4. Shrinking the class of coverings. Let $\mathcal{C}$ be a site. Consider the power set $\mathcal{S} = P(\mathrm{Arrow}(\mathcal{C}))$ (power set) of the set of morphisms, i.e., the set of all sets of morphisms. Let $\mathcal{S}_\tau \subset \mathcal{S}$ be the subset consisting of those $T \in \mathcal{S}$ such that (a) all $\varphi \in T$ have the same target, (b) the collection $\{\varphi\}_{\varphi\in T}$ is tautologically equivalent (see Definition 9.8.2) to some covering in $\mathrm{Cov}(\mathcal{C})$. Clearly, considering the elements of $\mathcal{S}_\tau$ as the coverings, we do not get exactly the notion of a site as defined in Definition 9.6.2. The structure $(\mathcal{C}, \mathcal{S}_\tau)$ we get satisfies slightly modified conditions. The modified conditions are:

(0') $\mathrm{Cov}(\mathcal{C}) \subset P(\mathrm{Arrow}(\mathcal{C}))$,
(1') If $V \to U$ is an isomorphism then $\{V \to U\} \in \mathrm{Cov}(\mathcal{C})$.
(2') If $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ and for each i we have $\{V_{ij} \to U_i\}_{j\in J_i} \in \mathrm{Cov}(\mathcal{C})$, then $\{V_{ij} \to U\}_{i\in I, j\in J_i}$ is tautologically equivalent to an element of $\mathrm{Cov}(\mathcal{C})$.
(3') If $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ and $V \to U$ is a morphism of $\mathcal{C}$ then $U_i \times_U V$ exists for all i and $\{U_i \times_U V \to V\}_{i\in I}$ is tautologically equivalent to an element of $\mathrm{Cov}(\mathcal{C})$.

And it is easy to verify that, given a structure satisfying (0') -- (3') above, then after suitably enlarging $\mathrm{Cov}(\mathcal{C})$ (compare Sets, Section 3.11) we get a site. Obviously there is little difference between this notion and the actual notion of a site, at least from the point of view of the topology. There are two benefits: because of condition (0') above the coverings automatically form a set, and because of (0') the totality of all structures of this type forms a set as well. The price you pay for this is that you have to keep writing ``tautologically equivalent'' everywhere.

9.42. Sheafification in a topology

In this section we explain the analogue of the sheafification construction in a topology.

Let $\mathcal{C}$ be a category. Let J be a topology on $\mathcal{C}$. Let $\mathcal{F}$ be a presheaf of sets. For every $U \in Ob(\mathcal{C})$ we define

$$L\mathcal{F}(U) = colim_{S\in J(U)^{opp}}\, Mor_{PSh(\mathcal{C})}(S, \mathcal{F})$$

as a colimit. Here we think of $J(U)$ as a partially ordered set, ordered by inclusion, see Lemma 9.40.2. The transition maps in the system are defined as follows. If $S \subset S'$ are in $J(U)$, then $S \to S'$ is a morphism of presheaves. Hence there is a natural restriction mapping

$$Mor_{PSh(\mathcal{C})}(S, \mathcal{F}) \longrightarrow Mor_{PSh(\mathcal{C})}(S', \mathcal{F}).$$

Thus we see that $S \mapsto Mor_{PSh(\mathcal{C})}(S, \mathcal{F})$ is a directed system as in Categories, Definition 4.19.1 provided we reverse the ordering on $J(U)$ (which is what the superscript ^{opp} is supposed to indicate). In particular, since $h_U \in J(U)$ there is a canonical map

$$\ell : \mathcal{F}(U) \longrightarrow L\mathcal{F}(U)$$

coming from the identification $\mathcal{F}(U) = Mor_{PSh(\mathcal{C})}(h_U, \mathcal{F})$. In addition, the colimit defining $L\mathcal{F}(U)$ is directed since for any pair of covering sieves S, S' on U the sieve $S \cap S'$ is a covering sieve too, see Lemma 9.40.2.

Let $f : V \to U$ be a morphism in $\mathcal{C}$. Let $S \in J(U)$. There is a commutative diagram

$$\begin{tikzcd} S \times_U V \arrow[r] \arrow[d] & h_V \arrow[d] \\ S \arrow[r] & h_U \end{tikzcd}$$

We can use the left vertical map to get canonical restriction maps

$$Mor_{PSh(\mathcal{C})}(S, \mathcal{F}) \to Mor_{PSh(\mathcal{C})}(S \times_U V, \mathcal{F}).$$

Base change $S \mapsto S \times_U V$ induces an order preserving map $J(U) \to J(V)$. And the restriction maps define a transformation of functors as in Categories, Lemma categories-lemma-functorial-colimit. Hence we get a natural retriction map

$$L\mathcal{F}(U) \longrightarrow L\mathcal{F}(V).$$

Lemma 9.42.1. *In the situation above.*

(1) *The assignment $U \mapsto L\mathcal{F}(U)$ combined with the restriction mappings defined above is a presheaf.*
(2) *The maps ℓ glue to give a morphism of presheaves $\ell : \mathcal{F} \to L\mathcal{F}$.*
(3) *The rule $\mathcal{F} \mapsto (\mathcal{F} \xrightarrow{\ell} L\mathcal{F})$ is a functor.*
(4) *If $\mathcal{F}$ is a subpresheaf of $\mathcal{G}$, then $L\mathcal{F}$ is a subpresheaf of $L\mathcal{G}$.*
(5) *The map $\ell : \mathcal{F} \to L\mathcal{F}$ has the following property: For every section $s \in L\mathcal{F}(U)$ there exists a covering sieve S on U and an element $\varphi \in Mor_{PSh(\mathcal{C})}(S, \mathcal{F})$ such that $\ell(\varphi)$ equals the restriction of s to S.*

Proof. Omitted. □

Definition 9.42.2. Let $\mathcal{C}$ be a category. Let J be a topology on $\mathcal{C}$. We say that a presheaf of sets $\mathcal{F}$ is *separated* if for every object U and every covering sieve S on U the canonical map $\mathcal{F}(U) \to Mor_{PSh(\mathcal{C})}(S, \mathcal{F})$ is injective.

Theorem 9.42.3. *Let $\mathcal{C}$ be a category. Let J be a topology on $\mathcal{C}$. Let $\mathcal{F}$ be a presheaf of sets.*

(1) *The presheaf $L\mathcal{F}$ is separated.*
(2) *If $\mathcal{F}$ is separated, then $L\mathcal{F}$ is a sheaf and the map of presheaves $\mathcal{F} \to L\mathcal{F}$ is injective.*
(3) *If $\mathcal{F}$ is a sheaf, then $\mathcal{F} \to L\mathcal{F}$ is an isomorphism.*
(4) *The presheaf $LL\mathcal{F}$ is always a sheaf.*

Proof. Part (3) is trivial from the definition of L and the definition of a sheaf (Definition 9.40.10). Part (4) follows formally from the others.

We sketch the proof of (1). Suppose S is a covering sieve of the object U. Suppose that $\varphi_i \in L\mathcal{F}(U)$, $i = 1, 2$ map to the same element in $Mor_{PSh(\mathcal{C})}(S, L\mathcal{F})$. We may find a single covering sieve S' on U such that both φ_i are represented by elements $\varphi_i \in Mor_{PSh(\mathcal{C})}(S', \mathcal{F})$. We may assume that $S' = S$ by replacing both S and S' by $S' \cap S$ which is also a covering sieve, see Lemma 9.40.2. Suppose $V \in Ob(\mathcal{C})$, and $\alpha : V \to U$ in $S(V)$. Then we have $S \times_U V = h_V$, see Lemma 9.40.5. Thus the restrictions of φ_i via $V \to U$ correspond to sections $s_{i,V,\alpha}$ of $\mathcal{F}$ over V. The assumption is that there exist a covering sieve $S_{V,\alpha}$ of V such that $s_{i,V,\alpha}$ restrict to the same element of $Mor_{PSh(\mathcal{C})}(S_{V,\alpha}, \mathcal{F})$. Consider the sieve S'' on U defined by the rule

$$(9.42.3.1)\qquad \begin{aligned} (f : T \to U) \in S''(T) \quad \Leftrightarrow \quad & \exists\, V,\ \alpha : V \to U,\ \alpha \in S(V), \\ & \exists\, g : T \to V,\ g \in S_{V,\alpha}(T), \\ & f = \alpha \circ g \end{aligned}$$

By axiom (2) of a topology we see that S'' is a covering sieve on U. By construction we see that φ_1 and φ_2 restrict to the same element of $Mor_{PSh(\mathcal{C})}(S'', L\mathcal{F})$ as desired.

We sketch the proof of (2). Assume that $\mathcal{F}$ is a separated presheaf of sets on $\mathcal{C}$ with respect to the topology J. Let S be a covering sieve of the object U of $\mathcal{C}$. Suppose that $\varphi \in Mor_{\mathcal{C}}(S, L\mathcal{F})$. We have to find an element $s \in L\mathcal{F}(U)$ restricting to φ. Suppose $V \in Ob(\mathcal{C})$, and $\alpha : V \to U$ in $S(V)$. The value $\varphi(\alpha) \in L\mathcal{F}(V)$ is given by a covering sieve $S_{V,\alpha}$ of V and a morphism of presheaves $\varphi_{V,\alpha} : S_{V,\alpha} \to \mathcal{F}$. As in the proof above, define a covering sieve S'' on U by Equation (9.42.3.1). We define

$$\varphi'' : S'' \longrightarrow \mathcal{F}$$

by the following simple rule: For every $f : T \to U$, $f \in S''(T)$ choose V, α, g as in Equation (9.42.3.1). Then set

$$\varphi''(f) = \varphi_{V,\alpha}(g).$$

We claim this is independent of the choice of V, α, g. Consider a second such choice V', α', g'. The restrictions of $\varphi_{V,\alpha}$ and $\varphi_{V',\alpha'}$ to the intersection of the following covering sieves on T

$$(S_{V,\alpha} \times_{V,g} T) \cap (S_{V',\alpha'} \times_{V',g'} T)$$

agree. Namely, these restrictions both correspond to the restriction of φ to T (via f) and the desired equality follows because $\mathcal{F}$ is separated. Denote the common restriction ψ. The independence of choice follows because $\varphi_{V,\alpha}(g) = \psi(\mathrm{id}_T) = \varphi_{V',\alpha'}(g')$. OK, so now φ'' gives an element $s \in L\mathcal{F}(U)$. We leave it to the reader to check that s restricts to φ. □

Definition 9.42.4. Let $\mathcal{C}$ be a category endowed with a topology J. Let $\mathcal{F}$ be a presheaf of sets on $\mathcal{C}$. The sheaf $\mathcal{F}^\# := LL\mathcal{F}$ together with the canonical map $\mathcal{F} \to \mathcal{F}^\#$ is called the *sheaf associated to* $\mathcal{F}$.

Proposition 9.42.5. *Let $\mathcal{C}$ be a category endowed with a topology. Let $\mathcal{F}$ be a presheaf of sets on $\mathcal{C}$. The canonical map $\mathcal{F} \to \mathcal{F}^\#$ has the following universal property: For any map $\mathcal{F} \to \mathcal{G}$, where $\mathcal{G}$ is a sheaf of sets, there is a unique map $\mathcal{F}^\# \to \mathcal{G}$ such that $\mathcal{F} \to \mathcal{F}^\# \to \mathcal{G}$ equals the given map.*

Proof. Same as the proof of Proposition 9.10.12. □

9.43. Topologies and sheaves

Lemma 9.43.1. *Let $\mathcal{C}$ be a category endowed with a topology J. Let U be an object of $\mathcal{C}$. Let S be a sieve on U. The following are equivalent*

(1) *The sieve S is a covering sieve.*
(2) *The sheafifcation $S^\# \to h_U^\#$ of the map $S \to h_U$ is an isomorphism.*

Proof. First we make a couple of general remarks. We will use that $S^\# = LLS$, and $h_U^\# = LLh_U$. In particular, by Lemma 9.42.1, we see that $S^\# \to h_U^\#$ is injective. Note that $\mathrm{id}_U \in h_U(U)$. Hence it gives rise to sections of Lh_U and $h_U^\# = LLh_U$ over U which we will also denote id_U.

Suppose S is a covering sieve. It clearly suffices to find a morphism $h_U \to S^\#$ such that the composition $h_U \to h_U^\#$ is the canonical map. To find such a map it suffices to find a section $s \in S^\#(U)$ wich restricts to id_U. But since S is a covering sieve, the element $\mathrm{id}_S \in Mor_{PSh(\mathcal{C})}(S, S)$ gives rise to a section of LS over U which restricts to id_U in Lh_U. Hence we win.

Suppose that $S^\# \to h_U^\#$ is an isomorphism. Let $1 \in S^\#(U)$ be the element corresponding to id_U in $h_U^\#(U)$. Because $S^\# = LLS$ there exists a covering sieve S' on U such that 1 comes from a

$$\varphi \in Mor_{PSh(\mathcal{C})}(S', LS).$$

This in turn means that for every $\alpha : V \to U$, $\alpha \in S'(V)$ there exists a covering sieve $S_{V,\alpha}$ on V such that $\varphi(\mathrm{id}_V)$ corresponds to a morphism of presheaves $S_{V,\alpha} \to S$. In other words $S_{V,\alpha}$ is contained in $S \times_U V$. By the second axiom of a topology we see that S is a covering sieve. □

Theorem 9.43.2. *Let $\mathcal{C}$ be a category. Let J, J' be topologies on $\mathcal{C}$. The following are equivalent*

(1) $J = J'$,
(2) *sheaves for the topology J are the same as sheaves for the topology J'.*

Proof. It is a tautology that if $J = J'$ then the notions of sheaves are the same. Conversely, Lemma 9.43.1 characterizes covering sieves in terms of the sheafification functor. But the sheafification functor $PSh(\mathcal{C}) \to Sh(\mathcal{C}, J)$ is the right adjoint of the inclusion functor $Sh(\mathcal{C}, J) \to PSh(\mathcal{C})$. Hence if the subcategories $Sh(\mathcal{C}, J)$ and $Sh(\mathcal{C}, J')$ are the same, then the sheafification functors are the same and hence the collections of covering sieves are the same. □

Lemma 9.43.3. *Assumption and notation as in Theorem 9.43.2. Then $J \subset J'$ if and only if every sheaf for the topology J' is a sheaf for the topology J.*

Proof. One direction is clear. For the other direction suppose that $Sh(\mathcal{C}, J') \subset Sh(\mathcal{C}, J)$. By formal nonsense this implies that if $\mathcal{F}$ is a presheaf of sets, and $\mathcal{F} \to \mathcal{F}^\#$, resp. $\mathcal{F} \to \mathcal{F}^{\#,\prime}$ is the sheafification wrt J, resp. J' then there is a canonical map $\mathcal{F}^\# \to \mathcal{F}^{\#,\prime}$ such that $\mathcal{F} \to \mathcal{F}^\# \to \mathcal{F}^{\#,\prime}$ equals the canonical map $\mathcal{F} \to \mathcal{F}^{\#,\prime}$. Of course, $\mathcal{F}^\# \to \mathcal{F}^{\#,\prime}$ identifies the second sheaf as the sheafification of the first with respect to the topology J'. Apply this to the map $S \to h_U$ of Lemma 9.43.1. We get a commutative diagram

$$\begin{tikzcd} S \arrow[r] \arrow[d] & S^\# \arrow[r] \arrow[d] & S^{\#,\prime} \arrow[d] \\ h_U \arrow[r] & h_U^\# \arrow[r] & h_U^{\#,\prime} \end{tikzcd}$$

And clearly, if S is a covering sieve for the topology J then the middle vertical map is an isomorphism (by the lemma) and we conclude that the right vertical map is an isomorphism as it is the sheafification of the one in the middle wrt J'. By the lemma again we conclude that S is a covering sieve for J' as well. □

9.44. Topologies and continuous functors

Explain how a continous functor gives an adjoint pair of functors on sheaves.

9.45. Points and topologies

Recall from Section 9.28 that given a functor $p = u : \mathcal{C} \to \mathit{Sets}$ we can define a stalk functor

$$PSh(\mathcal{C}) \longrightarrow \mathit{Sets}, \mathcal{F} \longmapsto \mathcal{F}_p.$$

Definition 9.45.1. Let $\mathcal{C}$ be a category. Let J be a topology on $\mathcal{C}$. A *point p* of the topology is given by a functor $u : \mathcal{C} \to \mathit{Sets}$ such that

(1) For every covering sieve S on U the map $S_p \to (h_U)_p$ is surjective.
(2) The stalk functor $Sh(\mathcal{C}) \to \mathit{Sets}$, $\mathcal{F} \to \mathcal{F}_p$ is exact.

9.46. Other chapters

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CHAPTER 10

Homological Algebra

10.1. Introduction

Basic homological algebra will be explained in this document. We add as needed in the other parts, since there is clearly an infinite amount of this stuff around. A reference is [**Mac63**].

10.2. Basic notions

The following notions are considered basic and will not be defined, and or proved. This does not mean they are all necessarily easy or well known.

(1) Nothing yet.

10.3. Abelian categories

An abelian category will be a category satisfying just enough axioms so the snake lemma holds.

Definition 10.3.1. A category $\mathcal{A}$ is called *preadditive* if each morphism set $Mor_{\mathcal{A}}(x, y)$ is endowed with the structure of an abelian group such that the compositions

$$Mor(x, y) \times Mor(y, z) \longrightarrow Mor(x, z)$$

are bilinear. A functor $F : \mathcal{A} \to \mathcal{B}$ of preadditive categories is called *additive* if and only if $F : Mor(x, y) \to Mor(F(x), F(y))$ is a homomorphism of abelian groups for all $x, y \in Ob(\mathcal{A})$.

In particular for every x, y there exists at least one morphism $x \to y$, namely the zero map.

Lemma 10.3.2. *Let $\mathcal{A}$ be a preadditive category. Let x be an object of $\mathcal{A}$. The following are equivalent*

(1) *x is an initial object,*
(2) *x is a final object, and*
(3) *$id_x = 0$ in $Mor_{\mathcal{A}}(x, x)$.*

Furthermore, if such an object 0 exists, then a morphism $\alpha : x \to y$ factors through 0 if and only if $\alpha = 0$.

Proof. Omitted. $\square$

Definition 10.3.3. In a preadditive category $\mathcal{A}$ we call *zero object*, and we denote it 0 any final and initial object as in Lemma 10.3.2 above.

Lemma 10.3.4. *Let $\mathcal{A}$ be a preadditive category. Let $x, y \in Ob(\mathcal{A})$. If the product $x \times y$ exists, then so does the coproduct $x \coprod y$. If the coproduct $x \coprod y$ exists, then so does the product $x \times y$. In this case also $x \coprod y \cong x \times y$.*

Proof. Suppose that $z = x \times y$ with projections $p : z \to x$ and $q : z \to y$. Denote $i : x \to z$ the morphism corresponding to $(1, 0)$. Denote $j : y \to z$ the morphism corresponding to $(0, 1)$. Thus we have the commutative diagram

$$\begin{tikzcd}
x \arrow[rr, "1"] \arrow[dr, "i"] & & x \\
& z \arrow[ur, "p"] \arrow[dr, "q"] & \\
y \arrow[rr, "1"] \arrow[ur, "j"] & & y
\end{tikzcd}$$

where the diagonal compositions are zero. It follows that $i \circ p + j \circ q : z \to z$ is the identity since it is a morphism which upon composing with p gives p and upon composing with q gives q. Suppose given morphisms $a : x \to w$ and $b : y \to w$. Then we can form the map $a \circ p + b \circ q : z \to w$. In this way we get a bijection $Mor(z, w) = Mor(x, w) \times Mor(y, w)$ which show that $z = x \coprod y$.

We leave it to the reader to construct the morphisms p, q given a coproduct $x \coprod y$ instead of a product. □

Definition 10.3.5. Given a pair of objects x, y in a preadditive category $\mathcal{A}$ we call *direct sum*, and we denote it $x \oplus y$ the product $x \times y$ endowed with the morphisms i, j, p, q as in Lemma 10.3.4 above.

Remark 10.3.6. Note that the proof of Lemma 10.3.4 shows that given p and q the morphisms i, j are uniquely determined by the rules $p \circ i = \mathrm{id}_x$, $q \circ j = \mathrm{id}_y$, $p \circ j = 0$, $q \circ i = 0$. Moreover, we automatically have $i \circ p + j \circ q = \mathrm{id}_{x \oplus y}$. Similarly, given i, j the morphisms p and q are uniquely determined. Finally, given objects x, y, z and morphisms $i : x \to z$, $j : y \to z$, $p : z \to x$ and $q : z \to y$ such that $p \circ i = \mathrm{id}_x$, $q \circ j = \mathrm{id}_y$, $p \circ j = 0$, $q \circ i = 0$ and $i \circ p + j \circ q = \mathrm{id}_z$, then z is the direct sum of x and y with the four morphisms equal to i, j, p, q.

Lemma 10.3.7. *Let $\mathcal{A}$, $\mathcal{B}$ be preadditive categories. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor. Then F transforms direct sums to direct sums and zero to zero.*

Proof. Suppose F is additive. A direct sum z of x and y is characterized by having morphisms $i : x \to z$, $j : y \to z$, $p : z \to x$ and $q : z \to y$ such that $p \circ i = \mathrm{id}_x$, $q \circ j = \mathrm{id}_y$, $p \circ j = 0$, $q \circ i = 0$ and $i \circ p + j \circ q = \mathrm{id}_z$, according to Remark 10.3.6. Clearly $F(x)$, $F(y)$, $F(z)$ and the morphisms $F(i)$, $F(j)$, $F(p)$, $F(q)$ satisfy exactly the same relations (by additivity) and we see that $F(z)$ is a direct sum of $F(x)$ and $F(y)$. □

Definition 10.3.8. A category $\mathcal{A}$ is called *additive* if it is preadditive and finite products exist, in other words it has a zero object and direct sums.

Namely the empty product is a finite product and if it exists, then it is a final object.

Definition 10.3.9. Let $\mathcal{A}$ be a preadditive category. Let $f : x \to y$ be a morphism.

(1) A *kernel* of f is a morphism $i : z \to x$ such that (a) $f \circ i = 0$ and (b) for any $i' : z' \to x$ such that $f \circ i' = 0$ there exists a unique morphism $g : z' \to z$ such that $i' = i \circ g$.
(2) If the kernel of f exists, then we denote this $\mathrm{Ker}(f) \to x$.

(3) A *cokernel* of f is a morphism $p : y \to z$ such that (a) $p \circ f = 0$ and (b) for any $p' : y \to z'$ such that $p' \circ f = 0$ there exists a unique morphism $g : z \to z'$ such that $p' = g \circ p$.
(4) If a cokernel of f exists we denote this $y \to \mathrm{Coker}(f)$.
(5) If a kernel of f exists, then a *coimage of f* is a cokernel for the morphism $\mathrm{Ker}(f) \to x$.
(6) If a kernel and coimage exist then we denote this $x \to \mathrm{Coim}(f)$.
(7) If a cokernel of f exists, then the *image of f* is a kernel of the morphism $y \to \mathrm{Coker}(f)$.
(8) If a cokernel and image of f exist then we denote this $\mathrm{Im}(f) \to y$.

Lemma 10.3.10. *Let $f : x \to y$ be a morphism in a preadditive category such that the kernel, cokernel, image and coimage all exist. Then f can be factored uniquely as $x \to Coim(f) \to Im(f) \to y$.*

Proof. There is a canonical morphism $\mathrm{Coim}(f) \to y$ because $\mathrm{Ker}(f) \to x \to y$ is zero. The composition $\mathrm{Coim}(f) \to y \to \mathrm{Coker}(f)$ is zero, because it is the unique morphism which gives rise to the morphism $x \to y \to \mathrm{Coker}(f)$ which is zero. Hence $\mathrm{Coim}(f) \to y$ factors uniquely through $\mathrm{Im}(f) \to y$, which gives us the desired map. □

Example 10.3.11. Let k be a field. Consider the category of filtered vector spaces over k. (See Definition 10.13.1.) Consider the filtered vector spaces (V, F) and (W, F) with $V = W = k$ and

$$F^iV = \begin{cases} V & \text{if} \quad i < 0 \\ 0 & \text{if} \quad i \geq 0 \end{cases} \text{ and } F^iW = \begin{cases} W & \text{if} \quad i \leq 0 \\ 0 & \text{if} \quad i > 0 \end{cases}$$

The map $f : V \to W$ corresponding to id_k on the underlying vector spaces has trivial kernel and cokernel but is not an isomorphism. Note also that $\mathrm{Coim}(f) = V$ and $\mathrm{Im}(f) = W$. This means that the category of filtered vector spaces over k is not abelian.

Definition 10.3.12. A category $\mathcal{A}$ is *abelian* if it is additive, if all kernels and cokernels exist, and if the natural map $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism for all morphisms f of $\mathcal{A}$.

Lemma 10.3.13. *Let $\mathcal{A}$ be a preadditive category. The additions on sets of morphisms make $\mathcal{A}^{opp}$ into a preadditive category. Furthermore, $\mathcal{A}$ is additive if and only if $\mathcal{A}^{opp}$ is additive, and $\mathcal{A}$ is abelian if and only if $\mathcal{A}^{opp}$ is abelian.*

Proof. Omitted. □

Definition 10.3.14. Let $f : x \to y$ be a morphism in an abelian category.

(1) We say f is *injective* if $\mathrm{Ker}(f) = 0$.
(2) We say f is *surjective* if $\mathrm{Coker}(f) = 0$.

If $x \to y$ is injective, then we say that x is a *subobject* of y and we use the notation $x \subset y$. If $x \to y$ is surjective, then we say that y is a *quotient* of x.

Lemma 10.3.15. *Let $f : x \to y$ be a morphism in an abelian category. Then*

(1) *f is injective if and only if f is a monomorphism, and*
(2) *f is surjective if and only if f is an epimorphism.*

Proof. Omitted. □

In an abelian category, if $x \subset y$ is a subobject, then we denote

$$x/y = \mathrm{Coker}(x \to y).$$

Lemma 10.3.16. *Let $\mathcal{A}$ be an abelian category. All finite limits and finite colimits exist in $\mathcal{A}$.*

Proof. To show that finite limits exist it suffices to show that finite products and equalizers exist, see Categories, Lemma 4.16.4. Finite products exist by definition and the equalizer of $a, b : x \to y$ is the kernel of $a - b$. The argument for finite colimits is similar but dual to this. $\square$

Example 10.3.17. Let $\mathcal{A}$ be an abelian category. Pushouts and fibre products in $\mathcal{A}$ have the following simple descriptions:

(1) If $a : x \to y$, $b : z \to y$ are morphisms in $\mathcal{A}$, then we have the fibre product: $x \times_y z = \text{Ker}((a, -b) : x \oplus z \to y)$.
(2) If $a : y \to x$, $b : y \to z$ are morphisms in $\mathcal{A}$, then we have the pushout: $x \amalg_y z = \text{Coker}((a, -b) : y \to x \oplus z)$.

Definition 10.3.18. Let $\mathcal{A}$ be an additive category. We say a sequence of morphisms

$$\ldots \to x \to y \to z \to \ldots$$

in $\mathcal{A}$ is a *complex* if the composition of any two (drawn) arrows is zero. If $\mathcal{A}$ is abelian then we say a sequence as above is *exact at y* if $\text{Im}(x \to y) = \text{Ker}(y \to z)$. We say it is *exact* if it is exact at every object. A *short exact sequence* is an exact complex of the form

$$0 \to A \to B \to C \to 0.$$

In the following lemma we asssume the reader knows what it means for a sequence of abelian groups to be exact.

Lemma 10.3.19. *Let $\mathcal{A}$ be an abelian category. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a complex of $\mathcal{A}$.*

(1) *$M_1 \to M_2 \to M_3 \to 0$ is exact if and only if*
$$0 \to Hom_{\mathcal{A}}(M_3, N) \to Hom_{\mathcal{A}}(M_2, N) \to Hom_{\mathcal{A}}(M_1, N)$$
is an exact sequence of abelian groups for all objects N of $\mathcal{A}$, and
(2) *$0 \to M_1 \to M_2 \to M_3$ is exact if and only if*
$$0 \to Hom_{\mathcal{A}}(N, M_1) \to Hom_{\mathcal{A}}(N, M_2) \to Hom_{\mathcal{A}}(N, M_1)$$
is an exact sequence of abelian groups for all objects N of $\mathcal{A}$.

Proof. Omitted. Hint: See Algebra, Lemma 7.10.1. $\square$

Definition 10.3.20. Let $\mathcal{A}$ be an abelian category. Let $i : A \to B$ and $q : B \to C$ be morphisms of $\mathcal{A}$ such that $0 \to A \to B \to C \to 0$ is a short exact sequence. We say the short exact sequence is *split* if there exist morphisms $j : C \to B$ and $p : B \to A$ such that (B, i, j, p, q) is the direct sum of A and C.

Lemma 10.3.21. *Let $\mathcal{A}$ be an abelian category. Let $0 \to A \to B \to C \to 0$ be a short exact sequence.*

(1) *Given a morphism $s : C \to B$ left inverse to $B \to C$, there exists a unique $\pi : B \to A$ such that (s, π) splits the short exact sequence as in Definition 10.3.20.*
(2) *Given a morphism $\pi : B \to A$ right inverse to $A \to B$, there exists a unique $s : C \to B$ such that (s, π) splits the short exact sequence as in Definition 10.3.20.*

Proof. Omitted. $\square$

Lemma 10.3.22. *Let $\mathcal{A}$ be an abelian category.*

(1) *If $x \to y$ is surjective, then for every $z \to y$ the projection $z \times_y z \to z$ is surjective.*
(2) *If $x \to y$ is injective, then for every $x \to z$ the morpism $z \to z \amalg_x y$ is injective.*

Proof. We prove (1). Assume $a : x \to y$ surjective and $b : z \to y$ arbitrary. Let $c : z \to t$ be a morphism of $\mathcal{A}$ such that $z \times_y z \to z \to t$ is zero. Note that

$$0 \to x \times_y z \to x \oplus z \to y \to 0$$

is a short exact sequence, use Example 10.3.17 and the fact that a is surjective. Consider the map $\tilde{c} = (0, c) : x \oplus z \to t$. By assumption the composition $x \times_y z \to x \oplus z \to t$ is zero hence we see that $\tilde{c}$ can be factored as $x \oplus z \to y \to t$ for some morphism $c' : y \to t$, see Lemma 10.3.19. This means that $c = c' \circ b$ and that $0 = c' \circ a$. As a is surjective we conclude that $c' = 0$, hence $c = 0$ as desired.

The proof of (2) is dual to the proof of (1) and is omitted. □

Lemma 10.3.23. *Let $\mathcal{A}$ be an abelian category. Suppose given a commutative diagram*

$$\begin{tikzcd} & x \arrow[r] \arrow[d, "\alpha"] & y \arrow[r] \arrow[d, "\beta"] & z \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & u \arrow[r] & v \arrow[r] & w & \end{tikzcd}$$

with exact rows, then there is a canonical exact sequence

$$Ker(\alpha) \to Ker(\beta) \to Ker(\gamma) \to Coker(\alpha) \to Coker(\beta) \to Coker(\gamma)$$

Moreover, if $x \to y$ is injective, then the first map is injective, and if $v \to w$ is surjective, then the last map is surjective.

Proof. Omitted. Let us sketch the construction of the map $\delta : \text{Ker}(\gamma) \to \text{Coker}(\alpha)$ is. Let $T \in Ob(\mathcal{A})$. Consider a morphism $a : T \to z$ with $\gamma \circ a = 0$. In other words a maps T into $\text{Ker}(\gamma)$. We have to construct $\delta \circ a : T \to \text{Coker}(\alpha)$. Because $y \to z$ is surjective, the fibre product $T' = T \times_z y$ surjects onto T, see Lemma 10.3.22. Denote $a' : T' \to y$ the second projection. Consider the morphism $\beta \circ a' : T' \to v$. Composing this morphism with $v \to w$ gives the same morphism as the composition $T' \to T \to z \to w$ in other words, it gives the zero morphism. Because $u \to v$ is the kernel of $v \to w$ we conclude that a' factors through a morphism $a'' : T' \to u$. Note that the kernel T'' of $T' \to T$ maps to zero under the composition $T' \to y \to z$, and hence maps into $\text{Im}(x \to y)$. Thus $a''|_{T''} : T'' \to u$ maps into the image of α. We conclude that there exists a factorization

$$\begin{tikzcd} T' \arrow[r, "a''"'] \arrow[d, "\mathrm{pr}_1"'] & u \arrow[d] \\ T \arrow[r, "\delta\circ a"] & \text{Coker}(\alpha) \end{tikzcd}$$

which gives the desired map $\delta \circ a : T \to \text{Coker}(\alpha)$. □

Lemma 10.3.24. *Let $\mathcal{A}$ be an abelian category. Let*

$$\begin{tikzcd} w \arrow[r] \arrow[d, "\alpha"] & x \arrow[r] \arrow[d, "\beta"] & y \arrow[r] \arrow[d, "\gamma"] & z \arrow[d, "\delta"] \\ w' \arrow[r] & x' \arrow[r] & y' \arrow[r] & z' \end{tikzcd}$$

be a commutative diagram with exact rows.

(1) *If α, γ are surjective and δ is injective, then β is surjective.*
(2) *If β, δ are injective and α is surjective, then γ is injective.*

Proof. Assume α, γ are surjective and δ is injective. We may replace w' by $\mathrm{Im}(w' \to x')$, i.e., we may assume that $w' \to x'$ is injective. We may replace z by $\mathrm{Im}(y \to z)$, i.e., we may assume that $y \to z$ is surjective. Then we may apply Lemma 10.3.23 to

$$\begin{tikzcd} & \mathrm{Ker}(y \to z) \arrow[r] \arrow[d] & y \arrow[r] \arrow[d] & z \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathrm{Ker}(y' \to z') \arrow[r] & y' \arrow[r] & z' & \end{tikzcd}$$

to conclude that $\mathrm{Ker}(y \to z) \to \mathrm{Ker}(y' \to z')$ is surjective. Finally, we apply Lemma 10.3.23 to

$$\begin{tikzcd} & w \arrow[r] \arrow[d] & x \arrow[r] \arrow[d] & \mathrm{Ker}(y \to z) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & w' \arrow[r] & x' \arrow[r] & \mathrm{Ker}(y' \to z') & \end{tikzcd}$$

to conclude that $x \to x'$ is surjective. This proves (1). The proof of (2) is dual to this. □

Lemma 10.3.25. *Let $\mathcal{A}$ be an abelian category. Let*

$$\begin{tikzcd} v \arrow[r] \arrow[d, "\alpha"] & w \arrow[r] \arrow[d, "\beta"] & x \arrow[r] \arrow[d, "\gamma"] & y \arrow[r] \arrow[d, "\delta"] & z \arrow[d, "\epsilon"] \\ v' \arrow[r] & w' \arrow[r] & x' \arrow[r] & y' \arrow[r] & z' \end{tikzcd}$$

be a commutative diagram with exact rows. If β, δ are isomorphisms, ϵ is injective, and α is surjective then γ is an isomorphism.

Proof. Immediate consequence of Lemma 10.3.24. □

10.4. Extensions

Definition 10.4.1. Let $\mathcal{A}$ be an abelian category. Let $A, C \in Ob(\mathcal{A})$. An *extension E of B by A* is a short exact sequence

$$0 \to A \to E \to B \to 0.$$

By abuse of language we often omit mention of the morphisms $A \to E$ and $E \to B$, allthough they are definitively part of the structure of an extension.

Definition 10.4.2. Let $\mathcal{A}$ be an abelian category. Let $A, C \in Ob(\mathcal{A})$. The set of isomorphism classes of extensions of B by A is denoted

$$\mathrm{Ext}_{\mathcal{A}}(B, A).$$

This is called the *Ext-group*.

This definition works, because by our conventions $\mathcal{A}$ is a set, and hence $\mathrm{Ext}_{\mathcal{A}}(B, A)$ is a set. In any of the cases of ``big'' abelian categories listed in Categories, Remark 4.2.2. one can check by hand that $\mathrm{Ext}_{\mathcal{A}}(B, A)$ is a set as well. Also, we will see later that this is always the case when $\mathcal{A}$ has either enough projectives or enough injectives. Insert future reference here.

Actually we can turn $\mathrm{Ext}_{\mathcal{A}}(-,-)$ into a functor

$$\mathcal{A}^{opp} \times \mathcal{A} \longrightarrow Sets, \quad (A, B) \longmapsto \mathrm{Ext}_{\mathcal{A}}(A, B)$$

as follows:

(1) Given a morphism $B' \to B$ and an extension E of B by A we define $E' = E \times_B B'$ so that we have the following commutative diagram of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & E' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A \arrow[r] & E \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

The extension E' is called the *pullback of E via $B' \to B$*.

(2) Given a morphism $A \to A'$ and an extension E of B by A we define $E' = A' \coprod_A E$ so that we have the following commutative diagram of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A' \arrow[r] & E' \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

The extension E' is called the *pushout of E via $A \to A'$*.

To see that this defines a functor as indicated above there are several things to verify. First of all functoriality in the variable B requires that $(E \times_B B') \times_{B'} B'' = E \times_B B''$ which is a general property of fibre products. Dually one deals with functoriality in the variable A. Finally, given $A \to A'$ and $B' \to B$ we have to show that

$$A' \coprod\nolimits_A (E \times_B B') \cong (A' \coprod\nolimits_A E) \times_B B'$$

as extensions of B' by A'. Recall that $A' \coprod_A E$ is a quotient of $A' \oplus E$. Thus the right hand side is a quotient of $A' \oplus E \times_B B'$, and it is straightforward to see that the kernel is exactly what you need in order to get the left hand side.

Note that if E_1 and E_2 are extensions of B by A, then $E_1 \oplus E_2$ is an extension of $B \oplus B$ by $A \oplus A$. We pull back by the diagonal map $B \to B \oplus B$ and we push out by the sum map $A \oplus A \to A$ to get an extension $E_1 + E_2$ of B by A.

$$\begin{tikzcd} 0 \arrow[r] & A \oplus A \arrow[r] \arrow[d, "\Sigma"] & E_1 \oplus E_2 \arrow[r] \arrow[d] & B \oplus B \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A \arrow[r] & E' \arrow[r] & B \oplus B \arrow[r] & 0 \\ 0 \arrow[r] & A \arrow[r] \arrow[u] & E_1 + E_2 \arrow[r] \arrow[u] & B \arrow[r] \arrow[u, "\Delta"'] & 0 \end{tikzcd}$$

The extension $E_1 + E_2$ is called the *Baer sum* of the given extensions.

Lemma 10.4.3. *The construction $(E_1, E_2) \mapsto E_1 + E_2$ above defines a commutative group law on $\mathrm{Ext}_{\mathcal{A}}(B, A)$ which is functorial in both variables.*

Proof. Omitted. □

Lemma 10.4.4. *Let $\mathcal{A}$ be an abelian category. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence in $\mathcal{A}$.*

(1) *There is a canonical six term exact sequence of abelian groups*

$$\begin{tikzcd} 0 \arrow[r] & Hom_{\mathscr{A}}(M_3, N) \arrow[r] & Hom_{\mathscr{A}}(M_2, N) \arrow[r] & Hom_{\mathscr{A}}(M_1, N) \arrow[dll] \\ & Ext_{\mathscr{A}}(M_3, N) \arrow[r] & Ext_{\mathscr{A}}(M_2, N) \arrow[r] & Ext_{\mathscr{A}}(M_1, N) \end{tikzcd}$$

for all objects N of $\mathscr{A}$, and

(2) *there is a canonical six term exact sequence of abelian groups*

$$\begin{tikzcd} 0 \arrow[r] & Hom_{\mathscr{A}}(N, M_1) \arrow[r] & Hom_{\mathscr{A}}(N, M_2) \arrow[r] & Hom_{\mathscr{A}}(N, M_1) \arrow[dll] \\ & Ext_{\mathscr{A}}(N, M_1) \arrow[r] & Ext_{\mathscr{A}}(N, M_2) \arrow[r] & Ext_{\mathscr{A}}(N, M_1) \end{tikzcd}$$

for all objects N of $\mathscr{A}$.

Proof. Omitted. Hint: The boundary maps are defined by using either push out or pull back of the given short exact sequence. □

10.5. Additive functors

Recall that we defined, in Categories, Definition 4.21.1 the notion of a ``right exact", ``left exact" and ``exact" functor in the setting of a functor between categories that have finite (co)limits. Thus this applies in particular to functors between abelian categories.

Lemma 10.5.1. *Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories. Let $F : \mathscr{A} \to \mathscr{B}$ be a functor.*

(1) *If F is either left or right exact, then it is additive.*
(2) *If F is additive then it is left exact if and only if for every short exact sequence $0 \to A \to B \to C \to 0$ the sequence $0 \to F(A) \to F(B) \to F(C)$ is exact.*
(3) *If F is additive then it is right exact if and only if for every short exact sequence $0 \to A \to B \to C \to 0$ the sequence $F(A) \to F(B) \to F(C) \to 0$ is exact.*
(4) *If F is additive then it is exact if and only if for every short exact sequence $0 \to A \to B \to C \to 0$ the sequence $0 \to F(A) \to F(B) \to F(C) \to 0$ is exact.*

Proof. Let us first note that if F commutes with the empty limit or the empty colimit, then $F(0) = 0$. In particular F applied to the zero morphism is zero. We will use this below without mention.

Suppose that F is left exact, i.e., commutes with finite limits. Then $F(A \times A) = F(A) \times F(A)$ with projections $F(p)$ and $F(q)$. Hence $F(A \oplus A) = F(A) \oplus F(A)$ with all four morphisms $F(i), F(j), F(p), F(q)$ equal to their counterparts in $\mathscr{B}$ as they satisfy the same relations, see Remark 10.3.6. Then $f = F(p + q)$ is a morphism $f : F(A) \oplus F(A) \to F(A)$ such that $f \circ F(i) = F(p \circ i + q \circ i) = F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$. And similarly $f \circ F(j) = \mathrm{id}_A$. We conclude that $F(p+q) = F(p)+F(q)$. For any pair of morphisms $a, b : B \to A$ the map $g = F(i \circ a + j \circ b) : F(B) \to F(A) \oplus F(A)$ is a morphism such that $F(p) \circ g = F(p \circ (i \circ a + j \circ b)) = F(a)$ and similarly $F(q) \circ g = F(b)$. Hence $g = F(i) \circ F(a) + F(j) \circ F(b)$. The sum of a and b is the composition

$$B \xrightarrow{i \circ a + j \circ b} A \oplus A \xrightarrow{p+q} A.$$

Applying F we get

$$F(B) \xrightarrow{F(i) \circ F(a) + F(j) \circ F(b)} F(A) \oplus F(A) \xrightarrow{F(p)+F(q)} A.$$

where we used the expressions for f and g obtained above. Hence F is additive.[1]

Denote $f : B \to C$ a map from B to C. Exactness of $0 \to A \to B \to C$ just means that $A = \mathrm{Ker}(f)$. Clearly the kernel of f is the equalizer of the two maps f and 0 from B to C. Hence if F commutes with limits, then $F(\mathrm{Ker}(f)) = \mathrm{Ker}(F(f))$ which exactly means that $0 \to F(A) \to F(B) \to F(C)$ is exact.

Conversely, suppose that F is additive and transforms any short exact sequence $0 \to A \to B \to C$ into an exact sequence $0 \to F(A) \to F(B) \to F(C)$. Because it is additive it commutes with direct sums and hence finite products in $\mathcal{A}$. To show it commutes with finite limits it therefore suffices to show that it commutes with equalizers. But equalizers in an abelian category are the same as the kernel of the difference map, hence it suffices to show that F commutes with taking kernels. Let $f : A \to B$ be a morphism. Factor f as $A \to I \to B$ with $f' : A \to I$ surjective and $i : I \to B$ injective. (This is possible by the definition of an abelian category.) Then it is clear that $\mathrm{Ker}(f) = \mathrm{Ker}(f')$. Also $0 \to \mathrm{Ker}(f') \to A \to I \to 0$ and $0 \to I \to B \to B/I \to 0$ are short exact. By the condition imposed on F we see that $0 \to F(\mathrm{Ker}(f')) \to F(A) \to F(I)$ and $0 \to F(I) \to F(B) \to F(B/I)$ are exact. Hence it is also the case that $F(\mathrm{Ker}(f'))$ is the kernel of the map $F(A) \to F(B)$, and we win.

The proof of (3) is similar to the proof of (2). Statement (4) is a combination of (2) and (3). □

Lemma 10.5.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor. For every pair of objects A, B of $\mathcal{A}$ the functor F induces an abelian group homomorphism*

$$Ext_{\mathcal{A}}(B, A) \longrightarrow Ext_{\mathcal{B}}(F(B), F(A))$$

which maps the extension E to $F(E)$.

Proof. Omitted. □

The following lemma is used in the proof that the category of abelian sheaves on a site is abelian, where the functor b is sheafification.

Lemma 10.5.3. *Let $a : \mathcal{A} \to \mathcal{B}$ and $b : \mathcal{B} \to \mathcal{A}$ be functors. Assume that*

1. *$\mathcal{A}$, $\mathcal{B}$ are additive categories, a, b are additive functors, and a is right adjoint to b,*
2. *$\mathcal{B}$ is abelian and b is left exact, and*
3. *$ba \cong id_{\mathcal{A}}$.*

Then $\mathcal{A}$ is abelian.

Proof. As $\mathcal{B}$ is abelian we see that all finite limits and colimits exist in $\mathcal{B}$ by Lemma 10.3.16. Since b is a left adjoint we see that b is also right exact and hence exact, see Categories, Lemma 4.22.3. Let $\varphi : B_1 \to B_2$ be a morphism of $\mathcal{B}$. In particular, if $K = \mathrm{Ker}(B_1 \to B_2)$, then K is the equalizer of 0 and φ and hence bK is the equalizer of 0 and $b\varphi$, hence bK is the kernel of $b\varphi$. Similarly, if $Q = \mathrm{Coker}(B_1 \to B_2)$, then Q is the coequalizer of 0 and φ and hence bQ is the coequalizer of 0 and $b\varphi$, hence bQ is the cokernel of $b\varphi$. Thus we see that every morphism of the form $b\varphi$ in $\mathcal{A}$ has a kernel and a cokernel. However, since $ba \cong \mathrm{id}$ we see that every morphism of $\mathcal{A}$ is of this form,

[1] I'm sure there is an infinitely slicker proof of this.

and we conclude that kernels and cokernels exist in $\mathcal{A}$. In fact, the argument shows that if $\psi : A_1 \to A_2$ is a morphism then

$$\mathrm{Ker}(\psi) = b\mathrm{Ker}(a\psi), \quad \text{and} \quad \mathrm{Coker}(\psi) = b\mathrm{Coker}(a\psi).$$

Now we still have to show that $\mathrm{Coim}(\psi) = \mathrm{Im}(\psi)$. We do this as follows. First note that since $\mathcal{A}$ has kernels and cokernels it has all finite limits and colimits (see proof of Lemma 10.3.16). Hence we see by Categories, Lemma 4.22.3 that a is left exact and hence transforms kernels (=equalizers) into kernels.

$$\begin{aligned}
\mathrm{Coim}(\psi) &= \mathrm{Coker}(\mathrm{Ker}(\psi) \to A_1) && \text{by definition} \\
&= b\mathrm{Coker}(a(\mathrm{Ker}(\psi) \to A_1)) && \text{by formula above} \\
&= b\mathrm{Coker}(\mathrm{Ker}(a\psi) \to aA_1)) && a \text{ preserves kernels} \\
&= b\mathrm{Coim}(a\psi) && \text{by definition} \\
&= b\mathrm{Im}(a\psi) && \mathcal{B} \text{ is abelian} \\
&= b\mathrm{Ker}(aA_2 \to \mathrm{Coker}(a\psi)) && \text{by definition} \\
&= \mathrm{Ker}(baA_2 \to b\mathrm{Coker}(a\psi)) && b \text{ preserves kernels} \\
&= \mathrm{Ker}(A_2 \to b\mathrm{Coker}(a\psi)) && ba = \mathrm{id}_{\mathcal{A}} \\
&= \mathrm{Ker}(A_2 \to \mathrm{Coker}(\psi)) && \text{by formula above} \\
&= \mathrm{Im}(\psi) && \text{by definition}
\end{aligned}$$

Thus the lemma holds. □

10.6. Localization

In this section we note how Gabriel-Zisman localization interacts with the additive structure on a category.

Lemma 10.6.1. *Let $\mathcal{C}$ be a preadditive category. Let S be a left or right multiplicative system. There exists a canonical preadditive structure on $S^{-1}\mathcal{C}$ such that the localization functor $Q : \mathcal{C} \to S^{-1}\mathcal{C}$ is additive.*

Proof. We will prove this in the case S is a left multiplicative system. The case where S is a right multiplicative system is dual. Suppose that X, Y are objects of $\mathcal{C}$ and that $\alpha, \beta : X \to Y$ are morphisms in $S^{-1}\mathcal{C}$. According to Categories, Lemma 4.24.3 we may represent these by pairs $s^{-1}f, s^{-1}g$ with common denominator s. In this case we define $\alpha + \beta$ to be the equivalence class of $s^{-1}(f + g)$. In the rest of the proof we show that this is well defined and that composition is bilinear. Once this is done it is clear that Q is an additive functor.

Let us show construction above is well defined. An abstract way of saying this is that filtered colimits of abelian groups agree with filtered colimits of sets and to use Categories, Equation (4.24.5.1). We can work this out in a bit more detail as follows. Say $s : Y \to Y_1$ and $f, g : X \to Y_1$. Suppose we have a second representation of α, β as $(s')^{-1}f', (s')^{-1}g'$ with $s' : Y \to Y_2$ and $f', g' : X \to Y_2$. By Categories, Remark 4.24.5 we can find a morphism $s_3 : Y \to Y_3$ and morphisms $a_1 : Y_1 \to Y_3$, $a_2 : Y_2 \to Y_3$ such that $a_1 \circ s = s_3 = a_2 \circ s'$ and also $a_1 \circ f = a_2 \circ f'$ and $a_1 \circ g = a_2 \circ g'$. Hence we see that $s^{-1}(f + g)$ is equivalent to

$$\begin{aligned}
s_3^{-1}(a_1 \circ (f + g)) &= s_3^{-1}(a_1 \circ f + a_2 \circ g) \\
&= s_3^{-1}(a_2 \circ f' + a_2 \circ g') \\
&= s_3^{-1}(a_2 \circ (f' + g'))
\end{aligned}$$

which is equivalent to $(s')^{-1}(f' + g')$.

Fix $s : Y \to Y'$ and $f, g : X \to Y'$ with $\alpha = s^{-1}f$ and $\beta = s^{-1}g$ as morphisms $X \to Y$ in $S^{-1}\mathscr{C}$. To show that composition is bilinear first consider the case of a morphism $\gamma : Y \to Z$ in $S^{-1}\mathscr{C}$. Say $\gamma = t^{-1}h$ for some $h : Y \to Z'$ and $t : Z \to Z'$ in S. Using LMS2 we choose morphisms $a : Y' \to Z''$ and $t' : Z' \to Z''$ in S such that $a \circ s = t' \circ h$. Picture

$$\begin{tikzcd}
 & & Z \arrow[d, "t"] \\
 & Y \arrow[r, "h"] \arrow[d, "s"] & Z' \arrow[d, "t"] \\
X \arrow[r, "{f,g}"] & Y' \arrow[r, "a"] & Z''
\end{tikzcd}$$

Then $\gamma \circ \alpha = (t' \circ t)^{-1}(a \circ f)$ and $\gamma \circ \beta = (t' \circ t)^{-1}(a \circ g)$. Hence we see that $\gamma \circ (\alpha + \beta)$ is represented by $(t' \circ t)^{-1}(a \circ (f + g)) = (t' \circ t)^{-1}(a \circ f + a \circ g)$ which represents $\gamma \circ \alpha + \gamma \circ \beta$.

Finally, assume that $\delta : W \to X$ is another morphism of $S^{-1}\mathscr{C}$. Say $\delta = r^{-1}i$ for some $i : W \to X'$ and $r : X \to X'$ in S. We claim that we can find a morphism $s : Y' \to Y''$ in S and morphisms $a'', b'' : X' \to Y''$ such that the following diagram commutes

$$\begin{tikzcd}
 & & Y \arrow[d, "s"] \\
 & X \arrow[r, "{f,g,f+g}"] \arrow[d, "s"] & Y' \arrow[d, "s'"] \\
W \arrow[r, "i"] & X' \arrow[r, "{a'',b'',a''+b''}"] & Y''
\end{tikzcd}$$

Namely, using LMS2 we can first choose $s_1 : Y' \to Y_1$, $s_2 : Y' \to Y_2$ in S and $a : X' \to Y_1$, $b : X' \to Y_2$ such that $a \circ s = s_1 \circ f$ and $b \circ s = s_2 \circ f$. Then using that the category Y'/S is filtered (see Categories, Remark 4.24.5), we can find a $s' : Y' \to Y''$ and morphisms $a' : Y_1 \to Y''$, $b' : Y_2 \to Y''$ such that $s' = a' \circ s_1$ and $s' = b' \circ s_2$. Setting $a'' = a' \circ a$ and $b'' = b' \circ b$ works. At this point we see that the compositions $\alpha \circ \delta$ and $\beta \circ \delta$ are represented by $(s' \circ s)^{-1}a''$ and $(s' \circ s)^{-1}b''$. Hence $\alpha \circ \delta + \beta \circ \delta$ is represented by $(s' \circ s)^{-1}(a'' + b'')$ which by the diagram again is a representative of $(\alpha + \beta) \circ \delta$. $\square$

Lemma 10.6.2. *Let $\mathscr{C}$ be an additive category. Let S be a left or right multiplicative system. Then $S^{-1}\mathscr{C}$ is an additive category and the localization functor $Q : \mathscr{C} \to S^{-1}\mathscr{C}$ is additive.*

Proof. By Lemma 10.6.1 we see that $S^{-1}\mathscr{C}$ is preadditive and that Q is additive. Recall that the functor Q commutes with finite colimits (resp. finite limits), see Categories, Lemmas 4.24.7 and 4.24.14. We conclude that $S^{-1}\mathscr{C}$ has a zero object and direct sums, see Lemmas 10.3.2 and 10.3.4. $\square$

The following lemma describes the kernel (see Definition 10.7.5) of the localization functor in case we invert a multiplicative system.

Lemma 10.6.3. *Let $\mathscr{C}$ be an additive category. Let S be a multiplicative system compatible with the triangulated structure. Let X be an object of $\mathscr{D}$. The following are equivalent*

(1) $Q(X) = 0$ *in* $S^{-1}\mathscr{C}$,
(2) *there exists* $Y \in Ob(\mathscr{C})$ *such that* $0 : X \to Y$ *is an element of S, and*

(3) *there exists* $Z \in Ob(\mathcal{C})$ *such that* $0 : Z \to X$ *is an element of* S.

Proof. If (2) holds we see that $0 = Q(0) : Q(X) \to Q(Y)$ is an isomorphism. In the additive category $S^{-1}\mathcal{C}$ this implies that $Q(X) = 0$. Hence (2) ⇒ (1). Similarly, (3) ⇒ (1). Suppose that $Q(X) = 0$. This implies that the morphism $f : 0 \to X$ is transformed into an isomorphism in $S^{-1}\mathcal{C}$. Hence by Categories, Lemma 4.24.18 there exists a morphism $g : Z \to 0$ such that $fg \in S$. This proves (1) ⇒ (3). Similarly, (1) ⇒ (2). □

Lemma 10.6.4. *Let* $\mathcal{A}$ *be an abelian category.*

(1) *If* S *is a left multiplicative system, then the category* $S^{-1}\mathcal{A}$ *has cokernels and the functor* $Q : \mathcal{A} \to S^{-1}\mathcal{A}$ *commutes with them.*
(2) *If* S *is a right multiplicative system, then the category* $S^{-1}\mathcal{A}$ *has kernels and the functor* $Q : \mathcal{A} \to S^{-1}\mathcal{A}$ *commutes with them.*
(3) *If* S *is a multiplicative system, then the category* $S^{-1}\mathcal{A}$ *is abelian and the functor* $Q : \mathcal{A} \to S^{-1}\mathcal{A}$ *is exact.*

Proof. Assume S is a left multiplicative system. Let $a : X \to Y$ be a morphism of $S^{-1}\mathcal{A}$. Then $a = s^{-1}f$ for some $s : Y \to Y'$ in S and $f : X \to Y'$. Since $Q(s)$ is an isomorphism we see that the existence of $\mathrm{Coker}(a : X \to Y)$ is equivalent to the existence of $\mathrm{Coker}(Q(f) : X \to Y')$. Since $\mathrm{Coker}(Q(f))$ is the coequalizer of 0 and $Q(f)$ we see that $\mathrm{Coker}(Q(f))$ is represented by $Q(\mathrm{Coker}(f))$ by Categories, Lemma 4.24.7. This proves (1).

Part (2) is dual to part (1).

If S is a multiplicative system, then S is both a left and a right multiplicative system. Thus we see that $S^{-1}\mathcal{A}$ has kernels and cokernels and Q commutes with kernels and cokernels. To finish the proof of (3) we have to show that Coim = Im in $S^{-1}\mathcal{A}$. Again using that any arrow in $S^{-1}\mathcal{A}$ is isomorphic to an arrow $Q(f)$ we see that the result follows from the result for $\mathcal{A}$. □

10.7. Serre subcategories

In [**Ser53**, Chapter I, Section 1] a notion of a ``class'' of abelian groups is defined. This notion has been extended to abelian categories by many authors (in slightly different ways). We will use the following variant which is virtually identical to Serre's original definition.

Definition 10.7.1. Let $\mathcal{A}$ be an abelian category.

(1) A *Serre subcategory* of $\mathcal{A}$ is a nonempty full subcategory $\mathcal{C}$ of $\mathcal{A}$ such that given an exact sequence
$$A \to B \to C$$
with $A, C \in Ob(\mathcal{C})$, then also $B \in Ob(\mathcal{C})$.
(2) A *weak Serre subcategory* of $\mathcal{A}$ is a nonempty full subcategory $\mathcal{C}$ of $\mathcal{A}$ such that given an exact sequence
$$A_0 \to A_1 \to A_2 \to A_3 \to A_4$$
with A_0, A_1, A_3, A_4 in $\mathcal{C}$, then also A_2 in $\mathcal{C}$.

In some references the second notion is called a ``thick'' subcategory and in other references the first notion is called a ``thick'' subcategory. However, it seems that the notion of a Serre subcagegory is universally accepted to be the one defined above. Note that in both cases the category $\mathcal{C}$ is abelian and that the inclusion functor $\mathcal{C} \to \mathcal{A}$ is a fully faithful exact functor. Let's characterize these types of subcategories in more detail.

Lemma 10.7.2. *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{C}$ be a subcategory of $\mathcal{A}$. Then $\mathcal{C}$ is a Serre subcategory if and only if the following conditions are satisfied:*

(1) $0 \in Ob(\mathcal{C})$,
(2) *$\mathcal{C}$ is a strictly full subcategory of $\mathcal{A}$,*
(3) *any subobject or quotient of an object of $\mathcal{C}$ is an object of $\mathcal{C}$,*
(4) *if $A \in Ob(\mathcal{A})$ is an extension of objects of $\mathcal{C}$ then also $A \in Ob(\mathcal{C})$.*

Moreover, a Serre subcategory is an abelian category and the inclusion functor is exact.

Proof. Omitted. □

Lemma 10.7.3. *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{C}$ be a subcategory of $\mathcal{A}$. Then $\mathcal{C}$ is a weak Serre subcategory if and only if the following conditions are satisfied:*

(1) $0 \in Ob(\mathcal{C})$,
(2) *$\mathcal{C}$ is a strictly full subcategory of $\mathcal{A}$,*
(3) *kernels and cokernels in $\mathcal{A}$ of morphisms between objects of $\mathcal{C}$ are in $\mathcal{C}$,*
(4) *if $A \in Ob(\mathcal{A})$ is an extension of objects of $\mathcal{C}$ then also $A \in Ob(\mathcal{C})$.*

Moreover, a weak Serre subcategory is an abelian category and the inclusion functor is exact.

Proof. Omitted. □

Lemma 10.7.4. *Let $\mathcal{A}$, $\mathcal{B}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor. Then the full subcategory of objects C of $\mathcal{A}$ such that $F(C) = 0$ forms a Serre subcategory of $\mathcal{A}$.*

Proof. Omitted. □

Definition 10.7.5. Let $\mathcal{A}$, $\mathcal{B}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor. Then the full subcategory of objects C of $\mathcal{A}$ such that $F(C) = 0$ is called the *kernel of the functor F*, and is sometimes denoted $\text{Ker}(F)$.

Lemma 10.7.6. *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{C} \subset \mathcal{A}$ be a Serre subcategory. There exists an abelian category $\mathcal{A}/\mathcal{C}$ and an exact functor*

$$F : \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{C}$$

which is essentially surjective and whose kernel is $\mathcal{C}$. The category $\mathcal{A}/\mathcal{C}$ and the functor F are characterized by the following universal property: For any exact functor $G : \mathcal{A} \to \mathcal{B}$ such that $\mathcal{C} \subset Ker(G)$ there exists a factorization $G = H \circ F$ for a unique exact functor $H : \mathcal{A}/\mathcal{C} \to \mathcal{B}$.

Proof. Consider the set of arrows of $\mathcal{A}$ defined by the following formula

$$S = \{f \in \text{Arrows}(\mathcal{A}) \mid \text{Ker}(f), \text{Coker}(f) \in Ob(\mathcal{C})\}.$$

We claim that S is a multiplicative system. To prove this we have to check MS1, MS2, MS3, see Categories, Definition 4.24.1.

It is clear that identities are elements of S. Suppose that $f : A \to B$ and $g : B \to C$ are elements of S. There are exact sequences

$$0 \to \text{Ker}(f) \to \text{Ker}(gf) \to \text{Ker}(g) \to 0$$
$$0 \to \text{Coker}(f) \to \text{Coker}(gf) \to \text{Coker}(g) \to 0$$

Hence it follows that $gf \in S$. This proves MS1.

Consider a solid diagram

$$\begin{tikzcd} A \arrow[r, "g"'] \arrow[d, "t"'] & B \arrow[d, dotted, "s"] \\ C \arrow[r, dotted, "f"] & C \amalg_A B \end{tikzcd}$$

with $t \in S$. Set $W = C \amalg_A B = \mathrm{Coker}((1,-1) : A \to C \oplus B)$. Then $\mathrm{Ker}(t) \to \mathrm{Ker}(s)$ is surjective and $\mathrm{Coker}(t) \to \mathrm{Coker}(s)$ is an isomorphism. Hence s is an element of S. This proves LMS2 and the proof of RMS2 is dual.

Finally, consider morphisms $f, g : B \to C$ and a morphism $s : A \to B$ in S such that $f \circ s = g \circ s$. This means that $(f - g) \circ s = 0$. In turn this means that $I = \mathrm{Im}(f - g) \subset C$ is a quotient of $\mathrm{Coker}(s)$ hence an object of $\mathscr{C}$. Thus $t : C \to C' = C/I$ is an element of S such that $t \circ (f - g) = 0$, i.e., such that $t \circ f = t \circ g$. This proves LMS3 and the proof of RMS3 is dual.

Having proved that S is a multiplicative system we set $\mathscr{A}/\mathscr{C} = S^{-1}\mathscr{A}$, and we set F equal to the localization functor Q. By Lemma 10.6.4 the category $\mathscr{A}/\mathscr{C}$ is abelian and F is exact. If X is in the kernel of $F = Q$, then by Lemma 10.6.3 we see that $0 : X \to Z$ is an element of S and hence X is an object of $\mathscr{C}$, i.e., the kernel of F is $\mathscr{C}$. Finally, if G is as in the statement of the lemma, then G turns every element of S into an isomorphism. Hence we obtain the functor $H : \mathscr{A}/\mathscr{C} \to \mathscr{B}$ from the universal property of localization, see Categories, Lemma 4.24.6. □

Lemma 10.7.7. *Let $\mathscr{A}$, $\mathscr{B}$ be abelian categories. Let $F : \mathscr{A} \to \mathscr{B}$ be an exact functor. Let $\mathscr{C} = Ker(F)$. Then the induced functor $\overline{F} : \mathscr{A}/\mathscr{C} \to \mathscr{B}$ is faithful.*

Proof. This is true because the kernel of $\overline{F}$ is zero by construction. Namely, if $f : X \to Y$ is a morphism in $\mathscr{A}/\mathscr{C}$ such that $\overline{F}(f) = 0$, then $\mathrm{Ker}(f) \to X$ and $Y \to \mathrm{Coker}(f)$ are transformed into isomorphisms by $\overline{F}$, hence are isomorphisms by the remark on the kernel of $\overline{F}$. Thus $f = 0$. □

10.8. K-groups

Definition 10.8.1. Let $\mathscr{A}$ be an abelian category. We denote $K_0(\mathscr{A})$ the *zeroth K-group of $\mathscr{A}$*. It is the abelian group constructed as follows. Take the free abelian group on the objects on $\mathscr{A}$ and for every short exact sequence $0 \to A \to B \to C \to 0$ impose the relation $[B] - [A] - [C] = 0$.

Another way to say this is that there is a presentation

$$\bigoplus_{A \to B \to C \text{ ses}} \mathbf{Z}[A \to B \to C] \longrightarrow \bigoplus_{A \in Ob(\mathscr{A})} \mathbf{Z}[A] \longrightarrow K_0(\mathscr{A}) \longrightarrow 0$$

with $[A \to B \to C] \mapsto [B] - [A] - [C]$ of $K_0(\mathscr{A})$. The short exact sequence $0 \to 0 \to 0 \to 0 \to 0$ leads to the relation $[0] = 0$ in $K_0(\mathscr{A})$. There are no set-theoretical issues as all of our categories are ``small'' if not mentioned otherwise. Some examples of K-groups for categories of modules over rings where computed in Algebra, Section 7.51.

Lemma 10.8.2. *Let $F : \mathscr{A} \to \mathscr{B}$ be an exact functor between abelian categories. Then F induces a homomorphism of K-groups $K_0(F) : K_0(\mathscr{A}) \to K_0(\mathscr{B})$ by simply setting $K_0(F)([A]) = [F(A)]$.*

Proof. Proves itself. □

Suppose we are given an object M of an abelian category $\mathcal{A}$ and a complex of the form

(10.8.2.1) $$\ldots \longrightarrow M \xrightarrow{\varphi} M \xrightarrow{\psi} M \xrightarrow{\varphi} M \longrightarrow \ldots$$

In this situation we define

$$H^0(M, \varphi, \psi) = \mathrm{Ker}(\psi)/\mathrm{Im}(\varphi), \quad \text{and} \quad H^1(M, \varphi, \psi) = \mathrm{Ker}(\varphi)/\mathrm{Im}(\psi).$$

Lemma 10.8.3. *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{C} \subset \mathcal{A}$ be a Serre subcategory and set $\mathcal{B} = \mathcal{A}/\mathcal{C}$.*

(1) *The exact functors $\mathcal{C} \to \mathcal{A}$ and $\mathcal{A} \to \mathcal{B}$ induce an exact sequence*
$$K_0(\mathcal{C}) \to K_0(\mathcal{A}) \to K_0(\mathcal{B}) \to 0$$
of K-groups, and

(2) *the kernel of $K_0(\mathcal{C}) \to K_0(\mathcal{A})$ is equal to the collection of elements of the form*
$$[H^0(M, \varphi, \psi)] - [H^1(M, \varphi, \psi)]$$
where (M, φ, ψ) is a complex as in (10.8.2.1) with the property that it becomes exact in $\mathcal{B}$; in other words that $H^0(M, \varphi, \psi)$ and $H^1(M, \varphi, \psi)$ are objects of $\mathcal{C}$.

Proof. We omit the proof of (1). The proof of (2) is in a sense completely combinatorial. First we remark that any class of the type $[H^0(M, \varphi, \psi)] - [H^1(M, \varphi, \psi)]$ is zero in $K_0(\mathcal{A})$ by the following calculation

$$\begin{aligned} 0 &= [M] - [M] \\ &= [\mathrm{Ker}(\varphi)] + [\mathrm{Im}(\varphi)] - [\mathrm{Ker}(\psi)] - [\mathrm{Im}(\psi)] \\ &= [\mathrm{Ker}(\varphi)/\mathrm{Im}(\psi)] - [\mathrm{Ker}(\psi)/\mathrm{Im}(\varphi)] \\ &= [H^1(M, \varphi, \psi)] - [H^0(M, \varphi, \psi)] \end{aligned}$$

as desired. Hence it suffices to show that any element in the kernel of $K_0(\mathcal{C}) \to K_0(\mathcal{A})$ is of this form.

Any element x in $K_0(\mathcal{C})$ can be represented as the difference $x = [P] - [Q]$ of two objects of $\mathcal{C}$ (fun exercise). Suppose that this element maps to zero in $K_0(\mathcal{A})$. This means that there exist

(1) a finite set $I = I^+ \coprod I^-$,

(2) for each $i \in I$ a short exact sequence
$$0 \to A_i \to B_i \to C_i \to 0$$
in the abelian category $\mathcal{A}$

such that

$$[P] - [Q] = \sum\nolimits_{i \in I^+} ([B_i] - [A_i] - [C_i]) - \sum\nolimits_{i \in I^-} ([B_i] - [A_i] - [C_i])$$

in the free abelian group on the objects of $\mathcal{A}$. We can rewrite this as

$$[P] + \sum\nolimits_{i \in I^+} ([A_i] + [C_i]) + \sum\nolimits_{i \in I^-} [B_i] = [Q] + \sum\nolimits_{i \in I^-} ([A_i] + [C_i]) + \sum\nolimits_{i \in I^+} [B_i].$$

Since the right and left hand side should contain the same objects of $\mathcal{A}$ counted with multiplicity, this means there should be a bijection τ between the terms which occur above. Set

$$T^+ = \{p\} \coprod \{a, c\} \times I^+ \coprod \{b\} \times I^-$$

and

$$T^- = \{q\} \coprod \{a, c\} \times I^- \coprod \{b\} \times I^+.$$

Set $T = T^+ \coprod T^- = \{p, q\} \coprod \{a, b, c\} \times I$. For $t \in T$ define

$$O(t) = \begin{cases} P & \text{if} \quad t = p \\ Q & \text{if} \quad t = q \\ A_i & \text{if} \quad t = (a, i) \\ B_i & \text{if} \quad t = (b, i) \\ C_i & \text{if} \quad t = (c, i) \end{cases}$$

Hence we can view $\tau : T^+ \to T^-$ as a bijection such that $O(t) = O(\tau(t))$ for all $t \in T^+$. Let $t_0^- = \tau(p)$ and let $t_0^+ \in T^+$ be the unique element such that $\tau(t_0^+) = q$. Consider the object

$$M^+ = \bigoplus\nolimits_{t \in T^+} O(t)$$

By using τ we see that it is equal to the object

$$M^- = \bigoplus\nolimits_{t \in T^-} O(t)$$

Consider the map

$$\varphi : M^+ \longrightarrow M^-$$

which on the summand $O(t) = A_i$ corresponding to $t = (a, i)$, $i \in I^+$ uses the map $A_i \to B_i$ into the summand $O((b, i)) = B_i$ of M^- and on the summand $O(t) = B_i$ corresponding to (b, i), $i \in I^-$ uses the map $B_i \to C_i$ into the summand $O((c, i)) = C_i$ of M^-. The map is zero on the summands corresponding to p and (c, i), $i \in I^+$. Similarly, consider the map

$$\psi : M^- \longrightarrow M^+$$

which on the summand $O(t) = A_i$ corresponding to $t = (a, i)$, $i \in I^-$ uses the map $A_i \to B_i$ into the summand $O((b, i)) = B_i$ of M^+ and on the summand $O(t) = B_i$ corresponding to (b, i), $i \in I^+$ uses the map $B_i \to C_i$ into the summand $O((c, i)) = C_i$ of M^+. The map is zero on the summands corresponding to q and (c, i), $i \in I^-$.

Note that the kernel of φ is equal to the direct sum of the summand P and the summands $O((c, i)) = C_i$, $i \in I^+$ and the subobjects A_i inside the summands $O((b, i)) = B_i$, $i \in I^-$. The image of ψ is equal to the direct sum of the summands $O((c, i)) = C_i$, $i \in I^+$ and the subobjects A_i inside the summands $O((b, i)) = B_i$, $i \in I^-$. In other words we see that

$$P \cong \mathrm{Ker}(\varphi)/\mathrm{Im}(\psi).$$

In exactly the same way we see that

$$Q \cong \mathrm{Ker}(\psi)/\mathrm{Im}(\varphi).$$

Since as we remarked above the existence of the bijection τ shows that $M^+ = M^-$ we see that the lemma follows. □

10.9. Cohomological delta-functors

Definition 10.9.1. Let $\mathcal{A}, \mathcal{B}$ be abelian categories. A *cohomological δ-functor* or simply a *δ-functor* from $\mathcal{A}$ to $\mathcal{B}$ is given by the following data:

(1) a collection $F^n : \mathcal{A} \to \mathcal{B}$, $n \geq 0$ of additive functors, and
(2) for every short exact sequence $0 \to A \to B \to C \to 0$ of $\mathcal{A}$ a collection $\delta_{A \to B \to C} : F^n(C) \to F^{n+1}(A)$, $n \geq 0$ of morphisms of $\mathcal{B}$.

These data are assumed to satisfy the following axioms

(1) for every short exact sequence as above the sequence

$$\begin{tikzcd}
0 \arrow[r] & F^0(A) \arrow[r] & F^0(B) \arrow[r] & F^0(C) \arrow[dll, "\delta_{A\to B\to C}"'] \\
& F^1(A) \arrow[r] & F^1(B) \arrow[r] & F^1(C) \arrow[dll, "\delta_{A\to B\to C}"'] \\
& F^2(A) \arrow[r] & F^2(B) \arrow[r] & \ldots
\end{tikzcd}$$

is exact, and

(2) for every morphism $(A \to B \to C) \to (A' \to B' \to C')$ of short exact sequences of $\mathscr{A}$ the diagrams

$$\begin{tikzcd}
F^n(C) \arrow[r, "\delta_{A\to B\to C}"'] \arrow[d] & F^{n+1}(A) \arrow[d] \\
F^n(C') \arrow[r, "\delta_{A'\to B'\to C'}"] & F^{n+1}(A')
\end{tikzcd}$$

are commutative.

Note that this in particular implies that F^0 is left exact.

Definition 10.9.2. Let $\mathscr{A}, \mathscr{B}$ be abelian categories. Let (F^n, δ_F) and (G^n, δ_G) be δ-functors from $\mathscr{A}$ to $\mathscr{B}$. A *morphism of δ-functors from F to G* is a collection of transformation of functors $t^n : F^n \to G^n$, $n \geq 0$ such that for every short exact sequence $0 \to A \to B \to C \to 0$ of $\mathscr{A}$ the diagrams

$$\begin{tikzcd}
F^n(C) \arrow[r, "\delta_{F,A\to B\to C}"'] \arrow[d, "t^n"'] & F^{n+1}(A) \arrow[d, "t^{n+1}"] \\
G^n(C) \arrow[r, "\delta_{G,A\to B\to C}"] & G^{n+1}(A)
\end{tikzcd}$$

are commutative.

Definition 10.9.3. Let $\mathscr{A}, \mathscr{B}$ be abelian categories. Let $F = (F^n, \delta_F)$ be a δ-functor from $\mathscr{A}$ to $\mathscr{B}$. We say F is a *universal δ-functor* if an only if for every δ-functor $G = (G^n, \delta_G)$ and any morphism of functors $t : F^0 \to G^0$ there exists a unique morphism of δ-functors $\{t^n\}_{n\geq 0} : F \to G$ such that $t = t^0$.

Lemma 10.9.4. *Let $\mathscr{A}, \mathscr{B}$ be abelian categories. Let $F = (F^n, \delta_F)$ be a δ-functor from $\mathscr{A}$ to $\mathscr{B}$. Suppose that for every $n > 0$ and any $A \in Ob(\mathscr{A})$ there exists an injective morphism $u : A \to B$ (depending on A and n) such that $F^n(u) : F^n(A) \to F^n(B)$ is zero. Then F is a universal δ-functor.*

Proof. Let $G = (G^n, \delta_G)$ be a δ-functor from $\mathscr{A}$ to $\mathscr{B}$ and let $t : F^0 \to G^0$ be a morphism of functors. We have to show there exists a unique morphism of δ-functors $\{t^n\}_{n\geq 0} : F \to G$ such that $t = t^0$. We construct t^n by induction on n. For $n = 0$ we set $t^0 = t$. Suppose we have already constructed a unique sequence of transformation of functors t^i for $i \leq n$ compatible with the maps δ in degrees $\leq n$.

Let $A \in Ob(\mathscr{A})$. By assumption we may choose a embedding $u : A \to B$ such that $F^{n+1}(u) = 0$. Let $C = B/u(A)$. The long exact cohomology sequence for the short exact sequence $0 \to A \to B \to C \to 0$ and the δ-functor F gives that $F^{n+1}(A) = \mathrm{Coker}(F^n(B) \to F^n(C))$ by our choice of u. Since we have already defined t^n we can set

$$t^{n+1}_A : F^{n+1}(A) \to G^{n+1}(A)$$

equal to the unique map such that

$$\begin{tikzcd}
\mathrm{Coker}(F^n(B) \to F^n(C)) \arrow[r, "t^n"'] \arrow[d, "\delta_{F,A\to B\to C}"'] & \mathrm{Coker}(G^n(B) \to G^n(C)) \arrow[d, "\delta_{G,A\to B\to C}"] \\
F^{n+1}(A) \arrow[r, "t^{n+1}_A"] & G^{n+1}(A)
\end{tikzcd}$$

commutes. This is clearly uniquely determined by the requirements imposed. We omit the verification that this defines a transformation of functors. □

Lemma 10.9.5. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor. If there exists a universal δ-functor (F^n, δ_F) from $\mathcal{A}$ to $\mathcal{B}$ with $F^0 = F$, then it is determined up to unique isomorphism of δ-functors.*

Proof. Immediate from the definitions. □

10.10. Complexes

Of course the notions of a chain complex and a cochain complex are dual and you only have to read one of the two parts of this section. So pick the one you like. (Actually, this doesn't quite work right since the conventions on numbering things are not adapted to an easy transition between chain and cochain complexes.)

A *chain complex* $A_\bullet$ in an additive category $\mathcal{A}$ is a complex

$$\ldots \to A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \to \ldots$$

of $\mathcal{A}$. In other words, we are given an object A_i of $\mathcal{A}$ for all $i \in \mathbf{Z}$ and for all $i \in \mathbf{Z}$ a morphism $d_i : A_i \to A_{i-1}$ such that $d_{i-1} \circ d_i = 0$ for all i. A *morphism of chain complexes* $f : A_\bullet \to B_\bullet$ is given by a family of morphisms $f_i : A_i \to B_i$ such that all the diagrams

$$\begin{tikzcd}
A_i \arrow[r, "d_i"'] \arrow[d, "f_i"'] & A_{i-1} \arrow[d, "f_{i-1}"] \\
B_i \arrow[r, "d_i"] & B_{i-1}
\end{tikzcd}$$

commute. The *category of chain complexes of $\mathcal{A}$* is denoted $\mathrm{Ch}(\mathcal{A})$. The full subcategory consisting of objects of the form

$$\ldots \to A_2 \to A_1 \to A_0 \to 0 \to 0 \to \ldots$$

is denoted $\mathrm{Ch}_{\geq 0}(\mathcal{A})$. In other words, a chain complex $A_\bullet$ belongs to $\mathrm{Ch}_{\geq 0}(\mathcal{A})$ if and only if $A_i = 0$ for all $i < 0$. A *homotopy* h between a pair of morphisms of chain complexes $f, g : A_\bullet \to B_\bullet$ is is a collection of morphisms $h_i : A_i \to B_{i+1}$ such that we have

$$f_i - g_i = d_{i+1} \circ h_i + h_{i-1} \circ d_i$$

for all i. Clearly, the notions of chain complex, morphism of chain complexes, and homotopies between morphisms of chain complexes makes sense even in a preadditive category.

Lemma 10.10.1. *Let $\mathscr{A}$ be an additive category. Let $f, g : B_\bullet \to C_\bullet$ be morphisms of chain complexes. Suppose given morphisms of chain complexes $a : A_\bullet \to B_\bullet$, and $c : C_\bullet \to D_\bullet$. If $\{h_i : B_i \to C_{i+1}\}$ defines a homotopy between f and g, then $\{c_{i+1} \circ h_i \circ a_i\}$ defines a homotopy between $c \circ f \circ a$ and $c \circ g \circ a$.*

Proof. Omitted. □

In particular this means that it makes sense to define the category of chain complexes with maps up to homotopy. We'll return to this later.

Definition 10.10.2. Let $\mathscr{A}$ be an additive category. We say a morphism $a : A_\bullet \to B_\bullet$ is a *homotopy equivalence* if there exists a morphism $b : B_\bullet \to A_\bullet$ such that there exists a homotopy between $a \circ b$ and id_A and there exists a homotopy between $b \circ a$ and id_B. If there exists such a morphism between $A_\bullet$ and $B_\bullet$, then we say that $A_\bullet$ and $B_\bullet$ are *homotopy equivalent*.

In other words, two complexes are homotopy equivalent if they become isomorphic in the category of complexes up to homotopy.

Lemma 10.10.3. *Let $\mathscr{A}$ be an abelian category.*

(1) *The category of chain complexes in $\mathscr{A}$ is abelian.*
(2) *A morphism of complexes $f : A_\bullet \to B_\bullet$ is injective if and only if each $f_n : A_n \to B_n$ is injective.*
(3) *A morphism of complexes $f : A_\bullet \to B_\bullet$ is surjective if and only if each $f_n : A_n \to B_n$ is surjective.*
(4) *A sequence of chain complexes*
$$A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet$$
is exact at $B_\bullet$ if and only if each sequence
$$A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i$$
is exact at B_i.

Proof. Omitted. □

For any $i \in \mathbf{Z}$ the ith *homology group* of a chain complex $A_\bullet$ in an abelian category is defined by the following formula
$$H_i(A_\bullet) = \mathrm{Ker}(d_i)/\mathrm{Im}(d_{i+1}).$$
If $f : A_\bullet \to B_\bullet$ is a morphism of chain complexes of $\mathscr{A}$ then we get an induced morphism $H_i(f) : H_i(A_\bullet) \to H_i(B_\bullet)$ because clearly $f_i(\mathrm{Ker}(d_i : A_i \to A_{i-1})) \subset \mathrm{Ker}(d_i : B_i \to B_{i-1})$, and similarly for $\mathrm{Im}(d_{i+1})$. Thus we obtain a functor
$$H_i : \mathrm{Ch}(\mathscr{A}) \longrightarrow \mathscr{A}.$$

Definition 10.10.4. Let $\mathscr{A}$ be an abelian category.

(1) A morphism of chain complexes $f : A_\bullet \to B_\bullet$ is called a *quasi-isomorphism* if the induced maps $H_i(f) : H_i(A_\bullet) \to H_i(B_\bullet)$ is an isomorphism for all $i \in \mathbf{Z}$.
(2) A chain complex $A_\bullet$ is called *acyclic* if all of its homology objects $H_i(A_\bullet)$ are zero.

Lemma 10.10.5. *Let $\mathscr{A}$ be an abelian category.*

(1) *If the maps $f, g : A_\bullet \to B_\bullet$ are homotopic, then the induced maps $H_i(f)$ and $H_i(g)$ are equal.*

(2) *If the map $f : A_\bullet \to B_\bullet$ is a homotopy equivalence, then f is a quisi-isomorphism.*

Proof. Omitted. □

Lemma 10.10.6. *Let $\mathscr{A}$ be an abelian category. Suppose that*

$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$$

is a short exact sequence of chain complexes of $\mathscr{A}$. Then there is a canonical long exact homology sequence

$$\begin{tikzcd}
\ldots & \ldots & \ldots \arrow[dll] \\
H_i(A_\bullet) \arrow[r] & H_i(B_\bullet) \arrow[r] & H_i(C_\bullet) \arrow[dll] \\
H_{i-1}(A_\bullet) \arrow[r] & H_{i-1}(B_\bullet) \arrow[r] & H_{i-1}(C_\bullet) \arrow[dll] \\
\ldots & \ldots & \ldots
\end{tikzcd}$$

Proof. Omitted. The maps come from the Snake Lemma 10.3.23 applied to the diagrams

$$\begin{tikzcd}
 & A_i/\mathrm{Im}(d_{A,i+1}) \arrow[r] \arrow[d, "d_{A,i}"] & B_i/\mathrm{Im}(d_{B,i+1}) \arrow[r] \arrow[d, "d_{B,i}"] & C_i/\mathrm{Im}(d_{C,i+1}) \arrow[r] \arrow[d, "d_{C,i}"] & 0 \\
0 \arrow[r] & \mathrm{Ker}(d_{A,i-1}) \arrow[r] & \mathrm{Ker}(d_{B,i-1}) \arrow[r] & \mathrm{Ker}(d_{C,i-1}) &
\end{tikzcd}$$

□

A *cochain complex* $A^\bullet$ in an additive category $\mathscr{A}$ is a complex

$$\ldots \to A^{n-1} \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} A^{n+1} \to \ldots$$

of $\mathscr{A}$. In other words, we are given an object A^i of $\mathscr{A}$ for all $i \in \mathbf{Z}$ and for all $i \in \mathbf{Z}$ a morphism $d^i : A^i \to A^{i+1}$ such that $d^{i+1} \circ d^i = 0$ for all i. A *morphism of cochain complexes* $f : A^\bullet \to B^\bullet$ is given by a family of morphisms $f^i : A^i \to B^i$ such that all the diagrams

$$\begin{tikzcd}
A^i \arrow[r, "d^i"'] \arrow[d, "f^i"'] & A^{i+1} \arrow[d, "f^{i+1}"] \\
B^i \arrow[r, "d^i"] & B^{i+1}
\end{tikzcd}$$

commute. The *category of cochain complexes of $\mathscr{A}$* is denoted $\mathrm{CoCh}(\mathscr{A})$. The full subcategory consisting of objects of the form

$$\ldots \to 0 \to 0 \to A^0 \to A^1 \to A^2 \to \ldots$$

is denoted $\mathrm{CoCh}_{\geq 0}(\mathscr{A})$. In other words, a cochain complex $A^\bullet$ belongs to the subcategory $\mathrm{CoCh}_{\geq 0}(\mathscr{A})$ if and only if $A^i = 0$ for all $i < 0$. A *homotopy* h between a pair of morphisms of cochain complexes $f, g : A^\bullet \to B^\bullet$ is is a collection of morphisms $h^i : A^i \to B^{i-1}$ such that we have

$$f^i - g^i = d^{i-1} \circ h^i + h^{i+1} \circ d^i$$

for all i. Clearly, the notions of cochain complex, morphism of cochain complexes, and homotopies between morphisms of cochain complexes makes sense even in a preadditive category.

Lemma 10.10.7. *Let $\mathcal{A}$ be an additive category. Let $f, g : B^\bullet \to C^\bullet$ be morphisms of cochain complexes. Suppose given morphisms of cochain complexes $a : A^\bullet \to B^\bullet$, and $c : C^\bullet \to D^\bullet$. If $\{h^i : B^i \to C^{i-1}\}$ defines a homotopy between f and g, then $\{c^{i-1} \circ h^i \circ a^i\}$ defines a homotopy between $c \circ f \circ a$ and $c \circ g \circ a$.*

Proof. Omitted. □

In particular this means that it makes sense to define the category of cochain complexes with maps up to homotopy. We'll return to this later.

Definition 10.10.8. Let $\mathcal{A}$ be an additive category. We say a morphism $a : A^\bullet \to B^\bullet$ is a *homotopy equivalence* if there exists a morphism $b : B^\bullet \to A^\bullet$ such that there exists a homotopy between $a \circ b$ and id_A and there exists a homotopy between $b \circ a$ and id_B. If there exists such a morphism between $A^\bullet$ and $B^\bullet$, then we say that $A^\bullet$ and $B^\bullet$ are *homotopy equivalent*.

In other words, two complexes are homotopy equivalent if they become isomorphic in the category of complexes up to homotopy.

Lemma 10.10.9. *Let $\mathcal{A}$ be an abelian category.*

(1) *The category of cochain complexes in $\mathcal{A}$ is abelian.*
(2) *A morphism of cochain complexes $f : A^\bullet \to B^\bullet$ is injective if and only if each $f^n : A^n \to B^n$ is injective.*
(3) *A morphism of cochain complexes $f : A^\bullet \to B^\bullet$ is surjective if and only if each $f^n : A^n \to B^n$ is surjective.*
(4) *A sequence of cochain complexes*
$$A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet$$
is exact at $B^\bullet$ if and only if each sequence
$$A^i \xrightarrow{f^i} B^i \xrightarrow{g^i} C^i$$
is exact at B^i.

Proof. Omitted. □

For any $i \in \mathbf{Z}$ the ith *cohomology group* of a cochain complex $A^\bullet$ is defined by the following formula
$$H^i(A^\bullet) = \mathrm{Ker}(d^i)/\mathrm{Im}(d^{i-1}).$$
If $f : A^\bullet \to B^\bullet$ is a morphism of cochain complexes of $\mathcal{A}$ then we get an induced morphism $H^i(f) : H^i(A^\bullet) \to H^i(B^\bullet)$ because clearly $f^i(\mathrm{Ker}(d^i : A^i \to A^{i+1})) \subset \mathrm{Ker}(d^i : B^i \to B^{i+1})$, and similarly for $\mathrm{Im}(d^{i-1})$. Thus we obtain a functor
$$H^i : \mathrm{CoCh}(\mathcal{A}) \longrightarrow \mathcal{A}.$$

Definition 10.10.10. Let $\mathcal{A}$ be an abelian category.

(1) A morphism of cochain complexes $f : A^\bullet \to B^\bullet$ of $\mathcal{A}$ is called a *quasi-isomorphism* if the induced maps $H^i(f) : H^i(A^\bullet) \to H^i(B^\bullet)$ is an isomorphism for all $i \in \mathbf{Z}$.
(2) A cochain complex $A^\bullet$ is called *acyclic* if all of its cohomology objects $H^i(A^\bullet)$ are zero.

Lemma 10.10.11. *Let $\mathcal{A}$ be an abelian category.*

(1) *If the maps $f, g : A^\bullet \to B^\bullet$ are homotopic, then the induced maps $H^i(f)$ and $H^i(g)$ are equal.*
(2) *If $f : A^\bullet \to B^\bullet$ is a homotopy equivalence, then f is a quasi-isomorphism.*

Proof. Omitted. $\square$

Lemma 10.10.12. *Let $\mathcal{A}$ be an abelian category. Suppose that*

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$$

is a short exact sequence of chain complexes of $\mathcal{A}$. Then there is a canonical long exact homology sequence

$$\begin{tikzcd}
\ldots & \ldots & \ldots \arrow[dll] \\
H^i(A^\bullet) \arrow[r] & H^i(B^\bullet) \arrow[r] & H^i(C^\bullet) \arrow[dll] \\
H^{i+1}(A^\bullet) \arrow[r] & H^{i+1}(B^\bullet) \arrow[r] & H^{i+1}(C^\bullet) \arrow[dll] \\
\ldots & \ldots & \ldots
\end{tikzcd}$$

Proof. Omitted. The maps come from the Snake Lemma 10.3.23 applied to the diagrams

$$\begin{tikzcd}
 & A^i/\mathrm{Im}(d_A^{i-1}) \arrow[r] \arrow[d, "d_A^i"] & B^i/\mathrm{Im}(d_B^{i-1}) \arrow[r] \arrow[d, "d_B^i"] & C^i/\mathrm{Im}(d_C^{i-1}) \arrow[r] \arrow[d, "d_C^i"] & 0 \\
0 \arrow[r] & \mathrm{Ker}(d_A^{i+1}) \arrow[r] & \mathrm{Ker}(d_B^{i+1}) \arrow[r] & \mathrm{Ker}(d_C^{i+1}) &
\end{tikzcd}$$

$\square$

10.11. Truncation of complexes

Let $\mathcal{A}$ be an abelian category. Let $A_\bullet$ be a chain complex. There are several ways to *truncate* the complex $A_\bullet$.

(1) The *``stupid'' truncation* $\sigma_{\leq n}$ is the the subcomplex $\sigma_{\leq n}A_\bullet$ defined by the rule $(\sigma_{\leq n}A_\bullet)_i = 0$ if $i > n$ and $(\sigma_{\leq n}A_\bullet)_i = A_i$ if $i \leq n$. In a picture

$$\begin{tikzcd}
\sigma_{\leq n}A_\bullet \arrow[d] & \ldots \arrow[r] & 0 \arrow[r] \arrow[d] & A_n \arrow[r] \arrow[d] & A_{n-1} \arrow[r] \arrow[d] & \ldots \\
A_\bullet & \ldots \arrow[r] & A_{n+1} \arrow[r] & A_n \arrow[r] & A_{n-1} \arrow[r] & \ldots
\end{tikzcd}$$

Note the property $\sigma_{\leq n}A_\bullet/\sigma_{\leq n-1}A_\bullet = A_n[-n]$.

(2) The ``*stupid'' truncation* $\sigma_{\geq n}$ is the the quotient complex $\sigma_{\geq n}A_\bullet$ defined by the rule $(\sigma_{\geq n}A_\bullet)_i = A_i$ if $i \geq n$ and $(\sigma_{\geq n}A_\bullet)_i = 0$ if $i < n$. In a picture

$$\begin{tikzcd}
A_\bullet \arrow[d] & \ldots \arrow[r] & A_{n+1} \arrow[r] \arrow[d] & A_n \arrow[r] \arrow[d] & A_{n-1} \arrow[r] \arrow[d] & \ldots \\
\sigma_{\geq n}A_\bullet & \ldots \arrow[r] & A_{n+1} \arrow[r] & A_n \arrow[r] & 0 \arrow[r] & \ldots
\end{tikzcd}$$

The map of complexes $\sigma_{\geq n}A_\bullet \to \sigma_{\geq n+1}A_\bullet$ is surjective with kernel $A_n[-n]$.

(3) The *canonical truncation* $\tau_{\geq n}A_\bullet$ is defined by the picture

$$\begin{tikzcd}
\tau_{\geq n}A_\bullet \arrow[d] & \ldots \arrow[r] & A_{n+1} \arrow[r] \arrow[d] & \mathrm{Ker}(d_n) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \ldots \\
A_\bullet & \ldots \arrow[r] & A_{n+1} \arrow[r] & A_n \arrow[r] & A_{n-1} \arrow[r] & \ldots
\end{tikzcd}$$

Note that these complexes have the property that

$$H_i(\tau_{\geq n}A_\bullet) = \begin{cases} H_i(A_\bullet) & \text{if} \quad i \geq n \\ 0 & \text{if} \quad i < n \end{cases}$$

(4) The *canonical truncation* $\tau_{\leq n}A_\bullet$ is defined by the picture

$$\begin{tikzcd}
A_\bullet \arrow[d] & \ldots \arrow[r] & A_{n+1} \arrow[r] \arrow[d] & A_n \arrow[r] \arrow[d] & A_{n-1} \arrow[r] \arrow[d] & \ldots \\
\tau_{\leq n}A_\bullet & \ldots \arrow[r] & 0 \arrow[r] & \mathrm{Coker}(d_{n+1}) \arrow[r] & A_{n-1} \arrow[r] & \ldots
\end{tikzcd}$$

Note that these complexes have the property that

$$H_i(\tau_{\leq n}A_\bullet) = \begin{cases} H_i(A_\bullet) & \text{if} \quad i \leq n \\ 0 & \text{if} \quad i > n \end{cases}$$

Let $\mathcal{A}$ be an abelian category. Let $A^\bullet$ be a cochain complex. There are four ways to truncate the complex $A^\bullet$.

(1) The ``*stupid'' truncation* $\sigma_{\geq n}$ is the subcomplex $\sigma_{\geq n}A^\bullet$ defined by the rule $(\sigma_{\geq n}A^\bullet)^i = 0$ if $i < n$ and $(\sigma_{\geq n}A^\bullet)^i = A_i$ if $i \geq n$. In a picture

$$\begin{tikzcd}
\sigma_{\geq n}A^\bullet \arrow[d] & \ldots \arrow[r] & 0 \arrow[r] \arrow[d] & A^n \arrow[r] \arrow[d] & A^{n+1} \arrow[r] \arrow[d] & \ldots \\
A^\bullet & \ldots \arrow[r] & A^{n-1} \arrow[r] & A^n \arrow[r] & A^{n+1} \arrow[r] & \ldots
\end{tikzcd}$$

Note the property $\sigma_{\geq n}A^\bullet/\sigma_{\geq n+1}A^\bullet = A^n[-n]$.

(2) The ``*stupid'' truncation* $\sigma_{\leq n}$ is the quotient complex $\sigma_{\leq n}A^\bullet$ defined by the rule $(\sigma_{\geq n}A^\bullet)^i = 0$ if $i > n$ and $(\sigma_{\geq n}A^\bullet)^i = A^i$ if $i \leq n$. In a picture

$$\begin{tikzcd}
A^\bullet \arrow[d] & \ldots \arrow[r] & A^{n-1} \arrow[r] \arrow[d] & A^n \arrow[r] \arrow[d] & A^{n+1} \arrow[r] \arrow[d] & \ldots \\
\sigma_{\leq n}A^\bullet & \ldots \arrow[r] & A^{n-1} \arrow[r] & A^n \arrow[r] & 0 \arrow[r] & \ldots
\end{tikzcd}$$

The map of complexes $\sigma_{\leq n}A^\bullet \to \sigma_{\leq n-1}A^\bullet$ is surjective with kernel $A^n[-n]$.

(3) The *canonical truncation* $\tau_{\le n}A^\bullet$ is defined by the picture

$$\begin{tikzcd} \tau_{\le n}A^\bullet \arrow[d] & \ldots \arrow[r] & A^{n-1} \arrow[r] \arrow[d] & \mathrm{Ker}(d^n) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \ldots \\ A^\bullet & \ldots \arrow[r] & A^{n-1} \arrow[r] & A^n \arrow[r] & A^{n+1} \arrow[r] & \ldots \end{tikzcd}$$

Note that these complexes have the property that

$$H^i(\tau_{\le n}A^\bullet) = \begin{cases} H^i(A^\bullet) & \text{if} \quad i \le n \\ 0 & \text{if} \quad i > n \end{cases}$$

(4) The *canonical truncation* $\tau_{\ge n}A^\bullet$ is defined by the picture

$$\begin{tikzcd} A^\bullet \arrow[d] & \ldots \arrow[r] & A^{n-1} \arrow[r] \arrow[d] & A^n \arrow[r] \arrow[d] & A^{n+1} \arrow[r] \arrow[d] & \ldots \\ \tau_{\ge n}A^\bullet & \ldots \arrow[r] & 0 \arrow[r] & \mathrm{Coker}(d^{n-1}) \arrow[r] & A^{n+1} \arrow[r] & \ldots \end{tikzcd}$$

Note that these complexes have the property that

$$H^i(\tau_{\le n}A^\bullet) = \begin{cases} 0 & \text{if} \quad i < n \\ H^i(A^\bullet) & \text{if} \quad i \ge n \end{cases}$$

10.12. Homotopy and the shift functor

It is an annoying feature that signs and indices have to be part of any discussion of homological algebra[2].

Definition 10.12.1. Let $\mathcal{A}$ be an additive category. Let $A_\bullet$ be a chain complex with boundary maps $d_{A,n} : A_n \to A_{n-1}$. For any $k \in \mathbf{Z}$ we define the *k-shifted chain complex* $A[k]_\bullet$ as follows:

(1) we set $A[k]_n = A_{n+k}$, and
(2) we set $d_{A[k],n} : A[k]_n \to A[k]_{n-1}$ equal to $d_{A[k],n} = (-1)^k d_{A,n+k}$.

If $f : A_\bullet \to B_\bullet$ is a morphism of chain complexes, then we let $f[k] : A[k]_\bullet \to B[k]_\bullet$ be the morphism of chain complexes with $f[k]_n = f_{k+n}$.

Of course this means we have functors $[k] : \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{A})$ which mutually commute (on the nose, without any intervening isomorphisms of functors), such that $A[k][l]_\bullet = A[k+l]_\bullet$ and with $[0] = \mathrm{id}_{\mathrm{Ch}(\mathcal{A})}$.

Definition 10.12.2. Let $\mathcal{A}$ be an abelian category. Let $A_\bullet$ be a chain complex with boundary maps $d_{A,n} : A_n \to A_{n-1}$. For any $k \in \mathbf{Z}$ we identify $H_{i+k}(A_\bullet) \to H_i(A[k]_\bullet)$ via the identification $A_{i+k} = A[k]_i$.

This identification is functorial in $A_\bullet$. Note that since no signs are involved in this definition we actually get a compatible system of identifications of all the homology objects $H_{i-k}(A[k]_\bullet)$, which are further compatible with the identifications $A[k][l]_\bullet = A[k+l]_\bullet$ and with $[0] = \mathrm{id}_{\mathrm{Ch}(\mathcal{A})}$.

Let $\mathcal{A}$ be an additive category. Suppose that $A_\bullet$ and $B_\bullet$ are chain complexes, $a, b : A_\bullet \to B_\bullet$ are morphsms of chain complexes, and $\{h_i : A_i \to B_{i+1}\}$ is a homotopy between a and b.

[2] I am sure you think that my conventions are wrong. If so and if you feel strongly about it then drop me an email with an explanation.

Recall that this means that $a_i - b_i = d_{i+1} \circ h_i + h_{i-1} \circ d_i$. What if $a = b$? Then we obtain the formula $0 = d_{i+1} \circ h_i + h_{i-1} \circ d_i$, in other words, $-d_{i+1} \circ h_i = h_{i-1} \circ d_i$. By definition above this means the collection $\{h_i\}$ above defines a morphism of chain complexes

$$A_\bullet \longrightarrow B[1]_\bullet.$$

Such a thing is the same as a morphism $A[-1]_\bullet \to B_\bullet$ by our remarks above. This proves the following lemma.

Lemma 10.12.3. *Let $\mathcal{A}$ be an additive category. Suppose that $A_\bullet$ and $B_\bullet$ are chain complexes. Given any morphism of chain complexes $a : A_\bullet \to B_\bullet$ there is a bijection between the set of homotopies from a to a and $Mor_{Ch(\mathcal{A})}(A_\bullet, B[1]_\bullet)$. More generally, the set of homotopies between a and b is either empty or a principal homogenous space under the group $Mor_{Ch(\mathcal{A})}(A_\bullet, B[1]_\bullet)$.*

Proof. See above. ☐

Lemma 10.12.4. *Let $\mathcal{A}$ be an abelian category. Let*

$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$$

be a sort exact sequence of complexes. Suppose that $\{s_n : C_n \to B_n\}$ is a family of morphisms which split the short exact sequences $0 \to A_n \to B_n \to C_n \to 0$. Let $\pi_n : B_n \to A_n$ be the associated projections, see Lemma 10.3.21. Then the family of morphisms

$$\pi_{n-1} \circ d_{B,n} \circ s_n : C_n \to A_{n-1}$$

define a morphism of complexes $\delta(s) : C_\bullet \to A[-1]_\bullet$.

Proof. Denote $i : A_\bullet \to B_\bullet$ and $q : B_\bullet \to C_\bullet$ the maps of complexes in the short exact sequence. Then $i_{n-1} \circ \pi_{n-1} \circ d_{B,n} \circ s_n = d_{B,n} \circ s_n - s_{n-1} \circ d_{C,n}$. Hence $i_{n-2} \circ d_{A,n-1} \circ \pi_{n-1} \circ d_{B,n} \circ s_n = d_{B,n-1} \circ (d_{B,n} \circ s_n - s_{n-1} \circ d_{C,n}) = -d_{B,n-1} \circ s_{n-1} \circ d_{C,n}$ as desired. ☐

Lemma 10.12.5. *Notation and assumptions as in Lemma 10.12.4 above. The morphism of complexes $\delta(s) : C_\bullet \to A[-1]_\bullet$ induces the maps*

$$H_i(\delta(s)) : H_i(C_\bullet) \longrightarrow H_i(A[-1]_\bullet) = H_{i-1}(A_\bullet)$$

which occur in the long exact homology sequence associated to the short exact sequence of chain complexes by Lemma 10.10.6.

Proof. Omitted. ☐

Lemma 10.12.6. *Notation and assumptions as in Lemma 10.12.4 above. Suppose $\{s'_n : C_n \to B_n\}$ is a second choice of splittings. Write $s'_n = s_n + \pi_n \circ h_n$ for some unique morphisms $h_n : C_n \to A_n$. The family of maps $\{h_n : C_n \to A[-1]_{n+1}\}$ is a homotopy between the associated morphisms $\delta(s), \delta(s') : C_\bullet \to A[-1]_\bullet$.*

Proof. Omitted. ☐

Definition 10.12.7. Let $\mathcal{A}$ be an additive category. Let $A^\bullet$ be a cochain complex with boundary maps $d_A^n : A^n \to A^{n-1}$. For any $k \in \mathbf{Z}$ we define the *k-shifted cochain complex* $A[k]^\bullet$ as follows:

(1) we set $A[k]^n = A^{n+k}$, and
(2) we set $d_{A[k]}^n : A[k]^n \to A[k]^{n-1}$ equal to $d_{A[k]}^n = (-1)^k d_A^{n+k}$.

If $f : A^\bullet \to B^\bullet$ is a morphism of cochain complexes, then we let $f[k] : A[k]^\bullet \to B[k]^\bullet$ be the morphism of cochain complexes with $f[k]^n = f^{k+n}$.

Of course this means we have functors $[k] : \text{CoCh}(\mathscr{A}) \to \text{CoCh}(\mathscr{A})$ which mutually commute (on the nose, without any intervening isomorphisms of functors) and such that $A[k][l]^\bullet = A[k+l]^\bullet$ and with $[0] = \text{id}_{\text{CoCh}(\mathscr{A})}$.

Definition 10.12.8. Let $\mathscr{A}$ be an abelian category. Let $A^\bullet$ be a cochain complex with boundary maps $d_A^n : A^n \to A^{n+1}$. For any $k \in \mathbf{Z}$ we identify $H^{i+k}(A^\bullet) \longrightarrow H^i(A[k]^\bullet)$ via the identification $A^{i+k} = A[k]^i$.

This identification is functorial in $A^\bullet$. Note that since no signs are involved in this definition we actually get a compatible system of identifications of all the homology objects $H^{i-k}(A[k]^\bullet)$, which are further compatible with the identifications $A[k][l]^\bullet = A[k+l]^\bullet$ and with $[0] = \text{id}_{\text{CoCh}(\mathscr{A})}$.

Let $\mathscr{A}$ be an additive category. Suppose that $A^\bullet$ and $B^\bullet$ are cochain complexes, $a, b : A^\bullet \to B^\bullet$ are morphsms of cochain complexes, and $\{h^i : A^i \to B^{i-1}\}$ is a homotopy between a and b. Recall that this means that $a^i - b^i = d^{i-1} \circ h^i + h^{i+1} \circ d^i$. What if $a = b$? Then we obtain the formula $0 = d^{i-1} \circ h^i + h^{i+1} \circ d^i$, in other words, $-d^{i-1} \circ h^i = h^{i+1} \circ d^i$. By definition above this means the collection $\{h^i\}$ above defines a morphism of cochain complexes

$$A^\bullet \longrightarrow B[-1]^\bullet.$$

Such a thing is the same as a morphism $A[1]^\bullet \to B^\bullet$ by our remarks above. This proves the following lemma.

Lemma 10.12.9. *Let $\mathscr{A}$ be an additive category. Suppose that $A^\bullet$ and $B^\bullet$ are cochain complexes. Given any morphism of cochain complexes $a : A^\bullet \to B^\bullet$ there is a bijection between the set of homotopies from a to a and $Mor_{CoCh(\mathscr{A})}(A^\bullet, B[-1]^\bullet)$. More generally, the set of homotopies between a and b is either empty or a principal homogenous space under the group $Mor_{CoCh(\mathscr{A})}(A^\bullet, B[-1]^\bullet)$.*

Proof. See above. □

Lemma 10.12.10. *Let $\mathscr{A}$ be an additive category. Let*

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$$

be a complex (!) of complexes. Suppose that we are given splittings $B^n = A^n \oplus C^n$ compatible with the maps in the displayed sequence. Let $s^n : C^n \to B^n$ and $\pi^n : B^n \to A^n$ be the corresponding maps. Then the family of morphisms

$$\pi^{n+1} \circ d_B^n \circ s^n : C^n \to A^{n+1}$$

define a morphism of complexes $\delta : C^\bullet \to A[1]^\bullet$.

Proof. Denote $i : A^\bullet \to B^\bullet$ and $q : B^\bullet \to C^\bullet$ the maps of complexes in the short exact sequence. Then $i^{n+1} \circ \pi^{n+1} \circ d_B^n \circ s^n = d_B^n \circ s^n - s^{n+1} \circ d_C^n$. Hence $i^{n+2} \circ d_A^{n+1} \circ \pi^{n+1} \circ d_B^n \circ s^n = d_B^{n+1} \circ (d_B^n \circ s^n - s^{n+1} \circ d_C^n) = -d_B^{n+1} \circ s^{n+1} \circ d_C^n$ as desired. □

Lemma 10.12.11. *Notation and assumptions as in Lemma 10.12.10 above. Assume in addition that $\mathscr{A}$ is abelian. The morphism of complexes $\delta : C^\bullet \to A[1]^\bullet$ induces the maps*

$$H^i(\delta) : H^i(C^\bullet) \longrightarrow H^i(A[1]^\bullet) = H^{i+1}(A^\bullet)$$

which occur in the long exact homology sequence associated to the short exact sequence of cochain complexes by Lemma 10.10.12.

Proof. Omitted. □

Lemma 10.12.12. *Notation and assumptions as in Lemma 10.12.10 above. Let* $\alpha : A^\bullet$, $\beta : B^\bullet \to C^n$ *be the given morphisms of complexes. Suppose* $(s')^n : C^n \to B^n$ *and* $(\pi')^n : B^n \to A^n$ *is a second choice of splittings. Write* $(s')^n = s^n + \alpha^n \circ h^n$ *and* $(\pi')^n = \pi^n + g^n \circ \beta^n$ *for some unique morphisms* $h^n : C^n \to A^n$ *and* $g^n : C^n \to A^n$. *Then*

1. $g^n = -h^n$, *and*
2. *the family of maps* $\{g^n : C^n \to A[1]^{n-1}\}$ *is a homotopy between* $\delta, \delta' : C^\bullet \to A[1]^\bullet$, *more precisely* $(\delta')^n = \delta^n + g^{n+1} \circ d_C^n + d_{A[1]}^{n-1} \circ g^n$.

Proof. As $(s')^n$ and $(\pi')^n$ are splittings we have $(\pi')^n \circ (s')^n = 0$. Hence

$$0 = (\pi^n + g^n \circ \beta^n) \circ (s^n + \alpha^n \circ h^n) = g^n \circ \beta^n \circ s^n + \pi^n \circ \alpha^n \circ h^n = g^n + h^n$$

which proves (1). We compute $(\delta')^n$ as follows

$$(\pi^{n+1} + g^{n+1} \circ \beta^{n+1}) \circ d_B^n \circ (s^n + \alpha^n \circ h^n) = \delta^n + g^{n+1} \circ d_C^n + d_A^n \circ h^n$$

Since $h^n = -g^n$ and since $d_{A[1]}^{n-1} = -d_A^n$ we conclude that (2) holds. $\square$

10.13. Filtrations

A nice reference for this material is [**Del71**, Section 1]. (Note that our conventions regarding abelian categories are different.)

Definition 10.13.1. Let $\mathcal{A}$ be an abelian category.

1. A *decreasing filtration* F on an object A is a family $(F^n A)_{n \in \mathbf{Z}}$ of subobjects of A such that
$$A \supset \ldots \supset F^n A \supset F^{n+1} A \supset \ldots \supset 0$$
2. A *filtered object of* $\mathcal{A}$ is pair (A, F) consisting of an object A of $\mathcal{A}$ and a decreasing filtration F on A.
3. A *morphism* $(A, F) \to (B, F)$ *of filtered objects* is given by a morphism $\varphi : A \to B$ of $\mathcal{A}$ such that $\varphi(F^i A) \subset F^i B$ for all $i \in \mathbf{Z}$.
4. The category of filtered objects is denoted $\text{Fil}(\mathcal{A})$.
5. Given a filtered object (A, F) and a subobject $X \subset A$ the *induced filtration* on X is the filtration with $F^n X = X \cap F^n A$.
6. Given a filtered object (A, F) and a surjection $\pi : A \to Y$ the *quotient filtration* is the filtration with $F^n Y = \pi(F^n A)$.
7. A filtration F on an object A is said to be *finite* if there exist n, m such that $F^n A = A$ and $F^m A = 0$.
8. The filtration on a filtered object (A, F) is said to be *separated* if $\bigcap_i F^i A = 0$ and *exhaustive* if $\bigcup F^i A = A$.

By abuse of notation we say that a morphism $f : (A, F) \to (B, F)$ of filtered objects is *injective* if $f : A \to B$ is injective in the abelian category $\mathcal{A}$. Similarly we say f is *surjective* if $f : A \to B$ is surjective in the category $\mathcal{A}$. Being injective (resp. surjective) is equivalent to being a monomorphism (resp. epimorphism) in $\text{Fil}(\mathcal{A})$. By Lemma 10.13.2 this is also equivalent to having zero kernel (resp. cokernel).

Lemma 10.13.2. *Let* $\mathcal{A}$ *be an abelian category. The category of filtered objects* $\text{Fil}(\mathcal{A})$ *has the following properties:*

1. *It is an additive category.*
2. *It has a zero object.*
3. *It has kernels and cokernels, images and coimages.*
4. *In general it is not an abelian category.*

Proof. It is clear that $\text{Fil}(\mathcal{A})$ is additive with direct sum given by $(A, F) \oplus (B, F) = (A \oplus B, F)$ where $F^p(A \oplus B) = F^p A \oplus F^p B$. The kernel of a morphism $f : (A, F) \to (B, F)$ of filtered objects is the injection $\text{Ker}(f) \subset A$ where $\text{Ker}(f)$ is endowed with the induced filtration. The cokernel of a morphism $f : A \to B$ of filtered objects is the surjection $B \to \text{Coker}(f)$ where $\text{Coker}(f)$ is endowed with the quotient filtration. Since all kernels and cokernels exist, so do all coimages and images. See Example 10.3.11 for the last statement. $\square$

Definition 10.13.3. Let $\mathcal{A}$ be an abelian category. A morphism $f : A \to B$ of filtered objects of $\mathcal{A}$ is said to be *strict* if $f(F^i A) = f(A) \cap F^i B$ for all $i \in \mathbf{Z}$.

This also equivalent to requiring that $f^{-1}(F^i B) = F^i A + \text{Ker}(f)$ for all $i \in \mathbf{Z}$. We characterize strict morphisms as follows.

Lemma 10.13.4. *Let $\mathcal{A}$ be an abelian category. Let $f : A \to B$ be a morphism of filtered objects of $\mathcal{A}$. The following are equivalent*

(1) *f is strict,*
(2) *the morphism $Coim(f) \to Im(f)$ of Lemma 10.3.10 is an isomorphism.*

Proof. Note that $\text{Coim}(f) \to \text{Im}(f)$ is an isomorphism of objects of $\mathcal{A}$, and that part (2) signifies that it is an isomorphism of filtered objects. By the description of kernels and cokernels in the proof of Lemma 10.13.2 we see that the filtration on $\text{Coim}(f)$ is the quotient filtration coming from $A \to \text{Coim}(f)$. Similarly, the filtration on $\text{Im}(f)$ is the induced filtration coming from the injection $\text{Im}(f) \to B$. The definition of strict is exactly that the quotient filtration is the induced filtration. $\square$

Lemma 10.13.5. *Let $\mathcal{A}$ be an abelian category. A composition of strict morphisms of filtered objects is strict.*

Proof. Let $f : A \to B$, $g : B \to C$ be strict morphisms of filtered objects. Then

$$\begin{aligned} g(f(F^p A)) &= g(f(A) \cap F^p B) \\ &\supset g(f(A)) \cap g(F^p(B)) \\ &= (g \circ f)(A) \cap (g(B) \cap F^p C) \\ &= (g \circ f)(A) \cap F^p C. \end{aligned}$$

The inclusion $g(f(F^p A)) \subset (g \circ f)(A) \cap F^p C$ is always true. $\square$

Lemma 10.13.6. *Let $\mathcal{A}$ be an abelian category. Let $f : A \to B$ be a strict monomorphism of filtered objects. Let $g : A \to C$ be a morphism of filtered objects. Then $f \oplus g : A \to B \oplus C$ is a strict monomorphism.*

Proof. Clear from the definitions. $\square$

Lemma 10.13.7. *Let $\mathcal{A}$ be an abelian category. Let $f : B \to A$ be a strict epimorphism of filtered objects. Let $g : C \to A$ be a morphism of filtered objects. Then $f \oplus g : B \oplus C \to A$ is a strict epimorphism.*

Proof. Clear from the definitions. $\square$

Lemma 10.13.8. *Let $\mathcal{A}$ be an abelian category. Let (A, F), (B, F) be filtered objects. Let $u : A \to B$ be a morphism of filtered objects. If u is injective then u is strict if and only if the filtration on A is the induced filtration. If u is surjective then u is strict if and only if the filtration on B is the quotient filtration.*

Proof. This is immediate from the definition. □

The following lemma says that subobjects of a filtered object have a well defined filtration independent of a choice of writing the object as a cokernel.

Lemma 10.13.9. *Let $\mathcal{A}$ be an abelian category. Let (A, F) be a filtered object of $\mathcal{A}$. Let $X \subset Y \subset A$ be subobjects of A. On the object*

$$Y/X = Ker(A/X \to A/Y)$$

the quotient filtration coming from the induced filtration on Y and the induced filtration coming from the quotient filtration on A/X agree. Any of the morphisms $X \to Y$, $X \to A$, $Y \to A$, $Y \to A/X$, $Y \to Y/X$, $Y/X \to A/X$ are strict (with induced/quotient filtrations).

Proof. The quotient filtration Y/X is given by $F^p(Y/X) = F^pY/(X \cap F^pY) = F^pY/F^pX$ because $F^pY = Y \cap F^pA$ and $F^pX = X \cap F^pA$. The induced filtration from the injection $Y/X \to A/X$ is given by

$$\begin{aligned} F^p(Y/X) &= Y/X \cap F^p(A/X) \\ &= Y/X \cap (F^pA + X)/X \\ &= (Y \cap F^pA)/(X \cap F^pA) \\ &= F^pY/F^pX. \end{aligned}$$

Hence the first statement of the lemma. The proof of the other cases is similar. □

Lemma 10.13.10. *Let $\mathcal{A}$ be an abelian category. Let $A, B, C \in Fil(\mathcal{A})$. Let $f : A \to B$ and $g : A \to C$ be morphisms Then there exists a push out*

$$\begin{tikzcd} A \arrow[r, "f"'] \arrow[d, "g"'] & B \arrow[d, "g'"] \\ C \arrow[r, "f'"] & C \amalg_A B \end{tikzcd}$$

in $Fil(\mathcal{A})$. If f is strict, so is f'.

Proof. Set $C \amalg_A B$ equal to $\text{Coker}((1, -1) : A \to C \oplus B)$ in $\text{Fil}(\mathcal{A})$. This cokernel exists, by Lemma 10.13.2. It is a pushout, see Example 10.3.17. Note that $F^p(C \times_A B)$ is the image of $F^pC \oplus F^pB$. Hence

$$(f')^{-1}(F^p(C \times_A B)) = g(f^{-1}(F^pB))) + F^pC$$

Whence the last statement. □

Lemma 10.13.11. *Let $\mathcal{A}$ be an abelian category. Let $A, B, C \in Fil(\mathcal{A})$. Let $f : B \to A$ and $g : C \to A$ be morphisms Then there exists a push out*

$$\begin{tikzcd} B \times_A C \arrow[r, "f'"'] \arrow[d, "g'"'] & B \arrow[d, "g"] \\ C \arrow[r, "f"] & A \end{tikzcd}$$

in $Fil(\mathcal{A})$. If f is strict, so is f'.

Proof. This lemma is dual to Lemma 10.13.10. □

Definition 10.13.12. Let $\mathcal{A}$ be an abelian category. A *graded object of* $\mathcal{A}$ is pair (A, k) consisting of an object A of $\mathcal{A}$ and a direct sum decomposition

$$A = \bigoplus\nolimits_{i \in \mathbf{Z}} k^i A$$

by subobjects indexed by $\mathbf{Z}$. A *morphism* $(A, k) \to (B, k)$ *of graded objects* is given by a morphism $\varphi : A \to B$ of $\mathcal{A}$ such that $\varphi(k^i A) \subset k^i B$ for all $i \in \mathbf{Z}$. The category of graded objects is denoted $\mathrm{Gr}(\mathcal{A})$.

With our definitions an abelian category does not necessarily have all (countable) direct sums. Of course the definition above still makes sense, but may be a little misleading in case $\mathcal{A}$ does not have infinite direct sums. For example, if $\mathcal{A} = \mathrm{Vect}_k$ is the category of finite dimensional vector spaces over a field k, then $\mathrm{Gr}(\mathrm{Vect}_k)$ is the category of finite dimensional vector spaces with a given gradation, and not the category of graded vector spaces all of whose graded pieces are finite dimensional.

Lemma 10.13.13. *Let $\mathcal{A}$ be an abelian category. The category of graded objects $Gr(\mathcal{A})$ is abelian.*

Proof. Omitted. □

Let $\mathcal{A}$ be an abelian category. Let (A, F) be a filtered object of $\mathcal{A}$. We denote $\mathrm{gr}^p_F(A) = \mathrm{gr}^p(A)$ the object $F^pA/F^{p+1}A$ of $\mathcal{A}$. This defines an additive functor

$$\mathrm{gr}^p : \mathrm{Fil}(\mathcal{A}) \longrightarrow \mathcal{A}, \quad (A, F) \longmapsto \mathrm{gr}^p(A).$$

Assume $\mathcal{A}$ has countable direct sums. For (A, F) in $\mathrm{Fil}(\mathcal{A})$ we may set

$$\mathrm{gr}(A) = \bigoplus\nolimits_{p \in \mathbf{Z}} \mathrm{gr}^p(A) = \bigoplus\nolimits_{p \in \mathbf{Z}} F^pA/F^{p+1}A.$$

This defines an additive functor

$$\mathrm{gr} : \mathrm{Fil}(\mathcal{A}) \longrightarrow \mathrm{Gr}(\mathcal{A}), \quad (A, F) \longmapsto \mathrm{gr}(A).$$

If $\mathcal{A}$ does not have all countable direct sums this functor is still defined on the subcategory of $\mathrm{Fil}(\mathcal{A})$ consisting of all filtered objects whose filtrations are finite.

Lemma 10.13.14. *Let $\mathcal{A}$ be an abelian category.*

(1) *Let A be a filtered object and $X \subset A$. Then for each p the sequence*
$$0 \to gr^p(X) \to gr^p(A) \to gr^p(A/X) \to 0$$
is exact (with induced filtration on X and quotient filtration on A/X).

(2) *Let $f : A \to B$ be a morphism of filtered objects of $\mathcal{A}$. Then for each p the sequences*
$$0 \to gr^p(Ker(f)) \to gr^p(A) \to gr^p(Coim(f)) \to 0$$
and
$$0 \to gr^p(Im(f)) \to gr^p(B) \to gr^p(Coker(f)) \to 0$$
are exact.

Proof. We have $F^{p+1}X = X \cap F^{p+1}A$, hence map $\mathrm{gr}^p(X) \to \mathrm{gr}^p(A)$ is injective. Dually the map $\mathrm{gr}^p(A) \to \mathrm{gr}^p(A/X)$ is surjective. The kernel of $F^pA/F^{p+1}A \to A/X + F^{p+1}A$ is clearly $F^{p+1}A + X \cap F^pA/F^{p+1}A = F^pX/F^{p+1}X$ hence exactness in the middle. The two short exact sequence of (2) are special cases of the short exact sequence of (1). □

Lemma 10.13.15. *Let $\mathcal{A}$ be an abelian category. Let $f : A \to B$ be a morphism of finite filtered objects of $\mathcal{A}$. The following are equivalent*

(1) *f is strict,*
(2) *the morphism $Coim(f) \to Im(f)$ is an isomorphism,*
(3) *$gr(Coim(f)) \to gr(Im(f))$ is an isomorphism,*
(4) *the sequence $gr(Ker(f)) \to gr(A) \to gr(B)$ is exact,*
(5) *the sequence $gr(A) \to gr(B) \to gr(Coker(f))$ is exact, and*
(6) *the sequence*

$$0 \to gr(Ker(f)) \to gr(A) \to gr(B) \to gr(Coker(f)) \to 0$$

is exact.

Proof. The equivalence of (1) and (2) is Lemma 10.13.4. By Lemma 10.13.14 we see that (4), (5), (6) imply (3) and that (3) implies (4), (5), (6). Hence it suffices to show that (3) implies (2). Thus we have to show that if $f : A \to B$ is an injective and surjective map of finite filtered objects which induces and isomorphism $\text{gr}(A) \to \text{gr}(B)$, then f induces an isomorphism of filtered objects. In other words, we have to show that $f(F^pA) = F^pB$ for all p. As the filtrations are finite we may prove this by descending induction on p. Suppose that $f(F^{p+1}A) = F^{p+1}B$. Then commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & F^{p+1}A \arrow[r] \arrow[d, "f"] & F^pA \arrow[r] \arrow[d, "f"] & \text{gr}^p(A) \arrow[r] \arrow[d, "\text{gr}^p(f)"] & 0 \\
0 \arrow[r] & F^{p+1}B \arrow[r] & F^pB \arrow[r] & \text{gr}^p(B) \arrow[r] & 0
\end{tikzcd}$$

and the five lemma imply that $f(F^pA) = F^pB$. □

Lemma 10.13.16. *Let $\mathcal{A}$ be an abelian category. Let $A \to B \to C$ be a complex of filtered objects of $\mathcal{A}$. Assume $\alpha : A \to B$ and $\beta : B \to C$ are strict morphisms of filtered objects. Then $gr(Ker(\beta)/Im(\alpha)) = Ker(gr(\beta))/Im(gr(\alpha)))$.*

Proof. This follows formally from Lemma 10.13.14 and the fact that $\text{Coim}(\alpha) \cong \text{Im}(\alpha)$ and $\text{Coim}(\beta) \cong \text{Im}(\beta)$ by Lemma 10.13.4. □

Lemma 10.13.17. *Let $\mathcal{A}$ be an abelian category. Let $A \to B \to C$ be a complex of filtered objects of $\mathcal{A}$. Assume A, B, C have finite filtrations and that $gr(A) \to gr(B) \to gr(C)$ is exact. Then*

(1) *for each $p \in \mathbf{Z}$ the sequence $gr^p(A) \to gr^p(B) \to gr^p(C)$ is exact,*
(2) *for each $p \in \mathbf{Z}$ the sequence $F^p(A) \to F^p(B) \to F^p(C)$ is exact,*
(3) *for each $p \in \mathbf{Z}$ the sequence $A/F^p(A) \to B/F^p(B) \to C/F^p(C)$ is exact,*
(4) *the maps $A \to B$ and $B \to C$ are strict, and*
(5) *$A \to B \to C$ is exact (as a sequence in $\mathcal{A}$).*

Proof. Part (1) is immediate from the definitions. We will prove (3) by induction on the length of the filtrations. If each of A, B, C has only one nonzero graded part, then (3) holds as $\text{gr}(A) = A$, etc. Let n be the largest integer such that at least one of F^nA, F^nB, F^nC is nonzero. Set $A' = A/F^nA$, $B' = B/F^nB$, $C' = C/F^nC$ with induced filtrations. Note that $\text{gr}(A) = F^nA \oplus \text{gr}(A')$ and similarly for B and C. The induction hypothesis applies to $A' \to B' \to C'$, which implies that $A/F^p(A) \to B/F^p(B) \to C/F^p(C)$ is exact for $p \geq n$. To conclude the same for $p = n + 1$, i.e., to prove that $A \to B \to C$ is exact we use the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & F^nA \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F^nB \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F^nC \arrow[r] & C \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

whose rows are short exact sequences of objects of $\mathcal{A}$. The proof of (2) is dual. Of course (5) follows from (2).

To prove (4) denote $f : A \to B$ and $g : B \to C$ the given morphisms. We know that $f(F^p(A)) = \mathrm{Ker}(F^p(B) \to F^p(C))$ by (2) and $f(A) = \mathrm{Ker}(g)$ by (5). Hence $f(F^p(A)) = \mathrm{Ker}(F^p(B) \to F^p(C)) = \mathrm{Ker}(g) \cap F^p(B) = f(A) \cap F^p(B)$ which proves that f is strict. The proof that g is strict is dual to this. □

10.14. Spectral sequences

A nice discussion of spectral sequences may be found in [**Eis95**]. See also [**McC01**], [**Lan02**], etc.

Definition 10.14.1. Let $\mathcal{A}$ be an abelian category.

(1) A *spectral sequence in* $\mathcal{A}$ is given by a system $(E_r, d_r)_{r\geq 1}$ where each E_r is an object of $\mathcal{A}$, each $d_r : E_r \to E_r$ is a morphism such that $d_r \circ d_r = 0$ and $E_{r+1} = \mathrm{Ker}(d_r)/\mathrm{Im}(d_r)$ for $r \geq 1$.

(2) A *morphism of spectral sequences* $f : (E_r, d_r)_{r\geq 1} \to (E'_r, d'_r)_{r\geq 1}$ is given by a family of morphisms $f_r : E_r \to E'_r$ such that $f_r \circ d_r = d'_r \circ f_r$ and such that f_{r+1} is the morphism induced by f_r via the identifications $E_{r+1} = \mathrm{Ker}(d_r)/\mathrm{Im}(d_r)$ and $E'_{r+1} = \mathrm{Ker}(d'_r)/\mathrm{Im}(d'_r)$.

We will sometimes loosen this definition somewhat and allow E_{r+1} to be an object with a given isomorphism $E_{r+1} \to \mathrm{Ker}(d_r)/\mathrm{Im}(d_r)$. In addition we sometimes have a system $(E_r, d_r)_{r\geq r_0}$ for some r_0 satsifying the properties of the definition above for indices $\geq r$. We will also call this a spectral sequence since by a simple renumbering it falls under the definition anyway. In fact, sometimes it makes sense to allow $r_0 = 0$ or even $r_0 = -1$ due to conventions in the literature.

Given a spectral sequence $(E_r, d_r)_{r\geq 1}$ we define

$$0 = B_1 \subset B_2 \subset \ldots \subset B_r \subset \ldots \subset Z_r \subset \ldots \subset Z_2 \subset Z_1 = E_1$$

by the following simple procedure. Set $B_2 = \mathrm{Im}(d_1)$ and $Z_2 = \mathrm{Ker}(d_1)$. Then it is clear that $d_2 : Z_2/B_2 \to Z_2/B_2$. Hence we can define B_3 as the unique subobject of E_1 containing B_2 such that B_3/B_2 is the image of d_2. Similarly we can define Z_3 as the unique subobject of E_1 containing B_2 such that Z_3/B_2 is the kernel of d_2. And so on and so forth. In particular we have

$$E_r = Z_r/B_r$$

for all $r \geq 1$. I case the spectral sequence starts at $r = r_0$ then we can similarly construct B_i, Z_i as subobjects in E_{r_0}.

Definition 10.14.2. Let $\mathcal{A}$ be an abelian category. Let $(E_r, d_r)_{r\geq 1}$ be a spectral sequence.

(1) If the subobjects $Z_\infty = \bigcap Z_r$ and $B_\infty = \bigcup B_r$ of E_1 exist then we define the *limit* of the spectral sequence to be the object

$$E_\infty = Z_\infty / B_\infty.$$

(2) We say that the spectral sequence *collapses at E_r*, or *degenerates at E_r* if the differentials $d_r, d_{r+1}, \ldots$ are all zero.

Note that if the spectral sequence collapses at E_r, then we have $E_r = E_{r+1} = \ldots = E_\infty$ (and the limit exists of course). Also, almost any abelian category we will encounter has countable sums and intersections.

10.15. Spectral sequences: exact couples

Definition 10.15.1. Let $\mathcal{A}$ be an abelian category.

(1) An *exact couple* is a datum (A, E, α, f, g) where A, E are objects of $\mathcal{A}$ and α, f, g are morphisms as in the following diagram

$$\begin{tikzcd} A \arrow[rr, "\alpha"'] & & A \arrow[dl, "g"] \\ & E \arrow[ul, "f"] & \end{tikzcd}$$

with the property that the kernel of each arrow is the image of its predecessor. So $\mathrm{Ker}(\alpha) = \mathrm{Im}(f)$, $\mathrm{Ker}(f) = \mathrm{Im}(g)$, and $\mathrm{Ker}(g) = \mathrm{Im}(\alpha)$.

(2) A *morphism of exact couples* $t : (A, E, \alpha, f, g) \to (A', E', \alpha', f', g')$ is given by morphisms $t_A : A \to A'$ and $t_E : E \to E'$ such that $\alpha' \circ t_A = t_A \circ \alpha$, $f' \circ t_E = t_A \circ f$, and $g' \circ t_A = t_E \circ g$.

Lemma 10.15.2. *Let $\mathcal{A}$ be an abelian category. Let (A, E, α, f, g) be an exact couple. Set*

(1) $d = g \circ f : E \to E$ *so that* $d \circ d = 0$,
(2) $E' = Ker(d)/Im(d)$,
(3) $A' = Im(\alpha)$,
(4) $\alpha' : A' \to A'$ *induced by* α,
(5) $f' : E' \to A'$ *induced by* f,
(6) $g' : A' \to E'$ *induced by* ``$g \circ \alpha^{-1}$''.

Then we have

(1) $Ker(d) = f^{-1}(ker(g)) = f^{-1}(Im(\alpha))$,
(2) $Im(d) = g(Im(f)) = g(Ker(\alpha))$,
(3) $(A', E', \alpha', f', g')$ *is an exact couple.*

Proof. Omitted. □

Hence it is clear that given an exact couple (A, E, α, f, g) we get a spectral sequence by setting $E_1 = E$, $d_1 = d$, $E_2 = E'$, $d_2 = d' = g' \circ f'$, $E_3 = E''$, $d_3 = d'' = g'' \circ f''$, and so on.

Definition 10.15.3. Let $\mathcal{A}$ be an abelian category. Let (A, E, α, f, g) be an exact couple. The *spectral sequence associated to the exact couple* is the spectral sequence $(E_r, d_r)_{r \geq 1}$ with $E_1 = E$, $d_1 = d$, $E_2 = E'$, $d_2 = d' = g' \circ f'$, $E_3 = E''$, $d_3 = d'' = g'' \circ f''$, and so on.

Lemma 10.15.4. *Let $\mathcal{A}$ be an abelian category. Let (A, E, α, f, g) be an exact couple. Let $(E_r, d_r)_{r \geq 1}$ be the spectral sequence associated to the exact couple. In this case we have*

$$0 = B_1 \subset \ldots \subset B_{r+1} = g(ker(\alpha^r)) \subset \ldots \subset Z_{r+1} = f^{-1}(Im(\alpha^r)) \subset \ldots \subset Z_1 = E$$

and the map $d_{r+1} : E_{r+1} \to E_{r+1}$ is described by the following rule: For any (test) object T of $\mathcal{A}$ and any elements $x : T \to Z_{r+1}$ and $y : T \to A$ such that $f \circ x = \alpha^r \circ y$ we have

$$d_r \circ \overline{x} = \overline{g \circ y}$$

where $\overline{x} : T \to E_{r+1}$ is the induced morphism.

Proof. Omitted. $\square$

Note that in the situation of the lemma we obviously have

$$B_\infty = g\left(\bigcup\nolimits_r \mathrm{Ker}(\alpha^r)\right) \subset Z_\infty = f^{-1}\left(\bigcap\nolimits_r \mathrm{Im}(\alpha^r)\right)$$

provided this exist and in this case $E_\infty = Z_\infty / B_\infty$.

10.16. Spectral sequences: differential objects

Definition 10.16.1. Let $\mathcal{A}$ be an abelian category. A *differential object* of $\mathcal{A}$ is a pair (A, d) consisting of an object A of $\mathcal{A}$ endowed with a selfmap d such that $d \circ d = 0$. A *morphism of differential objects* $(A, d) \to (B, d)$ is given by a morphism $\alpha : A \to B$ such that $d \circ \alpha = \alpha \circ d$.

Lemma 10.16.2. *Let $\mathcal{A}$ be an abelian category. The category of differential objects of $\mathcal{A}$ is abelian.*

Proof. Omitted. $\square$

Definition 10.16.3. For a differential object (A, d) we denote

$$H(A, d) = \mathrm{Ker}(d)/\mathrm{Im}(d)$$

its *homology*.

Lemma 10.16.4. *Let $\mathcal{A}$ be an abelian category. Let $0 \to (A, d) \to (B, d) \to (C, d) \to 0$ be a short exact sequence of differential objects. Then we get an exact homology sequence*

$$\ldots \to H(C, d) \to H(A, d) \to H(B, d) \to H(C, d) \to \ldots$$

Proof. Apply Lemma 10.10.12 to the short exact sequence of complexes

$$\begin{array}{ccccccccc}
0 & \to & A & \to & B & \to & C & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A & \to & B & \to & C & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A & \to & B & \to & C & \to & 0
\end{array}$$

$\square$

We come to an important example of a spectral sequence. Let $\mathcal{A}$ be an abelian category. Let (A, d) be a differential object of $\mathcal{A}$. Let $\alpha : (A, d) \to (A, d)$ be an endomorphism of this differential object. If we assume α injective, then we get a short exact sequence

$$0 \to (A, d) \to (A, d) \to (A/\alpha A, d) \to 0$$

of differential objects. By the Lemma 10.16.4 we get an exact couple

$$\begin{tikzcd}
H(A, d) \arrow[rr, "\overline{\alpha}"'] & & H(A, d) \arrow[dl, "g"] \\
& H(A/\alpha A, d) \arrow[ul, "f"] &
\end{tikzcd}$$

where g is the canonical map and f is the map defined in the snake lemma. Thus we get an associated spectral sequence! Since in this case we have $E_1 = H(A/\alpha A, d)$ we see that it makes sense to define $E_0 = A/\alpha A$ and $d_0 = d$. In other words, we start the spectral sequence with $r = 0$. According to our conventions in Section 10.14 we define a sequence of subobjects

$$0 = B_0 \subset \ldots \subset B_r \subset \ldots \subset Z_r \subset \ldots \subset Z_0 = E_0$$

with the property that $E_r = Z_r/B_r$. Namely we have for $r \geq 1$ that

(1) B_r is the image of $(\alpha^{r-1})^{-1}(dA)$ under the natural map $A \to A/\alpha A$,
(2) Z_r is the image of $d^{-1}(\alpha^r A)$ under the natural map $A \to A/\alpha A$, and
(3) $d_r : E_r \to E_r$ is given as follows: given an element $z \in Z_r$ choose an element $y \in A$ such that $d(z) = \alpha^r(y)$. Then $d_r(z + B_r + \alpha A) = y + B_r + \alpha A$.

Warning: It is not necessarily the case that $\alpha A \subset (\alpha^{r-1})^{-1}(dA)$, nor $\alpha A \subset d^{-1}(\alpha^r A)$. It is true that $(\alpha^{r-1})^{-1}(dA) \subset d^{-1}(\alpha^r A)$. We have

$$E_r = \frac{d^{-1}(\alpha^r A) + \alpha A}{(\alpha^{r-1})^{-1}(dA) + \alpha A}.$$

It is not hard to verify directly that (1) -- (3) give a spectral sequence.

Definition 10.16.5. Let $\mathcal{A}$ be an abelian category. Let (A, d) be a differential object of $\mathcal{A}$. Let $\alpha : A \to A$ be an injective selfmap of A which commutes with d. The *spectral sequence associated to* (A, d, α) is the spectral sequence $(E_r, d_r)_{r\geq 0}$ described above.

10.17. Spectral sequences: filtered differential objects

Definition 10.17.1. Let $\mathcal{A}$ be an abelian category. A *filtered differential object* (K, F, d) is a filtered object (K, F) of $\mathcal{A}$ endowed with an endomorphism $d : (K, F) \to (K, F)$ whose square is zero: $d \circ d = 0$.

Let $\mathcal{A}$ be an abelian category. Let (K, F, d) be a filtered differential object of $\mathcal{A}$. Note that each $F^n K$ is a differential object by itself. Assume $\mathcal{A}$ has countable direct sums. In this case set $A = \bigoplus F^n K$ and endow it with a differential d by using d on each summand. Consider the map

$$\alpha : A \to A$$

which maps the summand $F^n A$ into the summand $F^{n-1} A$. This is clearly an injective morphism of differential modules $\alpha : (A, d) \to (A, d)$. Hence, by Definition 10.16.5 we get a spectral sequence. We will call this *the spectral sequence associated to the filtered differential object* (K, F, d).

Let us figure out the terms of this spectral sequence. First, note that $A/\alpha A = \mathrm{gr}(K)$ endowed with its differential $d = \mathrm{gr}(d)$. Hence we see that

$$E_0 = \mathrm{gr}(K), \quad d_0 = \mathrm{gr}(d).$$

Hence the homology of the graded differential object $\mathrm{gr}(K)$ is the next term:

$$E_1 = H(\mathrm{gr}(K), \mathrm{gr}(d)).$$

In addition we see that E_0 is a graded object of $\mathcal{A}$ and that d_0 is compatible with the grading. Hence clearly E_1 is a graded object as well. But it turns out that the differential d_1 does not preserve this grading; instead it shifts the degree by 1.

To work this out precisely, we define

$$Z_r^p = \frac{F^p K \cap d^{-1}(F^{p+r} K) + F^{p+1} K}{F^{p+1} K}$$

and

$$B_r^p = \frac{F^pK \cap d(F^{p-r+1}K) + F^{p+1}K}{F^{p+1}K}.$$

This notation, allthough quite natural, seems to be different from the notation in most places in the literature. Perhaps it does not matter, since the literature does not seem to have a consistent choice of notation either. With these choices we see that $B_r \subset E_0$, resp. $Z_r \subset E_0$ (as defined in Section 10.16) is equal to $\bigoplus_p B_r^p$, resp. $\bigoplus_p Z_r^p$. Hence if we define

$$E_r^p = Z_r^p/B_r^p$$

for $r \geq 0$ and $p \in \mathbf{Z}$, then we have $E_r = \bigoplus_p E_r^p$. We can define a differential $d_r^p : E_r^p \to E_r^{p+r}$ by the rule

$$z + F^{p+1}K \longmapsto dz + F^{p+r+1}K$$

where $z \in F^pK \cap d^{-1}(F^{p+r}K)$.

Lemma 10.17.2. *Let $\mathcal{A}$ be an abelian category. Let (K, F, d) be a filtered differential object of $\mathcal{A}$. Assume $\mathcal{A}$ has countable direct sums. The spectral sequence $(E_r, d_r)_{r\geq 0}$ associated to (K, F, d) has terms*

$$E_r = \bigoplus\nolimits_{p\in\mathbf{Z}} E_r^p, \quad d_r = \bigoplus\nolimits_{p\in\mathbf{Z}} d_r^p.$$

Furthermore, we have $E_0^p = gr^pK$, $d_0 = gr(d)$, and $E_1^p = H(gr^p(K), d)$.

Proof. Follows from the discussion above. □

Lemma 10.17.3. *Let $\mathcal{A}$ be an abelian category. Let (K, F, d) be a filtered differential object of $\mathcal{A}$. Assume $\mathcal{A}$ has countable direct sums. The spectral sequence $(E_r, d_r)_{r\geq 0}$ associated to (K, F, d) has*

$$d_1^p : E_1^p = H(gr^p(K)) \longrightarrow E_1^{p+1} = H(gr^{p+1}(K))$$

equal to the boundary map in homology associated to the short exact sequence of differential objects

$$0 \to gr^{p+1}(K) \to F^pK/F^{p+2}K \to gr^{p+1}(K) \to 0.$$

Proof. Omitted. □

Definition 10.17.4. Let $\mathcal{A}$ be an abelian category. Let (K, F, d) be a filtered differential object of $\mathcal{A}$. The *induced filtration* on $H(K, d)$ is the filtration defined by $F^pH(K, d) = \mathrm{Im}(H(F^pK, d) \to H(K, d))$.

Lemma 10.17.5. *Let $\mathcal{A}$ be an abelian category. Let (K, F, d) be a filtered differential object of $\mathcal{A}$. The associated graded $gr(H(K))$ of the cohomology of K is a graded subquotient of the graded object $E_\infty = \bigoplus E_\infty^p$.*

Proof. Recall that $E_\infty = Z_\infty/B_\infty$ by definition, with $B_\infty = \bigcup B_r$ and $Z_\infty = \bigcap Z_r$. Hence $E_\infty = \bigoplus E_\infty^p$ with $E_\infty^p = Z_\infty^p/B_\infty^p$ with $B_\infty^p = \bigcup B_r^p$ and $Z_\infty^p = \bigcap Z_r^p$. Thus

$$E_\infty^p = \frac{\bigcap_r (F^pK \cap d^{-1}(F^{p+r}K) + F^{p+1}K)}{\bigcup_r (F^pK \cap d(F^{p-r+1}K) + F^{p+1}K)}.$$

On the other hand, we have

$$\mathrm{gr}^p H(K) = \frac{\mathrm{Ker}(d) \cap F^pK + F^{p+1}K}{\mathrm{Im}(d) \cap F^pK + F^{p+1}K}$$

The result follows since

$$\mathrm{Ker}(d) \cap F^pK + F^{p+1}K \subset \bigcup\nolimits_r \left(F^pK \cap d^{-1}(F^{p+r}K) + F^{p+1}K\right) \tag{10.17.5.1}$$

and

$$\bigcap\nolimits_r \left(F^pK \cap d(F^{p-r+1}K) + F^{p+1}K\right) \subset \mathrm{Im}(d) \cap F^pK + F^{p+1}K. \tag{10.17.5.2}$$

□

Definition 10.17.6. Let $\mathcal{A}$ be an abelian category. Let (K, F, d) be a filtered differential object of $\mathcal{A}$. We say the spectral sequence associated to (K, F, d) *converges* if $\mathrm{gr}(H(K)) = E_\infty$ via Lemma 10.17.5. In this case we also say that $(E_r, d_r)_{r\geq 0}$ *abuts to* or *converges to* $H(K)$.

In the literature one finds more refined notions distinguishing between ``weakly converging", ``abutting" and ``converging". Namely, one can require the filtration on $H(K)$ to be either ``arbitrary", or ``exhaustive and separated", or ``exhaustive and complete" in addition to the condition that $\mathrm{gr}(H(K)) = E_\infty$. We try to avoid introducing this notation by simply adding the relevant information in the statements of the results.

Lemma 10.17.7. *Let $\mathcal{A}$ be an abelian category. Let (K, F, d) be a filtered differential object of $\mathcal{A}$. The associated spectral sequence converges if and only if for every $p \in \mathbf{Z}$ we have equality in equations (10.17.5.2) and (10.17.5.1).*

Proof. Immediate from the discussions above. □

10.18. Spectral sequences: filtered complexes

Definition 10.18.1. Let $\mathcal{A}$ be an abelian category. A *filtered complex $K^\bullet$ of $\mathcal{A}$* is a complex of $\mathrm{Fil}(\mathcal{A})$ (see Definition 10.13.1).

We will denote the filtration on the objects by F. Thus F^pK^n denotes the pth step in the filtration of the nth term of the complex. Note that each $F^pK^\bullet$ is a complex of $\mathcal{A}$. Hence we could also have defined a filtered complex as a filtered object in the (abelian) category of complexes of $\mathcal{A}$. In particular $\mathrm{gr}K^\bullet$ is a graded object of the category of complexes of $\mathcal{A}$.

Let us denote d the differential of K. Forgetting the grading we can think of $\bigoplus K^n$ as a filtered differential object of $\mathcal{A}$. Hence according to Section 10.17 we obtain a spectral sequence $(E_r, d_r)_{r\geq 0}$. In this section we work out the terms of this spectral sequence, and we endow the terms of this spectral seqeunce with additional structure coming from the grading of K.

First we point out that $E_0^p = \mathrm{gr}^pK^\bullet$ is a complex and hence is graded. Thus E_0 is bigraded in a natural way. It is customary to use the bigrading

$$E_0 = \bigoplus\nolimits_{p,q} E_0^{p,q}, \quad E_0^{p,q} = \mathrm{gr}^pK^{p+q}$$

The idea is that $p+q$ should be thought of as the *total degree* of the (co)homology classes. Also, p is called the *filtration degree*, and q is called the *complementary degree*. The differential d_0 is compatible with this bigrading in the following way

$$d_0 = \bigoplus d_0^{p,q}, \quad d_0^{p,q} : E_0^{p,q} \to E_0^{p,q+1}.$$

Namely, d_0^p is just the differential on the complex $\mathrm{gr}^pK^\bullet$ (which occurs as gr^pE_0 just shifted a bit).

To go further we identify the objects B_r^p and Z_r^p introduced in Section 10.17 as graded objects and we work out the corresponding decompositions of the differentials. We do this in a completely straightforward manner, but again we warn the reader that our notation is not the same as notation found elsewhere. We define

$$Z_r^{p,q} = \frac{F^pK^{p+q} \cap d^{-1}(F^{p+r}K^{p+q+1}) + F^{p+1}K^{p+q}}{F^{p+1}K^{p+q}}$$

and

$$B_r^{p,q} = \frac{F^pK^{p+q} \cap d(F^{p-r+1}K^{p+q-1}) + F^{p+1}K^{p+q}}{F^{p+1}K^{p+q}}.$$

and of course $E_r^{p,q} = Z_r^{p,q}/B_r^{p,q}$. With these definitions it is completely clear that $Z_r^p = \bigoplus_q Z_r^{p,q}$, $B_r^p = \bigoplus_q B_r^{p,q}$, and $E_r^p = \bigoplus_q E_r^{p,q}$. Moreover,

$$0 \subset \ldots \subset B_r^{p,q} \subset \ldots \subset Z_r^{p,q} \subset \ldots \subset E_0^{p,q}$$

and hence it makes sense to define $Z_\infty^{p,q} = \bigcap_r Z_r^{p,q}$ and $B_\infty^{p,q} = \bigcup_r B_r^{p,q}$ and $E_\infty^{p,q} = Z_\infty^{p,q}/B_\infty^{p,q}$. Also, the map d_r^p decomposes as the direct sum of the maps

$$d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}, \quad z + F^{p+1}K^{p+q} \mapsto dz + F^{p+r+1}K^{p+q+1}$$

where $z \in F^pK^{p+q} \cap d^{-1}(F^{p+r}K^{p+q+1})$.

Lemma 10.18.2. *Let $\mathcal{A}$ be an abelian category. Let $(K^\bullet, F)$ be a filtered complex of $\mathcal{A}$. Assume $\mathcal{A}$ has countable direct sums. The spectral sequence $(E_r, d_r)_{r\geq 0}$ associated to $(K^\bullet, F)$ has bigraded terms*

$$E_r = \bigoplus E_r^{p,q}, \quad d_r = \bigoplus d_r^{p,q}.$$

with d_r of bidegree $(r, -r+1)$. Furthermore, we have $E_0^{p,q} = gr^p(K^{p+q})$, $d_0^{p,q} = gr^p(d^{p+q})$, and $E_1^{p,q} = H^{p+q}(gr^p(K^\bullet))$.

Proof. Follows from the discussion above. □

Lemma 10.18.3. *Let $\mathcal{A}$ be an abelian category. Let $(K^\bullet, F)$ be a filtered complex of $\mathcal{A}$. Assume $\mathcal{A}$ has countable direct sums. Let $(E_r, d_r)_{r\geq 0}$ be the spectral sequence associated to $(K^\bullet, F)$.*

(1) *The map*

$$d_1^{p,q} : E_1^{p,q} = H^{p+q}(gr^p(K^\bullet)) \longrightarrow E_1^{p+1,q} = H^{p+q+1}(gr^{p+1}(K^\bullet))$$

is equal to the boundary map in cohomology associated to the short exact sequence of complexes

$$0 \to gr^{p+1}(K^\bullet) \to F^pK^\bullet/F^{p+2}K^\bullet \to gr^{p+1}(K^\bullet) \to 0.$$

(2) *Assume that $d(F^pK) \subset F^{p+1}K$ for all $p \in \mathbf{Z}$. Then d induces the zero differential on $gr^p(K^\bullet)$ and hence $E_1^{p,q} = gr^p(K^\bullet)^{p+q}$. Furthermore, in this case*

$$d_1^{p,q} : E_1^{p,q} = gr^p(K^\bullet)^{p+q} \longrightarrow E_1^{p,q} = gr^{p+1}(K^\bullet)^{p+q+1}$$

is the morphism induced by d.

Proof. Omitted. But compare Lemma 10.17.3. □

Lemma 10.18.4. *Let $\mathcal{A}$ be an abelian category. Let $\alpha : (K^\bullet, F) \to (L^\bullet, F)$ be a morphism of filtered complexes of $\mathcal{A}$. Assume $\mathcal{A}$ has countable direct sums. Let $(E_r(K), d_r)_{r\geq 0}$, resp. $(E_r(L), d_r)_{r\geq 0}$ be the spectral sequence associated to $(K^\bullet, F)$, resp. $(L^\bullet, F)$. The morphism α induces a canonical morphism of spectral sequences $\{\alpha_r : E_r(K) \to E_r(L)\}_{r\geq 0}$ compatible with the bigradings.*

Proof. Obvious from the explicit representation of the terms of the spectral sequences. □

Definition 10.18.5. Let $\mathcal{A}$ be an abelian category. Let $(K^\bullet, F)$ be a filtered complex of $\mathcal{A}$. The *induced filtration* on $H^n(K^\bullet)$ is the filtration defined by $F^pH^n(K^\bullet) = \mathrm{Im}(H^n(F^pK^\bullet) \to H^n(K^\bullet))$.

Lemma 10.18.6. *Let $\mathcal{A}$ be an abelian category. Let $(K^\bullet, F)$ be a filtered complex of $\mathcal{A}$. The associated graded $gr(H^n(K^\bullet))$ of the cohomology of $K^\bullet$ is a graded subquotient of the graded object $\bigoplus_{p+q=n} E_\infty^{p,q}$.*

Proof. Let $q = n - p$. As in the proof of Lemma 10.17.5 we see that

$$E_\infty^{p,q} = \frac{\bigcap_r (F^pK^n \cap d^{-1}(F^{p+r}K^{n+1}) + F^{p+1}K^n)}{\bigcup_r (F^pK^n \cap d(F^{p-r+1}K^{n-1}) + F^{p+1}K^n)}.$$

On the other hand, we have

$$\mathrm{gr}^pH^n(K) = \frac{\mathrm{Ker}(d) \cap F^pK^n + F^{p+1}K^n}{\mathrm{Im}(d) \cap F^pK^n + F^{p+1}K^n} \tag{10.18.6.1}$$

The result follows since

$$\mathrm{Ker}(d) \cap F^pK^n + F^{p+1}K^n \subset \bigcup\nolimits_r \left(F^pK^n \cap d^{-1}(F^{p+r}K^{n+1}) + F^{p+1}K^n\right) \tag{10.18.6.2}$$

and

$$\bigcap\nolimits_r \left(F^pK^n \cap d(F^{p-r+1}K^{n-1}) + F^{p+1}K^n\right) \subset \mathrm{Im}(d) \cap F^pK^n + F^{p+1}K^n. \tag{10.18.6.3}$$

□

Definition 10.18.7. Let $\mathcal{A}$ be an abelian category. Let $(K^\bullet, F)$ be a filtered complex of $\mathcal{A}$. We say the spectral sequence associated to $(K^\bullet, F)$ *converges* if $\mathrm{gr}H^n(K^\bullet) = \bigoplus_{p+q=n} E_\infty^{p,q}$ for every $n \in \mathbf{Z}$.

This is often symbolized by the notation $E_r^{p,q} \Rightarrow H^{p+q}(K^\bullet)$. Please read the remarks following Definition 10.17.6.

Lemma 10.18.8. *Let $\mathcal{A}$ be an abelian category. Let $(K^\bullet, F)$ be a filtered complex of $\mathcal{A}$. The associated spectral sequence converges if and only if for every $p, q \in \mathbf{Z}$ we have equality in equations (10.18.6.3) and (10.18.6.2).*

Proof. Immediate from the discussions above. □

Lemma 10.18.9. *Let $\mathcal{A}$ be an abelian category. Let $(K^\bullet, F)$ be a filtered complex of $\mathcal{A}$. Assume that the filtration on each K^n is finite (see Definition 10.13.1). Then*

1. *the filtration on each $H^n(K^\bullet)$ is finite, and*
2. *the spectral sequence associated to $(K^\bullet, F)$ converges.*

Proof. Part (1) is clear from Equation (10.18.6.1). We will use Lemma 10.18.8 to prove part (2). Fix $p, n \in \mathbf{Z}$. Look at the left hand side of Equation (10.18.6.3). The expression is equal to the right hand side since $F^mK^{n-1} = 0$ for $m \ll 0$. Similarly, use $F^mK^{n+1} = K^{n+1}$ for $m \gg 0$ to prove equality in Equation (10.18.6.2). □

10.19. Spectral sequences: double complexes

Definition 10.19.1. Let $\mathcal{A}$ be an additive category. A *double complex* in $\mathcal{A}$ is given by a system $(\{A^{p,q}, d_1^{p,q}, d_2^{p,q}\}_{p,q\in\mathbf{Z}})$, where each $A^{p,q}$ is an object of $\mathcal{A}$ and $d_1^{p,q} : A^{p,q} \to A^{p+1,q}$ and $d_2^{p,q} : A^{p,q} \to A^{p,q+1}$ are morphisms of $\mathcal{A}$ such that the following rules hold:

1. $d_1^{p+1,q} \circ d_1^{p,q} = 0$

(2) $d_2^{p,q+1} \circ d_2^{p,q} = 0$
(3) $d_1^{p,q+1} \circ d_2^{p,q} = d_2^{p+1,q} \circ d_1^{p,q}$

for all $p, q \in \mathbf{Z}$.

This is just the cochain version of the definition. It says that each $A^{p,\bullet}$ is a cochain complex and that each $d_1^{p,\bullet}$ is a morphism of complexes $A^{p,\bullet} \to A^{p+1,\bullet}$ such that $d_1^{p+1,\bullet} \circ d_1^{p,\bullet} = 0$ as morphisms of complexes. In other words a double complex can be seen as a complex of complexes. So in the diagram

$$\begin{tikzcd}
\ldots & \ldots & \ldots & \ldots \\
\ldots \arrow[r] & A^{p,q+1} \arrow[u] \arrow[r, "d_1^{p,q+1}"] & A^{p+1,q+1} \arrow[u] \arrow[r] & \ldots \\
\ldots \arrow[r] & A^{p,q} \arrow[u, "d_2^{p,q}"] \arrow[r, "d_1^{p,q}"] & A^{p+1,q} \arrow[u, "d_2^{p+1,q}"'] \arrow[r] & \ldots \\
\ldots & \ldots \arrow[u] & \ldots \arrow[u] & \ldots
\end{tikzcd}$$

any square commutes. Warning: In the literature one encouters a different definition where a ``bicomplex" or a ``double complex" has the property that the squares in the diagram anti-commute.

It is customary to denote $H_I^p(K^{\bullet,\bullet})$ the complex with terms $\mathrm{Ker}(d_1^{p,q})/\mathrm{Im}(d_1^{p-1,q})$ (varying q) and differential induced by d_2. Then $H_{II}^q(H_I^p(K^{\bullet,\bullet}))$ denotes its cohomology in degree q. It is also customary to denote $H_{II}^q(K^{\bullet,\bullet})$ the complex with terms $\mathrm{Ker}(d_2^{p,q})/\mathrm{Im}(d_2^{p,q-1})$ (varying p) and differential induced by d_1. Then $H_I^p(H_{II}^q(K^{\bullet,\bullet}))$ denotes its cohomology in degree q.

Definition 10.19.2. Let $\mathcal{A}$ be an additive category. Let $A^{\bullet,\bullet}$ be a double complex. The *associated simple complex* $sA^\bullet$, also sometimes called the *associated total complex* is given by

$$sA^n = \bigoplus\nolimits_{n=p+q} A^{p,q}$$

(if it exists) with differential

$$d_{sA}^n = \sum\nolimits_{n=p+q} (d_1^{p,q} + (-1)^p d_2^{p,q})$$

Alternatively, we sometimes write $\mathrm{Tot}(A^{\bullet,\bullet})$ to denote this complex.

If countable direct sums exist in $\mathcal{A}$ or if for each n at most finitely many $A^{p,n-p}$ are nonzero, then $sA^\bullet$ exists. Note that the definition is *not* symmetric in the indices (p, q).

There are two natural filtrations on the simple complex $sA^\bullet$ associated to the double complex $A^{\bullet,\bullet}$. Namely, we define

$$F_I^p(sA^n) = \bigoplus\nolimits_{i+j=n,\ i\geq p} A^{i,j} \quad \text{and} \quad F_{II}^p(sA^n) = \bigoplus\nolimits_{i+j=n,\ j\geq p} A^{i,j}.$$

It is immediately verified that $(sA^\bullet, F_I)$ and $(sA^\bullet, F_{II})$ are filtered complexes. By Section 10.18 we obtain two spectral sequences. It is customary to denote $({}'E_r, {}'d_r)_{r\geq 0}$ the spectral sequence associated to the filtration F_I and to denote $({}''E_r, {}''d_r)_{r\geq 0}$ the spectral sequence associated to the filtration F_{II}. Here is a description of these spectral sequences.

Lemma 10.19.3. *Let $\mathcal{A}$ be an abelian category. Let $K^{\bullet,\bullet}$ be a double complex. The spectral sequences associated to $K^{\bullet,\bullet}$ have the following terms:*

(1) $'E_0^{p,q} = K^{p,q}$ *with* $'d_0^{p,q} = (-1)^p d_2^{p,q} : K^{p,q} \to K^{p,q+1}$,
(2) $''E_0^{p,q} = K^{q,p}$ *with* $''d_0^{p,q} = d_1^{q,p} : K^{q,p} \to K^{q+1,p}$,
(3) $'E_1^{p,q} = H^q(K^{p,\bullet})$ *with* $'d_1^{p,q} = H^q(d_1^{p,\bullet})$,
(4) $''E_1^{p,q} = H^q(K^{\bullet,p})$ *with* $''d_1^{p,q} = (-1)^q H^q(d_2^{\bullet,p})$,
(5) $'E_2^{p,q} = H_I^p(H_{II}^q(K^{\bullet,\bullet}))$,
(6) $''E_2^{p,q} = H_{II}^p(H_I^q(K^{\bullet,\bullet}))$.

Proof. Omitted. □

These spectral sequences define two filtrations on $H^n(sK^\bullet)$. We will denote these F_I and F_{II}.

Definition 10.19.4. Let $\mathcal{A}$ be an abelian category. Let $K^{\bullet,\bullet}$ be a double complex. We say the spectral sequence $('E_r, 'd_r)_{r\geq 0}$ *converges* if Definition 10.18.7 applies. In other words, for all n

$$\mathrm{gr}_{F_I}(H^n(sK^\bullet)) = \oplus_{p+q=n}\, 'E_\infty^{p,q}$$

via the canonical comparison of Lemma 10.18.6. Similarly we say the spectral sequence $(''E_r, ''d_r)_{r\geq 0}$ *converges* if Definition 10.18.7 applies. In other words for all n

$$\mathrm{gr}_{F_{II}}(H^n(sK^\bullet)) = \oplus_{p+q=n}\, ''E_\infty^{p,q}$$

via the canonical comparison of Lemma 10.18.6.

Same caveats as those following Definition 10.17.6.

Lemma 10.19.5. *(First quadrant spectral sequence.) Let $\mathcal{A}$ be an abelian category. Let $K^{\bullet,\bullet}$ be a double complex. Assume that for some $i \ll 0$ we have $K^{p,q} = 0$ whenever either $p < i$ or $q < i$. Then*

(1) *the filtrations F_I, F_{II} on each $H^n(K^\bullet)$ are finite,*
(2) *the spectral sequence $('E_r, 'd_r)_{r\geq 0}$ converges, and*
(3) *the spectral sequence $(''E_r, ''d_r)_{r\geq 0}$ converges.*

Proof. Follows immediately from Lemma 10.18.9. □

Here is our first application of spectral sequences.

Lemma 10.19.6. *Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$ be a complex. Let $A^{\bullet,\bullet}$ be a double complex. Let $\alpha^p : K^p \to A^{p,0}$ be morphisms. Assume that*

(1) *There exists a $i \ll 0$ such that $K^p = A^{p,q} = 0$ for all $p < i$ and all q.*
(2) *We have $A^{p,q} = 0$ if $q < 0$.*
(3) *The morphisms α^p give rise to a morphism of complexes $\alpha : K^\bullet \to A^{\bullet,0}$.*
(4) *The complex $A^{p,\bullet}$ is exact in all degrees $q\neq 0$ and the morphism $K^p \to A^{p,0}$ induces an isomorphism $K^p \to Ker(d_2^{p,0})$.*

Then α induces a quasi-isomorphism

$$K^\bullet \longrightarrow sA^\bullet$$

of complexes. Moreover, there is a variant of this lemma involving the second variable q instead of p.

Proof. The map is simply the map given by the morphisms $K^n \to A^{n,0} \to sA^n$, which are easily seen to define a morphism of complexes. Consider the spectral sequence $({}'E_r, {}'d_r)_{r\geq 0}$ associated to the double complex $A^{\bullet,\bullet}$. By Lemma 10.19.5 this spectral sequence converges and the induced filtration on $H^n(sA^\bullet)$ is finite for each n. By Lemma 10.19.3 and assumption (4) we have ${}'E_1^{p,q} = 0$ unless $q = 0$ and ${}'E_1^{p,0} = K^p$ with differential ${}'d_1^{p,0}$ identified with d_K^p. Hence ${}'E_2^{p,0} = H^p(K^\bullet)$ and zero otherwise. This clearly implies $d_2^{p,q} = d_3^{p,q} = \ldots = 0$ for degree reasons. Hence we conclude that $H^n(sA^\bullet) = H^n(K^\bullet)$. We omit the verification that this identification is given by the morphism of complexes $K^\bullet \to sA^\bullet$ introduced above. □

Remark 10.19.7. Let $\mathcal{A}$ be an abelian category. Let $\mathcal{C} \subset \mathcal{A}$ be a weak Serre subcategory (see Definition 10.7.1). Suppose that $K^{\bullet,\bullet}$ is a double complex to which Lemma 10.19.5 applies such that for some $r \geq 0$ all the objects ${}'E_r^{p,q}$ belong to $\mathcal{C}$. We claim all the cohomology groups $H^n(sK^\bullet)$ belong to $\mathcal{C}$. Namely, the assumptions imply that the kernels and images of ${}'d_r^{p,q}$ are in $\mathcal{C}$. Whereupon we see that each ${}'E_{r+1}^{p,q}$ is in $\mathcal{C}$. By induction we see that each ${}'E_\infty^{p,q}$ is in $\mathcal{C}$. Hence each $H^n(sK^\bullet)$ has a finite filtration whose subquotients are in $\mathcal{C}$. Using that $\mathcal{C}$ is closed under extensions we conclude that $H^n(sK^\bullet)$ is in $\mathcal{C}$ as claimed.

The same result holds for the second spectral sequence associated to $K^{\bullet,\bullet}$. Similarly, if $(K^\bullet, F)$ is a filtered complex to which Lemma 10.18.9 applies and for some $r \geq 0$ all the objects $E_r^{p,q}$ belong to $\mathcal{C}$, then each $H^n(K^\bullet)$ is an object of $\mathcal{C}$.

10.20. Injectives

Definition 10.20.1. Let $\mathcal{A}$ be an abelian category. An object $J \in Ob(\mathcal{A})$ is called *injective* if for every injection $A \hookrightarrow B$ and every morphism $A \to J$ there exists a morphism $B \to J$ making the following diagram commute

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[dl, dashed] \\ J & \end{tikzcd}$$

Here is the obligatory characterization of injective objects.

Lemma 10.20.2. *Let $\mathcal{A}$ be an abelian category. Let I be an object of $\mathcal{A}$. The following are equivalent:*

(1) *The object I is injective.*
(2) *The functor $B \mapsto Hom_{\mathcal{A}}(B, I)$ is exact.*
(3) *Any short exact sequence*
$$0 \to I \to A \to B \to 0$$
in $\mathcal{A}$ is split.
(4) *We have $Ext_{\mathcal{A}}(B, I) = 0$ for all $B \in Ob(\mathcal{A})$.*

Proof. Omitted. □

Lemma 10.20.3. *Let $\mathcal{A}$ be an abelian category. Suppose I_ω, $\omega \in \Omega$ is a set of injective objects of $\mathcal{A}$. If $\prod_{\omega\in\Omega} I_\omega$ exists then it is injective.*

Proof. Omitted. □

Definition 10.20.4. Let $\mathcal{A}$ be an abelian category. We say $\mathcal{A}$ has *enough injectives* if every object A has an injective morphism $A \to J$ into an injective object J.

Definition 10.20.5. Let $\mathcal{A}$ be an abelian category. We say that $\mathcal{A}$ has *functorial injective embeddings* if there exists a functor

$$J : \mathcal{A} \longrightarrow \text{Arrows}(\mathcal{A})$$

such that

(1) $s \circ J = \text{id}_{\mathcal{A}}$,
(2) for any object $A \in Ob(\mathcal{A})$ the morphism $J(A)$ is injective, and
(3) for any object $A \in Ob(\mathcal{A})$ the object $t(J(A))$ is an injective object of $\mathcal{A}$.

We will denote such a functor by $A \mapsto (A \to J(A))$.

10.21. Projectives

Definition 10.21.1. Let $\mathcal{A}$ be an abelian category. An object $P \in Ob(\mathcal{A})$ is called *projective* if for every surjection $A \to B$ and every morphism $P \to B$ there exists a morphism $P \to A$ making the following diagram commute

$$\begin{tikzcd} A \arrow[r] & B \\ P \arrow[u, dashed] \arrow[ur] & \end{tikzcd}$$

Here is the obligatory characterization of projective objects.

Lemma 10.21.2. *Let $\mathcal{A}$ be an abelian category. Let P be an object of $\mathcal{A}$. The following are equivalent:*

(1) *The object P is projective.*
(2) *The functor $B \mapsto Hom_{\mathcal{A}}(P, B)$ is exact.*
(3) *Any short exact sequence*

$$0 \to A \to B \to P \to 0$$

in $\mathcal{A}$ is split.

(4) *We have $Ext_{\mathcal{A}}(P, A) = 0$ for all $A \in Ob(\mathcal{A})$.*

Proof. Omitted. □

Lemma 10.21.3. *Let $\mathcal{A}$ be an abelian category. Suppose P_ω, $\omega \in \Omega$ is a set of projective objects of $\mathcal{A}$. If $\coprod_{\omega\in\Omega} P_\omega$ exists then it is projective.*

Proof. Omitted. □

Definition 10.21.4. Let $\mathcal{A}$ be an abelian category. We say $\mathcal{A}$ has *enough projectives* if every object A has an surjective morphism $P \to A$ from an projective object P onto it.

Definition 10.21.5. Let $\mathcal{A}$ be an abelian category. We say that $\mathcal{A}$ has *functorial projective surjections* if there exists a functor

$$P : \mathcal{A} \longrightarrow \text{Arrows}(\mathcal{A})$$

such that

(1) $t \circ J = \text{id}_{\mathcal{A}}$,
(2) for any object $A \in Ob(\mathcal{A})$ the morphism $P(A)$ is surjective, and
(3) for any object $A \in Ob(\mathcal{A})$ the object $s(P(A))$ is an projective object of $\mathcal{A}$.

We will denote such a functor by $A \mapsto (P(A) \to A)$.

10.22. Injectives and adjoint functors

Here are some lemmas on adjoint functors and their relationship with injectives. See also Lemma 10.5.3.

Lemma 10.22.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $u : \mathcal{A} \to \mathcal{B}$ and $v : \mathcal{B} \to \mathcal{A}$ be additive functors.*

(1) *u is right adjoint to v, and*
(2) *v transforms injective maps into injective maps.*

Then u transforms injectives into injectives.

Proof. Let I be an injective object of $\mathcal{A}$. Let $\varphi : N \to M$ be an injective map in $\mathcal{B}$ and let $\alpha : N \to uI$ be a morphism. By adjointness we get a morphism $\alpha : vN \to I$ and by assumption $v\varphi : vN \to vM$ is injective. Hence as I is an injective object we get a morphism $\beta : vM \to I$ extending α. By adjointness again this corresponds to a morphism $\beta : M \to uI$ as desired. ☐

Remark 10.22.2. Let $\mathcal{A}$, $\mathcal{B}$, $u : \mathcal{A} \to \mathcal{B}$ and $v : \mathcal{B} \to \mathcal{A}$ be as in Lemma 10.22.1. In the presence of assumption (1) assumption (2) is equivalent to requiring that v is exact. Moreover, condition (2) is necessary. Here is an example. Let $A \to B$ be a ring map. Let $u : \text{Mod}_B \to \text{Mod}_A$ be $u(N) = N_A$ and let $v : \text{Mod}_A \to \text{Mod}_B$ be $v(M) = M \otimes_A B$. Then u is right adjoint to v, and u is exact and v is right exact, but v does not transform injective maps into injective maps in general (i.e., v is not left exact). Moreover, it is **not** the case that u transforms injective B-modules into injective A-modules. For example, if $A = \mathbf{Z}$ and $B = \mathbf{Z}/p\mathbf{Z}$, then the injective B-module $\mathbf{Z}/p\mathbf{Z}$ is not an injective $\mathbf{Z}$-module. In fact, the lemma applies to this example if and only if the ring map $A \to B$ is flat.

Lemma 10.22.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $u : \mathcal{A} \to \mathcal{B}$ and $v : \mathcal{B} \to \mathcal{A}$ be additive functors. Assume*

(1) *u is right adjoint to v,*
(2) *v transforms injective maps into injective maps,*
(3) *$\mathcal{A}$ has enough injectives, and*
(4) *$vB = 0$ implies $B = 0$ for any $B \in Ob(\mathcal{B})$.*

Then $\mathcal{B}$ has enough injectives.

Proof. Pick $B \in Ob(\mathcal{B})$. Pick an injection $vB \to I$ for I an injective object of $\mathcal{A}$. According to Lemma 10.22.1 and the assumptions the corresponding map $B \to uI$ is the injection of B into an injective object. ☐

Remark 10.22.4. Let $\mathcal{A}$, $\mathcal{B}$, $u : \mathcal{A} \to \mathcal{B}$ and $v : \mathcal{B} \to \mathcal{A}$ be as In Lemma 10.22.3. In the presence of conditions (1) and (2) condition (4) is equivalent to v being faithful. Moreover, condition (4) is needed. An example is to consider the case where the functors u and v are both the zero functor.

Lemma 10.22.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $u : \mathcal{A} \to \mathcal{B}$ and $v : \mathcal{B} \to \mathcal{A}$ be additive functors. Assume*

(1) *u is right adjoint to v,*
(2) *v transforms injective maps into injective maps,*
(3) *$\mathcal{A}$ has enough injectives,*
(4) *$vB = 0$ implies $B = 0$ for any $B \in Ob(\mathcal{B})$, and*
(5) *$\mathcal{A}$ has functorial injective hulls.*

Then $\mathcal{B}$ has functorial injective hulls.

Proof. Let $A \mapsto (A \to J(A))$ be a functorial injective hull on $\mathcal{A}$. Then $B \mapsto (B \to uJ(vB))$ is a functorial injective hull on $\mathcal{B}$. Compare with the proof of Lemma 10.22.3. $\square$

Lemma 10.22.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $u : \mathcal{A} \to \mathcal{B}$ be a functor. If there exists a subset $\mathcal{P} \subset Ob(\mathcal{B})$ such that*

(1) *every object of $\mathcal{B}$ is a quotient of an element of $\mathcal{P}$, and*
(2) *for every $P \in \mathcal{P}$ there exists an object Q of $\mathcal{A}$ such that $Hom_{\mathcal{A}}(Q, A) = Hom_{\mathcal{B}}(P, u(A))$ functorially in A,*

then there exists a left adjoint v of u.

Proof. By the Yoneda lemma the object Q of $\mathcal{A}$ corresponding to P is defined up to unique isomorphism by the formula $Hom_{\mathcal{A}}(Q, A) = Hom_{\mathcal{B}}(P, u(A))$. Let us write $Q = v(P)$. Denote $i_P : P \to u(v(P))$ the map corresponding to $id_{v(P)}$ in $Hom_{\mathcal{A}}(v(P), v(P))$. Functoriality in (2) implies that the bijection is given by

$$Hom_{\mathcal{A}}(v(P), A) \to Hom_{\mathcal{B}}(P, u(A)), \quad \varphi \mapsto u(\varphi) \circ i_P$$

For any pair of elements $P_1, P_2 \in \mathcal{P}$ there is a canonical map

$$Hom_{\mathcal{B}}(P_2, P_1) \to Hom_{\mathcal{A}}(v(P_2), v(P_1)), \quad \varphi \mapsto v(\varphi)$$

which is characterized, using by $u(v(\varphi)) \circ i_{P_2} = i_{P_1} \circ \varphi$ in $Hom_{\mathcal{B}}(P_2, u(v(P_1)))$. Note that $\varphi \mapsto v(\varphi)$ is additive and compatible with composition; this can be seen directly from the characterization. Hence $P \mapsto v(P)$ is a functor from the full subcategory opf $\mathcal{B}$ whose objects are the elements of $\mathcal{P}$.

Given an arbitrary object B of $\mathcal{B}$ choose an exact sequence

$$P_2 \to P_1 \to B \to 0$$

which is possible by assumption (1). Define $v(B)$ to be the object of $\mathcal{A}$ fitting into the exact sequence

$$v(P_2) \to v(P_1) \to v(B) \to 0$$

Then

$$\begin{aligned} Hom_{\mathcal{A}}(v(B), A) &= \mathrm{Ker}(Hom_{\mathcal{A}}(v(P_1), A) \to Hom_{\mathcal{A}}(v(P_2), A)) \\ &= \mathrm{Ker}(Hom_{\mathcal{B}}(P_1, u(A)) \to Hom_{\mathcal{B}}(P_2, u(A))) \\ &= Hom_{\mathcal{B}}(B, u(A)) \end{aligned}$$

Hence we see that we may take $\mathcal{P} = Ob(\mathcal{B})$, i.e., we see that v is everywhere defined. $\square$

10.23. Inverse systems

Let $\mathcal{C}$ be a category. In Categories, Section 4.19 we defined the notion of an inverse system over a partially ordered set (with values in the category $\mathcal{C}$). If the partially ordered set is $\mathbf{N} = \{1, 2, 3, \ldots\}$ with the usual ordering such an inverse system over $\mathbf{N}$ is often simply called an *inverse system*. It consists quite simply of a pair $(M_i, f_{ii'})$ where each M_i, $i \in \mathbf{N}$ is an object of $\mathcal{C}$, and for each $i > i'$, $i, i' \in \mathbf{N}$ a morphism $f_{ii'} : M_i \to M_{i'}$ such that moreover $f_{i'i''} \circ f_{ii'} = f_{ii''}$ whenever this makes sense. It is clear that in fact it suffices to give the morphisms $M_2 \to M_1$, $M_3 \to M_2$, and so on. Hence an inverse system is frequently pictured as follows

$$M_1 \xleftarrow{\varphi_2} M_2 \xleftarrow{\varphi_3} M_3 \leftarrow \ldots$$

Moreover, we often omit the transition maps φ_i from the notation and we simply say ``let (M_i) be an inverse system''.

The collection of all inverse systems with values in $\mathcal{C}$ forms a category with the obvious notion of morphism.

Lemma 10.23.1. *Let $\mathcal{C}$ be a category.*

1. *If $\mathcal{C}$ is an additive category, then the category of inverse systems with values in $\mathcal{C}$ is an additive cateogry.*
2. *If $\mathcal{C}$ is an abelian category, then the category of inverse systems with values in $\mathcal{C}$ is an abelian cateogry. A sequence $(K_i) \to (L_i) \to (M_i)$ of inverse systems is exact if and only if each $K_i \to L_i \to N_i$ is exact.*

Proof. Omitted. □

The limit (see Categories, Section 4.19) of such an inverse system is denoted $lim\, M_i$, or $lim_i\, M_i$. If $\mathcal{C}$ is the category of abelian groups (or sets), then the limit always exists and in fact can be described as follows

$$lim_i\, M_i = \{(x_i) \in \prod M_i \mid \varphi_i(x_i) = x_{i-1},\ i = 2, 3, \ldots\}$$

see Categories, Section 4.14. However, given a short exact sequence

$$0 \to (A_i) \to (B_i) \to (C_i) \to 0$$

of inverse systems of abelian groups it is not always the case that the associated system of limits is exact. In order to discuss this further we introduce the following notion.

Definition 10.23.2. Let $\mathcal{C}$ be an abelian category. We say the inverse system (A_i) satisfies the *Mittag-Leffler condition*, or for short is *ML*, if for every i there exists a $c = c(i) \geq i$ such that

$$\mathrm{Im}(A_k \to A_i) = \mathrm{Im}(A_c \to A_i)$$

for all $k \geq c$.

It turns out that the Mittag-Leffler condition is good enough to ensure that the lim-functor is exact, provided one works within the abelian category of abelian groups, or abelian sheaves, etc. It is shown in a paper by A. Neeman (see [**Nee02**]) that this condition is not strong enough in a general abelian category (where limits of inverse systems exist).

Lemma 10.23.3. *Let*

$$0 \to (A_i) \to (B_i) \to (C_i) \to 0$$

be a short exact sequence of inverse systems of abelian groups.

1. *In any case the sequence*

   $$0 \to lim_i\, A_i \to lim_i\, B_i \to lim_i\, C_i$$

   is exact.
2. *If (B_i) is ML, then also (C_i) is ML.*
3. *If (A_i) is ML, then*

   $$0 \to lim_i\, A_i \to lim_i\, B_i \to lim_i\, C_i \to 0$$

   is exact.

Proof. Nice exercise. See Algebra, Lemma 7.81.1 for part (3). □

Lemma 10.23.4. *Let*
$$(A_i) \to (B_i) \to (C_i) \to (D_i)$$
be an exact sequence of inverse systems of abelian groups. If the system (A_i) is ML, then the sequence
$$lim_i\, B_i \to lim_i\, C_i \to lim_i\, D_i$$
is exact.

Proof. Let $Z_i = \text{Ker}(C_i \to D_i)$ and $I_i = \text{Im}(A_i \to B_i)$. Then $lim\, Z_i = \text{Ker}(lim\, C_i \to lim\, D_i)$ and we get a short exact sequence of systems
$$0 \to (I_i) \to (B_i) \to (Z_i) \to 0$$
Moreover, by Lemma 10.23.3 we see that (I_i) has (ML), thus another application of Lemma 10.23.3 shows that $lim\, B_i \to lim\, Z_i$ is surjective which proves the lemma. $\square$

The following characterization of essentially constant inverse systems shows in particular that they have ML.

Lemma 10.23.5. *Let $\mathcal{A}$ be an abelian category. Let (A_i) be an inverse system in $\mathcal{A}$ with limit $A = lim\, A_i$. Then (A_i) is essentially constant (see Categories, Definition 4.20.1) if and only if there exists an i and for all $j \geq i$ a direct sum decomposition $A_j = A \oplus Z_j$ such that (a) the maps $A_{j'} \to A_j$ are compatible with the direct sum decompositions, (b) for all j there exists some $j' \geq j$ such that $Z_{j'} \to Z_j$ is zero.*

Proof. Assume (A_i) is essentially constant. Then there exists an i and a morphism $A_i \to A$ such that for all $j \geq i$ there exists a $j' \geq j$ such that $A_{j'} \to A_j$ factors as $A_{j'} \to A_i \to A \to A_j$ (the last map comes from $A = lim\, A_i$). Hence setting $Z_j = \text{Ker}(A_j \to A)$ for all $j \geq i$ works. Proof of the converse is omitted. $\square$

Lemma 10.23.6. *Let*
$$0 \to (A_i) \to (B_i) \to (C_i) \to 0$$
be an exact sequence of inverse systems of abelian groups. If (A_i) has ML and (C_i) is essentially constant, then (B_i) has ML.

Proof. After renumbering we may assume that $C_i = C \oplus Z_i$ compatible with transition maps and that for all i there exists an $i' \geq i$ such that $Z_{i'} \to Z_i$ is zero, see Lemma 10.23.5. Pick i. Let $c \geq i$ by an integer such that $\text{Im}(A_c \to A) = \text{Im}(A_{i'} \to A_i)$ for all $i' \geq c$. Let $c' \geq c$ be an integer such that $Z_{c'} \to Z_c$ is zero. For $i' \geq c'$ consider the maps
$$\begin{tikzcd}
0 \arrow[r] & A_{i'} \arrow[r] \arrow[d] & B_{i'} \arrow[r] \arrow[d] & C \oplus Z_{i'} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{c'} \arrow[r] \arrow[d] & B_{c'} \arrow[r] \arrow[d] & C \oplus Z_{c'} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{c} \arrow[r] \arrow[d] & B_{c} \arrow[r] \arrow[d] & C \oplus Z_{c} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{i} \arrow[r] & B_{i} \arrow[r] & C \oplus Z_{i} \arrow[r] & 0
\end{tikzcd}$$
Because $Z_{c'} \to Z_c$ is zero the image $\text{Im}(B_{c'} \to B_c)$ is an extension C by a subgroup $A' \subset A_c$ which contains the image of $A_{c'} \to A_c$. Hence $\text{Im}(B_{c'} \to B_i)$ is an extension of C

by the image of A' which is the image of $A_c \to A_i$ by our choice of c. In exactly the same way one shows that $\mathrm{Im}(B_{i'} \to B_i)$ is an extension of C by the image of $A_c \to A_i$. Hence $\mathrm{Im}(B_{c'} \to B_i) = \mathrm{Im}(B_{i'} \to B_i)$ and we win. $\square$

Lemma 10.23.7. *Let*

$$(A_i^{-2} \to A_i^{-1} \to A_i^0 \to A_i^1)$$

be an inverse system of complexes of abelian groups and denote $A^{-2} \to A^{-1} \to A^0 \to A^1$ its limit. Denote (H_i^{-1}), (H_i^0) the inverse systems of cohomologies, and denote H^{-1}, H^0 the cohomologies of $A^{-2} \to A^{-1} \to A^0 \to A^1$. If (A_i^{-2}) and (A_i^{-1}) are ML and (H_i^{-1}) is essentially constant, then $H^0 = \lim H_i^0$.

Proof. Let $Z_i^j = \mathrm{Ker}(A_i^j \to A_i^{j+1})$ and $I_i^j = \mathrm{Im}(A_i^{j-1} \to A_i^j)$. Note that $\lim Z_i^0 = \mathrm{Ker}(\lim A_i^0 \to \lim A_i^1)$ as taking kernels commutes with limits. The systems (I_i^{-1}) and (I_i^0) have ML as quotients of the systems (A_i^{-2}) and (A_i^{-1}), see Lemma 10.23.3. Thus an exact sequence

$$0 \to (I_i^{-1}) \to (Z_i^{-1}) \to (H_i^{-1}) \to 0$$

of inverse systems where (I_i^{-1}) has ML and where (H_i^{-1}) is essentially constant by assumption. Hence (Z_i^{-1}) has ML by Lemma 10.23.6. The exact sequence

$$0 \to (Z_i^{-1}) \to (A_i^{-1}) \to (I_i^0) \to 0$$

and an application of Lemma 10.23.3 shows that $\lim A_i^{-1} \to \lim I_i^0$ is surjective. Finally, the exact sequence

$$0 \to (I_i^0) \to (Z_i^0) \to (H_i^0) \to 0$$

and Lemma 10.23.3 show that $\lim I_i^0 \to \lim Z_i^0 \to \lim H_i^0 \to 0$ is exact. Putting everything together we win. $\square$

10.24. Exactness of products

Lemma 10.24.1. *Let I be a set. For $i \in I$ let $L_i \to M_i \to N_i$ be a complex of abelian groups. Let $H_i = Ker(M_i \to N_i)/Im(L_i \to M_i)$ be the cohomology. Then*

$$\prod L_i \to \prod M_i \to \prod N_i$$

is a complex of abelian groups with homology $\prod H_i$.

Proof. Omitted. $\square$

10.25. Differential graded algebras

Definition 10.25.1. Let R be a (commutative) ring. A *differential graded algebra* is either

(1) a chain complex $A_\bullet$ of R-modules endowed with R-bilinear maps $A_n \times A_m \to A_{n+m}$, $(a, b) \mapsto ab$ such that

$$d_{n+m}(ab) = d_n(a)b + (-1)^n a d_m(b)$$

and such that $\bigoplus A_n$ becomes an associative and unital R-algebra, or

(2) a cochain complex $A^\bullet$ of R-modules endowed with R-bilinear maps $A^n \times A^m \to A^{n+m}$, $(a, b) \mapsto ab$ such that

$$d^{n+m}(ab) = d^n(a)b + (-1)^n a d^m(b)$$

and such that $\bigoplus A^n$ becomes an associative and unital R-algebra.

We often just write $A = \bigoplus A_n$ or $A = \bigoplus A^n$ and think of this as an associative unital R-algebra endowed with a $\mathbf{Z}$-grading and an additive operator d whose square is zero and which satisfies the Leibniz rule as explained above. In this case we often say ``Let (A, d) be a differential graded algebra".

Definition 10.25.2. A *homomorphism of differential graded algebras* $f : (A, d) \to (B, d)$ is an algebra map $f : A \to B$ compatible with the gradings and d.

Definition 10.25.3. A differential graded algebra (A, d) is *commutative* if $ab = (-1)^{nm}ba$ for a in degree n and b in degree m. We say A is *strictly commutative* if in addition $a^2 = 0$ for $\deg(a)$ odd.

The following definition makes sense in general but is perhaps ``correct" only when tensoring commutative differential graded algebras.

Definition 10.25.4. Let R be a ring. Let (A, d), (B, d) be differential graded algebras over R. The *tensor product differential graded algebra* of A and B is the algebra $A \otimes_R B$ with multiplication defined by

$$(a \otimes b)(a' \otimes b') = (-1)^{\deg(a')\deg(b)} aa' \otimes bb'$$

endowed with differential d defined by the rule $d(a \otimes b) = d(a) \otimes b + (-1)^m a \otimes d(b)$ where $m = \deg(b)$.

Lemma 10.25.5. *Let R be a ring. Let (A, d), (B, d) be differential graded algebras over R. Denote $A^\bullet$, $B^\bullet$ the underlying cochain complexes. As cochain complexes of R-modules we have*

$$(A \otimes_R B)^\bullet = Tot(A^\bullet \otimes_A B^\bullet).$$

Proof. Recall that the differential of the total complex is given by $d_1^{p,q} + (-1)^p d_2^{p,q}$ on $A^p \otimes_R B^q$. And this is exactly the same as the rule for the differential on $A \otimes_R B$ in Definition 10.25.4. □

10.26. Other chapters

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
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CHAPTER 11

Derived Categories

11.1. Introduction

We first discuss triangulated categories and localization in triangulated categories. Next, we prove that the homotopy category of complexes in an additive category is a triangulated category. Once this is done we define the derived category of an abelian category as the localization of the of homotopy category with respect to quasi-isomorphisms. A good reference is Verdier's thesis [**Ver96**].

11.2. Triangulated categories

Triangulated categories are a convenient tool to describe the type of structure inherent in the derived category of an abelian category. Some references are [**Ver96**] and [**Nee01**].

11.3. The definition of a triangulated category

In this section we collect most of the definitions concerning triangulated and pre-triangulated categories.

Definition 11.3.1. Let $\mathcal{D}$ be an additive category. Let $[n] : \mathcal{D} \to \mathcal{D}$, $E \mapsto E[n]$ be a collection of additive functors indexed by $n \in \mathbf{Z}$ such that $[n] \circ [m] = [n+m]$ and $[0] = \text{id}$ (equality as functors). In this situation we call *triangle* a sixtuple (X, Y, Z, f, g, h) where $X, Y, Z \in Ob(\mathcal{D})$ and $f : X \to Y$, $g : Y \to Z$ and $h : Z \to X[1]$ are morphisms of $\mathcal{D}$. A *morphism of triangles* $(X, Y, Z, f, g, h) \to (X', Y', Z', f', g', h')$ is given by morphisms $a : X \to X'$, $b : Y \to Y'$ and $c : Z \to Z'$ of $\mathcal{D}$ such that $b \circ f = f' \circ a$, $c \circ g = g' \circ b$ and $a[1] \circ h = h' \circ c$.

A morphism of triangles is visualized by the following commutative diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "a"] & Y \arrow[r] \arrow[d, "b"] & Z \arrow[r] \arrow[d, "c"] & X[1] \arrow[d, "a[1]"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1]
\end{tikzcd}$$

Here is the definition of a triangulated category as given in Verdier's thesis.

Definition 11.3.2. A *triangulated category* consists of a triple $(\mathcal{D}, \{[n]\}_{n \in \mathbf{Z}}, \mathcal{T})$ where

(1) $\mathcal{D}$ is an additive category,
(2) $[n] : \mathcal{D} \to \mathcal{D}$, $E \mapsto E[n]$ be a collection of additive functors indexed by $n \in \mathbf{Z}$ such that $[n] \circ [m] = [n+m]$ and $[0] = \text{id}$ (equality as functors), and
(3) $\mathcal{T}$ is a set of triangles called the *distinguished triangles*

subject to the following conditions

TR1 Any triangle isomorphic to a distinguished triangle is a distinguished triangle. Any triangle of the form $(X, X, 0, \mathrm{id}, 0, 0)$ is distinguished. For any morphism $f : X \to Y$ of $\mathcal{D}$ there exists a distinguished triangle of the form (X, Y, Z, f, g, h).

TR2 The triangle (X, Y, Z, f, g, h) is distinguished if and only if the triangle $(Y, Z, X[1], g, h, -f[1])$ is.

TR3 Given a solid commutative square

$$\begin{tikzcd} X \arrow[r] \arrow[d, "a"] & Y \arrow[r] \arrow[d, "b"] & Z \arrow[r] \arrow[d, dashed] & X[1] \arrow[d, "a[1]"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd}$$

whose rows are distinguished triangles there exists a morphism $c : Z \to Z'$ such that (a, b, c) is a morphism of triangles.

TR4 Given objects X, Y, Z of $\mathcal{D}$, and morphisms $f : X \to Y$, $g : Y \to Z$, and distinguished triangles (X, Y, Q_1, f, p_1, d_1), $(X, Z, Q_2, g \circ f, p_2, d_2)$, and (Y, Z, Q_3, g, p_3, d_3), there exist morphisms $a : Q_1 \to Q_2$ and $b : Q_2 \to Q_3$ such that
  (a) $(Q_1, Q_2, Q_3, a, b, p_1[1] \circ d_3)$ is a distinguished triangle,
  (b) the triple (id_X, g, a) is a morphism of triangles $(X, Y, Q_1, f, p_1, d_1) \to (X, Z, Q_2, g \circ f, p_2, d_2)$, and
  (c) the triple (f, id_Z, b) is a morphism of triangles $(X, Z, Q_2, g \circ f, p_2, d_2) \to (Y, Z, Q_3, g, p_3, d_3)$.

We will call $(\mathcal{D}, [\ ], \mathcal{T})$ a *pre-triangulated category* if TR1, TR2 and TR3 hold.

The explanation of TR4 is that if you think of Q_1 as Y/X, Q_2 as Z/X and Q_3 as Z/Y, then TR4(a) expresses the isomorphism $(Z/X)/(Y/Z) \cong Z/Y$ and TR(b) and TR(c) express that we can compare the triangles $X \to Y \to Q_1 \to X[1]$ etc with morphisms of triangles. For a more precise reformuation of this idea see the proof of Lemma 11.9.2.

The sign in TR2 means that if (X, Y, Z, f, g, h) is a distinguished triangle then in the long sequence

$$(11.3.2.1) \quad \ldots \to Z[-1] \xrightarrow{-h[-1]} X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1] \xrightarrow{-g[1]} Z[1] \to \ldots$$

each four term sequence gives a distinguished triangle.

As usual we abuse notation and we simply speak of a (pre-)triangulated category $\mathcal{D}$ without explicitly introducing notation for the additional data. The notion of a pre-triangulated category is useful in finding statements equivalent to TR4.

We have the following definition of a triangulated functor.

Definition 11.3.3. Let $\mathcal{D}, \mathcal{D}'$ be pre-triangulated categories. An *exact functor*, or a *triangulated functor* from $\mathcal{D}$ to $\mathcal{D}'$ is a functor $F : \mathcal{D} \to \mathcal{D}'$ together with given functorial isomorphisms $\xi_X : F(X[1]) \to F(X)[1]$ such that for every distinguished triangle (X, Y, Z, f, g, h) of $\mathcal{D}$ the triangle $(F(X), F(Y), F(Z), F(f), F(g), \xi_X \circ F(h))$ is a distinguished triangle of $\mathcal{D}'$.

An exact functor is additive, see Lemma 11.4.15. When we say two triangulated categories are equivalent we mean that they are equivalent in the 2-category of triangulated categories. A 2-morphism $a : (F, \xi) \to (F', \xi')$ in this 2-category is simply a transformation of functors

$a : F \to F'$ which is compatible with ξ and ξ', i.e.,

$$\begin{tikzcd} F \circ [1] \arrow[r, "\xi"'] \arrow[d, "a\star 1"'] & {[1] \circ F} \arrow[d, "1\star a"] \\ F' \circ [1] \arrow[r, "\xi'"] & {[1] \circ F'} \end{tikzcd}$$

commutes.

Definition 11.3.4. Let $(\mathcal{D}, [\ ], \mathcal{T})$ be a pre-triangulated category. A *pre-triangulated subcategory*[1] is a pair $(\mathcal{D}', \mathcal{T}')$ such that

(1) $\mathcal{D}'$ is an additive subcategory of $\mathcal{D}$ which is preserved under $[1]$ and $[-1]$,
(2) $\mathcal{T}' \subset \mathcal{T}$ is a subset such that for every $(X, Y, Z, f, g, h) \in \mathcal{T}'$ we have $X, Y, Z \in Ob(\mathcal{D}')$ and $f, g, h \in \text{Arrows}(\mathcal{D}')$, and
(3) $(\mathcal{D}', [\ ], \mathcal{T}')$ is a pre-triangulated category.

If $\mathcal{D}$ is a triangulated category, then we say $(\mathcal{D}', \mathcal{T}')$ is a *triangulated subcategory* if it is a pre-triangulated subcategory and $(\mathcal{D}', [\ ], \mathcal{T}')$ is a triangulated category.

In this situation the inclusion functor $\mathcal{D}' \to \mathcal{D}$ is an exact functor with $\xi_X : X[1] \to X[1]$ given by the identity on $X[1]$.

We will see in Lemma 11.4.1 that for a distinguished triangle (X, Y, Z, f, g, h) in a pre-triangulated category the composition $g \circ f : X \to Z$ is zero. Thus the sequence (11.3.2.1) is a complex. A homological functor is one that turns this complex into a long exact sequence.

Definition 11.3.5. Let $\mathcal{D}$ be a pre-triangulated category. Let $\mathcal{A}$ be an abelian category. An additive functor $H : \mathcal{D} \to \mathcal{A}$ is called *homological* if for every distinguished triangle (X, Y, Z, f, g, h) the sequence

$$H(X) \to H(Y) \to H(Z)$$

is exact in the abelian category $\mathcal{A}$. An additive functor $H : \mathcal{D}^{opp} \to \mathcal{A}$ is called *cohomological* if the corresponding functor $\mathcal{D} \to \mathcal{A}^{opp}$ is homological.

If $H : \mathcal{D} \to \mathcal{A}$ is a homological functor we often write $H^n(X) = H(X[n])$ so that $H(X) = H^0(X)$. Our discussion of TR2 above implies that says that a distinguished triangle (X, Y, Z, f, g, h) determines a long exact sequence

$$H^{-1}(Z) \xrightarrow{h[-1]} H^0(X) \xrightarrow{f} H^0(Y) \xrightarrow{g} H^0(Z) \xrightarrow{h} H^1(X) \tag{11.3.5.1}$$

This will be called the *long exact sequence* associated to the distinguished triangle and the homological functor. As indicated we will not use any signs for the morphisms in the long exact sequence. This has the side effect that maps in the long exact sequence associated to the rotation (TR2) of a distinguished triangle differ from the maps in the sequence above by some signs.

Definition 11.3.6. Let $\mathcal{A}$ be an abelian category. Let $\mathcal{D}$ be a triangulated category. A *δ-functor from $\mathcal{A}$ to $\mathcal{D}$* is given by a functor $G : \mathcal{A} \to \mathcal{D}$ and a rule which assigns to every short exact sequence

$$0 \to A \xrightarrow{a} B \xrightarrow{b} C \to 0$$

a morphism $\delta = \delta_{A \to B \to C} : G(C) \to G(A)[1]$ such that

[1] This definition may be nonstandard. If $\mathcal{D}'$ is a full subcategory then $\mathcal{T}'$ is the intersection of the set of triangles in $\mathcal{D}'$ with $\mathcal{T}$, see Lemma 11.4.14. In this case we drop $\mathcal{T}'$ from the notation.

(1) the triangle $(G(A), G(B), G(C), G(a), G(b), \delta_{A \to B \to C})$ is a distinguished triangle of $\mathcal{D}$ for any short exact sequence as above, and
(2) for every morphism $(A \to B \to C) \to (A' \to B' \to C')$ of short exact sequences the diagram

$$\begin{tikzcd} G(C) \arrow[r, "\delta_{A \to B \to C}"'] \arrow[d] & G(A)[1] \arrow[d] \\ G(C') \arrow[r, "\delta_{A' \to B' \to C'}"] & G(A')[1] \end{tikzcd}$$

is commutative.

In this situation we call $(G(A), G(B), G(C), G(a), G(b), \delta_{A \to B \to C})$ the *image of the short exact sequence under the given δ-functor*.

Note how a δ-functor comes equipped with additional structure. Strictly speaking it does not make sense to say that a given functor $\mathcal{A} \to \mathcal{D}$ is a δ-functor, but we will often do so anyway.

11.4. Elementary results on triangulated categories

Most of the results in this section are proved for pre-triangulated categories and a fortiori hold in any triangulated category.

Lemma 11.4.1. *Let $\mathcal{D}$ be a pre-triangulated category. Let (X, Y, Z, f, g, h) be a distinguished triangle. Then $g \circ f = 0$, $h \circ g = 0$ and $f[1] \circ h = 0$.*

Proof. By TR1 we know $(X, X, 0, 1, 0, 0)$ is a distinguished triangle. Apply TR3 to

$$\begin{tikzcd} X \arrow[r] \arrow[d, "1"] & X \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d, dashed] & X[1] \arrow[d, "1[1]"] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & X[1] \end{tikzcd}$$

Of course the dotted arrow is the zero map. Hence the commutativity of the diagram implies that $g \circ f = 0$. For the other cases rotate the triangle, i.e., apply TR2. □

Lemma 11.4.2. *Let $\mathcal{D}$ be a pre-triangulated category. For any object W of $\mathcal{D}$ the functor $Hom_{\mathcal{D}}(W, -)$ is homological, and the functor $Hom_{\mathcal{D}}(-, W)$ is cohomological.*

Proof. Consider a distinguished triangle (X, Y, Z, f, g, h). We have already seen that $g \circ f = 0$, see Lemma 11.4.1. Suppose $a : W \to Y$ is a morphism such that $g \circ a = 0$. Then we get a commutative diagram

$$\begin{tikzcd} W \arrow[r, "1"'] \arrow[d, dotted, "b"] & W \arrow[r] \arrow[d, "a"] & 0 \arrow[r] \arrow[d, "0"] & W[1] \arrow[d, dotted, "b[1]"] \\ X \arrow[r] & Y \arrow[r] & Z \arrow[r] & X[1] \end{tikzcd}$$

Both rows are distinguished triangles (use TR1 for the top row). Hence we can fill the dotted arrow b (first rotate using TR2, then apply TR3, and then rotate back). This proves the lemma. □

Lemma 11.4.3. *Let $\mathcal{D}$ be a pre-triangulated category. Let*

$$(a, b, c) : (X, Y, Z, f, g, h) \to (X', Y', Z', f', g', h')$$

be a morphism of distinguished triangles. If two among a, b, c are isomorphisms so is the third.

Proof. Assume that a and c are isomorphisms. For any object W of $\mathcal{D}$ write $H_W(-) = Hom_{\mathcal{D}}(W, -)$. Then we get a commutative diagram of abelian groups

$$\begin{tikzcd} H_W(Z[-1]) \arrow[r] \arrow[d] & H_W(X) \arrow[r] \arrow[d] & H_W(Y) \arrow[r] \arrow[d] & H_W(Z) \arrow[r] \arrow[d] & H_W(X[1]) \arrow[d] \\ H_W(Z'[-1]) \arrow[r] & H_W(X') \arrow[r] & H_W(Y') \arrow[r] & H_W(Z') \arrow[r] & H_W(X'[1]) \end{tikzcd}$$

By assumption the right two and left two vertical arrows are bijective. As H_W is homological by Lemma 11.4.2 and the five lemma (Homology, Lemma 10.3.25) it follows that the middle vertical arrow is an isomorphism. Hence by Yoneda's lemma, see Categories, Lemma 4.3.5 we see that b is an isomorphism. This implies the other cases by rotating (using TR2). $\square$

Lemma 11.4.4. *Let $\mathcal{D}$ be a pre-triangulated category. Let*

$$(0, b, 0), (0, b', 0) : (X, Y, Z, f, g, h) \to (X, Y, Z, f, g, h)$$

be endomorphisms of a distinguished triangle. Then $bb' = 0$.

Proof. Picture

$$\begin{tikzcd} X \arrow[r] \arrow[d, "0"] & Y \arrow[r] \arrow[d, "{b,b'}"] \arrow[dl, dotted, "\alpha"] & Z \arrow[r] \arrow[d, "0"] \arrow[dl, dotted, "\beta"] & X[1] \arrow[d, "0"] \\ X \arrow[r] & Y \arrow[r] & Z \arrow[r] & X[1] \end{tikzcd}$$

Applying Lemma 11.4.3 we find dotted arrows α and β such that $b' = f \circ \alpha$ and $b = \beta \circ g$. Then $bb' = \beta \circ g \circ f \circ \alpha = 0$ as $g \circ f = 0$ by Lemma 11.4.1. $\square$

Lemma 11.4.5. *Let $\mathcal{D}$ be a pre-triangulated category. Let (X, Y, Z, f, g, h) be a distinguished triangle. If*

$$\begin{tikzcd} Z \arrow[r, "f"'] \arrow[d, "c"'] & X[1] \arrow[d, "{a[1]}"] \\ Z \arrow[r, "f"] & X[1] \end{tikzcd}$$

is commutative and $a^2 = a$, $c^2 = c$, then there exists a morphism $b : Y \to Y$ with $b^2 = b$ such that (a, b, c) is an endomorphism of the triangle (X, Y, Z, f, g, h).

Proof. By TR3 there exists a morphism b' such that (a, b', c) is an endormorphism of (X, Y, Z, f, g, h). Then $(0, (b')^2 - b', 0)$ is also an endomorphism. By Lemma 11.4.4 we see that $(b')^2 - b'$ has square zero. Set $b = b' - (2b' - 1)((b')^2 - b') = 3(b')^2 - 2(b')^3$. A computation shows that (a, b, c) is an endomorphism and that $b^2 - b = (4(b')^2 - 4b' - 3)((b')^2 - b')^2 = 0$. $\square$

Lemma 11.4.6. *Let $\mathcal{D}$ be a pre-triangulated category. Let $f : X \to Y$ be a morphism of $\mathcal{D}$. There exists a distinguished triangle (X, Y, Z, f, g, h) which is unique up to (nonunique) isomorphism of triangles. More precisely, given a second such distinguished triangle (X, Y, Z', f, g', h') there exists an isomorphism*

$$(1, 1, c) : (X, Y, Z, f, g, h) \longrightarrow (X, Y, Z', f, g', h')$$

Proof. Existence by TR1. Uniqueness up to isomorphism by TR3 and Lemma 11.4.3. $\square$

Lemma 11.4.7. *Let $\mathcal{D}$ be a pre-triangulated category. Let $f : X \to Y$ be a morphism of $\mathcal{D}$. The following are equivalent*

(1) *f is an isomorphism,*
(2) *$(X, Y, 0, f, 0, 0)$ is a distinguished triangle, and*
(3) *for any distinguished triangle (X, Y, Z, f, g, h) we have $Z = 0$.*

Proof. Immediate from Lemma 11.4.6 and TR1. $\square$

Lemma 11.4.8. *Let $\mathcal{D}$ be a pre-triangulated category. Let (X, Y, Z, f, g, h) and (X', Y', Z', f', g', h') be triangles. The following are equivalent*

(1) *$(X \oplus X', Y \oplus Y', Z \oplus Z', f \oplus f', g \oplus g', h \oplus h')$ is a distinguished triangle,*
(2) *both (X, Y, Z, f, g, h) and (X', Y', Z', f', g', h') are distinguished triangles.*

Proof. Assume (2). By TR1 we may choose a distinguished triangle $(X \oplus X', Y \oplus Y', Q, f \oplus f', g'', h'')$. By TR3 we can find morphisms of distinguished triangles $(X, Y, Z, f, g, h) \to (X \oplus X', Y \oplus Y', Q, f \oplus f', g'', h'')$ and $(X', Y', Z', f', g', h') \to (X \oplus X', Y \oplus Y', Q, f \oplus f', g'', h'')$. Taking the direct sum of these morphisms we obtain a morphism of triangles

$$\begin{array}{c} (X \oplus X', Y \oplus Y', Z \oplus Z', f \oplus f', g \oplus g', h \oplus h') \\ \downarrow{\scriptstyle (1,1,c)} \\ (X \oplus X', Y \oplus Y', Q, f \oplus f', g'', h''). \end{array}$$

Let W be any object in $\mathcal{D}$ and apply the functor $H_W = Hom_{\mathcal{D}}(W, -)$ to this diagram. By Lemma 11.4.2 (applied three times) we deduce that $H_W(c) : H_W(Z \oplus Z') \to H_W(Q)$ is an isomorphism. Hence c is an isomorphism and we conclude that (1) holds.

Assume (1). We will show that (X, Y, Z, f, g, h) is a distinguished triangle. Let W be any object in $\mathcal{D}$ and set $H_W = Hom_{\mathcal{D}}(W, -)$. By Lemma 11.4.2 we see that $H_W(X) \to H_W(Y) \to H_W(Z) \to H_W(Z[1])$ is exact as it is a direct summand of the exact sequence associated to the distinguished triangle $(X \oplus X', Y \oplus Y', Z \oplus Z', f \oplus f', g \oplus g', h \oplus h')$. Using TR1 let (X, Y, Q, f, g'', h'') be a distinguished triangle. By TR3 there exists a morphism of distinguished triangles $(X \oplus X', Y \oplus Y', Z \oplus Z', f \oplus f', g \oplus g', h \oplus h') \to (X, Y, Q, f, g'', h'')$. Composing this with the inclusion map we get a morphism of triangles

$$(1, 1, c) : (X, Y, Z, f, g, h) \longrightarrow (X, Y, Q, f, g'', h'')$$

Applying H_W and using the above we once again see that $H_W(c) : H_W(Z) \to H_W(Q)$ is an isomorphism and we conclude that c is an isomorphism. Hence we win. $\square$

Lemma 11.4.9. *Let $\mathcal{D}$ be a pre-triangulated category. Let (X, Y, Z, f, g, h) be a distinguished triangle.*

(1) *If $h = 0$, then there exists a left inverse $s : Z \to Y$ to g.*
(2) *For any left inverse $s : Z \to Y$ of g the map $f \oplus s : X \oplus Z \to Y$ is an isomorphism.*
(3) *For any objects X', Z' of $\mathcal{D}$ the triangle $(X', X' \oplus Z', Z', (1, 0), (0, 1), 0)$ is distinguished.*

Proof. To see (1) use that $Hom_{\mathcal{D}}(Z, Y) \to Hom_{\mathcal{D}}(Z, Z) \to Hom_{\mathcal{D}}(Z, X[1])$ is exact by Lemma 11.4.2. By the same token, if s is as in (2), then $h = 0$ and the sequence

$$0 \to Hom_{\mathcal{D}}(W, X) \to Hom_{\mathcal{D}}(W, Y) \to Hom_{\mathcal{D}}(W, Z) \to 0$$

is split exact (split by $s : Z \to Y$). Hence by Yoneda's lemma we see that $X \oplus Z \to Y$ is an isomorphism. The last assertion follows from TR1 and Lemma 11.4.8. $\square$

Lemma 11.4.10. *Let $\mathcal{D}$ be a pre-triangulated category. Let $f : X \to Y$ be a morphism of $\mathcal{D}$. The following are equivalent*

(1) *f has a kernel,*
(2) *f has a cokernel,*
(3) *f is isomorphic to a map $K \oplus Z \to Z \oplus Q$ induced by id_Z.*

Proof. Any morphism isomorphic to a map of the form $X' \oplus Z \to Z \oplus Y'$ has both a kernel and a cokernel. Hence (3) ⇒ (1), (2). Next we prove (1) ⇒ (3). Suppose first that $f : X \to Y$ is a monomorphism, i.e., its kernel is zero. By TR1 there exists a distinguished triangle (X, Y, Z, f, g, h) and by Lemma 11.4.2 we see that $h = 0$. Then Lemma 11.4.9 implies that $Y = X \oplus Z$, i.e., we see that (3) holds. Next, assume f has a kernel K. As $K \to X$ is a monomorphism we conclude $X = K \oplus X'$ and $f|_{X'} : X' \to Y$ is a monomorphism. Hence $Y = X' \oplus Y'$ and we win. The implication (2) ⇒ (3) is dual to this. □

Let $\mathcal{D}$ be an additive category. Let $e : A \to A$ be an idempotent endomorphism of an object of $\mathcal{D}$. If $\text{Ker}(e)$ and $\text{Ker}(1 - e)$ exist, then $A = \text{Ker}(e) \oplus \text{Ker}(1 - e)$ and moreover $\text{Ker}(e) = \text{Coker}(1 - e)$. Dually, if $\text{Coker}(e)$ and $\text{Coker}(1 - e)$ exist, then $A = \text{Coker}(e) \oplus \text{Coker}(1 - e)$ and moreover $\text{Ker}(e) = \text{Coker}(1 - e)$.

Lemma 11.4.11. *Let $\mathcal{D}$ be an additive category.*

(1) *If $\mathcal{D}$ has countable products and kernels of maps which have a right inverse, then $\mathcal{D}$ has kernels of idempotents.*
(2) *If $\mathcal{D}$ has countable coproducts and cokernels of maps which have a left inverse, then $\mathcal{D}$ has cokernels of idempotents.*

Proof. Let X be an object of $\mathcal{D}$ and let $e : X \to X$ be an idempotent. The functor

$$W \longmapsto \text{Ker}(Mor_{\mathcal{D}}(W, X) \xrightarrow{e} Mor_{\mathcal{D}}(W, X))$$

if representable if and only if e has a kernel. Note that for any abelian group A and idempotent endomorphism $e : A \to A$ we have

$$\text{Ker}(e : A \to A) = \text{Ker}(\Phi : \prod\nolimits_{n \in \mathbf{N}} A \to \prod\nolimits_{n \in \mathbf{N}} A)$$

where

$$\Phi(a_1, a_2, a_3, \ldots) = (ea_1 + (1 - e)a_2, ea_2 + (1 - e)a_3, \ldots)$$

Moreover, Φ has the right inverse

$$\Psi(a_1, a_2, a_3, \ldots) = (a_1, (1 - e)a_1 + ea_2, (1 - e)a_2 + ea_3, \ldots).$$

Hence (1) holds. The proof of (2) is dual. □

Lemma 11.4.12. *Let $\mathcal{D}$ be a pre-triangulated category. If $\mathcal{D}$ has countable products, then $\mathcal{D}$ has kernels of idempotents. If $\mathcal{D}$ has countable coproducts, then $\mathcal{D}$ has kernels of idempotents.*

Proof. Assume $\mathcal{D}$ has countable products. By Lemma 11.4.11 it suffices to check that morphisms which have a right inverse have kernels. Any morphism which has a right inverse is an epimorphism, hence has a kernel by Lemma 11.4.10. The second statement is dual to the first (see also remark preceding Lemma 11.4.11). □

The following lemma makes it slightly easier to prove that a pre-triangulated category is triangulated.

Lemma 11.4.13. *Let $\mathcal{D}$ be a pre-triangulated category. In order to prove TR4 it suffices to show that given any pair of composable morphisms $f : X \to Y$ and $g : Y \to Z$ there exist*

(1) *isomorphisms $i : X' \to X$, $j : Y' \to Y$ and $k : Z' \to Z$, and then setting $f' = j^{-1}fi : X' \to Y'$ and $g' = k^{-1}gj : Y' \to Z'$ there exist*
(2) *distinguished triangles $(X', Y', Q_1, f', p_1, d_1)$, $(X', Z', Q_2, g' \circ f', p_2, d_2)$ and $(Y', Z', Q_3, g', p_3, d_3)$, such that the assertion of TR4 holds.*

Proof. The replacement of X, Y, Z by X', Y', Z' is harmless by our definition of distinguished triangles and their isomorphisms. The lemma follows from the fact that the distinguished triangles $(X', Y', Q_1, f', p_1, d_1)$, $(X', Z', Q_2, g' \circ f', p_2, d_2)$ and $(Y', Z', Q_3, g', p_3, d_3)$ are unique up to isomorphism by Lemma 11.4.6. □

Lemma 11.4.14. *Let $\mathcal{D}$ be a pre-triangulated category. Assume that $\mathcal{D}'$ is an additive full subcategory of $\mathcal{D}$. The following are equivalent*

(1) *there exists a set of triangles $\mathcal{T}'$ such that $(\mathcal{D}', \mathcal{T}')$ is a pre-triangulated subcategory of $\mathcal{D}$,*
(2) *$\mathcal{D}'$ is preserved under $[1], [-1]$ and given any morphism $f : X \to Y$ in $\mathcal{D}'$ there exists a distinguished triangle (X, Y, Z, f, g, h) in $\mathcal{D}$ such that Z is isomorphic to an object of $\mathcal{D}'$.*

In this case $\mathcal{T}'$ is the set of distinguished triangles (X, Y, Z, f, g, h) of $\mathcal{D}$ such that $X, Y, Z \in Ob(\mathcal{D}')$ and $f, g, h \in Arrows(\mathcal{D}')$. Finally, if $\mathcal{D}$ is a triangulated category, then (1) and (2) are also equivalent to

(3) *$\mathcal{D}'$ is a triangulated subcategory.*

Proof. Omitted. □

Lemma 11.4.15. *An exact functor of pre-triangulated categories is additive.*

Proof. Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor of pre-triangulated categories. Since $(0, 0, 0, 1_0, 1_0, 0)$ is a distinguished triangle of $\mathcal{D}$ the triangle

$$(F(0), F(0), F(0), 1_{F(0)}, 1_{F(0)}, F(0))$$

is distinguished in $\mathcal{D}'$. This implies that $1_{F(0)} \circ 1_{F(0)}$ is zero, see Lemma 11.4.1. Hence $F(0)$ is the zero object of $\mathcal{D}'$. This also implies that F applied to any zero morphism is zero (since a morphism in an additive category is zero if and only if it factors through the zero objet). Next, using that $(X, X \oplus Y, Y, (1, 0), (0, 1), 0)$ is a distinguished triangle, we see that $(F(X), F(X \oplus Y), F(Y), F(1, 0), F(0, 1), 0)$ is one too. This implies that the map $F(1, 0) \oplus F(0, 1) : F(X) \oplus F(Y) \to F(X \oplus Y)$ is an isomorphism, see Lemma 11.4.9. We omit the rest of the argument. □

Lemma 11.4.16. *Let $F : \mathcal{D} \to \mathcal{D}'$ be a fully faithful exact functor of pre-triangulated categories. Then a triangle (X, Y, Z, f, g, h) of $\mathcal{D}$ is distinguished if and only if $(F(X), F(Y), F(Z), F(f), F(g), F(h))$ is distinguished in $\mathcal{D}'$.*

Proof. The ``if" part is clear. Assume $(F(X), F(Y), F(Z))$ is distinguished in $\mathcal{D}'$. Pick a distinguished triangle (X, Y, Z', f, g', h') in $\mathcal{D}$. By Lemma 11.4.6 there exists an isomorphism of triangles

$$(1, 1, c') : (F(X), F(Y), F(Z)) \longrightarrow (F(X), F(Y), F(Z')).$$

Since F is fully faithful, there exists a morphism $c : Z \to Z'$ such that $F(c) = c'$. Then $(1, 1, c)$ is an isomorphism between (X, Y, Z) and (X, Y, Z'). Hence (X, Y, Z) is distinguished by TR1. □

Lemma 11.4.17. *Let $\mathcal{D}, \mathcal{D}', \mathcal{D}''$ be pre-triangulated categories. Let $F : \mathcal{D} \to \mathcal{D}'$ and $F' : \mathcal{D}' \to \mathcal{D}''$ be exact functors. Then $F' \circ F$ is an exact functor.*

Proof. Omitted. □

Lemma 11.4.18. *Let $\mathcal{D}$ be a pre-triangulated category. Let $\mathcal{A}$ be an abelian category. Let $H : \mathcal{D} \to \mathcal{A}$ be a homological functor.*

(1) *Let $\mathcal{D}'$ be a pre-triangulated category. Let $F : \mathcal{D}' \to \mathcal{D}$ be an exact functor. Then the composition $G \circ F$ is a homological functor as well.*
(2) *Let $\mathcal{A}'$ be an abelian category. Hence $G : \mathcal{A} \to \mathcal{A}'$ be an exact functor. Hence $G \circ H$ is a homological functor as well.*

Proof. Omitted. □

Lemma 11.4.19. *Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{A}$ be an abelian category. Let $G : \mathcal{A} \to \mathcal{D}$ be a δ-functor.*

(1) *Let $\mathcal{D}'$ be a triangulated category. Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor. Then the composition $F \circ G$ is a δ-functor as well.*
(2) *Let $\mathcal{A}'$ be an abelian category. Hence $H : \mathcal{A}' \to \mathcal{A}$ be an exact functor. Hence $G \circ H$ is a δ-functor as well.*

Proof. Omitted. □

Lemma 11.4.20. *Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{A}$ be an abelian category. Let $G : \mathcal{A} \to \mathcal{D}$ be a δ-functor. Let $H : \mathcal{D} \to \mathcal{B}$ be a homological functor. Assume that $H^{-1}(G(A)) = 0$ for all A in $\mathcal{A}$. Then the collection*

$$\{H^n \circ G, H^n(\delta_{A \to B \to C})\}_{n \geq 0}$$

is a δ-functor from $\mathcal{A} \to \mathcal{B}$, see Homology, Definition 10.9.1.

Proof. The notation signifies the following. If $0 \to A \xrightarrow{a} B \xrightarrow{b} C \to 0$ is a short exact sequence in $\mathcal{A}$, then

$$\delta = \delta_{A \to B \to C} : G(C) \to G(A)[1]$$

is a morphism in $\mathcal{D}$ such that $(G(A), G(B), G(C), a, b, \delta)$ is a distinguished triangle, see Definition 11.3.6. Then $H^n(\delta) : H^n(G(C)) \to H^n(G(A)[1]) = H^{n+1}(G(A))$ is clearly functorial in the short exact sequence. Finally, the long exact cohomology sequence (11.3.5.1) combined with the vanishing of $H^{-1}(G(C))$ gives a long exact sequence

$$0 \to H^0(G(A)) \to H^0(G(B)) \to H^0(G(C)) \xrightarrow{H^0(\delta)} H^1(G(A)) \to \ldots$$

in $\mathcal{B}$ as desired. □

The proof of the following result uses TR4.

Proposition 11.4.21. *Let $\mathcal{D}$ be a triangulated category. Any commutative diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ X' \arrow[r] & Y' \end{tikzcd}$$

can be extended to a diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & X[1] \arrow[d] \\
X' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d] & X'[1] \arrow[d] \\
X'' \arrow[r] \arrow[d] & Y'' \arrow[r] \arrow[d] & Z'' \arrow[r] \arrow[d] & X''[1] \arrow[d] \\
X[1] \arrow[r] & Y[1] \arrow[r] & Z[1] \arrow[r] & X[2]
\end{tikzcd}$$

where all the squares are commutative, except for the lower right square which is anticommutative. Moreover, each of the rows and columns are distinguished triangles. Finally, the morphisms on the bottom row (resp. right column) are obtained from the morphisms of the top row (resp. left column) by applying [1].

Proof. During this proof we avoid writing the arrows in order to make the proof legible. Choose distinguished triangles (X, Y, Z), (X', Y', Z'), (X, X', X''), (Y, Y', Y''), and and (X, Y', A). Note that the morphism $X \to Y'$ is both equal to the composition $X \to Y \to Y'$ and equal to the composition $X \to X' \to Y'$. Hence, we can find morphisms

(1) $a : Z \to A$ and $b : A \to Y''$, and
(2) $a' : X'' \to A$ and $b' : A \to Z'$

as in TR4. Denote $c : Y'' \to Z[1]$ the composition $Y'' \to Y[1] \to Z[1]$ and denote $c' : Z' \to X''[1]$ the composition $Z' \to X'[1] \to X''[1]$. The conclusion of our application TR4 are that

(1) (Z, A, Y'', a, b, c), (X'', A, Z', a', b', c') are distinguished triangles,
(2) $(X, Y, Z) \to (X, Y', A)$, $(X, Y', A) \to (Y, Y', Y'')$, $(X, X', X'') \to (X, Y', A)$, $(X, Y', A) \to (X', Y', Z')$ are morphisms of triangles.

First using that $(X, X', X'') \to (X, Y', A)$ and $(X, Y', A) \to (Y, Y', Y'')$. are morphisms of triangles we see the first of the commutative diagrams

$$\begin{tikzcd}
X' \arrow[r] \arrow[d] & Y' \arrow[d] \\
X'' \arrow[r, "b\circ a'"] \arrow[d] & Y'' \arrow[d] \\
X[1] \arrow[r] & Y[1]
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, "b'\circ a"] & X[1] \arrow[d] \\
Y' \arrow[r] & Z' \arrow[r] & X'[1]
\end{tikzcd}$$

is commutative. The second is commutative too using that $(X, Y, Z) \to (X, Y', A)$ and $(X, Y', A) \to (X', Y', Z')$ are morphisms of triangles. At this point we choose a distinguished triangle (X'', Y'', Z'') starting with the map $b \circ a' : X'' \to Y''$.

Next we apply TR4 one more time to the morphisms $X'' \to A \to Y''$ and the triangles (X'', A, Z', a', b', c'), (X'', Y'', Z''), and $(A, Y'', Z[1], b, c, -a[1])$ to get morphisms $a'' : Z' \to Z''$ and $b'' : Z'' \to Z[1]$. Then $(Z', Z'', Z[1], a'', b'', -b'[1] \circ a[1])$ is a distinguished triangle, hence also $(Z, Z', Z'', -b' \circ a, a'', -b'')$ and hence also $(Z, Z', Z'', b' \circ a, a'', b'')$. Moreover, $(X'', A, Z') \to (X'', Y'', Z'')$ and $(X'', Y'', Z'') \to (A, Y'', Z[1], b, c, -a[1])$ are morphisms of triangles. At this point we have defined all the distinguished triangles and all the morphisms, and all that's left is to verify some commutativity relations.

To see that the middle square in the diagram commutes, note that the arrow $Y' \to Z'$ factors as $Y' \to A \to Z'$ because $(X, Y', A) \to (X', Y', Z')$ is a morphism of triangles. Similarly, the morphism $Y' \to Y''$ factors as $Y' \to A \to Y''$ because $(X, Y', A) \to (Y, Y', Y'')$ is a morphism of triangles. Hence the middle square commutes because the square with sides (A, Z', Z'', Y'') commutes as $(X'', A, Z') \to (X'', Y'', Z'')$ is a morphism of triangles (by TR4). The square with sides $(Y'', Z'', Y[1], Z[1])$ commutes because $(X'', Y'', Z'') \to (A, Y'', Z[1], b, c, -a[1])$ is a morphism of triangles and $c : Y'' \to Z[1]$ is the composition $Y'' \to Y[1] \to Z[1]$. The square with sides $(Z', X'[1], X''[1], Z'')$ is commutative because $(X'', A, Z') \to (X'', Y'', Z'')$ is a morphism of triangles and $c' : Z' \to X''[1]$ is the composition $Z' \to X'[1] \to X''[1]$. Finally, we have to show that the square with sides $(Z'', X''[1], Z[1], X[2])$ anticommutes. This holds because $(X'', Y'', Z'') \to (A, Y'', Z[1], b, c, -a[1])$ is a morphism of triangles and we're done. □

11.5. Localization of triangulated categories

In order to construct the derived category starting from the homotopy category of compexes, we will use a localization process.

Definition 11.5.1. Let $\mathcal{D}$ be a pre-triangulated category. We say a multiplicative system S is *compatible with the triangulated structure* if the following two conditions hold:

MS5 For $s \in S$ we have $s[n] \in S$ for all $n \in \mathbf{Z}$.

MS6 Given a solid commutative square

$$\begin{tikzcd} X \arrow[r] \arrow[d, "s"] & Y \arrow[r] \arrow[d, "s'"] & Z \arrow[r] \arrow[d, dashed] & X[1] \arrow[d, "s[1]"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd}$$

whose rows are distinguished triangles with $s, s' \in S$ there exists a morphism $s'' : Z \to Z'$ in S such that (s, s', s'') is a morphism of triangles.

It turns out that these axioms are not independent of the axioms defining multiplicative systems.

Lemma 11.5.2. *Let $\mathcal{D}$ be a pre-triangulated category. Let S be a set of morphisms of $\mathcal{D}$ and assume that axioms MS1, MS5, MS6 hold (see Categories, Definition 4.24.1 and Definition 11.5.1). Then MS2 holds.*

Proof. Suppose that $f : X \to Y$ is a morphism of $\mathcal{D}$ and $t : X \to X'$ an element of S. Choose a distinguished triangle (X, Y, Z, f, g, h). Next, choose a distinguished triangle $(X', Y', Z, f', g', t[1] \circ h)$ (here we use TR1 and TR2). By MS5, MS6 (and TR2 to rotate) we can find the dotted arrow in the commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "t"] & Y \arrow[r] \arrow[d, dotted, "s'"] & Z \arrow[r] \arrow[d, "1"] & X[1] \arrow[d, "t[1]"] \\ X' \arrow[r] & Y' \arrow[r] & Z \arrow[r] & X'[1] \end{tikzcd}$$

with moreover $s' \in S$. This proves LMS2. The proof of RMS2 is dual. □

Lemma 11.5.3. *Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor of pre-triangulated categories. Let*

$$S = \{f \in Arrows(\mathcal{D}) \mid F(f) \text{ is an isomorphism}\}$$

Then S is a saturated (see Categories, Definition 4.24.17) multiplicative system compatible with the triangulated structure on $\mathcal{D}$.

Proof. We have to prove axioms MS1 -- MS6, see Categories, Definitions 4.24.1 and 4.24.17 and Definition 11.5.1. MS1, MS4, and MS5 are direct from the definitions. MS6 follows from TR3 and Lemma 11.4.3. By Lemma 11.5.2 we conclude that MS2 holds. To finish the proof we have to show that MS3 holds. To do this let $f, g : X \to Y$ be morphisms of $\mathcal{D}$, and let $t : Z \to X$ be an element of S such that $f \circ t = g \circ t$. As $\mathcal{D}$ is additive this simply means that $a \circ t = 0$ with $a = f - g$. Choose a distinguished triangle (Z, X, Q, t, g, h) using TR1 and TR2. Since $a \circ t = 0$ we see by Lemma 11.4.2 there exists a morphism $i : Q \to Y$ such that $i \circ g = a$. Finally, using TR1 again we can choose a triangle (Q, Y, W, i, j, k). Here is a picture

$$\begin{tikzcd} Z \arrow[r, "t"'] & X \arrow[r, "g"'] \arrow[d, "1"] & Q \arrow[r] \arrow[d, "i"] & Z[1] \\ & X \arrow[r, "a"'] & Y \arrow[d, "j"] & \\ & & W & \end{tikzcd}$$

OK, and now we apply the functor F to this diagram. Since $t \in S$ we see that $F(Q) = 0$, see Lemma 11.4.7. Hence $F(j)$ is an isomorphism by the same lemma, i.e., $j \in S$. Finally, $j \circ a = j \circ i \circ g = 0$ as $j \circ i = 0$. Thus $j \circ f = j \circ g$ and we see that LMS3 holds. The proof of RMS3 is dual. $\square$

Lemma 11.5.4. *Let $H : \mathcal{D} \to \mathcal{A}$ be a homological functor between a pre-triangulated category and an abelian category. Let*

$$S = \{f \in Arrows(\mathcal{D}) \mid H^i(f) \text{ is an isomorphism for all } i \in \mathbf{Z}\}$$

Then S is a saturated (see Categories, Definition 4.24.17) multiplicative system compatible with the triangulated structure on $\mathcal{D}$.

Proof. We have to prove axioms MS1 -- MS6, see Categories, Definitions 4.24.1 and 4.24.17 and Definition 11.5.1. MS1, MS4, and MS5 are direct from the definitions. MS6 follows from TR3 and the long exact cohomology sequence (11.3.5.1). By Lemma 11.5.2 we conclude that MS2 holds. To finish the proof we have to show that MS3 holds. To do this let $f, g : X \to Y$ be morphisms of $\mathcal{D}$, and let $t : Z \to X$ be an element of S such that $f \circ t = g \circ t$. As $\mathcal{D}$ is additive this simply means that $a \circ t = 0$ with $a = f - g$. Choose a distinguished triangle (Z, X, Q, t, g, h) using TR1 and TR2. Since $a \circ t = 0$ we see by Lemma 11.4.2 there exists a morphism $i : Q \to Y$ such that $i \circ g = a$. Finally, using TR1 again we can choose a triangle (Q, Y, W, i, j, k). Here is a picture

$$\begin{tikzcd} Z \arrow[r, "t"'] & X \arrow[r, "g"'] \arrow[d, "1"] & Q \arrow[r] \arrow[d, "i"] & Z[1] \\ & X \arrow[r, "a"'] & Y \arrow[d, "j"] & \\ & & W & \end{tikzcd}$$

OK, and now we apply the functors H^i to this diagram. Since $t \in S$ we see that $H^i(Q) = 0$ by the long exact cohomology sequence (11.3.5.1). Hence $H^i(j)$ is an isomorphism for all i by the same argument, i.e., $j \in S$. Finally, $j \circ a = j \circ i \circ g = 0$ as $j \circ i = 0$. Thus $j \circ f = j \circ g$ and we see that LMS3 holds. The proof of RMS3 is dual. □

Proposition 11.5.5. *Let $\mathcal{D}$ be a pre-triangulated category. Let S be a multiplicative system compatible with the triangulated structure. Then there exists a unique structure of a pre-triangulated category on $S^{-1}\mathcal{D}$ such that the localization functor $Q : \mathcal{D} \to S^{-1}\mathcal{D}$ is exact. Moreover, if $\mathcal{D}$ is a triangulated category, so is $S^{-1}\mathcal{D}$.*

Proof. We have seen that $S^{-1}\mathcal{D}$ is an additive category and that the localization functor Q is additive in Homology, Lemma 10.6.2, It is clear that we may define $Q(X)[n] = Q(X[n])$ since $\mathcal{S}$ is preserved under the shift functors $[n]$ by MS5. Finally, we say a triangle of $S^{-1}\mathcal{D}$ is distinguished if it is isomorphic to the image of a distinguished triangle under the localization functor Q.

Proof of TR1. The only thing to prove here is that if $a : Q(X) \to Q(Y)$ is a morphism of $S^{-1}\mathcal{D}$, then a fits into a distinguish triangle. Write $a = Q(s)^{-1} \circ Q(f)$ for some $s : Y \to Y'$ in S and $f : X \to Y'$. Choose a distinguished triangle (X, Y', Z, f, g, h) in $\mathcal{D}$. Then we see that $(Q(X), Q(Y), Q(Z), a, Q(g) \circ Q(s), Q(h))$ is a distinguished triangle of $S^{-1}\mathcal{D}$.

Proof of TR2. This is immediate from the definitions.

Proof of TR3. Note that the existence of the dotted arrow which is required to exist may be proven after replacing the two triangles by isomorphic triangles. Hence we may assume given distinguished triangles (X, Y, Z, f, g, h) and (X', Y', Z', f', g', h') of $\mathcal{D}$ and a commutative diagram

$$\begin{tikzcd} Q(X) \arrow[r, "Q(f)"'] \arrow[d, "a"'] & Q(Y) \arrow[d, "b"] \\ Q(X') \arrow[r, "Q(f')"] & Q(Y') \end{tikzcd}$$

in $S^{-1}\mathcal{D}$. Now we apply Categories, Lemma 4.24.8 to find a morphism $f'' : X'' \to Y''$ in $\mathcal{D}$ and a commutative diagram

$$\begin{tikzcd} X \arrow[r, "k"'] \arrow[d, "f"'] & X'' \arrow[d, "f''"] & X' \arrow[l, "s"'] \arrow[d, "f'"] \\ Y \arrow[r, "l"] & Y'' & Y' \arrow[l, "t"] \end{tikzcd}$$

in $\mathcal{D}$ with $s, t \in S$ and $a = s^{-1}k$, $b = t^{-1}l$. At this point we can use TR3 for $\mathcal{D}$ and MS6 to find a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "k"] & Y \arrow[r] \arrow[d, "l"] & Z \arrow[r] \arrow[d, "m"] & X[1] \arrow[d, "g[1]"] \\ X'' \arrow[r] & Y'' \arrow[r] & Z'' \arrow[r] & X''[1] \\ X' \arrow[r] \arrow[u, "s"'] & Y' \arrow[r] \arrow[u, "t"'] & Z' \arrow[r] \arrow[u, "r"'] & X'[1] \arrow[u, "s[1]"'] \end{tikzcd}$$

with $r \in S$. It follows that setting $c = Q(r)^{-1}Q(m)$ we obtain the desired morphism of triangles

$$\begin{array}{c} (Q(X), Q(Y), Q(Z), Q(f), Q(g), Q(h)) \\ \downarrow{\scriptstyle (a,b,c)} \\ (Q(X'), Q(Y'), Q(Z'), Q(f'), Q(g'), Q(h')) \end{array}$$

This proves the first statement of the lemma. If $\mathcal{D}$ is also a triangulated category, then we still have to prove TR4 in order to show that $S^{-1}\mathcal{D}$ is triangulated as well. To do this we reduce by Lemma 11.4.13 to the following statement: Given composable morphisms $a : Q(X) \to Q(Y)$ and $b : Q(Y) \to Q(Z)$ we have to produce an octahedron after possibly replacing $Q(X), Q(Y), Q(Z)$ by isomorphic objects. To do this we may first replace Y by an object such that $a = Q(f)$ for some morphism $f : X \to Y$ in $\mathcal{D}$. (More precisely, write $a = s^{-1}f$ with $s : Y \to Y'$ in S and $f : X \to Y'$. Then replace Y by Y'.) After this we similarly replace Z by an object such that $b = Q(g)$ for some morphism $g : Y \to Z$. Now we can find distinguished triangles (X, Y, Q_1, f, p_1, d_1), $(X, Z, Q_2, g \circ f, p_2, d_2)$, and (Y, Z, Q_3, g, p_3, d_3) in $\mathcal{D}$ (by TR1), and morphisms $a : Q_1 \to Q_2$ and $b : Q_2 \to Q_3$ as in TR4. Then it is immediately verified that applying the functor Q to all these data gives a corresponding structure in $S^{-1}\mathcal{D}$ □

The universal property of the localization of a triangulated category is as follows (we formulate this for pre-triangulated categories, hence it holds a fortiori for triangulated categories).

Lemma 11.5.6. *Let $\mathcal{D}$ be a pre-triangulated category. Let S be a multiplicative system compatible with the triangulated category. Let $Q : \mathcal{D} \to S^{-1}\mathcal{D}$ be the localization functor, see Proposition 11.5.5.*

(1) *If $H : \mathcal{D} \to \mathcal{A}$ is a homological functor into an abelian category $\mathcal{A}$ such that $H(s)$ is an isomorphism for all $s \in S$, then the unique factorization $H' : S^{-1}\mathcal{D} \to \mathcal{A}$ such that $H = H' \circ Q$ (see Categories, Lemma 4.24.6) is a homological functor too.*
(2) *If $F : \mathcal{D} \to \mathcal{D}'$ is an exact functor into a pre-triangulated category $\mathcal{D}'$ such that $F(s)$ is an isomorphism for all $s \in S$, then the unique factorization $F' : S^{-1}\mathcal{D} \to \mathcal{D}'$ such that $F = F' \circ Q$ (see Categories, Lemma 4.24.6) is an exact functor too.*

Proof. This lemma proves itself. Details omitted. □

The following lemma describes the kernel (see Definition 11.6.5) of the localization functor.

Lemma 11.5.7. *Let $\mathcal{D}$ be a pre-triangulated category. Let S be a multiplicative system compatible with the triangulated structure. Let Z be an object of $\mathcal{D}$. The following are equivalent*

(1) *$Q(Z) = 0$ in $S^{-1}\mathcal{D}$,*
(2) *there exists $Z' \in Ob(\mathcal{D})$ such that $0 : Z \to Z'$ is an element of S,*
(3) *there exists $Z' \in Ob(\mathcal{D})$ such that $0 : Z' \to Z$ is an element of S, and*
(4) *there exists an object Z' and a distinguished triangle $(X, Y, Z \oplus Z', f, g, h)$ such that $f \in S$.*

If S is saturated, then these are also equivalent to

(4) *the morphism $0 \to Z$ is an element of S,*
(5) *the morphism $Z \to 0$ is an element of S,*
(6) *there exists a distinguished triangle (X, Y, Z, f, g, h) such that $f \in S$.*

Proof. The equivalence of (1), (2), and (3) is Homology, Lemma 10.6.3. If (2) holds, then $(Z'[-1], Z'[-1] \oplus Z, Z, (1, 0), (0, 1), 0)$ is a distinguised triangle (see Lemma 11.4.9) with ``$0 \in S$". By rotating we conclude that (4) holds. If $(X, Y, Z \oplus Z', f, g, h)$ is a distinguished triangle with $f \in S$ then $Q(f)$ is an isomorphism hence $Q(Z \oplus Z') = 0$ hence $Q(Z) = 0$. Thus (1) -- (4) are all equivalent.

Next, assume that S is saturated. Note that each of (4), (5), (6) implies one of the equivalent conditions (1) -- (4). Suppose that $Q(Z) = 0$. Then $0 \to Z$ is a morphism of $\mathcal{D}$ which becomes an isomorphism in $S^{-1}\mathcal{D}$. According to Categories, Lemma 4.24.18 the fact that S is saturated implies that $0 \to Z$ is in S. Hence (1) $\Rightarrow$ (4). Dually (1) $\Rightarrow$ (5). Finally, if $0 \to Z$ is in S, then the triangle $(0, Z, Z, 0, \mathrm{id}_Z, 0)$ is distinguished by TR1 and TR2 and is a triangle as in (4). □

Lemma 11.5.8. *Let $\mathcal{D}$ be a triangulated category. Let S be a saturated multiplicative system in $\mathcal{D}$. Let (X, Y, Z, f, g, h) be a distinguished triangle in $\mathcal{D}$. Consider the category of morphisms of triangles*

$$\mathcal{I} = \{(s, s', s'') : (X, Y, Z, f, g, h) \to (X', Y', Z', f', g', h') \mid (s, s', s'') \in S\}$$

Then $\mathcal{I}$ is a filtered category and the functors $\mathcal{I} \to X/S$, $\mathcal{I} \to Y/S$, and $\mathcal{I} \to Z/S$ are surjective on objects.

Proof. We strongly suggest the reader skip the proof of this lemma and instead works it out on a napkin. The category $\mathcal{I}$ is nonempty as the identity provides an object. This proves the first condition of the definition of a filtered category, see Categories, Definition 4.17.1.

Note that if $s : X \to X'$ is a morphism of S, then using MS2 we can find $s' : Y \to Y'$ and $f' : X' \to Y'$ such that $f' \circ s = s' \circ f$, whereupon we can use MS6 to complete this into an object of $\mathcal{I}$. Hence certainly the surjectivity statement is correct.

Next we check condition (3) of Categories, Definition 4.17.1. Suppose $(s_1, s_1', s_1'') : (X, Y, Z) \to (X_1, Y_1, Z_1)$ and $(s_2, s_2', s_2'') : (X, Y, Z) \to (X_2, Y_2, Z_2)$ are objects of $\mathcal{I}$, and suppose $(a, b, c), (a', b', c')$ are two morphisms between them. Since $a \circ s_1 = a' \circ s_1$ there exists a morphism $s_3 : X_2 \to X_3$ such that $s_3 \circ a = s_3 \circ a'$. Using the surjectivity statement we can complete this to a morphism of triangles $(s_3, s_3', s_3'') : (X_2, Y_2, Z_2) \to (X_3, Y_3, Z_3)$ with $s_3, s_3', s_3'' \in S$. Thus $(s_3 \circ s_2, s_3' \circ s_2', s_3'' \circ s_2'') : (X, Y, Z) \to (X_3, Y_3, Z_3)$ is also an object of $\mathcal{I}$ and after composing the maps $(a, b, c), (a', b', c')$ with (s_3, s_3', s_3'') we obtain $a = a'$. By rotating we may do the same to get $b = b'$ and $c = c'$.

Finally, we check condition (2) of Categories, Definition 4.17.1. Suppose we are given two objects $(s_1, s_1', s_1'') : (X, Y, Z) \to (X_1, Y_1, Z_1)$ and $(s_2, s_2', s_2'') : (X, Y, Z) \to (X_2, Y_2, Z_2)$ of $\mathcal{I}$. Pick a morphism $s_3 : X \to X_3$ in S such that there exist morphisms $a : X_1 \to X_3$ and $a' : X_2 \to X_3$ with $s_3 = a \circ s_1$ and $s_3 = a' \circ s_2$. Because S is a saturated multiplicative system we see that $a' \in S$ (because S is the set of arrows of $\mathcal{D}$ which are turned into isomorphisms in $S^{-1}\mathcal{D}$, see Categories, Lemma 4.24.18). Hence, by the essential surjectivity above, we can find a morphism $(a, b, c) : (X_2, Y_2, Z_2) \to (X_3, Y_3, Z_3)$ with $a, b, c \in S$ such that $X \to X_3$ factors through $X \to X_1$. Replacing (X_2, Y_2, Z_2) by (X_3, Y_3, Z_3) and repeating this argument twice more, we may assume that $s_2 = a \circ s_1$, $s_2' = b \circ s_1'$, and $s_2'' = c \circ s_1''$. The problem is that it may not be the case that $b \circ f_1 = f_2 \circ a$, etc, i.e., we don't know that (a, b, c) is a morphism of triangles from (X_1, Y_1, Z_1) to (X_2, Y_2, Z_2). On the other hand, we do know that $(a, b, c) \circ (s_1, s_1', s_1'')$ is a morphism of triangles. Using MS3 one more time this means there exist morphisms $s_3 : X_2 \to X_3$, $s_3' : Y_2 \to Y_3$, and $s_3'' : Z_2 \to Z_3$ in S such that the required equalities hold after post-composing with them, e.g., $s_3' \circ b \circ f_1 = s_3' \circ f_2 \circ a$, etc. Using the essential surjectivity above once more we see that we may find a morphism of triangles $(s_4, s_4', s_4'') : (X_2, Y_2, Z_2) \to (X_4, Y_4, Z_4)$ with $s_4, s_4', s_4'' \in S$ such that s_4 factors through s_3, s_4' factors through s_3', and s_4'' factors through s_3''. We conclude that $(s_4 \circ a, s_4' \circ b, s_4'' \circ c)$ is a morphism of triangles from (X_1, Y_1, Z_1) to (X_4, Y_4, Z_4). □

11.6. Quotients of triangulated categories

Given a triangulated category and a triangulated subcategory we can construct another triangulated category by taking the ``quotient". The construction uses a localization. This is similar to the quotient of an abelian category by a Serre subcategory, see Homology, Section 10.7. Before we do the actual construction we briefly discuss kernels of exact functors.

Definition 11.6.1. Let $\mathcal{D}$ be a pre-triangulated category. We say a full pre-triangulated subcategory $\mathcal{D}'$ of $\mathcal{D}$ is *saturated* if whenever $X \oplus Y$ is isomorphic to an object of $\mathcal{D}'$ then both X and Y are isomorphic to objects of $\mathcal{D}'$.

Lemma 11.6.2. *Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor of pre-triangulated categories. Let $\mathcal{D}''$ be the full subcategory of $\mathcal{D}$ with objects*

$$Ob(\mathcal{D}'') = \{X \in Ob(\mathcal{D}) \mid F(X) = 0\}$$

Then $\mathcal{D}''$ is a strictly full saturated pre-triangulated subcategory of $\mathcal{D}$. If $\mathcal{D}$ is a triangulated category, then $\mathcal{D}''$ is a triangulated subcategory.

Proof. It is clear that $\mathcal{D}''$ is preserved under [1] and [−1]. If (X, Y, Z, f, g, h) is a distinguished triangle of $\mathcal{D}$ and $F(X) = F(Y) = 0$, then also $F(Z) = 0$ as $(F(X), F(Y), F(Z), F(f), F(g), F(h))$ is distinguished. Hence we may apply Lemma 11.4.14 to see that $\mathcal{D}''$ is a pre-triangulated subcategory (respectively a triangulated subcategory if $\mathcal{D}$ is a triangulated category). The final assertion of being saturated follows from $F(X) \oplus F(Y) = 0 \Rightarrow F(X) = F(Y) = 0$. □

Lemma 11.6.3. *Let $H : \mathcal{D} \to \mathcal{A}$ be a homological functor of a pre-triangulated category into an abelian category. Let $\mathcal{D}'$ be the full subcategory of $\mathcal{D}$ with objects*

$$Ob(\mathcal{D}') = \{X \in Ob(\mathcal{D}) \mid H(X[n]) = 0 \textit{ for all } n \in \mathbf{Z}\}$$

Then $\mathcal{D}'$ is a strictly full saturated pre-triangulated subcategory of $\mathcal{D}$. If $\mathcal{D}$ is a triangulated category, then $\mathcal{D}'$ is a triangulated subcategory.

Proof. It is clear that $\mathcal{D}'$ is preserved under [1] and [−1]. If (X, Y, Z, f, g, h) is a distinguished triangle of $\mathcal{D}$ and $H(X[n]) = H(Y[n]) = 0$ for all n, then also $H(Z[n]) = 0$ for all n by the long exact sequence (11.3.5.1). Hence we may apply Lemma 11.4.14 to see that $\mathcal{D}'$ is a pre-triangulated subcategory (respectively a triangulated subcategory if $\mathcal{D}$ is a triangulated category). The assertion of being saturated follows from

$$\begin{aligned} H((X \oplus Y)[n]) = 0 &\Rightarrow H(X[n] \oplus Y[n]) = 0 \\ &\Rightarrow H(X[n]) \oplus H(Y[n]) = 0 \\ &\Rightarrow H(X[n]) = H(Y[n]) = 0 \end{aligned}$$

for all $n \in \mathbf{Z}$. □

Lemma 11.6.4. *Let $H : \mathcal{D} \to \mathcal{A}$ be a homological functor of a pre-triangulated category into an abelian category. Let $\mathcal{D}^+_H, \mathcal{D}^-_H, \mathcal{D}^b_H$ be the full subcategory of $\mathcal{D}$ with objects*

$$\begin{aligned} Ob(\mathcal{D}^+_H) &= \{X \in Ob(\mathcal{D}) \mid H(X[n]) = 0 \text{ for all } n \ll 0\} \\ Ob(\mathcal{D}^-_H) &= \{X \in Ob(\mathcal{D}) \mid H(X[n]) = 0 \text{ for all } n \gg 0\} \\ Ob(\mathcal{D}^b_H) &= \{X \in Ob(\mathcal{D}) \mid H(X[n]) = 0 \text{ for all } |n| \gg 0\} \end{aligned}$$

Each of these is a strictly full saturated pre-triangulated subcategory of $\mathcal{D}$. If $\mathcal{D}$ is a triangulated category, then each is a triangulated subcategory.

Proof. Let us prove this for $\mathcal{D}^+_H$. It is clear that it is preserved under $[1]$ and $[-1]$. If (X, Y, Z, f, g, h) is a distinguished triangle of $\mathcal{D}$ and $H(X[n]) = H(Y[n]) = 0$ for all $n \ll 0$, then also $H(Z[n]) = 0$ for all $n \ll 0$ by the long exact sequence (11.3.5.1). Hence we may apply Lemma 11.4.14 to see that $\mathcal{D}^+_H$ is a pre-triangulated subcategory (respectively a triangulated subcategory if $\mathcal{D}$ is a triangulated category). The assertion of being saturated follows from

$$\begin{aligned} H((X \oplus Y)[n]) = 0 &\Rightarrow H(X[n] \oplus Y[n]) = 0 \\ &\Rightarrow H(X[n]) \oplus H(Y[n]) = 0 \\ &\Rightarrow H(X[n]) = H(Y[n]) = 0 \end{aligned}$$

for all $n \in \mathbf{Z}$. □

Definition 11.6.5. Let $\mathcal{D}$ be a (pre-)triangulated category.

(1) Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor. The *kernel of F* is the strictly full saturated (pre-)triangulated subcategory described in Lemma 11.6.2.
(2) Let $H : \mathcal{D} \to \mathcal{A}$ be a homological functor. The *kernel of H* is the strictly full saturated (pre-)triangulated subcategory described in Lemma 11.6.3.

These are sometimes denoted $\text{Ker}(F)$ or $\text{Ker}(H)$.

The proof of the following lemma uses TR4.

Lemma 11.6.6. *Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{D}' \subset \mathcal{D}$ be a full triangulated subcategory. Set*

$$(11.6.6.1) \qquad S = \left\{ \begin{array}{c} f \in Arrows(\mathcal{D}) \text{ such that there exists a distinguished triangle} \\ (X, Y, Z, f, g, h) \text{ of } \mathcal{D} \text{ with } Z \text{ isomorphic to an object of } \mathcal{D}' \end{array} \right\}$$

Then S is a multiplicative system compatible with the triangulated structure on $\mathcal{D}$. In this situation the following are equivalent

(1) *S is a saturated multiplicative system,*
(2) *$\mathcal{D}'$ is a saturated triangulated subcategory.*

Proof. To prove the first assertion we have to prove that MS1, MS2, MS3 and MS5, MS6 hold.

Proof of MS1. It is clear that identities are in S because $(X, X, 0, 1, 0, 0)$ is distinguished for every object X of $\mathcal{D}$ and because 0 is an object of $\mathcal{D}'$. Let $f : X \to Y$ and $g : Y \to Z$ be composable morphisms contained in S. Choose distinguished triangles (X, Y, Q_1, f, p_1, d_1), $(X, Z, Q_2, g \circ f, p_2, d_2)$, and (Y, Z, Q_3, g, p_3, d_3). By assumption we know that Q_1 and Q_3 are isomorphic to objects of $\mathcal{D}'$. By TR4 we know there exists a distinguished triangle (Q_1, Q_2, Q_3, a, b, c). Since $\mathcal{D}'$ is a triangulated subcategory we conclude that Q_2 is isomorphic to an object of $\mathcal{D}'$. Hence $g \circ f \in S$.

Proof of MS3. Let $a : X \to Y$ be a morphism and let $t : Z \to X$ be an element of S such that $a \circ t = 0$. To prove LMS3 we have to find a $s \in S$ such that $s \circ a = 0$. Choose a distinguished triangle (Z, X, Q, t, g, h) using TR1 and TR2. Since $a \circ t = 0$ we see by Lemma 11.4.2 there exists a morphism $i : Q \to Y$ such that $i \circ g = a$. Finally, using TR1 again we can choose a triangle (Q, Y, W, i, j, k). Here is a picture

$$\begin{tikzcd}
Z \arrow[r, "t"'] & X \arrow[r, "g"'] \arrow[d, "1"] & Q \arrow[r] \arrow[d, "i"] & Z[1] \\
& X \arrow[r, "a"'] & Y \arrow[d, "j"] & \\
& & W &
\end{tikzcd}$$

Since $t \in S$ we see that Q is isomorphic to an object of $\mathcal{D}'$. Hence $j \in S$. Finally, $j \circ a = j \circ i \circ g = 0$ as $j \circ i = 0$. Thus $j \circ f = j \circ g$ and we see that LMS3 holds. The proof of RMS3 is dual.

Proof of MS5. Follows as distinguished triangles and $\mathcal{D}'$ are stable under translations

Proof of MS6. Suppose given a commutative diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "s"] & Y \arrow[d, "s'"] \\
X' \arrow[r] & Y'
\end{tikzcd}$$

with $s, s' \in S$. By Proposition 11.4.21 we can extend this to a nine square diagram. As s, s' are elments of S we see that X'', Y'' are isomorphic to objects of $\mathcal{D}'$. Since $\mathcal{D}'$ is a full triangulated subcategory we see that Z'' is also an object of $\mathcal{D}'$. Whence the morphism $Z' \to Z''$ is an element of S. This proves MS6.

MS2 is a formal consquence of MS1, MS5, and MS6, see Lemma 11.5.2. This finishes the proof of the first assertion of the lemma.

Let's assume that S is saturated. (In the following we will use rotation of distinguished triangles without further mention.) Let $X \oplus Y$ be an object isomorphic to an object of $\mathcal{D}'$. Consider the morphism $f : 0 \to X$. The composition $0 \to X \to X \oplus Y$ is an element of S as $(0, X \oplus Y, X \oplus Y, 0, 1, 0)$ is a distinguished triangle. The composition $Y[-1] \to 0 \to X$ is an element of S as $(X, X \oplus Y, Y, (1, 0), (0, 1), 0)$ is a distinguished triangle, see Lemma 11.4.9. Hence $0 \to X$ is an element of S (as S is saturated). Thus X is isomorphic to an object of $\mathcal{D}'$ as desired.

Finally, assume $\mathcal{D}'$ satisfies condition (2) of the lemma. Let

$$W \xrightarrow{h} X \xrightarrow{g} Y \xrightarrow{f} Z$$

be composable morphisms of $\mathcal{D}$ such that $fg, gh \in S$. We will build up a picture of objects as in the diagram below.

$$
\begin{tikzcd}
 & & Q_{12} \arrow[dl, "+1"'] & & Q_{23} \arrow[ll, "+1"'] \arrow[dl, "+1"'] & & \\
 & Q_1 \arrow[dl, "+1"'] & & Q_2 \arrow[ll, "+1"'] \arrow[ul] \arrow[dl, "+1"'] & & Q_3 \arrow[ll, "+1"'] \arrow[ul] \arrow[dl, "+1"'] & \\
W \arrow[rr] & & X \arrow[rr] \arrow[ul] & & Y \arrow[rr] \arrow[ul] & & Z \arrow[ul]
\end{tikzcd}
$$

First choose distinguished triangles (W, X, Q_1), (X, Y, Q_2), (Y, Z, Q_3) (W, Y, Q_{12}), and (X, Z, Q_{23}). Denote $s : Q_2 \to Q_1[1]$ the composition $Q_2 \to X[1] \to Q_1[1]$. Denote $t : Q_3 \to Q_2[1]$ the composition $Q_3 \to Y[1] \to Q_2[1]$. By TR4 applied to the composition $W \to X \to Y$ and the composition $X \to Y \to Z$ there exist a distinguished triangles (Q_1, Q_{12}, Q_2) and (Q_2, Q_{23}, Q_3) which use the morphisms s and t. The objects Q_{12} and Q_{23} are isomorphic to objects of $\mathcal{D}'$ as $W \to Y$ and $X \to Z$ are assumed in S. Hence also $s[1]t$ is an element of S as S is closed under compositions and shifts. Note that $s[1]t = 0$ as $Y[1] \to Q_2[1] \to X[2]$ is zero, see Lemma 11.4.1. Hence $Q_3 \oplus Q_1[2]$ is isomorphic to an object of $\mathcal{D}'$, see Lemma 11.4.9. By assumption on $\mathcal{D}'$ we conclude that Q_3, Q_1 are isomorphic to objects of $\mathcal{D}'$. Looking at the distinguished triangle (Q_1, Q_{12}, Q_2) we conclude that Q_2 is also isomorphic to an object of $\mathcal{D}'$. Looking at the distinguished triangle (X, Y, Q_2) we finally conclude that $g \in S$. (It is also follows that $h, f \in S$, but we don't need this.) □

Definition 11.6.7. Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{B}$ be a full triangulated subcategory. We define the *quotient category* $\mathcal{D}/\mathcal{B}$ by the formula $\mathcal{D}/\mathcal{B} = S^{-1}\mathcal{D}$, where S is the multiplicative system of $\mathcal{D}$ associated to $\mathcal{B}$ via Lemma 11.6.6. The localization functor $Q : \mathcal{D} \to \mathcal{D}/\mathcal{B}$ is called the *quotient functor* in this case.

Note that the quotient functor $Q : \mathcal{D} \to \mathcal{D}/\mathcal{B}$ is an exact functor of triangulated categories, see Proposition 11.5.5. The universal property of this construction is the following.

Lemma 11.6.8. *Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{B}$ be a full triangulated subcategory of $\mathcal{D}$. Let $Q : \mathcal{D} \to \mathcal{D}/\mathcal{B}$ be the quotient functor.*

(1) *If $H : \mathcal{D} \to \mathcal{A}$ is a homological functor into an abelian category $\mathcal{A}$ such that $\mathcal{B} \subset Ker(H)$ then there exists a unique factorization $H' : \mathcal{D}/\mathcal{B} \to \mathcal{A}$ such that $H = H' \circ Q$ and H' is a homological functor too.*
(2) *If $F : \mathcal{D} \to \mathcal{D}'$ is an exact functor into a pre-triangulated category $\mathcal{D}'$ such that $\mathcal{B} \subset Ker(F)$ then there exists a unique factorization $F' : \mathcal{D}/\mathcal{B} \to \mathcal{D}'$ such that $F = F' \circ Q$ and F' is an exact functor too.*

Proof. This lemma follows from Lemma 11.5.6. Namely, if $f : X \to Y$ is a morphism of $\mathcal{D}$ such that for some distinguished triangle (X, Y, Z, f, g, h) the object Z is isomorphic to an object of $\mathcal{B}$, then $H(f)$, resp. $F(f)$ is an isomorphism under the assumptions of (1), resp. (2). Details omitted. □

The kernel of the quotient functor can be described as follows.

Lemma 11.6.9. *Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{B}$ be a full triangulated subcategory. The kernel of the quotient functor $Q : \mathcal{D} \to \mathcal{D}/\mathcal{B}$ is the strictly full subcategory of*

$\mathcal{D}$ whose objects are

$$Ob(Ker(Q)) = \left\{ \begin{array}{c} Z \in Ob(\mathcal{D}) \text{ such that there exists a } Z' \in Ob(\mathcal{D}) \\ \text{such that } Z \oplus Z' \text{ is isomorphic to an object of } \mathcal{B} \end{array} \right\}$$

In other words it is the smallest strictly full saturated triangulated subcategory of $\mathcal{D}$ containing $\mathcal{B}$.

Proof. First note that the kernel is automatically a strictly full triangulated subcategory stable containing summands of any of its objects, see Lemma 11.6.2. The description of its objects follows from the definitions and Lemma 11.5.7 part (4). □

Let $\mathcal{D}$ be a triangulated category. At this point we have constructions which induce order preserving maps between

(1) the partially ordered set of multiplicative systems S in $\mathcal{D}$ compatible with the triangulated structure, and
(2) the partially ordered set of full triangulated subcategories $\mathcal{B} \subset \mathcal{D}$.

Namely, the constructions are given by $S \mapsto \mathcal{B}(S) = \text{Ker}(Q : \mathcal{D} \to S^{-1}\mathcal{D})$ and $\mathcal{B} \mapsto S(\mathcal{B})$ where $S(\mathcal{B})$ is the multiplicative set of (11.6.6.1), i.e.,

$$S(\mathcal{B}) = \left\{ \begin{array}{c} f \in \text{Arrows}(\mathcal{D}) \text{ such that there exists a distinguished triangle} \\ (X, Y, Z, f, g, h) \text{ of } \mathcal{D} \text{ with } Z \text{ isomorphic to an object of } \mathcal{B} \end{array} \right\}$$

Note that it is not the case that these operations are mutually inverse.

Lemma 11.6.10. *Let $\mathcal{D}$ be a triangulated category. The operations described above have the following properties*

(1) *$S(\mathcal{B}(S))$ is the ``saturation'' of S, i.e., it is the smallest saturated multiplicative system in $\mathcal{D}$ containing S, and*
(2) *$\mathcal{B}(S(\mathcal{B}))$ is the ``saturation'' of $\mathcal{B}$, i.e., it is the smallest strictly full saturated triangulated subcategory of $\mathcal{D}$ containing $\mathcal{B}$.*

In particular, the constructions define mutually inverse maps between the (partially ordered) set of saturated multiplicative systems in $\mathcal{D}$ compatible with the triangulated structure on $\mathcal{D}$ and the (partially ordered) set of strictly full saturated triangulated subcategories of $\mathcal{D}$.

Proof. First, let's start with a full triangulated subcategory $\mathcal{B}$. Then $\mathcal{B}(S(\mathcal{B})) = \text{Ker}(Q : \mathcal{D} \to \mathcal{D}/\mathcal{B})$ and hence (2) is the content of Lemma 11.6.9.

Next, suppose that S is multiplicative system in $\mathcal{D}$ compatible with the triangulation on $\mathcal{D}$. Then $\mathcal{B}(S) = \text{Ker}(Q : \mathcal{D} \to S^{-1}\mathcal{D})$. Hence (using Lemma 11.4.7 in the localized category)

$$\begin{aligned} S(\mathcal{B}(S)) &= \left\{ \begin{array}{c} f \in \text{Arrows}(\mathcal{D}) \text{ such that there exists a distinguished} \\ \text{triangle } (X, Y, Z, f, g, h) \text{ of } \mathcal{D} \text{ with } Q(Z) = 0 \end{array} \right\} \\ &= \{ f \in \text{Arrows}(\mathcal{D}) \mid Q(f) \text{ is an isomorphism} \} \\ &= \hat{S} = S' \end{aligned}$$

in the notation of Categories, Lemma 4.24.18. The final statement of that lemma finishes the proof. □

Lemma 11.6.11. *Let $H : \mathcal{D} \to \mathcal{A}$ be a homological functor from a triangulated category $\mathcal{D}$ to an abelian category $\mathcal{A}$, see Definition 11.3.5. The subcategory $Ker(H)$ of $\mathcal{D}$ is a strictly* *full saturated triangulated subcategory of $\mathcal{D}$ whose corresponding saturated multiplicative system (see Lemma 11.6.10) is the set*

$$S = \{f \mid H^i(f) \text{ is an isomorphism for all } i \in \mathbf{Z}\}.$$

The functor H factors through the quotient functor $Q : \mathcal{D} \to \mathcal{D}/Ker(H)$.

Proof. The category $\mathrm{Ker}(H)$ is a strictly full saturated triangulated subcategory of $\mathcal{D}$ by Lemma 11.6.3. The set S is a saturated multiplicative system compatible with the triangulated structure by Lemma 11.5.4. Recall that the multiplicative system corresponding to $\mathrm{Ker}(H)$ is the set

$$\left\{ \begin{matrix} f \in \mathrm{Arrows}(K(\mathcal{A})) \text{ such that there exists a distinguished triangle} \\ (X, Y, Z, f, g, h) \text{ with } H^i(Z) = 0 \text{ for all } i \end{matrix} \right\}$$

By the long exact cohomology sequence, see (11.3.5.1), it is clear that f is an element of this set if and only if f is an element of S. Finally, the factorization of H through Q is a consequence of Lemma 11.6.8. □

It is clear that in the lemma above the factorization of H through $\mathcal{D}/\mathrm{Ker}(H)$ is the universal factorization. Namely, if $F : \mathcal{D} \to \mathcal{D}'$ is an exact functor of triangulated categories and if there exists a homological functor $H' : \mathcal{D}' \to \mathcal{A}$ such that $H \cong H' \circ F$, then F factors through the quotient functor $Q : \mathcal{D} \to \mathcal{D}/\mathrm{Ker}(H)$.

11.7. The homotopy category

Let $\mathcal{A}$ be an additive category. The homotopy category $K(\mathcal{A})$ of $\mathcal{A}$ is the the category of complexes of $\mathcal{A}$ with morphisms given by morphisms of complexes up to homotopy. Here is the formal definition.

Definition 11.7.1. Let $\mathcal{A}$ be an additive category.

(1) We set $\mathrm{Comp}(\mathcal{A}) = \mathrm{CoCh}(\mathcal{A})$ be the *category of (cochain) complexes*.
(2) A complex $K^\bullet$ is said to be *bounded below* if $K^n = 0$ for all $n \ll 0$.
(3) A complex $K^\bullet$ is said to be *bounded above* if $K^n = 0$ for all $n \gg 0$.
(4) A complex $K^\bullet$ is said to be *bounded* if $K^n = 0$ for all $|n| \gg 0$.
(5) We let $\mathrm{Comp}^+(\mathcal{A})$, $\mathrm{Comp}^-(\mathcal{A})$, resp. $\mathrm{Comp}^b(\mathcal{A})$ be the full subcategory of $\mathrm{Comp}(\mathcal{A})$ whose objects are the complexes which are bounded below, bounded above, resp. bounded.
(6) We let $K(\mathcal{A})$ be the category with the same objects as $\mathrm{Comp}(\mathcal{A})$ but as morphisms homotopy classes of maps of complexes (see Homology, Lemma 10.10.7).
(7) We let $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, resp. $K^b(\mathcal{A})$ be the full subcategory of $K(\mathcal{A})$ whose objects are bounded below, bounded above, resp. bounded complexes of $\mathcal{A}$.

It will turn out that the categories $K(\mathcal{A})$, $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, and $K^b(\mathcal{A})$ are triangulated categories. To prove this we first develop some machinery related to cones and split exact sequences.

11.8. Cones and termwise split sequences

Let $\mathcal{A}$ be an additive category, and let $K(\mathcal{A})$ denote the category of complexes of $\mathcal{A}$ with morphisms given by morphisms of complexes up to homotopy. Note that the shift functors $[n]$ on complexes, see Homology, Definition 10.12.7, give rise to functors $[n] : K(\mathcal{A}) \to K(\mathcal{A})$ such that $[n] \circ [m] = [n + m]$ and $[0] = \mathrm{id}$.

Definition 11.8.1. Let $\mathcal{A}$ be an additive category. Let $f : K^\bullet \to L^\bullet$ be a morphism of complexes of $\mathcal{A}$. The *cone* of f is the complex $C(f)^\bullet$ given by $C(f)^n = L^n \oplus K^{n+1}$ and differential

$$d^n_{C(f)} = \begin{pmatrix} d^n_L & f^{n+1} \\ 0 & -d^{n+1}_K \end{pmatrix}$$

It comes equipped with canonical morphisms of complexes $i : L^\bullet \to C(f)^\bullet$ and $p : C(f)^\bullet \to K^\bullet[1]$ induced by the obvious maps $L^n \to C(f)^n \to K^{n+1}$.

In other words $(K, L, C(f), f, i, p)$ forms a triangle:

$$K^\bullet \to L^\bullet \to C(f)^\bullet \to K^\bullet[1]$$

The formation of this triangle is functorial in the following sense.

Lemma 11.8.2. *Suppose that*

$$\begin{tikzcd} K_1^\bullet \arrow[r, "f_1"'] \arrow[d, "a"'] & L_1^\bullet \arrow[d, "b"] \\ K_2^\bullet \arrow[r, "f_2"] & L_2^\bullet \end{tikzcd}$$

is a diagram of morphisms of complexes which is commutative up to homotopy. Then there exists a morphism $c : C(f_1)^\bullet \to C(f_2)^\bullet$ which gives rise to a morphism of triangles $(a, b, c) : (K_1^\bullet, L_1^\bullet, C(f_1)^\bullet, f_1, i_1, p_1) \to (K_1^\bullet, L_1^\bullet, C(f_1)^\bullet, f_2, i_2, p_2)$ of $K(\mathcal{A})$.

Proof. Let $h^n : K_1^n \to L_2^{n-1}$ be a family of morphisms such that $f_2 \circ a - b \circ f_1 = d \circ h + h \circ d$. Define c^n by the matrix

$$c^n = \begin{pmatrix} a^n & h^{n+1} \\ 0 & b^n \end{pmatrix} : L_1^n \oplus K_1^{n+1} \to L_2^n \oplus K_2^{n+1}$$

A matrix computation show that c is a morphism of complexes. It is trivial that $c \circ i_1 = i_2 \circ b$, and it is trivial also to check that $p_2 \circ c = a \circ p_1$. □

Note that the morphism $c : C(f_1)^\bullet \to C(f_2)^\bullet$ constructed in the proof of Lemma 11.8.2 in general depends on the chosen homotopy h between $f_2 \circ a$ and $b \circ f_1$.

Definition 11.8.3. Let $\mathcal{A}$ be an additive category. A *termwise split injection $\alpha : A^\bullet \to B^\bullet$* is a morphism of complexes such that each $A^n \to B^n$ is isomorphic to the inclusion of a direct summand. A *termwise split surjection $\beta : B^\bullet \to C^\bullet$* is a morphism of complexes such that each $B^n \to C^n$ is isomorphic to the projection onto a direct summand.

Lemma 11.8.4. *Let $\mathcal{A}$ be an additive category. Let*

$$\begin{tikzcd} A^\bullet \arrow[r, "f"'] \arrow[d, "a"'] & B^\bullet \arrow[d, "b"] \\ C^\bullet \arrow[r, "g"] & D^\bullet \end{tikzcd}$$

be a diagram of morphisms of complexes commuting up to homotopy. If f is a split injection, then b is homotopic to a morphism which makes the diagram commute. If g is a split surjection, then a is homotopic to a morphism which makes the diagram commute.

Proof. Let $h^n : A^n \to D^{n-1}$ be a collection of morphisms such that $bf - ga = dh + hd$. Suppose that $\pi^n : B^n \to A^n$ are morphisms splitting the morphisms f^n. Take $b' = b + dh\pi + h\pi d$. Suppose $s^n : D^n \to C^n$ are morphisms splitting the morphisms $g^n : C^n \to D^n$. Take $a' = a + dsh + shd$. Computations omitted. □

The following lemma can be used to replace an morphism of complexes by a morphism where in each degree the map is the injection of a direct summand.

Lemma 11.8.5. *Let $\mathcal{A}$ be an additive category. Let $\alpha : K^\bullet \to L^\bullet$ be a morphism of complexes of $\mathcal{A}$. There exists a factorization*

$$\begin{tikzcd} K^\bullet \arrow[r, "\tilde{\alpha}"] \arrow[rr, bend right, "\alpha"'] & \tilde{L}^\bullet \arrow[r, "\pi"] & L^\bullet \end{tikzcd}$$

such that

(1) *$\tilde{\alpha}$ is a termwise split injection (see Definition 11.8.3),*
(2) *there is a map of complexes $s : L^\bullet \to \tilde{L}^\bullet$ such that $\pi \circ s = id_{L^\bullet}$ and such that $s \circ \pi$ is homotopic to $id_{\tilde{L}^\bullet}$.*

Moreover, if both $K^\bullet$ and $L^\bullet$ are in $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, or $K^b(\mathcal{A})$, then so is $\tilde{L}^\bullet$.

Proof. We set

$$\tilde{L}^n = L^n \oplus K^n \oplus K^{n+1}$$

and we define

$$d^n_{\tilde{L}} = \begin{pmatrix} d^n_L & 0 & 0 \\ 0 & d^n_K & \text{id}_{K^{n+1}} \\ 0 & 0 & -d^{n+1}_K \end{pmatrix}$$

Moreover, we set

$$\tilde{\alpha} = \begin{pmatrix} \alpha \\ \text{id}_{K^n} \\ 0 \end{pmatrix}$$

which is clearly a split injection. It is also clear that it defines a morphism of complexes. We define

$$\pi = \begin{pmatrix} \text{id}_{L^n} & 0 & 0 \end{pmatrix}$$

so that clearly $\pi \circ \tilde{\alpha} = \alpha$. We set

$$s = \begin{pmatrix} \text{id}_{L^n} \\ 0 \\ 0 \end{pmatrix}$$

so that $\pi \circ s = \text{id}_{L^\bullet}$. Finally, let $h^n : \tilde{L}^n \to \tilde{L}^{n-1}$ be the map which maps the summand K^n of $\tilde{L}^n$ via the identity morphism to the summand K^n of $\tilde{L}^{n-1}$. Then it is a trivial matter (see computations in remark below) to prove that

$$\text{id}_{\tilde{L}^\bullet} - s \circ \pi = d \circ h + h \circ d$$

which finishes the proof of the lemma. $\square$

Remark 11.8.6. To see the last displayed equality in the proof above we can argue with elements as follows. We have $s\pi(l, k, k^+) = (l, 0, 0)$. Hence the morphism of the left hand side maps (l, k, k^+) to $(0, k, k^+)$. On the other hand $h(l, k, k^+) = (0, 0, k)$ and $d(l, k, k^+) = (dl, dk + k^+, -dk^+)$. Hence $(dh + hd)(l, k, k^+) = d(0, 0, k) + h(dl, dk + k^+, -dk^+) = (0, k, -dk) + (0, 0, dk + k^+) = (0, k, k^+)$ as desired.

Lemma 11.8.7. *Let $\mathcal{A}$ be an additive category. Let $\alpha : K^\bullet \to L^\bullet$ be a morphism of complexes of $\mathcal{A}$. There exists a factorization*

$$\begin{tikzcd} K^\bullet \arrow[r, "i"] \arrow[rr, bend right, "\alpha"'] & \tilde{K}^\bullet \arrow[r, "\tilde{\alpha}"] & L^\bullet \end{tikzcd}$$

such that

(1) $\tilde{\alpha}$ *is a termwise split surjection (see Definition 11.8.3),*
(2) *there is a map of complexes* $s : \tilde{K}^\bullet \to K^\bullet$ *such that* $s \circ i = id_{K^\bullet}$ *and such that* $i \circ s$ *is homotopic to* $id_{\tilde{K}^\bullet}$*.*

Moreover, if both $K^\bullet$ *and* $L^\bullet$ *are in* $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, *or* $K^b(\mathcal{A})$, *then so is* $\tilde{K}^\bullet$*.*

Proof. Dual to Lemma 11.8.5. Take

$$\tilde{K}^n = K^n \oplus L^n \oplus L^{n+1}$$

and we define

$$d^n_{\tilde{K}} = \begin{pmatrix} d^n_K & 0 & 0 \\ 0 & d^n_L & \mathrm{id}_{L^{n+1}} \\ 0 & 0 & -d^{n+1}_L \end{pmatrix}$$

Moreover, we set

$$\tilde{\alpha} = \begin{pmatrix} \alpha & \mathrm{id}_{L^n} & 0 \end{pmatrix}$$

which is clearly a split surjection. It is also clear that it defines a morphism of complexes. We define

$$i = \begin{pmatrix} \mathrm{id}_{K^n} \\ 0 \\ 0 \end{pmatrix}$$

so that clearly $\tilde{\alpha} \circ i = \alpha$. We set

$$s = \begin{pmatrix} \mathrm{id}_{K^n} & 0 & 0 \end{pmatrix}$$

so that $s \circ i = \mathrm{id}_{K^\bullet}$. Finally, let $h^n : \tilde{K}^n \to \tilde{K}^{n-1}$ be the map which maps the summand L^n of $\tilde{K}^n$ via the identity morphism to the summand L^n of $\tilde{K}^{n-1}$. Then it is a trivial matter to prove that

$$\mathrm{id}_{\tilde{K}^\bullet} - i \circ s = d \circ h + h \circ d$$

which finishes the proof of the lemma. □

Definition 11.8.8. Let $\mathcal{A}$ be an additive category. A *termwise split sequence of complexes of* $\mathcal{A}$ is a complex of complexes

$$0 \to A^\bullet \xrightarrow{\alpha} B^\bullet \xrightarrow{\beta} C^\bullet \to 0$$

together with given direct sum decompositions $B^n = A^n \oplus C^n$ compatible with α^n and β^n. We often write $s^n : C^n \to B^n$ and $\pi^n : B^n \to A^n$ for the maps induced by the direct sum decompositions. According to Homology, Lemma 10.12.10 we get an associated morphism of complexes

$$\delta : C^\bullet \longrightarrow A^\bullet[1]$$

which in degree n is the map $\pi^{n+1} \circ d^n_C \circ s^n$. In other words $(A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta)$ forms a triangle

$$A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$$

This will be the *triangle associated to the termwise split sequence of complexes*.

Lemma 11.8.9. *Let* $\mathcal{A}$ *be an additive category. Let* $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ *be termwise split exact sequences as in Definition 11.8.8. Let* $(\pi')^n$, $(s')^n$ *be a second collection of splittings. Denote* $\delta' : C^\bullet \longrightarrow A^\bullet[1]$ *the morphism associated to this second set of splittings. Then*

$$(1, 1, 1) : (A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta) \longrightarrow (A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta')$$

is an isomorphism of triangles in $K(\mathcal{A})$*.*

Proof. The statement simply means that δ and δ' are homotopic maps of complexes. This is Homology, Lemma 10.12.12. □

Lemma 11.8.10. *Let $\mathcal{A}$ be an additive category. Let $0 \to A_i^\bullet \to B_i^\bullet \to C_i^\bullet \to 0$, $i = 1, 2$ be termwise split exact sequences. Suppose that $a : A_1^\bullet \to A_2^\bullet$, $b : B_1^\bullet \to B_2^\bullet$, and $c : C_1^\bullet \to C_2^\bullet$ are morphisms of complexes such that*

$$\begin{tikzcd} A_1^\bullet \arrow[r] \arrow[d, "a"'] & B_1^\bullet \arrow[r] \arrow[d, "b"'] & C_1^\bullet \arrow[d, "c"'] \\ A_2^\bullet \arrow[r] & B_2^\bullet \arrow[r] & C_2^\bullet \end{tikzcd}$$

commutes in $K(\mathcal{A})$. Then there exists a morphism $b' : B_1^\bullet \to B_2^\bullet$ which is homotopic to b such that the diagram above commutes in the category of complexes.

Proof. Let $f^n : A_1^n \to B_2^{n-1}$ be a collection of morphisms such that $b \circ \alpha_1 - \alpha_2 \circ a = d \circ f + f \circ d$. Let $g^n : B_1^n \to C_2^{n-1}$ be a collection of morphisms such that $c \circ \beta_1 - \beta_2 \circ b = d \circ g + g \circ d$. Suppose that $\pi^n : B_1^n \to A_1^n$ (resp. $s^n : C_2^n \to B_2^n$) are the morphisms splitting the morphisms α_1^n (resp. β_2^n). Set $h^n = -f^n \circ \pi^n + s^{n-1} \circ g^n$. Take $b' = b + d \circ h + h \circ d$. Computation omitted. □

Lemma 11.8.11. *Let $\mathcal{A}$ be an additive category. Let $f_1 : K_1^\bullet \to L_1^\bullet$ and $f_2 : K_2^\bullet \to L_2^\bullet$ be morphisms of complexes. Let*

$$(a, b, c) : (K_1^\bullet, L_1^\bullet, C(f_1)^\bullet, f_1, i_1, p_1) \longrightarrow (K_1^\bullet, L_1^\bullet, C(f_1)^\bullet, f_2, i_2, p_2)$$

be any morphism of triangles of $K(\mathcal{A})$. If a and b are homotopy equivalences then so is c.

Proof. Let $a^{-1} : K_2^\bullet \to K_1^\bullet$ be a morphism of complexes which is inverse to a in $K(\mathcal{A})$. Let $b^{-1} : L_2^\bullet \to L_1^\bullet$ be a morphism of complexes which is inverse to b in $K(\mathcal{A})$. Let $c' : C(f_2)^\bullet \to C(f_1)^\bullet$ be the morphism from Lemma 11.8.2 applied to $f_1 \circ a^{-1} = b^{-1} \circ f_2$. If we can show that $c \circ c'$ and $c' \circ c$ are isomorphisms in $K(\mathcal{A})$ then we win. Hence it suffices to prove the following: Given a morphism of triangles $(1, 1, c) : (K^\bullet, L^\bullet, C(f)^\bullet, f, i, p)$ in $K(\mathcal{A})$ the morphism c is an isomorphism in $K(\mathcal{A})$. By assumption the two squares in the diagram

$$\begin{tikzcd} L^\bullet \arrow[r] \arrow[d, "1"'] & C(f)^\bullet \arrow[r] \arrow[d, "c"'] & K^\bullet[1] \arrow[d, "1"'] \\ L^\bullet \arrow[r] & C(f)^\bullet \arrow[r] & K^\bullet[1] \end{tikzcd}$$

commute up to homotopy. By construction of $C(f)^\bullet$ the rows form termwise split sequences of complexes. By Lemma 11.8.10 we may replace c by a morphism homotopic to c such that the diagram commutes in the category of complexes. In this case each c^n is an isomorphism (because an upper triangular matrix with 1's on the diagonal is invertible). □

Hence if a and b are homotopy equivalences then the resulting morphism of triangles is an isomorphism of triangles in $K(\mathcal{A})$. It turns out that the collection of triangles of $K(\mathcal{A})$ given by cones and the collection of triangles of $K(\mathcal{A})$ given by termwise split sequences of complexes are the same up to isomorphisms, at least up to sign!

Lemma 11.8.12. *Let $\mathcal{A}$ be an additive category.*

(1) *Given a termwise split sequence of complexes* $(\alpha : A^\bullet \to B^\bullet, \beta : B^\bullet \to C^\bullet, s^n, \pi^n)$ *there exists a homotopy equivalence* $C(\alpha)^\bullet \to C^\bullet$ *such that the diagram*

$$\begin{tikzcd} A^\bullet \arrow[r] \arrow[d] & B^\bullet \arrow[r] \arrow[d] & C(\alpha)^\bullet \arrow[r, "-p"'] \arrow[d] & A^\bullet[1] \arrow[d] \\ A^\bullet \arrow[r] & B^\bullet \arrow[r] & C^\bullet \arrow[r, "\delta"] & A^\bullet[1] \end{tikzcd}$$

defines an isomorphism of triangles in $K(\mathcal{A})$.

(2) *Given a morphism of complexes* $f : K^\bullet \to L^\bullet$ *there exists an isomorphism of triangles*

$$\begin{tikzcd} K^\bullet \arrow[r] \arrow[d] & \tilde{L}^\bullet \arrow[r] \arrow[d] & M^\bullet \arrow[r, "\delta"'] \arrow[d] & K^\bullet[1] \arrow[d] \\ K^\bullet \arrow[r] & L^\bullet \arrow[r] & C(f)^\bullet \arrow[r, "-p"] & K^\bullet[1] \end{tikzcd}$$

where the upper triangle is the triangle associated to a termwise split exact sequence $K^\bullet \to \tilde{L}^\bullet \to M^\bullet$.

Proof. Proof of (1). We have $C(\alpha)^n = B^n \oplus A^{n+1}$ and we simply define $C(\alpha)^n \to C^n$ via the projection onto B^n followed by β^n. This defines a morphism of complexes because the compositions $A^{n+1} \to B^{n+1} \to B^n \to C^n$ are zero. To get a homotopy inverse we take $C^\bullet \to C(\alpha)^\bullet$ given by $(s^n, -\delta^n)$ in degree n. This is a morphism of complexes because the morphism δ^n can be characterized as the unique morphism $C^n \to A^{n+1}$ such that $d \circ s^n - s^{n+1} \circ d = \alpha \circ \delta^n$, see proof of Homology, Lemma 10.12.10. The composition $C^\bullet \to C(f)^\bullet \to C^\bullet$ is the identity. The composition $C(f)^\bullet \to C^\bullet \to C(f)^\bullet$ is equal to the morphism

$$\begin{pmatrix} s^n \circ \beta^n & 0 \\ -\delta^n \circ \beta^n & 0 \end{pmatrix}$$

To see that this is homotopic to the identity map use the homotopy $h^n : C(\alpha)^n \to C(\alpha)^{n-1})$ given by the matrix

$$\begin{pmatrix} 0 & 0 \\ \pi^n & 0 \end{pmatrix} : C(\alpha)^n = B^n \oplus A^{n+1} \to B^{n-1} \oplus A^n = C(\alpha)^{n-1}$$

It is trivial to verify that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} s^n & -\delta^n \end{pmatrix} \begin{pmatrix} \beta^n \\ 0 \end{pmatrix} = \begin{pmatrix} d & \alpha^{n+1} \\ 0 & -d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \pi^n & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \pi^{n+1} & 0 \end{pmatrix} \begin{pmatrix} d & \alpha^{n+1} \\ 0 & -d \end{pmatrix}$$

To finish the proof of (1) we have to show that the morphisms $-p : C(\alpha)^\bullet \to A^\bullet[1]$ (see Definition 11.8.1) and $C(\alpha)^\bullet \to C^\bullet \to A^\bullet[1]$ agree up to homotopy. This is clear from the above. Namely, we can use the homotopy inverse $(s, -\delta) : C^\bullet \to C(\alpha)^\bullet$ and check instead that the two maps $C^\bullet \to A^\bullet[1]$ agree. And note that $p \circ (s, -\delta) = -\delta$ as desired.

Proof of (2). We let $\tilde{f} : K^\bullet \to \tilde{L}^\bullet$, $s : L^\bullet \to \tilde{L}^\bullet$ and $\pi : L^\bullet \to L^\bullet$ be as in Lemma 11.8.5. By Lemmas 11.8.2 and 11.8.11 the triangles $(K^\bullet, L^\bullet, C(f), i, p)$ and $(K^\bullet, \tilde{L}^\bullet, C(\tilde{f}), \tilde{i}, \tilde{p})$ are isomorphic. Note that we can compose isomorphisms of triangles. Thus we may replace $L^\bullet$ by $\tilde{L}^\bullet$ and f by $\tilde{f}$. In other words we may assume that f is a termwise split injection. In this case the result follows from part (1). □

Lemma 11.8.13. *Let $\mathcal{A}$ be an additive category. Let $A_1^\bullet \to A_2^\bullet \to \ldots \to A_n^\bullet$ be a sequence of composable morphisms of complexes. There exists a commutative diagram*

$$\begin{tikzcd} A_1^\bullet \arrow[r] & A_2^\bullet \arrow[r] & \ldots \arrow[r] & A_n^\bullet \\ B_1^\bullet \arrow[r] \arrow[u] & B_2^\bullet \arrow[r] \arrow[u] & \ldots \arrow[r] & B_n^\bullet \arrow[u] \end{tikzcd}$$

such that each morphism $B_i^\bullet \to B_{i+1}^\bullet$ is a split injection and each $B_i^\bullet \to A_i^\bullet$ is a homotopy equivalence. Moreover, if all $A_i^\bullet$ are in $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, or $K^b(\mathcal{A})$, then so are the $B_i^\bullet$.

Proof. The case $n = 1$ is without content. Lemma 11.8.5 is the case $n = 2$. Suppose we have constructed the diagram except for B_n. Applying Lemma 11.8.5 to the composition $B_{n-1} \to A_{n-1} \to A_n$. The result is a factorization $B_{n-1} \to \tilde{B}_n \to A_n$ as desired. □

Lemma 11.8.14. *Let $\mathcal{A}$ be an additive category. Let $(\alpha : A^\bullet \to B^\bullet, \beta : B^\bullet \to C^\bullet, s^n, \pi^n)$ be a termwise split sequence of complexes. Let $(A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta)$ be the associated triangle. Then the triangle $(C^\bullet[-1], A^\bullet, B^\bullet, \delta[-1], \alpha, \beta)$ is isomorphic to the triangle $(C^\bullet[-1], A^\bullet, C(\delta[-1])^\bullet, \delta[-1], i, p)$.*

Proof. We write $B^n = A^n \oplus C^n$ and we identify α^n and β^n with the natural inclusion and projection maps. By construction of δ we have

$$d_B^n = \begin{pmatrix} d_A^n & \delta^n \\ 0 & d_C^n \end{pmatrix}$$

On the other hand the cone of $\delta[-1] : C^\bullet[-1] \to A^\bullet$ is given as $C(\delta[-1])^n = A^n \oplus C^n$ with differential identical with the matrix above! Whence the lemma. □

Lemma 11.8.15. *Let $\mathcal{A}$ be an additive category. Let $f : K^\bullet \to L^\bullet$ be a morphism of complexes. The triangle $(L^\bullet, C(f)^\bullet, K^\bullet[1], i, p, f[1])$ is the triangle associated to the termwise split sequence*

$$0 \to L^\bullet \to C(f)^\bullet \to K^\bullet[1] \to 0$$

coming from the definition of the cone of f.

Proof. Immediate from the definitions. □

11.9. Distinguished triangles in the homotopy category

Since we want our boundary maps in long exact sequences of cohomology to be given by the maps in the snake lemma without signs we define distinguished triangles in the homotopy category as follows.

Definition 11.9.1. Let $\mathcal{A}$ be an additive category. A triangle (X, Y, Z, f, g, h) of $K(\mathcal{A})$ is called a *distinguished triangle of* $K(\mathcal{A})$ if it is isomorphic to the triangle associated to a termwise split exact sequence of complexes, see Definition 11.8.8. Same definition for $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, and $K^b(\mathcal{A})$.

Note that according to Lemma 11.8.12 a triangle of the form $(K^\bullet, L^\bullet, C(f)^\bullet, f, i, -p)$ is a distinguished triangle. This does indeed lead to a triangulated category, see Lemma 11.8.12. Before we can prove the proposition we need one more lemma in order to be able to prove TR4.

Lemma 11.9.2. *Let $\mathcal{A}$ be an additive category. Suppose that $\alpha : A^\bullet \to B^\bullet$ and $\beta : B^\bullet \to C^\bullet$ are split injections of complexes. Then there exist distinguished triangles $(A^\bullet, B^\bullet, Q_1^\bullet, \alpha, p_1, d_1)$, $(A^\bullet, C^\bullet, Q_2^\bullet, \beta \circ \alpha, p_2, d_2)$ and $(B^\bullet, C^\bullet, Q_3^\bullet, \beta, p_3, d_3)$ for which TR4 holds.*

Proof. Say $\pi_1^n : B^n \to A^n$, and $\pi_3^n : C^n \to B^n$ are the splittings. Then also $A^\bullet \to C^\bullet$ is a split injection with splittings $\pi_2^n = \pi_1^n \circ \pi_3^n$. Let us write $Q_1^\bullet$, $Q_2^\bullet$ and $Q_3^\bullet$ for the ``quotient'' complexes. In other words, $Q_1^n = \mathrm{Ker}(\pi_1^n)$, $Q_3^n = \mathrm{Ker}(\pi_3^n)$ and $Q_2^n = \mathrm{Ker}(\pi_2^n)$. Note that the kernels exist. Then $B^n = A^n \oplus Q_1^n$ and $C_n = B^n \oplus Q_3^n$, where we think of A^n as a subobject of B^n and so on. This implies $C^n = A^n \oplus Q_1^n \oplus Q_3^n$. Note that $\pi_2^n = \pi_1^n \circ \pi_3^n$ is zero on both Q_1^n and Q_3^n. Hence $Q_2^n = Q_1^n \oplus Q_3^n$. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & A^\bullet & \to & B^\bullet & \to & Q_1^\bullet & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A^\bullet & \to & C^\bullet & \to & Q_2^\bullet & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & B^\bullet & \to & C^\bullet & \to & Q_3^\bullet & \to & 0 \end{array}$$

The rows of this diagram are termwise split exact sequences, and hence determine distinguished triangles by definition. Moreover downward arrows in the diagram above are compatible with the chosen splittings and hence define morphisms of triangles

$$(A^\bullet \to B^\bullet \to Q_1^\bullet \to A^\bullet[1]) \longrightarrow (A^\bullet \to C^\bullet \to Q_2^\bullet \to A^\bullet[1])$$

and

$$(A^\bullet \to C^\bullet \to Q_2^\bullet \to A^\bullet[1]) \longrightarrow (B^\bullet \to C^\bullet \to Q_3^\bullet \to B^\bullet[1]).$$

Note that the splittings $Q_3^n \to C^n$ of the bottom split sequence in the diagram provides a splitting for the split sequence $0 \to Q_1^\bullet \to Q_2^\bullet \to Q_3^\bullet \to 0$ upon composing with $C^n \to Q_2^n$. It follows easily from this that the morphism $\delta : Q_3^\bullet \to Q_1^\bullet[1]$ in the corresponding distinguished triangle

$$(Q_1^\bullet \to Q_2^\bullet \to Q_3^\bullet \to Q_1^\bullet[1])$$

is equal to the composition $Q_3^\bullet \to B^\bullet[1] \to Q_1^\bullet[1]$. Hence we get a structure as in the conclusion of axiom TR4. □

Proposition 11.9.3. *Let $\mathcal{A}$ be an additive category. The category $K(\mathcal{A})$ of complexes up to homotopy with its natural translation functors and distinguished triangles as defined above is a triangulated category.*

Proof. Proof of TR1. By definition every triangle isomorphic to a distinguished one is distinguished. Also, any triangle $(A^\bullet, A^\bullet, 0, 1, 0, 0)$ is distinguished since $0 \to A^\bullet \to A^\bullet \to 0 \to 0$ is a termwise split sequence of complexes. Finally, given any morphism of complexes $f : K^\bullet \to L^\bullet$ the triangle $(K, L, C(f), f, i, -p)$ is distinguished by Lemma 11.8.12.

Proof of TR2. Let (X, Y, Z, f, g, h) be a triangle. Assume $(Y, Z, X[1], g, h, -f[1])$ is distinguished. Then there exists a termwise split sequence of complexes $A^\bullet \to B^\bullet \to C^\bullet$ such that the associated triangle $(A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta)$ is isomorphic to $(Y, Z, X[1], g, h, -f[1])$. Rotating back we see that (X, Y, Z, f, g, h) is isomorphic to $(C^\bullet[-1], A^\bullet, B^\bullet, -\delta[-1], \alpha, \beta)$. It follows from Lemma 11.8.14 that the triangle $(C^\bullet[-1], A^\bullet, B^\bullet, \delta[-1], \alpha, \beta)$ is isomorphic to $(C^\bullet[-1], A^\bullet, C(\delta[-1])^\bullet, \delta[-1], i, p)$. Precomposing the previous isomorphism of triangles with -1 on Y it follows that (X, Y, Z, f, g, h) is isomorphic to $(C^\bullet[-1], A^\bullet, C(\delta[-1])^\bullet, \delta[-1], i, -p)$. Hence it is distinguished by Lemma 11.8.12. On the other hand, suppose that (X, Y, Z, f, g, h) is distinguished. By Lemma 11.8.12 this means that it is isomorphic to a triangle of the form $(K^\bullet, L^\bullet, C(f), f, i, -p)$ for some morphism of complexes f. Then the rotated triangle $(Y, Z, X[1], g, h, -f[1])$ is isomorphic to $(L^\bullet, C(f), K^\bullet[1], i, -p, -f[1])$ which is isomorphic to the triangle $(L^\bullet, C(f), K^\bullet[1], i, p, f[1])$. By Lemma 11.8.15 this triangle is distinguished. Hence $(Y, Z, X[1], g, h, -f[1])$ is distinguished as desired.

Proof of TR3. Let (X, Y, Z, f, g, h) and (X', Y', Z', f', g', h') be distinguished triangles of $K(\mathcal{A})$ and let $a : X \to X'$ and $b : Y \to Y'$ be morphisms such that $f' \circ a = b \circ f$. By Lemma 11.8.2 we may assume that $(X, Y, Z, f, g, h) = (X, Y, C(f), f, i, p)$ and $(X', Y', Z', f', g', h') = (X', Y', C(f'), f', i', p')$. At this point we simply apply Lemma 11.8.2 to the commutative diagram given by f, f', a, b.

Proof of TR4. At this point we know that $K(\mathcal{A})$ is a pre-triangulated category. Hence we can use Lemma 11.4.13. Let $A^\bullet \to B^\bullet$ and $B^\bullet \to C^\bullet$ be composable morphisms of $K(\mathcal{A})$. By Lemma 11.8.13 we may assume that $A^\bullet \to B^\bullet$ and $B^\bullet \to C^\bullet$ are split injective morphisms. In this case the result follows from Lemma 11.9.2. □

Remark 11.9.4. Let $\mathcal{A}$ be an additive category. Exactly the same proof as the proof of Proposition 11.9.3 shows that the categories $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, and $K^b(\mathcal{A})$ are triangulated categories. Namely, the cone of a morphisms between bounded (above, below) is bounded (above, below). But we prove below that these are triangulated subcategories of $K(\mathcal{A})$ which gives another proof.

Lemma 11.9.5. *Let $\mathcal{A}$ be an additive subcategory. The categories $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, and $K^b(\mathcal{A})$ are full triangulated subcategories of $K(\mathcal{A})$.*

Proof. Each of the categories mentioned is a full additive subcategory. We use the criterion of Lemma 11.4.14 to show that they are triangulated subcategories. It is clear that each of the categories $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, and $K^b(\mathcal{A})$ is preserved under the shift functors $[1], [-1]$. Finally, suppose that $f : A^\bullet \to B^\bullet$ is a morphism in $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, or $K^b(\mathcal{A})$. Then $(A^\bullet, B^\bullet, C(f)^\bullet, f, i, -p)$ is a distinguished triangle of $K(\mathcal{A})$ with $C(f)^\bullet \in K^+(\mathcal{A})$, $K^-(\mathcal{A})$, or $K^b(\mathcal{A})$ as is clear from the construction of the cone. Thus the lemma is proved. (Alternatively, $K^\bullet \to L^\bullet$ is isomorphic to an termwise split injection of complexes in $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, or $K^b(\mathcal{A})$, see Lemma 11.8.5 and then one can directly take the associated distinguished triangle.) □

Lemma 11.9.6. *Let $\mathcal{A}$, $\mathcal{B}$ be additive categories. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor. The induced functors*

$$\begin{aligned} F &: K(\mathcal{A}) \longrightarrow K(\mathcal{B}) \\ F &: K^+(\mathcal{A}) \longrightarrow K^+(\mathcal{B}) \\ F &: K^-(\mathcal{A}) \longrightarrow K^-(\mathcal{B}) \\ F &: K^b(\mathcal{A}) \longrightarrow K^b(\mathcal{B}) \end{aligned}$$

are exact functors of triangulated categories.

Proof. Suppose $A^\bullet \to B^\bullet \to C^\bullet$ is a termwise split sequence of complexes of $\mathcal{A}$ with splittings (s^n, π^n) and associated morphism $\delta : C^\bullet \to A^\bullet[1]$, see Definition 11.8.8. Then $F(A^\bullet) \to F(B^\bullet) \to F(C^\bullet)$ is a termwise split sequence of complexes with splittings $(F(s^n), F(\pi^n))$ and associated morphism $F(\delta) : F(C^\bullet) \to F(A^\bullet)[1]$. Thus F transforms distinguished triangles into distinguished triangles. □

11.10. Derived categories

In this section we construct the derived category of an abelian category $\mathcal{A}$ by inverting the quasi-isomorphisms in $K(\mathcal{A})$. Before we do this recall that the functors $H^i : \text{Comp}(\mathcal{A}) \to \mathcal{A}$ factor through $K(\mathcal{A})$, see Homology, Lemma 10.10.11. Moreover, in Homology, Definition 10.12.8 we have defined identifications $H^i(K^\bullet[n]) = H^{i+n}(K^\bullet)$. At this point it makes sense to redefine

$$H^i(K^\bullet) = H^0(K^\bullet[i])$$

in order to avoid confusion and possible sign errors.

Lemma 11.10.1. *Let $\mathcal{A}$ be an abelian category. The functor*

$$H^0 : K(\mathcal{A}) \longrightarrow \mathcal{A}$$

is homological.

Proof. Because H^0 is a functor, and by our definition of distinguished triangles it suffices to prove that given a termwise split short exact sequence of complexes $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ the sequence $H^0(A^\bullet) \to H^0(B^\bullet) \to H^0(C^\bullet)$ is exact. This follows from Homology, Lemma 10.10.12. $\square$

In particular, this lemma implies that a distinguished triangle (X, Y, Z, f, g, h) in $K(\mathcal{A})$ gives rise to a long exact cohomology sequence

$$(11.10.1.1) \qquad \ldots \longrightarrow H^i(X) \xrightarrow{H^i(f)} H^i(Y) \xrightarrow{H^i(g)} H^i(Z) \xrightarrow{H^i(h)} H^{i+1}(X) \longrightarrow \ldots$$

see (11.3.5.1). Moreover, there is a compatibility with the long exact sequence of cohomology associated to a short exact sequence of complexes (insert future reference here). For example, if $(A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta)$ is the distinguished triangle associated to a termwise split exact sequence of complexes (see Definition 11.8.8), then the cohomology sequence above agrees with the one defined using the snake lemma, see Homology, Lemma 10.10.12 and for agreement of sequences, see Homology, Lemma 10.12.11.

Recall that a complex $K^\bullet$ is *acyclic* if $H^i(K^\bullet) = 0$ for all $i \in \mathbf{Z}$. Moreover, recall that a morphism of complexes $f : K^\bullet \to L^\bullet$ is a *quasi-isomorphism* if and only if $H^i(f)$ is an isomorphism for all i. See Homology, Definition 10.10.10.

Lemma 11.10.2. *Let $\mathcal{A}$ be an abelian category. The full subcategory $Ac(\mathcal{A})$ of $K(\mathcal{A})$ consisting of acyclic complexes is a strictly full saturated triangulated subcategory of $K(\mathcal{A})$. The corresponding saturated multiplicative system (see Lemma 11.6.10) of $K(\mathcal{A})$ is the set $Qis(\mathcal{A})$ of quasi-isomorphisms. In particular, the kernel of the localization functor $Q : K(\mathcal{A}) \to Qis(\mathcal{A})^{-1}K(\mathcal{A})$ is $Ac(\mathcal{A})$ and the functor H^0 factors through Q.*

Proof. We know that H^0 is a homological functor by Lemma 11.10.1. Thus this lemma is a special case of Lemma 11.6.11. $\square$

Definition 11.10.3. Let $\mathcal{A}$ be an abelian category. Let $\mathrm{Ac}(\mathcal{A})$ and $\mathrm{Qis}(\mathcal{A})$ be as in Lemma 11.10.2. The *derived category of $\mathcal{A}$* is the triangulated category

$$D(\mathcal{A}) = K(\mathcal{A})/\mathrm{Ac}(\mathcal{A}) = \mathrm{Qis}(\mathcal{A})^{-1}K(\mathcal{A}).$$

We denote $H^0 : D(\mathcal{A}) \to \mathcal{A}$ the unique functor whose composition with the quotient functor gives back the functor H^0 defined above. Using Lemma 11.6.4 we introduce the strictly full saturated triangulated subcategories $D^+(\mathcal{A}), D^-(\mathcal{A}), D^b(\mathcal{A})$ whose sets of objects are

$$\begin{aligned} Ob(D^+(\mathcal{A})) &= \{X \in Ob(D(\mathcal{A})) \mid H^n(X) = 0 \text{ for all } n \ll 0\} \\ Ob(D^-(\mathcal{A})) &= \{X \in Ob(D(\mathcal{A})) \mid H^n(X) = 0 \text{ for all } n \gg 0\} \\ Ob(D^b(\mathcal{A})) &= \{X \in Ob(D(\mathcal{A})) \mid H^n(X) = 0 \text{ for all } |n| \gg 0\} \end{aligned}$$

The category $D^b(\mathcal{A})$ is called the *bounded derived category* of $\mathcal{A}$.

Each of the variants $D^+(\mathcal{A}), D^-(\mathcal{A}), D^b(\mathcal{A})$ can be constructed as a localization of the corresponding homotopy category. This relies on the following simple lemma.

Lemma 11.10.4. *Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$ be a complex.*

(1) *If* $H^n(K^\bullet) = 0$ *for all* $n \ll 0$*, then there exists a quasi-isomorphism* $K^\bullet \to L^\bullet$ *with* $L^\bullet$ *bounded below.*
(2) *If* $H^n(K^\bullet) = 0$ *for all* $n \gg 0$*, then there exists a quasi-isomorphism* $M^\bullet \to K^\bullet$ *with* $M^\bullet$ *bounded above.*
(3) *If* $H^n(K^\bullet) = 0$ *for all* $|n| \gg 0$*, then there exists a commutative diagram of morphisms of complexes*

$$\begin{tikzcd} K^\bullet \arrow[r] & L^\bullet \\ M^\bullet \arrow[u] \arrow[r] & N^\bullet \arrow[u] \end{tikzcd}$$

where all the arrows are quasi-isomorphisms, $L^\bullet$ *bounded below,* $M^\bullet$ *bounded above, and* $N^\bullet$ *a bounded complex.*

Proof. Pick $a \ll 0 \ll b$ and set $M^\bullet = \tau_{\leq a}K^\bullet$, $L^\bullet = K^\bullet/\tau_{\leq b}K^\bullet$, and $N^\bullet = L^\bullet/M^\bullet$. See Homology, Section 10.11 for the truncation functors. □

To state the following lemma denote $\text{Ac}^+(\mathcal{A})$, $\text{Ac}^-(\mathcal{A})$, resp. $\text{Ac}^b(\mathcal{A})$ the intersection of $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, resp. $K^b(\mathcal{A})$ with $\text{Ac}(\mathcal{A})$. Denote $\text{Qis}^+(\mathcal{A})$, $\text{Qis}^-(\mathcal{A})$, resp. $\text{Qis}^b(\mathcal{A})$ the intersection of $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, resp. $K^b(\mathcal{A})$ with $\text{Qis}(\mathcal{A})$.

Lemma 11.10.5. *Let* $\mathcal{A}$ *be an abelian category. The subcategories* $Ac^+(\mathcal{A})$*,* $Ac^-(\mathcal{A})$*, resp.* $Ac^b(\mathcal{A})$ *are strictly full saturated triangulated subcategories of* $K^+(\mathcal{A})$*,* $K^-(\mathcal{A})$*, resp.* $K^b(\mathcal{A})$*. The corresponding saturated multiplicative systems (see Lemma 11.6.10) are the sets* $Qis^+(\mathcal{A})$*,* $Qis^-(\mathcal{A})$*, resp.* $Qis^b(\mathcal{A})$*.*
(1) *The kernel of the functor* $K^+(\mathcal{A}) \to D^+(\mathcal{A})$ *is* $Ac^+(\mathcal{A})$ *and this induces an equivalence of triangulated categories*
$$K^+(\mathcal{A})/Ac^+(\mathcal{A}) = Qis^+(\mathcal{A})^{-1}K^+(\mathcal{A}) \longrightarrow D^+(\mathcal{A})$$
(2) *The kernel of the functor* $K^-(\mathcal{A}) \to D^-(\mathcal{A})$ *is* $Ac^-(\mathcal{A})$ *and this induces an equivalence of triangulated categories*
$$K^-(\mathcal{A})/Ac^-(\mathcal{A}) = Qis^-(\mathcal{A})^{-1}K^-(\mathcal{A}) \longrightarrow D^-(\mathcal{A})$$
(3) *The kernel of the functor* $K^b(\mathcal{A}) \to D^b(\mathcal{A})$ *is* $Ac^b(\mathcal{A})$ *and this induces an equivalence of triangulated categories*
$$K^b(\mathcal{A})/Ac^b(\mathcal{A}) = Qis^b(\mathcal{A})^{-1}K^b(\mathcal{A}) \longrightarrow D^b(\mathcal{A})$$

Proof. The initial statements follow from Lemma 11.6.11 by considering the restriction of the homological functor H^0. The statement on kernels in (1), (2), (3) is a consequence of the definitions in each case. Each of the functors is essentially surjective by Lemma 11.10.4. To finish the proof we have to show the functors are fully faithful. We first do this for the bounded below version.

Suppose that $K^\bullet, L^\bullet$ are bounded above complexes. A morphism between these in $D(\mathcal{A})$ is of the form $s^{-1}f$ for a pair $f : K^\bullet \to (L')^\bullet$, $s : L^\bullet \to (L')^\bullet$ where s is a quasi-isomorphism. This implies that $(L')^\bullet$ has cohomology bounded below. Hence by Lemma 11.10.4 we can choose a quasi-isomorphism $s' : (L')^\bullet \to (L'')^\bullet$ with $(L'')^\bullet$ bounded below. Then the pair $(s' \circ f, s' \circ s)$ defines a morphism in $\text{Qis}^+(\mathcal{A})^{-1}K^+(\mathcal{A})$. Hence the functor is ``full''. Finally, suppose that the pair $f : K^\bullet \to (L')^\bullet$, $s : L^\bullet \to (L')^\bullet$ defines a morphism in $\text{Qis}^+(\mathcal{A})^{-1}K^+(\mathcal{A})$ which is zero in $D(\mathcal{A})$. This means that there exists a quasi-isomorphism $s' : (L')^\bullet \to (L'')^\bullet$ such that $s' \circ f = 0$. Using Lemma 11.10.4 once more we obtain a quasi-isomorphism $s'' : (L'')^\bullet \to (L''')^\bullet$ with $(L''')^\bullet$ bounded below. Thus we see that $s'' \circ s' \circ f = 0$ which implies that $s^{-1}f$ is zero in $\text{Qis}^+(\mathcal{A})^{-1}K^+(\mathcal{A})$. This finishes the proof that the functor in (1) is an equivalence.

The proof of (2) is dual to the proof of (1). To prove (3) we may use the result of (2). Hence it suffices to prove that the functor $\text{Qis}^b(\mathcal{A})^{-1}K^b(\mathcal{A}) \to \text{Qis}^-(\mathcal{A})^{-1}K^-(\mathcal{A})$ is fully faithful. The argument given in the previous paragraph applies directly to show this where we consistently work with complexes which are already bounded above. $\square$

11.11. The canonical delta-functor

The derived category should be the receptacle for the universal cohomology functor. In order to state the result we use the notion of a δ-functor from an abelian category into a triangulated category, see Definition 11.3.6.

Consider the functor $\text{Comp}(\mathcal{A}) \to K(\mathcal{A})$. This functor is **not** a δ-functor in general. The easiest way to see this is to consider a nonsplit short exact sequence $0 \to A \to B \to C \to 0$ of objects of $\mathcal{A}$. Since $Hom_{K(\mathcal{A})}(C[0], A[1]) = 0$ we see that any distinguished triangle arising from this short exact sequence would look like $(A[0], B[0], C[0], a, b, 0)$. But the existence of such a distinguished triangle in $K(\mathcal{A})$ implies that the extension is split. A contradiction.

It turns out that the functor $\text{Comp}(\mathcal{A}) \to D(\mathcal{A})$ is a δ-functor. In order to see this we have to define the morphisms δ associated to a short exact sequence

$$0 \to A^\bullet \xrightarrow{a} B^\bullet \xrightarrow{b} C^\bullet \to 0$$

of complexes in the abelian category $\mathcal{A}$. Consider the cone $C(a)^\bullet$ of the morphism a. We have $C(a)^n = B^n \oplus A^{n+1}$ and we define $q^n : C(a)^n \to C^n$ via the projection to B^n followed by b^n. Hence a morphism of complexes

$$q : C(a)^\bullet \longrightarrow C^\bullet.$$

It is clear that $q \circ i = b$ where i is as in Definition 11.8.1. Note that, as $a^\bullet$ is injective in each degree, the kernel of q is identified with the cone of $\text{id}_{A^\bullet}$ which is acyclic. Hence we see that q is a quasi-isomorphism. According to Lemma 11.8.12 the triangle

$$(A, B, C(a), a, i, -p)$$

is a distinguished triangle in $K(\mathcal{A})$. As the localization functor $K(\mathcal{A}) \to D(\mathcal{A})$ is exact we see that $(A, B, C(a), a, i, -p)$ is a distinguished triangle in $D(\mathcal{A})$. Since q is a quasi-isomorphism we see that q is an isomorphism in $D(\mathcal{A})$. Hence we deduce that

$$(A, B, C, a, b, -p \circ q^{-1})$$

is a distinguished triangle of $D(\mathcal{A})$. This suggests the following lemma.

Lemma 11.11.1. *Let $\mathcal{A}$ be an abelian category. The functor Comp$(\mathcal{A}) \to D(\mathcal{A})$ defined has the natural structure of a δ-functor, with*

$$\delta_{A^\bullet \to B^\bullet \to C^\bullet} = -p \circ q^{-1}$$

with p and q as explained above. The same construction turns the functors Comp$^+(\mathcal{A}) \to D^+(\mathcal{A})$, Comp$^-(\mathcal{A}) \to D^-(\mathcal{A})$, and Comp$^b(\mathcal{A}) \to D^b(\mathcal{A})$ into δ-functors.

Proof. We have already seen that this choice leads to a distinguished triangle whenever given a short exact sequence of complexes. We have to show that given a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A^\bullet \arrow[r, "a"'] \arrow[d, "f"'] & B^\bullet \arrow[r, "b"'] \arrow[d, "g"'] & C^\bullet \arrow[r] \arrow[d, "h"'] & 0 \\ 0 \arrow[r] & (A')^\bullet \arrow[r, "a'"] & (B')^\bullet \arrow[r, "b'"] & (C')^\bullet \arrow[r] & 0 \end{tikzcd}$$

we get the desired commutative diagram of Definition 11.3.6 (2). By Lemma 11.8.2 the pair (f, g) induces a canonical morphism $c : C(a)^\bullet \to C(a')^\bullet$. It is a simple computation to show that $q' \circ c = h \circ q$ and $f[1] \circ p = p' \circ c$. From this the result follows directly. $\square$

Lemma 11.11.2. *Let $\mathcal{A}$ be an abelian category. Let*

$$\begin{tikzcd} 0 \arrow[r] & A^\bullet \arrow[r] \arrow[d] & B^\bullet \arrow[r] \arrow[d] & C^\bullet \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & D^\bullet \arrow[r] & E^\bullet \arrow[r] & F^\bullet \arrow[r] & 0 \end{tikzcd}$$

be a commutative diagram of morphisms of complexes such that the rows are short exact sequences of complexes, and the vertical arrows are quasi-isomorphisms. The δ-functor of Lemma 11.11.1 above maps the to short exact sequences $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ and $0 \to D^\bullet \to E^\bullet \to F^\bullet \to 0$ to isomorphic distinguished triangles.

Proof. Trivial from the fact that $K(\mathcal{A}) \to D(\mathcal{A})$ transforms quasi-isomorphisms into isomorphisms and that the associated distinguished triangles are functorial. $\square$

Lemma 11.11.3. *Let $\mathcal{A}$ be an abelian category. Let*

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$$

be a short exact sequences of complexes. Assume this short exact sequence is termwise split. Let $(A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta)$ be the distinguished triangle of $K(\mathcal{A})$ associated to the sequence. The δ-functor of Lemma 11.11.1 above maps the short exact sequences $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ to a triangle isomorphic to the distinguished triangle

$$(A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta).$$

Proof. Follows from Lemma 11.8.12. $\square$

11.12. Triangulated subcategories of the derived category

Let $\mathcal{A}$ be an abelian category. In this section we are going to look for strictly full saturated triangulated subcategories $\mathcal{D}' \subset D(\mathcal{A})$ and in the bounded versions.

Here is a simple construction. Let $\mathcal{B} \subset \mathcal{A}$ be a weak Serre subcategory, see Homology, Section 10.7. We let $D_{\mathcal{B}}(\mathcal{A})$ the full subcategory of $D(\mathcal{A})$ whose objects are

$$Ob(D_{\mathcal{B}}(\mathcal{A})) = \{X \in Ob(D(\mathcal{A})) \mid H^n(X) \text{ is an object of } \mathcal{B} \text{ for all } n\}$$

We also define $D^+_{\mathcal{B}}(\mathcal{A}) = D^+(\mathcal{A}) \cap D_{\mathcal{B}}(\mathcal{A})$ and similarly for the other bounded versions.

Lemma 11.12.1. *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{B} \subset \mathcal{A}$ be a weak Serre subcategory. The category $D_{\mathcal{B}}(\mathcal{A})$ is a strictly full saturated abelian subcategory of $D(\mathcal{A})$. Similarly for the bounded versions.*

Proof. It is clear that $D_{\mathcal{B}}(\mathcal{A})$ is an additive subcategory preserved under the translation functors. If $X \oplus Y$ is in $D_{\mathcal{B}}(\mathcal{A})$, then it is clear that both X and Y are in $D_{\mathcal{B}}$ because $H^n(X \oplus Y) = H^n(X) \oplus H^n(Y)$, hence both $H^n(X)$ and $H^n(Y)$ are kernels of maps between maps of an object of $\mathcal{B}$, hence objects of $\mathcal{B}$. By Lemma 11.4.14 it therefore suffices to show that given a distinguished triangle (X, Y, Z, f, g, h) such that X and Y are in $D_{\mathcal{B}}(\mathcal{A})$ then Z is an object of $D_{\mathcal{B}}(\mathcal{A})$. The long exact cohomology sequence (11.10.1.1) and the definition of a weak Serre subcategory (see Homology, Definition 10.7.1) show that $H^n(Z)$ is an object of $\mathcal{B}$ for all n. Thus Z is an object of $D_{\mathcal{B}}(\mathcal{A})$. □

An interesting feature of the situation of the lemma is that the functor $D(\mathcal{B}) \to D(\mathcal{A})$ factors through a canonical exact functor

$$D(\mathcal{B}) \longrightarrow D_{\mathcal{B}}(\mathcal{A}) \tag{11.12.1.1}$$

After all a complex made from objects of $\mathcal{B}$ certainly gives rise to an object of $D_{\mathcal{B}}(\mathcal{A})$ and as distinguished triangles in $D_{\mathcal{B}}(\mathcal{A})$ are exactly the distinguished triangles of $D(\mathcal{A})$ whose vertices are in $D_{\mathcal{B}}(\mathcal{A})$ we see that the functor is exact since $D(\mathcal{B}) \to D(\mathcal{A})$ is exact. Similarly we obtain functors $D^+(\mathcal{B}) \longrightarrow D^+_{\mathcal{B}}(\mathcal{A})$ etc for the bounded versions. A key question in many cases is whether the displayed functor is an equivalence.

Now, suppose that $\mathcal{B}$ is a Serre subcategory of $\mathcal{A}$. In this case we have the quotient functor $\mathcal{A} \to \mathcal{A}/\mathcal{B}$, see Homology, Lemma 10.7.6. In this case $D_{\mathcal{B}}(\mathcal{A})$ is the kernel of the functor $D(\mathcal{A}) \to D(\mathcal{A}/\mathcal{B})$. Thus we obtain a canonical functor

$$D(\mathcal{A})/D_{\mathcal{B}}(\mathcal{A}) \longrightarrow D(\mathcal{A}/\mathcal{B})$$

by Lemma 11.6.8. Similarly for the bounded versions.

Lemma 11.12.2. *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{B} \subset \mathcal{A}$ be a Serre subcategory. Then $D(\mathcal{A}) \to D(\mathcal{A}/\mathcal{B})$ is essentially surjective.*

Proof. We will use the description of the category $\mathcal{A}/\mathcal{B}$ in the proof of Homology, Lemma 10.7.6. Let $(X^\bullet, d^\bullet)$ be a complex of $\mathcal{A}/\mathcal{B}$. For each i we have an object X^i of $\mathcal{A}$ and $d^i = (s^i, f^i)$ where $s^i : Y^i \to X^i$ is a morphism of $\mathcal{A}$ whose kernel and cokernel are in $\mathcal{B}$ and $f^i : Y^i \to X^{i+1}$ is an arbitrary morphism of $\mathcal{A}$. Next, consider the complex

$$\ldots \to X^i \oplus Y^i \oplus Y^{i+1} \to X^{i+1} \oplus Y^{i+1} \oplus Y^{i+2} \to \ldots$$

in $\mathcal{A}$ with differential given by

$$\begin{pmatrix} 0 & f^i & s^{i+1} \\ 0 & 0 & -\mathrm{id}_{Y^{i+1}} \\ 0 & 0 & 0 \end{pmatrix}.$$

This complex becomes quasi-isomorphic to the complex $(X^\bullet, d^\bullet)$ in $\mathcal{A}/\mathcal{B}$ by the maps

$$(\mathrm{id}_{X^i}, s^i, 0) : X^i \oplus Y^i \oplus Y^{i+1} \to X^i$$

Calculation omitted. □

Lemma 11.12.3. *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{B} \subset \mathcal{A}$ be a Serre subcategory. Suppose that the functor $v : \mathcal{A} \to \mathcal{A}/\mathcal{B}$ has a left adjoint $u : \mathcal{A}/\mathcal{B} \to \mathcal{A}$ such that $vu \cong id$. Then*

$$D(\mathcal{A})/D_{\mathcal{B}}(\mathcal{A}) = D(\mathcal{A}/\mathcal{B})$$

and similarly for the bounded versions.

Proof. The functor $D(v) : D(\mathcal{A}) \to D(\mathcal{A}/\mathcal{B})$ is essentially surjective by Lemma 11.12.2. For an object X of $D(\mathcal{A})$ the adjunction mapping $c_X : uvX \to X$ maps to an isomorphism in $D(\mathcal{A}/\mathcal{B})$ because $vuv \cong v$ by the assumption that $vu \cong \text{id}$. Thus in a distinguished triangle (uvX, X, Z, c_X, g, h) the object Z is an object of $D_{\mathcal{B}}(\mathcal{A})$ as we see by looking at the long exact cohomology sequence. Hence c_X is an element of the multiplicative sytem used to define the quotient category $D(\mathcal{A})/D_{\mathcal{B}}(\mathcal{A})$. Thus $uvX \cong X$ in $D(\mathcal{A})/D_{\mathcal{B}}(\mathcal{A})$. For $X, Y \in Ob(\mathcal{A}))$ the map

$$Hom_{D(\mathcal{A})/D_{\mathcal{B}}(\mathcal{A})}(X, Y) \longrightarrow Hom_{D(\mathcal{A}/\mathcal{B})}(vX, vY)$$

is bijective because u gives an inverse (by the remarks above). □

11.13. Filtered derived categories

A reference for this section is [**Ill72**, I, Chapter V]. Let $\mathcal{A}$ be an abelian category. The goal is to define the filtered derived category $DF(\mathcal{A})$ of $\mathcal{A}$. In some sense this is the derived category of the category $\text{Fil}^f(\mathcal{A})$ of objects with a finite filtration in $\mathcal{A}$. We will slightly generalize Illusie's discussion by allowing our filtered complexes to have infinitely many nonzero $\text{gr}^p(K^\bullet)$ but we retaining the requirement that each term has a finite filtration. The rationale for this generalization is that it is not harder and it allows us to apply the discussion to the spectral sequences of Lemma 11.20.3, see also Remark 11.20.4.

We will use the notation regarding filtered objects introduced in Homology, Section 10.13. The category of filtered objects of $\mathcal{A}$ is denoted $\text{Fil}(\mathcal{A})$. All filtrations will be decreasing by fiat.

Definition 11.13.1. Let $\mathcal{A}$ be an abelian category. The *category of finite filtered objects of* $\mathcal{A}$ is the category of filtered objects (A, F) of $\mathcal{A}$ whose filtration F is finite. We denote it $\text{Fil}^f(\mathcal{A})$.

Thus $\text{Fil}^f(\mathcal{A})$ is a full subcategory of $\text{Fil}(\mathcal{A})$. For each $p \in \mathbf{Z}$ there is a functor $\text{gr}^p : \text{Fil}^f(\mathcal{A}) \to \mathcal{A}$. There is a functor $\text{gr} = \bigoplus_{p\in\mathbf{Z}} \text{gr}^p : \text{Fil}^f(\mathcal{A}) \to \mathcal{A}$. Finally, there is a functor

$$(\text{forget } F) : \text{Fil}^f(\mathcal{A}) \longrightarrow \mathcal{A}$$

which associates to the filtered object (A, F) the underlying object of $\mathcal{A}$. The category $\text{Fil}^f(\mathcal{A})$ is an additive category, but not abelian in general, see Homology, Example 10.3.11. The construction in this section is a special case of a more general construction of the derived category of an ``exact category'', see for example [**Büh10**], [**Kel90**].

Because the functors gr^p, gr, (forget F) are additive they induce exact functors of triangulated categories

$$\text{gr}^p, \text{gr}, (\text{forget } F) : K(\text{Fil}^f(\mathcal{A})) \longrightarrow K(\mathcal{A})$$

by Lemma 11.9.6. By analogy with the case of the homotopy category of an abelian category we make the following definitions.

Definition 11.13.2. Let $\mathcal{A}$ be an abelian category.

(1) Let $\alpha : K^\bullet \to L^\bullet$ be a morphism of $K(\text{Fil}^f(\mathcal{A}))$. We say that α is a *filtered quasi-isomorphism* if the morphism $\text{gr}(\alpha)$ is a quasi-isomorphism.
(2) Let $K^\bullet$ be an object of $K(\text{Fil}^f(\mathcal{A}))$. We say that $K^\bullet$ is *filtered acyclic* if the complex $\text{gr}(K^\bullet)$ is acyclic.

Note that $\alpha : K^\bullet \to L^\bullet$ is a filtered quasi-isomorphism if and only if each $\text{gr}^p(\alpha)$ is a quasi-isomorphism. Similarly a complex $K^\bullet$ is filtered acyclic if and only if each $\text{gr}^p(K^\bullet)$ is acyclic.

Lemma 11.13.3. *Let $\mathcal{A}$ be an abelian category.*

(1) *The functor $K(Fil^f(\mathcal{A})) \longrightarrow \mathcal{A}$, $K^\bullet \longmapsto H^0(gr(K^\bullet))$ is homological.*
(2) *The functor $K(Fil^f(\mathcal{A})) \to \mathcal{A}$, $K^\bullet \longmapsto H^0(gr^p(K^\bullet))$ is homological.*
(3) *The functor $K(Fil^f(\mathcal{A})) \longrightarrow \mathcal{A}$, $K^\bullet \longmapsto H^0((forget\ F)K^\bullet)$ is homological.*

Proof. This follows from the fact that $H^0 : K(\mathcal{A}) \to \mathcal{A}$ is homological, see Lemma 11.10.1 and the fact that the functors gr, gr^p, (forget F) are exact functors of triangulated categories. See Lemma 11.4.18. □

Lemma 11.13.4. *Let $\mathcal{A}$ be an abelian category. The full subcategory $FAc(\mathcal{A})$ of $K(Fil^f(\mathcal{A}))$ consisting of filtered acyclic complexes is a strictly full saturated triangulated subcategory of $K(Fil^f(\mathcal{A}))$. The corresponding saturated multiplicative system (see Lemma 11.6.10) of $K(Fil^f(\mathcal{A}))$ is the set $FQis(\mathcal{A})$ of filtered quasi-isomorphisms. In particular, the kernel of the localization functor*

$$Q : K(Fil^f(\mathcal{A})) \longrightarrow FQis(\mathcal{A})^{-1}K(Fil^f(\mathcal{A}))$$

is $FAc(\mathcal{A})$ and the functor $H^0 \circ gr$ factors through Q.

Proof. We know that $H^0 \circ \text{gr}$ is a homological functor by Lemma 11.13.3. Thus this lemma is a special case of Lemma 11.6.11. □

Definition 11.13.5. Let $\mathcal{A}$ be an abelian category. Let $\text{FAc}(\mathcal{A})$ and $\text{FQis}(\mathcal{A})$ be as in Lemma 11.13.4. The *filtered derived category of $\mathcal{A}$* is the triangulated category

$$DF(\mathcal{A}) = K(\text{Fil}^f(\mathcal{A}))/\text{FAc}(\mathcal{A}) = \text{FQis}(\mathcal{A})^{-1}K(\text{Fil}^f(\mathcal{A})).$$

Lemma 11.13.6. *The functors $gr^p, gr, (forget\ F)$ induce canonical exact functors*

$$gr^p, gr, (forget\ F) : DF(\mathcal{A}) \longrightarrow D(\mathcal{A})$$

which commute with the localization functors.

Proof. This follows from the universal propery of localization, see Lemma 11.5.6, provided we can show that a filtered quasi-isomorphism is turned into a quasi-isomorphism by each of the functors gr^p, gr, (forget F). This is true by definition for the first two. For the last one the statement we have to do a little bit of work. Let $f : K^\bullet \to L^\bullet$ be a filtered quasi-isomorphism in $K(\text{Fil}^f(\mathcal{A}))$. Choose a distinguished triangle $(K^\bullet, L^\bullet, M^\bullet, f, g, h)$ which contains f. Then $M^\bullet$ is filtered acyclic, see Lemma 11.13.4. Hence by the corresponding lemma for $K(\mathcal{A})$ it suffices to show that a filtered acyclic complex is an acyclic complex if we forget the filtration. This follows from Homology, Lemma 10.13.17. □

Definition 11.13.7. Let $\mathcal{A}$ be an abelian category. The *bounded filtered derived category* $DF^b(\mathcal{A})$ is the full subcategory of $DF(\mathcal{A})$ with objects those X such that $\text{gr}(X) \in D^b(\mathcal{A})$. Similarly for the bounded below filtered derived category $DF^+(\mathcal{A})$ and the bounded above filtered derived category $DF^-(\mathcal{A})$.

Lemma 11.13.8. *Let $\mathcal{A}$ be an abelian category. Let $K^\bullet \in K(Fil^f(\mathcal{A}))$.*

(1) *If $H^n(gr(K^\bullet)) = 0$ for all $n < a$, then there exists a filtered quasi-isomorphism $K^\bullet \to L^\bullet$ with $L^n = 0$ for all $n < a$.*
(2) *If $H^n(gr(K^\bullet)) = 0$ for all $n > b$, then there exists a filtered quasi-isomorphism $M^\bullet \to K^\bullet$ with $M^n = 0$ for all $n > b$.*

(3) *If $H^n(gr(K^\bullet)) = 0$ for all $|n| \gg 0$, then there exists a commutative diagram of morphisms of complexes*

$$\begin{tikzcd} K^\bullet \arrow[r] & L^\bullet \\ M^\bullet \arrow[r] \arrow[u] & N^\bullet \arrow[u] \end{tikzcd}$$

where all the arrows are filtered quasi-isomorphisms, $L^\bullet$ bounded below, $M^\bullet$ bounded above, and $N^\bullet$ a bounded complex.

Proof. Suppose that $H^n(\mathrm{gr}(K^\bullet)) = 0$ for all $n < a$. By Homology, Lemma 10.13.17 the sequence

$$K^{a-1} \xrightarrow{d^{a-2}} K^{a-1} \xrightarrow{d^{a-1}} K^a$$

is an exact sequence of objects of $\mathcal{A}$ and the morphisms d^{a-2} and d^{a-1} are strict. Hence $\mathrm{Coim}(d^{a-1}) = \mathrm{Im}(d^{a-1})$ in $\mathrm{Fil}^f(\mathcal{A})$ and the map $\mathrm{gr}(\mathrm{Im}(d^{a-1})) \to \mathrm{gr}(K^a)$ is injective with image equal to the image of $\mathrm{gr}(K^{a-1}) \to \mathrm{gr}(K^a)$, see Homology, Lemma 10.13.15. This means that the map $K^\bullet \to \tau_{\geq a}K^\bullet$ into the truncation

$$\tau_{\geq a}K^\bullet = (\ldots \to 0 \to K^a/\mathrm{Im}(d^{a-1}) \to K^{a+1} \to \ldots)$$

is a filtered quasi-isomorphism. This proves (1). The proof of (2) is dual to the proof of (1). Part (3) follows formally from (1) and (2). $\square$

To state the following lemma denote $\mathrm{FAc}^+(\mathcal{A})$, $\mathrm{FAc}^-(\mathcal{A})$, resp. $\mathrm{FAc}^b(\mathcal{A})$ the intersection of $K^+(\mathrm{Fil}^f\mathcal{A})$, $K^-(\mathrm{Fil}^f\mathcal{A})$, resp. $K^b(\mathrm{Fil}^f\mathcal{A})$ with $\mathrm{FAc}(\mathcal{A})$. Denote $\mathrm{FQis}^+(\mathcal{A})$, $\mathrm{FQis}^-(\mathcal{A})$, resp. $\mathrm{FQis}^b(\mathcal{A})$ the intersection of $K^+(\mathrm{Fil}^f\mathcal{A})$, $K^-(\mathrm{Fil}^f\mathcal{A})$, resp. $K^b(\mathrm{Fil}^f\mathcal{A})$ with $\mathrm{FQis}(\mathcal{A})$.

Lemma 11.13.9. *Let $\mathcal{A}$ be an abelian category. The subcategories $FAc^+(\mathcal{A})$, $FAc^-(\mathcal{A})$, resp. $FAc^b(\mathcal{A})$ are strictly full saturated triangulated subcategories of $K^+(Fil^f\mathcal{A})$, $K^-(Fil^f\mathcal{A})$, resp. $K^b(Fil^f\mathcal{A})$. The corresponding saturated multiplicative systems (see Lemma 11.6.10) are the sets $FQis^+(\mathcal{A})$, $FQis^-(\mathcal{A})$, resp. $FQis^b(\mathcal{A})$.*

(1) *The kernel of the functor $K^+(Fil^f\mathcal{A}) \to DF^+(\mathcal{A})$ is $FAc^+(\mathcal{A})$ and this induces an equivalence of triangulated categories*

$$K^+(Fil^f\mathcal{A})/FAc^+(\mathcal{A}) = FQis^+(\mathcal{A})^{-1}K^+(Fil^f\mathcal{A}) \longrightarrow DF^+(\mathcal{A})$$

(2) *The kernel of the functor $K^-(Fil^f\mathcal{A}) \to DF^-(\mathcal{A})$ is $FAc^-(\mathcal{A})$ and this induces an equivalence of triangulated categories*

$$K^-(Fil^f\mathcal{A})/FAc^-(\mathcal{A}) = FQis^-(\mathcal{A})^{-1}K^-(Fil^f\mathcal{A}) \longrightarrow DF^-(\mathcal{A})$$

(3) *The kernel of the functor $K^b(Fil^f\mathcal{A}) \to DF^b(\mathcal{A})$ is $FAc^b(\mathcal{A})$ and this induces an equivalence of triangulated categories*

$$K^b(Fil^f\mathcal{A})/FAc^b(\mathcal{A}) = FQis^b(\mathcal{A})^{-1}K^b(Fil^f\mathcal{A}) \longrightarrow DF^b(\mathcal{A})$$

Proof. This follows from the results above, in particular Lemma 11.13.8, by exactly the same arguments as used in the proof of Lemma 11.10.5. $\square$

11.14. Derived functors in general

A reference for this section is Deligne's exposée XVII in [**MA71**]. A very general notion of right and left derived functors exists where we have an exact functor between triangulated categories, a multiplicative system in the source category and we want to find the ``correct'' extension of the exact functor to the localized category.

Situation 11.14.1. Here $F : \mathcal{D} \to \mathcal{D}'$ is an exact functor of triangulated categories and S is a saturated multiplicative system in $\mathcal{D}$ compatible with the structure of triangulated category on $\mathcal{D}$.

Let $X \in Ob(\mathcal{D})$. Recall from Categories, Remark 4.24.5 the filtered category X/S of arrows $s : X \to X'$ in S with source X. Dually, in Categories, Remark 4.24.12 we defined the cofiltered category S/X of arrows $s : X' \to X$ in S with target X.

Definition 11.14.2. Assumptions and notation as in Situation 11.14.1. Let $X \in Ob(\mathcal{D})$.

(1) we say the *right derived functor RF is defined at X* if the ind-object
$$(X/S) \longrightarrow \mathcal{D}', \quad (s : X \to X') \longmapsto F(X')$$
is essentially constant[2]; in this case the value Y in $\mathcal{D}'$ is called the *value of RF at X*.

(2) we say the *left derived functor LF is defined at X* if the pro-object
$$(S/X) \longrightarrow \mathcal{D}', \quad (s : X' \to X) \longmapsto F(X')$$
is essentially constant; in this case the value Y in $\mathcal{D}'$ is called the *value of LF at X*.

By abuse of notation we often denote the values simply $RF(X)$ or $LF(X)$.

It will turn out that the full subcategory of $\mathcal{D}$ consisting of objects where RF is defined is a triangulated subcategory, and RF will define functor on this subcategory which transforms morphisms of s into isomorphisms.

Lemma 11.14.3. *Assumptions and notation as in Situation 11.14.1. Let $f : X \to Y$ be a morphism of $\mathcal{D}$.*

(1) *If RF is defined at X and Y then there exists a unique morphism $RF(f) : RF(X) \to RF(Y)$ between the values such that for any commutative diagram*
$$\begin{tikzcd} X \arrow[r, "s"'] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r, "s'"] & Y' \end{tikzcd}$$
with $s, s' \in S$ the diagram
$$\begin{tikzcd} F(X) \arrow[r] \arrow[d] & F(X') \arrow[r] \arrow[d] & RF(X) \arrow[d] \\ F(Y) \arrow[r] & F(Y') \arrow[r] & RF(Y) \end{tikzcd}$$
commutes.

[2] For a discussion of when an ind-object or pro-object of a category is essentially constant we refer to Categories, Section 4.20.

(2) *If LF is defined at X and Y then there exists a unique morphism $LF(f) : LF(X) \to LF(Y)$ between the values such that for any commutative diagram*

$$\begin{tikzcd} X' \arrow[r, "s"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "s'"] & Y \end{tikzcd}$$

with s, s' in S the diagram

$$\begin{tikzcd} LF(X) \arrow[r] \arrow[d] & F(X') \arrow[r] \arrow[d] & F(X) \arrow[d] \\ LF(Y) \arrow[r] & F(Y') \arrow[r] & F(Y) \end{tikzcd}$$

commutes.

Proof. Part (1) holds if we only assume that the colimits

$$RF(X) = colim_{s : X' \to X} F(X') \quad \text{and} \quad RF(Y) = colim_{s' : Y' \to Y} F(Y')$$

exist. Namely, to give a morphism $RF(X) \to RF(Y)$ between the colimits is the same thing as giving for each $s : X \to X'$ in $Ob(X/S)$ a morphism $F(X') \to RF(Y)$ compatible with morphisms in the category X/S. To get the morphism we choose a commutative diagram

$$\begin{tikzcd} X \arrow[r, "s"'] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r, "s'"] & Y' \end{tikzcd}$$

with s, s' in S as is possible by MS2 and we set $F(X') \to RF(Y)$ equal to the composition $F(X') \to F(Y') \to RF(Y)$. To see that this is independent of the choice of the diagram above use MS3. Details omitted. The proof of (2) is dual. □

Lemma 11.14.4. *Assumptions and notation as in Situation 11.14.1. Let $s : X \to Y$ be an element of S.*

(1) *RF is defined at X if and only if it is defined at Y. In this case the map $RF(s) : RF(X) \to RF(Y)$ between values is an isomorphism.*
(2) *LF is defined at X if and only if it is defined at Y. In this case the map $LF(s) : LF(X) \to LF(Y)$ between values is an isomorphism.*

Proof. Omitted. □

Lemma 11.14.5. *Assumptions and notation as in Situation 11.14.1. Let X be an object of $\mathcal{D}$ and $n \in \mathbf{Z}$.*

(1) *RF is defined at X if and only if it is defined at $X[n]$. In this case there is a canonical isomorphism $RF(X)[n] = RF(X[n])$ between values.*
(2) *LF is defined at X if and only if it is defined at $X[n]$. In this case there is a canonical isomorphism $LF(X)[n] \to LF(X[n])$ between values.*

Proof. Omitted. □

Lemma 11.14.6. *Assumptions and notation as in Situation 11.14.1. Let (X, Y, Z, f, g, h) be a distinguished triangle of $\mathcal{D}$. If RF is defined at two out of three of X, Y, Z, then it is defined at the third. Moreover, in this case*

$$(RF(X), RF(Y), RF(Z), RF(f), RF(g), RF(h))$$

is a distinguished triangle in $\mathcal{D}'$. Similarly for LF.

Proof. Say RF is defined at X, Y with values A, B. Let $RF(f) : A \to B$ be the induced morphism, see Lemma 11.14.3. We may choose a distinguished triangle $(A, B, C, RF(f), b, c)$ in $\mathcal{D}'$. We claim that C is a value of RF at Z.

To see this pick $s : X \to X'$ in S such that there exists a morphism $\alpha : A \to F(X')$ as in Categories, Definition 4.20.1. We may choose a commutative diagram

$$\begin{tikzcd} X \arrow[r, "s"'] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r, "s'"] & Y' \end{tikzcd}$$

with $s' \in S$ by MS2. Using that Y/S is filtered we can (after replacing s' by some $s'' : Y \to Y''$ in S) assume that there exists a morphism $\beta : B \to F(Y')$ as in Categories, Definition 4.20.1. Picture

$$\begin{tikzcd} A \arrow[r, "\alpha"'] \arrow[d, "RF(f)"'] & F(X') \arrow[r] \arrow[d, "F(f')"] & A \arrow[d, "RF(f)"] \\ B \arrow[r, "\beta"] & F(Y') \arrow[r] & B \end{tikzcd}$$

It may not be true that the left square commutes, but the outer and right squares commute. The assumption that the ind-object $\{F(Y')\}_{s' : Y' \to Y}$ is essentially constant means that there exists a $s'' : Y \to Y''$ in S and a morphism $h : Y' \to Y''$ with such that $s'' = h \circ s'$ and $F(h)$ equal to $F(Y') \to B \to F(Y') \to F(Y'')$. Hence after replacing Y' by Y'' and β by $F(h) \circ \beta$ the diagram will commute (by direct computation with arrows).

Using MS6 choose a morphism of triangles

$$(s, s', s'') : (X, Y, Z, f, g, h) \longrightarrow (X', Y', Z', f', g', h')$$

with $s'' \in S$. By TR3 choose a morphism of triangles

$$(\alpha, \beta, \gamma) : (A, B, C, RF(f), b, c) \longrightarrow (F(X'), F(Y'), F(Z'), F(f'), F(g'), F(h'))$$

By Lemma 11.14.4 it suffices to prove that $RF(Z')$ is defined and has value C. Consider the category $\mathcal{I}$ of Lemma 11.5.8 of triangles

$$\mathcal{I} = \{(t, t', t'') : (X', Y', Z', f', g', h') \to (X'', Y'', Z'', f'', g'', h'') \mid (t, t', t'') \in S\}$$

To show that the system $F(Z'')$ is essentially constant over the category Z'/S is equivalent to showing that the system of $F(Z'')$ is essentially constant over $\mathcal{I}$ by using the surjectivity of the functor $\mathcal{I} \to Z'/S$. For any object W in $\mathcal{D}'$ we can consider the diagram

$$
\begin{tikzcd}
lim_{\mathcal{I}} Mor_{\mathcal{D}'}(F(X''), W) \arrow[r] & Mor_{\mathcal{D}'}(A, W) \\
lim_{\mathcal{I}} Mor_{\mathcal{D}'}(F(Y''), W) \arrow[r] \arrow[u] & Mor_{\mathcal{D}'}(B, W) \arrow[u] \\
lim_{\mathcal{I}} Mor_{\mathcal{D}'}(F(Z''), W) \arrow[r] \arrow[u] & Mor_{\mathcal{D}'}(C, W) \arrow[u] \\
lim_{\mathcal{I}} Mor_{\mathcal{D}'}(F(X''[1]), W) \arrow[r] \arrow[u] & Mor_{\mathcal{D}'}(A[1], W) \arrow[u] \\
lim_{\mathcal{I}} Mor_{\mathcal{D}'}(F(Y''[1]), W) \arrow[r] \arrow[u] & Mor_{\mathcal{D}'}(B[1], W) \arrow[u]
\end{tikzcd}
$$

which shows that the middle arrow is an isomorphism by the 5 lemma. In this way we conclude that C is the colimit $colim_{\mathcal{I}} F(Z'')$. To see that the ind-object is essentially constant it now suffices to show that for any object W in $\mathcal{D}'$ the map

$$colim_{\mathcal{I}} Mor_{\mathcal{D}'}(W, F(Z'')) \longrightarrow Mor_{\mathcal{D}'}(W, C)$$

is bijective, see Categories, Lemma 4.20.6. To see this we can use again the 5 lemma and the commutative diagram

$$
\begin{tikzcd}
colim_{\mathcal{I}} Mor_{\mathcal{D}'}(W, F(X'')) \arrow[r] & Mor_{\mathcal{D}'}(W, A) \\
colim_{\mathcal{I}} Mor_{\mathcal{D}'}(W, F(Y'')) \arrow[r] \arrow[u] & Mor_{\mathcal{D}'}(W, B) \arrow[u] \\
colim_{\mathcal{I}} Mor_{\mathcal{D}'}(W, F(Z'')) \arrow[r] \arrow[u] & Mor_{\mathcal{D}'}(W, C) \arrow[u] \\
colim_{\mathcal{I}} Mor_{\mathcal{D}'}(W, F(X''[1])) \arrow[r] \arrow[u] & Mor_{\mathcal{D}'}(W, A[1]) \arrow[u] \\
colim_{\mathcal{I}} Mor_{\mathcal{D}'}(W, F(Y''[1])) \arrow[r] \arrow[u] & Mor_{\mathcal{D}'}(W, B[1]) \arrow[u]
\end{tikzcd}
$$

and the fact that Categories, Lemma 4.20.6 guarantees that the other horizontal arrows are isomorphisms. □

Lemma 11.14.7. *Assumptions and notation as in Situation 11.14.1. Let X, Y be objects of $\mathcal{D}$. If RF is defined at $X \oplus Y$, then it is defined at X and Y. Moreover, in this case $RF(X \oplus Y) = RF(X) \oplus RF(Y)$. Similarly for LF.*

Proof. Since S is a saturated system for any $s : X \to X'$ and $s' : Y \to Y'$ in S the morphism $s \oplus s' : X \oplus Y \to X' \oplus Y'$ is an element of S (as S is the set of arrows which become invertible under the additive localization functor $Q : \mathcal{D} \to S^{-1}\mathcal{D}$, see Categories, Lemma 4.24.18). To prove the lemma for RF it suffices to show that these arrows $s \oplus s'$

are cofinal in the filtered category $(X \oplus Y)/S$. To do this pick any $t : X \oplus Y \to Z$ in S. Using MS2 we can find morphisms $Z \to X'$, $Z \to Y'$ and $s : X \to X'$, $s' : Y \to Y'$ in S such that

$$\begin{tikzcd} X \arrow[d, "s"] & X \oplus Y \arrow[l] \arrow[r] \arrow[d] & Y \arrow[d, "s'"'] \\ X' & Z \arrow[l] \arrow[r] & Y' \end{tikzcd}$$

commutes. Hence the desired result. The proof for LF is dual. □

Proposition 11.14.8. *Assumptions and notation as in Situation 11.14.1. The full subcategory $\mathcal{E}$ of $\mathcal{D}$ consisting of objects at which RF is defined is a strictly full saturated triangulated subcategory of $\mathcal{D}$. Choosing values using the axiom of choice gives rise to an exact functor*

$$RF : \mathcal{E} \longrightarrow \mathcal{D}'$$

of triangulated categories. Elements of S with either source or target in $\mathcal{E}$ are morphisms of $\mathcal{E}$. Any element of $S_\mathcal{E} = Arrows(\mathcal{E}) \cap S$ is transformed into an isomorphism by RF. Hence an exact functor

$$RF : S_\mathcal{E}^{-1}\mathcal{E} \longrightarrow \mathcal{D}'.$$

A similar result holds for LF.

Proof. This is just a summary of the results obtained in Lemmas 11.14.3, 11.14.4, 11.14.5, 11.14.6, and 11.14.7. □

Definition 11.14.9. In Situation 11.14.1. We say F is *right deriveable*, or that RF *everywhere defined* if RF is defined at every object of $\mathcal{D}$. We say F is *left deriveable*, or that LF *everywhere defined* if LF is defined at every object of $\mathcal{D}$.

In this case we obtain a right (resp. left) derived functor

(11.14.9.1) $$RF : S^{-1}\mathcal{D} \longrightarrow \mathcal{D}', \quad (\text{resp. } LF : S^{-1}\mathcal{D} \longrightarrow \mathcal{D}'),$$

see Proposition 11.14.8. In most interesting situations it is not the case that $RF \circ Q$ is equal to F. In fact, it might happen that the canonical map $F(X) \to RF(X)$ is never an isomorphism. In practice this does not happen, because in practice we only know how to prove F is right deriveable by showing that RF can be computed by evaluating F at judiciously chosen objects of the triangulated category $\mathcal{D}$. This warrents a definition.

Definition 11.14.10. In Situation 11.14.1.

(1) An object X of $\mathcal{D}$ *computes* RF if RF is defined at X and the canonical map $F(X) \to RF(X)$ is an isomorphism.
(2) An object X of $\mathcal{D}$ *computes* LF if LF is defined at X and the canonical map $LF(X) \to F(X)$ is an isomorphism.

Lemma 11.14.11. *Assumptions and notation as in Situation 11.14.1. Let X be an object of $\mathcal{D}$ and $n \in \mathbf{Z}$.*

(1) *X computes RF if and only if $X[n]$ computes RF.*
(2) *X computes LF if and only if $X[n]$ computes LF.*

Proof. Omitted. □

Lemma 11.14.12. *Assumptions and notation as in Situation 11.14.1. Let (X, Y, Z, f, g, h) be a distinguished triangle of $\mathcal{D}$. If X, Y compute RF then so does Z. Similar for LF.*

Proof. By Lemma 11.14.6 we know that RF is defined at Z and that RF applied to the triangle produces a distinguished triangle. Consider the morphism of distinguished triangles

$$\begin{array}{c}(F(X), F(Y), F(Z), F(f), F(g), F(h)) \\ \downarrow \\ (RF(X), RF(Y), RF(Z), RF(f), RF(g), RF(h))\end{array}$$

Two out of three maps are isomorphisms, hence so is the third. □

Lemma 11.14.13. *Assumptions and notation as in Situation 11.14.1. Let X, Y be objects of $\mathcal{D}$. If $X \oplus Y$ computes RF, then X and Y compute RF. Similarly for LF.*

Proof. By Lemma 11.14.7 we know that RF is defined at X and Y and that $RF(X \oplus Y) = RF(X) \oplus RF(Y)$. Since the map

$$F(X) \oplus F(Y) = F(X \oplus Y) \longrightarrow RF(X \oplus Y) = RF(X) \oplus RF(Y)$$

is compatible with direct sum decompositions we win. □

Lemma 11.14.14. *Assumptions and notation as in Situation 11.14.1.*

(1) *If for every object $X \in Ob(\mathcal{D})$ there exists an arrow $s : X \to X'$ in S such that X' computes RF, then RF is everywhere defined.*
(2) *If for every object $X \in Ob(\mathcal{D})$ there exists an arrow $s : X' \to X$ in S such that X' computes LF, then LF is everywhere defined.*

Proof. This is clear from the definitions. □

Lemma 11.14.15. *Assumptions and notation as in Situation 11.14.1. If there exists a subset $\mathcal{I} \subset Ob(\mathcal{D})$ such that*

(1) *for all $X \in Ob(\mathcal{D})$ there exists $s : X \to X'$ in S with $X' \in \mathcal{I}$, and*
(2) *for every arrow $s : X \to X'$ in S with $X, X' \in \mathcal{I}$ the map $F(s) : F(X) \to F(X')$ is an isomorphism,*

then RF is everywhere defined and every $X \in \mathcal{I}$ computes RF. Dually, if there exists a subset $\mathcal{P} \subset Ob(\mathcal{D})$ such that

(1) *for all $X \in Ob(\mathcal{D})$ there exists $s : X' \to X$ in S with $X' \in \mathcal{P}$, and*
(2) *for every arrow $s : X \to X'$ in S with $X, X' \in \mathcal{P}$ the map $F(s) : F(X) \to F(X')$ is an isomorphism,*

then LF is everywhere defined and every $X \in \mathcal{P}$ computes LF.

Proof. Let X be an object of $\mathcal{D}$. Assumption (1) implies that the arrows $s : X \to X'$ in S with $X' \in \mathcal{I}$ are cofinal in the category X/S. Assumption (2) implies that F is constant on this cofinal subcategory. Clearly this implies that $F : (X/S) \to \mathcal{D}'$ is essentially constant with value $F(X')$ for any $s : X \to X'$ in S with $X' \in \mathcal{I}$. □

Lemma 11.14.16. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be triangulated categories. Let S, resp. S' be a saturated multiplicative system in $\mathcal{A}$, resp. $\mathcal{B}$ compatible with the triangulated structure. Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be exact functors. Denote $F' : \mathcal{A} \to (S')^{-1}\mathcal{B}$ the composition of F with the localization functor.*

(1) *If RF', RG, $R(G \circ F)$ are everywhere defined, then there is a canonical transformation of functors $t : R(G \circ F) \longrightarrow RG \circ RF'$.*
(2) *If LF', LG, $L(G \circ F)$ are everywhere defined, then there is a canonical transformation of functors $t : LG \circ LF' \to L(G \circ F)$.*

Proof. In this proof we try to be careful. Hence let us think of the derived functors as the functors

$$RF' : S^{-1}\mathcal{A} \to (S')^{-1}\mathcal{B}, \quad R(G \circ F) : S^{-1}\mathcal{A} \to \mathcal{C}, \quad RG : (S')^{-1}\mathcal{B} \to \mathcal{C}.$$

Let us denote $Q_A : \mathcal{A} \to S^{-1}\mathcal{A}$ and $Q_B : \mathcal{B} \to (S')^{-1}\mathcal{B}$ the localization functors. Then $F' = Q_B \circ F$. Note that for every object Y of $\mathcal{B}$ there is a canonical map

$$G(Y) \longrightarrow RG(Q_B(Y))$$

in other words, there is a transformation of functors $t' : G \to RG \circ Q_B$. Let X be an object of $\mathcal{A}$. We have

$$\begin{aligned} R(G \circ F)(Q_A(X)) &= colim_{s : X \to X' \in S}\, G(F(X')) \\ &\xrightarrow{t'} colim_{s : X \to X' \in S}\, RG(Q_B(F(X'))) \\ &= colim_{s : X \to X' \in S}\, RG(F'(X')) \\ &= RG(colim_{s : X \to X' \in S}\, F'(X')) \\ &= RG(RF'(X)). \end{aligned}$$

The system $F'(X')$ is essentially constant in the category $(S')^{-1}\mathcal{B}$. Hence we may pull the colimit inside the functor RG in the third equality of the diagram above, see Categories, Lemma 4.20.5 and its proof. We omit the proof this this defines a transformation of functors. The case of left derived functors is similar. □

11.15. Derived functors on derived categories

In practice derived functors come about most often when given an additive functor between abelian categories.

Situation 11.15.1. Here $F : \mathcal{A} \to \mathcal{B}$ is an additive functor between abelian categories. This induces exact functors

$$F : K(\mathcal{A}) \to K(\mathcal{B}), \quad K^+(\mathcal{A}) \to K^+(\mathcal{B}), \quad K^-(\mathcal{A}) \to K^-(\mathcal{B}).$$

We also denote F the composition $K(\mathcal{A}) \to D(\mathcal{B})$, $K^+(\mathcal{A}) \to D^+(\mathcal{B})$, and $K^-(\mathcal{A}) \to D(\mathcal{B})$ of F with the localization functor $K(\mathcal{B}) \to D(\mathcal{B})$, etc. This situation leads to four derived functors we will consider in the following.

(1) The right derived functor of $F : K(\mathcal{A}) \to D(\mathcal{B})$ relative to the multiplicative system $\text{Qis}(\mathcal{A})$.
(2) The right derived functor of $F : K^+(\mathcal{A}) \to D^+(\mathcal{B})$ relative to the multiplicative system $\text{Qis}^+(\mathcal{A})$.
(3) The left derived functor of $F : K(\mathcal{A}) \to D(\mathcal{B})$ relative to the multiplicative system $\text{Qis}(\mathcal{A})$.
(4) The left derived functor of $F : K^-(\mathcal{A}) \to D^-(\mathcal{B})$ relative to the multiplicative system $\text{Qis}(\mathcal{A})$.

Each of these cases is an example of Situation 11.14.1.

Some of the ambiguity that may arise is alleviated by the following.

Lemma 11.15.2. *In Situation 11.15.1.*

(1) *Let X be an object of $K^+(\mathcal{A})$. The right derived functor of $K(\mathcal{A}) \to D(\mathcal{B})$ is defined at X if and only if the right derived functor of $K^+(\mathcal{A}) \to D^+(\mathcal{B})$ is defined at X. Moreover, the values are canonically isomorphic.*

(2) *Let X be an object of $K^+(\mathcal{A})$. Then X computes the right derived functor of $K(\mathcal{A}) \to D(\mathcal{B})$ if and only if X computes the right derived functor of $K^+(\mathcal{A}) \to D^+(\mathcal{B})$.*
(3) *Let X be an object of $K^-(\mathcal{A})$. The left derived functor of $K(\mathcal{A}) \to D(\mathcal{B})$ is defined at X if and only if the left derived functor of $K^-(\mathcal{A}) \to D^-(\mathcal{B})$ is defined at X. Moreover, the values are canonically isomorphic.*
(4) *Let X be an object of $K^-(\mathcal{A})$. Then X computes the left derived functor of $K(\mathcal{A}) \to D(\mathcal{B})$ if and only if X computes the left derived functor of $K^-(\mathcal{A}) \to D^-(\mathcal{B})$.*

Proof. Let X be an object of $K^+(\mathcal{A})$. Consider a quasi-isomorphism $s : X \to X'$ in $K(\mathcal{A})$. By Lemma 11.10.4 there exists quasi-isomorphism $X' \to X''$ with X'' bounded below. Hence we see that $X/\text{Qis}^+(\mathcal{A})$ is cofinal in $X/\text{Qis}(\mathcal{A})$. Thus it is clear that (1) holds. Part (2) follows directly from part (1). Parts (3) and (4) are dual to parts (1) and (2). □

Given an object A of an abelian category $\mathcal{A}$ we get a complex

$$A[0] = (\ldots \to 0 \to A \to 0 \to \ldots)$$

where A is placed in degree zero. Hence a functor $\mathcal{A} \to K(\mathcal{A})$, $A \mapsto A[0]$. Let us temporarily say that a partial functor is one that is defined on a subcategory.

Definition 11.15.3. In Situation 11.15.1.

(1) The *right derived functors of F* are the partial functors RF associated to cases (1) and (2) of Situation 11.15.1.
(2) The *left derived functors of F* are the partial functors LF associated to cases (3) and (4) of Situation 11.15.1.
(3) An object A of $\mathcal{A}$ is said to be *right acyclic for F*, or *acyclic for RF* if $A[0]$ computes RF.
(4) An object A of $\mathcal{A}$ is said to be *left acyclic for F*, or *acyclic for LF* if $A[0]$ computes RF.

The following few lemmas give some criteria for the existence of enough acyclics.

Lemma 11.15.4. *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{I} \subset Ob(\mathcal{A})$ be a subset such that every object of $\mathcal{A}$ is a subobject of an element of $\mathcal{I}$. For every $K^\bullet$ with $K^n = 0$ for $n < a$ there exists a quasi-isomorphism $K^\bullet \to I^\bullet$ with $I^n = 0$ for $n < a$, each $I^n \in \mathcal{I}$, and each $K^n \to I^n$ injective.*

Proof. Consider the following induction hypothesis IH_n: There are $I^j \in \mathcal{I}$, $j \leq n$ almost all zero, maps $d^j : I^j \to I^{j+1}$ for $j < n$ and injective maps $\alpha^j : K^j \to I^j$ for $j \leq n$ such that the diagram

$$\begin{tikzcd}
\ldots \arrow[r] & K^{n-1} \arrow[r] \arrow[d, "\alpha"] & K^n \arrow[r] \arrow[d, "\alpha"] & K^{n+1} \arrow[r] & \ldots \\
\ldots \arrow[r] & I^{n-1} \arrow[r] & I^n & &
\end{tikzcd}$$

is commutative, such that $d^j \circ d^{j-1} = 0$ for $j < n$ and such that α induces isomorphisms $H^j(K^\bullet) \to \text{Ker}(d^j)/\text{Im}(d^{j-1})$ for $j < n$. Note that this implies

$$\alpha(\text{Im}(d_K^{n-1})) \subset \alpha(\text{Ker}(d_K^n)) \cap \text{Im}(d^{n-1}) \subset \alpha(K^n) \cap \text{Im}(d^{n-1}). \tag{11.15.4.1}$$

If these inclusions are not equalities, then choose an injection

$$I^n \oplus K^n/\text{Im}(d_K^{n-1}) \longrightarrow I$$

with $I \in \mathcal{I}$. Denote $\alpha' : K^n \to I$ the map obtained by composing $\alpha \oplus 1 : K^n \to I^n \oplus K^n/\mathrm{Im}(d_K^{n-1})$ with the displayed injection. Denote $d' : I^{n-1} \to I$ the composition $I^{n-1} \to I^n \to I$ of d^{n-1} by the inclusion of the first summand. Then $\alpha'(K^n) \cap \mathrm{Im}(d') = \alpha'(\mathrm{Im}(d_K^{n-1}))$ simply because the intersection of $\alpha'(K^n)$ with the first summand of $I^n \oplus K^n/\mathrm{Im}(d_K^{n-1})$ is equal to $\alpha'(\mathrm{Im}(d_K^{n-1}))$. Hence, after replacing I^n by I, α by α' and d^{n-1} by d' we may assume that we have equality in Equation (11.15.4.1). Once this is the case consider the solid diagram

$$\begin{tikzcd} K^n/\mathrm{Ker}(d_K^n) \arrow[r] \arrow[d] & K^{n+1} \arrow[d, dotted] \\ I^n/(\mathrm{Im}(d^{n-1}) + \alpha(\mathrm{Ker}(d_K^n))) \arrow[r, dotted] & M \end{tikzcd}$$

The horizontal arrow is injective by fiat and the vertical arrow is injective as we have equality in (11.15.4.1). Hence the push-out M of this diagram contains both K^{n+1} and $I^n/(\mathrm{Im}(d^{n-1}) + \alpha(\mathrm{Ker}(d_K^n)))$ as subobjects. Choose an injection $M \to I^{n+1}$ with $I^{n+1} \in \mathcal{I}$. By construction we get $d^n : I^n \to I^{n+1}$ and an injective map $\alpha^{n+1} : K^{n+1} \to I^{n+1}$. The equality in Equation (11.15.4.1) and the construction of d^n guarantee that $\alpha : H^n(K^\bullet) \to \mathrm{Ker}(d^n)/\mathrm{Im}(d^{n-1})$ is an isomorphism. In other words IH_{n+1} holds.

We finish the proof of by the following observations. First we note that IH_n is true for $n = a$ since we can just take $I^j = 0$ for $j < a$ and $K^a \to I^a$ an injection of K^a into an element of $\mathcal{A}$. Next, we note that in the proof of $IH_n \Rightarrow IH_{n+1}$ we only modified the object I^n, the map d^{n-1} and the map α^n. Hence we see that proceeding by induction we produce a complex $I^\bullet$ with $I^n = 0$ for $n < a$ consisting of objects from $\mathcal{I}$, and a termwise injective quasi-isomorphism $\alpha : K^\bullet \to I^\bullet$ as desired. □

Lemma 11.15.5. *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{P} \subset Ob(\mathcal{A})$ be a subset such that every object of $\mathcal{A}$ is a quotient of an element of $\mathcal{P}$. Then for every bounded above complex $K^\bullet$ there exists a quasi-isomorphism $P^\bullet \to K^\bullet$ with $P^\bullet$ bounded above and each $P^n \in \mathcal{P}$.*

Proof. This lemma is dual to Lemma 11.15.4. □

Lemma 11.15.6. *In Situation 11.15.1. Let $\mathcal{I} \subset Ob(\mathcal{A})$ be a subset with the following properties:*

1. *every object of $\mathcal{A}$ is a subobject of an element of $\mathcal{I}$,*
2. *for any short exact sequence $0 \to P \to Q \to R \to 0$ of $\mathcal{A}$ with $P, Q \in \mathcal{I}$, then $R \in \mathcal{I}$, and $0 \to F(P) \to F(Q) \to F(R) \to 0$ is exact.*

Then every object of $\mathcal{I}$ is acyclic for RF.

Proof. Pick $A \in \mathcal{I}$. Let $A[0] \to K^\bullet$ be a quasi-isomorphism with $L^\bullet$ bounded below. Then we can find a quasi-isomorphism $K^\bullet \to I^\bullet$ with $I^\bullet$ bounded below and each $I^n \in \mathcal{I}$, see Lemma 11.15.4. Hence we see that these resolutions are cofinal in the category $A[0]/\mathrm{Qis}^+(\mathcal{A})$. To finish the proof it therefore suffices to show that for any quasi-isomorphism $A[0] \to I^\bullet$ with $I^\bullet$ bounded above and $I^n \in \mathcal{I}$ we have $F(A)[0] \to F(I^\bullet)$ is a quasi-isomorphism. To see this suppose that $I^n = 0$ for $n < n_0$. Of course we may assume that $n_0 < 0$. Starting with $n = n_0$ we prove inductively that $\mathrm{Im}(d^{n-1}) = \mathrm{Ker}(d^n)$ and $\mathrm{Im}(d^{-1})$ are elements of $\mathcal{I}$ using propert (2) and the exact sequences

$$0 \to \mathrm{Ker}(d^n) \to I^n \to \mathrm{Im}(d^n) \to 0.$$

Moreover, property (2) also guarantees that the complex

$$0 \to F(I^{n_0}) \to F(I^{n_0+1}) \to \ldots \to F(I^{-1}) \to F(\mathrm{Im}(d^{-1})) \to 0$$

is exact. The exact sequence $0 \to \mathrm{Im}(d^{-1}) \to I^0 \to I^0/\mathrm{Im}(d^{-1}) \to 0$ implies that $I^0/\mathrm{Im}(d^{-1})$ is an element of $\mathcal{I}$. The exact sequence $0 \to A \to I^0/\mathrm{Im}(d^{-1}) \to \mathrm{Im}(d^0) \to 0$ then implies that $\mathrm{Im}(d^0) = \mathrm{Ker}(d^1)$ is an elements of $\mathcal{I}$ and from then on one continues as before to show that $\mathrm{Im}(d^{n-1}) = \mathrm{Ker}(d^n)$ is an element of $\mathcal{I}$ for all $n > 0$. Applying F to each of the short exact sequences mentioned above and using (2) we observe that $F(A)[0] \to F(I^\bullet)$ is an isomorphism as desired. □

Lemma 11.15.7. *In Situation 11.15.1. Let $\mathcal{P} \subset Ob(\mathcal{A})$ be a subset with the following properties:*

1. *every object of $\mathcal{A}$ is a quotient of an element of $\mathcal{P}$,*
2. *for any short exact sequence $0 \to P \to Q \to R \to 0$ of $\mathcal{A}$ with $Q, R \in \mathcal{P}$, then $P \in \mathcal{P}$, and $0 \to F(P) \to F(Q) \to F(R) \to 0$ is exact.*

Then every object of $\mathcal{P}$ is acyclic for LF.

Proof. Dual to the proof of Lemma 11.15.6. □

Proposition 11.15.8. *In Situation 11.15.1.*

1. *If every object of $\mathcal{A}$ injects into an object acyclic for RF, then RF is defined on all of $K^+(\mathcal{A})$ and we obtain an exact functor*
$$RF : D^+(\mathcal{B}) \longrightarrow D^+(\mathcal{A})$$
see (11.14.9.1). Moreover, any bounded below complex $K^\bullet$ whose terms are acyclic for RF computes RF.
2. *If every object of $\mathcal{A}$ is quotient of an object acyclic for LF, then LF is defined on all of $K^-(\mathcal{A})$ and we obtain an exact functor*
$$LF : D^-(\mathcal{B}) \longrightarrow D^-(\mathcal{A})$$
see (11.14.9.1). Moreover, any bounded above complex $K^\bullet$ whose terms are acyclic for LF computes LF.

Proof. Suppose every object of $\mathcal{A}$ injects into an object acyclic for RF. Let $\mathcal{I}$ be the set of objects acyclic for RF. Let $K^\bullet$ be a bounded below complex with $K^n \in \mathcal{I}$. By Lemma 11.15.4 the quasi-isomorphisms $\alpha : K^\bullet \to I^\bullet$ with $I^\bullet$ bounded below and $I^n \in \mathcal{I}$ are cofinal in the category $K^\bullet/\mathrm{Qis}^+(\mathcal{A})$. Hence in order to show that $K^\bullet$ computes RF it suffices to show that $F(K^\bullet) \to F(I^\bullet)$ is an isomorphism. Note that $C(\alpha)^\bullet$ is an acyclic bounded below complex all of whose terms are in $\mathcal{I}$. Hence it suffices to show: given an acyclic bounded below complex $I^\bullet$ all of whose terms are in $\mathcal{I}$ the complex $F(I^\bullet)$ is acyclic.

Say $I^n = 0$ for $n < n_0$. Then we break $I^\bullet$ into short exact sequences $0 \to \mathrm{Im}(d^n) \to I^{n+1} \to \mathrm{Im}(d^{n+1}) \to 0$ for $n \geq n_0$. These sequences induce distinguished triangles $(\mathrm{Im}(d^n), I^{n+1}, \mathrm{Im}(d^{n+1}))$ by Lemma 11.11.1. This implies inductively that each $\mathrm{Im}(d^n)$ is acyclic for RF by Lemma 11.14.12. Moreover, the long exact cohomology sequences (11.10.1.1) associated to the distinguished triangles $(F(\mathrm{Im}(d^n)), F(I^{n+1}), F(\mathrm{Im}(d^{n+1})))$ of $D^+(\mathcal{B})$ imply that

$$0 \to F(\mathrm{Im}(d^n)) \to F(I^{n+1}) \to F(\mathrm{Im}(d^{n+1})) \to 0$$

is short exact, and this in turn proves that $F(I^\bullet)$ is exact.

Finally, since by Lemma 11.15.4 every object of $K^+(\mathcal{A})$ is quasi-isomorphic to such a bounded below complex with terms in $\mathcal{I}$ we see that RF is everywhere defined, see Lemma 11.14.14. The proof in the case of LF is dual. $\square$

11.16. Higher derived functors

The following simple lemma shows that right derived functors ``move to the right".

Lemma 11.16.1. *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. Let $K^\bullet \in K^+(\mathcal{A})$ and $a \in \mathbf{Z}$ such that $H^i(K^\bullet) = 0$ for all $i < a$. If RF is defined at $K^\bullet$, then $H^i(RF(K^\bullet)) = 0$ for all $i < a$.*

Proof. Let $K^\bullet \to L^\bullet$ be any quasi-isomorphism. Then it is also true that $K^\bullet \to \tau_{\geq a} L^\bullet$ is a quasi-isomorphism by our assumption on $K^\bullet$. Hence in the category $K^\bullet/\text{Qis}^+(\mathcal{A})$ the quasi-isomorphisms $s : K^\bullet \to L^\bullet$ with $L^n = 0$ for $n < a$ are cofinal. Thus RF is the value of the essentially constant ind-object $F(L^\bullet)$ for these s it follows that $H^i(RF(K^\bullet)) = 0$ for $i < 0$. $\square$

Definition 11.16.2. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. Assume $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is everywhere defined. Let $i \in \mathbf{Z}$. The *ith right derived functor R^iF of F* is the functor

$$R^iF = H^i \circ RF : \mathcal{A} \longrightarrow \mathcal{B}$$

The following lemma shows that it really does not make a lot of sense to take the right derived functor unless the functor is left exact.

Lemma 11.16.3. *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories and assume $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is everywhere defined.*

(1) *We have $R^iF = 0$ for $i < 0$,*
(2) *R^0F is left exact,*
(3) *the map $F \to R^0F$ is an isomorphism if and only if F is left exact.*

Proof. Let A be an object of $\mathcal{A}$. Let $A[0] \to K^\bullet$ be any quasi-isomorphism. Then it is also true that $A[0] \to \tau_{\geq 0} K^\bullet$ is a quasi-isomorphism. Hence in the category $A[0]/\text{Qis}^+(\mathcal{A})$ the quasi-isomorphisms $s : A[0] \to K^\bullet$ with $K^n = 0$ for $n < 0$ are cofinal. Thus it is clear that $H^i(RF(A[0])) = 0$ for $i < 0$. Moreover, for such an s the sequence

$$0 \to A \to K^0 \to K^1$$

is exact. Hence if F is left exact, then $0 \to F(A) \to F(K^0) \to F(K^1)$ is exact as well, and we see that $F(A) \to H^0(F(K^\bullet))$ is an isomorphism for every $s : A[0] \to K^\bullet$ as above which implies that $H^0(RF(A[0])) = F(A)$.

Let $0 \to A \to B \to C \to 0$ be a short exact sequence of $\mathcal{A}$. By Lemma 11.11.1 we obtain a distinguished triangle $(A[0], B[0], C[0], a, b, c)$ in $K^+(\mathcal{A})$. From the long exact cohomology sequence (and the vanishing for $i < 0$ proved above) we deduce that $0 \to R^0F(A) \to R^0F(B) \to R^0F(C)$ is exact. Hence R^0F is left exact. Of course this also proves that if $F \to R^0F$ is an isomorphism, then F is left exact. $\square$

Lemma 11.16.4. *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories and assume $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is everywhere defined. Let A be an object of $\mathcal{A}$.*

(1) *A is right acyclic for F if and only if $F(A) \to R^0F(A)$ is an isomorphism and $R^iF(A) = 0$ for all $i > 0$,*

(2) *if F is left exact, then A is right acyclic for F if and only if $R^iF(A) = 0$ for all $i > 0$.*

Proof. If A is right acyclic for F, then $RF(A[0]) = F(A)[0]$ and in particular $F(A) \to R^0F(A)$ is an isomorphism and $R^iF(A) = 0$ for $i \neq 0$. Conversely, if $F(A) \to R^0F(A)$ is an isomorphism and $R^iF(A) = 0$ for all $i > 0$ then $F(A[0]) \to RF(A[0])$ is a quasi-isomorphism by Lemma 11.16.3 part (1) and hence A is acyclic. If F is left exact then $F = R^0F$, see Lemma 11.16.3. □

Lemma 11.16.5. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor between abelian categories and assume $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is everywhere defined. Let $0 \to A \to B \to C \to 0$ be a short exact sequence of $\mathcal{A}$.*

(1) *If A and C are right acyclic for F then so is B.*
(2) *If A and B are right acyclic for F then so is C.*
(3) *If B and C are right acyclic for F and $F(B) \to F(C)$ is surjective then A is right acyclic for F.*

In each of the three cases

$$0 \to F(A) \to F(B) \to F(C) \to 0$$

is a short exact sequence of $\mathcal{B}$.

Proof. By Lemma 11.11.1 we obtain a distinguished triangle $(A[0], B[0], C[0], a, b, c)$ in $K^+(\mathcal{A})$. As RF is an exact functor and since $R^iF = 0$ for $i < 0$ and $R^0F = F$ (Lemma 11.16.3) we obtain an exact cohomology sequence

$$0 \to F(A) \to F(B) \to F(C) \to R^1F(A) \to \ldots$$

in the abelian category $\mathcal{B}$. Thus the lemma follows from the characterization of acyclic objects in Lemma 11.16.4. □

Lemma 11.16.6. *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories and assume $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is everywhere defined.*

(1) *The functors R^iF, $i \geq 0$ come equipped with a canonical structure of a δ-functor from $\mathcal{A} \to \mathcal{B}$, see Homology, Definition 10.9.1.*
(2) *If every object of $\mathcal{A}$ is a subobject of a right acyclic object for F, then $\{R^iF, \delta\}_{i \geq 0}$ is a universal δ-functor, see Homology, Definition 10.9.3.*

Proof. The functor $\mathcal{A} \to \text{Comp}^+(\mathcal{A})$, $A \mapsto A[0]$ is exact. The functor $\text{Comp}^+(\mathcal{A}) \to D^+(\mathcal{A})$ is a δ-functor, see Lemma 11.11.1. The functor $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is exact. Finally, the functor $H^0 : D^+(\mathcal{B}) \to \mathcal{B}$ is a homological functor, see Definition 11.10.3. Hence we get the structure of a δ-functor from Lemma 11.4.20 and Lemma 11.4.19. Part (2) follows from Homology, Lemma 10.9.4 and the description of acyclics in Lemma 11.16.4. □

Lemma 11.16.7. *(Leray's acyclicity lemma) Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories and assume $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is everywhere defined. Let $A^\bullet$ be a bounded below complex of F-acyclic objects. The canonical map*

$$F(A^\bullet) \longrightarrow RF(A^\bullet)$$

is an isomorphism in $D^+(\mathcal{B})$, i.e., $A^\bullet$ computes RF.

Proof. First we claim the lemma holds for a bounded complex of acyclic objects. Namely, it holds for complexes with at most one nonzero object by definition. Suppose that $A^\bullet$ is a complex with $A^n = 0$ for $n \not\in [a, b]$. Using the ``stupid'' truncations we obtain a termwise split short exact sequence of complexes

$$0 \to \sigma_{\geq a+1} A^\bullet \to A^\bullet \to \sigma_{\leq a} A^\bullet \to 0$$

see Homology, Section 10.11. Thus a distinguished triangle $(\sigma_{\geq a+1} A^\bullet, A^\bullet, \sigma_{\leq a} A^\bullet)$. By induction hypothesis the two outer complexes compute RF. Then the middle one does too by Lemma 11.14.12.

Suppose that $A^\bullet$ is a bounded below complex of acyclic objects. To show that $F(A) \to RF(A)$ is an isomorphism in $D^+(\mathcal{B})$ it suffices to show that $H^i(F(A)) \to H^i(RF(A))$ is an isomorphism for all i. Pick i. Consider the termwise split short exact sequence of complexes

$$0 \to \sigma_{\geq i+2} A^\bullet \to A^\bullet \to \sigma_{\leq i+1} A^\bullet \to 0.$$

Note that this induces a termwise split short exact sequence

$$0 \to \sigma_{\geq i+2} F(A^\bullet) \to F(A^\bullet) \to \sigma_{\leq i+1} F(A^\bullet) \to 0.$$

Hence we get distinguished triangles

$$\begin{gathered}
(\sigma_{\geq i+2} A^\bullet, A^\bullet, \sigma_{\leq i+1} A^\bullet) \\
(\sigma_{\geq a+1} F(A^\bullet), F(A^\bullet), \sigma_{\leq a} F(A^\bullet)) \\
(RF(\sigma_{\geq a+1} A^\bullet), RF(A^\bullet), RF(\sigma_{\leq a} A^\bullet))
\end{gathered}$$

Using the last two we obtain a map of exact sequences

$$\begin{tikzcd}
H^i(\sigma_{\geq i+2} F(A^\bullet)) \arrow[r] \arrow[d] & H^i(F(A^\bullet)) \arrow[r] \arrow[d, "\alpha"] & H^i(\sigma_{\leq i+1} F(A^\bullet)) \arrow[r] \arrow[d, "\beta"] & H^{i+1}(\sigma_{\geq i+2} F(A^\bullet)) \arrow[d] \\
R^i F(\sigma_{\geq i+2} A^\bullet) \arrow[r] & R^i F(A^\bullet) \arrow[r] & R^i F(\sigma_{\leq i+1} A^\bullet) \arrow[r] & R^{i+1} F(\sigma_{\geq i+2} A^\bullet)
\end{tikzcd}$$

By the results of the first paragraph the map β is an isomorphism. By inspection the objects on the upper left and the upper right are zero. Hence to finish the proof we have to show that $R^i F(\sigma_{\geq i+2} A^\bullet) = 0$ and $R^{i+1} F(\sigma_{\geq i+2} A^\bullet) = 0$. This follows immediately from Lemma 11.16.1. $\square$

Lemma 11.16.8. *Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor of abelian categories. Then*

(1) *every object of $\mathcal{A}$ is right acyclic for F,*
(2) *$RF : D^+(\mathcal{A}) \to D^+(\mathcal{A})$ is everywhere defined,*
(3) *$RF : D(\mathcal{A}) \to D(\mathcal{A})$ is everywhere defined,*
(4) *every complex computes RF, in other words, the canonical map $F(K^\bullet) \to RF(K^\bullet)$ is an isomorphism for all complexes, and*
(5) *$R^i F = 0$ for $i \neq 0$.*

Proof. This is true because F transforms acyclic complexes into acyclic complexes and quasi-isomorphisms into quasi-isomorphisms. Details omitted. $\square$

11.17. Injective resolutions

In this section we prove some lemmas regarding the existence of injective resolutions in abelian categories having enough injectives.

Definition 11.17.1. Let $\mathcal{A}$ be an abelian category. Let $A \in Ob(\mathcal{A})$. An *injective resolution of A* is a complex $I^\bullet$ together with a map $A \to I^0$ such that:

(1) We have $I^n = 0$ for $n < 0$.
(2) Each I^n is an injective object of $\mathcal{A}$.
(3) The map $A \to I^0$ is an isomorphism onto $\mathrm{Ker}(d^0)$.
(4) We have $H^i(I^\bullet) = 0$ for $i > 0$.

Hence $A[0] \to I^\bullet$ is a quasi-isomorphism. In other words the complex

$$\ldots \to 0 \to A \to I^0 \to I^1 \to \ldots$$

is acyclic. Let $K^\bullet$ be a complex in $\mathcal{A}$. An *injective resolution of* $K^\bullet$ is a complex $I^\bullet$ together with a map $\alpha : K^\bullet \to I^\bullet$ of complexes such that

(1) We have $I^n = 0$ for $n \ll 0$, i.e., $I^\bullet$ is bounded below.
(2) Each I^n is an injective object of $\mathcal{A}$.
(3) The map $\alpha : K^\bullet \to I^\bullet$ is a quasi-isomorphism.

In other words an injective resolution $K^\bullet \to I^\bullet$ gives rise to a diagram

$$\begin{tikzcd}
\ldots \arrow[r] & K^{n-1} \arrow[r] \arrow[d] & K^n \arrow[r] \arrow[d] & K^{n+1} \arrow[r] \arrow[d] & \ldots \\
\ldots \arrow[r] & I^{n-1} \arrow[r] & I^n \arrow[r] & I^{n+1} \arrow[r] & \ldots
\end{tikzcd}$$

which induces an isomorphism on cohomology objects in each degree. An injective resolution of an object A of $\mathcal{A}$ is almost the same thing as an injective resolution of the complex $A[0]$.

Lemma 11.17.2. *Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$ be a complex of $\mathcal{A}$.*

(1) *If $K^\bullet$ has an injective resolution then $H^n(K^\bullet) = 0$ for $n \ll 0$.*
(2) *If $H^n(K^\bullet) = 0$ for all $n \ll 0$ then there exists a quasi-isomorphism $K^\bullet \to L^\bullet$ with $L^\bullet$ bounded below.*

Proof. Omitted. For the second statement use $L^\bullet = K^\bullet/\tau_{\leq n}K^\bullet$ for some $n \ll 0$. See Homology, Section 10.11 for the definition of the truncation $\tau_{\leq n}$. □

Lemma 11.17.3. *Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has enough injectives.*

(1) *Any object of $\mathcal{A}$ has an injective resolution.*
(2) *If $H^n(K^\bullet) = 0$ for all $n \ll 0$ then $K^\bullet$ has an injective resolution.*
(3) *If $K^\bullet$ is a complex with $K^n = 0$ for $n < a$, then there exists an injective resolution $\alpha : K^\bullet \to I^\bullet$ with $I^n = 0$ for $n < a$ such that each $\alpha^n : K^n \to I^n$ is injective.*

Proof. Proof of (1). First choose an injection $A \to I^0$ of A into an injective object of $\mathcal{A}$. Next, choose an injection $I_0/A \to I^1$ into an injective object of $\mathcal{A}$. Denote d^0 the induced map $I^0 \to I^1$. Next, choose an injection $I^1/\mathrm{Im}(d^0) \to I^2$ into an injective object of $\mathcal{A}$. Denote d^1 the induced map $I^1 \to I^2$. And so on. By Lemma 11.17.2 part (2) follows from part (3). Part (3) is a special case of Lemma 11.15.4. □

Lemma 11.17.4. *Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$ be an acyclic complex. Let $I^\bullet$ be bounded below and consisting of injective objects. Any morphism $K^\bullet \to I^\bullet$ is homotopic to zero.*

Proof. Let $\alpha : K^\bullet \to I^\bullet$ be a morphism of complexes. Assume that $\alpha^j = 0$ for $j < n$. We will show that there exists a morphism $h : K^{n+1} \to I^n$ such that $\alpha^n = h \circ d$. Thus α will be homotopic to the morphism of complexes β defined by

$$\beta^j = \begin{cases} 0 & \text{if} \quad j \leq n \\ \alpha^{n+1} - d \circ h & \text{if} \quad j = n+1 \\ \alpha^j & \text{if} \quad j > n+1 \end{cases}$$

This will clearly prove the lemma (by induction). To prove the existence of h note that $\alpha^n|_{d^{n-1}(K^{n-1})} = 0$ since $\alpha^{n-1} = 0$. Since $K^\bullet$ is acyclic we have $d^{n-1}(K^{n-1}) = \mathrm{Ker}(K^n \to K^{n+1})$. Hence we can think of α^n as a map into I^n defined on the subobject $\mathrm{Im}(K^n \to K^{n+1})$ of K^{n+1}. By injectivity of the object I^n we can extend this to a map $h : K^{n+1} \to I^n$ as desired. □

Remark 11.17.5. Let $\mathcal{A}$ be an abelian category. Using the fact that $K(\mathcal{A})$ is a triangulated category we may use Lemma 11.17.4 to obtain proofs of some of the lemmas below which are usually proved by chasing through diagrams. Namely, suppose that $\alpha : K^\bullet \to L^\bullet$ is a quasi-isomorphism of complexes. Then

$$(K^\bullet, L^\bullet, C(\alpha)^\bullet, \alpha, i, -p)$$

is a distinguished triangle in $K(\mathcal{A})$ (Lemma 11.8.12) and $C(f)^\bullet$ is an acyclic complex (Lemma 11.10.2). Next, let $I^\bullet$ be a bounded below complex of injective objects. Then

$$\begin{tikzcd} Hom_{K(\mathcal{A})}(C(\alpha)^\bullet, I^\bullet) \arrow[r] & Hom_{K(\mathcal{A})}(L^\bullet, I^\bullet) \arrow[r] & Hom_{K(\mathcal{A})}(K^\bullet, I^\bullet) \arrow[dll] \\ Hom_{K(\mathcal{A})}(C(\alpha)^\bullet[-1], I^\bullet) & & \end{tikzcd}$$

is an exact sequence of abelian groups, see Lemma 11.4.2. At this point Lemma 11.17.4 guarantees that the outer two groups are zero and hence $Hom_{K(\mathcal{A})}(L^\bullet, I^\bullet) = Hom_{K(\mathcal{A})}(K^\bullet, I^\bullet)$.

Lemma 11.17.6. *Let $\mathcal{A}$ be an abelian category. Consider a solid diagram*

$$\begin{tikzcd} K^\bullet \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & L^\bullet \arrow[dl, dashed, "\beta"] \\ I^\bullet & \end{tikzcd}$$

where $I^\bullet$ is bounded below and consists of injective objects, and α is a quasi-isomorphism.

(1) *There exists a map of complexes β making the diagram commute up to homotopy.*
(2) *If α is injective in every degree then we can find a β which makes the diagram commute.*

Proof. The ``correct" proof of part (1) is explained in Remark 11.17.5. We also give a direct proof here.

We first show that (2) implies (1). Namely, let $\tilde{\alpha} : K \to \tilde{L}^\bullet$, π, s be as in Lemma 11.8.5. Since $\tilde{\alpha}$ is injective by (2) there exists a morphism $\tilde{\beta} : \tilde{L}^\bullet \to I^\bullet$ such that $\gamma = \tilde{\beta} \circ \tilde{\alpha}$. Set $\beta = \tilde{\beta} \circ s$. Then we have

$$\beta \circ \alpha = \tilde{\beta} \circ s \circ \pi \circ \tilde{\alpha} \sim \tilde{\beta} \circ \tilde{\alpha} = \gamma$$

as desired.

Assume that $\alpha : K^\bullet \to L^\bullet$ is injective. Suppose we have already defined β in all degrees $\leq n-1$ compatible with differentials and such that $\gamma^j = \beta^j \circ \alpha^j$ for all $j \leq n-1$. Consider the commutative solid diagram

$$\begin{tikzcd} K^{n-1} \arrow[r] \arrow[d, "\alpha"] \arrow[dd, bend right, "\gamma"'] & K^n \arrow[d, "\alpha"] \arrow[dd, bend left, "\gamma"] \\ L^{n-1} \arrow[r] \arrow[d, "\beta"] & L^n \arrow[d, dashed] \\ I^{n-1} \arrow[r] & I^n \end{tikzcd}$$

Thus we see that the dotted arrow is prescribed on the subobjects $\alpha(K^n)$ and $d^{n-1}(L^{n-1})$. Moreover, these two arrows agree on $\alpha(d^{n-1}(K^{n-1}))$. Hence if

$$\alpha(d^{n-1}(K^{n-1})) = \alpha(K^n) \cap d^{n-1}(L^{n-1}) \tag{11.17.6.1}$$

then these morphisms glue to a morphism $\alpha(K^n) + d^{n-1}(L^{n-1}) \to I^n$ and, using the injectivity of I^n, we can extend this to a morphism from all of L^n into I^n. After this by induction we get the morphism β for all n simultaneously (note that we can set $\beta^n = 0$ for all $n \ll 0$ since $I^\bullet$ is bounded below -- in this way starting the induction).

It remains to prove the equality (11.17.6.1). The reader is encouraged to argue this for themselves with a suitable diagram chase. Nonetheless here is our argument. Note that the inclusion $\alpha(d^{n-1}(K^{n-1})) \subset \alpha(K^n) \cap d^{n-1}(L^{n-1})$ is obvious. Take an object T of $\mathcal{A}$ and a morphism $x : T \to L^n$ whose image is contained in the subobject $\alpha(K^n) \cap d^{n-1}(L^{n-1})$. Since α is injective we see that $x = \alpha \circ x'$ for some $x' : T \to K^n$. Moreover, since x lies in $d^{n-1}(L^{n-1})$ we see that $d^n \circ x = 0$. Hence using injectivity of α again we see that $d^n \circ x' = 0$. Thus x' gives a morphism $[x'] : T \to H^n(K^\bullet)$. On the other hand the corresponding map $[x] : T \to H^n(L^\bullet)$ induced by x is zero by assumption. Since α is a quasi-isomorphism we conclude that $[x'] = 0$. This of course means exactly that the image of x' is contained in $d^{n-1}(K^{n-1})$ and we win. □

Lemma 11.17.7. *Let $\mathcal{A}$ be an abelian category. Consider a solid diagram*

$$\begin{tikzcd} K^\bullet \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & L^\bullet \arrow[dl, dashed, "\beta_i"] \\ I^\bullet & \end{tikzcd}$$

where $I^\bullet$ is bounded below and consists of injective objects, and α is a quasi-isomorphism. Any two morphisms β_1, β_2 making the diagram commute up to homotopy are homotopic.

Proof. This follows from Remark 11.17.5. We also give a direct argument here.

Let $\tilde{\alpha} : K \to \tilde{L}^\bullet$, π, s be as in Lemma 11.8.5. If we can show that $\beta_1 \circ \pi$ is homotopic to $\beta_2 \circ \pi$, then we deduce that $\beta_1 \sim \beta_2$ because $\pi \circ s$ is the identity. Hence we may assume $\alpha^n : K^n \to L^n$ is the inclusion of a direct summand for all n. Thus we get a short exact sequence of complexes

$$0 \to K^\bullet \to L^\bullet \to M^\bullet \to 0$$

which is termwise split and such that $M^\bullet$ is acyclic. We choose splittings $L^n = K^n \oplus M^n$, so we have $\beta_i^n : K^n \oplus M^n \to I^n$ and $\gamma^n : K^n \to I^n$. In this case the condition on β_i is that there are morphisms $h_i^n : K^n \to I^{n-1}$ such that

$$\gamma^n - \beta_i^n|_{K^n} = d \circ h_i^n + h_i^{n+1} \circ d$$

Thus we see that

$$\beta_1^n|_{K^n} - \beta_2^n|_{K^n} = d \circ (h_1^n - h_2^n) + (h_1^{n+1} - h_2^{n+1}) \circ d$$

Consider the map $h^n : K^n \oplus M^n \to I^{n-1}$ which equals $h_1^n - h_2^n$ on the first summand and zero on the second. Then we see that

$$\beta_1^n - \beta_2^n - (d \circ h^n + h^{n+1}) \circ d)$$

is a morphism of complexes $L^\bullet \to I^\bullet$ which is identically zero on the subcomplex $K^\bullet$. Hence it factors as $L^\bullet \to M^\bullet \to I^\bullet$. Thus the result of the lemma follows from Lemma 11.17.4. □

Lemma 11.17.8. *Let $\mathcal{A}$ be an abelian category. Let $I^\bullet$ be bounded below complex consisting of injective objects. Let $L^\bullet \in K(\mathcal{A})$. Then*

$$Mor_{K(\mathcal{A})}(L^\bullet, I^\bullet) = Mor_{D(\mathcal{A})}(L^\bullet, I^\bullet).$$

Proof. Let a be an element of the right hand side. We may represent $a = \gamma\alpha^{-1}$ where $\alpha : K^\bullet \to L^\bullet$ is a quasi-isomorphism and $\gamma : K^\bullet \to I^\bullet$ is a map of complexes. By Lemma 11.17.6 we can find a morphism $\beta : L^\bullet \to I^\bullet$ such that $\beta \circ \alpha$ is homotopic to γ. This proves that the map is surjective. Let b be an element of the left hand side which maps to zero in the right hand side. Then b is the homotopy class of a morphism $\beta : L^\bullet \to I^\bullet$ such that there exists a quasi-iomorphism $\alpha : K^\bullet \to L^\bullet$ with $\beta \circ \alpha$ homotopic to zero. Then Lemma 11.17.7 shows that β is homotopic to zero also, i.e., $b = 0$. □

Lemma 11.17.9. *Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has enough injectives. For any short exact sequence $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ of $Comp^+(\mathcal{A})$ there exists a commutative diagram in $Comp^+(\mathcal{A})$*

$$\begin{tikzcd}
0 \arrow[r] & A^\bullet \arrow[r] \arrow[d] & B^\bullet \arrow[r] \arrow[d] & C^\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I_1^\bullet \arrow[r] & I_2^\bullet \arrow[r] & I_3^\bullet \arrow[r] & 0
\end{tikzcd}$$

where the vertical arrows are injective resolutions and the rows are short exact sequences of complexes. In fact, given any injective resolution $A^\bullet \to I^\bullet$ we may assume $I_1^\bullet = I^\bullet$.

Proof. Step 1. Choose an injective resolution $A^\bullet \to I^\bullet$ (see Lemma 11.17.3) or use the given one. Recall that $\text{Comp}^+(\mathcal{A})$ is an abelian category, see Homology, Lemma 10.10.9. Hence we may form the pushout along the injective map $A^\bullet \to I^\bullet$ to get

$$\begin{tikzcd}
0 \arrow[r] & A^\bullet \arrow[r] \arrow[d] & B^\bullet \arrow[r] \arrow[d] & C^\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^\bullet \arrow[r] & E^\bullet \arrow[r] & C^\bullet \arrow[r] & 0
\end{tikzcd}$$

Note that the lower short exact sequence is termwise split, see Homology, Lemma 10.20.2. Hence it suffices to prove the lemma when $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ is termwise split.

Step 2. Choose splittings. In other words, write $B^n = A^n \oplus C^n$. Denote $\delta : C^\bullet \to A^\bullet[1]$ the morphism as in Homology, Lemma 10.12.10. Choose injective resolutions $f_1 : A^\bullet \to I_1^\bullet$ and $f_3 : C^\bullet \to I_3^\bullet$. (If $A^\bullet$ is a complex of injectives, then use $I_1^\bullet = A^\bullet$.) We may assume f_3

is injective in every degree. By Lemma 11.17.6 we may find a morphism $\delta' : I_3^\bullet \to I_1^\bullet[1]$ such that $\delta' \circ f_3 = f_1[1] \circ \delta$ (equality of morphisms of complexes). Set $I_2^n = I_1^n \oplus I_3^n$. Define

$$d_{I_2}^n = \begin{pmatrix} d_{I_1}^n & (\delta')^n \\ 0 & d_{I_3}^n \end{pmatrix}$$

and define the maps $B^n \to I_2^n$ to be given as the sum of the maps $A^n \to I_1^n$ and $C^n \to I_3^n$. Everything is clear. □

11.18. Projective resolutions

This section is dual to Section 11.17. We give definitions and state results, but we do not reprove the lemmas.

Definition 11.18.1. Let $\mathcal{A}$ be an abelian category. Let $A \in Ob(\mathcal{A})$. An *projective resolution of A* is a complex $P^\bullet$ together with a map $P^0 \to A$ such that:

(1) We have $P^n = 0$ for $n > 0$.
(2) Each P^n is an projective object of $\mathcal{A}$.
(3) The map $P^0 \to A$ induces an isomorphism $\text{Coker}(d^{-1}) \to A$.
(4) We have $H^i(P^\bullet) = 0$ for $i < 0$.

Hence $P^\bullet \to A[0]$ is a quasi-isomorphism. In other words the complex

$$\ldots \to P^{-1} \to P^0 \to A \to 0 \to \ldots$$

is acyclic. Let $K^\bullet$ be a complex in $\mathcal{A}$. An *projective resolution of $K^\bullet$* is a complex $P^\bullet$ together with a map $\alpha : P^\bullet \to K^\bullet$ of complexes such that

(1) We have $P^n = 0$ for $n \gg 0$, i.e., $P^\bullet$ is bounded above.
(2) Each P^n is an projective object of $\mathcal{A}$.
(3) The map $\alpha : P^\bullet \to K^\bullet$ is a quasi-isomorphism.

Lemma 11.18.2. *Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$ be a complex of $\mathcal{A}$.*

(1) *If $K^\bullet$ has a projective resolution then $H^n(K^\bullet) = 0$ for $n \gg 0$.*
(2) *If $H^n(K^\bullet) = 0$ for $n \gg 0$ then there exists a quasi-isomorphism $L^\bullet \to K^\bullet$ with $L^\bullet$ bounded above.*

Proof. Dual to Lemma 11.17.2. □

Lemma 11.18.3. *Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has enough projectives.*

(1) *Any object of $\mathcal{A}$ has a projective resolution.*
(2) *If $H^n(K^\bullet) = 0$ for all $n \gg 0$ then $K^\bullet$ has a projective resolution.*
(3) *If $K^\bullet$ is a complex with $K^n = 0$ for $n > a$, then there exists a projective resolution $\alpha : P^\bullet \to K^\bullet$ with $P^n = 0$ for $n > a$ such that each $\alpha^n : P^n \to K^n$ is surjective.*

Proof. Dual to Lemma 11.17.3. □

Lemma 11.18.4. *Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$ be an acyclic complex. Let $P^\bullet$ be bounded above and consisting of projective objects. Any morphism $P^\bullet \to K^\bullet$ is homotopic to zero.*

Proof. Dual to Lemma 11.17.4. □

Remark 11.18.5. Let $\mathcal{A}$ be an abelian category. Suppose that $\alpha : K^\bullet \to L^\bullet$ is a quasi-isomorphism of complexes. Let $P^\bullet$ be a bounded above complex of projectives. Then

$$Hom_{K(\mathcal{A})}(P^\bullet, K^\bullet) \longrightarrow Hom_{K(\mathcal{A})}(P^\bullet, L^\bullet)$$

is an isomorphism. This is dual to Remark 11.17.5.

Lemma 11.18.6. *Let $\mathcal{A}$ be an abelian category. Consider a solid diagram*

$$\begin{tikzcd} K^\bullet & L^\bullet \arrow[l, "\alpha"'] \\ P^\bullet \arrow[u] \arrow[ur, dashed, "\beta"'] & \end{tikzcd}$$

where $P^\bullet$ is bounded above and consists of projective objects, and α is a quasi-isomorphism.

(1) *There exists a map of complexes β making the diagram commute up to homotopy.*
(2) *If α is surjective in every degree then we can find a β which makes the diagram commute.*

Proof. Dual to Lemma 11.17.6. □

Lemma 11.18.7. *Let $\mathcal{A}$ be an abelian category. Consider a solid diagram*

$$\begin{tikzcd} K^\bullet & L^\bullet \arrow[l, "\alpha"'] \\ P^\bullet \arrow[u] \arrow[ur, dashed, "\beta_i"'] & \end{tikzcd}$$

where $P^\bullet$ is bounded above and consists of projective objects, and α is a quasi-isomorphism. Any two morphisms β_1, β_2 making the diagram commute up to homotopy are homotopic.

Proof. Dual to Lemma 11.17.7. □

Lemma 11.18.8. *Let $\mathcal{A}$ be an abelian category. Let $P^\bullet$ be bounded above complex consisting of projective objects. Let $L^\bullet \in K(\mathcal{A})$. Then*

$$Mor_{K(\mathcal{A})}(P^\bullet, L^\bullet) = Mor_{D(\mathcal{A})}(P^\bullet, L^\bullet).$$

Proof. Dual to Lemma 11.17.8. □

Lemma 11.18.9. *Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has enough projectives. For any short exact sequence $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ of $Comp^+(\mathcal{A})$ there exists a commutative diagram in $Comp^+(\mathcal{A})$*

$$\begin{tikzcd} 0 \arrow[r] & P_1^\bullet \arrow[r] \arrow[d] & P_2^\bullet \arrow[r] \arrow[d] & P_3^\bullet \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A^\bullet \arrow[r] & B^\bullet \arrow[r] & C^\bullet \arrow[r] & 0 \end{tikzcd}$$

where the vertical arrows are projective resolutions and the rows are short exact sequences of complexes. In fact, given any projective resolution $P^\bullet \to C^\bullet$ we may assume $P_3^\bullet = P^\bullet$.

Proof. Dual to Lemma 11.17.9. □

Lemma 11.18.10. *Let $\mathcal{A}$ be an abelian category. Let $P^\bullet$, $K^\bullet$ be complexes. Let $n \in \mathbf{Z}$. Assume that*

(1) *$P^\bullet$ is a bounded complex consisting of projective objects,*
(2) *$P^i = 0$ for $i < n$, and*

(3) $H^i(K^\bullet) = 0$ *for* $i \geq n$.

Then $Hom_{K(\mathcal{A})}(P^\bullet, K^\bullet) = Hom_{D(\mathcal{A})}(P^\bullet, K^\bullet) = 0$.

Proof. The first equality follows from Lemma 11.18.8. Note that there is a distinguished triangle

$$(\tau_{\leq n-1}K^\bullet, K^\bullet, \tau_{\geq n}K^\bullet, f, g, h)$$

because the complex $K^\bullet/\tau_{\leq n-1}K^\bullet$ is quasi-isomorphic to $\tau_{\geq n}K^\bullet$. Hence, by Lemma 11.4.2 it suffices to prove $Hom_{K(\mathcal{A})}(P^\bullet, \tau_{\leq n-1}K^\bullet) = 0$ and $Hom_{K(\mathcal{A})}(P^\bullet, \tau_{\geq n}K^\bullet) = 0$. The first vanishing is trivial and the second is Lemma 11.18.4. □

Lemma 11.18.11. *Let $\mathcal{A}$ be an abelian category. Let $\beta : P^\bullet \to L^\bullet$ and $\alpha : E^\bullet \to L^\bullet$ be maps of complexes. Let $n \in \mathbf{Z}$. Assume*

(1) *$P^\bullet$ is a bounded complex of projectives and $P^i = 0$ for $i < n$,*
(2) *$H^i(\alpha)$ is an isomorphism for $i > n$ and surjective for $i = n$.*

Then there exists a map of complexes $\gamma : P^\bullet \to E^\bullet$ such that $\alpha \circ \gamma$ and β are homotopic.

Proof. Consider the cone $C^\bullet = C(\alpha)^\bullet$ with map $i : L^\bullet \to C^\bullet$. Note that $i \circ \beta$ is zero by Lemma 11.18.10. Hence we can lift β to $E^\bullet$ by Lemma 11.4.2. □

11.19. Right derived functors and injective resolutions

At this point we can use the material above to define the right derived functors of an additive functor between an abelian category having enough injectives and a general abelian category.

Lemma 11.19.1. *Let $\mathcal{A}$ be an abelian category. Let $I \in Ob(\mathcal{A})$ be an injective object. Let $I^\bullet$ be a bounded below complex of injectives in $\mathcal{A}$.*

(1) *$I^\bullet$ computes RF relative to $Qis^+(\mathcal{A})$ for any exact functor $F : K^+(\mathcal{A}) \to \mathcal{D}$ into any triangulated category $\mathcal{D}$.*
(2) *I is right acyclic for any additive functor $F : \mathcal{A} \to \mathcal{B}$ into any abelian category $\mathcal{B}$.*

Proof. Part (2) is a direct consequences of part (1) and Definition 11.15.3. To prove (1) let $\alpha : I^\bullet \to K^\bullet$ be a quasi-isomorphism into a complex. By Lemma 11.17.7 we see that α has a left inverse. Hence the category $I^\bullet/\text{Qis}^+(\mathcal{A})$ is essentially constant with value $\text{id} : I^\bullet \to I^\bullet$. Thus also the ind-object

$$I^\bullet/\text{Qis}^+(\mathcal{A}) \longrightarrow \mathcal{D}, \quad (I^\bullet \to K^\bullet) \longmapsto F(K^\bullet)$$

is essentially constant with value $F(I^\bullet)$. This proves (1), see Definitions 11.14.2 and 11.14.10. □

Lemma 11.19.2. *Let $\mathcal{A}$ be an abelian category with enough injectives.*

(1) *For any exact functor $F : K^+(\mathcal{A}) \to \mathcal{D}$ into a triangulated category $\mathcal{D}$ the right derived functor*
$$RF : D^+(\mathcal{A}) \longrightarrow \mathcal{D}$$
is everywhere defined.
(2) *For any additive functor $F : \mathcal{A} \to \mathcal{B}$ into an abelian category $\mathcal{B}$ the right derived functor*
$$RF : D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B})$$
is everywhere defined.

Proof. Combine Lemma 11.19.1 and Proposition 11.15.8 for the second assertion. To see the first assertion combine Lemma 11.17.3, Lemma 11.19.1, Lemma 11.14.14, and Equation (11.14.9.1). □

Lemma 11.19.3. *Let $\mathcal{A}$ be an abelian category with enough injectives. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor.*

(1) *The functor RF is an exact functor $D^+(\mathcal{A}) \to D^+(\mathcal{B})$.*
(2) *The functor RF induces an exact functor $K^+(\mathcal{A}) \to D^+(\mathcal{B})$.*
(3) *The functor RF induces a δ-functor $Comp^+(\mathcal{A}) \to D^+(\mathcal{B})$.*
(4) *The functor RF induces a δ-functor $\mathcal{A} \to D^+(\mathcal{B})$.*

Proof. This lemma simply reviews some of the results obtained sofar. Note that by Lemma 11.19.2 RF is everywhere defined. Here are some references:

(1) The derived functor is exact: This boils down to Lemma 11.14.6.
(2) This is true because $K^+(\mathcal{A}) \to D^+(\mathcal{A})$ is exact and compositions of exact functors are exact.
(3) This is true because $\text{Comp}^+(\mathcal{A}) \to D^+(\mathcal{A})$ is a δ-functor, see Lemma 11.11.1.
(4) This is true because $\mathcal{A} \to \text{Comp}^+(\mathcal{A})$ is exact and precomposing a δ-functor by an exact functor gives a δ-functor.

□

Lemma 11.19.4. *Let $\mathcal{A}$ be an abelian category with enough injectives. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor.*

(1) *For any short exact sequence $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ of complexes in $Comp^+(\mathcal{A})$ there is an associated long exact sequence*

$$\ldots \to H^i(RF(A^\bullet)) \to H^i(RF(B^\bullet)) \to H^i(RF(C^\bullet)) \to H^{i+1}(RF(A^\bullet)) \to \ldots$$

(2) *The functors $R^iF : \mathcal{A} \to \mathcal{B}$ are zero for $i < 0$. Also $R^0F = F : \mathcal{A} \to \mathcal{B}$.*
(3) *We have $R^iF(I) = 0$ for $i > 0$ and I injective.*
(4) *The sequence (R^iF, δ) forms a universal δ-functor (see Homology, Definition 10.9.3) from $\mathcal{A}$ to $\mathcal{B}$.*

Proof. This lemma simply reviews some of the results obtained sofar. Note that by Lemma 11.19.2 RF is everywhere defined. Here are some references:

(1) This follows from Lemma 11.19.3 part (3) combined with the long exact cohomology sequence (11.10.1.1) for $D^+(\mathcal{B})$.
(2) This is Lemma 11.16.3.
(3) This is the fact that injective objects are acyclic.
(4) This is Lemma 11.16.6.

□

11.20. Cartan-Eilenberg resolutions

This section can be expanded. The material can be generalized and applied in more cases. Resolutions need not use injectives and the method also works in the unbounded case in some situations.

Definition 11.20.1. Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$ be a bounded below complex. A *Cartan-Eilenberg resolution* of $K^\bullet$ is given by a double complex $I^{\bullet,\bullet}$ and a morphism of complexes $\epsilon : K^\bullet \to I^{\bullet,0}$ with the following properties:

(1) There exists a $i \ll 0$ such that $I^{p,q} = 0$ for all $p < i$ and all q.

(2) We have $I^{p,q} = 0$ if $q < 0$.
(3) The complex $I^{p,\bullet}$ is an injective resolution of K^p.
(4) The complex $\mathrm{Ker}(d_1^{p,\bullet})$ is an injective resolution of $\mathrm{Ker}(d_K^p)$.
(5) The complex $\mathrm{Im}(d_1^{p,\bullet})$ is an injective resolution of $\mathrm{Im}(d_K^p)$.
(6) The complex $H_I^p(I^{\bullet,\bullet})$ is an injective resolution of $H^p(K^\bullet)$.

Lemma 11.20.2. *Let $\mathcal{A}$ be an abelian category with enough injectives. Let $K^\bullet$ be a bounded below complex. There exists a Cartan-Eilenberg resolution of $K^\bullet$.*

Proof. Suppose that $K^p = 0$ for $p < n$. Decompose $K^\bullet$ into short exact sequences as follows: Set $Z^p = \mathrm{Ker}(d^p)$, $B^p = \mathrm{Im}(d^{p-1})$, $H^p = Z^p/B^p$, and consider

$$\begin{gathered} 0 \to Z^n \to K^n \to B^{n+1} \to 0 \\ 0 \to B^{n+1} \to Z^{n+1} \to H^{n+1} \to 0 \\ 0 \to Z^{n+1} \to K^{n+1} \to B^{n+2} \to 0 \\ 0 \to B^{n+2} \to Z^{n+2} \to H^{n+2} \to 0 \\ \ldots \end{gathered}$$

Set $I^{p,q} = 0$ for $p < n$. Inductively we choose injective resolutions as follows:

(1) Choose an injective resolution $Z^n \to J_Z^{n,\bullet}$.
(2) Using Lemma 11.17.9 choose injective resolutions $K^n \to I^{n,\bullet}$, $B^{n+1} \to J_B^{n+1,\bullet}$, and an exact sequence of complexes $0 \to J_Z^{n,\bullet} \to I^{n,\bullet} \to J_B^{n+1,\bullet} \to 0$ compatible with the short exact sequence $0 \to Z^n \to K^n \to B^{n+1} \to 0$.
(3) Using Lemma 11.17.9 choose injective resolutions $Z^{n+1} \to J_Z^{n+1,\bullet}$, $H^{n+1} \to J_H^{n+1,\bullet}$, and an exact sequence of complexes $0 \to J_B^{n+1,\bullet} \to J_Z^{n+1,\bullet} \to J_H^{n+1,\bullet} \to 0$ compatible with the short exact sequence $0 \to B^{n+1} \to Z^{n+1} \to H^{n+1} \to 0$.
(4) Etc.

Taking as maps $d_1^\bullet : I^{p,\bullet} \to I^{p+1,\bullet}$ the compositions $I^{p,\bullet} \to J_B^{p+1,\bullet} \to J_Z^{p+1,\bullet} \to I^{p+1,\bullet}$ everything is clear. □

Lemma 11.20.3. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor of abelian categories. Let $K^\bullet$ be a bounded below complex of $\mathcal{A}$. Let $I^{\bullet,\bullet}$ be a Cartan-Eilenberg resolution for $K^\bullet$. The spectral sequences $({}'E_r, {}'d_r)_{r\geq 0}$ and $({}''E_r, {}''d_r)_{r\geq 0}$ associated to the double complex $F(I^{\bullet,\bullet})$ satisfy the relations*

$$ {}'E_2^{p,q} = H^p(R^qF(K^\bullet))^3 \quad \textit{and} \quad {}''E_2^{p,q} = R^pF(H^q(K^\bullet)) $$

Moreover, these spectral sequences converge to $H^{p+q}(RF(K^\bullet))$ and the associated induced filtrations on $H^{p+q}(RF(K^\bullet))$ are finite.

Proof. We will use the following remarks without further mention:

(1) As $I^{p,\bullet}$ is an injective resolution of K^p we see that RF is defined at $K^p[0]$ with value $F(I^{p,\bullet})$.
(2) As $H_I^p(I^{\bullet,\bullet})$ is an injective resolution of $H^p(K^\bullet)$ the derived functor RF is defined at $H^p(K^\bullet)[0]$ with value $F(H_I^p(I^{\bullet,\bullet}))$.
(3) By Homology, Lemma 10.19.6 the total complex $sI^\bullet$ is an injective resolution of $K^\bullet$. Hence RF is defined at $K^\bullet$ with value $F(sI^\bullet)$.

[3] This notation sucks! It really means the pth cohomology group of the complex with terms $R^qF(K^n)$. Not the pth cohomology of the qth derived functor of F applied to $K^\bullet$...

Consider the spectral sequences associated to the double complex $K^{\bullet,\bullet} = F(I^{\bullet,\bullet})$, see Homology, Lemma 10.19.3. These both converge, see Homology, Lemma 10.19.5, to the cohomology groups of the associated total complex $s(F(I^{\bullet,\bullet}) = F(sI^\bullet)$ which computes $H^n(RF(K^\bullet))$.

Computation of the first spectral sequence. We have ${}'E_1^{p,q} = H^q(K^{p,\bullet})$ in other words

$$ {}'E_1^{p,q} = H^q(F(I^{p,\bullet})) = R^qF(K^p) $$

and the maps ${}'E_1^{p,q} \to {}'E^{p+1,q}$ are the maps $R^qF(K^p) \to R^qF(K^{p+1})$ as desired.

Computation of the second spectral sequence. We have ${}''E_1^{p,q} = H^q(K^{\bullet,p}) = H^q(F(I^{\bullet,p}))$. Note that the complex $I^{\bullet,p}$ is bounded below, consists of injectives, and moreover each kernel, image, and cohomology group of the differentials is an injective object of $\mathcal{A}$. Hence we can split the differentials, i.e., each differential is a split surjection onto a direct summand. It follows that the same is true after applying F. Hence ${}''E_1^{p,q} = F(H^q(I^{\bullet,p})) = F(H^q_I(I^{\bullet,p}))$. The differentials on this are $(-1)^q$ times F applied to the differential of the complex $H^p_I(I^{\bullet,\bullet})$ which is an injective resolution of $H^p(K^\bullet)$. Hence the description of the E_2 terms. □

Remark 11.20.4. The spectral sequences of Lemma 11.20.3 are functorial in the complex $K^\bullet$. This follows from functoriality properties of Cartan-Eilenberg resolutions. On the other hand, they are both examples of a more general spectral sequence which may be associated to a filtered complex of $\mathcal{A}$. The functoriality will follow from its construction. We will return to this in the section on the filtered derived category, see Remark 11.25.15.

11.21. Composition of right derived functors

Sometimes we can compute the right derived functor of a composition. Suppose that $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be left exact functors. Assume that the right derived functors $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$, $RG : D^+(\mathcal{B}) \to D^+(\mathcal{C})$, and $R(G \circ F) : D^+(\mathcal{A}) \to D^+(\mathcal{C})$ are everywhere defined. Then there exists a canonical transformation

$$ t : R(G \circ F) \longrightarrow RG \circ RF, $$

see Lemma 11.14.16. This transformation need not always be an isomorphism.

Lemma 11.21.1. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be left exact functors. Assume $\mathcal{A}$, $\mathcal{B}$ have enough injectives. If $F(I)$ is right acyclic for G for each injective object I of $\mathcal{A}$, then we have an isomorphism of functors*

$$ t : R(G \circ F) \longrightarrow RG \circ RF. $$

Proof. Let $A^\bullet$ be a bounded below complex of $\mathcal{A}$. Choose an injective resolution $A^\bullet \to I^\bullet$. The map t is given (see proof of Lemma 11.14.16) by the maps

$$ R(G \circ F)(A^\bullet) = (G \circ F)(I^\bullet) = G(F(I^\bullet))) \to RG(F(I^\bullet)) = RG(RF(A^\bullet)) $$

where the arrow is an isomorphism by Lemma 11.16.7. □

Lemma 11.21.2. *(Grothendieck spectral sequence.) With assumptions as in Lemma 11.21.1. Let A be an object of $\mathcal{A}$. There exists a spectral sequence $(E_r^{p,q}, d_r^{p,q})_{r\geq 0}$ associated to a filtered complex with*

$$ E_2^{p,q} = R^pG(R^qF(A)) $$

converging to $R^{p+q}(G \circ F)(A)$. Moreover, the induced filtration on each $R^n(G \circ F)(A)$ is finite.

Proof. Choose an injective resolution $A \to I^\bullet$. Choose a Cartan-Eilenberg resolution $F(I^\bullet) \to I^{\bullet,\bullet}$ using Lemma 11.20.2. Apply Lemma 11.20.3 (use the second spectral sequence). Details omitted. □

11.22. Resolution functors

Let $\mathcal{A}$ be an abelian category with enough injectives. Denote $\mathcal{I}$ the full additive subcategory of $\mathcal{A}$ whose objects are the injective objects of $\mathcal{A}$. It turns out that $K^+(\mathcal{I})$ and $D^+(\mathcal{A})$ are equivalent in this case (see Proposition 11.22.1). For many purposes it therefore makes sense to think of $D^+(\mathcal{A})$ as the (easier to grok) category $K^+(\mathcal{I})$ in this case.

Proposition 11.22.1. *Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has enough injectives. Denote $\mathcal{I} \subset \mathcal{A}$ the strictly full additive subcategory whose objects are the injective objects of $\mathcal{A}$. The functor*

$$K^+(\mathcal{I}) \longrightarrow D^+(\mathcal{A})$$

is exact, fully faithful and essentially surjective, i.e., an equivalence of triangulated categories.

Proof. It is clear that the functor is exact. It is essentially surjective by Lemma 11.17.3. Fully faithfulness is a consequence of Lemma 11.17.8. □

Proposition 11.22.1 implies that we can find resolution functors. It turns out that we can prove resolution functors exist even in some cases where the abelian category $\mathcal{A}$ is a ``big'' category, i.e., has a class of objects.

Definition 11.22.2. Let $\mathcal{A}$ be an abelian category with enough injectives. A *resolution functor*[4] for $\mathcal{A}$ is given by the following data:

(1) for all $K^\bullet \in Ob(K^+(\mathcal{A}))$ a bounded below complex of injectives $j(K^\bullet)$, and
(2) for all $K^\bullet \in Ob(K^+(\mathcal{A}))$ a quasi-isomorphism $i_{K^\bullet} : K^\bullet \to j(K^\bullet)$.

Lemma 11.22.3. *Let $\mathcal{A}$ be an abelian category with enough injectives. Given a resolution functor (j, i) there is a unique way to turn j into a functor and i into a 2-isomorphism producing a 2-commutative diagram*

$$\begin{tikzcd} K^+(\mathcal{A}) \arrow[rr, "j"'] \arrow[dr] & & K^+(\mathcal{I}) \arrow[dl] \\ & D^+(\mathcal{A}) & \end{tikzcd}$$

where $\mathcal{I}$ is the full additive subcategory of $\mathcal{A}$ consisting of injective objects.

Proof. For every morphism $\alpha : K^\bullet \to L^\bullet$ of $K^+(\mathcal{A})$ there is a unique morphism $j(\alpha) : j(K^\bullet) \to j(L^\bullet)$ in $K^+(\mathcal{I})$ such that

$$\begin{tikzcd} K^\bullet \arrow[r, "\alpha"'] \arrow[d, "i_{K^\bullet}"'] & L^\bullet \arrow[d, "i_{L^\bullet}"] \\ j(K^\bullet) \arrow[r, "j(\alpha)"] & j(L^\bullet) \end{tikzcd}$$

is commutative in $K^+(\mathcal{A})$. To see this either use Lemmas 11.17.6 and 11.17.7 or the equivalent Lemma 11.17.8. The uniqueness implies that j is a functor, and the commutativity of the diagram implies that i gives a 2-morphism which witnesses the 2-commutativity of the diagram of categories in the statement of the lemma. □

[4] This is likely nonstandard terminology.

Lemma 11.22.4. *Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has enough injectives. Then a resolution functor j exists and is unique up to unique isomorphism of functors.*

Proof. Consider the set of all objects $K^\bullet$ of $K^+(\mathcal{A})$. (Recall that by our conventions any category has a set of objects unless mentioned otherwise.) By Lemma 11.17.3 every object has an injective resolution. By the axiom of choice we can choose for each $K^\bullet$ an injective resolution $i_{K^\bullet} : K^\bullet \to j(K^\bullet)$. □

Lemma 11.22.5. *Let $\mathcal{A}$ be an abelian category with enough injectives. Any resolution functor $j : K^+(\mathcal{A}) \to K^+(\mathcal{I})$ is exact.*

Proof. Denote $i_{K^\bullet} : K^\bullet \to j(K^\bullet)$ the canonical maps of Definition 11.22.2. First we discuss the existence of the functorial isomorphism $j(K^\bullet[1]) \to j(K^\bullet)[1]$. Consider the diagram

$$\begin{tikzcd} K^\bullet[1] \arrow[r, equal] \arrow[d, "i_{K^\bullet[1]}"] & K^\bullet[1] \arrow[d, "i_{K^\bullet}[1]"] \\ j(K^\bullet[1]) \arrow[r, dotted, "\xi_{K^\bullet}"] & j(K^\bullet)[1] \end{tikzcd}$$

By Lemmas 11.17.6 and 11.17.7 there exists a unique dotted arrow $\xi_{K^\bullet}$ in $K^+(\mathcal{I})$ making the diagram commute in $K^+(\mathcal{A})$. We omit the verification that this gives a functorial isomorphism. (Hint: use Lemma 11.17.7 again.)

Let $(K^\bullet, L^\bullet, M^\bullet, f, g, h)$ be a distinguished triangle of $K^+(\mathcal{A})$. We have to show that $(j(K^\bullet), j(L^\bullet), j(M^\bullet), j(f), j(g), \xi_{K^\bullet} \circ j(h))$ is a distinguished triangle of $K^+(\mathcal{I})$. Note that we have a commutative diagram

$$\begin{tikzcd} K^\bullet \arrow[r, "f"'] \arrow[d] & L^\bullet \arrow[r, "g"'] \arrow[d] & M^\bullet \arrow[r, "h"'] \arrow[d] & K^\bullet[1] \arrow[d] \\ j(K^\bullet) \arrow[r, "j(f)"] & j(L^\bullet) \arrow[r, "j(g)"] & j(M^\bullet) \arrow[r, "\xi_{K^\bullet} \circ j(h)"] & j(K^\bullet)[1] \end{tikzcd}$$

in $K^+(\mathcal{A})$ whose vertical arrows are the quasi-isomorphisms i_K, i_L, i_M. Hence we see that the image of $(j(K^\bullet), j(L^\bullet), j(M^\bullet), j(f), j(g), \xi_{K^\bullet} \circ j(h))$ in $D^+(\mathcal{A})$ is isomorphic to a distinguished triangle and hence a distinguished triangle by TR1. Thus we see from Lemma 11.4.16 that $(j(K^\bullet), j(L^\bullet), j(M^\bullet), j(f), j(g), \xi_{K^\bullet} \circ j(h))$ is a distinguished triangle in $K^+(\mathcal{I})$. □

Lemma 11.22.6. *Let $\mathcal{A}$ be an abelian category which has enough injectives. Let j be a resolution functor. Write $Q : K^+(\mathcal{A}) \to D^+(\mathcal{A})$ for the natural functor. Then $j = j' \circ Q$ for a unique functor $j' : D^+(\mathcal{A}) \to K^+(\mathcal{I})$ which is quasi-inverse to the canonical functor $K^+(\mathcal{I}) \to D^+(\mathcal{A})$.*

Proof. By Lemma 11.10.5 Q is a localization functor. To prove the existence of j' it suffices to show that any element of $\text{Qis}^+(\mathcal{A})$ is mapped to an isomorphism under the functor j, see Lemma 11.5.6. This is true by the remarks following Definition 11.22.2. □

Remark 11.22.7. Suppose that $\mathcal{A}$ is a ``big'' abelian category with enough injectives such as the category of abelian groups. In this case we have to be slightly more careful in constructing our resolution functor since we cannot use the axiom of choice with a quantifier ranging over a class. But note that the proof of the lemma does show that any two localization functors are canonically isomorphic. Namely, given quasi-isomorphisms $i : K^\bullet \to I^\bullet$ and $i' : K^\bullet \to J^\bullet$ of a bounded below complex $K^\bullet$ into bounded below complexes of injectives there exists a unique(!) morphism $a : I^\bullet \to J^\bullet$ in $K^+(\mathcal{I})$ such that $i' = i \circ a$ as morphisms in $K^+(\mathcal{I})$. Hence the only issue is existence, and we will see how to deal with this in the next section.

11.23. Functorial injective embeddings and resolution functors

In this section we redo the construction of a resolution functor $K^+(\mathcal{A}) \to K^+(\mathcal{I})$ in case the category $\mathcal{A}$ has functorial injective embeddings. There are two reasons for this: (1) the proof is easier and (2) the construction also works if $\mathcal{A}$ is a ``big" abelian category. See Remark 11.23.3 below.

Let $\mathcal{A}$ be an abelian category. As before denote $\mathcal{I}$ the additive full subcategory of $\mathcal{A}$ consisting of injective objects. Consider the category $\text{InjRes}(\mathcal{A})$ of arrows $\alpha : K^\bullet \to I^\bullet$ where $K^\bullet$ is a bounded below complex of $\mathcal{A}$, $I^\bullet$ is a bounded below complex of injectives of $\mathcal{A}$ and α is a quasi-isomorphism. In other words, α is an injective resolution and $K^\bullet$ is bounded below. There is an obvious functor

$$s : \text{InjRes}(\mathcal{A}) \longrightarrow \text{Comp}^+(\mathcal{A})$$

defined by $(\alpha : K^\bullet \to I^\bullet) \mapsto K^\bullet$. There is also a functor

$$t : \text{InjRes}(\mathcal{A}) \longrightarrow K^+(\mathcal{I})$$

defined by $(\alpha : K^\bullet \to I^\bullet) \mapsto I^\bullet$.

Lemma 11.23.1. *Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has functorial injective embeddings, see Homology, Definition 10.20.5.*

1. *There exists a functor $inj : Comp^+(\mathcal{A}) \to InjRes(\mathcal{A})$ such that $s \circ inj = id$.*
2. *For any functor $inj : Comp^+(\mathcal{A}) \to InjRes(\mathcal{A})$ such that $s \circ inj = id$ we obtain a resolution functor, see Definition 11.22.2.*

Proof. Let $A \mapsto (A \to J(A))$ be a functorial injective embedding, see Homology, Definition 10.20.5. We first note that we may assume $J(0) = 0$. Namely, if not then for any object A we have $0 \to A \to 0$ which gives a direct sum decomposition $J(A) = J(0) \oplus \text{Ker}(J(A) \to J(0))$. Note that the functorial morphism $A \to J(A)$ has to map into the second summand. Hence we can replace our functor by $J'(A) = \text{Ker}(J(A) \to J(0))$ if needed.

Let $K^\bullet$ be a bounded below complex of $\mathcal{A}$. Say $K^p = 0$ if $p < B$. We are going to construct a double complex $I^{\bullet,\bullet}$ of injectives, together with a map $\alpha : K^\bullet \to I^{\bullet,0}$ such that α induces a quasi-isomorphism of $K^\bullet$ with the associated total complex of $I^{\bullet,\bullet}$. First we set $I^{p,q} = 0$ whenever $q < 0$. Next, we set $I^{p,0} = J(K^p)$ and $\alpha^p : K^p \to I^{p,0}$ the functorial embedding. Since J is a functor we see that $I^{\bullet,0}$ is a complex and that α is a morphism of complexes. Each α^p is injective. And $I^{p,0} = 0$ for $p < B$ because $J(0) = 0$. Next, we set $I^{p,1} = J(\text{Coker}(K^p \to I^{p,0}))$. Again by functoriality we see that $I^{\bullet,1}$ is a complex. And again we get that $I^{p,1} = 0$ for $p < B$. It is also clear that K^p maps isomorphically onto $\text{Ker}(I^{p,0} \to I^{p,1})$. As our third step we take $I^{p,2} = J(\text{Coker}(I^{p,0} \to I^{p,1}))$. And so on and so forth.

At this point we can apply Homology, Lemma 10.19.6 to get that the map

$$\alpha : K^\bullet \to sI^\bullet$$

is a quasi-isomorphism. To prove we get a functor inj it rests to show that the construction above is functorial. This verification is omitted.

Suppose we have a functor inj such that $s \circ inj = \text{id}$. For every object $K^\bullet$ of $\text{Comp}^+(\mathcal{A})$ we can write

$$inj(K^\bullet) = (i_{K^\bullet} : K^\bullet \to j(K^\bullet))$$

This provides us with a resolution functor as in Definition 11.22.2. □

Remark 11.23.2. Suppose inj is a functor such that $s \circ inj = \mathrm{id}$ as in part (2) of Lemma 11.23.1. Write $inj(K^\bullet) = (i_{K^\bullet} : K^\bullet \to j(K^\bullet))$ as in the proof of that lemma. Suppose $\alpha : K^\bullet \to L^\bullet$ is a map of bounded below complexes. Consider the map $inj(\alpha)$ in the category InjRes($\mathcal{A}$). It induces a commutative diagram

$$\begin{tikzcd} K^\bullet \arrow[r, "\alpha"] \arrow[d, "i_K"] & L^\bullet \arrow[d, "i_L"] \\ j(K)^\bullet \arrow[r, "inj(\alpha)"] & j(L)^\bullet \end{tikzcd}$$

of morphisms of complexes. Hence, looking at the proof of Lemma 11.22.3 we see that the functor $j : K^+(\mathcal{A}) \to K^+(\mathcal{I})$ is given by the rule

$$j(\alpha \text{ up to homotopy}) = inj(\alpha) \text{ up to homotpy} \in Hom_{K^+(\mathcal{I})}(j(K^\bullet), j(L^\bullet))$$

Hence we see that j matches $t \circ inj$ in this case, i.e., the diagram

$$\begin{tikzcd} \mathrm{Comp}^+(\mathcal{A}) \arrow[rr, "t\circ inj"] \arrow[dr] & & K^+(\mathcal{I}) \\ & K^+(\mathcal{A}) \arrow[ur, "j"'] & \end{tikzcd}$$

is commutative.

Remark 11.23.3. Let $Mod(\mathcal{O}_X)$ be the category of $\mathcal{O}_X$-modules on a ringed space $(X, \mathcal{O}_X)$ (or more generally on a ringed site). We will see later that $Mod(\mathcal{O}_X)$ has enough injectives and in fact functorial injective embeddings, see Injectives, Theorem 17.12.4. Note that the proof of Lemma 11.22.4 does not apply to $Mod(\mathcal{O}_X)$. But the proof of Lemma 11.23.1 does apply to $Mod(\mathcal{O}_X)$. Thus we obtain

$$j : K^+(Mod(\mathcal{O}_X)) \longrightarrow K^+(\mathcal{I})$$

which is a resolution functor where $\mathcal{I}$ is the additive category of injective $\mathcal{O}_X$-modules. This argument also works in the following cases:

(1) The category Mod_R of R-modules over a ring R.
(2) The category $PMod(\mathcal{O})$ of presheaves of $\mathcal{O}$-modules on a site endowed with a presheaf of rings.
(3) The category $Mod(\mathcal{O})$ of sheaves of $\mathcal{O}$-modules on a ringed site.
(4) Add more here as needed.

11.24. Right derived functors via resolution functors

The content of the following lemma is that we can simply define $RF(K^\bullet) = F(j(K^\bullet)$ if we are given a resolution functor j.

Lemma 11.24.1. *Let $\mathcal{A}$ be an abelian category with enough injectives Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor into an abelian category. Let (i, j) be a resolution functor, see Definition 11.22.2. The right derived functor RF of F fits into the following 2-commutative diagram*

$$\begin{tikzcd} D^+(\mathcal{A}) \arrow[rr, "j'"] \arrow[dr, "RF"'] & & K^+(\mathcal{I}) \arrow[dl, "F"] \\ & D^+(\mathcal{B}) & \end{tikzcd}$$

where j' is the functor from Lemma 11.22.6.

Proof. By Lemma 11.19.1 we have $RF(K^\bullet) = F(j(K^\bullet))$. $\square$

Remark 11.24.2. In the situation of Lemma 11.24.1 we see that we have actually lifted the right derived functor to an exact functor $F \circ j' : D^+(\mathcal{A}) \to K^+(\mathcal{B})$. It is occasionally useful to use such a factorization.

11.25. Filtered derived category and injective resolutions

If the underlying abelian category $\mathcal{A}$ has enough injectives then the category $\text{Fil}^f(\mathcal{A})$ has enough right acyclic objects relative to any left exact functor.

Definition 11.25.1. Let $\mathcal{A}$ be an abelian category. We say an object I of $\text{Fil}^f(\mathcal{A})$ is *filtered injective* if each $\text{gr}^p(I)$ is an injective object of $\mathcal{A}$.

This category is an example of an exact category, see Injectives, Remark 17.13.6. A special role is played by the strict morphisms, see Homology, Definition 10.13.3, i.e., the morphisms f such that $\text{Coim}(f) = \text{Im}(f)$. We will say that a complex $A \to B \to C$ in $\text{Fil}^f(\mathcal{A})$ is *exact* if the sequence $\text{gr}(A) \to \text{gr}(B) \to \text{gr}(C)$ is exact in $\mathcal{A}$. This implies that $A \to B$ and $B \to C$ are strict morphisms, see Homology, Lemma 10.13.17.

Lemma 11.25.2. *Let $\mathcal{A}$ be an abelian category. An object I of $\text{Fil}^f(\mathcal{A})$ is filtered injective if and only if there exist $a \leq b$, injective objects I_n, $a \leq n \leq b$ of $\mathcal{A}$ and an isomorphism $I \cong \bigoplus_{a \leq n \leq b} I_n$ such that $F^p I = \bigoplus_{n \geq p} I_n$.*

Proof. Follows from the fact that any injection $J \to M$ of $\mathcal{A}$ is split if J is an injective object. Details omitted. $\square$

Lemma 11.25.3. *Let $\mathcal{A}$ be an abelian category. Any strict monomorphism $u : I \to A$ of $\text{Fil}^f(\mathcal{A})$ where I is a filtered injective object is a split injection.*

Proof. Let p be the largest integer such that $F^p I \neq 0$. In particular $\text{gr}^p(I) = F^p I$. Let I' be the object of $\text{Fil}^f(\mathcal{A})$ whose underlying object of $\mathcal{A}$ is $F^p I$ and with filtration given by $F^n I' = 0$ for $n > p$ and $F^n I' = I' = F^p I$ for $n \leq p$. Note that $I' \to I$ is a strict monomorphism too. The fact that u is a strict monomorphism implies that $F^p I \to A/F^{p+1}(A)$ is injective, see Homology, Lemma 10.13.15. Choose a splitting $s : A/F^{p+1}A \to F^p I$ in $\mathcal{A}$. The induced morphism $s' : A \to I'$ is a strict morphism of filtered objects splitting the composition $I' \to I \to A$. Hence we can write $A = I' \oplus \text{Ker}(s')$ and $I = I' \oplus \text{Ker}(s'|_I)$. Note that $\ker(s'|_I) \to \ker(s')$ is a strict monomorphism and that $\ker(s'|_I)$ is a filtered injective object. By induction on the length of the filtration on I the map $\ker(s'|_I) \to \ker(s')$ is a split injection. Thus we win. $\square$

Lemma 11.25.4. *Let $\mathcal{A}$ be an abelian category. Let $u : A \to B$ be a strict monomorphism of $\text{Fil}^f(\mathcal{A})$ and $f : A \to I$ a morphism from A into a filtered injective object in $\text{Fil}^f(\mathcal{A})$. Then there exists a morphism $g : B \to I$ such that $f = g \circ u$.*

Proof. The pushout $f' : I \to I \amalg_A B$ of f by u is a strict monomorphism, see Homology, Lemma 10.13.10. Hence the result follows formally from Lemma 11.25.3. $\square$

Lemma 11.25.5. *Let $\mathcal{A}$ be an abelian category with enough injectives. For any object A of $\text{Fil}^f(\mathcal{A})$ there exists a strict monomorphism $A \to I$ where I is a filtered injective object.*

Proof. Pick $a \leq b$ such that $\text{gr}^p(A) = 0$ unless $p \in \{a, a+1, \ldots, b\}$. For each $n \in \{a, a+1, \ldots, b\}$ choose an injection $u_n : A/F^nA \to I_n$ with I_n and injective object. Set $I = \bigoplus_{a \leq n \leq b} I_p$ with filtration $F^pI = \bigoplus_{n \geq p} I_n$ and set $u : A \to I$ equal to the direct sum of the maps u_n. $\square$

Lemma 11.25.6. *Let $\mathcal{A}$ be an abelian category with enough injectives. For any object A of $\text{Fil}^f(\mathcal{A})$ there exists a filtered quasi-isomorphism $A[0] \to I^\bullet$ where $I^\bullet$ is a complex of filtered injective objects with $I^n = 0$ for $n < 0$.*

Proof. First choose a strict monomorphism $u_0 : A \to I^0$ of A into a filtered injective object, see Lemma 11.25.5. Next, choose a strict monomorphism $u_1 : \text{Coker}(u_0) \to I^1$ into a filtered injective object of $\mathcal{A}$. Denote d^0 the induced map $I^0 \to I^1$. Next, choose a strict monomorphism $u_2 : \text{Coker}(u_1) \to I^2$ into a filtered injective object of $\mathcal{A}$. Denote d^1 the induced map $I^1 \to I^2$. And so on. This works because each of the sequences

$$0 \to \text{Coker}(u_n) \to I^{n+1} \to \text{Coker}(u_{n+1}) \to 0$$

is short exact, i.e., induces a short exact sequence on applying gr. To see this use Homology, Lemma 10.13.15. $\square$

Lemma 11.25.7. *Let $\mathcal{A}$ be an abelian category with enough injectives. Let $f : A \to B$ be a morphism of $\text{Fil}^f(\mathcal{A})$. Given filtered quasi-isomorphisms $A[0] \to I^\bullet$ and $B[0] \to J^\bullet$ where $I^\bullet, J^\bullet$ are complexes of filtered injective objects with $I^n = J^n = 0$ for $n < 0$, then there exists a commutative diagram*

$$\begin{tikzcd}
A[0] \arrow[r] \arrow[d] & B[0] \arrow[d] \\
I^\bullet \arrow[r] & J^\bullet
\end{tikzcd}$$

Proof. As $A[0] \to I^\bullet$ and $C[0] \to J^\bullet$ are filtered quasi-isomorphisms we conclude that $a : A \to I^0$, $b : B \to J^0$ and all the morphisms d_I^n, d_J^n are strict, see Homology, Lemma 11.13.4. We will inductively construct the maps f^n in the following commutative diagram

$$\begin{tikzcd}
A \arrow[r, "a"'] \arrow[d, "f"'] & I^0 \arrow[r] \arrow[d, "f^0"] & I^1 \arrow[r] \arrow[d, "f^1"] & I^2 \arrow[r] \arrow[d, "f^2"] & \ldots \\
B \arrow[r, "b"] & J^0 \arrow[r] & J^1 \arrow[r] & J^2 \arrow[r] & \ldots
\end{tikzcd}$$

Because $A \to I^0$ is a strict monomorphism and because J^0 is filtered injective, we can find a morphism $f^0 : I^0 \to J^0$ such that $f^0 \circ a = b \circ f$, see Lemma 11.25.4. The composition $d_J^0 \circ b \circ f$ is zero, hence $d_J^0 \circ f^0 \circ a = 0$, hence $d_J^0 \circ f^0$ factors through a unique morphism

$$\text{Coker}(a) = \text{Coim}(d_I^0) = \text{Im}(d_I^0) \longrightarrow J^1.$$

As $\text{Im}(d_I^0) \to I^1$ is a strict monomorphism we can extend the displayed arrow to a morphism $f^1 : I^1 \to J^1$ by Lemma 11.25.4 again. And so on. $\square$

Lemma 11.25.8. *Let $\mathcal{A}$ be an abelian category with enough injectives. Let $0 \to A \to B \to C \to 0$ be a short exact sequence in $\text{Fil}^f(\mathcal{A})$. Given filtered quasi-isomorphisms $A[0] \to I^\bullet$ and $C[0] \to J^\bullet$ where $I^\bullet, J^\bullet$ are complexes of filtered injective objects with $I^n = J^n = 0$*

for $n < 0$, then there exists a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A[0] \arrow[r] \arrow[d] & B[0] \arrow[r] \arrow[d] & C[0] \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & I^\bullet \arrow[r] & M^\bullet \arrow[r] & J^\bullet \arrow[r] & 0 \end{tikzcd}$$

where the lower row is a termwise split sequence of complexes.

Proof. As $A[0] \to I^\bullet$ and $C[0] \to J^\bullet$ are filtered quasi-isomorphisms we conclude that $a : A \to I^0$, $c : C \to J^0$ and all the morphisms d_I^n, d_J^n are strict, see Homology, Lemma 11.13.4. We are going to step by step construct the south-east and the south arrows in the following commutative diagram

$$\begin{tikzcd} B \arrow[r, "\beta"] \arrow[dr, "b"] & C \arrow[r, "c"] \arrow[dr, "\overline{b}"] & J^0 \arrow[r] \arrow[d, "\delta^0"] & J^1 \arrow[r] \arrow[d, "\delta^1"] & \ldots \\ A \arrow[u, "\alpha"] \arrow[r, "a"] & I^0 \arrow[r] & I^1 \arrow[r] & I^2 \arrow[r] & \ldots \end{tikzcd}$$

As $A \to B$ is a strict monomorphism, we can find a morphism $b : B \to I^0$ such that $b \circ \alpha = a$, see Lemma 11.25.4. As A is the kernel of the strict morphism $I^0 \to I^1$ and $\beta = \text{Coker}(\alpha)$ we obtain a unique morphism $\overline{b} : C \to I^1$ fitting into the diagram. As c is a strict monomorphism and I^1 is filtered injective we can find $\delta^0 : J^0 \to I^1$, see Lemma 11.25.4. Because $B \to C$ is a strict epimorphism and because $B \to I^0 \to I^1 \to I^2$ is zero, we see that $C \to I^1 \to I^2$ is zero. Hence $d_I^1 \circ \delta^0$ is zero on $C \cong \text{Im}(c)$. Hence $d_I^1 \circ \delta^0$ factors through a unique morphism

$$\text{Coker}(c) = \text{Coim}(d_J^0) = \text{Im}(d_J^0) \longrightarrow I^2.$$

As I^2 is filtered injective and $\text{Im}(d_J^0) \to J^1$ is a strict monomorphism we can extend the displayed morphism to a morphism $\delta^1 : J^1 \to I^2$, see Lemma 11.25.4. And so on. We set $M^\bullet = I^\bullet \oplus J^\bullet$ with differential

$$d_M^n = \begin{pmatrix} d_I^n & (-1)^{n+1}\delta^n \\ 0 & d_J^n \end{pmatrix}$$

Finally, the map $B[0] \to M^\bullet$ is given by $b \oplus c \circ \beta : M \to I^0 \oplus J^0$. □

Lemma 11.25.9. *Let $\mathcal{A}$ be an abelian category with enough injectives. For every $K^\bullet \in K^+(Fil^f(\mathcal{A}))$ there exists a filtered quasi-isomorphism $K^\bullet \to I^\bullet$ with $I^\bullet$ bounded below, each I^n a filtered injective object, and each $K^n \to I^n$ a strict monomorphism.*

Proof. After replacing $K^\bullet$ by a shift (which is harmless for the proof) we may assume that $K^n = 0$ for $n < 0$. Consider the short exact sequences

$$\begin{matrix} 0 \to \ker(d_K^0) \to K^0 \to \text{Coim}(d_K^0) \to 0 \\ 0 \to \ker(d_K^1) \to K^1 \to \text{Coim}(d_K^1) \to 0 \\ 0 \to \ker(d_K^2) \to K^2 \to \text{Coim}(d_K^2) \to 0 \\ \ldots \end{matrix}$$

of the exact category $\text{Fil}^f(\mathcal{A})$ and the maps $u_i : \text{Coim}(d_K^i) \to \text{Ker}(d_K^{i+1})$. For each $i \geq 0$ we may choose filtered quasi-isomorphisms

$$\begin{matrix} \ker(d_K^i)[0] \to I_{ker,i}^\bullet \\ \text{Coim}(d_K^i)[0] \to I_{coim,i}^\bullet \end{matrix}$$

with $I^n_{ker,i}, I^n_{coim,i}$ filtered injective and zero for $n < 0$, see Lemma 11.25.6. By Lemma 11.25.7 we may lift u_i to a morphism of complexes $u_i^\bullet : I^\bullet_{coim,i} \to I^\bullet_{ker,i+1}$. Finally, for each $i \geq 0$ we may complete the diagrams

$$\begin{tikzcd}
0 \arrow[r] & \ker(d^i_K)[0] \arrow[r] \arrow[d] & K^i[0] \arrow[r] \arrow[d] & \mathrm{Coim}(d^i_K)[0] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^\bullet_{ker,i} \arrow[r, "\alpha_i"] & I^\bullet_i \arrow[r, "\beta_i"] & I^\bullet_{coim,i} \arrow[r] & 0
\end{tikzcd}$$

with the lower sequence a termwise split exact sequence, see Lemma 11.25.8. For $i \geq 0$ set $d_i : I^\bullet_i \to I^\bullet_{i+1}$ equal to $d_i = \alpha_{i+1} \circ u_i^\bullet \circ \beta_i$. Note that $d_i \circ d_{i-1} = 0$ because $\beta_i \circ \alpha_i = 0$. Hence we have constructed a commutative diagram

$$\begin{tikzcd}
I^\bullet_0 \arrow[r] & I^\bullet_1 \arrow[r] & I^\bullet_2 \arrow[r] & \ldots \\
K^0[0] \arrow[r] \arrow[u] & K^1[0] \arrow[r] \arrow[u] & K^2[0] \arrow[r] \arrow[u] & \ldots
\end{tikzcd}$$

Here the vertical arrows are filtered quasi-isomorphisms. The upper row is a complex of complexes and each complex consists of filtered injective objects with no nonzero objects in degree < 0. Thus we obtain a double complex by setting $I^{a,b} = I^b_a$ and using

$$d_1^{a,b} : I^{a,b} = I^b_a \to I^b_{a+1} = I^{a+1,b}$$

the map d^b_a and using for

$$d_2^{a,b} : I^{a,b} = I^b_a \to I^{b+1}_a = I^{a,b+1}$$

the map $d^b_{I_a}$. Denote $\mathrm{Tot}(I^{\bullet,\bullet})$ the total complex associated to this double complex, see Homology, Definition 10.19.2. Observe that the maps $K^n[0] \to I^\bullet_n$ come from maps $K^n \to I^{n,0}$ which give rise to a map of complexes

$$K^\bullet \longrightarrow \mathrm{Tot}(I^{\bullet,\bullet})$$

We claim this is a filtered quasi-isomorphism. As $\mathrm{gr}(-)$ is an additive functor, we see that $\mathrm{gr}(\mathrm{Tot}(I^{\bullet,\bullet})) = \mathrm{Tot}(\mathrm{gr}(I^{\bullet,\bullet}))$. Thus we can use Homology, Lemma 10.19.6 to conclude that $\mathrm{gr}(K^\bullet) \to \mathrm{gr}(\mathrm{Tot}(I^{\bullet,\bullet}))$ is a quasi-isomorphism as desired. ☐

Lemma 11.25.10. *Let $\mathcal{A}$ be an abelian category. Let $K^\bullet, I^\bullet \in K(Fil^f(\mathcal{A}))$. Assume $K^\bullet$ is filtered acyclic and $I^\bullet$ bounded below and consisting of filtered injective objects. Any morphism $K^\bullet \to I^\bullet$ is homotopic to zero: $Hom_{K(Fil^f(\mathcal{A}))}(K^\bullet, I^\bullet) = 0$.*

Proof. Let $\alpha : K^\bullet \to I^\bullet$ be a morphism of complexes. Assume that $\alpha^j = 0$ for $j < n$. We will show that there exists a morphism $h : K^{n+1} \to I^n$ such that $\alpha^n = h \circ d$. Thus α will be homotopic to the morphism of complexes β defined by

$$\beta^j = \begin{cases} 0 & \text{if} \quad j \leq n \\ \alpha^{n+1} - d \circ h & \text{if} \quad j = n+1 \\ \alpha^j & \text{if} \quad j > n+1 \end{cases}$$

This will clearly prove the lemma (by induction). To prove the existence of h note that $\alpha^n \circ d_K^{n-1} = 0$ since $\alpha^{n-1} = 0$. Since $K^\bullet$ is filtered acyclic we see that d_K^{n-1} and d_K^n are strict and that

$$0 \to \mathrm{Im}(d_K^{n-1}) \to K^n \to \mathrm{Im}(d_K^n) \to 0$$

is an exact sequence of the exact category $\text{Fil}^f(\mathcal{A})$, see Homology, Lemma 10.13.17. Hence we can think of α^n as a map into I^n defined on $\text{Im}(d_K^n)$. Using that $\text{Im}(d_K^n) \to K^{n+1}$ is a strict monomorphism and that I^n is filtered injective we may lift this map to a map $h : K^{n+1} \to I^n$ as desired, see Lemma 11.25.4. $\square$

Lemma 11.25.11. *Let $\mathcal{A}$ be an abelian category. Let $I^\bullet \in K(Fil^f(\mathcal{A}))$ be a bounded below complex consisting of filtered injective objects.*

(1) *Let $\alpha : K^\bullet \to L^\bullet$ in $K(Fil^f(\mathcal{A}))$ be a filtered quasi-isomorphism. Then the map*
$$Hom_{K(Fil^f(\mathcal{A}))}(L^\bullet, I^\bullet) \to Hom_{K(Fil^f(\mathcal{A}))}(K^\bullet, I^\bullet)$$
is bijective.

(2) *Let $L^\bullet \in K(\mathcal{A})$. Then*
$$Hom_{K(Fil^f(\mathcal{A}))}(L^\bullet, I^\bullet) = Hom_{DF(\mathcal{A})}(L^\bullet, I^\bullet).$$

Proof. Proof of (1). Note that
$$(K^\bullet, L^\bullet, C(\alpha)^\bullet, \alpha, i, -p)$$
is a distinguished triangle in $K(\text{Fil}^f(\mathcal{A}))$ (Lemma 11.8.12) and $C(f)^\bullet$ is a filtered acyclic complex (Lemma 11.13.4). Then

$$\begin{tikzcd} Hom_{K(\text{Fil}^f(\mathcal{A}))}(C(\alpha)^\bullet, I^\bullet) \arrow[r] & Hom_{K(\text{Fil}^f(\mathcal{A}))}(L^\bullet, I^\bullet) \arrow[r] & Hom_{K(\text{Fil}^f(\mathcal{A}))}(K^\bullet, I^\bullet) \arrow[dll] \\ Hom_{K(\text{Fil}^f(\mathcal{A}))}(C(\alpha)^\bullet[-1], I^\bullet) & & \end{tikzcd}$$

is an exact sequence of abelian groups, see Lemma 11.4.2. At this point Lemma 11.25.10 guarantees that the outer two groups are zero and hence $Hom_{K(\mathcal{A})}(L^\bullet, I^\bullet) = Hom_{K(\mathcal{A})}(K^\bullet, I^\bullet)$.

Proof of (2). Let a be an element of the right hand side. We may represent $a = \gamma\alpha^{-1}$ where $\alpha : K^\bullet \to L^\bullet$ is a filtered quasi-isomorphism and $\gamma : K^\bullet \to I^\bullet$ is a map of complexes. By part (1) we can find a morphism $\beta : L^\bullet \to I^\bullet$ such that $\beta \circ \alpha$ is homotopic to γ. This proves that the map is surjective. Let b be an element of the left hand side which maps to zero in the right hand side. Then b is the homotopy class of a morphism $\beta : L^\bullet \to I^\bullet$ such that there exists a filtered quasi-iomorphism $\alpha : K^\bullet \to L^\bullet$ with $\beta \circ \alpha$ homotopic to zero. Then part (1) shows that β is homotopic to zero also, i.e., $b = 0$. $\square$

Lemma 11.25.12. *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{I}^f \subset Fil^f(\mathcal{A})$ denote the strictly full additive subcategory whose objects are the filtered injective objects. The canonical functor*
$$K^+(\mathcal{I}^f) \longrightarrow DF^+(\mathcal{A})$$
is exact, fully faithful and essentially surjective, i.e., an equivalence of triangulated categories. Furthermore the diagrams

$$\begin{tikzcd} K^+(\mathcal{I}^f) \arrow[r] \arrow[d, "gr^p"'] & DF^+(\mathcal{A}) \arrow[d, "gr^p"'] \\ K^+(\mathcal{I}) \arrow[r] & D^+(\mathcal{A}) \end{tikzcd} \quad \begin{tikzcd} K^+(\mathcal{I}^f) \arrow[r] \arrow[d, "forget\ F"] & DF^+(\mathcal{A}) \arrow[d, "forget\ F"] \\ K^+(\mathcal{I}) \arrow[r] & D^+(\mathcal{A}) \end{tikzcd}$$

are commutative, where $\mathcal{I} \subset \mathcal{A}$ is the strictly full additive subcategory whose objects are the injective objects.

Proof. The functor $K^+(\mathcal{I}^f) \to DF^+(\mathcal{A})$ is essentially surjective by Lemma 11.25.9. It is fully faithful by Lemma 11.25.11. It is an exact functor by our definitions regarding distinguished triangles. The commutativity of the squares is immediate. $\square$

Remark 11.25.13. We can invert the arrow of the lemma only if $\mathcal{A}$ is a category in our sense, namely if it has a set of objects. However, suppose given a big abelian category $\mathcal{A}$ with enough injectives, such as $Mod(\mathcal{O}_X)$ for example. Then for any given set of objects $\{A_i\}_{i\in I}$ there is an abelian subcategory $\mathcal{A}' \subset \mathcal{A}$ containing all of them and having enough injectives, see Sets, Lemma 3.12.1. Thus we may use the lemma above for $\mathcal{A}'$. This essentially means that if we use a set worth of diagrams, etc then we will never run into trouble using the lemma.

Let $\mathcal{A}, \mathcal{B}$ be abelian categories. Let $T : \mathcal{A} \to \mathcal{B}$ be a left exact functor. (We cannot use the letter F for the functor since this would conflict too much with our use of the letter F to indicate filtrations.) Note that T induces an additive functor

$$T : \text{Fil}^f(\mathcal{A}) \to \text{Fil}^f(\mathcal{B})$$

by the rule $T(A, F) = (T(A), F)$ where $F^pT(A) = T(F^pA)$ which makes sense as T is left exact. (Warning: It may not be the case that $\text{gr}(T(A)) = T(\text{gr}(A))$.) This induces functors of triangulated categories

$$T : K^+(\text{Fil}^f(\mathcal{A})) \longrightarrow K^+(\text{Fil}^f(\mathcal{B})) \tag{11.25.13.1}$$

The filtered right derived functor of T are is the right derived functor of Definition 11.14.2 for this exact functor composed with the exact functor $K^+(\text{Fil}^f(\mathcal{B})) \to DF^+(\mathcal{B})$ and the multiplicative set $\text{FQis}^+(\mathcal{A})$. Assume $\mathcal{A}$ has enough injectives. At this point we can redo the discussion of Section 11.19 to define the *filtered right derived functors*

$$RT : DF^+(\mathcal{A}) \longrightarrow DF^+(\mathcal{B}) \tag{11.25.13.2}$$

of our functor T.

However, instead we will proceed as in Section 11.24, and it will turn out that we can define RT even if T is just additive. Namely, we first choose a quasi-inverse $j' : DF^+(\mathcal{A}) \to K^+(\mathcal{I}^f)$ of the equivalence of Lemma 11.25.12. By Lemma 11.4.16 we see that j' is an exact functor of triangulated categories. Next, we note that for a filtered injective object I we have a (noncanonical) decomposition

$$I \cong \bigoplus\nolimits_{p\in\mathbf{Z}} I_p, \quad \text{with} \quad F^pI = \bigoplus\nolimits_{q\geq p} I_q \tag{11.25.13.3}$$

by Lemma 11.25.2. Hence if T is any additive functor $T : \mathcal{A} \to \mathcal{B}$ then we get an additive functor

$$T_{ext} : \mathcal{I}^f \to \text{Fil}^f(\mathcal{B}) \tag{11.25.13.4}$$

by setting $T_{ext}(I) = \bigoplus T(I_p)$ with $F^pT_{ext}(I) = \bigoplus_{q\geq p} T(I_q)$. Note that we have the property $\text{gr}(T_{ext}(I)) = T(\text{gr}(I))$ by construction. Hence we obtain a functor

$$T_{ext} : K^+(\mathcal{I}^f) \to K^+(\text{Fil}^f(\mathcal{B})) \tag{11.25.13.5}$$

which commutes with gr. Then we define (11.25.13.2) by the composition

$$RT = T_{ext} \circ j'. \tag{11.25.13.6}$$

Since $RT : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is computed by injective resolutions as well, see Lemmas 11.19.1, the commutation of T with gr, and the commutative diagrams of Lemma 11.25.12 imply that

$$\mathrm{gr}^p \circ RT \cong RT \circ \mathrm{gr}^p \tag{11.25.13.7}$$

and

$$(\text{forget } F) \circ RT \cong RT \circ (\text{forget } F) \tag{11.25.13.8}$$

as functors $DF^+(\mathcal{A}) \to D^+(\mathcal{B})$.

The filtered derived functor RT (11.25.13.2) induces functors

$$\begin{aligned} RT &: \mathrm{Fil}^f(\mathcal{A}) \to DF^+(\mathcal{B}), \\ RT &: \mathrm{Comp}^+(\mathrm{Fil}^f(\mathcal{A})) \to DF^+(\mathcal{B}), \\ RT &: KF^+(\mathcal{A}) \to DF^+(\mathcal{B}). \end{aligned}$$

Note that since $\mathrm{Fil}^f(\mathcal{A})$, and $\mathrm{Comp}^+(\mathrm{Fil}^f(\mathcal{A}))$ are no longer abelian it does not make sense to say that RT restricts to a δ-functor on them. (This can be repaired by thinking of these categories as exact categories and formulating the notion of a δ-functor from an exact category into a triangulated category.) But it does make sense, and it is true by construction, that RT is an exact functor on the triangulated category $KF^+(\mathcal{A})$.

Lemma 11.25.14. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories. Let $T : \mathcal{A} \to \mathcal{B}$ be a left exact functor. Assume $\mathcal{A}$ has enough injectives. Let $(K^\bullet, F)$ be an object of $Comp^+(Fil^f(\mathcal{A}))$. There exists a spectral sequence $(E_r^{p,q}, d_r)_{r\geq 0}$ which is the spectral sequence associated to an object of $Comp^+(Fil^f(\mathcal{B}))$ with*

$$E_1^{p,q} = R^{p+q}T(gr^p(K^\bullet))$$

which converges to $R^{p+q}T(K^\bullet)$ inducing a finite filtration on each $R^nT(K^\bullet)$. Moreover the construction of this spectral sequence is functorial in the object $K^\bullet$ of $Comp^+(Fil^f(\mathcal{A}))$. In fact the terms (E_r, d_r) for $r \geq 2$ do not depend on any choices.

Proof. Choose a filtered quasi-isomorphism $K^\bullet \to I^\bullet$ with $I^\bullet$ a bounded below complex of filtered injective objects, see Lemma 11.25.9. Consider the complex $RT(K^\bullet) = T_{ext}(I^\bullet)$, see (11.25.13.6). Thus we can consider the spectral sequence $(E_r, d_r)_{r\geq 0}$ associated to this as a filtered complex in $\mathcal{B}$, see Homology, Section 10.18. By Homology, Lemma 10.18.2 we have $E_1^{p,q} = H^{p+q}(\mathrm{gr}^p(T(I^\bullet)))$. By Equation (11.25.13.3) we have $E_1^{p,q} = H^{p+q}(T(\mathrm{gr}^p(I^\bullet)))$, and by definition of a filtered injective resolution the map $\mathrm{gr}^p(K^\bullet) \to \mathrm{gr}^p(I^\bullet)$ is an injective resolution. Hence $E_1^{p,q} = R^{p+q}T(\mathrm{gr}^p(K^\bullet))$.

On the other hand, each I^n has a finite filtration and hence each $T(I^n)$ has a finite filtration. Thus we may apply Homology, Lemma 10.18.9 to conclude that the spectral sequence converges to $H^n(T(I^\bullet)) = R^nT(K^\bullet)$ moreover inducing finite filtrations on each of the terms.

Suppose that $K^\bullet \to L^\bullet$ is a morphism of $\mathrm{Comp}^+(\mathrm{Fil}^f(\mathcal{A}))$. Choose a filtered quasi-isomorphism $L^\bullet \to J^\bullet$ with $J^\bullet$ a bounded below complex of filtered injective objects, see Lemma 11.25.9. By our results above, for example Lemma 11.25.11, there exists a diagram

$$\begin{tikzcd} K^\bullet \arrow[r] \arrow[d] & L^\bullet \arrow[d] \\ I^\bullet \arrow[r] & J^\bullet \end{tikzcd}$$

which commutes up to homotopy. Hence we get a morphism of filtered complexes $T(I^\bullet) \to T(J^\bullet)$ which gives rise to the morphism of spectral sequences, see Homology, Lemma 10.18.4. The last statement follows from this. □

Remark 11.25.15. As promised in Remark 11.20.4 we discuss the connection of the lemma above with the constructions using Cartan-Eilenberg resolutions. Namely, assume the notations of Lemma 11.20.3. In particular $K^\bullet$ is a bounded below complex of $\mathcal{A}$ and $T : \mathcal{A} \to \mathcal{B}$ is a left exact functor. We give an alternative construction of the spectral sequences $'E$ and $''E$

First spectral sequence. Consider the ``stupid'' filtration on $K^\bullet$ obtained by setting $F^p(K^\bullet) = \sigma_{\geq p}(K^\bullet)$, see Homology, Section 10.11. Note that this stupid in the sense that $d(F^p(K^\bullet)) \subset F^{p+1}(K^\bullet)$, compare Homology, Lemma 10.18.3. Note that $\text{gr}^p(K^\bullet) = K^p[p]$ with this filtration. According to the above there is a spectral sequence with E_1 term $E_1^{p,q} = R^{p+q}T(K^p[p])$. Then the E_2 term is clearly $E_2^{p,q} = H^p(R^{p+q}T(K^\bullet))$ as in the spectral sequence $'E_r$.

Second spectral sequence. Consider the filtration on the complex $K^\bullet$ obtained by setting $F^p(K^\bullet) = \tau_{\leq -p}(K^\bullet)$, see Homology, Section 10.11. The minus sign is necessary to get a decreasing filtration. Note that $\text{gr}^p(K^\bullet)$ is quasi-isomorphic to $H^{-p}(K^\bullet)[-p]$ with this filtration. According to the above there is a spectral sequence with E_1 term

$$E_1^{p,q} = R^{p+q}T(H^{-p}(K^\bullet)[-p]) = R^{2p+q}T(H^{-p}(K^\bullet)) = {''E}_2^{i,j}$$

with $i = 2p + q$ and $j = -p$. (This looks unnatural, but note that we could just have well developped the whole theory of filtered complexes using increasing filtrations, with the end result that this then looks natural, but the other one doesn't.) We leave it to the reader to see that the differentials match up.

Actually, given a Cartan-Eilenberg resolution $K^\bullet \to I^{\bullet,\bullet}$ the induced morphism $K^\bullet \to sI^\bullet$ into the associated simple complex will be a filtered injective resolution for either filtration using suitable filtrations on $sI^\bullet$. This can be used to match up the spectral sequences exactly.

11.26. Ext groups

In this section we start describing the ext groups of objects of an abelian category. First we have the following very general definition.

Definition 11.26.1. Let $\mathcal{A}$ be an abelian category. Let $i \in \mathbf{Z}$. Let X, Y be objects of $D(\mathcal{A})$. The *ith extension group* of X by Y is the group

$$\text{Ext}^i_{\mathcal{A}}(X, Y) = Hom_{D(\mathcal{A})}(X, Y[i]) = Hom_{D(\mathcal{A})}(X[-i], Y).$$

If $A, B \in Ob(\mathcal{A})$ we set $\text{Ext}^i_{\mathcal{A}}(A, B) = \text{Ext}^i_{\mathcal{A}}(A[0], B[0])$.

Since $Hom_{D(\mathcal{A})}(X, -)$, resp. $Hom_{D(\mathcal{A})}(-, Y)$ is a homological, resp. cohomological functor, see Lemma 11.4.2, we see that a distinguished triangle (Y, Y', Y''), resp. (X, X', X'') leads to a long exact sequence

$$\ldots \to \text{Ext}^i_{\mathcal{A}}(X, Y) \to \text{Ext}^i_{\mathcal{A}}(X, Y') \to \text{Ext}^i_{\mathcal{A}}(X, Y'') \to \text{Ext}^{i+1}_{\mathcal{A}}(X, Y) \to \ldots$$

respectively

$$\ldots \to \text{Ext}^i_{\mathcal{A}}(X'', Y) \to \text{Ext}^i_{\mathcal{A}}(X', Y) \to \text{Ext}^i_{\mathcal{A}}(X, Y) \to \text{Ext}^{i+1}_{\mathcal{A}}(X'', Y) \to \ldots$$

Note that since $D^+(\mathcal{A})$, $D^-(\mathcal{A})$, $D^b(\mathcal{A})$ are full subcategories we may compute the ext groups by Hom groups in these categories provided X, Y are contained in them.

In case the category $\mathcal{A}$ has enough injectives or enough projectives we can compute the Ext groups using injective or projective resolutions. To avoid confusion, recall that having an injective (resp. projective) resolution implies vanishing of homology in all low (resp. high) degrees, see Lemmas 11.17.2 and 11.18.2.

Lemma 11.26.2. *Let $\mathcal{A}$ be an abelian category. Let $X^\bullet, Y^\bullet \in Ob(K(\mathcal{A}))$.*

(1) *Let $Y^\bullet \to I^\bullet$ be an injective resolution (Definition 11.17.1). Then*

$$Ext^i_{\mathcal{A}}(X^\bullet, Y^\bullet) = Hom_{K(\mathcal{A})}(X^\bullet, I^\bullet[i]).$$

(2) *Let $P^\bullet \to X^\bullet$ be a projective resolution (Definition 11.18.1). Then*

$$Ext^i_{\mathcal{A}}(X^\bullet, Y^\bullet) = Hom_{K(\mathcal{A})}(P^\bullet[-i], Y^\bullet).$$

Proof. Follows immediately from Lemma 11.17.8 and Lemma 11.18.8. □

In the rest of this section we discuss extensions of objects of the abelian category itself. First we observe the following.

Lemma 11.26.3. *Let $\mathcal{A}$ be an abelian category and let $A, B \in Ob(\mathcal{A})$. For $i < 0$ we have $Ext^i_{\mathcal{A}}(B, A) = 0$. We have $Ext^0_{\mathcal{A}}(B, A) = Hom_{\mathcal{A}}(B, A)$.*

Proof. Let $L^\bullet \to B[0]$ be any quasi-isomorphism. Then it is also true that $\tau_{\leq 0}L^\bullet \to B[0]$ is a quasi-isomorphism. Hence a morphism $B[0] \to A[i]$ in $D(\mathcal{A})$ can be represented as fs^{-1} where $s : L^\bullet \to B[0]$ is a quasi-isomorphism, $f : L^\bullet \to A[i]$ a morphism, and $L^n = 0$ for $n > 0$. Thus $f = 0$ if $i < 0$. If $i = 0$, then f corresponds exactly to a morphism $B = \text{Coker}(L^{-1} \to L^0) \to A$. □

Let $\mathcal{A}$ be an abelian category. Suppose that $0 \to A \to A' \to A'' \to 0$ is a short exact sequence of objects of $\mathcal{A}$. Then $0 \to A[0] \to A'[0] \to A''[0] \to 0$ leads to a distiguished triangle in $D(\mathcal{A})$ (see Lemma 11.11.1) hence a long exact sequence of Ext groups

$$0 \to \text{Ext}^0_{\mathcal{A}}(B, A) \to \text{Ext}^0_{\mathcal{A}}(B, A') \to \text{Ext}^0_{\mathcal{A}}(B, A'') \to \text{Ext}^1_{\mathcal{A}}(B, A) \to \ldots$$

Similarly, given a short exact sequence $0 \to B \to B' \to B'' \to 0$ we obtain a long exact sequence of Ext groups

$$0 \to \text{Ext}^0_{\mathcal{A}}(B'', A) \to \text{Ext}^0_{\mathcal{A}}(B', A) \to \text{Ext}^0_{\mathcal{A}}(B, A) \to \text{Ext}^1_{\mathcal{A}}(B'', A) \to \ldots$$

We may view these Ext groups as an application of the construction of the derived category. It shows one can define Ext groups and construct the long exact sequence of Ext groups without needing the existence of enough injectives or projectives. There is an alternative construction of the Ext groups due to Yoneda which avoids the use of the derived category, see [**Yon60**].

Definition 11.26.4. Let $\mathcal{A}$ be an abelian category. Let $A, B \in Ob(\mathcal{A})$. A degree i *Yoneda extension* of B by A is an exact sequence

$$E : 0 \to A \to Z_{i-1} \to Z_{i-2} \to \ldots \to Z_0 \to B \to 0$$

in $\mathcal{A}$. We say two Yoneda extensions E and E' of the same degree are *equivalent* if there exists a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] & Z_{i-1} \arrow[r] & \ldots \arrow[r] & Z_0 \arrow[r] & B \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[u] \arrow[d] & Z''_{i-1} \arrow[r] \arrow[u] \arrow[d] & \ldots \arrow[r] & Z''_0 \arrow[r] \arrow[u] \arrow[d] & B \arrow[r] \arrow[u] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & Z'_{i-1} \arrow[r] & \ldots \arrow[r] & Z'_0 \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

where the middle row is a Yoneda extension as well.

It is not immediately clear that the equivalence of the definition is an equivalence relation. Although it is instructive to prove this directly this will also follow from Lemma 11.26.5 below.

Let $\mathcal{A}$ be an abelian category with objects A, B. Given a Yoneda extension $E : 0 \to A \to Z_{i-1} \to Z_{i-2} \to \ldots \to Z_0 \to B \to 0$ we define an associated element $\delta(E) \in \mathrm{Ext}^i(B, A)$ as the morphism $\delta(E) = fs^{-1} : B[0] \to A[i]$ where s is the quasi-isomorphism

$$(\ldots \to 0 \to A \to Z_{i-1} \to \ldots \to Z_0 \to 0 \to \ldots) \longrightarrow B[0]$$

and f is the morphism of complexes

$$(\ldots \to 0 \to A \to Z_{i-1} \to \ldots \to Z_0 \to 0 \to \ldots) \longrightarrow A[i]$$

We call $\delta(E) = fs^{-1}$ the *class* of the Yoneda extension. It turns out that this class characterizes the equivalence class of the Yoneda extension.

Lemma 11.26.5. *Let $\mathcal{A}$ be an abelian category with objects A, B. Any element in $\mathrm{Ext}^i_{\mathcal{A}}(B, A)$ is $\delta(E)$ for some degree i Yoneda extension of B by A. Given two Yoneda extensions E, E' of the same degree then E is equivalent to E' if and only if $\delta(E) = \delta(E')$.*

Proof. Let $\xi : B[0] \to A[i]$ be an element of $\mathrm{Ext}^i_{\mathcal{A}}(B, A)$. We may write $\xi = fs^{-1}$ for some quasi-isomorphism $s : L^\bullet \to B[0]$ and map $f : L^\bullet \to A[i]$. After replacing $L^\bullet$ by $\tau_{\leq 0}L^\bullet$ we may assume that $L^i = 0$ for $i > 0$. Picture

$$\begin{tikzcd}
L^{-i-1} \arrow[r] & L^{-i} \arrow[r] \arrow[d] & \ldots \arrow[r] & L^0 \arrow[r] & B \arrow[r] & 0 \\
& A & & & &
\end{tikzcd}$$

Then setting $Z_{i-1} = (L^{-i+1} \oplus A)/L^{-i}$ and $Z_j = L^{-j}$ for $j = i-2, \ldots, 0$ we see that we obtain a degree i extension E of B by A whose class $\delta(E)$ equals ξ.

It is immediate from the definitions that equivalent Yoneda extensions have the same class. Suppose that $E : 0 \to A \to Z_{i-1} \to Z_{i-2} \to \ldots \to Z_0 \to B \to 0$ and $E' : 0 \to A \to Z'_{i-1} \to Z'_{i-2} \to \ldots \to Z'_0 \to B \to 0$ are Yoneda extensions with the same class. By construction of $D(\mathcal{A})$ as the localization of $K(\mathcal{A})$ at the set of quasi-isomorphisms, this means there exists a complex $L^\bullet$ and quasi-isomorphisms

$$t : L^\bullet \to (\ldots \to 0 \to A \to Z_{i-1} \to \ldots \to Z_0 \to 0 \to \ldots)$$

and

$$t' : L^\bullet \to (\ldots \to 0 \to A \to Z'_{i-1} \to \ldots \to Z'_0 \to 0 \to \ldots)$$

such that $s \circ t = s' \circ t'$ and $f \circ t = f' \circ t'$, see Categories, Section 4.24. Let E'' be the degree i extension of B by A constructed from the pair $L^\bullet \to B[0]$ and $L^\bullet \to A[i]$ in the first paragraph of the proof. Then the reader sees readily that there exists ``morphisms" of degree i Yoneda extensions $E'' \to E$ and $E'' \to E'$ as in the definition of equivalent Yoneda extensions (details omitted). This finishes the proof. $\square$

Lemma 11.26.6. *Let $\mathcal{A}$ be an abelian category. Let A, B be objects of $\mathcal{A}$. Then $Ext^1_{\mathcal{A}}(B, A)$ is the group $Ext_{\mathcal{A}}(B, A)$ constructed in Homology, Definition 10.4.2.*

Proof. This is the case $i = 1$ of Lemma 11.26.5. $\square$

11.27. Unbounded complexes

A reference for the material in this section is [**Spa88**]. The following lemma is useful to find ``good" left resolutions of unbounded complexes.

Lemma 11.27.1. *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{P} \subset Ob(\mathcal{A})$ be a subset. Assume that every object of $\mathcal{A}$ is a quotient of an element of $\mathcal{P}$. Let $K^\bullet$ be a complex. There exists a commutative diagram*

$$\begin{tikzcd}
P_1^\bullet \arrow[r] \arrow[d] & P_2^\bullet \arrow[r] \arrow[d] & \ldots \\
\tau_{\leq 1}K^\bullet \arrow[r] & \tau_{\leq 2}K^\bullet \arrow[r] & \ldots
\end{tikzcd}$$

in the category of complexes such that

(1) *the vertical arrows are quasi-isomorphisms,*
(2) *$P_1^\bullet$ is a bounded above complex with terms in $\mathcal{P}$,*
(3) *the arrows $P_n^\bullet \to P_{n+1}^\bullet$ are termwise split injections and each cokernel P^i_{n+1}/P^i_n is an element of $\mathcal{P}$.*

Proof. By Lemma 11.15.5 any bounded above complex has a resolution by a bounded above complex whose terms are in $\mathcal{P}$. Thus we obtain the first complex $P_1^\bullet$. By induction it suffices, given $P_1^\bullet, \ldots, P_n^\bullet$ to construct $P_{n+1}^\bullet$ and the maps $P_n^\bullet \to P_{n+1}^\bullet$ and $P_n^\bullet \to \tau_{\leq n+1}K^\bullet$. Consider the cone $C_1^\bullet$ of the composition $P_n^\bullet \to \tau_{\leq n}K^\bullet \to \tau_{\leq n+1}K^\bullet$. This fits into the distinguished triangle

$$P_n^\bullet \to \tau_{\leq n+1}K^\bullet \to C_1^\bullet \to P_n^\bullet[1]$$

Note that $C_1^\bullet$ is bounded above, hence we can choose a quasi-isomorphism $Q^\bullet \to C_1^\bullet$ where $Q^\bullet$ is a bounded above complex whose terms are elements of $\mathcal{P}$. Take the cone $C_2^\bullet$ of the map of complexes $Q^\bullet \to P_n^\bullet[1]$ to get the distinguished triangle

$$Q^\bullet \to P_n^\bullet[1] \to C_2^\bullet \to Q^\bullet[1]$$

By the axioms of triangulated categories we obtain a map of distinguished triangles

$$\begin{tikzcd}
P_n^\bullet \arrow[r] \arrow[d] & C_2^\bullet[-1] \arrow[r] \arrow[d] & Q^\bullet \arrow[r] \arrow[d] & P_n^\bullet[1] \arrow[d] \\
P_n^\bullet \arrow[r] & \tau_{\leq n+1}K^\bullet \arrow[r] & C_1^\bullet \arrow[r] & P_n^\bullet[1]
\end{tikzcd}$$

in the triangulated category $K(\mathcal{A})$. Set $P_{n+1}^\bullet = C_2^\bullet[-1]$. Note that (3) holds by construction. Choose an actual morphism of complexes $f : P_{n+1}^\bullet \to \tau_{\leq n+1}K^\bullet$. The left square of the diagram above commutes up to homotopy, but as $P_n^\bullet \to P_{n+1}^\bullet$ is a termwise split injection we can lift the homotopy and modify our choice of f to make it commute. Finally, f is a quasi-isomorphism, because both $P_n^\bullet \to P_n^\bullet$ and $Q^\bullet \to C_1^\bullet$ are. $\square$

In some cases we can use the lemma above to show that a left derived functor is everywhere defined.

Proposition 11.27.2. *Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor of abelian categories. Let $\mathcal{P} \subset Ob(\mathcal{A})$ be a subset. Assume*

(1) *every object of $\mathcal{A}$ is a quotient of an element of $\mathcal{P}$,*
(2) *for any bounded above acyclic complex $P^\bullet$ of $\mathcal{A}$ with $P^n \in \mathcal{P}$ for all n the complex $F(P^\bullet)$ is exact,*
(3) *$\mathcal{A}$ and $\mathcal{B}$ have colimits of systems over* $\mathbf{N}$*,*
(4) *colimits over* $\mathbf{N}$ *are exact in both $\mathcal{A}$ and $\mathcal{B}$, and*
(5) *F commutes with colimits over* $\mathbf{N}$.

Then LF is defined on all of $D(\mathcal{A})$.

Proof. By (1) and Lemma 11.15.5 for any bounded above complex $K^\bullet$ there exists a quasi-isomorphism $P^\bullet \to K^\bullet$ with $P^\bullet$ bounded above and $P^n \in \mathcal{P}$ for all n. Suppose that $s : P^\bullet \to (P')^\bullet$ is a quasi-isomorphism of bounded above complexes consisting of objects of $\mathcal{P}$. Then $F(P^\bullet) \to F((P')^\bullet)$ is a quasi-isomorphism because $F(C(s)^\bullet)$ is acyclic by assumption (2). This already shows that LF is defined on $D^-(\mathcal{A})$ and that a bounded above complex consisting of objects of $\mathcal{P}$ computes LF, see Lemma 11.14.15.

Next, let $K^\bullet$ be an arbitrary complex of $\mathcal{A}$. Choose a diagram

$$\begin{tikzcd} P_1^\bullet \arrow[r] \arrow[d] & P_2^\bullet \arrow[r] \arrow[d] & \ldots \\ \tau_{\leq 1}K^\bullet \arrow[r] & \tau_{\leq 2}K^\bullet \arrow[r] & \ldots \end{tikzcd}$$

as in Lemma 11.27.1. Note that the map $colim\, P_n^\bullet \to K^\bullet$ is a quasi-isomorphism because colimits over $\mathbf{N}$ in $\mathcal{A}$ are exact and $H^i(P_n^\bullet) = H^i(K^\bullet)$ for $n > i$. We claim that

$$F(colim\, P_n^\bullet) = colim\, F(P_n^\bullet)$$

(termwise colimits) is $LF(K^\bullet)$, i.e., that $colim\, P_n^\bullet$ computes LF. To see this, by Lemma 11.14.15, it suffices to prove the following claim. Suppose that

$$colim\, Q_n^\bullet = Q^\bullet \xrightarrow{\alpha} P^\bullet = colim\, P_n^\bullet$$

is a quasi-isomorphism of complexes, such that each $P_n^\bullet, Q_n^\bullet$ is a bounded above complex whose terms are in $\mathcal{P}$ and the maps $P_n^\bullet \to \tau_{\leq n}P^\bullet$ and $Q_n^\bullet \to \tau_{\leq n}Q^\bullet$ are quasi-isomorphisma. Claim: $F(\alpha)$ is a quasi-isomorphism.

The problem is that we do not assume that α is given as a colimit of maps between the complexes $P_n^\bullet$ and $Q_n^\bullet$. However, for each n we know that the solid arrows in the diagram

$$\begin{tikzcd} & R^\bullet \arrow[d, dotted] & \\ P_n^\bullet \arrow[d] & L^\bullet \arrow[l, dotted] \arrow[r, dotted] & Q_n^\bullet \arrow[d] \\ \tau_{\leq n}P^\bullet \arrow[rr, "\tau_{\leq n}\alpha"] & & \tau_{\leq n}Q^\bullet \end{tikzcd}$$

are quasi-isomorphisms. Because quasi-isomorphisms form a multiplicative system in $K(\mathcal{A})$ (see Lemma 11.10.2) we can find a quasi-isomorphism $L^\bullet \to P_n^\bullet$ and map of complexes $L^\bullet \to Q_n^\bullet$ such that the diagram above commutes up to homotopy. Then $\tau_{\leq n}L^\bullet \to L^\bullet$ is a quasi-isomorphism. Hence (by the first part of the proof) we can find a bounded above complex $R^\bullet$ whose terms are in $\mathcal{P}$ and a quasi-isomorphism $R^\bullet \to L^\bullet$ (as indicated in the diagram). Using the result of the first paragraph of the proof we see that $F(R^\bullet) \to F(P_n^\bullet)$ and $F(R^\bullet) \to F(Q_n^\bullet)$ are quasi-isomorphisms. Thus we obtain a isomorphisms $H^i(F(P_n^\bullet)) \to H^i(F(Q_n^\bullet))$ fitting into the commutative diagram

$$\begin{tikzcd} H^i(F(P_n^\bullet)) \arrow[r] \arrow[d] & H^i(F(Q_n^\bullet)) \arrow[d] \\ H^i(F(P^\bullet)) \arrow[r] & H^i(F(Q^\bullet)) \end{tikzcd}$$

The exact same argument shows that these maps are also compatible as n varies. Since by (4) and (5) we have

$$H^i(F(P^\bullet)) = H^i(F(colim\ P_n^\bullet)) = H^i(colim\ F(P_n^\bullet)) = colim\ H^i(F(P_n^\bullet))$$

and similarly for $Q^\bullet$ we conclude that $H^i(\alpha) : H^i(F(P^\bullet) \to H^i(F(Q^\bullet)$ is an isomorphism and the claim follows. $\square$

Lemma 11.27.3. *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{I} \subset Ob(\mathcal{A})$ be a subset. Assume that every object of $\mathcal{A}$ is a subobject of an element of $\mathcal{I}$. Let $K^\bullet$ be a complex. There exists a commutative diagram*

$$\begin{tikzcd} \ldots \arrow[r] & \tau_{\geq -2}K^\bullet \arrow[r] \arrow[d] & \tau_{\geq -1}K^\bullet \arrow[d] \\ \ldots \arrow[r] & I_2^\bullet \arrow[r] & I_1^\bullet \end{tikzcd}$$

in the category of complexes such that

(1) *the vertical arrows are quasi-isomorphisms,*
(2) *$I_1^\bullet$ is a bounded above complex with terms in $\mathcal{I}$,*
(3) *the arrows $I_{n+1}^\bullet \to I_n^\bullet$ are termwise split surjections and $Ker(I_{n+1}^i \to I_n^i)$ is an element of $\mathcal{I}$.*

Proof. This lemma is dual to Lemma 11.27.1. $\square$

11.28. K-injective complexes

The following types of complexes can be used to compute right derived functors on the unbounded derived category.

Definition 11.28.1. Let $\mathcal{A}$ be an abelian category. A complex $I^\bullet$ is *K-injective* if for every acyclic complex $M^\bullet$ we have $Hom_{K(\mathcal{A})}(M^\bullet, I^\bullet) = 0$.

In the situation of the definition we have in fact $Hom_{K(\mathcal{A})}(M^\bullet[i], I^\bullet) = 0$ for all i as the translate of an acyclic complex is acyclic.

Lemma 11.28.2. *Let $\mathcal{A}$ be an abelian category. Let $I^\bullet$ be a complex. The following are equivalent*

(1) *$I^\bullet$ is K-injective,*

(2) *for every quasi-isomorphism* $M^\bullet \to N^\bullet$ *the map*

$$Hom_{K(\mathcal{A})}(N^\bullet, I^\bullet) \to Hom_{K(\mathcal{A})}(M^\bullet, I^\bullet)$$

is bijective, and

(3) *for every complex* $N^\bullet$ *the map*

$$Hom_{K(\mathcal{A})}(N^\bullet, I^\bullet) \to Hom_{D(\mathcal{A})}(N^\bullet, I^\bullet)$$

is an isomorphism.

Proof. Assume (1). Then (2) holds because the functor $Hom_{K(\mathcal{A})}(-, I^\bullet)$ is cohomological and the cone on a quasi-isomorphism is acyclic.

Assume (2). A morphism $N^\bullet \to I^\bullet$ in $D(\mathcal{A})$ is of the form $fs^{-1} : N^\bullet \to I^\bullet$ where $s : M^\bullet \to N^\bullet$ is a quasi-isomorphism and $f : M^\bullet \to I^\bullet$ is a map. By (2) this corresponds to a unique morphism $N^\bullet \to I^\bullet$ in $K(\mathcal{A})$, i.e., (3) holds.

Assume (3). If $M^\bullet$ is acyclic then $M^\bullet$ is isomorphic to the zero complex in $D(\mathcal{A})$ hence $Hom_{D(\mathcal{A})}(N^\bullet, I^\bullet) = 0$, whence $Hom_{K(\mathcal{A})}(N^\bullet, I^\bullet) = 0$ by (3), i.e., (1) holds. □

Lemma 11.28.3. *Let* $\mathcal{A}$ *be an abelian category. A bounded below complex of injectives is K-injective.*

Proof. Follows from Lemmas 11.28.2 and 11.17.8. □

Lemma 11.28.4. *Let* $\mathcal{A}$ *be an abelian category. Let* $F : K(\mathcal{A}) \to \mathcal{D}'$ *be an exact functor of triangulated categories. Then* RF *is defined at every complex in* $K(\mathcal{A})$ *which is quasi-isomorphic to a K-injective complex. In fact, every K-injective complex computes* RF.

Proof. By Lemma 11.14.4 it suffices to show that RF is defined at a K-injective complex, i.e., it suffices to show a K-injective complex $I^\bullet$ computes RF. Any quasi-isomorphism $I^\bullet \to N^\bullet$ is a homotopy equivalence as it has an inverse by Lemma 11.28.2. Thus $I^\bullet \to I^\bullet$ is a final object of $I^\bullet/\text{Qis}(\mathcal{A})$ and we win. □

Lemma 11.28.5. *Let* $\mathcal{A}$ *be an abelian category. Assume every complex has a quasi-isomorphism towards a K-injective complex. Then any exact functor* $F : K(\mathcal{A}) \to \mathcal{D}'$ *of triangulated categories has a right derived functor*

$$RF : D(\mathcal{A}) \longrightarrow \mathcal{D}'$$

and $RF(I^\bullet) = I^\bullet$ *for K-injective complexes* $I^\bullet$.

Proof. To see this we apply Lemma 11.14.15 with $\mathcal{I}$ the collection of K-injective complexes. Since (1) holds by assumption, it suffices to prove that if $I^\bullet \to J^\bullet$ is a quasi-isomorphism of K-injective complexes, then $F(I^\bullet) \to F(J^\bullet)$ is an isomorphism. This is clear because $I^\bullet \to J^\bullet$ is a homotopy equivalence, i.e., an isomorphism in $K(\mathcal{A})$, by Lemma 11.28.2. □

The following lemma can be generalized to limits over bigger ordinals.

Lemma 11.28.6. *Let* $\mathcal{A}$ *be an abelian category. Let*

$$\ldots \to I_3^\bullet \to I_2^\bullet \to I_1^\bullet$$

be an inverse system of K-injective complexes. Assume

(1) *each* $I_n^\bullet$ *is K-injective,*
(2) *each map* $I_{n+1}^m \to I_n^m$ *is a split surjection,*
(3) *the limits* $I^m = \lim I_n^m$ *exist.*

Then the complex $I^\bullet$ is K-injective.

Proof. Let $M^\bullet$ be an acyclic complex. Let us abbreviate $H_n(a, b) = Hom_{\mathcal{A}}(M^a, I_n^b)$. With this notation $Hom_{K(\mathcal{A})}(M^\bullet, I^\bullet)$ is the cohomology of the complex

$$\prod_m \lim_n H_n(m, m-2) \to \prod_m \lim_n H_n(m, m-1) \to \prod_m \lim_n H_n(m, m) \to \prod_m \lim_n H_n(m, m+1)$$

in the third spot from the left. We may exchange the order of $\prod$ and lim and each of the complexes

$$\prod_m H_n(m, m-2) \to \prod_m H_n(m, m-1) \to \prod_m H_n(m, m) \to \prod_m H_n(m, m+1)$$

is exact by assumption (1). By assumption (2) the maps in the systems

$$\ldots \to \prod_m H_3(m, m-2) \to \prod_m H_2(m, m-2) \to \prod_m H_1(m, m-2)$$

are surjective. Thus the lemma follows from Homology, Lemma 10.23.4. □

Remark 11.28.7. It appears that a combination of Lemmas 11.27.3, 11.28.5, and 11.28.6 produces ``enough K-injectives'' for any abelian category with enough injectives and countable limits. Actually, this may not work! Namely, suppose that $K^\bullet$ is a complex and $I_n^\bullet$ is the system of bounded above complexes of injectives produced by Lemma 11.27.3. Let $I^\bullet = lim\, I_n^\bullet$ be the termwise limit which is K-injective by Lemma 11.28.6. The problem is that the map $K^\bullet \to I^\bullet$ may not be a quasi-isomorphism. Namely, if lim_n is not exact in $\mathcal{A}$ then there is no reason to think that it is a quasi-isomorphism in general.

11.29. Bounded cohomological dimension

There is another case where the unbounded derived functor exists. Namely, when the functor has bounded cohomological dimension.

Lemma 11.29.1. *Let $\mathcal{A}$ be an abelian category. Let $d : Ob(\mathcal{A}) \to \{0, 1, 2, \ldots, \infty\}$ be a function. Assume that*

(1) *every object of $\mathcal{A}$ is a subobject of an object A with $d(A) = 0$,*
(2) *if $0 \to A \to B \to C \to 0$ is a short exact sequence then $d(C) \leq \max\{d(A) - 1, d(B)\}$.*

Let $K^\bullet$ be a complex such that $n + d(K^n)$ tends to $-\infty$ as $n \to -\infty$. Then there exists a quasi-isomorphism $K^\bullet \to L^\bullet$ with $d(L^n) = 0$ for all $n \in \mathbf{Z}$.

Proof. By Lemma 11.15.4 we can find a quasi-isomorphism $\sigma_{\geq 0}K^\bullet \to M^\bullet$ with $M^n = 0$ for $n < 0$ and $d(M^n) = 0$ for $n \geq 0$. Then $K^\bullet$ is quasi-isomorphic to the complex

$$\ldots \to K^{-2} \to K^{-1} \to M^0 \to M^1 \to \ldots$$

Hence we may assume that $d(K^n) = 0$ for $n \gg 0$. Note that the condition $n + d(K^n) \to -\infty$ as $n \to -\infty$ is not violated by this replacement.

We are going to improve $K^\bullet$ by an (infinite) sequence of elementary replacements. An *elementary replacement* is the following. Choose an index n such that $d(K^n) > 0$. Choose an injection $K^n \to M$ where $d(M) = 0$. Set $M' = \mathrm{Coker}(K^n \to M \oplus K^{n+1})$. Consider the map of complexes

$$
\begin{tikzcd}
K^\bullet : \arrow[d] & K^{n-1} \arrow[r] \arrow[d] & K^n \arrow[r] \arrow[d] & K^{n+1} \arrow[r] \arrow[d] & K^{n+2} \arrow[d] \\
(K')^\bullet : & K^{n-1} \arrow[r] & M \arrow[r] & M' \arrow[r] & K^{n+2}
\end{tikzcd}
$$

It is clear that $K^\bullet \to (K')^\bullet$ is a quasi-isomorphism. Moreover, it is clear that $d((K')^n) = 0$ and $d((K')^{n+1}) \leq \max\{d(K^{n+1}), d(K^n) - 1\}$ and the other values are unchanged.

To finish the proof we carefully choose the order in which to do the elementary replacements so that for every integer m the complex $\sigma_{\geq m}K^\bullet$ is changed only a finite number of times. To do this set

$$\xi(K^\bullet) = \max\{n + d(K^n) \mid d(K^n) > 0\}$$

and

$$I = \{n \in \mathbf{Z} \mid \xi(K^\bullet) = n + d(K^n) \wedge d(K^n) > 0\}$$

Our assumption that $n + d(K^n)$ tends to $-\infty$ as $n \to -\infty$ and the fact that $d(K^n) = 0$ for $n >> 0$ implies $\xi(K^\bullet) < +\infty$ and that I is a finite set. It is clear that $\xi((K')^\bullet) \leq \xi(K^\bullet)$ for an elementary transformation as above. An elementary transformation changes the complex in degrees $\leq \xi(K^\bullet) + 1$. Hence if we can find finite sequence of elementary transformations which decrease $\xi(K^\bullet)$, then we win. However, note that if we do an elementary transformation starting with the smallest element $n \in I$, then we either decrease the size of I, or we increase $\min I$. Since every element of I is $\leq \xi(K^\bullet)$ we see that we win after a finite number of steps. $\square$

Lemma 11.29.2. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor of abelian categories. If*

(1) *every object of $\mathcal{A}$ is a subobject of an object which is right acyclic for F,*
(2) *there exists an integer n such that $R^nF = 0$,*

then $RF : D(\mathcal{A}) \to D(\mathcal{B})$ exists. Any complex consisting of right acyclic objects for F computes RF and any complex is the source of a quasi-isomorphism into such a complex.

Proof. Note that the first condition implies that $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ exists, see Proposition 11.15.8. Let A be an object of $\mathcal{A}$. Choose an injection $A \to A'$ with A' acyclic. Then we see that $R^{n+1}F(A) = R^nF(A'/A) = 0$ by the long exact cohomology sequence. Hence we conclude that $R^{n+1}F = 0$. Continuing like this using induction we find that $R^mF = 0$ for all $m \geq n$.

We are going to use Lemma 11.29.1 with the function $d : Ob(\mathcal{A}) \to \{0, 1, 2, \ldots\}$ given by $d(A) = \min\{0\} \cup \{i \mid R^iF(A) \neq 0\}$. The first assumption of Lemma 11.29.1 is our assumption (3) and the second assumption of Lemma 11.29.1 follows from the long exact cohomology sequence. Hence for every complex $K^\bullet$ there exists a quasi-isomorphism $K^\bullet \to L^\bullet$ with L^n right acyclic for F. We claim that if $L^\bullet \to M^\bullet$ is a quasi-isomorphism of complexes of right acyclic objects for F, then $F(L^\bullet) \to F(M^\bullet)$ is a quasi-isomorphism. If we prove this claim then we are done by Lemma 11.14.15. To prove the claim pick an integer $i \in \mathbf{Z}$. Consider the distinguished triangle

$$\sigma_{\geq i-n-1}L^\bullet \to \sigma_{\geq i-n-1}M^\bullet \to Q^\bullet,$$

i.e., let $Q^\bullet$ be the cone of the first map. Note that $Q^\bullet$ is bounded below and that $H^j(Q^\bullet)$ is zero except possibly for $j = i - n - 1$ or $j = i - n - 2$. We may apply RF to $Q^\bullet$. Using the spectral sequence of Lemma 11.20.3 and the assumed vanishing of cohomology (2) we conclude that $R^jF(Q^\bullet)$ is zero except possibly for $j \in \{i - n - 2, \ldots, i - 1\}$. Hence we see that $RF(\sigma_{\geq i-n-1}L^\bullet) \to RF(\sigma_{\geq i-n-1}L^\bullet)$ induces an isomorphism of cohomology objects in degrees $\geq i$. By Proposition 11.15.8 we know that $RF(\sigma_{\geq i-n-1}L^\bullet) = \sigma_{\geq i-n-1}F(L^\bullet)$ and $RF(\sigma_{\geq i-n-1}M^\bullet) = \sigma_{\geq i-n-1}F(M^\bullet)$. We conclude that $F(L^\bullet) \to F(M^\bullet)$ is an isomorphism in degree i as desired. □

Lemma 11.29.3. *Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor of abelian categories. If*

1. *every object of $\mathcal{A}$ is a quotient of an object which is left acyclic for F,*
2. *there exists an integer n such that $L^nF = 0$,*

then $LF : D(\mathcal{A}) \to D(\mathcal{B})$ exists. Any complex consisting of left acyclic objects for F computes LF and any complex is the target of a quasi-isomorphism into such a complex.

Proof. This is dual to Lemma 11.29.2. □

11.30. Other chapters

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CHAPTER 12

More on Algebra

12.1. Introduction

In this chapter we prove some results in commutative algebra which are less elementary than those in the first chapter on commutative algebra, see Algebra, Section 7.1. A reference is [**Mat70**].

12.2. Computing Tor

Let R be a ring. We denote $D(R)$ the derived category of the abelian category Mod_R of R-modules. Note that Mod_R has enough projectives as every free R-module is projective. Thus we can define the left derived functors of any additive functor from Mod_R to any abelian category.

This implies in particular to the functor $- \otimes_R M : \text{Mod}_R \to \text{Mod}_R$ whose right derived functors are the Tor functors $\text{Tor}_i^R(-, M)$, see Algebra, Section 7.69. There is also a total right derived functor

$$- \otimes_R^{\mathbf{L}} M : D^-(R) \longrightarrow D^-(R) \tag{12.2.0.1}$$

which is denoted $- \otimes_R^{\mathbf{L}} M$. Its satellites are the Tor modules, i.e., we have

$$H^{-p}(N \otimes_R^{\mathbf{L}} M) = \text{Tor}_p^R(N, M).$$

A special situation occurs when we consider the tensor product with an R-algebra A. In this case we think of $- \otimes_R A$ as a functor from Mod_R to Mod_A. Hence the total right derived functor

$$- \otimes_R^{\mathbf{L}} A : D^-(R) \longrightarrow D^-(A) \tag{12.2.0.2}$$

which is denoted $- \otimes_R^{\mathbf{L}} A$. Its satellites are the tor groups, i.e., we have

$$H^{-p}(N \otimes_R^{\mathbf{L}} A) = \text{Tor}_p^R(N, A).$$

In particular these Tor groups naturally have the structure of A-modules.

12.3. Derived tensor product

We can construct the derived tensor product in greater generality. In fact, it turns out that the boundedness assumptions are not necessary, provided we choose K-flat resolutions.

Lemma 12.3.1. *Let R be a ring. Let $P^\bullet$ be a complex of R-modules. Let $\alpha, \beta : L^\bullet \to M^\bullet$ be homotopy equivalent maps of complexes. Then α and β induce homotopy equivalent maps*

$$Tot(\alpha \otimes id_P), Tot(\beta \otimes id_P) : Tot(L^\bullet \otimes_R P^\bullet) \longrightarrow Tot(M^\bullet \otimes_R P^\bullet).$$

In particular the construction $L^\bullet \mapsto Tot(L^\bullet \otimes_R P^\bullet)$ defines an endo-functor of the homotopy category of complexes.

Proof. Say $\alpha = \beta + dh + hd$ for some homotopy h defined by $h^n : L^n \to M^{n-1}$. Set

$$H^n = \bigoplus\nolimits_{a+b=n} h^a \otimes \mathrm{id}_{P^b} : \bigoplus\nolimits_{a+b=n} L^a \otimes_R P^b \longrightarrow \bigoplus\nolimits_{a+b=n} M^{a-1} \otimes_R P^b$$

Then a straightforward computation shows that

$$\mathrm{Tot}(\alpha \otimes \mathrm{id}_P) = \mathrm{Tot}(\beta \otimes \mathrm{id}_P) + dH + Hd$$

as maps $\mathrm{Tot}(L^\bullet \otimes_R P^\bullet) \to \mathrm{Tot}(M^\bullet \otimes_R P^\bullet)$. □

Lemma 12.3.2. *Let R be a ring. Let $P^\bullet$ be a complex of R-modules. The functor*

$$K(Mod_R) \longrightarrow K(Mod_R), \quad L^\bullet \longmapsto Tot(L^\bullet \otimes_R P^\bullet)$$

is an exact functor of triangulated categories.

Proof. By our definition of the triangulated structure on $K(\mathrm{Mod}_R)$ we have to check that our functor maps a termwise split short exact sequence of complexes to a termwise split short exact sequence of complexes. As the terms of $\mathrm{Tot}(L^\bullet \otimes_R P^\bullet)$ are direct sums of the tensor products $L^a \otimes_R P^b$ this is clear. □

The following definition will allow us to think intelligently about derived tensor products of unbounded complexes.

Definition 12.3.3. Let R be a ring. A complex $K^\bullet$ is called *K-flat* if for every acyclic complex $M^\bullet$ the total complex $\mathrm{Tot}(M^\bullet \otimes_R K^\bullet)$ is acyclic.

Lemma 12.3.4. *Let R be a ring. Let $K^\bullet$ be a K-flat complex. Then the functor*

$$K(Mod_R) \longrightarrow K(Mod_R), \quad L^\bullet \longmapsto Tot(L^\bullet \otimes_R K^\bullet)$$

transforms quasi-isomorphisms into quasi-isomorphisms.

Proof. Follows from Lemma 12.3.2 and the fact that quasi-isomorphisms in $K(\mathrm{Mod}_R)$ and $K(\mathrm{Mod}_A)$ are characterized by having acyclic cones. □

Lemma 12.3.5. *Let $R \to R'$ be a ring map. If $K^\bullet$ is a K-flat complex of R-modules, then $K^\bullet \otimes_R R'$ is a K-flat complex of R'-modules.*

Proof. Follows from the definitions and the fact that $(K^\bullet \otimes_R R') \otimes_{R'} L^\bullet = K^\bullet \otimes_R L^\bullet$ for any complex $L^\bullet$ of R'-modules. □

Lemma 12.3.6. *Let R be a ring. If $K^\bullet$, $L^\bullet$ are K-flat complexes of R-modules, then $Tot(K^\bullet \otimes_R L^\bullet)$ is a K-flat complex of R-modules.*

Proof. Follows from the isomorphism

$$\mathrm{Tot}(M^\bullet \otimes_R \mathrm{Tot}(K^\bullet \otimes_R L^\bullet)) = \mathrm{Tot}(\mathrm{Tot}(M^\bullet \otimes_R K^\bullet) \otimes_R L^\bullet)$$

and the definition. □

Lemma 12.3.7. *Let R be a ring. Let $(K_1^\bullet, K_2^\bullet, K_3^\bullet)$ be a distinguished triangle in $K(Mod_R)$. If two out of three of $K_i^\bullet$ are K-flat, so is the third.*

Proof. Follows from Lemma 12.3.2 and the fact that in a distinguished triangle in $K(\mathrm{Mod}_A)$ if two out of three are acyclic, so is the third. □

Lemma 12.3.8. *Let R be a ring. Let $P^\bullet$ be a bounded above complex of flat R-modules. Then $P^\bullet$ is K-flat.*

Proof. Let $L^\bullet$ be an acyclic complex of R-modules. Let $\xi \in H^n(\text{Tot}(L^\bullet \otimes_R P^\bullet))$. We have to show that $\xi = 0$. Since $\text{Tot}^n(L^\bullet \otimes_R P^\bullet)$ is a direct sum with terms $L^a \otimes_R P^b$ we see that ξ comes from an element in $H^n(\text{Tot}(\tau_{\leq m} L^\bullet \otimes_R P^\bullet))$ for some $m \in \mathbf{Z}$. Since $\tau_{\leq m} L^\bullet$ is also acyclic we may replace $L^\bullet$ by $\tau_{\leq m} L^\bullet$. Hence we may assume that $L^\bullet$ is bounded above. In this case the spectral sequence of Homology, Lemma 10.19.5 has

$$'E_1^{p,q} = H^p(L^\bullet \otimes_R P^q)$$

which is zero as P^q is flat and $L^\bullet$ acyclic. Hence $H^*(\text{Tot}(L^\bullet \otimes_R P^\bullet)) = 0$. □

In the following lemma by a colimit of a system of complexes we mean the termwise colimit.

Lemma 12.3.9. *Let R be a ring. Let $K_1^\bullet \to K_2^\bullet \to \ldots$ be a system of K-flat complexes. Then $colim_i K_i^\bullet$ is K-flat.*

Proof. Because we are taking termwise colimits it is clear that

$$colim_i \text{Tot}(M^\bullet \otimes_R K_i^\bullet) = \text{Tot}(M^\bullet \otimes_R colim_i K_i^\bullet)$$

Hence the lemma follows from the fact that filtered colimits are exact. □

Lemma 12.3.10. *Let R be a ring. For any complex $M^\bullet$ there exists a K-flat complex $K^\bullet$ and a quasi-isomorphism $K^\bullet \to M^\bullet$. Moreover each K^n is a flat R-module.*

Proof. Let $\mathcal{P} \subset Ob(\text{Mod}_R)$ be the class of flat R-modules. By Derived Categories, Lemma 11.27.1 there exists a system $K_1^\bullet \to K_2^\bullet \to \ldots$ and a diagram

$$\begin{tikzcd} K_1^\bullet \arrow[r] \arrow[d] & K_2^\bullet \arrow[r] \arrow[d] & \ldots \\ \tau_{\leq 1} M^\bullet \arrow[r] & \tau_{\leq 2} M^\bullet \arrow[r] & \ldots \end{tikzcd}$$

with the properties (1), (2), (3) listed in that lemma. These properties imply each complex $K_i^\bullet$ is a bounded above complex of flat modules. Hence $K_i^\bullet$ is K-flat by Lemma 12.3.8. The induced map $colim_i K_i^\bullet \to M^\bullet$ is a quasi-isomorphism by construction. The complex $colim_i K_i^\bullet$ is K-flat by Lemma 12.3.9. The final assertion of the lemma is true because the colimit of a system of flat modules is flat, see Algebra, Lemma 7.35.2. □

Lemma 12.3.11. *Let R be a ring. Let $\alpha : P^\bullet \to Q^\bullet$ be a quasi-isomorphism of K-flat complexes of R-modules. For every complex $L^\bullet$ of R-modules the induced map*

$$Tot(id_L \otimes \alpha) : Tot(L^\bullet \otimes_R P^\bullet) \longrightarrow Tot(L^\bullet \otimes_R Q^\bullet)$$

is a quasi-isomorphism.

Proof. Choose a quasi-isomorphism $K^\bullet \to L^\bullet$ with $K^\bullet$ a K-flat complex, see Lemma 12.3.10. Consider the commutative diagram

$$\begin{tikzcd} \text{Tot}(K^\bullet \otimes_R P^\bullet) \arrow[r] \arrow[d] & \text{Tot}(K^\bullet \otimes_R Q^\bullet) \arrow[d] \\ \text{Tot}(L^\bullet \otimes_R P^\bullet) \arrow[r] & \text{Tot}(L^\bullet \otimes_R Q^\bullet) \end{tikzcd}$$

The result follows as by Lemma 12.3.4 the vertical arrows and the top horizontal arrow are quasi-isomorphisms. □

Let R be a ring. Let $M^\bullet$ be an object of $D(R)$. Choose a K-flat resolution $K^\bullet \to M^\bullet$, see Lemma 12.3.10. By Lemmas 12.3.1 and 12.3.2 we obtain an exact functor of triangulated categories

$$K(\text{Mod}_R) \longrightarrow K(\text{Mod}_R), \quad L^\bullet \longmapsto \text{Tot}(L^\bullet \otimes_R K^\bullet)$$

By Lemma 12.3.4 this functor induces a functor $D(R) \to D(R)$ simply because $D(R)$ is the localization of $K(\text{Mod}_R)$ at quasi-isomorphism. By Lemma 12.3.11 the resulting functor (up to isomorphism) does not depend on the choice of the K-flat resolution.

Definition 12.3.12. Let R be a ring. Let $M^\bullet$ be an object of $D(R)$. The *derived tensor product*

$$- \otimes_R^{\mathbf{L}} M^\bullet : D(R) \longrightarrow D(R)$$

is the exact functor of triangulated categories described above.

This functor extends the functor (12.2.0.1). It is clear from our explicit constructions that there is a canonical isomorphism

$$M^\bullet \otimes_R^{\mathbf{L}} L^\bullet \cong L^\bullet \otimes_R^{\mathbf{L}} M^\bullet$$

whenever both $L^\bullet$ and $M^\bullet$ are in $D(R)$. Hence when we write $M^\bullet \otimes_R^{\mathbf{L}} L^\bullet$ we will usually be agnostic about which variable we are using to define the derived tensor product with.

12.4. Derived change of rings

Let $R \to A$ be a ring map. We can also use K-flat resolutions to define a derived base change functor

$$- \otimes_R^{\mathbf{L}} A : D(R) \to D(A)$$

extending the functor (12.2.0.2). Namely, for every complex of R-modules $M^\bullet$ we can choose a K-flat resolution $K^\bullet \to M^\bullet$ and set $M^\bullet \otimes_R^{\mathbf{L}} A = K^\bullet \otimes_R A$. You can use Lemmas 12.3.10 and 12.3.11 to see that this is well defined. However, to cross all the t's and dot all the i's it is perhaps more convenient to use some general theory.

Lemma 12.4.1. *The construction above is independent of choices and defines an exact functor of triangulated categories $D(R) \to D(A)$.*

Proof. To see this we use the general theory developed in Derived Categories, Section 11.14. Set $\mathcal{D} = K(\text{Mod}_R)$ and $\mathcal{D}' = D(A)$. Let us write $F : \mathcal{D} \to \mathcal{D}'$ the exact functor of triangulated categories defined by the rule $F(M^\bullet) = M^\bullet \otimes_R A$. We let S be the set of quasi-isomorphisms in $\mathcal{D} = K(\text{Mod}_R)$. This gives a situation as in Derived Categories, Situation 11.14.1 so that Derived Categories, Definition 11.14.2 applies. We claim that LF is everywhere defined. This follows from Derived Categories, Lemma 11.14.15 with $\mathcal{P} \subset Ob(\mathcal{D})$ the collection of K-flat complexes: (1) follows from Lemma 12.3.10 and (2) follows from Lemma 12.3.11. Thus we obtain a derived functor

$$LF : D(R) = S^{-1}\mathcal{D} \longrightarrow \mathcal{D}' = D(A)$$

see Derived Categories, Equation (11.14.9.1). Finally, Derived Categories, Lemma 11.14.15 also guarantees that $LF(K^\bullet) = F(K^\bullet) = K^\bullet \otimes_R A$ when $K^\bullet$ is K-flat, i.e., LF is indeed computed in the way described above. $\square$

12.5. Tor independence

We often encounter the following situation. Suppose that

$$\begin{tikzcd} A \arrow[r] & A' \\ R \arrow[u] \arrow[r] & R' \arrow[u] \end{tikzcd}$$

is a ``base change'' diagram of rings, i.e., $A' = A \otimes_R R'$. In this situation, for any A-module M we have $M \otimes_A A' = M \otimes_R R'$. Thus $- \otimes_R R'$ is equal to $- \otimes_A A'$ as a functor $\mathrm{Mod}_A \to \mathrm{Mod}_{A'}$. In general this equality **does not extend** to derived tensor products. Let $K^\bullet \in D^-(A)$. We have

$$K^\bullet \otimes_A^{\mathbf{L}} A' = P^\bullet \otimes_A A'$$

where $P^\bullet \to K^\bullet$ is a projective resolution in the category of A-modules. Pick a projective resolution $E^\bullet \to P^\bullet$ in the category of R-modules. Then it is also the case that $E^\bullet \to K^\bullet$ is a projective resolution in the category of R-modules. Hence

$$K^\bullet \otimes_R^{\mathbf{L}} R' = E^\bullet \otimes_R R'$$

The map $E^\bullet \to P^\bullet$ and the map $R' \to A'$ combined determine a comparison map

$$K^\bullet \otimes_R^{\mathbf{L}} R' = E^\bullet \otimes_R R' \longrightarrow P^\bullet \otimes_A A' = K^\bullet \otimes_A^{\mathbf{L}} A' \tag{12.5.0.1}$$

A simple example where this is not an isomorphism is to take $R = k[x]$, $A = R' = A' = k[x]/(x) = k$ and $K^\bullet = A[0]$. Clearly, a necessary condition is that $\mathrm{Tor}_p^R(A, R') = 0$ for all $p > 0$.

Definition 12.5.1. Let R be a ring. Let A, B be R-algebras. We say A and B are *Tor independent over R* if $\mathrm{Tor}_p^R(A, B) = 0$ for all $p > 0$.

Lemma 12.5.2. *The comparison map (12.5.0.1) is an isomorphism if A and R' are Tor independent over R.*

Proof. To prove this we choose a free resolution $F^\bullet \to R'$ of R' as an R-module. Because A and R' are Tor independent over R we see that $F^\bullet \otimes_R A$ is a free A-module resolution of A' over A. By our general construction of the derived tensor product above we see that

$$P^\bullet \otimes_A A' \cong \mathrm{Tot}(P^\bullet \otimes_A (F^\bullet \otimes_R A)) = \mathrm{Tot}(P^\bullet \otimes_R F^\bullet) \cong \mathrm{Tot}(E^\bullet \otimes_R F^\bullet) \cong E^\bullet \otimes_R R'$$

as desired. □

12.6. Spectral sequences for Tor

In this section we collect various spectral sequences that come up when considering the Tor functors.

Example 12.6.1. Let R be a ring. Let $K_\bullet$ be a bounded above chain complex of R-modules. Let M be an R-module. Then there is a spectral sequence with E_2-page

$$\mathrm{Tor}_i^R(H_j(K_\bullet), M) \Rightarrow H_{i+j}(K_\bullet \otimes_R^{\mathbf{L}} M)$$

and another spectral sequence with E_1-page

$$\mathrm{Tor}_i^R(K_j, M) \Rightarrow H_{i+j}(K_\bullet \otimes_R^{\mathbf{L}} M)$$

This follows from the dual to Derived Categories, Lemma 11.20.3.

Example 12.6.2. Let $R \to S$ be a ring map. Let M be an R-module and let N be an S-module. Then there is a spectral sequence

$$\mathrm{Tor}_n^S(\mathrm{Tor}_m^R(M, S), N) \Rightarrow \mathrm{Tor}_{n+m}^R(M, N).$$

To construct it choose a R-free resolution $P_\bullet$ of M. Then we have

$$M \otimes_R^{\mathbf{L}} N = P^\bullet \otimes_R N = (P^\bullet \otimes_R S) \otimes_S N$$

and then apply the first spectral sequence of Example 12.6.1.

Example 12.6.3. Consider a commutative diagram

$$\begin{tikzcd} B \arrow[r] & B' = B \otimes_A A' \\ A \arrow[u] \arrow[r] & A' \arrow[u] \end{tikzcd}$$

and B-modules M, N. Set $M' = M \otimes_A A' = M \otimes_B B'$ and $N' = N \otimes_A A' = N \otimes_B B'$. *Assume that $A \to B$ is flat and that M and N are A-flat.* Then there is a spectral sequence

$$\mathrm{Tor}_i^A(\mathrm{Tor}_j^B(M, N), A') \Rightarrow \mathrm{Tor}_{i+j}^{B'}(M', N')$$

The reason is as follows. Choose free resolution $F_\bullet \to M$ as a B-module. As B and M are A-flat we see that $F_\bullet \otimes_A A'$ is a free B'-resolution of M'. Hence we see that the groups $\mathrm{Tor}_n^{B'}(M', N')$ are computed by the complex

$$(F_\bullet \otimes_A A') \otimes_{B'} N' = (F_\bullet \otimes_B N) \otimes_A A' = (F_\bullet \otimes_B N) \otimes_A^{\mathbf{L}} A'$$

the last equality because $F_\bullet \otimes_B N$ is a complex of flat A-modules as N is flat over A. Hence we obtain the spectral sequence by applying the spectral sequence of Example 12.6.1.

Example 12.6.4. Let $K^\bullet, L^\bullet$ be objects of $D^-(R)$. Then there are spectral sequences

$$H^p(K^\bullet \otimes_R^{\mathbf{L}} H^q(L^\bullet)) \Rightarrow H^{p+q}(K^\bullet \otimes_R^{\mathbf{L}} L^\bullet)$$

and

$$H^q(H^p(K^\bullet) \otimes_R^{\mathbf{L}} L^\bullet) \Rightarrow H^{p+q}(K^\bullet \otimes_R^{\mathbf{L}} L^\bullet)$$

After replacing $K^\bullet$ and $L^\bullet$ by bounded above complexes of projectives, these spectral sequences are simply the two spectral sequences for computing the cohomology of $\mathrm{Tot}(K^\bullet \otimes L^\bullet)$ discussed in Homology, Section 10.19.

12.7. Products and Tor

The simplest example of the product maps comes from the following situation. Suppose that $K^\bullet, L^\bullet \in D(R)$ with one of them contained in $D^-(R)$. Then there are maps

(12.7.0.1)
$$H^i(K^\bullet) \otimes_R H^j(L^\bullet) \longrightarrow H^{i+j}(K^\bullet \otimes_R^{\mathbf{L}} L^\bullet)$$

Namely, to define these maps we may assume that one of $K^\bullet, L^\bullet$ is a bounded above complex of projective R-modules. In that case $K^\bullet \otimes_R^{\mathbf{L}} L^\bullet$ is represented by the complex $\mathrm{Tot}(K^\bullet \otimes_R L^\bullet)$, see Section 12.2. Next, suppose that $\xi \in H^i(K^\bullet)$ and $\zeta \in H^j(L^\bullet)$. Choose $k \in \mathrm{Ker}(K^i \to K^{i+1})$ and $l \in \mathrm{Ker}(L^j \to L^{j+1})$ representing ξ and ζ. Then we set

$$\xi \cup \zeta = \text{class of } k \otimes l \text{ in } H^{i+j}(\mathrm{Tot}(K^\bullet \otimes_R L^\bullet)).$$

This make sense because the formula (see Homology, Definition 10.19.2) for the differential d on the total complex shows that $k \otimes l$ is a cocycle. Moreover, if $k' = d_K(k'')$ for some $k'' \in K^{i-1}$, then $k' \otimes l = \mathrm{d}(k'' \otimes l)$ because l is a cocycle. Similarly, altering the choice of

l representing ζ does not change the class of $k \otimes l$. It is equally clear that $\cup$ is bilinear, and hence to a general element of $H^i(K^\bullet) \otimes_R H^j(L^\bullet)$ we assign

$$\sum \xi_i \otimes \zeta_i \longmapsto \sum \xi_i \cup \zeta_i$$

in $H^{i+j}(\mathrm{Tot}(K^\bullet \otimes_R L^\bullet))$.

Let $R \to A$ be a ring map. Let $K^\bullet, L^\bullet \in D^-(R)$. Then we have a canonical identification

$$(K^\bullet \otimes^{\mathbf{L}}_R A) \otimes^{\mathbf{L}}_A (L^\bullet \otimes^{\mathbf{L}}_R A) = (K^\bullet \otimes^{\mathbf{L}}_R L^\bullet) \otimes^{\mathbf{L}}_R A \tag{12.7.0.2}$$

in $D(A)$. It is constructed as follows. First, choose projective resolutions $P^\bullet \to K^\bullet$ and $Q^\bullet \to L^\bullet$ over R. Then the left hand side is represented by the complex $\mathrm{Tot}((P^\bullet \otimes_R A) \otimes_A (Q^\bullet \otimes_R A))$ and the right hand side by the complex $\mathrm{Tot}(P^\bullet \otimes_R Q^\bullet) \otimes_R A$. These complexes are canonically isomorphic. Thus the construction above induces products

$$\mathrm{Tor}_n^R(K^\bullet, A) \otimes_A \mathrm{Tor}_m^R(L^\bullet, A) \longrightarrow \mathrm{Tor}_{n+m}^R(K^\bullet \otimes_R L^\bullet, A)$$

which are occasionally usefull.

Let M, N be R-modules. Using the general construction above and functoriality of Tor we obtain canonical maps

$$\mathrm{Tor}_n^R(M, A) \otimes_A \mathrm{Tor}_m^R(N, A) \longrightarrow \mathrm{Tor}_{n+m}^R(M \otimes_R N, A) \tag{12.7.0.3}$$

Here is a direct construction using projective resolutions. First, choose projective resolutions

$$P_\bullet \to M, \quad Q_\bullet \to N, \quad T_\bullet \to M \otimes_R N$$

over R. We have $H_0(\mathrm{Tot}(P_\bullet \otimes_R Q_\bullet)) = M \otimes_R N$ by right exactness of $\otimes_R$. Hence Derived Categories, Lemmas 11.18.6 and 11.18.7 guarantee the existence and uniqueness of a map of complexes $\mu : \mathrm{Tot}(P_\bullet \otimes_R Q_\bullet) \to T_\bullet$ such that $H_0(\mu) = \mathrm{id}_{M \otimes_R N}$. This induces a canonical map

$$\begin{aligned}(M \otimes^{\mathbf{L}}_R A) \otimes^{\mathbf{L}}_A (N \otimes^{\mathbf{L}}_R A) &= \mathrm{Tot}((P_\bullet \otimes_R A) \otimes_A (Q_\bullet \otimes_R A)) \\ &= \mathrm{Tot}(P_\bullet \otimes_R Q_\bullet) \otimes_R A \\ &\to T_\bullet \otimes_R A \\ &= (M \otimes_R N) \otimes^{\mathbf{L}}_R A\end{aligned}$$

in $D(A)$. Hence the products (12.7.0.3) above are constructed using (12.7.0.1) over A to construct

$$\mathrm{Tor}_n^R(M, A) \otimes_A \mathrm{Tor}_m^R(N, A) \to H^{-n-m}((M \otimes^{\mathbf{L}}_R A) \otimes^{\mathbf{L}}_A (N \otimes^{\mathbf{L}}_R A))$$

and then composing by the displayed map above to end up in $\mathrm{Tor}_{n+m}^R(M \otimes_R N, A)$.

An interesting special case of the above occurs when $M = N = B$ where B is an R-algebra. In this case we obtain maps

$$\mathrm{Tor}_n^R(B, A) \otimes_A \mathrm{Tor}_m^R(B, A) \longrightarrow \mathrm{Tor}_n^R(B \otimes_R B, A) \longrightarrow \mathrm{Tor}_n^R(B, A)$$

the second arrow being induced by the multiplication map $B \otimes_R B \to B$ via functoriality for Tor. In other words we obtain an A-algebra structure on $\mathrm{Tor}_\star^R(B, A)$. This algebra structure has many intriguing properties (associativity, graded commutative, B-algebra structure, divided powers in some case, etc) which we will discuss elsewhere (insert future reference here).

Lemma 12.7.1. *Let R be a ring. Let A, B, C be R-algebras and let $B \to C$ be an R-algebra map. Then the induced map*

$$Tor_\star^R(B, A) \longrightarrow Tor_\star^R(C, A)$$

is an A-algebra homomorphism.

Proof. Omitted. Hint: You can prove this by working through the definitions, writing all the complexes explicitly. $\square$

12.8. Formal glueing of module categories

Fix a noetherian scheme X, and a closed subscheme Z with complement U. Our goal is to explain a result of Artin that describes how coherent sheaves on X can be constructed (uniquely) from coherent sheaves on the formal completion of X along Z, and those on U with a suitable compatibility on the overlap.

Definition 12.8.1. Let R be a ring. Let M be an R-module.

(1) Let $I \subset R$ be an ideal. We say M is an *I-power torsion module* if for every $m \in M$ there exists an $n > 0$ such that $I^n m = 0$.
(2) Let $f \in R$. We say M is *an f-power torsion module* if for each $m \in M$, there exists an $n > 0$ such that $f^n m = 0$.

Thus an f-power torsion module is the same thing as a I-power torsion module for $I = (f)$. We sometimes use the notation $M[I^n] = \{m \in M \mid I^n m = 0\}$ and $M[I^\infty] = \bigcup M[I^n]$ for an R-module M. Thus M is I-power torsion if and only if $M = M[I^\infty]$ if and only if $M = \bigcup M[I^n]$.

Lemma 12.8.2. *Let $\varphi : R \to S$ be a ring map. Let $I \subset R$ be an ideal. The following are equivalent*

(1) *φ is flat and $R/I \to S/IS$ is faithfully flat,*
(2) *φ is flat, and the map $Spec(S/IS) \to Spec(R/I)$ is surjective.*
(3) *φ is flat, and the base change functor $M \mapsto M \otimes_R S$ is faithful on modules annihilated by I, and*
(4) *φ is flat, and the base change functor $M \mapsto M \otimes_R S$ is faithful on I-power torsion modules.*

Proof. If $R \to S$ is flat, then $R/I^n \to S/I^n S$ is flat for every n, see Algebra, Lemma 7.35.6. Hence (1) and (2) are equivalent by Algebra, Lemma 7.35.15. The equivalence of (1) with (3) follows by identifying I-torsion R-modules with R/I-modules, using that

$$M \otimes_R S = M \otimes_{R/I} S/IS$$

for R-modules M annihilated by I, and Algebra, Lemma 7.35.13. The implication (4) $\Rightarrow$ (3) is immediate. Assume (3). We have seen above that $R/I^n \to S/I^n S$ is flat, and by assumption it induces a surjection on spectra, as $Spec(R/I^n) = Spec(R/I)$ and similarly for S. Hence the base change functor is faithful on modules annihilated by I^n. Since any I-power torsion module M is the union $M = \bigcup M_n$ where M_n is annihilated by I^n we see that the base change functor is faithful on the category of all I-power torsion modules (as tensor product commutes with colimits). $\square$

Lemma 12.8.3. *Let R be a ring. Let I be an ideal of R. Let M be an I-power torsion module. Then M admits a resolution*

$$\ldots \to K_2 \to K_1 \to K_0 \to M \to 0$$

with each K_i a direct sum of copies of R/I^n for n variable.

Proof. There is a canonical surjection

$$\bigoplus_{m \in M} R/I^{n_m} \to M \to 0$$

where n_m is the smallest positive integer such that $I^{n_m} \cdot m = 0$. The kernel of the preceding surjection is also an I-power torsion module. Proceeding inductively, we construct the desired resolution of M. □

Lemma 12.8.4. *Assume $(\varphi : R \to S, I)$ satisfies the equivalent conditions of Lemma 12.8.2. The following are equivalent*

1. *for any I-power torsion module M, the natural map $M \to M \otimes_R S$ is an isomorphism, and*
2. *$R/I \to S/IS$ is an isomorphism.*

Proof. The implication (1) ⇒ (2) is immediate. Assume (2). First assume that M is annihilated by I. In this case, M is an R/I-module. Hence, we have an isomorphism

$$M \otimes_R S = M \otimes_{R/I} S/IS = M \otimes_{R/I} R/I = M$$

proving the claim. Next we prove by induction that $M \to M \otimes_R S$ is an isomorphism for any module M is annihilated by I^n. Assume the induction hypothesis holds for n and assume M is annihilated by I^{n+1}. Then we have a short exact sequence

$$0 \to I^n M \to M \to M/I^n M \to 0$$

and as $R \to S$ is flat this gives rise to a short exact sequence

$$0 \to I^n M \otimes_R S \to M \otimes_R S \to M/I^n M \otimes_R S \to 0$$

Using that the canonical map is an isomorphism for $M' = I^n M$ and $M'' = M/I^n M$ (by induction hypothesis) we conclude the same thing is true for M. Finally, suppose that M is a general I-power torsion module. Then $M = \bigcup M_n$ where M_n is annihilated by I^n and we conclude using that tensor products commute with colimits. □

Lemma 12.8.5. *Let R be a ring. Let I be an ideal of R. For any R-module M set $M[I^n] = \{m \in M \mid I^n m = 0\}$. If I is finitely generated then the following are equivalent*

1. *$M[I] = 0$,*
2. *$M[I^n] = 0$ for all $n \geq 1$, and*
3. *if $I = (f_1, \ldots, f_t)$, then the map $M \to \bigoplus M_{f_i}$ is injective.*

Proof. This follows from Algebra, Lemma 7.20.4. □

Lemma 12.8.6. *Let R be a ring. Let I be an ideal of R. For any R-module M set $M[I^\infty] = \bigcup_{n \geq 1} M[I^n]$. If I is finitely generated, then $(M/M[I^\infty])[I] = 0$.*

Proof. Let $m \in M$. If m maps to an element of $(M/M[I^\infty])[I]$ then $Im \subset M[I^\infty]$. Write $I = (f_1, \ldots, f_t)$. Then we see that $f_i m \in M[I^\infty]$, i.e., $I^{n_i} f_i m = 0$ for some $n_i > 0$. Thus we see that $I^N m = 0$ with $N = \sum n_i + 2$. Hence m maps to zero in $(M/M[I^\infty])$ which proves the lemma. □

Lemma 12.8.7. *Assume $\varphi : R \to S$ is a flat ring map and $I \subset R$ is a finitely generated ideal such that $R/I \to S/IS$ is an isomorphism. Then*

1. *for any R-module M the map $M \to M \otimes_R S$ induces an isomorphism $M[I^\infty] \to (M \otimes_R S)[(IS)^\infty]$ of I-power torsion submodules,*

(2) *the natural map*

$$Hom_R(M, N) \longrightarrow Hom_S(M \otimes_R S, N \otimes_R S)$$

is an isomorphism if either M or N is I-power torsion, and

(3) *the base change functor $M \mapsto M \otimes_R S$ defines an equivalence of categories between I-power torsion modules and IS-power torsion modules.*

Proof. Note that the equivalent conditions of both Lemma 12.8.2 and Lemma 12.8.4 are satisfied. We will use these without further mention. We first prove (1). Let M be any R-module. Set $M' = M/M[I^\infty]$ and consider the exact sequence

$$0 \to M[I^\infty] \to M \to M' \to 0$$

As $M[I^\infty] = M[I^\infty] \otimes_R S$ we see that it suffices to show that $(M' \otimes_R S)[(IS)^\infty] = 0$. Write $I = (f_1, \ldots, f_t)$. By Lemma 12.8.6 we see that $M'[I^\infty] = 0$. Hence for every $n > 0$ the map

$$M' \longrightarrow \bigoplus\nolimits_{i=1,\ldots t} M', \quad x \longmapsto (f_1^n x, \ldots, f_t^n x)$$

is injective. As S is flat over R also the corresponding map $M' \otimes_R S \to \bigoplus_{i=1,\ldots t} M' \otimes_R S$ is injective. This means that $(M' \otimes_R S)[I^n] = 0$ as desired.

Next we prove (2). If N is I-power torsion, then $N \otimes_R S = N$ and the displayed map of (2) is an isomorphism by Algebra, Lemma 7.11.17. If M is I-power torsion, then the image of any map $M \to N$ factors through $M[I^\infty]$ and the image of any map $M \otimes_R S \to N \otimes_R S$ factors through $(N \otimes_R S)[(IS)^\infty]$. Hence in this case part (1) guarantees that we may replace N by $N[I^\infty]$ and the result follows from the case where N is I-power torsion we just discussed.

Next we prove (3). The functor is fully faithful by (2). For essential surjectivity, we simply note that for any IS-power torsion S-module N, the natural map $N \otimes_R S \to N$ is an isomorphism. □

Lemma 12.8.8. *Let R be a ring. Let $I = (f_1, \ldots, f_n)$ be a finitely generated ideal of R. Let M be the R-module generated by elements $e_1, \ldots, e_n$ subject to the relations $f_i e_j - f_j e_i = 0$. There exists a short exact sequence*

$$0 \to K \to M \to I \to 0$$

such that K is annihilated by I.

Proof. This is just a truncation of the Koszul complex, see (insert future reference here). The map $M \to I$ is is determined by the rule $e_i \mapsto f_i$. If $m = \sum a_i e_i$ is in the kernel of $M \to I$, i.e., $\sum a_i f_i = 0$, then $f_j m = \sum f_j a_i e_i = (\sum f_i a_i) e_j = 0$. □

Lemma 12.8.9. *Let R be a ring. Let $I = (f_1, \ldots, f_n)$ be a finitely generated ideal of R. For any R-module N set*

$$H_1(N, f_\bullet) = \frac{\{(x_1, \ldots, x_n) \in N^{\oplus n} \mid f_i x_j = f_j x_i\}}{\{(f_1 x, \ldots, f_n x) \mid x \in N\}}$$

For any R-module N there exists a canonical short exact sequence

$$0 \to Ext_R(R/I, N) \to H_1(N, f_\bullet) \to Hom_R(K, N)$$

where K is as in Lemma 12.8.8.

Proof. The notation above indicates the Ext-groups in Mod_R as defined in Homology, Section 10.4. These are denoted $\text{Ext}_R(M, N)$. Using the long exact sequence of Homology, Lemma 10.4.4 associated to the short exact sequence $0 \to I \to R \to R/I \to 0$ and the fact that $\text{Ext}_R(R, N) = 0$ we see that

$$\text{Ext}_R(R/I, N) = \text{Coker}(N \longrightarrow Hom(I, N))$$

Using the short exact sequence of Lemma 12.8.8 we see that we get a complex

$$N \to Hom(M, N) \to Hom_R(K, N)$$

whose homology in the middle is canonically isomorphic to $\text{Ext}_R(R/I, N)$. The proof of the lemma is now complete as the cokernel of the first map is canonically isomorphic to $H_1(N, f_\bullet)$. □

Lemma 12.8.10. *Let R be a ring. Let $I = (f_1, \ldots, f_n)$ be a finitely generated ideal of R. For any R-module N the Koszul homology group $H_1(N, f_\bullet)$ defined in Lemma 12.8.9 is annihilated by I.*

Proof. Let $(x_1, \ldots, x_n) \in N^{\oplus n}$ with $f_i x_j = f_j x_i$. Then we have $f_i(x_1, \ldots, x_n) = (f_i x_i, \ldots, f_i x_n)$. In other words f_i annihilates $H_1(N, f_\bullet)$. □

We can improve on the full faithfulness of Lemma 12.8.7 by showing that Ext-groups whose source is I-power torsion are insensitive to passing to S as well. See Remark 12.8.12 below for a more highbrow version of the following lemma.

Lemma 12.8.11. *Assume $\varphi : R \to S$ is a flat ring map and $I \subset R$ is a finitely generated ideal such that $R/I \to S/IS$ is an isomorphism. Let M, N be R-modules. Assume M is I-power torsion. Given an short exact sequence*

$$0 \to N \otimes_R S \to \tilde{E} \to M \otimes_R S \to 0$$

there exists a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N \otimes_R S \arrow[r] & \tilde{E} \arrow[r] & M \otimes_R S \arrow[r] & 0
\end{tikzcd}$$

with exact rows.

Proof. As M is I-power torsion we see that $M \otimes_R S = M$, see Lemma 12.8.4. We will use this identification without further mention. As $R \to S$ is flat, the base change functor is exact and we obtain a functorial map of Ext-groups

$$\text{Ext}_R(M, N) \longrightarrow \text{Ext}_S(M \otimes_R S, N \otimes_R S),$$

see Homology, Lemma 10.5.2. The claim of the lemma is that this map is surjective when M is I-power torsion. In fact we will show that it is an isomorphism. By Lemma 12.8.3 we can find a surjection $M' \to M$ with M' a direct sum of modules of the form R/I^n. Using the long exact sequence of Homology, Lemma 10.4.4 we see that it suffices to prove the lemma for M'. Using compatibility of Ext with direct sums (details omitted) we reduce to the case where $M = R/I^n$ for some n.

Let $f_1, \ldots, f_t$ be generators for I^n. By Lemma 12.8.9 we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ext}_R(R/I^n, N) \arrow[r] \arrow[d] & H_1(N, f_\bullet) \arrow[r] \arrow[d] & Hom_R(K, N) \arrow[d] \\ 0 \arrow[r] & \mathrm{Ext}_S(S/I^nS, N \otimes S) \arrow[r] & H_1(N \otimes S, f_\bullet) \arrow[r] & Hom_S(K \otimes S, N \otimes S) \end{tikzcd}$$

with exact rows where K is as in Lemma 12.8.8. Hence it suffices to prove that the two right vertical arrows are isomorphisms. Since K is annihilated by I^n we see that $Hom_R(K, N) = Hom_S(K \otimes_R S, N \otimes_R S)$ by Lemma 12.8.7. As $R \to S$ is flat we have $H_1(N, f_\bullet) \otimes_R S = H_1(N \otimes_R S, f_\bullet)$. As $H_1(N, f_\bullet)$ is annihilated by I^n, see Lemma 12.8.10 we have $H_1(N, f_\bullet) \otimes_R S = H_1(N, f_\bullet)$ by Lemma 12.8.4. □

Remark 12.8.12. Assume $\varphi : R \to S$ is a flat ring map and $I \subset R$ is a finitely generated ideal such that $R/I \to S/IS$ is an isomorphism. Let M, N be R-modules and assume M is I-power torsion. Then the canonical map

$$\mathrm{Ext}^i_R(M, N) \longrightarrow \mathrm{Ext}^i_S(M \otimes_R S, N \otimes_R S)$$

is an isomorphism for all i. We sketch a proof of this strengthening of Lemma 12.8.11. Consider the Koszul complex $K_\bullet = K_\bullet(R, f_\bullet)$ which is the complex

$$0 \to \wedge^n R^n \to \wedge^{n-1} R^n \to \ldots \to \wedge^i R^n \to \ldots \to R^n \to R \to 0$$

where the last term R is placed in degree 0 with maps given by

$$e_{j_1} \wedge \ldots \wedge e_{j_i} \longmapsto \sum\nolimits_{a=1}^{i} (-1)^{i+1} f_{j_a} e_{j_1} \wedge \ldots \wedge \hat{e}_{j_a} \wedge \ldots \wedge e_{j_i}$$

Then $H_0(K_\bullet) = R/I$ and every homology module $H_i(K_\bullet)$ is annihilated by I. Having said this, we prove the statement on Ext-groups by induction on i. The case $i = 0$ is Lemma 12.8.7. Assume that the result holds for all $i \leq i_0$ and all modules N, M with M being I-power torsion. Pick a pair of modules N and M with M being I-power torsion and let's prove that the map $\mathrm{Ext}^{i_0+1}_R(M, N) \to \mathrm{Ext}^{i_0+1}_S(M \otimes_R S, N \otimes_R S)$ is an isomorphism. By Lemma 12.8.3 and the long exact sequence of Ext-groups and compatibility of Ext with direct sums we reduce to the case that $M = R/I^n$. Since I^n is finitely generated we can choose finitely many generators $f_1, \ldots, f_t \in I^n$ and consider the Koszul complex $K_\bullet = K_\bullet(R, f_\bullet)$. Note that $K_\bullet \otimes_R S = K_\bullet(S, f_\bullet)$. As $K_\bullet$ is a finite complex of finite free R-modules we see that the map

$$Hom_R(K_\bullet, N) \otimes_R S \longrightarrow Hom_S(K_\bullet \otimes_R S, N \otimes_R S) \tag{12.8.12.1}$$

is an isomorphism of complexes. As $R \to S$ is flat and using Lemmas 12.8.7 we see that

$$H_b(K_\bullet) = H_b(K_\bullet) \otimes_R S = H_b(K_\bullet \otimes_R S).$$

Below we will use the spectral sequences

$$E(R)_2^{a,b} = \mathrm{Ext}^a_R(H_b(K_\bullet), N) \Rightarrow H^{a+b}(Hom_R(K_\bullet, N)),$$
$$E(S)_2^{a,b} = \mathrm{Ext}^a_R(H_b(K_\bullet \otimes_R S), N \otimes_R S) \Rightarrow H^{a+b}(Hom_R(K_\bullet \otimes_R S, N \otimes_R S))$$

see (insert future reference here). The first one combined with the fact that each $H_b(K_\bullet)$ is annihilated by I^n implies that $H^c(Hom_R(K_\bullet, N))$ is annihilated by $I^{n(t+1)}$. Hence using Lemma 12.8.7 once more we see that

$$H^c(Hom_R(K_\bullet, N)) = H^c(Hom_R(K_\bullet, N)) \otimes_R S = H^c(Hom_S(K_\bullet \otimes_R S, N \otimes_R S))$$

because (12.8.12.1) is an isomorphism and $R \to S$ is flat. Combined we see that the map $E(R)_r^{a,b} \to E(S)_r^{a,b}$ of spectral sequences is an isomorphism for $r = 2$ and $a \leq i_0$ (induction hypothesis) and an isomorphism on abutments in all degrees. Then a formal argument on spectral sequences (insert future reference here) implies that $E(R)_2^{i_0+1,0} \to E(R)_2^{i_0+1,0}$ is an isomorphism as well, which is the result we wanted to prove. This ends the sketch of the proof of the result on Ext-groups; if we ever need to use this result in the stacks project we will put in a detailed proof.

Let $R \to S$ be a ring map. Let $f_1, \ldots, f_t \in R$ and $I = (f_1, \ldots, f_t)$. Then for any R-module M we can define a complex

$$0 \to M \xrightarrow{\alpha} M \otimes_R S \times \prod M_{f_i} \xrightarrow{\beta} \prod (M \otimes_R S)_{f_i} \times \prod M_{f_i f_j} \tag{12.8.12.2}$$

where $\alpha(m) = (m \otimes 1, m/1, \ldots, m/1)$ and

$$\beta(m', m_1, \ldots, m_t) = ((m'/1 - m_1 \otimes 1, \ldots, m'/1 - m_t \otimes 1), (m_1 - m_2, \ldots, m_{t-1} - m_t)).$$

We would like to know when this complex is exact.

Lemma 12.8.13. *Assume $\varphi : R \to S$ is a flat ring map and $I = (f_1, \ldots, f_t) \subset R$ is an ideal such that $R/I \to S/IS$ is an isomorphism. Let M be an R-module. Then the complex (12.8.12.2) is exact.*

Proof. Let $m \in M$. If $\alpha(m) = 0$, then $m \in M[I^\infty]$, see Lemma 12.8.5. Pick n such that $I^n m = 0$ and consider the map $\varphi : R/I^n \to M$. If $m \otimes 1 = 0$, then $\varphi \otimes 1_S = 0$, hence $\varphi = 0$ (see Lemma 12.8.7) hence $m = 0$. In this way we see that α is injective.

Let $(m', m'_1, \ldots, m'_t) \in \mathrm{Ker}(\beta)$. Write $m'_i = m_i/f_i^n$ for some $n > 0$ and $m_i \in M$. We may, after possibly enlarging n assume that $f_i^n m' = m_i \otimes 1$ in $M \otimes_R S$ and $f_j^n m_i - f_i^n m_j = 0$ in M. In particular we see that $(m_1, \ldots, m_t)$ defines an element ξ of $H_1(M, (f_1^n, \ldots, f_t^n))$. Since $H_1(M, (f_1^n, \ldots, f_t^n))$ is annihilated by I^{tn+1} (see Lemma 12.8.10) and since $R \to S$ is flat we see that

$$H_1(M, (f_1^n, \ldots, f_t^n)) = H_1(M, (f_1^n, \ldots, f_t^n)) \otimes_R S = H_1(M \otimes_R S, (f_1^n, \ldots, f_t^n))$$

by Lemma 12.8.4 The existence of m' implies that ξ maps to zero in the last group, i.e., the element ξ is zero. Thus there exists an $m \in M$ such that $m_i = f_i^n m$. Then $(m', m'_1, \ldots, m'_t) - \alpha(m) = (m'', 0, \ldots, 0)$ for some $m'' \in (M \otimes_R S)[(IS)^\infty]$. By Lemma 12.8.7 we conclude that $m'' \in M[I^\infty]$ and we win. □

Remark 12.8.14. In this remark we define a category of glueing data. Let $R \to S$ be a ring map. Let $f_1, \ldots, f_t \in R$ and $I = (f_1, \ldots, f_t)$. Consider the category $\mathrm{Glue}(R \to S, f_1, \ldots, f_t)$ as the category whose

1. objects are systems $(M', M_i, \alpha_i, \alpha_{ij})$, where M' is an S-module, M_i is an R_{f_i}-module, $\alpha_i : (M')_{f_i} \to M_i \otimes_R S$ is an isomorphism, and $\alpha_{ij} : (M_i)_{f_j} \to (M_j)_{f_i}$ are isomorphisms such that
   (a) $\alpha_{ij} \circ \alpha_i = \alpha_j$ as maps $(M')_{f_i f_j} \to (M_j)_{f_i}$, and
   (b) $\alpha_{jk} \circ \alpha_{ij} = \alpha_{ik}$ as maps $(M_i)_{f_j f_k} \to (M_k)_{f_i f_j}$ (cocycle condition).
2. morphisms $(M', M_i, \alpha_i, \alpha_{ij}) \to (N', N_i, \beta_i, \beta_{ij})$ are given by maps $\varphi' : M' \to N'$ and $\varphi_i : M_i \to N_i$ compatible with the given maps $\alpha_i, \beta_i, \alpha_{ij}, \beta_{ij}$.

There is a canonical functor

$$\mathrm{Can} : \mathrm{Mod}_R \longrightarrow \mathrm{Glue}(R \to S, f_1, \ldots, f_t), \quad M \longmapsto (M \otimes_R S, M_{f_i}, \mathrm{can}_i, \mathrm{can}_{ij})$$

where $\text{can}_i : (M \otimes_R S)_{f_i} \to M_{f_i} \otimes_R S$ and $\text{can}_{ij} : (M_{f_i})_{f_j} \to (M_{f_j})_{f_i}$ are the canonical isomorphisms. For any object $\mathbf{M} = (M', M_i, \alpha_i, \alpha_{ij})$ of the category $\text{Glue}(R \to S, f_1, \ldots, f_t)$ we define

$$H^0(\mathbf{M}) = \{(m', m_i) \mid \alpha_i(m') = m_i \otimes 1, \alpha_{ij}(m_i) = m_j\}$$

in other words defined by the exact sequence

$$0 \to H^0(\mathbf{M}) \to M' \times \prod M_i \to \prod M'_{f_i} \times \prod (M_i)_{f_j}$$

similar to (12.8.12.2). We think of $H^0(\mathbf{M})$ as an R-module. Thus we also get a functor

$$H^0 : \text{Glue}(R \to S, f_1, \ldots, f_t) \longrightarrow \text{Mod}_R$$

Our next goal is to show that the functors Can and H^0 are sometimes quasi-inverse to each other.

Lemma 12.8.15. *Assume $\varphi : R \to S$ is a flat ring map and $I = (f_1, \ldots, f_t) \subset R$ is an ideal such that $R/I \to S/IS$ is an isomorphism. Then the functor H^0 is a left quasi-inverse to the functor Can of Remark 12.8.14.*

Proof. This is a reformulation of Lemma 12.8.13. $\square$

Lemma 12.8.16. *Assume $\varphi : R \to S$ is a flat ring map and let $I = (f_1, \ldots, f_t) \subset R$ be an ideal. Then $Glue(R \to S, f_1, \ldots, f_t)$ is an abelian category, and the functor Can is exact and commutes with arbitrary colimits.*

Proof. Given a morphism $(\varphi', \varphi_i) : (M', M_i, \alpha_i, \alpha_{ij}) \to (N', N_i, \beta_i, \beta_{ij})$ of the category $\text{Glue}(R \to S, f_1, \ldots, f_t)$ we see that its kernel exists and is equal to the object $(\text{Ker}(\varphi'), \text{Ker}(\varphi_i), \alpha_i, \alpha_{ij})$ and its cokernel exists and is equal to the object $(\text{Coker}(\varphi'), \text{Coker}(\varphi_i), \beta_i, \beta_{ij})$. This works because $R \to S$ is flat, hence taking kernels/cokernels commutes with $- \otimes_R S$. Details omitted. The exactness follows from the R-flatness of R_{f_i} and S, while commuting with colimits follows as tensor products commute with colimits. $\square$

Lemma 12.8.17. *Let $\varphi : R \to S$ be a flat ring map and $(f_1, \ldots, f_t) = R$. Then Can and H^0 are quasi-inverse equivalences of categories*

$$Mod_R = Glue(R \to S, f_1, \ldots, f_t)$$

Proof. Consider an object $\mathbf{M} = (M', M_i, \alpha_i, \alpha_{ij})$ of $\text{Glue}(R \to S, f_1, \ldots, f_t)$. By Algebra, Lemma 7.21.4 there exists a unique module M and isomorphisms $M_{f_i} \to M_i$ which recover the glueing data α_{ij}. Then both M' and $M \otimes_R S$ are S-modules which recover the modules $M_i \otimes_R S$ upon localizing at f_i. Whence there is a canonical isomorphism $M \otimes_R S \to M'$. This shows that $\mathbf{M}$ is in the essential image of Can. Combined with Lemma 12.8.15 the lemma follows. $\square$

Lemma 12.8.18. *Let $\varphi : R \to S$ be a flat ring map and $I = (f_1, \ldots, f_t)$ and ideal. Let $R \to R'$ be a flat ring map, and set $S' = S \otimes_R R'$. Then we obtain a commutative diagram of categories and functors*

$$\begin{tikzcd}
Mod_R \arrow[r, "Can"'] \arrow[d, "-\otimes_R R'"'] & Glue(R \to S, f_1, \ldots, f_t) \arrow[r, "H^0"'] \arrow[d, "-\otimes_R R'"] & Mod_R \arrow[d, "-\otimes_R R'"] \\
Mod_{R'} \arrow[r, "Can"] & Glue(R' \to S', f_1, \ldots, f_t) \arrow[r, "H^0"] & Mod_{R'}
\end{tikzcd}$$

Proof. Omitted. $\square$

Proposition 12.8.19. *Assume $\varphi : R \to S$ is a flat ring map and $I = (f_1, \ldots, f_t) \subset R$ is an ideal such that $R/I \to S/IS$ is an isomorphism. Then Can and H^0 are quasi-inverse equivalences of categories*

$$Mod_R = Glue(R \to S, f_1, \ldots, f_t)$$

Proof. We have already seen that $H^0 \circ \mathrm{Can}$ is isomorphic to the identity functor, see Lemma 12.8.15. Consider an object $\mathbf{M} = (M', M_i, \alpha_i, \alpha_{ij})$ of $\mathrm{Glue}(R \to S, f_1, \ldots, f_t)$. We get a natural morphism

$$\Psi : (H^0(\mathbf{M}) \otimes_R S, H^0(\mathbf{M})_{f_i}, \mathrm{can}_i, \mathrm{can}_{ij}) \longrightarrow (M', M_i, \alpha_i, \alpha_{ij}).$$

Namely, by definition $H^0(\mathbf{M})$ comes equipped with compatible R-module maps $H^0(\mathbf{M}) \to M'$ and $H^0(\mathbf{M}) \to M_i$. We have to show that this map is an isomorphism.

Pick an index i and set $R' = R_{f_i}$. Combining Lemmas 12.8.18 and 12.8.17 we see that $\Psi \otimes_R R'$ is an isomorphism. Hence the kernel, resp. cokernel of Ψ is a system of the form $(K, 0, 0, 0)$, resp. $(Q, 0, 0, 0)$. Note that $H^0((K, 0, 0, 0)) = K$, that H^0 is left exact, and that by construction $H^0(\Psi)$ is bijective. Hence we see $K = 0$, i.e., the kernel of Ψ is zero.

The conclusion of the above is that we obtain a short exact sequence

$$0 \to H^0(\mathbf{M}) \otimes_R S \to M' \to Q \to 0$$

and that $M_i = H^0(\mathbf{M})_{f_i}$. Note that we may think of Q as an R-module which is I-power torsion so that $Q = Q \otimes_R S$. By Lemma 12.8.11 we see that there exists a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathbf{M}) \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^0(\mathbf{M}) \otimes_R S \arrow[r] & M' \arrow[r] & Q \arrow[r] & 0
\end{tikzcd}$$

with exact rows. This clearly determines an isomorphism $\mathrm{Can}(E) \to (M', M_i, \alpha_i, \alpha_{ij})$ in the category $\mathrm{Glue}(R \to S, f_1, \ldots, f_t)$ and we win. (Of course, a posteriori we have $Q = 0$.) □

Next, we specialize this very general proposition to get something more useable. Namely, if $I = (f)$ is a principal ideal then the objects of $\mathrm{Glue}(R \to S, f)$ are simply triples (M', M_1, α_1) and there is *no* cocycle condition to check!

Theorem 12.8.20. *Let R be a ring, and let $f \in R$. Let $\varphi : R \to S$ be a flat ring map inducing an isomorphism $R/fR \to S/fS$. Then the functor*

$$Mod_R \longrightarrow Mod_S \times_{Mod_{S_f}} Mod_{R_f}, \quad M \longmapsto (M \otimes_R S, M_f, can)$$

is an equivalence.

Proof. The category appearing on the right side of the arrow is the category of triples (M', M_1, α_1) where M' is an S-module, M_1 is a R_f-module, and $\alpha_1 : M'_f \to M_1 \otimes_R S$ is a S_f-isomorphism, see Categories, Example 4.28.3. Hence this theorem is a special case of Proposition 12.8.19. □

A useful special case of Theorem 12.8.20 is when R is noetherian, and S is a completion of R at an element f. The completion $R \to S$ is flat, and the functor $M \mapsto M \otimes_R S$ can be identified with the f-adic completion functor when M is finitely generated. To state this more precisely, let $\mathrm{Mod}_{fg}(R)$ denote the category of finitely generated R-modules.

Proposition 12.8.21. *Let R be a noetherian ring. Let $f \in R$ be an element. Let $R^\wedge$ be the f-adic completion of R. Then the functor $M \mapsto (M^\wedge, M_f, can)$ defines an equivalence*

$$Mod_{fg}(R) \longrightarrow Mod_{fg}(R^\wedge) \times_{Mod_{fg}(R^\wedge_f)} Mod_{fg}(R_f)$$

Proof. The ring map $R \to R^\wedge$ is flat by Algebra, Lemma 7.90.3. It is clear that $R/fR = R^\wedge/fR^\wedge$. By Algebra, Lemma 7.90.2 the completion of a finite R-module M is equal to $M \otimes_R R^\wedge$. Hence the displayed functor of the proposition is equal to the functor occuring in Theorem 12.8.20. In particular it is fully faithful. Let (M_1, M_2, ψ) be an object of the right hand side. By Theorem 12.8.20 there exists an R-module M such that $M_1 = M \otimes_R R^\wedge$ and $M_2 = M_f$. As $R \to R^\wedge \times R_f$ is faithfully flat we conclude from Algebra, Lemma 7.21.2 that M is finitely generated, i.e., $M \in \mathrm{Mod}_{fg}(R)$. This proves the proposition. □

Remark 12.8.22. The equivalences of Proposition 12.8.19, Theorem 12.8.20, and Proposition 12.8.21 preserve the $\otimes$-structures on either side. Thus, it defines equivalences of various categories built out of the pair $(\mathrm{Mod}_R, \otimes)$, such as the category of R-algebras.

Remark 12.8.23. Given a differential manifold X with a compact closed submanifold Z having complement U, specifying a sheaf on X is the same as specifying a sheaf on U, a sheaf on an unspecified tubular neighbourhood T of Z in X, and an isomorphism between the two resulting sheaves along $T \cap U$. Tubular neighbourhoods do not exist in algebraic geometry as such, but results such as Proposition 12.8.19, Theorem 12.8.20, and Proposition 12.8.21 allow us to work with formal neighbourhoods instead.

12.9. Lifting

In this section we collection some lemmas concerning lifting statements of the following kind: If A is a ring and $I \subset A$ is an ideal, and $\overline{\xi}$ is some kind of structure over A/I, then we can lift $\overline{\xi}$ to a similar kind of structure ξ over A or over some étale extension of A. Here are some types of structure for which we have already proved some results:

(1) idempotents, see Algebra, Lemmas 7.49.5 and 7.49.6,
(2) projective modules, see Algebra, Lemma 7.71.4,
(3) basis elements, see Algebra, Lemmas 7.93.1 and 7.93.3,
(4) ring maps, i.e., proving certain algebras are formally smooth, see Algebra, Lemma 7.127.4, Proposition 7.127.13, and Lemma 7.127.16,
(5) syntomic ring maps, see Algebra, Lemma 7.125.19,
(6) smooth ring maps, see Algebra, Lemma 7.126.19,
(7) étale ring maps, see Algebra, Lemma 7.132.10,
(8) factoring polynomials, see Algebra, Lemma 7.132.19, and
(9) Algebra, Section 7.139 discusses henselian local rings.

The interested reader will find more results of this nature in Smoothing Ring Maps, Section 13.4 in particular Smoothing Ring Maps, Proposition 13.4.2.

Let A be a ring and let $I \subset A$ be an ideal. Let $\overline{\xi}$ be some kind of structure over A/I. In the following lemmas we look for étale ring maps $A \to A'$ which induce isomorphisms $A/I \to A'/IA'$ and objects ξ' over A' lifting $\overline{\xi}$. A general remark is that given étale ring maps $A \to A' \to A''$ such that $A/I \cong A'/IA'$ and $A'/IA' \cong A''/IA''$ the composition $A \to A''$ is also étale (Algebra, Lemma 7.132.3) and also satisfies $A/I \cong A''/IA''$. We will frequently use this in the following lemmas without further mention. Here is a trivial example of the type of result we are looking for.

Lemma 12.9.1. *Let A be a ring, let $I \subset A$ be an ideal, let $\overline{u} \in A/I$ be an invertible element. There exists an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and an invertible element $u' \in A'$ lifting $\overline{u}$.*

Proof. Choose any lift $f \in A$ of $\overline{u}$ and set $A' = A_f$ and u the image of f in A'. □

Lemma 12.9.2. *Let A be a ring, let $I \subset A$ be an ideal, let $\overline{e} \in A/I$ be an idempotent. There exists an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and an idempotent $e' \in A'$ lifting $\overline{e}$.*

Proof. Choose any lift $x \in A$ of $\overline{e}$. Set

$$A' = A[t]/(t^2 - t)\left[\frac{1}{t - 1 + x}\right].$$

The ring map $A \to A'$ is étale because $(2t - 1)\mathrm{d}t = 0$ and $(2t - 1)(2t - 1) = 1$ which is invertible. We have $A'/IA' = A/I[t]/(t^2 - t)[\frac{1}{t-1+\overline{e}}] \cong A/I$ the last map sending t to $\overline{e}$ which works as $\overline{e}$ is a root of $t^2 - t$. This also shows that setting e' equal to the class of t in A' works. □

Lemma 12.9.3. *Let A be a ring, let $I \subset A$ be an ideal. Let $Spec(A/I) = \coprod_{j \in J} \overline{U}_j$ be a finite disjoint open covering. Then there exists an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and a finite disjoint open covering $Spec(A') = \coprod_{j \in J} U'_j$ lifting the given covering.*

Proof. This follows from Lemma 12.9.2 and the fact that open and closed subsets of Spectra correspond to idempotents, see Algebra, Lemma 7.18.3. □

Lemma 12.9.4. *Let $A \to B$ be a ring map and $J \subset B$ an ideal. If $A \to B$ is étale at every prime of $V(J)$, then there exists a $g \in B$ mapping to an invertible element of B/J such that $A' = B_g$ is étale over A.*

Proof. The set of points of $Spec(B)$ where $A \to B$ is not étale is a closed subset of $Spec(B)$, see Algebra, Definition 7.132.1. Write this as $V(J')$ for some ideal $J' \subset B$. Then $V(J') \cap V(J) = \emptyset$ hence $J + J' = B$ by Algebra, Lemma 7.16.2. Write $1 = f + g$ with $f \in J$ and $g \in J'$. Then g works. □

The assumption on the leading coefficient in the following lemma will be removed in Lemma 12.9.6.

Lemma 12.9.5. *Let A be a ring, let $I \subset A$ be an ideal. Let $f \in A[x]$ be a monic polynomial. Let $\overline{f} = \overline{g}\overline{h}$ be a factorization of f in $A/I[x]$ and assume*

1. *the leading coefficient of $\overline{g}$ is an invertible element of A/I, and*
2. *$\overline{g}$, $\overline{h}$ generate the unit ideal in $A/I[x]$.*

Then there exists an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and a factorization $f = g'h'$ in $A'[x]$ lifting the given factorization over A/I.

Proof. Applying Lemma 12.9.1 we may assume that the leading coefficient of $\overline{g}$ is the reduction of an invertible element $u \in A$. Then we may replace $\overline{g}$ by $\overline{u}^{-1}\overline{g}$ and $\overline{h}$ by $\overline{u}\overline{h}$. Thus we may assume that $\overline{g}$ is monic. Since f is monic we conclude that $\overline{h}$ is monic too. Say $\deg(\overline{g}) = n$ and $\deg(\overline{h}) = m$ so that $\deg(f) = n + m$. Write $f = x^{n+m} + \sum \alpha_i x^{n+m-i}$ for some $\alpha_1, \ldots, \alpha_{n+m} \in A$. Consider the ring map

$$R = \mathbf{Z}[a_1, \ldots, a_{n+m}] \longrightarrow S = \mathbf{Z}[b_1, \ldots, b_n, c_1, \ldots, c_m]$$

of Algebra, Example 7.132.12. Let $R \to A$ be the ring map which sends a_i to α_i. Set

$$B = A \otimes_R S$$

By construction the image of f in $B[x]$ factors. Write $\overline{g} = x^n + \sum \overline{\beta}_i x^{n-i}$ and $\overline{h} = x^m + \sum \overline{\gamma}_i x^{m-i}$. The A-algebra map

$$B \longrightarrow A/I, \quad 1 \otimes b_i \mapsto \overline{\beta}_i, \quad 1 \otimes c_i \mapsto \overline{\gamma}_i$$

maps the factorization of f over B to the given factorization over A/I. The displayed map is surjective; denote $J \subset B$ its kernel. From the discussion in Algebra, Example 7.132.12 it is clear that $A \to B$ is etale at all points of $V(J) \subset Spec(B)$. Choose $g \in B$ as in Lemma 12.9.4 and set $A' = B_g$. □

Lemma 12.9.6. *Let A be a ring, let $I \subset A$ be an ideal. Let $f \in A[x]$ be a monic polynomial. Let $\overline{f} = \overline{g}\overline{h}$ be a factorization of f in $A/I[x]$ and assume that $\overline{g}$, $\overline{h}$ generate the unit ideal in $A/I[x]$. Then there exists an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and a factorization $f = g'h'$ in $A'[x]$ lifting the given factorization over A/I.*

Proof. Say $f = x^d + a_1 x^{d-1} + \ldots + a_d$ has degree d. Write $\overline{g} = \sum \overline{b}_j x^j$ and $\overline{h} = \sum \overline{c}_j x^j$. Then we see that $1 = \sum \overline{b}_j \overline{c}_{d-j}$. It follows that $Spec(A/I)$ is covered by the standard opens $D(\overline{b}_j \overline{c}_{d-j})$. However, each point $\mathfrak{p}$ of $Spec(A/I)$ is contained in at most one of these as by looking at the induced factorization of f over the field $\kappa(\mathfrak{p})$ we see that $\deg(\overline{g} \bmod \mathfrak{p}) + \deg(\overline{h} \bmod \mathfrak{p}) = d$. Hence our open covering is a disjoint open covering. Applying Lemma 12.9.3 (and replacing A by A') we see that we may assume there is a corresponding disjoint open covering of $Spec(A)$. This disjoint open covering corresponds to a product decomposition of A, see Algebra, Lemma 7.20.3. It follows that

$$A = A_0 \times \ldots \times A_d, \quad I = I_0 \times \ldots \times I_d,$$

where the image of $\overline{g}$, resp. $\overline{h}$ in A_j/I_j has degree j, resp. $d - j$ with invertible leading coefficient. Clearly, it suffices to prove the result for each factor A_j separatedly. Hence the lemma follows from Lemma 12.9.5. □

Lemma 12.9.7. *Let $R \to S$ be a ring map. Let $I \subset R$ be an ideal of R and let $J \subset S$ be an ideal of S. If the closure of the image of $V(J)$ in $Spec(R)$ is disjoint from $V(I)$, then there exists an element $f \in R$ which maps to 1 in R/I and to an element of J in S.*

Proof. Let $I' \subset R$ be an ideal such that $V(I')$ is the closure of the image of $V(J)$. Then $V(I) \cap V(I') = \emptyset$ by assumption and hence $I + I' = R$ by Algebra, Lemma 7.16.2. Write $1 = g + f$ with $g \in I$ and $f \in I'$. We have $V(f') \supset V(J)$ where f' is the image of f in S. Hence $(f')^n \in J$ for some n, see Algebra, Lemma 7.16.2. Replacing f by f^n we win. □

Lemma 12.9.8. *Let A be a ring, let $I \subset A$ be an ideal. Let $A \to B$ be an integral ring map. Let $\overline{e} \in B/IB$ be an idempotent. Then there exists an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and an idempotent $e' \in B \otimes_A A'$ lifting $\overline{e}$.*

Proof. Choose an element $y \in B$ lifting $\overline{e}$. Then $z = y^2 - y$ is an element of IB. By Algebra, Lemma 7.34.4 there exist a monic polynomial $g(x) = x^d + \sum a_j x^j$ of degree d with $a_j \in I$ such that $g(z) = 0$ in B. Hence $f(x) = g(x^2 - x) \in A[x]$ is a monic polynomial such that $f(x) \equiv x^d (x-1)^d \bmod I$ and such that $f(y) = 0$ in B. By Lemma 12.9.5 we can find an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and such that $f = gh$ in $A[x]$ with $g(x) = x^d \bmod IA'$ and $h(x) = (x-1)^d \bmod IA'$. After replacing A by A' we may assume that the factorization is defined over A. In that case we see that $b_1 = g(y) \in B$ is a lift of $\overline{e}^d = \overline{e}$ and $b_2 = h(y) \in B$ is a lift of $(\overline{e}-1)^d = (-1)^d(1-\overline{e})^d = (-1)^d(1-\overline{e})$ and moreover $b_1 b_2 = 0$. Thus $(b_1, b_2)B/IB = B/IB$ and $V(b_1, b_2) \subset Spec(B)$ is disjoint from $V(IB)$. Since $Spec(B) \to Spec(A)$ is closed (see Algebra, Lemmas 7.32.20 and 7.36.6) we can find an $a \in A$ which maps to an invertible element of A/I whose image in B lies in (b_1, b_2), see Lemma 12.9.7. After replacing A by the localization A_a we get that $(b_1, b_2) = B$. Then $Spec(B) = D(b_1) \amalg D(b_2)$; disjoint union because $b_1 b_2 = 0$. Let $e \in B$ be the idempotent corresponding to the open and closed subset $D(b_1)$, see Algebra, Lemma 7.18.3. Since b_1 is a lift of $\overline{e}$ and b_2 is a lift of $\pm(1-\overline{e})$ we conclude that e is a lift of $\overline{e}$ by the uniqueness statement in Algebra, Lemma 7.18.3. □

Lemma 12.9.9. *Let A be a ring, let $I \subset A$ be an ideal. Let $\overline{P}$ be finite projective A/I-module. Then there exists an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and a finite projective A'-module P' lifting $\overline{P}$.*

Proof. We can choose an integer n and a direct sum decomposition $(A/I)^{\oplus n} = \overline{P} \oplus \overline{K}$ for some R/I-module $\overline{K}$. Choose a lift $\varphi : A^{\oplus n} \to A^{\oplus n}$ of the projector $\overline{p}$ associated to the direct summand $\overline{P}$. Let $f \in A[x]$ be the characteristic polynomial of φ. Set $B = A[x]/(f)$. By Cayley-Hamilton (Algebra, Lemma 7.15.1) there is a map $B \to \text{End}_A(A^{\oplus n})$ mapping x to φ. For every prime $\mathfrak{p} \supset I$ the image of f in $\kappa(\mathfrak{p})$ is $(x-1)^r x^{n-r}$ where r is the dimension of $\overline{P} \otimes_{A/I} \kappa(\mathfrak{p})$. Hence $(x-1)^n x^n$ maps to zero in $B \otimes_A \kappa(\mathfrak{p})$ for all $\mathfrak{p} \supset I$. Hence the image of $(x-1)^n x^n$ in B is contained in

$$\bigcup\nolimits_{\mathfrak{p} \supset I} \mathfrak{p}B = (\bigcup\nolimits_{\mathfrak{p} \supset I} \mathfrak{p})B = \sqrt{I}B$$

the first equality because B is a free A-module and the second by Algebra, Lemma 7.16.2. Thus $(x-1)^N x^N$ is contained in IB for some N. It follows that $x^N + (1-x)^N$ is a unit in B/IB and that

$$\overline{e} = \text{image of } \frac{x^N}{x^N + (1-x)^N} \text{ in } B/IB$$

is an idempotent as both assertions hold in $\mathbf{Z}[x]/(x^n(x-1)^N)$. The image of $\overline{e}$ in $\text{End}_{A/I}((A/I)^{\oplus n})$ is

$$\frac{\overline{p}^N}{\overline{p}^N + (1-\overline{p})^N} = \overline{p}$$

as $\overline{p}$ is an idempotent. After replacing A by an étale extension A' as in the lemma, we may assume there exists an idempotent $e \in B$ which maps to $\overline{e}$ in B/IB, see Lemma 12.9.8. Then the image of e under the map

$$B = A[x]/(f) \longrightarrow \text{End}_A(A^{\oplus n}).$$

is an idempotent element p which lifts $\overline{p}$. Setting $P = \text{Im}(p)$ we win. □

Lemma 12.9.10. *Let A be a ring. Let $0 \to K \to A^{\oplus m} \to M \to 0$ be a sequence of A-modules. Consider the A-algebra $C = Sym_A^*(M)$ with its presentation $\alpha : A[y_1, \ldots, y_m] \to C$ coming from the surjection $A^{\oplus m} \to M$. Then*

$$NL(\alpha) = (K \otimes_A C \to \bigoplus\nolimits_{j=1,\ldots,m} C dy_j)$$

(see Algebra, Section 7.123) in particular $\Omega_{C/A} = M \otimes_A C$.

Proof. Let $J = \mathrm{Ker}(\alpha)$. The lemma asserts that $J/J^2 \cong K \otimes_A C$. Note that α is a homomorphism of graded algebras. We will prove that in degree d we have $(J/J^2)_d = K \otimes_A C_{d-1}$. Note that

$$J_d = \mathrm{Ker}(\mathrm{Sym}^d_A(A^{\oplus m}) \to \mathrm{Sym}^d_A(M)) = \mathrm{Im}(K \otimes_A \mathrm{Sym}^{d-1}_A(A^{\oplus m}) \to \mathrm{Sym}^d_A(A^{\oplus m})),$$

see Algebra, Lemma 7.12.2. It follows that $(J^2)_d = \sum_{a+b=d} J_a \cdot J_b$ is the image of

$$K \otimes_A K \otimes_A \mathrm{Sym}^{d-2}_A(A^{\otimes m}) \to \mathrm{Sym}^d_A(A^{\oplus m}).$$

The cokernel of the map $K \otimes_A \mathrm{Sym}^{d-2}_A(A^{\otimes m}) \to \mathrm{Sym}^{d-1}_A(A^{\oplus m})$ is $\mathrm{Sym}^{d-1}_A(M)$ by the lemma referenced above. Hence it is clear that $(J/J^2)_d = J_d/(J^2)_d$ is equal to

$$\begin{aligned}\mathrm{Coker}(K \otimes_A K \otimes_A \mathrm{Sym}^{d-2}_A(A^{\otimes m}) \to K \otimes_A \mathrm{Sym}^{d-1}_A(A^{\otimes m})) &= K \otimes_A \mathrm{Sym}^{d-1}_A(M) \\ &= K \otimes_A C_{d-1}\end{aligned}$$

as desired. □

Lemma 12.9.11. *Let A be a ring. Let M be an A-module. Then $C = Sym^*_A(M)$ is smooth over A if and only if M is a finite projective A-module.*

Proof. Let $\sigma : C \to A$ be the projection onto the degree 0 part of C. Then $J = \mathrm{Ker}(\sigma)$ is the part of degree > 0 and we see that $J/J^2 = M$ as an A-module. Hence if $A \to C$ is smooth then M is a finite projective A-module by Algebra, Lemma 7.128.4.

Conversely, assume that M is finite projective and choose a surjection $A^{\oplus n} \to M$ with kernel K. Of course the sequence $0 \to K \to A^{\oplus n} \to M \to 0$ is split as M is projective. In particular we see that K is a finite A-module and hence C is of finite presentation over A as C is a quotient of $A[x_1, \ldots, x_n]$ by the ideal generated by $K \subset \bigoplus Ax_i$. The computation of Lemma 12.9.10 shows that $NL_{C/A}$ is homotopy equivalent to $(K \to M) \otimes_A C$. Hence $NL_{C/A}$ is quasi-isomorphic to $C \otimes_A M$ placed in degree 0 which means that C is smooth over A by Algebra, Definition 7.126.1. □

Lemma 12.9.12. *Let A be a ring, let $I \subset A$ be an ideal. Consider a commutative diagram*

$$\begin{tikzcd} B \arrow[rd] & \\ A \arrow[u] \arrow[r] & A/I \end{tikzcd}$$

where B is a smooth A-algebra. Then there exists an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and an A-algebra map $B \to A'$ lifting the ring map $B \to A/I$.

Proof. Let $J \subset B$ be the kernel of $B \to A/I$ so that $B/J = A/I$. By Algebra, Lemma 7.128.3 the sequence

$$0 \to I/I^2 \to J/J^2 \to \Omega_{B/A} \otimes_B B/J \to 0$$

is split exact. Thus $\overline{P} = J/(J^2 + IB) = \Omega_{B/A} \otimes_B B/J$ is a finite projective A/I-module. Choose an integer n and a direct sum decomposition $A/I^{\oplus n} = \overline{P} \oplus \overline{K}$. By Lemma 12.9.9 we can find an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and a finite projective A-module K which lifts $\overline{K}$. We may and do replace A by A'. Set $B' = B \otimes_A \mathrm{Sym}^*_A(K)$. Since $A \to \mathrm{Sym}^*_A(K)$ is smooth by Lemma 12.9.11 we see that $B \to B'$ is smooth which in turn implies that $A \to B'$ is smooth (see Algebra, Lemmas 7.126.4 and 7.126.13). Moreover the section $\mathrm{Sym}^*_A(K) \to A$ determines a section $B' \to B$

and we let $B' \to A/I$ be the composition $B' \to B \to A/I$. Let $J' \subset B'$ be the kernel of $B' \to A/I$. We have $JB' \subset J'$ and $B \otimes_A K \subset J'$. These maps combine to give an isomorphism

$$(A/I)^{\oplus n} \cong J/J^2 \oplus \overline{K} \longrightarrow J'/((J')^2 + IB')$$

Thus, after replacing B by B' we may assume that $J/(J^2 + IB) = \Omega_{B/A} \otimes_B B/J$ is a free A/I-module of rank n.

In this case, choose $f_1, \ldots, f_n \in J$ which map to a basis of $J/(J^2 + IB)$. Consider the finitely presented A-algebra $C = B/(f_1, \ldots, f_n)$. Note that we have an exact sequence

$$0 \to H_1(L_{C/A}) \to (f_1, \ldots, f_n)/(f_1, \ldots, f_n)^2 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to 0$$

see Algebra, Lemma 7.123.3 (note that $H_1(L_{B/A}) = 0$ and that $\Omega_{B/A}$ is finite projective, in particular flat so the Tor group vanishes). For any prime $\mathfrak{q} \supset J$ of B the module $\Omega_{B/A,\mathfrak{q}}$ is free of rank n because $\Omega_{B/A}$ is finite projective and because $\Omega_{B/A} \otimes_B B/J$ is free of rank n. By our choice of $f_1, \ldots, f_n$ the map

$$\big((f_1, \ldots, f_n)/(f_1, \ldots, f_n)^2\big)_{\mathfrak{q}} \to \Omega_{B/A,\mathfrak{q}}$$

is surjective modulo I. Hence we see that this map of modules over the local ring $C_{\mathfrak{q}}$ has to be an isomorphism. Thus $H_1(L_{C/A})_{\mathfrak{q}} = 0$ and $\Omega_{C/A,\mathfrak{q}} = 0$. By Algebra, Lemma 7.126.12 we see that $A \to C$ is smooth at the prime $\overline{\mathfrak{q}}$ of C corresponding to $\mathfrak{q}$. Since $\Omega_{C/A,\mathfrak{q}} = 0$ it is actually étale at $\overline{\mathfrak{q}}$. Thus $A \to C$ is étale at all primes of C containing JC. By Lemma 12.9.4 we can find an $f \in C$ mapping to an invertible element of C/JC such that $A \to C_f$ is étale. By our choice of f it is still true that $C_f/JC_f = A/I$. The map $C_f/IC_f \to A/I$ is surjective and étale by Algebra, Lemma 7.132.8. Hence A/I is isomorphic to the localization of C_f/IC_f at some element $g \in C$, see Algebra, Lemma 7.132.9. Set $A' = C_{fg}$ to conclude the proof. $\square$

12.10. Auto-associated rings

Some of this material is in [**Laz69**].

Definition 12.10.1. A ring R is said to be *auto-associated* if R is local and its maximal ideal $\mathfrak{m}$ is weakly associated to R.

Lemma 12.10.2. *An auto-associated ring R has the following property: (P) Every proper finitely generated ideal $I \subset R$ has a nonzero annihilator.*

Proof. By assumption there exists a nonzero element $x \in R$ such that for every $f \in \mathfrak{m}$ we have $f^n x = 0$. Say $I = (f_1, \ldots, f_r)$. Then x is in the kernel of $R \to \bigoplus R_{f_i}$. Hence we see that there exists a nonzero $y \in R$ such that $f_i y = 0$ for all i, see Algebra, Lemma 7.20.4. As $y \in \text{Ann}_R(I)$ we win. $\square$

Lemma 12.10.3. *Let R be a ring having property (P) of Lemma 12.10.2. Let $u : N \to M$ be a homomorphism of projective R-modules. Then u is universally injective if and only if u is injective.*

Proof. Assume u is injective. Our goal is to show u is universally injective. First we choose a module Q such that $N \oplus Q$ is free. On considering the map $N \oplus Q \to M \oplus Q$ we see that it suffices to prove the lemma in case N is free. In this case N is a directed colimit of finite free R-modules. Thus we reduce to the case that N is a finite free R-module, say $N = R^{\oplus n}$. We prove the lemma by induction on n. The case $n = 0$ is trivial.

Let $u : R^{\oplus n} \to M$ be an injective module map with M projective. Choose an R-module Q such that $M \oplus Q$ is free. After replacing u by the composition $R^{\oplus n} \to M \to M \oplus Q$ we see that we may assume that M is free. Then we can find a direct summand $R^{\oplus m} \subset M$ such that $u(R^{\oplus n}) \subset R^{\oplus m}$. Hence we may assume that $M = R^{\oplus m}$. In this case u is given by a matrix $A = (a_{ij})$ so that $u(x_1, \ldots, x_n) = (\sum x_i a_{i1}, \ldots, \sum x_i a_{im})$. As u is injective, in particular $u(x, 0, \ldots, 0) = (xa_{11}, xa_{12}, \ldots, xa_{1m}) \neq 0$ if $x \neq 0$, and as R has property (P) we see that $a_{11}R + a_{12}R + \ldots + a_{1m}R = R$. Hence see that $R(a_{11}, \ldots, a_{1m}) \subset R^{\oplus m}$ is a direct summand of $R^{\oplus m}$, in particular $R^{\oplus m}/R(a_{11}, \ldots, a_{1m})$ is a projective R-module. We get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r] \arrow[d, "1"] & R^{\oplus n} \arrow[r] \arrow[d, "u"] & R^{\oplus n-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R \arrow[r, "(a_{11}, \ldots, a_{1m})"] & R^{\oplus m} \arrow[r] & R^{\oplus m}/R(a_{11}, \ldots, a_{1m}) \arrow[r] & 0
\end{tikzcd}$$

with split exact rows. Thus the right vertical arrow is injective and we may apply the induction hypothesis to conclude that the right verical arrow is universally injective. It follows that the middle vertical arrow is universally injective. □

Lemma 12.10.4. *Let R be a ring. The following are equivalent*

1. *R has property (P) of Lemma 12.10.2,*
2. *any injective map of projective R-modules is universally injective,*
3. *if $u : N \to M$ is injective and N, M are finite projective R-modules then Coker(u) is a finite projective R-module,*
4. *if $N \subset M$ and N, M are finite projective as R-modules, then N is a direct summand of M, and*
5. *any injective map $R \to R^{\oplus n}$ is a split injection.*

Proof. The implication (1) ⇒ (2) is Lemma 12.10.3. It is clear that (3) and (4) are equivalent. We have (2) ⇒ (3), (4) by Algebra, Lemma 7.76.4. Part (5) is a special case of (4). Assume (5). Let $I = (a_1, \ldots, a_n)$ be a proper finitely generated ideal of R. As $I \neq R$ we see that $R \to R^{\oplus n}$, $x \mapsto (xa_1, \ldots, xa_n)$ is not a split injection. Hence it has a nonzero kernel and we conclude that $\text{Ann}_R(I) \neq 0$. Thus (1) holds. □

Example 12.10.5. If the equivalent conditions of Lemma 12.10.4 hold, then it is not always the case that every injective map of free R-modules is a split injection. For example suppose that $R = k[x_1, x_2, x_3, \ldots]/(x_i^2)$. This is an auto-associated ring. Consider the map of free R-modules

$$u : \bigoplus\nolimits_{i \geq 1} Re_i \longrightarrow \bigoplus\nolimits_{i \geq 1} Rf_i, \quad e_i \longmapsto f_i - x_i f_{i+1}.$$

For any integer n the restriction of u to $\bigoplus_{i=1,\ldots,n} Re_i$ is injective as the images $u(e_1), \ldots, u(e_n)$ are R-linearly independent. Hence u is injective and hence universally injective by the lemma. Since $u \otimes \text{id}_k$ is bijective we see that if u were a split injection then u would be surjective. But u is not surjective because the inverse image of f_1 would be the element

$$\sum\nolimits_{i \geq 0} x_1 \ldots x_i e_{i+1} = e_1 + x_1 e_2 + x_1 x_2 e_3 + \ldots$$

which is not an element of the direct sum. A side remark is that Coker(u) is a flat (because u is universally injective), countably generated R-module which is not projective (as u is not split), hence not Mittag-Leffler (see Algebra, Lemma 7.87.1).

12.11. Flattening stratification

Let $R \to S$ be a ring map and let N be an S-module. For any R-algebra R' we can consider the base changes $S' = S \otimes_R R'$ and $M' = M \otimes_R R'$. We say $R \to R'$ *flattens* M if the module M' is flat over R'. We would like to understand the structure of the collection of ring maps $R \to R'$ which flatten M. In particular we would like to know if there exists a *universal flattening* $R \to R_{univ}$ *of* M, i.e., a ring map $R \to R_{univ}$ which flattens M and has the property that any ring map $R \to R'$ which flattens M factors through $R \to R_{univ}$. It turns out that such a universal solution usually does not exist.

We will discuss *universal flattenings* and *flattening stratifications* in a scheme theoretic setting $\mathcal{F}/X/S$ in More on Flatness, Section 34.21. If the universal flattening $R \to R_{univ}$ exists then the morphism of schemes $Spec(R_{univ}) \to Spec(R)$ is the universal flattening of the quasi-coherent module $\widetilde{M}$ on $Spec(S)$.

In this and the next few sections we prove some basic algebra facts related to this. The most basic result is perhaps the following.

Lemma 12.11.1. *Let R be a ring. Let M be an R-module. Let I_1, I_2 be ideals of R. If M/I_1M is flat over R/I_1 and M/I_2M is flat over R/I_2, then $M/(I_1 \cap I_2)M$ is flat over $R/(I_1 \cap I_2)$.*

Proof. By replacing R with $R/(I_1 \cap I_2)$ and M by $M/(I_1 \cap I_2)M$ we may assume that $I_1 \cap I_2 = 0$. Let $J \subset R$ be an ideal. To prove that M is flat over R we have to show that $J \otimes_R M \to M$ is injective, see Algebra, Lemma 7.35.4. By flatness of M/I_1M over R/I_1 the map

$$J/(J \cap I_1) \otimes_R M = (J + I_1)/I_1 \otimes_{R/I_1} M/I_1M \longrightarrow M/I_1M$$

is injective. As $0 \to (J \cap I_1) \to J \to J/(J \cap I_1) \to 0$ is exact we obtain a diagram

$$\begin{tikzcd}
(J \cap I_1) \otimes_R M \arrow[r] \arrow[d] & J \otimes_R M \arrow[r] \arrow[d] & J/(J \cap I_1) \otimes_R M \arrow[r] \arrow[d] & 0 \\
M \arrow[r, equal] & M \arrow[r] & M/I_1M &
\end{tikzcd}$$

hence it suffices to show that $(J \cap I_1) \otimes_R M \to M$ is injective. Since $I_1 \cap I_2 = 0$ the ideal $J \cap I_1$ maps isomorphically to an ideal $J' \subset R/I_2$ and we see that $(J \cap I_1) \otimes_R M = J' \otimes_{R/I_2} M/I_2M$. By flatness of M/I_2M over R/I_2 the map $J' \otimes_{R/I_2} M/I_2M \to M/I_2M$ is injective, which clearly implies that $(J \cap I_1) \otimes_R M \to M$ is injective. $\square$

12.12. Flattening over an Artinian ring

A universal flattening exists when the base ring is an Artinian local ring. It exists for an arbitrary module. Hence, as we will see later, a flatting stratification exists when the base scheme is the spectrum of an Artinian local ring.

Lemma 12.12.1. *Let R be an Artinian ring. Let M be an R-module. Then there exists a smallest ideal $I \subset R$ such that M/IM is flat over R/I.*

Proof. This follows directly from Lemma 12.11.1 and the Artinian property. $\square$

This ideal has the following universal property.

Lemma 12.12.2. *Let R be an Artinian ring. Let M be an R-module. Let $I \subset R$ be the smallest ideal $I \subset R$ such that M/IM is flat over R/I. Then I has the following universal property: For every ring map $\varphi : R \to R'$ we have*

$$R' \otimes_R M \text{ is flat over } R' \Leftrightarrow \text{we have } \varphi(I) = 0.$$

Proof. Note that I exists by Lemma 12.12.1. The implication $\Rightarrow$ follows from Algebra, Lemma 7.35.6. Let $\varphi : R \to R'$ be such that $M \otimes_R R'$ is flat over R'. Let $J = \text{Ker}(\varphi)$. By Algebra, Lemma 7.93.7 and as $R' \otimes_R M = R' \otimes_{R/J} M/JM$ is flat over R' we conclude that M/JM is flat over R/J. Hence $I \subset J$ as desired. □

12.13. Flattening over a closed subset of the base

Let $R \to S$ be a ring map. Let $I \subset R$ be an ideal. Let M be an S-module. In the following we will consider the following condition

$$\forall \mathfrak{q} \in V(IS) \subset Spec(S) : M_{\mathfrak{q}} \text{ is flat over } R. \tag{12.13.0.1}$$

Geometrically, this means that M is flat over R along the inverse image of $V(I)$ in $Spec(S)$. If R and S are Noetherian rings and M is a finite S-module, then (12.13.0.1) is equivalent to the condition that M/I^nM is flat over R/I^n for all $n \geq 1$, see Algebra, Lemma 7.91.10.

Lemma 12.13.1. *Let $R \to S$ be a ring map. Let $I \subset R$ be an ideal. Let M be an S-module. Let $R \to R'$ be a ring map and $IR' \subset I' \subset R'$ an ideal. If (12.13.0.1) holds for $(R \to S, I, M)$, then (12.13.0.1) holds for $(R' \to S \otimes_R R', I', M \otimes_R R')$.*

Proof. Assume (12.13.0.1) holds for $(R \to S, I \subset R, M)$. Let $I'(S \otimes_R R') \subset \mathfrak{q}'$ be a prime of $S \otimes_R R'$. Let $\mathfrak{q} \subset S$ be the corresponding prime of S. Then $IS \subset \mathfrak{q}$. Note that $(M \otimes_R R')_{\mathfrak{q}'}$ is a localization of the base change $M_{\mathfrak{q}} \otimes_R R'$. Hence $(M \otimes_R R')_{\mathfrak{q}'}$ is flat over R' as a localization of a flat module, see Algebra, Lemmas 7.35.6 and 7.35.19. □

Lemma 12.13.2. *Let $R \to S$ be a ring map. Let $I \subset R$ be an ideal. Let M be an S-module. Let $R \to R'$ be a ring map and $IR' \subset I' \subset R'$ an ideal such that*

1. *the map $V(I') \to V(I)$ induced by $Spec(R') \to Spec(R)$ is surjective, and*
2. *$R'_{\mathfrak{p}'}$ is flat over R for all primes $\mathfrak{p}' \in V(I')$.*

If (12.13.0.1) holds for $(R' \to S \otimes_R R', I', M \otimes_R R')$, then (12.13.0.1) holds for $(R \to S, I, M)$.

Proof. Assume (12.13.0.1) holds for $(R' \to S \otimes_R R', IR', M \otimes_R R')$. Pick a prime $IS \subset \mathfrak{q} \subset S$. Let $I \subset \mathfrak{p} \subset R$ be the corresponding prime of R. By assumption there exists a prime $\mathfrak{p}' \in V(I')$ of R' lying over $\mathfrak{p}$ and $R_{\mathfrak{p}} \to R'_{\mathfrak{p}'}$ is flat. Choose a prime $\overline{\mathfrak{q}}' \subset \kappa(\mathfrak{q}) \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p}')$ which corresponds to a prime $\mathfrak{q}' \subset S \otimes_R R'$ which lies over $\mathfrak{q}$ and over $\mathfrak{p}'$. Note that $(S \otimes_R R')_{\mathfrak{q}'}$ is a localization of $S_{\mathfrak{q}} \otimes_{R_{\mathfrak{p}}} R'_{\mathfrak{p}'}$. By assumption the module $(M \otimes_R R')_{\mathfrak{q}'}$ is flat over $R'_{\mathfrak{p}'}$. Hence Algebra, Lemma 7.92.1 implies that $M_{\mathfrak{q}}$ is flat over $R_{\mathfrak{p}}$ which is what we wanted to prove. □

Lemma 12.13.3. *Let $R \to S$ be a ring map of finite presentation. Let M be an S-module of finite presentation. Let $R' = colim_{\lambda \in \Lambda} R_\lambda$ be a directed colimit of R-algebras. Let $I_\lambda \subset R_\lambda$ be ideals such that $I_\lambda R_\mu \subset I_\mu$ for all $\mu \geq \lambda$ and set $I' = colim_\lambda I_\lambda$. If (12.13.0.1) holds for $(R' \to S \otimes_R R', I', M \otimes_R R')$, then there exists a $\lambda \in \Lambda$ such that (12.13.0.1) holds for $(R_\lambda \to S \otimes_R R_\lambda, I_\lambda, M \otimes_R R_\lambda)$.*

Proof. We are going to write $S_\lambda = S \otimes_R R_\lambda$, $S' = S \otimes_R R'$, $M_\lambda = M \otimes_R R_\lambda$, and $M' = M \otimes_R R'$. The base change S' is of finite presentation over R' and M' is of finite presentation over S' and similarly for the versions with subscript λ, see Algebra, Lemma 7.13.2. By Algebra, Theorem 7.120.4 the set

$$U' = \{\mathfrak{q}' \in Spec(S') \mid M'_{\mathfrak{q}'} \text{ is flat over } R'\}$$

is open in $Spec(S')$. Note that $V(I'S')$ is a quasi-compact space which is contained in U' by assumption. Hence there exist finitely many $g'_j \in S'$, $j = 1, \ldots, m$ such that $D(g'_j) \subset U'$ and such that $V(I'S') \subset \bigcup D(g'_j)$. Note that in particular $(M')_{g'_j}$ is a flat module over R'.

We are going to pick increasingly large elements $\lambda \in \Lambda$. First we pick it large enough so that we can find $g_{j,\lambda} \in S_\lambda$ mapping to g'_j. The inclusion $V(I'S') \subset \bigcup D(g'_j)$ means that $I'S' + (g'_1, \ldots, g'_m) = S'$ which can be expressed as $1 = \sum z_s h_s + \sum f_j g'_j$ for some $z_s \in I'$, $h_s, f_j \in S'$. After increasing λ we may assume such an equation holds in S_λ. Hence we may assume that $V(I_\lambda S_\lambda) \subset \bigcup D(g_{j,\lambda})$. By Algebra, Lemma 7.120.5 we see that for some sufficiently large λ the modules $(M_\lambda)_{g_{j,\lambda}}$ are flat over R_λ. In particular the module M_λ is flat over R_λ at all the primes lying over the ideal I_λ. $\square$

12.14. Flattening over a closed subsets of source and base

In this section we slightly generalize the discussion in Section 12.13. We strongly suggest the reader first read and understand that section.

Situation 12.14.1. Let $R \to S$ be a ring map. Let $J \subset S$ be an ideal. Let M be an S-module.

In this situation, given an R-algebra R' and an ideal $I' \subset R'$ we set $S' = S \otimes_R R'$ and $M' = M \otimes_R R'$. We will consider the condition

$$\forall \mathfrak{q}' \in V(I'S' + JS') \subset Spec(S') : M'_{\mathfrak{q}'} \text{ is flat over } R'. \tag{12.14.1.1}$$

Geometrically, this means that M' is flat over R' along the intersection of the inverse image of $V(I')$ with the inverse image of $V(J)$. Since $(R \to S, J, M)$ are fixed, condition (12.14.1.1) only depends on the pair (R', I') where R' is viewed as an R-algebra.

Lemma 12.14.2. *In Situation 12.14.1 let $R' \to R''$ be an R-algebra map. Let $I' \subset R'$ and $I'R'' \subset I'' \subset R''$ be ideals. If (12.14.1.1) holds for (R', I'), then (12.14.1.1) holds for (R'', I'').*

Proof. Assume (12.14.1.1) holds for (R', I'). Let $I''S'' + JS'' \subset \mathfrak{q}''$ be a prime of S''. Let $\mathfrak{q}' \subset S'$ be the corresponding prime of S'. Then both $I'S' \subset \mathfrak{q}'$ and $JS' \subset \mathfrak{q}'$ because the corresponding conditions hold for $\mathfrak{q}''$. Note that $(M'')_{\mathfrak{q}''}$ is a localization of the base change $M'_{\mathfrak{q}'} \otimes_R R''$. Hence $(M'')_{\mathfrak{q}''}$ is flat over R'' as a localization of a flat module, see Algebra, Lemmas 7.35.6 and 7.35.19. $\square$

Lemma 12.14.3. *In Situation 12.14.1 let $R' \to R''$ be an R-algebra map. Let $I' \subset R'$ and $I'R'' \subset I'' \subset R''$ be ideals. Assume*

1. *the map $V(I'') \to V(I')$ induced by $Spec(R'') \to Spec(R')$ is surjective, and*
2. *$R''_{\mathfrak{p}''}$ is flat over R' for all primes $\mathfrak{p}'' \in V(I'')$.*

If (12.14.1.1) holds for (R'', I''), then (12.14.1.1) holds for (R', I').

Proof. Assume (12.14.1.1) holds for (R'', I''). Pick a prime $I'S' + JS' \subset \mathfrak{q}' \subset S'$. Let $I' \subset \mathfrak{p}' \subset R'$ be the corresponding prime of R'. By assumption there exists a prime $\mathfrak{p}'' \in V(I'')$ of R'' lying over $\mathfrak{p}'$ and $R'_{\mathfrak{p}'} \to R''_{\mathfrak{p}''}$ is flat. Choose a prime $\overline{\mathfrak{q}}'' \subset \kappa(\mathfrak{q}') \otimes_{\kappa(\mathfrak{p}')} \kappa(\mathfrak{p}'')$. This corresponds to a prime $\mathfrak{q}'' \subset S'' = S' \otimes_{R'} R''$ which lies over $\mathfrak{q}'$ and over $\mathfrak{p}''$. In particular we see that $I''S'' \subset \mathfrak{q}''$ and that $JS'' \subset \mathfrak{q}''$. Note that $(S' \otimes_{R'} R'')_{\mathfrak{q}''}$ is a localization of $S'_{\mathfrak{q}'} \otimes_{R'_{\mathfrak{p}'}} R''_{\mathfrak{p}''}$. By assumption the module $(M' \otimes_{R'} R'')_{\mathfrak{q}''}$ is flat over $R''_{\mathfrak{p}''}$. Hence Algebra, Lemma 7.92.1 implies that $M'_{\mathfrak{q}'}$ is flat over $R'_{\mathfrak{p}'}$ which is what we wanted to prove. □

Lemma 12.14.4. *In Situation 12.14.1 assume $R \to S$ is essentially of finite presentation and M is an S-module of finite presentation. Let $R' = colim_{\lambda \in \Lambda} R_\lambda$ be a directed colimit of R-algebras. Let $I_\lambda \subset R_\lambda$ be ideals such that $I_\lambda R_\mu \subset I_\mu$ for all $\mu \geq \lambda$ and set $I' = colim_\lambda I_\lambda$. If (12.14.1.1) holds for (R', I'), then there exists a $\lambda \in \Lambda$ such that (12.14.1.1) holds for (R_λ, I_λ).*

Proof. We first prove the lemma in case $R \to S$ is of finite presentation and then we explain what needs to be changed in the general case. We are going to write $S_\lambda = S \otimes_R R_\lambda$, $S' = S \otimes_R R'$, $M_\lambda = M \otimes_R R_\lambda$, and $M' = M \otimes_R R'$. The base change S' is of finite presentation over R' and M' is of finite presentation over S' and similarly for the versions with subscript λ, see Algebra, Lemma 7.13.2. By Algebra, Theorem 7.120.4 the set

$$U' = \{\mathfrak{q}' \in Spec(S') \mid M'_{\mathfrak{q}'} \text{ is flat over } R'\}$$

is open in $Spec(S')$. Note that $V(I'S' + JS')$ is a quasi-compact space which is contained in U' by assumption. Hence there exist finitely many $g'_j \in S'$, $j = 1, \ldots, m$ such that $D(g'_j) \subset U'$ and such that $V(I'S' + JS') \subset \bigcup D(g'_j)$. Note that in particular $(M')_{g'_j}$ is a flat module over R'.

We are going to pick increasingly large elements $\lambda \in \Lambda$. First we pick it large enough so that we can find $g_{j,\lambda} \in S_\lambda$ mapping to g'_j. The inclusion $V(I'S' + JS') \subset \bigcup D(g'_j)$ means that $I'S' + JS' + (g'_1, \ldots, g'_m) = S'$ which can be expressed as

$$1 = \sum y_t k_t + \sum z_s h_s + \sum f_j g'_j$$

for some $z_s \in I'$, $y_t \in J$, $k_t, h_s, f_j \in S'$. After increasing λ we may assume such an equation holds in S_λ. Hence we may assume that $V(I_\lambda S_\lambda + JS_\lambda) \subset \bigcup D(g_{j,\lambda})$. By Algebra, Lemma 7.120.5 we see that for some sufficiently large λ the modules $(M_\lambda)_{g_{j,\lambda}}$ are flat over R_λ. In particular the module M_λ is flat over R_λ at all the primes corresponding to points of $V(I_\lambda S_\lambda + JS_\lambda)$.

In the case that S is essentially of finite presentation, we can write $S = \Sigma^{-1}C$ where $R \to C$ is of finite presentation and $\Sigma \subset C$ is a multiplicative subset. We can also write $M = \Sigma^{-1}N$ for some finitely presented C-module N, see Algebra, Lemma 7.117.3. At this point we introduce $C_\lambda, C', N_\lambda, N'$. Then in the discussion above we obtain an open $U' \subset Spec(C')$ over which N' is flat over R'. The assumption that (12.14.1.1) is true means that $V(I'S' + JS')$ maps into U', because for a prime $\mathfrak{q}' \subset S'$, corresponding to a prime $\mathfrak{r}' \subset C'$ we have $M'_{\mathfrak{q}'} = N'_{\mathfrak{r}'}$. Thus we can find $g'_j \in C'$ such that $\bigcup D(g'_j)$ contains the image of $V(I'S' + JS')$. The rest of the proof is exactly the same as before. □

Lemma 12.14.5. *In Situation 12.14.1. Let $I \subset R$ be an ideal. Assume*

(1) *R is a Noetherian ring,*
(2) *S is a Noetherian ring,*

(3) *M is a finite S-module, and*
(4) *for each $n \geq 1$ and any prime $\mathfrak{q} \in V(J + IS)$ the module $(M/I^n M)_{\mathfrak{q}}$ is flat over R/I^n.*

Then (12.14.1.1) holds for (R, I), i.e., for every prime $\mathfrak{q} \in V(J + IS)$ the localization $M_{\mathfrak{q}}$ is flat over R.

Proof. Let $\mathfrak{q} \in V(J + IS)$. Then Algebra, Lemma 7.91.10 applied to $R \to S_{\mathfrak{q}}$ and $M_{\mathfrak{q}}$ implies that $M_{\mathfrak{q}}$ is flat over R. □

12.15. Flattening over a Noetherian complete local ring

The following three lemmas give a completely algebraic proof of the existence of the ``local" flattening stratification when the base is a complete local Noetherian ring R and the given module is finite over a finite type R-algebra S.

Lemma 12.15.1. *Let $R \to S$ be a ring map. Let M be an S-module. Assume*
(1) *$(R, \mathfrak{m})$ is a complete local Noetherian ring,*
(2) *S is a Noetherian ring, and*
(3) *M is finite over S.*

Then there exists an ideal $I \subset \mathfrak{m}$ such that
(1) *$(M/IM)_{\mathfrak{q}}$ is flat over R/I for all primes $\mathfrak{q}$ of S/IS lying over $\mathfrak{m}$, and*
(2) *if $J \subset R$ is an ideal such that $(M/JM)_{\mathfrak{q}}$ is flat over R/J for all primes $\mathfrak{q}$ lying over $\mathfrak{m}$, then $I \subset J$.*

In other words, I is the smallest ideal of R such that (12.13.0.1) holds for $(\overline{R} \to \overline{S}, \overline{\mathfrak{m}}, \overline{M})$ where $\overline{R} = R/I$, $\overline{S} = S/IS$, $\overline{\mathfrak{m}} = \mathfrak{m}/I$ and $\overline{M} = M/IM$.

Proof. Let $J \subset R$ be an ideal. Apply Algebra, Lemma 7.91.10 to the module M/JM over the ring R/J. Then we see that $(M/JM)_{\mathfrak{q}}$ is flat over R/J for all primes $\mathfrak{q}$ of S/JS if and only if $M/(J + \mathfrak{m}^n)M$ is flat over $R/(J + \mathfrak{m}^n)$ for all $n \geq 1$. We will use this remark below.

For every $n \geq 1$ the local ring $R/\mathfrak{m}^n$ is Artinian. Hence, by Lemma 12.12.1 there exists a smallest ideal $I_n \supset \mathfrak{m}^n$ such that M/I_nM is flat over R/I_n. It is clear that $I_{n+1} + \mathfrak{m}^n$ is contains I_n and applying Lemma 12.11.1 we see that $I_n = I_{n+1} + \mathfrak{m}^n$. Since $R = lim_n\ R/\mathfrak{m}^n$ we see that $I = lim_n\ I_n/\mathfrak{m}^n$ is an ideal in R such that $I_n = I + \mathfrak{m}^n$ for all $n \geq 1$. By the initial remarks of the proof we see that I verifies (1) and (2). Some details omitted. □

Lemma 12.15.2. *With notation $R \to S$, M, and I and assumptions as in Lemma 12.15.1. Consider a local homomorphism of local rings $\varphi : (R, \mathfrak{m}) \to (R', \mathfrak{m}')$ such that R' is Noetherian. Then the following are equivalent*
(1) *condition (12.13.0.1) holds for $(R' \to S \otimes_R R', \mathfrak{m}', M \otimes_R R')$, and*
(2) *$\varphi(I) = 0$.*

Proof. The implication (2) ⇒ (1) follows from Lemma 12.13.1. Let $\varphi : R \to R'$ be as in the lemma satisfying (1). We have to show that $\varphi(I) = 0$. This is equivalent to the condition that $\varphi(I)R' = 0$. By Artin-Rees in the Noetherian local ring R' (see Algebra, Lemma 7.47.6) this is equivalent to the condition that $\varphi(I)R' + (\mathfrak{m}')^n = (\mathfrak{m}')^n$ for all $n > 0$. Hence this is equivalent to the condition that the composition $\varphi_n : R \to R' \to R'/(\mathfrak{m}')^n$ annihilates I for each n. Now assumption (1) for φ implies assumption (1) for φ_n by Lemma 12.13.1. This reduces us to the case where R' is Artinian local.

Assume R' Artinian. Let $J = \mathrm{Ker}(\varphi)$. We have to show that $I \subset J$. By the construction of I in Lemma 12.15.1 it suffices to show that $(M/JM)_{\mathfrak{q}}$ is flat over R/J for every prime $\mathfrak{q}$ of

S/JS lying over $\mathfrak{m}$. As R' is Artinian, condition (1) signifies that $M \otimes_R R'$ is flat over R'. As R' is Artinian and $R/J \to R'$ is a local injective ring map, it follows that R/J is Artinian too. Hence the flatness of $M \otimes_R R' = M/JM \otimes_{R/J} R'$ over R' implies that M/JM is flat over R/J by Algebra, Lemma 7.93.7. This concludes the proof. □

Lemma 12.15.3. *With notation $R \to S$, M, and I and assumptions as in Lemma 12.15.1. In addition assume that $R \to S$ is of finite type. Then for any local homomorphism of local rings $\varphi : (R, \mathfrak{m}) \to (R', \mathfrak{m}')$ the following are equivalent*

1. *condition (12.13.0.1) holds for $(R' \to S \otimes_R R', \mathfrak{m}', M \otimes_R R')$, and*
2. *$\varphi(I) = 0$.*

Proof. The implication (2) ⇒ (1) follows from Lemma 12.13.1. Let $\varphi : R \to R'$ be as in the lemma satisfying (1). As R is Noetherian we see that $R \to S$ is of finite presentation and M is an S-module of finite presentation. Write $R' = colim_\lambda R_\lambda$ as a directed colimit of local R-subalgebras $R_\lambda \subset R'$, with maximal ideals $\mathfrak{m}_\lambda = R_\lambda \cap \mathfrak{m}'$ such that each R_λ is essentially of finite type over R. By Lemma 12.13.3 we see that condition (12.13.0.1) holds for $(R_\lambda \to S \otimes_R R_\lambda, \mathfrak{m}_\lambda, M \otimes_R R_\lambda)$ for some λ. Hence Lemma 12.15.2 applies to the ring map $R \to R_\lambda$ and we see that I maps to zero in R_λ, a fortiori it maps to zero in R'. □

12.16. Descent flatness along integral maps

First a few simple lemmas.

Lemma 12.16.1. *Let R be a ring. Let $P(T)$ be a monic polynomial with coefficients in R. If there exists an $\alpha \in R$ such that $P(\alpha) = 0$, then $P(T) = (T - \alpha)Q(T)$ for some monic polynomial $Q(T) \in R[T]$.*

Proof. By induction on the degree of P. If $\deg(P) = 1$, then $P(T) = T - \alpha$ and the result is true. If $\deg(P) > 1$, then we can write $P(T) = (T - \alpha)Q(T) + r$ for some polynomial $Q \in R[T]$ of degree $< \deg(P)$ and some $r \in R$ by long division. By assumption $0 = P(\alpha) = (\alpha - \alpha)Q(\alpha) + r = r$ and we conclude that $r = 0$ as desired. □

Lemma 12.16.2. *Let R be a ring. Let $P(T)$ be a monic polynomial with coefficients in R. There exists a finite free ring map $R \to R'$ such that $P(T) = (T - \alpha)Q(T)$ for some $\alpha \in R'$ and some monic polynomial $Q(T) \in R'[T]$.*

Proof. Write $P(T) = T^d + a_1 T^{d-1} + \ldots + a_0$. Set $R' = R[x]/(x^d + a_1 x^{d-1} + \ldots + a_0)$. Set α equal to the congruence class of x. Then it is clear that $P(\alpha) = 0$. Thus we win by Lemma 12.16.1. □

Lemma 12.16.3. *Let $R \to S$ be a finite ring map. There exists a finite free ring extension $R \subset R'$ such that $S \otimes_R R'$ is a quotient of a ring of the form*

$$R'[T_1, \ldots, T_n]/(P_1(T_1), \ldots, P_n(T_n))$$

with $P_i(T) = \prod_{j=1,\ldots,d_i}(T - \alpha_{ij})$ for some $\alpha_{ij} \in R'$.

Proof. Let $x_1, \ldots, x_n \in S$ be generators of S over R. For each i we can choose a monic polynomial $P_i(T) \in R[T]$ such that $P(x_i) = 0$ in S, see Algebra, Lemma 7.32.3. Say $\deg(P_i) = d_i$. By Lemma 12.16.2 (applied $\sum d_i$ times) there exists a finite free ring extension $R \subset R'$ such that each P_i splits completely:

$$P_i(T) = \prod_{j=1,\ldots,d_i}(T - \alpha_{ij})$$

for certain $\alpha_{ik} \in R'$. Let $R'[T_1, \ldots, T_n] \to S \otimes_R R'$ be the R'-algebra map which maps T_i to $x_i \otimes 1$. As this maps $P_i(T_i)$ to zero, this induces the desired surjection. □

Lemma 12.16.4. *Let R be a ring. Let $S = R[T_1, \ldots, T_n]/J$. Assume J contains elements of the form $P_i(T_i)$ with $P_i(T) = \prod_{j=1,\ldots,d_i}(T - \alpha_{ij})$ for some $\alpha_{ij} \in R$. For $\underline{k} = (k_1, \ldots, k_n)$ with $1 \leq k_i \leq d_i$ consider the ring map*

$$\Phi_{\underline{k}} : R[T_1, \ldots, T_n] \to R, \quad T_i \longmapsto \alpha_{ik_i}$$

Set $J_{\underline{k}} = \Phi_{\underline{k}}(J)$. Then the image of $Spec(S) \to Spec(R)$ is equal to $V(\bigcap J_{\underline{k}})$.

Proof. This lemma proves itself. Hint: $V(\bigcap J_{\underline{k}}) = \bigcup V(J_{\underline{k}})$. □

The following result is due to Ferrand, see [**Fer69**].

Lemma 12.16.5. *Let $R \to S$ be a finite injective homomorphism of Noetherian rings. Let M be an R-module. If $M \otimes_R S$ is a flat S-module, then M is a flat R-module.*

Proof. Let M be an R-module such that $M \otimes_R S$ is flat over S. By Algebra, Lemma 7.35.7 in order to prove that M is flat we may replace R by any faithfully flat ring extension. By Lemma 12.16.3 we can find a finite locally free ring extension $R \subset R'$ such that $S' = S \otimes_R R' = R'[T_1, \ldots, T_n]/J$ for some ideal $J \subset R'[T_1, \ldots, T_n]$ which contains the elements of the form $P_i(T_i)$ with $P_i(T) = \prod_{j=1,\ldots,d_i}(T - \alpha_{ij})$ for some $\alpha_{ij} \in R'$. Note that R' is Noetherian and that $R' \subset S'$ is a finite extension of rings. Hence we may replace R by R' and assume that S has a presentation as in Lemma 12.16.4. Note that $Spec(S) \to Spec(R)$ is surjective, see Algebra, Lemma 7.32.15. Thus, using Lemma 12.16.4 we conclude that $I = \bigcap J_{\underline{k}}$ is an ideal such that $V(I) = Spec(R)$. This means that $I \subset \sqrt{(0)}$, and since R is Noetherian that I is nilpotent. The maps $\Phi_{\underline{k}}$ induce commutative diagrams

$$\begin{tikzcd} S \arrow[rr] & & R/J_{\underline{k}} \\ & R \arrow[ul] \arrow[ur] & \end{tikzcd}$$

from which we conclude that $M/J_{\underline{k}}M$ is flat over $R/J_{\underline{k}}$. By Lemma 12.11.1 we see that M/IM is flat over R/I. Finally, applying Algebra, Lemma 7.93.5 we conclude that M is flat over R. □

Lemma 12.16.6. *Let $R \to S$ be an injective integral ring map. Let M be a finitely presented module over $R[x_1, \ldots, x_n]$. If $M \otimes_R S$ is flat over S, then M is flat over R.*

Proof. Choose a presentation

$$R[x_1, \ldots, x_n]^{\oplus t} \to R[x_1, \ldots, x_n]^{\oplus r} \to M \to 0.$$

Let's say that the first map is given by the $r \times t$-matrix $T = (f_{ij})$ with $f_{ij} \in R[x_1, \ldots, x_n]$. Write $f_{ij} = \sum f_{ij,I} x^I$ with $f_{ij,I} \in R$ (multi-index notation). Consider diagrams

$$\begin{tikzcd} R \arrow[r] & S \\ R_\lambda \arrow[u] \arrow[r] & S_\lambda \arrow[u] \end{tikzcd}$$

where R_λ is a finitely generated $\mathbf{Z}$-subalgebra of R containing all $f_{ij,I}$ and S_λ is a finite R_λ-subalgebra of S. Let M_λ be the finite $R_\lambda[x_1, \ldots, x_n]$-module defined by a presentation as above, using the same matrix T but now viewed as a matrix over $R_\lambda[x_1, \ldots, x_n]$. Note that S is the directed colimit of the S_λ (details omitted). By Algebra, Lemma 7.120.5 we see that for some λ the module $M_\lambda \otimes_{R_\lambda} S_\lambda$ is flat over S_λ. By Lemma 12.16.5 we conclude that M_λ is flat over R_λ. Since $M = M_\lambda \otimes_{R_\lambda} R$ we win by Algebra, Lemma 7.35.6. □

12.17. Torsion and flatness

In this section we discuss the relationship between torsion and flatness.

Definition 12.17.1. Let R be a domain. Let M be an R-module.

(1) We say an element $x \in M$ is *torsion* if there exists a nonzero $f \in R$ such that $fx = 0$.
(2) We say M is *torsion free* if the only torsion element of M is 0.

Lemma 12.17.2. *Let R be a domain. Let M be an R-module. The set of torsion elements of M forms a submodule $M_{tors} \subset M$. The quotient module M/M_{tors} is torsion free.*

Proof. Omitted. □

Lemma 12.17.3. *Let R be a domain. Any flat R-module is torsion free.*

Proof. If $x \in R$ is nonzero, then $x : R \to R$ is injective, and hence if M is flat over R, then $x : M \to M$ is injective. Thus if M is flat over R, then M is torsion free. □

Lemma 12.17.4. *Let A be a valuation ring. An A-module M is flat over A if and only if M is torsion free.*

Proof. The implication ``flat ⇒ torsion free'' is Lemma 12.17.3. For the converse, assume M is torsion free. By the equational criterion of flatness (see Algebra, Lemma 7.35.10) we have to show that every relation in M is trivial. To do this assume that $\sum_{i=1,\ldots,n} a_i x_i = 0$ with $x_i \in M$ and $f_i \in A$. After renumbering we may assume that $v(a_1) \leq v(a_i)$ for all i. Hence we can write $a_i = a'_i a_1$ for some $a'_i \in A$. Note that $a'_1 = 1$. As A is torsion free we see that $x_1 = -\sum_{i\geq 2} a'_i x_i$. Thus, if we choose $y_i = x_i$, $i = 2, \ldots, n$ then

$$x_1 = \sum\nolimits_{j\geq 2} -a'_j y_j, \quad x_i = y_i, (i \geq 2) \quad 0 = a_1 \cdot (-a'_j) + a_j \cdot 1 (j \geq 2)$$

shows that the relation was trivial (to be explicit the elements a_{ij} are defined by setting $a_{1j} = -a'_j$ and $a_{ij} = \delta_{ij}$ for $i, j \geq 2$). □

12.18. Flatness and finiteness conditions

In this section we discuss some implications of the type ``flat + finite type ⇒ finite presentation''. We will revisit this result in the chapter on flatness, see More on Flatness, Section 34.1. A first result of this type was proved in Algebra, Lemma 7.100.6.

Lemma 12.18.1. *Let R be a ring. Let $S = R[x_1, \ldots, x_n]$ be a polynomial ring over R. Let M be an S-module. Assume*

(1) *there exist finitely many primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ of R such that the map $R \to \prod R_{\mathfrak{p}_j}$ is injective,*
(2) *M is a finite S-module,*
(3) *M flat over R, and*
(4) *for every prime $\mathfrak{p}$ of R the module $M_{\mathfrak{p}}$ is of finite presentation over $S_{\mathfrak{p}}$.*

Then M is of finite presentation over S.

Proof. Choose a presentation

$$0 \to K \to S^{\oplus r} \to M \to 0$$

of M as an S-module. Let $\mathfrak{q}$ be a prime ideal of S lying over a prime $\mathfrak{p}$ of R. By assumption there exist finitely many elements $k_1, \ldots, k_t \in K$ such that if we set $K' = \sum Sk_j \subset K$ then $K'_{\mathfrak{p}} = K_{\mathfrak{p}}$ and $K'_{\mathfrak{p}_j} = K_{\mathfrak{p}_j}$ for $j = 1, \ldots, m$. Setting $M' = S^{\oplus r}/K'$ we deduce that in particular $M'_{\mathfrak{q}} = M_{\mathfrak{q}}$. By openness of flatness, see Algebra, Theorem 7.120.4 we conclude that there exists a $g \in S$, $g \notin \mathfrak{q}$ such that M'_g is flat over R. Thus $M'_g \to M_g$ is a surjective map of flat R-modules. Consider the commutative diagram

$$\begin{tikzcd} M'_g \arrow[r] \arrow[d] & M_g \arrow[d] \\ \prod (M'_g)_{\mathfrak{p}_j} \arrow[r] & \prod (M_g)_{\mathfrak{p}_j} \end{tikzcd}$$

The bottom arrow is an isomorphism by choice of $k_1, \ldots, k_t$. The left vertical arrow is an injective map as $R \to \prod R_{\mathfrak{p}_j}$ is injective and M'_g is flat over R. Hence the top horizontal arrow is injective, hence an isomorphism. This proves that M_g is of finite presentation over S_g. We conclude by applying Algebra, Lemma 7.21.2. $\square$

Lemma 12.18.2. *Let $R \to S$ be a ring homomorphism. Assume*

(1) *there exist finitely many primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ of R such that the map $R \to \prod R_{\mathfrak{p}_j}$ is injective,*
(2) *$R \to S$ is of finite type,*
(3) *S flat over R, and*
(4) *for every prime $\mathfrak{p}$ of R the ring $S_{\mathfrak{p}}$ is of finite presentation over $R_{\mathfrak{p}}$.*

Then S is of finite presentation over R.

Proof. By assumption S is a quotient of a polynomial ring over R. Thus the result follows directly from Lemma 12.18.1. $\square$

Lemma 12.18.3. *Let R be a ring. Let $S = R[x_1, \ldots, x_n]$ be a graded polynomial algebra over R, i.e.,* $\deg(x_i) > 0$ *but not necessarily equal to* 1. *Let M be a graded S-module. Assume*

(1) *R is a local ring,*
(2) *M is a finite S-module, and*
(3) *M is flat over R.*

Then M is finitely presented as an S-module.

Proof. Let $M = \bigoplus M_d$ be the grading on M. Pick homogeneous generators $m_1, \ldots, m_r \in M$ of M. Say $\deg(m_i) = d_i \in \mathbf{Z}$. This gives us a presentation

$$0 \to K \to \bigoplus\nolimits_{i=1,\ldots,r} S(-d_i) \to M \to 0$$

which in each degree d leads to the short exact sequence

$$0 \to K_d \to \bigoplus\nolimits_{i=1,\ldots,r} S_{d-d_i} \to M_d \to 0.$$

By assumption each M_d is a finite flat R-module. By Algebra, Lemma 7.72.4 this implies each M_d is a finite free R-module. Hence we see each K_d is a finite R-module. Also each K_d is flat over R by Algebra, Lemma 7.35.12. Hence we conclude that each K_d is finite free by Algebra, Lemma 7.72.4 again.

Let $\mathfrak{m}$ be the maximal ideal of R. By the flatness of M over R the short exact sequences above remain short exact after tensoring with $\kappa = \kappa(\mathfrak{m})$. As the ring $S \otimes_R \kappa$ is Noetherian we see that there exist homogeneous elements $k_1, \ldots, k_t \in K$ such that the images $\overline{k}_j$ generate $K \otimes_R \kappa$ over $S \otimes_R \kappa$. Say $\deg(k_j) = e_j$. Thus for any d the map

$$\bigoplus\nolimits_{j=1,\ldots,t} S_{d-e_j} \longrightarrow K_d$$

becomes surjective after tensoring with κ. By Nakayama's lemma (Algebra, Lemma 7.14.5) this implies the map is surjective over R. Hence K is generated by $k_1, \ldots, k_t$ over S and we win. $\square$

Lemma 12.18.4. *Let R be a ring. Let $S = \bigoplus_{n\geq 0} S_n$ be a graded R-algebra. Let $M = \bigoplus_{d\in\mathbf{Z}} M_d$ be a graded S-module. Assume S is finitely generated as an R-algebra, assume S_0 is a finite R-algebra, and assume there exist finitely many primes $\mathfrak{p}_j$, $i = 1, \ldots, m$ such that $R \to \prod R_{\mathfrak{p}_j}$ is injective.*

(1) *If S is flat over R, then S is a finitely presented R-algebra.*
(2) *If M is flat as an R-module and finite as an S-module, then M is finitely presented as an S-module.*

Proof. As S is finitely generated as an R-algebra, it is finitely generated as an S_0 algebra, say by homogeneous elements $t_1, \ldots, t_n \in S$ of degrees $d_1, \ldots, d_n > 0$. Set $P = R[x_1, \ldots, x_n]$ with $\deg(x_i) = d_i$. The ring map $P \to S$, $x_i \to t_i$ is finite as S_0 is a finite R-module. To prove (1) it suffices to prove that S is a finitely presented P-module. To prove (2) it suffices to prove that M is a finitely presented P-module. Thus it suffices to prove that if $S = P$ is a graded polynomial ring and M is a finite S-module flat over R, then M is finitely presented as an S-module. By Lemma 12.18.3 we see $M_{\mathfrak{p}}$ is a finitely presented $S_{\mathfrak{p}}$-module for every prime $\mathfrak{p}$ of R. Thus the result follows from Lemma 12.18.1. $\square$

Remark 12.18.5. Let R be a ring. When does R satisfy the condition mentioned in Lemmas 12.18.1, 12.18.2, and 12.18.4? This holds if

(1) R is local,
(2) R is Noetherian,
(3) R is a domain,
(4) R is a reduced ring with finitely many minimal primes, or
(5) R has finitely many weakly associated primes, see Algebra, Lemma 7.63.16.

Thus these lemmas hold in all cases listed above.

The following lemma wil be improved below, see Proposition 12.18.8.

Lemma 12.18.6. *Let A be a valuation ring. Let $A \to B$ be a ring map of finite type. Let M be a finite B-module.*

(1) *If B is flat over A, then B is a finitely presented A-algebra.*
(2) *If M is flat as an A-module, then M is finitely presented as a B-module.*

Proof. We are going to use that an A-module is flat if and only if it is torsion free, see Lemma 12.17.4. By Algebra, Lemma 7.53.9 we can find a graded A-algebra S with $S_0 = A$ and generated by finitely many elements in degree 1, an element $f \in S_1$ and a finite graded S-module N such that $B \cong S_{(f)}$ and $M \cong N_{(f)}$. If M is torsion free, then we can take N torsion free by replacing it by N/N_{tors}, see Lemma 12.17.2. Similarly, if B is torsion free, then we can take S torsion free by replacing it by S/S_{tors}. Hence in case (1), we may apply Lemma 12.18.4 to see that S is a finitely presented A-algebra, which implies that $B = S_{(f)}$

is a finitely presented A-algebra. To see (2) we may first replace S by a graded polynomial ring, and then we may apply Lemma 12.18.3 to conclude. □

Lemma 12.18.7. *Let R be a domain with fraction field K. Let $S = R[x_1, \ldots, x_n]$ be a polynomial ring over R. Let M be a finite S-module. Assume that M is flat over R. If for every subring $R \subset R' \subset K$, $R \neq R'$ the module $M \otimes_R R'$ is finitely presented over $S \otimes_R R'$, then M is finitely presented over S.*

Proof. Suppose that $f_1, \ldots, f_n \in R$ are elements which generate the unit ideal. If $R \neq R_{f_i}$ for each $i = 1, \ldots, n$, then we conclude that M_{f_i} is finitely presented over S_{f_i} for each i, and hence M is finitely presented over S by Algebra, Lemma 7.21.2. Thus we are done if such a sequence of elements exists. Assume this is not the case. In particular, for every $x \in R$ we have either $x \in R^*$, or $1 - x \in R^*$. This implies that R is local, see Algebra, Lemma 7.17.2.

Choose a presentation

$$0 \to K \to R[x_1, \ldots, x_n]^{\oplus r} \to M \to 0.$$

Throughout the rest of the proof we will use that this sequence stays exact after tensoring with any R-algebra, see Algebra, Lemma 7.35.11. Let R' be the integral closure of R in its fraction field. If $R \neq R'$, then we see that $M \otimes_R R'$ is finitely presented over $R'[x_1, \ldots, x_n]$. In particular, the module $K \otimes_R R'$ is finitely generated. Thus we may pick $k_1, \ldots, k_t \in K$ such that $k_1 \otimes 1, \ldots, k_t \otimes 1$ generate $K \otimes_R R'$. Set $K' = \sum R[x_1, \ldots, x_n]k_i \subset K$. Set $M' = R[x_1, \ldots, x_n]^{\oplus t}/K'$. Then M' is a finitely presented module over $R[x_1, \ldots, x_n]$ such that $M' \otimes_R R' \cong M \otimes_R R'$ is flat over R'. By Lemma 12.16.6 we conclude that M' is flat over R. Hence the surjective map $M' \to M$ is also injective as M' is torsion free, see Lemma 12.17.3. In other words, $M' \cong M$ and we conclude that M is finitely presented. Thus we are done if R is not a normal domain. Assume this is not the case. This reduces us to the case where R is a normal local domain.

Pick any pair of nonzero elements $x, y \in R$. Consider the inclusions $R \subset R[x/y]$ and $R[y/x]$. As R is a normal domain we get a short exact sequence

$$0 \to R \xrightarrow{(-1,1)} R[x/y] \oplus R[y/x] \xrightarrow{(1,1)} R[x/y, y/x] \to 0$$

see Algebra, Lemma 7.32.21. If $R \neq R[x/y]$ and $R \neq R[y/x]$ then we see that $K \otimes_R R[x/y]$ and $K \otimes_R R[y/x]$ are finitely generated as $R[x/y][x_1, \ldots, x_n]$ and $R[y/x][x_1, \ldots, x_n]$ modules. Thus we can find $k_1, \ldots, k_t \in K$ such that the elements $k_i \otimes 1$ generate $K \otimes_R R[x/y]$ and $K \otimes_R R[y/x]$ as $R[x/y][x_1, \ldots, x_n]$ and $R[y/x][x_1, \ldots, x_n]$ modules. Set $K' = \sum R[x_1, \ldots, x_n]k_i \subset K$. Tensoring the sequence above with $K' \subset K$ we get the diagram

$$\begin{tikzcd}
 & K' \arrow[r] \arrow[d] & K' \otimes_R R[x/y] \oplus K' \otimes_R R[y/x] \arrow[r] \arrow[d] & K' \otimes_R R[x/y, y/x] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] & K \otimes_R R[x/y] \oplus K \otimes_R R[y/x] \arrow[r] & K \otimes_R R[x/y, y/x] \arrow[r] & 0
\end{tikzcd}$$

Now we know that the vertical arrows in the middle and on the right are isomorphisms. The lower row is exact as K is flat over R. Hence the left vertical arrow is surjective, i.e., an isomorphism. Thus we win if there exists a pair of nonzero elements such that neither x/y nor y/x is an element of R. Assume this is not the case. Then we know that $R \subset f.f.(R)$ is a normal local domain such that for every $x \in f.f.(R)$ either $x \in R$, or $x^{-1} \in R$. In other words, R is a valuation ring, see Algebra, Lemma 7.46.4. In this case M is finitely presented by Lemma 12.18.6 and we win. □

The following result is a special case of results in [**GR71**] which we discuss in great detail in More on Flatness, Section 34.1.

Proposition 12.18.8. *Let R be a domain. Let $R \to S$ be a ring map of finite type. Let M be a finite S-module.*

(1) *If S is flat over R, then S is a finitely presented R-algebra.*
(2) *If M is flat as an R-module, then M is finitely presented as a S-module.*

Proof. It suffices to prove part (2) in case $S = R[x_1, \ldots, x_n]$. Choose a presentation

$$0 \to K \to R[x_1, \ldots, x_n]^{\oplus r} \to M \to 0.$$

Throughout the rest of the proof we will use that this sequence stays exact after tensoring with any R-algebra, see Algebra, Lemma 7.35.11. Let L be the fraction field of R. Consider the set

$$\mathcal{R} = \{R' \mid R \subset R' \subset L \text{ and } M \otimes_R R' \text{ not of finite presentation over } S \otimes_R R'\}$$

We order $\mathcal{R}$ by inclusion. Suppose that $\{R_i\}_{i \in I}$ is a totally ordered subset of $\mathcal{R}$. Set $R_\infty = \bigcup_{i \in I} R_i$. We claim that $R_\infty \in \mathcal{R}$. Namely, if $M \otimes_R R_\infty$ is finitely presented over $S \otimes_R R_\infty$, then $K \otimes_R R_\infty$ is finitely generated, say by $k_1, \ldots, k_t$. Then for some $i \in I$ we have $k_1, \ldots, k_t \in K \otimes_R R_i$. For any $i' \geq i$ set $M_{i'} = R_i[x_1, \ldots, x_n]^{\oplus r}/\sum R_i[x_1, \ldots, x_n]k_i$. By Algebra, Lemma 7.120.5 we see that $M_{i'}$ is flat over R_i for some sufficiently large $i' \in I$. For such an i' the surjective map $M_{i'} \to M \otimes_R R_i$ is also injective as $M_{i'}$ is torsion free. Hence we conclude that $M \otimes_R R_i$ is finitely presented which is a contradiction. In other words $R_\infty \in \mathcal{R}$. This shows that Zorn's lemma applies to $\mathcal{R}$ if $\mathcal{R}$ is not empty. But Lemma 12.18.7 shows that $\mathcal{R}$ does not have any maximal elements and the proposition is proved. □

12.19. Blowing up and flatness

In this section we begin our discussion of results of the form: ``After a blow up the strict transform becomes flat''.

Definition 12.19.1. Let R be a domain. Let M be an R-module. Let $R \subset R'$ be an extension of domains. The *strict transform of M along $R \to R'$* is the torsion free R'-module

$$M' = (M \otimes_R R')/(M \otimes_R R')_{tors}.$$

The following is a very weak version of flattening by blowing up, but it is already sometimes a useful result.

Lemma 12.19.2. *Let $(R, \mathfrak{m})$ be a local domain with fraction field K. Let S be a finite type R-algebra. Let M be a finite S-module. For every valuation ring $A \subset K$ dominating R there exists an ideal $I \subset \mathfrak{m}$ and a nonzero element $a \in I$ such that*

(1) *I is finitely generated,*
(2) *A has center on $R[\frac{I}{a}]$,*
(3) *the fibre ring of $R \to R[\frac{I}{a}]$ at $\mathfrak{m}$ is not zero, and*
(4) *the strict transform $S_{I,a}$ of S along $R \to R[\frac{I}{a}]$ is flat and of finite presentation over R, and the strict transform $M_{I,a}$ of M along $R \to R[\frac{I}{a}]$ is flat over R and finitely presented over $S_{I,a}$.*

Proof. Note that the assertion makes sense as $R[\frac{I}{a}]$ is a domain, and $R \to R[\frac{I}{a}]$ is injective, see Algebra, Lemmas 7.54.2 and 7.54.4. Before we start the proof of the Lemma, note that there is no loss in generality assuming that $S = R[x_1, \ldots, x_n]$ is a polynomial ring over R. We also fix a presentation

$$0 \to K \to S^{\oplus r} \to M \to 0.$$

Let M_A be the strict transform of M along $R \to A$. It is a finite module over $S_A = A[x_1, \ldots, x_n]$. By Lemma 12.17.4 we see that M_A is flat over A. By Lemma 12.18.6 we see that M_A is finitely presented. Hence there exist finitely many elements $k_1, \ldots, k_t \in S_A^{\oplus r}$ which generate the kernel of the presentation $S_A^{\oplus r} \to M_A$ as an S_A-module. For any choice of $a \in I \subset \mathfrak{m}$ satisfying (1), (2), and (3) we denote $M_{I,a}$ the strict transform of M along $R \to R[\frac{I}{a}]$. It is a finite module over $S_{I,a} = R[\frac{I}{a}][x_1, \ldots, x_n]$. By Algebra, Lemma 7.46.13 we have $A = colim_{I,a} R[\frac{I}{a}]$. This implies that $S_A = colim\, S_{I,a}$ and $M_A = colim_{I,a} M_{I,a}$. Thus we may choose $a \in I \subset R$ such that $k_1, \ldots, k_t$ are elements of $S_{I,a}^{\oplus r}$ and map to zero in $M_{I,a}$. For any such pair (I, a) we set

$$M'_{I,a} = S_{I,a}^{\oplus r} / \sum S_{I,a} k_j.$$

Since $M_A = S_A^{\oplus r} / \sum S_A k_j$ we see that also $M_A = colim_{I,a} M'_{I,a}$. At this point we may apply Algebra, Lemma 7.120.5 (3) to conclude that $M'_{I,a}$ is flat for some pair (I, a). (This lemma does not apply a priori to the system $M_{I,a}$ as the transition maps may not satisfy the assumptions of the lemma.) Since flatness implies torsion free (Lemma 12.17.3), we also conclude that $M'_{I,a} = M_{I,a}$ for such a pair and we win. $\square$

12.20. Completion and flatnes

In this section we discuss when the completion of a ``big'' flat module is flat.

Lemma 12.20.1. *Let R be a ring. Let $I \subset R$ be an ideal. Let A be a set. Assume R is Noetherian and complete with respect to I. There is a canonical map*

$$\left(\bigoplus\nolimits_{\alpha \in A} R\right)^\wedge \longrightarrow \prod\nolimits_{\alpha \in A} R$$

from the I-adic completion of the direct sum into the product which is universally injective.

Proof. By definition an element x of the left hand side is $x = (x_n)$ where $x_n = (x_{n,\alpha}) \in \bigoplus_{\alpha \in A} R/I^n$ such that $x_{n,\alpha} = x_{n+1,\alpha} \bmod I^n$. As $R = R^\wedge$ we see that for any α there exists a $y_\alpha \in R$ such that $x_{n,\alpha} = y_\alpha \bmod I^n$. Note that for each n there are only finitely many α such that the elements $x_{n,\alpha}$ are nonzero. Conversely, given $(y_\alpha) \in \prod_\alpha R$ such that for each n there are only finitely many α such that $y_\alpha \bmod I^n$ is nonzero, then this defines an element of the left hand side. Hence we can think of an element of the left hand side as infinite ``convergent sums'' $\sum_\alpha y_\alpha$ with $y_\alpha \in R$ such that for each n there are only finitely many y_α which are nonzero modulo I^n. The displayed map maps this element to the element to (y_α) in the product. In particular the map is injective.

Let Q be a finite R-module. We have to show that the map

$$Q \otimes_R \left(\bigoplus\nolimits_{\alpha \in A} R\right)^\wedge \longrightarrow Q \otimes_R \left(\prod\nolimits_{\alpha \in A} R\right)$$

is injective, see Algebra, Theorem 7.76.3. Choose a presentation $R^{\oplus k} \to R^{\oplus m} \to Q \to 0$ and denote $q_1, \ldots, q_m \in Q$ the corresponding generators for Q. By Artin-Rees (Algebra, Lemma 7.47.4) there exists a constant c such that $\mathrm{Im}(R^{\oplus k} \to R^{\oplus m}) \cap (I^N)^{\oplus m} \subset$

$\mathrm{Im}((I^{N-c})^{\oplus k} \to R^{\oplus m})$. Let us contemplate the diagram

$$\begin{tikzcd} \bigoplus_{l=1}^{k} \left(\bigoplus_{\alpha \in A} R\right)^\wedge \arrow[r] \arrow[d] & \bigoplus_{j=1}^{m} \left(\bigoplus_{\alpha \in A} R\right)^\wedge \arrow[r] \arrow[d] & Q \otimes_R \left(\bigoplus_{\alpha \in A} R\right)^\wedge \arrow[r] \arrow[d] & 0 \\ \bigoplus_{l=1}^{k} \left(\prod_{\alpha \in A} R\right) \arrow[r] & \bigoplus_{j=1}^{m} \left(\prod_{\alpha \in A} R\right) \arrow[r] & Q \otimes_R \left(\prod_{\alpha \in A} R\right) \arrow[r] & 0 \end{tikzcd}$$

with exact rows. Pick an element $\sum_j \sum_\alpha y_{j,\alpha}$ of $\bigoplus_{j=1,\ldots,m} \left(\bigoplus_{\alpha \in A} R\right)^\wedge$. If this element maps to zero in the module $Q \otimes_R \left(\prod_{\alpha \in A} R\right)$, then we see in particular that $\sum_j q_j \otimes y_{j,\alpha} = 0$ in Q for each α. Thus we can find an element $(z_{1,\alpha}, \ldots, z_{k,\alpha}) \in \bigoplus_{l=1,\ldots,k} R$ which maps to $(y_{1,\alpha}, \ldots, y_{m,\alpha}) \in \bigoplus_{j=1,\ldots,m} R$. Moreover, if $y_{j,\alpha} \in I^{N_\alpha}$ for $j = 1, \ldots, m$, then we may assume that $z_{l,\alpha} \in I^{N_\alpha - c}$ for $l = 1, \ldots, k$. Hence the sum $\sum_l \sum_\alpha z_{l,\alpha}$ is ``convergent'' and defines an element of $\bigoplus_{l=1,\ldots,k} \left(\bigoplus_{\alpha \in A} R\right)^\wedge$ which maps to the element $\sum_j \sum_\alpha y_{j,\alpha}$ we started out with. Thus the right vertical arrow is injective and we win. □

Lemma 12.20.2. *Let R be a ring. Let $I \subset R$ be an ideal. Let A be a set. Assume R is Noetherian. The completion $(\bigoplus_{\alpha \in A} R)^\wedge$ is a flat R-module.*

Proof. Denote $R^\wedge$ the completion of R with respect to I. As $R \to R^\wedge$ is flat by Algebra, Lemma 7.90.3 it suffices to prove that $(\bigoplus_{\alpha \in A} R)^\wedge$ is a flat $R^\wedge$-module (use Algebra, Lemma 7.35.3). Since

$$(\bigoplus\nolimits_{\alpha \in A} R)^\wedge = (\bigoplus\nolimits_{\alpha \in A} R^\wedge)^\wedge$$

we may replace R by $R^\wedge$ and assume that R is complete with respect to I (see Algebra, Lemma 7.90.8). In this case Lemma 12.20.1 tells us the map $(\bigoplus_{\alpha \in A} R)^\wedge \to \prod_{\alpha \in A} R$ is universally injective. Thus, by Algebra, Lemma 7.76.7 it suffices to show that $\prod_{\alpha \in A} R$ is flat. By Algebra, Proposition 7.84.5 (and Algebra, Lemma 7.84.4) we see that $\prod_{\alpha \in A} R$ is flat. □

12.21. The Koszul complex

We define the Koszul complex as follows.

Definition 12.21.1. Let R be a ring. Let $\varphi : E \to R$ be an R-module map. The *Koszul complex* $K_\bullet(\varphi)$ associated to φ is the commutative differential graded algebra defined as follows:

1. the underlying graded algebra is the exterior algebra $K_\bullet(\varphi) = \wedge(E)$,
2. the differential $d : K_\bullet(\varphi) \to K_\bullet(\varphi)$ is the unique derivation such that $d(e) = \varphi(e)$ for all $e \in E = K_1(\varphi)$.

Explicitly, if $e_1 \wedge \ldots \wedge e_n$ is one of the generators of degree n in $K_\bullet(\varphi)$, then

$$d(e_1 \wedge \ldots \wedge e_n) = \sum\nolimits_{i=1,\ldots,n} (-1)^{i+1} \varphi(e_i) e_1 \wedge \ldots \wedge \widehat{e_i} \wedge \ldots \wedge e_n.$$

It is straightforward to see that this gives a well defined derivation on the tensor algebra, which annihilates $e \otimes e$ and hence factors through the exterior algebra.

We often assume that E is a finite free module, say $E = R^{\oplus n}$. In this case the map φ is given by a sequence of elements $f_1, \ldots, f_n \in R$.

Definition 12.21.2. Let R be a ring and let $f_1, \ldots, f_n \in R$. The *Koszul complex on* $f_1, \ldots, f_r$ is the Koszul complex associated to the map $(f_1, \ldots, f_n) : R^{\oplus n} \to R$. Notation $K_\bullet(f_\bullet)$, $K_\bullet(f_1, \ldots, f_n)$, $K_\bullet(R, f_1, \ldots, f_n)$, or $K_\bullet(R, f_\bullet)$.

Of course, if E is finite locally free, then $K_\bullet(\varphi)$ is locally on $Spec(R)$ isomorphic to a Koszul complex $K_\bullet(f_1, \ldots, f_n)$. This complex has many interesting formal properties.

Lemma 12.21.3. *Let $\varphi : E \to R$ and $\varphi : E' \to R$ be an R-module maps. Let $\psi : E \to E'$ be an R-module map such that $\varphi' \circ \psi = \varphi$. Then ψ induces a homomorphism of differential graded algebras $K_\bullet(\varphi) \to K_\bullet(\varphi')$.*

Proof. This is immediate from the definitions. □

Lemma 12.21.4. *Let $f_1, \ldots, f_c \in R$ be a sequence. Let (x_{ij}) be an invertible $c \times c$-matrix with coefficients in R. Then the complexes $K_\bullet(f_\bullet)$ and*

$$K_\bullet(\sum x_{1j}f_j, \sum x_{2j}f_j, \ldots, \sum x_{cj}f_j)$$

are isomorphic.

Proof. Set $g_i = \sum x_{ij}f_j$. The matrix (x_{ij}) gives an isomorphism $x : R^{\oplus c} \to R^{\oplus c}$ such that $(g_1, \ldots, g_c) \circ x = (f_1, \ldots, f_c)$. Hence this follows from the functoriality of the Koszul complex described in Lemma 12.21.3. □

Lemma 12.21.5. *Let R be a ring. Let $\varphi : E \to R$ be an R-module map. Let $e \in E$ with image $f = \varphi(e)$ in R. Then*

$$f = de + ed$$

as endomorphisms of $K_\bullet(\varphi)$.

Proof. This is true because $d(ea) = d(e)a - ed(a) = fa - ed(a)$. □

Lemma 12.21.6. *Let R be a ring. Let $f_1, \ldots, f_c \in R$ be a sequence. Multiplication by f_i on $K_\bullet(f_\bullet)$ is homotopic to zero, and in particular the cohomology modules $H_i(K_\bullet(f_\bullet))$ are annihilated by the ideal $(f_1, \ldots, f_r)$.*

Proof. Special case of Lemma 12.21.5. □

In Derived Categories, Section 11.8 we defined the cone of a morphism of cochain complexes. The cone $C(f)_\bullet$ of a morphism of chain complexes $f : A_\bullet \to B_\bullet$ is the complex $C(f)_\bullet$ given by $C(f)_n = B_n \oplus A_{n-1}$ and differential

$$d_{C(f),n} = \begin{pmatrix} d_{B,n} & f_{n-1} \\ 0 & -d_{A,n-1} \end{pmatrix} \tag{12.21.6.1}$$

It comes equipped with canonical morphisms of complexes $i : B_\bullet \to C(f)_\bullet$ and $p : C(f)_\bullet \to A_\bullet[-1]$ induced by the obvious maps $B_n \to C(f)_n \to A_{n-1}$.

Lemma 12.21.7. *Let R be a ring. Let $\varphi : E \to R$ be an R-module map. Let $f \in R$. Set $E' = E \oplus R$ and define $\varphi' : E' \to R$ by φ on E and multiplication by f on R. The complex $K_\bullet(\varphi')$ is isomorphic to the cone of the map of complexes*

$$f : K_\bullet(\varphi) \longrightarrow K_\bullet(\varphi).$$

Proof. Denote $e_0 \in E'$ the element $1 \in R \subset R \oplus E$. By our definition of the cone above we see that

$$C(f)_n = K_n(\varphi) \oplus K_{n-1}(\varphi) = \wedge^n(E) \oplus \wedge^{n-1}(E) = \wedge^n(E')$$

where in the last = we map $(0, e_1 \wedge \ldots \wedge e_{n-1})$ to $e_0 \wedge e_1 \wedge \ldots \wedge e_{n-1}$ in $\wedge^n(E')$. A computation shows that this isomorphism is compatible with differentials. Namely, this is clear for elements of the first summand as $\varphi'|_E = \varphi$ and $d_{C(f)}$ restricted to the first summand is just $d_{K_\bullet(\varphi)}$. On the other hand, if $e_1 \wedge \ldots \wedge e_{n-1}$ is in the first summand, then

$$d_{C(f)}(0, e_1 \wedge \ldots \wedge e_{n-1}) = fe_1 \wedge \ldots \wedge e_{n-1} - d_{K_\bullet(\varphi)}(e_1 \wedge \ldots \wedge e_{n-1})$$

and on the other hand

$$\begin{aligned}
&d_{K_\bullet(\varphi')}(e_0 \wedge e_1 \wedge \ldots \wedge e_{n-1}) \\
&= \textstyle\sum_{i=0,\ldots,n-1}(-1)^i \varphi'(e_i)e_0 \wedge \ldots \wedge \widehat{e_i} \wedge \ldots \wedge e_{n-1} \\
&= fe_1 \wedge \ldots \wedge e_{n-1} + \textstyle\sum_{i=1,\ldots,n-1}(-1)^i \varphi(e_i)e_0 \wedge \ldots \wedge \widehat{e_i} \wedge \ldots \wedge e_{n-1} \\
&= fe_1 \wedge \ldots \wedge e_{n-1} - e_0 \left(\textstyle\sum_{i=1,\ldots,n-1}(-1)^{i+1} \varphi(e_i)e_1 \wedge \ldots \wedge \widehat{e_i} \wedge \ldots \wedge e_{n-1}\right)
\end{aligned}$$

which is the image of the result of the previous computation. □

Lemma 12.21.8. *Let R be a ring. Let $f_1, \ldots, f_n$ be a sequence of elements of R. The complex $K_\bullet(f_1, \ldots, f_n)$ is isomorphic to the cone of the map of complexes*

$$f_n : K_\bullet(f_1, \ldots, f_{n-1}) \longrightarrow K_\bullet(f_1, \ldots, f_{n-1}).$$

Proof. Special case of Lemma 12.21.7. □

Lemma 12.21.9. *Let R be a ring. Let $A_\bullet$ be a complex of R-modules. Let $f, g \in R$. Let $C(f)_\bullet$ be the cone of $f : A_\bullet \to A_\bullet$. Define similarly $C(g)_\bullet$ and $C(fg)_\bullet$. Then $C(fg)_\bullet$ is homotopy equivalent to the cone of a map*

$$C(f)_\bullet[1] \longrightarrow C(g)_\bullet$$

Proof. We first prove this if $A_\bullet$ is the complex consisting of R placed in degree 0. In this case the map we use is

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & R \arrow[r, "f"] \arrow[d, "1"] & R \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & R \arrow[r, "g"] & R \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

The cone of this is the chain complex consisting of $R \oplus R$ placed in degrees 1 and 0 and differential (12.21.6.1)

$$\begin{pmatrix} g & 1 \\ 0 & -f \end{pmatrix} : R^{\oplus 2} \longrightarrow R^{\oplus 2}$$

We leave it to the reader to show this this chain complex is homotopic to the complex $fg : R \to R$. In general we write $C(f)_\bullet$ and $C(g)_\bullet$ as the total complex of the double complexes

$$(R \xrightarrow{f} R) \otimes_R A_\bullet \quad \text{and} \quad (R \xrightarrow{g} R) \otimes_R A_\bullet$$

and in this way we deduce the result from the special case discussed above. Some details omitted. □

Lemma 12.21.10. *Let R be a ring. Let $\varphi : E \to R$ be an R-module map. Let $f, g \in R$. Set $E' = E \oplus R$ and define $\varphi'_f, \varphi'_g, \varphi'_{fg} : E' \to R$ by φ on E and multiplication by f, g, fg on R. The complex $K_\bullet(\varphi'_{fg})$ is isomorphic to the cone of a map of complexes*

$$K_\bullet(\varphi'_f)[1] \longrightarrow K_\bullet(\varphi'_g).$$

Proof. By Lemma 12.21.7 the complex $K_\bullet(\varphi'_f)$ is isomorphic to the cone of multiplication by f on $K_\bullet(\varphi)$ and similarly for the other two cases. Hence the the lemma follows from Lemma 12.21.9. □

Lemma 12.21.11. *Let R be a ring. Let $f_1, \ldots, f_{n-1}$ be a sequence of elements of R. Let $f, g \in R$. The complex $K_\bullet(f_1, \ldots, f_{n-1}, fg)$ is homotopy equivalent to the cone of a map of complexes*

$$K_\bullet(f_1, \ldots, f_{n-1}, f)[1] \longrightarrow K_\bullet(f_1, \ldots, f_{n-1}, g)$$

Proof. Special case of Lemma 12.21.10. □

Lemma 12.21.12. *Let A be a ring. Let $f_1, \ldots, f_n$, $g_1, \ldots, g_m$ be elements of A. Then there is an isomorphism of Koszul complexes*

$$K_\bullet(A, f_1, \ldots, f_n, g_1, \ldots, g_m) = Tot(K_\bullet(A, f_1, \ldots, f_n) \otimes_A K_\bullet(A, g_1, \ldots, g_m)).$$

Proof. We first show that $K_\bullet(A, f_1, \ldots, f_n, g_1, \ldots, g_m)$ is isomorphic to the tensor product $K_\bullet(A, f_1, \ldots, f_n) \otimes_A K_\bullet(A, g_1, \ldots, g_m)$ as a differential graded A-algebra. This is clear as the multiplication map

$$\wedge(A^{\oplus n}) \otimes_A \wedge(A^{\oplus m}) \longrightarrow \wedge(A^{\oplus n} \oplus A^{\oplus m})$$

is an isomorphism and the fact that the d of generators agree. Thus the lemma follows from Homology, Lemma 10.25.5. □

12.22. Koszul regular sequences

Please take a look at Algebra, Sections 7.65 and 7.66 before looking at this one.

Definition 12.22.1. Let R be a ring. A sequence of elements $f_1, \ldots, f_c$ of R is called *Koszul-regular* if $H_i(K_\bullet(f_1, \ldots, f_r)) = 0$ for all $i \neq 0$. A sequence of elements $f_1, \ldots, f_c$ of R is called H_1*-regular* if $H_1(K_\bullet(f_1, \ldots, f_r)) = 0$.

Clear a Koszul-regular sequence is H_1-regular. If $f = f_1 \in R$ is a length 1 sequence then it is clear that the following are all equivalent

(1) f is a regular sequence of length one,
(2) f is a Koszul-regular sequence of length one, and
(3) f is a H_1-regular sequence of length one.

It is also clear that these imply that f is a quasi-regular sequence of length one. But there do exist quasi-regular sequences of length 1 which are not regular sequences. Namely, let

$$R = k[x, y_0, y_1, \ldots]/(xy_0, xy_1 - y_0, xy_2 - y_1, \ldots)$$

and let f be the image of x in R. Then f is a zero divisor, but $\bigoplus_{n \geq 0}(f^n)/(f^{n+1}) \cong k[x]$ is a polynomial ring.

Lemma 12.22.2. *A regular sequence is Koszul-regular.*

Proof. Let $f_1, \ldots, f_c$ be a regular sequence. Then f_1 is a nonzero divisor in R. Hence

$$0 \to K_\bullet(R, f_2, \ldots, f_c) \xrightarrow{f_1} K_\bullet(R, f_2, \ldots, f_c) \to K_\bullet(R/(f_1), \overline{f}_2, \ldots, \overline{f}_c) \to 0$$

is a short exact sequence of complexes. By Lemma 12.21.8 the complex $K_\bullet(R, f_1, \ldots, f_c)$ is isomorphic to the cone of the first map. Hence $K_\bullet(R/(f_1), \overline{f}_2, \ldots, \overline{f}_c)$ is quasi-isomorphic to $K_\bullet(R, f_1, \ldots, f_c)$. As $\overline{f}_2, \ldots, \overline{f}_c$ is a regular sequence in $R/(f_1)$ the result follows from the case $c = 1$ discussed above and induction. □

Lemma 12.22.3. *Let $f_1, \ldots, f_{c-1} \in R$ be a sequence and $f, g \in R$.*

(1) *If $f_1, \ldots, f_{c-1}, f$ and $f_1, \ldots, f_{c-1}, g$ are H_1-regular then $f_1, \ldots, f_{c-1}, fg$ is an H_1-regular sequence too.*
(2) *If $f_1, \ldots, f_{c-1}, f$ and $f_1, \ldots, f_{c-1}, f$ are Koszul-regular then $f_1, \ldots, f_{c-1}, fg$ is a Koszul-regular sequence too.*

Proof. By Lemma 12.21.11 we have exact sequences

$$H_i(K_\bullet(f_1, \ldots, f_{c-1}, f)) \to H_i(K_\bullet(f_1, \ldots, f_{c-1}, fg)) \to H_i(K_\bullet(f_1, \ldots, f_{c-1}, g))$$

for all i. □

Lemma 12.22.4. *Let $\varphi : R \to S$ be a flat ring map.*

1. *If $f_1, \ldots, f_r$ is a H_1-regular sequence in R, then $\varphi(f_1), \ldots, \varphi(f_r)$ is a H_1-regular sequence in S.*
2. *If $f_1, \ldots, f_r$ is a Koszul-regular sequence in R, then $\varphi(f_1), \ldots, \varphi(f_r)$ is a Koszul-regular sequence in S.*

Proof. This is true because $K_\bullet(f_1, \ldots, f_r) \otimes_R S = K_\bullet(\varphi(f_1), \ldots, \varphi(f_r))$. □

Lemma 12.22.5. *An H_1-regular sequence is quasi-regular.*

Proof. Let $f_1, \ldots, f_c$ be an H_1-regular sequence. Denote $J = (f_1, \ldots, f_c)$. The assumption means that we have an exact sequence

$$\wedge^2(R^c) \to R^{\oplus c} \to J \to 0$$

where the first arrow is given by $e_i \wedge e_j \mapsto f_i e_j - f_j e_i$. In particular this implies that

$$J/J^2 = J \otimes_R R/J = (R/J)^c$$

is a finite free module. To finish the proof we have to prove for every $n \geq 2$ the following: if

$$\xi = \sum\nolimits_{|I|=n, I=(i_1,\ldots,i_c)} a_I f_1^{i_1} \ldots f_c^{i_c} \in J^{n+1}$$

then $a_I \in J$ for all I. Note that $f_1, \ldots, f_{c-1}, f_c^n$ is a H_1-regular sequence by Lemma 12.22.3. Hence we see that the required result holds for the multi-index $I = (0, \ldots, 0, n)$. It turns out that we can reduce the general case to this case as follows.

Let $S = R[x_1, x_2, \ldots, x_c, 1/x_c]$. The ring map $R \to S$ is faithfully flat, hence $f_1, \ldots, f_c$ is an H_1-regular sequence in S, see Lemma 12.22.4. By Lemma 12.21.4 we see that

$$g_1 = f_1 - x_1/x_c f_c, \ldots g_{c-1} = f_{c-1} - x_{c-1}/x_c f_c, g_c = (1/x_c) f_c$$

is an H_1-regular sequence in S. Finally, note that our element ξ can be rewritten

$$\xi = \sum\nolimits_{|I|=n, I=(i_1,\ldots,i_c)} a_I (g_1 + x_c g_c)^{i_1} \ldots (g_{c-1} + x_c g_c)^{i_{c-1}} (x_c g_c)^{i_c}$$

and the coefficient of g_c^n in this expression is

$$\sum a_I x_1^{i_1} \ldots x_c^{i_c} \in JS.$$

Since the monomials $x_1^{i_1} \ldots x_c^{i_c}$ form part of an R-basis of S over R we conclude that $a_I \in J$ for all I as desired. □

Lemma 12.22.6. *Let A be a ring. Let $I \subset A$ be an ideal. Let $g_1, \ldots, g_m$ be a sequence in A whose image in A/I is H_1-regular. Then $I \cap (g_1, \ldots, g_m) = I(g_1, \ldots, g_m)$.*

Proof. Consider the exact sequence of complexes

$$0 \to I \otimes_A K_\bullet(A, g_1, \ldots, g_m) \to K_\bullet(A, g_1, \ldots, g_m) \to K_\bullet(A/I, g_1, \ldots, g_m) \to 0$$

Since the complex on the right has $H_1 = 0$ by assumption we see that

$$\mathrm{Coker}(I^{\oplus m} \to I) \longrightarrow \mathrm{Coker}(A^{\oplus m} \to A)$$

is injective. This is equivalent to the assertion of the lemma. □

Lemma 12.22.7. *Let A be a ring. Let $I \subset J \subset A$ be ideals. Assume that $J/I \subset A/I$ is generated by an H_1-regular sequence. Then $I \cap J^2 = IJ$.*

Proof. To prove this choose $g_1, \ldots, g_m \in J$ whose images in A/I form a H_1-regular sequence which generates J/I. In particular $J = I + (g_1, \ldots, g_m)$. Suppose that $x \in I \cap J^2$. Because $x \in J^2$ can write

$$x = \sum a_{ij} g_i g_j + \sum a_j g_j + a$$

with $a_{ij} \in A$, $a_j \in I$ and $a \in I^2$. Then $\sum a_{ij} g_i g_j \in I \cap (g_1, \ldots, g_m)$ hence by Lemma 12.22.6 we see that $\sum a_{ij} g_i g_j \in I(g_1, \ldots, g_m)$. Thus $x \in IJ$ as desired. □

Lemma 12.22.8. *Let A be a ring. Let I be an ideal generated by a quasi-regular sequence $f_1, \ldots, f_n$ in A. Let $g_1, \ldots, g_m \in A$ be elements whose images $\overline{g}_1, \ldots, \overline{g}_m$ form an H_1-regular sequence in A/I. Then $f_1, \ldots, f_n, g_1, \ldots, g_m$ is a quasi-regular sequence in A.*

Proof. We claim that $g_1, \ldots, g_m$ forms an H_1-regular sequence in A/I^d for every d. By induction assume that this holds in A/I^{d-1}. We have a short exact sequence of complexes

$$0 \to K_\bullet(A, g_\bullet) \otimes_A I^{d-1}/I^d \to K_\bullet(A/I^d, g_\bullet) \to K_\bullet(A/I^{d-1}, g_\bullet) \to 0$$

Since $f_1, \ldots, f_n$ is quasi-regular we see that the first complex is a direct sum of copies of $K_\bullet(A/I, g_1, \ldots, g_m)$ hence acyclic in degree 1. By induction hypothesis the last complex is acyclic in degree 1. Hence also the middle complex is. In particular, the sequence $g_1, \ldots, g_m$ forms a quasi-regular sequence in A/I^d for every $d \geq 1$, see Lemma 12.22.5. Now we are ready to prove that $f_1, \ldots, f_n, g_1, \ldots, g_m$ is a quasi-regular sequence in A. Namely, set $J = (f_1, \ldots, f_n, g_1, \ldots, g_m)$ and suppose that (with multinomial notation)

$$\sum\nolimits_{|N|+|M|=d} a_{N,M} f^N g^M \in J^{d+1}$$

for some $a_{N,M} \in A$. We have to show that $a_{N,M} \in J$ for all N, M. Let $e \in \{0, 1, \ldots, d\}$. Then

$$\sum\nolimits_{|N|=d-e,\ |M|=e} a_{N,M} f^N g^M \in (g_1, \ldots, g_m)^{e+1} + I^{d-e+1}$$

Because $g_1, \ldots, g_m$ is a quasi-regular sequence in A/I^{d-e+1} we deduce

$$\sum\nolimits_{|N|=d-e} a_{N,M} f^N \in (g_1, \ldots, g_m) + I^{d-e+1}$$

for each M with $|M| = e$. By Lemma 12.22.6 applied to I^{d-e}/I^{d-e+1} in the ring A/I^{d-e+1} this implies $\sum_{|N|=d-e} a_{N,M} f^N \in I^{d-e}(g_1, \ldots, g_m)$. Since $f_1, \ldots, f_n$ is quasi-regular in A this implies that $a_{N,M} \in J$ for each N, M with $|N| = d - e$ and $|M| = e$. This proves the lemma. □

Lemma 12.22.9. *Let A be a ring. Let I be an ideal generated by an H_1-regular sequence $f_1, \ldots, f_n$ in A. Let $g_1, \ldots, g_m \in A$ be elements whose images $\overline{g}_1, \ldots, \overline{g}_m$ form an H_1-regular sequence in A/I. Then $f_1, \ldots, f_n, g_1, \ldots, g_m$ is an H_1-regular sequence in A.*

Proof. We have to show that $H_1(A, f_1, \ldots, f_n, g_1, \ldots, g_m) = 0$. To do this consider the commutative diagram

$$\begin{tikzcd}
\wedge^2(A^{\oplus n+m}) \arrow[r] \arrow[d] & A^{\oplus n+m} \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
\wedge^2(A/I^{\oplus m}) \arrow[r] & A/I^{\oplus m} \arrow[r] & A/I \arrow[r] & 0
\end{tikzcd}$$

Consider an element $(a_1, \ldots, a_{n+m}) \in A^{\oplus n+m}$ which maps to zero in A. Because $\overline{g}_1, \ldots, \overline{g}_m$ form an H_1-regular sequence in A/I we see that $(\overline{a}_{n+1}, \ldots, \overline{a}_{n+m})$ is the image of some element $\overline{\alpha}$ of $\wedge^2(A/I^{\oplus m})$. We can lift $\overline{\alpha}$ to an element $\alpha \in \wedge^2(A^{\oplus n+m})$ and substract the image of it in $A^{\oplus n+m}$ from our element $(a_1, \ldots, a_{n+m})$. Thus we may assume that $a_{n+1}, \ldots, a_{n+m} \in I$. Since $I = (f_1, \ldots, f_n)$ we can modify our element $(a_1, \ldots, a_{n+m})$ by linear combinations of the elements

$$(0, \ldots, g_j, 0, \ldots, 0, f_i, 0, \ldots, 0)$$

in the image of the top left horizontal arrow to reduce to the case that $a_{n+1}, \ldots, a_{n+m}$ are zero. In this case $(a_1, \ldots, a_n, 0, \ldots, 0)$ defines an element of $H_1(A, f_1, \ldots, f_n)$ which we assumed to be zero. $\square$

Lemma 12.22.10. *Let A be a ring. Let $f_1, \ldots, f_n, g_1, \ldots, g_m \in A$ be an H_1-regular sequence. Then the images $\overline{g}_1, \ldots, \overline{g}_m$ in $A/(f_1, \ldots, f_n)$ form an H_1-regular sequence.*

Proof. Set $I = (f_1, \ldots, f_n)$. We have to show that any relation $\sum_{j=1,\ldots,m} \overline{a}_j \overline{g}_j$ in A/I is a linear combination of trivial relations. Because $I = (f_1, \ldots, f_n)$ we can lift this relation to a relation

$$\sum\nolimits_{j=1,\ldots,m} a_j g_j + \sum\nolimits_{i=1,\ldots,n} b_i f_i = 0$$

in A. By assumption this relation in A is a linear combination of trivial relations. Taking the image in A/I we obtain what we want. $\square$

Lemma 12.22.11. *Let A be a ring. Let I be an ideal generated by a Koszul-regular sequence $f_1, \ldots, f_n$ in A. Let $g_1, \ldots, g_m \in A$ be elements whose images $\overline{g}_1, \ldots, \overline{g}_m$ form a Koszul-regular sequence in A/I. Then $f_1, \ldots, f_n, g_1, \ldots, g_m$ is a Koszul-regular sequence in A.*

Proof. Our assumptions say that $K_\bullet(A, f_1, \ldots, f_n)$ is a finite free resolution of A/I and $K_\bullet(A/I, \overline{g}_1, \ldots, \overline{g}_m)$ is a finite free resolution of $A/(f_i, g_j)$ over A/I. Then

$$\begin{aligned} K_\bullet(A, f_1, \ldots, f_n, g_1, \ldots, g_m) &= \text{Tot}(K_\bullet(A, f_1, \ldots, f_n) \otimes_A K_\bullet(A, g_1, \ldots, g_m)) \\ &\cong \text{Tot}(A/I \otimes_A K_\bullet(A, g_1, \ldots, g_m)) \\ &= K_\bullet(A/I, \overline{g}_1, \ldots, \overline{g}_m) \\ &\cong A/(f_i, g_j) \end{aligned}$$

The first equality by Lemma 12.21.12. The first quasi-isomorphism by Lemma 12.3.8. The second equality is clear. The last quasi-isomorphism by assumption. Hence we win. $\square$

To conclude in the following lemma it is necessary to assume that both $f_1, \ldots, f_n$ and $f_1, \ldots, f_n, g_1, \ldots, g_m$ are Koszul-regular. A counter example to dropping the assumption that $f_1, \ldots, f_n$ is Koszul-regular is Examples, Lemma 64.6.1.

Lemma 12.22.12. *Let A be a ring. Let $f_1, \ldots, f_n, g_1, \ldots, g_m \in A$. If both $f_1, \ldots, f_n$ and $f_1, \ldots, f_n, g_1, \ldots, g_m$ are Koszul-regular sequences in A, then $\overline{g}_1, \ldots, \overline{g}_m$ in $A/(f_1, \ldots, f_n)$ form a Koszul-regular sequence.*

Proof. Set $I = (f_1, \ldots, f_n)$. Our assumptions say that $K_\bullet(A, f_1, \ldots, f_n)$ is a finite free resolution of A/I and $K_\bullet(A, f_1, \ldots, f_n, g_1, \ldots, g_m)$ is a finite free resolution of $A/(f_i, g_j)$

over A. Then

$$\begin{aligned} A/(f_i, g_j) &\cong K_\bullet(A, f_1, \ldots, f_n, g_1, \ldots, g_m) \\ &= \text{Tot}(K_\bullet(A, f_1, \ldots, f_n) \otimes_A K_\bullet(A, g_1, \ldots, g_m)) \\ &\cong \text{Tot}(A/I \otimes_A K_\bullet(A, g_1, \ldots, g_m)) \\ &= K_\bullet(A/I, \overline{g}_1, \ldots, \overline{g}_m) \end{aligned}$$

The first quasi-isomorphism by assumption. The first equality by Lemma 12.21.12. The second quasi-isomorphism by Lemma 12.3.8. The second equality is clear. Hence we win. □

Lemma 12.22.13. *Let R be a ring. Let I be an ideal generated by $f_1, \ldots, f_r \in R$.*

(1) *If I can be generated by a quasi-regular sequence of length r, then $f_1, \ldots, f_r$ is a quasi-regular sequence.*
(2) *If I can be generated by an H_1-regular sequence of length r, then $f_1, \ldots, f_r$ is an H_1-regular sequence.*
(3) *If I can be generated by a Koszul-regular sequence of length r, then $f_1, \ldots, f_r$ is a Koszul-regular sequence.*

In other words, a minimal set of generators of an ideal generated by a quasi-regular (resp. H_1-regular, Koszul-regular) sequence is a quasi-regular (resp. H_1-regular, Koszul-regular) sequence.

Proof. If I can be generated by a quasi-regular sequence of length r, then I/I^2 is free of rank r over R/I. Since $f_1, \ldots, f_r$ generate by assumption we see that the images $\overline{f_i}$ form a basis of I/I^2 over R/I. It follows that $f_1, \ldots, f_r$ is a quasi-regular sequence as all this means, besides the freeness of I/I^2, is that the maps $\text{Sym}^n_{R/I}(I/I^2) \to I^n/I^{n+1}$ are isomorphisms.

We continue to assume that I can be generated by a quasi-regular sequence, say $g_1, \ldots, g_r$. Write $g_j = \sum a_{ij} f_i$. As $f_1, \ldots, f_r$ is quasi-regular according to the previous paragraph, we see that $\det(a_{ij})$ is invertible mod I. The matrix a_{ij} gives a map $R^{\oplus r} \to R^{\oplus r}$ which induces a map of Koszul complexes $\alpha : K_\bullet(R, f_1, \ldots, f_r) \to K_\bullet(R, g_1, \ldots, g_r)$, see Lemma 12.21.3. This map becomes an isomorphism on inverting $\det(a_{ij})$. Since the cohomology modules of both $K_\bullet(R, f_1, \ldots, f_r)$ and $K_\bullet(R, g_1, \ldots, g_r)$ are annihilated by I, see Lemma 12.21.6, we see that α is a quasi-isomorphism. Hence if $g_1, \ldots, g_r$ is H_1-regular, then so is $f_1, \ldots, f_r$. Similarly for Koszul-regular. □

Lemma 12.22.14. *Let $A \to B$ be a ring map. Let $f_1, \ldots, f_r$ be a sequence in B such that $B/(f_1, \ldots, f_r)$ is A-flat. Let $A \to A'$ be a ring map. Then the canonical map*

$$H_1(K_\bullet(B, f_1, \ldots, f_r)) \otimes_A A' \longrightarrow H_1(K_\bullet(B', f'_1, \ldots, f'_r))$$

is surjective, where $B' = B \otimes_A A'$ and $f'_i \in B'$ is the image of f_i.

Proof. The sequence

$$\wedge^2(B^{\oplus r}) \to B^{\oplus r} \to B \to B/J \to 0$$

is a complex of A-modules with B/J flat over A and cohomology group $H_1 = H_1(K_\bullet(B, f_1, \ldots, f_r))$ in the spot $B^{\oplus r}$. If we tensor this with A' we obtain a complex

$$\wedge^2((B')^{\oplus r}) \to (B')^{\oplus r} \to B' \to B'/J' \to 0$$

which is exact at B' and B'/J'. In order to compute its cohomology group $H'_1 = H_1(K_\bullet(B', f'_1, \ldots, f'_r))$ at $(B')^{\oplus r}$ we split the first sequence above into short exact sequences $0 \to J \to B \to B/J \to$

0 and $0 \to K \to B^{\oplus r} \to J \to 0$ and $\wedge^2(B^{\oplus r}) \to K \to H_1 \to 0$. Tensoring with A' over A we obtain the exact sequences

$$\begin{gathered} 0 \to J \otimes_A A' \to B \otimes_A A' \to (B/J) \otimes_A A' \to 0 \\ K \otimes_A A' \to B^{\oplus r} \otimes_A A' \to J \otimes_A A' \to 0 \\ \wedge^2(B^{\oplus r}) \otimes_A A' \to K \otimes_A A' \to H_1 \otimes_A A' \to 0 \end{gathered}$$

where the first one is exact as B/J is flat over A, see Algebra, Lemma 7.35.11. Hence we conclude what we want. □

Lemma 12.22.15. *Let R be a ring. Let $a_1, \ldots, a_n \in R$ be elements such that $R \to R^{\oplus n}$, $x \mapsto (xa_1, \ldots, xa_n)$ is injective. Then the element $\sum a_i t_i$ of the polynomial ring $R[t_1, \ldots, t_n]$ is a nonzero divisor.*

Proof. If one of the a_i is a unit this is just the statement that any element of the form $t_1 + a_2 t_2 + \ldots + a_n t_n$ is a nonzero divisor in the polynomial ring over R.

Case I: R is Noetherian. Let $\mathfrak{q}_j$, $j = 1, \ldots, m$ be the associated primes of R. We have to show that each of the maps

$$\sum a_i t_i : \mathrm{Sym}^d(R^{\oplus n}) \longrightarrow \mathrm{Sym}^{d+1}(R^{\oplus n})$$

is injective. As $\mathrm{Sym}^d(R^{\oplus n})$ is a free R-module its associated primes are $\mathfrak{q}_j$, $j = 1, \ldots, m$. For each j there exists an $i = i(j)$ such that $a_i \not\in \mathfrak{q}_j$ because there exists an $x \in R$ with $\mathfrak{q}_j x = 0$ but $a_i x \neq 0$ for some i by assumption. Hence a_i is a unit in $R_{\mathfrak{q}_j}$ and the map is injective after localizing at $\mathfrak{q}_j$. Thus the map is injective, see Algebra, Lemma 7.60.18.

Case II: R general. We can write R as the union of Noetherian rings R_λ with $a_1, \ldots, a_n \in R_\lambda$. For each R_λ the result holds, hence the result holds for R. □

Lemma 12.22.16. *Let R be a ring. Let $f_1, \ldots, f_n$ be a Koszul-regular sequence in R. Consider the faithfully flat, smooth ring map*

$$R \longrightarrow S = R[\{t_{ij}\}_{i \leq j}, t_{11}^{-1}, t_{22}^{-1}, \ldots, t_{nn}^{-1}]$$

For $1 \leq i \leq n$ set

$$g_i = \sum\nolimits_{i \leq j} t_{ij} f_j \in S.$$

Then $g_1, \ldots, g_n$ is a regular sequence in S and $(f_1, \ldots, f_n)S = (g_1, \ldots, g_n)$.

Proof. The equality of ideals is obvious as the matrix

$$\begin{pmatrix} t_{11} & t_{12} & t_{13} & \ldots \\ 0 & t_{22} & t_{23} & \ldots \\ 0 & 0 & t_{33} & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

is invertible in S. Because $f_1, \ldots, f_n$ is a Koszul-regular sequence we see that the kernel of $R \to R^{\oplus n}$, $x \mapsto (xf_1, \ldots, xf_n)$ is zero (as it computes the $n$the Koszul homology of R w.r.t. $f_1, \ldots, f_n$). Hence by Lemma 12.22.15 we see that $g_1 = f_1 t_{11} + \ldots + f_n t_{1n}$ is a nonzero divisor in $S' = R[t_{11}, t_{12}, \ldots, t_{1n}, t_{11}^{-1}]$. We see that $g_1, f_2, \ldots, f_n$ is a Koszul-sequence in S' by Lemma 12.22.4 and 12.22.13. We conclude that $\overline{f}_2, \ldots, \overline{f}_n$ is a Koszul-regular sequence in $S'/(g_2)$ by Lemma 12.22.12. Hence by induction on n we see that the images $\overline{g}_2, \ldots, \overline{g}_n$ of $g_2, \ldots, g_n$ in $S'/(g_2)[\{t_{ij}\}_{2 \leq i \leq j}, t_{22}^{-1}, \ldots, t_{nn}^{-1}]$ form a regular sequence. This in turn means that $g_1, \ldots, g_n$ forms a regular sequence in S. □

12.23. Regular ideals

We will discuss the notion of a regular ideal sheaf in great generality in Divisors, Section 26.12. Here we define the corresponding notion in the affine case, i.e., in the case of an ideal in a ring.

Definition 12.23.1. Let R be a ring and let $I \subset R$ be an ideal.

(1) We say I is a *regular ideal* if for every $\mathfrak{p} \in V(I)$ there exists a $g \in R$, $g \notin \mathfrak{p}$ and a regular sequence $f_1, \ldots, f_r \in R_g$ such that I_g is generated by $f_1, \ldots, f_r$.
(2) We say I is a *Koszul-regular ideal* if for every $\mathfrak{p} \in V(I)$ there exists a $g \in R$, $g \notin \mathfrak{p}$ and a Koszul-regular sequence $f_1, \ldots, f_r \in R_g$ such that I_g is generated by $f_1, \ldots, f_r$.
(3) We say I is a *H_1-regular ideal* if for every $\mathfrak{p} \in V(I)$ there exists a $g \in R$, $g \notin \mathfrak{p}$ and an H_1-regular sequence $f_1, \ldots, f_r \in R_g$ such that I_g is generated by $f_1, \ldots, f_r$.
(4) We say I is a *quasi-regular ideal* if for every $\mathfrak{p} \in V(I)$ there exists a $g \in R$, $g \notin \mathfrak{p}$ and a quasi-regular sequence $f_1, \ldots, f_r \in R_g$ such that I_g is generated by $f_1, \ldots, f_r$.

It is clear that given $I \subset R$ we have the implications

$$\begin{aligned} I \text{ is a regular ideal} &\Rightarrow I \text{ is a Koszul-regular ideal} \\ &\Rightarrow I \text{ is a } H_1\text{-regular ideal} \\ &\Rightarrow I \text{ is a quasi-regular ideal} \end{aligned}$$

see Lemmas 12.22.2 and 12.22.5. Such an ideal is always finitely generated.

Lemma 12.23.2. *A quasi-regular ideal is finitely generated.*

Proof. Let $I \subset R$ be a quasi-regular ideal. Since $V(I)$ is quasi-compact, there exist $g_1, \ldots, g_m \in R$ such that $V(I) \subset D(g_1) \cup \ldots \cup D(g_m)$ and such that I_{g_j} is generated by a quasi-regular sequence $g_{j1}, \ldots, g_{jr_j} \in R_{g_j}$. Write $g_{ji} = g'_{ji}/g_j^{e_{ij}}$ for some $g'_{ij} \in I$. Write $1 + x = \sum g_j h_j$ for some $x \in I$ which is possible as $V(I) \subset D(g_1) \cup \ldots \cup D(g_m)$. Note that $Spec(R) = D(g_1) \cup \ldots \cup D(g_m) \bigcup D(x)$ Then I is generated by the elements g'_{ij} and x as these generate on each of the pieces of the cover, see Algebra, Lemma 7.21.2. ☐

We prove flat descent for Koszul-regular, H_1-regular, quasi-regular ideals.

Lemma 12.23.3. *Let $A \to B$ be a faithfully flat ring map. Let $I \subset A$ be an ideal. If IB is a Koszul-regular (resp. H_1-regular, resp. quasi-regular) ideal in B, then I is a Koszul-regular (resp. H_1-regular, resp. quasi-regular) ideal in A.*

Proof. We fix the prime $\mathfrak{p} \supset I$ throughout the proof. Assume IB is quasi-regular. By Lemma 12.23.2 IB is a finite module, hence I is a finite A-module by Algebra, Lemma 7.77.2. As $A \to B$ is flat we see that

$$I/I^2 \otimes_{A/I} B/IB = I/I^2 \otimes_A B = IB/(IB)^2.$$

As IB is quasi-regular, the B/IB-module $IB/(IB)^2$ is finite locally free. Hence I/I^2 is finite projective, see Algebra, Proposition 7.77.3. In particular, after replacing A by A_f for some $f \in A$, $f \notin \mathfrak{p}$ we may assume that I/I^2 is free of rank r. Pick $f_1, \ldots, f_r \in I$ which give a basis of I/I^2. By Nakayama's lemma (see Algebra, Lemma 7.14.5) we see that, after another replacement $A \rightsquigarrow A_f$ as above, I is generated by $f_1, \ldots, f_r$.

Proof of the ``quasi-regular'' case. Above we have seen that I/I^2 is free on the r-generators $f_1, \ldots, f_r$. To finish the proof in this case we have to show that the maps $\mathrm{Sym}^d(I/I^2) \to$

I^d/I^{d+1} are isomorphisms for each $d \geq 2$. This is clear as the faithfully flat base changes $\text{Sym}^d(IB/(IB)^2) \to (IB)^d/(IB)^{d+1}$ are isomorphisms locally on B by assumption. Details omitted.

Proof of the ``$H_1$-regular" and ``Koszul-regular" case. Consider the sequence of elements $f_1, \ldots, f_r$ generating I we constructed above. By Lemma 12.22.13 we see that $f_1, \ldots, f_r$ map to a H_1-regular or Koszul-regular sequence in B_g for any $g \in B$ such that IB is generated by an H_1-regular or Koszul-regular sequence. Hence $K_\bullet(A, f_1, \ldots, f_r) \otimes_A B_g$ has vanishing H_1 or H_i, $i > 0$. Since the homology of $K_\bullet(B, f_1, \ldots, f_r) = K_\bullet(A, f_1, \ldots, f_r) \otimes_A B$ is annihilated by IB (see Lemma 12.21.6) and since $V(IB) \subset \bigcup_{g \text{ as above}} D(g)$ we conclude that $K_\bullet(A, f_1, \ldots, f_r) \otimes_A B$ has vanishing homology in degree 1 or all positive degrees. Using that $A \to B$ is faithfully flat we conclude that the same is true for $K_\bullet(A, f_1, \ldots, f_r)$. □

Lemma 12.23.4. *Let A be a ring. Let $I \subset J \subset A$ be ideals. Assume that $J/I \subset A/I$ is a H_1-regular ideal. Then $I \cap J^2 = IJ$.*

Proof. Follows immediately from Lemma 12.22.7 by localizing. □

12.24. Local complete intersection maps

We can use the material above to define a local complete intersection map between rings using presentations by (finite) polynomial algebras.

Lemma 12.24.1. *Let $A \to B$ be a finite type ring map. If for some presentation $\alpha : A[x_1, \ldots, x_n] \to B$ the kernel I is a Koszul-regular ideal then for any presentation $\beta : A[y_1, \ldots, y_m] \to B$ the kernel J is a Koszul-regular ideal.*

Proof. Choose $f_j \in A[x_1, \ldots, x_n]$ with $\alpha(f_j) = \beta(y_j)$ and $g_i \in A[y_1, \ldots, y_m]$ with $\beta(g_i) = \alpha(x_i)$. Then we get a commutative diagram

$$\begin{tikzcd} A[x_1, \ldots, x_n, y_1, \ldots, y_m] \arrow[r, "y_j \mapsto f_j"'] \arrow[d, "x_i \mapsto g_i"] & A[x_1, \ldots, x_n] \arrow[d] \\ A[y_1, \ldots, y_m] \arrow[r] & B \end{tikzcd}$$

Note that the kernel K of $A[x_i, y_j] \to B$ is equal to $K = (I, y_j - f_j) = (J, x_i - f_i)$. In particular, as I is finitely generated by Lemma 12.23.2 we see that $J = K/(x_i - f_i)$ is finitely generated too.

Pick a prime $\mathfrak{q} \subset B$. Since $I/I^2 \oplus B^{\oplus m} = J/J^2 \oplus B^{\oplus n}$ (Algebra, Lemma 7.123.12) we see that

$$\dim J/J^2 \otimes_B \kappa(\mathfrak{q}) + n = \dim I/I^2 \otimes_B \kappa(\mathfrak{q}) + m.$$

Pick $p_1, \ldots, p_t \in I$ which map to a basis of $I/I^2 \otimes \kappa(\mathfrak{q}) = I \otimes_{A[x_i]} \kappa(\mathfrak{q})$. Pick $q_1, \ldots, q_s \in J$ which map to a basis of $J/J^2 \otimes \kappa(\mathfrak{q}) = J \otimes_{A[y_j]} \kappa(\mathfrak{q})$. So $s + n = t + m$. By Nakayama's lemma there exist $h \in A[x_i]$ and $h' \in A[y_j]$ both mapping to a nonzero element of $\kappa(\mathfrak{q})$ such that $I_h = (p_1, \ldots, p_t)$ in $A[x_i, 1/h]$ and $J_{h'} = (q_1, \ldots, q_s)$ in $A[y_j, 1/h']$. As I is Koszul-regular we may also assume that I_h is generated by a Koszul regular sequence. This sequence must necessarily have length $t = \dim I/I^2 \otimes_B \kappa(\mathfrak{q})$, hence we see that $p_1, \ldots, p_t$ is a Koszul-regular sequence by Lemma 12.22.13. As also $y_1 - f_1, \ldots, y_m - f_m$ is a regular sequence we conclude

$$y_1 - f_1, \ldots, y_m - f_m, p_1, \ldots, p_t$$

is a Koszul-regular sequence in $A[x_i, y_j, 1/h]$ (see Lemma 12.22.11). This sequence generates the ideal K_h. Hence the ideal $K_{hh'}$ is generated by a Koszul-regular sequence of length $m + t = n + s$. But it is also generated by the sequence

$$x_1 - g_1, \ldots, x_n - g_n, q_1, \ldots, q_s$$

of the same length which is thus a Koszul-regular sequence by Lemma 12.22.13. Finally, by Lemma 12.22.12 we conclude that the images of $q_1, \ldots, q_s$ in

$$A[x_i, y_j, 1/hh']/(x_1 - g_1, \ldots, x_n - g_n) \cong A[y_j, 1/h'']$$

form a Koszul-regular sequence generating $J_{h''}$. Since h'' is the image of hh' it doesn't map to zero in $\kappa(\mathfrak{q})$ and we win. □

This lemma allows us to make the following definition.

Definition 12.24.2. A ring map $A \to B$ is called a *local complete intersection* if it is of finite type and for some (equivalently any) presentation $B = A[x_1, \ldots, x_n]/I$ the ideal I is Koszul-regular.

This notion is local.

Lemma 12.24.3. *Let $R \to S$ be a ring map. Let $g_1, \ldots, g_m \in S$ generate the unit ideal. If each $R \to S_{g_j}$ is a local complete intersection so is $R \to S$.*

Proof. Let $S = R[x_1, \ldots, x_n]/I$ be a presentation. Pick $h_j \in R[x_1, \ldots, x_n]$ mapping to g_j in S. Then $R[x_1, \ldots, x_n, x_{n+1}]/(I, x_{n+1}h_j - 1)$ is a presentation of S_{g_j}. Hence $I_j = (I, x_{n+1}h_j - 1)$ is a Koszul-regular ideal in $R[x_1, \ldots, x_n, x_{n+1}]$. Pick a prime $I \subset \mathfrak{q} \subset R[x_1, \ldots, x_n]$. Then $h_j \not\in \mathfrak{q}$ for some j and $\mathfrak{q}_j = (\mathfrak{q}, x_{n+1}h_j - 1)$ is a prime ideal of $V(I_j)$ lying over $\mathfrak{q}$. Pick $f_1, \ldots, f_r \in I$ which map to a basis of $I/I^2 \otimes \kappa(\mathfrak{q})$. Then $x_{n+1}h_j - 1, f_1, \ldots, f_r$ is a sequence of elements of I_j which map to a basis of $I_j \otimes \kappa(\mathfrak{q}_j)$. By Nakayma's lemma there exists an $h \in R[x_1, \ldots, x_n, x_{n+1}]$ such that $(I_j)_h$ is generated by $x_{n+1}h_j - 1, f_1, \ldots, f_r$. We may also assume that $(I_j)_h$ is generated by a Koszul regular sequence of some length e. Looking at the dimension of $I_j \otimes \kappa(\mathfrak{q}_j)$ we see that $e = r + 1$. Hence by Lemma 12.22.13 we see that $x_{n+1}h_j - 1, f_1, \ldots, f_r$ is a Koszul-regular sequence generating $(I_j)_h$ for some $h \in R[x_1, \ldots, x_n, x_{n+1}]$, $h \not\in \mathfrak{q}_j$. By Lemma 12.22.12 we see that $I_{h'}$ is generated by a Koszul-regular sequence for some $h' \in R[x_1, \ldots, x_n]$, $h' \not\in \mathfrak{q}$ as desired. □

Lemma 12.24.4. *Let R be a ring. Let $R[x_1, \ldots, x_n]$. If $R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ be a relative global complete intersection. Then $f_1, \ldots, f_c$ is a Koszul regular sequence.*

Proof. Recall that the homology groups $H_i(K_\bullet(f_\bullet))$ are annihilated by the ideal $(f_1, \ldots, f_c)$. Hence it suffices to show that $H_i(K_\bullet(f_\bullet))_\mathfrak{q}$ is zero for all primes $\mathfrak{q} \subset R[x_1, \ldots, x_n]$ containing $(f_1, \ldots, f_c)$. This follows from Algebra, Lemma 7.125.13 and the fact that a regular sequence is Koszul regular (Lemma 12.22.2). □

Lemma 12.24.5. *A syntomic ring map is a local complete intersection.*

Proof. Combine Lemmas 12.24.4 and 12.24.3 and Algebra, Lemma 7.125.16. □

For a local complete intersection $R \to S$ we have $H_n(L_{S/R}) = 0$ for $n \geq 2$. Since we haven't (yet) defined the full cotangent complex we can't state and prove this, but we can deduce one of the consequences.

Lemma 12.24.6. *Let $A \to B \to C$ be ring maps. Assume $B \to C$ is a local complete intersection homomorphism. Choose a presentation $\alpha : A[x_s, s \in S] \to B$ with kernel I. Choose a presentation $\beta : B[y_1, \ldots, y_m] \to C$ with kernel J. Let $\gamma : A[x_s, y_t] \to C$ be the induced presentation of C with kernel K. Then we get a canonical commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_{A[x_s]/A} \otimes C & \longrightarrow & \Omega_{A[x_s,y_t]/A} \otimes C & \longrightarrow & \Omega_{B[y_t]/B} \otimes C & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & I/I^2 \otimes C & \longrightarrow & K/K^2 & \longrightarrow & J/J^2 & \longrightarrow & 0 \end{array}$$

with exact rows. In particular, the six term exact sequence of Algebra, Lemma 7.123.3 can be completed with a zero on the left, i.e., the sequence

$$0 \to H_1(NL_{B/A} \otimes_B C) \to H_1(L_{C/A}) \to H_1(L_{C/B}) \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

is exact.

Proof. The only thing to prove is the injectivity of the map $I/I^2 \otimes C \to K/K^2$. By assumption the ideal J is Koszul-regular. Hence we have $IA[x_s, y_j] \cap K^2 = IK$ by Lemma 12.23.4. This means that the kernel of $K/K^2 \to J/J^2$ is isomorphic to $IA[x_s, y_j]/IK$. Since $I/I^2 \otimes_A C = IA[x_s, y_j]/IK$ this provides us with the desired injectivity of $I/I^2 \otimes_A C \to K/K^2$ so that the result follows from the snake lemma, see Homology, Lemma 10.3.23. $\square$

Lemma 12.24.7. *Let $A \to B \to C$ be ring maps. If $B \to C$ is a filtered colimit of local complete intersection homomorphisms then the conclusion of Lemma 12.24.6 remains valid.*

Proof. Follows from Lemma 12.24.6 and Algebra, Lemma 7.123.7. $\square$

12.25. Cartier's equality and geometric regularity

A reference for this section and the next is [**Mat70**, Section 39]. In order to comfortably read this section the reader should be familiar with the naive cotangent complex and its properties, see Algebra, Section 7.123.

Lemma 12.25.1 (Cartier equality)**.** *Let K/k be a finitely generated field extension. Then $\Omega_{K/k}$ and $H_1(L_{K/k})$ are finite dimensional and $trdeg_k(K) = \dim_K \Omega_{K/k} - \dim_k H_1(L_{K/k})$.*

Proof. We can find a global complete intersection $A = k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ over k such that K is isomorphic to the fraction field of A, see Algebra, Lemma 7.141.10 and its proof. In this case we see that $NL_{K/k}$ is homotopy equivalent to the complex

$$\bigoplus\nolimits_{j=1,\ldots,c} K \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} K \mathrm{d}x_i$$

by Algebra, Lemmas 7.123.2 and 7.123.10. The transcendence degree of K over k is the dimension of A (by Algebra, Lemma 7.107.1) which is $n - c$ and we win. $\square$

Lemma 12.25.2. *Let $K \subset L \subset M$ be field extensions. Then the Jacobi-Zariski sequence*

$$0 \to H_1(L_{L/K}) \otimes_L M \to H_1(L_{M/K}) \to H_1(L_{M/L}) \to \Omega_{L/K} \otimes_L M \to \Omega_{M/K} \to \Omega_{M/L} \to 0$$

is exact.

Proof. Combine Lemma 12.24.7 with Algebra, Lemma 7.141.10. $\square$

Lemma 12.25.3. *Given a commutative diagram of fields*

$$\begin{tikzcd} K \arrow[r] & K' \\ k \arrow[u] \arrow[r] & k' \arrow[u] \end{tikzcd}$$

with $k \subset k'$ and $K \subset K'$ finitely generated field extensions the kernel and cokernel of the maps

$$\alpha : \Omega_{K/k} \otimes_K K' \to \Omega_{K'/k'} \quad \textit{and} \quad \beta : H_1(L_{K/k}) \otimes_K K' \to H_1(L_{K'/k'})$$

are finite dimensional and

$$\dim Ker(\alpha) - \dim Coker(\alpha) - \dim Ker(\beta) + \dim Coker(\beta) = trdeg_k(k') - trdeg_K(K')$$

Proof. The Jacobi-Zariski sequences for $k \subset k' \subset K'$ and $k \subset K \subset K'$ are

$$0 \to H_1(L_{k'/k}) \otimes K' \to H_1(L_{K'/k}) \to H_1(L_{K'/k'}) \to \Omega_{k'/k} \otimes K' \to \Omega_{K'/k} \to \Omega_{K'/k} \to 0$$

and

$$0 \to H_1(L_{K/k}) \otimes K' \to H_1(L_{K'/k}) \to H_1(L_{K'/K}) \to \Omega_{K/k} \otimes K' \to \Omega_{K'/k} \to \Omega_{K'/K} \to 0$$

By Lemma 12.25.1 the vector spaces $\Omega_{k'/k}$, $\Omega_{K'/K}$, $H_1(L_{K'/K})$, and $H_1(L_{k'/k})$ are finite dimensional and the alternating sum of their dimensions is $\mathrm{trdeg}_k(k') - \mathrm{trdeg}_K(K')$. The lemma follows. □

12.26. Geometric regularity

Let k be a field. Let $(A, \mathfrak{m}, K)$ be a Noetherian local k-algebra. The Jacobi-Zariski sequence (Algebra, Lemma 7.123.3) is a canonical exact sequence

$$H_1(L_{K/k}) \to \mathfrak{m}/\mathfrak{m}^2 \to \Omega_{A/k} \otimes_A K \to \Omega_{K/k} \to 0$$

because $H_1(L_{K/A}) = \mathfrak{m}/\mathfrak{m}^2$ by Algebra, Lemma 7.123.4. We will show that exactness on the left of this sequence characterizes whether or not a regular local ring A is geometrically regular over k. We will link this to the notion of formal smoothness in Section 12.30.

Proposition 12.26.1. *Let k be a field of characteristic $p > 0$. Let $(A, \mathfrak{m}, K)$ be a Noetherian local k-algebra. The following are equivalent*

(1) *A is geometrically regular over k,*
(2) *for all $k \subset k' \subset k^{1/p}$ finite over k the ring $A \otimes_k k'$ is regular,*
(3) *A is regular and the canonical map $H_1(L_{K/k}) \to \mathfrak{m}/\mathfrak{m}^2$ is injective, and*
(4) *A is regular and the map $\Omega_{k/\mathbf{F}_p} \otimes_k K \to \Omega_{A/\mathbf{F}_p} \otimes_A K$ is injective.*

Proof. Proof of (3) ⇒ (1). Assume (3). Let $k \subset k'$ be a finite purely inseparable extension. Set $A' = A \otimes_k k'$. This is a local ring with maximal ideal $\mathfrak{m}'$. Set $K' = A'/\mathfrak{m}'$. We get a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H_1(L_{K/k}) \otimes K' \arrow[r] \arrow[d, "\beta"] & \mathfrak{m}/\mathfrak{m}^2 \otimes K' \arrow[r] \arrow[d] & \Omega_{A/k} \otimes_A K' \arrow[r] \arrow[d, "\cong"] & \Omega_{K/k} \otimes K' \arrow[r] \arrow[d, "\alpha"] & 0 \\ & H_1(L_{K'/k'}) \arrow[r] & \mathfrak{m}'/(\mathfrak{m}')^2 \arrow[r] & \Omega_{A'/k'} \otimes_{A'} K' \arrow[r] & \Omega_{K'/k'} \arrow[r] & 0 \end{tikzcd}$$

with exact rows. The third vertical arrow is an isomorphism by base change for modules of differentials (Algebra, Lemma 7.122.12). Thus α is surjective. By Lemma 12.25.3 we have

$$\dim \mathrm{Ker}(\alpha) - \dim \mathrm{Ker}(\beta) + \dim \mathrm{Coker}(\beta) = 0$$

(and these dimensions are all finite). A diagram chase shows that $\dim \mathfrak{m}'/(\mathfrak{m}')^2 \leq \dim \mathfrak{m}/\mathfrak{m}^2$. However, since $A \to A'$ is finite flat we see that $\dim(A) = \dim(A')$, see Algebra, Lemma 7.103.6. Hence A' is regular by definition.

Equivalence of (3) and (4). Consider the Jacobi-Zariski sequences for rows of the commutative diagram

$$\begin{tikzcd} \mathbf{F}_p \arrow[r] & A \arrow[r] & K \\ \mathbf{F}_p \arrow[r] \arrow[u] & k \arrow[r] \arrow[u] & K \arrow[u] \end{tikzcd}$$

to get a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{m}/\mathfrak{m}^2 \arrow[r] & \Omega_{A/\mathbf{F}_p} \otimes_A K \arrow[r] & \Omega_{K/\mathbf{F}_p} \arrow[r] & 0 \\ 0 \arrow[r] & H_1(L_{K/k}) \arrow[r] \arrow[u] & \Omega_{k/\mathbf{F}_p} \otimes_k K \arrow[r] \arrow[u] & \Omega_{K/\mathbf{F}_p} \arrow[r] \arrow[u] & \Omega_{K/k} \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

with exact rows. We have used that $H_1(L_{K/A}) = \mathfrak{m}/\mathfrak{m}^2$ and that $H_1(L_{K/\mathbf{F}_p}) = 0$ as $K/\mathbf{F}_p$ is separable, see Algebra, Proposition 7.141.8. Thus it is clear that the kernels of $H_1(L_{K/k}) \to \mathfrak{m}/\mathfrak{m}^2$ and $\Omega_{k/\mathbf{F}_p} \otimes_k K \to \Omega_{A/\mathbf{F}_p} \otimes_A K$ have the same dimension.

Proof of (2) $\Rightarrow$ (4) following Faltings, see [**Fal78**]. Let $a_1, \ldots, a_n \in k$ be elements such that $\mathrm{d}a_1, \ldots, \mathrm{d}a_n$ are linearly independent in $\Omega_{k/\mathbf{F}_p}$. Consider the field extension $k' = k(a_1^{1/p}, \ldots, a_n^{1/p})$. By Algebra, Lemma 7.141.2 we see that $k' = k[x_1, \ldots, x_n]/(x_1^p - a_1, \ldots, x_n^p - a_n)$. In particular we see that the naive cotangent complex of k'/k is homotopic to the complex $\bigoplus_{j=1,\ldots,n} k' \to \bigoplus_{i=1,\ldots,n} k'$ with the zero differential as $\mathrm{d}(x_j^p - a_j) = 0$ in $\Omega_{k[x_1,\ldots,x_n]/k}$. Set $A' = A \otimes_k k'$ and $K' = A'/\mathfrak{m}'$ as above. By Algebra, Lemma 7.123.6 we see that $NL_{A'/A}$ is homotopy equivalent to the complex $\bigoplus_{j=1,\ldots,n} A' \to \bigoplus_{i=1,\ldots,n} A'$ with the zero differential, i.e., $H_1(L_{A'/A})$ and $\Omega_{A'/A}$ are free of rank n. The Jacobi-Zariski sequence for $\mathbf{F}_p \to A \to A'$ is

$$H_1(L_{A'/A}) \to \Omega_{A/\mathbf{F}_p} \otimes_A A' \to \Omega_{A'/\mathbf{F}_p} \to \Omega_{A'/A} \to 0$$

Using the presentation $A[x_1, \ldots, x_n] \to A'$ with kernel $(x_j^p - a_j)$ we see, unwinding the maps in Algebra, Lemma 7.123.3, that the jth basis vector of $H_1(L_{A'/A})$ maps to $\mathrm{d}a_j \otimes 1$ in $\Omega_{A/\mathbf{F}_p} \otimes A'$. As $\Omega_{A'/A}$ is free (hence flat) we get on tensoring with K' an exact sequence

$$K'^{\oplus n} \to \Omega_{A/\mathbf{F}_p} \otimes_A K' \xrightarrow{\beta} \Omega_{A'/\mathbf{F}_p} \otimes_{A'} K' \to K'^{\oplus n} \to 0$$

We conclude that the elements $\mathrm{d}a_j \otimes 1$ generate $\mathrm{Ker}(\beta)$ and we have to show that are linearly independent, i.e., we have to show $\dim(\ker(\beta)) = n$. Consider the following big diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{m}'/(\mathfrak{m}')^2 \arrow[r] & \Omega_{A'/\mathbf{F}_p} \otimes K' \arrow[r] & \Omega_{K'/\mathbf{F}_p} \arrow[r] & 0 \\
0 \arrow[r] & \mathfrak{m}/\mathfrak{m}^2 \otimes K' \arrow[r] \arrow[u, "\alpha"] & \Omega_{A/\mathbf{F}_p} \otimes K' \arrow[r] \arrow[u, "\beta"] & \Omega_{K/\mathbf{F}_p} \otimes K' \arrow[r] \arrow[u, "\gamma"] & 0
\end{tikzcd}$$

By Lemma 12.25.1 and the Jacobi-Zariski sequence for $\mathbf{F}_p \to K \to K'$ we see that the kernel and cokernel of γ have the same finite dimension. By assumption A' is regular (and of the same dimension as A, see above) hence the kernel and cokernel of α have the same dimension. It follows that the kernel and cokernel of β have the same dimension which is what we wanted to show.

The implication (1) $\Rightarrow$ (2) is trivial. This finishes the proof of the proposition. $\square$

Lemma 12.26.2. *Let k be a field of characteristic $p > 0$. Let $(A, \mathfrak{m}, K)$ be a Noetherian local k-algebra. Assume A is geometrically regular over k. Let $k \subset F \subset K$ be a finitely generated subextension. Let $\varphi : k[y_1, \ldots, y_m] \to A$ be a k-algebra map such that y_i maps to an element of F in K and such that $dy_1, \ldots, dy_m$ map to a basis of $\Omega_{F/k}$. Set $\mathfrak{p} = \varphi^{-1}(\mathfrak{m})$. Then*

$$k[y_1, \ldots, y_m]_{\mathfrak{p}} \to A$$

is flat and $A/\mathfrak{p}A$ is regular.

Proof. Set $A_0 = k[y_1, \ldots, y_m]_{\mathfrak{p}}$ with maximal ideal $\mathfrak{m}_0$ and residue field K_0. Note that $\Omega_{A_0/k}$ is free of rank m and $\Omega_{A_0/k} \otimes K_0 \to \Omega_{K_0/k}$ is an isomorphism. It is clear that A_0 is geometrically regular over k. Hence $H_1(L_{K_0/k}) \to \mathfrak{m}_0/\mathfrak{m}_0^2$ is an isomorphism, see Proposition 12.26.1. Now consider

$$\begin{tikzcd}
H_1(L_{K_0/k}) \otimes K \arrow[r] \arrow[d] & \mathfrak{m}_0/\mathfrak{m}_0^2 \otimes K \arrow[d] \\
H_1(L_{K/k}) \arrow[r] & \mathfrak{m}/\mathfrak{m}^2
\end{tikzcd}$$

Since the left vertical arrow is injective by Lemma 12.25.2 and the lower horizontal by Proposition 12.26.1 we conclude that the right vertical one is too. Hence a regular system of parameters in A_0 maps to part of a regular system of parameters in A. We win by Algebra, Lemmas 7.119.2 and 7.98.3. $\square$

12.27. Topological rings and modules

Let's quickly discuss some properties of topological abelian groups. An abelian group M is a *topological abelian group* if M is endowed with a topology such that addition $M \times M \to M$ is continuous. A *homomorphism of topological abelian groups* is just a homomorphism of abelian groups which is continuous. The category of commutative topological groups is additive and has kernels and cokernels, but is not abelian (as the axiom Im = Coim doesn't hold). If $N \subset M$ is a subgroup, then we think of N and M/N as topological groups also, namely using the induced topology on N and the quotient topology on M/N (i.e., such that $M \to M/N$ is submersive). Note that if $N \subset M$ is an open subgroup, then the topology on M/N is discrete.

We say the topology on M is *linear* if there exists a fundamental system of neighbourhoods of 0 consisting of subgroups. If so then these subgroups are also open. An example is the following. Let I be a directed partially ordered set and let G_i be an inverse system of (discrete) abelian groups over I. Then

$$G = lim_{i \in I} G_i$$

with the inverse limit topology is linearly topologized with a fundamental system of neighbourhoods of 0 given by $\text{Ker}(G \to G_i)$. Conversely, let M be a linearly topologized abelian group. Choose any fundamental system of open subgroups $U_i \subset M$, $i \in I$ (i.e., the U_i form a fundamental system of open neighbourhoods and each U_i is a subgroup of M). Setting $i \geq i' \Leftrightarrow U_i \subset U_{i'}$ we see that I is a directed partially ordered set. We obtain a homomorphism of linearly topologized abelian groups

$$c : M \longrightarrow lim_{i \in I} M/U_i.$$

It is clear that M is *separated* (as a topological space) if and only if c is injective. We say that M is *complete* if c is an isomorphism[1]. We leave it to the reader to check that this condition is independent of the choice of fundamental system of open subgroups $\{U_i\}_{i \in I}$ chosen above. In fact the topological abelian group $M^\wedge = lim_{i \in I} M/U_i$ is independent of this choice and is sometimes called the *completion* of M. Any $G = lim\, G_i$ as above is complete, in particular, the completion $M^\wedge$ is always complete.

Definition 12.27.1 (Topological rings)**.** Let R be a ring and let M be an R-module.

(1) We say R is a *topological ring* if R is endowed with a topology such that both addition and multiplication are continuous as maps $R \times R \to R$ where $R \times R$ has the product topology. In this case we say M is a *topological module* if M is endowed with a topology such that addition $M \times M \to M$ and scalar multiplication $R \times M \to M$ are continuous.
(2) A *homomorphism of topological modules* is just a continuous R-module map. A *homomorphism of topological rings* is a ring homomorphism which is continuous for the given topologies.
(3) We say M is *linearly topologized* if 0 has a fundamental system of neighbourhoods consisting of submodules. We say R is *linearly topologized* if 0 has a fundamental system of neighbourhoods consisting of ideals.
(4) If R is linearly topologized, we say that $I \subset R$ is an *ideal of definition* if I is open and if every neighbourhood of 0 contains I^n for some n.
(5) If R is linearly topologized, we say that R is *pre-admissible* if R has an ideal of definition.
(6) If R is linearly topologized, we say that R is *admissible* if it is pre-admissible and complete[2].
(7) If R is linearly topologized, we say that R is *pre-adic* if there exists an ideal of definition I such that $\{I^n\}_{n \geq 0}$ forms a fundamental system of neighbourhoods of 0.
(8) If R is linearly topologized, we say that R is *adic* if R is pre-adic and complete.

Note that a (pre)adic ring is the same thing as a (pre)admissible ring which has an ideal of definition I such that I^n is open for all $n \geq 1$.

[1] We include being separated as part of being complete as we'd like to have a unique limits in complete groups. There is a definition of completeness for any topological group, agreeing, modulo the separation issue, with this one in our special case.

[2] By our conventions this includes separated.

Let R be a ring and let M be an R-module. Let $I \subset R$ be an ideal. Then we can consider the linear topology on R which has $\{I^n\}_{n\geq 0}$ as a fundamental system of neighbourhoods of 0. This topology is called the I-adic topology; R is a pre-adic toplogical ring in the I-adic topology[3]. Moreover, the linear topology on M which has $\{I^n M\}_{n\geq 0}$ as a fundamental system of open neighbourhoods of 0 turns M into a topological R-module. This is called the *I-adic topology* on M. We see that M is I-adically complete (as defined in Algebra, Definition 7.90.5) if and only M is complete in the I-adic topology[4]. In particular, we see that R is I-adically complete if and only if R is an adic topological ring in the I-adic topology.

As a special case, note that the discrete topology is the 0-adic topology and that any ring in the discrete topology is adic.

Lemma 12.27.2. *Let $\varphi : R \to S$ be a ring map. Let $I \subset R$ and $J \subset S$ be ideals and endow R with the I-adic topology and S with the J-adic topology. Then φ is a homomorphism of topological rings if and only if $\varphi(I^n) \subset J$ for some $n \geq 1$.*

Proof. Omitted. □

12.28. Formally smooth maps of topological rings

There is a version of formal smoothness which applies to homomorphisms of topological rings.

Definition 12.28.1. Let $R \to S$ be a homomorphism of topological rings with R and S linearly topologized. We say S is *formally smooth over R* if for every commutative solid diagram

$$\begin{tikzcd} S \arrow[r] \arrow[dr, dashed] & A/J \\ R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

of homomorphisms of topological rings where A is a discrete ring and $J \subset A$ is an ideal of square zero, a dotted arrow exists which makes the diagram commute.

We will mostly use this notion when given ideals $\mathfrak{m} \subset R$ and $\mathfrak{n} \subset S$ and we endow R with the $\mathfrak{m}$-adic topology and S with the $\mathfrak{n}$-adic topology. Continuity of $\varphi : R \to S$ holds if and only if $\varphi(\mathfrak{m}^m) \subset \mathfrak{n}$ for some $m \geq 1$, see Lemma 12.27.2. It turns out that in this case only the topology on S is relevant.

Lemma 12.28.2. *Let $\varphi : R \to S$ be a ring map.*

(1) *If $R \to S$ is formally smooth in the sense of Algebra, Definition 7.127.1, then $R \to S$ is formally smooth for any linear topology on R and any pre-adic topology on S such that $R \to S$ is continuous.*
(2) *Let $\mathfrak{n} \subset S$ and $\mathfrak{m} \subset R$ ideals such that φ is continuous for the $\mathfrak{m}$-adic topology on R and the $\mathfrak{n}$-adic topology on S. Then the following are equivalent*
    (a) *φ is formally smooth for the $\mathfrak{m}$-adic topology on R and the $\mathfrak{n}$-adic topology on S, and*

[3] Thus the I-adic topology is sometimes called the I-pre-adic topology.

[4] It may happen that the I-adic completion $M^\wedge$ is not I-adically complete, even though $M^\wedge$ is always complete with respect to the limit topology. If I is finitely generated then the I-adic topology and the limit topology on $M^\wedge$ agree, see Algebra, Lemma 7.90.7 and its proof.

(b) *φ is formally smooth for the discrete topology on R and the $\mathfrak{n}$-adic topology on S.*

Proof. Assume $R \to S$ is formally smooth in the sense of Algebra, Definition 7.127.1. If S has a pre-adic topology, then there exists an ideal $\mathfrak{n} \subset S$ such that S has the $\mathfrak{n}$-adic topology. Suppose given a solid commutative diagram as in Definition 12.28.1. Continuity of $S \to A/J$ means that $\mathfrak{n}^k$ maps to zero in A/J for some $k \geq 1$, see Lemma 12.27.2. We obtain a ring map $\psi : S \to A$ from the assumed formal smoothness of S over R. Then $\psi(\mathfrak{n}^k) \subset J$ hence $\psi(\mathfrak{n}^{2k}) = 0$ as $J^2 = 0$. Hence ψ is continuous by Lemma 12.27.2. This proves (1).

The proof of (2)(b) $\Rightarrow$ (2)(a) is the same as the proof of (1). Assume (2)(a). Suppose given a solid commutative diagram as in Definition 12.28.1 where we use the discrete topology on R. Since φ is continuos we see that $\varphi(\mathfrak{m}^n) \subset \mathfrak{n}$ for some $m \geq 1$. As $S \to A/J$ is continuous we see that $\mathfrak{n}^k$ maps to zero in A/J for some $k \geq 1$. Hence $\mathfrak{m}^{nk}$ maps in J under the map $R \to A$. Thus $\mathfrak{m}^{2nk}$ maps to zero in A and we see that $R \to A$ is continuous in the $\mathfrak{m}$-adic topology. Thus (2)(a) gives a dotted arrow as desired. $\square$

Definition 12.28.3. Let $R \to S$ be a ring map. Let $\mathfrak{n} \subset S$ be an ideal. If the equivalent conditions (2)(a) and (2)(b) of Lemma 12.28.2 hold, then we say $R \to S$ is *formally smooth for the $\mathfrak{n}$-adic topology*.

This property is inherited by the completions.

Lemma 12.28.4. *Let $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ be rings endowed with finitely generated ideals. Endow R and S with the $\mathfrak{m}$-adic and $\mathfrak{n}$-adic topologies. Let $R \to S$ be a homomorphism of topological rings. The following are equivalent*

(1) *$R \to S$ is formally smooth for the $\mathfrak{n}$-adic topology,*
(2) *$R \to S^\wedge$ is formally smooth for the $\mathfrak{n}^\wedge$-adic topology,*
(3) *$R^\wedge \to S^\wedge$ is formally smooth for the $\mathfrak{n}^\wedge$-adic topology.*

Here $R^\wedge$ and $S^\wedge$ are the $\mathfrak{m}$-adic and $\mathfrak{n}$-adic completions of R and S.

Proof. The assumption that $\mathfrak{m}$ is finitely generated implies that $R^\wedge$ is $\mathfrak{m}R^\wedge$-adically complete, that $\mathfrak{m}R^\wedge = \mathfrak{m}^\wedge$ and that $R^\wedge/\mathfrak{m}^n R^\wedge = R/\mathfrak{m}^n$, see Algebra, Lemma 7.90.7 and its proof. Similarly for $(S, \mathfrak{n})$. Thus it is clear that diagrams as in Definition 12.28.1 for the cases (1), (2), and (3) are in 1-to-1 correspondence. $\square$

The advantage of working with adic rings is that one gets a stronger lifting property.

Lemma 12.28.5. *Let $R \to S$ be a ring map. Let $\mathfrak{n}$ be an ideal of S. Assume that $R \to S$ is formally smooth in the $\mathfrak{n}$-adic topology. Consider a solid commutative diagram*

$$\begin{tikzcd} S \arrow[r] \arrow[dr, dashed, "\psi"] & A/J \\ R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

of homomorphisms of topological rings where A is adic and A/J is the quotient (as topological ring) of A by a closed ideal $J \subset A$ such that J^t is contained in an ideal of definition of A for some $t \geq 1$. Then there exists a dotted arrow in the category of topological rings which makes the diagram commute.

Proof. Let $I \subset A$ be an ideal of definition so that $I \supset J^t$ for some n. Then $A = lim\, A/I^n$ and $A/J = lim\, A/J + I^n$ because J is assumed closed. Consider the following diagram of discrete R algebras $A_{n,m} = A/J^n + I^m$:

$$\begin{tikzcd}
A/J^3 + I^3 \arrow[r] \arrow[d] & A/J^2 + I^3 \arrow[r] \arrow[d] & A/J + I^3 \arrow[d] \\
A/J^3 + I^2 \arrow[r] \arrow[d] & A/J^2 + I^2 \arrow[r] \arrow[d] & A/J + I^2 \arrow[d] \\
A/J^3 + I \arrow[r] & A/J^2 + I \arrow[r] & A/J + I
\end{tikzcd}$$

Note that each of the commutative squares defines a surjection

$$A_{n+1,m+1} \longrightarrow A_{n+1,m} \times_{A_{n,m}} A_{n,m+1}$$

of R-algebras whose kernel has square zero. We will inductively construct R-algebra maps $\varphi_{n,m} : S \to A_{n,m}$. Namely, we have the maps $\varphi_{1,m} = \psi \bmod J + I^m$. Note that each of these maps is continuous as ψ is. We can inductively choose the maps $\varphi_{n,1}$ by starting with our choice of $\varphi_{1,1}$ and lifting up, using the formal smoothness of S over R, along the right column of the diagram above. We construct the remaining maps $\varphi_{n,m}$ by induction on $n+m$. Namely, we choose $\varphi_{n+1,m+1}$ by lifting the pair $(\varphi_{n+1,m}, \varphi_{n,m+1})$ along the displayed surjection above (again using the formal smoothness of S over R). In this way all of the maps $\varphi_{n,m}$ are compatible with the transition maps of the system. As $J^t \subset I$ we see that for example $\varphi_n = \varphi_{nt,n} \bmod I^n$ induces a map $S \to A/I^n$. Taking the limit $\varphi = lim\, \varphi_n$ we obtain a map $S \to A = lim\, A/I^n$. The composition into A/J agrees with ψ as we have seen that $A/J = lim\, A/J + I^n$. Finally we show that φ is continuous. Namely, we know that $\psi(\mathfrak{n}^r) \subset J + I^r/J$ for some r by our assumption that ψ is a morphism of topological rings, see Lemma 12.27.2. Hence $\varphi(\mathfrak{n}^r) \subset J + I$ hence $\varphi(\mathfrak{n}^{rt}) \subset I$ as desired. $\square$

Lemma 12.28.6. *Let $R \to S$ be a ring map. Let $\mathfrak{n} \subset \mathfrak{n}' \subset S$ be ideals. If $R \to S$ is formally smooth for the $\mathfrak{n}$-adic topology, then $R \to S$ is formally smooth for the $\mathfrak{n}'$-adic topology.*

Proof. Omitted. $\square$

Lemma 12.28.7. *A composition of formally smooth continuous homomorphisms of linearly topologized rings is formally smooth.*

Proof. Omitted. (Hint: This is completely formal, and follows from considering a suitable diagram.) $\square$

Lemma 12.28.8. *Let R, S be rings. Let $\mathfrak{n} \subset S$ be an ideal. Let $R \to S$ be formally smooth for the $\mathfrak{n}$-adic topology. Let $R \to R'$ be any ring map. Then $R' \to S' = S \otimes_R R'$ is formally smooth in the $\mathfrak{n}' = \mathfrak{n}S'$-adic topology.*

Proof. Let a solid diagram

$$\begin{tikzcd}
S \arrow[r] \arrow[rrd, dashed] & S' \arrow[r] \arrow[rd, dashed] & A/J \\
R \arrow[r] \arrow[u] & R' \arrow[r] \arrow[u] & A \arrow[u]
\end{tikzcd}$$

as in Definition 12.28.1 be given. Then the composition $S \to S' \to A/J$ is continuous. By assumption the longer dotted arrow exists. By the universal property of tensor product we obtain the shorter dotted arrow. □

We have seen descent for formal smoothness along faithfully flat ring maps in Algebra, Lemma 7.127.15. Something similar holds in the current setting of topological rings. However, here we just prove the following very simple and easy to prove version which is already quite useful.

Lemma 12.28.9. *Let R, S be rings. Let $\mathfrak{n} \subset S$ be an ideal. Let $R \to R'$ be a ring map. Set $S' = S \otimes_R R'$ and $\mathfrak{n}' = \mathfrak{n}S$. If*

(1) *the map $R \to R'$ embeds R as a direct summand of R' as an R-module, and*
(2) *$R' \to S'$ is formally smooth for the $\mathfrak{n}'$-adic topology,*

then $R \to S$ is formally smooth in the $\mathfrak{n}$-adic topology.

Proof. Let a solid diagram

$$\begin{tikzcd} S \arrow[r] & A/J \\ R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

as in Definition 12.28.1 be given. Set $A' = A \otimes_R R'$ and $J' = \mathrm{Im}(J \otimes_R R' \to A')$. The base change of the diagram above is the diagram

$$\begin{tikzcd} S' \arrow[r] \arrow[rd, dashed, "\psi'"] & A'/J' \\ R' \arrow[u] \arrow[r] & A' \arrow[u] \end{tikzcd}$$

with continuous arrows. By condition (2) we obtain the dotted arrow $\psi' : S' \to A'$. Using condition (1) choose a direct summand decomposition $R' = R \oplus C$ as R-modules. (Warning: C isn't an ideal in R'.) Then $A' = A \oplus A \otimes_R C$. Set

$$J'' = \mathrm{Im}(J \otimes_R C \to A \otimes_R C) \subset J' \subset A'.$$

Then $J' = J \oplus J''$ as A-modules. The image of the composition $\psi : S \to A'$ of ψ' with $S \to S'$ is contained in $A + J' = A \oplus J''$. However, in the ring $A + J' = A \oplus J''$ the A-submodule J'' is an ideal! (Use that $J^2 = 0$.) Hence the composition $S \to A + J' \to (A + J')/J'' = A$ is the arrow we were looking for. □

The following lemma will be improved on in Section 12.30.

Lemma 12.28.10. *Let k be a field and let $(A, \mathfrak{m}, K)$ be a Noetherian local k-algebra. If $k \to A$ is formally smooth for the $\mathfrak{m}$-adic topology, then A is a regular local ring.*

Proof. Let $k_0 \subset k$ be the prime field. Then k_0 is perfect, hence k/k_0 is separable, hence formally smooth by Algebra, Lemma 7.141.6. By Lemmas 12.28.2 and 12.28.7 we see that $k_0 \to A$ is formally smooth for the $\mathfrak{m}$-adic topology on A. Hence we may assume $k = \mathbf{Q}$ or $k = \mathbf{F}_p$.

By Algebra, Lemmas 7.90.4 and 7.102.8 it suffices to prove the completion $A^\wedge$ is regular. By Lemma 12.28.4 we may replace A by $A^\wedge$. Thus we may assume that A is a Noetherian complete local ring. By the Cohen structure theorem (Algebra, Theorem 7.143.8) there exist a map $K \to A$. As k is the prime field we see that $K \to A$ is a k-algebra map.

Let $x_1, \ldots, x_n \in \mathfrak{m}$ be elements whose images form a basis of $\mathfrak{m}/\mathfrak{m}^2$. Set $T = K[[X_1, \ldots, X_n]]$. Note that

$$A/m^2 \cong K[x_1, \ldots, x_n]/(x_i x_j)$$

and

$$T/\mathfrak{m}_T^2 \cong K[X_1, \ldots, X_n]/(X_i X_j).$$

Let $A/\mathfrak{m}^2 \to T/m_T^2$ be the local K-algebra isomorphism given by mapping the class of x_i to the class of X_i. Denote $f_1 : A \to T/\mathfrak{m}_T^2$ the composition of this isomorphism with the quotient map $A \to A/\mathfrak{m}^2$. The assumption that $k \to A$ is formally smooth in the $\mathfrak{m}$-adic topology means we can lift f_1 to a map $f_2 : A \to T/\mathfrak{m}_T^3$, then to a map $f_3 : A \to T/\mathfrak{m}_T^4$, and so on, for all $n \geq 1$. Warning: the maps f_n are continuous k-algebra maps and may not be K-algebra maps. We get an induced map $f : A \to T = lim\, T/\mathfrak{m}_T^n$ of local k-algebras. By our choice of f_1, the map f induces an isomorphism $\mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}_T/\mathfrak{m}_T^2$ hence each f_n is surjective and we conclude f is surjective as A is complete. This implies $\dim(A) \geq \dim(T) = n$. Hence A is regular by definition. (It also follows that f is an isomorphism.) $\square$

The following result will be improved on in Section 12.30

Lemma 12.28.11. *Let k be a field. Let $(A, \mathfrak{m}, K)$ be a regular local k-algebra such that K/k is separable. Then $k \to A$ is formally smooth in the $\mathfrak{m}$-adic topology.*

Proof. It suffices to prove that the completion of A is formally smooth over k, see Lemma 12.28.4. Hence we may assume that A is a complete local regular k-algebra with residue field K separable over k. Since K is formally smooth over k by Algebra, Proposition 7.141.8 we can successively find maps

$$\begin{tikzcd}
 & & & K \arrow[d] \arrow[dl, dashed] \arrow[dll, dashed] \arrow[dlll, dashed] & \\
\ldots \arrow[r] & A/\mathfrak{m}^4 \arrow[r] & A/\mathfrak{m}^3 \arrow[r] & A/\mathfrak{m}^2 \arrow[r] & K
\end{tikzcd}$$

of k-algebras. Since A is complete this defines a k-algebra map $K \to A$. Pick $a_1, \ldots, a_n \in \mathfrak{m}$ which map to a K-basis of $\mathfrak{m}/\mathfrak{m}^2$. Consider the K-algebra map

$$c : K[[x_1, \ldots, x_n]] \longrightarrow A$$

which maps x_i to a_i (existence of c follows from the universal property of the powerseries ring). By construction the maps $K[[x_1, \ldots, x_n]] \to A/\mathfrak{m}^e$ are surjective for all $e \geq 1$. Since $K[[x_1, \ldots, x_n]]$ is complete we see that c is surjective. Since $\dim(A) = n$ as A is regular and since $K[[x_1, \ldots, x_n]]$ is a domain of dimension n we see that the kernel of c is zero. Hence c is an isomorphism.

We win because the power series ring $K[[x_1, \ldots, x_n]]$ is formally smooth over k. Namely, K is formally smooth over k and $K[x_1, \ldots, x_n]$ is formally smooth over K as a polynomial algebra. Hence $K[x_1, \ldots, x_n]$ is formally smooth over k by Algebra, Lemma 7.127.3. It follows that $k \to K[x_1, \ldots, x_n]$ is formally smooth for the $(x_1, \ldots, x_n)$-adic topology by Lemma 12.28.2. Finally, it follows that $k \to K[[x_1, \ldots, x_n]]$ is formally smooth for the $(x_1, \ldots, x_n)$-adic topology by Lemma 12.28.4. $\square$

12.29. Some results on power series rings

Questions on formally smooth maps between Noetherian local rings can often be reduced to questions on maps between power series rings. In this section we prove some helper lemmas to facilitate this kind of argument.

Lemma 12.29.1. *Let K be a field of characteristic 0 and $A = K[[x_1, \ldots, x_n]]$. Let L be a field of characteristic $p > 0$ and $B = L[[x_1, \ldots, x_n]]$. Let Λ be a Cohen ring. Let $C = \Lambda[[x_1, \ldots, x_n]]$.*

1. *$\mathbf{Q} \to A$ is formally smooth in the $\mathfrak{m}$-adic topology.*
2. *$\mathbf{F}_p \to B$ is formally smooth in the $\mathfrak{m}$-adic topology.*
3. *$\mathbf{Z} \to C$ is formally smooth in the $\mathfrak{m}$-adic topology.*

Proof. By the universal property of power series rings it suffices to prove:

1. $\mathbf{Q} \to K$ is formally smooth.
2. $\mathbf{F}_p \to L$ is formally smooth.
3. $\mathbf{Z} \to \Lambda$ is formally smooth in the $\mathfrak{m}$-adic topology.

The first two are Algebra, Proposition 7.141.8. The third follows from Algebra, Lemma 7.143.7 since for any test diagram as in Definition 12.28.1 some power of p will be zero in A/J and hence some power of p will be zero in A. □

Lemma 12.29.2. *Let K be a field and $A = K[[x_1, \ldots, x_n]]$. Let Λ be a Cohen ring and let $B = \Lambda[[x_1, \ldots, x_n]]$.*

1. *If $y_1, \ldots, y_n \in A$ is a regular system of parameters then $K[[y_1, \ldots, y_n]] \to A$ is an isomorphism.*
2. *If $z_1, \ldots, z_r \in A$ form part of a regular system of parameters for A, then $r \leq n$ and $A/(z_1, \ldots, z_r) \cong K[[y_1, \ldots, y_{n-r}]]$.*
3. *If $p, y_1, \ldots, y_n \in B$ is a regular system of parameters then $\Lambda[[y_1, \ldots, y_n]] \to B$ is an isomorphism.*
4. *If $p, z_1, \ldots, z_r \in B$ form part of a regular system of parameters for B, then $r \leq n$ and $B/(z_1, \ldots, z_r) \cong \Lambda[[y_1, \ldots, y_{n-r}]]$.*

Proof. Proof of (1). Set $A' = K[[y_1, \ldots, y_n]]$. It is clear that the map $A' \to A$ induces an isomorphism $A'/\mathfrak{m}^n_{A'} \to A/\mathfrak{m}^n_A$ for all $n \geq 1$. Since A and A' are both complete we deduce that $A' \to A$ is an isomorphism. Proof of (2). Extend $z_1, \ldots, z_r$ to a regular system of parameters $z_1, \ldots, z_r, y_1, \ldots, y_{n-r}$ of A. Consider the map $A' = K[[z_1, \ldots, z_r, y_1, \ldots, y_{n-r}]] \to A$. This is an isomorphism by (1). Hence (2) follows as it is clear that $A'/(z_1, \ldots, z_r) \cong K[[y_1, \ldots, y_{n-r}]]$. The proofs of (3) and (4) are exactly the same as the proofs of (1) and (2). □

Lemma 12.29.3. *Let $A \to B$ be a local homomorphism of Noetherian complete local rings. Then there exists a commutative diagram*

$$\begin{tikzcd} S \arrow[r] & B \\ R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

with the following properties:

1. *the horizontal arrows are surjective,*
2. *if the characteristic of $A/\mathfrak{m}_A$ is zero, then S and R are power series rings over fields,*

(3) *if the characteristic of $A/\mathfrak{m}_A$ is $p > 0$, then S and R are power series rings over Cohen rings, and*
(4) *$R \to S$ maps a regular system of parameters of R to part of a regular system of parameters of S.*

In particular $R \to S$ is flat (see Algebra, Lemma 7.119.2) with regular fibre $S/\mathfrak{m}_R S$ (see Algebra, Lemma 7.98.3).

Proof. Use the Cohen structure theorem (Algebra, Theorem 7.143.8) to choose a surjection $S \to B$ as in the statement of the lemma where we choose S to be a power series over a Cohen ring if the residue characteristic is $p > 0$ and a power series over a field else. Let $J \subset S$ be the kernel of $S \to B$. Next, choose a surjection $R = \Lambda[[x_1, \ldots, x_n]] \to A$ where we choose Λ to be a Cohen ring if the residue characteristic of A is $p > 0$ and Λ equal to the residue field of A otherwise. We lift the composition $\Lambda[[x_1, \ldots, x_n]] \to A \to B$ to a map $\varphi : R \to S$. This is possible because $\Lambda[[x_1, \ldots, x_n]]$ is formally smooth over $\mathbf{Z}$ in the $\mathfrak{m}$-adic topology (see Lemma 12.29.1) by an application of Lemma 12.28.5. Finally, we replace φ by the map $\varphi' : R = \Lambda[[x_1, \ldots, x_n]] \to S' = S[[y_1, \ldots, y_n]]$ with $\varphi'|_\Lambda = \varphi|_\Lambda$ and $\varphi'(x_i) = \varphi(x_i) + y_i$. We also replace $S \to B$ by the map $S' \to B$ which maps y_i to zero. After this replacement it is clear that a regular system of parameters of R maps to part of a regular sequence in S' and we win. □

12.30. Geometric regularity and formal smoothness

In this section we combine the results of the previous sections to prove the following characterization of geometrically regular local rings over fields. We then recycle some of our arguments to prove a characterization of formally smooth maps in the $\mathfrak{m}$-adic topology between Noetherian local rings.

Theorem 12.30.1. *Let k be a field. Let $(A, \mathfrak{m}, K)$ be a Noetherian local k-algebra. If the characteristic of k is zero then the following are equivalent*

(1) *A is a regular local ring, and*
(2) *$k \to A$ is formally smooth in the $\mathfrak{m}$-adic topology.*

If the characteristic of k is $p > 0$ then the following are equivalent

(1) *A is geometrically regular over k,*
(2) *$k \to A$ is formally smooth in the $\mathfrak{m}$-adic topology.*
(3) *for all $k \subset k' \subset k^{1/p}$ finite over k the ring $A \otimes_k k'$ is regular,*
(4) *A is regular and the canonical map $H_1(L_{K/k}) \to \mathfrak{m}/\mathfrak{m}^2$ is injective, and*
(5) *A is regular and the map $\Omega_{k/\mathbf{F}_p} \otimes_k K \to \Omega_{A/\mathbf{F}_p} \otimes_A K$ is injective.*

Proof. If the characteristic of k is zero, then the equivalence of (1) and (2) follows from Lemmas 12.28.10 and 12.28.11.

If the characteristic of k is $p > 0$, then it follows from Proposition 12.26.1 that (1), (3), (4), and (5) are equivalent. Assume (2) holds. By Lemma 12.28.8 we see that $k' \to A' = A \otimes_k k'$ is formally smooth for the $\mathfrak{m}' = \mathfrak{m}A$-adic topology. Hence if $k \subset k'$ is finite purely inseparable, then A' is a regular local ring by Lemma 12.28.10. Thus we see that (1) holds.

Finally, we will prove that (5) implies (2). Choose a solid diagram

$$\begin{tikzcd} A \arrow[r] \arrow[dr, dashed, "\bar{\psi}"] & B/J \\ k \arrow[u, "i"] \arrow[r, "\varphi"] & B \arrow[u, "\pi"'] \end{tikzcd}$$

as in Definition 12.28.1. As $J^2 = 0$ we see that J has a canonical B/J module structure and via $\bar{\psi}$ an A-module structure. As $\bar{\psi}$ is continuous for the $\mathfrak{m}$-adic topology we see that $\mathfrak{m}^n J = 0$ for some n. Hence we can filter J by B/J-submodules $0 \subset J_1 \subset J_2 \subset \ldots \subset J_n = J$ such that each quotient J_{t+1}/J_t is annihilated by $\mathfrak{m}$. Considering the sequence of ring maps $B \to B/J_1 \to B/J_2 \to \ldots \to B/J$ we see that it suffices to prove the existence of the dotted arrow when J is annihilated by $\mathfrak{m}$, i.e., when J is a K-vector space.

Assume given a diagram as above such that J is annihilated by $\mathfrak{m}$. By Lemma 12.28.11 we see that $\mathbf{F}_p \to A$ is formally smooth in the $\mathfrak{m}$-adic topology. Hence we can find a ring map $\psi : A \to B$ such that $\pi \circ \psi = \bar{\psi}$. Then $\psi \circ i, \varphi : k \to B$ are two maps whose compositions with π are equal. Hence $D = \psi \circ i - \varphi : k \to J$ is a derivation. By Algebra, Lemma 7.122.3 we can write $D = \xi \circ \mathrm{d}$ for some k-linear map $\xi : \Omega_{k/\mathbf{F}_p} \to J$. Using the K-vector space structure on J we extend ξ to a K-linear map $\xi' : \Omega_{k/\mathbf{F}_p} \otimes_k K \to J$. Using (5) we can find a K-linear map $\xi'' : \Omega_{A/\mathbf{F}_p} \otimes_A K$ whose restriction to $\Omega_{k/\mathbf{F}_p} \otimes_k K$ is ξ'. Write

$$D' : A \xrightarrow{\mathrm{d}} \Omega_{A/\mathbf{F}_p} \to \Omega_{A/\mathbf{F}_p} \otimes_A K \xrightarrow{\xi''} J.$$

Finally, set $\psi' = \psi - D' : A \to B$. The reader verifies that ψ' is a ring map such that $\pi \circ \psi' = \bar{\psi}$ and such that $\psi' \circ i = \varphi$ as desired. $\square$

Example 12.30.2. Let k be a field of characteristic $p > 0$. Suppose that $a \in k$ is an element which is not a pth power. A standard example of a geometrically regular local k-algebra whose residue field is purely inseparable over k is the ring

$$A = k[x, y]_{(x, y^p - a)}/(y^p - a - x)$$

Namely, A is a localization of a smooth algebra over k hence $k \to A$ is formally smooth, hence $k \to A$ is formally smooth for the $\mathfrak{m}$-adic topology. A closely related example is the following. Let $k = \mathbf{F}_p(s)$ and $K = \mathbf{F}_p(t)^{perf}$. We claim the ring map

$$k \longrightarrow A = K[[x]], \quad s \longmapsto t + x$$

is formally smooth for the (x)-adic topology on A. Namely, $\Omega_{k/\mathbf{F}_p}$ is 1-dimensional with basis $\mathrm{d}s$. It maps to the element $\mathrm{d}x + \mathrm{d}t = \mathrm{d}x$ in $\Omega_{A/\mathbf{F}_p}$. We leave it to the reader to show that $\Omega_{A/\mathbf{F}_p}$ is free on $\mathrm{d}x$ as an A-module. Hence we see that condition (5) of Theorem 12.30.1 holds and we conclude that $k \to A$ is formally smooth in the (x)-adic topology.

Lemma 12.30.3. *Let $A \to B$ be a local homomorphism of Noetherian local rings. Assume $A \to B$ is formally smooth in the $\mathfrak{m}_B$-adic topology. Then $A \to B$ is flat.*

Proof. We may assume that A and B a Noetherian complete local rings by Lemma 12.28.4 and Algebra, Lemma 7.90.10 (this also uses Algebra, Lemma 7.35.8 and 7.90.4 to see that flatness of the map on completions implies flatness of $A \to B$). Choose a commutative

diagram

$$\begin{array}{ccc} S & \longrightarrow & B \\ \uparrow & & \uparrow \\ R & \longrightarrow & A \end{array}$$

as in Lemma 12.29.3 with $R \to S$ flat. Let $I \subset R$ be the kernel of $R \to A$. Because B is formally smooth over A we see that the A-algebra map

$$S/IS \longrightarrow B$$

has a section, see Lemma 12.28.5. Hence B is a direct summand of the flat A-module S/IS (by base change of flatness, see Algebra, Lemma 7.35.6), whence flat. □

Proposition 12.30.4. *Let $A \to B$ be a local homomorphism of Noetherian local rings. Let k be the residue field of A and $\overline{B} = B \otimes_A k$ the special fibre. The following are equivalent*

(1) *$A \to B$ is flat and $\overline{B}$ is geometrically regular over k,*
(2) *$A \to B$ is flat and $k \to \overline{B}$ is formally smooth in the $\mathfrak{m}_{\overline{B}}$-adic topology, and*
(3) *$A \to B$ is formally smooth in the $\mathfrak{m}_B$-adic topology.*

Proof. The equivalence of (1) and (2) follows from Theorem 12.30.1.

Assume (3). By Lemma 12.30.3 we see that $A \to B$ is flat. By Lemma 12.28.8 we see that $k \to \overline{B}$ is formally smooth in the $\mathfrak{m}_{\overline{B}}$-adic topology. Thus (2) holds.

Assume (2). Lemma 12.28.4 tells us formal smoothness is preserved under completion. The same is true for flatness by Algebra, Lemma 7.90.4. Hence we may replace A and B by their respective completions and assume that A and B are Noetherian complete local rings. In this case choose a diagram

$$\begin{array}{ccc} S & \longrightarrow & B \\ \uparrow & & \uparrow \\ R & \longrightarrow & A \end{array}$$

as in Lemma 12.29.3. We will use all of the properties of this diagram without further mention. Fix a regular system of parameters $t_1, \ldots, t_d$ of R with $t_1 = p$ in case the characteristic of k is $p > 0$. Set $\overline{S} = S \otimes_R k$. Consider the short exact sequence

$$0 \to J \to S \to B \to 0$$

Since B is flat over A we see that $J \otimes_R k$ is the kernel of $\overline{S} \to \overline{B}$. As $\overline{B}$ and $\overline{S}$ are regular we see that $J \otimes_R k$ is generated by elements $\overline{x}_1, \ldots, \overline{x}_r$ which form part of a regular system of parameters of $\overline{S}$, see Algebra, Lemma 7.98.4. Lift these elements to $x_1, \ldots, x_r \in J$. Then $t_1, \ldots, t_d, x_1, \ldots, x_r$ is part of a regular system of parameters for S. Hence $S/(x_1, \ldots, x_r)$ is a power series ring over a field (if the characteristic of k is zero) or a power series ring over a Cohen ring (if the characteristic of k is $p > 0$), see Lemma 12.29.2. Moreover, it is still the case that $R \to S/(x_1, \ldots, x_r)$ maps $t_1, \ldots, t_d$ to a part of a regular system of parameters of $S/(x_1, \ldots, x_r)$. In other words, we may replace S by $S/(x_1, \ldots, x_r)$ and assume we have a diagram

$$\begin{array}{ccc} S & \longrightarrow & B \\ \uparrow & & \uparrow \\ R & \longrightarrow & A \end{array}$$

as in Lemma 12.29.3 with moreover $\overline{S} = \overline{B}$. In this case the map

$$S \otimes_R A \longrightarrow B$$

is an isomorphism as it is surjective and an isomorphism on special fibres, see Algebra, Lemma 7.91.1. Thus by Lemma 12.28.8 it suffices to show that $R \to S$ is formally smooth in the $\mathfrak{m}_S$-adic topology. Of course, since $\overline{S} = \overline{B}$, we have that $\overline{S}$ is formally smooth over $k = R/\mathfrak{m}_R$.

Choose elements $y_1, \ldots, y_m \in S$ such that $t_1, \ldots, t_d, y_1, \ldots, y_m$ is a regular system of parameters for S. If the characteristic of k is zero, choose a coefficient field $K \subset S$ and if the characteristic of k is $p > 0$ choose a Cohen ring $\Lambda \subset S$ with residue field K. At this point the map $K[[t_1, \ldots, t_d, y_1, \ldots, y_m]] \to S$ (characteristic zero case) or $\Lambda[[t_2, \ldots, t_d, y_1, \ldots, y_m]] \to S$ (characteristic $p > 0$ case) is an isomorphism, see Lemma 12.29.2. From now on we think of S as the above power series ring.

The rest of the proof is analogous to the argument in the proof of Theorem 12.30.1. Choose a solid diagram

$$\begin{tikzcd} S \arrow[r] \arrow[dr, dashed, "\bar{\psi}"] & N/J \\ R \arrow[u, "i"] \arrow[r, "\varphi"] & N \arrow[u, "\pi"] \end{tikzcd}$$

as in Definition 12.28.1. As $J^2 = 0$ we see that J has a canonical N/J module structure and via $\bar{\psi}$ a S-module structure. As $\bar{\psi}$ is continuous for the $\mathfrak{m}_S$-adic topology we see that $\mathfrak{m}_S^n J = 0$ for some n. Hence we can filter J by N/J-submodules $0 \subset J_1 \subset J_2 \subset \ldots \subset J_n = J$ such that each quotient J_{t+1}/J_t is annihilated by $\mathfrak{m}_S$. Considering the sequence of ring maps $N \to N/J_1 \to N/J_2 \to \ldots \to N/J$ we see that it suffices to prove the existence of the dotted arrow when J is annihilated by $\mathfrak{m}_S$, i.e., when J is a K-vector space.

Assume given a diagram as above such that J is annihilated by $\mathfrak{m}_S$. As $\mathbf{Q} \to S$ (characteristic zero case) or $\mathbf{Z} \to S$ (characteristic $p > 0$ case) is formally smooth in the $\mathfrak{m}_S$-adic topology (see Lemma 12.29.1), we can find a ring map $\psi : S \to N$ such that $\pi \circ \psi = \bar{\psi}$. Since S is a power series ring in $t_1, \ldots, t_d$ (characteristic zero) or $t_2, \ldots, t_d$ (characteristic $p > 0$) over a subring, it follows from the universal property of power series rings that we can change our choice of ψ so that $\psi(t_i)$ equals $\varphi(t_i)$ (automatic for $t_1 = p$ in the characteristic p case). Then $\psi \circ i$ and $\varphi : R \to N$ are two maps whose compositions with π are equal and which agree on $t_1, \ldots, t_d$. Hence $D = \psi \circ i - \varphi : R \to J$ is a derivation which annihilates $t_1, \ldots, t_d$. By Algebra, Lemma 7.122.3 we can write $D = \xi \circ \mathrm{d}$ for some R-linear map $\xi : \Omega_{R/\mathbf{Z}} \to J$ which annihilates $\mathrm{d}t_1, \ldots, \mathrm{d}t_d$ (by construction) and $\mathfrak{m}_R \Omega_{R/\mathbf{Z}}$ (as J is annihilated by $\mathfrak{m}_R$). Hence ξ factors as a composition

$$\Omega_{R/\mathbf{Z}} \to \Omega_{k/\mathbf{Z}} \xrightarrow{\xi'} J$$

where ξ' is k-linear. Using the K-vector space structure on J we extend ξ' to a K-linear map

$$\xi'' : \Omega_{k/\mathbf{Z}} \otimes_k K \longrightarrow J.$$

Using that $\overline{S}/k$ is formally smooth we see that

$$\Omega_{k/\mathbf{Z}} \otimes_k K \to \Omega_{\overline{S}/\mathbf{Z}} \otimes_S K$$

is injective by Theorem 12.30.1 (this is true also in the characteristic zero case as it is even true that $\Omega_{k/\mathbf{Z}} \to \Omega_{K/\mathbf{Z}}$ is injective in characteristic zero, see Algebra, Proposition 7.141.8).

Hence we can find a K-linear map $\xi''' : \Omega_{\overline{S}/\mathbf{Z}} \otimes_S K \to J$ whose restriction to $\Omega_{k/\mathbf{Z}} \otimes_k K$ is ξ''. Write

$$D' : S \xrightarrow{\mathrm{d}} \Omega_{S/\mathbf{Z}} \to \Omega_{\overline{S}/\mathbf{Z}} \to \Omega_{\overline{S}/\mathbf{Z}} \otimes_S K \xrightarrow{\xi'''} J.$$

Finally, set $\psi' = \psi - D' : S \to N$. The reader verifies that ψ' is a ring map such that $\pi \circ \psi' = \bar{\psi}$ and such that $\psi' \circ i = \varphi$ as desired. □

As an application of the result above we prove that deformations of formally smooth algebras are unobstructed.

Lemma 12.30.5. *Let A be a Noetherian complete local ring with residue field k. Let B be a Noetherian complete local k-algebra. Assume $k \to B$ is formally smooth in the $\mathfrak{m}_B$-adic topology. Then there exists a Noetherian complete local ring C and a local homomorphism $A \to C$ which is formally smooth in the $\mathfrak{m}_C$-adic topology such that $C \otimes_A k \cong B$.*

Proof. Choose a diagram

$$\begin{tikzcd} S \arrow[r] & B \\ R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

as in Lemma 12.29.3. Let $t_1, \ldots, t_d$ be a regular system of parameters for R with $t_1 = p$ in case the characteristic of k is $p > 0$. As B and $\overline{S} = S \otimes_A k$ are regular we see that $\mathrm{Ker}(\overline{S} \to B)$ is generated by elements $\overline{x}_1, \ldots, \overline{x}_r$ which form part of a regular system of parameters of $\overline{S}$, see Algebra, Lemma 7.98.4. Lift these elements to $x_1, \ldots, x_r \in S$. Then $t_1, \ldots, t_d, x_1, \ldots, x_r$ is part of a regular system of parameters for S. Hence $S/(x_1, \ldots, x_r)$ is a power series ring over a field (if the characteristic of k is zero) or a power series ring over a Cohen ring (if the characteristic of k is $p > 0$), see Lemma 12.29.2. Moreover, it is still the case that $R \to S/(x_1, \ldots, x_r)$ maps $t_1, \ldots, t_d$ to a part of a regular system of parameters of $S/(x_1, \ldots, x_r)$. In other words, we may replace S by $S/(x_1, \ldots, x_r)$ and assume we have a diagram

$$\begin{tikzcd} S \arrow[r] & B \\ R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

as in Lemma 12.29.3 with moreover $\overline{S} = B$. In this case $R \to S$ is formally smooth in the $\mathfrak{m}_S$-adic topology by Proposition 12.30.4. Hence the base change $C = S \otimes_R A$ is formally smooth over A in the $\mathfrak{m}_C$-adic topology by Lemma 12.28.8. □

Remark 12.30.6. The assertion of Lemma 12.30.5 is quite strong. Namely, suppose that we have a diagram

$$\begin{tikzcd} & B \\ A \arrow[r] & A' \arrow[u] \end{tikzcd}$$

of local homomorphisms of Noetherian complete local rings where $A \to A'$ induces an isomorphism of residue fields $k = A/\mathfrak{m}_A = A'/\mathfrak{m}_{A'}$ and with $B \otimes_{A'} k$ formally smooth over k. Then we can extend this to a commutative diagram

$$\begin{tikzcd} C \arrow[r] & B \\ A \arrow[u] \arrow[r] & A' \arrow[u] \end{tikzcd}$$

of local homomorphisms of Noetherian complete local rings where $A \to C$ is formally smooth in the $\mathfrak{m}_C$-adic topology and where $C \otimes_A k \cong B \otimes_{A'} k$. Namely, pick $A \to C$ as in Lemma 12.30.5 lifting $B \otimes_{A'} k$ over k. By formal smoothness we can find the arrow $C \to B$, see Lemma 12.28.5. Denote $C \otimes_A^\wedge A'$ the completion of $C \otimes_A A'$ with respect to the ideal $C \otimes_A \mathfrak{m}_{A'}$. Note that $C \otimes_A^\wedge A'$ is a Noetherian complete local ring (see Algebra, Lemma 7.90.9) which is flat over A' (see Algebra, Lemma 7.91.10). We have moreover

(1) $C \otimes_A^\wedge A' \to B$ is surjective,
(2) if $A \to A'$ is surjective, then $C \to B$ is surjective,
(3) if $A \to A'$ is finite, then $C \to B$ is finite, and
(4) if $A' \to B$ is flat, then $C \otimes_A^\wedge A' \cong B$.

Namely, by Nakayama's lemma for nilpotent ideals (see Algebra, Lemma 7.14.5) we see that $C \otimes_A k \cong B \otimes_{A'} k$ implies that $C \otimes_A A'/\mathfrak{m}_{A'}^n \to B/\mathfrak{m}_{A'}^n B$ is surjective for all n. This proves (1). Parts (2) and (3) follow from part (1). Part (4) follows from Algebra, Lemma 7.91.1.

12.31. Regular ring maps

Let k be a field. Recall that a Noetherian k-algebra A is said to be *geometrically regular* over k if and only if $A \otimes_k k'$ is regular for all finite purely inseparable extensions k' of k, see Algebra, Definition 7.148.2. Moreover, if this is the case then $A \otimes_k k'$ is regular for every finitely generated field extension $k \subset k'$, see Algebra, Lemma 7.148.1. We use this notion in the following definition.

Definition 12.31.1. A ring map $R \to \Lambda$ is *regular* if it is flat and for every prime $\mathfrak{p} \subset R$ the fibre ring

$$\Lambda \otimes_R \kappa(\mathfrak{p}) = \Lambda_\mathfrak{p}/\mathfrak{p}\Lambda_\mathfrak{p}$$

is Noetherian and geometrically regular over $\kappa(\mathfrak{p})$.

If $R \to \Lambda$ is a ring map with Λ Noetherian, then the fibre rings are always Noetherian.

Lemma 12.31.2 (Regular is a local property)**.** *Let $R \to \Lambda$ be a ring map with Λ Noetherian. Then $R \to \Lambda$ is regular if and only if the local ring maps $R_\mathfrak{p} \to \Lambda_\mathfrak{q}$ are regular for all $\mathfrak{q} \subset \Lambda$ lying over $\mathfrak{p} \subset R$.*

Proof. This is true because a Noetherian ring is regular if and only if all the local rings are regular local rings, see Algebra, Definition 12.31.1 and a ring map is flat if and only if all the induced maps of local rings are flat, see Algebra, Lemma 7.35.19. □

Lemma 12.31.3 (Regular maps and base change)**.** *Let $R \to \Lambda$ be a regular ring map. For any finite type ring map $R \to R'$ the base change $R' \to \Lambda \otimes_R R'$ is regular too.*

Proof. Flatness is preserved under any base change, see Algebra, Lemma 7.35.6. Consider a prime $\mathfrak{p}' \subset R'$ lying over $\mathfrak{p} \subset R$. The residue field extension $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{p}')$ is finitely generated as R' is of finite type over R. Hence the fibre ring

$$(\Lambda \otimes_R R') \otimes_{R'} \kappa(\mathfrak{p}') = \Lambda \otimes_R \kappa(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p}')$$

is Noetherian by Algebra, Lemma 7.28.7 and the assumption on the fibre rings of $R \to \Lambda$. Geometric regularity of the fibres is preserved by Algebra, Lemma 7.148.1. □

Lemma 12.31.4. *Let R be a ring. Let $(A_i, \varphi_{ii'})$ be a directed system of smooth R-algebras. Set $\Lambda = colim A_i$. If the fibre rings $\Lambda \otimes_R \kappa(\mathfrak{p})$ are Noetherian for all $\mathfrak{p} \subset R$, then $R \to \Lambda$ is regular.*

Proof. Note that Λ is flat over R by Algebra, Lemmas 7.35.2 and 7.126.10. Let $\kappa(\mathfrak{p}) \subset k$ be a finite purely inseparable extension. Note that

$$\Lambda \otimes_R \kappa(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})} k = \Lambda \otimes_R k = colim A_i \otimes_R k$$

is a colimit of smooth k-algebras, see Algebra, Lemma 7.126.4. Since each local ring of a smooth k-algebra is regular by Algebra, Lemma 7.129.3 we conclude that all local rings of $\Lambda \otimes_R k$ are regular by Algebra, Lemma 7.98.8. This proves the lemma. □

Let's see when a field extension defines a regular ring map.

Lemma 12.31.5. *Let $k \subset K$ be a field extension. Then $k \to K$ is a regular ring map if and only if K is a separable field extension of k.*

Proof. If $k \to K$ is regular, then K is geometrically reduced over k, hence K is separable over k by Algebra, Proposition 7.141.8. Conversely, if K/k is separable, then K is a colimit of smooth k-algebras, see Algebra, Lemma 7.141.10 hence is regular by Lemma 12.31.4. □

Lemma 12.31.6. *Let $A \to B \to C$ be ring maps. If $A \to C$ is regular and $B \to C$ is flat and surjective on spectra, then $A \to B$ is regular.*

Proof. By Algebra, Lemma 7.35.9 we see that $A \to B$ is flat. Let $\mathfrak{p} \subset A$ be a prime. The ring map $B \otimes_A \kappa(\mathfrak{p}) \to C \otimes_A \kappa(\mathfrak{p})$ is flat and surjective on spectra. Hence $B \otimes_A \kappa(\mathfrak{p})$ is geometrically regular by Algebra, Lemma 7.148.3. □

12.32. Ascending properties along regular ring maps

This section is the analogue of Algebra, Section 7.145 but where the ring map $R \to S$ is regular.

Lemma 12.32.1. *Let $\varphi : R \to S$ be a ring map. Assume*

(1) *φ is regular,*
(2) *S is Noetherian, and*
(3) *R is Noetherian and reduced.*

Then S is reduced.

Proof. For Noetherian rings being reduced is the same as having properties (S_1) and (R_0), see Algebra, Lemma 7.140.3. Hence we may apply Algebra, Lemmas 7.145.4 and 7.145.5. □

12.33. Permanence of properties under completion

Given a Noetherian local ring A we denote $A^\wedge$ the completion of A with respect to its maximal ideal. We will use without further mention that $A^\wedge$ is a Noetherian complete local ring (Algebra, Lemmas 7.90.10 and 7.90.7) and that $A \to A^\wedge$ is flat (Algebra, Lemma 7.90.3).

Lemma 12.33.1. *Let A be a Noetherian local ring. Then* $\dim(A) = \dim(A^\wedge)$.

Proof. See for example Algebra, Lemma 7.103.7. □

Lemma 12.33.2. *Let A be a Noetherian local ring. Then depth(A) = depth$(A^\wedge)$.*

Proof. See Algebra, Lemma 7.145.1. □

Lemma 12.33.3. *Let A be a Noetherian local ring. Then A is Cohen-Macaulay if and only if $A^\wedge$ is so.*

Proof. A local ring A is Cohen-Macaulay if and only $\dim(A) = \text{depth}(A)$. As both of these invariants are preserved under completion (Lemmas 12.33.1 and 12.33.2) the claim follows. □

Lemma 12.33.4. *Let A be a Noetherian local ring. Then A is regular if and only if $A^\wedge$ is so.*

Proof. If $A^\wedge$ is regular, then A is regular by Algebra, Lemma 7.102.8. Assume A is regular. Let $\mathfrak m$ be the maximal ideal of A. Then $\dim_{\kappa(\mathfrak m)} \mathfrak m/\mathfrak m^2 = \dim(A) = \dim(A^\wedge)$ (Lemma 12.33.1). On the other hand, $\mathfrak m A^\wedge$ is the maximal ideal of $A^\wedge$ and hence $\mathfrak m_{A^\wedge}$ is generated by at most $\dim(A^\wedge)$ elements. Thus $A^\wedge$ is regular. (You can also use Algebra, Lemma 7.103.8.) □

Lemma 12.33.5. *Let A be a Noetherian local ring.*

1. *If $A^\wedge$ is reduced, then so is A.*
2. *In general A reduced does not imply $A^\wedge$ is reduced.*
3. *If A is Nagata, then A is reduced if and only if $A^\wedge$ is reduced.*

Proof. As $A \to A^\wedge$ is faithfully flat we have (1) by Algebra, Lemma 7.146.2. For (2) see Algebra, Example 7.110.4 (there are also examples in characteristic zero, see Algebra, Remark 7.110.5). For (3) see Algebra, Lemmas 7.144.27 and 7.144.24. □

12.34. Field extensions, revisited

In this section we study some peculiarities of field extensions in characteristic $p > 0$.

Definition 12.34.1. Let p be a prime number. Let $k \to K$ be an extension of fields of characteristic p. Denote kK^p the compositum of k and K^p in K.

1. A subset $\{x_i\} \subset K$ is called *p-independent over k* if the elements $x^E = \prod x_i^{e_i}$ where $0 \leq e_i < p$ are linearly independent over kK^p.
2. A subset $\{x_i\}$ of K is called a *p-basis of K over k* if the elements x^E form a basis of K over kK^p.

This is related to the notion of a p-basis of a $\mathbf{F}_p$-algebra which we will discuss later (insert future reference here).

Lemma 12.34.2. *Let $k \subset K$ be a field extension. Assume k has characteristic $p > 0$. Let $\{x_i\}$ be a subset of K. The following are equivalent*

1. *the elements $\{x_i\}$ are p-independent over k, and*
2. *the elements dx_i are K-linearly independent in $\Omega_{K/k}$.*

Any p-independent collection can be extended to a p-basis of K over k. In particular, the field K has a p-basis over k. Moreover, the following are equivalent:

(a) *$\{x_i\}$ is a p-basis of K over k, and*
(b) *dx_i is a basis of the K-vector space $\Omega_{K/k}$.*

Proof. Assume (2) and suppose that $\sum a_E x^E = 0$ is a linear relation with $a_E \in kK^p$. Let $\theta_i : K \to K$ be a k-derivation such that $\theta_i(x_j) = \delta_{ij}$ (Kronecker delta). Note that any k-derivation of K annihilates kK^p. Applying θ_i to the given relation we obtain new relations

$$\sum\nolimits_{E, e_i > 0} e_i a_E x_1^{e_1} \ldots x_i^{e_i - 1} \ldots x_n^{e_n} = 0$$

Hence if we pick $\sum a_E x^E$ as the relation with minimal total degree $|E| = \sum e_i$ for some $a_E \not= 0$, then we get a contradiction. Hence (2) holds.

If $\{x_i\}$ is a p-basis for K over k, then $K \cong kK^p[X_i]/(X_i^p - x_i^p)$. Hence we see that $\mathrm{d}x_i$ forms a basis for $\Omega_{K/k}$ over K. Thus (a) implies (b).

Let $\{x_i\}$ be a p-independent subset of K over k. An application of Zorn's lemma shows that we can enlarge this to a maximal p-independent subset of K over k. We claim that any maximal p-independent subset $\{x_i\}$ of K is a p-basis of K over k. The claim will imply that (1) implies (2) and establish the existence of p-bases. To prove the claim let L be the subfield of K generated by kK^p and the x_i. We have to show that $L = K$. If $x \in K$ but $x \not\in L$, then $x^p \in L$ and $L(x) \cong L[z]/(z^p - x)$. Hence $\{x_i\} \cup \{x\}$ is p-independent over k, a contradiction.

Finally, we have to show that (b) implies (a). By the equivalence of (1) and (2) we see that $\{x_i\}$ is a maximal p-independent subset of K over k. Hence by the claim above it is a p-basis. $\square$

Lemma 12.34.3. *Let $k \subset K$ be a field extension. Let $\{K_\alpha\}_{\alpha \in A}$ be a collection of subfields of K with the following properties*

1. *$k \subset K_\alpha$ for all $\alpha \in A$,*
2. *$k = \bigcap_{\alpha \in A} K_\alpha$,*
3. *for $\alpha, \alpha' \in A$ there exists an $\alpha'' \in A$ such that $K_{\alpha''} \subset K_\alpha \cap K_{\alpha'}$.*

Then for $n \geq 1$ and $V \subset K^{\oplus n}$ a K-vector space we have $V \cap k^{\oplus n} \not= 0$ if and only if $V \cap K_\alpha^{\oplus n} \not= 0$ for all $\alpha \in A$.

Proof. By induction on n. The case $n = 1$ follows from the assumptions. Assume the result proven for subspaces of $K^{\oplus n-1}$. Assume that $V \subset K^{\oplus n}$ has nonzero intersection with $K_\alpha^{\oplus n}$ for all $\alpha \in A$. If $V \cap 0 \oplus k^{\oplus n-1}$ is nonzero then we win. Hence we may assume this is not the case. By induction hypothesis we can find an α such that $V \cap 0 \oplus K_\alpha^{\oplus n-1}$ is zero. Let $v = (x_1, \ldots, x_n) \in V \cap K_\alpha$ be a nonzero element. By our choice of α we see that x_1 is not zero. Replace v by $x_1^{-1}v$ so that $v = (1, x_2, \ldots, x_n)$. Note that if $v' = (x_1', \ldots, x_n') \in V \cap K_\alpha$, then $v' - x_1' v = 0$ by our choice of α. Hence we see that $V \cap K_\alpha^{\oplus n} = K_\alpha v$. If we choose some α' such that $K_{\alpha'} \subset K_\alpha$, then we see that necessarily $v \in V \cap K_{\alpha'}^{\oplus n}$ (by the same arguments applied to α'). Hence

$$x_2, \ldots, x_n \in \bigcap\nolimits_{\alpha' \in A, K_{\alpha'} \subset K_\alpha} K_{\alpha'}$$

which equals k by (2) and (3). $\square$

Lemma 12.34.4. *Let K be a field of characteristic p. Let $\{K_\alpha\}_{\alpha \in A}$ be a collection of subfields of K with the following properties*

1. *$K^p \subset K_\alpha$ for all $\alpha \in A$,*
2. *$K^p = \bigcap_{\alpha \in A} K_\alpha$,*
3. *for $\alpha, \alpha' \in A$ there exists an $\alpha'' \in A$ such that $K_{\alpha''} \subset K_\alpha \cap K_{\alpha'}$.*

Then

(1) *the intersection of the kernels of the maps* $\Omega_{K/\mathbf{F}_p} \to \Omega_{K/K_\alpha}$ *is zero,*
(2) *for any finite extension* $K \subset L$ *we have* $L^p = \bigcap_{\alpha \in A} L^p K_\alpha$.

Proof. Proof of (1). Choose a p-basis $\{x_i\}$ for K over $\mathbf{F}_p$. Suppose that $\eta = \sum_{i \in I'} y_i \mathrm{d}x_i$ maps to zero in Ω_{K/K_α} for every $\alpha \in A$. Here the index set I' is finite. By Lemma 12.34.2 this means that for every α there exists a relation

$$\sum\nolimits_E a_{E,\alpha} x^E, \quad a_{E,\alpha} \in K_\alpha$$

where E runs over multi-indices $E = (e_i)_{i \in I'}$ with $0 \leq e_i < p$. On the other hand, Lemma 12.34.2 guarantees there is no such relation $\sum a_E x^E = 0$ with $a_E \in K^p$. This is a contradiction by Lemma 12.34.3.

Proof of (2). Suppose that we have a tower $K \subset M \subset L$ of finite extensions of fields. Set $M_\alpha = M^p K_\alpha$ and $L_\alpha = L^p K_\alpha = L^p M_\alpha$. Then we can first prove that $M^p = \bigcap_{\alpha \in A} M_\alpha$, and after that prove that $L^p = \bigcap_{\alpha \in A} L_\alpha$. Hence it suffices to prove (2) for primitive field extensions having no nontrivial subfields. First, assume that $L = K(\theta)$ is separable over K. Then L is generated by θ^p over K, hence we may assume that $\theta \in L^p$. In this case we see that

$$L^p = K^p \oplus K^p\theta \oplus \ldots K^p\theta^{d-1} \quad \text{and} \quad L^pK_\alpha = K_\alpha \oplus K_\alpha\theta \oplus \ldots K_\alpha\theta^{d-1}$$

where $d = [L : K]$. Thus the conclusion is clear in this case. The other case is where $L = K(\theta)$ with $\theta^p = t \in K$, $t \notin K^p$. In this case we have

$$L^p = K^p \oplus K^p t \oplus \ldots K^p t^{p-1} \quad \text{and} \quad L^pK_\alpha = K_\alpha \oplus K_\alpha t \oplus \ldots K_\alpha t^{p-1}$$

Again the result is clear. □

Lemma 12.34.5. *Let k be a field of characteristic $p > 0$. Let $n, m \geq 0$. As k' ranges through all subfields $k^p \subset k' \subset k$ with $[k : k'] < \infty$ the subfields*

$$f.f.(k'[[x_1^p, \ldots, x_n^p]][y_1^p, \ldots, y_m^p]) \subset f.f.(k[[x_1, \ldots, x_d]][y_1, \ldots, y_m])$$

form a family of subfields as in Lemma 12.34.4. Moreover, each of the ring extensions $k'[[x_1^p, \ldots, x_n^p]][y_1^p, \ldots, y_m^p] \subset k[[x_1, \ldots, x_n]][y_1, \ldots, y_m]$ is finite.

Proof. Write $A = k[[x_1, \ldots, x_n]][y_1, \ldots, y_m]$ and $A' = k'[[x_1^p, \ldots, x_n^p]][y_1^p, \ldots, y_m^p]$. We also set $K = f.f.(A)$ and $K' = f.f.(A')$. The ring extension $k'[[x_1^p, \ldots, x_d^p]] \subset k[[x_1, \ldots, x_d]]$ is finite by Algebra, Lemma 7.90.16 which implies that $A \to A'$ is finite. For $f \in A$ we see that $f^p \in A'$. Hence $K^p \subset K'$. Any element of K' can be written as a/b^p with $a \in A'$ and $b \in A$ nonzero. Suppose that $f/g^p \in K$, $f, g \in A$, $g \neq 0$ is contained in K' for every choice of k'. Fix a choice of k' for the moment. By the above we see $f/g^p = a/b^p$ for some $a \in A'$ and some nonzero $b \in A$. Hence $b^p f \in A'$. For any A'-derivation $D : A \to A$ we see that $0 = D(b^p f) = b^p D(f)$ hence $D(f) = 0$ as A is a domain. Taking $D = \partial_{x_i}$ and $D = \partial_{y_j}$ we conclude that that $f \in k[[x_1^p, \ldots, x_n^p]][y_1^p, \ldots, y_d^p]$. Applying a k'-derivation $\theta : k \to k$ we similarly conclude that all coefficients of f are in k', i.e., $f \in A'$. Since it is clear that $A = \bigcap_{k'} A'$ where k' ranges over all subfields as in the lemma we win. □

12.35. The singular locus

Let R be a Noetherian ring. The *regular locus* $\mathrm{Reg}(X)$ of $X = \mathit{Spec}(R)$ is the set of primes $\mathfrak{p}$ such that $R_\mathfrak{p}$ is a regular local ring. The *singular locus* $\mathrm{Sing}(X)$ of $X = \mathit{Spec}(R)$ is the complement $X \setminus \mathrm{Reg}(X)$, i.e., the set of primes $\mathfrak{p}$ such that $R_\mathfrak{p}$ is not a regular local ring. By the discussion preceding Algebra, Definition 7.102.6 we see that $\mathrm{Reg}(X)$ is stable under generalization In the section we study conditions that guarantee that $\mathrm{Reg}(X)$ is open.

Definition 12.35.1. Let R be a Noetherian ring. Let $X = Spec(R)$.

(1) We say R is *J-0* if $\mathrm{Reg}(X)$ contains a nonempty open.
(2) We say R is *J-1* if $\mathrm{Reg}(X)$ is open.
(3) We say R is *J-2* if any finite type R-algebra is J-1.

The ring $\mathbf{Q}[x]/(x^2)$ does not satisfy J-0. On the other hand J-1 implies J-0 for domains and even reduced rings as such a ring is regular at the minimal primes. Here is a characterization of the J-1 property.

Lemma 12.35.2. *Let R be a Noetherian ring. Let $X = Spec(R)$. The ring R is J-1 if and only if $V(\mathfrak p) \cap Reg(X)$ contains a nonempty open subset of $V(\mathfrak p)$ for all $\mathfrak p \in Reg(X)$.*

Proof. This follows immediately from Topology, Lemma 5.11.5. $\square$

Lemma 12.35.3. *Let R be a Noetherian ring. Let $X = Spec(R)$. Assume that for all $\mathfrak p \subset R$ the ring $R/\mathfrak p$ is J-0. Then R is J-1.*

Proof. We will show that the criterion of Lemma 12.35.2 applies. Let $\mathfrak p \in \mathrm{Reg}(X)$ be a prime of height r. Pick $f_1, \ldots, f_r \in \mathfrak p$ which map to generators of $\mathfrak p R_{\mathfrak p}$. Since $\mathfrak p \in \mathrm{Reg}(X)$ we see that $f_1, \ldots, f_r$ maps to a regular sequence in $R_{\mathfrak p}$, see Algebra, Lemma 7.98.3. Thus by Algebra, Lemma 7.65.8 we see that after replacing R by R_g for some $g \in R$, $g \not\in \mathfrak p$ the sequence $f_1, \ldots, f_r$ is a regular sequence in R. Next, let $\mathfrak p \subset \mathfrak q$ be a prime ideal such that $(R/\mathfrak p)_{\mathfrak q}$ is a regular local ring. By the assumption of the lemma there exists a non-empty open subset of $V(\mathfrak p)$ consisting of such primes, hence it suffices to prove $R_{\mathfrak q}$ is regular. Note that $f_1, \ldots, f_r$ is a regular sequence in $R_{\mathfrak q}$ such that $R_{\mathfrak q}/(f_1, \ldots, f_r)R_{\mathfrak q}$ is regular. Hence $R_{\mathfrak q}$ is regular by Algebra, Lemma 7.98.7. $\square$

Lemma 12.35.4. *Let $R \to S$ be a ring map. Assume that*

(1) *R is a Noetherian domain,*
(2) *$R \to S$ is injective and of finite type, and*
(3) *S is a domain and J-0.*

Then R is J-0.

Proof. After replacing S by S_g for some nonzero $g \in S$ we may assume that S is a regular ring. By generic flatness we may assume that also $R \to S$ is faithfully flat, see Algebra, Lemma 7.109.1. Then R is regular by Algebra, Lemma 7.146.4. $\square$

Lemma 12.35.5. *Let $R \to S$ be a ring map. Assume that*

(1) *R is a Noetherian domain and J-0,*
(2) *$R \to S$ is injective and of finite type, and*
(3) *S is a domain and $f.f.(R) \to f.f.(S)$ is separable.*

Then S is J-0.

Proof. We may replace R by a principal localization and assume R is a regular ring. By Algebra, Lemma 7.129.9 the ring map $R \to S$ is smooth at (0). Hence after replacing S by a principal localization we may assume that S is smooth over R. Then S is regular too, see Algebra, Lemma 7.145.8. $\square$

Lemma 12.35.6. *Let R be a Noetherian ring. The following are equivalent*

(1) *R is J-2,*
(2) *every finite type R-algebra which is a domain is J-0,*
(3) *every finite R-algebra is J-1,*

(4) *for every prime $\mathfrak{p}$ and every finite purely inseparable extension $\kappa(\mathfrak{p}) \subset L$ there exists a finite R-algebra R' which is a domain, which is J-0, and whose field of fractions is L.*

Proof. It is clear that we have the implications (1) ⇒ (2) and (2) ⇒ (4). Recall that a domain which is J-1 is J-0. Hence we also have the implications (1) ⇒ (3) and (3) ⇒ (4).

Let $R \to S$ be a finite type ring map and let's try to show S is J-1. By Lemma 12.35.3 it suffices to prove that $S/\mathfrak{q}$ is J-0 for every prime $\mathfrak{q}$ of S. In this way we see (2) ⇒ (1).

Assume (4). We will show that (2) holds which will finish the proof. Let $R \to S$ be a finite type ring map with S a domain. Let $\mathfrak{p} = \text{Ker}(R \to S)$. Set $K = f.f.(S)$. There exists a diagram of fields

$$\begin{tikzcd} K \arrow[r] & K' \\ \kappa(\mathfrak{p}) \arrow[u] \arrow[r] & L \arrow[u] \end{tikzcd}$$

where the horizontal arrows are finite purely inseparable field extensions and where K'/L is separable, see Algebra, Lemma 7.39.4. Choose $R' \subset L$ as in (4) and let S' be the image of the map $S \otimes_R R' \to K'$. Then S' is a domain whose fraction field is K', hence S' is J-0 by Lemma 12.35.5 and our choice of R'. Then we apply Lemma 12.35.4 to see that S is J-0 as desired. □

12.36. Regularity and derivations

Let $R \to S$ be a ring map. Let $D : R \to R$ be a derivation. We say that D *extends to S* if there exists a derivation $D' : S \to S$ such that

$$\begin{tikzcd} S \arrow[r, "D'"'] & S \\ R \arrow[u] \arrow[r, "D"] & R \arrow[u] \end{tikzcd}$$

is commutative.

Lemma 12.36.1. *Let R be a ring. Let $D : R \to R$ be a derivation.*

(1) *For any ideal $I \subset R$ the derivation D extends canonically to a derication $D^\wedge : R^\wedge \to R^\wedge$ on the I-adic completion.*
(2) *For any mulitplicative subset $S \subset R$ the derivation D extends uniquely to the localization $S^{-1}R$ of R.*

If $R \subset R'$ is an finite type extension of rings such that $R_g \cong R'_g$ for some nonzero divisor $g \in R$, then $g^N D$ extends to R' for some $N \geq 0$.

Proof. Proof of (1). For $n \geq 2$ we have $D(I^n) \subset I^{n-1}$ by the Leibniz rule. Hence D induces maps $D_n : R/I^n \to R/I^{n-1}$. Taking the limit we obtain $D^\wedge$. We omit the verification that $D^\wedge$ is a derivation.

Proof of (2). To extend D to $S^{-1}R$ just set $D(r/s) = D(r)/s - rD(s)/s^2$ and check the axioms.

Proof of the final statement. Let $x_1, \ldots, x_n \in R'$ be generators of R' over R. Choose an N such that $g^N x_i \in R$. Consider $g^{N+1}D$. By (2) this extends to R_g. Moreover, by the Leibniz rule and our construction of the extension above we have

$$g^{N+1}D(x_i) = g^{N+1}D(g^{-N}g^Nx_i) = -Ng^Nx_iD(g) + gD(g^Nx_i)$$

and both terms are in R. This implies that

$$g^{N+1}D(x_1^{e_1}\ldots x_n^{e_n}) = \sum e_i x_1^{e_1}\ldots x_i^{e_i-1}\ldots x_n^{e_n}g^{N+1}D(x_i)$$

is an element of R'. Hence every element of R' (which can be written as a sum of monomials in the x_i with coefficients in R) is mapped to an element of R' by $g^{N+1}D$ and we win. □

Lemma 12.36.2. *Let R be a regular ring. Let $f \in R$. Assume there exists a derivation $D : R \to R$ such that $D(f)$ is a unit of $R/(f)$. Then $R/(f)$ is regular.*

Proof. It suffices to prove this when R is a local ring with maximal ideal $\mathfrak{m}$ and residue field κ. In this case it suffices to prove that $f \not\in \mathfrak{m}^2$, see Algebra, Lemma 7.98.3. However, if $f \in \mathfrak{m}^2$ then $D(f) \in \mathfrak{m}$ by the Leibniz rule, a contradiction. □

Lemma 12.36.3. *Let R be a regular $\mathbf{F}_p$-algebra. Let $f \in R$. Assume there exists a derivation $D : R \to R$ such that $D(f)$ is a unit of R. Then $R[z]/(z^p - f)$ is regular.*

Proof. Apply Lemma 12.36.2 to the extension of D to $R[z]$ which maps z to zero. □

Lemma 12.36.4. *Let p be a prime number. Let B be a domain with $p = 0$ in B. Let $f \in B$ be an element which is not a pth power in the fraction field of B. If B is of finite type over a Noetherian complete local ring, then there exists a derivation $D : B \to B$ such that $D(f)$ is not zero.*

Proof. Let R be a Noetherian complete local ring such that there exists a finite type ring map $R \to B$. Of course we may replace R by its image in B, hence we may assume R is a domain of characteristic $p > 0$ (as well as Noetherian complete local). By Algebra, Lemma 7.143.10 we can write R as a finite extension of $k[[x_1, \ldots, x_n]]$ for some field k and integer n. Hence we may replace R by $k[[x_1, \ldots, x_n]]$. Next, we use Algebra, Lemma 7.106.7 to factor $R \to B$ as

$$R \subset R[y_1, \ldots, y_d] \subset B' \subset B$$

with B' finite over $R[y_1, \ldots, y_d]$ and $B'_g \cong B_g$ for some nonzero $g \in R$. Note that $f' = g^{pN}f \in B'$ for some large integer N. It is clear that f' is not a pth power in $f.f.(B') = f.f.(B)$. If we can find a derivation $D' : B' \to B'$ with $D'(f') \neq 0$, then Lemma 12.36.1 guarantees that $D = g^MD'$ extends to S for some $M > 0$. Then $D(f) = g^ND'(f) = g^MD'(g^{-pN}f') = g^{M-pN}D'(f')$ is nonzero. Thus it suffices to prove the lemma in case B is a finite exteion of $A = k[[x_1, \ldots, x_n]][y_1, \ldots, y_m]$.

Note that $\mathrm{d}f$ is not zero in $\Omega_{f.f.(B)/\mathbf{F}_p}$, see Algebra, Lemma 7.141.1. We apply Lemma 12.34.5 to find a subfield $k' \subset k$ of finite index such that with $A' = k'[[x_1^p, \ldots, x_n^p]][y_1^p, \ldots, y_m^p]$ the element $\mathrm{d}f$ does not map to zero in $\Omega_{f.f.(B)/f.f.(A')}$. Thus we can choose a $f.f.(A')$-derivation $D' : f.f.(B) \to f.f.(B)$ with $D'(f) \neq 0$. Since $A' \subset A$ and $A \subset B$ are finite by construction we see that $A' \subset B$ is finite. Choose $b_1, \ldots, b_t \in B$ which generate B as an A'-module. Then $D'(b_i) = f_i/g_i$ for some $f_i, g_i \in B$ with $g_i \neq 0$. Setting $D = g_1 \ldots g_tD'$ we win. □

Lemma 12.36.5. *Let A be a Noetherian complete local domain. Then A is J-0.*

Proof. By Algebra, Lemma 7.143.10 we can find a regular subring $A_0 \subset A$ with A finite over A_0. If $f.f.(A_0) \subset f.f.(A)$ is separable, then we are done by Lemma 12.35.5. If not, then A_0 and A have characteristic $p > 0$. For any subextension $f.f.(A_0) \subset M \subset f.f.(A)$ there exists a finite subextension $A_0 \subset B \subset A$ such that $f.f.(B) = M$. Hence, arguing by induction on $[f.f.(A) : f.f.(A_0)]$ we may assume there exists $A_0 \subset B \subset A$ such that B is J-0 and $f.f.(B) \subset f.f.(A)$ has no nontrivial subextensions. In this case, if $f.f.(B) \subset f.f.(A)$ is separable, then we see that A is J-0 by Lemma 12.35.5. If not, then $f.f.(A) = f.f.(B)[z]/(z^p - b)$ for some $b \in B$ which is not a pth power in $f.f.(B)$. By Lemma 12.36.4 we can find a derivation $D : B \to B$ with $D(f) \neq 0$. Applying Lemma 12.36.3 we see that $A_{\mathfrak p}$ is regular for any prime $\mathfrak p$ of A lying over a regular prime of B and not containing $D(f)$. As B is J-0 we conclude A is too. □

Proposition 12.36.6. *The following types of rings are J-2:*

(1) *fields,*
(2) *Noetherian complete local rings,*
(3) $\mathbf{Z}$*,*
(4) *Dedekind domains with fraction field of characteristic zero,*
(5) *finite type ring extensions of any of the above.*

Proof. For fields, $\mathbf{Z}$ and Dedekind domains of characteristic zero you just check condition (4) of Lemma 12.35.6. In the case of Noetherian complete local rings, note that if $R \to R'$ is finite and R is a Noetherian complete local ring, then R' is a product of Noetherian complete local rings, see Algebra, Lemma 7.143.2. Hence it suffices to prove that a Noetherian complete local ring which is a domain is J-0, which is Lemma 12.36.5. □

12.37. Formal smoothness and regularity

The title of this section refers to Proposition 12.37.2.

Lemma 12.37.1. *Let $A \to B$ be a local homomorphism of Noetherian local rings. Let $D : A \to A$ be a derivation. Assume that B is complete and $A \to B$ is formally smooth in the $\mathfrak m_B$-adic topology. Then there exists an extension $D' : B \to B$ of D.*

Proof. Denote $B[\epsilon] = B[x]/(x^2)$ the ring of dual numbers over B. Consider the ring map $\psi : A \to B[\epsilon]$, $a \mapsto a + \epsilon D(a)$. Consider the commutative diagram

$$\begin{tikzcd} B \arrow[r, "1"'] & B \\ A \arrow[u] \arrow[r, "\psi"] & B[\epsilon] \arrow[u] \end{tikzcd}$$

By Lemma 12.28.5 and the assumption of formal smoothness of B/A we find a map $\varphi : B \to B[\epsilon]$ fitting into the diagram. Write $\varphi(b) = b + \epsilon D'(b)$. Then $D' : B \to B$ is the desired extension. □

Proposition 12.37.2. *Let $A \to B$ be a local homomorphism of Noetherian complete local rings. The following are equivalent*

(1) *$A \to B$ is regular,*
(2) *$A \to B$ is flat and $\overline{B}$ is geometrically regular over k,*
(3) *$A \to B$ is flat and $k \to \overline{B}$ is formally smooth in the $\mathfrak m_{\overline{B}}$-adic topology, and*
(4) *$A \to B$ is formally smooth in the $\mathfrak m_B$-adic topology.*

Proof. We have seen the equivalence of (2), (3), and (4) in Proposition 12.30.4. It is clear that (1) implies (2). Thus we assume the equivalent conditions (2), (3), and (4) hold and we prove (1).

Let $\mathfrak{p}$ be a prime of A. We will show that $B \otimes_A \kappa(\mathfrak{p})$ is geometrically regular over $\kappa(\mathfrak{p})$. By Lemma 12.28.8 we may replace A by $A/\mathfrak{p}$ and B by $B/\mathfrak{p}B$. Thus we may assume that A is a domain and that $\mathfrak{p} = (0)$.

Choose $A_0 \subset A$ as in Algebra, Lemma 7.143.10. We will use all the properties stated in that lemma without further mention. As $A_0 \to A$ induces an isomorphism on residue fields, and as $B/\mathfrak{m}_A B$ is geometrically regular over $A/\mathfrak{m}_A$ we can find a diagram

$$\begin{tikzcd} C \arrow[r] & B \\ A_0 \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

with $A_0 \to C$ formally smooth in the $\mathfrak{m}_C$-adic topology such that $B = C \otimes_{A_0} A$, see Remark 12.30.6. (Completion in the tensor product is not needed as $A_0 \to A$ is finite, see Algebra, Lemma 7.90.2.) Hence it suffices to show that $C \otimes_{A_0} f.f.(A_0)$ is a geometrically regular algebra over $f.f.(A_0)$.

The upshot of the preceding paragraph is that we may assume that $A = k[[x_1, \ldots, x_n]]$ where k is a field or $A = \Lambda[[x_1, \ldots, x_n]]$ where Λ is a Cohen ring. In this case B is a regular ring, see Algebra, Lemma 7.103.8. Hence $B \otimes_A f.f.(A)$ is a regular ring too and we win if the characteristic of $f.f.(A)$ is zero.

Thus we are left with the case where $A = k[[x_1, \ldots, x_n]]$ and k is a field of characteristic $p > 0$. Set $K = f.f.(A)$. Let $L \supset K$ be a finite purely inseparable field extension. We will show by induction on $[L : K]$ that $B \otimes_A L$ is regular. The base case is $L = K$ which we've seen above. Let $K \subset M \subset L$ be a subfield such that L is a degree p extension of M obtained by adjoining a pth root of an element $f \in M$. Let A' be a finite A-subalgebra of M with fraction field M. Clearing denominators, we may and do assume $f \in A'$. Set $A'' = A'[z]/(z^p - f)$ and note that $A' \subset A''$ is finite and that the fraction field of A'' is L. By induction we know that $B \otimes_A M$ ring is regular. We have

$$B \otimes_A L = B \otimes_A M[z]/(z^p - f)$$

By Lemma 12.36.4 we know there exists a derivation $D : A' \to A'$ such that $D(f) \neq 0$. As $A' \to B \otimes_A A'$ is formally smooth in the $\mathfrak{m}$-adic topology by Lemma 12.28.9 we can use Lemma 12.37.1 to extend D to a derivation $D' : B \otimes_A A' \to B \otimes_A A'$. Note that $D'(f) = D(f)$ is a unit in $B \otimes_A M$ as $D(f)$ is not zero in $A' \subset M$. Hence $B \otimes_A L$ is regular by Lemma 12.36.3 and we win. □

12.38. G-rings

Let A be a Noetherian local ring A. In Section 12.33 we have seen that some but not all properties of A are reflected in the completion $A^\wedge$ of A. To study this further we introduce some terminology. For a prime $\mathfrak{q}$ of A the fibre ring

$$(A^\wedge) \otimes_A \kappa(\mathfrak{q}) = (A^\wedge)_\mathfrak{q}/\mathfrak{q}(A^\wedge)_\mathfrak{q}$$

is called a *formal fibre* of A. We think of the formal fibre as an algebra over $\kappa(\mathfrak{q})$. Thus $A \to A^\wedge$ is a regular ring homomorphism if and only if all the formal fibres are geometrically regular algebras.

Definition 12.38.1. A ring R is called a *G-ring* if R is Noetherian and for every prime $\mathfrak{p}$ of R the ring map $R_{\mathfrak{p}} \to (R_{\mathfrak{p}})^\wedge$ is regular.

By the discussion above we see that R is a G-ring if and only if every local ring $R_{\mathfrak{p}}$ has geometrically regular formal fibres. Note that if $\mathbf{Q} \subset R$, then it suffices to check the formal fibres are regular. Another way to express the G-ring condition is described in the following lemma.

Lemma 12.38.2. *Let R be a Noetherian ring. Then R is a G-ring if and only if for every pair of primes $\mathfrak{q} \subset \mathfrak{p} \subset R$ the algebra*

$$(R/\mathfrak{q})^\wedge_{\mathfrak{p}} \otimes_{R/\mathfrak{q}} \kappa(\mathfrak{q})$$

is geometrically regular over $\kappa(\mathfrak{q})$.

Proof. This follows from the fact that

$$R^\wedge_{\mathfrak{p}} \otimes_R \kappa(\mathfrak{q}) = (R/\mathfrak{q})^\wedge_{\mathfrak{p}} \otimes_{R/\mathfrak{q}} \kappa(\mathfrak{q})$$

as algebras over $\kappa(\mathfrak{q})$. □

Lemma 12.38.3. *Let $R \to R'$ be a finite type map of Noetherian rings and let*

$$\begin{tikzcd} \mathfrak{q}' \arrow[r] \arrow[d, dash] & \mathfrak{p}' \arrow[r] \arrow[d, dash] & R' \\ \mathfrak{q} \arrow[r] & \mathfrak{p} \arrow[r] & R \arrow[u] \end{tikzcd}$$

be primes. Assume $R \to R'$ is quasi-finite at $\mathfrak{p}'$.

(1) *If the formal fibre $R^\wedge_{\mathfrak{p}} \otimes_R \kappa(\mathfrak{q})$ is geometrically regular over $\kappa(\mathfrak{q})$, then the formal fibre $R'_{\mathfrak{p}'} \otimes_{R'} \kappa(\mathfrak{q}')$ is geometrically regular over $\kappa(\mathfrak{q}')$.*
(2) *If the formal fibres of $R_{\mathfrak{p}}$ are geometrically regular, then the formal fibres of $R'_{\mathfrak{p}'}$ are geometrically regular.*
(3) *If $R \to R'$ is quasi-finite and R is a G-ring, then R' is a G-ring.*

Proof. It is clear that (1) ⇒ (2) ⇒ (3). Assume $R^\wedge_{\mathfrak{p}} \otimes_R \kappa(\mathfrak{q})$ is geometrically regular over $\kappa(\mathfrak{q})$. By Algebra, Lemma 7.115.3 we see that

$$R^\wedge_{\mathfrak{p}} \otimes_R R' = (R'_{\mathfrak{p}'})^\wedge \times B$$

for some $R^\wedge_{\mathfrak{p}}$-algebra B. Hence $R'_{\mathfrak{p}'} \to (R'_{\mathfrak{p}'})^\wedge$ is a factor of a base change of the map $R_{\mathfrak{p}} \to R^\wedge_{\mathfrak{p}}$. It follows that $(R'_{\mathfrak{p}'})^\wedge \otimes_{R'} \kappa(\mathfrak{q}')$ is a factor of

$$R^\wedge_{\mathfrak{p}} \otimes_R R' \otimes_{R'} \kappa(\mathfrak{q}') = R^\wedge_{\mathfrak{p}} \otimes_R \kappa(\mathfrak{q}) \otimes_{\kappa(\mathfrak{q})} \kappa(\mathfrak{q}').$$

Thus the result follows as extension of base field preserves geometric regularity, see Algebra, Lemma 7.148.1. □

Lemma 12.38.4. *Let R be a Noetherian ring. Then R is a G-ring if and only if for every finite free ring map $R \to S$ the formal fibres of S are regular rings.*

Proof. Assume that for any finite free ring map $R \to S$ the ring S has regular formal fibres. Let $\mathfrak{q} \subset \mathfrak{p} \subset R$ be primes and let $\kappa(\mathfrak{q}) \subset L$ be a finite purely inseparable extension. To show that R is a G-ring it suffices to show that

$$R^\wedge_{\mathfrak{p}} \otimes_R \kappa(\mathfrak{q}) \otimes_{\kappa(\mathfrak{q})} L$$

is a regular ring. Choose a finite free extension $R \to R'$ such that $\mathfrak{q}' = \mathfrak{q}R'$ is a prime and such that $\kappa(\mathfrak{q}')$ is isomorphic to L over $\kappa(\mathfrak{q})$, see Algebra, Lemma 7.142.2. By Algebra, Lemma 7.90.17 we have

$$R_{\mathfrak{p}}^\wedge \otimes_R R' = \prod (R'_{\mathfrak{p}'_i})^\wedge$$

where $\mathfrak{p}'_i$ are the primes of R' lying over $\mathfrak{p}$. Thus we have

$$R_{\mathfrak{p}}^\wedge \otimes_R \kappa(\mathfrak{q}) \otimes_{\kappa(\mathfrak{q})} L = R_{\mathfrak{p}}^\wedge \otimes_R R' \otimes_{R'} \kappa(\mathfrak{q}') = \prod (R'_{\mathfrak{p}'_i})^\wedge \otimes_{R'_{\mathfrak{p}'_i}} \kappa(\mathfrak{q}')$$

Our assumption is that the rings on the right are regular, hence the ring on the left is regular too. Thus R is a G-ring. The converse follows from Lemma 12.38.3. □

Lemma 12.38.5. *Let k be a field of characteristic p. Let $A = k[[x_1, \ldots, x_n]][y_1, \ldots, y_n]$ and denote $K = f.f.(A)$. Let $\mathfrak{p} \subset A$ be a prime. Then $A_{\mathfrak{p}}^\wedge \otimes_A K$ is geometrically regular over K.*

Proof. Let $L \supset K$ be a finite purely inseparable field extension. We will show by induction on $[L : K]$ that $A_{\mathfrak{p}}^\wedge \otimes L$ is regular. The base case is $L = K$: as A is regular, $A_{\mathfrak{p}}^\wedge$ is regular (Lemma 12.33.4), hence the localization $A_{\mathfrak{p}}^\wedge \otimes K$ is regular. Let $K \subset M \subset L$ be a subfield such that L is a degree p extension of M obtained by adjoining a pth root of an element $f \in M$. Let B be a finite A-subalgebra of M with fraction field M. Clearing denominators, we may and do assume $f \in B$. Set $C = B[z]/(z^p - f)$ and note that $B \subset C$ is finite and that the fraction field of C is L. Since $A \subset B \subset C$ are finite and $L/M/K$ are purely inseparable we see that for every element of B or C some power of it lies in A. Hence there is a unique prime $\mathfrak{r} \subset B$, resp. $\mathfrak{q} \subset C$ lying over $\mathfrak{p}$. Note that

$$A_{\mathfrak{p}}^\wedge \otimes_A M = B_{\mathfrak{r}}^\wedge \otimes_B M$$

see Algebra, Lemma 7.90.17. By induction we know that this ring is regular. In the same manner we have

$$A_{\mathfrak{p}}^\wedge \otimes_A L = C_{\mathfrak{r}}^\wedge \otimes_C L = B_{\mathfrak{r}}^\wedge \otimes_B M[z]/(z^p - f)$$

the last equality because the completion of $C = B[z]/(z^p - f)$ equals $B_{\mathfrak{r}}^\wedge[z]/(z^p - f)$. By Lemma 12.36.4 we know there exists a derivation $D : B \to B$ such that $D(f) \neq 0$. In other words, $g = D(f)$ is a unit in M! By Lemma 12.36.1 D extends to a derivation of $B_{\mathfrak{r}}$, $B_{\mathfrak{r}}^\wedge$ and $B_{\mathfrak{r}}^\wedge \otimes_B M$ (successively extending through a localization, a completion, and a localization). Since it is an extension we end up with a derivation of $B_{\mathfrak{r}}^\wedge \otimes_B M$ which maps f to g and g is a unit of the ring $B_{\mathfrak{r}}^\wedge \otimes_B M$. Hence $A_{\mathfrak{p}}^\wedge \otimes_A L$ is regular by Lemma 12.36.3 and we win. □

Proposition 12.38.6. *A Noetherian complete local ring is a G-ring.*

Proof. Let A be a Noetherian complete local ring. By Lemma 12.38.2 it suffices to check that $B = A/\mathfrak{q}$ has geometrically regular formal fibres over the minimal prime (0) of B. Thus we may assume that A is a domain and it suffices to check the condition for the formal fibres over the minimal prime (0) of A. Set $K = f.f(A)$.

We can choose a subring $A_0 \subset A$ which is a regular complete local ring such that A is finite over A_0, see Algebra, Lemma 7.143.10. Moreover, we may assume that A_0 is a power series ring over a field or a Cohen ring. By Lemma 12.38.3 we see that it suffices to prove the result for A_0.

Assume that A is a power series ring over a field or a Cohen ring. Since A is regular the localizations $A_{\mathfrak{p}}$ are regular (see Algebra, Definition 7.102.6 and the discussion preceding it). Hence the completions $A_{\mathfrak{p}}^\wedge$ are regular, see Lemma 12.33.4. Hence the fibre $A_{\mathfrak{p}}^\wedge \otimes_A K$

is, as a localization of $A^\wedge_\mathfrak{p}$, also regular. Thus we are done if the characteristic of K is 0. The positive characteristic case is the case $A = k[[x_1, \ldots, x_d]]$ which is a special case of Lemma 12.38.5. $\square$

Lemma 12.38.7. *Let R be a Noetherian ring. Then R is a G-ring if and only if $R_\mathfrak{m}$ has geometrically regular formal fibres for every maximal ideal $\mathfrak{m}$ of R.*

Proof. Assume $R_\mathfrak{m} \to R_\mathfrak{m}^\wedge$ is regular for every maximal ideal $\mathfrak{m}$ of R. Let $\mathfrak{p}$ be a prime of R and choose a maximal ideal $\mathfrak{p} \subset \mathfrak{m}$. Since $R_\mathfrak{m} \to R_\mathfrak{m}^\wedge$ is faithfully flat we can choose a prime $\mathfrak{p}'$ if $R_\mathfrak{m}^\wedge$ lying over $\mathfrak{p}R_\mathfrak{m}$. Consider the commutative diagram

$$\begin{tikzcd} R_\mathfrak{m}^\wedge \arrow[r] & (R_\mathfrak{m}^\wedge)_{\mathfrak{p}'} \arrow[r] & (R_\mathfrak{m}^\wedge)_{\mathfrak{p}'}^\wedge \\ R_\mathfrak{m} \arrow[r] \arrow[u] & R_\mathfrak{p} \arrow[r] \arrow[u] & R_\mathfrak{p}^\wedge \arrow[u] \end{tikzcd}$$

By assumption the ring map $R_\mathfrak{m} \to R_\mathfrak{m}^\wedge$ is regular. By Proposition 12.38.6 $(R_\mathfrak{m}^\wedge)_{\mathfrak{p}'} \to (R_\mathfrak{m}^\wedge)_{\mathfrak{p}'}^\wedge$ is regular. Hence $R_\mathfrak{m} \to (R_\mathfrak{m}^\wedge)_{\mathfrak{p}'}^\wedge$ is regular and since it factors through the localization $R_\mathfrak{p}$, also the ring map $R_\mathfrak{p} \to (R_\mathfrak{m}^\wedge)_{\mathfrak{p}'}^\wedge$ is regular. Thus we may apply Lemma 12.31.6 to see that $R_\mathfrak{p} \to R_\mathfrak{p}^\wedge$ is regular. $\square$

Lemma 12.38.8. *Let p be a prime number. Let A be a Noetherian complete local domain with fraction field K of characteristic p. Let $\mathfrak{q} \subset A[x]$ be a maximal ideal lying over the maximal ideal of A and let $\mathfrak{r} \subset \mathfrak{q}$ be a prime lying over $(0) \subset A$. Then $A[x]^\wedge_\mathfrak{q} \otimes_{A[x]} \kappa(\mathfrak{r})$ is geometrically regular over $\kappa(\mathfrak{r})$.*

Proof. Note that $K \subset \kappa(r)$ is finite. Hence, given a finite purely inseparable extension $\kappa(r) \subset L$ there exists a finite extension of Noetherian complete local domains $A \subset B$ such that $\kappa(r) \otimes_A B$ surjects onto L. Namely, you take $B \subset L$ a finite A-subalgebra whose field of fractions is L. Denote $\mathfrak{r}' \subset B[x]$ the kernel of the map $B[x] = A[x] \otimes_A B \to \kappa(\mathfrak{r}) \otimes_A B \to L$ so that $\kappa(\mathfrak{r}') = L$. Then

$$A[x]^\wedge_\mathfrak{q} \otimes_{A[x]} L = A[x]^\wedge_\mathfrak{q} \otimes_{A[x]} B[x] \otimes_{B[x]} \kappa(\mathfrak{r}') = \prod B[x]^\wedge_{\mathfrak{q}_i} \otimes_{B[x]} \kappa(\mathfrak{r}')$$

where $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ are the primes of $B[x]$ lying over $\mathfrak{q}$, see Algebra, Lemma 7.90.17. Thus we see that it suffices to prove the rings $B[x]^\wedge_{\mathfrak{q}_i} \otimes_{B[x]} \kappa(\mathfrak{r}')$ are regular. This reduces us to showing that $A[x]^\wedge_\mathfrak{q} \otimes_{A[x]} \kappa(\mathfrak{r})$ is regular in the special case that $K = \kappa(\mathfrak{r})$.

Assume $K = \kappa(\mathfrak{r})$. In this case we see that $\mathfrak{r}K[x]$ is generated by $x - f$ for some $f \in K$ and

$$A[x]^\wedge_\mathfrak{q} \otimes_{A[x]} \kappa(\mathfrak{r}) = (A[x]^\wedge_\mathfrak{q} \otimes_A K)/(x - f)$$

The derivation $D = \mathrm{d}/\mathrm{d}x$ of $A[x]$ extends to $K[x]$ and maps $x - f$ to a unit of $K[x]$. Moreover D extends to $A[x]^\wedge_\mathfrak{q} \otimes_A K$ by Lemma 12.36.1. As $A \to A[x]^\wedge_\mathfrak{q}$ is formally smooth (see Lemmas 12.28.2 and 12.28.4) the ring $A[x]^\wedge_\mathfrak{q} \otimes_A K$ is regular by Proposition 12.37.2 (the arguments of the proof of that proposition simplify significantly in this particular case). We conclude by Lemma 12.36.2. $\square$

Proposition 12.38.9. *Let R be a G-ring. If $R \to S$ is essentially of finite type then S is a G-ring.*

Proof. Since being a G-ring is a property of the local rings it is clear that a localization of a G-ring is a G-ring. Conversely, if every localization at a prime is a G-ring, then the ring is a G-ring. Thus it suffices to show that $S_\mathfrak{q}$ is a G-ring for every finite type R-algebra S and every prime $\mathfrak{q}$ of S. Writing S as a quotient of $R[x_1, \ldots, x_n]$ we see from Lemma 12.38.3 that it suffices to prove that $R[x_1, \ldots, x_n]$ is a G-ring. By induction on n it suffices to prove that $R[x]$ is a G-ring. Let $\mathfrak{q} \subset R[x]$ be a maximal ideal. By Lemma 12.38.7 it suffices to show that

$$R[x]_{\mathfrak{q}} \longrightarrow R[x]_{\mathfrak{q}}^\wedge$$

is regular. If $\mathfrak{q}$ lies over $\mathfrak{p} \subset R$, then we may replace R by $R_{\mathfrak{p}}$. Hence we may assume that R is a Noetherian local G-ring with maximal ideal $\mathfrak{m}$ and that $\mathfrak{q} \subset R[x]$ lies over $\mathfrak{m}$. Note that there is a unique prime $\mathfrak{q}' \subset R^\wedge[x]$ lying over $\mathfrak{q}$. Consider the diagram

$$\begin{tikzcd} R[x]_{\mathfrak{q}}^\wedge \arrow[r] & (R^\wedge[x]_{\mathfrak{q}'})^\wedge \\ R[x]_{\mathfrak{q}} \arrow[u] \arrow[r] & R^\wedge[x]_{\mathfrak{q}'} \arrow[u] \end{tikzcd}$$

Since R is a G-ring the lower horizontal arrow is regular (as a localization of a base change of the regular ring map $R \to R^\wedge$). Suppose we can prove the right vertical arrow is regular. Then it follows that the composition $R[x]_{\mathfrak{q}} \to (R^\wedge[x]_{\mathfrak{q}'})^\wedge$ is regular, and hence the left vertical arrow is regular by Lemma 12.31.6. Hence we see that we may assume R is a Noetherian complete local ring and $\mathfrak{q}$ a prime lying over the maximal ideal of R.

Let R be a Noetherian complete local ring and let $\mathfrak{q} \subset R[x]$ be a maximal ideal lying over the maximal ideal of R. Let $\mathfrak{r} \subset \mathfrak{q}$ be a prime ideal. We want to show that $R[x]_{\mathfrak{q}}^\wedge \otimes_{R[x]} \kappa(\mathfrak{r})$ is a geometrically regular algebra over $\kappa(\mathfrak{r})$. Set $\mathfrak{p} = R \cap \mathfrak{r}$. Then we can replace R by $R/\mathfrak{p}$ and $\mathfrak{q}$ and $\mathfrak{r}$ by their images in $R/\mathfrak{p}[x]$, see Lemma 12.38.2. Hence we may assume that R is a domain and that $\mathfrak{r} \cap R = (0)$.

By Algebra, Lemma 7.143.10 we can find $R_0 \subset R$ which is regular and such that R is finite over R_0. Applying Lemma 12.38.3 we see that it suffices to prove $R[x]_{\mathfrak{q}}^\wedge \otimes_{R[x]} \kappa(\mathfrak{r})$ is geometrically regular over $\kappa(r)$ when, in addition to the above, R is a regular complete local ring.

Now R is a regular complete local ring, we have $\mathfrak{q} \subset \mathfrak{r} \subset R[x]$, we have $(0) = R \cap \mathfrak{r}$ and $\mathfrak{q}$ is a maximal ideal lying over the maximal ideal of R. Since R is regular the ring $R[x]$ is regular (Algebra, Lemma 7.145.8). Hence the localization $R[x]_{\mathfrak{q}}$ is regular. Hence the completions $R[x]_{\mathfrak{q}}^\wedge$ are regular, see Lemma 12.33.4. Hence the fibre $R[x]_{\mathfrak{q}}^\wedge \otimes_{R[x]} \kappa(\mathfrak{r})$ is, as a localization of $R[x]_{\mathfrak{q}}^\wedge$, also regular. Thus we are done if the characteristic of $f.f.(R)$ is 0.

If the characteristic of R is positive, then $R = k[[x_1, \ldots, x_n]]$. In this case we split the argument in two subcases:

(1) The case $\mathfrak{r} = (0)$. The result is a direct consequence of Lemma 12.38.5.
(2) The case $\mathfrak{r} \neq (0)$. This is Lemma 12.38.8.

□

Remark 12.38.10. Let R be a G-ring and let $I \subset R$ be an ideal. In general it is not the case that the I-adic completion $R^\wedge$ is a G-ring. An example was given by Nishimura in [**Nis81**]. A generalization and, in some sense, clarification of this example can be found in the last section of [**Dum00**].

Proposition 12.38.11. *The following types of rings are G-rings:*

(1) *fields,*

(2) *Noetherian complete local rings,*
(3) **Z***,*
(4) *Dedekind domains with fraction field of characteristic zero,*
(5) *finite type ring extensions of any of the above.*

Proof. For fields, **Z** and Dedekind domains of characteristic zero this follows immediately from the definition and the fact that the completion of a discrete valuation ring is a discrete valuation ring. A Noetherian complete local ring is a G-ring by Proposition 12.38.6. The statement on finite type overrings is Proposition 12.38.9. □

12.39. Excellent rings

In this section we discuss Grothendieck's notion of excellent rings. For the definitions of G-rings, J-2 rings, and universally catenary rings we refer to Definition 12.38.1, Definition 12.35.1, and Algebra, Definition 7.97.5.

Definition 12.39.1. Let R be a ring.

(1) We say R is *quasi-excellent* if R is Noetherian, a G-ring, and J-2.
(2) We say R is *excellent* if R is quasi-excellent and universally catenary.

Thus a Noetherian ring is quasi-excellent if it has geometrically regular formal fibres and if any finite type algebra over it has closed singular set. For such a ring to be excellent we require in addition that there exists (locally) a good dimension function.

Lemma 12.39.2. *Any localization of a finite type ring over a (quasi-)excellent ring is (quasi-)excellent.*

Proof. For finite type algebras this follows from the definitions for the properties J-2 and universally catenary. For G-rings, see Proposition 12.38.9. We omit the proof that localization preserves (quasi-)excellency. □

Lemma 12.39.3. *A quasi-excellent ring is Nagata.*

Proof. Let R be quasi-excellent. Using that a finite type algebra over R is quasi-excellent (Lemma 12.39.2) we see that it suffices to show that any quasi-excellent domain is N-1, see Algebra, Lemma 7.144.17. Applying Algebra, Lemma 7.144.29 (and using that a quasi-excellent ring is J-2) we reduce to showing that a quasi-excellent local domain R is N-1. As $R \to R^\wedge$ is regular we see that $R^\wedge$ is reduced by Lemma 12.32.1. In other words, R is analytically unramified. Hence R is N-1 by Algebra, Lemma 7.144.24. □

Proposition 12.39.4. *The following types of rings are excellent:*

(1) *fields,*
(2) *Noetherian complete local rings,*
(3) **Z***,*
(4) *Dedekind domains with fraction field of characteristic zero,*
(5) *finite type ring extensions of any of the above.*

Proof. See Propositions 12.38.11 and 12.36.6 to see that these rings are G-rings and have J-2. Any Cohen-Macaulay ring is universally catenary (in particular fields, Dedekind rings, and more generally regular rings are universally catenary). Via the Cohen structure theorem we see that complete local rings are universally catenary, see Algebra, Remark 7.143.9. □

12.40. Pseudo-coherent modules

Suppose that R is a ring. Recall that an R-module M is of finite type if there exists a surjection $R^{\oplus a} \to M$ and of finite presentation if there exists a presentation $R^{\oplus a_1} \to R^{\oplus a_0} \to M \to 0$. Similarly, we can consider those R-modules for which there exists a length n resolution

$$R^{\oplus a_n} \to R^{\oplus a_{n-1}} \to \ldots \to R^{\oplus a_0} \to M \to 0 \tag{12.40.0.1}$$

by finite free R-modules. A module is called *pseudo-coherent* of we can find such a resolution for every n. Here is the formal definition.

Definition 12.40.1. Let R be a ring. Denote $D(R)$ its derived category. Let $m \in \mathbf{Z}$.

(1) An object $K^\bullet$ of $D(R)$ is *m-pseudo-coherent* if there exists a bounded complex $E^\bullet$ of finite free R-modules and a morphism $\alpha : E^\bullet \to K^\bullet$ such that $H^i(\alpha)$ is an isomorphism for $i > m$ and $H^m(\alpha)$ is surjective.
(2) An object $K^\bullet$ of $D(R)$ is *pseudo-coherent* if it is quasi-isomorphic to a bounded above complex of finite free R-modules.
(3) An R-module M is called *m-pseudo-coherent* if if $M[0]$ is an m-pseudo-coherent object of $D(R)$.
(4) An R-module M is called *pseudo-coherent*[5] if $M[0]$ is a pseudo-coherent object of $D(R)$.

As usual we apply this terminology also to complexes of R-modules. Since any morphism $E^\bullet \to K^\bullet$ in $D(R)$ is represented by an actual map of complexes, see Derived Categories, Lemma 11.18.8, there is no ambiguity. It turns out that $K^\bullet$ is pseudo-coherent if and only if $K^\bullet$ is m-pseudo-coherent for all $m \in \mathbf{Z}$, see Lemma 12.40.5. Also, if the ring is Noetherian the condition can be understood as a finite generation condition on the cohomology, see Lemma 12.40.16. Let us first relate this to the informal discussion above.

Lemma 12.40.2. *Let R be a ring and $m \in \mathbf{Z}$. Let $(K^\bullet, L^\bullet, M^\bullet, f, g, h)$ be a distinguished triangle in $D(R)$.*

(1) *If $K^\bullet$ is $(m+1)$-pseudo-coherent and $L^\bullet$ is m-pseudo-coherent then $M^\bullet$ is m-pseudo-coherent.*
(2) *If $K^\bullet, M^\bullet$ are m-pseudo-coherent, then $L^\bullet$ is m-pseudo-coherent.*
(3) *If $L^\bullet$ is $(m+1)$-pseudo-coherent and $M^\bullet$ is m-pseudo-coherent, then $K^\bullet$ is $(m+1)$-pseudo-coherent.*

Proof. Proof of (1). Choose $\alpha : P^\bullet \to K^\bullet$ with $P^\bullet$ a bounded complex of finite free modules such that $H^i(\alpha)$ is an isomorphism for $i > m+1$ and surjective for $i = m+1$. We may replace $P^\bullet$ by $\sigma_{\geq m+1}P^\bullet$ and hence we may assume that $P^i = 0$ for $i < m+1$. Choose $\beta : E^\bullet \to L^\bullet$ with $E^\bullet$ a bounded complex of finite free modules such that $H^i(\beta)$ is an isomorphism for $i > m$ and surjective for $i = m$. By Derived Categories, Lemma 11.18.11 we can find a map $\alpha : P^\bullet \to E^\bullet$ such that the diagram

$$\begin{tikzcd} K^\bullet \arrow[r] & L^\bullet \\ P^\bullet \arrow[u] \arrow[r, "\alpha"] & E^\bullet \arrow[u] \end{tikzcd}$$

[5] This clashes with what is meant by a pseudo-coherent module in [**Bou61**].

is commutative in $D(R)$. The cone $C(\alpha)^\bullet$ is a bounded complex of finite free R-modules, and the commutativity of the diagram implies that there exists a morphism of distinguished triangles

$$(P^\bullet, E^\bullet, C(\alpha)^\bullet) \longrightarrow (K^\bullet, L^\bullet, M^\bullet).$$

It follows from the induced map on long exact cohomology sequences and Homology, Lemmas 10.3.24 and 10.3.25 that $C(\alpha)^\bullet \to M^\bullet$ induces an isomorpism on cohomology in degrees $> m$ and a surjection in degree m. Hence $M^\bullet$ is m-pseudo-coherent.

Assertions (2) and (3) follow from (1) by rotating the distinguished triangle. □

Lemma 12.40.3. *Let R be a ring. Let $K^\bullet$ be a complex of R-modules. Let $m \in \mathbf{Z}$.*

(1) *If $K^\bullet$ is m-pseudo-coherent and $H^i(K^\bullet) = 0$ for $i > m$, then $H^m(K^\bullet)$ is a finite type R-module.*
(2) *If $K^\bullet$ is m-pseudo-coherent and $H^i(K^\bullet) = 0$ for $i > m + 1$, then $H^{m+1}(K^\bullet)$ is a finitely presented R-module.*

Proof. Proof of (1). Choose a bounded complex $E^\bullet$ of finite projective R-modules and a map $\alpha : E^\bullet \to K^\bullet$ which induces an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. It is clear that it suffices to prove the result for $E^\bullet$. Let n be the largest integer such that $E^n \neq 0$. If $n = m$, then the result is clear. If $n > m$, then $E^{n-1} \to E^n$ is surjective as $H^n(E^\bullet) = 0$. As E^n is finite projective we see that $E^{n-1} = E' \oplus E^n$. Hence it suffices to prove the result for the complex $(E')^\bullet$ which is the same as $E^\bullet$ except has E' in degree $n - 1$ and 0 in degree n. We win by induction on n.

Proof of (2). Choose a bounded complex $E^\bullet$ of finite projective R-modules and a map $\alpha : E^\bullet \to K^\bullet$ which induces an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. As in the proof of (1) we can reduce to the case that $E^i = 0$ for $i > m+1$. Then we see that $H^{m+1}(K^\bullet) \cong H^{m+1}(E^\bullet) = \mathrm{Coker}(E^m \to E^{m+1})$ which is of finite presentation. □

Lemma 12.40.4. *Let R be a ring. Let M be an R-module. Then*

(1) *M is 0-pseudo-coherent if and only if M is a finite type R-module,*
(2) *M is (-1)-pseudo-coherent if and only if M is a finitely presented R-module,*
(3) *M is $(-d)$-pseudo-coherent if and only if there exists a resolution*
$$R^{\oplus a_d} \to R^{\oplus a_{d-1}} \to \ldots \to R^{\oplus a_0} \to M \to 0$$
of length d, and
(4) *M is pseudo-coherent if and only if there exists an infinite resolution*
$$\ldots \to R^{\oplus a_1} \to R^{\oplus a_0} \to M \to 0$$
by finite free R-modules.

Proof. If M is of finite type (resp. of finite presentation), then M is 0-pseudo-coherent (resp. (-1)-pseudo-coherent) as follows from the discussion preceding Definition 12.40.1. Conversely, if M is 0-pseudo-coherent, then $M = H^0(M[0])$ is of finite type by Lemma 12.40.3. If M is (-1)-pseudo-coherent, then it is 0-pseudo-coherent hence of finite type. Choose a surjection $R^{\oplus a} \to M$ and denote $K = \mathrm{Ker}(R^{\oplus a} \to M)$. By Lemma 12.40.2 we see that K is 0-pseudo-coherent, hence of finite type, whence M is of finite presentation.

To prove the third and fourth statement use induction and an argument similar to the above (details omitted). □

Lemma 12.40.5. *Let R be a ring. Let $K^\bullet$ be a complex of R-modules. The following are equivalent*

(1) *$K^\bullet$ is pseudo-coherent,*
(2) *$K^\bullet$ is m-pseudo-coherent for every $m \in \mathbf{Z}$, and*
(3) *$K^\bullet$ is quasi-isomorphic to a bounded above complex of finite projective R-modules.*

Proof. We see that (1) $\Rightarrow$ (3) as a finite free module is a finite projective R-module. Conversely, suppose $P^\bullet$ is a bounded above complex of finite projective R-modules. Say $P^i = 0$ for $i > n_0$. We choose a direct sum decompositions $F^{n_0} = P^{n_0} \oplus C^{n_0}$ with F^{n_0} a finite free R-module, and inductively

$$F^{n-1} = P^{n-1} \oplus C^n \oplus C^{n-1}$$

for $n \leq n_0$ with F^{n_0} a finite free R-module. As a complex $F^\bullet$ has maps $F^{n-1} \to F^n$ which agree with $P^{n-1} \to P^n$, induce the identity $C^n \to C^n$, and are zero on C^{n-1}. The map $F^\bullet \to P^\bullet$ is a quasi-isomorphism (even a homotopy equivalence) and hence (3) implies (1).

Assume (1). Let $E^\bullet$ be a bounded above complex of finite free R-modules and let $E^\bullet \to K^\bullet$ be a quasi-isomorphism. Then the induced maps $\sigma_{\geq m}E^\bullet \to K^\bullet$ from the stupid truncation of $E^\bullet$ to $K^\bullet$ show that $K^\bullet$ is m-pseudo-coherent. Hence (1) implies (2).

Assume (2). We first apply (2) for $n = 0$ to obtain a map of complexes $\alpha : F^\bullet \to K^\bullet$ where $F^\bullet$ is bounded above, consists of finite free R-modules and such that $H^i(\alpha)$ is an isomorphism for $i > 0$ and surjective for $i = 0$. Note that these conditions remain satisfied after replacing $F^\bullet$ by $\sigma_{\geq 0}F^\bullet$. Picture

$$\begin{tikzcd} & F^0 \arrow[r] \arrow[d, "\alpha"] & F^1 \arrow[r] \arrow[d, "\alpha"] & \ldots \\ K^{-1} \arrow[r] & K^0 \arrow[r] & K^1 \arrow[r] & \ldots \end{tikzcd}$$

By induction on $n < 0$ we are going to extend $F^\bullet$ to a complex $F^n \to F^{n+1} \to \ldots \to F^{-1} \to F^0 \to \ldots$ of finite free R-modules and extend α such that $H^i(\alpha)$ is an isomorphism for $i > n$ and surjective for $i = n$. By shifting it suffices to prove the induction step for $n = -1$. By Lemma 12.40.3 the kernel of $H^0(F^\bullet) = \mathrm{Ker}(d^0_F) \to H^0(K^\bullet)$ is a finitely generated R-module. Hence we can choose a finite free R-module F^{-1} and a map $d_F^{-1} : F^{-1} \to F^0$ whose image is this kernel. Then $\alpha(\mathrm{Im}(d_F^{-1})) \subset \mathrm{Im}(d_K^{-1})$ and as F^{-1} is projective we can a lift $\alpha : F^{-1} \to K^{-1}$ fitting into the diagram

$$\begin{tikzcd} F^{-1} \arrow[r] \arrow[d] & F^0 \arrow[r] \arrow[d, "\alpha"] & F^1 \arrow[r] \arrow[d, "\alpha"] & \ldots \\ K^{-1} \arrow[r] & K^0 \arrow[r] & K^1 \arrow[r] & \ldots \end{tikzcd}$$

By Lemma 12.40.3 the cokernel of $H^{-1}(F^\bullet) \to H^{-1}(K^\bullet)$ is a finitely generated R-module. Hence we can add a finite free summand to F^{-1} which is annihilated by d_F^{-1} but via α maps to generators of this cokernel. This proves the lemma. $\square$

Lemma 12.40.6. *Let R be a ring. Let $(K^\bullet, L^\bullet, M^\bullet, f, g, h)$ be a distinguished triangle in $D(R)$. If two out of three of $K^\bullet, L^\bullet, M^\bullet$ are pseudo-coherent then the third is also pseudo-coherent.*

Proof. Combine Lemmas 12.40.2 and 12.40.5. $\square$

Lemma 12.40.7. *Let R be a ring. Let $K^\bullet$ be a complex of R-modules. Let $m \in \mathbf{Z}$.*

(1) *If $H^i(K^\bullet) = 0$ for all $i \geq m$, then $K^\bullet$ is m-pseudo-coherent.*

(2) *If $H^i(K^\bullet) = 0$ for $i > m$ and $H^m(K^\bullet)$ is a finite R-module, then $K^\bullet$ is m-pseudo-coherent.*
(3) *If $H^i(K^\bullet) = 0$ for $i > m + 1$, the module $H^{m+1}(K^\bullet)$ is of finite presentation, and $H^m(K^\bullet)$ is of finite type, then $K^\bullet$ is m-pseudo-coherent.*

Proof. It suffices to prove (3). Set $M = H^{m+1}(K^\bullet)$. Note that $\tau_{\geq m+1}K^\bullet$ is quasi-isomorphic to $M[-m-1]$. By Lemma 12.40.4 we see that $M[-m-1]$ is m-pseudo-coherent. Since we have the distinguished triangle

$$(\tau_{\leq m}K^\bullet, K^\bullet, \tau_{\geq m+1}K^\bullet)$$

by Lemma 12.40.2 it suffices to prove that $\tau_{\leq m}K^\bullet$ is pseudo-coherent. By assumption $H^m(\tau_{\leq m}K^\bullet)$ is a finite type R-module. Hence we can find a finite free R-module E and a map $E \to \mathrm{Ker}(d^m_K)$ such that the composition $E \to \mathrm{Ker}(d^m_K) \to H^m(\tau_{\leq m}K^\bullet)$ is surjective. Then $E[-m] \to \tau_{\leq m}K^\bullet$ witnesses the fact that $\tau_{\leq m}K^\bullet$ is m-pseudo-coherent. □

Lemma 12.40.8. *Let R be a ring. Let $m \in \mathbf{Z}$. If $K^\bullet \oplus L^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) so are $K^\bullet$ and $L^\bullet$.*

Proof. In this proof we drop the superscript $^\bullet$. Assume that $K \oplus L$ is m-pseudo-coherent. It is clear that $K, L \in D^-(R)$. Note that there is a distinguished triangle

$$(K \oplus L, K \oplus L, K \oplus L[1]) = (K, K, 0) \oplus (L, L, L \oplus L[1])$$

see Derived Categories, Lemma 11.4.8. By Lemma 12.40.2 we see that $L \oplus L[1]$ is m-pseudo-coherent. Hence also $L[1] \oplus L[2]$ is m-pseudo-coherent. By induction $L[n] \oplus L[n+1]$ is m-pseudo-coherent. By Lemma 12.40.7 we see that $L[n]$ is m-pseudo-coherent for large n. Hence working backwards, using the distinguished triangles

$$(L[n], L[n] \oplus L[n-1], L[n-1])$$

we conclude that $L[n], L[n-1], \ldots, L$ are m-pseudo-coherent as desired. The pseudo-coherent case follows from this and Lemma 12.40.5. □

Lemma 12.40.9. *Let R be a ring. Let $m \in \mathbf{Z}$. Let $K^\bullet$ be a bounded above complex of R-modules such that K^i is $(m-i)$-pseudo-coherent for all i. Then $K^\bullet$ is m-pseudo-coherent. In particular, if $K^\bullet$ is a bounded above complex of pseudo-coherent R-modules, then $K^\bullet$ is pseudo-coherent.*

Proof. We may replace $K^\bullet$ by $\sigma_{\geq m-1}K^\bullet$ (for example) and hence assume that $K^\bullet$ is bounded. Then the complex $K^\bullet$ is m-pseudo-coherent as each $K^i[-i]$ is m-pseudo-coherent by induction on the length of the complex: use Lemma 12.40.6 and the stupid truncations. For the final statement, it suffices to prove that $K^\bullet$ is m-pseudo-coherent for all $m \in \mathbf{Z}$, see Lemma 12.40.5. This follows from the first part. □

Lemma 12.40.10. *Let R be a ring. Let $m \in \mathbf{Z}$. Let $K^\bullet \in D^-(R)$ such that $H^i(K^\bullet)$ is $(m-i)$-pseudo-coherent (resp. pseudo-coherent) for all i. Then $K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent).*

Proof. Assume $K^\bullet$ is an object of $D^-(R)$ such that each $H^i(K^\bullet)$ is $(m-i)$-pseudo-coherent. Let n be the largest integer such that $H^n(K^\bullet)$ is nonzero. We will prove the lemma by induction on n. If $n < m$, then $K^\bullet$ is m-pseudo-coherent by Lemma 12.40.7. If $n \geq m$, then we have the distinguished triangle

$$(\tau_{\leq n-1}K^\bullet, K^\bullet, H^n(K^\bullet)[-n])$$

Since $H^n(K^\bullet)[-n]$ is m-pseudo-coherent by assumption, we can use Lemma 12.40.2 to see that it suffices to prove that $\tau_{\leq n-1}K^\bullet$ is m-pseudo-coherent. By induction on n we win. (The pseudo-coherent case follows from this and Lemma 12.40.5.) □

Lemma 12.40.11. *Let $A \to B$ be a ring map. Assume that B is pseudo-coherent as an A-module. Let $K^\bullet$ be a complex of B-modules. The following are equivalent*

1. *$K^\bullet$ is m-pseudo-coherent as a complex of B-modules, and*
2. *$K^\bullet$ is m-pseudo-coherent as a complex of A-modules.*

The same equivalence holds for pseudo-coherence.

Proof. Assume (1). Choose a bounded complex of finite free B-modules $E^\bullet$ and a map $\alpha : E^\bullet \to K^\bullet$ which is an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. Consider the distinguished triangle $(E^\bullet, K^\bullet, C(\alpha)^\bullet)$. By Lemma 12.40.7 $C(\alpha)^\bullet$ is m-pseudo-coherent as a complex of A-modules. Hence it suffices to prove that $E^\bullet$ is pseudo-coherent as a complex of A-modules, which follows from Lemma 12.40.9. The pseudo-coherent case of (1) ⇒ (2) follows from this and Lemma 12.40.5.

Assume (2). Let n be the largest integer such that $H^n(K^\bullet)\neq 0$. We will prove that $K^\bullet$ is m-pseudo-coherent as a complex of B-modules by induction on $n - m$. The case $n < m$ follows from Lemma 12.40.7. Choose a bounded complex of finite free A-modules $E^\bullet$ and a map $\alpha : E^\bullet \to K^\bullet$ which is an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. Consider the induced map of complexes

$$\alpha \otimes 1 : E^\bullet \otimes_A B \to K^\bullet.$$

Note that $C(\alpha \otimes 1)^\bullet$ is acyclic in degrees $\geq n$ as $H^n(E) \to H^n(E^\bullet \otimes_A B) \to H^n(K^\bullet)$ is surjective by construction and since $H^i(E^\bullet \otimes_A B) = 0$ for $i > n$ by the spectral sequence of Example 12.6.4. On the other hand, $C(\alpha \otimes 1)^\bullet$ is m-pseudo-coherent as a complex of A-modules because both $K^\bullet$ and $E^\bullet \otimes_A B$ (see Lemma 12.40.9) are so, see Lemma 12.40.2. Hence by induction we see that $C(\alpha\otimes 1)^\bullet$ is m-pseudo-coherent as a complex of B-modules. Finally another application of Lemma 12.40.2 shows that $K^\bullet$ is m-pseudo-coherent as a complex of B-modules (as clearly $E^\bullet \otimes_A B$ is pseudo-coherent as a complex of B-modules). The pseudo-coherent case of (2) ⇒ (1) follows from this and Lemma 12.40.5. □

Lemma 12.40.12. *Let $A \to B$ be a ring map. Let $K^\bullet$ be an m-pseudo-coherent (resp. pseudo-coherent) complex of A-modules. Then $K^\bullet \otimes_A^{\mathbf{L}} B$ is an m-pseudo-coherent (resp. pseudo-coherent) complex of B-modules.*

Proof. First we note that the statement of the lemma makes sense as $K^\bullet$ is bounded above and hence $K^\bullet \otimes_A^{\mathbf{L}} B$ is defined by Equation (12.2.0.2). Having said this, choose a bounded complex $E^\bullet$ of finite free A-modules and $\alpha : E^\bullet \to K^\bullet$ with $H^i(\alpha)$ an isomorphism for $i > m$ and surjective for $i = m$. Then the cone $C(\alpha)^\bullet$ is acyclic in degrees $\geq m$. Since $- \otimes_A^{\mathbf{L}} B$ is an exact functor we get a distinguished triangle

$$(E^\bullet \otimes_A^{\mathbf{L}} B, K^\bullet \otimes_A^{\mathbf{L}} B, C(\alpha)^\bullet \otimes_A^{\mathbf{L}} B)$$

of complexes of B-modules. By the dual to Derived Categories, Lemma 11.16.1 we see that $H^i(C(\alpha)^\bullet \otimes_A^{\mathbf{L}} B) = 0$ for $i \geq m$. Since $E^\bullet$ is a complex of projective R-modules we see that $E^\bullet \otimes_A^{\mathbf{L}} B = E^\bullet \otimes_A B$ and hence

$$E^\bullet \otimes_A B \longrightarrow K^\bullet \otimes_A^{\mathbf{L}} B$$

is a morphism of complexes of B-modules that witnesses the fact that $K^\bullet \otimes_A^{\mathbf{L}} B$ is m-pseudo-coherent. The case of pseudo-coherent complexes follows from the case of m-pseudo-coherent complexes via Lemma 12.40.5. $\square$

Lemma 12.40.13. *Let $A \to B$ be a flat ring map. Let M be an m-pseudo-coherent (resp. pseudo-coherent) A-module. Then $M \otimes_A B$ is an m-pseudo-coherent (resp. pseudo-coherent) B-module.*

Proof. Immediate consequence of Lemma 12.40.12 and the fact that $M \otimes_A^{\mathbf{L}} B = M \otimes_A B$ because B is flat over A. $\square$

The following lemma also follows from the stronger Lemma 12.40.14.

Lemma 12.40.14. *Let R be a ring. Let $f_1, \ldots, f_r \in R$ be elements which generate the unit ideal. Let $m \in \mathbf{Z}$. Let $K^\bullet$ be a complex of R-modules. If for each i the complex $K^\bullet \otimes_R R_{f_i}$ is m-pseudo-coherent (resp. pseudo-coherent), then $K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent).*

Proof. We will use without further mention that $- \otimes_R R_{f_i}$ is an exact functor and that therefore

$$H^i(K^\bullet)_{f_i} = H^i(K^\bullet) \otimes_R R_{f_i} = H^i(K^\bullet \otimes_R R_{f_i}).$$

Assume $K^\bullet \otimes_R R_{f_i}$ is m-pseudo-coherent for $i = 1, \ldots, r$. Let $n \in \mathbf{Z}$ be the largest integer such that $H^n(K^\bullet \otimes_R R_{f_i})$ is nonzero for some i. This implies in particular that $H^i(K^\bullet) = 0$ for $i > n$ (and that $H^n(K^\bullet) \neq 0$) see Algebra, Lemma 7.21.2. We will prove the lemma by induction on $n - m$. If $n < m$, then the lemma is true by Lemma 12.40.7. If $n \geq m$, then $H^n(K^\bullet)_{f_i}$ is a finite R_{f_i}-module for each i, see Lemma 12.40.3. Hence $H^n(K^\bullet)$ is a finite R-module, see Algebra, Lemma 7.21.2. Choose a finite free R-module E and a surjection $E \to H^n(K^\bullet)$. As E is projective we can lift this to a map of complexes $\alpha : E[-n] \to K^\bullet$. Then the cone $C(\alpha)^\bullet$ has vanishing cohomology in degrees $\geq n$. On the other hand, the complexes $C(\alpha)^\bullet \otimes_R R_{f_i}$ are m-pseudo-coherent for each i, see Lemma 12.40.2. Hence by induction we see that $C(\alpha)^\bullet$ is m-pseudo-coherent as a complex of R-modules. Applying Lemma 12.40.2 once more we conclude. $\square$

Lemma 12.40.15. *Let R be a ring. Let $m \in \mathbf{Z}$. Let $K^\bullet$ be a complex of R-modules. Let $R \to R'$ be a faithfully flat ring map. If the complex $K^\bullet \otimes_R R'$ is m-pseudo-coherent (resp. pseudo-coherent), then $K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent).*

Proof. We will use without further mention that $- \otimes_R R'$ is an exact functor and that therefore

$$H^i(K^\bullet) \otimes_R R' = H^i(K^\bullet \otimes_R R').$$

Assume $K^\bullet \otimes_R R'$ is m-pseudo-coherent. Let $n \in \mathbf{Z}$ be the largest integer such that $H^n(K^\bullet)$ is nonzero; then n is also the largest integer such that $H^n(K^\bullet \otimes_R R')$ is nonzero. We will prove the lemma by induction on $n - m$. If $n < m$, then the lemma is true by Lemma 12.40.7. If $n \geq m$, then $H^n(K^\bullet) \otimes_R R'$ is a finite R'-module, see Lemma 12.40.3. Hence $H^n(K^\bullet)$ is a finite R-module, see Algebra, Lemma 7.77.2. Choose a finite free R-module E and a surjection $E \to H^n(K^\bullet)$. As E is projective we can lift this to a map of complexes $\alpha : E[-n] \to K^\bullet$. Then the cone $C(\alpha)^\bullet$ has vanishing cohomology in degrees $\geq n$. On the other hand, the complex $C(\alpha)^\bullet \otimes_R R'$ is m-pseudo-coherent, see Lemma 12.40.2. Hence by induction we see that $C(\alpha)^\bullet$ is m-pseudo-coherent as a complex of R-modules. Applying Lemma 12.40.2 once more we conclude. $\square$

Lemma 12.40.16. *Let R be a Noetherian ring. Then*

(1) *A complex of R-modules $K^\bullet$ is m-pseudo-coherent if and only if $K^\bullet \in D^-(R)$ and $H^i(K^\bullet)$ is a finite R-module for $i \geq m$.*
(2) *A complex of R-modules $K^\bullet$ is pseudo-coherent if and only if $K^\bullet \in D^-(R)$ and $H^i(K^\bullet)$ is a finite R-module for all i.*
(3) *An R-module is pseudo-coherent if and only if it is finite.*

Proof. In Algebra, Lemma 7.67.1 we have seen that any finite R-module is pseudo-coherent. On the other hand, a pseudo-coherent module is finite, see Lemma 12.40.4. Hence (3) holds. Suppose that $K^\bullet$ is an m-pseudo-coherent complex. Then there exists a bounded complex of finite free R-modules $E^\bullet$ such that $H^i(K^\bullet)$ is isomorphic to $H^i(E^\bullet)$ for $i > m$ and such that $H^m(K^\bullet)$ is a quotient of $H^m(E^\bullet)$. Thus it is clear that each $H^i(K^\bullet)$, $i \geq m$ is a finite module. The converse implication in (1) follows from Lemma 12.40.10 and part (3). Part (2) follows from (1) and Lemma 12.40.5. $\square$

12.41. Tor dimension

Instead of resolving by projective modules we can look at resolutions by flat modules. This leads to the following concept.

Definition 12.41.1. Let R be a ring. Denote $D(R)$ its derived category. Let $a, b \in \mathbf{Z}$.
(1) An object $K^\bullet$ of $D(R)$ has *tor-amplitude in* $[a, b]$ if $H^i(K^\bullet \otimes_R^{\mathbf{L}} M) = 0$ for all R-modules M and all $i \not\in [a, b]$.
(2) An object $K^\bullet$ of $D(R)$ has *finite tor dimension* if it has tor-amplitude in $[a, b]$ for some a, b.
(3) An R-module M has *tor dimension* $\leq d$ if if $M[0]$ as an object of $D(R)$ has tor-amplitude in $[-d, 0]$.
(4) An R-module M has *finite tor dimension* if $M[0]$ as an object of $D(R)$ has finite tor dimension.

We observe that if $K^\bullet$ has finite tor dimension, then $K^\bullet \in D^b(R)$.

Lemma 12.41.2. *Let R be a ring. Let $K^\bullet$ be a bounded above complex of flat R-modules with tor-amplitude in $[a, b]$. Then Coker(d_K^{a-1}) is a flat R-module.*

Proof. As $K^\bullet$ is a bounded above complex of flat modules we see that $K^\bullet \otimes_R M = K^\bullet \otimes_R^{\mathbf{L}} M$. Hence for every R-module M the sequence

$$K^{a-2} \otimes_R M \to K^{a-1} \otimes_R M \to K^a \otimes_R M$$

is exact in the middle. Since $K^{a-2} \to K^{a-1} \to K^a \to \text{Coker}(d_K^{a-1}) \to 0$ is a flat resolution this implies that $\text{Tor}_1^R(\text{Coker}(d_K^{a-1}), M) = 0$ for all R-modules M. This means that $\text{Coker}(d_K^{a-1})$ is flat, see Algebra, Lemma 7.69.7. $\square$

Lemma 12.41.3. *Let R be a ring. Let $K^\bullet$ be an object of $D(R)$. Let $a, b \in \mathbf{Z}$. The following are equivalent*
(1) *$K^\bullet$ has tor-amplitude in $[a, b]$.*
(2) *$K^\bullet$ is quasi-isomorphic to a complex $E^\bullet$ of flat R-modules with $E^i = 0$ for $i \not\in [a, b]$.*

Proof. If (2) holds, then we may compute $K^\bullet \otimes_R^{\mathbf{L}} M = E^\bullet \otimes_R M$ and it is clear that (1) holds. Assume that (1) holds. We may replace $K^\bullet$ by a projective resolution. Let n be the largest integer such that $K^n \neq 0$. If $n > b$, then $K^{n-1} \to K^n$ is surjective as $H^n(K^\bullet) = 0$. As K^n is projective we see that $K^{n-1} = K' \oplus K^n$. Hence it suffices to prove the result for the complex $(K')^\bullet$ which is the same as $K^\bullet$ except has K' in degree $n-1$ and 0 in degree n. Thus, by induction on n, we reduce to the case that $K^\bullet$ is a complex of projective R-modules with $K^i = 0$ for $i > b$.

Set $E^\bullet = \tau_{\geq a}K^\bullet$. Everything is clear except that E^a is flat which follows immediately from Lemma 12.41.2 and the definitions. □

Lemma 12.41.4. *Let R be a ring and $m \in \mathbf{Z}$. Let $(K^\bullet, L^\bullet, M^\bullet, f, g, h)$ be a distinguished triangle in $D(R)$. Let $a, b \in \mathbf{Z}$.*

1. *If $K^\bullet$ has tor-amplitude in $[a+1, b+1]$ and $L^\bullet$ has tor-amplitude in $[a, b]$ then $M^\bullet$ has tor-amplitude in $[a, b]$.*
2. *If $K^\bullet, M^\bullet$ have tor-amplitude in $[a, b]$, then $L^\bullet$ has tor-amplitude in $[a, b]$.*
3. *If $L^\bullet$ has tor-amplitude in $[a+1, b+1]$ and $M^\bullet$ has tor-amplitude in $[a, b]$, then $K^\bullet$ has tor-amplitude in $[a+1, b+1]$.*

Proof. Omitted. Hint: This just follows from the long exact cohomology sequence associated to a distinguished triangle and the fact that $- \otimes_R^{\mathbf{L}} M$ preserves distinguished triangles. The easiest one to prove is (2) and the others follow from it by translation. □

Lemma 12.41.5. *Let R be a ring. Let M be an R-module. Let $d \geq 0$. The following are equivalent*

1. *M has tor dimension $\leq d$, and*
2. *there exists a resolution*
$$0 \to F_d \to \ldots \to F_1 \to F_0 \to M \to 0$$
with F_i a flat R-module.

In particular an R-module has tor dimension 0 if and only if it is a flat R-module.

Proof. Assume (2). Then the complex $E^\bullet$ with $E^{-i} = F_i$ is quasi-isomorphic to M. Hence the Tor dimension of M is at most d by Lemma 12.41.3. Conversely, assume (1). Let $P^\bullet \to M$ be a projective resolution of M. By Lemma 12.41.2 we see that $\tau_{\geq -d}P^\bullet$ is a flat resolution of M of length d, i.e., (2) holds. □

Lemma 12.41.6. *Let R be a ring. Let $a, b \in \mathbf{Z}$. If $K^\bullet \oplus L^\bullet$ has tor amplitude in $[a, b]$ so do $K^\bullet$ and $L^\bullet$.*

Proof. Clear from the fact that the Tor functors are additive. □

Lemma 12.41.7. *Let R be a ring. Let $K^\bullet$ be a bounded complex of R-modules such that K^i has tor amplitude in $[a-i, b-i]$ for all i. Then $K^\bullet$ has tor amplitude in $[a, b]$. In particular if $K^\bullet$ is a finite complex of R-modules of finite tor dimension, then $K^\bullet$ has finite tor dimension.*

Proof. Follows by induction on the length of the finite complex: use Lemma 12.41.4 and the stupid truncations. □

Lemma 12.41.8. *Let R be a ring. Let $a, b \in \mathbf{Z}$. Let $K^\bullet \in D^b(R)$ such that $H^i(K^\bullet)$ has tor amplitude in $[a-i, b-i]$ for all i. Then $K^\bullet$ has tor amplitude in $[a, b]$. In particular if $K^\bullet \in D^-(R)$ and all its cohomology groups have finite tor dimension then $K^\bullet$ has finite tor dimension.*

Proof. Follows by induction on the length of the finite complex: use Lemma 12.41.4 and the canonical truncations. □

Lemma 12.41.9. *Let $A \to B$ be a ring map. Assume that B is flat as an A-module. Let $K^\bullet$ be a complex of B-modules. Let $a, b \in \mathbf{Z}$. If $K^\bullet$ as a complex of B-modules has tor amplitude in $[a, b]$, then $K^\bullet$ as a complex of A-modules has tor amplitude in $[a, b]$.*

Proof. This is true because $K^\bullet \otimes_A^{\mathbf{L}} M = K^\bullet \otimes_B^{\mathbf{L}} (M \otimes_A B)$ since any projective resolution of $K^\bullet$ as a complex of B-modules is a flat resolution of $K^\bullet$ as a complex of A-modules and can be used to compute $K^\bullet \otimes_A^{\mathbf{L}} M$. $\square$

Lemma 12.41.10. *Let $A \to B$ be a ring map. Assume that B has tor dimension $\leq d$ as an A-module. Let $K^\bullet$ be a complex of B-modules. Let $a, b \in \mathbf{Z}$. If $K^\bullet$ as a complex of B-modules has tor amplitude in $[a, b]$, then $K^\bullet$ as a complex of A-modules has tor amplitude in $[a - d, b]$.*

Proof. Let M be an A-module. Choose a free resolution $F^\bullet \to M$. Then

$$K^\bullet \otimes_A^{\mathbf{L}} M = \mathrm{Tot}(K^\bullet \otimes_A F^\bullet) = \mathrm{Tot}(K^\bullet \otimes_B (F^\bullet \otimes_A B)) = K^\bullet \otimes_B^{\mathbf{L}} (M \otimes_A^{\mathbf{L}} B).$$

By our assumption on B as an A-module we see that $M \otimes_A^{\mathbf{L}} B$ has cohomology only in degrees $-d, -d+1, \ldots, 0$. Because $K^\bullet$ has tor amplitude in $[a, b]$ we see from the spectral sequence in Example 12.6.4 that $K^\bullet \otimes_B^{\mathbf{L}} (M \otimes_A^{\mathbf{L}} B)$ has cohomology only in degrees $[-d + a, b]$ as desired. $\square$

Lemma 12.41.11. *Let $A \to B$ be a ring map. Let $a, b \in \mathbf{Z}$. Let $K^\bullet$ be a complex of A-modules with tor amplitude in $[a, b]$. Then $K^\bullet \otimes_A^{\mathbf{L}} B$ as a complex of B-modules has tor amplitude in $[a, b]$.*

Proof. By Lemma 12.41.3 we can find a quasi-isomorphism $E^\bullet \to K^\bullet$ where $E^\bullet$ is a complex of flat A-modules with $E^i = 0$ for $i \not\in [a, b]$. Then $E^\bullet \otimes_A B$ computes $K^\bullet \otimes_A^{\mathbf{L}} B$ by construction and each $E^i \otimes_A B$ is a flat B-module by Algebra, Lemma 7.35.6. Hence we conclude by Lemma 12.41.3. $\square$

Lemma 12.41.12. *Let $A \to B$ be a flat ring map. Let $d \geq 0$. Let M be an A-module of tor dimension $\leq d$. Then $M \otimes_A B$ is a B-module of tor dimension $\leq d$.*

Proof. Immediate consequence of Lemma 12.41.11 and the fact that $M \otimes_A^{\mathbf{L}} B = M \otimes_A B$ because B is flat over A. $\square$

Lemma 12.41.13. *Let R be a ring. Let $f_1, \ldots, f_r \in R$ be elements which generate the unit ideal. Let $a, b \in \mathbf{Z}$. Let $K^\bullet$ be a complex of R-modules. If for each i the complex $K^\bullet \otimes_R R_{f_i}$ has tor amplitude in $[a, b]$, then $K^\bullet$ has tor amplitude in $[a, b]$.*

Proof. Note that $- \otimes_R R_{f_i}$ is an exact functor and that therefore

$$H^i(K^\bullet)_{f_i} = H^i(K^\bullet) \otimes_R R_{f_i} = H^i(K^\bullet \otimes_R R_{f_i}).$$

and similarly for every R-module M we have

$$H^i(K^\bullet \otimes_R^{\mathbf{L}} M)_{f_i} = H^i(K^\bullet \otimes_R^{\mathbf{L}} M) \otimes_R R_{f_i} = H^i(K^\bullet \otimes_R R_{f_i} \otimes_{R_{f_i}}^{\mathbf{L}} M_{f_i}).$$

Hence the result follows from the fact that an R-module N is zero if and only if N_{f_i} is zero for each i, see Algebra, Lemma 7.21.2. $\square$

Lemma 12.41.14. *Let R be a ring. Let $a, b \in \mathbf{Z}$. Let $K^\bullet$ be a complex of R-modules. Let $R \to R'$ be a faithfully flat ring map. If the complex $K^\bullet \otimes_R R'$ has tor amplitude in $[a, b]$, then $K^\bullet$ has tor amplitude in $[a, b]$.*

Proof. Let M be an R-module. Since $R \to R'$ is flat we see that

$$(M \otimes_R^{\mathbf{L}} K^\bullet) \otimes_R R' = ((M \otimes_R R') \otimes_{R'}^{\mathbf{L}} (K^\bullet \otimes_R R')$$

and taking cohomology commutes with tensoring with R'. Hence $\mathrm{Tor}_i^R(M, K^\bullet) = \mathrm{Tor}_i^{R'}(M \otimes_R R', K^\bullet \otimes_R R')$. Since $R \to R'$ is faithfully flat, the vanishing of $\mathrm{Tor}_i^{R'}(M \otimes_R R', K^\bullet \otimes_R R')$ for $i \not\in [a, b]$ implies the same thing for $\mathrm{Tor}_i^R(M, K^\bullet)$. $\square$

Lemma 12.41.15. *Let R be a ring of finite global dimension d. Then*

(1) *every module has finite tor dimension $\leq d$,*
(2) *a complex of R-modules $K^\bullet$ with $H^i(K^\bullet) \neq 0$ only if $i \in [a, b]$ has tor amplitude in $[a - d, b]$, and*
(3) *a complex of R-modules $K^\bullet$ has finite tor dimension if and only if $K^\bullet \in D^b(R)$.*

Proof. The assumption on R means that every module has a finite projective resolution of length at most d, in particular every module has finite tor dimension. The second statement follows from Lemma 12.41.8 and the definitions. The third statement is a rephrasing of the second. $\square$

12.42. Perfect complexes

A perfect complex is a pseudo-coherent complex of finite tor dimension. But we can also define the directly as follows.

Definition 12.42.1. Let R be a ring. Denote $D(R)$ the derived category of the abelian category of R-modules.

(1) An object K of $D(R)$ is *perfect* if it is quasi-isomorphic to a bounded complex of finite projective R-modules.
(2) An R-module M is *perfect* if $M[0]$ is a perfect object in $D(R)$.

Lemma 12.42.2. *Let $K^\bullet$ be an object of $D(R)$. The following are equivalent*

(1) *$K^\bullet$ is perfect, and*
(2) *$K^\bullet$ is pseudo-coherent and has finite tor dimension.*

Proof. It is clear that (1) implies (2), see Lemmas 12.40.5 and 12.41.3. Assume (2). Choose a bounded above complex $F^\bullet$ of finite free R-modules and a quasi-isomorphism $F^\bullet \to K^\bullet$. Assume that $K^\bullet$ has tor-amplitude in $[a, b]$. Set $E^\bullet = \tau_{\geq a} F^\bullet$. Note that E^i is finite free except E^a which is a finitely presented R-module. By Lemma 12.41.2 E^a is flat. Hence by Algebra, Lemma 7.72.2 we see that E^a is finite projective. $\square$

Lemma 12.42.3. *Let M be a module over a ring R. The following are equivalent*

(1) *M is a perfect module, and*
(2) *there exists a resolution*

$$0 \to F_d \to \ldots \to F_1 \to F_0 \to M \to 0$$

with each F_i a finite projective R-module.

Proof. Assume (2). Then the complex $E^\bullet$ with $E^{-i} = F_i$ is quasi-isomorphic to $M[0]$. Hence M is perfect. Conversely, assume (1). By Lemmas 12.42.2 and 12.40.4 we can find resolution $E^\bullet \to M$ with E^{-i} a finite free R-module. By Lemma 12.41.2 we see that $F_d = \mathrm{Coker}(E^{d-1} \to E^d)$ is flat for some d sufficiently large. By Algebra, Lemma 7.72.2 we see that F_d is finite projective. Hence

$$0 \to F_d \to E^{-d+1} \to \ldots \to E^0 \to M \to 0$$

is the desired resolution. □

Lemma 12.42.4. *Let R be a ring. Let $(K^\bullet, L^\bullet, M^\bullet, f, g, h)$ be a distinguished triangle in $D(R)$. If two out of three of $K^\bullet, L^\bullet, M^\bullet$ are perfect then the third is also perfect.*

Proof. Combine Lemmas 12.42.2, 12.40.6, and 12.41.4. □

Lemma 12.42.5. *Let R be a ring. If $K^\bullet \oplus L^\bullet$ is perfect, then so are $K^\bullet$ and $L^\bullet$.*

Proof. Follows from Lemmas 12.42.2, 12.40.8, and 12.41.6. □

Lemma 12.42.6. *Let R be a ring. Let $K^\bullet$ be a bounded complex of perfect R-modules. Then $K^\bullet$ is a perfect complex.*

Proof. Follows by induction on the length of the finite complex: use Lemma 12.42.4 and the stupid truncations. □

Lemma 12.42.7. *Let R be a ring. If $K^\bullet \in D^-(R)$ and all its cohomology modules are perfect, then $K^\bullet$ is perfect.*

Proof. Follows by induction on the length of the finite complex: use Lemma 12.42.4 and the canonical truncations. □

Lemma 12.42.8. *Let $A \to B$ be a ring map. Assume that B is perfect as an A-module. Let $K^\bullet$ be a perfect complex of B-modules. Then $K^\bullet$ is perfect as a complex of A-modules.*

Proof. Using Lemma 12.42.2 this translates into the corresponding results for pseudo-coherent modules and modules of finite tor dimension. See Lemma 12.41.10 and Lemma 12.40.11 for those results. □

Lemma 12.42.9. *Let $A \to B$ be a ring map. Let $K^\bullet$ be a perfect complex of A-modules. Then $K^\bullet \otimes_A^{\mathbf{L}} B$ is a perfect complex of B-modules.*

Proof. Using Lemma 12.42.2 this translates into the corresponding results for pseudo-coherent modules and modules of finite tor dimension. See Lemma 12.41.11 and Lemma 12.40.12 for those results. □

Lemma 12.42.10. *Let $A \to B$ be a flat ring map. Let M be a perfect A-module. Then $M \otimes_A B$ is a perfect B-module.*

Proof. By Lemma 12.42.3 the assumption implies that M has a finite resolution $F_\bullet$ by finite projective R-modules. As $A \to B$ is flat the complex $F_\bullet \otimes_A B$ is a finite length resolution of $M \otimes_A B$ by finite projective modules over B. Hence $M \otimes_A B$ is perfect. □

Lemma 12.42.11. *Let R be a ring. Let $f_1, \ldots, f_r \in R$ be elements which generate the unit ideal. Let $K^\bullet$ be a complex of R-modules. If for each i the complex $K^\bullet \otimes_R R_{f_i}$ is perfect, then $K^\bullet$ is perfect.*

Proof. Using Lemma 12.42.2 this translates into the corresponding results for pseudo-coherent modules and modules of finite tor dimension. See Lemma 12.41.13 and Lemma 12.40.14 for those results. □

Lemma 12.42.12. *Let R be a ring. Let $a, b \in \mathbf{Z}$. Let $K^\bullet$ be a complex of R-modules. Let $R \to R'$ be a faithfully flat ring map. If the complex $K^\bullet \otimes_R R'$ has tor amplitude in $[a, b]$, then $K^\bullet$ has tor amplitude in $[a, b]$.*

Proof. Using Lemma 12.42.2 this translates into the corresponding results for pseudo-coherent modules and modules of finite tor dimension. See Lemma 12.41.14 and Lemma 12.40.15 for those results. □

Lemma 12.42.13. *Let R be a regular ring of finite dimension. Then*

(1) *an R-module is perfect if and only if it is a finite R-module, and*
(2) *a complex of R-modules $K^\bullet$ is perfect if and only if $K^\bullet \in D^b(R)$ and each $H^i(K^\bullet)$ is a finite R-module.*

Proof. By Algebra, Lemma 7.102.7 the assumption on R means that R has finite global dimension. Hence every module has finite tor dimension, see Lemma 12.41.15. On the other hand, as R is Noetherian, a module is pseudo-coherent if and only if it is finite, see Lemma 12.40.16. This proves part (1).

Let $K^\bullet$ be a complex of R-modules. If $K^\bullet$ is perfect, then it is in $D^b(R)$ and it is quasi-isomorphic to a finite complex of finite projective R-modules so certainly each $H^i(K^\bullet)$ is a finite R-module (as R is Noetherian). Conversely, suppose that $K^\bullet$ is in $D^b(R)$ and each $H^i(K^\bullet)$ is a finite R-module. Then by (1) each $H^i(K^\bullet)$ is a perfect R-module, whence $K^\bullet$ is perfect by Lemma 12.42.7 □

Lemma 12.42.14. *Let R be a ring. Let $\mathfrak{p} \subset R$ be a prime ideal. Let $i \in \mathbf{Z}$. Let $K^\bullet$ be a pseudo-coherent complex of R-modules such that $H^i(K^\bullet \otimes_R^{\mathbf{L}} \kappa(\mathfrak{p})) = 0$. Then there exists an $f \in R$, $f \not\in \mathfrak{p}$ such that*

$$K^\bullet \otimes_R R_f = \tau_{\geq i+1} K^\bullet \otimes_R R_f \oplus \tau_{\leq i-1} K^\bullet \otimes_R R_f$$

in $D(R_f)$ with $\tau_{\geq i+1} K^\bullet \otimes_R R_f$ a perfect complex with tor amplitude in $[i+1, j]$ for some $j \in \mathbf{Z}$.

Proof. We may assume that $K^\bullet$ is a bounded above complex of finite free R-modules. Let us inspect what is happening in degree i:

$$\ldots \to K^{i-2} \to R^{\oplus l} \to R^{\oplus m} \to R^{\oplus n} \to K^{i+2} \to \ldots$$

Let A be the $m \times l$ matrix corresponding to $K^{i-1} \to K^i$ and let B be the $n \times m$ matrix corresponding to $K^i \to K^{i+1}$. The assumption is that A mod $\mathfrak{p}$ has rank r and that B mod $\mathfrak{p}$ has rank $m - r$. In other words, there is some $r \times r$ minor a of A which is not in $\mathfrak{p}$ and there is some $(m-r) \times (m-r)$-minor b of B which is not in $\mathfrak{p}$. Set $f = ab$. Then after inverting f we can find direct sum decompositions $K^{i-1} = R^{\oplus l-r} \oplus R^{\oplus r}$, $K^i = R^{\oplus r} \oplus R^{\oplus m-r}$, $K^{i+1} = R^{\oplus m-r} \oplus R^{\oplus n-m+r}$ such that the module map $K^{i-1} \to K^i$ kills of $R^{\oplus l-r}$ and induces an isomorphism of $R^{\oplus r}$ onto the corresponding summand of K^i and such that the module map $K^i \to K^{i+1}$ kills of $R^{\oplus r}$ and induces an isomorphism of $R^{\oplus m-r}$ onto the corresponding summand of K^{i+1}. Thus $K^\bullet$ becomes quasi-isomorphic to

$$\ldots \to K^{i-2} \to R^{\oplus l-r} \to 0 \to R^{\oplus n-m+r} \to K^{i+2} \to \ldots$$

and everything is clear. □

Lemma 12.42.15. *Let R be a ring. Let $a, b \in \mathbf{Z}$. Let $K^\bullet$ be a pseudo-coherent complex of R-modules. The following are equivalent*

(1) *$K^\bullet$ is perfect with tor amplitude in $[a, b]$,*
(2) *for every prime $\mathfrak{p}$ we have $H^i(K^\bullet \otimes_R^{\mathbf{L}} \kappa(\mathfrak{p})) = 0$ for all $i \not\in [a, b]$, and*
(3) *for every maximal ideal $\mathfrak{m}$ we have $H^i(K^\bullet \otimes_R^{\mathbf{L}} \kappa(\mathfrak{m})) = 0$ for all $i \not\in [a, b]$.*

Proof. We omit the proof of the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Assume (3). Let $i \in \mathbf{Z}$ with $i \notin [a, b]$. By Lemma 12.42.14 we see that the assumption implies that $H^i(K^\bullet)_\mathfrak{m} = 0$ for all maximal ideals of R. Hence $H^i(K^\bullet) = 0$, see Algebra, Lemma 7.21.1. Moreover, Lemma 12.42.14 now also implies that for every maximal ideal $\mathfrak{m}$ there exists an element $f \in R$, $f \notin \mathfrak{m}$ such that $K^\bullet \otimes_R R_f$ is perfect with tor amplitude in $[a, b]$. Hence we conclude by applealing to Lemmas 12.42.11 and 12.41.13. $\square$

Lemma 12.42.16. *Let R be a ring. Let $K^\bullet$ be a pseudo-coherent complex of R-modules. The following are equivalent*

(1) *$K^\bullet$ is perfect,*
(2) *for every prime ideal $\mathfrak{p}$ the complex $K^\bullet \otimes_R R_\mathfrak{p}$ is perfect,*
(3) *for every prime $\mathfrak{p}$ we have $H^i(K^\bullet \otimes_R^{\mathbf{L}} \kappa(\mathfrak{p})) = 0$ for all $i \ll 0$,*
(4) *for every maximal ideal $\mathfrak{m}$ the complex $K^\bullet \otimes_R R_\mathfrak{m}$ is perfect,*
(5) *for every maximal ideal $\mathfrak{m}$ we have $H^i(K^\bullet \otimes_R^{\mathbf{L}} \kappa(\mathfrak{m})) = 0$ for all $i \ll 0$.*

Proof. Assume (5). Pick a maximal ideal $\mathfrak{m}$ of R. By Lemma 12.42.14 we see that the assumption implies that $K^\bullet \otimes_R R_f$ is a perfect complex for some $f \in R$, $f \notin \mathfrak{m}$. Since $Spec(R)$ is quasi-compact we conclude that $K^\bullet$ is perfect by Lemmas 12.42.11. The proof of the other implications is omitted. $\square$

The following lemma useful in order to find perfect complexes over a polynomial ring $B = A[x_1, \ldots, x_d]$.

Lemma 12.42.17. *Let $A \to B$ be a ring map. Let $a, b \in \mathbf{Z}$. Let $d \geq 0$. Let $K^\bullet$ be a complex of B-modules. Assume*

(1) *the ring map $A \to B$ is flat,*
(2) *for every prime $\mathfrak{p} \subset A$ the ring $B \otimes_A \kappa(\mathfrak{p})$ has finite global dimension $\leq d$,*
(3) *$K^\bullet$ is pseudo-coherent as a complex of B-modules, and*
(4) *$K^\bullet$ has tor amplitude in $[a, b]$ as a complex of A-modules.*

Then $K^\bullet$ is perfect as a complex of B-modules with tor amplitude in $[a - d, b]$.

Proof. We may assume that $K^\bullet$ is a bounded above complex of finite free B-modules. In particular, $K^\bullet$ is flat as a complex of A-modules and $K^\bullet \otimes_A M = K^\bullet \otimes_A^{\mathbf{L}}$ for any A-module M. For every prime $\mathfrak{p}$ of A the complex

$$K^\bullet \otimes_A \kappa(\mathfrak{p})$$

is a bounded above complex of finite free modules over $B \otimes_A \kappa(\mathfrak{p})$ with vanishing H^i except for $i \in [a, b]$. As $B \otimes_A \kappa(\mathfrak{p})$ has global dimension d we see from Lemma 12.41.15 that $K^\bullet \otimes_A \kappa(\mathfrak{p})$ has tor amplitude in $[a - d, b]$. Let $\mathfrak{q}$ be a prime of B lying over $\mathfrak{p}$. Since $K^\bullet \otimes_A \kappa(\mathfrak{p})$ is a bounded above complex of free $B \otimes_A \kappa(\mathfrak{q})$-modules we see that

$$\begin{aligned} K^\bullet \otimes_B^{\mathbf{L}} \kappa(\mathfrak{q}) &= K^\bullet \otimes_B \kappa(\mathfrak{q}) \\ &= (K^\bullet \otimes_A \kappa(\mathfrak{p})) \otimes_{B \otimes_A \kappa(\mathfrak{q})} \kappa(\mathfrak{q}) \\ &= (K^\bullet \otimes_A \kappa(\mathfrak{p})) \otimes_{B \otimes_A \kappa(\mathfrak{q})}^{\mathbf{L}} \kappa(\mathfrak{q}) \end{aligned}$$

Hence the arguments above imply that $H^i(K^\bullet \otimes_B^{\mathbf{L}} \kappa(\mathfrak{q})) = 0$ for $i \notin [a - d, b]$. We conclude by Lemma 12.42.15. $\square$

12.43. Characterizing perfect complexes

Let R be a ring. Recall that $D(R)$ has direct sums which are given simply by taking direct sums of complexes, see Injectives, Lemma 17.17.4. We will use this in the lemmas of this section without further mention.

Lemma 12.43.1. *Let R be a ring. Let $K \in D(R)$ be an object such that for every countable set of objects $E_n \in D(R)$ the canonical map*

$$\bigoplus Hom_{D(R)}(K, E_n) \longrightarrow Hom_{D(R)}(K, \bigoplus E_n)$$

is a bijection. Then, given any system $L_n^\bullet$ of complexes over $\mathbf{N}$ we have that

$$colim\ Hom_{D(R)}(K, L_n^\bullet) \longrightarrow Hom_{D(R)}(K, L^\bullet)$$

is a bijection, where $L^\bullet$ is the termwise colimit, i.e., $L^m = colim\ L_n^m$ for all $m \in \mathbf{Z}$.

Proof. Consider the short exact sequence of complexes

$$0 \to \bigoplus L_n^\bullet \to \bigoplus L_n^\bullet \to L^\bullet \to 0$$

where the first map is given by $1 - t_n$ in degree n where $t_n : L_n^\bullet \to L_{n+1}^\bullet$ is the transition map. By Derived Categories, Lemma 11.11.1 this is a distinguished triangle in $D(R)$. Apply the homological functor $Hom_{D(R)}(K, -)$, see Derived Categories, Lemma 11.4.2. Thus a long exact cohomology sequence

$$\begin{tikzcd}
 & \ldots \arrow[r] & Hom_{D(R)}(K, colim\ L_n^\bullet[-1]) \arrow[dll] \\
Hom_{D(R)}(K, \bigoplus L_n^\bullet) \arrow[r] & Hom_{D(R)}(K, \bigoplus L_n^\bullet) \arrow[r] & Hom_{D(R)}(K, colim\ L_n^\bullet) \arrow[dll] \\
Hom_{D(R)}(K, \bigoplus L_n^\bullet[1]) \arrow[r] & \ldots &
\end{tikzcd}$$

Since we have assumed that $Hom_{D(R)}(K, \bigoplus L_n^\bullet)$ is equal to $\bigoplus Hom_{D(R)}(K, L_n^\bullet)$ we see that the first map on every row of the diagram is injective (by the explicit description of this map as the sum of the maps induced by $1 - t_n$). Hence we conclude that $Hom_{D(R)}(K, colim\ L_n^\bullet)$ is the cokernel of the first map of the middle row in the diagram above which is what we had to show. $\square$

Definition 12.43.2. Let $\mathcal{D}$ be an additive category with arbitrary direct sums. A *compact object* of $\mathcal{D}$ is an object K such that the map

$$\bigoplus\nolimits_{i \in I} Hom_{\mathcal{D}}(K, E_i) \longrightarrow Hom_{\mathcal{D}}(K, \bigoplus\nolimits_{i \in I} E_i)$$

is bijective for any set I and objects $E_i \in Ob(\mathcal{D})$ parametrized by $i \in I$.

The following proposition shows up in various places. See for example [**Ric89**, proof of Proposition 6.3] (this treats the bounded case), [**TT90**, Theorem 2.4.3] (the statement doesn't match exactly), and [**BN93**, Proposition 6.4] (watch out for horrendous notational conventions).

Proposition 12.43.3. *Let R be a ring. For an object K of $D(R)$ the following are equivalent*

(1) *K is perfect, and*
(2) *K is a compact object of $D(R)$.*

Proof. Assume K is perfect, i.e., K is quasi-isomorphic to a bounded complex $P^\bullet$ of finite projective modules, see Definition 12.42.1. If E_i is represented by the complex $E_i^\bullet$, then $\bigoplus E_i$ is represented by the complex whose degree n term is $\bigoplus E_i^n$. On the other hand, as P^n is projective for all n we have $Hom_{D(R)}(P^\bullet, K^\bullet) = Hom_{K(R)}(P^\bullet, K^\bullet)$ for every complex of R-modules $K^\bullet$, see Derived Categories, Lemma 11.18.8. Thus $Hom_{D(R)}(P^\bullet, E^\bullet)$ is the cohomology of the complex

$$\prod Hom_R(P^n, E^{n-1}) \to \prod Hom_R(P^n, E^n) \to \prod Hom_R(P^n, E^{n+1}).$$

Since $P^\bullet$ is bounded we see that we may replace the $\prod$ signs by $\bigoplus$ signs in the complex above. Since each P^n is a finite R-module we see that $Hom_R(P^n, \bigoplus_i E_i^m) = \bigoplus_i Hom_R(P^n, E_i^m)$ for all n, m. Combining these remarks we see that the map of Definition 12.43.2 is a bijection.

Conversely, assume K is compact. Represent K by a complex $K^\bullet$ and consider the map

$$K^\bullet \longrightarrow \bigoplus\nolimits_{n\geq 0} \tau_{\geq n}K^\bullet$$

where we have used the canonical truncations, see Homology, Section 10.11. This makes sense as in each degree the direct sum on the right is finite. By assumption this map factors through a finite direct sum. We conclude that $K \to \tau_{\geq n}K$ is zero for at least one n, i.e., K is in $D^-(R)$.

Since $K \in D^-(R)$ and since every R-module is a quotient of a free module, we may represent K by a bounded above complex $K^\bullet$ of free R-modules, see Derived Categories, Lemma 11.15.5. Note that we have

$$K^\bullet = \bigcup\nolimits_{n\leq 0} \sigma_{\geq n}K^\bullet$$

where we have used the stupid truncations, see Homology, Section 10.11. Hence by Lemma 12.43.1 we see that $1 : K^\bullet \to K^\bullet$ factors through $\sigma_{\geq n}K^\bullet \to K^\bullet$ in $D(R)$. Thus we see that $1 : K^\bullet \to K^\bullet$ factors as

$$K^\bullet \xrightarrow{\varphi} L^\bullet \xrightarrow{\psi} K^\bullet$$

in $D(R)$ for some complex $L^\bullet$ which is bounded and whose terms are free R-modules. Say $L^i = 0$ for $i \not\in [a, b]$. Fix a, b from now on. Let c be the largest integer $\leq b + 1$ such that we can find a factorization of $1_{K^\bullet}$ as above with L^i is finite free for $i < c$. We will show by induction that $c = b + 1$. Namely, write $L^c = \bigoplus_{\lambda\in\Lambda} R$. Since L^{c-1} is finite free we can find a finite subset $\Lambda' \subset \Lambda$ such that $L^{c-1} \to L^c$ factors through $\bigoplus_{\lambda\in\Lambda'} R \subset L^c$. Consider the map of complexes

$$\pi : L^\bullet \longrightarrow (\bigoplus\nolimits_{\lambda\in\Lambda\setminus\Lambda'} R)[-i]$$

given by the projection onto the factors corresponding to $\Lambda \setminus \Lambda'$ in degree i. By our assumption on K we see that, after possibly replacing Λ' by a larger finite subset, we may assume that $\pi \circ \varphi = 0$ in $D(R)$. Let $(L')^\bullet \subset L^\bullet$ be the kernel of π. Since π is surjective we get a short exact sequence of complexes, which gives a distinguished triangle in $D(R)$ (see Derived Categories, Lemma 11.11.1). Since $Hom_{D(R)}(K, -)$ is homological (see Derived Categories, Lemma 11.4.2) and $\pi \circ \varphi = 0$, we can find a morphism $\varphi' : K^\bullet \to (L')^\bullet$ in $D(R)$ whose composition with $(L')^\bullet \to L^\bullet$ gives φ. Setting ψ' equal to the composition of ψ with $(L')^\bullet \to L^\bullet$ we obtain a new factorization. Since $(L')^\bullet$ agrees with $L^\bullet$ except in degree c and since $(L')^c = \bigoplus_{\lambda\in\Lambda'} R$ the induction step is proved.

The conclusion of the discussion of the preceding paragraph is that $1_K : K \to K$ factors as

$$K \xrightarrow{\varphi} L \xrightarrow{\psi} K$$

in $D(R)$ where L can be represented by a finite complex of free R-modules. In particular we see that L is perfect. Note that $e = \varphi \circ \psi \in \text{End}_{D(R)}(L)$ is an idempotent. By Derived Categories, Lemma 11.4.12 we see that $L = \text{Ker}(e) \oplus \text{Ker}(1 - e)$ (see also discussion preceding Derived Categories, Lemma 11.4.11). The map $\varphi : K \to L$ induces an isomorphism with $\text{Ker}(1 - e)$ in $D(R)$. Hence we finally conclude that K is perfect by Lemma 12.42.5. $\square$

Lemma 12.43.4. *Let R be a ring. Let $I \subset R$ be an ideal. Let K be an object of $D(R)$. Assume that*

(1) *$K \otimes_R^{\mathbf{L}} R/I$ is perfect in $D(R/I)$, and*
(2) *I is a nilpotent ideal.*

Then K is perfect in $D(R)$.

Proof. Assumption (2) means that $I^n = 0$ for some n. The result holds if $n = 1$. Below we will prove the result holds for $n = 2$. This will imply that the complex $K \otimes_R^{\mathbf{L}} R/I^2$ is perfect in $D(R/I^2)$. Since $(I^2)^{\lceil n/2 \rceil} = 0$ we see that we win by induction on n.

We prove the lemma in case $I^2 = 0$. First, we may represent K by a K-flat complex $K^\bullet$ with all K^n flat, see Lemma 12.3.10. Then we see that we have a short exact sequence of complexes

$$0 \to K^\bullet \otimes_R I \to K^\bullet \to K^\bullet \otimes_R R/I \to 0$$

Note that $K^\bullet \otimes_R R/I$ represents $K \otimes_R^{\mathbf{L}} R/I$ by constuction of the derived tensor product. Also

$$K^\bullet \otimes_R I \cong K^\bullet \otimes_R R/I \otimes_{R/I} I$$

represents $K \otimes_R^{\mathbf{L}} R/I \otimes_{R/I}^{\mathbf{L}} I$ because $K^\bullet \otimes_R R/I$ is a K-flat complex over R/I, see Lemma 12.3.5. By assumption (1) we see that both $K^\bullet \otimes_R R/I$ and $K^\bullet \otimes_R I$ have finitely many nonzero cohomology groups (since a perfect complex has finite Tor-amplitude, see Lemma 12.42.2). We conclude that $K \in D^b(R)$ by the long exact cohomology sequence associated to short exact sequence of complexes displayed above. In particular we can represent K by a bounded above complex $K^\bullet$ of free R-modules (see Derived Categories, Lemma 11.15.5). Then for any complex $E^\bullet$ of R-modules we have

$$Hom_{D(R)}(K, E^\bullet) = Hom_{K(R)}(K^\bullet, E^\bullet)$$

see Derived Categories, Lemma 11.18.8.

We will now show that K is perfect using the criterion of Proposition 12.43.3. Thus we let $E_j \in D(R)$ be a family of objects parametrized by a set J. We choose complexes $E_j^\bullet$ with flat terms representing E_j, see for example Lemma 12.3.10. It is clear that

$$0 \to E_j^\bullet \otimes_R I \to E_j^\bullet \to E_j^\bullet \otimes_R R/I \to 0$$

is a short exact sequence of complexes. Taking direct sums we obtain a similar short exact sequence

$$0 \to \bigoplus E_j^\bullet \otimes_R I \to \bigoplus E_j^\bullet \to \bigoplus E_j^\bullet \otimes_R R/I \to 0$$

(Note that $- \otimes_R I$ and $- \otimes_R R/I$ commute with direct sums.) This short exact sequence determines a distinguished triangle in $D(R)$, see Derived Categories, Lemma 11.11.1. Apply the homological functor $Hom_{D(R)}(K, -)$ (see Derived Categories, Lemma 11.4.2) to get a commutative diagram

$$\begin{tikzcd}
\bigoplus Hom_{D(R)}(K^\bullet, E_j^\bullet \otimes_R R/I)[-1] \arrow[r] \arrow[d] & Hom_{D(R)}(K^\bullet, \bigoplus E_j^\bullet \otimes_R R/I)[-1] \arrow[d] \\
\bigoplus Hom_{D(R)}(K^\bullet, E_j^\bullet \otimes_R I) \arrow[r] \arrow[d] & Hom_{D(R)}(K^\bullet, \bigoplus E_j^\bullet \otimes_R I) \arrow[d] \\
\bigoplus Hom_{D(R)}(K^\bullet, E_j^\bullet) \arrow[r] \arrow[d] & Hom_{D(R)}(K^\bullet, \bigoplus E_j^\bullet) \arrow[d] \\
\bigoplus Hom_{D(R)}(K^\bullet, E_j^\bullet \otimes_R R/I) \arrow[r] \arrow[d] & Hom_{D(R)}(K^\bullet, \bigoplus E_j^\bullet \otimes_R R/I) \arrow[d] \\
\bigoplus Hom_{D(R)}(K^\bullet, E_j^\bullet \otimes_R I)[1] \arrow[r] & Hom_{D(R)}(K^\bullet, \bigoplus E_j^\bullet \otimes_R I)[1]
\end{tikzcd}$$

with exact columns. Note that the complexes $E_j^\bullet \otimes_R R/I$ and $E_j^\bullet \otimes_R I$ have terms annihilated by I. Hence the 5 lemma (see Homology, Lemma 10.3.25) shows it suffices to show that given any collection of complexes $M_j^\bullet$ with $IM_j^n = 0$ for all n, j the map

$$\bigoplus Hom_{D(R)}(K^\bullet, M_j^\bullet) \longrightarrow Hom_{D(R)}(K^\bullet, \bigoplus M_j^\bullet)$$

is a bijection. By our choice of $K^\bullet$ we can rewrite this as

$$\bigoplus Hom_{K(R)}(K^\bullet, M_j^\bullet) \longrightarrow Hom_{K(R)}(K^\bullet, \bigoplus M_j^\bullet)$$

Since I annihilates M_j^n we see this is equal to

$$\bigoplus Hom_{K(R/I)}(K^\bullet \otimes_R R/I, M_j^\bullet) \longrightarrow Hom_{K(R)}(K^\bullet \otimes_R R/I, \bigoplus M_j^\bullet).$$

Using that $K^\bullet \otimes_R R/I$ is a bounded above complex of free R/I-modules we see this is equal to the map

$$\bigoplus Hom_{D(R/I)}(K^\bullet \otimes_R R/I, M_j^\bullet) \longrightarrow Hom_{D(R)}(K^\bullet \otimes_R R/I, \bigoplus M_j^\bullet)$$

by Derived Categories, Lemma 11.18.8 as before. The complex $K^\bullet \otimes_R R/I$ represents $K \otimes_R^{\mathbf{L}} R/I$ since $K^\bullet$ is K-flat (Lemma 12.3.8). We conclude that $K^\bullet \otimes_R R/I$ is compact (by Proposition 12.43.3). Hence the last displayed map is a bijection and we win. $\square$

12.44. Relatively finitely presented modules

Let R be a ring. Let $A \to B$ be a finite map of finite type R-algebras. Let M be a finite B-module. In this case it is **not true** that

$$M \text{ of finite presentation over } B \Leftrightarrow M \text{ of finite presentation over } A$$

A counter example is $R = k[x_1, x_2, x_3, \ldots]$, $A = R$, $B = R/(x_i)$, and $M = B$. To ``fix" this we introduce a relative notion of finite presentation.

Lemma 12.44.1. *Let $R \to A$ be a ring map of finite type. Let M be an A-module. The following are equivalent*

(1) *for some presentation $\alpha : R[x_1, \ldots, x_n] \to A$ the module M is a finitely presented $R[x_1, \ldots, x_n]$-module,*

(2) *for all presentations* $\alpha : R[x_1, \ldots, x_n] \to A$ *the module* M *is a finitely presented* $R[x_1, \ldots, x_n]$*-module, and*
(3) *for any surjection* $A' \to A$ *where* A' *is a finitely presented* R*-algebra, the module* M *is finitely presented as* A'*-module.*

In this case M *is a finitely presented* A*-module.*

Proof. If $\alpha : R[x_1, \ldots, x_n] \to A$ and $\beta : R[y_1, \ldots, y_m] \to A$ are presentations. Choose $f_j \in R[x_1, \ldots, x_n]$ with $\alpha(f_j) = \beta(y_j)$ and $g_i \in R[y_1, \ldots, y_m]$ with $\beta(g_i) = \alpha(x_i)$. Then we get a commutative diagram

$$\begin{tikzcd} R[x_1, \ldots, x_n, y_1, \ldots, y_m] \arrow[r, "y_j \mapsto f_j"'] \arrow[d, "x_i \mapsto g_i"] & R[x_1, \ldots, x_n] \arrow[d] \\ R[y_1, \ldots, y_m] \arrow[r] & A \end{tikzcd}$$

Hence the equivalence of (1) and (2) follows by applying Algebra, Lemmas 7.6.4 and 7.7.4. The equivalence of (2) and (3) follows by choosing a presentation $A' = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ and using Algebra, Lemma 7.7.4 to show that M is finitely presented as A'-module if and only if M is finitely presented as a $R[x_1, \ldots, x_n]$-module. $\square$

Definition 12.44.2. Let $R \to A$ be a finite type ring map. Let M be an A-module. We say M is *an* A*-module finitely presented relative to* R if the equivalent conditions of Lemma 12.44.1 hold.

Note that if $R \to A$ is of finite presentation, then M is an A-module finitely presented relative to R if and only if M is a finitely presented A-module. It is equally clear that A as an A-module is finitely presented relative to R if and only if A is of finite presentation over R. If R is Noetherian the notion is uninteresting. Now we can formulate the result we were looking for.

Lemma 12.44.3. *Let* R *be a ring. Let* $A \to B$ *be a finite map of finite type* R*-algebras. Let* M *be a* B*-module. Then* M *is an* A*-module finitely presented relative to* R *if and only if* M *is a* B*-module finitely presented relative to* R*.*

Proof. Choose a surjection $R[x_1, \ldots, x_n] \to A$. Choose $y_1, \ldots, y_m \in B$ which generate B over A. As $A \to B$ is finite each y_i satisfies a monic equation with coefficients in A. Hence we can find monic polynomials $P_j(T) \in R[x_1, \ldots, x_n][T]$ such that $P_j(y_j) = 0$ in B. Then we get a commutative diagram

$$\begin{tikzcd} R[x_1, \ldots, x_n] \arrow[r] \arrow[d] & R[x_1, \ldots, x_n, y_1, \ldots, y_m]/(P_j(y_j)) \arrow[d] \\ A \arrow[r] & B \end{tikzcd}$$

Since the top arrow is a finite and finitely presented ring map we conclude by Algebra, Lemma 7.7.4 and the definition. $\square$

With this result in hand we see that the relative notion makes sense and behaves well with regards to finite maps of rings of finite type over R. It is also stable under localization, stable under base change, and "glues" well.

Lemma 12.44.4. *Let* R *be a ring,* $f \in R$ *an element,* $R_f \to A$ *is a finite type ring map,* $g \in A$*, and* M *an* A*-module. If* M *of finite presentation relative to* R_f*, then* M_g *is an* A_g*-module of finite presentation relative to* R*.*

Proof. Choose a presentation $R_f[x_1, \ldots, x_n] \to A$. We write $R_f = R[x_0]/(fx_0 - 1)$. Consider the presentation $R[x_0, x_1, \ldots, x_n, x_{n+1}] \to A_g$ which extends the given map, maps x_0 to the image of $1/f$, and maps x_{n+1} to $1/g$. Choose $g' \in R[x_0, x_1, \ldots, x_n]$ which maps to g (this is possible). Suppose that

$$R_f[x_1, \ldots, x_n]^{\oplus s} \to R_f[x_1, \ldots, x_n]^{\oplus t} \to M \to 0$$

is a presentation of M given by a matrix (h_{ij}). Pick $h'_{ij} \in R[x_0, x_1, \ldots, x_n]$ which map to h_{ij}. Then

$$R[x_0, x_1, \ldots, x_n, x_{n+1}]^{\oplus s+2t} \to R[x_0, x_1, \ldots, x_n, x_{n+1}]^{\oplus t} \to M_g \to 0$$

is a presentation of M_f. Here the $t \times (s+2t)$ matrix defining the map has a first $t \times s$ block consisting of the matrix h'_{ij}, a second $t \times t$ block which is $(x_0 f -)I_t$, and a third block which is $(x_{n+1}g' - 1)I_t$. □

Lemma 12.44.5. *Let $R \to A$ be a finite type ring map. Let M be an A-module finitely presented relative to R. For any ring map $R \to R'$ the $A \otimes_R R'$-module*

$$M \otimes_A A' = M \otimes_R R'$$

is finitely presented relative to R'.

Proof. Choose a surjection $R[x_1, \ldots, x_n] \to A$. Choose a presentation

$$R[x_1, \ldots, x_n]^{\oplus s} \to R[x_1, \ldots, x_n]^{\oplus t} \to M \to 0$$

Then

$$R'[x_1, \ldots, x_n]^{\oplus s} \to R'[x_1, \ldots, x_n]^{\oplus t} \to M \otimes_R R' \to 0$$

is a presentation of the base change and we win. □

Lemma 12.44.6. *Let $R \to A$ be a finite type ring map. Let M be an A-module finitely presented relative to R. Let $A \to A'$ be a ring map of finite presentation. A'-module $M \otimes_A A'$ is finitely presented relative to R.*

Proof. Choose a surjection $R[x_1, \ldots, x_n] \to A$. Choose a presentation $A' = A[y_1, \ldots, y_m]/(g_1, \ldots, g_l)$. Pick $g'_i \in R[x_1, \ldots, x_n, y_1, \ldots, y_m]$ mapping to g_i. Say

$$R[x_1, \ldots, x_n]^{\oplus s} \to R[x_1, \ldots, x_n]^{\oplus t} \to M \to 0$$

is a presentation of M given by a matrix (h_{ij}). Then

$$R[x_1, \ldots, x_n, y_1, \ldots, y_m]^{\oplus s+tl} \to R[x_0, x_1, \ldots, x_n, y_1, \ldots, y_m]^{\oplus t} \to M \otimes_A A' \to 0$$

is a presentation of $M \otimes_A A'$. Here the $t \times (s + lt)$ matrix defining the map has a first $t \times s$ block consisting of the matrix h_{ij}, followed by l blocks of size $t \times t$ which are $g'_i I_t$. □

Lemma 12.44.7. *Let $R \to A \to B$ be finite type ring maps. Let M be a B-module. If M is finitely presented relative to A and A is of finite presentation over R, then M is finitely presented relative to R.*

Proof. Choose a surjection $A[x_1, \ldots, x_n] \to B$. Choose a presentation

$$A[x_1, \ldots, x_n]^{\oplus s} \to A[x_1, \ldots, x_n]^{\oplus t} \to M \to 0$$

given by a matrix (h_{ij}). Choose a presentation

$$A = R[y_1, \ldots, y_m]/(g_1, \ldots, g_u).$$

Choose $h'_{ij} \in R[y_1, \ldots, y_m, x_1, \ldots, x_n]$ mapping to h_{ij}. Then we obtain the presentation

$$R[y_1, \ldots, y_m, x_1, \ldots, x_n]^{\oplus s+tu} \to R[y_1, \ldots, y_m, x_1, \ldots, x_n]^{\oplus t} \to M \to 0$$

where the $t \times (s + tu)$-matrix is given by a first $t \times s$ block consisting of h'_{ij} followed by u blocks of size $t \times t$ given by $g_i I_t$, $i = 1, \ldots, u$. □

Lemma 12.44.8. *Let $R \to A$ be a finite type ring map. Let M be an A-module. Let $f_1, \ldots, f_r \in A$ generate the unit ideal. The following are equivalent*

(1) *each M_{f_i} is finitely presented relative to R, and*
(2) *M is finitely presented relative to R.*

Proof. The implication (2) ⇒ (1) is in Lemma 12.44.4. Assume (1). Write $1 = \sum f_i g_i$ in A. Choose a surjection $R[x_1, \ldots, x_n, y_1, \ldots, y_r, z_1, \ldots, z_r] \to A$. such that y_i maps to f_i and z_i maps to g_i. Then we see that there exists a surjection

$$P = R[x_1, \ldots, x_n, y_1, \ldots, y_r, z_1, \ldots, z_r]/(\sum y_i z_i - 1) \longrightarrow A.$$

By Lemma 12.44.1 we see that M_{f_i} is a finitely presented A_{f_i}-module, hence by Algebra, Lemma 7.21.2 we see that M is a finitely presented A-module. Hence M is a finite P-module (with P as above). Choose a surjection $P^{\oplus t} \to M$. We have to show that the kernel K of this map is a finite P-module. Since P_{y_i} surjects onto A_{f_i} we see by Lemma 12.44.1 and Algebra, Lemma 7.5.3 that the localization K_{y_i} is a finitely generated P_{y_i}-module. Choose elements $k_{i,j} \in K$, $i = 1, \ldots, r$, $j = 1, \ldots, s_i$ such that the images of $k_{i,j}$ in K_{y_i} generate. Set $K' \subset K$ equal to the P-module generated by the elements $k_{i,j}$. Then K/K' is a module whose localization at y_i is zero for all i. Since $(y_1, \ldots, y_r) = P$ we see that $K/K' = 0$ as desired. □

Lemma 12.44.9. *Let $R \to A$ be a finite type ring map. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of A-modules.*

(1) *If M', M'' are finitely presented relative to R, then so is M.*
(2) *If M' is finite a type A-module and M is finitely presented relative to R, then M'' is finitely presented relative to R.*

Proof. Follows immediately from Algebra, Lemma 7.5.4. □

Lemma 12.44.10. *Let $R \to A$ be a finite type ring map. Let M, M' be A-modules. If $M \oplus M'$ is finitely presented relative to R, then so are M and M'.*

Proof. Omitted. □

12.45. Relatively pseudo-coherent modules

This section is the analogue of Section 12.44 for pseudo-coherence.

Lemma 12.45.1. *Let R be a ring. Let $K^\bullet$ be an object of $D^-(R)$. Consider the R-algebra map $R[x] \to R$ which maps x to zero. Then*

$$K^\bullet \otimes^{\mathbf{L}}_{R[x]} R \cong K^\bullet \oplus K^\bullet[1]$$

in $D(R)$.

Proof. Choose a projective resolution $P^\bullet \to K^\bullet$ over R. Then

$$P^\bullet \otimes_R R[x] \xrightarrow{x} P^\bullet \otimes_R R[x]$$

is a double complex of projective $R[x]$-modules whose associated total complex is quasi-isomorphic to $P^\bullet$. Hence

$$\begin{aligned} K^\bullet \otimes^{\mathbf{L}}_{R[x]} R &\cong \mathrm{Tot}(P^\bullet \otimes_R R[x] \xrightarrow{x} P^\bullet \otimes_R R[x]) \otimes_{R[x]} R = \mathrm{Tot}(P^\bullet \xrightarrow{0} P^\bullet) \\ &= P^\bullet \oplus P^\bullet[1] \cong K^\bullet \oplus K^\bullet[1] \end{aligned}$$

as desired. □

Lemma 12.45.2. *Let R be a ring and $K^\bullet$ a complex of R-modules. Let $m \in \mathbf{Z}$. Consider the R-algebra map $R[x] \to R$ which maps x to zero. Then $K^\bullet$ is m-pseudo-coherent as a complex of R-modules if and only if $K^\bullet$ is m-pseudo-coherent as a complex of $R[x]$-modules.*

Proof. This is a special case of Lemma 12.40.11. We also prove it in another way as follows.

Note that $0 \to R[x] \to R[x] \to R \to 0$ is exact. Hence R is pseudo-coherent as an $R[x]$-module. Thus one implication of the lemma follows from Lemma 12.40.11. To prove the other implication, assume that $K^\bullet$ is m-pseudo-coherent as a complex of $R[x]$-modules. By Lemma 12.40.12 we see that $K^\bullet \otimes^{\mathbf{L}}_{R[x]} R$ is m-pseudo-coherent as a comples of R-modules. By Lemma 12.45.1 we see that $K^\bullet \oplus K^\bullet[1]$ is m-pseudo-coherent as a complex of R-modules. Finally, we conclude that $K^\bullet$ is m-pseudo-coherent as a complex of R-modules from Lemma 12.40.8. □

Lemma 12.45.3. *Let $R \to A$ be a ring map of finite type. Let $K^\bullet$ be a complex of A-modules. Let $m \in \mathbf{Z}$. The following are equivalent*

(1) *for some presentation $\alpha : R[x_1, \ldots, x_n] \to A$ the complex $K^\bullet$ is an m-pseudo-coherent complex of $R[x_1, \ldots, x_n]$-modules,*
(2) *for all presentations $\alpha : R[x_1, \ldots, x_n] \to A$ the complex $K^\bullet$ is an m-pseudo-coherent complex of $R[x_1, \ldots, x_n]$-modules.*

In particular the same equivalence holds for pseudo-coherence.

Proof. If $\alpha : R[x_1, \ldots, x_n] \to A$ and $\beta : R[y_1, \ldots, y_m] \to A$ are presentations. Choose $f_j \in R[x_1, \ldots, x_n]$ with $\alpha(f_j) = \beta(y_j)$ and $g_i \in R[y_1, \ldots, y_m]$ with $\beta(g_i) = \alpha(x_i)$. Then we get a commutative diagram

$$\begin{tikzcd} R[x_1, \ldots, x_n, y_1, \ldots, y_m] \arrow[r, "y_j \mapsto f_j"'] \arrow[d, "x_i \mapsto g_i"] & R[x_1, \ldots, x_n] \arrow[d] \\ R[y_1, \ldots, y_m] \arrow[r] & A \end{tikzcd}$$

After a change of coordinates the ring homomorphism $R[x_1, \ldots, x_n, y_1, \ldots, y_m] \to R[x_1, \ldots, x_n]$ is isomorphic to the ring homomorphism which maps each y_i to zero. Similarly for the left vertical map in the diagram. Hence, by induction on the number of variables this lemma follows from Lemma 12.45.2. The pseudo-coherent case follows from this and Lemma 12.40.5. □

Definition 12.45.4. Let $R \to A$ be a finite type ring map. Let $K^\bullet$ be a complex of A-modules. Let M be an A-module. Let $m \in \mathbf{Z}$.

(1) We say $K^\bullet$ is *m-pseudo-coherent relative to R* if the equivalent conditions of Lemma 12.45.3 hold.
(2) We say $K^\bullet$ is *pseudo-coherent relative to R* if $K^\bullet$ is m-pseudo-coherent relative to R for all $m \in \mathbf{Z}$.

(3) We say M is *m-pseudo-coherent relative to R* if $M[0]$ is m-pseudo-coherent.
(4) We say M is *pseudo-coherent relative to R* if $M[0]$ is pseudo-coherent relative to R.

Part (2) means that $K^\bullet$ is pseudo-coherent as a complex of $R[x_1, \ldots, x_n]$-modules for any surjection $R[y_1, \ldots, y_m] \to A$, see Lemma 12.40.5. This definition has the following pleasing property.

Lemma 12.45.5. *Let R be a ring. Let $A \to B$ be a finite map of finite type R-algebras. Let $m \in \mathbf{Z}$. Let $K^\bullet$ be a complex of B-modules. Then $K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) relative to R if and only if $K^\bullet$ seen as a complex of A-modules is m-pseudo-coherent (pseudo-coherent) relative to R.*

Proof. Choose a surjection $R[x_1, \ldots, x_n] \to A$. Choose $y_1, \ldots, y_m \in B$ which generate B over A. As $A \to B$ is finite each y_i satisfies a monic equation with coefficients in A. Hence we can find monic polynomials $P_j(T) \in R[x_1, \ldots, x_n][T]$ such that $P_j(y_j) = 0$ in B. Then we get a commutative diagram

$$\begin{tikzcd} & R[x_1, \ldots, x_n, y_1, \ldots, y_m] \arrow[d] \\ R[x_1, \ldots, x_n] \arrow[r] \arrow[d] & R[x_1, \ldots, x_n, y_1, \ldots, y_m]/(P_j(y_j)) \arrow[d] \\ A \arrow[r] & B \end{tikzcd}$$

The top horizontal arrow and the top right vertial arrow satisfy the assumptions of Lemma 12.40.11. Hence $K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) as a complex of $R[x_1, \ldots, x_n]$-modules if and only if $K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) as a complex of $R[x_1, \ldots, x_n, y_1, \ldots, y_m]$-modules. $\square$

Lemma 12.45.6. *Let R be a ring. Let $R \to A$ be a finite type ring map. Let $m \in \mathbf{Z}$. Let $(K^\bullet, L^\bullet, M^\bullet, f, g, h)$ be a distinguished triangle in $D(A)$.*

(1) *If $K^\bullet$ is $(m+1)$-pseudo-coherent relative to R and $L^\bullet$ is m-pseudo-coherent relative to R then $M^\bullet$ is m-pseudo-coherent relative to R.*
(2) *If $K^\bullet, M^\bullet$ are m-pseudo-coherent relative to R, then $L^\bullet$ is m-pseudo-coherent relative to R.*
(3) *If $L^\bullet$ is $(m+1)$-pseudo-coherent relative to R and $M^\bullet$ is m-pseudo-coherent relative to R, then $K^\bullet$ is $(m+1)$-pseudo-coherent relative to R.*

Moreover, if two out of three of $K^\bullet, L^\bullet, M^\bullet$ are pseudo-coherent relative to R, the so is the third.

Proof. Follows immediately from Lemma 12.40.2 and the definitions. $\square$

Lemma 12.45.7. *Let $R \to A$ be a finite type ring map. Let M be an A-module. Then*

(1) *M is 0-pseudo-coherent relative to R if and only if M is a finite type A-module,*
(2) *M is (-1)-pseudo-coherent relative to R if and only if M is a finitely presented relative to R,*
(3) *M is $(-d)$-pseudo-coherent relative to R if and only if for every surjection $R[x_1, \ldots, x_n] \to A$ there exists a resolution*

$$R[x_1, \ldots, x_n]^{\oplus a_d} \to R[x_1, \ldots, x_n]^{\oplus a_{d-1}} \to \ldots \to R[x_1, \ldots, x_n]^{\oplus a_0} \to M \to 0$$

of length d, and

(4) *M is pseudo-coherent relative to R if and only if for every presentation $R[x_1, \ldots, x_n] \to A$ there exists an infinite resolution*
$$\ldots \to R[x_1, \ldots, x_n]^{\oplus a_1} \to R[x_1, \ldots, x_n]^{\oplus a_0} \to M \to 0$$
by finite free $R[x_1, \ldots, x_n]$-modules.

Proof. Follows immediately from Lemma 12.40.4 and the definitions. $\square$

Lemma 12.45.8. *Let $R \to A$ be a finite type ring map. Let $m \in \mathbf{Z}$. Let $K^\bullet, L^\bullet \in D(A)$. If $K^\bullet \oplus L^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) relative to R so are $K^\bullet$ and $L^\bullet$.*

Proof. Immediate from Lemma 12.40.8 and the definitions. $\square$

Lemma 12.45.9. *Let $R \to A$ be a finite type ring map. Let $m \in \mathbf{Z}$. Let $K^\bullet$ be a bounded above complex of A-modules such that K^i is $(m - i)$-pseudo-coherent relative to R for all i. Then $K^\bullet$ is m-pseudo-coherent relative to R. In particular, if $K^\bullet$ is a bounded above complex of A-modules pseudo-coherent relative to R, then $K^\bullet$ is pseudo-coherent relative to R.*

Proof. Immediate from Lemma 12.40.9 and the definitions. $\square$

Lemma 12.45.10. *Let $R \to A$ be a finite type ring map. Let $m \in \mathbf{Z}$. Let $K^\bullet \in D^-(A)$ such that $H^i(K^\bullet)$ is $(m-i)$-pseudo-coherent (resp. pseudo-coherent) relative to R for all i. Then $K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) relative to R.*

Proof. Immediate from Lemma 12.40.10 and the definitions. $\square$

Lemma 12.45.11. *Let R be a ring, $f \in R$ an element, $R_f \to A$ is a finite type ring map, $g \in A$, and $K^\bullet$ a complex of A-modules. If $K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) relative to R_f, then $K^\bullet \otimes_A A_g$ is m-pseudo-coherent (resp. pseudo-coherent) relative to R.*

Proof. First we show that $K^\bullet$ is m-pseudo-coherent relative to R. Namely, suppose $R_f[x_1, \ldots, x_n] \to A$ is surjective. Write $R_f = R[x_0]/(fx_0 - 1)$. Then $R[x_0, x_1, \ldots, x_n] \to A$ is surjective, and $R_f[x_1, \ldots, x_n]$ is pseudo-coherent as an $R[x_0, \ldots, x_n]$-module. Hence by Lemma 12.40.11 we see that $K^\bullet$ is m-pseudo-coherent as a complex of $R[x_0, x_1, \ldots, x_n]$-modules.

Choose an element $g' \in R[x_0, x_1, \ldots, x_n]$ which maps to $g \in A$. By Lemma 12.40.12 we see that
$$\begin{aligned} K^\bullet \otimes^{\mathbf{L}}_{R[x_0,x_1,\ldots,x_n]} R[x_0, x_1, \ldots, x_n, \frac{1}{g'}] &= K^\bullet \otimes_{R[x_0,x_1,\ldots,x_n]} R[x_0, x_1, \ldots, x_n, \frac{1}{g'}] \\ &= K^\bullet \otimes_A A_f \end{aligned}$$
is m-pseudo-coherent as a complex of $R[x_0, x_1, \ldots, x_n, \frac{1}{g'}]$-modules. write
$$R[x_0, x_1, \ldots, x_n, \frac{1}{g'}] = R[x_0, \ldots, x_n, x_{n+1}]/(x_{n+1}g' - 1).$$
As $R[x_0, x_1, \ldots, x_n, \frac{1}{g'}]$ is pseudo-coherent as a $R[x_0, \ldots, x_n, x_{n+1}]$-module we conclude (see Lemma 12.40.11) that $K^\bullet \otimes_A A_g$ is m-pseudo-coherent as a complex of $R[x_0, \ldots, x_n, x_{n+1}]$-modules as desired. $\square$

Lemma 12.45.12. *Let $R \to A$ be a finite type ring map. Let $m \in \mathbf{Z}$. Let $K^\bullet$ be a complex of A-modules which is m-pseudo-coherent (resp. pseudo-coherent) relative to R. Let $R \to R'$ be a ring map such that A and R' are Tor independent over R. Set $A' = A \otimes_R R'$. Then $K^\bullet \otimes^{\mathbf{L}}_A A'$ is is m-pseudo-coherent (resp. pseudo-coherent) relative to R'.*

Proof. Choose a surjection $R[x_1, \ldots, x_n] \to A$. Note that

$$K^\bullet \otimes_A^{\mathbf{L}} A' = K^\bullet \otimes_R^{\mathbf{L}} R' = K^\bullet \otimes^{\mathbf{L}}_{R[x_1,\ldots,x_n]} R'[x_1, \ldots, x_n]$$

by Lemma 12.5.2 applied twice. Hence we win by Lemma 12.40.12. $\square$

Lemma 12.45.13. *Let $R \to A \to B$ be finite type ring maps. Let $m \in \mathbf{Z}$. Let $K^\bullet$ be a complex of A-modules. Assume B as a B-module is pseudo-coherent relative to A. If $K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) relative to R, then $K^\bullet \otimes_A^{\mathbf{L}} B$ is m-pseudo-coherent (resp. pseudo-coherent) relative to R.*

Proof. Choose a surjection $A[y_1, \ldots, y_m] \to B$. Choose a surjection $R[x_1, \ldots, x_n] \to A$. Combined we get a surjection $R[x_1, \ldots, x_n, y_1, \ldots y_m] \to B$. Choose a resolution $E^\bullet \to B$ of B by a complex of finite free $A[y_1, \ldots, y_n]$-modules (which is possible by our assumption on the ring map $A \to B$). We may assume that $K^\bullet$ is a bounded above complex of flat A-modules. Then

$$\begin{aligned} K^\bullet \otimes_A^{\mathbf{L}} B &= \mathrm{Tot}(K^\bullet \otimes_A B[0]) \\ &= \mathrm{Tor}(K^\bullet \otimes_A A[y_1, \ldots, y_m] \otimes_{A[y_1,\ldots,y_m]} B[0]) \\ &\cong \mathrm{Tot}\left((K^\bullet \otimes_A A[y_1, \ldots, y_m]) \otimes_{A[y_1,\ldots,y_m]} E^\bullet\right) \\ &= \mathrm{Tot}(K^\bullet \otimes_A E^\bullet) \end{aligned}$$

in $D(A[y_1, \ldots, y_m])$. The quasi-isomorphism $\cong$ comes from an application of Lemma 12.3.8. Thus we have to show that $\mathrm{Tot}(K^\bullet \otimes_A E^\bullet)$ is m-pseudo-coherent as a complex of $R[x_1, \ldots, x_n, y_1, \ldots y_m]$-modules. Note that $\mathrm{Tot}(K^\bullet \otimes_A E^\bullet)$ has a filtration by subcomplexes with successive quotients the complexes $K^\bullet \otimes_A E^i[-i]$. Note that for $i \ll 0$ the complexes $K^\bullet \otimes_A E^i[-i]$ have zero cohomology in degrees $\leq m$ and hence are m-pseudo-coherent (over any ring). Hence, applying Lemma 12.45.6 and induction, it suffices to show that $K^\bullet \otimes_A E^i[-i]$ is pseudo-coherent relative to R for all i. Note that $E^i = 0$ for $i > 0$. Since also E^i is finite free this reduces to proving that $K^\bullet \otimes_A A[y_1, \ldots, y_m]$ is m-pseudo-coherent relative to R which follows from Lemma 12.45.12 for instance. $\square$

Lemma 12.45.14. *Let $R \to A \to B$ be finite type ring maps. Let $m \in \mathbf{Z}$. Let M be an A-module. Assume B as a B-module is flat and pseudo-coherent relative to A. If M is m-pseudo-coherent (resp. pseudo-coherent) relative to R, then $M \otimes_A B$ is m-pseudo-coherent (resp. pseudo-coherent) relative to R.*

Proof. Immediate from Lemma 12.45.13. $\square$

Lemma 12.45.15. *Let R be a ring. Let $A \to B$ be a map of finite type R-algebras. Let $m \in \mathbf{Z}$. Let $K^\bullet$ be a complex of B-modules. Assume A is pseudo-coherent relative to R. Then the following are equivalent*

(1) *$K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) relative to A, and*
(2) *$K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) relative to R.*

Proof. Choose a surjection $R[x_1, \ldots, x_n] \to A$. Choose a surjection $A[y_1, \ldots, y_m] \to B$. Then we get a surjection

$$R[x_1, \ldots, x_n, y_1, \ldots, y_m] \to A[y_1, \ldots, y_m]$$

which is a flat base change of $R[x_1, \ldots, x_n] \to A$. By assumption A is a pseudo-coherent module over $R[x_1, \ldots, x_n]$ hence by Lemma 12.40.13 we see that $A[y_1, \ldots, y_m]$ is pseudo-coherent over $R[x_1, \ldots, x_n, y_1, \ldots, y_m]$. Thus the lemma follows from Lemma 12.40.11 and the definitions. $\square$

Lemma 12.45.16. *Let $R \to A$ be a finite type ring map. Let $K^\bullet$ be a complex of A-modules. Let $m \in \mathbf{Z}$. Let $f_1, \ldots, f_r \in A$ generate the unit ideal. The following are equivalent*

1. *each $K^\bullet \otimes_A A_{f_i}$ is m-pseudo-coherent relative to R, and*
2. *$K^\bullet$ is m-pseudo-coherent relative to R.*

The same equivalence holds for pseudo-coherence.

Proof. The implication (2) $\Rightarrow$ (1) is in Lemma 12.45.11. Assume (1). Write $1 = \sum f_i g_i$ in A. Choose a surjection $R[x_1, \ldots, x_n, y_1, \ldots, y_r, z_1, \ldots, z_r] \to A$. such that y_i maps to f_i and z_i maps to g_i. Then we see that there exists a surjection

$$P = R[x_1, \ldots, x_n, y_1, \ldots, y_r, z_1, \ldots, z_r]/(\sum y_i z_i - 1) \longrightarrow A.$$

Note that P is pseudo-coherent as an $R[x_1, \ldots, x_n, y_1, \ldots, y_r, z_1, \ldots, z_r]$-module and that $P[1/y_i]$ is pseudo-coherent as an $R[x_1, \ldots, x_n, y_1, \ldots, y_r, z_1, \ldots, z_r, 1/y_i]$-module. Hence by Lemma 12.40.11 we see that $K^\bullet \otimes_A A_{f_i}$ is an m-pseudo-coherent complex of $P[1/y_i]$-modules for each i. Thus by Lemma 12.40.14 we see that $K^\bullet$ is pseudo-coherent as a complex of P-modules, and Lemma 12.40.11 shows that $K^\bullet$ is pseudo-coherent as a complex of $R[x_1, \ldots, x_n, y_1, \ldots, y_r, z_1, \ldots, z_r]$-modules. □

Lemma 12.45.17. *Let R be a Noetherian ring. Let $R \to A$ be a finite type ring map. Then*

1. *A complex of A-modules $K^\bullet$ is m-pseudo-coherent relative to R if and only if $K^\bullet \in D^-(A)$ and $H^i(K^\bullet)$ is a finite A-module for $i \geq m$.*
2. *A complex of A-modules $K^\bullet$ is pseudo-coherent relative to R if and only if $K^\bullet \in D^-(A)$ and $H^i(K^\bullet)$ is a finite A-module for all i.*
3. *An A-module is pseudo-coherent relative to R if and only if it is finite.*

Proof. Immediate consequence of Lemma 12.40.16 and the definitions. □

12.46. Pseudo-coherent and perfect ring maps

We can define these types of ring maps as follows.

Definition 12.46.1. Let $A \to B$ be a ring map.

1. We say $A \to B$ is a *pseudo-coherent ring map* if it is of finite type and B, as a B-module, is pseudo-coherent relative to A.
2. We say $A \to B$ is a *perfect ring map* if it is a pseudo-coherent ring map such that B as an A-module has finite tor dimension.

This terminology may be nonstandard. Using Lemma 12.45.7 we see that $A \to B$ is pseudo-coherent if and only if $B = A[x_1, \ldots, x_n]/I$ and B as an $A[x_1, \ldots, x_n]$-module has a resolution by finite free $A[x_1, \ldots, x_n]$-modules. The motivation for the definition of a perfect ring map is Lemma 12.42.2. The following lemmas gives a more useful and intuitive characterization of a perfect ring map.

Lemma 12.46.2. *A ring map $A \to B$ is perfect if and only if $B = A[x_1, \ldots, x_n]/I$ and B as an $A[x_1, \ldots, x_n]$-module has a finite resolution by finite projective $A[x_1, \ldots, x_n]$-modules.*

Proof. If $A \to B$ is perfect, then $B = A[x_1, \ldots, x_n]/I$ and B is pseudo-coherent as an $A[x_1, \ldots, x_n]$-module and has finite tor dimension as an A-module. Hence Lemma 12.42.17 implies that B is perfect as a $A[x_1, \ldots, x_n]$-module, i.e., it has a finite resolution by finite projective $A[x_1, \ldots, x_n]$-modules (Lemma 12.42.3). Conversely, if $B = A[x_1, \ldots, x_n]/I$ and B as an $A[x_1, \ldots, x_n]$-module has a finite resolution by finite projective $A[x_1, \ldots, x_n]$-modules then B is pseudo-coherent as an $A[x_1, \ldots, x_n]$-module, hence $A \to B$ is pseudo-coherent.

Moreover, the given resolution over $A[x_1, \ldots, x_n]$ is a finite resolution by flat A-modules and hence B has finite tor dimension as an A-module. □

Lots of the results of the preceding sections can be reformulated in terms of this terminology. We also refer to More on Morphisms, Sections 33.36 and 33.37 for the corresponding discussion concerning morphisms of schemes.

Lemma 12.46.3. *A finite type ring map of Noetherian rings is pseudo-coherent.*

Proof. See Lemma 12.45.17. □

Lemma 12.46.4. *A ring map which is flat and of finite presentation is perfect.*

Proof. Let $A \to B$ be a ring map which is flat and of finite presentation. It is clear that B has finite tor dimension. By Algebra, Lemma 7.120.5 there exists a finite type $\mathbf{Z}$-algebra $A_0 \subset A$ and a flat finite type ring map $A_0 \to B_0$ such that $B = B_0 \otimes_{A_0} A$. By Lemma 12.45.17 we see that $A_0 \to B_0$ is pseudo-coherent. As $A_0 \to B_0$ is flat we see that B_0 and A are tor independent over A_0, hence we may use Lemma 12.45.12 to conclude that $A \to B$ is pseudo-coherent. □

Lemma 12.46.5. *Let $A \to B$ be a finite type ring map with A a regular ring of finite dimension. Then $A \to B$ is perfect.*

Proof. By Algebra, Lemma 7.102.7 the assumption on A means that A has finite global dimension. Hence every module has finite tor dimension, see Lemma 12.41.15, in particular B does. By Lemma 12.46.3 the map is pseudo-coherent. □

Lemma 12.46.6. *A local complete intersection homomorphism is perfect.*

Proof. Let $A \to B$ he a local complete intersection homomorphism. By Definition 12.24.2 this means that $B = A[x_1, \ldots, x_n]/I$ where I is a Koszul ideal in $A[x_1, \ldots, x_n]$. By Lemmas 12.46.2 and 12.42.3 it suffices to show that I is a perfect module over $A[x_1, \ldots, x_n]$. By Lemma 12.42.11 this is a local question. Hence we may assume that I is generated by a Koszul-regular sequence (by Definition 12.23.1). Of course this means that I has a finite free resolution and we win. □

12.47. Other chapters

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CHAPTER 13

Smoothing Ring Maps

13.1. Introduction

The main result of this chapter is the following:

$\boxed{\text{A regular map of Noetherian rings is a filtered colimit of smooth ones.}}$

This theorem is due to Popescu, see [**Pop90**]. A readable exposition of Popescu's proof was given by Richard Swan, see [**Swa98**] who used notes by André and a paper of Ogoma, see [**Ogo94**].

Our exposition follows Swan's, but we first prove an intermediate result which let's us work in a slightly simpler situation. Here is an overview. We first solve the following ``lifting problem'': A flat infinitesimal deformation of a filtered colimit of smooth algebras is a filtered colimit of smooth algebras. This result essentially says that it suffices to prove the main theorem for maps between reduced Noetherian rings. Next we prove two very clever lemmas called the ``lifting lemma'' and the ``desingularization lemma''. We show that these lemmas combined reduce the main theorem to proving a Noetherian, geometrically regular k-algebra Λ is a filtered limit of smooth k-algebras. Next, we discuss the necessary local tricks that go into the Popescu-Ogoma-Swan-André proof. Finally, in the last three sections we give the proof.

We end this introduction with some pointers to references. Let A be a henselian Noetherian local ring. We say A has the *approximation property* if for any $f_1, \ldots, f_m \in A[x_1, \ldots, x_n]$ the system of equations $f_1 = 0, \ldots, f_m = 0$ has a solution in the completion of A if and only if it has a solution in A. This definition is due to Artin. Artin first proved the approximation property for analytic systems of equations, see [**Art68**]. In [**Art69a**] Artin proved the approximation property for local rings essentially of finite type over an excellent discrete valuation ring. Artin conjectured (page 26 of [**Art69a**]) that every excellent henselian local ring should have the approximation property.

At some point in time it became a conjecture that that every regular homomorphism of Noetherian rings is a filtered colimit of smooth algebras (see for example [**Ray72**], [**Pop81**], [**Art82**], [**AD83**]). We're not sure who this conjecture[1] is due to. The relationship with the approximation property is that if $A \to A^\wedge$ is a colimit of smooth algebras, then the approximation property holds (insert future reference here). Moreover, the main theorem applies to the map $A \to A^\wedge$ if A is an excellent local ring, as one of the conditions of an excellent local ring is that the formal fibres are geometrically regular. Note that excellent local rings were defined by Grothendieck and their definition appeared in print in 1965.

In [**Art82**] it was shown that $R \to R^\wedge$ is a filtered colimit of smooth algebras for any local ring R essentially of finite type over a field. In [**AR88**] it was shown that $R \to R^\wedge$ is a

[1]The question/conjecture as formulated in [**Art82**], [**AD83**], and [**Pop81**] is stronger and was shown to be equivalent to the original version in [**CP84**].

filtered colimit of smooth algebras for any local ring R essentially of finite type over an excellent discrete valution ring. Finally, the main theorem was shown in [**Pop85**], [**Pop86**], [**Pop90**], [**Ogo94**], and [**Swa98**] as discussed above.

Conversely, using some of the results above, in [**Rot90**] it was shown that any local ring with the approximation property is excellent.

The paper [**Spi99**] provides an alternative approach to the main theorem, but it seems hard to read (for example [**Spi99**, Lemma 5.2] appears to be an incorrectly reformulated version of [**Elk73**, Lemma 3]). There is also a Bourbaki lecture about this material, see [**Tei95**].

13.2. Colimits

In Categories, Section 4.17 we discuss filtered colimits. In particular, note that Categories, Lemma 4.19.3 tells us that colimits over filtered index categories are the same thing as colimits over directed partially ordered sets.

Lemma 13.2.1. *Let $R \to \Lambda$ be a ring map. Let $\mathcal{E}$ be a set of R-algebras such that each $A \in \mathcal{E}$ is of finite presentation over R. Then the following two statements are equivalent*

(1) *Λ is a filtered colimit of elements of $\mathcal{E}$, and*
(2) *for any R algebra map $A \to \Lambda$ with A of finite presentation over R we can find a factorization $A \to B \to \Lambda$ with $B \in \mathcal{E}$.*

Proof. Suppose that $\mathcal{I} \to \mathcal{E}$, $i \mapsto A_i$ is a diagram such that $\Lambda = colim_i A_i$. Let $A \to \Lambda$ with A of finite presentation over R. Pick a presentation $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. Say $A \to \Lambda$ maps x_s to $\lambda_s \in \Lambda$. We can find an $i \in Ob(\mathcal{I})$ and elements $a_s \in A_i$ whose image in Λ is λ_s. Increasing i if necessary we may also assume that $f_t(a_1, \ldots, a_n) = 0$ in A_i. Hence we can factor $A \to \Lambda$ through A_i by mapping x_s to a_s.

Conversely, suppose that (2) holds. Consider the category $\mathcal{I}$ whose objects are R-algebra maps $A \to \Lambda$ with $A \in \mathcal{E}$ and whose morphisms are commutative diagrams

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] & & A' \arrow[dl] \\ & \Lambda & \end{tikzcd}$$

of R-algebras. We claim that $\mathcal{I}$ is a filtered index category and that $\Lambda = colim_{\mathcal{I}} A$. To see that $\mathcal{I}$ is filtered, let $A \to \Lambda$ and $A' \to \Lambda$ be two objects. Then we can factor $A \otimes_R A' \to \Lambda$ through an object of $\mathcal{I}$ by assumption (2) and the fact that the elements of $\mathcal{E}$ are of finite presentation over R. Suppose that $\varphi, \psi : A \to A'$ are two morphisms of $\mathcal{I}$. Let $x_1, \ldots, x_n$ be generators of A as an R-algebra. By assumption (2) we can factor the R-algebra map $A'/(\varphi(x_i) - \psi(x_i)) \to \Lambda$ through an object of $\mathcal{I}$. This proves that $\mathcal{I}$ is filtered. We omit the proof that $\Lambda = colim_{\mathcal{I}} A$. $\square$

13.3. Singular ideals

Let $R \to A$ be a ring map. The singular ideal of A over R is the radical ideal in A cutting out the singular locus of the morphism $Spec(A) \to Spec(R)$. Here is a formal definition.

Definition 13.3.1. Let $R \to A$ be a ring map. The *singular ideal of A over R*, denoted $H_{A/R}$ is the unique radical ideal $H_{A/R} \subset A$ with

$$V(H_{A/R}) = \{\mathfrak{q} \in Spec(A) \mid R \to A \text{ not smooth at } \mathfrak{q}\}$$

This makes sense because the set of primes where $R \to A$ is smooth is open, see Algebra, Definition 7.126.11. In order to find an explicit set of generators for the singular ideal we first prove the following lemma.

Lemma 13.3.2. *Let R be a ring. Let $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. Let $\mathfrak{q} \subset A$. Assume $R \to A$ is smooth at $\mathfrak{q}$. Then there exists an $a \in A$, $a \not\in \mathfrak{q}$, an integer c, $0 \leq c \leq \min(n, m)$, subsets $U \subset \{1, \ldots, n\}$, $V \subset \{1, \ldots, m\}$ of cardinality c such that*

$$a = a' \det(\partial f_j/\partial x_i)_{j \in V, i \in U}$$

for some $a' \in A$ and

$$af_\ell \in (f_j, j \in V) + (f_1, \ldots, f_m)^2$$

for all $\ell \in \{1, \ldots, m\}$.

Proof. Set $I = (f_1, \ldots, f_m)$ so that the the naive cotangent complex of A over R is homotopy equivalent to $I/I^2 \to \bigoplus A\mathrm{d}x_i$, see Algebra, Lemma 7.123.2. We will use the formation of the naive cotangent complex commutes with localization, see Algebra, Section 7.123, especially Algebra, Lemma 7.123.10. By Algebra, Definitions 7.126.1 and 7.126.11 we see that $(I/I^2)_a \to \bigoplus A_a\mathrm{d}x_i$ is a split injection for some $a \in A$, $a \not\in \mathfrak{p}$. After renumbering $x_1, \ldots, x_n$ and $f_1, \ldots, f_m$ we may assume that $f_1, \ldots, f_c$ form a basis for the vector space $I/I^2 \otimes_A \kappa(\mathfrak{q})$ and that $\mathrm{d}x_{c+1}, \ldots, \mathrm{d}x_n$ map to a basis of $\Omega_{A/R} \otimes_A \kappa(\mathfrak{q})$. Hence after replacing a by aa' for some $a' \in A$, $a' \not\in \mathfrak{q}$ we may assume $f_1, \ldots, f_c$ form a basis for $(I/I^2)_a$ and that $\mathrm{d}x_{c+1}, \ldots, \mathrm{d}x_n$ map to a basis of $(\Omega_{A/R})_a$. In this situation a^N for some large integer N satisfies the conditions of the lemma (with $U = V = \{1, \ldots, c\}$). $\square$

We will use the notion of a *strictly standard* element in a A over R. Our notion is slightly weaker than the one in Swan's paper [**Swa98**]. We also define an *elementary standard* element to be one of the type we found in the lemma above. We compare the different types of elements in Lemma 13.4.7.

Definition 13.3.3. Let $R \to A$ be a ring map of finite presentation. We say an element $a \in A$ is *elementary standard in A over R* if there exists a presentation $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ and $0 \leq c \leq \min(n, m)$ such that

$$a = a' \det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c} \tag{13.3.3.1}$$

for some $a' \in A$ and

$$af_{c+j} \in (f_1, \ldots, f_c) + (f_1, \ldots, f_m)^2 \tag{13.3.3.2}$$

for $j = 1, \ldots, m - c$. We say $a \in A$ is *strictly standard in A over R* if there exists a presentation $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ and $0 \leq c \leq \min(n, m)$ such that

$$a = \sum\nolimits_{I \subset \{1,\ldots,n\},\ |I|=c} a_I \det(\partial f_j/\partial x_i)_{j=1,\ldots,c,\ i \in I} \tag{13.3.3.3}$$

for some $a_I \in A$ and

$$af_{c+j} \in (f_1, \ldots, f_c) + (f_1, \ldots, f_m)^2 \tag{13.3.3.4}$$

for $j = 1, \ldots, m - c$.

The following lemma is useful to find implications of (13.3.3.3).

Lemma 13.3.4. *Let R be a ring. Let $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ and write $I = (f_1, \ldots, f_n)$. Let $a \in A$. Then (13.3.3.3) implies there exists an A-linear map $\psi : \bigoplus_{i=1,\ldots,n} Adx_i \to A^{\oplus c}$ such that the composition*

$$A^{\oplus c} \xrightarrow{(f_1,\ldots,f_c)} I/I^2 \xrightarrow{f\mapsto df} \bigoplus\nolimits_{i=1,\ldots,n} Adx_i \xrightarrow{\psi} A^{\oplus c}$$

is multiplication by a. Conversely, if such a ψ exists, then a^c satisfies (13.3.3.3).

Proof. This is a special case of Algebra, Lemma 7.14.6. □

Lemma 13.3.5 (Elkik)**.** *Let $R \to A$ be a ring map of finite presentation. The singular ideal $H_{A/R}$ is the radical of the ideal generated by strictly standard elements in A over R and also the radical of the ideal generated by elementary standard elements in A over R.*

Proof. Assume a is strictly standard in A over R. We claim that A_a is smooth over R, which proves that $a \in H_{A/R}$. Namely, let $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$, c, and $a' \in A$ be as in Definition 13.3.3. Write $I = (f_1, \ldots, f_m)$ so that the naive cotangent complex of A over R is given by $I/I^2 \to \bigoplus A\mathrm{d}x_i$. Assumption (13.3.3.4) implies that $(I/I^2)_a$ is generated by the classes of $f_1, \ldots, f_c$. Assumption (13.3.3.3) implies that the differential $(I/I^2)_a \to \bigoplus A_a\mathrm{d}x_i$ has a left inverse, see Lemma 13.3.4. Hence $R \to A_a$ is smooth by definition and Algebra, Lemma 7.123.10.

Let $H_e, H_s \subset A$ be the radical of the ideal generated by elementary, resp. strictly standard elements of A over R. By definition and what we just proved we have $H_e \subset H_s \subset H_{A/R}$. The inclusion $H_{A/R} \subset H_e$ follows from Lemma 13.3.2. □

Example 13.3.6. The set of points where a finitely presented ring map is smooth needn't be a quasi-compact open. For example, let $R = k[x, y_1, y_2, y_3, \ldots]/(xy_i)$ and $A = R/(x)$. Then the smooth locus of $R \to A$ is $\bigcup D(y_i)$ which is not quasi-compact.

Lemma 13.3.7. *Let $R \to A$ be a ring map of finite presentation. Let $R \to R'$ be a ring map. If $a \in A$ is elementary, resp. strictly standard in A over R, then $a \otimes 1$ is elementary, resp. strictly standard in $A \otimes_R R'$ over R'.*

Proof. If $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ is a presentation of A over R, then $A \otimes_R R' = R'[x_1, \ldots, x_n]/(f'_1, \ldots, f'_m)$ is a presentation of $A \otimes_R R'$ over R'. Here f'_j is the image of f_j in $R'[x_1, \ldots, x_n]$. Hence the result follows from the definitions. □

Lemma 13.3.8. *Let $R \to A \to \Lambda$ be ring maps with A of finite presentation over R. Assume that $H_{A/R}\Lambda = \Lambda$. Then there exists a factorization $A \to B \to \Lambda$ with B smooth over R.*

Proof. Choose $f_1, \ldots, f_r \in H_{A/R}$ and $\lambda_1, \ldots, \lambda_r \in \Lambda$ such that $\sum f_i\lambda_i = 1$ in Λ. Set $B = A[x_1, \ldots, x_r]/(f_1x_1 + \ldots + f_rx_r - 1)$ and define $B \to \Lambda$ by mapping x_i to λ_i. Details omitted. □

13.4. Presentations of algebras

Some of the results in this section are due to Elkik. Note that the algebra C in the following lemma is a symmetric algebra over A. Moreover, if R is Noetherian, then C is of finite presentation over R.

Lemma 13.4.1. *Let R be a ring and let A be a finitely presented R-algebra. There exists finite type R-algebra map $A \to C$ which has a retraction with the following two properties*

(1) *for each $a \in A$ such that A_a is syntomic*[2] *over R the ring C_a is smooth over A_a and has a presentation $C_a = R[y_1, \ldots, y_m]/J$ such that J/J^2 is free over C_a, and*
(2) *for each $a \in A$ such that A_a is smooth over R the module $\Omega_{C_a/R}$ is free over C_a.*

Proof. Choose a presentation $A = R[x_1, \ldots, x_n]/I$ and write $I = (f_1, \ldots, f_m)$. Define the A-module K by the short exact sequence

$$0 \to K \to A^{\oplus m} \to I/I^2 \to 0$$

where the jth basis vector e_j in the middle is mapped to the class of f_j on the right. Set

$$C = \mathrm{Sym}^*_A(I/I^2).$$

The retraction is just the projection onto the degree 0 part of C. We have a surjection $R[x_1, \ldots, x_n, y_1, \ldots, y_m] \to C$ which maps y_j to the class of f_j in I/I^2. The kernel J of this map is generated by the elements $f_1, \ldots, f_m$ and by elements $\sum h_j y_j$ with $h_j \in R[x_1, \ldots, x_n]$ such that $\sum h_j e_j$ defines an element of K. By Algebra, Lemma 7.123.3 applied to $R \to A \to C$ and the presentations above and More on Algebra, Lemma 12.9.10 there is a short exact sequence

$$I/I^2 \otimes_A C \to J/J^2 \to K \otimes_A C \to 0 \tag{13.4.1.1}$$

of C-modules. Let $h \in R[x_1, \ldots, x_n]$ be an element with image $a \in A$. We will use as presentations for the localized rings

$$A_a = R[x_0, x_1, \ldots, x_n]/I' \quad \text{and} \quad C_a = R[x_0, x_1, \ldots, x_n, y_1, \ldots, y_m]/J'$$

where $I' = (hx_0 - 1, I)$ and $J' = (hx_0 - 1, J)$. Hence $I'/(I')^2 = C_a \oplus I/I^2 \otimes_A C_a$ and $J'/(J')^2 = C_a \oplus (J/J^2)_a$ as C_a-modules. Thus we obtain

$$C_a \oplus I/I^2 \otimes_A C_a \to C_a \oplus (J/J^2)_a \to K \otimes_A C_a \to 0 \tag{13.4.1.2}$$

as the sequence of Algebra, Lemma 7.123.3 corresponding to $R \to A_a \to C_a$ and the presentations above.

Next, assume that $a \in A$ is such that A_a is syntomic over R. Then $(I/I^2)_a$ is finite projective over A_a, see Algebra, Lemma 7.125.17. Hence we see $K_a \oplus (I/I^2)_a \cong A_a^{\oplus m}$ is free. In particular K_a is finite projective too. By More on Algebra, Lemma 12.24.6 the sequence (13.4.1.2) is exact on the left. Hence

$$J'/(J')^2 \cong C_a \oplus I/I^2 \otimes_A C_a \oplus K \otimes_A C_a \cong C_a^{\oplus m+1}$$

This proves (1). Finally, suppose that in addition A_a is smooth over R. Then the same presentation shows that $\Omega_{C_a/R}$ is the cokernel of the map

$$J'/(J')^2 \longrightarrow \bigoplus\nolimits_i C_a \mathrm{d}x_i \oplus \bigoplus\nolimits_j C_a \mathrm{d}y_j$$

The summand C_a of $J'/(J')^2$ in the decomposition above corresponds to $hx_0 - 1$ and hence maps isomorphically to the summand $C_a\mathrm{d}x_0$. The summand $I/I^2 \otimes_A C_a$ of $J'/(J')^2$ maps injectively to $\bigoplus_{i=1,\ldots,n} C_a \mathrm{d}x_i$ with quotient $\Omega_{A_a/R} \otimes_{A_a} C_a$. The summand $K \otimes_A C_a$ maps injectively to $\bigoplus_{j \geq 1} C_a \mathrm{d}y_j$ with quotient isomorphic to $I/I^2 \otimes_A C_a$. Thus the cokernel of the last displayed map is the module $I/I^2 \otimes_A C_a \oplus \Omega_{A_a/R} \otimes_{A_a} C_a$. Since $(I/I^2)_a \oplus \Omega_{A_a/R}$ is free (from the definition of smooth ring maps) we see that (2) holds. □

[2] Or just that $R \to A_a$ is a local complete intersection, see More on Algebra, Definition 12.24.2.

The following proposition was proved for henselian pairs by Elkik in [**Elk73**]. In the form stated below it can be found in [**Ara01**], where they also prove that ring maps between smooth algebras can be lifted.

Proposition 13.4.2. *Let R be a ring and let $I \subset R$ be an ideal. Let $R/I \to \overline{A}$ be a smooth ring map. Then there exists a smooth ring map $R \to A$ such that A/IA is isomorphic to $\overline{A}$.*

Proof. Choose a presentation $\overline{A} = (R/I)[x_1, \ldots, x_n]/\overline{J}$. Set $\overline{C} = \text{Sym}^*_{\overline{A}}(\overline{J}/\overline{J}^2)$. Note that $\overline{J}/\overline{J}^2$ is a finite projective $\overline{A}$-module (follows from the definition of smoothness). By Lemma 13.4.1 and its proof the ring map $\overline{A} \to \overline{C}$ is smooth and we can find a presentation $\overline{C} = R/I[y_1, \ldots, y_m]/\overline{K}$ with $\overline{K}/\overline{K}^2$ free over $\overline{C}$. By Algebra, Lemma 7.125.6 we can even assume that $\overline{C} = R/I[y_1, \ldots, y_m]/(\overline{f}_1, \ldots, \overline{f}_c)$ where $\overline{f}_1, \ldots, \overline{f}_c$ maps to a basis of $\overline{K}/\overline{K}^2$ over $\overline{C}$. Choose $f_1, \ldots, f_c \in R[y_1, \ldots, y_c]$ lifting $\overline{f}_1, \ldots, \overline{f}_c$ and set

$$C = R[y_1, \ldots, y_m]/(f_1, \ldots, f_c)$$

By construction $C/IC = \overline{C}$. Consider the naive cotangent complex

$$(f_1, \ldots, f_c)/(f_1, \ldots, f_c)^2 \longrightarrow \bigoplus\nolimits_{i=1,\ldots,m} C\mathrm{d}y_i$$

associated to the presentation of C. For every prime $\mathfrak{q} \supset IC$ of C the images $\mathrm{d}f_j$ are linearly independent in $\bigoplus \kappa(\mathfrak{q})\mathrm{d}y_i$ because $\overline{C}$ is smooth over R/I. Hence we conclude that $((f_1, \ldots, f_c)/(f_1, \ldots, f_c)^2)_{\mathfrak{q}}$ is free of rank c and maps to a direct summand of $\bigoplus C_{\mathfrak{q}}\mathrm{d}y_j$. Hence $R \to C$ is smooth at $\mathfrak{q}$, see Algebra, Lemma 7.126.12. Thus we can find a $g \in C$ mapping to an invertible element of C/IC such that $R \to C_g$ is smooth, see More on Algebra, Lemma 12.9.4. We conclude that there exists a finite projective $\overline{A}$-module $\overline{P}$ such that $\overline{C} = \text{Sym}^*_{\overline{A}}(\overline{P})$ is isomorphic to C/IC for some smooth R-algebra C.

Choose an integer n and a direct sum decomposition $\overline{A}^{\oplus n} = \overline{P} \oplus \overline{Q}$. By More on Algebra, Lemma 12.9.9 we can find an étale ring map $C \to C'$ which induces an isomorphism $C/IC \to C'/IC'$ and a finite projective C'-module Q such that Q/IQ is isomorphic to $\overline{Q} \otimes_{\overline{A}} C/IC$. Then $D = \text{Sym}^*_{C'}(Q)$ is a smooth C'-algebra (see More on Algebra, Lemma 12.9.11). Picture

$$\begin{tikzcd}
R \arrow[rr] \arrow[d] & & C \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & D \arrow[d] \\
R/I \arrow[r] & \overline{A} \arrow[r] & C/IC \arrow[r, "\cong"] & C'/IC' \arrow[r] & D/ID
\end{tikzcd}$$

Observe that our choice of Q gives

$$\begin{aligned}
D/ID &= \text{Sym}^*_{C/IC}(\overline{Q} \otimes_{\overline{A}} C/IC) \\
&= \text{Sym}^*_{\overline{A}}(\overline{Q}) \otimes_{\overline{A}} C/IC \\
&= \text{Sym}^*_{\overline{A}}(\overline{Q}) \otimes_{\overline{A}} \text{Sym}^*_{\overline{A}}(\overline{P}) \\
&= \text{Sym}^*_{\overline{A}}(\overline{Q} \oplus \overline{P}) \\
&= \text{Sym}^*_{\overline{A}}(\overline{A}^{\oplus n}) \\
&= \overline{A}[x_1, \ldots, x_n]
\end{aligned}$$

Choose $f_1, \ldots, f_n \in D$ which map to $x_1, \ldots, x_n$ in $D/ID = \overline{A}[x_1, \ldots, x_n]$. Set $A = D/(f_1, \ldots, f_n)$. Note that $\overline{A} = A/IA$. By an argument similar to the argument in the first paragraph of the proof we see that $R \to A$ is smooth at all primes of IA. Hence, after replacing A by A_f for a suitable $f \in A$ (see More on Algebra, Lemma 12.9.4) we win. $\square$

We know that any syntomic ring map $R \to A$ is locally a relative global complete intersection, see Algebra, Lemma 7.125.16. The next lemma says that a vector bundle over $Spec(A)$ is a relative global complete intersection.

Lemma 13.4.3. *Let $R \to A$ be a syntomic ring map. Then there exists a smooth R-algebra map $A \to C$ with a retraction such that C is a global relative complete intersection over R, i.e.,*

$$C \cong R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$$

flat over R and all fibres of dimension $n - c$.

Proof. Apply Lemma 13.4.1 to get $A \to C$. By Algebra, Lemma 7.125.6 we can write $C = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ with f_i mapping to a basis of J/J^2. The ring map $R \to C$ is syntomic (hence flat) as it is a composition of a syntomic and a smooth ring map. The dimension of the fibres is $n - c$ by Algebra, Lemma 7.124.4 (the fibres are local complete intersections, so the lemma applies). $\square$

Lemma 13.4.4. *Let $R \to A$ be a smooth ring map. Then there exists a smooth R-algebra map $A \to B$ with a retraction such that B is standard smooth over R, i.e.,*

$$B \cong R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$$

and $\det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c}$ is invertible in B.

Proof. Apply Lemma 13.4.3 to get a smooth R-algebra map $A \to C$ with a retraction such that $C = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a relative global complete intersection over R. As C is smooth over R we have a short exact sequence

$$0 \to \bigoplus\nolimits_{j=1,\ldots,c} Cf_j \to \bigoplus\nolimits_{i=1,\ldots,n} C\mathrm{d}x_i \to \Omega_{C/R} \to 0$$

Since $\Omega_{C/R}$ is a projective C-module this sequence is split. Choose a left inverse t to the first map. Say $t(\mathrm{d}x_i) = \sum c_{ij} f_j$ so that $\sum_i \frac{\partial f_j}{\partial x_i} c_{i\ell} = \delta_{j\ell}$ (Kronecker delta). Let

$$B' = C[y_1, \ldots, y_c] = R[x_1, \ldots, x_n, y_1, \ldots, y_c]/(f_1, \ldots, f_c)$$

The R-algebra map $C \to B'$ has a retraction given by mapping y_j to zero. We claim that the map

$$R[z_1, \ldots, z_n] \longrightarrow B', \quad z_i \longmapsto x_i - \sum\nolimits_j c_{ij} y_j$$

is étale at every point in the image of $Spec(C) \to Spec(B')$. In $\Omega_{B'/R[z_1,\ldots,z_n]}$ we have

$$0 = \mathrm{d}f_j - \sum\nolimits_i \frac{\partial f_j}{\partial x_i}\mathrm{d}z_i \equiv \sum\nolimits_{i,\ell} \frac{\partial f_j}{\partial x_i} c_{i\ell}\mathrm{d}y_\ell \equiv \mathrm{d}y_j \bmod (y_1, \ldots, y_c)\Omega_{B'/R[z_1,\ldots,z_n]}$$

Since $0 = \mathrm{d}z_i = \mathrm{d}x_i$ modulo $\sum B'\mathrm{d}y_j + (y_1, \ldots, y_c)\Omega_{B'/R[z_1,\ldots,z_n]}$ we conclude that

$$\Omega_{B'/R[z_1,\ldots,z_n]}/(y_1, \ldots, y_c)\Omega_{B'/R[z_1,\ldots,z_n]} = 0.$$

As $\Omega_{B'/R[z_1,\ldots,z_n]}$ is a finite B'-module by Nakayama's lemma there exists a $g \in 1 + (y_1, \ldots, y_c)$ that $(\Omega_{B'/R[z_1,\ldots,z_n]})_g = 0$. This proves that $R[z_1, \ldots, z_n] \to B'_g$ is unramified, see Algebra, Definition 7.138.1. For any ring map $R \to k$ where k is a field we obtain an unramified ring map $k[z_1, \ldots, z_n] \to (B'_g) \otimes_R k$ between smooth k-algebras of dimension n. It follows that $k[z_1, \ldots, z_n] \to (B'_g) \otimes_R k$ is flat by Algebra, Lemmas 7.119.1 and 7.129.2. By the critère de platitude par fibre (Algebra, Lemma 7.119.8) we conclude that $R[z_1, \ldots, z_n] \to B'_g$ is flat. Finally, Algebra, Lemma 7.132.7 implies that $R[z_1, \ldots, z_n] \to B'_g$ is étale. Set $B = B'_g$. Note that $C \to B$ is smooth and has a retraction, so also $A \to B$ is smooth and has a retraction. Moreover, $R[z_1, \ldots, z_n] \to B$ is étale. By Algebra, Lemma 7.132.2 we can write

$$B = R[z_1, \ldots, z_n, w_1, \ldots, w_c]/(g_1, \ldots, g_c)$$

with $\det(\partial g_j/\partial w_i)$ invertible in B. This proves the lemma. ☐

Lemma 13.4.5. *Let $R \to \Lambda$ be a ring map. If Λ is a filtered colimit of smooth R-algebras, then Λ is a filtered colimit of standard smooth R-algebras.*

Proof. Let $A \to \Lambda$ be an R-algebra map with A of finite presentation over R. According to Lemma 13.2.1 we have to factor this map through a standard smooth algebra, and we know we can factor it as $A \to B \to \Lambda$ with B smooth over R. Choose an R-algebra map $B \to C$ with a retraction $C \to B$ such that C is standard smooth over R, see Lemma 13.4.4. Then the desired factorization is $A \to B \to C \to B \to \Lambda$. ☐

Lemma 13.4.6. *Let $R \to A$ be a standard smooth ring map. Let $E \subset A$ be a finite subset of order $|E| = n$. Then there exists a presentation $A = R[x_1, \ldots, x_{n+m}]/(f_1, \ldots, f_c)$ with $c \geq n$, with $\det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c}$ invertible in A, and such that E is the set of congruence classes of $x_1, \ldots, x_n$.*

Proof. Choose a presentation $A = R[y_1, \ldots, y_m]/(g_1, \ldots, g_d)$ such that the image of $\det(\partial g_j/\partial y_i)_{i,j=1,\ldots,d}$ is invertible in A. Choose an enumerations $E = \{a_1, \ldots, a_n\}$ and choose $h_i \in R[y_1, \ldots, y_m]$ whose image in A is a_i. Consider the presentation

$$A = R[x_1, \ldots, x_n, y_1, \ldots, y_m]/(x_1 - h_1, \ldots, x_n - h_n, g_1, \ldots, g_d)$$

and set $c = n + d$. ☐

Lemma 13.4.7. *Let $R \to A$ be a ring map of finite presentation. Let $a \in A$. Consider the following conditions on a:*

(1) *A_a is smooth over R,*
(2) *A_a is smooth over R and $\Omega_{A_a/R}$ is stably free,*
(3) *A_a is smooth over R and $\Omega_{A_a/R}$ is free,*
(4) *A_a is standard smooth over R,*
(5) *a is strictly standard in A over R,*
(6) *a is elementary standard in A over R.*

Then we have

(a) *(4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1),*
(b) *(6) $\Rightarrow$ (5),*
(c) *(6) $\Rightarrow$ (4),*
(d) *(5) $\Rightarrow$ (2),*
(e) *(2) $\Rightarrow$ the elements a^e, $e \geq e_0$ are strictly standard in A over R,*
(f) *(4) $\Rightarrow$ the elements a^e, $e \geq e_0$ are elementary standard in A over R.*

Proof. Part (a) is clear from the definitions and Algebra, Lemma 7.126.7. Part (b) is clear from Definition 13.3.3.

Proof of (c). Choose a presentation $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ such that (13.3.3.1) and (13.3.3.2) hold. Choose $h \in R[x_1, \ldots, x_n]$ mapping to a. Then

$$A_a = R[x_0, x_1, \ldots, x_n]/(x_0h - 1, f_1, \ldots, f_n).$$

Write $J = (x_0h - 1, f_1, \ldots, f_n)$. By (13.3.3.2) we see that the A_a-module J/J^2 is generated by $x_0h - 1, f_1, \ldots, f_c$ over A_a. Hence, as in the proof of Algebra, Lemma 7.125.6, we can choose a $g \in 1 + J$ such that

$$A_a = R[x_0, \ldots, x_n, x_{n+1}]/(x_0h - 1, f_1, \ldots, f_n, gx_{n+1} - 1).$$

At this point (13.3.3.1) implies that $R \to A_a$ is standard smooth (use the coordinates $x_0, x_1, \ldots, x_c, x_{n+1}$ to take derivatives).

Proof of (d). Choose a presentation $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ such that (13.3.3.3) and (13.3.3.4) hold. We already know that A_a is smooth over R, see Lemma 13.3.5. As above we get a presentation $A_a = R[x_0, x_1, \ldots, x_n]/J$ with J/J^2 free. Then $\Omega_{A_a/R} \oplus J/J^2 \cong A_a^{\oplus n+1}$ by the definition of smooth ring maps, hence we see that $\Omega_{A_a/R}$ is stably free.

Proof of (e). Choose a presentation $A = R[x_1, \ldots, x_n]/I$ with I finitely generated. By assumption we have a short exact sequence

$$0 \to (I/I^2)_a \to \bigoplus\nolimits_{i=1,\ldots,n} A_a \mathrm{d}x_i \to \Omega_{A_a/R} \to 0$$

which is split exact. Hence we see that $(I/I^2)_a \oplus \Omega_{A_a/R}$ is a free A_a-module. Since $\Omega_{A_a/R}$ is stably free we see that $(I/I^2)_a$ is stably free as well. Thus replacing the presentation chosen above by $A = R[x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+r}]/J$ with $J = (I, x_{n+1}, \ldots, x_{n+r})$ for some r we get that $(J/J^2)_a$ is (finite) free. Choose $f_1, \ldots, f_c \in J$ which map to a basis of $(J/J^2)_a$. Extend this to a list of generators $f_1, \ldots, f_m \in J$. Consider the presentation $A = R[x_1, \ldots, x_{n+r}]/(f_1, \ldots, f_m)$. Then (13.3.3.4) holds for a^e for all sufficiently large e by construction. Moreover, since $(J/J^2)_a \to \bigoplus_{i=1,\ldots,n} A_a \mathrm{d}x_i$ is a split injection we can find an A_a-linear left inverse. Writing this left inverse in terms of the basis $f_1, \ldots, f_c$ and clearing denominators we find a linear map $\psi_0 : A^{\oplus n} \to A^{\oplus c}$ such that

$$A^{\oplus c} \xrightarrow{(f_1,\ldots,f_c)} J/J^2 \xrightarrow{f \mapsto \mathrm{d}f} \bigoplus\nolimits_{i=1,\ldots,n} A\mathrm{d}x_i \xrightarrow{\psi_0} A^{\oplus c}$$

is multiplication by a^{e_0} for some $e_0 \geq 1$. By Lemma 13.3.4 we see (13.3.3.3) holds for all a^{ce_0} and hence for a^e for all e with $e \geq ce_0$.

Proof of (f). Choose a presentation $A_a = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ such that $\det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c}$ is invertible in A_a. We may assume that for some $m < n$ the classes of the elements $x_1, \ldots, x_m$ correspond $a_i/1$ where $a_1, \ldots, a_m \in A$ are generators of A over R, see Lemma 13.4.6. After replacing x_i by $a^N x_i$ for $m < i \leq n$ we may assume the class of x_i is $a_i/1 \in A_a$ for some $a_i \in A$. Consider the ring map

$$\Psi : R[x_1, \ldots, x_n] \longrightarrow A, \quad x_i \longmapsto a_i.$$

This is a surjective ring map. By replacing f_j by $a^N f_j$ we may assume that $f_j \in R[x_1, \ldots, x_n]$ and that $\Psi(f_j) = 0$ (since after all $f_j(a_1/1, \ldots, a_n/1) = 0$ in A_a). Let $J = \mathrm{Ker}(\Psi)$. Then $A = R[x_1, \ldots, x_n]/J$ is a presentation and $f_1, \ldots, f_c \in J$ are elements such that $(J/J^2)_a$ is freely generated by $f_1, \ldots, f_c$ and such that $\det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c}$ maps to an invertible element of A_a. It follows that (13.3.3.1) and (13.3.3.2) hold for a^e and all large enough e as desired. $\square$

13.5. The lifting problem

The goal in this section is to prove (Proposition 13.5.3) that the collection of algebras which are filtered colimits of smooth algebras is closed under infinitesimal flat deformations. The proof is elementary and only uses the results on presentations of smooth algebras from Section 13.4.

Lemma 13.5.1. *Let $R \to \Lambda$ be a ring map. Let $I \subset R$ be an ideal. Assume that*

(1) $I^2 = 0$*, and*
(2) $\Lambda/I\Lambda$ *is a filtered colimit of smooth* R/I*-algebras.*

Let $\varphi : A \to \Lambda$ be an R-algebra map with A of finite presentation over R. Then there exists a factorization

$$A \to B/J \to \Lambda$$

where B is a smooth R-algebra and $J \subset IB$ is a finitely generated ideal.

Proof. Choose a factorization

$$A/IA \to \bar{B} \to \Lambda/I\Lambda$$

with $\bar{B}$ standard smooth over R/I; this is possible by assumption and Lemma 13.4.5. Write

$$\bar{B} = A/IA[t_1, \ldots, t_r]/(\bar{g}_1, \ldots, \bar{g}_s)$$

and say $\bar{B} \to \Lambda/I\Lambda$ maps t_i to the class of λ_i modulo $I\Lambda$. Choose $g_1, \ldots, g_s \in A[t_1, \ldots, t_r]$ lifting $\bar{g}_1, \ldots, \bar{g}_s$. Write $\varphi(g_i)(\lambda_1, \ldots, \lambda_r) = \sum \epsilon_{ij}\mu_{ij}$ for some $\epsilon_{ij} \in I$ and $\mu_{ij} \in \Lambda$. Define

$$A' = A[t_1, \ldots, t_r, \delta_{i,j}]/(g_i - \sum \epsilon_{ij}\delta_{ij})$$

and consider the map

$$A' \longrightarrow \Lambda, \quad a \longmapsto \varphi(a), \quad t_i \longmapsto \lambda_i, \quad \delta_{ij} \longmapsto \mu_{ij}$$

We have

$$A'/IA' = A/IA[t_1, \ldots, t_r]/(\bar{g}_1, \ldots, \bar{g}_s)[\delta_{ij}] \cong \bar{B}[\delta_{ij}]$$

This is a standard smooth algebra over R/I as $\bar{B}$ is standard smooth. Choose a presentation $A'/IA' = R/I[x_1, \ldots, x_n]/(\bar{f}_1, \ldots, \bar{f}_c)$ with $\det(\partial \bar{f}_j/\partial x_i)_{i,j=1,\ldots,c}$ invertible in A'/IA'. Choose lifts $f_1, \ldots, f_c \in R[x_1, \ldots, x_n]$ of $\bar{f}_1, \ldots, \bar{f}_c$. Then

$$B = R[x_1, \ldots, x_n, x_{n+1}]/(f_1, \ldots, f_c, x_{n+1} \det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c} - 1)$$

is smooth over R. Since smooth ring maps are formally smooth (Algebra, Proposition 7.127.13) there exists an R-algebra map $B \to A'$ which is an isomorphism modulo I. Then $B \to A'$ is surjective by Nakayama's lemma (Algebra, Lemma 7.14.5). Thus $A' = B/J$ with $J \subset IB$ finitely generated (see Algebra, Lemma 7.6.3). □

Lemma 13.5.2. *Let $R \to \Lambda$ be a ring map. Let $I \subset R$ be an ideal. Assume that*

(1) $I^2 = 0$*,*
(2) $\Lambda/I\Lambda$ *is a filtered colimit of smooth* R/I*-algebras, and*
(3) $R \to \Lambda$ *is flat.*

Let $\varphi : B \to \Lambda$ be an R-algebra map with B smooth over R. Let $J \subset IB$ be a finitely generated ideal. Then there exists R-algebra maps

$$B \xrightarrow{\alpha} B' \xrightarrow{\beta} \Lambda$$

such that B' is smooth over R, such that $\alpha(J) = 0$ and such that $\beta \circ \alpha = \varphi \bmod I\Lambda$.

Proof. If we can prove the lemma in case $J = (h)$, then we can prove the lemma by induction on the number of generators of J. Namely, suppose that J can be generated by n elements $h_1, \ldots, h_n$ and the lemma holds for all cases where J is generated by $n-1$ elements. Then we apply the case $n = 1$ to produce $B \to B' \to \Lambda$ where the first map kills of h_n. Then we let J' be the ideal of B' generated by the images of $h_1, \ldots, h_{n-1}$ and we apply the case for $n-1$ to produce $B' \to B'' \to \Lambda$. It is easy to verify that $B \to B'' \to \Lambda$ does the job.

Assume $J = (h)$ and write $h = \sum \epsilon_i b_i$ for some $\epsilon_i \in I$ and $b_i \in B$. Note that $0 = \varphi(h) = \sum \epsilon_i \varphi(b_i)$. As Λ is flat over R, the equational criterion for flatness (Algebra, Lemma 7.35.10) implies that we can find $\lambda_j \in \Lambda$, $j = 1, \ldots, m$ and $a_{ij} \in R$ such that $\varphi(b_i) = \sum_j a_{ij}\lambda_j$ and $\sum_i \epsilon_i a_{ij} = 0$. Set

$$C = B[x_1, \ldots, x_m]/(b_i - \sum a_{ij}x_j)$$

with $C \to \Lambda$ given by φ and $x_j \mapsto \lambda_j$. Choose a factorization

$$C \to B'/J' \to \Lambda$$

as in Lemma 13.5.1. Since B is smooth over R we can lift the map $B \to C \to B'/J'$ to a map $\psi : B \to B'$. We claim that $\psi(h) = 0$. Namely, the fact that ψ agrees with $B \to C \to B'/J'$ mod I implies that

$$\psi(b_i) = \sum a_{ij}\xi_j + \theta_i$$

for some $\xi_i \in B'$ and $\theta_i \in IB'$. Hence we see that

$$\psi(h) = \psi(\sum \epsilon_i b_i) = \sum \epsilon_i a_{ij}\xi_j + \sum \epsilon_i\theta_i = 0$$

because of the relations above and the fact that $I^2 = 0$. □

Proposition 13.5.3. *Let $R \to \Lambda$ be a ring map. Let $I \subset R$ be an ideal. Assume that*

(1) *I is nilpotent,*
(2) *$\Lambda/I\Lambda$ is a filtered colimit of smooth R/I-algebras, and*
(3) *$R \to \Lambda$ is flat.*

Then Λ is a colimit of smooth R-algebras.

Proof. Since $I^n = 0$ for some n, it follows by induction on n that it suffices to consider the case where $I^2 = 0$. Let $\varphi : A \to \Lambda$ be an R-algebra map with A of finite presentation over R. We have to find a factorization $A \to B \to \Lambda$ with B smooth over R, see Lemma 13.2.1. By Lemma 13.5.1 we may assume that $A = B/J$ with B smooth over R and $J \subset IB$ a finitely generated ideal. By Lemma 13.5.2 we can find a (possibly noncommutative) diagram

$$\begin{tikzcd} B \arrow[rr, "\alpha"] \arrow[dr, "\varphi"'] & & B' \arrow[dl, "\beta"] \\ & \Lambda & \end{tikzcd}$$

of R-algebras which commutes modulo I and such that $\alpha(J) = 0$. The map

$$D : B \longrightarrow I\Lambda, \quad b \longmapsto \varphi(b) - \beta(\alpha(b))$$

is a derivation over R hence we can write it as $D = \xi \circ \mathrm{d}_{B/R}$ for some B-linear map $\xi : \Omega_{B/R} \to I\Lambda$. Since $\Omega_{B/R}$ is a finite projective B-module we can write $\xi = \sum_{i=1,\ldots,n} \epsilon_i \Xi_i$ for some $\epsilon_i \in I$ and B-linear maps $\Xi_i : \Omega_{B/R} \to \Lambda$. (Details omitted. Hint: write $\Omega_{B/R}$ as a direct sum of a finite free module to reduce to the finite free case.) We define

$$B'' = \mathrm{Sym}^*_{B'}\left(\bigoplus\nolimits_{i=1,\ldots,n} \Omega_{B/R} \otimes_{B,\alpha} B'\right)$$

and we define $\beta' : B'' \to \Lambda$ by β on B' and by

$$\beta'|_{i\text{th summand } \Omega_{B/R} \otimes_{B,\alpha} B'} = \Xi_i \otimes \beta$$

and $\alpha' : B \to B''$ by

$$\alpha'(b) = \alpha(b) \oplus \sum \epsilon_i \mathrm{d}_{B/R}(b) \otimes 1 \oplus 0 \oplus \ldots$$

At this point the diagram

$$\begin{tikzcd} B \arrow[rr, "\alpha'"] \arrow[dr, "\varphi"'] & & B'' \arrow[dl, "\beta'"] \\ & \Lambda & \end{tikzcd}$$

does commute. Moreover, it is direct from the definitions that $\alpha'(J) = 0$ as $I^2 = 0$. Hence the desired factorization. $\square$

13.6. The lifting lemma

Here is a fiendishly clever lemma.

Lemma 13.6.1. *Let R be a Noetherian ring. Let Λ be an R-algebra. Let $\pi \in R$ and assume that $Ann_R(\pi) = Ann_R(\pi^2)$ and $Ann_\Lambda(\pi) = Ann_\Lambda(\pi^2)$. Suppose we have R-algebra maps $R/\pi^2R \to \bar{C} \to \Lambda/\pi^2\Lambda$ with $\bar{C}$ of finite presentation. Then there exists an R-algebra homomorphism $D \to \Lambda$ and a commutative diagram*

$$\begin{tikzcd} R/\pi^2R \arrow[r] \arrow[d] & \bar{C} \arrow[r] \arrow[d] & \Lambda/\pi^2\Lambda \arrow[d] \\ R/\pi R \arrow[r] & D/\pi D \arrow[r] & \Lambda/\pi\Lambda \end{tikzcd}$$

with the following properties

(a) *D is of finite presentation,*
(b) *$R \to D$ is smooth at any prime $\mathfrak{q}$ with $\pi \notin \mathfrak{q}$,*
(c) *$R \to D$ is smooth at any prime $\mathfrak{q}$ with $\pi \in \mathfrak{q}$ lying over a prime of $\bar{C}$ where $R/\pi^2R \to \bar{C}$ is smooth, and*
(d) *$\bar{C}/\pi\bar{C} \to D/\pi D$ is smooth at any prime lying over a prime of $\bar{C}$ where $R/\pi^2R \to \bar{C}$ is smooth.*

Proof. We choose a presentation

$$\bar{C} = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$$

We also denote $I = (f_1, \ldots, f_m)$ and $\bar{I}$ the image of I in $R/\pi^2R[x_1, \ldots, x_n]$. Since R is Noetherian, so is $\bar{C}$. Hence the smooth locus of $R/\pi^2R \to \bar{C}$ is quasi-compact, see Topology, Lemma 5.6.2. Applying Lemma 13.3.2 we may choose a finite list of elements $a_1, \ldots, a_r \in R[x_1, \ldots, x_n]$ such that

(1) the union of the open subspaces $Spec(\bar{C}_{a_k}) \subset Spec(\bar{C})$ cover the smooth locus of $R/\pi^2R \to \bar{C}$, and

(2) for each $k = 1, \ldots, r$ there exists a finite subset $E_k \subset \{1, \ldots, m\}$ such that $(\bar{I}/\bar{I}^2)_{a_k}$ is freely generated by the classes of f_j, $j \in E_k$.

Set $I_k = (f_j, j \in E_k) \subset I$ and denote $\bar{I}_k$ the image of I_k in $R/\pi^2 R[x_1, \ldots, x_n]$. By (2) and Nakayama's lemma we see that $(\bar{I}/\bar{I}_k)_{a_k}$ is annihilated by $1 + b'_k$ for some $b'_k \in \bar{I}_{a_k}$. Suppose b'_k is the image of $b_k/(a_k)^N$ for some $b_k \in I$ and some integer N. After replacing a_k by $a_k b_k$ we get

(3) $(\bar{I}_k)_{a_k} = (\bar{I})_{a_k}$.

Thus, after possibly replacing a_k by a high power, we may write

(4) $a_k f_\ell = \sum_{j \in E_k} h^j_{k,\ell} f_j + \pi^2 g_{k,\ell}$

for any $\ell \in \{1, \ldots, m\}$ and some $h^j_{i,\ell}, g_{i,\ell} \in R[x_1, \ldots, x_n]$. If $\ell \in E_k$ we choose $h^j_{k,\ell} = a_k \delta_{\ell,j}$ (Kronecker delta) and $g_{k,\ell} = 0$. Set

$$D = R[x_1, \ldots, x_n, z_1, \ldots, z_m]/(f_j - \pi z_j, p_{k,\ell}).$$

Here $j \in \{1, \ldots, m\}$, $k \in \{1, \ldots, r\}$, $\ell \in \{1, \ldots, m\}$, and

$$p_{k,\ell} = a_k z_\ell - \sum\nolimits_{j \in E_k} h^j_{k,\ell} z_j - \pi g_{k,\ell}.$$

Note that for $\ell \in E_k$ we have $p_{k,\ell} = 0$ by our choices above.

The map $R \to D$ is the given one. Say $\bar{C} \to \Lambda/\pi^2\Lambda$ maps x_i to the class of λ_i modulo π^2. For an element $f \in R[x_1, \ldots, x_n]$ we denote $f(\lambda) \in \Lambda$ the result of subsituting λ_i for x_i. Then we know that $f_j(\lambda) = \pi^2 \mu_j$ for some $\mu_j \in \Lambda$. Define $D \to \Lambda$ by the rules $x_i \mapsto \lambda_i$ and $z_j \mapsto \pi \mu_j$. This is well defined because

$$\begin{aligned} p_{k,\ell} \mapsto & \ a_k(\lambda)\pi\mu_\ell - \sum\nolimits_{j \in E_k} h^j_{k,\ell}(\lambda)\pi\mu_j - \pi g_{k,\ell}(\lambda) \\ & = \pi \left(a_k(\lambda)\mu_\ell - \sum\nolimits_{j \in E_k} h^j_{k,\ell}(\lambda)\mu_j - g_{k,\ell}(\lambda) \right) \end{aligned}$$

Substituting $x_i = \lambda_i$ in (4) above we see that the expression inside the brackets is annihilated by π^2, hence it is annihilated by π as we have assumed $\text{Ann}_\Lambda(\pi) = \text{Ann}_\Lambda(\pi^2)$. The map $\bar{C} \to D/\pi D$ is determined by $x_i \mapsto x_i$ (clearly well defined). Thus we are done if we can prove (b), (c), and (d).

Using (4) we obtain the following key equality

$$\begin{aligned} \pi p_{k,\ell} & = \pi a_k z_\ell - \sum\nolimits_{j \in E_k} \pi h^j_{k,\ell} z_j - \pi^2 g_{k,\ell} \\ & = -a_k(f_\ell - \pi z_\ell) + a_k f_\ell + \sum\nolimits_{j \in E_k} h^j_{k,\ell}(f_j - \pi z_j) - \sum\nolimits_{j \in E_k} h^j_{k,\ell} f_j - \pi^2 g_{k,\ell} \\ & = -a_k(f_\ell - \pi z_\ell) + \sum\nolimits_{j \in E_k} h^j_{k,\ell}(f_j - \pi z_j) \end{aligned}$$

The end result is an element of the ideal generated by $f_j - \pi z_j$. In particular, we see that $D[1/\pi]$ is isomorphic to $R[1/\pi][x_1, \ldots, x_n, z_1, \ldots, z_m]/(f_j - \pi z_j)$ which is isomorphic to $R[1/\pi][x_1, \ldots, x_n]$ hence smooth over R. This proves (b).

For fixed $k \in \{1, \ldots, r\}$ consider the ring

$$D_k = R[x_1, \ldots, x_n, z_1, \ldots, z_m]/(f_j - \pi z_j, j \in E_k, p_{k,\ell})$$

The number of equations is $m = |E_k| + (m - |E_k|)$ as $p_{k,\ell}$ is zero if $\ell \in E_k$. Also, note that

$$\begin{aligned}(D_k/\pi D_k)_{a_k} &= R/\pi R[x_1, \ldots, x_n, 1/a_k, z_1, \ldots, z_m]/(f_j, j \in E_k, p_{k,\ell}) \\ &= (\bar{C}/\pi\bar{C})_{a_k}[z_1, \ldots, z_m]/(a_k z_\ell - \sum\nolimits_{j \in E_k} h^j_{k,\ell} z_j) \\ &\cong (\bar{C}/\pi\bar{C})_{a_k}[z_j, j \in E_k]\end{aligned}$$

In particular $(D_k/\pi D_k)_{a_k}$ is smooth over $(\bar{C}/\pi\bar{C})_{a_k}$. By our choice of a_k we have that $(\bar{C}/\pi\bar{C})_{a_k}$ is smooth over $R/\pi R$ of relative dimension $n - |E_k|$, see (2). Hence for a prime $\mathfrak{q}_k \subset D_k$ containing π and lying over $Spec(\bar{C}_{a_k})$ the fibre ring of $R \to D_k$ is smooth at $\mathfrak{q}_k$ of dimension n. Thus $R \to D_k$ is syntomic at $\mathfrak{q}_k$ by our count of the number of equations above, see Algebra, Lemma 7.125.11. Hence $R \to D_k$ is smooth at $\mathfrak{q}_k$, see Algebra, Lemma 7.126.16.

To finish the proof, let $\mathfrak{q} \subset D$ be a prime containing π lying over a prime where $R/\pi^2 R \to \bar{C}$ is smooth. Then $a_k \not\in \mathfrak{q}$ for some k by (1). We will show that the surjection $D_k \to D$ induces an isomorphism on local rings at $\mathfrak{q}$. Since we know that the ring maps $\bar{C}/\pi\bar{C} \to D_k/\pi D_k$ and $R \to D_k$ are smooth at the corresponding prime $\mathfrak{q}_k$ by the preceding paragraph this will prove (c) and (d) and thus finish the proof.

First, note that for any ℓ the equation $\pi p_{k,\ell} = -a_k(f_\ell - \pi z_\ell) + \sum_{j \in E_k} h^j_{k,\ell}(f_j - \pi z_j)$ proved above shows that $f_\ell - \pi z_\ell$ maps to zero in $(D_k)_{a_k}$ and in particular in $(D_k)_{\mathfrak{q}_k}$. The relations (4) imply that $a_k f_\ell = \sum_{j \in E_k} h^j_{k,\ell} f_j$ in I/I^2. Since $(\bar{I}_k/\bar{I}_k^2)_{a_k}$ is free on f_j, $j \in E_k$ we see that

$$a_{k'} h^j_{k,\ell} - \sum\nolimits_{j' \in E_{k'}} h^{j'}_{k',\ell} h^j_{k,j'}$$

is zero in $\bar{C}_{a_k}$ for every k, k', ℓ and $j \in E_k$. Hence we can find a large integer N such that

$$a_k^N \left(a_{k'} h^j_{k,\ell} - \sum\nolimits_{j' \in E_{k'}} h^{j'}_{k',\ell} h^j_{k,j'} \right)$$

is in $I_k + \pi^2 R[x_1, \ldots, x_n]$. Computing modulo π we have

$$\begin{aligned}&a_k p_{k',\ell} - a_{k'} p_{k,\ell} + \sum h^{j'}_{k',\ell} p_{k,j'} \\ &= -a_k \sum h^{j'}_{k',\ell} z_{j'} + a_{k'} \sum h^j_{k,\ell} z_j + \sum h^{j'}_{k',\ell} a_k z_{j'} - \sum \sum h^{j'}_{k',\ell} h^j_{k,j'} z_j \\ &= \sum \left(a_{k'} h^j_{k,\ell} - \sum h^{j'}_{k',\ell} h^j_{k,j'} \right) z_j\end{aligned}$$

with Einstein summation convention. Combining with the above we see $a_k^{N+1} p_{k',\ell}$ is contained in the ideal generated by I_k and π in $R[x_1, \ldots, x_n, z_1, \ldots, z_m]$. Thus $p_{k',\ell}$ maps into $\pi(D_k)_{a_k}$. On the other hand, the equation

$$\pi p_{k',\ell} = -a_{k'}(f_\ell - \pi z_\ell) + \sum\nolimits_{j' \in E_{k'}} h^{j'}_{k',\ell}(f_{j'} - \pi z_{j'})$$

shows that $\pi p_{k',\ell}$ is zero in $(D_k)_{a_k}$. Since we have assumed that $\mathrm{Ann}_R(\pi) = \mathrm{Ann}_R(\pi^2)$ and since $(D_k)_{\mathfrak{q}_k}$ is smooth hence flat over R we see that $\mathrm{Ann}_{(D_k)_{\mathfrak{q}_k}}(\pi) = \mathrm{Ann}_{(D_k)_{\mathfrak{q}_k}}(\pi^2)$. We conclude that $p_{k',\ell}$ maps to zero as well, hence $D_\mathfrak{q} = (D_k)_{\mathfrak{q}_k}$ and we win. $\square$

13.7. The desingularization lemma

Here is another fiendishly clever lemma.

Lemma 13.7.1. *Let R be a Noetherian ring. Let Λ be an R-algebra. Let $\pi \in R$ and assume that $Ann_\Lambda(\pi) = Ann_\Lambda(\pi^2)$. Let $A \to \Lambda$ be an R-algebra map with A of finite presentation. Assume*

(1) *the image of π is strictly standard in A over R, and*
(2) *there exists a section $\rho : A/\pi^4 A \to R/\pi^4 R$ which is compatible with the map to $\Lambda/\pi^4\Lambda$.*

Then we can find R-algebra maps $A \to B \to \Lambda$ with B of finite presentation such that $\mathfrak{a}B \subset H_{B/R}$ where $\mathfrak{a} = Ann_R(Ann_R(\pi^2)/Ann_R(\pi))$.

Proof. Choose a presentation

$$A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$$

and $0 \le c \le \min(n, m)$ such that (13.3.3.3) holds for π and such that

$$\pi f_{c+j} \in (f_1, \ldots, f_c) + (f_1, \ldots, f_m)^2 \tag{13.7.1.1}$$

for $j = 1, \ldots, m-c$. Say ρ maps x_i to the class of $r_i \in R$. Then we can replace x_i by $x_i - r_i$. Hence we may assume $\rho(x_i) = 0$ in $R/\pi^4 R$. This implies that $f_j(0) \in \pi^4 R$ and that $A \to \Lambda$ maps x_i to $\pi^4\lambda_i$ for some $\lambda_i \in \Lambda$. Write

$$f_j = f_j(0) + \sum\nolimits_{i=1,\ldots,n} r_{ji}x_i + \text{h.o.t.}$$

This implies that the constant term of $\partial f_j/\partial x_i$ is r_{ji}. Apply ρ to (13.3.3.3) for π and we see that

$$\pi = \sum\nolimits_{I\subset\{1,\ldots,n\},\ |I|=c} r_I \det(r_{ji})_{j=1,\ldots,c,\ i\in I} \bmod \pi^4 R$$

for some $r_I \in R$. Thus we have

$$u\pi = \sum\nolimits_{I\subset\{1,\ldots,n\},\ |I|=c} r_I \det(r_{ji})_{j=1,\ldots,c,\ i\in I}$$

for some $u \in 1 + \pi^3 R$. By Algebra, Lemma 7.14.6 this implies there exists a $n \times c$ matrix (s_{ik}) such that

$$u\pi\delta_{jk} = \sum\nolimits_{i=1,\ldots,n} r_{ji}c_{ik} \quad \text{for all } j,k = 1,\ldots,c$$

(Kronecker delta). We introduce auxiliary variables $v_1, \ldots, v_c, w_1, \ldots, w_n$ and we set

$$h_i = x_i - \pi^2 \sum\nolimits_{j=1,\ldots c} s_{ij}v_j - \pi^3 w_i$$

In the following we will use that

$$R[x_1, \ldots, x_n, v_1, \ldots, v_c, w_1, \ldots, w_n]/(h_1, \ldots, h_n) = R[v_1, \ldots, v_c, w_1, \ldots, w_n]$$

without further mention. In $R[x_1, \ldots, x_n, v_1, \ldots, v_c, w_1, \ldots, w_n]/(h_1, \ldots, h_n)$ we have

$$\begin{aligned} f_j &= f_j(x_1 - h_1, \ldots, x_n - h_n) \\ &= \sum\nolimits_i \pi^2 r_{ji}s_{ik}v_k + \sum\nolimits_i \pi^3 r_{ji}w_i \bmod \pi^4 \\ &= \pi^3 v_j + \sum \pi^3 r_{ji}w_i \bmod \pi^4 \end{aligned}$$

for $1 \le j \le c$. Hence we can choose elements $g_j \in R[v_1, \ldots, v_c, w_1, \ldots, w_n]$ such that $g_j = v_j + \sum r_{ji}w_i \bmod \pi$ and such that $f_j = \pi^3 g_j$ in the R-algebra $R[x_1, \ldots, x_n, v_1, \ldots, v_c, w_1, \ldots, w_n]/(h_1, \ldots, h_n)$. We set

$$B = R[x_1, \ldots, x_n, v_1, \ldots, v_c, w_1, \ldots, w_n]/(f_1, \ldots, f_n, h_1, \ldots, h_n, g_1, \ldots, g_c).$$

The map $A \to B$ is clear. We define $B \to \Lambda$ by mapping $x_i \to \pi^4\lambda_i$, $v_i \mapsto 0$, and $w_i \mapsto \pi\lambda_i$. Then it is clear that the elements f_j and h_i are mapped to zero in Λ. Moreover, it is clear that g_i is mapped to an element t of $\pi\Lambda$ such that $\pi^3 t = 0$ (as $f_i = \pi^3 g_i$ modulo the ideal generated by the h's). Hence our assumption that $\mathrm{Ann}_\Lambda(\pi) = \mathrm{Ann}_\Lambda(\pi^2)$ implies that $t = 0$. Thus we are done if we can prove the statement about smoothness.

Note that $B_\pi \cong A_\pi[v_1, \ldots, v_c]$ because the equations $g_i = 0$ are implied by $f_i = 0$. Hence B_π is smooth over R as A_π is smooth over R by the assumption that π is strictly standard in A over R, see Lemma 13.3.5.

Set $B' = R[v_1, \ldots, v_c, w_1, \ldots, w_n]/(g_1, \ldots, g_c)$. As $g_i = v_i + \sum r_{ji} w_i \bmod \pi$ we see that $B'/\pi B' = R/\pi R[w_1, \ldots, w_n]$. Hence $R \to B'$ is smooth of relative dimension n at every point of $V(\pi)$ by Algebra, Lemmas 7.125.11 and 7.126.16 (the first lemma shows it is syntomic at those primes, in particular flat, whereupon the second lemma shows it is smooth).

Let $\mathfrak{q} \subset B$ be a prime with $\pi \in \mathfrak{q}$ and for some $r \in \mathfrak{a}$, $r \notin \mathfrak{q}$. Denote $\mathfrak{q}' = B' \cap \mathfrak{q}$. We claim the surjection $B' \to B$ induces an isomorphism of local rings $(B')_{\mathfrak{q}'} \to B_{\mathfrak{q}}$. This will conclude the proof of the lemma. Note that $B_{\mathfrak{q}}$ is the quotient of $(B')_{\mathfrak{q}'}$ by the ideal generated by f_{c+j}, $j = 1, \ldots, m-c$. We observe two things: first the image of f_{c+j} in $(B')_{\mathfrak{q}'}$ is divisible by π^2 and second the image of πf_{c+j} in $(B')_{\mathfrak{q}'}$ can be written as $\sum b_{j_1 j_2} f_{c+j_1} f_{c+j_2}$ by (13.7.1.1). Thus we see that the image of each πf_{c+j} is contained in the ideal generated by the elements $\pi^2 f_{c+j'}$. Hence $\pi f_{c+j} = 0$ in $(B')_{\mathfrak{q}'}$ as this is a Noetherian local ring, see Algebra, Lemma 7.47.6. As $R \to (B')_{\mathfrak{q}'}$ is flat we see that

$$\left(\mathrm{Ann}_R(\pi^2)/\mathrm{Ann}_R(\pi)\right) \otimes_R (B')_{\mathfrak{q}'} = \mathrm{Ann}_{(B')_{\mathfrak{q}'}}(\pi^2)/\mathrm{Ann}_{(B')_{\mathfrak{q}'}}(\pi)$$

Because $r \in \mathfrak{a}$ is invertible in $(B')_{\mathfrak{q}'}$ we see that this module is zero. Hence we see that the image of f_{c+j} is zero in $(B')_{\mathfrak{q}'}$ as desired. □

Lemma 13.7.2. *Let R be a Noetherian ring. Let Λ be an R-algebra. Let $\pi \in R$ and assume that $\mathrm{Ann}_R(\pi) = \mathrm{Ann}_R(\pi^2)$ and $\mathrm{Ann}_\Lambda(\pi) = \mathrm{Ann}_\Lambda(\pi^2)$. Let $A \to \Lambda$ and $D \to \Lambda$ be R-algebra maps with A and D of finite presentation. Assume*

1. *π is strictly standard in A over R, and*
2. *there exists an R-algebra map $A/\pi^4 A \to D/\pi^4 D$ compatible with the maps to $\Lambda/\pi^4\Lambda$.*

Then we can find an R-algebra map $B \to \Lambda$ with B of finite presentation and R-algebra maps $A \to B$ and $D \to B$ compatible with the maps to Λ such that $H_{D/R}B \subset H_{B/D}$ and $H_{D/R}B \subset H_{B/R}$.

Proof. We apply Lemma 13.7.1 to

$$D \longrightarrow A \otimes_R D \longrightarrow \Lambda$$

and the image of π in D. By Lemma 13.3.7 we see that π is strictly standard in $A \otimes_R D$ over D. As our section $\rho : (A \otimes_R D)/\pi^4(A \otimes_R D) \to D/\pi^4 D$ we take the map induced by the map in (2). Thus Lemma 13.7.1 applies and we obtain a factorization $A \otimes_R D \to B \to \Lambda$ with B of finite presentation and $\mathfrak{a}B \subset H_{B/D}$ where

$$\mathfrak{a} = \mathrm{Ann}_D(\mathrm{Ann}_D(\pi^2)/\mathrm{Ann}_D(\pi)).$$

For any prime $\mathfrak{q}$ of D such that $D_{\mathfrak{q}}$ is flat over R we have $\mathrm{Ann}_{D_{\mathfrak{q}}}(\pi^2)/\mathrm{Ann}_{D_{\mathfrak{q}}}(\pi) = 0$ because annihilators of elements commutes with flat base change and we assumed $\mathrm{Ann}_R(\pi) = \mathrm{Ann}_R(\pi^2)$. Because D is Noetherian we see that $\mathrm{Ann}_D(\pi^2)/\mathrm{Ann}_D(\pi)$ is a finite D-module, hence formation of its annihilator commutes with localization. Thus we see that $\mathfrak{a} \not\subset \mathfrak{q}$. Hence we see that $D \to B$ is smooth at any prime of B lying over $\mathfrak{q}$. Since any prime of D where $R \to D$ is smooth is one where $D_{\mathfrak{q}}$ is flat over R we conclude that $H_{D/R}B \subset H_{B/D}$. The final inclusion $H_{D/R}B \subset H_{B/R}$ follows because compositions of smooth ring maps are smooth (Algebra, Lemma 7.126.14). $\square$

Lemma 13.7.3. *Let R be a Noetherian ring. Let Λ be an R-algebra. Let $\pi \in R$ and assume that $Ann_R(\pi) = Ann_R(\pi^2)$ and $Ann_\Lambda(\pi) = Ann_\Lambda(\pi^2)$. Let $A \to \Lambda$ be an R-algebra map with A of finite presentation and assume π is strictly standard in A over R. Let*

$$A/\pi^8 A \to \bar{C} \to \Lambda/\pi^8\Lambda$$

be a factorization with $\bar{C}$ of finite presentation. Then we can find a factorization $A \to B \to \Lambda$ with B of finite presentation such that $R_\pi \to B_\pi$ is smooth and such that

$$H_{\bar{C}/(R/\pi^8 R)} \cdot \Lambda/\pi^8\Lambda \subset \sqrt{H_{B/R}\Lambda} \bmod \pi^8\Lambda.$$

Proof. Apply Lemma 13.6.1 to get $R \to D \to \Lambda$ with a factorization $\bar{C}/\pi^4\bar{C} \to D/\pi^4 D \to \Lambda/\pi^4\Lambda$ such that $R \to D$ is smooth at any prime not containing π and at any prime lying over a prime of $\bar{C}/\pi^4\bar{C}$ where $R/\pi^8 R \to \bar{C}$ is smooth. By Lemma 13.7.2 we can find a finitely presented R-algebra B and factorizations $A \to B \to \Lambda$ and $D \to B \to \Lambda$ such that $H_{D/R}B \subset H_{B/R}$. We omit the verification that this is a solution to the problem posed by the lemma. $\square$

13.8. Warmup: reduction to a base field

In this section we apply the lemmas in the previous sections to prove that it suffices to prove the main result when the base ring is a field, see Lemma 13.8.4.

Situation 13.8.1. Here $R \to \Lambda$ is a regular ring map of Noetherian rings.

Let $R \to \Lambda$ be as in Situation 13.8.1. We say *PT holds for $R \to \Lambda$* if Λ is a filtered colimit of smooth R-algebras.

Lemma 13.8.2. *Let $R_i \to \Lambda_i$, $i = 1, 2$ be as in Situation 13.8.1. If PT holds for $R_i \to \Lambda_i$, $i = 1, 2$, then PT holds for $R_1 \times R_2 \to \Lambda_1 \times \Lambda_2$.*

Proof. Omitted. Hint: A product of colimits is a colimit. $\square$

Lemma 13.8.3. *Let $R \to A \to \Lambda$ be ring maps with A of finite presentation over R. Let $S \subset R$ be a multiplicative set. Let $S^{-1}A \to B' \to S^{-1}\Lambda$ be a factorization with B' smooth over $S^{-1}R$. Then we can find a factorization $A \to B \to \Lambda$ such that some $s \in S$ maps to an elementary standard element in B over R.*

Proof. We first apply Lemma 13.4.4 to $S^{-1}R \to B'$. Thus we may assume B' is standard smooth over $S^{-1}R$. Write $A = R[x_1, \ldots, x_n]/(g_1, \ldots, g_t)$ and say $x_i \mapsto \lambda_i$ in Λ. We may write $B' = S^{-1}R[x_1, \ldots, x_{n+m}]/(f_1, \ldots, f_c)$ for some $c \geq n$ where $\det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c}$ is invertible in B' and such that $A \to B'$ is given by $x_i \mapsto x_i$, see Lemma 13.4.6. After multiplying x_i, $i > n$ by an element of S and correspondingly modifying the equations f_j we may assume $B' \to S^{-1}\Lambda$ maps x_i to $\lambda_i/1$ for some $\lambda_i \in \Lambda$ for $i > n$. Choose a relation

$$1 = a_0 \det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c} + \sum\nolimits_{j=1,\ldots,c} a_j f_j$$

for some $a_j \in S^{-1}R[x_1, \ldots, x_{n+m}]$. Since each element of S is invertible in B' we may (by clearing denominators) assume that $f_j, a_j \in R[x_1, \ldots, x_{n+m}]$ and that

$$s_0 = a_0 \det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c} + \sum\nolimits_{j=1,\ldots,c} a_j f_j$$

for some $s_0 \in S$. Since g_j maps to zero in $S^{-1}R[x_1, \ldots, x_{n+m}]/(f_1, \ldots, x_c)$ we can find elements $s_j \in S$ such that $s_j g_j = 0$ in $R[x_1, \ldots, x_{n+m}]/(f_1, \ldots, f_c)$. Since f_j maps to zero in $S^{-1}\Lambda$ we can find $s'_j \in S$ such that $s'_j f_j(\lambda_1, \ldots, \lambda_{n+m}) = 0$ in Λ. Consider the ring

$$B = R[x_1, \ldots, x_{n+m}]/(s'_1 f_1, \ldots, s'_c f_c, g_1, \ldots, g_t)$$

and the factorization $A \to B \to \Lambda$ with $B \to \Lambda$ given by $x_i \mapsto \lambda_i$. We claim that $s = s_0 s_1 \ldots s_t s'_1 \ldots s'_c$ is elementary standard in B over R which finishes the proof. Namely, $s_j g_j \in (f_1, \ldots, f_c)$ and hence $s g_j \in (s'_1 f_1, \ldots, s'_c f_c)$. Finally, we have

$$a_0 \det(\partial s'_j f_j/\partial x_i)_{i,j=1,\ldots,c} + \sum\nolimits_{j=1,\ldots,c} (s'_1 \ldots \hat{s'_j} \ldots s'_c) a_j s'_j f_j = s_0 s'_1 \ldots s'_c$$

which divides s as desired. □

Lemma 13.8.4. *If for every Situation 13.8.1 where R is a field PT holds, then PT holds in general.*

Proof. Assume PT holds for any Situation 13.8.1 where R is a field. Let $R \to \Lambda$ be as in Situation 13.8.1 arbitrary. Note that $R/I \to \Lambda/I\Lambda$ is another regular ring map of Noetherian rings, see More on Algebra, Lemma 12.31.3. Consider the set of ideals

$$\mathcal{I} = \{I \subset R \mid R/I \to \Lambda/I\Lambda \text{ does not have PT}\}$$

We have to show that $\mathcal{I}$ is empty. If this set is nonempty, then it contains a maxmimal element because R is Noetherian. Replacing R by R/I and Λ by Λ/I we obtain a situation where PT holds for $R/I \to \Lambda/I\Lambda$ for any nonzero ideal of R. In particular, we see by applying Proposition 13.5.3 that R is a reduced ring.

Let $A \to \Lambda$ be an R-algebra homomorphism with A of finite presentation. We have to find a factorization $A \to B \to \Lambda$ with B smooth over R, see Lemma 13.2.1.

Let $S \subset R$ be the set of nonzero divisors and consider the total ring of fractions $Q = S^{-1}R$ of R. We know that $Q = K_1 \times \ldots \times K_n$ is a product of fields, see Algebra, Lemmas 7.22.2 and 7.28.6. By Lemma 13.8.2 and our assumption PT holds for the ring map $S^{-1}R \to S^{-1}\Lambda$. Hence we can find a factorization $S^{-1}A \to B' \to \Lambda$ with B' smooth over $S^{-1}R$.

We apply Lemma 13.8.3 and find a factorization $A \to B \to \Lambda$ such that some $\pi \in S$ is elementary standard in B over R. After replacing A by B we may assume that π is elementary standard, hence strictly standard in A. We know that $R/\pi^8 R \to \Lambda/\pi^8\Lambda$ satisfies PT. Hence we can find a factorization $R/\pi^8 R \to A/\pi^8 A \to \bar{C} \to \Lambda/\pi^8\Lambda$ with $R/\pi^8 R \to \bar{C}$ smooth. By Lemma 13.6.1 we can find an R-algebra map $D \to \Lambda$ with D smooth over R and a factorization $R/\pi^4 R \to A/\pi^4 A \to D/\pi^4 D \to \Lambda/\pi^4\Lambda$. By Lemma 13.7.2 we can find $A \to B \to \Lambda$ with B smooth over R which finishes the proof. □

13.9. Local tricks

Situation 13.9.1. We are given a Noetherian ring R and an R-algebra map $A \to \Lambda$ and a prime $\mathfrak{q} \subset \Lambda$. We assume A is of finite presentation over R. In this situation we denote $\mathfrak{h}_A = \sqrt{H_{A/R}\Lambda}$.

Let $R \to A \to \Lambda \supset \mathfrak{q}$ be as in Situation 13.9.1. We say $R \to A \to \Lambda \supset \mathfrak{q}$ *can be resolved* if there exists a factorization $A \to B \to \Lambda$ with B of finite presentation and $\mathfrak{h}_A \subset \mathfrak{h}_B \not\subset \mathfrak{q}$. In this case we will call the factorization $A \to B \to \Lambda$ a *resolution of* $R \to A \to \Lambda \supset \mathfrak{q}$.

Lemma 13.9.2. *Let $R \to A \to \Lambda \supset \mathfrak{q}$ be as in Situation 13.9.1. Let $r \geq 1$ and $\pi_1, \ldots, \pi_r \in R$ map to elements of $\mathfrak{q}$. Assume*

(1) *for $i = 1, \ldots, r$ we have*

$$Ann_{R/(\pi_1^8, \ldots, \pi_{i-1}^8)R}(\pi_i) = Ann_{R/(\pi_1^8, \ldots, \pi_{i-1}^8)R}(\pi_i^2)$$

and

$$Ann_{\Lambda/(\pi_1^8, \ldots, \pi_{i-1}^8)\Lambda}(\pi_i) = Ann_{\Lambda/(\pi_1^8, \ldots, \pi_{i-1}^8)\Lambda}(\pi_i^2)$$

(2) *for $i = 1, \ldots, r$ the element π_i maps to a strictly standard element in A over R.*

Then, if

$$R/(\pi_1^8, \ldots, \pi_r^8)R \to A/(\pi_1^8, \ldots, \pi_r^8)A \to \Lambda/(\pi_1^8, \ldots, \pi_r^8)\Lambda \supset \mathfrak{q}/(\pi_1^8, \ldots, \pi_r^8)\Lambda$$

can be resolved, so can $R \to A \to \Lambda \supset \mathfrak{q}$.

Proof. We are going to prove this by induction on r.

The case $r = 1$. Here the assumption is that there exists a factorization $A/\pi_1^8 \to \bar{C} \to \Lambda/\pi_1^8$ which resolves the situation modulo π_1^8. Conditions (1) and (2) are the assumptions needed to apply Lemma 13.7.3. Thus we can ``lift" the resolution $\bar{C}$ to a resolution of $R \to A \to \Lambda \supset \mathfrak{q}$.

The case $r > 1$. In this case we apply the induction hypothesis for $r - 1$ to the situation $R/\pi_1^8 \to A/\pi_1^8 \to \Lambda/\pi_1^8 \supset \mathfrak{q}/\pi_1^8\Lambda$. Note that property (2) is preserved by Lemma 13.3.7. □

Lemma 13.9.3. *Let $R \to A \to \Lambda \supset \mathfrak{q}$ be as in Situation 13.9.1. Let $\mathfrak{p} = R \cap \mathfrak{q}$. Assume that $\mathfrak{q}$ is minimal over $\mathfrak{h}_A$ and that $R_\mathfrak{p} \to A_\mathfrak{p} \to \Lambda_\mathfrak{q} \supset \mathfrak{q}\Lambda_\mathfrak{q}$ can be resolved. Then there exists a factorization $A \to C \to \Lambda$ with C of finite presentation such that $H_{C/R}\Lambda \not\subset \mathfrak{q}$.*

Proof. Let $A_\mathfrak{p} \to C \to \Lambda_\mathfrak{q}$ be a resolution of $R_\mathfrak{p} \to A_\mathfrak{p} \to \Lambda_\mathfrak{q} \supset \mathfrak{q}\Lambda_\mathfrak{q}$. By our assumption that $\mathfrak{q}$ is minimal over $\mathfrak{h}_A$ this means that $H_{C/R_\mathfrak{p}}\Lambda_\mathfrak{q} = \Lambda_\mathfrak{q}$. By Lemma 13.3.8 we may assume that C is smooth over $\Lambda_\mathfrak{p}$. By Lemma 13.4.4 we may assume that C is standard smooth over $R_\mathfrak{p}$. Write $A = R[x_1, \ldots, x_n]/(g_1, \ldots, g_t)$ and say $A \to \Lambda$ is given by $x_i \mapsto \lambda_i$. Write $C = R_\mathfrak{p}[x_1, \ldots, x_{n+m}]/(f_1, \ldots, f_c)$ for some $c \geq n$ such that $A \to C$ maps x_i to x_i and such that $\det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c}$ is invertible in C, see Lemma 13.4.6. After clearing denominators we may assume $f_1, \ldots, f_c$ are elements of $R[x_1, \ldots, x_{n+m}]$. Of course $\det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c}$ is not invertible in $R[x_1, \ldots, x_{n+m}]/(f_1, \ldots, f_c)$ but it becomes invertible after inverting some element $s_0 \in R$, $s_0 \notin \mathfrak{p}$. As g_j maps to zero under $R[x_1, \ldots, x_n] \to A \to C$ we can find $s_j \in R$, $s_j \notin \mathfrak{p}$ such that $s_j g_j$ is zero in $R[x_1, \ldots, x_{n+m}]/(f_1, \ldots, f_c)$. Write $f_j = F_j(x_1, \ldots, x_{n+m}, 1)$ for some polynomial $F_j \in R[x_1, \ldots, x_n, X_{n+1}, \ldots, X_{n+m+1}]$ homogeneous in $X_{n+1}, \ldots, X_{n+m+1}$. Pick $\lambda_{n+i} \in \Lambda$, $i = 1, \ldots, m+1$ with $\lambda_{n+m+1} \notin \mathfrak{q}$ such that x_{n+i} maps to $\lambda_{n+i}/\lambda_{n+m+1}$ in $\Lambda_\mathfrak{q}$. Then

$$\begin{aligned} F_j(\lambda_1, \ldots, \lambda_{n+m+1}) &= (\lambda_{n+m+1})^{\deg(F_j)} F_j(\lambda_1, \ldots, \lambda_n, \frac{\lambda_{n+1}}{\lambda_{n+m+1}}, \ldots, \frac{\lambda_{n+m}}{\lambda_{n+m+1}}, 1) \\ &= (\lambda_{n+m+1})^{\deg(F_j)} f_j(\lambda_1, \ldots, \lambda_n, \frac{\lambda_{n+1}}{\lambda_{n+m+1}}, \ldots, \frac{\lambda_{n+m}}{\lambda_{n+m+1}}) \\ &= 0 \end{aligned}$$

in $\Lambda_\mathfrak{q}$. Thus we can find $\lambda_0 \in \Lambda$, $\lambda_0 \not\in \mathfrak{q}$ such that $\lambda_0 F_j(\lambda_1, \ldots, \lambda_{n+m+1}) = 0$ in Λ. Now we set B equal to

$$R[x_0, \ldots, x_{n+m+1}]/(g_1, \ldots, g_t, x_0F_1(x_1, \ldots, x_{n+m+1}), \ldots, x_0F_c(x_1, \ldots, x_{n+m+1}))$$

which we map to Λ by mapping x_i to λ_i. Let b be the image of $x_0x_1s_0s_1 \ldots s_t$ in B. Then B_b is isomorphic to

$$R_{s_0s_1}[x_0, x_1, \ldots, x_{n+m+1}, 1/x_0x_{n+m+1}]/(f_1, \ldots, f_c)$$

which is smooth over R by construction. Since b does not map to an element of $\mathfrak{q}$, we win. $\square$

Lemma 13.9.4. *Let $R \to A \to \Lambda \supset \mathfrak{q}$ be as in Situation 13.9.1. Let $\mathfrak{p} = R \cap \mathfrak{q}$. Assume*

1. *$\mathfrak{q}$ is minimal over $\mathfrak{h}_A$,*
2. *$R_\mathfrak{p} \to A_\mathfrak{p} \to \Lambda_\mathfrak{q} \supset \mathfrak{q}\Lambda_\mathfrak{q}$ can be resolved, and*
3. *$\dim(\Lambda_\mathfrak{q}) = 0$.*

Then $R \to A \to \Lambda \supset \mathfrak{q}$ can be resolved.

Proof. By (3) the ring $\Lambda_\mathfrak{q}$ is Artinian local hence $\mathfrak{q}\Lambda_\mathfrak{q}$ is nilpotent. Thus $(\mathfrak{h}_A)^N\Lambda_\mathfrak{q} = 0$ for some $N > 0$. Thus there exists a $\lambda \in \Lambda$, $\lambda \not\in \mathfrak{q}$ such that $\lambda(\mathfrak{h}_A)^N = 0$ in Λ. Say $H_{A/R} = (a_1, \ldots, a_r)$ so that $\lambda a_i^N = 0$ in Λ. By Lemma 13.9.3 we can find a factorization $A \to C \to \Lambda$ with C of finite presentation such that $\mathfrak{h}_C \not\subset \mathfrak{q}$. Write $C = A[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. Set

$$B = A[x_1, \ldots, x_n, y_1, \ldots, y_r, z, t_{ij}]/(f_j - \sum y_it_{ij}, zy_i)$$

where t_{ij} is a set of rm variables. Note that there is a map $B \to C[y_i, z]/(y_iz)$ given by setting t_{ij} equal to zero. The map $B \to \Lambda$ is the composition $B \to C[y_i, z]/(y_iz) \to \Lambda$ where $C[y_i, z]/(y_iz) \to \Lambda$ is the given map $C \to \Lambda$, maps z to λ, and maps y_i to the image of a_i^N in Λ.

We claim that B is a solution for $R \to A \to \Lambda \supset \mathfrak{q}$. First note that B_z is isomorphic to $C[y_1, \ldots, y_r, z, z^{-1}]$ and hence is smooth. On the other hand, $B_{y_\ell} \cong A[x_i, y_i, y_\ell^{-1}, t_{ij}, i{\neq}\ell]$ which is smooth over A. Thus we see that z and $a_\ell y_\ell$ (compositions of smooth maps are smooth) are all elements of $H_{B/R}$. This proves the lemma. $\square$

13.10. Separable residue fields

In this section we explain how to solve a local problem in the case of a separable residue field extension.

Lemma 13.10.1 (Ogoma)**.** *Let A be a Noetherian ring and let M be a finite A-module. Let $S \subset A$ be a multiplicative set. If $\pi \in A$ and $Ker(\pi : S^{-1}M \to S^{-1}M) = Ker(\pi^2 : S^{-1}M \to S^{-1}M)$ then there exists an $s \in S$ such that for any $n > 0$ we have $Ker(s^n\pi : M \to M) = Ker((s^n\pi)^2 : M \to M)$.*

Proof. Let $K = \mathrm{Ker}(\pi : M \to M)$ and $K' = \{m \in M \mid \pi^2m = 0 \text{ in } S^{-1}M\}$ and $Q = K'/K$. Note that $S^{-1}Q = 0$ by assumption. Since A is Noetherian we see that Q is a finite A-module. Hence we can find an $s \in S$ such that s annihilates Q. Then s works. $\square$

Lemma 13.10.2. *Let Λ be a Noetherian ring. Let $I \subset \Lambda$ be an ideal. Let $I \subset \mathfrak{q}$ be a prime. Let n, e be positive integers Assume that $\mathfrak{q}^n\Lambda_\mathfrak{q} \subset I\Lambda_\mathfrak{q}$ and that $\Lambda_\mathfrak{q}$ is a regular local ring of dimension d. Then there exists an $n > 0$ and $\pi_1, \ldots, \pi_d \in \Lambda$ such that*

1. *$(\pi_1, \ldots, \pi_d)\Lambda_\mathfrak{q} = \mathfrak{q}\Lambda_\mathfrak{q}$,*
2. *$\pi_1^n, \ldots, \pi_d^n \in I$, and*

(3) *for* $i = 1, \ldots, d$ *we have*

$$Ann_{\Lambda/(\pi_1^e, \ldots, \pi_{i-1}^e)\Lambda}(\pi_i) = Ann_{\Lambda/(\pi_1^e, \ldots, \pi_{i-1}^e)\Lambda}(\pi_i^2).$$

Proof. Set $S = \Lambda \setminus \mathfrak{q}$ so that $\Lambda_\mathfrak{q} = S^{-1}\Lambda$. First pick $\pi_1, \ldots, \pi_d$ with (1) which is possible as $\Lambda_\mathfrak{q}$ is regular. By assumption $\pi_i^n \in I\Lambda_\mathfrak{q}$. Thus we can find $s_1, \ldots, s_d \in S$ such that $s_i\pi_i^n \in I$. Replacing π_i by $s_i\pi_i$ we get (2). Note that (1) and (2) are preserved by further multiplying by elements of S. Suppose that (3) holds for $i = 1, \ldots, t$ for some $t \in \{0, \ldots, d\}$. Note that $\pi_1, \ldots, \pi_d$ is a regular sequence in $S^{-1}\Lambda$, see Algebra, Lemma 7.98.3. In particular $\pi_1^e, \ldots, \pi_t^e, \pi_{t+1}$ is a regular sequence in $S^{-1}\Lambda = \Lambda_\mathfrak{q}$ by Algebra, Lemma 7.65.10. Hence we see that

$$\mathrm{Ann}_{S^{-1}\Lambda/(\pi_1^e, \ldots, \pi_{i-1}^e)}(\pi_i) = \mathrm{Ann}_{S^{-1}\Lambda/(\pi_1^e, \ldots, \pi_{i-1}^e)}(\pi_i^2).$$

Thus we get (3) for $i = t + 1$ after replacing π_{t+1} by $s\pi_{t+1}$ for some $s \in S$ by Lemma 13.10.1. By induction on t this produces a sequence satisfying (1), (2), and (3). $\square$

Lemma 13.10.3. *Let* $k \to A \to \Lambda \supset \mathfrak{q}$ *be as in Situation 13.9.1 where*

(1) k *is a field,*
(2) Λ *is Noetherian,*
(3) $\mathfrak{q}$ *is minimal over* $\mathfrak{H}_A$*,*
(4) $\Lambda_\mathfrak{q}$ *is a regular local ring, and*
(5) *the field extension* $k \subset \kappa(\mathfrak{q})$ *is separable.*

Then $k \to A \to \Lambda \supset \mathfrak{q}$ *can be resolved.*

Proof. Set $d = \dim \Lambda_\mathfrak{q}$. Set $R = k[x_1, \ldots, x_d]$. Choose $n > 0$ such that $\mathfrak{q}^n\Lambda_\mathfrak{q} \subset \mathfrak{H}_A\Lambda_\mathfrak{q}$ which is possible as $\mathfrak{q}$ is minimal over $\mathfrak{H}_A$. Choose generators $a_1, \ldots, a_r$ of $H_{A/R}$. Set

$$B = A[x_1, \ldots, x_d, z_{ij}]/(x_i^n - \sum z_{ij}a_j)$$

Each B_{a_j} is smooth over R it is a polynomial algebra over $A_{a_j}[x_1, \ldots, x_d]$ and A_{a_j} is smooth over k. Hence B_{x_i} is smooth over R. Let $B \to C$ be the R-algebra map constructed in Lemma 13.4.1 which comes with a R-algebra retraction $C \to B$. In particular a map $C \to \Lambda$ fitting into the diagram above. By construction C_{x_i} is a smooth R-algebra with $\Omega_{C_{x_i}/R}$ free. Hence we can find $c > 0$ such that x_i^c is strictly standard in C/R, see Lemma 13.4.7. Now choose $\pi_1, \ldots, \pi_d \in \Lambda$ as in Lemma 13.10.2 where $n = n$, $e = 8c$, $\mathfrak{q} = \mathfrak{q}$ and $I = \mathfrak{H}_A$. Write $\pi_i^n = \sum \lambda_{ij}a_j$ for some $\pi_{ij} \in \Lambda$. There is a map $B \to \Lambda$ given by $x_i \mapsto \pi_i$ and $z_{ij} \mapsto \lambda_{ij}$. Set $R = k[x_1, \ldots, x_d]$. Diagram

$$\begin{tikzcd} R \arrow[r] & B \arrow[dr] & \\ k \arrow[u] \arrow[r] & A \arrow[u] \arrow[r] & \Lambda \end{tikzcd}$$

Now we apply Lemma 13.9.2 to $R \to C \to \Lambda \supset \mathfrak{q}$ and the sequence of elements $x_1^c, \ldots, x_d^c$ of R. Assumption (2) is clear. Assumption (1) holds for R by inspection and for Λ by our choice of $\pi_1, \ldots, \pi_d$. (Note that if $\mathrm{Ann}_\Lambda(\pi) = \mathrm{Ann}_\Lambda(\pi^2)$, then we have $\mathrm{Ann}_\Lambda(\pi) = \mathrm{Ann}_\Lambda(\pi^c)$ for all $c > 0$.) Thus it suffices to resolve

$$R/(x_1^e, \ldots, x_d^e) \to C/(x_1^e, \ldots, x_d^e) \to \Lambda/(\pi_1^e, \ldots, \pi_d^e) \supset \mathfrak{q}/(\pi_1^e, \ldots, \pi_d^e)$$

for $e = 8c$. By Lemma 13.9.4 it suffices to resolve this after localizing at $\mathfrak{q}$. But since $x_1, \ldots, x_d$ map to a regular sequence in $\Lambda_\mathfrak{q}$ we see that $R \to \Lambda$ is flat, see Algebra, Lemma 7.119.2. Hence

$$R/(x_1^e, \ldots, x_d^e) \to \Lambda_\mathfrak{q}/(\pi_1^e, \ldots, \pi_d^e)$$

is a flat ring map of Artinian local rings. Moreover, this map induces a separable field extension on residue fields by assumption. Thus this map is a filtered colimit of smooth algebras by Algebra, Lemma 7.141.10 and Proposition 13.5.3. Existence of the desired solution follows from Lemma 13.2.1. □

13.11. Inseparable residue fields

In this section we explain how to solve a local problem in the case of an inseparable residue field extension.

Lemma 13.11.1. *Let k be a field of characteristic $p > 0$. Let $(\Lambda, \mathfrak{m}, K)$ be an Artinian local k-algebra. Assume that $\dim H_1(L_{K/k}) < \infty$. Then Λ is a filtered colimit of Artinian local k-algebras A with each map $A \to \Lambda$ flat, with $\mathfrak{m}_A\Lambda = \mathfrak{m}$, and with A essentially of finite type over k.*

Proof. Note that the flatness of $A \to \Lambda$ implies that $A \to \Lambda$ is injective, so the lemma really tells us that Λ is a directed union of these types of subrings $A \subset \Lambda$. Let n be the minimal integer such that $\mathfrak{m}^n = 0$. We will prove this lemma by induction on n. The case $n = 1$ is clear as a field extension is a union of finitely generated field extensions.

Pick $\lambda_1, \ldots, \lambda_d \in \mathfrak{m}$ which generate $\mathfrak{m}$. As K is formally smooth over $\mathbf{F}_p$ (see Algebra, Lemma 7.141.6) we can find a ring map $\sigma : K \to \Lambda$ which is a section of the quotient map $\Lambda \to K$. In general σ is **not** a k-algebra map. Given σ we define

$$\Psi_\sigma : K[x_1, \ldots, x_d] \longrightarrow \Lambda$$

using σ on elements of K and mapping x_i to λ_i. Claim: there exists a $\sigma : K \to \Lambda$ and a subfield $k \subset F \subset K$ finitely generated over k such that the image of k in Λ is contained in $\Psi_\sigma(F[x_1, \ldots, x_d])$.

We will prove the claim by induction on the least integer n such that $\mathfrak{m}^n = 0$. It is clear for $n = 1$. If $n > 1$ set $I = \mathfrak{m}^{n-1}$ and $\Lambda' = \Lambda/I$. By induction we may assume given $\sigma' : K \to \Lambda'$ and $k \subset F' \subset K$ finitely generated such that the image of $k \to \Lambda \to \Lambda'$ is contained in $A' = \Psi_{\sigma'}(F'[x_1, \ldots, x_d])$. Denote $\tau' : k \to A'$ the induced map. Choose a lift $\sigma : K \to \Lambda$ of σ' (this is possible by the formal smoothness of $K/\mathbf{F}_p$ we mentioned above). For later reference we note that we can change σ to $\sigma + D$ for some derivation $D : K \to I$. Set $A = F[x_1, \ldots, x_d]/(x_1, \ldots, x_d)^n$. Then Ψ_σ induces a ring map $\Psi_\sigma : A \to \Lambda$. The composition with the quotient map $\Lambda \to \Lambda'$ induces a surjective map $A \to A'$ with nilpotent kernel. Choose a lift $\tau : k \to A$ of τ' (possible as $k/\mathbf{F}_p$ is formally smooth). Thus we obtain two maps $k \to \Lambda$, namely $\Psi_\sigma \circ \tau : k \to \Lambda$ and the given map $i : k \to \Lambda$. These maps agree modulo I, whence the difference is a derivation $\theta = i - \Psi_\sigma \circ \tau : k \to I$. Note that if we change σ into $\sigma + D$ then we change θ into $\theta - D|_k$.

Choose a set of elements $\{y_j\}_{j \in J}$ of k whose differentials $\text{d}y_j$ form a basis of $\Omega_{k/\mathbf{F}_p}$. The Jacobi-Zariski sequence for $\mathbf{F}_p \subset k \subset K$ is

$$0 \to H_1(L_{K/k}) \to \Omega_{k/\mathbf{F}_p} \otimes K \to \Omega_{K/\mathbf{F}_p} \to \Omega_{K/k} \to 0$$

As $\dim H_1(L_{K/k}) < \infty$ we can find a finite subset $J_0 \subset J$ such that the image of the first map is contained in $\bigoplus_{j \in J_0} K\text{d}y_j$. Hence the elements $\text{d}y_j$, $j \in J \setminus J_0$ map to K-linearly independent elements of $\Omega_{K/\mathbf{F}_p}$. Therefore we can choose a $D : K \to I$ such that $\theta - D|_k = \xi \circ \text{d}$ where ξ is a composition

$$\Omega_{k/\mathbf{F}_p} = \bigoplus\nolimits_{j \in J} k\text{d}y_j \longrightarrow \bigoplus\nolimits_{j \in J_0} k\text{d}y_j \longrightarrow I$$

Let $f_j = \xi(\mathrm{d}y_j) \in I$ for $j \in J_0$. Change σ into $\sigma + D$ as above. Then we see that $\theta(a) = \sum_{j\in J_0} a_j f_j$ for $a \in k$ where $\mathrm{d}a = \sum a_j \mathrm{d}y_j$ in $\Omega_{k/\mathbf{F}_p}$. Note that I is generated by the monomials $\lambda^E = \lambda_1^{e_1} \ldots \lambda_d^{e_d}$ of total degree $|E| = \sum e_i = n - 1$ in $\lambda_1, \ldots, \lambda_d$. Write $f_j = \sum_E c_{j,E}\lambda^E$ with $c_{j,E} \in K$. Replace F' by $F = F'(c_{j,E})$. Then the claim holds.

Choose σ and F as in the claim. The kernel of Ψ_σ is generated by finitely many polynomials $g_1, \ldots, g_t \in K[x_1, \ldots, x_d]$ and we may assume their coefficients are in F after enlarging F by adjoining finitely many elements. In this case it is clear that the map $A = F[x_1, \ldots, x_d]/(g_1, \ldots, g_t) \to K[x_1, \ldots, x_d]/(g_1, \ldots, g_t) = \Lambda$ is flat. By the claim A is a k-subalgebra of Λ. It is clear that Λ is the filtered colimit of these algebras, as K is the filtered union of the subfields F. Finally, these algebras are essentially of finite type over k by Algebra, Lemma 7.50.3. □

Lemma 13.11.2. *Let k be a field of characteristic $p > 0$. Let Λ be a Noetherian geometrically regular k-algebra. Let $\mathfrak{q} \subset \Lambda$ be a prime ideal. Let $n \geq 1$ be an integer and let $E \subset \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$ be a finite subset. Then we can find $m \geq 0$ and $\varphi : k[y_1, \ldots, y_m] \to \Lambda$ with the following properties*

(1) *setting $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$ we have $\mathfrak{q}\Lambda_\mathfrak{q} = \mathfrak{p}\Lambda_\mathfrak{q}$ and $k[y_1, \ldots, y_m]_\mathfrak{p} \to \Lambda_\mathfrak{q}$ is flat,*
(2) *there is a factorization by homomorphisms of local Artinian rings*

$$k[y_1, \ldots, y_m]_\mathfrak{p}/\mathfrak{p}^n k[y_1, \ldots, y_m]_\mathfrak{p} \to D \to \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$$

where the first arrow is essentially smooth and the second is flat,
(3) *E is contained in D modulo $\mathfrak{q}^n\Lambda_\mathfrak{q}$.*

Proof. Set $\bar{\Lambda} = \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$. Note that $\dim H_1(L_{\kappa(\mathfrak{q})/k}) < \infty$ by More on Algebra, Proposition 12.26.1. Pick $A \subset \bar{\Lambda}$ containing E such that A is local Artinian, essentially of finite type over k, the map $A \to \bar{\Lambda}$ is flat, and $\mathfrak{m}_A$ generates the maximal ideal of $\bar{\Lambda}$, see Lemma 13.11.1. Denote $F = A/\mathfrak{m}_A$ the residue field so that $k \subset F \subset K$. Pick $\lambda_1, \ldots, \lambda_t \in \Lambda$ which map to elements of A in $\bar{\Lambda}$ such that moreover the images of $\mathrm{d}\lambda_1, \ldots, \mathrm{d}\lambda_t$ form a basis of $\Omega_{F/k}$. Consider the map $\varphi' : k[y_1, \ldots, y_t] \to \Lambda$ sending y_j to λ_j. Set $\mathfrak{p}' = (\varphi')^{-1}(\mathfrak{q})$. By More on Algebra, Lemma 12.26.2 the ring map $k[y_1, \ldots, y_t]_{\mathfrak{p}'} \to \Lambda_\mathfrak{q}$ is flat and $\Lambda_\mathfrak{q}/\mathfrak{p}'\Lambda_\mathfrak{q}$ is regular. Thus we can choose further elements $\lambda_{t+1}, \ldots, \lambda_m \in \Lambda$ which map into $A \subset \bar{\Lambda}$ and which map to a regular system of parameters of $\Lambda_\mathfrak{q}/\mathfrak{p}'\Lambda_\mathfrak{q}$. We obtain $\varphi : k[y_1, \ldots, y_m] \to \Lambda$ having property (1) such that $k[y_1, \ldots, y_m]_\mathfrak{p}/\mathfrak{p}^n k[y_1, \ldots, y_m]_\mathfrak{p} \to \bar{\Lambda}$ factors through A. Thus $k[y_1, \ldots, y_m]_\mathfrak{p}/\mathfrak{p}^n k[y_1, \ldots, y_m]_\mathfrak{p} \to A$ is flat by Algebra, Lemma 7.35.8. By construction the residue field extension $\kappa(\mathfrak{p}) \subset F$ is finitely generated and $\Omega_{F/\kappa(\mathfrak{p})} = 0$. Hence it is finite separable by More on Algebra, Lemma 12.25.1. Thus $k[y_1, \ldots, y_m]_\mathfrak{p}/\mathfrak{p}^n k[y_1, \ldots, y_m]_\mathfrak{p} \to A$ is finite by Algebra, Lemma 7.50.3. Finally, we conclude that it is étale by Algebra, Lemma 7.132.7. Since an étale ring map is certainly essentially smooth we win. □

Lemma 13.11.3. *Let $\varphi : k[y_1, \ldots, y_m] \to \Lambda$, n, $\mathfrak{q}$, $\mathfrak{p}$ and*

$$k[y_1, \ldots, y_m]_\mathfrak{p}/\mathfrak{p}^n \to D \to \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$$

be as in Lemma 13.11.2. Then for any $\lambda \in \Lambda \setminus \mathfrak{q}$ there exists an integer $q > 0$ and a factorization

$$k[y_1, \ldots, y_m]_\mathfrak{p}/\mathfrak{p}^n \to D \to D' \to \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$$

such that $D \to D'$ is an essentially smooth map of local Artinian rings, the last arrow is flat, and λ^q is in D'.

Proof. Set $\bar{\Lambda} = \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$. Let $\bar{\lambda}$ be the image of λ in $\bar{\Lambda}$. Let $\alpha \in \kappa(\mathfrak{q})$ be the image of λ in the residue field. Let $k \subset F \subset \kappa(\mathfrak{q})$ be the residue field of D. If α is in F then we can find an $x \in D$ such that $x\bar{\lambda} = 1 \bmod \mathfrak{q}$. Hence $(x\bar{\lambda})^q = 1 \bmod (\mathfrak{q})^q$ if q is divisible by p. Hence $\bar{\lambda}^q$ is in D. If α is transcendental over F, then we can take $D' = (D[\bar{\lambda}])_\mathfrak{m}$ equal to the subring generated by D and $\bar{\lambda}$ localized at $\mathfrak{m} = D[\bar{\lambda}] \cap \mathfrak{q}\bar{\Lambda}$. This works because $D[\bar{\lambda}]$ is in fact a polynomial algebra over D in this case. Finally, if $\lambda \bmod \mathfrak{q}$ is algebraic over F, then we can find a p-power q such that α^q is separable algebraic over F, see Algebra, Section 7.38. Note that D and $\bar{\Lambda}$ are henselian local rings, see Algebra, Lemma 7.139.11. Let $D \to D'$ be a finite étale extension whose residue field extension is $F \subset F(\alpha^q)$, see Algebra, Lemma 7.139.8. Since $\bar{\Lambda}$ is henselian and $F(\alpha^q)$ is contained in its residue field we can find a factorization $D' \to \bar{\Lambda}$. By the first part of the argument we see that $\bar{\lambda}^{qq'} \in D'$ for some $q' > 0$. □

Lemma 13.11.4. *Let $k \to A \to \Lambda \supset \mathfrak{q}$ be as in Situation 13.9.1 where*

1. *k is a field of characteristic $p > 0$,*
2. *Λ is Noetherian and geometrically regular over k,*
3. *$\mathfrak{q}$ is minimal over $\mathfrak{h}_A$.*

Then $k \to A \to \Lambda \supset \mathfrak{q}$ can be resolved.

Proof. The lemma is proven by the following steps in the given order. We will justify each of these steps below.

1. Pick an integer $N > 0$ such that $\mathfrak{q}^N\Lambda_\mathfrak{q} \subset H_{A/k}\Lambda_\mathfrak{q}$.
2. Pick generators $a_1, \ldots, a_t \in A$ of the ideal $H_{A/R}$.
3. Set $d = \dim(\Lambda_\mathfrak{q})$.
4. Set $B = A[x_1, \ldots, x_d, z_{ij}]/(x_i^{2N} - \sum z_{ij}a_j)$.
5. Consider B as a $k[x_1, \ldots, x_d]$-algebra and let $B \to C$ be as in Lemma 13.4.1. We also obtain a section $C \to B$.
6. Choose $c > 0$ such that each x_i^c is strictly standard in C over $k[x_1, \ldots, x_d]$.
7. Set $n = N + dc$ and $e = 8c$.
8. Let $E \subset \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$ be the images of generators of A as a k-algebra.
9. Choose an integer m and a k-algebra map $\varphi : k[y_1, \ldots, y_m] \to \Lambda$ and a factorization by local Artinian rings
$$k[y_1, \ldots, y_m]_\mathfrak{p}/\mathfrak{p}^n k[y_1, \ldots, y_m]_\mathfrak{p} \to D \to \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$$
such that the first arrow is essentially smooth, the second is flat, E is contained in D, with $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$ the map $k[y_1, \ldots, y_m]_\mathfrak{p} \to \Lambda_\mathfrak{q}$ is flat, and $\mathfrak{p}\Lambda_\mathfrak{q} = \mathfrak{q}\Lambda_\mathfrak{q}$.
10. Choose $\pi_1, \ldots, \pi_d \in \mathfrak{p}$ which map to a regular system of parameters of $k[y_1, \ldots, y_m]_\mathfrak{p}$.
11. Let $R = k[y_1, \ldots, y_m, t_1, \ldots, t_m]$ and $\gamma_i = \pi_i t_i$.
12. If necessary modify the choice of π_i such that for $i = 1, \ldots, d$ we have
$$\text{Ann}_{R/(\gamma_1^e, \ldots, \gamma_{i-1}^e)R}(\gamma_i) = \text{Ann}_{R/(\gamma_1^e, \ldots, \gamma_{i-1}^e)R}(\gamma_i^2)$$
13. There exist $\delta_1, \ldots, \delta_d \in \Lambda$, $\delta_i \not\in \mathfrak{q}$ and a factorization $D \to D' \to \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$ with D' local Artinian, $D \to D'$ essentially smooth, the map $D' \to \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$ flat such that, with $\pi'_i = \delta_i\pi_i$, we have for $i = 1, \ldots, d$
    (a) $(\pi'_i)^{2N} = \sum a_j\lambda_{ij}$ in Λ where $\lambda_{ij} \bmod \mathfrak{q}^n\Lambda_\mathfrak{q}$ is an element of D',
    (b) $\text{Ann}_{\Lambda/(\pi'^e_1, \ldots, \pi'^e_{i-1})}(\pi'_i) = \text{Ann}_{\Lambda/(\pi'^e_1, \ldots, \pi'^e_{i-1})}(\pi'^2_i)$,
    (c) $\delta_i \bmod \mathfrak{q}^n\Lambda_\mathfrak{q}$ is an element of D'.

(14) Define $B \to \Lambda$ by sending x_i to π'_i and z_{ij} to λ_{ij} found above. Define $C \to \Lambda$ by composing the map $B \to \Lambda$ with the retraction $C \to B$.
(15) Map $R \to \Lambda$ by φ on $k[y_1, \ldots, y_m]$ and by sending t_i to δ_i. Further introduce a map
$$k[x_1, \ldots, x_d] \longrightarrow R = k[y_1, \ldots, y_m, t_1, \ldots, t_d]$$
by sending x_i to $\gamma_i = \pi_i t_i$.
(16) It suffices to resolve
$$R \to C \otimes_{k[x_1,\ldots,x_d]} R \to \Lambda \supset \mathfrak{q}$$
(17) Set $I = (\gamma_1^e, \ldots, \gamma_d^e) \subset R$.
(18) It suffices to resolve
$$R/I \to C \otimes_{k[x_1,\ldots,x_d]} R/I \to \Lambda/I\Lambda \supset \mathfrak{q}/I\Lambda$$
(19) We denote $\mathfrak{r} \subset R = k[y_1, \ldots, y_m, t_1, \ldots, t_d]$ the inverse image of $\mathfrak{q}$.
(20) It suffices to resolve
$$(R/I)_\mathfrak{r} \to C \otimes_{k[x_1,\ldots,x_d]} (R/I)_\mathfrak{r} \to \Lambda_\mathfrak{q}/I\Lambda_\mathfrak{q} \supset \mathfrak{q}\Lambda_\mathfrak{q}/I\Lambda_\mathfrak{q}$$
(21) Set $J = (\pi_1^e, \ldots, \pi_d^e)$ in $k[y_1, \ldots, y_m]$.
(22) It suffices to resolve
$$(R/JR)_\mathfrak{p} \to C \otimes_{k[x_1,\ldots,x_d]} (R/JR)_\mathfrak{p} \to \Lambda_\mathfrak{q}/J\Lambda_\mathfrak{q} \supset \mathfrak{q}\Lambda_\mathfrak{q}/J\Lambda_\mathfrak{q}$$
(23) It suffices to resolve
$$(R/\mathfrak{p}^n R)_\mathfrak{p} \to C \otimes_{k[x_1,\ldots,x_d]} (R/\mathfrak{p}^n R)_\mathfrak{p} \to \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q} \supset \mathfrak{q}\Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$$
(24) It suffices to resolve
$$(R/\mathfrak{p}^n R)_\mathfrak{p} \to B \otimes_{k[x_1,\ldots,x_d]} (R/\mathfrak{p}^n R)_\mathfrak{p} \to \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q} \supset \mathfrak{q}\Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$$
(25) The ring $D'[t_1, \ldots, t_d]$ is given the structure of an $R_\mathfrak{p}/\mathfrak{p}^n R_\mathfrak{p}$-algebra by the given map $k[y_1, \ldots, y_m]_\mathfrak{p}/\mathfrak{p}^n k[y_1, \ldots, y_m]_\mathfrak{p} \to D'$ and by sending t_i to t_i. It suffices to find a factorization
$$B \otimes_{k[x_1,\ldots,x_d]} (R/\mathfrak{p}^n R)_\mathfrak{p} \to D'[t_1, \ldots, t_d] \to \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$$
where the second arrow sends t_i to δ_i and induces the given homomorphism $D' \to \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$.
(26) Such a factorization exists by our choice of D' above.

We now give the justification for each of the steps, except that we skip justifying the steps which just introduce notation.

Ad (1). This is possible as $\mathfrak{q}$ is minimal over $\mathfrak{h}_A = \sqrt{H_{A/k}\Lambda}$.

Ad (6). Note that A_{a_i} is smooth over k. Hence B_{a_j}, which is isomorphic to a polynomial algebra over $A_{a_j}[x_1, \ldots, x_d]$, is smooth over $k[x_1, \ldots, x_d]$. Thus B_{x_i} is smooth over $k[x_1, \ldots, x_d]$. By Lemma 13.4.1 we see that C_{x_i} is smooth over $k[x_1, \ldots, x_d]$ with finite free module of differentials. Hence some power of x_i is strictly standard in C over $k[x_1, \ldots, x_n]$ by Lemma 13.4.7.

Ad (9). This follows by applying Lemma 13.11.2.

Ad (10). Since $k[y_1, \ldots, y_m]_\mathfrak{p} \to \Lambda_\mathfrak{q}$ is flat and $\mathfrak{p}\Lambda_\mathfrak{q} = \mathfrak{q}\Lambda_\mathfrak{q}$ by construction we see that $\dim(k[y_1, \ldots, y_m]_\mathfrak{p}) = d$ by Algebra, Lemma 7.103.7. Thus we can find $\pi_1, \ldots, \pi_d \in \Lambda$ which map to a regular system of parameters in $\Lambda_\mathfrak{q}$.

Ad (12). By Algebra, Lemma 7.98.3 any permutation of the sequence $\pi_1, \ldots, \pi_d$ is a regular sequence in $k[y_1, \ldots, y_m]_\mathfrak{p}$. Hence $\gamma_1 = \pi_1 t_1, \ldots, \gamma_d = \pi_d t_d$ is a regular sequence in $R_\mathfrak{p} = k[y_1, \ldots, y_m]_\mathfrak{p}[t_1, \ldots, t_d]$, see Algebra, Lemma 7.65.11. Let $S = k[y_1, \ldots, y_m] \setminus \mathfrak{p}$ so that $R_\mathfrak{p} = S^{-1}R$. Note that $\pi_1, \ldots, \pi_d$ and $\gamma_1, \ldots, \gamma_d$ remain regular sequences if we multiply our π_i by elements of S. Suppose that

$$\mathrm{Ann}_{R/(\gamma_1^e, \ldots, \gamma_{i-1}^e)R}(\gamma_i) = \mathrm{Ann}_{R/(\gamma_1^e, \ldots, \gamma_{i-1}^e)R}(\gamma_i^2)$$

holds for $i = 1, \ldots, t$ for some $t \in \{0, \ldots, d\}$. Note that $\gamma_1^e, \ldots, \gamma_t^e, \gamma_{t+1}$ is a regular sequence in $S^{-1}R$ by Algebra, Lemma 7.65.10. Hence we see that

$$\mathrm{Ann}_{S^{-1}R/(\gamma_1^e, \ldots, \gamma_{i-1}^e)}(\gamma_i) = \mathrm{Ann}_{S^{-1}R/(\gamma_1^e, \ldots, \gamma_{i-1}^e)}(\gamma_i^2).$$

Thus we get

$$\mathrm{Ann}_{R/(\gamma_1^e, \ldots, \gamma_t^e)R}(\gamma_{t+1}) = \mathrm{Ann}_{R/(\gamma_1^e, \ldots, \gamma_t^e)R}(\gamma_{t+1}^2)$$

after replacing π_{t+1} by $s\pi_{t+1}$ for some $s \in S$ by Lemma 13.10.1. By induction on t this produces the desired sequence.

Ad (13). Let $S = \Lambda \setminus \mathfrak{q}$ so that $\Lambda_\mathfrak{q} = S^{-1}\Lambda$. Set $\bar{\Lambda} = \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$. Suppose that we have a $t \in \{0, \ldots, d\}$ and $\delta_1, \ldots, \delta_t \in S$ and a factorization $D \to D' \to \bar{\Lambda}$ as in (13) such that (a), (b), (c) hold for $i = 1, \ldots, t$. We have $\pi_{t+1}^N \in H_{A/k}\Lambda_\mathfrak{q}$ as $\mathfrak{q}^N\Lambda_\mathfrak{q} \subset H_{A/k}\Lambda_\mathfrak{q}$ by (1). Hence $\pi_{t+1}^N \in H_{A/k}\bar{\Lambda}$. Hence $\pi_{t+1}^N \in H_{A/k}D'$ as $D' \to \bar{\Lambda}$ is faithfully flat, see Algebra, Lemma 7.76.11. Recall that $H_{A/k} = (a_1, \ldots, a_t)$. Say $\pi_{t+1}^N = \sum a_j d_j$ in D' and choose $c_j \in \Lambda_\mathfrak{q}$ lifting $d_j \in D'$. Then $\pi_{t+1}^N = \sum c_j a_j + \epsilon$ with $\epsilon \in \mathfrak{q}^n\Lambda_\mathfrak{q} \subset \mathfrak{q}^{n-N}H_{A/k}\Lambda_\mathfrak{q}$. Write $\epsilon = \sum a_j c'_j$ for some $c'_j \in \mathfrak{q}^{n-N}\Lambda_\mathfrak{q}$. Hence $\pi_{t+1}^{2N} = \sum(\pi_{t+1}^N c_j + \pi_{t+1}^N c'_j)a_j$. Note that $\pi_{t+1}^N c'_j$ maps to zero in $\bar{\Lambda}$; this trivial but key observation will ensure later that (a) holds. Now we choose $s \in S$ such that there exist $\mu_{t+1j} \in \Lambda$ such that on the one hand $\pi_{t+1}^N c_j + \pi_{t+1}^N c'_j = \mu_{t+1j}/s^{2N}$ in $S^{-1}\Lambda$ and on the other $(s\pi_{t+1})^{2N} = \sum \mu_{t+1j}a_j$ in Λ (minor detail omitted). We may further replace s by a power and enlarge D' such that s maps to an element of D'. With these choices μ_{t+1j} maps to $s^{2N}d_j$ which is an element of D'. Note that $\pi_1, \ldots, \pi_d$ are a regular sequence of parameters in $S^{-1}\Lambda$ by our choice of φ. Hence $\pi_1, \ldots, \pi_d$ forms a regular sequence in $\Lambda_\mathfrak{q}$ by Algebra, Lemma 7.98.3. It follows that ${\pi'}_1^e, \ldots, {\pi'}_t^e, s\pi_{t+1}$ is a regular sequence in $S^{-1}\Lambda$ by Algebra, Lemma 7.65.10. Thus we get

$$\mathrm{Ann}_{S^{-1}\Lambda/({\pi'}_1^e, \ldots, {\pi'}_t^e)}(s\pi_{t+1}) = \mathrm{Ann}_{S^{-1}\Lambda/({\pi'}_1^e, \ldots, {\pi'}_t^e)}((s\pi_{t+1})^2).$$

Hence we may apply Lemma 13.10.1 to find an $s' \in S$ such that

$$\mathrm{Ann}_{\Lambda/({\pi'}_1^e, \ldots, {\pi'}_t^e)}((s')^q s\pi_{t+1}) = \mathrm{Ann}_{\Lambda/({\pi'}_1^e, \ldots, {\pi'}_t^e)}(((s')^q s\pi_{t+1})^2).$$

for any $q > 0$. By Lemma 13.11.3 we can choose q and enlarge D' such that $(s')^q$ maps to an element of D'. Setting $\delta_{t+1} = (s')^q s$ and we conclude that (a), (b), (c) hold for $i = 1, \ldots, t+1$. For (a) note that $\lambda_{t+1j} = (s')^{2Nq}\mu_{t+1j}$ works. By induction on t we win.

Ad (16). By construction the radical of $H_{(C\otimes_{k[x_1,\ldots,x_d]}R)/R}\Lambda$ contains $\mathfrak{h}_A$. Namely, the elements $a_j \in H_{A/k}$ map to elements of $H_{B/k[x_1,\ldots,x_n]}$, hence map to elements of $H_{C/k[x_1,\ldots,x_n]}$, hence $a_j \otimes 1$ map to elements of $H_{C\otimes_{k[x_1,\ldots,x_d]}R/R}$. Moreover, if we have a solution $C\otimes_{k[x_1,\ldots,x_n]} R \to T \to \Lambda$ of

$$R \to C \otimes_{k[x_1,\ldots,x_d]} R \to \Lambda \supset \mathfrak{q}$$

then $H_{T/R} \subset H_{T/k}$ as R is smooth over k. Hence T will also be a solution for the original situation $k \to A \to \Lambda \supset \mathfrak{q}$.

Ad (18). Follows on applying Lemma 13.9.2 to $R \to C \otimes_{k[x_1,\ldots,x_d]} R \to \Lambda \supset \mathfrak{q}$ and the sequence of elements $\gamma_1^c, \ldots, \gamma_d^c$. We note that since x_i^c are strictly standard in C over $k[x_1, \ldots, x_d]$ the elements γ_i^c are strictly standard in $C \otimes_{k[x_1,\ldots,x_d]} R$ over R by Lemma 13.3.7. The other assumption of Lemma 13.9.2 holds by steps (12) and (13).

Ad (20). Apply Lemma 13.9.4 to the situation in (18). In the rest of the arguments the target ring is local Artinian, hence we are looking for a factorization by a smooth algebra T over the source ring.

Ad (22). Suppose that $C \otimes_{k[x_1,\ldots,x_d]} (R/JR)_\mathfrak{p} \to T \to \Lambda_\mathfrak{q}/J\Lambda_\mathfrak{q}$ is a solution to

$$(R/JR)_\mathfrak{p} \to C \otimes_{k[x_1,\ldots,x_d]} (R/JR)_\mathfrak{p} \to \Lambda_\mathfrak{q}/J\Lambda_\mathfrak{q} \supset \mathfrak{q}\Lambda_\mathfrak{q}/J\Lambda_\mathfrak{q}$$

Then $C \otimes_{k[x_1,\ldots,x_d]} (R/I)_\mathfrak{r} \to T_\mathfrak{r} \to \Lambda_\mathfrak{q}/I\Lambda_\mathfrak{q}$ is a solution to the situation in (20).

Ad (23). Our $n = N + dc$ is large enough so that $\mathfrak{p}^n k[y_1, \ldots, y_m]_\mathfrak{p} \subset J_\mathfrak{p}$ and $\mathfrak{q}^n\Lambda_\mathfrak{q} \subset J\Lambda_\mathfrak{q}$. Hence if we have a solution $C \otimes_{k[x_1,\ldots,x_d]} (R/\mathfrak{p}^n R)_\mathfrak{p} \to T \to \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$ of (22 then we can take T/JT as the solution for (23).

Ad (24). This is true because we have a section $C \to B$ in the category of R-algebras.

Ad (25). This is true because D' is essentially smooth over the local Artinian ring $k[y_1, \ldots, y_m]_\mathfrak{p}/\mathfrak{p}^n k[y_1, \ldots, y_m]_\mathfrak{p}$ and

$$R_\mathfrak{p}/\mathfrak{p}^n R_\mathfrak{p} = k[y_1, \ldots, y_m]_\mathfrak{p}/\mathfrak{p}^n k[y_1, \ldots, y_m]_\mathfrak{p}[t_1, \ldots, t_d].$$

Hence $D'[t_1, \ldots, t_d]$ is a filtered colimit of smooth $R_\mathfrak{p}/\mathfrak{p}^n R_\mathfrak{p}$-algebras and $B \otimes_{k[x_1,\ldots,x_d]} (R_\mathfrak{p}/\mathfrak{p}^n R_\mathfrak{p})$ factors through one of these.

Ad (26). The final twist of the proof is that we cannot just use the map $B \to D'$ which maps x_i to the image of π_i' in D' and z_{ij} to the image of λ_{ij} in D' because we need the diagram

$$\begin{tikzcd}
B \arrow[r] & D'[t_1, \ldots, t_d] \\
k[x_1, \ldots, x_d] \arrow[u] \arrow[r] & R_\mathfrak{p}/\mathfrak{p}^n R_\mathfrak{p} \arrow[u]
\end{tikzcd}$$

to commute and we need the compostion $B \to D'[t_1, \ldots, t_d] \to \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$ to be the map of (14). This requires us to map x_i to the image of $\pi_i t_i$ in $D'[t_1, \ldots, t_d]$. Hence we map z_{ij} to the image of $\lambda_{ij} t_i^{2N}/\delta_i^{2N}$ in $D'[t_1, \ldots, t_d]$ and everything is clear. □

13.12. The main theorem

In this section we wrap up the discussion.

Theorem 13.12.1 (Popescu)**.** *Any regular homomorphism of Noetherian rings is a filtered colimit of smooth ring maps.*

Proof. By Lemma 13.8.4 it suffices to prove this for $k \to \Lambda$ where Λ is Noetherian and geometrically regular over k. Let $k \to A \to \Lambda$ be a factorization with A a finite type k-algebra. It suffices to construct a factorization $A \to B \to \Lambda$ with B of finite type such that $\mathfrak{h}_B = \Lambda$, see Lemma 13.3.8. Hence we may perform Noetherian induction on the ideal $\mathfrak{h}_A$. Pick a prime $\mathfrak{q} \supset \mathfrak{h}_A$ such that $\mathfrak{q}$ is minimal over $\mathfrak{h}_A$. It now suffices to resolve $k \to A \to \Lambda \supset \mathfrak{q}$ (as defined in the text following Situation 13.9.1). If the characteristic of k is zero, this follows from Lemma 13.10.3. If the characteristic of k is $p > 0$, this follows from Lemma 13.11.4. □

13.13. Other chapters

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(2) Conventions
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CHAPTER 14

Simplicial Methods

14.1. Introduction

This is a minimal introduction to simplicial methods. We just add here whenever something is needed later on. A general reference to this material is perhaps [**GJ99**]. An example of the things you can do is the paper by Quillen on Homotopical Algebra, see [**Qui67**] or the paper on Etale Homotopy by Artin and Mazur, see [**AM69**].

14.2. The category of finite ordered sets

The category Δ is the category with

(1) objects $[0], [1], [2], \ldots$ with $[n] = \{0, 1, 2, \ldots, n\}$ and
(2) a morphism $[n] \to [m]$ is the set of nondecreasing maps of the corresponding sets $\{0, 1, 2, \ldots, n\} \to \{0, 1, 2, \ldots, m\}$.

Here *nondecreasing* for a map $\varphi : [n] \to [m]$ means by definition that $\varphi(i) \geq \varphi(j)$ if $i \geq j$. In other words, Δ is a category equivalent to the ``big" category of finite totally ordered sets and nondecreasing maps. There are exactly $n + 1$ morphisms $[0] \to [n]$ and there is exactly 1 morphism $[n] \to [0]$. There are exactly $(n + 1)(n + 2)/2$ morphisms $[1] \to [n]$ and there are exactly $n + 2$ morphisms $[n] \to [1]$. And so on and so forth.

Definition 14.2.1. For any integer $n \geq 1$, and any $0 \leq j \leq n$ we let $\delta^n_j : [n-1] \to [n]$ denote the injective order preserving map skipping j. For any integer $n \geq 0$, and any $0 \leq j \leq n$ we denote $\sigma^n_j : [n+1] \to [n]$ the surjective order preserving map with $(\sigma^n_j)^{-1}(\{j\}) = \{j, j+1\}$.

Lemma 14.2.2. *Any morphism in Δ can be written as a composition of an identity morphism, and the morphisms δ^n_j and σ^n_j.*

Proof. Let $\varphi : [n] \to [m]$ be a morphism of Δ. If $j \notin \mathrm{Im}(\varphi)$, then we can write φ as $\delta^m_j \circ \psi$ for some morphism $\psi : [n] \to [m - 1]$. If $\varphi(j) = \varphi(j + 1)$ then we can write φ as $\psi \circ \sigma^{n-1}_j$ for some morphism $\psi : [n - 1] \to [m]$. The result follows because each replacement as above lowers $n + m$ and hence at some point φ is both injective and surjective, hence an identity morphism. $\square$

Lemma 14.2.3. *The morphisms δ^n_j and σ^n_j satisfy the following relations.*

(1) *If* $0 \leq i < j \leq n+1$, *then* $\delta_j^{n+1} \circ \delta_i^n = \delta_i^{n+1} \circ \delta_{j-1}^n$. *In other words the diagram*

$$\begin{tikzcd}
& {[n]} \arrow[dr, "\delta_j^{n+1}"] & \\
{[n-1]} \arrow[ur, "\delta_i^n"] \arrow[dr, "\delta_{j-1}^n"'] & & {[n+1]} \\
& {[n]} \arrow[ur, "\delta_i^{n+1}"'] &
\end{tikzcd}$$

commutes.

(2) *If* $0 \leq i < j \leq n-1$, *then* $\sigma_j^{n-1} \circ \delta_i^n = \delta_i^{n-1} \circ \sigma_{j-1}^{n-2}$. *In other words the diagram*

$$\begin{tikzcd}
& {[n]} \arrow[dr, "\sigma_j^{n-1}"] & \\
{[n-1]} \arrow[ur, "\delta_i^n"] \arrow[dr, "\sigma_{j-1}^{n-2}"'] & & {[n-1]} \\
& {[n-2]} \arrow[ur, "\delta_i^{n-1}"'] &
\end{tikzcd}$$

commutes.

(3) *If* $0 \leq j \leq n-1$, *then* $\sigma_j^{n-1} \circ \delta_j^n = id_{[n-1]}$ *and* $\sigma_j^{n-1} \circ \delta_{j+1}^n = id_{[n-1]}$. *In other words the diagram*

$$\begin{tikzcd}
& {[n]} \arrow[dr, "\sigma_j^{n-1}"] & \\
{[n-1]} \arrow[ur, "\delta_j^n"] \arrow[dr, "\delta_{j+1}^n"'] \arrow[rr, "id_{[n-1]}"] & & {[n-1]} \\
& {[n]} \arrow[ur, "\sigma_j^{n-1}"'] &
\end{tikzcd}$$

commutes.

(4) *If* $0 < j+1 < i \leq n$, *then* $\sigma_j^{n-1} \circ \delta_i^n = \delta_{i-1}^{n-1} \circ \sigma_j^{n-2}$. *In other words the diagram*

$$\begin{tikzcd}
& {[n]} \arrow[dr, "\sigma_j^{n-1}"] & \\
{[n-1]} \arrow[ur, "\delta_i^n"] \arrow[dr, "\sigma_j^{n-2}"'] & & {[n-1]} \\
& {[n-2]} \arrow[ur, "\delta_{i-1}^{n-1}"'] &
\end{tikzcd}$$

commutes.

(5) *If* $0 \leq i \leq j \leq n-1$, *then* $\sigma^{n-1}_j \circ \sigma^n_i = \sigma^{n-1}_i \circ \sigma^n_{j+1}$. *In other words the diagram*

$$\begin{tikzcd} & {[n]} \arrow[rd, "\sigma^{n-1}_j"] & \\ {[n+1]} \arrow[ru, "\sigma^n_i"] \arrow[rd, "\sigma^n_{j+1}"'] & & {[n-1]} \\ & {[n]} \arrow[ru, "\sigma^{n-1}_i"'] & \end{tikzcd}$$

commutes.

Proof. Omitted. ☐

Lemma 14.2.4. *The category* Δ *is the universal category with objects* $[n]$, $n \geq 0$ *and morphisms* δ^n_j *and* σ^n_j *such that (a) every morphism is a composition of these morphisms, (b) the relations listed in Lemma 14.2.3 are satisfied, and (c) any relation among the morphisms is a consquence of those relations.*

Proof. Omitted. ☐

14.3. Simplicial objects

Definition 14.3.1. Let $\mathcal{C}$ be a category.

(1) A *simplicial object U of $\mathcal{C}$* is a contravariant functor U from Δ to $\mathcal{C}$, in a formula:
$$U : \Delta^{opp} \longrightarrow \mathcal{C}$$
(2) If $\mathcal{C}$ is the category of sets, then we call U a *simplicial set*.
(3) If $\mathcal{C}$ is the category of abelian groups, then we call U a *simplicial abelian group*.
(4) A *morphism of simplicial objects* $U \to U'$ is a transformation of functors.
(5) The *category of simplicial objects of* $\mathcal{C}$ is denoted $\text{Simp}(\mathcal{C})$.

This means there are objects $U([0]), U([1]), U([2]), \ldots$ and for φ any nondecreasing map $\varphi : [m] \to [n]$ a morphism $U(\varphi) : U([n]) \to U([m])$, satisfying $U(\varphi \circ \psi) = U(\psi) \circ U(\varphi)$.

In particular there is a unique morphism $U([0]) \to U([n])$ and there are exactly $n+1$ morphisms $U([n]) \to U([0])$ corresponding to the $n+1$ maps $[0] \to [n]$. Obviously we need some more notation to be able to talk intelligently about these simplicial objects. We do this by considering the morphisms we singled out in Section 14.2 above.

Lemma 14.3.2. *Let $\mathcal{C}$ be a category.*

(1) *Given a simplicial object U in $\mathcal{C}$ we obtain a sequence of objects $U_n = U([n])$ endowed with the morphisms $d^n_j = U(\delta^n_j) : U_n \to U_{n-1}$ and $s^n_j = U(\sigma^n_j) : U_n \to U_{n+1}$. These morphisms satisfy the opposites of the relations displayed in Lemma 14.2.3.*
(2) *Conversely, given a sequence of objects U_n and morphisms d^n_j, s^n_j satisfying these relations there exists a unique simplicial object U in $\mathcal{C}$ such that $U_n = U([n])$, $d^n_j = U(\delta^n_j)$, and $s^n_j = U(\sigma^n_j)$.*
(3) *A morphism between simplicial objects U and U' is given by a family of morphisms $U_n \to U'_n$ commuting with the morphisms d^n_j and s^n_j.*

Proof. This follows from Lemma 14.2.4. ☐

Remark 14.3.3. By abuse of notation we sometimes write $d_i : U_n \to U_{n-1}$ instead of d_i^n, and similarly for $s_i : U_n \to U_{n+1}$. The relations among the morphisms d_i^n and s_i^n may be expressed as follows:

(1) If $i < j$, then $d_i \circ d_j = d_{j-1} \circ d_i$.
(2) If $i < j$, then $d_i \circ s_j = s_{j-1} \circ d_i$.
(3) We have $\mathrm{id} = d_j \circ s_j = d_{j+1} \circ s_j$.
(4) If $i > j + 1$, then $d_i \circ s_j = s_j \circ d_{i-1}$.
(5) If $i \leq j$, then $s_i \circ s_j = s_{j+1} \circ s_i$.

This means that whenever the compositions on both the left and the right are defined then the corresponding equality should hold.

We get a unique morphism $s_0^0 = U(\sigma_0^0) : U_0 \to U_1$ and two morphisms $d_0^1 = U(\delta_0^1)$, and $d_1^1 = U(\delta_1^1)$ which are morphisms $U_1 \to U_0$. There are two morphisms $s_0^1 = U(\sigma_0^1)$, $s_1^1 = U(\sigma_1^1)$ which are morphsms $U_1 \to U_2$. Three morphisms $d_0^2 = U(\delta_0^2)$, $d_1^2 = U(\delta_1^2)$, $d_2^2 = U(\delta_2^2)$ which are morphisms $U_3 \to U_2$. And so on.

Pictorially we think of U as follows:

$$\begin{tikzcd} U_2 \arrow[r, shift left=3] \arrow[r] \arrow[r, shift right=3] & U_1 \arrow[l, shift left=1.5] \arrow[l, shift right=1.5] \arrow[r, shift left] \arrow[r, shift right] & U_0 \arrow[l] \end{tikzcd}$$

Here the d-morphisms are the arrows pointing right and the s-morphisms are the arrows pointing left.

Example 14.3.4. The simplest example is the *constant* simplicial object with value $X \in Ob(\mathcal{C})$. In other words, $U_n = X$ and all maps are id_X.

Example 14.3.5. Suppose that $Y \to X$ is a morphism of $\mathcal{C}$ such that all the fibred products $Y \times_X Y \times_X \ldots \times_X Y$ exist. Then we set U_n equal to the $(n+1)$-fold fibre product, and we let $\varphi : [n] \to [m]$ correspond to the map (on ``coordinates") $(y_0, \ldots, y_m) \mapsto (y_{\varphi(0)}, \ldots, y_{\varphi(n)})$. In other words, the map $U_0 = Y \to U_1 = Y \times_X Y$ is the diagonal map. The two maps $U_1 = Y \times_X Y \to U_0 = Y$ are the projection maps.

Geometrically Example 14.3.5 above is an important example. It tells us that it is a good idea to think of the maps $d_j^n : U_{n+1} \to U_n$ as projection maps (forgetting the jth component), and to think of the maps $s_j^n : U_n \to U_{n+1}$ as diagonal maps (repeating the jth coordinate). We will return to this in the sections below.

Lemma 14.3.6. *Let $\mathcal{C}$ be a category. Let U be a simplicial object of $\mathcal{C}$. Each of the morphisms $s_i^n : U_n \to U_{n+1}$ has a left inverse. In particular s_i^n is a monomorphism.*

Proof. This is true because $d_i^{n+1} \circ s_i^n = \mathrm{id}_{U_n}$. □

14.4. Simplicial objects as presheaves

Another observation is that we may think of a simplicial object of $\mathcal{C}$ as a presheaf with values in $\mathcal{C}$ over Δ. See Sites, Definition 9.2.2. And in fact, if U, U' are simplicial objects of $\mathcal{C}$, then we have

$$Mor(U, U') = Mor_{PSh(\Delta)}(U, U'). \tag{14.4.0.1}$$

Some of the material below could be replace by the more general constructions in the chapter on sites. However, it seems a clearer picture arises from the arguments specific to simplicial objects.

14.5. Cosimplicial objects

A cosimplicial object of a category $\mathcal{C}$ could be defined simply as a simplicial object of the opposite category $\mathcal{C}^{opp}$. This is not really how the human brain works, so we introduce them separately here and point out some simple properties.

Definition 14.5.1. Let $\mathcal{C}$ be a category.

(1) A *cosimplicial object U of $\mathcal{C}$* is a covariant functor U from Δ to $\mathcal{C}$, in a formula:

$$U : \Delta \longrightarrow \mathcal{C}$$

(2) If $\mathcal{C}$ is the category of sets, then we call U a *cosimplicial set*.
(3) If $\mathcal{C}$ is the category of abelian groups, then we call U a *cosimplicial abelian group*.
(4) A *morphism of cosimplicial objects* $U \to U'$ is a transformation of functors.
(5) The *category of cosimplicial objects of $\mathcal{C}$* is denoted $\text{CoSimp}(\mathcal{C})$.

This means there are objects $U([0]), U([1]), U([2]), \ldots$ and for φ any nondecreasing map $\varphi : [m] \to [n]$ a morphism $U(\varphi) : U([m]) \to U([n])$, satisfying $U(\varphi \circ \psi) = U(\varphi) \circ U(\psi)$.

In particular there is a unique morphism $U([n]) \to U([0])$ and there are exactly $n+1$ morphisms $U([0]) \to U([n])$ corresponding to the $n+1$ maps $[0] \to [n]$. Obviously we need some more notation to be able to talk intelligently about these simplicial objects. We do this by considering the morphisms we singled out in Section 14.2 above.

Lemma 14.5.2. *Let $\mathcal{C}$ be a category.*

(1) *Given a cosimplicial object U in $\mathcal{C}$ we obtain a sequence of objects $U_n = U([n])$ endowed with the morphisms $\delta^n_j = U(\delta^n_j) : U_{n-1} \to U_n$ and $\sigma^n_j = U(\sigma^n_j) : U_{n+1} \to U_n$. These morphisms satisfy the relations displayed in Lemma 14.2.3.*
(2) *Conversely, given a sequence of objects U_n and morphisms δ^n_j, σ^n_j satisfying these relations there exists a unique cosimplicial object U in $\mathcal{C}$ such that $U_n = U([n])$, $\delta^n_j = U(\delta^n_j)$, and $\sigma^n_j = U(\sigma^n_j)$.*
(3) *A morphism between simplicial objects U and U' is given by a family of morphisms $U_n \to U'_n$ commuting with the morphisms δ^n_j and σ^n_j.*

Proof. This follows from Lemma 14.2.4. □

Remark 14.5.3. By abuse of notation we sometimes write $\delta_i : U_{n-1} \to U_n$ instead of δ^n_i, and similarly for $\sigma_i : U_{n+1} \to U_n$. The relations among the morphisms δ^n_i and σ^n_i may be expressed as follows:

(1) If $i < j$, then $\delta_j \circ \delta_i = \delta_i \circ \delta_{j-1}$.
(2) If $i < j$, then $\sigma_j \circ \delta_i = \delta_i \circ \sigma_{j-1}$.
(3) We have $\text{id} = \sigma_j \circ \delta_j = \sigma_j \circ \delta_{j+1}$.
(4) If $i > j+1$, then $\sigma_j \circ \delta_i = \delta_{i-1} \circ \sigma_j$.
(5) If $i \leq j$, then $\sigma_j \circ \sigma_i = \sigma_i \circ \sigma_{j+1}$.

This means that whenever the compositions on both the left and the right are defined then the corresponding equality should hold.

We get a unique morphism $\sigma^0_0 = U(\sigma^0_0) : U_1 \to U_0$ and two morphisms $\delta^1_0 = U(\delta^1_0)$, and $\delta^1_1 = U(\delta^1_1)$ which are morphisms $U_0 \to U_1$. There are two morphisms $\sigma^1_0 = U(\sigma^1_0)$, $\sigma^1_1 = U(\sigma^1_1)$ which are morphsms $U_2 \to U_1$. Three morphisms $\delta^2_0 = U(\delta^2_0)$, $\delta^2_1 = U(\delta^2_1)$, $\delta^2_2 = U(\delta^2_2)$ which are morphisms $U_2 \to U_3$. And so on.

Pictorially we think of U as follows:

$$U_0 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; U_1 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; U_2$$

Here the δ-morphisms are the arrows pointing right and the σ-morphisms are the arrows pointing left.

Example 14.5.4. The simplest example is the *constant* cosimplicial object with value $X \in Ob(\mathcal{C})$. In other words, $U_n = X$ and all maps are id_X.

Example 14.5.5. Suppose that $Y \to X$ is a morphism of $\boldsymbol{C}$ such that all the push outs $Y \coprod_X Y \coprod_X \ldots \coprod_X Y$ exist. Then we set U_n equal to the $(n+1)$-fold push out, and we let $\varphi : [n] \to [m]$ correspond to the map

$$(y \text{ in } i\text{th component}) \mapsto (y \text{ in } \varphi(i)\text{th component})$$

on ``coordinates". In other words, the map $U_1 = Y \coprod_X Y \to U_0 = Y$ is the identity on each component. The two maps $U_0 = Y \to U_1 = Y \coprod_X Y$ are the two natural maps.

Lemma 14.5.6. *Let $\mathcal{C}$ be a category. Let U be a cosimplicial object of $\mathcal{C}$. Each of the morphisms $\delta^n_i : U_{n-1} \to U_n$ has a left inverse. In particular δ^n_i is a monomorphism.*

Proof. This is true because $\sigma^{n-1}_i \circ \delta^n_i = \mathrm{id}_{U_n}$ for $j < n$. □

14.6. Products of simplicial objects

Of course we should define the product of simplicial objects as the product in the category of simplicial objects. This may lead to the potentially confusing situation where the product exists but is not described as below. To avoid this we define the product directly as follows.

Definition 14.6.1. Let $\mathcal{C}$ be a category. Let U and V be simplicial objects of $\mathcal{C}$. Assume the products $U_n \times V_n$ exist in $\mathcal{C}$. The *product of U and V* is the simplicial object $U \times V$ defined as follows:

(1) $(U \times V)_n = U_n \times V_n$,
(2) $d^n_i = (d^n_i, d^n_i)$, and
(3) $s^n_i = (s^n_i, s^n_i)$.

In other words, $U \times V$ is the product of the presheaves U and V on Δ.

Lemma 14.6.2. *If U and V are simplicial objects in the category $\mathcal{C}$, and if $U \times V$ exists, then we have*

$$Mor(W, U \times V) = Mor(W, U) \times Mor(W, V)$$

for any third simplicial object W of $\mathcal{C}$.

Proof. Omitted. □

14.7. Fibre products of simplicial objects

Of course we should define the fibre product of simplicial objects as the fibre product in the category of simplicial objects. This may lead to the potentially confusing situation where the fibre product exists but is not described as below. To avoid this we define the fibre product directly as follows.

Definition 14.7.1. Let $\mathcal{C}$ be a category. Let U, V, W be simplicial objects of $\mathcal{C}$. Let $a : V \to U$, $b : W \to U$ be morphisms. Assume the fibre products $V_n \times_{U_n} W_n$ exist in $\mathcal{C}$. The *fibre product of V and W over U* is the simplicial object $V \times_U W$ defined as follows:

(1) $(V \times_U W)_n = V_n \times_{U_n} W_n$,
(2) $d_i^n = (d_i^n, d_i^n)$, and
(3) $s_i^n = (s_i^n, s_i^n)$.

In other words, $V \times_U W$ is the fibre product of the presheaves V and W over the presheaf U on Δ.

Lemma 14.7.2. *If U, V, W are simplicial objects in the category $\mathcal{C}$, and if $a : V \to U$, $b : W \to U$ are morphisms and if $V \times_U W$ exists, then we have*

$$Mor(T, V \times_U V) = Mor(T, V) \times_{Mor(T,U)} Mor(T, W)$$

for any fourth simplicial object T of $\mathcal{C}$.

Proof. Omitted. □

14.8. Push outs of simplicial objects

Of course we should define the push out of simplicial objects as the push out in the category of simplicial objects. This may lead to the potentially confusing situation where the push outs exist but are not as described below. To avoid this we define the push out directly as follows.

Definition 14.8.1. Let $\mathcal{C}$ be a category. Let U, V, W be simplicial objects of $\mathcal{C}$. Let $a : U \to V$, $b : U \to W$ be morphisms. Assume the psuh outs $V_n \amalg_{U_n} W_n$ exist in $\mathcal{C}$. The *push out of V and W over U* is the simplicial object $V \amalg_U W$ defined as follows:

(1) $(V \amalg_U W)_n = V_n \amalg_{U_n} W_n$,
(2) $d_i^n = (d_i^n, d_i^n)$, and
(3) $s_i^n = (s_i^n, s_i^n)$.

In other words, $V \amalg_U W$ is the push out of the presheaves V and W over the presheaf U on Δ.

Lemma 14.8.2. *If U, V, W are simplicial objects in the category $\mathcal{C}$, and if $a : U \to V$, $b : U \to W$ are morphisms and if $V \amalg_U W$ exists, then we have*

$$Mor(V \amalg_U W, T) = Mor(V, T) \times_{Mor(U,T)} Mor(W, T)$$

for any fourth simplicial object T of $\mathcal{C}$.

Proof. Omitted. □

14.9. Products of cosimplicial objects

Of course we should define the product of cosimplicial objects as the product in the category of cosimplicial objects. This may lead to the potentially confusing situation where the product exists but is not described as below. To avoid this we define the product directly as follows.

Definition 14.9.1. Let $\mathcal{C}$ be a category. Let U and V be cosimplicial objects of $\mathcal{C}$. Assume the products $U_n \times V_n$ exist in $\mathcal{C}$. The *product of U and V* is the cosimplicial object $U \times V$ defined as follows:

(1) $(U \times V)_n = U_n \times V_n$,
(2) for any $\varphi : [n] \to [m]$ the map $(U \times V)(\varphi) : U_n \times V_n \to U_m \times V_m$ is the product $U(\varphi) \times V(\varphi)$.

Lemma 14.9.2. *If U and V are cosimplicial objects in the category $\mathcal{C}$, and if $U \times V$ exists, then we have*

$$Mor(W, U \times V) = Mor(W, U) \times Mor(W, V)$$

for any third cosimplicial object W of $\mathcal{C}$.

Proof. Omitted. □

14.10. Fibre products of cosimplicial objects

Of course we should define the fibre product of cosimplicial objects as the fibre product in the category of cosimplicial objects. This may lead to the potentially confusing situation where the product exists but is not described as below. To avoid this we define the fibre product directly as follows.

Definition 14.10.1. Let $\mathcal{C}$ be a category. Let U, V, W be cosimplicial objects of $\mathcal{C}$. Let $a : V \to U$ and $b : W \to U$ be morphisms. Assume the fibre products $V_n \times_{U_n} W_n$ exist in $\mathcal{C}$. The *fibre product of V and W over U* is the cosimplicial object $V \times_U W$ defined as follows:

(1) $(V \times_U W)_n = V_n \times_{U_n} W_n$,
(2) for any $\varphi : [n] \to [m]$ the map $(V \times_U W)(\varphi) : V_n \times_{U_n} W_n \to V_m \times_{U_m} W_m$ is the product $V(\varphi) \times_{U(\varphi)} W(\varphi)$.

Lemma 14.10.2. *If U, V, W are cosimplicial objects in the category $\mathcal{C}$, and if $a : V \to U$, $b : W \to U$ are morphisms and if $V \times_U W$ exists, then we have*

$$Mor(T, V \times_U W) = Mor(T, V) \times_{Mor(T,U)} Mor(T, W)$$

for any fourth cosimplicial object T of $\mathcal{C}$.

Proof. Omitted. □

14.11. Simplicial sets

Let U be a simplical set. It is a good idea to think of U_0 as the 0-*simplices*, the set U_1 as the 1-*simplices*, the set U_2 as the 2-*simplices*, and so on.

We think of the maps $s^n_j : U_n \to U_{n+1}$ as the map that associates to an n-simplex A the degenerate $(n+1)$-simplex B whose $(j, j+1)$-edge is collapsed to the vertex j of A. We think of the map $d^n_j : U_n \to U_{n-1}$ as the map that associates to an n-simplex A one of the faces, namely the face that omits the vertex j. In this way it become possible to visualize the relations among the maps s^n_j and d^n_j geometrically.

Definition 14.11.1. Let U be a simplicial set. We say x is an *n-simplex of U* to signify that x is an element of U_n. We say that y is the $j$the *face of* x to signify that $d^n_j x = y$. We say that z is the jth *degeneracy of* x if $z = s^n_j x$. A simplex is called *degenerate* if it is the degeneracy of another simplex.

Here are a few fundamental examples.

Example 14.11.2. For every $n \geq 0$ we denote $\Delta[n]$ the simplicial set

$$\begin{aligned} \Delta^{opp} &\longrightarrow Sets \\ [k] &\longmapsto Mor_\Delta([k], [n]) \end{aligned}$$

We leave it to the reader to verify the following statements. Every m-simplex of $\Delta[n]$ with $m > n$ is degenerate. There is a unique nondegenerate n-simplex of $\Delta[n]$, namely $\mathrm{id}_{[n]}$.

Lemma 14.11.3. *Let U be a simplicial set. Let $n \geq 0$ be an integer. There is a canonical bijection*

$$Mor(\Delta[n], U) \longrightarrow U_n$$

which maps a morphism φ to the value of φ on the unique nondegenerate n-simplex of $\Delta[n]$.

Proof. Omitted. □

Example 14.11.4. Consider the category $\Delta/[n]$ of objects over $[n]$ in Δ, see Categories, Example 4.2.13. There is a functor $p : \Delta/[n] \to \Delta$. The fibre category of p over $[k]$, see Categories, Section 4.32, has as objects the set $\Delta[n]_k$ of k-simplices in $\Delta[n]$, and as morphisms only identities. For every morphism $\varphi : [k] \to [l]$ of Δ, and every object $\psi : [l] \to [n]$ in the fibre category over $[l]$ there is a unique object over $[k]$ with a morphism covering φ, namely $\psi \circ \varphi : [k] \to [n]$. Thus $\Delta/[n]$ is fibred in sets over Δ. In other words, we may think of $\Delta/[n]$ as a presheaf of sets over Δ. See also, Categories, Example 4.35.7. And this presheaf of sets agrees with the simplicial set $\Delta[n]$. In particular, from Equation (14.4.0.1) and Lemma 14.11.3 above we get the formula

$$Mor_{PSh(\Delta)}(\Delta/[n], U) = U_n$$

for any simplicial set U.

Lemma 14.11.5. *Let U, V be simplicial sets. Let $a, b \geq 0$ be integers. Assume every n-simplex of U is degenerate if $n > a$. Assume every n-simplex of V is degenerate if $n > b$. Then every n-simplex of $U \times V$ is degenerate if $n > a + b$.*

Proof. Suppose $n > a + b$. Let $(u, v) \in (U \times V)_n = U_n \times V_n$. By assumption, there exists a $\alpha : [n] \to [a]$ and a $u' \in U_a$ and a $\beta : [n] \to [b]$ and a $v' \in V_b$ such that $u = U(\alpha)(u')$ and $v = V(\beta)(v')$. Because $n > a + b$, there exists an $0 \leq i \leq a + b$ such that $\alpha(i) = \alpha(i + 1)$ and $\beta(i) = \beta(i + 1)$. It follows immediately that (u, v) is in the image of s^{n-1}_i. □

14.12. Products with simplicial sets

Let $\mathcal{C}$ be a category. Let U be a simplicial set. Let V be a simplicial object of $\mathcal{C}$. We can consider the covariant functor which associates to a simplicial object W of $\mathcal{C}$ the set

$$\left\{ (f_{n,u} : V_n \to W_n)_{n \geq 0, u \in U_n} \text{ such that } \begin{matrix} \forall \varphi : [m] \to [n] \\ f_{m,\varphi(u)} \circ V(\varphi) = W(\varphi) \circ f_{n,u} \end{matrix} \right\} \tag{14.12.0.1}$$

If this functor is of the form $Mor_{\text{Simp}(\mathcal{C})}(Q, -)$ then we can think of Q as the product of U with V. Instead of formalizing this in this way we just directly define the product as follows.

Definition 14.12.1. Let $\mathcal{C}$ be a category such that the coproduct of any two objects of $\mathcal{C}$ exists. Let U be a simplicial set. Let V be a simplicial object of $\mathcal{C}$. Assume that each U_n is finite nonempty. In this case we define the *product $U \times V$ of U and V* to be the simplicial object of $\mathcal{C}$ whose nth term is the object

$$(U \times V)_n = \coprod\nolimits_{u \in U_n} V_n$$

with maps for $\varphi : [m] \to [n]$ given by the morphism

$$\coprod\nolimits_{u \in U_n} V_n \longrightarrow \coprod\nolimits_{u' \in U_m} V_m$$

which maps the component V_n corresponding to u to the component V_m corresponding to $u' = U(\varphi)(u)$ via the morphism $V(\varphi)$. More loosely, if all of the coproducts displayed above exist (without assuming anything about $\mathcal{C}$) we will say that the *product $U \times V$ exists*.

Lemma 14.12.2. *Let $\mathcal{C}$ be a category such that the coproduct of any two objects of $\mathcal{C}$ exists. Let U be a simplicial set. Let V be a simplicial object of $\mathcal{C}$. Assume that each U_n is finite nonempty. The functor $W \mapsto Mor_{Simp(\mathcal{C})}(U \times V, W)$ is canonically isomorphic to the functor which maps W to the set in Equation (14.12.0.1).*

Proof. Omitted. $\square$

Lemma 14.12.3. *Let $\mathcal{C}$ be a category such that the coproduct of any two objects of $\mathcal{C}$ exists. Let us temporarily denote FSSets the category of simplicial sets all of whose components are finite nonempty.*

(1) *The rule $(U, V) \mapsto U \times V$ defines a functor $FSSets \times Simp(\mathcal{C}) \to Simp(\mathcal{C})$.*
(2) *For every U, V as above there is a canonical map of simplicial objects*
$$U \times V \longrightarrow V$$
defined by taking the identity on each component of $(U \times V)_n = \coprod_u V_n$.

Proof. Omitted. $\square$

We briefly study a special case of the construction above. Let $\mathcal{C}$ be a category. Let X be an object of $\mathcal{C}$. Let $k \geq 0$ be an integer. If all coproducts $X \coprod \ldots \coprod X$ exist then according to the definition above the product
$$X \times \Delta[k]$$
exists, where we think of X as the corresponding constant simplicial object.

Lemma 14.12.4. *With X and k as above. For any simplicial object V of $\mathcal{C}$ we have the following canonical bijection*
$$Mor_{Simp(\mathcal{C})}(X \times \Delta[k], V) \longrightarrow Mor_{\mathcal{C}}(X, V_k).$$
wich maps γ to the restriction of the morphism γ_k to the component corresponding to $id_{[k]}$. Similarly, for any $n \geq k$, if W is an n-truncated simplicial object of $\mathcal{C}$, then we have
$$Mor_{Simp_n(\mathcal{C})}(sk_n(X \times \Delta[k]), W) = Mor_{\mathcal{C}}(X, W_k).$$

Proof. A morphism $\gamma : X \times \Delta[k] \to V$ is given by a family of morphisms $\gamma_\alpha : X \to V_n$ where $\alpha : [n] \to [k]$. The morphisms have to satisfy the rules that for all $\varphi : [m] \to [n]$ the diagrams
$$\begin{tikzcd} X \arrow[r, "\gamma_\alpha"] \arrow[d, "\mathrm{id}_X"] & V_n \arrow[d, "V(\varphi)"] \\ X \arrow[r, "\gamma_{\alpha\circ\varphi}"] & V_m \end{tikzcd}$$
commute. Taking $\alpha = \mathrm{id}_{[k]}$, we see that for any $\varphi : [m] \to [k]$ we have $\gamma_\varphi = V(\varphi) \circ \gamma_{\mathrm{id}_{[k]}}$. Thus the morphism γ is determined by the value of γ on the component corresponding to $\mathrm{id}_{[k]}$. Conversely, given such a morphism $f : X \to V_k$ we easily construct a morphism γ by putting $\gamma_\alpha = V(\alpha) \circ f$.

The truncated case is similar, and left to the reader. $\square$

A particular example of this is the case $k = 0$. In this case the formula of the lemma just says that
$$Mor_{\mathcal{C}}(X, V_0) = Mor_{\mathrm{Simp}(\mathcal{C})}(X, V)$$
where on the right hand side X indicates the constant simplicial object with value X. We will use this formula without further mention in the following.

14.13. Hom from simplicial sets into cosimplicial objects

Let $\mathcal{C}$ be a category. Let U be a simplicial object of $\mathcal{C}$, and let V be a cosimplicial object of $\mathcal{C}$. Then we get a cosimplicial set $Hom_\mathcal{C}(U, V)$ as follows:

(1) we set $Hom_\mathcal{C}(U, V)_n = Mor_\mathcal{C}(U_n, V_n)$, and
(2) for $\varphi : [m] \to [n]$ we take the map $Hom_\mathcal{C}(U, V)_m \to Hom_\mathcal{C}(U, V)_n$ given by $f \mapsto V(\varphi) \circ f \circ U(\varphi)$.

This is our motivation for the following definition.

Definition 14.13.1. Let $\mathcal{C}$ be a category with finite products. Let V be a cosimplicial object of $\mathcal{C}$. Let U be a simplicial set such that each U_n is finite nonempty. We define $Hom(U, V)$ to be the cosimplicial object of $\mathcal{C}$ defined as follows:

(1) we set $Hom(U, V)_n = \prod_{u \in U_n} V_n$, in other words the unique object of $\mathcal{C}$ such that its X-valued points satisfy
$$Mor_\mathcal{C}(X, Hom(U, V)_n) = \text{Map}(U_n, Mor_\mathcal{C}(X, V_n))$$
and
(2) for $\varphi : [m] \to [n]$ we take the map $Hom(U, V)_m \to Hom(U, V)_n$ given by $f \mapsto V(\varphi) \circ f \circ U(\varphi)$ on X-valued points as above.

We leave it to the reader to spell out the definition in terms of maps between products. We also point out that the construction is functorial in both U (contravariantly) and V (covariantly), exactly as in Lemma 14.12.3 in the case of products of simplicial sets with simplicial objects.

14.14. Internal Hom

Let $\mathcal{C}$ be a category with finite nonempty products. Let U, V be simplicial objects $\mathcal{C}$. In some cases the functor
$$\begin{array}{ccc} \text{Simp}(\mathcal{C})^{opp} & \longrightarrow & Sets \\ W & \longmapsto & Mor_{\text{Simp}(\mathcal{C})}(W \times V, U) \end{array}$$
is representable. In this case we denote $\mathcal{H}om(V, U)$ the resulting simplicial object of $\mathcal{C}$, and we say that the *internal hom of V into U exists*. Moreover, in this case we would have
$$\begin{aligned} Mor_\mathcal{C}(X, \mathcal{H}om(V, U)_n) & = Mor_{\text{Simp}(\mathcal{C})}(X \times \Delta[n], \mathcal{H}om(V, U)) \\ & = Mor_{\text{Simp}(\mathcal{C})}(X \times \Delta[n] \times V, U) \\ & = Mor_{\text{Simp}(\mathcal{C})}(X, \mathcal{H}om(\Delta[n] \times V, U)) \\ & = Mor_\mathcal{C}(X, \mathcal{H}om(\Delta[n] \times V, U)_0) \end{aligned}$$
provided that $\mathcal{H}om(\Delta[n] \times V, U)$ exists also. Here we have used the material from Section 14.12.

The lesson we learn from this is that, given U and V, if we want to construct the internal hom then we should try to construct the objects
$$\mathcal{H}om(\Delta[n] \times V, U)_0$$
because these should be the nth term of $\mathcal{H}om(V, U)$. In the next section we study a construction of simplicial objects ``$Hom(\Delta[n], U)$''.

14.15. Hom from simplicial sets into simplicial objects

Motivated by the discussion on internal hom we define what should be the simplicial object classifying morphisms from a simplicial set into a given simplicial object of the category $\mathcal{C}$.

Definition 14.15.1. Let $\mathcal{C}$ be a category such that the coproduct of any two objects exists. Let U be a simplicial set, with U_n finite nonempty for all $n \geq 0$. Let V be a simplicial object of $\mathcal{C}$. We denote $Hom(U, V)$ any simplicial object of $\mathcal{C}$ such that

$$Mor_{\text{Simp}(\mathcal{C})}(W, Hom(U, V)) = Mor_{\text{Simp}(\mathcal{C})}(W \times U, V)$$

functorially in the simplicial object W of $\mathcal{C}$.

Of course $Hom(U, V)$ need not exist. Also, by the discussion in Section 14.14 we expect that if it does exist, then $Hom(U, V)_n = Hom(U \times \Delta[n], V)_0$. We do not use the italic notation for these Hom objects since $Hom(U, V)$ is not an internal hom.

Lemma 14.15.2. *Assume the category $\mathcal{C}$ has coproducts of any two objects and countable limits. Let U be a simplicial set, with U_n finite nonempty for all $n \geq 0$. Let V be a simplicial object of $\mathcal{C}$. Then the functor*

$$\begin{array}{ccc} \mathcal{C}^{opp} & \longrightarrow & Sets \\ X & \longmapsto & Mor_{Simp(\mathcal{C})}(X \times U, V) \end{array}$$

is representable.

Proof. A morphism from $X \times U$ into V is given by a collection of morphisms $f_u : X \to V_n$ with $n \geq 0$ and $u \in U_n$. And such a collection actually defines a morphism if and only if for all $\varphi : [m] \to [n]$ all the diagrams

$$\begin{tikzcd} X \arrow[r, "f_u"] \arrow[d, "\mathrm{id}_X"'] & V_n \arrow[d, "V(\varphi)"] \\ X \arrow[r, "f_{U(\varphi)(u)}"] & V_m \end{tikzcd}$$

commute. Thus it is natural to introduce a category $\mathcal{U}$ and a functor $\mathcal{V} : \mathcal{U}^{opp} \to \mathcal{C}$ as follows:

1. The set of objects of $\mathcal{U}$ is $\coprod_{n \geq 0} U_n$,
2. a morphism from $u' \in U_m$ to $u \in U_n$ is a $\varphi : [m] \to [n]$ such that $U(\varphi)(u) = u'$
3. for $u \in U_n$ we set $\mathcal{V}(u) = V_n$, and
4. for $\varphi : [m] \to [n]$ such that $U(\varphi)(u) = u'$ we set $\mathcal{V}(\varphi) = V(\varphi) : V_n \to V_m$.

At this point it is clear that our functor is nothing but the functor defining

$$lim_{\mathcal{U}^{opp}} \mathcal{V}$$

Thus if $\mathcal{C}$ has countable limits then this limit and hence an object representing the functor of the lemma exist. $\square$

Lemma 14.15.3. *Assume the category $\mathcal{C}$ has coproducts of any two objects and finite limits. Let U be a simplicial set, with U_n finite nonempty for all $n \geq 0$. Assume that all n-simplices of U are degenerate for all $n \gg 0$. Let V be a simplicial object of $\mathcal{C}$. Then the functor*

$$\begin{array}{ccc} \mathcal{C}^{opp} & \longrightarrow & Sets \\ X & \longmapsto & Mor_{Simp(\mathcal{C})}(X \times U, V) \end{array}$$

is representable.

Proof. We have to show that the category $\mathcal{U}$ described in the proof of Lemma 14.15.2 has a finite subcategory $\mathcal{U}'$ such that the limit of $\mathcal{V}$ over $\mathcal{U}'$ is the same as the limit of $\mathcal{V}$ over $\mathcal{U}$. We will use Categories, Lemma 4.15.4. For $m > 0$ let $\mathcal{U}_{\leq m}$ denote the full subcategory with objects $\coprod_{0\leq n\leq m} U_m$. Let m_0 be an integer such that every n-simplex of the simplicial set U is degenerate if $n > m_0$. For any $m \geq m_0$ large enough, the subcategory $\mathcal{U}_{\leq m}$ satisfies property (1) of the lemma cited above.

Suppose that $u \in U_n$ and $u' \in U_{n'}$ with $n, n' \leq m_0$ and suppose that $\varphi : [k] \to [n]$, $\varphi' : [k] \to [n']$ are morphisms such that $U(\varphi)(u) = U(\varphi')(u')$. A simple combinatorial argument shows that if $k > 2m_0$, then there exists an index $0 \leq i \leq 2m_0$ such that $\varphi(i) = \varphi(i+1)$ and $\varphi'(i) = \varphi'(i+1)$. (The pidgeon hole principle would tell you this works if $k > m_0^2$ which is good enough for the argument below anyways.) Hence, if $k > 2m_0$, we may write $\varphi = \psi \circ \sigma^{k-1}_i$ and $\varphi' = \psi' \circ \sigma^{k-1}_i$ for some $\psi : [k-1] \to [n]$ and some $\psi' : [k-1] \to [n']$. Since $s^{k-1}_i : U_{k-1} \to U_k$ is injective, see Lemma 14.3.6, we conclude that $U(\psi)(u) = U(\psi')(u')$ also. Continuing in this fashion we conclude that given morphisms $u \to z$ and $u' \to z$ of $\mathcal{U}$ with $u, u' \in \mathcal{U}_{\leq m_0}$, there exists a commutative diagram

$$\begin{tikzcd}
u \arrow[dr] \arrow[drr] & & \\
& a \arrow[r] & z \\
u' \arrow[ur] \arrow[urr] & &
\end{tikzcd}$$

with $a \in \mathcal{U}_{\leq 2m_0}$.

It is easy to deduce from this that the finite subcategory $\mathcal{U}_{\leq 2m_0}$ works. Namely, suppose given $x' \in U_n$ and $x'' \in U_{n'}$ with $n, n' \leq 2m_0$ as well as morphisms $x' \to x$ and $x'' \to x$ of $\mathcal{U}$ with the same target. By our choice of m_0 we can find objects u, u' of $\mathcal{U}_{\leq m_0}$ and morphisms $u \to x'$, $u' \to x''$. By the above we can find $a \in \mathcal{U}_{\leq 2m_0}$ and morphisms $u \to a$, $u' \to a$ such that

$$\begin{tikzcd}
u \arrow[r] \arrow[dr] \arrow[drr] & x' \arrow[dr] & \\
& a \arrow[r] & x \\
u' \arrow[r] \arrow[ur] \arrow[urr] & x'' \arrow[ur] &
\end{tikzcd}$$

is commutative. Turning this diagram 90 degrees clockwise we get the desired diagram as in (2) of the cited lemma. ☐

Lemma 14.15.4. *Assume the category $\mathcal{C}$ has coproducts of any two objects and finite limits. Let U be a simplicial set, with U_n finite nonempty for all $n \geq 0$. Assume that all n-simplices of U are degenerate for all $n \gg 0$. Let V be a simplicial object of $\mathcal{C}$. Then $Hom(U, V)$ exists, moreover we have the expected equalities*

$$Hom(U, V)_n = Hom(U \times \Delta[n], V)_0.$$

Proof. We construct this simplicial object as follows. For $n \geq 0$ let $Hom(U, V)_n$ denote the object of $\mathcal{C}$ representing the functor

$$X \longmapsto Mor_{\text{Simp}(\mathcal{C})}(X \times U \times \Delta[n], V)$$

This exists by Lemma 14.15.3 because $U \times \Delta[n]$ is a simplicial set with finite sets of simplices and no nondegenerate simplices in high enough degree, see Lemma 14.11.5. For $\varphi : [m] \to [n]$ we obtain an induced map of simplicial sets $\varphi : \Delta[m] \to \Delta[n]$. Hence we obtain a morphism $X \times U \times \Delta[m] \to X \times U \times \Delta[n]$ functorial in X, and hence a transformation of functors, which in turn gives

$$Hom(U, V)(\varphi) : Hom(U, V)_n \longrightarrow Hom(U, V)_m.$$

Clearly this defines a contravariant functor $Hom(U, V)$ from Δ into the category $\mathcal{C}$. In other words, we have a simplicial object of $\mathcal{C}$.

We have to show that $Hom(U, V)$ satisfies the desired universal property

$$Mor_{\text{Simp}(\mathcal{C})}(W, Hom(U, V)) = Mor_{\text{Simp}(\mathcal{C})}(W \times U, V)$$

To see this, let $f : W \to Hom(U, V)$ be given. We want to construct the element $f' : W \times U \to V$ of the right hand side. By construction, each $f_n : W_n \to Hom(U, V)_n$ corresponds to a morphism $f_n : W_n \times U \times \Delta[n] \to V$. Further, for every morphism $\varphi : [m] \to [n]$ the diagram

$$\begin{tikzcd}
W_n \times U \times \Delta[m] \arrow[r, "W(\varphi)\times\mathrm{id}\times\mathrm{id}"'] \arrow[d, "\mathrm{id}\times\mathrm{id}\times\varphi"'] & W_m \times U \times \Delta[m] \arrow[d, "f_m"] \\
W_n \times U \times \Delta[n] \arrow[r, "f_n"] & V
\end{tikzcd}$$

is commutative. For $\psi : [n] \to [k]$ in $(\Delta[n])_k$ we denote $(f_n)_{k,\psi} : W_n \times U_k \to V_k$ the component of $(f_n)_k$ corresponding to the element ψ. We define $f'_n : W_n \times U_n \to V_n$ as $f'_n = (f_n)_{n,\mathrm{id}}$, in other words, as the restriction of $(f_n)_n : W_n \times U_n \times (\Delta[n])_n \to V_n$ to $W_n \times U_n \times \mathrm{id}_{[n]}$. To see that the collection (f'_n) defines a morphism of simplicial objects, we have to show for any $\varphi : [m] \to [n]$ that $V(\varphi) \circ f'_n = f'_m \circ W(\varphi) \times U(\varphi)$. The commutative diagram above says that $(f_n)_{m,\varphi} : W_n \times U_m \to V_m$ is equal to $(f_m)_{m,\mathrm{id}} \circ W(\varphi) : W_n \times U_m \to V_m$. But then the fact that f_n is a morphism of simplicial objects implies that the diagram

$$\begin{tikzcd}
W_n \times U_n \times (\Delta[n])_n \arrow[r, "(f_n)_n"'] \arrow[d, "\mathrm{id}\times U(\varphi)\times\varphi"'] & V_n \arrow[d, "V(\varphi)"] \\
W_n \times U_m \times (\Delta[n])_m \arrow[r, "(f_n)_m"] & V_m
\end{tikzcd}$$

is commutative. And this implies that $(f_n)_{m,\varphi} \circ U(\varphi)$ is equal to $V(\varphi) \circ (f_n)_{n,\mathrm{id}}$. Alltogether we obtain $V(\varphi) \circ (f_n)_{n,\mathrm{id}} = (f_n)_{m,\varphi} \circ U(\varphi) = (f_m)_{m,\mathrm{id}} \circ W(\varphi) \circ U(\varphi) = (f_m)_{m,\mathrm{id}} \circ W(\varphi) \times U(\varphi)$ as desired.

On the other hand, given a morphism $f' : W \times U \to V$ we define a morphism $f : W \to Hom(U, V)$ as follows. By Lemma 14.12.4 the morphisms $\mathrm{id} : W_n \to W_n$ corresponds to a unique morphism $c_n : W_n \times \Delta[n] \to W$. Hence we can consider the composition

$$W_n \times \Delta[n] \times U \xrightarrow{c_n} W \times U \xrightarrow{f'} V.$$

By construction this corresponds to a unique morphism $f_n : W_n \to Hom(U, V)_n$. We leave it to the reader to see that these define a morphism of simplicial sets as desired.

We also leave it to the reader to see that $f \mapsto f'$ and $f' \mapsto f$ are mutually inverse operations. $\square$

We spell out the construction above in a special case. Let X be an object of a category $\mathcal{C}$. Assume that self products $X \times \ldots \times X$ exist. Let k be an integer. Consider the simplicial object U with terms

$$U_n = \prod\nolimits_{\alpha \in Mor([k],[n])} X$$

and maps given $\varphi : [m] \to [n]$

$$\begin{matrix} U(\varphi) : \prod_{\alpha \in Mor([k],[n])} X & \longrightarrow & \prod_{\alpha' \in Mor([k],[m])} X \\ (f_\alpha)_\alpha & \longmapsto & (f_{\varphi \circ \alpha'})_{\alpha'} \end{matrix}$$

In terms of ``coordinates", the element $(x_\alpha)_\alpha$ is mapped to the element $(x_{\varphi \circ \alpha'})_{\alpha'}$. We claim this object is equal to

$$Hom(\Delta[k], X)$$

where we think of X as the constant simplicial object X.

Lemma 14.15.5. *With X, k and U as above.*

1. *For any simplicial object V of $\mathcal{C}$ we have the following canonical bijection*
$$Mor_{Simp(\mathcal{C})}(V, U) \longrightarrow Mor_{\mathcal{C}}(V_k, X).$$
wich maps γ to the morphism γ_k composed with the projection onto the factor corresponding to $id_{[k]}$.
2. *Similarly, if W is an k-truncated simplicial object of $\mathcal{C}$, then we have*
$$Mor_{Simp_k(\mathcal{C})}(W, sk_k U) = Mor_{\mathcal{C}}(W_k, X).$$
3. *The object U constructed above is an incarnation of $Hom(\Delta[k], X)$.*

Proof. We first prove (1). Suppose that $\gamma : V \to U$ is a morphism. This is given by a family of morphisms $\gamma_\alpha : V_n \to X$ for $\gamma : [k] \to [n]$. The morphisms have to satisfy the rules that for all $\varphi : [m] \to [n]$ the diagrams

$$\begin{tikzcd} X \arrow[d, "\mathrm{id}_X"] & V_n \arrow[l, "\gamma_{\varphi \circ \alpha'}"'] \arrow[d, "V(\varphi)"] \\ X & V_m \arrow[l, "\gamma_{\alpha'}"] \end{tikzcd}$$

commute for all $\alpha' : [k] \to [m]$. Taking $\alpha' = \mathrm{id}_{[k]}$, we see that for any $\varphi : [k] \to [n]$ we have $\gamma_\varphi = \gamma_{\mathrm{id}_{[k]}} \circ V(\varphi)$. Thus the morphism γ is determined by the component of γ_k corresponding to $\mathrm{id}_{[k]}$. Conversely, given such a morphism $f : V_k \to X$ we easily construct a morphism γ by putting $\gamma_\alpha = f \circ V(\alpha)$.

The truncated case is similar, and left to the reader.

To see (3) we argue as follows:

$$\begin{aligned} Mor(V, Hom(\Delta[k], X)) &= Mor(V \times \Delta[k], X) \\ &= \{(f_n : V_n \times \Delta[k]_n \to X) \mid f_n \text{ compatible}\} \\ &= \{(f_n : V_n \to \prod\nolimits_{\Delta[k]_n} X) \mid f_n \text{ compatible}\} \\ &= Mor(V, U) \end{aligned}$$

Thus U and $Hom(\Delta[k], X)$ define the same functor on the category of simplicial objects and hence are canonically isomorphic. $\square$

Lemma 14.15.6. *Assume the category $\mathcal{C}$ has coproducts of any two objects and finite limits. Let $a : U \to V$, $b : U \to W$ be morphisms of simplicial sets. Assume U_n, V_n, W_n finite nonempty for all $n \geq 0$. Assume that all n-simplices of U, V, W are degenerate for all $n \gg 0$. Let T be a simplicial object of $\mathcal{C}$. Then*

$$Hom(V,T) \times_{Hom(U,T)} Hom(W,T) = Hom(V \amalg_U W, T)$$

In other words, the fibre product on the left hand side is represented by the Hom object on the right hand side.

Proof. By Lemma 14.15.4 all the required Hom objects exist and satisfy the correct functorial properties. Now we can identify the nth term on the left hand side as the object representing the functor that associates to X the first set of the following sequence of functorial equalities

$$\begin{aligned}
&Mor(X \times \Delta[n], Hom(V,T) \times_{Hom(U,T)} Hom(W,T)) \\
&= Mor(X \times \Delta[n], Hom(V,T)) \times_{Mor(X\times\Delta[n], Hom(U,T))} Mor(X \times \Delta[n], Hom(W,T)) \\
&= Mor(X \times \Delta[n] \times V, T) \times_{Mor(X\times\Delta[n]\times U,T)} Mor(X \times \Delta[n] \times W, T) \\
&= Mor(X \times \Delta[n] \times (V \amalg_U W), T))
\end{aligned}$$

Here we have used the fact that

$$(X \times \Delta[n] \times V) \times_{X\times\Delta[n]\times U} (X \times \Delta[n] \times W) = X \times \Delta[n] \times (V \amalg_U W)$$

which is easy to verify term by term. The result of the lemma follows as the last term in the displayed sequence of equalities corresponds to $Hom(V \amalg_U W, T)_n$. $\square$

14.16. Splitting simplicial objects

A subobject N of an object X of the category $\mathcal{C}$ is an object N of $\mathcal{C}$ together with a monomorphism $N \to X$. Of course we say (by abouse of notation) that the subobjects N, N' are equal if there exists an isomorphism $N \to N'$ compatible with the morphisms to X. The collection of subobjects forms a partially ordered set. (Because of our conventions on categories; not true for category of spaces up to homotopy for example.)

Definition 14.16.1. Let $\mathcal{C}$ be a category which admits finite nonempty coproducts. We say a simplicial object U of $\mathcal{C}$ is *split* if there exist subobjects $N(U_m)$ of U_m, $m \geq 0$ with the property that

$$\coprod\nolimits_{\varphi : [n]\to[m] \text{ surjective}} N(U_m) \longrightarrow U_n \tag{14.16.1.1}$$

is an isomorphism for all $n \geq 0$.

If this is the case, then $N(U_0) = U_0$. Next, we have $U_1 = U_0 \coprod N(U_1)$. Second we have

$$U_2 = U_0 \coprod N(U_1) \coprod N(U_1) \coprod N(U_2).$$

It turns out that in many categories $\mathcal{C}$ every simplicial object is split.

Lemma 14.16.2. *Let U be a simplicial set. Then U has a splitting with $N(U_m)$ equal to the set of nondegenerate m-simplices.*

Proof. Let $x \in U_n$. Suppose that there are surjections $\varphi : [n] \to [k]$ and $\psi : [n] \to [l]$ and nondegenerate simplices $y \in U_k$, $z \in U_l$ such that $x = U(\varphi)(y)$ and $x = U(\psi)(z)$. Choose a right inverse $\xi : [l] \to [n]$ of ψ, i.e., $\psi \circ \xi = \text{id}_{[l]}$. Then $z = U(\xi)(x)$. Hence $z = U(\xi)(x) = U(\varphi \circ \xi)(y)$. Since z is nondegenerate we conclude that $\varphi \circ \xi : [l] \to [k]$ is surjective, and hence $l \geq k$. Similarly $k \geq l$. Hence we see that $\varphi \circ \xi : [l] \to [k]$ has to be the identity map for any choice of right inverse ξ of ψ. This easily implies that $\psi = \varphi$. $\square$

Of course it can happen that a map of simplicial sets maps a nondegenerate n-simplex to a degenerate n-simplex. Thus the splitting of Lemma 14.16.2 is not functorial. Here is a case where it is functorial.

Lemma 14.16.3. *Let $f : U \to V$ be a morphism of simplicial sets. Suppose that (a) the image of every nondegenerate simplex of U is a nondegerate simplex of V and (b) no two nondegenerate simplices of U are mapped to the same simplex of V. Then f_n is injective for all n. Same holds with ``injective'' replaced by ``surjective'' or ``bijective''.*

Proof. Under hypothesis (a) we see that the map f preserves the disjoint union decompositions of the splitting of Lemma 14.16.2, in other words that we get commutative diagrams

$$\begin{tikzcd} \coprod_{\varphi : [n] \to [m] \text{ surjective}} N(U_m) \arrow[r] \arrow[d] & U_n \arrow[d] \\ \coprod_{\varphi : [n] \to [m] \text{ surjective}} N(V_m) \arrow[r] & V_n. \end{tikzcd}$$

And then (b) clearly shows that the left vertical arrow is injective (resp. surjective, resp. bijective). $\square$

Lemma 14.16.4. *Let U be a simplicial set. Let $n \geq 0$ be an integer. The rule*

$$U'_m = \bigcup\nolimits_{\varphi : [m] \to [i],\ i \leq n} Im(U(\varphi))$$

defines a sub simplicial set $U' \subset U$ with $U'_i = U_i$ for $i \leq n$. Moreover, all m-simplices of U' are degenerate for all $m > n$.

Proof. If $x \in U_m$ and $x = U(\varphi)(y)$ for some $y \in U_i$, $i \leq n$ and some $\varphi : [m] \to [i]$ then any image $U(\psi)(x)$ for any $\psi : [m'] \to [m]$ is equal to $U(\varphi \circ \psi)(y)$ and $\varphi \circ \psi : [m'] \to [i]$. Hence U' is a simplicial set. By construction all simplices in dimension $n + 1$ and higher are degenerate. $\square$

Lemma 14.16.5. *Let U be a simplicial abelian group. Then U has a splitting obtained by taking $N(U_0) = U_0$ and for $m \geq 1$ taking*

$$N(U_m) = \bigcap\nolimits_{i=0}^{m-1} Ker(d_i^m).$$

Moreover, this splitting is functorial on the category of simplicial abelian groups.

Proof. By induction on n we will show that the choice of $N(U_m)$ in the lemma garantees that (14.16.1.1) is an isomorphism for $m \leq n$. This is clear for $n = 0$. In the rest of this proof we are going to drop the superscripts from the maps d_i and s_i in order to improve readability. We will also repeatedly use the relations from Remark 14.3.3.

First we make a general remark. For $0 \leq i \leq m$ and $z \in U_m$ we have $d_i(s_i(z)) = z$. Hence we can write any $x \in U_{m+1}$ uniquely as $x = x' + x''$ with $d_i(x') = 0$ and $x'' \in \text{Im}(s_i)$ by taking $x' = (x - s_i(d_i(x)))$ and $x'' = s_i(d_i(x))$. Moreover, the element $z \in U_m$ such that $x'' = s_i(z)$ is unique because s_i is injective.

Here is a procedure for decomposing any $x \in U_{n+1}$. First, write $x = x_0 + s_0(z_0)$ with $d_0(x_0) = 0$. Next, write $x_0 = x_1 + s_1(z_1)$ with $d_n(x_1) = 0$. Continue like this to get

$$\begin{array}{rcl} x & = & x_0 + s_0(z_0), \\ x_0 & = & x_1 + s_1(z_1), \\ x_1 & = & x_2 + s_2(z_2), \\ \ldots & \ldots & \ldots \\ x_{n-1} & = & x_n + s_n(z_n) \end{array}$$

where $d_i(x_i) = 0$ for all $i = n, \ldots, 0$. By our general remark above all of the x_i and z_i are determined uniquely by x. We claim that $x_i \in \mathrm{Ker}(d_0) \cap \mathrm{Ker}(d_1) \cap \ldots \cap \mathrm{Ker}(d_i)$ and $z_i \in \mathrm{Ker}(d_0) \cap \ldots \cap \mathrm{Ker}(d_{i-1})$ for $i = n, \ldots, 0$. Here and in the following an empty intersection of kernels indicates the whole space; i.e., the notation $z_0 \in \mathrm{Ker}(d_0) \cap \ldots \cap \mathrm{Ker}(d_{i-1})$ when $i = 0$ means $z_0 \in U_n$ with no restriction.

We prove this by ascending induction on i. It is clear for $i = 0$ by construction of x_0 and z_0. Let us prove it for $0 < i \leq n$ assuming the result for $i-1$. First of all we have $d_i(x_i) = 0$ by construction. So pick a j with $0 \leq j < i$. We have $d_j(x_{i-1}) = 0$ by induction. Hence

$$0 = d_j(x_{i-1}) = d_j(x_i) + d_j(s_i(z_i)) = d_j(x_i) + s_{i-1}(d_j(z_i)).$$

The last equality by the relations of Remark 14.3.3. These relations also imply that $d_{i-1}(d_j(x_i)) = d_j(d_i(x_i)) = 0$ because $d_i(x_i) = 0$ by construction. Then the uniqueness in the general remark above shows the equality $0 = x' + x'' = d_j(x_i) + s_{i-1}(d_j(z_i))$ can only hold if both terms are zero. We conclude that $d_j(x_i) = 0$ and by injectivity of s_{i-1} we also conclude that $d_j(z_i) = 0$. This proves the claim.

The claim implies we can uniquely write

$$x = s_0(z_0) + s_1(z_1) + \ldots + s_n(z_n) + x_0$$

with $x_0 \in N(U_{n+1})$ and $z_i \in \mathrm{Ker}(d_0) \cap \ldots \cap \mathrm{Ker}(d_{i-1})$. We can reformulate this as saying that we have found a direct sum decomposition

$$U_{n+1} = N(U_{n+1}) \oplus \bigoplus\nolimits_{i=0}^{i=n} s_i\Big(\mathrm{Ker}(d_0) \cap \ldots \cap \mathrm{Ker}(d_{i-1})\Big)$$

with the property that

$$\mathrm{Ker}(d_0) \cap \ldots \cap \mathrm{Ker}(d_j) = N(U_{n+1}) \oplus \bigoplus\nolimits_{i=j+1}^{i=n} s_i\Big(\mathrm{Ker}(d_n) \cap \ldots \cap \mathrm{Ker}(d_{i-1})\Big)$$

for $j = 0, \ldots, n$. The result follows from this statement as follows. Each of the z_i in the expression for x can be written uniquely as

$$z_i = s_i(z'_{i,i}) + \ldots + s_{n-1}(z'_{i,n-1}) + z_{i,0}$$

with $z_{i,0} \in N(U_n)$ and $z'_{i,j} \in \mathrm{Ker}(d_0) \cap \ldots \cap \mathrm{Ker}(d_{j-1})$. The first few steps in the decomposition of z_i are zero because z_i already is in the kernel of $d_0, \ldots, d_i$. This in turn uniquely gives

$$x = x_0 + s_0(z_{0,0}) + s_1(z_{1,0}) + \ldots + s_n(z_{n,0}) + \sum\nolimits_{0 \leq i \leq j \leq n-1} s_i(s_j(z'_{i,j})).$$

Continuing in this fashion we see that we in the end obtain a decomposition of x as a sum of terms of the form

$$s_{i_1} s_{i_2} \ldots s_{i_k}(z)$$

with $0 \leq i_1 \leq i_2 \leq \ldots \leq i_k \leq n-k+1$ and $z \in N(U_{n+1-k})$. This is exactly the required decomposition, because any surjective map $[n+1] \to [n+1-k]$ can be uniquely expressed in the form

$$\sigma^{n-k}_{i_k} \ldots \sigma^{n-1}_{i_2} \sigma^n_{i_1}$$

with $0 \leq i_1 \leq i_2 \leq \ldots \leq i_k \leq n-k+1$. □

Lemma 14.16.6. *Let $\mathcal{A}$ be an abelian category. Let U be a simplicial object in $\mathcal{A}$. Then U has a splitting obtained by taking $N(U_0) = U_0$ and for $m \geq 1$ taking*

$$N(U_m) = \bigcap\nolimits_{i=0}^{m-1} Ker(d^m_i).$$

Moreover, this splitting is functorial on the category of simplicial objects of $\mathcal{A}$.

Proof. For any object A of $\mathcal{A}$ we obtain a simplicial abelian group $Mor_{\mathcal{A}}(A, U)$. Each of these are canonically split by Lemma 14.16.5. Moreover,

$$N(Mor_{\mathcal{A}}(A, U_m)) = \bigcap\nolimits_{i=0}^{m-1} \mathrm{Ker}(d^m_i) = Mor_{\mathcal{A}}(A, N(U_m)).$$

Hence we see that the morphism (14.16.1.1) becomes an isomorphism after applying the functor $Mor_{\mathcal{A}}(A, -)$ for any object of $\mathcal{A}$. Hence it is an isomorphism by the Yoneda lemma. □

Lemma 14.16.7. *Let $\mathcal{A}$ be an abelian category. Let $f : U \to V$ be a morphism of simplicial objects of $\mathcal{A}$. If the induced morphisms $N(f)_i : N(U)_i \to N(V)_i$ are injective for all i, then f_i is injective for all i. Same holds with ``injective'' replaced with ``surjective'', or ``isomorphism''.*

Proof. This is clear from Lemma 14.16.6 and the definition of a splitting. □

Lemma 14.16.8. *Let $\mathcal{A}$ be an abelian category. Let U be a simplicial object in $\mathcal{A}$. Let $N(U_m)$ as in Lemma 14.16.6 above. Then $d^m_m(N(U_m)) \subset N(U_{m-1})$.*

Proof. For $j = 0, \ldots, m-2$ we have $d^{m-1}_j d^m_m = d^{m-1}_{m-1} d^m_j$ by the relations in Remark 14.3.3. The result follows. □

Lemma 14.16.9. *Let $\mathcal{A}$ be an abelian category. Let U be a simplicial object of $\mathcal{A}$. Let $n \geq 0$ be an integer. The rule*

$$U'_m = \sum\nolimits_{\varphi : [m] \to [i],\ i \leq n} Im(U(\varphi))$$

defines a sub simplicial object $U' \subset U$ with $U'_i = U_i$ for $i \leq n$. Moreover, $N(U'_m) = 0$ for all $m > n$.

Proof. Pick m, $i \leq n$ and some $\varphi : [m] \to [i]$. The image under $U(\psi)$ of $\mathrm{Im}(U(\varphi))$ for any $\psi : [m'] \to [m]$ is equal to the image of $U(\varphi \circ \psi)$ and $\varphi \circ \psi : [m'] \to [i]$. Hence U' is a simplicial object. Pick $m > n$. We have to show $N(U'_m) = 0$. By definition of $N(U_m)$ and $N(U'_m)$ we have $N(U'_m) = U'_m \cap N(U_m)$ (intersection of subobjects). Since U is split by Lemma 14.16.6, it suffices to show that U'_m is contained in the sum

$$\sum\nolimits_{\varphi : [m] \to [m'] \text{ surjective},\ m' < m} \mathrm{Im}(U(\varphi)|_{N(U_{m'})}).$$

By the splitting each $U_{m'}$ is the sum of images of $N(U_{m''})$ via $U(\psi)$ for surjective maps $\psi : [m'] \to [m'']$. Hence the displayed sum above is the same as

$$\sum\nolimits_{\varphi : [m] \to [m'] \text{ surjective},\ m' < m} \mathrm{Im}(U(\varphi)).$$

Clearly U'_m is contained in this by the simple fact that any $\varphi : [m] \to [i]$, $i \leq n$ occuring in the definition of U'_m may be factored as $[m] \to [m'] \to [i]$ with $[m] \to [m']$ surjective and $m' < m$ as in the last displayed sum above. $\square$

14.17. Skelet and coskelet functors

Let $\Delta_{\leq n}$ denote the full subcategory of Δ with objects $[0], [1], [2], \ldots, [n]$. Let $\mathcal{C}$ be a category.

Definition 14.17.1. An *n-truncated simplicial object of $\mathcal{C}$* is a contravariant functor from $\Delta_{\leq n}$ to $\mathcal{C}$. A *morphism of n-truncated simplicial objects* is a transformation of functors. We denote the category of n-truncated simplicial objects of $\mathcal{C}$ by the symbol $\text{Simp}_n(\mathcal{C})$.

Given a simplicial object U of $\mathcal{C}$ the truncation $\text{sk}_n U$ is the restriction of U to the subcategory $\Delta_{\leq n}$. This defines a *skelet functor*

$$\text{sk}_n : \text{Simp}(\mathcal{C}) \longrightarrow \text{Simp}_n(\mathcal{C})$$

from the category of simplicial objects of $\mathcal{C}$ to the category of n-truncated simplicial objects of $\mathcal{C}$. See Remark 14.19.6 to avoid possible confusion with other functors in the literature.

The *coskelet functor* (if it exists) is a functor

$$\text{cosk}_n : \text{Simp}(\mathcal{C}) \longrightarrow \text{Simp}_n(\mathcal{C})$$

which is right adjoint to the skelet functor. In a formula

$$Mor_{\text{Simp}(\mathcal{C})}(U, \text{cosk}_n V) = Mor_{\text{Simp}_n(\mathcal{C})}(\text{sk}_n U, V) \tag{14.17.1.1}$$

Given a n-truncated simplicial object V we say that *$\text{cosk}_n V$ exists* if there exists a $\text{cosk}_n V \in Ob(\text{Simp}(\mathcal{C}))$ and a morphism $\text{sk}_n \text{cosk}_n V \to V$ such that the displayed formula holds, in other words if the functor $U \mapsto Mor_{\text{Simp}_n(\mathcal{C})}(\text{sk}_n U, V)$ is representable. If it exists it is unique up to unique isomorphism by the Yoneda lemma. See Categories, Section 4.3.

Example 14.17.2. Suppose the category $\mathcal{C}$ has finite nonempty self products. A 0-truncated simplicial object of $\mathcal{C}$ is the same as an object X of $\mathcal{C}$. In this case we claim that $\text{cosk}_0(X)$ is the simplicial object U with $U_n = X^{n+1}$ the $(n+1)$-fold self product of X, and structure of simplicial object as in Example 14.3.5. Namely, a morphism $V \to U$ where V is a simplicial object is given by morphisms $V_n \to X^{n+1}$, such that all the diagrams

$$\begin{tikzcd}
V_n \arrow[r] \arrow[d, "{V([0]\to[n], 0\mapsto i)}"'] & X^{n+1} \arrow[d, "\text{pr}_i"] \\
V_0 \arrow[r] & X
\end{tikzcd}$$

commute. Clearly this means that the map determines and is determined by a unique morphism $V_0 \to X$. This proves that formula (14.17.1.1) holds.

Recall the category $\Delta/[n]$, see Example 14.11.4. We let $(\Delta/[n])_{\leq m}$ denote the full subcategory of $\Delta/[n]$ consisting of objects $[k] \to [n]$ of $\Delta/[n]$ with $k \leq m$. In other words we have the following commutative diagram of categories and functors

$$\begin{tikzcd}
(\Delta/[n])_{\leq m} \arrow[r] \arrow[d] & \Delta/[n] \arrow[d] \\
\Delta_{\leq m} \arrow[r] & \Delta
\end{tikzcd}$$

Given a m-truncated simplicial object U of $\mathcal{C}$ we define a functor

$$U(n) : (\Delta/[n])^{opp}_{\leq m} \longrightarrow \mathcal{C}$$

by the rules

$$\begin{matrix} ([k] \to [n]) & \longmapsto & U_k \\ (\psi : ([k'] \to [n]) \to ([k] \to [n])) & \longmapsto & U(\psi) : U_k \to U_{k'} \end{matrix}$$

For a given morphism $\varphi : [n] \to [n']$ of Δ we have an associated functor

$$"\varphi" : (\Delta/[n])_{\leq m} \longrightarrow (\Delta/[n'])_{\leq m}$$

which maps $\alpha : [k] \to [n]$ to $\varphi \circ \alpha : [k] \to [n']$. The composition $U(n') \circ "\varphi"$ is equal to the functor $U(n)$.

Lemma 14.17.3. *If the category $\mathcal{C}$ has finite limits, then $cosk_m$ functors exist for all m. Moreover, for any m-truncated simplicial object U the simplicial object $cosk_m U$ is described by the formula*

$$(cosk_m U)_n = lim_{(\Delta/[n])^{opp}_{\leq m}} U(n)$$

and for $\varphi : [n] \to [n']$ the map $cosk_m U(\varphi)$ comes from the identification $U(n') \circ "\varphi" = U(n)$ above via Categories, Lemma 4.13.8.

Proof. During the proof of this lemma we denote $\text{cosk}_m U$ the simplicial object with $(\text{cosk}_m U)_n$ equal to $lim_{(\Delta/[n])^{opp}_{\leq m}} U(n)$. We will conclude at the end of the proof that it does satsify the required mapping property.

Suppose that V is a simplicial object. A morphism $\gamma : V \to \text{cosk}_m U$ is given by a sequence of morphisms $\gamma_n : V_n \to (\text{cosk}_m U)_n$. By definition of a limit, this is given by a collection of morphisms $\gamma(\alpha) : V_n \to U_k$ where α ranges over all $\alpha : [k] \to [n]$ with $k \leq m$. These morphisms then also satisfy the rules that

$$\begin{tikzcd} V_n \arrow[r, "\gamma(\alpha)"'] & U_k \\ V_{n'} \arrow[u, "V(\varphi)"] \arrow[r, "\gamma(\alpha')"] & U_{k'} \arrow[u, "U(\psi)"'] \end{tikzcd}$$

are commutative, given any $0 \leq k, k' \leq m$, $0 \leq n, n'$ and any $\psi : [k] \to [k']$, $\varphi : [n] \to [n']$, $\alpha : [k] \to [n]$ and $\alpha' : [k'] \to [n']$ in Δ such that $\varphi \circ \alpha = \alpha' \circ \psi$. Taking $n = k$, $\varphi = \alpha'$, and $\alpha = \psi = \text{id}_{[k]}$ we deduce that $\gamma(\alpha') = \gamma(\text{id}_{[k]}) \circ V(\alpha')$. In other words, the morphisms $\gamma(\text{id}_{[k]})$, $k \leq m$ determine the morphism γ. And it is easy to see that these morphisms form a morphism $\text{sk}_m V \to U$.

Conversely, given a morphism $\gamma : \text{sk}_m V \to U$, we obtain a family of morphsms $\gamma(\alpha)$ where α ranges over all $\alpha : [k] \to [n]$ with $k \leq m$ by setting $\gamma(\alpha) = \gamma(\text{id}_{[k]}) \circ V(\alpha)$. These morphisms satisfy all the displayed commutativity restraints pictured above, and hence give rise to a morphism $V \to \text{cosk}_m U$. $\square$

Lemma 14.17.4. *Let $\mathcal{C}$ be a category. Let U be an m-truncated simplicial object of $\mathcal{C}$. For $n \leq m$ the limit $lim_{(\Delta/[n])^{opp}_{\leq m}} U(n)$ exists and is canonically isomorphic to U_n.*

Proof. This is true because the category $(\Delta/[n])_{\leq m}$ has an final object in this case, namely the identity map $[n] \to [n]$. $\square$

Lemma 14.17.5. *Let $\mathcal{C}$ be a category with finite limits. Let U be an n-truncated simplicial object of $\mathcal{C}$. The morphism $sk_n cosk_n U \to U$ is an isomorphism.*

Proof. Combine Lemmas 14.17.3 and 14.17.4. □

Let us describe a particular instance of the coskelet functor in more detail. By abuse of notation we will denote sk_n also the restriction functor $\text{Simp}_{n'}(\mathcal{C}) \to \text{Simp}_n(\mathcal{C})$ for any $n' \geq n$. We are going to describe a right adjoint of the functor $\text{sk}_n : \text{Simp}_{n+1}(\mathcal{C}) \to \text{Simp}_n(\mathcal{C})$. For $n \geq 1$, $0 \leq i < j \leq n+1$ define $\delta^{n+1}_{i,j} : [n-1] \to [n+1]$ to be the increasing map omitting i and j. Note that $\delta^{n+1}_{i,j} = \delta^{n+1}_j \circ \delta^n_i = \delta^{n+1}_i \circ \delta^n_{j-1}$, see Lemma 14.2.3. This motivates the following lemma.

Lemma 14.17.6. *Let n be an integer ≥ 1. Let U be a n-truncated simplicial object of $\mathcal{C}$. Consider the contravariant functor from $\mathcal{C}$ to Sets which associates to an object T the set*

$$\{(f_0, \ldots, f_{n+1}) \in Mor_{\mathcal{C}}(T, U_n) \mid d^n_{j-1} \circ f_i = d^n_i \circ f_j \ \forall\ 0 \leq i < j \leq n+1\}$$

If this functor is representable by some object U_{n+1} of $\mathcal{C}$, then

$$U_{n+1} = lim_{(\Delta/[n+1])^{opp}_{\leq n}} U(n)$$

Proof. The limit, if it exists, represents the functor that associates to an object T the set

$$\{(f_\alpha)_{\alpha : [k] \to [n+1], k \leq n} \mid f_{\alpha \circ \psi} = U(\psi) \circ f_\alpha \ \forall\ \psi : [k'] \to [k], \alpha : [k] \to [n+1]\}.$$

In fact we will show this functor is isomorphic to the one displayed in the lemma. The map in one direction is given by the rule

$$(f_\alpha)_\alpha \longmapsto (f_{\delta^{n+1}_0}, \ldots, f_{\delta^{n+1}_{n+1}}).$$

This satisfies the conditions of the lemma because

$$d^n_{j-1} \circ f_{\delta^{n+1}_i} = f_{\delta^{n+1}_i \circ \delta^n_{j-1}} = f_{\delta^{n+1}_j \circ \delta^n_i} = d^n_i \circ f_{\delta^{n+1}_j}$$

by the relations we recalled above the lemma. To construct a map in the other direction we have to associate to a system $(f_0, \ldots, f_{n+1})$ as in the displayed formula of the lemma a system of maps f_α. Let $\alpha : [k] \to [n+1]$ be given. Since $k \leq n$ the map α is not surjective. Hence we can write $\alpha = \delta^{n+1}_i \circ \psi$ for some $0 \leq i \leq n+1$ and some $\psi : [k] \to [n]$. We have no choice but to define

$$f_\alpha = U(\psi) \circ f_i.$$

Of course we have to check that this is independent of the choice of the pair (i, ψ). First, observe that given i there is a unique ψ which works. Second, suppose that (j, ϕ) is another pair. Then $i \neq j$ and we may assume $i < j$. Since both i, j are not in the image of α we may actually write $\alpha = \delta^{n+1}_{i,j} \circ \xi$ and then we see that $\psi = \delta^n_{j-1} \circ \xi$ and $\phi = \delta^n_i \circ \xi$. Thus

$$\begin{aligned} U(\psi) \circ f_i &= U(\delta^n_{j-1} \circ \xi) \circ f_i \\ &= U(\xi) \circ d^n_{j-1} \circ f_i \\ &= U(\xi) \circ d^n_i \circ f_j \\ &= U(\delta^n_i \circ \xi) \circ f_j \\ &= U(\phi) \circ f_j \end{aligned}$$

as desired. We still have to verify that the maps f_α so defined satisfy the rules of a system of maps $(f_\alpha)_\alpha$. To see this suppose that $\psi : [k'] \to [k]$, $\alpha : [k] \to [n+1]$ with $k, k' \leq n$. Set $\alpha' = \alpha \circ \psi$. Choose i not in the image of α. Then clearly i is not in the image of α' also. Write $\alpha = \delta^n_i \circ \phi$ (we cannot use the letter ψ here because we've already used it). Then obviously $\alpha' = \delta^n_i \circ \phi \circ \psi$. By construction above we then have

$$U(\psi) \circ f_\alpha = U(\psi) \circ U(\phi) \circ f_i = U(\phi \circ \psi) \circ f_i = f_{\alpha \circ \psi} = f_{\alpha'}$$

as desired. We leave to the reader the pleasant task of verifying that our constructions are mutually inverse bijections, and are functorial in T. $\square$

Lemma 14.17.7. *Let n be an integer ≥ 1. Let U be a n-truncated simplicial object of $\mathcal{C}$. Consider the contravariant functor from $\mathcal{C}$ to Sets which associates to an object T the set*

$$\{(f_0, \ldots, f_{n+1}) \in Mor_{\mathcal{C}}(T, U_n) \mid d^n_{j-1} \circ f_i = d^n_i \circ f_j \ \forall \ 0 \leq i < j \leq n+1\}$$

If this functor is representable by some object U_{n+1} of $\mathcal{C}$, then there exists an $(n+1)$-truncated simplicial object $\tilde{U}$, with $sk_n\tilde{U} = U$ and $\tilde{U}_{n+1} = U_{n+1}$ such that the following adjointness holds

$$Mor_{Simp_{n+1}(\mathcal{C})}(V, \tilde{U}) = Mor_{Simp_n(\mathcal{C})}(sk_n V, U)$$

Proof. By Lemma 14.17.4 there are identifications

$$U_i = lim_{(\Delta/[i])^{opp}_{\leq n}} U(i)$$

for $0 \leq i \leq n$. By Lemma 14.17.6 we have

$$U_{n+1} = lim_{(\Delta/[n+1])^{opp}_{\leq n}} U(n).$$

Thus we may define for any $\varphi : [i] \to [j]$ with $i, j \leq n+1$ the corresponding map $\tilde{U}(\varphi) : \tilde{U}_j \to \tilde{U}_i$ exactly as in Lemma 14.17.3. This defines an $(n+1)$-truncated simplicial object $\tilde{U}$ with $\text{sk}_n\tilde{U} = U$.

To see the adjointness we argue as follows. Given any element $\gamma : \text{sk}_n V \to U$ of the right hand side of the formula consider the morphisms $f_i = \gamma_n \circ d^{n+1}_i : V_{n+1} \to V_n \to U_n$. These clearly satisfy the relations $d^n_{j-1} \circ f_i = d^n_i \circ f_j$ and hence define a unique morphism $V_{n+1} \to U_{n+1}$ by our choice of U_{n+1}. Conversely, given a morphsm $\gamma' : V \to \tilde{U}$ of the left hand side we can simply restrict to $\Delta_{\leq n}$ to get an element of the right hand side. We leave it to the reader to show these are mutually inverse constructions. $\square$

Remark 14.17.8. Let U, and U_{n+1} be as in Lemma 14.17.7. On T-valued points we can easily describe the face and degeneracy maps of $\tilde{U}$. Explicitly, the maps $d^{n+1}_i : U_{n+1} \to U_n$ are given by

$$(f_0, \ldots, f_{n+1}) \longmapsto f_i.$$

And the maps $s^n_j : U_n \to U_{n+1}$ are given by

$$\begin{aligned} f \quad \longmapsto \quad & (s^{n-1}_{j-1} \circ d^{n-1}_0 \circ f, \\ & s^{n-1}_{j-1} \circ d^{n-1}_1 \circ f, \\ & \ldots \\ & s^{n-1}_{j-1} \circ d^{n-1}_{j-1} \circ f, \\ & f, \\ & f, \\ & s^{n-1}_j \circ d^{n-1}_{j+1} \circ f, \\ & s^{n-1}_j \circ d^{n-1}_{j+2} \circ f, \\ & \ldots \\ & s^{n-1}_j \circ d^{n-1}_n \circ f) \end{aligned}$$

where we leave it to the reader to verify that the RHS is an element of the displayed set of Lemma 14.17.7. For $n = 0$ there is one map, namely $f \mapsto (f, f)$. For $n = 1$ there are two maps, namely $f \mapsto (f, f, s_0 d_1 f)$ and $f \mapsto (s_0 d_0 f, f, f)$. For $n = 2$ there are three maps, namely $f \mapsto (f, f, s_0 d_1 f, s_0 d_2 f)$, $f \mapsto (s_0 d_0 f, f, f, s_1 d_2 f)$, and $f \mapsto (s_1 d_0 f, s_1 d_1 f, f, f)$. And so on and so forth.

Remark 14.17.9. The construction of Lemma 14.17.7 above in the case of simplicial sets is the following. Given an n-truncated simplicial set U, we make a canonical $(n+1)$-truncated simplicial set $\tilde{U}$ as follows. We add a set of $(n+1)$-simplices U_{n+1} by the formula of the lemma. Namely, an element of U_{n+1} is a numbered collection of $(f_0, \ldots, f_{n+1})$ of n-simplices, with the property that they glue as they would in a $(n+1)$-simplex. In other words, the ith face of f_j is the $(j-1)$st face of f_i for $i < j$. Geometrically it is obvious how to define the face and degeneracy maps for $\tilde{U}$. If V is an $(n+1)$-truncated simplicial set, then its $(n+1)$-simplices give rise to compatible collections of n-simplices $(f_0, \ldots, f_{n+1})$ with $f_i \in V_n$. Hence there is a natural map $Mor(\text{sk}_n V, U) \to Mor(V, \tilde{U})$ which is inverse to the canonical restriction mapping the other way.

Also, it is enough to do the combinatorics of the construction in the case of truncated simplicial sets. Namely, for any object T of the category $\mathcal{C}$, and any n-truncated simplicial object U of $\mathcal{C}$ we can consider the n-truncated simplicial set $Mor(T, U)$. We may apply the construction to this, and take its set of $(n+1)$-simplices, and require this to be representable. This is a good way to think about the result of Lemma 14.17.7.

Remark 14.17.10. *Inductive construction of coskelets.* Suppose that $\mathcal{C}$ is a category with finite limits. Suppose that U is an m-truncated simplicial object in $\mathcal{C}$. Then we can inductively construct n-truncated objects U^n as follows:

(1) To start, set $U^m = U$.
(2) Given U^n for $n \geq m$ set $U^{n+1} = \tilde{U}^n$, where $\tilde{U}^n$ is constructed from U^n as in Lemma 14.17.7.

Since the construction of Lemma 14.17.7 has the property that it leaves the n-skeleton of U^n unchanged, we can then define $\text{cosk}_m U$ to be the simplicial object with $(\text{cosk}_m U)_n = U^n_n = U^{n+1}_n = \ldots$. And it follows formally from Lemma 14.17.7 that U^n satisfies the formula

$$Mor_{\text{Simp}_n(\mathcal{C})}(V, U^n) = Mor_{\text{Simp}_m(\mathcal{C})}(\text{sk}_m V, U)$$

for all $n \geq m$. It also then follows formally from this that

$$Mor_{\text{Simp}(\mathcal{C})}(V, \text{cosk}_m U) = Mor_{\text{Simp}_m(\mathcal{C})}(\text{sk}_m V, U)$$

with $\text{cosk}_m U$ chosen as above.

Lemma 14.17.11. *Let $\mathcal{C}$ be a category which has finite limits.*

(1) *For every n the functor $sk_n : Simp(\mathcal{C}) \to Simp_n(\mathcal{C})$ has a right adjoint $cosk_n$.*
(2) *For every $n' \geq n$ the functor $sk_n : Simp_{n'}(\mathcal{C}) \to Simp_n(\mathcal{C})$ has a right adjoint, namely $sk_{n'} cosk_n$.*
(3) *For every $m \geq n \geq 0$ and every n-truncated simplicial object U of $\mathcal{C}$ we have $cosk_m sk_m cosk_n U = cosk_n U$.*
(4) *If U is a simplicial object of $\mathcal{C}$ such that the canonical map $U \to cosk_n sk_n U$ is an isomorphism for some $n \geq 0$, then the canonical map $U \to cosk_m sk_m U$ is an isomorphism for all $m \geq n$.*

Proof. The existence in (1) follows from Lemma 14.17.3 above and the equality in (2), and (3) follows from the discussion in Remark 14.17.10. After this (4) is obvious. $\square$

Lemma 14.17.12. *Let U, V be n-truncated simplicial objects of a category $\mathcal{C}$. Then*

$$cosk_n(U \times V) = cosk_n U \times cosk_n V$$

whenever the left and right hand sides exist.

Proof. Let W be a simplicial object. We have

$$\begin{aligned} Mor(W, \text{cosk}_n(U \times V)) &= Mor(\text{sk}_n W, U \times V) \\ &= Mor(\text{sk}_n W, U) \times Mor(\text{sk}_n W, V) \\ &= Mor(W, \text{cosk}_n U) \times Mor(W, \text{cosk}_n V) \\ &= Mor(W, \text{cosk}_n U \times \text{cosk}_n V) \end{aligned}$$

The lemma follows. □

Lemma 14.17.13. *Assume $\mathcal{C}$ has fibre products. Let U, V, W be n-truncated simplicial objects of the category $\mathcal{C}$. Then*

$$cosk_n(V \times_U W) = cosk_n U \times_{cosk_n U} cosk_n V$$

Proof. Omitted, but very similar to the proof of Lemma 14.17.12 above. □

Lemma 14.17.14. *The canonical map $\Delta[n] \to cosk_1 sk_1 \Delta[n]$ is an isomorphism.*

Proof. Consider a simplicial set U and a morphism $f : U \to \Delta[n]$. This is a rule that associates to each $u \in U_i$ a map $f_u : [i] \to [n]$ in Δ. Furthermore, these maps should have the property that $f_u \circ \varphi = f_{U(\varphi)(u)}$ for any $\varphi : [j] \to [i]$. Denote $\epsilon^i_j : [0] \to [i]$ the map which maps 0 to j. Denote $F : U_0 \to [n]$ the map $u \mapsto f_u(0)$. Then we see that

$$f_u(j) = F(\epsilon^i_j(u))$$

for all $0 \leq j \leq i$ and $u \in U_i$. In particular, if we know the function F then we know the maps f_u for all $u \in U_i$ all i. Conversely, given a map $F : U_0 \to [n]$, we can set for any i, and any $u \in U_i$ and any $0 \leq j \leq i$

$$f_u(j) = F(\epsilon^i_j(u))$$

This does not in general define a morphism f of simplicial sets as above. Namely, the condition is that all the maps f_u are nondecreasing. This clearly is equivalent to the condition that $F(\epsilon^i_j(u)) \leq F(\epsilon^i_{j'}(u))$ whenever $0 \leq j \leq j' \leq i$ and $u \in U_i$. But in this case the morphisms

$$\epsilon^i_j, \epsilon^i_{j'} : [0] \to [i]$$

both factor through the map $\epsilon^i_{j,j'} : [1] \to [i]$ defined by the rules $0 \mapsto j$, $1 \mapsto j'$. In other words, it is enough to check the inequalities for $i = 1$ and $u \in X_1$. In other words, we have

$$Mor(U, \Delta[n]) = Mor(\text{sk}_1 U, \text{sk}_1 \Delta[n])$$

as desired. □

14.18. Augmentations

Definition 14.18.1. Let $\mathcal{C}$ be a category. Let U be a simplicial object of $\mathcal{C}$. An *augmentation $\epsilon : U \to X$ of U towards an object X of $\mathcal{C}$* is a morphism from U into the constant simplicial object X.

Lemma 14.18.2. *Let $\mathcal{C}$ be a category. Let $x \in Ob(\mathcal{C})$. Let U be a simplicial object of $\mathcal{C}$. To give an augmentation of U towards X is the same as giving a morphism $\epsilon_0 : U_0 \to X$ such that $\epsilon_0 \circ d^1_0 = \epsilon_0 \circ d^1_1$.*

Proof. Given a morphism $\epsilon : U \to X$ we certainly obtain an ϵ_0 as in the lemma. Conversely, given ϵ_0 as in the lemma, define $\epsilon_n : U_n \to X$ by choosing any morphism $\alpha : [0] \to [n]$ and taking $\epsilon_n = \epsilon_0 \circ U(\alpha)$. Namely, if $\beta : [0] \to [n]$ is another choice, then there exists a morphism $\gamma : [1] \to [n]$ such that α and β both factor as $[0] \to [1] \to [n]$. Hence the condition on ϵ_0 shows that ϵ_n is well defined. Then it is easy to show that $(\epsilon_n) : U \to X$ is a morphism of simplicial objects. □

Lemma 14.18.3. *Let $\mathcal{C}$ be a category with fibred products. Let $f : Y \to X$ be a morphism of $\mathcal{C}$. Let U be the simplicial object of $\mathcal{C}$ whose nth term is the $(n+1)$fold fibred product $Y \times_X Y \times_X \ldots \times_X Y$. See Example 14.3.5. For any simplicial object V of $\mathcal{C}$ we have*

$$\begin{aligned} Mor_{Simp(\mathcal{C})}(V, U) &= Mor_{Simp_{\leq 1}(\mathcal{C})}(sk_1 V, sk_1 U) \\ &= \{g_0 : V_0 \to Y \mid f \circ g_0 \circ d^1_0 = f \circ g_0 \circ d^1_1\} \end{aligned}$$

In particular we have $U = cosk_1 sk_1 U$.

Proof. Suppose that $g : \text{sk}_1 V \to \text{sk}_1 U$ is a morphism of 1-truncated simplicial objects. Then the diagram

$$\begin{tikzcd} V_1 \arrow[r, shift left, "d^1_0"] \arrow[r, shift right, "d^1_1"'] \arrow[d, "g_1"'] & V_0 \arrow[d, "g_0"] & \\ Y \times_X Y \arrow[r, shift left, "pr_1"] \arrow[r, shift right, "pr_0"'] & Y \arrow[r] & X \end{tikzcd}$$

is commutative, which proves that the relation shown in the lemma holds. We have to show that, conversely, given a morphism g_0 satisfying the relation $f \circ g_0 \circ d^1_0 = f \circ g_0 \circ d^1_1$ we get a unique morphism of simplicial objects $g : V \to U$. This is done as follows. For any $n \geq 1$ let $g_{n,i} = g_0 \circ V([0] \to [n], 0 \mapsto i) : V_n \to Y$. The equality above implies that $f \circ g_{n,i} = f \circ g_{n,i+1}$ because of the commutative diagram

$$\begin{tikzcd} {[0]} \arrow[dr, "0\mapsto 0"'] \arrow[drr, "0\mapsto i"] & & \\ & {[1]} \arrow[r, "{0\mapsto i, 1\mapsto i+1}"] & {[n]} \\ {[0]} \arrow[ur, "0\mapsto 1"] \arrow[urr, "0\mapsto i+1"'] & & \end{tikzcd}$$

Hence we get $(g_{n,0}, \ldots, g_{n,n}) : V_n \to Y \times_X \ldots \times_X Y = U_n$. We leave it to the reader to see that this is a morphism of simplicial objects. The last assertion of the lemma is equivalent to the first equality in the displayed formula of the lemma. □

Remark 14.18.4. Let $\mathcal{C}$ be a category with fibre products. Let V be a simplicial object. Let $\epsilon : V \to X$ be an augmentation. Let U be the simplicial object whose nth term is the $(n+1)$st fibred product of V_0 over X. By a simple combination of Lemmas 14.18.2 and 14.18.3 we obtain a canonical morphism $V \to U$.

14.19. Left adjoints to the skeleton functors

In this section we construct a left adjoint $i_{m!}$ of the skeleton functor sk_m in certain cases. The adjointness formula is

$$Mor_{\text{Simp}_m(\mathcal{C})}(U, \text{sk}_m V) = Mor_{\text{Simp}(\mathcal{C})}(i_{m!} U, V).$$

It turns out that this left adjoint exists when the category $\mathcal{C}$ has finite colimits.

We use a similar construction as in Section 14.17. Recall the category $[n]/\Delta$ of objects under $[n]$, see Categories, Example 4.2.14. Its objects are morphisms $\alpha : [n] \to [k]$ and its morphisms are commutative triangles. We let $([n]/\Delta)_{\leq m}$ denote the full subcategory of $[n]/\Delta$ consisting of objects $[n] \to [k]$ with $k \leq m$. Given a m-truncated simplicial object U of $\mathcal{C}$ we define a functor

$$U(n) : ([n]/\Delta)_{\leq m}^{opp} \longrightarrow \mathcal{C}$$

by the rules

$$\begin{array}{rcl} ([n] \to [k]) & \longmapsto & U_k \\ (\psi : ([n] \to [k']) \to ([n] \to [k])) & \longmapsto & U(\psi) : U_k \to U_{k'} \end{array}$$

For a given morphism $\varphi : [n] \to [n']$ of Δ we have an associated functor

$$"\varphi" : ([n']/\Delta)_{\leq m} \longrightarrow ([n]/\Delta)_{\leq m}$$

which maps $\alpha : [n'] \to [k]$ to $\varphi \circ \alpha : [n] \to [k]$. The composition $U(n) \circ "\varphi"$ is equal to the functor $U(n')$.

Lemma 14.19.1. *Let $\mathcal{C}$ be a category which has finite colimits. The functors $i_{m!}$ exist for all m. Let U be an m-truncated simplicial object of $\mathcal{C}$. The simplicial object $i_{m!}U$ is described by the formula*

$$(i_{m!}U)_n = colim_{([n]/\Delta)_{\leq m}^{opp}} U(n)$$

and for $\varphi : [n] \to [n']$ the map $i_{m!}U(\varphi)$ comes from the identification $U(n) \circ "\varphi" = U(n')$ above via Categories, Lemma 4.13.7.

Proof. In this proof we denote $i_{m!}U$ the simplicial object whose nth term is given by the displayed formula of the lemma. We will show it satisfies the adjointness property.

Let V be a simplicial object of $\mathcal{C}$. Let $\gamma : U \to \text{sk}_m V$ be given. A morphism

$$colim_{([n]/\Delta)_{\leq m}^{opp}} U(n) \to T$$

is given by a compatible system of morphisms $f_\alpha : U_k \to T$ where $\alpha : [n] \to [k]$ with $k \leq m$. Certainly, we have such a system of morphisms by taking the compositions

$$U_k \xrightarrow{\gamma_k} V_k \xrightarrow{V(\alpha)} V_n.$$

Hence we get an induced morphism $(i_{m!}U)_n \to V_n$. We leave it to the reader to see that these form a morphism of simplicial objects $\gamma' : i_{m!}U \to V$.

Coversely, given a morphism $\gamma' : i_{m!}U \to V$ we obtain a morphism $\gamma : U \to \text{sk}_m V$ by setting $\gamma_i : U_i \to V_i$ equal to the composition

$$U_i \xrightarrow{\text{id}_{[i]}} colim_{([i]/\Delta)_{\leq m}^{opp}} U(i) \xrightarrow{\gamma_i'} V_i$$

for $0 \leq i \leq n$. We leave it to the reader to see that this is the inverse of the construction above. $\square$

Lemma 14.19.2. *Let $\mathcal{C}$ be a category. Let U be an m-truncated simplicial object of $\mathcal{C}$. For any $n \leq m$ the colimit*

$$colim_{([n]/\Delta)_{\leq m}^{opp}} U(n)$$

exists and is equal to U_n.

Proof. This is so because the category $([n]/\Delta)_{\leq m}$ has an initial object, namely $\text{id} : [n] \to [n]$. $\square$

Lemma 14.19.3. *Let $\mathcal{C}$ be a category which has finite colimits. Let U be an m-truncated simplicial object of $\mathcal{C}$. The map $U \to sk_m i_{m!} U$ is an isomorphism.*

Proof. Combine Lemmas 14.19.1 and 14.19.2. $\square$

Lemma 14.19.4. *If U is an m-truncated simplicial set and $n > m$ then all n-simplices of $i_{m!}U$ are degenerate.*

Proof. This can be seen from the construction of $i_{m!}U$ in Lemma 14.19.1, but we can also argue directly as follows. Write $V = i_{m!}U$. Let $V' \subset V$ be the simplicial subset with $V'_i = V_i$ for $i \leq m$ and all i simplices degenerate for $i > m$, see Lemma 14.16.4. By the adjunction formula, since $\text{sk}_m V' = U$, there is an inverse to the injection $V' \to V$. Hence $V' = V$. $\square$

Lemma 14.19.5. *Let U be a simplicial set. Let $n \geq 0$ be an integer. The morphism $i_{n!}sk_n U \to U$ identifies $i_{n!}sk_n U$ with the simplicial set $U' \subset U$ defined in Lemma 14.16.4.*

Proof. By Lemma 14.19.4 the only nondegenerate simplices of $i_{n!}\text{sk}_n U$ are in degrees $\leq n$. The map $i_{n!}\text{sk}_n U \to U$ is an isomorphism in degrees $\leq n$. Combined we conclude that the map $i_{n!}\text{sk}_n U \to U$ maps nondegenerate simplices to nondegenerate simplices and no two nondegenerate simplices have the same image. Hence Lemma 14.16.3 applies. Thus $i_{n!}\text{sk}_n U \to U$ is injective. The result follows easily from this. $\square$

Remark 14.19.6. In some texts the composite functor

$$\text{Simp}(\mathcal{C}) \xrightarrow{\text{sk}_m} \text{Simp}_m(\mathcal{C}) \xrightarrow{i_{m!}} \text{Simp}(\mathcal{C})$$

is denoted sk_m. This makes sense because Lemma 14.19.5 says that $i_{m!}\text{sk}_m V$ is just the sub simplicial set of V consisting of all i-simplices of V, $i \leq m$ and their degeneracies. In those texts it is also customary to denote the composition

$$\text{Simp}(\mathcal{C}) \xrightarrow{\text{sk}_m} \text{Simp}_m(\mathcal{C}) \xrightarrow{\text{cosk}_m} \text{Simp}(\mathcal{C})$$

by cosk_m.

Lemma 14.19.7. *Let $U \subset V$ be simplicial sets. Suppose $n \geq 0$ and $x \in V_n$, $x \notin U_n$ are such that*

1. *$V_i = U_i$ for $i < n$,*
2. *$V_n = U_n \cup \{x\}$,*
3. *any $z \in V_j$, $z \notin U_j$ for $j > n$ is degenerate.*

Let $\Delta[n] \to V$ be the unique morphism mapping the nondegenerate n-simplex of $\Delta[n]$ to x. In this case the diagram

$$\begin{tikzcd} \Delta[n] \arrow[r] & V \\ i_{(n-1)!}sk_{n-1}\Delta[n] \arrow[r] \arrow[u] & U \arrow[u] \end{tikzcd}$$

is a push out diagram.

Proof. Let us denote $\partial\Delta[n] = i_{(n-1)!}\text{sk}_{n-1}\Delta[n]$ for convenience. There is a natural map $U \amalg_{\partial\Delta[n]} \Delta[n] \to V$. We have to show that it is bijective in degree j for all j. This is clear for $j \leq n$. Let $j > n$. The third condition means that any $z \in V_j$, $z \notin U_j$ is a degenerate simplex, say $z = s^{j-1}_i(z')$. Of course $z' \notin U_{j-1}$. By induction it follows that z' is a degeneracy of x. Thus we conclude that all j-simplices of V are either in U or degeneracies of x. This implies that the map $U \amalg_{\partial\Delta[n]} \Delta[n] \to V$ is surjective. Note that a nondegerate simplex of $U \amalg_{\partial\Delta[n]} \Delta[n]$ is either the image of a nondegenerate simplex of U, or the image of the (unique) nondegenerate n-simplex of $\Delta[n]$. Since clearly x is nondegenerate we deduce that $U \amalg_{\partial\Delta[n]} \Delta[n] \to V$ maps nondegenerate simplices to nondegenerate simplices and is injective on nondegenerate simplices. Hence it is injective, by Lemma 14.16.3. □

Lemma 14.19.8. *Let $U \subset V$ be simplicial sets, with U_n, V_n finite nonempty for all n. Assume that U and V have finitely many nondegenerate simplices. Then there exists a sequence of sub simplicial sets*

$$U = W^0 \subset W^1 \subset W^2 \subset \ldots W^r = V$$

such that Lemma 14.19.7 applies to each of the inclusions $W^i \subset W^{i+1}$.

Proof. Let n be the smallest integer such that V has a nondegenerate simplex that does not belong to U. Let $x \in V_n$, $x \notin U_n$ be such a nondegenerate simplex. Let $W \subset V$ be the set of elements which are either in U, or are a (repeated) degeneracy of x (in other words, are of the form $V(\varphi)(x)$ with $\varphi : [m] \to [n]$ surjective). It is easy to see that W is a simplicial set. The inclusion $U \subset W$ satisfies the conditions of Lemma 14.19.7. Moreover the number of nondegenerate simplices of V which are not contained in W is exactly one less than the number of nondegenerate simplices of V which are not contained in U. Hence we win by induction on this number. □

Lemma 14.19.9. *Let $\mathcal{A}$ be an abelian category Let U be an m-truncated simplicial object of $\mathcal{A}$. For $n > m$ we have $N(i_{m!}U)_n = 0$.*

Proof. Write $V = i_{m!}U$. Let $V' \subset V$ be the simplicial subobject of V with $V'_i = V_i$ for $i \leq m$ and $N(V'_i) = 0$ for $i > m$, see Lemma 14.16.9. By the adjunction formula, since $\text{sk}_m V' = U$, there is an inverse to the injection $V' \to V$. Hence $V' = V$. □

Lemma 14.19.10. *Let $\mathcal{A}$ be an abelian category. Let U be a simplicial object of $\mathcal{A}$. Let $n \geq 0$ be an integer. The morphism $i_{n!}sk_nU \to U$ identifies $i_{n!}sk_nU$ with the simplicial subobject $U' \subset U$ defined in Lemma 14.16.9.*

Proof. By Lemma 14.19.9 we have $N(i_{n!}\text{sk}_nU)_i = 0$ for $i > n$. The map $i_{n!}\text{sk}_nU \to U$ is an isomorphism in degrees $\leq n$, see Lemma 14.19.3. Combined we conclude that the map $i_{n!}\text{sk}_nU \to U$ induces injective maps $N(i_{n!}\text{sk}_nU)_i \to N(U)_i$ for all i. Hence Lemma 14.16.7 applies. Thus $i_{n!}\text{sk}_nU \to U$ is injective. The result follows easily from this. □

Here is another way to think about the coskelet functor using the material above.

Lemma 14.19.11. *Let $\mathcal{C}$ be a category with finite coproducts and finite limits. Let V be a simplicial object of $\mathcal{C}$. In this case*

$$(cosk_n sk_n V)_{n+1} = Hom(i_{n!}sk_n\Delta[n+1], V)_0.$$

Proof. By Lemma 14.12.4 the object on the left represents the functor which assigns to X the first set of the following equalities

$$\begin{aligned} Mor(X \times \Delta[n+1], \text{cosk}_n\text{sk}_n V) &= Mor(X \times \text{sk}_n\Delta[n+1], \text{sk}_n V) \\ &= Mor(X \times i_{n!}\text{sk}_n\Delta[n+1], V). \end{aligned}$$

The object on the right in the formula of the lemma is represented by the functor which assigns to X the last set in the sequence of equalities. This proves the result.

In the sequence of equalities we have used that $\text{sk}_n(X \times \Delta[n+1]) = X \times \text{sk}_n\Delta[n+1]$ and that $i_{n!}(X \times \text{sk}_n\Delta[n+1]) = X \times i_{n!}\text{sk}_n\Delta[n+1]$. The first equality is obvious. For any (possibly truncated) simplicial object W of $\mathcal{C}$ and any object X of $\mathcal{C}$ denote temporarily $Mor_{\mathcal{C}}(X, W)$

the (possibly truncted) simplicial set $[n] \mapsto Mor_{\mathcal{C}}(X, W_n)$. From the definitions it follows that $Mor(U \times X, W) = Mor(U, Mor_{\mathcal{C}}(X, W))$ for any (possibly truncted) simplicial set U. Hence

$$\begin{aligned} Mor(X \times i_{n!}\text{sk}_n\Delta[n+1], W) &= Mor(i_{n!}\text{sk}_n\Delta[n+1], Mor_{\mathcal{C}}(X, W)) \\ &= Mor(\text{sk}_n\Delta[n+1], \text{sk}_n\, Mor_{\mathcal{C}}(X, W)) \\ &= Mor(X \times \text{sk}_n\Delta[n+1], \text{sk}_n W) \\ &= Mor(i_{n!}(X \times \text{sk}_n\Delta[n+1]), W). \end{aligned}$$

This proves the second equality used, and ends the proof of the lemma. □

Lemma 14.19.12. *Let $\mathcal{C}$ be a category with finite coproducts and finite limits. Let X be an object of $\mathcal{C}$. Let $k \geq 0$. The canonical map*

$$Hom(\Delta[k], X) \longrightarrow cosk_1 sk_1\, Hom(\Delta[k], X)$$

is an isomorphism.

Proof. For any simplicial object V we have

$$\begin{aligned} Mor(V, \text{cosk}_1\text{sk}_1\, Hom(\Delta[k], X)) &= Mor(\text{sk}_1 V, \text{sk}_1\, Hom(\Delta[k], X)) \\ &= Mor(i_{1!}\text{sk}_1 V, Hom(\Delta[k], X)) \\ &= Mor(i_{1!}\text{sk}_1 V \times \Delta[k], X) \end{aligned}$$

The first equality by the adjointness of sk and cosk, the second equality by the adjointness of $i_{1!}$ and sk_1, and the first equality by Definition 14.15.1 where the last X denotes the constant simplicial object with value X. By Lemma 14.18.2 an element in this set depends only on the terms of degree 0 and 1 of $i_{1!}\text{sk}_1 V \times \Delta[k]$. These agree with the degree 0 and 1 terms of $V \times \Delta[k]$, see Lemma 14.19.3. Thus the set above is equal to $Mor(V \times \Delta[k], X) = Mor(V, Hom(\Delta[k], X))$. □

Lemma 14.19.13. *Let $\mathcal{C}$ be a category with finite coproducts and finite limits. Let X be an object of $\mathcal{C}$. Let $k \geq 0$. The canonical map*

$$Hom(\Delta[k], X)_1 \longrightarrow (cosk_0 sk_0\, Hom(\Delta[k], X))_1$$

is identified with the map

$$\prod\nolimits_{\alpha : [k] \to [1]} X \longrightarrow X \times X$$

which is the projection onto the factors where α is a constant map.

Proof. It is shown in Example 14.17.2 that $\text{cosk}_0 Z$ equals $Z \times Z$ in degree 1. Moreover, it is true in general that the morphism $V_1 \to (\text{cosk}_0\text{sk}_0 V)_1$ is the morphism $(d^1_0, d^1_1) : V_1 \to V_0 \times V_0$ (left to the reader). Thus we simply have to compute the 0th and 1st term of $Hom(\Delta[k], X)$. According to Lemma 14.15.5 we have $Hom(\Delta[k], X)_0 = \prod_{\alpha : [k] \to [0]} X = X$, and $Hom(\Delta[k], X)_0 = \prod_{\alpha : [k] \to [1]} X$. The lemma follows from the description of the morphisms of the simplicial object just above Lemma 14.15.5. □

14.20. Simplicial objects in abelian categories

Recall that an abelian category is defined in Homology, Section 10.3.

Lemma 14.20.1. *Let $\mathcal{A}$ be an abelian category.*

1. *The categories $Simp(\mathcal{A})$ and $CoSimp(\mathcal{A})$ are abelian.*
2. *A morphism of (co)simplicial objects $f : A \to B$ is injective if and only if each $f_n : A_n \to B_n$ is injective.*

(3) *A morphism of (co)simplicial objects $f : A \to B$ is surjective if and only if each $f_n : A_n \to B_n$ is surjective.*
(4) *A sequence of (co)simplicial objects*
$$A \xrightarrow{f} B \xrightarrow{g} C$$
is exact at B if and only if each sequence
$$A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i$$
is exact at B_i.

Proof. Pre-additivity is easy. A final object is given by $U_n = 0$ in all degrees. Existence of direct products we saw in Lemmas 14.6.2 and 14.9.2. Kernels and cokernels are obtained by taking termwise kernels and cokernels. $\square$

For an object A of $\mathscr{A}$ and an integer k consider the k-truncated simplicial object U with
(1) $U_i = 0$ for $i < k$,
(2) $U_k = A$,
(3) all morphisms $U(\varphi)$ equal to zero, except $U(\mathrm{id}_{[k]}) = \mathrm{id}_A$.

Since $\mathscr{A}$ has both finite limits and finite colimits we see that both $\mathrm{cosk}_k U$ and $i_{k!}U$ exist. We will describe both of these and the canonical map $i_{k!}U \to \mathrm{cosk}_k U$.

Lemma 14.20.2. *With A, k and U as above, so $U_i = 0$, $i < k$ and $U_k = A$.*
(1) *Given a k-truncated simplicial object V we have*
$$Mor(U, V) = \{f : A \to V_k \mid d^k_i \circ f = 0,\ i = 0, \ldots, k\}$$
and
$$Mor(V, U) = \{f : V_k \to A \mid f \circ s^{k-1}_i = 0,\ i = 0, \ldots, k-1\}.$$
(2) *The object $i_{k!}U$ has nth term equal to $\bigoplus_\alpha A$ where α runs over all surjective morphisms $\alpha : [n] \to [k]$.*
(3) *For any $\varphi : [m] \to [n]$ the map $i_{k!}U(\varphi)$ is described as the mapping $\bigoplus_\alpha A \to \bigoplus_{\alpha'} A$ which maps to component corresponding to $\alpha : [n] \to [k]$ to zero if $\alpha \circ \varphi$ is not surjective and by the identity to the component corresponding to $\alpha \circ \varphi$ if it is surjective.*
(4) *The object $cosk_k U$ has nth term equal to $\bigoplus_\beta A$, where β runs over all injective morphisms $\beta : [k] \to [n]$.*
(5) *For any $\varphi : [m] \to [n]$ the map $cosk_k U(\varphi)$ is described as the mapping $\bigoplus_\beta A \to \bigoplus_{\beta'} A$ which maps to component corresponding to $\beta : [k] \to [n]$ to zero if β does not factor through φ and by the identity to each of the components corresponding to β' such that $\beta = \varphi \circ \beta'$ if it does.*
(6) *The canonical map $c : i_{k!}U \to cosk_k U$ in degree n has (α, β) coefficient $A \to A$ equal to zero if $\alpha \circ \beta$ is not the identity and equal to id_A if it is.*
(7) *The canonical map $c : i_{k!}U \to cosk_k U$ is injective.*

Proof. The proof of (1) is left to the reader.

Let us take the rules of (2) and (3) as the definition of a simplicial object, call it $\tilde{U}$. We will show that it is an incarnation of $i_{k!}U$. This will prove (2), (3) at the same time. We have to show that given a morphism $f : U \to \mathrm{sk}_k V$ there exists a unique morphism $\tilde{f} : \tilde{U} \to V$ which recovers f upon taking the k-skeleton. From (1) we see that f corresponds with a morphism $f_k : A \to V_k$ which maps into the kernel of d^k_i for all i. For any surjective

$\alpha : [n] \to [k]$ we set $\tilde{f}_\alpha : A \to V_n$ equal to the composition $\tilde{f}_\alpha = V(\alpha) \circ f_k : A \to V_n$. We define $\tilde{f}_n : \tilde{U}_n \to V_n$ as the sum of the $\tilde{f}_\alpha$ over $\alpha : [n] \to [k]$ surjective. Such a collection of $\tilde{f}_\alpha$ defines a morphism of simplicial objects if and only if for any $\varphi : [m] \to [n]$ the diagram

$$\begin{tikzcd} \bigoplus_{\alpha : [n] \to [k] \text{ surjective}} A \arrow[r, "\tilde{f}_n"'] \arrow[d, "(3)"'] & V_n \arrow[d, "V(\varphi)"] \\ \bigoplus_{\alpha' : [m] \to [k] \text{ surjective}} A \arrow[r, "\tilde{f}_m"] & V_m \end{tikzcd}$$

is commutative. Choosing $\varphi = \alpha$ shows our choice of $\tilde{f}_\alpha$ is uniquely determined by f_k. The commutativity in general may be checked for each summand of the left upper corner separately. It is clear for the summands corresponding to α where $\alpha \circ \varphi$ is surjective, because those get mapped by id_A to the summand with $\alpha' = \alpha \circ \varphi$, and we have $\tilde{f}_{\alpha'} = V(\alpha') \circ f_k = V(\alpha \circ \varphi) \circ f_k = V(\varphi) \circ \tilde{f}_\alpha$. For those where $\alpha \circ \varphi$ is not surjective, we have to show that $V(\varphi) \circ \tilde{f}_\alpha = 0$. By definition this is equal to $V(\varphi) \circ V(\alpha) \circ f_k = V(\alpha \circ \varphi) \circ f_k$. Since $\alpha \circ \varphi$ is not surjective we can write it as $\delta^k_i \circ \psi$, and we deduce that $V(\varphi) \circ V(\alpha) \circ f_k = V(\psi) \circ d^k_i \circ f_k = 0$ see above.

Let us take the rules of (4) and (5) as the definition of a simplicial object, call it $\tilde{U}$. We will show that it is an incarnation of $\mathrm{cosk}_k U$. This will prove (4), (5) at the same time. The argument is completely dual to the proof of (2), (3) above, but we give it anyway. We have to show that given a morphism $f : \mathrm{sk}_k V \to U$ there exists a unique morphism $\tilde{f} : V \to \tilde{U}$ which recovers f upon taking the k-skeleton. From (1) we see that f corresponds with a morphism $f_k : V_k \to A$ which is zero on the image of s^{k-1}_i for all i. For any injective $\beta : [k] \to [n]$ we set $\tilde{f}_\beta : V_n \to A$ equal to the composition $\tilde{f}_\beta = f_k \circ V(\beta) : V_n \to A$. We define $\tilde{f}_n : V_n \to \tilde{U}_n$ as the sum of the $\tilde{f}_\beta$ over $\beta : [k] \to [n]$ injective. Such a collection of $\tilde{f}_\beta$ defines a morphism of simplicial objects if and only if for any $\varphi : [m] \to [n]$ the diagram

$$\begin{tikzcd} V_n \arrow[r, "\tilde{f}_n"'] \arrow[d, "V(\varphi)"'] & \bigoplus_{\beta : [k] \to [n] \text{ injective}} A \arrow[d, "(5)"] \\ V_m \arrow[r, "\tilde{f}_m"] & \bigoplus_{\beta' : [k] \to [m] \text{ injective}} A \end{tikzcd}$$

is commutative. Choosing $\varphi = \beta$ shows our choice of $\tilde{f}_\beta$ is uniquely determined by f_k. The commutativity in general may be checked for each summand of the right lower corner separately. It is clear for the summands corresponding to β' where $\varphi \circ \beta'$ is injective, because these summands get mapped into by exactly the summand with $\beta = \varphi \circ \beta'$ and we have in that case $\tilde{f}_{\beta'} \circ V(\varphi) = f_k \circ V(\beta') \circ V(\varphi) = f_k \circ V(\beta) = \tilde{f}_\beta$. For those where $\varphi \circ \beta'$ is not injective, we have to show that $\tilde{f}_{\beta'} \circ V(\varphi) = 0$. By definition this is equal to $f_k \circ V(\beta') \circ V(\varphi) = f_k \circ V(\varphi \circ \beta')$. Since $\varphi \circ \beta'$ is not injective we can write it as $\psi \circ \sigma^{k-1}_i$, and we deduce that $f_k \circ V(\beta') \circ V(\varphi) = f_k \circ s^{k-1}_i \circ V(\psi) = 0$ see above.

The composition $i_{k!} U \to \mathrm{cosk}_k U$ is the unique map of simplicial objects which is the identity on $A = U_k = (i_{k!} U)_k = (\mathrm{cosk}_k U)_k$. Hence it suffices to check that the proposed rule defines a morphism of simplicial objects. To see this we have to show that for any

$\varphi : [m] \to [n]$ the diagram

$$\begin{tikzcd} \bigoplus_{\alpha : [n] \to [k] \text{ surjective}} A \arrow[r, "(6)"'] \arrow[d, "(3)"'] & \bigoplus_{\beta : [k] \to [n] \text{ injective}} A \arrow[d, "(5)"] \\ \bigoplus_{\alpha' : [m] \to [k] \text{ surjective}} A \arrow[r, "(6)"] & \bigoplus_{\beta' : [k] \to [m] \text{ injective}} A \end{tikzcd}$$

is commutative. Now we can think of this in terms of matrices filled with only 0's and 1's as follows: The matrix of (3) has a nonzero (α', α) entry if and only if $\alpha' = \alpha \circ \varphi$. Likewise the matrix of (5) has a nonzero (β', β) entry if and only if $\beta = \varphi \circ \beta'$. The upper matrix of (6) has a nonzero (α, β) entry if and only if $\alpha \circ \beta = \mathrm{id}_{[k]}$. Similarly for the lower matrix of (6). The commutativity of the diagram then comes down to computing the (α, β') entry for both compositions and seeing they are equal. This comes down to the following equality

$$\#\left\{\beta \mid \beta = \varphi \circ \beta' \wedge \alpha \circ \beta = \mathrm{id}_{[k]}\right\} = \#\left\{\alpha' \mid \alpha' = \alpha \circ \varphi \wedge \alpha' \circ \beta' = \mathrm{id}_{[k]}\right\}$$

whose proof may safely be left to the reader.

Finally, we prove (7). This follows directly from Lemmas 14.16.7, 14.17.5, 14.19.3 and 14.19.9. □

Definition 14.20.3. Let $\mathcal{A}$ be an abelian category. Let A be an object of $\mathcal{A}$ and let k be an integer ≥ 0. The *Eilenberg-Maclane object* $K(A, k)$ is given by the object $K(A, k) = i_{k!}U$ which is described in Lemma 14.20.2 above.

Lemma 14.20.4. *Let $\mathcal{A}$ be an abelian category. Let A be an object of $\mathcal{A}$ and let k be an integer ≥ 0. Consider the simplicial object E defined by the following rules*

(1) $E_n = \bigoplus_\alpha A$, *where the sum is over $\alpha : [n] \to [k+1]$ whose image is either $[k]$ or $[k+1]$.*
(2) *Given $\varphi : [m] \to [n]$ the map $E_n \to E_m$ maps the summand corresponding to α via id_A to the summand corresponding to $\alpha \circ \varphi$, provided $Im(\alpha \circ \varphi)$ is equal to $[k]$ or $[k+1]$.*

Then there exists a short exact sequence

$$0 \to K(A, k) \to E \to K(A, k+1) \to 0$$

which is term by term split exact.

Proof. The maps $K(A, k)_n \to E_n$ resp. $E_n \to K(A, k+1)_n$ are given by the inclusion of direct sums, resp. projection of direct sums which is obvious from the inclusions of index sets. It is clear that these are maps of simplicial objects. □

Lemma 14.20.5. *Let $\mathcal{A}$ be an abelian category. For any simplicial object V of $\mathcal{A}$ we have*

$$V = colim_n\, i_{n!} sk_n V$$

where all the transition maps are injections.

Proof. This is true simply because each V_m is equal to $(i_{n!}\mathrm{sk}_n V)_m$ as soon as $n \geq m$. See also Lemma 14.19.10 for the transition maps. □

14.21. Simplicial objects and chain complexes

Let $\mathcal{A}$ be an abelian category. See Homology, Section 10.10 for conventions and notation regarding chain complexes. Let U be a simplicial object of $\mathcal{A}$. The *associated chain complex $s(U)$ of U*, sometimes called the *Moore complex*, is the chain complex

$$\ldots \to U_2 \to U_1 \to U_0 \to 0 \to 0 \to \ldots$$

with boundary maps $d_n : U_n \to U_{n-1}$ given by the formula

$$d_n = \sum\nolimits_{i=0}^{n} (-1)^i d^n_i.$$

This is a complex because, by the relations listed in Remark 14.3.3, we have

$$\begin{aligned} d_n \circ d_{n+1} &= (\sum\nolimits_{i=0}^{n} (-1)^i d^n_i) \circ (\sum\nolimits_{j=0}^{n+1} (-1)^j d^{n+1}_j) \\ &= \sum\nolimits_{0 \leq i < j \leq n+1} (-1)^{i+j} d^n_{j-1} \circ d^{n+1}_i + \sum\nolimits_{n \geq i \geq j \geq 0} (-1)^{i+j} d^n_i \circ d^{n+1}_j \\ &= 0. \end{aligned}$$

The signs cancel! We denote the associated chain complex $s(U)$. Clearly, the construction is functorial and hence defines a functor

$$s : \text{Simp}(\mathcal{A}) \longrightarrow \text{Ch}_{\geq 0}(\mathcal{A}).$$

Thus we have the confusing but correct formula $s(U)_n = U_n$.

Lemma 14.21.1. *The functor s is exact.*

Proof. Clear from Lemma 14.20.1. □

Lemma 14.21.2. *Let $\mathcal{A}$ be an abelian category. Let A be an object of $\mathcal{A}$ and let k be an integer. Let E be the object described in Lemma 14.20.4. Then the complex $s(E)$ is acyclic.*

Proof. For a morphism $\alpha : [n] \to [k+1]$ we define $\alpha' : [n+1] \to [k+1]$ to be the map such that $\alpha'|_{[n]} = \alpha$ and $\alpha'(n+1) = k+1$. Note that if the image of α is $[k]$ or $[k+1]$, then the image of α' is $[k+1]$. Consider the family of maps $h_n : E_n \to E_{n+1}$ which maps the summand corresponding to α to the summand corresponding to α' via the identity on A. Let us compute $d_{n+1} \circ h_n - h_{n-1} \circ d_n$. We will first do this in case the category $\mathcal{A}$ is the category of abelian groups. Let us use the notation x_α to indicate the element $x \in A$ in the summand of E_n corresponding to the map α occuring in the index set. Let us also adopt the convention that x_α designates the zero element of E_n whenever $\text{Im}(\alpha)$ is not $[k]$ or $[k+1]$. With these conventions we see that

$$d_{n+1}(h_n(x_\alpha)) = \sum\nolimits_{i=0}^{n+1} (-1)^i x_{\alpha' \circ \delta^{n+1}_i}$$

and

$$h_{n-1}(d_n(x_\alpha)) = \sum\nolimits_{i=0}^{n} (-1)^i x_{(\alpha \circ \delta^n_i)'}$$

It is easy to see that $\alpha' \circ \delta^{n+1}_i = (\alpha \circ \delta^n_i)'$ for $i = 0, \ldots, n$. It is also easy to see that $\alpha' \circ \delta^{n+1}_{n+1} = \alpha$. Thus we see that

$$(d_{n+1} \circ h_n - h_{n-1} \circ d_n)(x_\alpha) = (-1)^{n+1} x_\alpha$$

These identities continue to hold if $\mathcal{A}$ is any abelian category because they hold in the simplicial abelian group $[n] \mapsto \mathit{Hom}(A, E_n)$; details left to the reader. We conclude that the identity map on E is homotopic to zero, with homotopy given by the system of maps $h'_n = (-1)^{n+1} h_n : E_n \to E_{n+1}$. Hence we see that E is acyclic, for example by Homology, Lemma 10.10.5. □

Lemma 14.21.3. *Let $\mathcal{A}$ be an abelian category. Let A be an object of $\mathcal{A}$ and let k be an integer. We have $H_i(s(K(A, k))) = A$ if $i = k$ and 0 else.*

Proof. First, let us prove this if $k = 0$. In this case we have $K(A, 0)_n = A$ for all n. Furthermore, all the maps in this simplicial abelian group are id_A, in other words $K(A, 0)$ is the constant simplicial object with value A. The boundary maps $d_n = \sum_{i=0}^{n}(-1)^i \mathrm{id}_A = 0$ if n odd and $= \mathrm{id}_A$ if n is even. Thus $s(K(A, 0))$ looks like this

$$\ldots \to A \xrightarrow{0} A \xrightarrow{1} A \xrightarrow{0} A \to 0$$

and the result is clear.

Next, we prove the result for all k by induction. Given the result for k consider the short exact sequence

$$0 \to K(A, k) \to E \to K(A, k + 1) \to 0$$

from Lemma 14.20.4. By Lemma 14.20.1 the associated sequence of chain complexes is exact. By Lemma 14.21.2 we see that $s(E)$ is acyclic. Hence the result for $k + 1$ follows from the long exact sequence of homlogy, see Homology, Lemma 10.10.6. □

There is a second chain complex we can associate to a simplicial object of $\mathcal{A}$. Recall that by Lemma 14.16.6 any simplicial object U of $\mathcal{A}$ is canonically split with $N(U_m) = \bigcap_{i=0}^{m-1} \mathrm{Ker}(d_i^m)$. We define the *normalized chain complex* $N(U)$ to be the chain complex

$$\ldots \to N(U_2) \to N(U_1) \to N(U_0) \to 0 \to 0 \to \ldots$$

with boundary map $d_n : N(U_n) \to N(U_{n-1})$ given by the restriction of $(-1)^n d_n^n$ to the direct summand $N(U_n)$ of U_n. Note that Lemma 14.16.8 implies that $d_n^n(N(U_n)) \subset N(U_{n-1})$. It is a complex because $d_n^n \circ d_{n+1}^{n+1} = d_n^n \circ d_n^{n+1}$ and d_n^{n+1} is zero on $N(U_{n+1})$ by definition. Thus we obtain a second functor

$$N : \mathrm{Simp}(\mathcal{A}) \longrightarrow \mathrm{Ch}_{\geq 0}(\mathcal{A}).$$

Here is the reason for the sign in the differential.

Lemma 14.21.4. *Let $\mathcal{A}$ be an abelian category. Let U be a simplicial object of $\mathcal{A}$. The canonical map $N(U_n) \to U_n$ gives rise to a morphism of complexes $N(U) \to s(U)$.*

Proof. This is clear because the differential on $s(U)_n = U_n$ is $\sum(-1)^i d_i^n$ and the maps d_i^n, $i < n$ are zero on $N(U_n)$, whereas the restriction of $(-1)^n d_n^n$ is the boundary map of $N(U)$ by definition. □

Lemma 14.21.5. *Let $\mathcal{A}$ be an abelian category. Let A be an object of $\mathcal{A}$ and let k be an integer. We have $N(K(A, k))_i = A$ if $i = k$ and 0 else.*

Proof. It is clear that $N(K(A, k))_i = 0$ when $i < k$ because $K(A, k)_i = 0$ in that case. It is clear that $N(K(A, k))_k = A$ since $K(A, k)_{k-1} = 0$ and $K(A, k)_k = A$. For $i > k$ we have $N(K(A, k))_i = 0$ by Lemma 14.19.9 and the definition of $K(A, k)$, see Definition 14.20.3. □

Lemma 14.21.6. *Let $\mathcal{A}$ be an abelian category. Let U be a simplicial object of $\mathcal{A}$. The canonical morphism of chain complexes $N(U) \to s(U)$ is split. In fact,*

$$s(U) = N(U) \oplus A(U)$$

for some complex $A(U)$. The construction $U \mapsto A(U)$ is functorial.

Proof. Define $A(U)_n$ to be the image of

$$\bigoplus\nolimits_{\varphi : [n] \to [m] \text{ surjective, } m < n} N(U_m) \xrightarrow{\bigoplus U(\varphi)} U_n$$

which is a subobject of U_n complementary to $N(U_n)$ according to Lemma 14.16.6 and Definition 14.16.1. We show that $A(U)$ is a subcomplex. Pick a surjective map $\varphi : [n] \to [m]$ with $m < n$ and consider the composition

$$N(U_m) \xrightarrow{U(\varphi)} U_n \xrightarrow{d_n} U_{n-1}.$$

This composition is the sum of the maps

$$N(U_m) \xrightarrow{U(\varphi \circ \delta^n_i)} U_{n-1}$$

with sign $(-1)^i$, $i = 0, \ldots, n$.

First we will prove by ascending induction on m, $0 \leq m < n-1$ that all the maps $U(\varphi \circ \delta^n_i)$ map $N(U_m)$ into $A(U)_{n-1}$. (The case $m = n-1$ is treated below.) Whenever the map $\varphi \circ \delta^n_i : [n-1] \to [m]$ is surjective then the image of $N(U_m)$ under $U(\varphi \circ \delta^n_i)$ is contained in $A(U)_{n-1}$ by definition. If $\varphi \circ \delta^n_i : [n-1] \to [m]$ is not surjective, set $j = \varphi(i)$ and observe that i is the unique index whose image under φ is j. We may write $\varphi \circ \delta^n_i = \delta^m_j \circ \psi \circ \delta^n_i$ for some $\psi : [n-1] \to [m-1]$. Hence $U(\varphi \circ \delta^n_i) = U(\psi \circ \delta^n_i) \circ d^m_j$ which is zero on $N(U_m)$ unless $j = m$. If $j = m$, then $d^m_m(N(U_m)) \subset N(U_{m-1})$ and hence $U(\varphi \circ \delta^n_i)(N(U_m)) \subset U(\psi \circ \delta^n_i)(N(U_{m-1}))$ and we win by induction hypothesis.

To finish proving that $A(U)$ is a subcomplex we still have to deal with the composition

$$N(U_m) \xrightarrow{U(\varphi)} U_n \xrightarrow{d_n} U_{n-1}.$$

in case $m = n-1$. In this case $\varphi = \sigma^{n-1}_j$ for some $0 \leq j \leq n-1$ and $U(\varphi) = s^{n-1}_j$. Thus the composition is given by the sum

$$\sum (-1)^i d^n_i \circ s^{n-1}_j$$

Recall from Remark 14.3.3 that $d^n_j \circ s^{n-1}_j = d^n_{j+1} \circ s^{n-1}_j = \text{id}$ and these drop out because the corresponding terms have opposite signs. The map $d^n_n \circ s^{n-1}_j$, if $j < n-1$, is equal to $s^{n-2}_j \circ d^{n-1}_{n-1}$. Since d^{n-1}_{n-1} maps $N(U_{n-1})$ into $N(U_{n-2})$, we see that the image $d^n_n(s^{n-1}_j(N(U_{n-1}))$ is contained in $s^{n-2}_j(N(U_{n-2}))$ which is contained in $A(U_{n-1})$ by definition. For all other combinations of (i, j) we have either $d^n_i \circ s^{n-1}_j = s^{n-2}_{j-1} \circ d^{n-1}_i$ (if $i < j$), or $d^n_i \circ s^{n-1}_j = s^{n-2}_j \circ d^{n-1}_{i-1}$ (if $n > i > j+1$) and in these cases the map is zero because of the definition of $N(U_{n-1})$. □

Lemma 14.21.7. *The functor N is exact.*

Proof. By Lemma 14.21.1 and the functorial decomposition of Lemma 14.21.5. □

Lemma 14.21.8. *Let $\mathcal{A}$ be an abelian category. Let V be a simplicial object of $\mathcal{A}$. The canonical morphism of chain complexes $N(V) \to s(V)$ is a quasi-isomorphism. In other words, the complex $A(V)$ of Lemma 14.21.6 is acyclic.*

Proof. Note that the result holds for $K(A, k)$ for any object A and any $k \geq 0$, by Lemmas 14.21.3 and 14.21.5. Consider the hypothesis $IH_{n,m}$: for all V such that $V_j = 0$ for $j \leq m$ and all $i \leq n$ the map $N(V) \to s(V)$ induces an isomorphism $H_i(N(V)) \to H_i(s(V))$.

To start of the induction, note that $IH_{n,n}$ is trivially true, because in that case $N(V)_n = 0$ and $s(V)_n = 0$.

Assume $IH_{n,m}$, with $m \leq n$. Pick a simplicial object V such that $V_j = 0$ for $j < m$. By Lemma 14.20.2 and Definition 14.20.3 we have $K(V_m, m) = i_{m!}\mathrm{sk}_m V$. By Lemma 14.19.10 the natural morphism

$$K(V_m, m) = i_{m!}\mathrm{sk}_m V \to V$$

is injective. Thus we get a short exact sequence

$$0 \to K(V_m, m) \to V \to W \to 0$$

for some W with $W_i = 0$ for $i = 0, \ldots, m$. This short exact sequence induces a morphism of short exact sequence of associated complexes

$$\begin{tikzcd} 0 \arrow[r] & N(K(V_m, m)) \arrow[r] \arrow[d] & N(V) \arrow[r] \arrow[d] & N(W) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & s(K(V_m, m)) \arrow[r] & s(V) \arrow[r] & s(W) \arrow[r] & 0 \end{tikzcd}$$

see Lemmas 14.21.1 and 14.21.7. Hence we deduce the result for V from the result on the ends. □

14.22. Dold-Kan

Lemma 14.22.1. *Let $\mathcal{A}$ be an abelian category. The functor N is faithful, and reflects isomorphisms, injections and surjections.*

Proof. The faithfulness is immediate from the canonical splitting of Lemma 14.16.6. The statement on reflecting injections, surjections, and isomorphisms follows from Lemma 14.16.7. □

Lemma 14.22.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $N : \mathcal{A} \to \mathcal{B}$, and $S : \mathcal{B} \to \mathcal{A}$ be functors. Suppose that*

1. *the functors S and N are exact,*
2. *there is an isomorphism $g : N \circ S \to id_{\mathcal{B}}$ to the identity functor of $\mathcal{B}$,*
3. *N is faithful, and*
4. *S is essentially surjective.*

Proof. It suffices to construct a functorial isomorphism $S(N(A)) \cong A$. To do this choose B and an isomorphism $f : A \to S(B)$. Consider the map

$$f^{-1} \circ g_{S(B)} \circ S(N(f)) : S(N(A)) \to S(N(S(B))) \to S(B) \to A.$$

It is easy to show this does not depend on the choice of f, B and gives the desired isomorphism $S \circ N \to \mathrm{id}_{\mathcal{A}}$. □

Theorem 14.22.3. *Let $\mathcal{A}$ be an abelian category. The functor N induces an equivalence of categories*

$$N : Simp(\mathcal{A}) \longrightarrow Ch_{\geq 0}(\mathcal{A})$$

Proof. We will describe a functor in the reverse direction inspired by the construction of Lemma 14.20.4 (except that we throw in a sign to get the boundaries right). Let $A_\bullet$ be a chain complex with boundary maps $d_{A,n} : A_n \to A_{n-1}$. For each $n \geq 0$ denote

$$I_n = \left\{ \alpha : [n] \to \{0, 1, 2, \ldots\} \mid \mathrm{Im}(\alpha) = [k] \text{ for some } k \right\}.$$

For $\alpha \in I_n$ we denote $k(\alpha)$ the unique integer such that $\mathrm{Im}(\alpha) = [k]$. We define a simplicial object $S(A_\bullet)$ as follows:

(1) $S(A_\bullet)_n = \bigoplus_{\alpha \in I_n} A_{k(\alpha)}$, which we will write as $\bigoplus_{\alpha \in I_n} A_{k(\alpha)} \cdot \alpha$ to suggest thinking of ``α'' as a basis vector for the summand corresponding to it,
(2) given $\varphi : [m] \to [n]$ we define $S(A_\bullet)(\varphi)$ by its restriction to the direct summand $A_{k(\alpha)} \cdot \alpha$ of $S(A_\bullet)_n$ as follows
   (a) $\alpha \circ \varphi \not\in I_m$ then we set it equal to zero,
   (b) $\alpha \circ \varphi \in I_m$ but $k(\alpha \circ \varphi)$ not equal to either $k(\alpha)$ or $k(\alpha) - 1$ then we set it equal to zero as well,
   (c) if $\alpha \circ \varphi \in I_m$ and $k(\alpha \circ \varphi) = k(\alpha)$ then we use the identity map to the summand $A_{k(\alpha \circ \varphi)} \cdot (\alpha \circ \varphi)$ of $S(A_\bullet)_m$, and
   (d) if $\alpha \circ \varphi \in I_m$ and $k(\alpha \circ \varphi) = k(\alpha) - 1$ then we use $(-1)^{k(\alpha)} d_{A,k(\alpha)}$ to the summand $A_{k(\alpha \circ \varphi)} \cdot (\alpha \circ \varphi)$ of $S(A_\bullet)_m$.

It is an exercise (FIXME) to show that this is a simplicial complex; one has to use in particular that the compositions $d_{A,k} \circ d_{A,k-1}$ are all zero.

Having verified this, the correct way to proceed with the proof would be to prove directly that N and S are quasi-inverse functors (FIXME). Instead we prove this by an indirect method using Eilenberg-Maclane objects and truncations. It is clear that $A_\bullet \mapsto S(A_\bullet)$ is an exact functor from chain complexes to simplicial objects. If $A_i = 0$ for $i = 0, \ldots, n$ then $S(A_\bullet)_i = 0$ for $i = 0, \ldots, n$. The objects $K(A, k)$, see Definition 14.20.3, are equal to $S(A[-k])$ where $A[-k]$ is the chain complex with A in degree k and zero elsewhere.

Moreover, for each integer k we get a sub simplicial object $S_{\leq k}(A_\bullet)$ by considering only those α with $k(\alpha) \leq k$. In fact this is nothing but $S(\sigma_{\leq k} A_\bullet)$, where $\sigma_{\leq k} A_\bullet$ is the ``stupid'' truncation of $A_\bullet$ at k (which simply replaces A_i by 0 for $i > k$). Also, by Lemma 14.19.10 we see that it is equal to $i_{k!}\mathrm{sk}_k S(A_\bullet)$. Clearly, the quotient $S_{\leq k}(A_\bullet)/S_{\leq k-1}(A_\bullet) = K(A_k, k)$ and the quotient $S(A_\bullet)/S_{\leq k}(A_\bullet) = S(A/\sigma_{\leq k} A_\bullet)$ is a simplicial object whose ith term is zero for $i = 0, \ldots, k$. Since $S_{\leq k-1}(A_\bullet)$ is filtered with subquotients $K(A_i, i)$, $i < k$ we see that $N(S_{\leq k-1}(A_\bullet))_k = 0$ by exactness of the functor N, see Lemma 14.21.7. All in all we conclude that the maps

$$N(S(A_\bullet))_k \leftarrow N(S_{\leq k}(A_\bullet))_k \to N(S(A_k[-k])) = N(K(A_k, k))_k = A_k$$

are functorial isomorphisms.

It is actually easy to identify the map $A_k \to N(S(A_\bullet))_k$. Note that there is a unique map $A_k \to S(A_\bullet)_k$ corresponding to the summand $\alpha = \mathrm{id}_{[k]}$. Note that $\mathrm{Im}(\mathrm{id}_{[k]} \circ \delta^k_i)$ has cardinality $k - 1$ but does not have image $[k-1]$ unless $i = k$. Hence d^k_i kills the summand $A_k \cdot \mathrm{id}_{[k]}$ for $i = 0, \ldots, k-1$. From the abstract computation of $N(S(A_\bullet))_k$ above we conclude that the summand $A_k \cdot \mathrm{id}_{[k]}$ is equal to $N(S(A_\bullet))_k$.

In order to show that $N \circ S$ is the identity functor on $\mathrm{Ch}_{\geq 0}(\mathcal{A})$, the last thing we have to verify is that we recover the map $d_{A,k+1} : A_{k+1} \to A_k$ as the differential on the complex $N(S(A_\bullet))$ as follows

$$A_{k+1} = N(S(A_\bullet))_{k+1} \to N(S(A_\bullet))_k = A_k$$

By definition the map $N(S(A_\bullet))_{k+1} \to N(S(A_\bullet))_k$ corresponds to the restriction of $(-1)^{k+1} d^{k+1}_{k+1}$ to $N(S(A_\bullet))$ which is the summand $A_{k+1} \cdot \mathrm{id}_{[k+1]}$. And by the definition of $S(A_\bullet)$ above the map d^{k+1}_{k+1} maps $A_{k+1} \cdot \mathrm{id}_{[k+1]}$ into $A_k \cdot \mathrm{id}_{[k]}$ by $(-1)^{k+1} d_{A,k+1}$. The signs cancel and hence the desired equality.

We know that N is faithful, see Lemma 14.22.1. If we can show that S is essentially surjective, then it will follow that N is an equivalence, see Homology, Lemma 14.22.2.

Note that if $A_\bullet$ is a chain complex then $S(A_\bullet) = colim_n S_{\leq n}(A_\bullet) = colim_n S(\sigma_{\leq n} A_\bullet) = colim_n i_{n!}\text{sk}_n S(A_\bullet)$ by construction of S. By Lemma 14.20.5 it suffices to show that $i_{n!}V$ is in the essential image for any n-truncated simplicial object V. By induction on n it suffices to show that any extension

$$0 \to S(A_\bullet) \to V \to K(A, n) \to 0$$

where $A_i = 0$ for $i \geq n$ is in the essential image of S. By Homology, Lemma 10.5.2 we have abelian group homomorphisms

$$\text{Ext}_{\text{Simp}(\mathcal{A})}(K(A,n), S(A_\bullet)) \underset{S}{\overset{N}{\rightleftarrows}} \text{Ext}_{\text{Ch}_{\geq 0}(\mathcal{A})}(A[-n], A_\bullet)$$

between ext groups (see Homology, Definition 10.4.2). We want to show that S is surjective. We know that $N \circ S = \text{id}$. Hence it suffices to show that $\text{Ker}(N) = 0$. Clearly an extension

$$\begin{array}{cccccccccccccc} & 0 & \to & 0 & \to & A_{n-1} & \to & A_{n-2} & \to & \ldots & \to & A_0 & \to & 0 \\ & & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\ E: & 0 & \to & A & \to & A_{n-1} & \to & A_{n-2} & \to & \ldots & \to & A_0 & \to & 0 \\ & & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\ & 0 & \to & A & \to & 0 & \to & 0 & \to & \ldots & \to & 0 & \to & 0 \end{array}$$

of $A_\bullet$ by $A[-n]$ in $\text{Ch}(\mathcal{A})$ is zero if and only if the map $A \to A_{n-1}$ is zero. Thus we have to show that any extension

$$0 \to S(A_\bullet) \to V \to K(A, n) \to 0$$

such that $A = N(V)_n \to N(V)_{n-1}$ is zero is split. By Lemma 14.20.2 we have

$$Mor(K(A,n), V) = \left\{ f : A \to \bigcap\nolimits_{i=0}^{n} \ker(d_i^n : V_n \to V_{n-1}) \right\}$$

and if $A = N(V)_n \to N(V)_{n-1}$ is zero, then the intersection occuring in the formula above is equal to A. Let $i : K(A,n) \to V$ be the morphism corresponding to id_A on the right hand side of the displayed formula. Clearly this is a section to the map $V \to K(A, n)$ and the extension is split as desired. $\square$

14.23. Dold-Kan for cosimplicial objects

Let $\mathcal{A}$ be an abelian category. According to Homology, Lemma 10.3.13 also $\mathcal{A}^{opp}$ is abelian. It follows formally from the definitions that

$$\text{CoSimp}(\mathcal{A}) = \text{Simp}(\mathcal{A}^{opp})^{opp}.$$

Thus Dold-Kan (Theorem 14.22.3) implies that $\text{CoSimp}(\mathcal{A})$ is equivalent to the category $\text{Ch}_{\geq 0}(\mathcal{A}^{opp})^{opp}$. And it follows formally from the definitions that

$$\text{CoCh}_{\geq 0}(\mathcal{A}) = \text{Ch}_{\geq 0}(\mathcal{A}^{opp})^{opp}.$$

Putting these arrows together we obtain an equivalence

$$Q : \text{CoSimp}(\mathcal{A}) \longrightarrow \text{CoCh}_{\geq 0}(\mathcal{A}).$$

In this section we describe Q.

First we define the *cochain complex $s(U)$ associated to a cosimplicial object U*. It is the cochain complex with terms zero in negative degrees, and $s(U)^n = U_n$ for $n \geq 0$. As differentials we use the maps $d^n : s(U)^n \to s(U)^{n+1}$ defined by $d^n = \sum_{i=0}^{n+1}(-1)^i\delta_i^{n+1}$. In other words the complex $s(U)$ looks like

$$0 \longrightarrow U_0 \xrightarrow{\delta_0^1-\delta_1^1} U_1 \xrightarrow{\delta_0^2-\delta_1^2+\delta_2^2} U_2 \longrightarrow \ldots$$

This is sometimes also called the *Moore complex* associated to U.

On the other hand, given a cosimplicial object U of $\mathcal{A}$ set $Q(U)^0 = U_0$ and

$$Q(U)^n = \text{Coker}(\bigoplus\nolimits_{i=0}^{n-1} U_{n-1} \xrightarrow{\delta_i^n} U_n).$$

The differential $d^n : Q(U)^n \to Q(U)^{n+1}$ is induced by $(-1)^{n+1}\delta_{n+1}^{n+1}$, i.e., by fitting the morphism $(-1)^{n+1}\delta_{n+1}^{n+1}$ into a commutative diagram

$$\begin{tikzcd} U_n \arrow[r, "(-1)^{n+1}\delta_{n+1}^{n+1}"'] \arrow[d] & U_{n+1} \arrow[d] \\ Q(U)^n \arrow[r, "d_n"] & Q(U)^{n+1}. \end{tikzcd}$$

We leave it to the reader to show that this diagram makes sense, i.e., that the image of δ_i^n maps into the kernel of the right vertical arrow for $i = 0, \ldots, n-1$. (This is dual to Lemma 14.16.8.) Thus our cochain complex $Q(U)$ looks like this

$$0 \to Q(U)^0 \to Q(U)^1 \to Q(U)^2 \to \ldots$$

This is called the *normalized cochain complex associated to U*. The dual to the Dold-Kan Theorem 14.22.3 is the following.

Lemma 14.23.1. *Let $\mathcal{A}$ be an abelian category.*

(1) *The functor $s : CoSimp(\mathcal{A}) \to CoCh_{\geq 0}(\mathcal{A})$ is exact.*
(2) *The maps $s(U)^n \to Q(U)^n$ define a morphism of cochain complexes.*
(3) *There exists a functorial direct sum decomposition $s(U) = A(U) \oplus Q(U)$ in $CoCh_{\geq 0}(\mathcal{A})$.*
(4) *The functor Q is exact.*
(5) *The morphism of complexes $s(U) \to Q(U)$ is a quasi-isomorphism.*
(6) *The functor $U \mapsto Q(U)^\bullet$ defines an equivalence of categories $CoSimp(\mathcal{A}) \to CoCh_{\geq 0}(\mathcal{A})$.*

Proof. Omitted. But the results are the exact dual statements to Lemmas 14.21.1, 14.21.4, 14.21.6, 14.21.7, 14.21.8, and Theorem 14.22.3. □

14.24. Homotopies

Consider the simplicial sets $\Delta[0]$ and $\Delta[1]$. Recall that there are two morphisms

$$e_0, e_1 : \Delta[0] \longrightarrow \Delta[1],$$

coming from the morphisms $[0] \to [1]$ mapping 0 to an element of $[1] = \{0, 1\}$. Recall also that each set $\Delta[1]_k$ is finite. Hence, if the category $\mathcal{C}$ has finite coproducts, then we can form the product

$$U \times \Delta[1]$$

for any simplicial object U of $\mathcal{C}$, see Definition 14.12.1. Note that $\Delta[0]$ has the property that $\Delta[0]_k = \{*\}$ is a singleton for all $k \geq 0$. Hence $U \times \Delta[0] = U$. Thus e_0, e_1 above gives rise to morphisms

$$e_0, e_1 : U \to U \times \Delta[1].$$

Definition 14.24.1. Let $\mathcal{C}$ be a category having finite coproducts. Suppose that U and V are two simplicial objects of $\mathcal{C}$. We say morphisms $a, b : U \to V$ are *homotopic* if there exists a morphism

$$h : U \times \Delta[1] \longrightarrow V$$

such that $a = h \circ e_0$ and $b = h \circ e_1$. In this case h is called a *homotopy connecting a and b*.

It is possible to define this notion for pairs of maps between simplicial objects in any category. To do this you just work out what it means to have the morphisms $h_n : (U \times \Delta[1])_n \to V_n$ in terms of the mapping property of coproducts.

Let $\mathcal{C}$ be a category with finite coproducts. Let U, V be simplicial objects of $\mathcal{C}$. Let $a, b : U \to V$ be morphisms. Further, suppose that $h : U \times \Delta[1] \to V$ is a homotopy connecting a and b. For every $n \geq 0$ let us write

$$\Delta[1]_n = \{\alpha_0^n, \ldots, \alpha_{n+1}^n\}$$

where $\alpha_i : [n] \to [1]$ is the map such that

$$\alpha_i^n(j) = \begin{cases} 0 & \text{if} \quad j < i \\ 1 & \text{if} \quad j \geq i \end{cases}$$

Thus

$$h_n : (U \times \Delta[1])_n = \coprod U_n \cdot \alpha_i^n \longrightarrow V_n$$

has a component $h_{n,i} : U_n \to V_n$ which is the restriction to the summand corresponding to α_i^n for all $i = 0, \ldots, n+1$.

Lemma 14.24.2. *In the situation above, we have the following relations:*

(1) *We have $h_{n,0} = b_n$ and $h_{n,n+1} = a_n$.*
(2) *We have $d_j^n \circ h_{n,i} = h_{n-1,i-1} \circ d_j^n$ for $i > j$.*
(3) *We have $d_j^n \circ h_{n,i} = h_{n-1,i} \circ d_j^n$ for $i \leq j$.*
(4) *We have $s_j^n \circ h_{n,i} = h_{n+1,i+1} \circ s_j^n$ for $i > j$.*
(5) *We have $s_j^n \circ h_{n,i} = h_{n+1,i} \circ s_j^n$ for $i \leq j$.*

Conversely, given a system of maps $h_{n,i}$ satisfying the properties listed above, then these define a morphisms h which is a homotopy between a and b.

Proof. Omitted. You can prove the last statement using the fact, see Lemma 14.2.4 that to give a morphism of simplicial objects is the same as giving a sequence of morphisms h_n commuting with all d_j^n and s_j^n. ☐

Example 14.24.3. Suppose in the situation above $a = b$. Then there is a *trivial* homotopy between a and b, namely the one with $h_{n,i} = a_n = b_n$.

Remark 14.24.4. Let $\mathcal{C}$ be any category (no assumptions whatsoever). We say that a pair of morphisms $a, b : U \to V$ of simplicial objects are *homotopic* if there exist morphisms[1] $h_{n,i} : U_n \to V_n$, for $n \geq 0$, $i = 0, \ldots, n+1$ satisfying the relations of Lemma 14.24.2. This is a ``better'' definition, because it applies to any category. Also it has the following

[1]In the literature, often the maps $h_{n+1,i} \circ s_i : U_n \to V_{n+1}$ are used instead of the maps $h_{n,i}$. Of course the relations these maps satisfy are different from the ones in Lemma 14.24.2.

property: if $F : \mathcal{C} \to \mathcal{C}'$ is any functor then a homotopic to b implies trivially that $F(a)$ is homotopic to $F(b)$. Since the lemma says that the newer notion is the same as the old one in case finite coproduct exist, we deduce in particular that functors preserve the old notion whenever both categories have finite coproducts.

Definition 14.24.5. Let $\mathcal{C}$ be a category having finite coproducts. Suppose that U and V are two simplicial objects of $\mathcal{C}$. We say a morphism $a : U \to V$ is a *homotopy equivalence* if there exists a morphism $b : V \to U$ such that $a \circ b$ is homotopic to id_V and $b \circ a$ is homotopic to id_U. If there exists such a morphism between U and V, then we say that U and V are *homotopy equivalent*.

The following lemma says that $U \times \Delta[1]$ is homotopy equivalent to U.

Lemma 14.24.6. *Let $\mathcal{C}$ be a category with finite coproducts. Let U be a simplicial object of $\mathcal{C}$. Consider the maps $e_1, e_0 : U \to U \times \Delta[1]$, and $\pi : U \times \Delta[1] \to U$, see Lemma 14.12.3.*

(1) *We have $\pi \circ e_1 = \pi \circ e_0 = id_U$, and*
(2) *The morphisms $id_{U\times\Delta[1]}$, and $e_0 \circ \pi$ are homotopic.*
(3) *The morphisms $id_{U\times\Delta[1]}$, and $e_1 \circ \pi$ are homotopic.*

Proof. The first assertion is trivial. For the second, consider the map of simplicial sets $\Delta[1] \times \Delta[1] \longrightarrow \Delta[1]$ which in degree n assigns to a pair (β_1, β_2), $\beta_i : [n] \to [1]$ the morphism $\beta : [n] \to [1]$ defined by the rule

$$\beta(i) = \max\{\beta_1(i), \beta_2(i)\}.$$

It is a morphism of simplicial sets, because the action $\Delta[1](\varphi) : \Delta[1]_n \to \Delta[1]_m$ of $\varphi : [m] \to [n]$ is by precomposing. Clearly, using notation from Section 14.24, we have $\beta = \beta_1$ if $\beta_2 = \alpha^n_0$ and $\beta = \alpha^n_{n+1}$ if $\beta_2 = \alpha^n_{n+1}$. This implies easily that the induced morphism

$$U \times \Delta[1] \times \Delta[1] \longrightarrow U \times \Delta[1]$$

of Lemma 14.12.3 is a homotopy between $\mathrm{id}_{U\times\Delta[1]}$ and $e_0 \circ \pi$. Similarly for $e_1 \circ \pi$ (use minimum instead of maximum). □

Lemma 14.24.7. *Let $f : Y \to X$ be a morphism of a category $\mathcal{C}$ with fibre products. Assume f has a section s. Consider the simplicial object U constructed in Example 14.3.5 starting with f. The morphism $U \to U$ which in each degree is the self map $(s \circ f)^{n+1}$ of $Y \times_X \ldots \times_X Y$ given by $s \circ f$ on each factor is homotopic to the identity on U. In particular, U is homotopy equivalent to the constant simplicial object X.*

Proof. Set $g^0 = \mathrm{id}_Y$ and $g^1 = s \circ f$. We use the morphisms

$$\begin{array}{rcl} Y \times_X \ldots \times_X Y \times Mor([n],[1]) & \to & Y \times_X \ldots \times_X Y \\ (y_0, \ldots, y_n) \times \alpha & \mapsto & (g^{\alpha(0)}(y_0), \ldots, g^{\alpha(n)}(y_n)) \end{array}$$

where we use the functor of points point of view to define the maps. Another way to say this is to say that $h_{n,0} = \mathrm{id}$, $h_{n,n+1} = (s \circ f)^{n+1}$ and $h_{n,i} = \mathrm{id}_Y^{i+1} \times (s \circ f)^{n+1-i}$. We leave it to the reader to show that these satsify the relations of Lemma 14.24.2. Hence they define the desired homotopy. See also Remark 14.24.4 which shows that we do not need to assume anything else on the category $\mathcal{C}$. □

14.25. Homotopies in abelian categories

Let $\mathcal{A}$ be an abelian category. Let U, V be simplicial objects of $\mathcal{A}$. Let $a, b : U \to V$ be morphisms. Further, suppose that $h : U \times \Delta[1] \to V$ is a homotopy connecting a and b. Consider the two morphisms of chain complexes $s(a), s(b) : s(U) \longrightarrow s(V)$. Using the notation introduced above Lemma 14.24.2 we define

$$s(h)_n : U_n \longrightarrow V_{n+1}$$

by the formula

$$s(h)_n = \sum\nolimits_{i=0}^{n} (-1)^{i+1} h_{n+1,i+1} \circ s^n_i. \tag{14.25.0.1}$$

Let us compute $d_{n+1} \circ s(h)_n + s(h)_{n-1} \circ d_n$. We first compute

$$\begin{aligned} d_{n+1} \circ s(h)_n &= \sum\nolimits_{j=0}^{n+1} \sum\nolimits_{i=0}^{n} (-1)^{j+i+1} d^{n+1}_j \circ h_{n+1,i+1} \circ s^n_i \\ &= \sum\nolimits_{1 \leq i+1 \leq j \leq n+1} (-1)^{j+i+1} h_{n,i+1} \circ d^{n+1}_j \circ s^n_i \\ &\quad + \sum\nolimits_{n \geq i \geq j \geq 0} (-1)^{i+j+1} h_{n,i} \circ d^{n+1}_j \circ s^n_i \\ &= \sum\nolimits_{1 \leq i+1 < j \leq n+1} (-1)^{j+i+1} h_{n,i+1} \circ s^{n-1}_i \circ d^n_{j-1} \\ &\quad + \sum\nolimits_{1 \leq i+1 = j \leq n+1} (-1)^{j+i+1} h_{n,i+1} \\ &\quad + \sum\nolimits_{n \geq i = j \geq 0} (-1)^{i+j+1} h_{n,i} \\ &\quad + \sum\nolimits_{n \geq i > j \geq 0} (-1)^{i+j+1} h_{n,i} \circ s^{n-1}_{i-1} \circ d^n_j \end{aligned}$$

We leave it to the reader to see that the first and the last of the four sums cancel exactly against all the terms of

$$s(h)_{n-1} \circ d_n = \sum_{i=0}^{n-1} \sum_{j=0}^{n} (-1)^{i+1+j} h_{n,i+1} \circ s^{n-1}_i \circ d^n_j.$$

Hence we obtain

$$\begin{aligned} d_{n+1} \circ s(h)_n + s(h)_{n-1} \circ d_n &= \sum_{j=1}^{n+1} (-1)^{2j} h_{n,j} + \sum_{i=0}^{n} (-1)^{2i+1} h_{n,i} \\ &= h_{n,n+1} - h_{n,0} \\ &= a_n - b_n \end{aligned}$$

Thus we've proved part of the following lemma.

Lemma 14.25.1. *Let $\mathcal{A}$ be an abelian category. Let $a, b : U \to V$ be morphisms of simplicial objects of $\mathcal{A}$. If a, b are homotopic, then $s(a), s(b) : s(U) \to s(V)$, and $N(a), N(b) : N(U) \to N(V)$ are homotopic maps of chain complexes.*

Proof. The part about $s(a)$ and $s(b)$ is clear from the calculation above the lemma. On the other hand, if follows from Lemma 14.21.6 that $N(a)$, $N(b)$ are compositions

$$N(U) \to s(U) \to s(V) \to N(V)$$

where we use $s(a)$, $s(b)$ in the middle. Hence the assertion follows from Homology, Lemma 10.10.1. □

Lemma 14.25.2. *Let $\mathcal{A}$ be an abelian category. Let $a : U \to V$ be a morphism of simplicial objects of $\mathcal{A}$. If a is a homotopy equivalence, then $s(a) : s(U) \to s(V)$, and $N(a) : N(U) \to N(V)$ are homotopy equivalences of chain complexes.*

Proof. Omitted. See Lemma 14.25.1 above. □

14.26. Homotopies and cosimplicial objects

Let $\mathcal{C}$ be a category with finite products. Let V be a cosimplicial object and consider $Hom(\Delta[1], V)$, see Section 14.13. The morphisms $e_0, e_1 : \Delta[0] \to \Delta[1]$ produce two morphisms $e_0, e_1 : Hom(\Delta[1], V) \to V$.

Definition 14.26.1. Let $\mathcal{C}$ be a category having finite products. Suppose that U and V are two cosimplicial objects of $\mathcal{C}$. We say morphisms $a, b : U \to V$ are *homotopic* if there exists a morphism

$$h : U \longrightarrow Hom(\Delta[1], V)$$

such that $a = e_0 \circ h$ and $b = e_1 \circ h$. In this case h is called a *homotopy connecting a and b.*

This is really exactly the same as the notion we introduced for simplicial objects earlier. In particular, recall that $\Delta[1]_n$ is a finite set, and that

$$h_n = (h_{n,\alpha}) : U \longrightarrow \prod\nolimits_{\alpha \in \Delta[1]_n} V_n$$

is given by a collection of maps $h_{n,\alpha} : U_n \to V_n$ parametrized by elements of $\Delta[1]_n = Mor_\Delta([n], [1])$. As in Lemma 14.24.2 these morphisms satisfy some relations. Namely, for every $f : [n] \to [m]$ in Δ we should have

$$h_{m,\alpha} \circ U(f) = V(f) \circ h_{n,\alpha \circ f} \tag{14.26.1.1}$$

The condition that $a = e_0 \circ h$ means that $a_n = h_{n,0:[n]\to[1]}$ where $0 : [n] \to [1]$ is the constant map with value zero. Similarly, we should have $b_n = h_{n,1:[n]\to[1]}$. In particular we deduce once more that the notion of homotopy can be formulated between cosimplicial objects of any category, i.e., existence of products is not necessary. Here is a precise formulation of why this is dual to the notion of a homotopy between morphisms of simplicial objects.

Lemma 14.26.2. *Let $\mathcal{C}$ be a category having finite products. Suppose that U and V are two cosimplicial objects of $\mathcal{C}$. Let $a, b : U \to V$ be morphisms of cosimplicial objects. Recall that U, V correspond to simplicial objects U', V' of $\mathcal{C}^{opp}$. Moreover a, b correspond to morphisms $a', b' : V' \to U'$. The following are equivalent*

(1) *The morphisms $a, b : U \to V$ of cosimplicial objects are homotopic.*
(2) *The morphisms $a', b' : V' \to U'$ of simplicial objects of $\mathcal{C}^{opp}$ are homotopic.*

Proof. If $\mathcal{C}$ has finite products, then $\mathcal{C}^{opp}$ has finite coproducts. And the contravariant functor $(-)' : \mathcal{C} \to \mathcal{C}^{opp}$ transforms products into coproducts. Then it is immediate from the definitions that $(Hom(\Delta[1], V))' = V' \times \Delta[1]$. And so on and so forth. □

Lemma 14.26.3. *Let $\mathcal{C}, \mathcal{C}', \mathcal{D}\mathcal{D}'$ be categories such that $\mathcal{C}, \mathcal{C}'$ have finite products, and $\mathcal{D}, \mathcal{D}'$ have finite coproducts.*

(1) *Let $a, b : U \to V$ be morphisms of simplicial objects of $\mathcal{D}$. Let $F : \mathcal{D} \to \mathcal{D}'$ be a covariant functor. If a and b are homotopic, then $F(a)$, $F(b)$ are homotopic morphisms $F(U) \to F(V)$ of simplicial objects.*
(2) *Let $a, b : U \to V$ be morphisms of cosimplicial objects of $\mathcal{C}$. Let $F : \mathcal{C} \to \mathcal{C}'$ be a covariant functor. If a and b are homotopic, then $F(a)$, $F(b)$ are homotopic morphisms $F(U) \to F(V)$ of cosimplicial objects.*

(3) *Let $a, b : U \to V$ be morphisms of simplicial objects of $\mathcal{D}$. Let $F : \mathcal{D} \to \mathcal{C}$ be a contravariant functor. If a and b are homotopic, then $F(a)$, $F(b)$ are homotopic morphisms $F(V) \to F(U)$ of cosimplicial objects.*
(4) *Let $a, b : U \to V$ be morphisms of cosimplicial objects of $\mathcal{C}$. Let $F : \mathcal{C} \to \mathcal{D}$ be a contravariant functor. If a and b are homotopic, then $F(a)$, $F(b)$ are homotopic morphisms $F(V) \to F(U)$ of simplicial objects.*

Proof. By Lemma 14.26.2 above, we can turn F into a covariant functor between a pair of categories which have finite coproducts, and we have to show that the functor preserves homotopic pairs of maps. It is explained in Remark 14.24.4 how this is the case. Even if the functor does not commute with coproducts! □

Lemma 14.26.4. *Let $f : Y \to X$ be a morphism of a category $\mathcal{C}$ with push outs. Assume f has a section s. Consider the cosimplicial object U constructed in Example 14.5.5 starting with f. The morphism $U \to U$ which in each degree is the self map of $Y \amalg_X \ldots \amalg_X Y$ given by $s \circ f$ on each factor is homotopic to the identity on U. In particular, U is homotopy equivalent to the constant cosimplicial object X.*

Proof. The dual statement which is Lemma 14.24.7. Hence this lemma follows on applying Lemma 14.26.2. □

Lemma 14.26.5. *Let $\mathcal{A}$ be an abelian category. Let $a, b : U \to V$ be morphisms of cosimplicial objects of $\mathcal{A}$. If a, b are homotopic, then $s(a), s(b) : s(U) \to s(V)$, and $Q(a), Q(b) : Q(U) \to Q(V)$ are homotopic maps of cochain complexes.*

Proof. Let $(-)' : \mathcal{A} \to \mathcal{A}^{opp}$ be the contravariant functor $A \mapsto A$. By Lemma 14.26.4 the maps a' and b' are homotopic. By Lemma 14.25.1 we see that $s(a')$ and $s(b')$ are homotopic maps of chain complexes. Since $s(a') = (s(a))'$ and $s(b') = (s(b))'$ we conclude that also $s(a)$ and $s(b)$ are homotopic by applying the additive contravariant functor $(-)'' : \mathcal{A}^{opp} \to \mathcal{A}$. The result for the Q-complexes follows from the direct sum decomposition of Lemma 14.23.1 for example. □

14.27. More homotopies in abelian categories

Let $\mathcal{A}$ be an abelian category. In this section we show that a homotopy between morphisms in $\text{Ch}_{\geq 0}(\mathcal{A})$ always comes from a morphism $U \times \Delta[1] \to V$ in the category of simplicial objects. In some sense this will provide a converse to Lemma 14.25.1. We first develop some material on homotopies between morphisms of chain complexes.

Lemma 14.27.1. *Let $\mathcal{A}$ be an abelian category. Let A be a chain complex. Consider the covariant functor*

$$B \longmapsto \{(a, b, h) \mid a, b : A \to B \text{ and } h \text{ a homotopy between } a, b\}$$

There exists a chain complex $\diamond A$ such that $Mor_{Ch(\mathcal{A})}(\diamond A, -)$ is isomorphic to the displayed functor. The construction $A \mapsto \diamond A$ is functorial.

Proof. We set $\diamond A_n = A_n \oplus A_n \oplus A_{n-1}$, and we define $d_{\diamond A,n}$ by the matrix

$$d_{\diamond A,n} = \begin{pmatrix} d_{A,n} & 0 & \text{id}_{A_{n-1}} \\ 0 & d_{A,n} & -\text{id}_{A_{n-1}} \\ 0 & 0 & -d_{A,n-1} \end{pmatrix} : A_n \oplus A_n \oplus A_{n-1} \to A_{n-1} \oplus A_{n-1} \oplus A_{n-2}$$

If $\mathcal{A}$ is the category of abelian groups, and $(x, y, z) \in A_n \oplus A_n \oplus A_{n-1}$ then $d_{\diamond A,n}(x, y, z) = (d_n(x) + z, d_n(y) - z, -d_{n-1}(z))$. It is easy to verify that $d^2 = 0$. Clearly, there are two maps

$\diamond a, \diamond b : A \to \diamond A$ (first summand and second summand), and a map $\diamond A \to A[-1]$ which give a short exact sequence

$$0 \to A \oplus A \to \diamond A \to A[-1] \to 0$$

which is termwise split. Moreover, there is a sequence of maps $\diamond h_n : A_n \to \diamond A_{n+1}$, namely the identity from A_n to the summand A_n of $\diamond A_{n+1}$, such that $\diamond h$ is a homotopy between $\diamond a$ and $\diamond b$.

We conclude that any morphism $f : \diamond A \to B$ gives rise to a triple (a, b, h) by setting $a = f \circ \diamond a$, $b = f \circ \diamond b$ and $h_n = f_{n+1} \circ \diamond h_n$. Conversely, given a triple (a, b, h) we get a morphism $f : \diamond A \to B$ by taking

$$f_n = (a_n, b_n, h_{n-1}).$$

To see that this is a morphism of chain complexes you have to do a calculation. We only do this in case $\mathcal{A}$ is the category of abelian groups: Say $(x, y, z) \in \diamond A_n = A_n \oplus A_n \oplus A_{n-1}$. Then

$$\begin{aligned} f_{n-1}(d_n(x, y, z)) &= f_{n-1}(d_n(x) + z, d_n(y) - z, -d_{n-1}(z)) \\ &= a_n(d_n(x)) + a_n(z) + b_n(d_n(y)) - b_n(z) - h_{n-2}(d_{n-1}(z)) \end{aligned}$$

and

$$\begin{aligned} d_n(f_n(x, y, z) &= d_n(a_n(x) + b_n(y) + h_{n-1}(z)) \\ &= d_n(a_n(x)) + d_n(b_n(y)) + d_n(h_{n-1}(z)) \end{aligned}$$

which are the same by definition of a homotopy. ☐

Note that the extension

$$0 \to A \oplus A \to \diamond A \to A[-1] \to 0$$

comes with sections of the morphisms $\diamond A_n \to A[-1]_n$ with the property that the associated morphism $\delta : A[-1] \to (A \oplus A)[-1]$, see Homology, Lemma 10.12.4 equals the morphism $(1, -1) : A[-1] \to A[-1] \oplus A[-1]$.

Lemma 14.27.2. *Let $\mathcal{A}$ be an abelian category. Let*

$$0 \to A \oplus A \to B \to C \to 0$$

be a short exact sequence of chain complexes of $\mathcal{A}$. Suppose given in addition morphisms $s_n : C_n \to B_n$ splitting the associated short exact sequence in degree n. Let $\delta(s) : C \to (A \oplus A)[-1] = A[-1] \oplus A[-1]$ be the associated morphism of complexes, see Homology, Lemma 10.12.4. If $\delta(s)$ factors through the morphism $(1, -1) : A[-1] \to A[-1] \oplus A[-1]$, then there is a unique morphism $B \to \diamond A$ fitting into a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A \oplus A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A \oplus A \arrow[r] & \diamond A \arrow[r] & A[-1] \arrow[r] & 0 \end{tikzcd}$$

where the vertical maps are compatible with the splittings s_n and the splittings of $\diamond A_n \to A[-1]_n$ as well.

Proof. Denote $(p_n, q_n) : B_n \to A_n \oplus A_n$ the morphism π_n of Homology, Lemma 10.12.4. Also write $(a, b) : A \oplus A \to B$, and $r : B \to C$ for the maps in the short exact sequence. Write the factorization of $\delta(s)$ as $\delta(s) = (1, -1) \circ f$. This means that $p_{n-1} \circ d_{B,n} \circ s_n = f_n$, and $q_{n-1} \circ d_{B,n} \circ s_n = -f_n$, and Set $B_n \to \diamond A_n = A_n \oplus A_n \oplus A_{n-1}$ equal to $(p_n, q_n, f_n \circ r_n)$.

Now we have to check that this actually defines a morphism of complexes. We will only do this in the case of abelian grous. Pick $x \in B_n$. Then $x = a_n(x_1) + b_n(x_2) + s_n(x_3)$ and it suffices to show that our definition commutes with differential for each term separately. For the term $a_n(x_1)$ we have $(p_n, q_n, f_n \circ r_n)(a_n(x_1)) = (x_1, 0, 0)$ and the result is obvious. Similarly for the term $b_n(x_2)$. For the term $s_n(x_3)$ we have

$$\begin{aligned} (p_n, q_n, f_n \circ r_n)(d_n(s_n(x_3))) & = (p_n, q_n, f_n \circ r_n)(\\ & \quad a_n(f_n(x_3)) - b_n(f_n(x_3)) + s_n(d_n(x_3))) \\ & = (f_n(x_3), -f_n(x_3), f_n(d_n(x_3))) \end{aligned}$$

by definition of f_n. And

$$\begin{aligned} d_n(p_n, q_n, f_n \circ r_n)(s_n(x_3)) & = d_n(0, 0, f_n(x_3)) \\ & = (f_n(x_3), -f_n(x_3), d_{A[-1],n}(f_n(x_3))) \end{aligned}$$

The result follows as f is a morphism of complexes. □

Lemma 14.27.3. *Let $\mathcal{A}$ be an abelian category. Let U, V be simplicial objects of $\mathcal{A}$. Let $a, b : U \to V$ be a pair of morphisms. Assume the corresponding maps of chain complexes $N(a), N(b) : N(U) \to N(V)$ are homotopic by a homotopy $\{N_n : N(U)_n \to N(V)_{n+1}\}$. Then a, b are homotopic in the sense of Definition 14.24.1. Moreover, one can choose the homotopy $h : U \times \Delta[1] \to V$ such that $N_n = N(h)_n$ where $N(h)$ is the homotopy coming from h as in Section 14.25.*

Proof. Let $(\diamond N(U), \diamond a, \diamond b, \diamond h)$ be as in Lemma 14.27.1 and its proof. By that lemma there exists a morphism $\diamond N(U) \to N(V)$ representing the triple $(N(a), N(b), \{N_n\})$. We will show there exists a morphism $\psi : N(U \times \Delta[1]) \to \diamond N(U)$ such that $\diamond a = \psi \circ N(e_0)$, and $\diamond b = \psi \circ N(e_1)$. Moreover, we will show that the homotopy between $N(e_0), N(e_1) : N(U) \to N(U \times \Delta[1])$ coming from (14.25.0.1) and Lemma 14.25.1 with $h = \mathrm{id}_{U \times \Delta[1]}$ is mapped via ψ to the canonical homotopy $\diamond h$ between the two maps $\diamond a, \diamond b : N(U) \to \diamond N(U)$. Certainly this will imply the lemma.

Note that $N : \text{Simp}(\mathcal{A}) \to \text{Ch}_{\geq 0}(\mathcal{A})$ as a functor is a direct summand of the functor $N : \text{Simp}(\mathcal{A}) \to \text{Ch}_{\geq 0}(\mathcal{A})$. Also, the functor $\diamond$ is compatible with direct sums. Thus it suffices instead to construct a morphism $\Psi : s(U \times \Delta[1]) \to \diamond s(U)$ with the corresponding properties. This is what we do below.

By Definition 14.24.1 the morphisms $e_0 : U \to U \times \Delta[1]$ and $e_1 : U \to U \times \Delta[1]$ are homotopic with homotopy $\mathrm{id}_{U \times \Delta[1]}$. By Lemma 14.25.1 we get an explicit homotopy $\{h_n : s(U)_n \to s(U \times \Delta[1])_{n+1}\}$ between the morphisms of chain complexes $s(e_0) : s(U) \to s(U \times \Delta[1])$ and $s(e_1) : s(U) \to s(U \times \Delta[1])$. By Lemma 14.27.2 above we get a corresponding morphism

$$\Phi : \diamond s(U) \to s(U \times \Delta[1])$$

According to the construction, Φ_n restricted to the summand $s(U)[-1]_n = s(U)_{n-1}$ of $\diamond s(U)_n$ is equal to h_{n-1}. And

$$h_{n-1} = \sum\nolimits_{i=0}^{n-1} (-1)^{i+1} s_i^n \cdot \alpha_{i+1}^n : U_{n-1} \to \bigoplus\nolimits_j U_n \cdot \alpha_j^n.$$

with obvious notation.

On the other hand, the morphisms $e_i : U \to U \times \Delta[1]$ induce a morphism $(e_0, e_1) : U \oplus U \to U \times \Delta[1]$. Denote W the cokernel. Note that, if we write $(U \times \Delta[1])_n = \bigoplus_{\alpha : [n] \to [1]} U_n \cdot \alpha$, then we may identify $W_n = \bigoplus_{i=1}^n U_n \cdot \alpha_i^n$ with α_i^n as in Section 14.24. We have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & U \oplus U \arrow[r] \arrow[dr, "{(1,1)}"'] & U \times \Delta[1] \arrow[r] \arrow[d, "\pi"] & W \arrow[r] & 0 \\ & & U & & \end{tikzcd}$$

This implies we have a similar commutative diagram after applying the functor s. Next, we choose the splittings $\sigma_n : s(W)_n \to s(U \times \Delta[1])_n$ by mapping the summand $U_n \cdot \alpha_i^n \subset W_n$ via $(-1, 1)$ to the summands $U_n \cdot \alpha_0^n \oplus U_n \cdot \alpha_i^n \subset (U \times \Delta[1])_n$. Note that $s(\pi)_n \circ \sigma_n = 0$. It follows that $(1,1) \circ \delta(\sigma)_n = 0$. Hence $\delta(\sigma)$ factors as in Lemma 14.27.2. By that lemma we obtain a canonical morphism $\Psi : s(U \times \Delta[1]) \to \diamond s(U)$.

To compute Ψ we first compute the morphism $\delta(\sigma) : s(W) \to s(U)[-1] \oplus s(U)[-1]$. According to Homology, Lemma 10.12.4 and its proof, to do this we have compute

$$d_{s(U\times\delta[1]),n} \circ \sigma_n - \sigma_{n-1} \circ d_{s(W),n}$$

and write it as a morphism into $U_{n-1} \cdot \alpha_0^{n-1} \oplus U_{n-1} \cdot \alpha_n^{n-1}$. We only do this in case $\mathcal{A}$ is the category of abelian groups. We use the short hand notation x_α for $x \in U_n$ to denote the element x in the summand $U_n \cdot \alpha$ of $(U \times \Delta[1])_n$. Recall that

$$d_{s(U\times\delta[1]),n} = \sum\nolimits_{i=0}^n (-1)^i d_i^n$$

where d_i^n maps the summand $U_n \cdot \alpha$ to the summand $U_{n-1} \cdot (\alpha \circ \delta_i^n)$ via the morphism d_i^n of the simplicial object U. In terms of the notation above this means

$$d_{s(U\times\delta[1]),n}(x_\alpha) = \sum\nolimits_{i=0}^n (-1)^i (d_i^n(x))_{\alpha\circ\delta_i^n}$$

Starting with $x_\alpha \in W_n$, in other words $\alpha = \alpha_j^n$ for some $j \in \{1, \ldots, n\}$, we see that $\sigma_n(x_\alpha) = x_\alpha - x_{\alpha_0^n}$ and hence

$$(d_{s(U\times\delta[1]),n} \circ \sigma_n)(x_\alpha) = \sum\nolimits_{i=0}^n (-1)^i (d_i^n(x))_{\alpha\circ\delta_i^n} - \sum\nolimits_{i=0}^n (-1)^i (d_i^n(x))_{\alpha_0^n\circ\delta_i^n}$$

To compute $d_{s(W),n}(x_\alpha)$, we have to omit all terms where $\alpha \circ \delta_i^n = \alpha_0^{n-1}, \alpha_n^{n-1}$. Hence we get

$$(\sigma_{n-1} \circ d_{s(W),n})(x_\alpha) = \\ \sum\nolimits_{i=0,\ldots,n \text{ and } \alpha\circ\delta_i^n \neq \alpha_0^{n-1} \text{ or } \alpha_n^{n-1}} \left((-1)^i (d_i^n(x))_{\alpha\circ\delta_i^n} - (-1)^i (d_i^n(x))_{\alpha_0^{n-1}}\right)$$

Clearly the difference of the two terms is the sum

$$\sum\nolimits_{i=0,\ldots,n \text{ and } \alpha\circ\delta_i^n = \alpha_0^{n-1} \text{ or } \alpha_n^{n-1}} \left((-1)^i (d_i^n(x))_{\alpha\circ\delta_i^n} - (-1)^i (d_i^n(x))_{\alpha_0^{n-1}}\right)$$

Of course, if $\alpha \circ \delta_i^n = \alpha_0^{n-1}$ then the term drops out. Recall that $\alpha = \alpha_j^n$ for some $j \in \{1, \ldots, n\}$. The only way $\alpha_j^n \circ \delta_i^n = \alpha_n^{n-1}$ is if $j = n$ and $i = n$. Thus we actually get 0 unless $j = n$ and in that case we get $(-1)^n (d_n^n(x))_{\alpha_n^{n-1}} - (-1)^n (d_n^n(x))_{\alpha_0^{n-1}}$. In other words, we conclude the morphism

$$\delta(\sigma)_n : W_n \to (s(U)[-1] \oplus s(U)[-1])_n = U_{n-1} \oplus U_{n-1}$$

is zero on all summands except $U_n \cdot \alpha_n^n$ and on that summand it is equal to $((-1)^n d_n^n, -(-1)^n d_n^n)$. (Namely, the first summand of the two corresponds to the factor with α_n^{n-1} because that is the map $[n-1] \to [1]$ which maps everybody to 0, and hence corresponds to e_0.)

We obtain a canonical diagram

$$\begin{tikzcd}
0 \arrow[r] & s(U) \oplus s(U) \arrow[r] \arrow[d] & \diamond s(U) \arrow[r] \arrow[d, "\Phi"] & s(U)[-1] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & s(U) \oplus s(U) \arrow[r] \arrow[d] & s(U \times \Delta[1]) \arrow[r] \arrow[d, "\Psi"] & s(W) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & s(U) \oplus s(U) \arrow[r] & \diamond s(U) \arrow[r] & s(U)[-1] \arrow[r] & 0
\end{tikzcd}$$

We claim that $\Phi \circ \Psi$ is the identity. To see this it is enough to prove that the composition of Φ and $\delta(\sigma)$ as a map $s(U)[-1] \to s(W) \to s(U)[-1] \oplus s(U)[-1]$ is the identity in the first factor and minus identity in the second. By the computations above it is $((-1)^n d^n_0, -(-1)^n d^n_0) \circ (-1)^n s^n_n = (1, -1)$ as desired. □

14.28. A homotopy equivalence

Suppose that A, B are sets, and that $f : A \to B$ is a map. Consider the associated map of simplicial sets

$$\begin{tikzcd}
\mathrm{cosk}_0(A) \arrow[r, equal] & \Big(\ldots A \times A \times A \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] \arrow[d] & A \times A \arrow[l] \arrow[l, shift left=2] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[d] & A \Big) \arrow[l] \arrow[d] \\
\mathrm{cosk}_0(B) \arrow[r, equal] & \Big(\ldots B \times B \times B \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & B \times B \arrow[l] \arrow[l, shift left=2] \arrow[r, shift left=1] \arrow[r, shift right=1] & B \Big) \arrow[l]
\end{tikzcd}$$

See Example 14.17.2. The case $n = 0$ of the following lemma says that this map of simplicial sets has a section if f is surjective. The proof: choose a section of f.

Lemma 14.28.1. *Let $f : V \to U$ be a morphism of simplicial sets. Let $n \geq 0$ be an integer. Assume*

1. *The map $f_i : V_i \to U_i$ is a bijection for $i < n$.*
2. *The map $f_n : V_n \to U_n$ is a surjection.*
3. *The canonical morphism $U \to cosk_n sk_n U$ is an isomorphism.*
4. *The canonical morphism $V \to cosk_n sk_n V$ is an isomorphism.*

Then there exists a morphism of simplicial sets $g : U \to V$ such that $f \circ g = id_U$.

Proof. By Lemma 14.16.2 both U and V have canonical splittings with $N(U_i)$ and $N(V_i)$ equal to the sets of nondegenerate simplices. We have to find maps $g_m : U_m \to V_m$ for $m \geq 0$ such that

$$d^k_i \circ g_k = g_{k-1} \circ d^k_i \tag{14.28.1.1}$$

$$s^k_i \circ g_k = g_{k+1} \circ s^k_i \tag{14.28.1.2}$$

for all k. By induction on m we will show that we can find maps $g_0, \ldots, g_m$ such that (14.28.1.1) holds for $1 \leq k \leq m$ and (14.28.1.2) holds for $0 \leq k \leq m-1$. We set g_i equal to the inverse of f_i for $i = 0, \ldots, n-1$. Clearly the induction hypothesis holds for $m = n - 1$. We define $g_n : U_n \to V_n$ as follows. Pick $u \in U_n$, then

1. if u is degenerate, write $u = U(\varphi)(u')$ for some nondegenerate $u' \in U_m$ and some surjective $\varphi : [n] \to [m]$. We set $g_n(u) = V(\varphi)(g_m(u'))$. This is well defined as the pair (φ, u') is unique.
2. if u is nondegenerate, we choose any $v \in V_n$ mapping to u and we set $g_n(u) = v$.

This choice of g_n garantees that the induction hypothesis holds for $m = n$. Namely, we forced (14.28.1.2) with $k = n-1$ by our choice of g_n on degenerate simplices, and (14.28.1.1) with $k = n$ holds because the equality takes place in $V_{n-1} = U_{n-1}$.

One way to finish the proof at this point is to show that the family of maps $g_0, \ldots, g_n$ defines a morphism of n-truncated simplicial sets $\text{sk}_n U \to \text{sk}_n V$ which is a right inverse to $\text{sk}_n f$. Then since cosk_n is a functor and by the hypothesis of the lemma we get g as $\text{cosk}_n(g_0, \ldots, g_n)$. But we can also see this directly as follows.

Given the induction hypothesis for $m \geq n$ we inductively define g_{m+1} as follows. Since $U \to \text{cosk}_n \text{sk}_n U$ is an isomorphism, we see that also $U \to \text{cosk}_m \text{sk}_m U$ is an isomorphism. Hence elements of U_{m+1} are $(m+2)$-tuples $(u_0, \ldots, u_{m+1})$ with $u_i \in U_m$ satisfying the equalities $d^m_{j-1}(u_i) = d^m_i(u_j)\ \forall\ 0 \leq i < j \leq m+1$. Similarly for V_{m+1}. Thus we may simply map the element $(u_0, \ldots, u_{m+1})$ to the element $(g_m(u_0), \ldots, g_m(u_{m+1}))$. To verify the induction hypothesis for $m+1$ with this choice of g_{m+1} we will use the explicit form of the maps d_i and s_i as given in Remark 14.17.8. This remark shows immediately that the commutation of $g_0, \ldots, g_m$ with d_i and s_i implies the desired commutation for g_{m+1}. □

Let A, B be sets. Let $f^0, f^1 : A \to B$ be maps of sets. Consider the induced maps $f^0, f^1 : \text{cosk}_0(A) \to \text{cosk}_0(B)$ abusively denoted by the same symbols. The following lemma for $n = 0$ says that f_0 is homotopic to f_1. In fact, the homotopy is given by the map $h : \text{cosk}_0(A) \times \Delta[1] \to \text{cosk}_0(A)$ with components

$$\begin{array}{rcl} h_m : A \times \ldots \times A \times Mor_\Delta([m],[1]) & \longrightarrow & A \times \ldots \times A, \\ (a_0, \ldots, a_m, \alpha) & \longmapsto & (f^{\alpha(0)}(a_0), \ldots, f^{\alpha(m)}(a_m)) \end{array}$$

To check that this works, note that for a map $\varphi : [k] \to [m]$ the induced maps are $(a_0, \ldots, a_m) \mapsto (a_{\varphi(0)}, \ldots, a_{\varphi(k)})$ and $\alpha \mapsto \alpha \circ \varphi$. Thus $h = (h_m)_{m \geq 0}$ is clearly a map of simplicial sets as desired.

Lemma 14.28.2. *Let $f^0, f^1 : V \to U$ be maps of a simplicial sets. Let $n \geq 0$ be an integer. Assume*

1. *The maps $f^j_i : V_i \to V_i$, $j = 0, 1$ are equal for $i < n$.*
2. *The canonical morphism $U \to cosk_n sk_n U$ is an isomorphism.*
3. *The canonical morphism $V \to cosk_n sk_n V$ is an isomorphism.*

Then f^0 is homotopic to f^1.

Proof. We have to construct a morphism of simplicial sets $h : V \times \Delta[1] \to U$ which recovers f^i on composing with e_i. The case $n = 0$ was dealt with above the lemma. Thus we may assume that $n \geq 1$. The map $\Delta[1] \to \text{cosk}_1 \text{sk}_1 \Delta[1]$ is an isomorphism, see Lemma 14.17.14. Thus we see that $\Delta[1] \to \text{cosk}_n \text{sk}_n \Delta[1]$ is an isomorphism as $n \geq 1$, see Lemma 14.17.11. And hence $V \times \Delta[1] \to \text{cosk}_n \text{sk}_n(V \times \Delta[1])$ is an isomorphism too, see Lemma 14.17.12. In other words, in order to construct the homotopy it suffices to construct a suitable morphism of n-truncated simplicial sets $h : \text{sk}_n V \times \text{sk}_n \Delta[1] \to \text{sk}_n U$.

For $k = 0, \ldots, n-1$ we define h_k by the formula $h_k(v, \alpha) = f^0(v) = f^1(v)$. The map $h_n : V_n \times Mor_\Delta([k],[1]) \to U_n$ is defined as follows. Pick $v \in V_n$ and $\alpha : [n] \to [1]$:

1. If $\text{Im}(\alpha) = \{0\}$, then we set $h_n(v, \alpha) = f^0(v)$.
2. If $\text{Im}(\alpha) = \{0, 1\}$, then we set $h_n(v, \alpha) = f^0(v)$.
3. If $\text{Im}(\alpha) = \{1\}$, then we set $h_n(v, \alpha) = f^1(v)$.

Let $\varphi : [k] \to [l]$ be a morphism of $\Delta_{\leq n}$. We will show that the diagram

$$\begin{array}{ccc} V_{[l]} \times Mor([l],[1]) & \longrightarrow & U_{[l]} \\ \downarrow & & \downarrow \\ V_{[k]} \times Mor([k],[1]) & \longrightarrow & U_{[k]} \end{array}$$

commutes. Pick $v \in V_{[l]}$ and $\alpha : [l] \to [1]$. The commutativity means that

$$h_k(V(\varphi)(v), \alpha \circ \varphi) = U(\varphi)(h_l(v, \alpha)).$$

In almost every case this holds because $h_k(V(\varphi)(v), \alpha\circ\varphi) = f^0(V(\varphi)(v))$ and $U(\varphi)(h_l(v,\alpha)) = U(\varphi)(f^0(v))$, combined with the fact that f^0 is a morphism of simplicial sets. The only cases where this does not hold is when either (A) $\mathrm{Im}(\alpha) = \{1\}$ and $l = n$ or (B) $\mathrm{Im}(\alpha \circ \varphi) = \{1\}$ and $k = n$. Observe moreover that necessarily $f^0(v) = f^1(v)$ for any degenerate n-simplex of V. Thus we can narrow the cases above down even further to the cases (A) $\mathrm{Im}(\alpha) = \{1\}$, $l = n$ and v nondegenerate, and (B) $\mathrm{Im}(\alpha \circ \varphi) = \{1\}$, $k = n$ and $V(\varphi)(v)$ nondegenerate.

In case (A), we see that also $\mathrm{Im}(\alpha \circ \varphi) = \{1\}$. Hence we see that not only $h_l(v, \alpha) = f^1(v)$ but also $h_k(V(\varphi)(v), \alpha \circ \varphi) = f^1(V(\varphi)(v))$. Thus we see that the relation holds because f^1 is a morphism of simplicial sets.

In case (B) we conclude that $l = k = n$ and φ is bijective, since otherwise $V(\varphi)(v)$ is degenerate. Thus $\varphi = \mathrm{id}_{[n]}$, which is a trivial case. □

Lemma 14.28.3. *With assumptions and notation as in Lemma 14.28.1 above. The composition $g \circ f$ is homotopy equivalent to the identity on V. In particular, the morphism f is a homotopy equivalence.*

Proof. Immediate from Lemma 14.28.2 above. □

Lemma 14.28.4. *Let A, B be sets, and that $f : A \to B$ is a map. Consider the simplicial set U with n-simplices*

$$A \times_B A \times_B \ldots \times_B A \ (n + 1 \text{ factors}).$$

see Example 14.3.5. If f is surjective, the morphism

$$U \to \mathbf{B}$$

where $\mathbf{B}$ indicates the constant simplicial set with value B is a homotopy equivalence.

Proof. For $b \in B$, write $A_b = f^{-1}(b)$. It is a nonempty set. It is clear that $\mathbf{B} = \coprod_{b\in B}\{b\}$ and that $U = \coprod_{b\in B} \mathrm{cosk}_0 A_b$. Each of the morphisms $\mathrm{cosk}_0 A_b \to \{b\}$ is a homotopy equivalence by Lemma 14.28.3. It follows easily that $U \to \mathbf{B}$ is a homotopy equivalence. □

14.29. Other chapters

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CHAPTER 15

Sheaves of Modules

15.1. Introduction

In this chapter we work out basic notions of sheaves of modules. This in particular includes the case of abelian sheaves, since these may be viewed as sheaves of $\underline{\mathbf{Z}}$-modules. Basic references are [**Ser55b**], [**DG67**] and [**MA71**].

We work out what happens for sheaves of modules on ringed topoi in another chapter (see Modules on Sites, Section 16.1), allthough there we will mostly just duplicate the discussion from this chapter.

15.2. Pathology

A ringed space is a pair consisting of a topological space X and a sheaf of rings $\mathcal{O}$. We allow $\mathcal{O} = 0$ in the definition. In this case the category of modules has a single object (namely 0). It is still an abelian category etc, but it is a little degenerate. Similarly the sheaf $\mathcal{O}$ may be zero over open subsets of X, etc.

This doesn't happen when considering locally ringed spaces (as we will do later).

15.3. The abelian category of sheaves of modules

Let $(X, \mathcal{O}_X)$ be a ringed space, see Sheaves, Definition 6.25.1. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules, see Sheaves, Definition 6.10.1. Let $\varphi, \psi : \mathcal{F} \to \mathcal{G}$ be morphisms of sheaves of $\mathcal{O}_X$-modules. We define $\varphi + \psi : \mathcal{F} \to \mathcal{G}$ to be the map which on each open $U \subset X$ is the sum of the maps induced by φ, ψ. This is clearly again a map of sheaves of $\mathcal{O}_X$-modules. It is also clear that composition of maps of $\mathcal{O}_X$-modules is bilinear with respect to this addition. Thus $Mod(\mathcal{O}_X)$ is a pre-additive category, see Homology, Definition 10.3.1.

We will denote 0 the sheaf of $\mathcal{O}_X$-modules which has constant value $\{0\}$ for all open $U \subset X$. Clearly this is both a final and an initial object of $Mod(\mathcal{O}_X)$. Given a morphism of $\mathcal{O}_X$-modules $\varphi : \mathcal{F} \to \mathcal{G}$ the following are equivalent: (a) φ is zero, (b) φ factors through 0, (c) φ is zero on sections over each open U, and (d) $\varphi_x = 0$ for all $x \in X$. See Sheaves, Lemma 6.16.1.

Moreover, given a pair $\mathcal{F}$, $\mathcal{G}$ of sheaves of $\mathcal{O}_X$-modules we may define the direct sum as

$$\mathcal{F} \oplus \mathcal{G} = \mathcal{F} \times \mathcal{G}$$

with obvious maps (i, j, p, q) as in Homology, Definition 10.3.5. Thus $Mod(\mathcal{O}_X)$ is an additive category, see Homology, Definition 10.3.8.

Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of $\mathcal{O}_X$-modules. We may define $\mathrm{Ker}(\varphi)$ to be the subsheaf of $\mathcal{F}$ with sections

$$\mathrm{Ker}(\varphi)(U) = \{s \in \mathcal{F}(U) \mid \varphi(s) = 0 \text{ in } \mathcal{G}(U)\}$$

for all open $U \subset X$. It is easy to see that this is indeed a kernel in the category of $\mathcal{O}_X$-modules. In other words, a morphism $\alpha : \mathcal{H} \to \mathcal{F}$ factors through $\mathrm{Ker}(\varphi)$ if and only if $\varphi \circ \alpha = 0$. Moreover, on the level of stalks we have $\mathrm{Ker}(\varphi)_x = \mathrm{Ker}(\varphi_x)$.

On the other hand, we define $\mathrm{Coker}(\varphi)$ as the sheaf of $\mathcal{O}_X$-modules associated to the presheaf of $\mathcal{O}_X$-modules defined by the rule

$$U \longmapsto \mathrm{Coker}(\mathcal{G}(U) \to \mathcal{F}(U)) = \mathcal{F}(U)/\varphi(\mathcal{G}(U)).$$

Since taking stalks commutes with taking sheafification, see Sheaves, Lemma 6.17.2 we see that $\mathrm{Coker}(\varphi)_x = \mathrm{Coker}(\varphi_x)$. Thus the map $\mathcal{G} \to \mathrm{Coker}(\varphi)$ is surjective (as a map of sheaves of sets), see Sheaves, Section 6.16. To show that this is a cokernel, note that if $\beta : \mathcal{G} \to \mathcal{H}$ is a morphism of $\mathcal{O}_X$-modules such that $\beta \circ \varphi$ is zero, then you get for every open $U \subset X$ a map induced by β from $\mathcal{G}(U)/\varphi(\mathcal{F}(U))$ into $\mathcal{H}(U)$. By the universal property of sheafification (see Sheaves, Lemma 6.20.1) we obtain a canonical map $\mathrm{Coker}(\varphi) \to \mathcal{H}$ such that the original β is equal to the composition $\mathcal{G} \to \mathrm{Coker}(\varphi) \to \mathcal{H}$. The morphism $\mathrm{Coker}(\varphi) \to \mathcal{H}$ is unique because of the surjectivity mentioned above.

Lemma 15.3.1. *Let $(X, \mathcal{O}_X)$ be a ringed space. The category $Mod(\mathcal{O}_X)$ is an abelian category. Moreover a complex*

$$\mathcal{F} \to \mathcal{G} \to \mathcal{H}$$

is exact at $\mathcal{G}$ if and only if for all $x \in X$ the complex

$$\mathcal{F}_x \to \mathcal{G}_x \to \mathcal{H}_x$$

is exact at $\mathcal{G}_x$.

Proof. By Homology, Definition 10.3.12 we have to show that image and coimage agree. By Sheaves, Lemma 6.16.1 it is enough to show that image and coimage have the same stalk at every $x \in X$. By the constructions of kernels and cokernels above these stalks are the coimage and image in the categories of $\mathcal{O}_{X,x}$-modules. Thus we get the result from the fact that the category of modules over a ring is abelian. □

Actually the category $Mod(\mathcal{O}_X)$ has many more properties. Here are two constructions we can do.

(1) Given any set I and for each $i \in I$ a $\mathcal{O}_X$-module we can form the product

$$\prod\nolimits_{i \in I} \mathcal{F}_i$$

which is the sheaf that associates to each open U the product of the modules $\mathcal{F}_i(U)$. This is also the categorical product, as in Categories, Definition 4.13.5.

(2) Given any set I and for each $i \in I$ a $\mathcal{O}_X$-module we can form the direct sum

$$\bigoplus\nolimits_{i \in I} \mathcal{F}_i$$

which is the *sheafification* of the presheaf that associates to each open U the direct sum of the modules $\mathcal{F}_i(U)$. This is also the categorical coproduct, as in Categories, Definition 4.13.6. To see this you use the universal property of sheafification.

Since any abelian category has equalizers and coequalizers we conclude that all limits and colimits exist in $Mod(\mathcal{O}_X)$, see Categories, Lemmas 4.13.10 and 4.13.11.

Lemma 15.3.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. All limits and colimits exist in $Mod(\mathcal{O}_X)$. Limits are the same as the corresponding limits of presheaves of $\mathcal{O}_X$-modules (i.e., commute with taking sections over opens). Finite direct sums are the same as the correponding finite* *direct sums of pre-sheaves of $\mathcal{O}_X$-modules. A colimit is the sheafification of the corresponding colimit in the category of presheaves.*

Proof. Omitted. But see discussion above. $\square$

Lemma 15.3.3. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let I be a set. For $i \in I$, let $\mathcal{F}_i$ be a sheaf of $\mathcal{O}_X$-modules. For $U \subset X$ quasi-compact open the map*

$$\bigoplus\nolimits_{i\in I} \mathcal{F}_i(U) \longrightarrow \left(\bigoplus\nolimits_{i\in I} \mathcal{F}_i\right)(U)$$

is bijective.

Proof. If s is an element of the right hand side, then there exists an open covering $U = \bigcup_{j\in J} U_j$ such that $s|_{U_j}$ is a finite sum $\sum_{i\in I_j} s_{ji}$ with $s_{ji} \in \mathcal{F}_i(U_j)$. Because U is quasi-compact we may assume that the covering is finite, i.e., that J is finite. Then $I' = \bigcup_{j\in J} I_j$ is a finite subset of I. Clearly, s is a section of the subsheaf $\bigoplus_{i\in I'} \mathcal{F}_i$. The result follows from the fact that for a finite direct sum sheafification is not needed, see Lemma 15.3.2 above. $\square$

The existence of limits and colimits allows us to consider exactness properties of functors defined on the category of $\mathcal{O}$-modules in terms of limits and colimits, as in Categories, Section 4.21. See Homology, Lemma 10.5.1 for a description of exactness properties in terms of short exact sequences.

Lemma 15.3.4. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces.*

1. *The functor $f_* : Mod(\mathcal{O}_X) \to Mod(\mathcal{O}_Y)$ is left exact. In fact it commutes with all limits.*
2. *The functor $f^* : Mod(\mathcal{O}_Y) \to Mod(\mathcal{O}_X)$ is right exact. In fact it commutes with all colimits.*
3. *Pullback $f^{-1} : Ab(Y) \to Ab(X)$ on abelian sheaves is exact.*

Proof. Parts (1) and (2) hold because (f^*, f_*) is an adjoint pair of functors, see Sheaves, Lemma 6.26.2 and Categories, Section 4.22. Part (3) holds because exactness can be checked on stalks (Lemma 15.3.1) and the description of stalks of the pullback, see Sheaves, Lemma 6.22.1. $\square$

Lemma 15.3.5. *Let $j : U \to X$ be an open immersion of topological spaces. The functor $j_! : Ab(U) \to Ab(X)$ is exact.*

Proof. This is clear from the description of stalks given in Sheaves, Lemma 6.31.6. $\square$

15.4. Sections of sheaves of modules

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $s \in \Gamma(X, \mathcal{F}) = \mathcal{F}(X)$ be a global section. There is a unique *map of $\mathcal{O}_X$-modules*

$$\mathcal{O}_X \longrightarrow \mathcal{F}, \ f \longmapsto fs$$

associated to s. The notation above signifies that a local section f of $\mathcal{O}_X$, i.e., a section f over some open U, is mapped to the multiplication of f with the restriction of s to U. Conversely, any map $\varphi : \mathcal{O}_X \to \mathcal{F}$ gives rise to a section $s = \varphi(1)$ such that φ is the morphism associated to s.

Definition 15.4.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is *generated by global sections* if there exist a set I, and global sections $s_i \in \Gamma(X, \mathcal{F})$, $i \in I$ such that the map

$$\bigoplus\nolimits_{i \in I} \mathcal{O}_X \longrightarrow \mathcal{F}$$

which is the map associated to s_i on the summand corresponding to i, is surjective. In this case we say that the sections s_i *generate* $\mathcal{F}$.

We often use the abuse of notation introduced in Sheaves, Section 6.11 where, given a local section s of $\mathcal{F}$ defined in an open neighbourhood of a point $x \in X$, we denote s_x, or even s the image of s in the stalk $\mathcal{F}_x$.

Lemma 15.4.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let I be a set. Let $s_i \in \Gamma(X, \mathcal{F})$, $i \in I$. be global sections. The sections s_i generate $\mathcal{F}$ if and only if for all $x \in X$ the elements $s_{i,x} \in \mathcal{F}_x$ generate the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$.*

Proof. Omitted. □

Lemma 15.4.3. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules. If $\mathcal{F}$ and $\mathcal{G}$ are generated by global sections then so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.*

Proof. Omitted. □

Lemma 15.4.4. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let I be a set. Let s_i, $i \in I$ be a collection of local sections of $\mathcal{F}$, i.e., $s_i \in \mathcal{F}(U_i)$ for some opens $U_i \subset X$. There exists a unique smallest subsheaf of $\mathcal{O}_X$-modules $\mathcal{G}$ such that each s_i corresponds to a local section of $\mathcal{G}$.*

Proof. Consider the subpresheaf of $\mathcal{O}_X$-modules defined by the rule

$$U \longmapsto \{\text{sums } \sum\nolimits_{i \in J} f_i(s_i|_U) \text{ where } J \text{ is finite, } U \subset U_i \text{ for } i \in J, \text{ and } f_i \in \mathcal{O}_X(U)\}$$

Let $\mathcal{G}$ be the sheafification of this subpresheaf. This is a subsheaf of $\mathcal{F}$ by Sheaves, Lemma 6.16.3. Since all the finite sums clearly have to be in $\mathcal{G}$ this is the smallest subsheaf as desired. □

Definition 15.4.5. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Given a set I, and local sections s_i, $i \in I$ of $\mathcal{F}$ we say that the subsheaf $\mathcal{G}$ of Lemma 15.4.4 above is the *subsheaf generated by the s_i*.

Lemma 15.4.6. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Given a set I, and local sections s_i, $i \in I$ of $\mathcal{F}$. Let $\mathcal{G}$ be the subsheaf generated by the s_i and let $x \in X$. Then $\mathcal{G}_x$ is the $\mathcal{O}_{X,x}$-submodule of $\mathcal{F}_x$ generated by the elements $s_{i,x}$ for those i such that s_i is defined at x.*

Proof. This is clear from the construction of $\mathcal{G}$ in the proof of Lemma 15.4.4. □

15.5. Supports of modules and sections

Definition 15.5.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules.

(1) The *support of $\mathcal{F}$* is the set of points $x \in X$ such that $\mathcal{F}_x \neq 0$.
(2) We denote $\text{Supp}(\mathcal{F})$ the support of $\mathcal{F}$.
(3) Let $s \in \Gamma(X, \mathcal{F})$ be a global section. The *support of s* is the set of points $x \in X$ such that the image $s_x \in \mathcal{F}_x$ of s is not zero.

Of course the support of a local section is then defined also since a local section is a global section of the restriction of $\mathcal{F}$.

Lemma 15.5.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $U \subset X$ open.*

(1) *The support of $s \in \mathcal{F}(U)$ is closed in U.*
(2) *The support of fs is contained in the intersections of the supports of $f \in \mathcal{O}_X(U)$ and $s \in \mathcal{F}(U)$*
(3) *The support of $s + s'$ is contained in the union of the supports of $s, s' \in \mathcal{F}(U)$.*
(4) *The support of $\mathcal{F}$ is the union of the supports of all local sections of $\mathcal{F}$.*
(5) *If $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of $\mathcal{O}_X$-modules, then the support of $\varphi(s)$ is contained in the support of $s \in \mathcal{F}(U)$.*

Proof. This is true because if $s_x = 0$, then s is zero in an open neighbourhood of x by definition of stalks. Similarly for f. Details omitted. □

In general the support of a sheaf of modules is not closed. Namely, the sheaf could be an abelian sheaf on $\mathbf{R}$ (with the usual archimedean topology) which is the direct sum of infinitely many nonzero skyscraper sheaves each supported at a single point p_i of $\mathbf{R}$. Then the support would be the set of points p_i which may not be closed.

Another example is to consider the open immersion $j : U = (0, \infty) \to \mathbf{R} = X$, and the abelian sheaf $j_!\underline{\mathbf{Z}}_U$. By Sheaves, Section 6.31 the support of this sheaf is exactly U.

Lemma 15.5.3. *Let X be a topological space. The support of a sheaf of rings is closed.*

Proof. This is true because (according to our conventions) a ring is 0 if and only if $1 = 0$, and hence the support of a sheaf of rings is the support of the unit section. □

15.6. Closed immersions and abelian sheaves

Recall that we think of an abelian sheaf on a topological space X as a sheaf of $\underline{\mathbf{Z}}_X$-modules. Thus we may apply any results, definitions for sheaves of modules to abelian sheaves.

Lemma 15.6.1. *Let X be a topological space. Let $Z \subset X$ be a closed subset. Denote $i : Z \to X$ the inclusion map. The functor*

$$i_* : Ab(Z) \longrightarrow Ab(X)$$

is exact, fully faithful, with essential image exactly those abelian sheaves whose support is contained in Z.

Proof. Exactness follows from the description of stalks in Sheaves, Lemma 6.32.1 and Lemma 15.3.1. The rest was shown in Sheaves, Lemma 6.31.10. □

Let $\mathcal{F}$ be a sheaf on X. There is a canonical subsheaf of $\mathcal{F}$ which consists of exactly those sections whose support is contained in Z. Here is the exact statement.

Lemma 15.6.2. *Let X be a topological space. Let $Z \subset X$ be a closed subset. Let $\mathcal{F}$ be a sheaf on X. For $U \subset X$ open set*

$$\Gamma(U, \mathcal{H}_Z(\mathcal{F})) = \{s \in \mathcal{F}(U) \mid \text{ the support of } s \text{ is contained in } Z \cap U\}$$

Then $\mathcal{H}_Z(\mathcal{F})$ is an abelian subsheaf of $\mathcal{F}$. It is the largest abelian subsheaf of $\mathcal{F}$ whose support is contained in Z. The construction $\mathcal{F} \mapsto \mathcal{H}_Z(\mathcal{F})$ is functorial in the abelian sheaf $\mathcal{F}$.

Proof. This follows from Lemma 15.5.2. □

This seems like a good opportunity to show that the functor i_* has a right adjoint on abelian sheaves.

Lemma 15.6.3. *Denote*[1] $i^!: Ab(X) \to Ab(Z)$ *the functor* $\mathcal{F} \mapsto i^{-1}\mathcal{H}_Z(\mathcal{F})$. *Then* $i^!$ *is a right adjoint to* i_*, *in a formula*

$$Mor_{Ab(X)}(i_*\mathcal{G}, \mathcal{F}) = Mor_{Ab(Z)}(\mathcal{G}, i^!\mathcal{F}).$$

In particular i_* *commutes with arbitrary colimits.*

Proof. Note that $i_*i^!\mathcal{F} = \mathcal{H}_Z(\mathcal{F})$. Since i_* is fully faithful we are reduced to showing that

$$Mor_{Ab(X)}(i_*\mathcal{G}, \mathcal{F}) = Mor_{Ab(X)}(i_*\mathcal{G}, \mathcal{H}_Z(\mathcal{F})).$$

This follows since the support of the image via any homomorphism of a section of $i_*\mathcal{G}$ is supported on Z, see Lemma 15.5.2. □

Remark 15.6.4. In Sheaves, Remark 6.32.5 we showed that i_* as a functor on the categories of sheaves of sets does not have a right adjoint simply because it is not exact. However, it is very close to being true, in fact, the functor i_* is exact on sheaves of pointed sets, sections with support in Z can be defined for sheaves of pointed sets, and $i^!$ makes sense and is a right adjoint to i_*.

15.7. A canonical exact sequence

We give this exact sequence its own section.

Lemma 15.7.1. *Let X be a topological space. Let $U \subset X$ be an open subset with complement $Z \subset X$. Denote $j : U \to X$ the open immersion and $i : Z \to X$ the closed immersion. For any sheaf of abelian groups $\mathcal{F}$ on X the adjunction mappings $j_!j^*\mathcal{F} \to \mathcal{F}$ and $\mathcal{F} \to i_*i^*\mathcal{F}$ give a short exact sequence*

$$0 \to j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \to 0$$

of sheaves of abelian groups. For any morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of abelian sheaves on X we obtain a morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \to & j_!j^*\mathcal{F} & \to & \mathcal{F} & \to & i_*i^*\mathcal{F} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & j_!j^*\mathcal{G} & \to & \mathcal{G} & \to & i_*i^*\mathcal{G} & \to & 0 \end{array}$$

Proof. We may check exactness on stalks (Lemma 15.3.1). For a description of the stalks in question see Sheaves, Lemmas 6.31.6 and 6.32.1. We omit the proof of the functorial behaviour of the exact sequence. □

15.8. Modules locally generated by sections

Let $(X, \mathcal{O}_X)$ be a ringed space. In this and the following section we will often restrict sheaves to open subspaces $U \subset X$, see Sheaves, Section 6.31. In particular, we will often denote the open subspace by $(U, \mathcal{O}_U)$ instead of the more correct notation $(U, \mathcal{O}_X|_U)$, see Sheaves, Definition 6.31.2.

Consider the open immersion $j : U = (0, \infty) \to \mathbf{R} = X$, and the abelian sheaf $j_!\underline{\mathbf{Z}}_U$. By Sheaves, Section 6.31 the stalk of $j_!\underline{\mathbf{Z}}_U$ at $x = 0$ is 0. In fact the sections of this sheaf over any open interval containing 0 are 0. Thus there is no open neighbourhood of the point 0 over which the sheaf can be generated by sections.

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[1] This is likely nonstandard notation.

Definition 15.8.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is *locally generated by sections* if for every $x \in X$ there exists an open neighbourhood U such that $\mathcal{F}|_U$ is globally generated as a sheaf of $\mathcal{O}_U$-modules.

In other words there exists a set I and for each i a section $s_i \in \mathcal{F}(U)$ such that the associated map

$$\bigoplus\nolimits_{i \in I} \mathcal{O}_U \longrightarrow \mathcal{F}|_U$$

is surjective.

Lemma 15.8.2. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The pullback $f^*\mathcal{G}$ is locally generated by sectons if $\mathcal{G}$ is locally generated by sections.*

Proof. Given an open subspace V of X we may consider the commutative diagram of ringed spaces

$$\begin{tikzcd}
(f^{-1}V, \mathcal{O}_{f^{-1}V}) \arrow[r, "j'"'] \arrow[d, "f'"'] & (X, \mathcal{O}_X) \arrow[d, "f"] \\
(V, \mathcal{O}_V) \arrow[r, "j"] & (Y, \mathcal{O}_Y)
\end{tikzcd}$$

We know that $f^*\mathcal{G}|_{f^{-1}V} \cong (f')^*(\mathcal{G}|_V)$, see Sheaves, Lemma 6.26.3. Thus we may assume that $\mathcal{G}$ is globally generated.

We have seen that f^* commutes with all colimits, and is right exact, see Lemma 15.3.4. Thus if we have a surjection

$$\bigoplus\nolimits_{i \in I} \mathcal{O}_Y \to \mathcal{G} \to 0$$

then upon applying f^* we obtain the surjection

$$\bigoplus\nolimits_{i \in I} \mathcal{O}_X \to f^*\mathcal{G} \to 0.$$

This implies the lemma. $\square$

15.9. Modules of finite type

Definition 15.9.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is of *finite type* if for every $x \in X$ there exists an open neighbourhood U such that $\mathcal{F}|_U$ is generated by finitely many sections.

Lemma 15.9.2. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The pullback $f^*\mathcal{G}$ of a finite type $\mathcal{O}_Y$-module is a finite type $\mathcal{O}_X$-module.*

Proof. Arguing as in the proof of Lemma 15.8.2 we may assume $\mathcal{G}$ is globally generated by finitely many sections. We have seen that f^* commutes with all colimits, and is right exact, see Lemma 15.3.4. Thus if we have a surjection

$$\bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_Y \to \mathcal{G} \to 0$$

then upon applying f^* we obtain the surjection

$$\bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_X \to f^*\mathcal{G} \to 0.$$

This implies the lemma. $\square$

Lemma 15.9.3. *Let X be a ringed space. The image of a morphism of $\mathcal{O}_X$-modules of finite type is of finite type. Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of $\mathcal{O}_X$-modules. If $\mathcal{F}_1$ and $\mathcal{F}_3$ are of finite type, so is $\mathcal{F}_2$.*

Proof. The statement on images is trivial. The statement on short exact sequences comes from the fact that sections of $\mathcal{F}_3$ locally lift to sections of $\mathcal{F}_2$ and the corresponding result in the category of modules over a ring (applied to the stalks for example). $\square$

Lemma 15.9.4. *Let X be a ringed space. Let $\varphi : \mathcal{G} \to \mathcal{F}$ be a homomorphism of $\mathcal{O}_X$-modules. Let $x \in X$. Assume $\mathcal{F}$ of finite type and the map on stalks $\varphi_x : \mathcal{G}_x \to \mathcal{F}_x$ surjective. Then there exists an open neighbourhood $x \in U \subset X$ such that $\varphi|_U$ is surjective.*

Proof. Choose an open neighbourhood $U \subset X$ such that $\mathcal{F}$ is generated by $s_1, \ldots, s_n \in \mathcal{F}(U)$ over U. By assumption of surjectivity of φ_x, after shrinking V we may assume that $s_i = \varphi(t_i)$ for some $t_i \in \mathcal{G}(U)$. Then U works. $\square$

Lemma 15.9.5. *Let X be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let $x \in X$. Assume $\mathcal{F}$ of finite type and $\mathcal{F}_x = 0$. Then there exists an open neighbourhood $x \in U \subset X$ such that $\mathcal{F}|_U$ is zero.*

Proof. This is a special case of Lemma 15.9.4 applied to the morphism $0 \to \mathcal{F}$. $\square$

Lemma 15.9.6. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. If $\mathcal{F}$ is of finite type then support of $\mathcal{F}$ is closed.*

Proof. This is a reformulation of Lemma 15.9.5. $\square$

Lemma 15.9.7. *Let X be a ringed space. Let I be a partially ordered set and let $(\mathcal{F}_i, f_{ii'})$ be a system over I consisting of sheaves of $\mathcal{O}_X$-modules (see Categories, Section 4.19). Let $\mathcal{F} = colim\, \mathcal{F}_i$ be the colimit. Assume (a) I is directed, (b) $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module and (c) X is quasi-compact. Then there exists an i such that $\mathcal{F}_i \to \mathcal{F}$ is surjective. If the transition maps $f_{ii'}$ are injective then we conclude that $\mathcal{F} = \mathcal{F}_i$ for some $i \in I$.*

Proof. Let $x \in X$. There exists an open neighbourhood $U \subset X$ of x and finitely many sections $s_j \in \mathcal{F}(U)$, $j = 1, \ldots, m$ such that $s_1, \ldots, s_m$ generate $\mathcal{F}$ as $\mathcal{O}_U$-module. After possibly shrinking U to a smaller open neighbourhood of x we may assume that each s_j comes from a section of $\mathcal{F}_i$ for some $i \in I$. Hence, since X is quasi-compact we can find a finite open covering $X = \bigcup_{j=1,\ldots,m} U_j$, and for each j an index i_j and finitely many sections $s_{jl} \in \mathcal{F}_{i_j}(U_j)$ whose images generate the restriction of $\mathcal{F}$ to U_j. Clearly, the lemma holds for any index $i \in I$ which is $\geq$ all i_j. $\square$

Lemma 15.9.8. *Let X be a ringed space. There exists a set of $\mathcal{O}_X$-modules $\{\mathcal{F}_i\}_{i \in I}$ of finite type such that each finite type $\mathcal{O}_X$-module on X is isomorphic to exactly one of the $\mathcal{F}_i$.*

Proof. For each open covering $\mathcal{U} : X = \bigcup U_j$ consider the sheaves of $\mathcal{O}_X$-modules $\mathcal{F}$ such that each restriction $\mathcal{F}|_{U_j}$ is a quotient of $\mathcal{O}_{U_j}^{\oplus r}$ for some $r_j \geq 0$. These are parametrized by subsheaves $\mathcal{K}_i \subset \mathcal{O}_{U_j}^{\oplus r_j}$ and glueing data

$$\varphi_{jj'} : \mathcal{O}_{U_j \cap U_{j'}}^{\oplus r_j}/(\mathcal{K}_j|_{U_j \cap U_{j'}}) \longrightarrow \mathcal{O}_{U_j \cap U_{j'}}^{\oplus r_{j'}}/(\mathcal{K}_{j'}|_{U_j \cap U_{j'}})$$

see Sheaves, Section 6.33. Note that the collection of all glueing data forms a set. The collection of all coverings $\mathcal{U} : X = \bigcup_{j \in J} U_i$ where $J \to \mathcal{P}(X)$, $j \mapsto U_j$ is injective forms a set as well. Hence the collection of all sheaves of $\mathcal{O}_X$-modules gotten from glueing quotients as above forms a set $\mathcal{I}$. By definition every finite type $\mathcal{O}_X$-module is isomorphic to an element of $\mathcal{I}$. Choosing an element out of each isomorphism class inside $\mathcal{I}$ gives the desired set of sheaves (uses axiom of choice). $\square$

15.10. Quasi-coherent modules

In this section we introduce an abstract notion of quasi-coherent $\mathcal{O}_X$-module. This notion is very useful in algebraic geometry, since quasi-coherent modules on a scheme have a good description on any affine open. However, we warn the reader that in the general setting of (locally) ringed spaces this notion is not well behaved at all. The category of quasi-coherent sheaves is not abelian in general, infinite direct sums of quasi-coherent sheaves aren't quasi-coherent, etc, etc.

Definition 15.10.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is a *quasi-coherent sheaf of $\mathcal{O}_X$-modules* if for every point $x \in X$ there exists an open neighbourhood $x \in U \subset X$ such that $\mathcal{F}|_U$ is isomorphic to the cokernel of a map

$$\bigoplus\nolimits_{j \in J} \mathcal{O}_U \longrightarrow \bigoplus\nolimits_{i \in I} \mathcal{O}_U$$

The category of quasi-coherent $\mathcal{O}_X$-modules is denoted $QCoh(\mathcal{O}_X)$.

The definition means that X is covered by open sets U such that $\mathcal{F}|_U$ has a *presentation* of the form

$$\bigoplus\nolimits_{j \in J} \mathcal{O}_U \longrightarrow \bigoplus\nolimits_{i \in I} \mathcal{O}_U \to \mathcal{F}|_U \to 0.$$

Here presentation signifies that the displayed sequence is exact. In other words

(1) for every point x of X there exists an open neighbourhood such that $\mathcal{F}|_U$ is generated by global sections, and
(2) for a suitable choice of these sections the kernel of the associated surjection is also generated by global sections.

Lemma 15.10.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. The direct sum of two quasi-coherent $\mathcal{O}_X$-modules is a quasi-coherent $\mathcal{O}_X$-module*

Proof. Omitted. □

Remark 15.10.3. Warning: It is not true in general that an infinite direct sum of quasi-coherent $\mathcal{O}_X$-modules is quasi-coherent. For more esoteric behaviour of quasi-coherent modules see Example 15.10.9.

Lemma 15.10.4. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The pullback $f^*\mathcal{G}$ of a quasi-coherent $\mathcal{O}_Y$-module is quasi-coherent.*

Proof. Arguing as in the proof of Lemma 15.8.2 we may assume $\mathcal{G}$ has a global presentation by direct sums of copies of $\mathcal{O}_Y$. We have seen that f^* commutes with all colimits, and is right exact, see Lemma 15.3.4. Thus if we have an exact sequence

$$\bigoplus\nolimits_{j \in J} \mathcal{O}_Y \longrightarrow \bigoplus\nolimits_{i \in I} \mathcal{O}_Y \to \mathcal{G} \to 0$$

then upon applying f^* we obtain the exact sequence

$$\bigoplus\nolimits_{j \in J} \mathcal{O}_X \longrightarrow \bigoplus\nolimits_{i \in I} \mathcal{O}_X \to f^*\mathcal{G} \to 0.$$

This implies the lemma. □

This gives plenty of examples of quasi-coherent sheaves.

Lemma 15.10.5. *Let $(X, \mathcal{O}_X)$ be ringed space. Let $\alpha : R \to \Gamma(X, \mathcal{O}_X)$ be a ring homomorphism from a ring R into the ring of global sections on X. Let M be an R-module. The following three constructions give canonically isomorphic sheaves of $\mathcal{O}_X$-modules:*

(1) *Let* $\pi : (X, \mathcal{O}_X) \longrightarrow (\{*\}, R)$ *be the morphism of ringed spaces with* $\pi : X \to \{*\}$ *the unique map and with* π*-map* $\pi^\sharp$ *the given map* $\alpha : R \to \Gamma(X, \mathcal{O}_X)$*. Set* $\mathcal{F}_1 = \pi^* M$.

(2) *Choose a presentation* $\bigoplus_{j \in J} R \to \bigoplus_{i \in I} R \to M \to 0$*. Set*

$$\mathcal{F}_2 = Coker\left(\bigoplus\nolimits_{j \in J} \mathcal{O}_X \to \bigoplus\nolimits_{i \in I} \mathcal{O}_X\right).$$

Here the map on the component $\mathcal{O}_X$ *corresponding to* $j \in J$ *given by the section* $\sum_i \alpha(r_{ij})$ *where the* r_{ij} *are the matrix coefficients of the map in the presentation of* M.

(3) *Set* $\mathcal{F}_3$ *equal to the sheaf associated to the presheaf* $U \mapsto \mathcal{O}_X(U) \otimes_R M$*, where the map* $R \to \mathcal{O}_X(U)$ *is the composition of* α *and the restriction map* $\mathcal{O}_X(X) \to \mathcal{O}_X(U)$.

This construction has the following properties:

(1) *The resulting sheaf of* $\mathcal{O}_X$*-modules* $\mathcal{F}_M = \mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3$ *is quasi-coherent.*

(2) *The construction gives a functor from the category of* R*-modules to the category of quasi-coherent sheaves on* X *which commutes with arbitrary colimits.*

(3) *For any* $x \in X$ *we have* $\mathcal{F}_{M,x} = \mathcal{O}_{X,x} \otimes_R M$ *functorial in* M.

(4) *Given any* $\mathcal{O}_X$*-module* $\mathcal{G}$ *we have*

$$Mor_{\mathcal{O}_X}(\mathcal{F}_M, \mathcal{G}) = Hom_R(M, \Gamma(X, \mathcal{G}))$$

where the R*-module structure on* $\Gamma(X, \mathcal{G})$ *comes from the* $\Gamma(X, \mathcal{O}_X)$*-module structure via* α.

Proof. The isomorphism between $\mathcal{F}_1$ and $\mathcal{F}_2$ comes from the fact that π^* is defined as the sheafification of the presheaf in (3), see Sheaves, Section 6.26. The isomorphism between the constructions in (2) and (1) comes from the fact that the functor π^* is right exact, so $\pi^*(\bigoplus_{j \in J} R) \to \pi^*(\bigoplus_{i \in I} R) \to \pi^* M \to 0$ is exact, π^* commutes with arbitrary direct sums, see Lemma 15.3.4, and finally the fact that $\pi^*(R) = \mathcal{O}_X$.

Assertion (1) is clear from construction (2). Assertion (2) is clear since π^* has these properties. Assertion (3) follows from the description of stalks of pullback sheaves, see Sheaves, Lemma 6.26.4. Assertion (4) follows from adjointness of π_* and π^*. □

Definition 15.10.6. In the situation of Lemma 15.10.5 we say $\mathcal{F}_M$ is the *sheaf associated to the module* M *and the ring map* α. If $R = \Gamma(X, \mathcal{O}_X)$ and $\alpha = \mathrm{id}_R$ we simply say $\mathcal{F}_M$ is the *sheaf associated to the module* M.

Lemma 15.10.7. *Let* $(X, \mathcal{O}_X)$ *be ringed space. Set* $R = \Gamma(X, \mathcal{O}_X)$*. Let* M *be an* R*-module. Let* $\mathcal{F}_M$ *be the quasi-coherent sheaf of* $\mathcal{O}_X$*-modules associated to* M*. If* $g : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ *is a morphism of ringed spaces, then* $g^*\mathcal{F}_M$ *is the sheaf associated to the* $\Gamma(Y, \mathcal{O}_Y)$*-module* $\Gamma(Y, \mathcal{O}_Y) \otimes_R M$.

Proof. The assertion follows from the first description of $\mathcal{F}_M$ in Lemma 15.10.5 as $\pi^* M$, and the following commutative diagram of ringed spaces

$$\begin{tikzcd}
(X, \mathcal{O}_X) \arrow[r, "\pi"'] \arrow[d, "g"'] & (\{*\}, \Gamma(X, \mathcal{O}_X)) \arrow[d, "g^\sharp"] \\
(Y, \mathcal{O}_Y) \arrow[r, "\pi"] & (\{*\}, \Gamma(Y, \mathcal{O}_Y))
\end{tikzcd}$$

(Also use Sheaves, Lemma 6.26.3.) □

Lemma 15.10.8. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $x \in X$ be a point. Assume that x has a fundamental system of quasi-compact neighbourhoods. Consider any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$. Then there exists an open neighbourhood U of x such that $\mathcal{F}|_U$ is isomorphic to the sheaf of modules $\mathcal{F}_M$ on $(U, \mathcal{O}_U)$ associated to some $\Gamma(U, \mathcal{O}_U)$-module M.*

Proof. First we may replace X by an open neighbourhood of x and assume that $\mathcal{F}$ is isomorphic to the cokernel of a map

$$\Psi : \bigoplus\nolimits_{j\in J} \mathcal{O}_X \longrightarrow \bigoplus\nolimits_{i\in I} \mathcal{O}_X.$$

The problem is that this map may not be given by a ``matrix", because the module of global sections of a direct sum is in general different from the direct sum of the modules of global sections.

Let $x \in E \subset X$ be a quasi-compact neighbourhood of x (note: E may not be open). Let $x \in U \subset E$ be an open neighbourhood of x contained in E. Next, we proceed as in the proof of Lemma 15.3.3. For each $j \in J$ denote $s_j \in \Gamma(X, \bigoplus_{i\in I} \mathcal{O}_X)$ the image of the section 1 in the summand $\mathcal{O}_X$ corresponding to j. There exists a finite collection of opens U_{jk}, $k \in K_j$ such that $E \subset \bigcup_{k\in K_j} U_{jk}$ and such that each restriction $s_j|_{U_{jk}}$ is a finite sum $\sum_{i\in I_{jk}} f_{jki}$ with $I_{jk} \subset I$, and f_{jki} in the summand $\mathcal{O}_X$ corresponding to $i \in I$. Set $I_j = \bigcup_{k\in k_J} I_{jk}$. This is a finite set. Since $U \subset E \subset \bigcup_{k\in K_j} U_{jk}$ the section $s_j|_U$ is a section of the finite direct sum $\bigoplus_{i\in I_j} \mathcal{O}_X$. By Lemma 15.3.2 we see that actually $s_j|_U$ is a sum $\sum_{i\in I_j} f_{ij}$ and $f_{ij} \in \mathcal{O}_X(U) = \Gamma(U, \mathcal{O}_U)$.

At this point we can define a module M as the cokernel of the map

$$\bigoplus\nolimits_{j\in J} \Gamma(U, \mathcal{O}_U) \longrightarrow \bigoplus\nolimits_{i\in I} \Gamma(U, \mathcal{O}_U)$$

with matrix given by the (f_{ij}). By construction (2) of Lemma 15.10.5 we see that $\mathcal{F}_M$ has the same presentation as $\mathcal{F}|_U$ and therefore $\mathcal{F}_M \cong \mathcal{F}|_U$. □

Example 15.10.9. Let X be countably many copies $L_1, L_2, L_3, \ldots$ of the real line all glued together at 0; a fundamental system of neighbourhoods of 0 being the collection $\{U_n\}_{n\in\mathbf{N}}$, with $U_n \cap L_i = (-1/n, 1/n)$. Let $\mathcal{O}_X$ be the sheaf of continuous real valued functions. Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function which is identically zero on $(-1, 1)$ and identically 1 on $(-\infty, -2)\cup(2, \infty)$. Denote f_n the continuous function on X which is equal to $x \mapsto f(nx)$ on each $L_j = \mathbf{R}$. Let 1_{L_j} be the characteristic function of L_j. We consider the map

$$\bigoplus\nolimits_{j\in\mathbf{N}} \mathcal{O}_X \longrightarrow \bigoplus\nolimits_{j,i\in\mathbf{N}} \mathcal{O}_X, \quad e_j \longmapsto \sum\nolimits_{j\in\mathbf{N}} f_i 1_{L_j} e_{ij}$$

with obvious notation. This makes sense because this sum is locally finite. Clearly, there is no neighbourhood of $0 \in X$ such that this map is given by a ``matrix" as in the proof of Lemma 15.10.8 above.

Note that $\bigoplus_{j\in\mathbf{N}} \mathcal{O}_X$ is the sheaf associated to the free module with basis e_j and similarly for the other direct sum. Thus we see that a morphism of sheaves associated to modules in general even locally on X does not come from a morphism of modules. Similarly there should be an example of a ringed space X and a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ such that $\mathcal{F}$ is not locally of the form $\mathcal{F}_M$. (Please email if you find one.) Moreover, there should be examples of locally compact spaces X and maps $\mathcal{F}_M \to \mathcal{F}_N$ which also do not locally come from maps of modules (the proof of Lemma 15.10.8 shows this cannot happen if N is free).

15.11. Modules of finite presentation

Definition 15.11.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is of *finite presentation* if for every point $x \in X$ there exists an open neighbourhood $x \in U \subset X$, and $n, m \in \mathbf{N}$ such that $\mathcal{F}|_U$ is isomorphic to the cokernel of a map

$$\bigoplus\nolimits_{j=1,\ldots,m} \mathcal{O}_U \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_U$$

This means that X is covered by open sets U such that $\mathcal{F}|_U$ has a *presentation* of the form

$$\bigoplus\nolimits_{j=1,\ldots,m} \mathcal{O}_U \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_U \to \mathcal{F}|_U \to 0.$$

Here presentation signifies that the displayed sequence is exact. In other words

(1) for every point x of X there exists an open neighbourhood such that $\mathcal{F}|_U$ is generated by finitely many global sections, and
(2) for a suitable choice of these sections the kernel of the associated surjection is also generated by finitely many global sections.

Lemma 15.11.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Any $\mathcal{O}_X$-module of finite presentation is quasi-coherent.*

Proof. Immediate from defintions. □

Lemma 15.11.3. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a $\mathcal{O}_X$-module of finite presentation. Let $\psi : \mathcal{O}_X^{\oplus r} \to \mathcal{F}$ be a surjection. Then $Ker(\psi)$ is an $\mathcal{O}_X$-module of finite type.*

Proof. Let $x \in X$. Choose an open neighbourhood $U \subset X$ of x such that there exists a presentation

$$\mathcal{O}_U^{\oplus m} \xrightarrow{\chi} \mathcal{O}_U^{\oplus n} \xrightarrow{\varphi} \mathcal{F}|_U \to 0.$$

Let e_k be the section generating the kth factor of $\mathcal{O}_X^{\oplus r}$. For every $k = 1, \ldots, r$ we can, after shrinking U to a small neighbourhood of x, lift $\psi(e_k)$ to a section $\tilde{e}_k$ of $\mathcal{O}_U^{\oplus n}$ over U. This gives a morphism of sheaves $\alpha : \mathcal{O}_U^{\oplus r} \to \mathcal{O}_U^{\oplus n}$ such that $\varphi \circ \alpha = \psi$. Similarly, after shrinking U, we can find a morphism $\beta : \mathcal{O}_U^{\oplus n} \to \mathcal{O}_U^{\oplus r}$ such that $\psi \circ \beta = \varphi$. Then the map

$$\mathcal{O}_U^{\oplus m} \oplus \mathcal{O}_U^{\oplus r} \xrightarrow{\beta\circ\chi, 1-\beta\circ\alpha} \mathcal{O}_U^{\oplus r}$$

is a surjection onto the kernel of ψ. □

Lemma 15.11.4. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The pullback $f^*\mathcal{G}$ of a module of finite presentation is of finite presentation.*

Proof. Exactly the same as the proof of Lemma 15.10.4 but with finite index sets. □

Lemma 15.11.5. *Let $(X, \mathcal{O}_X)$ be a ringed space. Set $R = \Gamma(X, \mathcal{O}_X)$. Let M be an R-module. The $\mathcal{O}_X$-module $\mathcal{F}_M$ associated to M is a directed colimit of finitely presented $\mathcal{O}_X$-modules.*

Proof. This follows immediately from Lemma 15.10.5 and the fact that any module is a directed colimit of finitely presented modules, see Algebra, Lemma 7.8.13. □

Lemma 15.11.6. *Let X be a ringed space. Let I be a partially ordered set and let $(\mathcal{F}_i, \varphi_{ii'})$ be a system over I consisting of sheaves of $\mathcal{O}_X$-modules (see Categories, Section 4.19). Assume*

(1) *I is directed,*
(2) *$\mathcal{G}$ is an $\mathcal{O}_X$-module of finite presentation, and*

(3) *X has a cofinal system of open coverings $\mathcal{U} : X = \bigcup_{j \in J} U_j$ with J finite and $U_j \cap U_{j'}$ quasi-compact for all $j, j' \in J$*

Then we have

$$colim_i\ Hom_X(\mathcal{G}, \mathcal{F}_i) = Hom_X(\mathcal{G}, colim_i\ \mathcal{F}_i).$$

Proof. Let α be an element of the right hand side. For every point $x \in X$ we may choose an open neighbourhood $U \subset X$ and finitely many sections $s_j \in \mathcal{G}(U)$ which generate $\mathcal{G}$ over U and finitely many relations $\sum f_{kj} s_j = 0$, $k = 1, \ldots, n$ with $f_{kj} \in \mathcal{O}_X(U)$ which generate the kernel of $\bigoplus_{j=1,\ldots,m} \mathcal{O}_U \to \mathcal{G}$. After possibly shrinking U to a smaller open neighbourhood of x we may assume there exists an index $i \in I$ such that the sections $\alpha(s_j)$ all come from sections $s'_j \in \mathcal{F}_i(U)$. After possibly shrinking U to a smaller open neighbourhood of x and increasing i we may assume the relations $\sum f_{kj} s'_j = 0$ hold in $\mathcal{F}_i(U)$. Hence we see that $\alpha|_U$ lifts to a morphism $\mathcal{G}|_U \to \mathcal{F}_i|_U$ for some index $i \in I$.

By condition (3) and the preceding arguments, we may choose a finite open covering $X = \bigcup_{j=1,\ldots,m} U_j$ such that (a) $\mathcal{G}|_{U_j}$ is generated by finitely many sections $s_{jk} \in \mathcal{G}(U_j)$, (b) the restriction $\alpha|_{U_j}$ comes from a morphism $\alpha_j : \mathcal{G} \to \mathcal{F}_{i_j}$ for some $i_j \in I$, and (c) the intersections $U_j \cap U_{j'}$ are all quasi-compact. For every pair $(j, j') \in \{1, \ldots, m\}^2$ and any k we can find we can find an index $i \geq \max(i_j, i_{j'})$ such that

$$\varphi_{i_j i}(\alpha_j(s_{jk}|_{U_j \cap U_{j'}})) = \varphi_{i_{j'} i}(\alpha_{j'}(s_{jk}|_{U_j \cap U_{j'}}))$$

see Sheaves, Lemma 6.29.1 (2). Since there are finitely many of these pairs (j, j') and finitely many s_{jk} we see that we can find a single i which works for all of them. For this index i all of the maps $\varphi_{i_j i} \circ \alpha_j$ agree on the overlaps $U_j \cap U_{j'}$ as the sections s_{jk} generate $\mathcal{G}$ over this overlap. Hence we get a morphism $\mathcal{G} \to \mathcal{F}_i$ as desired. □

Remark 15.11.7. In the lemma above some condition beyond the condition that X is quasi-compact is necessary. See Sheaves, Example 6.29.2.

15.12. Coherent modules

The category of coherent sheaves on a ringed space X is a more reasonable object than the category of quasi-coherent sheaves, in the sense that it is at least an abelian subcategory of $Mod(\mathcal{O}_X)$ no matter what X is. On the other hand, the pull back of a coherent module is ``almost never'' coherent in the general setting of ringed spaces.

Definition 15.12.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is a *coherent $\mathcal{O}_X$-module* if the following two conditions hold:

(1) $\mathcal{F}$ is of finite type, and
(2) for every open $U \subset X$ and every finite collection $s_i \in \mathcal{F}(U)$, $i = 1, \ldots, n$ the kernel of the associated map $\bigoplus_{i=1,\ldots,n} \mathcal{O}_U \to \mathcal{F}|_U$ is of finite type.

The category of coherent $\mathcal{O}_X$-modules is denoted $\text{Coh}(\mathcal{O}_X)$.

Lemma 15.12.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Any coherent $\mathcal{O}_X$-module is of finite presentation and hence quasi-coherent.*

Proof. Let $\mathcal{F}$ be a coherent sheaf on X. Pick a point $x \in X$. By (1) of the definition of coherent, we may find an open neighbourhood U and sections s_i, $i = 1, \ldots, n$ of $\mathcal{F}$ over U such that $\Psi : \bigoplus_{i=1,\ldots,n} \mathcal{O}_U \to \mathcal{F}$ is surjective. By (2) of the definition of coherent, we may find an open neighbourhood V, $x \in V \subset U$ and sections $t_1, \ldots, t_m$ of $\bigoplus_{i=1,\ldots,n} \mathcal{O}_V$ which generate the kernel of $\Psi|_V$. Then over V we get the presentation

$$\bigoplus\nolimits_{j=1,\ldots,m} \mathcal{O}_V \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_V \to \mathcal{F}|_V \to 0$$

as desired. □

Example 15.12.3. Suppose that X is a point. In this case the definition above gives a notion for modules over rings. What does the definition of coherent mean? It is closely related to the notion of Noetherian, but it is not the same: Namely, the ring $R = \mathbf{C}[x_1, x_2, x_3, \ldots]$ is coherent as a module over itself but not Noetherian as a module over itself. See Algebra, Section 7.84 for more discussion.

Lemma 15.12.4. *Let $(X, \mathcal{O}_X)$ be a ringed space.*

1. *Any finite type subsheaf of a coherent sheaf is coherent.*
2. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism from a finite type sheaf $\mathcal{F}$ to a coherent sheaf $\mathcal{G}$. Then Ker(φ) is finite type.*
3. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of coherent $\mathcal{O}_X$-modules. Then Ker(φ) and Coker(φ) are coherent.*
4. *The category of coherent sheaves on X is abelian.*
5. *Given a short exact sequence of $\mathcal{O}_X$-modules $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ if two out of three are coherent so is the third.*

Proof. Condition (2) of Definition 15.12.1 holds for any subsheaf of a coherent sheaf. Thus we get the first statement.

Assume the hypotheses of (2). Let us show that Ker(φ) is of finite type. Pick $x \in X$. Choose an open neighbourhood U of x in X such that $\mathcal{F}|_U$ is generated by $s_1, \ldots, s_n$. By Definition 15.12.1 the kernel $\mathcal{K}$ of the induced map $\bigoplus_{i=1}^n \mathcal{O}_U \to \mathcal{G}$, $e_i \mapsto \varphi(s_i)$ is of finite type. Hence Ker(φ) which is the image of the composition $\mathcal{K} \to \bigoplus_{i=1}^n \mathcal{O}_U \to \mathcal{F}$ is of finite type.

Assume the hypotheses of (3). By (2) the kernel of φ is of finite type and hence by (1) it is coherent.

With the same hypotheses let us show that Coker(φ) is coherent. Since $\mathcal{G}$ is of finite type so is Coker(φ). Let $U \subset X$ be open and let $\overline{s}_i \in$ Coker$(\varphi)(U)$, $i = 1, \ldots, n$ be sections. We have to show that the kernel of the associated morphism $\overline{\Psi} : \bigoplus_{i=1}^n \mathcal{O}_U \to$ Coker(φ) has finite type. There exists an open covering of U such that on each open all the sections $\overline{s}_i$ lift to sections s_i of $\mathcal{G}$. Hence we may assume this is the case over U. Thus $\overline{\Psi}$ lifts to $\Psi : \bigoplus_{i=1}^n \mathcal{O}_U \to \mathcal{G}$ Consider the following diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ker}(\Psi) \arrow[r] \arrow[d] & \bigoplus_{i=1}^n \mathcal{O}_U \arrow[r] \arrow[d, equal] & \mathcal{G} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathrm{Ker}(\overline{\Psi}) \arrow[r] & \bigoplus_{i=1}^n \mathcal{O}_U \arrow[r] & \mathrm{Coker}(\varphi) \arrow[r] & 0 \end{tikzcd}$$

By the snake lemma we get a short exact sequence $0 \to \mathrm{Ker}(\Psi) \to \mathrm{Ker}(\overline{\Psi}) \to \mathrm{Im}(\varphi) \to 0$. Hence by Lemma 15.9.3 we see that Ker$(\overline{\Psi})$ has finite type.

Statement (4) follows from (3).

Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of $\mathcal{O}_X$-modules. It suffices to prove that if $\mathcal{F}_1$ and $\mathcal{F}_3$ are coherent so is $\mathcal{F}_2$. By Lemma 15.9.3 we see that $\mathcal{F}_2$ has finite type. Let $s_1, \ldots, s_n$ be finitely many local sections of $\mathcal{F}_2$ defined over a common open U of X. We have to show that the module of relations $\mathcal{K}$ between them is of finite type. Consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \bigoplus_{i=1}^n \mathcal{O}_U \arrow[r] \arrow[d] & \bigoplus_{i=1}^n \mathcal{O}_U \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F}_1 \arrow[r] & \mathcal{F}_2 \arrow[r] & \mathcal{F}_3 \arrow[r] & 0
\end{tikzcd}$$

with obvious notation. By the snake lemma we get a short exact sequence $0 \to \mathcal{K} \to \mathcal{K}_3 \to \mathcal{F}_1$ where $\mathcal{K}_3$ is the module of relations among the images of the sections s_i in $\mathcal{F}_3$. Since $\mathcal{F}_3$ is coherent we see that $\mathcal{K}_3$ is finite type. Since $\mathcal{F}_1$ is coherent we see that the image $\mathcal{I}$ of $\mathcal{K}_3 \to \mathcal{F}_1$ is coherent. Hence $\mathcal{K}$ is the kernel of the map $\mathcal{K}_3 \to \mathcal{I}$ between a finite type sheaf and a coherent sheaves and hence finite type by (2). □

Lemma 15.12.5. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Assume $\mathcal{O}_X$ is a coherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is coherent if and only if it is of finite presentation.*

Proof. Omitted. □

Lemma 15.12.6. *Let X be a ringed space. Let $\varphi : \mathcal{G} \to \mathcal{F}$ be a homomorphism of $\mathcal{O}_X$-modules. Let $x \in X$. Assume $\mathcal{G}$ of finite type, $\mathcal{F}$ coherent and the map on stalks $\varphi_x : \mathcal{G}_x \to \mathcal{F}_x$ injective. Then there exists an open neighbourhood $x \in U \subset X$ such that $\varphi|_U$ is injective.*

Proof. Denote $\mathcal{K} \subset \mathcal{G}$ the kernel of φ. By Lemma 15.12.4 we see that $\mathcal{K}$ is a finite type $\mathcal{O}_X$-module. Our assumption is that $\mathcal{K}_x = 0$. By Lemma 15.9.5 there exists an open neighbourhood U of x such that $\mathcal{K}|_U = 0$. Then U works. □

15.13. Closed immersions of ringed spaces

When do we declare a morphism of ringed spaces $i : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ to be a closed immersion? It depends on what types of sheaves of modules you want to consider. For example, we saw in Section 15.6 that if $i : Z \to X$ is the inclusion of a closed subspace and $\mathcal{O}_Z = \underline{\mathbf{Z}}_Z$, $\mathcal{O}_X = \underline{\mathbf{Z}}_X$ then we obtain a good notion in the setting of abelian sheaves.

On the other hand, if we want i_* and i^* to provide an equivalence between (certain) categories of quasi-coherent sheaves, then this doesn't work. Namely, typically the sheaf $i_*\underline{\mathbf{Z}}_Z$ isn't a quasi-coherent $\underline{\mathbf{Z}}_X$-module. This already happens in case $X = \mathbf{R}$ and Z is a point.

A minimal condition is that $i_*\mathcal{O}_Z$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules. On the other hand, it seems reasonable to assume that every local section of $i_*\mathcal{O}_Z$ comes (locally) from a local section of $\mathcal{O}_X$, in other words to assume that $\mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective. In this case the kernel $\mathcal{I}$ is a *sheaf of ideals of* $\mathcal{O}_X$, i.e., a submodule of $\mathcal{O}_X$. And an easy way to garantee that $i_*\mathcal{O}_Z$ is a quasi-coherent $\mathcal{O}_X$-module is to assume that $\mathcal{I}$ is locally generated by sections. This leads to the following (nonstandard) definition.

Definition 15.13.1. A *closed immersion of ringed spaces*[2] is a morphism $i : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ with the following properties:

(1) The map i is a closed immersion of topological spaces.
(2) The associated map $\mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective. Denote the kernel by $\mathcal{I}$.
(3) The $\mathcal{O}_X$-module $\mathcal{I}$ is locally generated by sections.

[2]This is likely nonstandard notation; we chose it because it works well for schemes.

Actually, this definition still does not garantee that i_* of a quasi-coherent $\mathcal{O}_Z$-module is a quasi-coherent $\mathcal{O}_X$-module. The problem is that it is not clear how to convert a local presentation of a quasi-coherent $\mathcal{O}_Z$-module into a local presentation for the pushforward. However, the following is trivial.

Lemma 15.13.2. *Let $i : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ be a closed immersion of locally ringed spaces. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_Z$-module. Then $i_*\mathcal{F}$ is locally on X the cokernel of a map of quasi-coherent $\mathcal{O}_X$-modules.*

Proof. This is true because $i_*\mathcal{O}_Z$ is quasi-coherent by definition. And locally on Z the sheaf $\mathcal{F}$ is a cokernel of a map between direct sums of copies of $\mathcal{O}_Z$. Moreover, any direct sum of copies of the *the same* quasi-coherent sheaf is quasi-coherent. And finally, i_* commutes with arbitrary colimits, see Lemma 15.6.3. Some details omitted. $\square$

Lemma 15.13.3. *Let $i : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ be a closed immersion of locally ringed spaces. Let $\mathcal{F}$ be an $\mathcal{O}_Z$-module. Then $i_*\mathcal{F}$ is of finite type if and only if $\mathcal{F}$ is of finite type.*

Proof. Suppose that $\mathcal{F}$ is of finite type. Pick $x \in X$. If $x \not\in Z$, then $i_*\mathcal{F}$ is zero in a neighbourhood of x and hence finitely generated in a neighbourhood of x. If $x = i(z)$, then choose an open neighbourhood $z \in V \subset Z$ and sections $s_1, \ldots, s_n \in \mathcal{F}(V)$ which generate $\mathcal{F}$ over V. Write $V = Z \cap U$ for some open $U \subset X$. Note that U is a neighbourhood of x. Clearly the sections s_i give sections s_i of $i_*\mathcal{F}$ over U. The resulting map

$$\bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_U \longrightarrow i_*\mathcal{F}|_U$$

is surjective by inspection of what it does on stalks (the only thing you use is that $\mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective). Hence $i_*\mathcal{F}$ is of finite type.

Conversely, suppose that $i_*\mathcal{F}$ is of finite type. Choose $z \in Z$. Set $x = i(z)$. By assumption there exists an open neighbourhood $U \subset X$ of x, and sections $s_1, \ldots, s_n \in (i_*\mathcal{F})(U)$ which generate $i_*\mathcal{F}$ over U. Set $V = Z \cap U$. By definition of i_* the sections s_i correspond to sections s_i of $\mathcal{F}$ over V. The resulting map

$$\bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_V \longrightarrow \mathcal{F}|_V$$

is surjective by inspection of what it does on stalks (the only thing you use is that $\mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective). Hence $\mathcal{F}$ is of finite type. $\square$

15.14. Locally free sheaves

Let $(X, \mathcal{O}_X)$ be a ringed space. Our conventions allow (some of) the stalks $\mathcal{O}_{X,x}$ to be the zero ring. This means we have to be a little carefull when defining the rank of a locally free sheaf.

Definition 15.14.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say $\mathcal{F}$ is *locally free* if for every point $x \in X$ there exists a set I and an open neighbourhood $x \in U \subset X$ such that $\mathcal{F}|_U$ is isomorphic to $\bigoplus_{i \in I} \mathcal{O}_X|_U$ as an $\mathcal{O}_X|_U$-module. We say $\mathcal{F}$ is *finite locally free* if we may choose the index set I to be finite always.

A finite direct sum of (finite) locally free sheaves is (finite) locally free. However, it may not be the case that an infinite direct sum of locally free sheaves is locally free.

Lemma 15.14.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. If $\mathcal{F}$ is locally free then it is quasi-coherent.*

Proof. Omitted. $\square$

Lemma 15.14.3. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. If $\mathcal{G}$ is a locally free $\mathcal{O}_Y$-module, then $f^*\mathcal{G}$ is a locally free $\mathcal{O}_X$-module.*

Proof. Omitted. $\square$

Lemma 15.14.4. *Let $(X, \mathcal{O}_X)$ be a ringed space. Suppose that the support of $\mathcal{O}_X$ is X, i.e., all stalk of $\mathcal{O}_X$ are nonzero rings. Let $\mathcal{F}$ be a locally free sheaf of $\mathcal{O}_X$-modules. There exists a locally constant function*

$$rank_{\mathcal{F}} : X \longrightarrow \{0, 1, 2, \ldots\} \cup \{\infty\}$$

such that for any point $x \in X$ the cardinality of any set I such that $\mathcal{F}$ is isomorphic to $\bigoplus_{i \in I} \mathcal{O}_X$ in a neighbourhood of x is $rank_{\mathcal{F}}(x)$.

Proof. Under the assumption of the lemma the cardinality of I can be read off from the rank of the free module $\mathcal{F}_x$ over the nonzero ring $\mathcal{O}_{X,x}$, and it is constant in a neighbourhood of x. $\square$

15.15. Tensor product

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_X$-modules. We have briefly discussed the tensor product in the setting of change of rings in Sheaves, Sections 6.6 and 6.20. In exactly the same way we define first the *tensor product presheaf*

$$\mathcal{F} \otimes_{p, \mathcal{O}_X} \mathcal{G}$$

as the rule which assigns to $U \subset X$ open the $\mathcal{O}_X(U)$-module $\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$. Having defined this we define the *tensor product sheaf* as the sheafification of the above:

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} = (\mathcal{F} \otimes_{p, \mathcal{O}_X} \mathcal{G})^\#$$

This can be characterized as the sheaf of $\mathcal{O}_X$-modules such that for any third sheaf of $\mathcal{O}_X$-modules $\mathcal{H}$ we have

$$Hom_{\mathcal{O}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}, \mathcal{H}) = \text{Bilin}_{\mathcal{O}_X}(\mathcal{F} \times \mathcal{G}, \mathcal{H}).$$

Here the right hand side indicates the set of bilinear maps of sheaves of $\mathcal{O}_X$-modules (definition omitted).

The tensor product of modules M, N over a ring R satisfies symmetry, namely $M \otimes_R N = N \otimes_R M$, hence the same holds for tensor products of sheaves of modules, i.e., we have

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} = \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F}$$

functorial in $\mathcal{F}, \mathcal{G}$. And since tensor product of modules satisfies associativity we also get canonical functorial isomorphisms

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}) \otimes_{\mathcal{O}_X} \mathcal{H} = \mathcal{F} \otimes_{\mathcal{O}_X} (\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{H})$$

functorial in $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$.

Lemma 15.15.1. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_X$-modules. Let $x \in X$. There is a canonical isomorphism of $\mathcal{O}_{X,x}$-modules*

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x$$

functorial in $\mathcal{F}$ and $\mathcal{G}$.

Proof. Omitted. $\square$

Lemma 15.15.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}'$, $\mathcal{G}'$ be presheaves of $\mathcal{O}_X$-modules with sheafifications $\mathcal{F}$, $\mathcal{G}$. Then $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} = (\mathcal{F}' \otimes_{p, \mathcal{O}_X} \mathcal{G}')^\#$.*

Proof. Omitted. $\square$

Lemma 15.15.3. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{G}$ be an $\mathcal{O}_X$-module. If $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is an exact sequence of $\mathcal{O}_X$-modules then the induced sequence*

$$\mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{F}_2 \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{F}_3 \otimes_{\mathcal{O}_X} \mathcal{G} \to 0$$

is exact.

Proof. This follows from the fact that exactness may be checked at stalks (Lemma 15.3.1), the description of stalks (Lemma 15.15.1) and the corresponding result for tensor products of modules (Algebra, Lemma 7.11.10). $\square$

Lemma 15.15.4. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_Y$-modules. Then $f^*(\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{G}) = f^*\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}$ functorially in $\mathcal{F}$, $\mathcal{G}$.*

Proof. Omitted. $\square$

Lemma 15.15.5. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_X$-modules.*

(1) *If $\mathcal{F}$, $\mathcal{G}$ are locally generated by sections, so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.*
(2) *If $\mathcal{F}$, $\mathcal{G}$ are of finite type, so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.*
(3) *If $\mathcal{F}$, $\mathcal{G}$ are quasi-coherent, so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.*
(4) *If $\mathcal{F}$, $\mathcal{G}$ are of finite presentation, so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.*
(5) *If $\mathcal{F}$ is of finite presentation and $\mathcal{G}$ is coherent, then $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is coherent.*
(6) *If $\mathcal{F}$, $\mathcal{G}$ are coherent, so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.*
(7) *If $\mathcal{F}$, $\mathcal{G}$ are locally free, so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.*

Proof. We first prove that the tensor product of locally free $\mathcal{O}_X$-modules is locally free. This follows if we show that $(\bigoplus_{i\in I} \mathcal{O}_X) \otimes_{\mathcal{O}_X} (\bigoplus_{j\in J} \mathcal{O}_X) \cong \bigoplus_{(i,j)\in I\times J} \mathcal{O}_X$. The sheaf $\bigoplus_{i\in I} \mathcal{O}_X$ is the sheaf associated to the presheaf $U \mapsto \bigoplus_{i\in I} \mathcal{O}_X(U)$. Hence the tensor product is the sheaf associated to the presheaf

$$U \longmapsto (\bigoplus\nolimits_{i\in I} \mathcal{O}_X(U)) \otimes_{\mathcal{O}_X(U)} (\bigoplus\nolimits_{j\in J} \mathcal{O}_X(U)).$$

We deduce what we want since for any ring R we have $(\bigoplus_{i\in I} R) \otimes_R (\bigoplus_{j\in J} R) = \bigoplus_{(i,j)\in I\times J} R$.

If $\mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F} \to 0$ is exact, then by Lemma 15.15.3 the complex $\mathcal{F}_2 \otimes \mathcal{G} \to \mathcal{F}_1 \otimes \mathcal{G} \to \mathcal{F} \otimes \mathcal{G} \to 0$ is exact. Using this we can prove (5). Namely, in this case there exists locally such an exact sequence with $\mathcal{F}_i$, $i = 1, 2$ finite free. Hence the two terms $\mathcal{F}_2 \otimes \mathcal{G}$ are isomorphic to finite direct sums of $\mathcal{G}$. Since finite direct sums are coherent sheaves, these are coherent and so is the cokernel of the map, see Lemma 15.12.4.

And if also $\mathcal{G}_2 \to \mathcal{G}_1 \to \mathcal{G} \to 0$ is exact, then we see that

$$\mathcal{F}_2 \otimes_{\mathcal{O}_X} \mathcal{G}_1 \oplus \mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{G}_2 \to \mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{G}_1 \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \to 0$$

is exact. Using this we can for example prove (3). Namely, the assumption means that we can locally find presentations as above with $\mathcal{F}_i$ and $\mathcal{G}_i$ free $\mathcal{O}_X$-modules. Hence the displayed presentation is a presentation of the tensor product by free sheaves as well.

The proof of the other statements is omitted. $\square$

Lemma 15.15.6. *Let $(X, \mathcal{O}_X)$ be a ringed space. For any $\mathcal{O}_X$-module $\mathcal{F}$ the functor*

$$Mod(\mathcal{O}_X) \longrightarrow Mod(\mathcal{O}_X), \quad \mathcal{G} \longmapsto \mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$$

commutes with arbitrary colimits.

Proof. Let I be a partially ordered set and let $\{\mathcal{G}_i\}$ be a system over I. Set $\mathcal{G} = colim_i \mathcal{G}_i$. Recall that $\mathcal{G}$ is the sheaf associated to the presheaf $\mathcal{G}' : U \mapsto colim_i \mathcal{G}_i(U)$, see Sheaves, Section 6.29. By Lemma 15.15.2 the tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is the sheafification of the presheaf

$$U \longmapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} colim_i \mathcal{G}_i(U) = colim_i \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}_i(U)$$

where the equality sign is Algebra, Lemma 7.11.8. Hence the lemma follows from the description of colimits in $Mod(\mathcal{O}_X)$. □

15.16. Flat modules

We can define flat modules exactly as in the case of modules over rings.

Definition 15.16.1. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ is *flat* if the functor

$$Mod(\mathcal{O}_X) \longrightarrow Mod(\mathcal{O}_X), \quad \mathcal{G} \mapsto \mathcal{G} \otimes_{\mathcal{O}} \mathcal{F}$$

is exact.

We can characterize flatness by looking at the stalks.

Lemma 15.16.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ is flat if and only if the stalk $\mathcal{F}_x$ is a flat $\mathcal{O}_{X,x}$-module for all $x \in X$.*

Proof. Assume $\mathcal{F}_x$ is a flat $\mathcal{O}_{X,x}$-module for all $x \in X$. In this case, if $\mathcal{G} \to \mathcal{H} \to \mathcal{K}$ is exact, then also $\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F}$ is exact because we can check exactness at stalks and because tensor product commutes with taking stalks, see Lemma 15.15.1. Conversely, suppose that $\mathcal{F}$ is flat, and let $x \in X$. Consider the skyscraper sheaves $i_{x,*}M$ where M is a $\mathcal{O}_{X,x}$-module. Note that

$$M \otimes_{\mathcal{O}_{X,x}} \mathcal{F}_x = \left(i_{x,*}M \otimes_{\mathcal{O}_X} \mathcal{F}\right)_x$$

again by Lemma 15.15.1. Since $i_{x,*}$ is exact, we see that the fact that $\mathcal{F}$ is flat implies that $M \mapsto M \otimes_{\mathcal{O}_{X,x}} \mathcal{F}_x$ is exact. Hence $\mathcal{F}_x$ is a flat $\mathcal{O}_{X,x}$-module. □

Thus the following definition makes sense.

Definition 15.16.3. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $x \in X$ An $\mathcal{O}_X$-module $\mathcal{F}$ is *flat at* x if $\mathcal{F}_x$ is a flat $\mathcal{O}_{X,x}$-module.

Hence we see that $\mathcal{F}$ is a flat $\mathcal{O}_X$-module if and only if it is flat at every point.

Lemma 15.16.4. *Let $(X, \mathcal{O}_X)$ be a ringed space. A filtered colimit of flat $\mathcal{O}_X$-modules is flat. A direct sum of flat $\mathcal{O}_X$-modules is flat.*

Proof. This follows from Lemma 15.15.6, Lemma 15.15.1, Algebra, Lemma 7.8.9, and the fact that we can check exactness at stalks. □

Lemma 15.16.5. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $U \subset X$ be open. The sheaf $j_{U!}\mathcal{O}_U$ is a flat sheaf of $\mathcal{O}_X$-modules.*

Proof. The stalks of $j_!\mathcal{O}_U$ are either zero of equal to $\mathcal{O}_{X,x}$. Apply Lemma 15.16.2. □

Lemma 15.16.6. *Let $(X, \mathcal{O}_X)$ be a ringed space.*

1. *Any sheaf of $\mathcal{O}_X$-modules is a quotient of a direct sum $\bigoplus j_{U_i!}\mathcal{O}_{U_i}$.*
2. *Any $\mathcal{O}_X$-module is a quotient of a flat $\mathcal{O}_X$-module.*

Proof. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. For every open $U \subset X$ and every $s \in \mathcal{F}(U)$ we get a morphism $j_{U!}\mathcal{O}_U \to \mathcal{F}$, namely the adjoint to the morphism $\mathcal{O}_U \to \mathcal{F}|_U$, $1 \mapsto s$. Clearly the map

$$\bigoplus\nolimits_{(U,s)} j_{U!}\mathcal{O}_U \longrightarrow \mathcal{F}$$

is surjective, and the source is flat by combining Lemmas 15.16.4 and 15.16.5. $\square$

Lemma 15.16.7. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let*

$$0 \to \mathcal{F}'' \to \mathcal{F}' \to \mathcal{F} \to 0$$

be a short exact sequence of $\mathcal{O}_X$-modules. Assume $\mathcal{F}$ is flat. Then for any $\mathcal{O}_X$-module $\mathcal{G}$ the sequence

$$0 \to \mathcal{F}'' \otimes_{\mathcal{O}} \mathcal{G} \to \mathcal{F}' \otimes_{\mathcal{O}} \mathcal{G} \to \mathcal{F} \otimes_{\mathcal{O}} \mathcal{G} \to 0$$

is exact.

Proof. Using that $\mathcal{F}_x$ is a flat $\mathcal{O}_{X,x}$-module for every $x \in X$ and that exactness can be checked on stalks, this follows from Algebra, Lemma 7.35.11. $\square$

Lemma 15.16.8. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let*

$$0 \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F}_0 \to 0$$

be a short exact sequence of $\mathcal{O}_X$-modules.

(1) *If $\mathcal{F}_2$ and $\mathcal{F}_0$ are flat so is $\mathcal{F}_1$.*
(2) *If $\mathcal{F}_1$ and $\mathcal{F}_0$ are flat so is $\mathcal{F}_2$.*

Proof. Since exactness and flatness may be checked at the level of stalks this follows from Algebra, Lemma 7.35.12. $\square$

Lemma 15.16.9. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let*

$$\ldots \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F}_0 \to \mathcal{Q} \to 0$$

be an exact complex of $\mathcal{O}_X$-modules. If $\mathcal{Q}$ and all $\mathcal{F}_i$ are flat $\mathcal{O}_X$-modules, then for any $\mathcal{O}_X$-module $\mathcal{G}$ the complex

$$\ldots \to \mathcal{F}_2 \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{F}_0 \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{Q} \otimes_{\mathcal{O}_X} \mathcal{G} \to 0$$

is exact also.

Proof. Follows from Lemma 15.16.7 by splitting the complex into short exact sequences and using Lemma 15.16.8 to prove inductively that $\mathrm{Im}(\mathcal{F}_{i+1} \to \mathcal{F}_i)$ is flat. $\square$

15.17. Flat morphisms of ringed spaces

The pointwise definition is motivated by Lemma 15.16.2 and Definition 15.16.3 above.

Definition 15.17.1. Let $f : X \to Y$ be a morphism of ringed spaces. Let $x \in X$. We say f is said to be *flat at x* if the map of rings $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat. We say f is *flat* if f is flat at every $x \in X$.

Consider the map of sheaves of rings $f^\sharp : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. We see that the stalk at x is the ring map $f_x^\sharp : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$. Hence f is flat at x if and only if $\mathcal{O}_X$ is flat at x as an $f^{-1}\mathcal{O}_Y$-module. And f is flat if and only if $\mathcal{O}_X$ is flat as an $f^{-1}\mathcal{O}_Y$-module. A very special case of a flat morphism is an open immersion.

Lemma 15.17.2. *Let $f : X \to Y$ be a flat morphism of ringed spaces. Then the pullback functor $f^* : Mod(\mathcal{O}_Y) \to Mod(\mathcal{O}_X)$ is exact.*

Proof. The functor f^* is the composition of the exact functor $f^{-1} : Mod(\mathcal{O}_Y) \to Mod(f^{-1}\mathcal{O}_Y)$ and the change of rings functor

$$Mod(f^{-1}\mathcal{O}_Y) \to Mod(\mathcal{O}_X), \quad \mathcal{F} \longmapsto \mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X.$$

Thus the result follows from the discussion following Definition 15.17.1. □

15.18. Symmetric and exterior powers

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-algebra. We define the *tensor algebra* of $\mathcal{F}$ to be the sheaf of noncommutative $\mathcal{O}_X$-algebras

$$\mathrm{T}(\mathcal{F}) = \mathrm{T}_{\mathcal{O}_X}(\mathcal{F}) = \bigoplus\nolimits_{n \geq 0} \mathrm{T}^n(\mathcal{F}).$$

Here $\mathrm{T}^0(\mathcal{F}) = \mathcal{O}_X$, $\mathrm{T}^1(\mathcal{F}) = \mathcal{F}$ and for $n \geq 2$ we have

$$\mathrm{T}^n(\mathcal{F}) = \mathcal{F} \otimes_{\mathcal{O}_X} \ldots \otimes_{\mathcal{O}_X} \mathcal{F} \ (n \text{ factors})$$

We define $\wedge(\mathcal{F})$ to be the quotient of $\mathrm{T}(\mathcal{F})$ by the two sided ideal generated by local sections $s \otimes s$ of $\mathrm{T}^2(\mathcal{F})$ where s is a local section of $\mathcal{F}$. This is called the *exterior algebra* of $\mathcal{F}$. Similarly, we define $\mathrm{Sym}(\mathcal{F})$ to be the quotient of $\mathrm{T}(\mathcal{F})$ by the two sided ideal generated by local sections of the form $s \otimes t - t \otimes s$ of $\mathrm{T}^2(\mathcal{F})$.

Both $\wedge(\mathcal{F})$ and $\mathrm{Sym}(\mathcal{F})$ are graded $\mathcal{O}_X$-algebras, with grading inherited from $\mathrm{T}(\mathcal{F})$. Moreover $\mathrm{Sym}(\mathcal{F})$ is commutative, and $\wedge(\mathcal{F})$ is graded commutative.

Lemma 15.18.1. *In the situation described above. The sheaf $\wedge^n\mathcal{F}$ is the sheafification of the presheaf*

$$U \longmapsto \wedge^n_{\mathcal{O}_X(U)}(\mathcal{F}(U)).$$

See Algebra, Section 7.12. Similarly, the sheaf $Sym^n\mathcal{F}$ is the sheafification of the presheaf

$$U \longmapsto Sym^n_{\mathcal{O}_X(U)}(\mathcal{F}(U)).$$

Proof. Omitted. It may be more efficient to define $\mathrm{Sym}(\mathcal{F})$ and $\wedge(\mathcal{F})$ in this way instead of the method given above. □

Lemma 15.18.2. *In the situation described above. Let $x \in X$. There are canonical isomorphisms of $\mathcal{O}_{X,x}$-modules $T(\mathcal{F})_x = T(\mathcal{F}_x)$, $Sym(\mathcal{F})_x = Sym(\mathcal{F}_x)$, and $\wedge(\mathcal{F})_x = \wedge(\mathcal{F}_x)$.*

Proof. Clear from Lemma 15.18.1 above, and Algebra, Lemma 7.12.4. □

Lemma 15.18.3. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_Y$-modules. Then $f^*T(\mathcal{F}) = T(f^*\mathcal{F})$, and similarly for the exterior and symmetric algebras associated to $\mathcal{F}$.*

Proof. Omitted. □

Lemma 15.18.4. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F} \to 0$ be an exact sequence of sheaves of $\mathcal{O}_X$-modules. For each $n \geq 1$ there is an exact sequence*

$$\mathcal{F}_2 \otimes_{\mathcal{O}_X} Sym^{n-1}(\mathcal{F}_1) \to Sym^n(\mathcal{F}_1) \to Sym^n(\mathcal{F}) \to 0$$

and similarly an exact sequence

$$\mathcal{F}_2 \otimes_{\mathcal{O}_X} \wedge^{n-1}(\mathcal{F}_1) \to \wedge^n(\mathcal{F}_1) \to \wedge^n(\mathcal{F}) \to 0$$

Proof. See Algebra, Lemma 7.12.2. □

Lemma 15.18.5. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules.*

(1) *If $\mathcal{F}$ is locally generated by sections, then so is each $T^n(\mathcal{F})$, $\wedge^n(\mathcal{F})$, and $Sym^n(\mathcal{F})$.*

(2) *If $\mathcal{F}$ is of finite type, then so is each $T^n(\mathcal{F})$, $\wedge^n(\mathcal{F})$, and $\mathrm{Sym}^n(\mathcal{F})$.*
(3) *If $\mathcal{F}$ is of finite presentation, then so is each $T^n(\mathcal{F})$, $\wedge^n(\mathcal{F})$, and $\mathrm{Sym}^n(\mathcal{F})$.*
(4) *If $\mathcal{F}$ is coherent, then for $n > 0$ each $T^n(\mathcal{F})$, $\wedge^n(\mathcal{F})$, and $\mathrm{Sym}^n(\mathcal{F})$ is coherent.*
(5) *If $\mathcal{F}$ is quasi-coherent, then so is each $T^n(\mathcal{F})$, $\wedge^n(\mathcal{F})$, and $\mathrm{Sym}^n(\mathcal{F})$.*
(6) *If $\mathcal{F}$ is locally free, then so is each $T^n(\mathcal{F})$, $\wedge^n(\mathcal{F})$, and $\mathrm{Sym}^n(\mathcal{F})$.*

Proof. These statements for $\mathrm{T}^n(\mathcal{F})$ follow from Lemma 15.15.5.

Statements (1) and (2) follow from the fact that $\wedge^n(\mathcal{F})$ and $\mathrm{Sym}^n(\mathcal{F})$ are quotients of $\mathrm{T}^n(\mathcal{F})$.

Statement (6) follows from Algebra, Lemma 7.12.1.

For (3) and (5) we will use Lemma 15.18.4 above. By locally choosing a presentation $\mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F} \to 0$ with $\mathcal{F}_i$ free, or finite free and applying the lemma we see that $\mathrm{Sym}^n(\mathcal{F})$, $\wedge^n(\mathcal{F})$ has a similar presentation; here we use (6) and Lemma 15.15.5.

To prove (4) we will use Algebra, Lemma 7.12.3. We may localize on X and assume that $\mathcal{F}$ is generated by a finite set $(s_i)_{i\in I}$ of global sections. The lemma mentioned above combined with Lemma 15.18.1 above implies that for $n \geq 2$ there exists an exact sequence

$$\bigoplus_{j\in J} \mathrm{T}^{n-2}(\mathcal{F}) \to \mathrm{T}^n(\mathcal{F}) \to \mathrm{Sym}^n(\mathcal{F}) \to 0$$

where the index set J is finite. Now we know that $\mathrm{T}^{n-2}(\mathcal{F})$ is finitely generated and hence the image of the first arrow is a coherent subsheaf of $\mathrm{T}^n(\mathcal{F})$, see Lemma 15.12.4. By that same lemma we conclude that $\mathrm{Sym}^n(\mathcal{F})$ is coherent. ☐

Lemma 15.18.6. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules.*
(1) *If $\mathcal{F}$ is quasi-coherent, then so is each $T(\mathcal{F})$, $\wedge(\mathcal{F})$, and $\mathrm{Sym}(\mathcal{F})$.*
(2) *If $\mathcal{F}$ is locally free, then so is each $T(\mathcal{F})$, $\wedge(\mathcal{F})$, and $\mathrm{Sym}(\mathcal{F})$.*

Proof. It is not true that an infinite direct sum $\bigoplus \mathcal{G}_i$ of locally free modules is locally free, or that an infinite direct sum of quasi-coherent modules is quasi-coherent. The problem is that given a point $x \in X$ the open neighbourhoods U_i of x on which $\mathcal{G}_i$ becomes free (resp. has a suitable presentation) may have an intersection which is not an open neighbourhood of x. However, in the proof of Lemma 15.18.5 we saw that once a suitable open neighbourhood for $\mathcal{F}$ has been chosen, then this open neighbourhood works for each of the sheaves $\mathrm{T}^n(\mathcal{F})$, $\wedge^n(\mathcal{F})$ and $\mathrm{Sym}^n(\mathcal{F})$. The lemma follows. ☐

15.19. Internal Hom

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}, \mathcal{G}$ be $\mathcal{O}_X$-modules. Consider the rule

$$U \longmapsto Hom_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U).$$

It follows from the discussion in Sheaves, Section 6.33 that this is a sheaf of abelian groups. In addition, given an element $\varphi \in Hom_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ and a section $f \in \mathcal{O}_X(U)$ then we can define $f\varphi \in Hom_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ by either precomposing with multiplication by f on $\mathcal{F}|_U$ or postcomposing with multiplication by f on $\mathcal{G}|_U$ (it gives the same result). Hence we in fact get a sheaf of $\mathcal{O}_X$-modules. We will denote this sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$. There is a canonical ``evaluation'' morphism

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathcal{G}.$$

For every $x \in X$ there is also a canonical morphism

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \to Hom_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$$

which is rarely an isomorphism.

Lemma 15.19.1. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ be $\mathcal{O}_X$-modules. There is a canonical isomorphism*

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}, \mathcal{H}) \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{H}))$$

which is functorial in all three entries (sheaf Hom in all three spots). In particular, to give a morphism $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{H}$ is the same as giving a morphism $\mathcal{F} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{H})$.

Proof. This is the analogue of Algebra, Lemma 7.11.9. The proof is the same, and is omitted. □

Lemma 15.19.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_X$-modules.*

1. *If $\mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F} \to 0$ is an exact sequence of $\mathcal{O}_X$-modules, then*
$$0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_1, \mathcal{G}) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_2, \mathcal{G})$$
is exact.
2. *If $0 \to \mathcal{G} \to \mathcal{G}_1 \to \mathcal{G}_2$ is an exact sequence of $\mathcal{O}_X$-modules, then*
$$0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}_1) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}_2)$$
is exact.

Proof. Omitted. □

Lemma 15.19.3. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_X$-modules. If $\mathcal{F}$ is finitely presented then the canonical map*

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \to Hom_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$$

is an isomorphism.

Proof. By localizing on X we may assume that $\mathcal{F}$ has a presentation

$$\bigoplus\nolimits_{j=1,\ldots,m} \mathcal{O}_X \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_X \to \mathcal{F} \to 0.$$

By Lemma 15.19.2 this gives an exact sequence $0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \bigoplus_{i=1,\ldots,n} \mathcal{G} \longrightarrow \bigoplus_{j=1,\ldots,m} \mathcal{G}$. Taking stalks we get an exact sequence $0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \to \bigoplus_{i=1,\ldots,n} \mathcal{G}_x \longrightarrow \bigoplus_{j=1,\ldots,m} \mathcal{G}_x$ and the result follows since $\mathcal{F}_x$ sits in an exact sequence $\bigoplus_{j=1,\ldots,m} \mathcal{O}_{X,x} \longrightarrow \bigoplus_{i=1,\ldots,n} \mathcal{O}_{X,x} \to \mathcal{F}_x \to 0$ which induces the exact sequence $0 \to Hom_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x) \to \bigoplus_{i=1,\ldots,n} \mathcal{G}_x \longrightarrow \bigoplus_{j=1,\ldots,m} \mathcal{G}_x$ which is the same as the one above. □

Lemma 15.19.4. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_X$-modules. If $\mathcal{F}$ is finitely presented then the sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is locally a kernel of a map between finite direct sums of copies of $\mathcal{G}$. In particular, if $\mathcal{G}$ is coherent then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is coherent too.*

Proof. The first assertion we saw in the proof of Lemma 15.19.3. And the result for coherent sheaves then follows from Lemma 15.12.4. □

15.20. Koszul complexes

We suggest first reading the section on Koszul complexes in More on Algebra, Section 12.21. We define the Koszul complex in the category of $\mathcal{O}_X$-modules as follows.

Definition 15.20.1. Let X be a ringed space. Let $\varphi : \mathcal{E} \to \mathcal{O}_X$ be an $\mathcal{O}_X$-module map. The *Koszul complex* $K_\bullet(\varphi)$ associated to φ is the sheaf of commutative differential graded algebras defined as follows:

(1) the underlying graded algebra is the exterior algebra $K_\bullet(\varphi) = \wedge(\mathcal{E})$,
(2) the differential $d : K_\bullet(\varphi) \to K_\bullet(\varphi)$ is the unique derivation such that $d(e) = \varphi(e)$ for all local sections e of $\mathcal{E} = K_1(\varphi)$.

Explicitly, if $e_1 \wedge \ldots \wedge e_n$ is a wedge product of local sections of $\mathcal{E}$, then

$$d(e_1 \wedge \ldots \wedge e_n) = \sum\nolimits_{i=1,\ldots,n} (-1)^{i+1}\varphi(e_i)e_1 \wedge \ldots \wedge \widehat{e_i} \wedge \ldots \wedge e_n.$$

It is straightforward to see that this gives a well defined derivation on the tensor algebra, which annihilates $e \wedge e$ and hence factors through the exterior algebra.

Definition 15.20.2. Let X be a ringed space and let $f_1, \ldots, f_n \in \Gamma(X, \mathcal{O}_X)$. The *Koszul complex on* $f_1, \ldots, f_r$ is the Koszul complex associated to the map $(f_1, \ldots, f_n) : \mathcal{O}_X^{\oplus n} \to \mathcal{O}_X$. Notation $K_\bullet(\mathcal{O}_X, f_1, \ldots, f_n)$, or $K_\bullet(\mathcal{O}_X, f_\bullet)$.

Of course, given an $\mathcal{O}_X$-module map $\varphi : \mathcal{E} \to \mathcal{O}_X$, if $\mathcal{E}$ is finite locally free, then $K_\bullet(\varphi)$ is locally on X isomorphic to a Koszul complex $K_\bullet(\mathcal{O}_X, f_1, \ldots, f_n)$.

15.21. Invertible sheaves

Definition 15.21.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Assume that all stalks $\mathcal{O}_{X,x}$ are local rings[3]. An *invertible $\mathcal{O}_X$-module* is a sheaf of $\mathcal{O}_X$-modules $\mathcal{L}$ such that for each point $x \in X$ there exists an open neighbourhood $U \subset X$ and an isomorphism $\mathcal{L}|_U \cong \mathcal{O}_X|_U$. We say that $\mathcal{L}$ is *trivial* if it is isomorphic as an $\mathcal{O}_X$-module to $\mathcal{O}_X$.

Lemma 15.21.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Assume that all stalks $\mathcal{O}_{X,x}$ are local rings.*

(1) *If $\mathcal{L}$, $\mathcal{N}$ are invertible $\mathcal{O}_X$-modules, then so is $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N}$.*
(2) *If $\mathcal{L}$ is an invertible $\mathcal{O}_X$-modules, then so is $\mathcal{L}^{\otimes -1} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$.*
(3) *If $\mathcal{L}$ is an invertible $\mathcal{O}_X$-modules, then the evaluation map $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1} \to \mathcal{O}_X$ is an isomorphism.*

Proof. Omitted. □

Definition 15.21.3. Let $(X, \mathcal{O}_X)$ be a ringed space. Assume that all stalks $\mathcal{O}_{X,x}$ are local rings. Given an invertible sheaf $\mathcal{L}$ on X we define the nth *tensor power of $\mathcal{L}$* by the rule

$$\mathcal{L}^{\otimes n} = \begin{cases} \mathcal{O}_X & \text{if } n = 0 \\ \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) & \text{if } n = -1 \\ \mathcal{L} \otimes_{\mathcal{O}_X} \ldots \otimes_{\mathcal{O}_X} \mathcal{L} & \text{if } n > 0 \\ \mathcal{L}^{\otimes -1} \otimes_{\mathcal{O}_X} \ldots \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1} & \text{if } n < -1 \end{cases}$$

[3]We should at least assume that they are nonzero. However, in this generality the stalks $\mathcal{O}_{X,x}$ can have nontrivial Picard groups, and then there are two possible definitions. One were we require $\mathcal{L}$ to be locally free of rank 1, and the other where we require $\mathcal{L}$ to be a flat, finite presentation $\mathcal{O}_X$-module such that there exists a second such sheaf $\mathcal{L}^{\otimes -1}$ with $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1} \cong \mathcal{O}_X$.

With this definition we have canonical isomorphisms $\mathcal{L}^{\otimes n} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes m} \to \mathcal{L}^{\otimes n+m}$, and these isomorphisms statisfy a commutativity and an associativity constraint (formulation omitted). Thus we can define a $\mathbf{Z}$-graded ring structure on $\bigoplus \Gamma(X, \mathcal{L}^{\otimes n})$ by mapping $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ and $t \in \Gamma(X, \mathcal{L}^{\otimes m})$ to the section corresponding to $s \otimes t$ in $\Gamma(X, \mathcal{L}^{\otimes n+m})$. We omit the verification that this defines a commutative and associative ring with 1. However, by our conventions in Algebra, Section 7.52 a graded ring has no nonzero elements in negative degrees. This leads to the following definition.

Definition 15.21.4. Let $(X, \mathcal{O}_X)$ be a ringed space. Assume that all stalks $\mathcal{O}_{X,x}$ are local rings. Given an invertible sheaf $\mathcal{L}$ on X we define the *associated graded ring* to be

$$\Gamma_*(X, \mathcal{L}) = \bigoplus\nolimits_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$$

Given a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ we set

$$\Gamma_*(X, \mathcal{L}, \mathcal{F}) = \bigoplus\nolimits_{n \in \mathbf{Z}} \Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$$

which we think of as a graded $\Gamma_*(X, \mathcal{L})$-module.

We often write simply $\Gamma_*(\mathcal{L})$ and $\Gamma_*(\mathcal{F})$ (allthough this is ambiguous if $\mathcal{F}$ is invertible). The multiplication of $\Gamma_*(\mathcal{L})$ on $\Gamma_*(\mathcal{F})$ is defined using the isomorphisms above. If $\gamma : \mathcal{F} \to \mathcal{G}$ is a $\mathcal{O}_X$-module map, then we get an $\Gamma_*(\mathcal{L})$-module homomorphism $\gamma : \Gamma_*(\mathcal{F}) \to \Gamma_*(\mathcal{G})$. If $\alpha : \mathcal{L} \to \mathcal{N}$ is an $\mathcal{O}_X$-module map between invertible $\mathcal{O}_X$-modules, then we obtain a graded ring homomorphism $\Gamma_*(\mathcal{L}) \to \Gamma_*(\mathcal{N})$. If $f : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a morphism of locally ringed spaces (see Schemes, Definition 21.2.1), and if $\mathcal{L}$ is invertible on X, then we get an invertible sheaf $f^*\mathcal{L}$ on Y and an induced homomorphism of graded rings

$$f^* : \Gamma_*(X, \mathcal{L}) \longrightarrow \Gamma_*(Y, f^*\mathcal{L})$$

Furthermore, there are some compatibilties between the constructions above whose statements we omit.

Lemma 15.21.5. *Let $(X, \mathcal{O}_X)$ be a ringed space. Assume that all stalks $\mathcal{O}_{X,x}$ are local rings. There exists a set of invertible modules $\{\mathcal{L}_i\}_{i \in I}$ such that each invertible module on X is isomorphic to exactly one of the $\mathcal{L}_i$.*

Proof. For each open covering $\mathcal{U} : X = \bigcup U_j$ consider the sheaves of $\mathcal{O}_X$-modules gotten from glueing the sheaves $\mathcal{O}_X|_{U_j}$, see Sheaves, Section 6.33. Note that the collection of all glueing data forms a set. The collection of all coverings $\mathcal{U} : X = \bigcup_{j \in J} U_i$ where $J \to \mathcal{P}(X)$, $j \mapsto U_j$ is injective forms a set as well. Hence the collection of all sheaves of of $\mathcal{O}_X$-modules gotten from glueing trivial invertible $\mathcal{O}_X$-modules forms a set $\mathcal{I}$. By definition every invertible $\mathcal{O}_X$-module is isomorphic to an element of $\mathcal{I}$. Choosing an element out of each isomorphism class inside $\mathcal{I}$ gives the desired set of invertible sheaves (uses axiom of choice). $\square$

This lemma says roughly speaking that the collection of isomorphism classes of invertible sheaves forms a set. Lemma 15.21.2 says that tensor product defines the structure of an abelian group on this set.

Definition 15.21.6. Let $(X, \mathcal{O}_X)$ be a ringed space. Assume all stalks $\mathcal{O}_{X,x}$ are local rings. The *Picard group* $\mathrm{Pic}(X)$ of X is the abelian group whose elements are isomorphism classes of invertible $\mathcal{O}_X$-modules, with addition corresponding to tensor product.

Lemma 15.21.7. *Let X be a ringed space. Assume that each stalk $\mathcal{O}_{X,x}$ is a local ring with maximal ideal $\mathfrak{m}_x$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. For any section $s \in \Gamma(X, \mathcal{L})$ the set*

$$X_s = \{x \in X \mid \text{image } s \notin \mathfrak{m}_x \mathcal{L}_x\}$$

is open in X. The map $s : \mathcal{O}_{X_s} \to \mathcal{L}|_{X_s}$ is an isomorphism, and there exists a section s' of $\mathcal{L}^{\otimes -1}$ over X_s such that $s'(s|_{X_s}) = 1$.

Proof. Suppose $x \in X_s$. We have an isomorphism

$$\mathcal{L}_x \otimes_{\mathcal{O}_{X,x}} (\mathcal{L}^{\otimes -1})_x \longrightarrow \mathcal{O}_{X,x}$$

by Lemma 15.21.2. Both $\mathcal{L}_x$ and $(\mathcal{L}^{\otimes -1})_x$ are free $\mathcal{O}_{X,x}$-modules of rank 1. We conclude from Algebra, Nakayama's Lemma 7.14.5 that s_x is a basis for $\mathcal{L}_x$. Hence there exists a basis element $t_x \in (\mathcal{L}^{\otimes -1})_x$ such that $s_x \otimes t_x$ maps to 1. Choose an open neighbourhood U of x such that t_x comes from a section t of $(\mathcal{L}^{\otimes -1})_x$ over U and such that $s \otimes t$ maps to $1 \in \mathcal{O}_X(U)$. Clearly, for every $x' \in U$ we see that s generates the module $\mathcal{L}_{x'}$. Hence $U \subset X_s$. This proves that X_s is open. Moreover, the section t constructed over U above is unique, and hence these glue to give te section s' of the lemma. □

It is also true that, given a morphism of locally ringed spaces $f : Y \to X$ (see Schemes, Definition 21.2.1) that the inverse image $f^{-1}(X_s)$ is equal to Y_{f^*s}, where $f^*s \in \Gamma(Y, f^*\mathcal{L})$ is the pull back of s.

15.22. Localizing sheaves of rings

Let X be a topological space and let $\mathcal{O}_X$ be a presheaf of rings. Let $\mathcal{S} \subset \mathcal{O}_X$ be a presheaf of sets contained in $\mathcal{O}_X$. Suppose that for every open $U \subset X$ the set $\mathcal{S}(U) \subset \mathcal{O}_X(U)$ is a multiplicative subset, see Algebra, Definition 7.9.1. In this case we can consider the presheaf of rings

$$\mathcal{S}^{-1}\mathcal{O}_X : U \longmapsto \mathcal{S}(U)^{-1}\mathcal{O}_X(U).$$

The restriction mapping sends the section f/s, $f \in \mathcal{O}_X(U)$, $s \in \mathcal{S}(U)$ to $(f|_V)/(s|_V)$ if $V \subset U$ are opens of X.

Lemma 15.22.1. *Let X be a topological space and let $\mathcal{O}_X$ be a presheaf of rings. Let $\mathcal{S} \subset \mathcal{O}_X$ be a pre-sheaf of sets contained in $\mathcal{O}_X$. Suppose that for every open $U \subset X$ the set $\mathcal{S}(U) \subset \mathcal{O}_X(U)$ is a multiplicative subset.*

1. *There is a map of presheaves of rings $\mathcal{O}_X \to \mathcal{S}^{-1}\mathcal{O}_X$ such that every local section of $\mathcal{S}$ maps to an invertible section of $\mathcal{O}_X$.*
2. *For any homomorphism of presheaves of rings $\mathcal{O}_X \to \mathcal{A}$ such that each local section of $\mathcal{S}$ maps to an invertible section of $\mathcal{A}$ there exists a unique factorization $\mathcal{S}^{-1}\mathcal{O}_X \to \mathcal{A}$.*
3. *For any $x \in X$ we have*
$$(\mathcal{S}^{-1}\mathcal{O}_X)_x = \mathcal{S}_x^{-1}\mathcal{O}_{X,x}.$$
4. *The sheafification $(\mathcal{S}^{-1}\mathcal{O}_X)^\#$ is a sheaf of rings with a map of sheaves of rings $(\mathcal{O}_X)^\# \to (\mathcal{S}^{-1}\mathcal{O}_X)^\#$ which is universal for maps of $(\mathcal{O}_X)^\#$ into sheaves of rings such that each local section of $\mathcal{S}$ maps to an invertible section.*
5. *For any $x \in X$ we have*
$$(\mathcal{S}^{-1}\mathcal{O}_X)^\#_x = \mathcal{S}_x^{-1}\mathcal{O}_{X,x}.$$

Proof. Omitted. □

Let X be a topological space and let $\mathcal{O}_X$ be a presheaf of rings. Let $\mathcal{S} \subset \mathcal{O}_X$ be a presheaf of sets contained in $\mathcal{O}_X$. Suppose that for every open $U \subset X$ the set $\mathcal{S}(U) \subset \mathcal{O}_X(U)$ is a multiplicative subset. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_X$-modules In this case we can consider the presheaf of $\mathcal{S}^{-1}\mathcal{O}_X$-modules

$$\mathcal{S}^{-1}\mathcal{F} : U \longmapsto \mathcal{S}(U)^{-1}\mathcal{F}(U).$$

The restriction mapping sends the section t/s, $t \in \mathcal{F}(U)$, $s \in \mathcal{S}(U)$ to $(t|_V)/(s|_V)$ if $V \subset U$ are opens of X.

Lemma 15.22.2. *Let X be a topological space. Let $\mathcal{O}_X$ be a presheaf of rings. Let $\mathcal{S} \subset \mathcal{O}_X$ be a pre-sheaf of sets contained in $\mathcal{O}_X$. Suppose that for every open $U \subset X$ the set $\mathcal{S}(U) \subset \mathcal{O}_X(U)$ is a multiplicative subset. For any presheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ we have*

$$\mathcal{S}^{-1}\mathcal{F} = \mathcal{S}^{-1}\mathcal{O}_X \otimes_{p,\mathcal{O}_X} \mathcal{F}$$

(see Sheaves, Section 6.6 for notation) and if $\mathcal{F}$ and $\mathcal{O}_X$ are sheaves then

$$(\mathcal{S}^{-1}\mathcal{F})^\# = (\mathcal{S}^{-1}\mathcal{O}_X)^\# \otimes_{\mathcal{O}_X} \mathcal{F}$$

(see Sheaves, Section 6.20 for notation).

Proof. Omitted. □

15.23. Other chapters

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(5) Topology
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CHAPTER 16

Modules on Sites

16.1. Introduction

In this document we work out basic notions of sheaves of modules on ringed topoi or ringed sites. We first work out some basic facts on abelian sheaves. After this we introduce ringed sites and ringed topoi. We work through some of the very basic notions on (pre)sheaves of $\mathcal{O}$-modules, analogous to the material on (pre)sheaves of $\mathcal{O}$-modules in the chapter on sheaves on spaces. Having done this, we duplicate much of the discussion in the chapter on sheaves of modules (see Modules, Section 15.1). Basic references are [**Ser55b**], [**DG67**] and [**MA71**].

16.2. Abelian presheaves

Let $\mathcal{C}$ be a category. Abelian presheaves were introduced in Sites, Sections 9.2 and 9.7 and discussed a bit more in Sites, Section 9.38. We will follow the convention of this last reference, in that we think of an abelian presheaf as a presheaf of sets endowed with addition rules on all sets of sections compatible with the restriction mappings. Recall that the category of abelian presheaves on $\mathcal{C}$ is denoted $PAb(\mathcal{C})$.

The category $PAb(\mathcal{C})$ is abelian as defined in Homology, Definition 10.3.12. Given a map of presheaves $\varphi : \mathcal{G}_1 \to \mathcal{G}_2$ the kernel of φ is the abelian presheaf $U \mapsto \mathrm{Ker}(\mathcal{G}_1(U) \to \mathcal{G}_2(U))$ and the cokernel of φ is the presheaf $U \mapsto \mathrm{Coker}(\mathcal{G}_1(U) \to \mathcal{G}_2(U))$. Since the category of abelian groups is abelian it follows that $\mathrm{Coim} = \mathrm{Im}$ because this holds over each U. A sequence of abelian presheaves

$$\mathcal{G}_1 \longrightarrow \mathcal{G}_2 \longrightarrow \mathcal{G}_3$$

is exact if and only if $\mathcal{G}_1(U) \to \mathcal{G}_2(U) \to \mathcal{G}_3(U)$ is an exact sequence of abelian groups for all $U \in Ob(\mathcal{C})$. We leave the verifications to the reader.

Lemma 16.2.1. *Let $\mathcal{C}$ be a category.*

1. *All limits and colimits exist in $PAb(\mathcal{C})$.*
2. *All limits and colimits commute with taking sections over objects of $\mathcal{C}$.*

Proof. Let $\mathcal{I} \to PAb(\mathcal{C})$, $i \mapsto \mathcal{F}_i$ be a diagram. We can simply define abelian presheaves L and C by the rules

$$L : U \longmapsto lim_i\, \mathcal{F}_i(U)$$

and

$$C : U \longmapsto colim_i\, \mathcal{F}_i(U).$$

It is clear that there are maps of abelian presheaves $L \to \mathcal{F}_i$ and $\mathcal{F}_i \to C$, by using the corresponding maps on groups of sections over each U. It is straightforward to check that L and C endowed with these maps are the limit and colimit of the diagram in $PAb(\mathcal{C})$. This proves (1) and (2). Details omitted. $\square$

16.3. Abelian sheaves

Let $\mathcal{C}$ be a site. The category of abelian sheaves on $\mathcal{C}$ is denoted $Ab(\mathcal{C})$. It is the full subcategory of $PAb(\mathcal{C})$ consisting of those abelian presheaves whose underlying presheaves of sets are sheaves. Properties (α) -- (ζ) of Sites, Section 9.38 hold, see Sites, Proposition 9.38.3. In particular the inclusion functor $Ab(\mathcal{C}) \to PAb(\mathcal{C})$ has a left adjoint, namely the sheafification functor $\mathcal{G} \mapsto \mathcal{G}^\#$.

We suggest the reader prove the lemma on a piece of scratch paper rather than reading the proof.

Lemma 16.3.1. *Let $\mathcal{C}$ be a site. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of abelian sheaves on $\mathcal{C}$.*

1. *The category $Ab(\mathcal{C})$ is an abelian category.*
2. *The kernel $Ker(\varphi)$ of φ is the same as the kernel of φ as a morphism of presheaves.*
3. *The morphism φ is injective (Homology, Definition 10.3.14) if and only if φ is injective as a map of presheaves (Sites, Definition 9.3.1), if and only if φ is injective as a map of sheaves (Sites, Definition 9.11.1).*
4. *The cokernel $Coker(\varphi)$ of φ is the sheafification of the cokernel of φ as a morphism of presheaves.*
5. *The morphism φ is surjective (Homology, Definition 10.3.14) if and only if φ is surjective as a map of sheaves (Sites, Definition 9.11.1).*
6. *A complex of abelian sheaves*
$$\mathcal{F} \to \mathcal{G} \to \mathcal{H}$$
is exact at $\mathcal{G}$ if and only if for all $U \in Ob(\mathcal{C})$ and all $s \in \mathcal{G}(U)$ mapping to zero in $\mathcal{H}(U)$ there exists a covering $\{U_i \to U\}_{i \in I}$ in $\mathcal{C}$ such that each $s|_{U_i}$ is in the image of $\mathcal{F}(U_i) \to \mathcal{G}(U_i)$.

Proof. We claim that Homology, Lemma 10.5.3 applies to the categories $\mathcal{A} = Ab(\mathcal{C})$ and $\mathcal{B} = PAb(\mathcal{C})$, and the functors $a : \mathcal{A} \to \mathcal{B}$ (inclusion), and $b : \mathcal{B} \to \mathcal{A}$ (sheafification). Let us check the assumptions of Homology, Lemma 10.5.3. Assumption (1) is that $\mathcal{A}$, $\mathcal{B}$ are additive categories, a, b are additive functors, and a is right adjoint to b. The first two statements are clear and adjointness is Sites, Section 9.38 (ϵ). Assumption (2) says that $PAb(\mathcal{C})$ is abelian which we saw in Section 16.2 and that sheafification is left exact, which is Sites, Section 9.38 (ζ). The final assumption is that $ba \cong \text{id}_{\mathcal{A}}$ which is Sites, Section 9.38 (δ). Hence Homology, Lemma 10.5.3 applies and we conclude that $Ab(\mathcal{C})$ is abelian.

In the proof of Homology, Lemma 10.5.3 it is shown that $\text{Ker}(\varphi)$ and $\text{Coker}(\varphi)$ are equal to the sheafification of the kernel and cokernel of φ as a morphism of abelian presheaves. This proves (4). Since the kernel is a equalizer (i.e., a limit) and since sheafifcation commutes with finite limits, we conclude that (2) holds.

Statement (2) implies (3). Statement (4) implies (5) by our description of sheafification. The characterization of exactness in (6) follows from (2) and (5), and the fact that the sequence is exact if and only if $\text{Im}(\mathcal{F} \to \mathcal{G}) = \text{Ker}(\mathcal{G} \to \mathcal{H})$. □

Another way to say part (6) of the lemma is that a sequence of abelian sheaves
$$\mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3$$
is exact if and only if the sheafification of $U \mapsto \mathcal{F}_2(U)/\mathcal{F}_1(U)$ is equal to the kernel of $\mathcal{F}_2 \to \mathcal{F}_3$.

Lemma 16.3.2. *Let $\mathcal{C}$ be a site.*

(1) *All limits and colimits exist in $Ab(\mathcal{C})$.*
(2) *Limits are the same as the corresponding limits of abelian presheaves over $\mathcal{C}$ (i.e., commute with taking sections over objects of $\mathcal{C}$).*
(3) *Finite direct sums are the same as the correponding finite direct sums in the category of abelian pre-sheaves over $\mathcal{C}$.*
(4) *A colimit is the sheafification of the corresponding colimit in the category of abelian presheaves.*

Proof. By Lemma 16.2.1 limits and colimits of abelian presheaves exist, and are described by taking limits and colimits on the level of sections over objects.

Let $\mathcal{I} \to Ab(\mathcal{C})$, $i \mapsto \mathcal{F}_i$ be a diagram. Let $lim_i \mathcal{F}_i$ be the limit of the diagram as an abelian presheaf. By Sites, Lemma 9.10.1 this is an abelian sheaf. Then it is quite easy to see that $lim_i \mathcal{F}_i$ is the limit of the diagram in $Ab(\mathcal{C})$. This proves limits exist and (2) holds.

By Categories, Lemma 4.22.2, and because sheafification is left adjoint to the inclusion functor we see that $colim_i \mathcal{F}$ exists and is the sheafification of the colimit in $PAb(\mathcal{C})$. This proves colimits exist and (4) holds.

Finite direct sums are the same thing as finite products in any abelian category. Hence (3) follows from (2). $\square$

16.4. Free abelian presheaves

In order to prepare notation for the following definition, let us agree to denote the free abelian group on a set S as[1] $\mathbf{Z}[S] = \bigoplus_{s \in S} \mathbf{Z}$. It is characterized by the property

$$Mor_{Ab}(\mathbf{Z}[S], A) = Mor_{Sets}(S, A)$$

In other words the construction $S \mapsto \mathbf{Z}[S]$ is a left adjoint to the forgetful functor $Ab \to Sets$.

Definition 16.4.1. Let $\mathcal{C}$ be a category. Let $\mathcal{G}$ be a presheaf of sets. The *free abelian presheaf* $\mathbf{Z}_\mathcal{G}$ on $\mathcal{G}$ is the abelian presheaf defined by the rule

$$U \longmapsto \mathbf{Z}[\mathcal{G}(U)].$$

In the special case $\mathcal{G} = h_X$ of a representable presheaf associated to an object X of $\mathcal{C}$ we use the notation $\mathbf{Z}_X = \mathbf{Z}_{h_X}$. In other words

$$\mathbf{Z}_X(U) = \mathbf{Z}[Mor_\mathcal{C}(U, X)].$$

This construction is clearly functorial in the presheaf $\mathcal{G}$. In fact it is adjoint to the forgetful functor $PAb(\mathcal{C}) \to PSh(\mathcal{C})$. Here is the precise statement.

Lemma 16.4.2. *Let $\mathcal{C}$ be a category. Let $\mathcal{G}$, $\mathcal{F}$ be a presheaves of sets. Let $\mathcal{A}$ be an abelian presheaf. Let U be an object of $\mathcal{C}$. Then we have*

$$\begin{aligned} Mor_{PSh(\mathcal{C})}(h_U, \mathcal{F}) &= \mathcal{F}(U), \\ Mor_{PAb(\mathcal{C})}(\mathbf{Z}_\mathcal{G}, \mathcal{A}) &= Mor_{PSh(\mathcal{C})}(\mathcal{G}, \mathcal{A}), \\ Mor_{PAb(\mathcal{C})}(\mathbf{Z}_U, \mathcal{A}) &= \mathcal{A}(U). \end{aligned}$$

All of these equalities are functorial.

Proof. Omitted. $\square$

[1] In other chapters the notation $\mathbf{Z}[S]$ sometimes indicates the polynomial ring over $\mathbf{Z}$ on S.

Lemma 16.4.3. *Let $\mathcal{C}$ be a category. Let I be a set. For each $i \in I$ let $\mathcal{G}_i$ be a presheaf of sets. Then*

$$\mathbf{Z}_{\coprod_i \mathcal{G}_i} = \bigoplus\nolimits_{i \in I} \mathbf{Z}_{\mathcal{G}_i}$$

in $PAb(\mathcal{C})$.

Proof. Omitted. $\square$

16.5. Free abelian sheaves

Here is the notion of a free abelian sheaf on a sheaf of sets.

Definition 16.5.1. Let $\mathcal{C}$ be a site. Let $\mathcal{G}$ be a presheaf of sets. The *free abelian sheaf* $\mathbf{Z}_{\mathcal{G}}^\#$ on $\mathcal{G}$ is the abelian sheaf $\mathbf{Z}_{\mathcal{G}}^\#$ which is the sheafification of the abelian presheaf on $\mathcal{G}$. In the special case $\mathcal{G} = h_X$ of a representable presheaf associated to an object X of $\mathcal{C}$ we use the notation $\mathbf{Z}_X^\#$.

This construction is clearly functorial in the presheaf $\mathcal{G}$. In fact it provides an adjoint to the forgetful functor $Ab(\mathcal{C}) \to Sh(\mathcal{C})$. Here is the precise statement.

Lemma 16.5.2. *Let $\mathcal{C}$ be a site. Let $\mathcal{G}$, $\mathcal{F}$ be a sheaves of sets. Let $\mathcal{A}$ be an abelian sheaf. Let U be an object of $\mathcal{C}$. Then we have*

$$\begin{aligned} Mor_{Sh(\mathcal{C})}(h_U^\#, \mathcal{F}) &= \mathcal{F}(U), \\ Mor_{Ab(\mathcal{C})}(\mathbf{Z}_{\mathcal{G}}^\#, \mathcal{A}) &= Mor_{Sh(\mathcal{C})}(\mathcal{G}, \mathcal{A}), \\ Mor_{Ab(\mathcal{C})}(\mathbf{Z}_U^\#, \mathcal{A}) &= \mathcal{A}(U). \end{aligned}$$

All of these equalities are functorial.

Proof. Omitted. $\square$

Lemma 16.5.3. *Let $\mathcal{C}$ be a site. Let $\mathcal{G}$ be a presheaf of sets. Then $\mathbf{Z}_{\mathcal{G}}^\# = (\mathbf{Z}_{\mathcal{G}^\#})^\#$.*

Proof. Omitted. $\square$

16.6. Ringed sites

In this chapter we mainly work with sheaves of modules on a ringed site. Hence we need to define this notion.

Definition 16.6.1. Ringed sites.

(1) A *ringed site* is a pair $(\mathcal{C}, \mathcal{O})$ where $\mathcal{C}$ is a site and $\mathcal{O}$ is a sheaf of rings on $\mathcal{C}$. The sheaf $\mathcal{O}$ is called the *structure sheaf* of the ringed site.
(2) Let $(\mathcal{C}, \mathcal{O})$, $(\mathcal{C}', \mathcal{O}')$ be ringed sites. A *morphism of ringed sites*

$$(f, f^\sharp) : (\mathcal{C}, \mathcal{O}) \longrightarrow (\mathcal{C}', \mathcal{O}')$$

is given by a morphism of sites $f : \mathcal{C} \to \mathcal{C}'$ (see Sites, Definition 9.14.1) together with a map of sheaves of rings $f^\sharp : f^{-1}\mathcal{O}' \to \mathcal{O}$, which by adjunction is the same thing as a map of sheaves of rings $f^\sharp : \mathcal{O}' \to f_*\mathcal{O}$.
(3) Let $(f, f^\sharp) : (\mathcal{C}_1, \mathcal{O}_1) \to (\mathcal{C}_2, \mathcal{O}_2)$ and $(g, g^\sharp) : (\mathcal{C}_2, \mathcal{O}_2) \to (\mathcal{C}_3, \mathcal{O}_3)$ be morphisms of ringed sites. Then we define the *composition of morphisms of ringed sites* by the rule

$$(g, g^\sharp) \circ (f, f^\sharp) = (g \circ f, f^\sharp \circ g^\sharp).$$

Here we use composition of morphisms of sites defined in Sites, Definition 9.14.4 and $f^\sharp \circ g^\sharp$ indicates the morphism of sheaves of rings

$$\mathcal{O}_3 \xrightarrow{g^\sharp} g_*\mathcal{O}_2 \xrightarrow{g_*f^\sharp} g_*f_*\mathcal{O}_1 = (g \circ f)_*\mathcal{O}_1$$

16.7. Ringed topoi

A ringed topos is just a ringed site, except that the notion of a morphism of ringed topoi is different from the notion of a morphism of ringed sites.

Definition 16.7.1. Ringed topoi.

(1) A *ringed topos* is a pair $(Sh(\mathcal{C}), \mathcal{O})$ where $\mathcal{C}$ is a site and $\mathcal{O}$ is a sheaf of rings on $\mathcal{C}$. The sheaf $\mathcal{O}$ is called the *structure sheaf* of the ringed site.
(2) Let $(Sh(\mathcal{C}), \mathcal{O})$, $(Sh(\mathcal{C}'), \mathcal{O}')$ be ringed topoi. A *morphism of ringed topoi*

$$(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \longrightarrow (Sh(\mathcal{C}'), \mathcal{O}')$$

is given by a morphism of topoi $f : \mathcal{C} \to \mathcal{C}'$ (see Sites, Definition 9.15.1) together with a map of sheaves of rings $f^\sharp : f^{-1}\mathcal{O}' \to \mathcal{O}$, which by adjunction is the same thing as a map of sheaves of rings $f^\sharp : \mathcal{O}' \to f_*\mathcal{O}$.
(3) Let $(f, f^\sharp) : (Sh(\mathcal{C}_1), \mathcal{O}_1) \to (Sh(\mathcal{C}_2), \mathcal{O}_2)$ and $(g, g^\sharp) : (Sh(\mathcal{C}_2), \mathcal{O}_2) \to (Sh(\mathcal{C}_3), \mathcal{O}_3)$ be morphisms of ringed topoi. Then we define the *composition of morphisms of ringed topoi* by the rule

$$(g, g^\sharp) \circ (f, f^\sharp) = (g \circ f, f^\sharp \circ g^\sharp).$$

Here we use composition of morphisms of topoi defined in Sites, Definition 9.15.1 and $f^\sharp \circ g^\sharp$ indicates the morphism of sheaves of rings

$$\mathcal{O}_3 \xrightarrow{g^\sharp} g_*\mathcal{O}_2 \xrightarrow{g_*f^\sharp} g_*f_*\mathcal{O}_1 = (g \circ f)_*\mathcal{O}_1$$

Every morphism of ringed topoi is the composition of an equivalence of ringed topoi with a morphism of ringed topoi associated to a morpism of ringed sites. Here is the precise statement.

Lemma 16.7.2. *Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. There exists a factorization*

$$\begin{tikzcd}
(Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \arrow[r, "(f,f^\sharp)"'] \arrow[d, "(g,g^\sharp)"'] & (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D}) \arrow[d, "(e,e^\sharp)"] \\
(Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) \arrow[r, "(h,h^\sharp)"] & (Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'})
\end{tikzcd}$$

where

(1) *$g : Sh(\mathcal{C}) \to Sh(\mathcal{C}')$ is an equivalence of topoi induced by a special cocontinuous functor $\mathcal{C} \to \mathcal{C}'$ (see Sites, Definition 9.25.2),*
(2) *$e : Sh(\mathcal{D}) \to Sh(\mathcal{D}')$ is an equivalence of topoi induced by a special cocontinuous functor $\mathcal{D} \to \mathcal{D}'$ (see Sites, Definition 9.25.2),*
(3) *$\mathcal{O}_{\mathcal{C}'} = g_*\mathcal{O}_\mathcal{C}$ and $g^\sharp$ is the obvious map,*
(4) *$\mathcal{O}_{\mathcal{D}'} = e_*\mathcal{O}_\mathcal{D}$ and $e^\sharp$ is the obvious map,*
(5) *the sites $\mathcal{C}'$ and $\mathcal{D}'$ have final objects and fibre products (i.e., all finite limits),*

(6) *h is a morphism of sites induced by a continuous functor $u : \mathcal{D}' \to \mathcal{C}'$ which commutes with all finite limits (i.e., it satisfies the assumptions of Sites, Proposition 9.14.6), and*
(7) *given any set of sheaves $\mathcal{F}_i$ (resp. $\mathcal{G}_j$) on $\mathcal{C}$ (resp. $\mathcal{D}$) we may assume each of these is a representable sheaf on $\mathcal{C}'$ (resp. $\mathcal{D}'$).*

Moreover, if $(f, f^\sharp)$ is an equivalence of ringed topoi, then we can choose the diagram such that $\mathcal{C}' = \mathcal{D}'$, $\mathcal{O}_{\mathcal{C}'} = \mathcal{O}_{\mathcal{D}'}$ and $(h, h^\sharp)$ is the identiy.

Proof. This follows from Sites, Lemma 9.25.6, and Sites, Remarks 9.25.7 and 9.25.8. You just have to carry along the sheaves of rings. Some details omitted. $\square$

16.8. 2-morphisms of ringed topoi

This is a brief section concerning the notion of a 2-morphism of ringed topoi.

Definition 16.8.1. Let $f, g : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be two morphisms of ringed topoi. A *2-morphism from f to g* is given by a transformation of functors $t : f_* \to g_*$ such that

$$\begin{tikzcd} & \mathcal{O}_\mathcal{D} \arrow[dl, "f^\sharp"'] \arrow[dr, "g^\sharp"] & \\ f_*\mathcal{O}_\mathcal{C} \arrow[rr, "t"] & & g_*\mathcal{O}_\mathcal{C} \end{tikzcd}$$

is commutative.

Pictorially we sometimes represent t as follows:

$$\begin{tikzcd} (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow t"] & (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D}) \end{tikzcd}$$

As in Sites, Section 9.32 giving a 2-morphism $t : f_* \to g_*$ is equivalent to giving $t : g^{-1} \to f^{-1}$ (usually denoted by the same symbol) such that the diagram

$$\begin{tikzcd} f^{-1}\mathcal{O}_\mathcal{D} \arrow[dr, "f^\sharp"'] & & g^{-1}\mathcal{O}_\mathcal{D} \arrow[ll, "t"'] \arrow[dl, "g^\sharp"] \\ & \mathcal{O}_\mathcal{C} & \end{tikzcd}$$

is commutative. As in Sites, Section 9.32 the axioms of a strict 2-category hold with horizontal and vertical compositions defined as explained in loc. cit.

16.9. Presheaves of modules

Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings on $\mathcal{C}$. At this point we have not yet defined a presheaf of $\mathcal{O}$-modules. Thus we do so right now.

Definition 16.9.1. Let $\mathcal{C}$ be a category, and let $\mathcal{O}$ be a presheaf of rings on $\mathcal{C}$.

(1) A *presheaf of $\mathcal{O}$-modules* is given by an abelian presheaf $\mathcal{F}$ together with a map of presheaves of sets
$$\mathcal{O} \times \mathcal{F} \longrightarrow \mathcal{F}$$
such that for every object U of $\mathcal{C}$ the map $\mathcal{O}(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ defines the structure of an $\mathcal{O}(U)$-module structure on the abelian group $\mathcal{F}(U)$.

(2) A *morphism* $\varphi : \mathcal{F} \to \mathcal{G}$ *of presheaves of* $\mathcal{O}$*-modules* is a morphism of abelian presheaves $\varphi : \mathcal{F} \to \mathcal{G}$ such that the diagram

$$\begin{tikzcd} \mathcal{O} \times \mathcal{F} \arrow[r] \arrow[d, "\mathrm{id}\times\varphi"'] & \mathcal{F} \arrow[d, "\varphi"] \\ \mathcal{O} \times \mathcal{G} \arrow[r] & \mathcal{G} \end{tikzcd}$$

commutes.

(3) The set of $\mathcal{O}$-module morphisms as above is denoted $Hom_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$.

(4) The category of presheaves of $\mathcal{O}$-modules is denoted $PMod(\mathcal{O})$.

Suppose that $\mathcal{O}_1 \to \mathcal{O}_2$ is a morphism of presheaves of rings on the category $\mathcal{C}$. In this case, if $\mathcal{F}$ is a presheaf of $\mathcal{O}_2$-modules then we can think of $\mathcal{F}$ as a presheaf of $\mathcal{O}_1$-modules by using the composition

$$\mathcal{O}_1 \times \mathcal{F} \to \mathcal{O}_2 \times \mathcal{F} \to \mathcal{F}.$$

We sometimes denote this by $\mathcal{F}_{\mathcal{O}_1}$ to indicate the restriction of rings. We call this the *restriction of* $\mathcal{F}$. We obtain the restriction functor

$$PMod(\mathcal{O}_2) \longrightarrow PMod(\mathcal{O}_1)$$

On the other hand, given a presheaf of $\mathcal{O}_1$-modules $\mathcal{G}$ we can construct a presheaf of $\mathcal{O}_2$-modules $\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}$ by the rule

$$U \longmapsto \left(\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}\right)(U) = \mathcal{O}_2(U) \otimes_{\mathcal{O}_1(U)} \mathcal{G}(U)$$

where $U \in Ob(\mathcal{C})$, with obvious restriction mappings. The index p stands for ``presheaf" and not ``point". This presheaf is called the tensor product presheaf. We obtain the *change of rings* functor

$$PMod(\mathcal{O}_1) \longrightarrow PMod(\mathcal{O}_2)$$

Lemma 16.9.2. *With* $\mathcal{C}$*,* $\mathcal{O}_1 \to \mathcal{O}_2$*,* $\mathcal{F}$ *and* $\mathcal{G}$ *as above there exists a canonical bijection*

$$Hom_{\mathcal{O}_1}(\mathcal{G}, \mathcal{F}_{\mathcal{O}_1}) = Hom_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}, \mathcal{F})$$

In other words, the restriction and change of rings functors defined above are adjoint to each other.

Proof. This follows from the fact that for a ring map $A \to B$ the restriction functor and the change of ring functor are adjoint to each other. □

16.10. Sheaves of modules

Definition 16.10.1. Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$.

(1) A *sheaf of* $\mathcal{O}$*-modules* is a presheaf of $\mathcal{O}$-modules $\mathcal{F}$, see Definition 16.9.1, such that the underlying presheaf of abelian groups $\mathcal{F}$ is a sheaf.

(2) A *morphism of sheaves of* $\mathcal{O}$*-modules* is a morphism of presheaves of $\mathcal{O}$-modules.

(3) Given sheaves of $\mathcal{O}$-modules $\mathcal{F}$ and $\mathcal{G}$ we denote $Hom_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$ the set of morphism of sheaves of $\mathcal{O}$-modules.

(4) The category of sheaves of $\mathcal{O}$-modules is denoted $Mod(\mathcal{O})$.

This definition kind of makes sense even if $\mathcal{O}$ is just a presheaf of rings, allthough we do not know any examples where this is useful, and we will avoid using the terminology ``sheaves of $\mathcal{O}$-modules" in case $\mathcal{O}$ is not a sheaf of rings.

16.11. Sheafification of presheaves of modules

Lemma 16.11.1. *Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a presheaf of rings on $\mathcal{C}$ Let $\mathcal{F}$ be a presheaf $\mathcal{O}$-modules. Let $\mathcal{O}^\#$ be the sheafification of $\mathcal{O}$ as a presheaf of rings, see Sites, Section 9.38. Let $\mathcal{F}^\#$ be the sheafification of $\mathcal{F}$ as a presheaf of abelian groups. There exists a map of sheaves of sets*

$$\mathcal{O}^\# \times \mathcal{F}^\# \longrightarrow \mathcal{F}^\#$$

which makes the diagram

$$\begin{tikzcd} \mathcal{O} \times \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\ \mathcal{O}^\# \times \mathcal{F}^\# \arrow[r] & \mathcal{F}^\# \end{tikzcd}$$

commute and which makes $\mathcal{F}^\#$ into a sheaf of $\mathcal{O}^\#$-modules. In addition, if $\mathcal{G}$ is a sheaf of $\mathcal{O}^\#$-modules, then any morphism of presheaves of $\mathcal{O}$-modules $\mathcal{F} \to \mathcal{G}$ (into the restriction of $\mathcal{G}$ to a $\mathcal{O}$-module) factors uniquely as $\mathcal{F} \to \mathcal{F}^\# \to \mathcal{G}$ where $\mathcal{F}^\# \to \mathcal{G}$ is a morphism of $\mathcal{O}^\#$-modules.

Proof. Omitted. ☐

This actually means that the functor $i : Mod(\mathcal{O}^\#) \to PMod(\mathcal{O})$ (combining restriction and including sheaves into presheaves) and the sheafification functor of the lemma $^\# : PMod(\mathcal{O}) \to Mod(\mathcal{O}^\#)$ are adjoint. In a formula

$$Mor_{PMod(\mathcal{O})}(\mathcal{F}, i\mathcal{G}) = Mor_{Mod(\mathcal{O}^\#)}(\mathcal{F}^\#, \mathcal{G})$$

An important case happens when $\mathcal{O}$ is already a sheaf of rings. In this case the formula reads

$$Mor_{PMod(\mathcal{O})}(\mathcal{F}, i\mathcal{G}) = Mor_{Mod(\mathcal{O})}(\mathcal{F}^\#, \mathcal{G})$$

because $\mathcal{O} = \mathcal{O}^\#$ in this case.

Lemma 16.11.2. *Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a presheaf of rings on $\mathcal{C}$ The sheafification functor*

$$PMod(\mathcal{O}) \longrightarrow Mod(\mathcal{O}^\#), \quad \mathcal{F} \longmapsto \mathcal{F}^\#$$

is exact.

Proof. This is true because it holds for sheafification $PAb(\mathcal{C}) \to Ab(\mathcal{C})$. See the discussion in Section 16.3. ☐

Let $\mathcal{C}$ be a site. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a morphism of sheaves of rings on $\mathcal{C}$. In Section 16.9 we defined a restriction functor and a change of rings functor on presheaves of modules associated to this situation.

If $\mathcal{F}$ is a sheaf of $\mathcal{O}_2$-modules then the restriction $\mathcal{F}_{\mathcal{O}_1}$ of $\mathcal{F}$ is clearly a sheaf of $\mathcal{O}_1$-modules. We obtain the restriction functor

$$Mod(\mathcal{O}_2) \longrightarrow Mod(\mathcal{O}_1)$$

On the other hand, given a sheaf of $\mathcal{O}_1$-modules $\mathcal{G}$ the presheaf of $\mathcal{O}_2$-modules $\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}$ is in general not a sheaf. Hence we define the *tensor product sheaf* $\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G}$ by the formula

$$\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G} = (\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G})^\#$$

as the sheafification of our construction for presheaves. We obtain the *change of rings* functor

$$Mod(\mathcal{O}_1) \longrightarrow Mod(\mathcal{O}_2)$$

Lemma 16.11.3. *With X, $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{F}$ and $\mathcal{G}$ as above there exists a canonical bijection*

$$Hom_{\mathcal{O}_1}(\mathcal{G}, \mathcal{F}_{\mathcal{O}_1}) = Hom_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G}, \mathcal{F})$$

In other words, the restriction and change of rings functors are adjoint to each other.

Proof. This follows from Lemma 16.9.2 and the fact that $Hom_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G}, \mathcal{F}) = Hom_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}, \mathcal{F})$ because $\mathcal{F}$ is a sheaf. □

16.12. Morphisms of topoi and sheaves of modules

All of this material is completely straightforward. We formulate everything in the case of morphisms of topoi, but of course the results also hold in the case of morphisms of sites.

Lemma 16.12.1. *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. There is a natural map of sheaves of sets*

$$f_*\mathcal{O} \times f_*\mathcal{F} \longrightarrow f_*\mathcal{F}$$

which turns $f_*\mathcal{F}$ into a sheaf of $f_*\mathcal{O}$-modules. This construction is functorial in $\mathcal{F}$.

Proof. Denote $\mu : \mathcal{O} \times \mathcal{F} \to \mathcal{F}$ the multiplication map. Recall that f_* (on sheaves of sets) is left exact and hence commutes with products. Hence $f_*\mu$ is a map as indicated. This proves the lemma. □

Lemma 16.12.2. *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{D}$. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}$-modules. There is a natural map of sheaves of sets*

$$f^{-1}\mathcal{O} \times f^{-1}\mathcal{G} \longrightarrow f^{-1}\mathcal{G}$$

which turns $f^{-1}\mathcal{G}$ into a sheaf of $f^{-1}\mathcal{O}$-modules. This construction is functorial in $\mathcal{G}$.

Proof. Denote $\mu : \mathcal{O} \times \mathcal{G} \to \mathcal{G}$ the multiplication map. Recall that f^{-1} (on sheaves of sets) is exact and hence commutes with products. Hence $f^{-1}\mu$ is a map as indicated. This proves the lemma. □

Lemma 16.12.3. *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{D}$. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}$-modules. Let $\mathcal{F}$ be a sheaf of $f^{-1}\mathcal{O}$-modules. Then*

$$Mor_{Mod(f^{-1}\mathcal{O})}(f^{-1}\mathcal{G}, \mathcal{F}) = Mor_{Mod(\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}).$$

Here we use Lemmas 16.12.2 and 16.12.1, and we think of $f_*\mathcal{F}$ as an $\mathcal{O}$-module by restriction via $\mathcal{O} \to f_*f^{-1}\mathcal{O}$.

Proof. First we note that we have

$$Mor_{Ab(\mathcal{C})}(f^{-1}\mathcal{G}, \mathcal{F}) = Mor_{Ab(\mathcal{D})}(\mathcal{G}, f_*\mathcal{F}).$$

by Sites, Proposition 9.38.3. Suppose that $\alpha : f^{-1}\mathcal{G} \to \mathcal{F}$ and $\beta : \mathcal{G} \to f_*\mathcal{F}$ are morphisms of abelian sheaves which correspond via the formula above. We have to show that α is $f^{-1}\mathcal{O}$-linear if and only if β is $\mathcal{O}$-linear. For example, suppose α is $f^{-1}\mathcal{O}$-linear, then clearly $f_*\alpha$ is $f_*f^{-1}\mathcal{O}$-linear, and hence (as restriction is a functor) is $\mathcal{O}$-linear. Hence it suffices to prove that the adjunction map $\mathcal{G} \to f_*f^{-1}\mathcal{G}$ is $\mathcal{O}$-linear. Using that both f_* and f^{-1} commute with products (on sheaves of sets) this comes down to showing that

$$\begin{tikzcd} \mathcal{O} \times \mathcal{G} \arrow[r] \arrow[d] & f_*f^{-1}(\mathcal{O} \times \mathcal{G}) \arrow[d] \\ \mathcal{G} \arrow[r] & f_*f^{-1}\mathcal{G} \end{tikzcd}$$

is commutative. This holds because the adjunction mapping $\mathrm{id}_{Sh(\mathcal{D})} \to f_*f^{-1}$ is a transformation of functors. We omit the proof of the implication β linear $\Rightarrow \alpha$ linear. □

Lemma 16.12.4. *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. Let $\mathcal{G}$ be a sheaf of $f_*\mathcal{O}$-modules. Then*

$$Mor_{Mod(\mathcal{O})}(\mathcal{O} \otimes_{f^{-1}f_*\mathcal{O}} f^{-1}\mathcal{G}, \mathcal{F}) = Mor_{Mod(f_*\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}).$$

Here we use Lemmas 16.12.2 and 16.12.1, and we use the canonical map $f^{-1}f_*\mathcal{O} \to \mathcal{O}$ in the definition of the tensor product.

Proof. Note that we have

$$Mor_{Mod(\mathcal{O})}(\mathcal{O} \otimes_{f^{-1}f_*\mathcal{O}} f^{-1}\mathcal{G}, \mathcal{F}) = Mor_{Mod(f^{-1}f_*\mathcal{O})}(f^{-1}\mathcal{G}, \mathcal{F}_{f^{-1}f_*\mathcal{O}})$$

by Lemma 16.11.3. Hence the result follows from Lemma 16.12.3. □

16.13. Morphisms of ringed topoi and modules

We have now introduced enough notation so that we are able to define the pullback and pushforward of modules along a morphism of ringed topoi.

Definition 16.13.1. Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi or ringed sites.

(1) Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_\mathcal{C}$-modules. We define the *pushforward* of $\mathcal{F}$ as the sheaf of $\mathcal{O}_\mathcal{D}$-modules which as a sheaf of abelian groups equals $f_*\mathcal{F}$ and with module structure given by the restriction via $f^\sharp : \mathcal{O}_\mathcal{D} \to f_*\mathcal{O}_\mathcal{C}$ of the module structure

$$f_*\mathcal{O}_\mathcal{C} \times f_*\mathcal{F} \longrightarrow f_*\mathcal{F}$$

from Lemma 16.12.1.

(2) Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_\mathcal{D}$-modules. We define the *pullback* $f^*\mathcal{G}$ to be the sheaf of $\mathcal{O}_\mathcal{C}$-modules defined by the formula

$$f^*\mathcal{F} = \mathcal{O}_\mathcal{C} \otimes_{f^{-1}\mathcal{O}_\mathcal{D}} f^{-1}\mathcal{F}$$

where the ring map $f^{-1}\mathcal{O}_\mathcal{D} \to \mathcal{O}_\mathcal{C}$ is $f^\sharp$, and where the module structure is given by Lemma 16.12.2.

Thus we have defined functors

$$\begin{aligned} f_* : Mod(\mathcal{O}_\mathcal{C}) &\longrightarrow Mod(\mathcal{O}_\mathcal{D}) \\ f^* : Mod(\mathcal{O}_\mathcal{D}) &\longrightarrow Mod(\mathcal{O}_\mathcal{C}) \end{aligned}$$

The final result on these functors is that they are indeed adjoint as expected.

Lemma 16.13.2. *Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi or ringed sites. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_\mathcal{C}$-modules. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_\mathcal{D}$-modules. There is a canonical bijection*

$$Hom_{\mathcal{O}_\mathcal{C}}(f^*\mathcal{G}, \mathcal{F}) = Hom_{\mathcal{O}_\mathcal{D}}(\mathcal{G}, f_*\mathcal{F}).$$

In other words: the functor f^* is the left adjoint to f_*.

Proof. This follows from the work we did before:

$$\begin{aligned} Hom_{\mathcal{O}_\mathcal{C}}(f^*\mathcal{G}, \mathcal{F}) &= Mor_{Mod(\mathcal{O}_\mathcal{C})}(\mathcal{O}_\mathcal{C} \otimes_{f^{-1}\mathcal{O}_\mathcal{D}} f^{-1}\mathcal{G}, \mathcal{F}) \\ &= Mor_{Mod(f^{-1}\mathcal{O}_\mathcal{D})}(f^{-1}\mathcal{G}, \mathcal{F}_{f^{-1}\mathcal{O}_\mathcal{D}}) \\ &= Hom_{\mathcal{O}_\mathcal{D}}(\mathcal{G}, f_*\mathcal{F}). \end{aligned}$$

Here we use Lemmas 16.11.3 and 16.12.3. □

Lemma 16.13.3. *$(f, f^\sharp) : (Sh(\mathcal{C}_1), \mathcal{O}_1) \to (Sh(\mathcal{C}_2), \mathcal{O}_2)$ and $(g, g^\sharp) : (Sh(\mathcal{C}_2), \mathcal{O}_2) \to (Sh(\mathcal{C}_3), \mathcal{O}_3)$ be morphisms of ringed topoi. There are canonical isomorphisms of functors $(g \circ f)_* \cong g_* \circ f_*$ and $(g \circ f)^* \cong f^* \circ g^*$.*

Proof. This is clear from the definitions. □

16.14. The abelian category of sheaves of modules

Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. Let $\mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}$-modules, see Sheaves, Definition 6.10.1. Let $\varphi, \psi : \mathcal{F} \to \mathcal{G}$ be morphisms of sheaves of $\mathcal{O}_X$-modules. We define $\varphi + \psi : \mathcal{F} \to \mathcal{G}$ to be the sum of φ and ψ as morphisms of abelian sheaves. This is clearly again a map of $\mathcal{O}$-modules. It is also clear that composition of maps of $\mathcal{O}$-modules is bilinear with respect to this addition. Thus $Mod(\mathcal{O})$ is a pre-additive category, see Homology, Definition 10.3.1.

We will denote 0 the sheaf of $\mathcal{O}$-modules which has constant value $\{0\}$ for all objects U of $\mathcal{C}$. Clearly this is both a final and an initial object of $Mod(\mathcal{O})$. Given a morphism of $\mathcal{O}$-modules $\varphi : \mathcal{F} \to \mathcal{G}$ the following are equivalent: (a) φ is zero, (b) φ factors through 0, (c) φ is zero on sections over each opject U.

Moreover, given a pair $\mathcal{F}, \mathcal{G}$ of sheaves of $\mathcal{O}$-modules we may define the direct sum as

$$\mathcal{F} \oplus \mathcal{G} = \mathcal{F} \times \mathcal{G}$$

with obvious maps (i, j, p, q) as in Homology, Definition 10.3.5. Thus $Mod(\mathcal{O})$ is an additive category, see Homology, Definition 10.3.8.

Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of $\mathcal{O}$-modules. We may define $\text{Ker}(\varphi)$ to be the kernel of φ as a map of abelian sheaves. By Section 16.3 this is the subsheaf of $\mathcal{F}$ with sections

$$\text{Ker}(\varphi)(U) = \{s \in \mathcal{F}(U) \mid \varphi(s) = 0 \text{ in } \mathcal{G}(U)\}$$

for all objects U of $\mathcal{C}$. It is easy to see that this is indeed a kernel in the category of $\mathcal{O}$-modules. In other words, a morphism $\alpha : \mathcal{H} \to \mathcal{F}$ factors through $\text{Ker}(\varphi)$ if and only if $\varphi \circ \alpha = 0$.

Similarly, we define $\text{Coker}(\varphi)$ as the cokernel of φ as a map of abelian sheaves. There is a unique multiplication map

$$\mathcal{O} \times \text{Coker}(\varphi) \longrightarrow \text{Coker}(\varphi)$$

such that the map $\mathcal{G} \to \text{Coker}(\varphi)$ becomes a morphism of $\mathcal{O}$-modules (verification omitted). The map $\mathcal{G} \to \text{Coker}(\varphi)$ is surjective (as a map of sheaves of sets, see Section 16.3). To show that $\mathrm{Coker}(\varphi)$ is a cokernel in $Mod(\mathcal{O})$, note that if $\beta : \mathcal{G} \to \mathcal{H}$ is a morphism of $\mathcal{O}$-modules such that $\beta \circ \varphi$ is zero, then you get for every object U of $\mathcal{C}$ a map induced by β from $\mathcal{G}(U)/\varphi(\mathcal{F}(U))$ into $\mathcal{H}(U)$. By the universal property of sheafification (see Sheaves, Lemma 6.20.1) we obtain a canonical map $\mathrm{Coker}(\varphi) \to \mathcal{H}$ such that the original β is equal to the composition $\mathcal{G} \to \mathrm{Coker}(\varphi) \to \mathcal{H}$. The morphism $\mathrm{Coker}(\varphi) \to \mathcal{H}$ is unique because of the surjectivity mentioned above.

Lemma 16.14.1. *Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. The category $Mod(\mathcal{O})$ is an abelian category. The forgetful functor $Mod(\mathcal{O}) \to Ab(\mathcal{C})$ is exact, hence kernels, cokernels and exactness of $\mathcal{O}$-modules, correspond to the corresponding notions for abelian sheaves.*

Proof. Above we have seen that $Mod(\mathcal{O})$ is an additive category, with kernels and cokernels and that $Mod(\mathcal{O}) \to Ab(\mathcal{C})$ preserves kernels and cokernels. By Homology, Definition 10.3.12 we have to show that image and coimage agree. This is clear because it is true in $Ab(\mathcal{C})$. The lemma follows. ☐

Lemma 16.14.2. *Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. All limits and colimits exist in $Mod(\mathcal{O})$, and the forgetful functor $Mod(\mathcal{O}) \to Ab(\mathcal{C})$ commutes with them.*

Proof. Let $\mathcal{I} \to Mod(\mathcal{C})$, $i \mapsto \mathcal{F}_i$ be a diagram. Let $lim_i \mathcal{F}_i$ be the limit of the diagram in $Ab(\mathcal{C})$. By the description of this limit in Lemma 16.3.2 we see immediately that there exists a multiplication

$$\mathcal{O} \times lim_i \mathcal{F}_i \longrightarrow lim_i \mathcal{F}_i$$

which turns $lim_i \mathcal{F}_i$ into a sheaf of $\mathcal{O}$-modules. It is easy to see that this is the limit of the diagram in $Mod(\mathcal{C})$. Let $colim_i \mathcal{F}_i$ be the colimit of the diagram in $PAb(\mathcal{C})$. By the description of this colimit in the proof of Lemma 16.2.1 we see immediately that there exists a multiplication

$$\mathcal{O} \times colim_i \mathcal{F}_i \longrightarrow colim_i \mathcal{F}_i$$

which turns $colim_i \mathcal{F}_i$ into a presheaf of $\mathcal{O}$-modules. Applying sheafification we get a sheaf of $\mathcal{O}$-modules $(colim_i \mathcal{F}_i)^\#$, see Lemma 16.11.1. It is easy to see that $(colim_i \mathcal{F}_i)^\#$ is the colimit of the diagram in $Mod(\mathcal{C})$, and by Lemma 16.3.2 forgetting the $\mathcal{O}$-module structure is the colimit in $Ab(\mathcal{C})$. ☐

The existence of limits and colimits allows us to consider exactness properties of functors defined on the category of $\mathcal{O}$-modules in terms of limits and colimits, as in Categories, Section 4.21. See Homology, Lemma 10.5.1 for a description of exactness properties in terms of short exact sequences.

Lemma 16.14.3. *Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi.*

(1) *The functor f_* is left exact. In fact it commutes with all limits.*
(2) *The functor f^* is right exact. In fact it commutes with all colimits.*

Proof. This is true because (f^*, f_*) is an adjoint pair of functors, see Lemma 16.13.2. See Categories, Section 4.22. ☐

Lemma 16.14.4. *Let $\mathcal{C}$ be a site with enough points. In this case exactness of a sequence of abelian sheaves may be checked on stalks.*

Proof. This is immediate from Sites, Lemma 9.34.2. ☐

16.15. Exactness of pushforward

Some technical lemmas concerning exactness properties of pushforward.

Lemma 16.15.1. *Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. The following are equivalent:*

(1) *$f^{-1}f_*\mathcal{F} \to \mathcal{F}$ is surjective for all $\mathcal{F}$ in $Ab(\mathcal{C})$, and*
(2) *$f_* : Ab(\mathcal{C}) \to Ab(\mathcal{D})$ reflects surjections.*

In this case the functor $f_* : Ab(\mathcal{C}) \to Ab(\mathcal{D})$ is faithful.

Proof. Assume (1). Suppose that $a : \mathcal{F} \to \mathcal{F}'$ is a map of abelian sheaves on $\mathcal{C}$ such that f_*a is surjective. As f^{-1} is exact this implies that $f^{-1}f_*a : f^{-1}f_*\mathcal{F} \to f^{-1}f_*\mathcal{F}'$ is surjective. Combined with (1) this implies that a is surjective. This means that (2) holds.

Assume (2). Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}$. We have to show that the map $f^{-1}f_*\mathcal{F} \to \mathcal{F}$ is surjective. By (2) it suffices to show that $f_*f^{-1}f_*\mathcal{F} \to f_*\mathcal{F}$ is surjective. And this is true because there is a canonical map $f_*\mathcal{F} \to f_*f^{-1}f_*\mathcal{F}$ which is a one-sided inverse.

We omit the proof of the final assertion. □

Lemma 16.15.2. *Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. Assume at least one of the following properties holds*

(1) *f_* transforms surjections of sheaves of sets into surjections,*
(2) *f_* transforms surjections of abelian sheaves into surjections,*
(3) *f_* commutes with coequalizers on sheaves of sets,*
(4) *f_* commutes with pushouts on sheaves of sets,*

Then $f_* : Ab(\mathcal{C}) \to Ab(\mathcal{D})$ is exact.

Proof. Since $f_* : Ab(\mathcal{C}) \to Ab(\mathcal{D})$ is a right adjoint we already know that it transforms a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ of abelian sheaves on $\mathcal{C}$ into an exact sequence

$$0 \to f_*\mathcal{F}_1 \to f_*\mathcal{F}_2 \to f_*\mathcal{F}_3$$

see Categories, Sections 4.21 and 4.22 and Homology, Section 10.5. Hence it suffices to prove that the map $f_*\mathcal{F}_2 \to f_*\mathcal{F}_3$ is surjective. If (1), (2) holds, then this is clear from the definitions. By Sites, Lemma 9.36.1 we see that (4) formally implies (1), hence in this case we are done also. Assume (3). Then $\mathcal{F}_3$ is the coequalizer of two maps $\mathcal{F}_1 \to \mathcal{F}_2$ (the zero map and the given map). Hence also $f_*\mathcal{F}_3$ is the coequalizer of two maps $f_*\mathcal{F}_1 \to f_*\mathcal{F}_2$. In particular we see that $f_*\mathcal{F}_2 \to f_*\mathcal{F}_3$ is surjective. □

Lemma 16.15.3. *Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites associated to the continuous functor $u : \mathcal{C} \to \mathcal{D}$. Assume u is almost cocontinuous. Then*

(1) *$f_* : Ab(\mathcal{D}) \to Ab(\mathcal{C})$ is exact.*
(2) *if $f^\sharp : f^{-1}\mathcal{O}_\mathcal{C} \to \mathcal{O}_\mathcal{D}$ is given so that f becomes a morphism of ringed sites, then $f_* : Mod(\mathcal{O}_\mathcal{D}) \to Mod(\mathcal{O}_\mathcal{C})$ is exact.*

Proof. Part (2) follows from part (1) by Lemma 16.14.2. Part (1) follows from Sites, Lemmas 9.37.6 and 9.36.1. □

16.16. Exactness of lower shriek

Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between sites. Assume that

(a) u is cocontinuous, and
(b) u is continuous.

Let $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be the morphism of topoi associated with u, see Sites, Lemma 9.19.1. Recall that $g^{-1} = u^p$, i.e., g^{-1} is given by the simple formula $(g^{-1}\mathcal{G})(U) = \mathcal{G}(u(U))$, see Sites, Lemma 9.19.5. We would like to show that $g^{-1} : Ab(\mathcal{D}) \to Ab(\mathcal{C})$ has a left adjoint $g_!$. By Sites, Lemma 9.19.5 the functor $g_!^{Sh} = (u_p\ )^\#$ is a left adjoint on sheaves of sets. Moreover, we know that $g_!^{Sh}\mathcal{F}$ is the sheaf associated to the presheaf

$$V \longmapsto colim_{V \to u(U)} \mathcal{F}(U)$$

where the colimit is over $(\mathcal{I}_V^u)^{opp}$ and is taken in the category of sets. Hence the following definition is natural.

Definition 16.16.1. With $u : \mathcal{C} \to \mathcal{D}$ satisfying (a), (b) above. For $\mathcal{F} \in PAb(\mathcal{C})$ we define $g_{p!}\mathcal{F}$ as the presheaf

$$V \longmapsto colim_{V \to u(U)} \mathcal{F}(U)$$

with colimits over $(\mathcal{I}_V^u)^{opp}$ taken in Ab. For $\mathcal{F} \in PAb(\mathcal{C})$ we set $g_!\mathcal{F} = (g_{p!}\mathcal{F})^\#$.

The reason for being so explicit with this is that the functors the functors $g_!^{Sh}$ and $g_!$ are different. Whenever we use both we have to be careful to make the distinction clear.

Lemma 16.16.2. *The functor $g_{p!}$ is a left adjoint to the functor u^p. The functor $g_!$ is a left adjoint to the functor g^{-1}. In other words the formulas*

$$Mor_{PAb(\mathcal{C})}(\mathcal{F}, u^p\mathcal{G}) = Mor_{PAb(\mathcal{D})}(g_{p!}\mathcal{F}, \mathcal{G}),$$
$$Mor_{Ab(\mathcal{C})}(\mathcal{F}, g^{-1}\mathcal{G}) = Mor_{Ab(\mathcal{D})}(g_!\mathcal{F}, \mathcal{G})$$

hold bifunctorially in $\mathcal{F}$ and $\mathcal{G}$.

Proof. The second formula follows formally from the first, since if $\mathcal{F}$ and $\mathcal{G}$ are abelian sheaves then

$$\begin{aligned} Mor_{Ab(\mathcal{C})}(\mathcal{F}, g^{-1}\mathcal{G}) &= Mor_{PAb(\mathcal{D})}(g_{p!}\mathcal{F}, \mathcal{G}) \\ &= Mor_{Ab(\mathcal{D})}(g_!\mathcal{F}, \mathcal{G}) \end{aligned}$$

by the universal property of sheafification.

To prove the first formula, let $\mathcal{F}$, $\mathcal{G}$ be abelian presheaves. To prove the lemma we will construct maps from the group on the left to the group on the right and omit the verification that these are mutually inverse.

Note that there is a canonical map of abelian presheaves $\mathcal{F} \to u^p g_{p!}\mathcal{F}$ which on sections over U is the natural map $\mathcal{F}(U) \to colim_{u(U) \to u(U')} \mathcal{F}(U')$, see Sites, Lemma 9.5.3. Given a map $\alpha : g_{p!}\mathcal{F} \to \mathcal{G}$ we get $u^p\alpha : u^p g_{p!}\mathcal{F} \to u^p\mathcal{G}$. which we can precompose by the map $\mathcal{F} \to u^p g_{p!}\mathcal{F}$.

Note that there is a canonical map of abelian presheaves $g_{p!}u^p\mathcal{G} \to \mathcal{G}$ which on sections over V is the natural map $colim_{V \to u(U)} \mathcal{G}(u(U)) \to \mathcal{G}(V)$. It maps a section $s \in u(U)$ in the summand corresponding to $t : V \to u(U)$ to $t^*s \in \mathcal{G}(V)$. Hence, given a map $\beta : \mathcal{F} \to u^p\mathcal{G}$ we get a map $g_{p!}\beta : g_{p!}\mathcal{F} \to g_{p!}u^p\mathcal{G}$ which we can postcompose with the map $g_{p!}u^p\mathcal{G} \to \mathcal{G}$ above. $\square$

Lemma 16.16.3. *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that*

(a) *u is cocontinuous,*
(b) *u is continuous, and*
(c) *fibre products and equalizers exist in $\mathcal{C}$ and u commutes with them.*

In this case the functor $g_! : Ab(\mathcal{C}) \to Ab(\mathcal{D})$ is exact.

Proof. Compare with Sites, Lemma 9.19.6. Assume (a), (b), and (c). We already know that $g_!$ is right exact as it is a left adjoint, see Categories, Lemma 4.22.3 and Homology, Section 10.5. We have $g_! = (g_{p!}\ )^\#$. We have to show that $g_!$ transforms injective maps of abelian sheaves into injective maps of abelian presheaves. Recall that sheafification of abelian presheaves is exact, see Lemma 16.3.2. Thus it suffices to show that $g_{p!}$ transforms injective maps of abelian presheaves into injective maps of abelian presheaves. To do this it suffices that colimits over the categories $(\mathcal{I}_V^u)^{opp}$ of Sites, Section 9.5 transform injective maps between diagrams into injections. This follows from Sites, Lemma 9.5.1 and Algebra, Lemma 7.8.11. □

Lemma 16.16.4. *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that*

(a) *u is cocontinuous,*
(b) *u is continuous, and*
(c) *u is fully faithful.*

For g and $g_!$ as above the canonical map $\mathcal{F} \to g^{-1}g_!\mathcal{F}$ is an isomorphism for all abelian sheaves $\mathcal{F}$ on $\mathcal{C}$.

Proof. Pick $U \in Ob(\mathcal{C})$. We will show that $g^{-1}g_1\mathcal{F}(U) = \mathcal{F}(U)$. First, note that $g^{-1}g_1\mathcal{F}(U) = g_!\mathcal{F}(u(U))$. Hence it suffices to show that $g_!\mathcal{F}(u(U)) = \mathcal{F}(U)$. We know that $g_!\mathcal{F}$ is the (abelian) sheaf associated to the presheaf $g_{p!}\mathcal{F}$ which is defined by the rule

$$V \longmapsto colim_{V \to u(U')}\, \mathcal{F}(U')$$

with colimit taken in Ab. If $V = u(U)$, then, as u is fully faithful this colimit is over $U \to U'$. Hence we conclude that $g_{p!}\mathcal{F}(u(U) = \mathcal{F}(U)$. Since u is cocontinuous and continuous any covering of $u(U)$ in $\mathcal{D}$ can be refined by a covering (!) $\{u(U_i) \to u(U)\}$ of $\mathcal{D}$ where $\{U_i \to U\}$ is a covering in $\mathcal{C}$. This implies that $(g_{p!}\mathcal{F})^+(u(U)) = \mathcal{F}(U)$ also, since in the colimit defining the value of $(g_{p!}\mathcal{F})^+$ on $u(U)$ we may restrict to the cofinal system of coverings $\{u(U_i) \to u(U)\}$ as above. Hence we see that $(g_{p!}\mathcal{F})^+(u(U)) = \mathcal{F}(U)$ for all objects U of $\mathcal{C}$ as well. Repeating this argument one more time gives the equality $(g_{p!}\mathcal{F})^\#(u(U)) = \mathcal{F}(U)$ for all objects U of $\mathcal{C}$. This produces the desired equality $g^{-1}g_!\mathcal{F} = \mathcal{F}$. □

Remark 16.16.5. In general the functor $g_!$ cannot be extended to categories of modules in case g is (part of) a morphism of ringed topoi. Namely, given any ring map $A \to B$ the functor $M \mapsto B \otimes_A M$ has a right adjoint (restriction) but not in general a left adjoint (because its existence would imply that $A \to B$ is flat). We will see in Section 16.19 below that it is possible to define $j_!$ on sheaves of modules in the case of a localization of sites. We will discuss this in greater generality in Section 16.35 below.

16.17. Global types of modules

Definition 16.17.1. Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules.

(1) We say $\mathcal{F}$ is a *free $\mathcal{O}$-module* if $\mathcal{F}$ is isomorphic as an $\mathcal{O}$-module to a sheaf of the form $\bigoplus_{i \in I} \mathcal{O}$.

(2) We say $\mathcal{F}$ is *finite free* if $\mathcal{F}$ is isomorphic as an $\mathcal{O}$-module to a sheaf of the form $\bigoplus_{i\in I}\mathcal{O}$ with a finite index set I.
(3) We say $\mathcal{F}$ is *generated by global sections* if there exists a surjection

$$\bigoplus\nolimits_{i\in I}\mathcal{O} \longrightarrow \mathcal{F}$$

from a free $\mathcal{O}$-module onto $\mathcal{F}$.
(4) We say $\mathcal{F}$ is *generated by finitely many global sections* if there exists a surjection

$$\bigoplus\nolimits_{i\in I}\mathcal{O} \longrightarrow \mathcal{F}$$

with I finite.
(5) We say $\mathcal{F}$ has a *global presentation* if there exists an exact sequence

$$\bigoplus\nolimits_{j\in J}\mathcal{O} \longrightarrow \bigoplus\nolimits_{i\in I}\mathcal{O} \longrightarrow \mathcal{F}$$

of $\mathcal{O}$-modules.
(6) We say $\mathcal{F}$ has a *global finite presentation* if there exists an exact sequence

$$\bigoplus\nolimits_{j\in J}\mathcal{O} \longrightarrow \bigoplus\nolimits_{i\in I}\mathcal{O} \longrightarrow \mathcal{F}$$

of $\mathcal{O}$-modules with I and J finite sets.

Note that for any set I the direct sum $\bigoplus_{i\in I}\mathcal{O}$ exists (Lemma 16.14.2) and is the sheafification of the presheaf $U \mapsto \bigoplus_{i\in I}\mathcal{O}(U)$. This module is called the *free $\mathcal{O}$-module on the set I*.

Lemma 16.17.2. *Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let $\mathcal{F}$ be an $\mathcal{O}_\mathcal{D}$-module.*

(1) *If $\mathcal{F}$ is free then $f^*\mathcal{F}$ is free.*
(2) *If $\mathcal{F}$ is finite free then $f^*\mathcal{F}$ is finite free.*
(3) *If $\mathcal{F}$ is generated by global sections then $f^*\mathcal{F}$ is generated by global sections.*
(4) *If $\mathcal{F}$ is generated by finitely many global sections then $f^*\mathcal{F}$ is generated by finitely many global sections.*
(5) *If $\mathcal{F}$ has a global presentation then $f^*\mathcal{F}$ has a global presentation.*
(6) *If $\mathcal{F}$ has a finite global presentation then $f^*\mathcal{F}$ has a finite global presentation.*

Proof. This is true because f^* commutes with arbitrary colimits (Lemma 16.14.3) and $f^*\mathcal{O}_\mathcal{D} = \mathcal{O}_\mathcal{C}$. □

16.18. Intrinsic properties of modules

Let $\mathcal{P}$ be a property of sheaves of modules on ringed topoi. We say $\mathcal{P}$ is an *intrinsic property* if we have $\mathcal{P}(\mathcal{F}) \Leftrightarrow \mathcal{P}(f^*\mathcal{F})$ whenever $(f, f^\sharp) : (Sh(\mathcal{C}'), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ is an equivalence of ringed topoi. For example, the property of being free is intrinsic. Indeed, the free $\mathcal{O}$-module on the set I is characterized by the property that

$$Mor_{Mod(\mathcal{O})}(\bigoplus\nolimits_{i\in I}\mathcal{O}, \mathcal{F}) = \prod\nolimits_{i\in I} Mor_{Sh(\mathcal{C})}(\{*\}, \mathcal{F})$$

for a variable $\mathcal{F}$ in $Mod(\mathcal{O})$. Alternatively, we can also use Lemma 16.17.2 to see that being free is intrinsic. In fact, each of the properties defined in Definition 16.17.1 is intrinsic for the same reason. How will we go about defining other intrinsic properties of $\mathcal{O}$-modules?

The upshot of Lemma 16.7.2 is the following: Suppose you want to define an intrinsic property $\mathcal{P}$ of an $\mathcal{O}$-module on a topos. Then you can proceed as follows:

(1) Given any site $\mathcal{C}$, any sheaf of rings $\mathcal{O}$ on $\mathcal{C}$ and any $\mathcal{O}$-module $\mathcal{F}$ define the corresponding property $\mathcal{P}(\mathcal{C}, \mathcal{O}, \mathcal{F})$.
(2) For any pair of sites $\mathcal{C}, \mathcal{C}'$, any special cocontinuous functor $u : \mathcal{C} \to \mathcal{C}'$, any sheaf of rings $\mathcal{O}$ on $\mathcal{C}$ any $\mathcal{O}$-module $\mathcal{F}$, show that

$$\mathcal{P}(\mathcal{C}, \mathcal{O}, \mathcal{F}) \Leftrightarrow \mathcal{P}(\mathcal{C}', g_*\mathcal{O}, g_*\mathcal{F})$$

where $g : Sh(\mathcal{C}) \to Sh(\mathcal{C}')$ is the equivalence of topoi associated to u.

In this case, given any ringed topos $(Sh(\mathcal{C}), \mathcal{O})$ and any sheaf of $\mathcal{O}$-modules $\mathcal{F}$ we simply say that $\mathcal{F}$ has property $\mathcal{P}$ if $\mathcal{P}(\mathcal{C}, \mathcal{O}, \mathcal{F})$ is true. And Lemma 16.7.2 combined with (2) above garantees that this is well defined.

Moreover, the same Lemma 16.7.2 also garantees that if in addition

(3) For any morphism of ringed sites $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ such that f is given by a functor $u : \mathcal{D} \to \mathcal{C}$ satisfying the assumptions of Sites, Proposition 9.14.6, and any $\mathcal{O}_\mathcal{D}$-module $\mathcal{G}$ we have

$$\mathcal{P}(\mathcal{D}, \mathcal{O}_\mathcal{D}, \mathcal{F}) \Rightarrow \mathcal{P}(\mathcal{C}, \mathcal{O}_\mathcal{C}, f^*\mathcal{F})$$

then it is true that $\mathcal{P}$ is preserved under pullback of modules w.r.t. arbitrary morphisms of ringed topoi.

We will use this method in the following sections to see that: locally free, locally generated by sections, finite type, finite presentation, quasi-coherent, and coherent are intrinsic properties of modules.

Perhaps a more satisfying method would be to find an intrinsic definition of these notions, rather than the laborious process sketched here. On the other hand, in many geometric situations where we want to apply these definitions we are given a definite ringed site, and a definite sheaf of modules, and it is nice to have a definition already adapted to this language.

16.19. Localization of ringed sites

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $U \in Ob(\mathcal{C})$. We explain the counterparts of the results in Sites, Section 9.21 in this setting.

Denote $\mathcal{O}_U = j_U^{-1}\mathcal{O}$ the restriction of $\mathcal{O}$ to the site $\mathcal{C}/U$. It is described by the simple rule $\mathcal{O}_U(V/U) = \mathcal{O}(V)$. With this notation the localization morphism j_U becomes a morphism of ringed topoi

$$(j_U, j_U^\sharp) : (Sh(\mathcal{C}/U), \mathcal{O}_U) \longrightarrow (Sh(\mathcal{C}), \mathcal{O})$$

namely, we take $j_U^\sharp : j_U^{-1}\mathcal{O} \to \mathcal{O}_U$ the identity map. Moreover, we obtain the following descriptions for pushforward and pullback of modules.

Definition 16.19.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $U \in Ob(\mathcal{C})$.

(1) The ringed site $(\mathcal{C}/U, \mathcal{O}_U)$ is called the *localization of the ringed site* $(\mathcal{C}, \mathcal{O})$ *at the object* U.
(2) The morphism of ringed topoi $(j_U, j_U^\sharp) : (Sh(\mathcal{C}/U), \mathcal{O}_U) \to (Sh(\mathcal{C}), \mathcal{O})$ is called the *localization morphism*.
(3) The functor $j_{U*} : Mod(\mathcal{O}_U) \to Mod(\mathcal{O})$ is called the *direct image functor*.
(4) For a sheaf of $\mathcal{O}$-modules $\mathcal{F}$ on $\mathcal{C}$ the sheaf $j_U^*\mathcal{F}$ is called the *restriction of* $\mathcal{F}$ *to* $\mathcal{C}/U$. We will sometimes denote it by $\mathcal{F}|_{\mathcal{C}/U}$ or even $\mathcal{F}|_U$. It is described by the simple rule $j_U^*(\mathcal{F})(X/U) = \mathcal{F}(X)$.

(5) The left adjoint $j_{U!} : Mod(\mathcal{O}_U) \to Mod(\mathcal{O})$ of restriction is called *extension by zero*. It exists and is exact by Lemmas 16.19.2 and 16.19.3.

As in the topological case, see Sheaves, Section 6.31, the extension by zero $j_{U!}$ functor is different from extension by the empty set $j_{U!}$ defined on sheaves of sets. Here is the lemma defining extension by zero.

Lemma 16.19.2. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $U \in Ob(\mathcal{C})$. The restriction functor $j_U^* : Mod(\mathcal{O}) \to Mod(\mathcal{O}_U)$ has a left adjoint $j_{U!} : Mod(\mathcal{O}_U) \to Mod(\mathcal{O})$. So*

$$Mor_{Mod(\mathcal{O}_U)}(\mathcal{G}, j_U^*\mathcal{F}) = Mor_{Mod(\mathcal{O})}(j_{U!}\mathcal{G}, \mathcal{F})$$

for $\mathcal{F} \in Ob(Mod(\mathcal{O}))$ and $\mathcal{G} \in Ob(Mod(\mathcal{O}_U))$. Moreover, the extension by zero $j_{U!}\mathcal{G}$ of $\mathcal{G}$ is the sheaf associated to the presheaf

$$V \longmapsto \bigoplus\nolimits_{\varphi \in Mor_{\mathcal{C}}(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U)$$

with obvious restriction mappings and an obvious $\mathcal{O}$-module structure.

Proof. The $\mathcal{O}$-module structure on the presheaf is defined as follows. If $f \in \mathcal{O}(V)$ and $s \in \mathcal{G}(V \xrightarrow{\varphi} U)$, then we define $f \cdot s = fs$ where $f \in \mathcal{O}_U(\varphi : V \to U) = \mathcal{O}(V)$ (because $\mathcal{O}_U$ is the restriction of $\mathcal{O}$ to $\mathcal{C}/U$).

Similarly, let $\alpha : \mathcal{G} \to \mathcal{F}|_U$ be a morphism of $\mathcal{O}_U$-modules. In this case we can define a map from the presheaf of the lemma into $\mathcal{F}$ by mapping

$$\bigoplus\nolimits_{\varphi \in Mor_{\mathcal{C}}(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U) \longrightarrow \mathcal{F}(V)$$

by the rule that $s \in \mathcal{G}(V \xrightarrow{\varphi} U)$ maps to $\alpha(s) \in \mathcal{F}(V)$. It is clear that this is $\mathcal{O}$-linear, and hence induces a morphism of $\mathcal{O}$-modules $\alpha' : j_{U!}\mathcal{G} \to \mathcal{F}$ by the properties of sheafification of modules (Lemma 16.11.1).

Conversely, let $\beta : j_{U!}\mathcal{G} \to \mathcal{F}$ by a map of $\mathcal{O}$-modules. Recall from Sites, Section 9.21 that there exists an extension by the empty set $j_{U!}^{Sh} : Sh(\mathcal{C}/U) \to Sh(\mathcal{C})$ on sheaves of sets which is left adjoint to j_U^{-1}. Moreover, $j_{U!}^{Sh}\mathcal{G}$ is the sheaf associated to the presheaf

$$V \longmapsto \coprod\nolimits_{\varphi \in Mor_{\mathcal{C}}(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U)$$

Hence there is a natural map $j_{U!}^{Sh}\mathcal{G} \to j_{U!}\mathcal{G}$ of sheaves of sets. Hence precomposing β by this map we get a map of sheaves of sets $j_{U!}^{Sh}\mathcal{G} \to \mathcal{F}$ which by adjunction corresponds to a map of sheaves of sets $\beta' : \mathcal{G} \to \mathcal{F}|_U$. We claim that β' is $\mathcal{O}_U$-linear. Namely, suppose that $\varphi : V \to U$ is an object of $\mathcal{C}/U$ and that $s, s' \in \mathcal{G}(\varphi : V \to U)$, and $f \in \mathcal{O}(V) = \mathcal{O}_U(\varphi : V \to U)$. Then by the discussion above we see that $\beta'(s + s')$, resp. $\beta'(fs)$ in $\mathcal{F}|_U(\varphi : V \to U)$ correspond to $\beta(s + s')$, resp. $\beta(fs)$ in $\mathcal{F}(V)$. Since β is a homomorphism we conclude.

To conclude the proof of the lemma we have to show that the constructions $\alpha \mapsto \alpha'$ and $\beta \mapsto \beta'$ are mutually inverse. We omit the verifications. $\square$

Lemma 16.19.3. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $U \in Ob(\mathcal{C})$. The functor $j_{U!} : Mod(\mathcal{O}_U) \to Mod(\mathcal{O})$ is exact.*

Proof. Since $j_{U!}$ is a left adjoint to j_U^* we see that it is right exact (see Categories, Lemma 4.22.3 and Homology, Section 10.5). Hence it suffices to show that if $\mathcal{G}_1 \to \mathcal{G}_2$ is an injective map of $\mathcal{O}_U$-modules, then $j_{U!}\mathcal{G}_1 \to j_{U!}\mathcal{G}_2$ is injective. The map on sections of presheaves over an object V (as in Lemma 16.19.2) is the map

$$\bigoplus\nolimits_{\varphi \in Mor_{\mathcal{C}}(V,U)} \mathcal{G}_1(V \xrightarrow{\varphi} U) \longrightarrow \bigoplus\nolimits_{\varphi \in Mor_{\mathcal{C}}(V,U)} \mathcal{G}_2(V \xrightarrow{\varphi} U)$$

which is injective by assumption. Since sheafification is exact by Lemma 16.11.2 we conclude $j_{U!}\mathcal{G}_1 \to j_{U!}\mathcal{G}_2$ is injective and we win. □

Lemma 16.19.4. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $f : V \to U$ be a morphism of $\mathcal{C}$. Then there exists a commutative diagram*

$$\begin{tikzcd} (Sh(\mathcal{C}/V), \mathcal{O}_V) \arrow[rr, "{(j,j^\sharp)}"'] \arrow[dr, "{(j_V, j_V^\sharp)}"'] & & (Sh(\mathcal{C}/U), \mathcal{O}_U) \arrow[dl, "{(j_U, j_U^\sharp)}"] \\ & (Sh(\mathcal{C}), \mathcal{O}) & \end{tikzcd}$$

of ringed topoi. Here $(j, j^\sharp)$ is the localization morphism associated to the object U/V of the ringed site $(\mathcal{C}/V, \mathcal{O}_V)$.

Proof. The only thing to check is that $j_V^\sharp = j^\sharp \circ j^{-1}(j_U^\sharp)$, since everything else follows directly from Sites, Lemma 9.21.7 and Sites, Equation (9.21.7.1). We omit the verification of the equality. □

Remark 16.19.5. Localization and presheaves of modules; see Sites, Remark 9.21.9. Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings. Let U be an object of $\mathcal{C}$. Strictly speaking the functors j_U^*, j_{U*} and $j_{U!}$ have not been defined for presheaves of $\mathcal{O}$-modules. But of course, we can think of a presheaf as a sheaf for the chaotic topology on $\mathcal{C}$ (see Sites, Examples 9.6.6). Hence we also obtain a functor

$$j_U^* : PMod(\mathcal{O}) \longrightarrow PMod(\mathcal{O}_U)$$

and functors

$$j_{U*}, j_{U!} : PMod(\mathcal{O}_U) \longrightarrow PMod(\mathcal{O})$$

which are right, left adjoint to j_U^*. Inspecting the proof of Lemma 16.19.2 we see that $j_{U!}\mathcal{G}$ is the presheaf

$$V \longmapsto \bigoplus\nolimits_{\varphi \in Mor_{\mathcal{C}}(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U)$$

In addition the functor $j_{U!}$ is exact (by Lemma 16.19.3 in the case of the discrete topologies). Moreover, if $\mathcal{C}$ is actually a site, and $\mathcal{O}$ is actually a sheaf of rings, then the diagram

$$\begin{tikzcd} Mod(\mathcal{O}_U) \arrow[r, "j_{U!}"'] \arrow[d, "forget"'] & Mod(\mathcal{O}) \\ PMod(\mathcal{O}_U) \arrow[r, "j_{U!}"] & PMod(\mathcal{O}) \arrow[u, "(\ )^\#"'] \end{tikzcd}$$

commutes.

16.20. Localization of morphisms of ringed sites

This section is the analogue of Sites, Section 9.24.

Lemma 16.20.1. *Let $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}) \longrightarrow (\mathcal{D}, \mathcal{O}')$ be a morphism of ringed sites where f is given by the continuous functor $u : \mathcal{D} \to \mathcal{C}$. Let V be an object of $\mathcal{D}$ and set $U = u(V)$. Then there is a canonical map of sheaves of rings $(f')^\sharp$ such that the diagram of Sites, Lemma 9.24.1 is turned into a commutative diagram of ringed topoi*

$$\begin{tikzcd}
(Sh(\mathcal{C}/U), \mathcal{O}_U) \arrow[r, "(j_U, j_U^\sharp)"'] \arrow[d, "(f', (f')^\sharp)"'] & (Sh(\mathcal{C}), \mathcal{O}) \arrow[d, "(f, f^\sharp)"] \\
(Sh(\mathcal{D}/V), \mathcal{O}'_V) \arrow[r, "(j_V, j_V^\sharp)"] & (Sh(\mathcal{D}), \mathcal{O}').
\end{tikzcd}$$

Moreover, in this situation we have $f'_* j_U^{-1} = j_V^{-1} f_*$ and $f'_* j_U^* = j_V^* f_*$.

Proof. Just take $(f')^\sharp$ to be

$$(f')^{-1}\mathcal{O}'_V = (f')^{-1} j_V^{-1}\mathcal{O}' = j_U^{-1} f^{-1}\mathcal{O}' \xrightarrow{j_U^{-1} f^\sharp} j_U^{-1}\mathcal{O} = \mathcal{O}_U$$

and everything else follows from Sites, Lemma 9.24.1. (Note that $j^{-1} = j^*$ on sheaves of modules if j is a localization morphism, hence the first equality of functors implies the second.) $\square$

Lemma 16.20.2. *Let $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}) \longrightarrow (\mathcal{D}, \mathcal{O}')$ be a morphism of ringed sites where f is given by the continuous functor $u : \mathcal{D} \to \mathcal{C}$. Let $V \in Ob(\mathcal{D})$, $U \in Ob(\mathcal{C})$ and $c : U \to u(V)$ a morphism of $\mathcal{C}$. There exists a commutative diagram of ringed topoi*

$$\begin{tikzcd}
(Sh(\mathcal{C}/U), \mathcal{O}_U) \arrow[r, "(j_U, j_U^\sharp)"'] \arrow[d, "(f_c, f_c^\sharp)"'] & (Sh(\mathcal{C}), \mathcal{O}) \arrow[d, "(f, f^\sharp)"] \\
(Sh(\mathcal{D}/V), \mathcal{O}'_V) \arrow[r, "(j_V, j_V^\sharp)"] & (Sh(\mathcal{D}), \mathcal{O}').
\end{tikzcd}$$

The morphism $(f_c, f_c^\sharp)$ is equal to the composition of the morphism

$$(f', (f')^\sharp) : (Sh(\mathcal{C}/u(V)), \mathcal{O}_{u(V)}) \longrightarrow (Sh(\mathcal{D}/V), \mathcal{O}'_V)$$

of Lemma 16.20.1 and the morphism

$$(j, j^\sharp) : (Sh(\mathcal{C}/U), \mathcal{O}_U) \to (Sh(\mathcal{C}/u(V)), \mathcal{O}_{u(V)})$$

of Lemma 16.19.4. Given any morphisms $b : V' \to V$, $a : U' \to U$ and $c' : U' \to u(V')$ such that

$$\begin{tikzcd}
U' \arrow[r, "c'"'] \arrow[d, "a"'] & u(V') \arrow[d, "u(b)"] \\
U \arrow[r, "c"] & u(V)
\end{tikzcd}$$

commutes, then the following diagram of ringed topoi

$$\begin{tikzcd}
(Sh(\mathcal{C}/U'), \mathcal{O}_{U'}) \arrow[r, "(j_{U'/U}, j_{U'/U}^\sharp)"'] \arrow[d, "(f_{c'}, f_{c'}^\sharp)"'] & (Sh(\mathcal{C}/U), \mathcal{O}_U) \arrow[d, "(f_c, f_c^\sharp)"] \\
(Sh(\mathcal{D}/V'), \mathcal{O}'_{V'}) \arrow[r, "(j_{V'/V}, j_{V'/V}^\sharp)"] & (Sh(\mathcal{D}/V), \mathcal{O}'_V)
\end{tikzcd}$$

commutes.

Proof. On the level of morphisms of topoi this is Sites, Lemma 9.24.3. To check that the diagrams commute as morphisms of ringed topoi use Lemmas 16.19.4 and 16.20.1 exactly as in the proof of Sites, Lemma 9.24.3. □

16.21. Localization of ringed topoi

This section is the analogue of Sites, Section 9.26 in the setting of ringed topoi.

Lemma 16.21.1. *Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. Let $\mathcal{F} \in Sh(\mathcal{C})$ be a sheaf. For a sheaf $\mathcal{H}$ on $\mathcal{C}$ denote $\mathcal{H}_\mathcal{F}$ the sheaf $\mathcal{H} \times \mathcal{F}$ seen as an object of the category $Sh(\mathcal{C})/\mathcal{F}$. The pair $(Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F})$ is a ringed topos and there is a canonical morphism of ringed topoi*

$$(j_\mathcal{F}, j_\mathcal{F}^\sharp) : (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F}) \longrightarrow (Sh(\mathcal{C}), \mathcal{O})$$

which is a localization as in Section 16.19 such that

1. *the functor $j_\mathcal{F}^{-1}$ is the functor $\mathcal{H} \mapsto \mathcal{H}_\mathcal{F}$,*
2. *the functor $j_\mathcal{F}^*$ is the functor $\mathcal{H} \mapsto \mathcal{H}_\mathcal{F}$,*
3. *the functor $j_{\mathcal{F}!}$ on sheaves of sets is the forgetful functor $\mathcal{G}/\mathcal{F} \mapsto \mathcal{G}$,*
4. *the functor $j_{\mathcal{F}!}$ on sheaves of modules associates to the $\mathcal{O}_\mathcal{F}$-module $\varphi : \mathcal{G} \to \mathcal{F}$ the $\mathcal{O}$-module which is the sheafification of the presheaf*
$$V \longmapsto \bigoplus\nolimits_{s \in \mathcal{F}(V)} \{\sigma \in \mathcal{G}(V) \mid \varphi(\sigma) = s\}$$
for $V \in Ob(\mathcal{C})$.

Proof. By Sites, Lemma 9.26.1 we see that $Sh(\mathcal{C})/\mathcal{F}$ is a topos and that (1) and (3) are true. In particular this shows that $j_\mathcal{F}^{-1}\mathcal{O} = \mathcal{O}_\mathcal{F}$ and shows that $\mathcal{O}_\mathcal{F}$ is a sheaf of rings. Thus we may choose the map $j_\mathcal{F}^\sharp$ to be the identity, in particular we see that (2) is true. Moreover, the proof of Sites, Lemma 9.26.1 shows that we may assume $\mathcal{C}$ is a site with all finite limits and a subcanonical topology and that $\mathcal{F} = h_U$ for some object U of $\mathcal{C}$. Then (4) follows from the description of $j_{\mathcal{F}!}$ in Lemma 16.19.2. Alternatively one could show directly that the functor described in (4) is a left adjoint to $j_\mathcal{F}^*$. □

Definition 16.21.2. Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. Let $\mathcal{F} \in Sh(\mathcal{C})$.

1. The ringed topos $(Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F})$ is called the *localization of the ringed topos $(Sh(\mathcal{C}), \mathcal{O})$ at $\mathcal{F}$*.
2. The morphism of ringed topoi $(j_\mathcal{F}, j_\mathcal{F}^\sharp) : (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F}) \to (Sh(\mathcal{C}), \mathcal{O})$ of Lemma 16.21.1 is called the *localization morphism*.

We continue the tradition, esthablished in the chapter on sites, that we check the localization constructions on topoi are compatible with the constructions of localization on sites, whenever this makes sense.

Lemma 16.21.3. *With $(Sh(\mathcal{C}), \mathcal{O})$ and $\mathcal{F} \in Sh(\mathcal{C})$ as in Lemma 16.21.1. If $\mathcal{F} = h_U^\#$ for some object U of $\mathcal{C}$ then via the identification $Sh(\mathcal{C}/U) = Sh(\mathcal{C})/h_U^\#$ of Sites, Lemma 9.21.4 we have*

1. *canonically $\mathcal{O}_U = \mathcal{O}_\mathcal{F}$, and*
2. *with these identifications we have $(j_\mathcal{F}, j_\mathcal{F}^\sharp) = (j_U, j_U^\sharp)$.*

Proof. The assertion for underlying topoi is Sites, Lemma 9.26.5. Note that $\mathcal{O}_U$ is the restriction of $\mathcal{O}$ which by Sites, Lemma 9.21.6 corresponds to $\mathcal{O} \times h_U^\#$ under the equivalence of Sites, Lemma 9.21.4. By definition of $\mathcal{O}_\mathcal{F}$ we get (1). What's left is to prove that $j_\mathcal{F}^\sharp = j_U^\sharp$ under this identification. We omit the verification. □

Localization is functorial in the following two ways: We can ``relocalize'' a localization (see Lemma 16.21.4) or we can given a morphism of ringed topoi, localize upstairs at the inverse image of a sheaf downstairs and get a commutative diagram of locally ringed spaces (see Lemma 16.22.1).

Lemma 16.21.4. *Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. If $s : \mathcal{G} \to \mathcal{F}$ is a morphism of sheaves on $\mathcal{C}$ then there exists a natural commutative diagram of morphisms of ringed topoi*

$$\begin{tikzcd} (Sh(\mathcal{C})/\mathcal{G}, \mathcal{O}_\mathcal{G}) \arrow[rr, "{(j,j^\sharp)}"'] \arrow[dr, "{(j_\mathcal{G}, j^\sharp_\mathcal{G})}"'] & & (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F}) \arrow[dl, "{(j_\mathcal{F}, j^\sharp_\mathcal{F})}"] \\ & (Sh(\mathcal{C}), \mathcal{O}) & \end{tikzcd}$$

where $(j, j^\sharp)$ is the localization morphism of the ringed topos $(Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F})$ at the object $\mathcal{G}/\mathcal{F}$.

Proof. All assertions follow from Sites, Lemma 9.26.6 except the assertion that $j^\sharp_\mathcal{G} = j^\sharp \circ j^{-1}(j^\sharp_\mathcal{F})$. We omit the verification. $\square$

Lemma 16.21.5. *With $(Sh(\mathcal{C}), \mathcal{O})$, $s : \mathcal{G} \to \mathcal{F}$ as in Lemma 16.21.4. If there exist a morphism $f : V \to U$ of $\mathcal{C}$ such that $\mathcal{G} = h_V^\#$ and $\mathcal{F} = h_U^\#$ and s is induced by f, then the diagrams of Lemma 16.19.4 and Lemma 16.21.4 agree via the identifications $(j_\mathcal{F}, j^\sharp_\mathcal{F}) = (j_U, j^\sharp_U)$ and $(j_\mathcal{G}, j^\sharp_\mathcal{G}) = (j_V, j^\sharp_V)$ of Lemma 16.21.3.*

Proof. All assertions follow from Sites, Lemma 9.26.7 except for the assertion that the two maps $j^\sharp$ agree. This holds since in both cases the map $j^\sharp$ is simply the identity. Some details omitted. $\square$

16.22. Localization of morphisms of ringed topoi

This section is the analogue of Sites, Section 9.27.

Lemma 16.22.1. *Let*

$$f : (Sh(\mathcal{C}), \mathcal{O}) \longrightarrow (Sh(\mathcal{D}), \mathcal{O}')$$

be a morphism of ringed topoi. Let $\mathcal{G}$ be a sheaf on $\mathcal{D}$. Set $\mathcal{F} = f^{-1}\mathcal{G}$. Then there exists a commutative diagram of ringed topoi

$$\begin{tikzcd} (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F}) \arrow[r, "{(j_\mathcal{F}, j^\sharp_\mathcal{F})}"'] \arrow[d, "{(f', (f')^\sharp)}"'] & (Sh(\mathcal{C}), \mathcal{O}) \arrow[d, "{(f, f^\sharp)}"] \\ (Sh(\mathcal{D})/\mathcal{G}, \mathcal{O}'_\mathcal{G}) \arrow[r, "{(j_\mathcal{G}, j^\sharp_\mathcal{G})}"] & (Sh(\mathcal{D}), \mathcal{O}') \end{tikzcd}$$

We have $f'_*j_\mathcal{F}^{-1} = j_\mathcal{G}^{-1}f_*$ and $f'_*j_\mathcal{F}^* = j_\mathcal{G}^*f_*$. Moreover, the morphism f' is characterized by the rule

$$(f')^{-1}(\mathcal{H} \xrightarrow{\varphi} \mathcal{G}) = (f^{-1}\mathcal{H} \xrightarrow{f^{-1}\varphi} \mathcal{F}).$$

Proof. By Sites, Lemma 9.27.1 we have the diagram of underlying topoi, the equality $f'_*j_\mathcal{F}^{-1} = j_\mathcal{G}^{-1}f_*$, and the description of $(f')^{-1}$. To define $(f')^\sharp$ we use the map

$$(f')^\sharp : \mathcal{O}'_\mathcal{G} = j_\mathcal{G}^{-1}\mathcal{O}' \xrightarrow{j_\mathcal{G}^{-1}f^\sharp} j_\mathcal{G}^{-1}f_*\mathcal{O} = f'_*j_\mathcal{F}^{-1}\mathcal{O} = f'_*\mathcal{O}_\mathcal{F}$$

or equivalently the map

$$(f')^\sharp : (f')^{-1}\mathcal{O}'_\mathcal{G} = (f')^{-1}j_\mathcal{G}^{-1}\mathcal{O}' = j_\mathcal{F}^{-1}f^{-1}\mathcal{O}' \xrightarrow{j_\mathcal{F}^{-1}f^\sharp} j_\mathcal{F}^{-1}\mathcal{O} = \mathcal{O}_\mathcal{F}.$$

We omit the verification that these two maps are indeed adjoint to each other. The second construction of $(f')^\sharp$ shows that the diagram commutes in the 2-category of ringed topoi (as the maps $j_\mathcal{F}^\sharp$ and $j_\mathcal{G}^\sharp$ are identities). Finally, the equality $f'_*j_\mathcal{F}^* = j_\mathcal{G}^*f_*$ follows from the equality $f'_*j_\mathcal{F}^{-1} = j_\mathcal{G}^{-1}f_*$ and the fact that pullbacks of sheaves of modules and sheaves of sets agree, see Lemma 16.21.1. $\square$

Lemma 16.22.2. *Let*

$$f : (Sh(\mathcal{C}), \mathcal{O}) \longrightarrow (Sh(\mathcal{D}), \mathcal{O}')$$

be a morphism of ringed topoi. Let $\mathcal{G}$ be a sheaf on $\mathcal{D}$. Set $\mathcal{F} = f^{-1}\mathcal{G}$. If f is given by a continuous functor $u : \mathcal{D} \to \mathcal{C}$ and $\mathcal{G} = h_V^\#$, then the commutative diagrams of Lemma 16.20.1 and Lemma 16.22.1 agree via the identifications of Lemma 16.21.3.

Proof. At the level of morphisms of topoi this is Sites, Lemma 9.27.2. This works also on the level of morphisms of ringed topoi since the formulas defining $(f')^\sharp$ in the proofs of Lemma 16.20.1 and Lemma 16.22.1 agree. $\square$

Lemma 16.22.3. *Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a morphism of ringed topoi. Let $\mathcal{G}$ be a sheaf on $\mathcal{D}$, let $\mathcal{F}$ be a sheaf on $\mathcal{C}$, and let $s : \mathcal{F} \to f^{-1}\mathcal{G}$ a morphism of sheaves. There exists a commutative diagram of ringed topoi*

$$\begin{tikzcd}
(Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F}) \arrow[r, "(j_\mathcal{F}, j_\mathcal{F}^\sharp)"'] \arrow[d, "(f_c, f_c^\sharp)"'] & (Sh(\mathcal{C}), \mathcal{O}) \arrow[d, "(f, f^\sharp)"] \\
(Sh(\mathcal{D})/\mathcal{G}, \mathcal{O}'_\mathcal{G}) \arrow[r, "(j_\mathcal{G}, j_\mathcal{G}^\sharp)"] & (Sh(\mathcal{D}), \mathcal{O}').
\end{tikzcd}$$

The morphism $(f_s, f_s^\sharp)$ is equal to the composition of the morphism

$$(f', (f')^\sharp) : (Sh(\mathcal{C})/f^{-1}\mathcal{G}, \mathcal{O}_{f^{-1}\mathcal{G}}) \longrightarrow (Sh(\mathcal{D})/\mathcal{G}, \mathcal{O}'_\mathcal{G})$$

of Lemma 16.22.1 and the morphism

$$(j, j^\sharp) : (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F}) \to (Sh(\mathcal{C})/f^{-1}\mathcal{G}, \mathcal{O}_{f^{-1}\mathcal{G}})$$

of Lemma 16.21.4. Given any morphisms $b : \mathcal{G}' \to \mathcal{G}$, $a : \mathcal{F}' \to \mathcal{F}$, and $s' : \mathcal{F}' \to f^{-1}\mathcal{G}'$ such that

$$\begin{tikzcd}
\mathcal{F}' \arrow[r, "s'"'] \arrow[d, "a"'] & f^{-1}\mathcal{G}' \arrow[d, "f^{-1}b"] \\
\mathcal{F} \arrow[r, "s"] & f^{-1}\mathcal{G}
\end{tikzcd}$$

commutes, then the following diagram of ringed topoi

$$\begin{tikzcd}
(Sh(\mathcal{C})/\mathcal{F}', \mathcal{O}_{\mathcal{F}'}) \arrow[r, "(j_{\mathcal{F}'/\mathcal{F}}, j_{\mathcal{F}'/\mathcal{F}}^\sharp)"'] \arrow[d, "(f_{s'}, f_{s'}^\sharp)"'] & (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F}) \arrow[d, "(f_s, f_s^\sharp)"] \\
(Sh(\mathcal{D})/\mathcal{G}', \mathcal{O}'_{\mathcal{G}'}) \arrow[r, "(j_{\mathcal{G}'/\mathcal{G}}, j_{\mathcal{G}'/\mathcal{G}}^\sharp)"] & (Sh(\mathcal{D})/\mathcal{G}, \mathcal{O}'_{\mathcal{G}'})
\end{tikzcd}$$

commutes.

Proof. On the level of morphisms of topoi this is Sites, Lemma 9.27.3. To check that the diagrams commute as morphisms of ringed topoi use the commutative diagrams of Lemmas 16.21.4 and 16.22.1. □

Lemma 16.22.4. *Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$, $s : \mathcal{F} \to f^{-1}\mathcal{G}$ be as in Lemma 16.22.3. If f is given by a continuous functor $u : \mathcal{D} \to \mathcal{C}$ and $\mathcal{G} = h_V^\#$, $\mathcal{F} = h_U^\#$ and s comes from a morphism $c : U \to u(V)$, then the commutative diagrams of Lemma 16.20.2 and Lemma 16.22.3 agree via the identifications of Lemma 16.21.3.*

Proof. This is formal using Lemmas 16.21.5 and 16.22.2. □

16.23. Local types of modules

According to our general strategy explained in Section 16.18 we first define the local types for sheaves of modules on a ringed site, and then we immediately show that these types are intrinsic, hence make sense for sheaves of modules on ringed topoi.

Definition 16.23.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. We will freely use the notions defined in Definition 16.17.1.

(1) We say $\mathcal{F}$ is *locally free* if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}_{i\in I}$ of $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/U_i}$ is a free $\mathcal{O}_{U_i}$-module.
(2) We say $\mathcal{F}$ is *finite locally free* if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}_{i\in I}$ of $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/U_i}$ is a finite free $\mathcal{O}_{U_i}$-module.
(3) We say $\mathcal{F}$ is *locally generated by sections* if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}_{i\in I}$ of $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/U_i}$ is an $\mathcal{O}_{U_i}$-module generated by global sections.
(4) We say $\mathcal{F}$ is *of finite type* if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}_{i\in I}$ of $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/U_i}$ is an $\mathcal{O}_{U_i}$-module generated by finitely many global sections.
(5) We say $\mathcal{F}$ is *quasi-coherent* if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}_{i\in I}$ of $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/U_i}$ is an $\mathcal{O}_{U_i}$-module which has a global presentation.
(6) We say $\mathcal{F}$ is *of finite presentation* if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}_{i\in I}$ of $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/U_i}$ is an $\mathcal{O}_{U_i}$-module which has a finite global presentation.
(7) We say $\mathcal{F}$ is *coherent* if and only if $\mathcal{F}$ is of finite type, and for every object U of $\mathcal{C}$ and any $s_1, \ldots, s_n \in \mathcal{F}(U)$ the kernel of the map $\bigoplus_{i=1,\ldots,n} \mathcal{O}_U \to \mathcal{F}|_U$ is of finite type on $(\mathcal{C}/U, \mathcal{O}_U)$.

Lemma 16.23.2. *Any of the properties (1) -- (7) of Definition 16.23.1 is intrinsic (see discussion in Section 16.18).*

Proof. Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a special cocontinuous functor. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules on $\mathcal{C}$. Let $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be the equivalence of topoi associated to u. Set $\mathcal{O}' = g_*\mathcal{O}$, and let $g^\sharp : \mathcal{O}' \to g_*\mathcal{O}$ be the identity. Finally, set $\mathcal{F}' = g_*\mathcal{F}$. Let $\mathcal{P}_l$ be one of the properties (1) -- (6) listed in Definition 16.23.1. (We will discuss the coherent case at the end of the proof.) Let $\mathcal{P}_g$ denote the corresponding property listed in Definition 16.17.1. We have already seen that $\mathcal{P}_g$ is intrinsic. We have to show that $\mathcal{P}_l(\mathcal{C}, \mathcal{O}, \mathcal{F})$ holds if and only if $\mathcal{P}_l(\mathcal{D}, \mathcal{O}', \mathcal{F}')$ holds.

Assume that $\mathcal{F}$ has $\mathcal{P}_l$. Let V be an object of $\mathcal{D}$. One of the properties of a special cocontinuous functor is that there exists a covering $\{u(U_i) \to V\}_{i\in I}$ in the site $\mathcal{D}$. By assumption, for each i there exists a covering $\{U_{ij} \to U_i\}_{j \in J_i}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{U_{ij}}$ is $\mathcal{P}_g$. By Sites, Lemma 9.25.3 we have commutative diagrams of ringed topoi

$$\begin{tikzcd} (Sh(\mathcal{C}/U_{ij}), \mathcal{O}_{U_{ij}}) \arrow[r] \arrow[d] & (Sh(\mathcal{C}), \mathcal{O}) \arrow[d] \\ (Sh(\mathcal{D}/u(U_{ij})), \mathcal{O}'_{u(U_{ij})}) \arrow[r] & (Sh(\mathcal{D}), \mathcal{O}') \end{tikzcd}$$

where the vertical arrows are equivalences. Hence we conclude that $\mathcal{F}'|_{u(U_{ij})}$ has property $\mathcal{P}_g$ also. And moreover, $\{u(U_{ij}) \to V\}_{i \in I, j \in J_i}$ is a covering of the site $\mathcal{D}$. Hence $\mathcal{F}'$ has property $\mathcal{P}_l$.

Assume that $\mathcal{F}'$ has $\mathcal{P}_l$. Let U be an object of $\mathcal{C}$. By assumption, there exists a covering $\{V_i \to u(U)\}_{i \in I}$ such that $\mathcal{F}'|_{V_i}$ has property $\mathcal{P}_g$. Because u is cocontinuous we can refine this covering by a family $\{u(U_j) \to u(U)\}_{j \in J}$ where $\{U_j \to U\}_{j \in J}$ is a covering in $\mathcal{C}$. Say the refinement is given by $\alpha : J \to I$ and $u(U_j) \to V_{\alpha(j)}$. Restricting is transitive, i.e., $(\mathcal{F}'|_{V_{\alpha(j)}})|_{u(U_j)} = \mathcal{F}'|_{u(U_j)}$. Hence by Lemma 16.17.2 we see that $\mathcal{F}'|_{u(U_j)}$ has property $\mathcal{P}_g$. Hence the diagram

$$\begin{tikzcd} (Sh(\mathcal{C}/U_j), \mathcal{O}_{U_j}) \arrow[r] \arrow[d] & (Sh(\mathcal{C}), \mathcal{O}) \arrow[d] \\ (Sh(\mathcal{D}/u(U_j)), \mathcal{O}'_{u(U_j)}) \arrow[r] & (Sh(\mathcal{D}), \mathcal{O}') \end{tikzcd}$$

where the vertical arrows are equivalences shows that $\mathcal{F}|_{U_j}$ has property $\mathcal{P}_g$ also. Thus $\mathcal{F}$ has property $\mathcal{P}_l$ as desired.

Finally, we prove the lemma in case $\mathcal{P}_l = \textit{coherent}$[2]. Assume $\mathcal{F}$ is coherent. This implies that $\mathcal{F}$ is of finite type and hence $\mathcal{F}'$ is of finite type also by the first part of the proof. Let V be an object of $\mathcal{D}$ and let $s_1, \ldots, s_n \in \mathcal{F}'(V)$. We have to show that the kernel $\mathcal{K}'$ of $\bigoplus_{j=1,\ldots,n} \mathcal{O}_V \to \mathcal{F}'|_V$ is of finite type on $\mathcal{D}/V$. This means we have to show that for any V'/V there exists a covering $\{V'_i \to V'\}$ such that $\mathcal{F}'|_{V'_i}$ is generated by finitely many sections. Replacing V by V' (and restricting the sections s_j to V') we reduce to the case where $V' = V$. Since u is a special cocontinuous functor, there exists a covering $\{u(U_i) \to V\}_{i \in I}$ in the site $\mathcal{D}$. Using the isomorphism of topoi $Sh(\mathcal{C}/U_i) = Sh(\mathcal{D}/u(U_i))$ we see that $\mathcal{K}'|_{u(U_i)}$ corresponds to the kernel $\mathcal{K}_i$ of a map $\bigoplus_{j=1,\ldots,n} \mathcal{O}_{U_i} \to \mathcal{F}|_{U_i}$. Since $\mathcal{F}$ is coherent we see that $\mathcal{K}_i$ is of finite type. Hence we conclude (by the first part of the proof again) that $\mathcal{K}|_{u(U_i)}$ is of finite type. Thus there exist coverings $\{V_{il} \to u(U_i)\}$ such that $\mathcal{K}|_{V_{il}}$ is generated by finitely many global sections. Since $\{V_{il} \to V\}$ is a covering of $\mathcal{D}$ we conclude that $\mathcal{K}$ is of finite type as desired.

Assume $\mathcal{F}'$ is coherent. This implies that $\mathcal{F}'$ is of finite type and hence $\mathcal{F}$ is of finite type also by the first part of the proof. Let U be an object of $\mathcal{C}$, and let $s_1, \ldots, s_n \in \mathcal{F}(U)$. We have to show that the kernel $\mathcal{K}$ of $\bigoplus_{j=1,\ldots,n} \mathcal{O}_U \to \mathcal{F}|_U$ is of finite type on $\mathcal{C}/U$. Using the isomorphism of topoi $Sh(\mathcal{C}/U) = Sh(\mathcal{D}/u(U))$ we see that $\mathcal{K}|_U$ corresponds to the kernel $\mathcal{K}'$ of a map $\bigoplus_{j=1,\ldots,n} \mathcal{O}_{u(U)} \to \mathcal{F}'|_{u(U)}$. As $\mathcal{F}'$ is coherent, we see that $\mathcal{K}'$ is of finite type. Hence, by the first part of the proof again, we conclude that $\mathcal{K}$ is of finite type. $\square$

[2]The mechanics of this are a bit awkward, and we suggest the reader skip this part of the proof.

Hence from now on we may refer to the properties of $\mathcal{O}$-modules defined in Definition 16.23.1 without specifying a site.

Lemma 16.23.3. *Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. Let $\mathcal{F}$ be an $\mathcal{O}$-module. Assume that the site $\mathcal{C}$ has a final object X. Then*

1. *The following are equivalent*
   1. (a) *$\mathcal{F}$ is locally free,*
   2. (b) *there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is a locally free $\mathcal{O}_{X_i}$-module, and*
   3. (c) *there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is a free $\mathcal{O}_{X_i}$-module.*
2. *The following are equivalent*
   1. (a) *$\mathcal{F}$ is finite locally free,*
   2. (b) *there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is a finite locally free $\mathcal{O}_{X_i}$-module, and*
   3. (c) *there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is a finite free $\mathcal{O}_{X_i}$-module.*
3. *The following are equivalent*
   1. (a) *$\mathcal{F}$ is locally generated by sections,*
   2. (b) *there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is an $\mathcal{O}_{X_i}$-module locally generated by sections, and*
   3. (c) *there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is an $\mathcal{O}_{X_i}$-module globally generated by sections.*
4. *The following are equivalent*
   1. (a) *$\mathcal{F}$ is of finite type,*
   2. (b) *there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is an $\mathcal{O}_{X_i}$-module of finite type, and*
   3. (c) *there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is an $\mathcal{O}_{X_i}$-module globally generated by finitely many sections.*
5. *The following are equivalent*
   1. (a) *$\mathcal{F}$ is quasi-coherent,*
   2. (b) *there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is a quasi-coherent $\mathcal{O}_{X_i}$-module, and*
   3. (c) *there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is an $\mathcal{O}_{X_i}$-module which has a global presentation.*
6. *The following are equivalent*
   1. (a) *$\mathcal{F}$ is of finite presentation,*
   2. (b) *there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is an $\mathcal{O}_{X_i}$-module of finite presentation, and*
   3. (c) *there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is an $\mathcal{O}_{X_i}$-module has a finite global presentation.*
7. *The following are equivalent*
   1. (a) *$\mathcal{F}$ is coherent, and*
   2. (b) *there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is a coherent $\mathcal{O}_{X_i}$-module.*

Proof. In each case we have (a) $\Rightarrow (b)$. In each of the cases (1) - (6) condition (b) implies condition (c) by axiom (2) of a site (see Sites, Definition 9.6.2) and the definition of the local types of modules. Suppose $\{X_i \to X\}$ is a covering. Then for every object U of $\mathcal{C}$ we get an induced covering $\{X_i \times_X U \to U\}$. Moreover, the global property for $\mathcal{F}|_{\mathcal{C}/X_i}$ in part (c) implies the corresponding global property for $\mathcal{F}|_{\mathcal{C}/X_i \times_X U}$ by Lemma 16.17.2, hence the sheaf has property (a) by definition. We omit the proof of (b) ⇒ (a) in case (7). □

Lemma 16.23.4. *Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let $\mathcal{F}$ be an $\mathcal{O}_\mathcal{D}$-module.*

(1) *If $\mathcal{F}$ is locally free then $f^*\mathcal{F}$ is locally free.*
(2) *If $\mathcal{F}$ is finite locally free then $f^*\mathcal{F}$ is finite locally free.*
(3) *If $\mathcal{F}$ is locally generated by sections then $f^*\mathcal{F}$ is locally generated by sections.*
(4) *If $\mathcal{F}$ is of finite type then $f^*\mathcal{F}$ is of finite type.*
(5) *If $\mathcal{F}$ is quasi-coherent then $f^*\mathcal{F}$ is quasi-coherent.*
(6) *If $\mathcal{F}$ is of finite presentation then $f^*\mathcal{F}$ is of finite presentation.*

Proof. According to the discussion in Section 16.18 we need only check preservation under pullback for a morphism of ringed sites $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ such that f is given by a left exact, continuous functor $u : \mathcal{D} \to \mathcal{C}$ between sites which have all finite limits. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_\mathcal{D}$-modules which has one of the properties (1) -- (6) of Definition 16.23.1. We know $\mathcal{D}$ has a final object Y and $X = u(Y)$ is a final object for $\mathcal{C}$. By assumption we have a covering $\{Y_i \to Y\}$ such that $\mathcal{G}|_{\mathcal{D}/Y_i}$ has the corresponding global property. Set $X_i = u(Y_i)$ so that $\{X_i \to X\}$ is a covering in $\mathcal{C}$. We get a commutative diagram of morphisms ringed sites

$$
\begin{tikzcd}
(\mathcal{C}/X_i, \mathcal{O}_\mathcal{C}|_{X_i}) \arrow[r] \arrow[d] & (\mathcal{C}, \mathcal{O}_\mathcal{C}) \arrow[d] \\
(\mathcal{D}/Y_i, \mathcal{O}_\mathcal{D}|_{Y_i}) \arrow[r] & (\mathcal{D}, \mathcal{O}_\mathcal{D})
\end{tikzcd}
$$

by Sites, Lemma 9.24.2. Hence by Lemma 16.17.2 that $f^*\mathcal{G}|_{X_i}$ has the corresponding global property. Hence we conclude that $\mathcal{G}$ has the local property we started out with by Lemma 16.23.3. □

16.24. Tensor product

In Sections 16.9 and 16.11 we defined the change of rings functor by a tensor product construction. To be sure this construction makes sense also to define the tensor product of presheaves of $\mathcal{O}$-modules. To be precise, suppose $\mathcal{C}$ is a category, $\mathcal{O}$ is a presheaf of rings, and $\mathcal{F}, \mathcal{G}$ are presheaves of $\mathcal{O}$-modules. In this case we define $\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G}$ to be the presheaf

$$U \longmapsto (\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G})(U) = \mathcal{F}(U) \otimes_{\mathcal{O}(U)} \mathcal{G}(U)$$

If $\mathcal{C}$ is a site, $\mathcal{O}$ is a sheaf of rings and $\mathcal{F}, \mathcal{G}$ are sheaves of $\mathcal{O}$-modules then we define

$$\mathcal{F} \otimes_\mathcal{O} \mathcal{G} = (\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G})^\#$$

to be the sheaf of $\mathcal{O}$-modules associated to the presheaf $\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G}$.

Here are some formulas which we will use below without further mention:

$$(\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G}) \otimes_{p,\mathcal{O}} \mathcal{H} = \mathcal{F} \otimes_{p,\mathcal{O}} (\mathcal{G} \otimes_{p,\mathcal{O}} \mathcal{H}),$$

and similarly for sheaves. If $\mathcal{O}_1 \to \mathcal{O}_2$ is a map of presheaves of rings, then

$$(\mathcal{F} \otimes_{p,\mathcal{O}_1} \mathcal{G}) \otimes_{p,\mathcal{O}_1} \mathcal{O}_2 = (\mathcal{F} \otimes_{p,\mathcal{O}_1} \mathcal{O}_2) \otimes_{p,\mathcal{O}_2} (\mathcal{G} \otimes_{p,\mathcal{O}_1} \mathcal{O}_2),$$

and similarly for sheaves. These follow from their algebraic counterparts and sheafification.

Let $\mathcal{C}$ be a site, let $\mathcal{O}$ be a sheaf of rings and let $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ be sheaves of $\mathcal{O}$-modules. In this case we define

$$\text{Bilin}_{\mathcal{O}}(\mathcal{F} \times \mathcal{G}, \mathcal{H}) = \{\varphi \in Mor_{Sh(\mathcal{C})}(\mathcal{F} \times \mathcal{G}, \mathcal{H}) \mid \varphi \text{ is } \mathcal{O}\text{-bilinear}\}.$$

With this definition we have

$$Hom_{\mathcal{O}}(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}, \mathcal{H}) = \text{Bilin}_{\mathcal{O}}(\mathcal{F} \times \mathcal{G}, \mathcal{H}).$$

In other words $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$ represents the functor which associates to $\mathcal{H}$ the set of bilinear maps $\mathcal{F} \times \mathcal{G} \to \mathcal{H}$. In particular, since the notion of a bilinear map makes sense for a pair of modules on a ringed topos, we see that the tensor product of sheaves of modules is intrinsic to the topos (compare the discussion in Section 16.18). In fact we have the following.

Lemma 16.24.1. *Let $f : (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \to (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$ be a morphism of ringed topoi. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_{\mathcal{D}}$-modules. Then $f^*(\mathcal{F} \otimes_{\mathcal{O}_{\mathcal{D}}} \mathcal{G}) = f^*\mathcal{F} \otimes_{\mathcal{O}_{\mathcal{C}}} f^*\mathcal{G}$ functorially in $\mathcal{F}$, $\mathcal{G}$.*

Proof. For a sheaf $\mathcal{H}$ of $\mathcal{O}_{\mathcal{C}}$ modules we have

$$\begin{aligned}
Hom_{\mathcal{O}_{\mathcal{C}}}(f^*(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}), \mathcal{H}) &= Hom_{\mathcal{O}_{\mathcal{D}}}(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}, f_*\mathcal{H}) \\
&= \text{Bilin}_{\mathcal{O}_{\mathcal{D}}}(\mathcal{F} \times \mathcal{G}, f_*\mathcal{H}) \\
&= \text{Bilin}_{f^{-1}\mathcal{O}_{\mathcal{D}}}(f^{-1}\mathcal{F} \times f^{-1}\mathcal{G}, \mathcal{H}) \\
&= Hom_{f^{-1}\mathcal{O}_{\mathcal{D}}}(f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_{\mathcal{D}}} f^{-1}\mathcal{G}, \mathcal{H}) \\
&= Hom_{\mathcal{O}_{\mathcal{C}}}(f^*\mathcal{F} \otimes_{f^*\mathcal{O}_{\mathcal{D}}} f^*\mathcal{G}, \mathcal{H})
\end{aligned}$$

The interesting ``='' in this sequence of equalities is the third equality. It follows from the definition and adjointness of f_* and f^{-1} (as discussed in previous sections) in a straightforward manner. □

Lemma 16.24.2. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of $\mathcal{O}$-modules.*

1. *If $\mathcal{F}$, $\mathcal{G}$ are locally free, so is $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$.*
2. *If $\mathcal{F}$, $\mathcal{G}$ are finite locally free, so is $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$.*
3. *If $\mathcal{F}$, $\mathcal{G}$ are locally generated by sections, so is $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$.*
4. *If $\mathcal{F}$, $\mathcal{G}$ are of finite type, so is $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$.*
5. *If $\mathcal{F}$, $\mathcal{G}$ are quasi-coherent, so is $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$.*
6. *If $\mathcal{F}$, $\mathcal{G}$ are of finite presentation, so is $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$.*
7. *If $\mathcal{F}$ is of finite presentation and $\mathcal{G}$ is coherent, then $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$ is coherent.*
8. *If $\mathcal{F}$, $\mathcal{G}$ are coherent, so is $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$.*

Proof. Omitted. Hint: Compare with Sheaves of Modules, Lemma 15.15.5. □

16.25. Internal Hom

Let $\mathcal{C}$ be a category and let $\mathcal{O}$ be a presheaf of rings. Let $\mathcal{F}$, $\mathcal{G}$ be presheaves of $\mathcal{O}$-modules. Consider the rule

$$U \longmapsto Hom_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U).$$

For $\varphi : V \to U$ in $\mathcal{C}$ we define a restriction mapping

$$Hom_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U) \longrightarrow Hom_{\mathcal{O}_V}(\mathcal{F}|_V, \mathcal{G}|_V)$$

by restricting via the relocalization morphism $j : \mathcal{C}/V \to \mathcal{C}/U$, see Sites, Lemma 9.21.7. Hence this defines a presheaf $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$. In addition, given an element $\varphi \in Hom_{\mathcal{O}|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ and a section $f \in \mathcal{O}(U)$ then we can define $f\varphi \in Hom_{\mathcal{O}|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ by either precomposing with multiplication by f on $\mathcal{F}|_U$ or postcomposing with multiplication by f on $\mathcal{G}|_U$ (it gives the same result). Hence we in fact get a presheaf of $\mathcal{O}$-modules. There is a canonical ``evaluation" morphism

$$\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathcal{G}.$$

Lemma 16.25.1. *If $\mathcal{C}$ is a site, $\mathcal{O}$ is a sheaf of rings, $\mathcal{F}$ is a presheaf of $\mathcal{O}$-modules, and $\mathcal{G}$ is a sheaf of $\mathcal{O}$-modules, then $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$ is a sheaf of $\mathcal{O}$-modules.*

Proof. Omitted. Hints: Note first that $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) = \mathcal{H}om_{\mathcal{O}}(\mathcal{F}^\#, \mathcal{G})$, which reduces the question to the case where both $\mathcal{F}$ and $\mathcal{G}$ are sheaves. The result for sheaves of sets is Sites, Lemma 9.22.1. $\square$

In the situation of the lemma the ``evaluation" morphism factors through the tensor product of sheaves of modules

$$\mathcal{F} \otimes_{\mathcal{O}} \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathcal{G}.$$

Lemma 16.25.2. *Internal hom and (co)limits. Let $\mathcal{C}$ be a category and let $\mathcal{O}$ be a presheaf of rings.*

(1) *For any presheaf of $\mathcal{O}$-modules $\mathcal{F}$ the functor*

$$PMod(\mathcal{O}) \longrightarrow PMod(\mathcal{O}), \quad \mathcal{G} \longmapsto \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$$

commutes with arbitrary limits.

(2) *For any presheaf of $\mathcal{O}$-modules $\mathcal{G}$ the functor*

$$PMod(\mathcal{O}) \longrightarrow PMod(\mathcal{O})^{opp}, \quad \mathcal{F} \longmapsto \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$$

commutes with arbitrary colimits, in a formula

$$\mathcal{H}om_{\mathcal{O}}(colim_i \mathcal{F}_i, \mathcal{G}) = lim_i \mathcal{H}om_{\mathcal{O}}(\mathcal{F}_i, \mathcal{G}).$$

Suppose that $\mathcal{C}$ is a site, and $\mathcal{O}$ is a sheaf of rings.

(3) *For any sheaf of $\mathcal{O}$-modules $\mathcal{F}$ the functor*

$$Mod(\mathcal{O}) \longrightarrow Mod(\mathcal{O}), \quad \mathcal{G} \longmapsto \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$$

commutes with arbitrary limits.

(4) *For any sheaf of $\mathcal{O}$-modules $\mathcal{G}$ the functor*

$$Mod(\mathcal{O}) \longrightarrow Mod(\mathcal{O})^{opp}, \quad \mathcal{F} \longmapsto \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$$

commutes with arbitrary colimits, in a formula

$$\mathcal{H}om_{\mathcal{O}}(colim_i \mathcal{F}_i, \mathcal{G}) = lim_i \mathcal{H}om_{\mathcal{O}}(\mathcal{F}_i, \mathcal{G}).$$

Proof. Let $\mathcal{I} \to PMod(\mathcal{O})$, $i \mapsto \mathcal{G}_i$ be a diagram. Let U be an object of the category $\mathcal{C}$. As j_U^* is both a left and a right adjoint we see that $lim_i j_U^* \mathcal{G}_i = j_U^* lim_i \mathcal{G}_i$. Hence we have

$$\begin{aligned} \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, lim_i \mathcal{G}_i)(U) &= Hom_{\mathcal{O}_U}(\mathcal{F}|_U, lim_i \mathcal{G}_i|_U) \\ &= lim_i Hom_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}_i|_U) \\ &= lim_i \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}_i)(U) \end{aligned}$$

by definition of a limit. This proves (1). Part (2) is proved in exactly the same way. Part (3) follows from (1) because the limit of a diagram of sheaves is the same as the limit in the category of presheaves. Finally, (4) follow because, in the formula we have

$$Mor_{Mod(\mathcal{O})}(colim_i \mathcal{F}_i, \mathcal{G}) = Mor_{PMod(\mathcal{O})}(colim_i^{PSh} \mathcal{F}_i, \mathcal{G})$$

as the colimit $colim_i \mathcal{F}_i$ is the sheafification of the colimit $colim_i^{PSh} \mathcal{F}_i$ in $PMod(\mathcal{O})$. Hence (4) follows from (2) (by the remark on limits above again). $\square$

Lemma 16.25.3. *Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings.*

(1) *Let $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ be presheaves of $\mathcal{O}$-modules. There is a canonical isomorphism*
$$\mathcal{H}om_{\mathcal{O}}(\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G}, \mathcal{H}) \longrightarrow \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}}(\mathcal{G}, \mathcal{H}))$$
which is functorial in all three entries (sheaf Hom in all three spots). In particular,
$$Mor_{PMod(\mathcal{O})}(\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G}, \mathcal{H}) = Mor_{PMod(\mathcal{O})}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}}(\mathcal{G}, \mathcal{H}))$$

(2) *Suppose that $\mathcal{C}$ is a site, $\mathcal{O}$ is a sheaf of rings, and $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ are sheaves of $\mathcal{O}$-modules. There is a canonical isomorphism*
$$\mathcal{H}om_{\mathcal{O}}(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}, \mathcal{H}) \longrightarrow \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}}(\mathcal{G}, \mathcal{H}))$$
which is functorial in all three entries (sheaf Hom in all three spots). In particular,
$$Mor_{Mod(\mathcal{O})}(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}, \mathcal{H}) = Mor_{Mod(\mathcal{O})}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}}(\mathcal{G}, \mathcal{H}))$$

Proof. This is the analogue of Algebra, Lemma 7.11.9. The proof is the same, and is omitted. ☐

Lemma 16.25.4. *Tensor product and (co)limits. Let $\mathcal{C}$ be a category and let $\mathcal{O}$ be a presheaf of rings.*

(1) *For any presheaf of $\mathcal{O}$-modules $\mathcal{F}$ the functor*
$$PMod(\mathcal{O}) \longrightarrow PMod(\mathcal{O}), \quad \mathcal{G} \longmapsto \mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G}$$
commutes with arbitrary colimits.

(2) *Suppose that $\mathcal{C}$ is a site, and $\mathcal{O}$ is a sheaf of rings. For any sheaf of $\mathcal{O}$-modules $\mathcal{F}$ the functor*
$$PMod(\mathcal{O}) \longrightarrow PMod(\mathcal{O}), \quad \mathcal{G} \longmapsto \mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$$
commutes with arbitrary colimits.

Proof. This is because tensor product is adjoint to internal hom according to Lemma 16.25.3. See Categories, Lemma 4.22.2. ☐

16.26. Flat modules

We can define flat modules exactly as in the case of modules over rings.

Definition 16.26.1. Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings.

(1) A presheaf $\mathcal{F}$ of $\mathcal{O}$-modules is called *flat* if the functor
$$PMod(\mathcal{O}) \longrightarrow PMod(\mathcal{O}), \quad \mathcal{G} \mapsto \mathcal{G} \otimes_{p,\mathcal{O}} \mathcal{F}$$
is exact.

(2) A map $\mathcal{O} \to \mathcal{O}'$ of presheaves of rings is called *flat* if $\mathcal{O}'$ is flat as a presheaf of $\mathcal{O}$-modules.

(3) If $\mathcal{C}$ is a site, $\mathcal{O}$ is a sheaf of rings and $\mathcal{F}$ is a sheaf of $\mathcal{O}$-modules, then we say $\mathcal{F}$ is *flat* if the functor
$$Mod(\mathcal{O}) \longrightarrow Mod(\mathcal{O}), \quad \mathcal{G} \mapsto \mathcal{G} \otimes_{\mathcal{O}} \mathcal{F}$$
is exact.

(4) A map $\mathcal{O} \to \mathcal{O}'$ of sheaves of rings on a site is called *flat* if $\mathcal{O}'$ is flat as a sheaf of $\mathcal{O}$-modules.

The notion of a flat module or flat ring map is intrinsic (Section 16.18).

Lemma 16.26.2. *Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules. If each $\mathcal{F}(U)$ is a flat $\mathcal{O}(U)$-module, then $\mathcal{F}$ is flat.*

Proof. This is immediate from the definitions. $\square$

Lemma 16.26.3. *Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules. If $\mathcal{F}$ is a flat $\mathcal{O}$-module, then $\mathcal{F}^\#$ is a flat $\mathcal{O}^\#$-module.*

Proof. Omitted. (Hint: Sheafification is exact.) $\square$

Lemma 16.26.4. *(Colimits and tensor product.)*

1. *A filtered colimit of flat presheaves of modules is flat. A direct sum of flat presheaves of modules is flat.*
2. *A filtered colimit of flat sheaves of modules is flat. A direct sum of flat sheaves of modules is flat.*

Proof. Part (1) follows from Lemma 16.25.4 and Algebra, Lemma 7.8.9 by looking at sections over objects. To see part (2), use Lemma 16.25.4 and the fact that a filtered colimit of exact complexes is an exact complex (this uses that sheafification is exact and commutes with colimits). Some details omitted. $\square$

Lemma 16.26.5. *Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings. Let U be an object of $\mathcal{C}$. Consider the functor $j_U : \mathcal{C}/U \to \mathcal{C}$.*

1. *The presheaf of $\mathcal{O}$-modules $j_{U!}\mathcal{O}_U$ (see Remark 16.19.5) is flat.*
2. *If $\mathcal{C}$ is a site, $\mathcal{O}$ is a sheaf of rings, $j_{U!}\mathcal{O}_U$ is a flat sheaf of $\mathcal{O}$-modules.*

Proof. Proof of (1). By the discussion in Remark 16.19.5 we see that

$$j_{U!}\mathcal{O}_U(V) = \bigoplus\nolimits_{\varphi \in Mor_\mathcal{C}(V,U)} \mathcal{O}(V)$$

which is a flat $\mathcal{O}(V)$-module. Hence (1) follows from Lemma 16.26.2. Then (2) follows as $j_{U!}\mathcal{O}_U = (j_{U!}\mathcal{O}_U)^\#$ (the first $j_{U!}$ on sheaves, the second on presheaves) and Lemma 16.26.3. $\square$

Lemma 16.26.6. *Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings.*

1. *Any presheaf of $\mathcal{O}$-modules is a quotient of a direct sum $\bigoplus j_{U_i!}\mathcal{O}_{U_i}$.*
2. *Any presheaf of $\mathcal{O}$-modules is a quotient of a flat presheaf of $\mathcal{O}$-modules.*
3. *If $\mathcal{C}$ is a site, $\mathcal{O}$ is a sheaf of rings, then any sheaf of $\mathcal{O}$-modules is a quotient of a direct sum $\bigoplus j_{U_i!}\mathcal{O}_{U_i}$.*
4. *If $\mathcal{C}$ is a site, $\mathcal{O}$ is a sheaf of rings, then any sheaf of $\mathcal{O}$-modules is a quotient of a flat sheaf of $\mathcal{O}$-modules.*

Proof. Proof of (1). For every object U of $\mathcal{C}$ and every $s \in \mathcal{F}(U)$ we get a morphism $j_{U!}\mathcal{O}_U \to \mathcal{F}$, namely the adjoint to the morphism $\mathcal{O}_U \to \mathcal{F}|_U$, $1 \mapsto s$. Clearly the map

$$\bigoplus\nolimits_{(U,s)} j_{U!}\mathcal{O}_U \longrightarrow \mathcal{F}$$

is surjective. The source is flat by combining Lemmas 16.26.4 and 16.26.5 which proves (2). The sheaf case follows from this either by sheafifying or repeating the same argument. $\square$

Lemma 16.26.7. *Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings. Let*

$$0 \to \mathcal{F}'' \to \mathcal{F}' \to \mathcal{F} \to 0$$

be a short exact sequence of presheaves of $\mathcal{O}$-modules. Assume $\mathcal{F}$ is flat. Then

(1) *For any presheaf $\mathcal{G}$ of $\mathcal{O}$-modules, the sequence*
$$0 \to \mathcal{F}'' \otimes_{p,\mathcal{O}} \mathcal{G} \to \mathcal{F}' \otimes_{p,\mathcal{O}} \mathcal{G} \to \mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G} \to 0$$
is exact.

(2) *If $\mathcal{C}$ is a site, and $\mathcal{O}$, $\mathcal{F}$, $\mathcal{F}'$, $\mathcal{F}''$, and $\mathcal{G}$ are all sheaves, the sequence*
$$0 \to \mathcal{F}'' \otimes_{\mathcal{O}} \mathcal{G} \to \mathcal{F}' \otimes_{\mathcal{O}} \mathcal{G} \to \mathcal{F} \otimes_{\mathcal{O}} \mathcal{G} \to 0$$
is exact.

Proof. Choose a flat presheaf of $\mathcal{O}$-modules $\mathcal{G}'$ which surjects onto $\mathcal{G}$. This is possible by Lemma 16.26.6. Let $\mathcal{G}'' = \text{Ker}(\mathcal{G}' \to \mathcal{G})$. The lemma follows by applying the snake lemma to the following diagram
$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \mathcal{F}'' \otimes_{p,\mathcal{O}} \mathcal{G} & \to & \mathcal{F}' \otimes_{p,\mathcal{O}} \mathcal{G} & \to & \mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \mathcal{F}'' \otimes_{p,\mathcal{O}} \mathcal{G}' & \to & \mathcal{F}' \otimes_{p,\mathcal{O}} \mathcal{G}' & \to & \mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G}' & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \mathcal{F}'' \otimes_{p,\mathcal{O}} \mathcal{G}'' & \to & \mathcal{F}' \otimes_{p,\mathcal{O}} \mathcal{G}'' & \to & \mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G}'' & \to & 0 \\
 & & & & & & \uparrow & & \\
 & & & & & & 0 & &
\end{array}$$
with exact rows and columns. The middle row is exact because tensoring with the flat module $\mathcal{G}'$ is exact. The sheaf case follows from the presheaf case as sheafification is exact. $\square$

Lemma 16.26.8. *Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings. Let*
$$0 \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F}_0 \to 0$$
be a short exact sequence of presheaves of $\mathcal{O}$-modules.

(1) *If $\mathcal{F}_2$ and $\mathcal{F}_0$ are flat so is $\mathcal{F}_1$.*
(2) *If $\mathcal{F}_1$ and $\mathcal{F}_0$ are flat so is $\mathcal{F}_2$.*

If $\mathcal{C}$ is a site and $\mathcal{O}$ is a sheaf of rings then the same result holds $Mod(\mathcal{O})$.

Proof. Let $\mathcal{G}^\bullet$ be an arbitrary exact complex of presheaves of $\mathcal{O}$-modules. Assume that $\mathcal{F}_0$ is flat. By Lemma 16.26.7 we see that
$$0 \to \mathcal{G}^\bullet \otimes_{p,\mathcal{O}} \mathcal{F}_2 \to \mathcal{G}^\bullet \otimes_{p,\mathcal{O}} \mathcal{F}_1 \to \mathcal{G}^\bullet \otimes_{p,\mathcal{O}} \mathcal{F}_0 \to 0$$
is a short exact sequence of complexes of presheaves of $\mathcal{O}$-modules. Hence (1) and (2) follow from the snake lemma. The case of sheaves of modules is proved in the same way. $\square$

Lemma 16.26.9. *Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings. Let*
$$\ldots \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F}_0 \to \mathcal{Q} \to 0$$
be an exact complex of presheaves of $\mathcal{O}$-modules. If $\mathcal{Q}$ and all $\mathcal{F}_i$ are flat $\mathcal{O}$-modules, then for any presheaf $\mathcal{G}$ of $\mathcal{O}$-modules the complex
$$\ldots \to \mathcal{F}_2 \otimes_{p,\mathcal{O}} \mathcal{G} \to \mathcal{F}_1 \otimes_{p,\mathcal{O}} \mathcal{G} \to \mathcal{F}_0 \otimes_{p,\mathcal{O}} \mathcal{G} \to \mathcal{Q} \otimes_{p,\mathcal{O}} \mathcal{G} \to 0$$
is exact also. If $\mathcal{C}$ is a site and $\mathcal{O}$ is a sheaf of rings then the same result holds $Mod(\mathcal{O})$.

Proof. Follows from Lemma 16.26.7 by splitting the complex into short exact sequences and using Lemma 16.26.8 to prove inductively that $\text{Im}(\mathcal{F}_{i+1} \to \mathcal{F}_i)$ is flat. $\square$

Lemma 16.26.10. *Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a map of sheaves of rings on a site $\mathcal{C}$. If $\mathcal{G}$ is a flat $\mathcal{O}_1$-module, then $\mathcal{G} \otimes_{\mathcal{O}_1} \mathcal{O}_2$ is a flat $\mathcal{O}_2$-module.*

Proof. This is true because

$$(\mathcal{G} \otimes_{\mathcal{O}_1} \mathcal{O}_2) \otimes_{\mathcal{O}_2} \mathcal{H} = \mathcal{G} \otimes_{\mathcal{O}_1} \mathcal{F}$$

(as sheaves of abelian groups for example). □

16.27. Flat morphisms

Definition 16.27.1. Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \longrightarrow (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. We say $(f, f^\sharp)$ is *flat* if the ring map $f^\sharp : f^{-1}\mathcal{O}' \to \mathcal{O}$ is flat. We say a morphism of ringed sites is *flat* if the associated morphism of ringed topoi is flat.

Lemma 16.27.2. *Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{C}')$ be a morphism of ringed topoi. Then*

$$f^{-1} : Ab(\mathcal{C}') \longrightarrow Ab(\mathcal{C}), \quad \mathcal{F} \longmapsto f^{-1}\mathcal{F}$$

is exact. If $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ is a flat morphism of ringed topoi then

$$f^* : Mod(\mathcal{O}') \longrightarrow Mod(\mathcal{O}), \quad \mathcal{F} \longmapsto f^*\mathcal{F}$$

is exact.

Proof. Given an abelian sheaf $\mathcal{G}$ on $\mathcal{C}'$ the underlying sheaf of sets of $f^{-1}\mathcal{G}$ is the same as f^{-1} of the underlying sheaf of sets of $\mathcal{G}$, see Sites, Section 9.38. Hence the exactness of f^{-1} for sheaves of sets (required in the definition of a morphism of topoi, see Sites, Definition 9.15.1) implies the exactness of f^{-1} as a functor on abelian sheaves.

To see the statement on modules recall that $f^*\mathcal{F}$ is defined as the tensor product $f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}', f^\sharp} \mathcal{O}$. Hence f^* is a composition of functors both of which are exact. □

16.28. Invertible modules

Here is the definition.

Definition 16.28.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site.

(1) A finite locally free $\mathcal{O}$-module $\mathcal{F}$ is said to have *rank r* if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ of U such that $\mathcal{F}|_{U_i}$ is isomorphic to $\mathcal{O}_{U_i}^{\oplus r}$ as an $\mathcal{O}_{U_i}$-module.
(2) An *invertible $\mathcal{O}$-module* is a finite locally free $\mathcal{O}$-module of rank 1.
(3) The sheaf $\mathcal{O}^*$ is the subsheaf of $\mathcal{O}$ defined by the rule

$$U \longmapsto \mathcal{O}^*(U) = \{f \in \mathcal{O}(U) \mid \exists g \in \mathcal{O}(U) \text{ such that } fg = 1\}$$

It is a sheaf of abelian groups with multiplication as the group law.

Lemma 16.28.2. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed space.*

(1) *If $\mathcal{L}$, $\mathcal{N}$ are invertible $\mathcal{O}$-modules, then so is $\mathcal{L} \otimes_{\mathcal{O}} \mathcal{N}$.*
(2) *If $\mathcal{L}$ is an invertible $\mathcal{O}$-modules, then so is $\mathcal{L}^{\otimes -1} = \mathcal{H}om_{\mathcal{O}}(\mathcal{L}, \mathcal{O})$.*
(3) *If $\mathcal{L}$ is an invertible $\mathcal{O}$-module, then the evaluation map $\mathcal{L} \otimes_{\mathcal{O}} \mathcal{L}^{\otimes -1} \to \mathcal{O}$ is an isomorphism.*

Proof. Omitted. □

Lemma 16.28.3. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed space. There exists a set of invertible modules $\{\mathcal{L}_i\}_{i \in I}$ such that each invertible module on $(\mathcal{C}, \mathcal{O})$ is isomorphic to exactly one of the $\mathcal{L}_i$.*

Proof. Omitted, but see Sheaves of Modules, Lemma 15.21.5. $\square$

This lemma says roughly speaking that the collection of isomorphism classes of invertible sheaves forms a set. Lemma 16.28.2 says that tensor product defines the structure of an abelian group on this set.

Definition 16.28.4. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. The *Picard group* $\mathrm{Pic}(\mathcal{O})$ the ringed site is the abelian group whose elements are isomorphism classes of invertible $\mathcal{O}$-modules, with addition corresponding to tensor product.

16.29. Modules of differentials

In this section we briefly explain how to define the module of relative differentials for a morphism of ringed topoi. We suggest the reader take a look at the corresponding section in the chapter on commutative algebra (Algebra, Section 7.122).

Definition 16.29.1. Let $\mathcal{C}$ be a site. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. Let $\mathcal{F}$ be an $\mathcal{O}_2$-module. A $\mathcal{O}_1$*-derivation* or more precisely a φ*-derivation* into $\mathcal{F}$ is a map $D : \mathcal{O}_2 \to \mathcal{F}$ which is additive, annihilates the image of $\mathcal{O}_1 \to \mathcal{O}_2$, and satisfies the *Leibniz rule*

$$D(ab) = aD(b) + D(a)b$$

for all a, b local sections of $\mathcal{O}_2$ (wherever they are both defined). We denote $\mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{F})$ the set of φ-derivations into $\mathcal{F}$.

This is the sheaf theoretic analogue of Algebra, Definition 16.29.1. Given a derivation $D : \mathcal{O}_2 \to \mathcal{F}$ as in the definition the map on global sections

$$D : \Gamma(\mathcal{O}_2) \longrightarrow \Gamma(\mathcal{F})$$

clearly is a $\Gamma(\mathcal{O}_1)$-derivation as in the algebra definition. Note that if $\alpha : \mathcal{F} \to \mathcal{G}$ is a map of $\mathcal{O}_2$-modules, then there is an induced map

$$\mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{F}) \longrightarrow \mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{G})$$

given by the rule $D \mapsto \alpha \circ D$. In other words we obtain a functor.

Lemma 16.29.2. *Let $\mathcal{C}$ be a site. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. The functor*

$$Mod(\mathcal{O}_2) \longrightarrow Ab, \quad \mathcal{F} \longmapsto Der_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{F})$$

is representable.

Proof. This is proved in exactly the same way as the analogous statement in algebra. During this proof, for any sheaf of sets $\mathcal{F}$ on $\mathcal{C}$, let us denote

$$\mathcal{O}_2[\mathcal{F}] = \bigoplus\nolimits_{U \in Ob(\mathcal{C}), s \in \mathcal{F}(U)} j_{U!}(\mathcal{O}_2|_U).$$

This is a sheaf of $\mathcal{O}_2$-modules. If $\mathcal{F}$ is actually a sheaf of $\mathcal{O}_2$-modules, then there is a canonical map

$$c : \mathcal{O}_2[\mathcal{F}] \longrightarrow \mathcal{F}$$

which maps the summand $j_{U!}(\mathcal{O}_2|_U)$ corresponding to $s \in \mathcal{O}_2(U)$ into $\mathcal{F}$ by the map which is adjoint to the map $\mathcal{O}_2|_U \to \mathcal{F}|_U$ determined by s. We will employ the short hand $[s] \mapsto s$

to describe this map and similarly for other maps below. OK, and now consider the map of $\mathcal{O}_2$-modules

$$\begin{array}{ccc} \mathcal{O}_2[\mathcal{O}_2 \times \mathcal{O}_2] \oplus \mathcal{O}_2[\mathcal{O}_2 \times \mathcal{O}_2] \oplus \mathcal{O}_2[\mathcal{O}_1] & \longrightarrow & \mathcal{O}_2[\mathcal{O}_2] \\ [(a,b)] \oplus [(f,g)] \oplus [h] & \longmapsto & [a+b] - [a] - [b] + \\ & & [fg] - g[f] - f[g] + \\ & & [\varphi(h)] \end{array} \tag{16.29.2.1}$$

with short hand notation as above. Set $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$ equal to the cokernel of this map. Then it is clear that there exists a map of sheaves of sets

$$\mathrm{d} : \mathcal{O}_2 \longrightarrow \Omega_{\mathcal{O}_2/\mathcal{O}_1}$$

mapping a local section f to the image of $[f]$ in $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$. By construction d is a $\mathcal{O}_1$-derivation. Next, let $\mathcal{F}$ be a sheaf of $\mathcal{O}_2$-modules and let $D : \mathcal{O}_2 \to \mathcal{F}$ be a $\mathcal{O}_1$-derivation. Then we can define

$$\alpha_D : \Omega_{\mathcal{O}_2/\mathcal{O}_1} \longrightarrow \mathcal{F}$$

by setting $\alpha_D(f[g]) = fD(g)$ for local sections f, g of $\mathcal{O}_2$. It follows from the definition of a derivation that this map annihilates sections in the image of the map (16.29.2.1), so that we get the desired map. Since it is clear that $\alpha_D \circ \mathrm{d} = D$ the lemma is proved. □

Definition 16.29.3. Let $\mathcal{C}$ be a site. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. The *module of differentials* of the ring map φ is the object representing the functor $\mathcal{F} \mapsto \mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{F})$ which exists by Lemma 16.29.2. It is denoted $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$, and the *universal φ-derivation* is denoted $\mathrm{d} : \mathcal{O}_2 \to \Omega_{\mathcal{O}_2/\mathcal{O}_1}$.

Since this module and the derivation form the universal object representing a functor, this notion is clearly intrinsic (i.e., does not depend on the choice of the site underlying the ringed topos, see Section 16.18). Note that $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$ is the cokernel of the map (16.29.2.1) of $\mathcal{O}_2$-modules. Moreover the map d is described by the rule that $\mathrm{d}f$ is the image of the local section $[f]$.

Lemma 16.29.4. *Let $\mathcal{C}$ be a site. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. For any object U of $\mathcal{C}$ there is a canonical isomorphism*

$$\Omega_{\mathcal{O}_2/\mathcal{O}_1}|_U = \Omega_{(\mathcal{O}_2|_U)/(\mathcal{O}_1|_U)}$$

compatible with universal derivations.

Proof. Let us denote $j : \mathcal{C}/U \to \mathcal{C}$ the usual localization functor. We are trying to show that $j^{-1}\Omega_{\mathcal{O}_2/\mathcal{O}_1} = \Omega_{j^{-1}\mathcal{O}_2/j^{-1}\mathcal{O}_1}$. Note that on the one hand

$$\mathcal{H}om_{j^{-1}\mathcal{O}_2}(j^{-1}\Omega_{\mathcal{O}_2/\mathcal{O}_1}, \mathcal{F}) = \mathcal{H}om_{\mathcal{O}_2}(\Omega_{\mathcal{O}_2/\mathcal{O}_1}, j_*\mathcal{F}) = \mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, j_*\mathcal{F})$$

and on the other hand

$$\mathcal{H}om_{j^{-1}\mathcal{O}_2}(\Omega_{j^{-1}\mathcal{O}_2/j^{-1}\mathcal{O}_1}, \mathcal{F}) = \mathrm{Der}_{j^{-1}\mathcal{O}_1}(j^{-1}\mathcal{O}_2, \mathcal{F})$$

Hence we have to show that $\mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, j_*\mathcal{F}) = \mathrm{Der}_{j^{-1}\mathcal{O}_1}(j^{-1}\mathcal{O}_2, \mathcal{F})$. By adjunction there is a natural identification

$$\mathcal{M}or_{Sh(\mathcal{C})}(\mathcal{O}_2, j_*\mathcal{F}) = \mathcal{M}or_{Sh(\mathcal{C}/U)}(j^{-1}\mathcal{O}_2, \mathcal{F})$$

and it suffices to prove that $\mathcal{O}_1$-derivations on the left hand side correspond to $j^{-1}\mathcal{O}_1$-derivations on the right hand side and vice versa. We omit the verification that this is so. □

Here is a particular situation where derivations come up naturally.

Lemma 16.29.5. *Let $\mathcal{C}$ be a site. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. Consider a short exact sequence*

$$0 \to \mathcal{F} \to \mathcal{A} \to \mathcal{O}_2 \to 0$$

Here $\mathcal{A}$ is a sheaf of $\mathcal{O}_1$-algebras, $\pi : \mathcal{A} \to \mathcal{O}_2$ is a surjection of sheaves of $\mathcal{O}_1$-algebras, and $\mathcal{F} = Ker(\pi)$ is its kernel. Assume $\mathcal{F}$ an ideal sheaf with square zero in $\mathcal{A}$. So $\mathcal{F}$ has a natural structure of an $\mathcal{O}_2$-module. A section $s : \mathcal{O}_2 \to \mathcal{A}$ of π is a $\mathcal{O}_1$-algebra map such that $\pi \circ s = id$. Given any section $s : \mathcal{O}_2 \to \mathcal{F}$ of π and any φ-derivation $D : \mathcal{O}_1 \to \mathcal{F}$ the map

$$s + D : \mathcal{O}_1 \to \mathcal{A}$$

is a section of π and every section s' is of the form $s + D$ for a unique φ-derivation D.

Proof. Recall that the $\mathcal{O}_2$-module structure on $\mathcal{F}$ is given by $h\tau = \tilde{h}\tau$ (multiplication in $\mathcal{A}$) where h is a local section of $\mathcal{O}_2$, and $\tilde{h}$ is a local lift of h to a local section of $\mathcal{A}$, and τ is a local section of $\mathcal{F}$. In particular, given s, we may use $\tilde{h} = s(h)$. To verify that $s + D$ is a homomorphism of sheaves of rings we compute

$$\begin{aligned} (s+D)(ab) &= s(ab) + D(ab) \\ &= s(a)s(b) + aD(b) + D(a)b \\ &= s(a)s(b) + s(a)D(b) + D(a)s(b) \\ &= (s(a) + D(a))(s(b) + D(b)) \end{aligned}$$

by the Leibniz rule. In the same manner one shows $s + D$ is a $\mathcal{O}_1$-algebra map because D is an $\mathcal{O}_1$-derivation. Conversely, given s' we set $D = s' - s$. Details omitted. □

Definition 16.29.6. Let $X = (Sh(\mathcal{C}), \mathcal{O})$ and $Y = (Sh(\mathcal{C}'), \mathcal{O}')$ be ringed topoi. Let $(f, f^\sharp) : X \to Y$ be a morphism of ringed topoi. In this situation

(1) for a sheaf $\mathcal{F}$ of $\mathcal{O}$-modules a *Y-derivation* $D : \mathcal{O} \to \mathcal{F}$ is just a $f^\sharp$-derivation, and
(2) the *sheaf of differentials* $\Omega_{X/Y}$ *of X over Y* is the module of differentials of $f^\sharp : f^{-1}\mathcal{O}' \to \mathcal{O}$, see Definition 16.29.3.

Thus $\Omega_{X/Y}$ comes equipped with a *universal Y-derivation* $\mathrm{d}_{X/Y} : \mathcal{O} \longrightarrow \Omega_{X/Y}$.

Recall that $f^\sharp : f^{-1}\mathcal{O}' \to \mathcal{O}$ so that this definition makes sense.

Lemma 16.29.7. *Let $X = (Sh(\mathcal{C}_X), \mathcal{O}_X)$, $Y = (Sh(\mathcal{C}_Y), \mathcal{O}_Y)$, $X' = (Sh(\mathcal{C}_{X'}), \mathcal{O}_{X'})$, and $Y' = (Sh(\mathcal{C}_{Y'}), \mathcal{O}_{Y'})$ be ringed topoi. Let*

$$\begin{tikzcd} X' \arrow[r, "f"'] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

be a commutative diagram of morphisms of ringed topoi. The map $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_{X'}$ composed with the map $f_*d_{X'/Y'} : f_*\mathcal{O}_{X'} \to f_*\Omega_{X'/Y'}$ is a Y-derivation. Hence we obtain a canonical map of $\mathcal{O}_X$-modules $\Omega_{X/Y} \to f_*\Omega_{X'/Y'}$, and by adjointness of f_* and f^* a canonical $\mathcal{O}_{X'}$-module homomorphism

$$c_f : f^*\Omega_{X/Y} \longrightarrow \Omega_{X'/Y'}.$$

It is uniquely characterized by the property that $f^*d_{X/Y}(t)$ mapsto $d_{X'/Y'}(f^*t)$ for any local section t of $\mathcal{O}_X$.

Proof. This is clear except for the last assertion. Let us explain the meaning of this. Let $U \in Ob(\mathcal{C}_X)$ and let $t \in \mathcal{O}_X(U)$. This is what it means for t to be a local section of $\mathcal{O}_X$. Now, we may think of t as a map of sheaves of sets $t : h_U^\# \to \mathcal{O}_X$. Then $f^{-1}t : f^{-1}h_U^\# \to f^{-1}\mathcal{O}_X$. By f^*t we mean the composition

$$\begin{tikzcd} f^{-1}h_U^\# \arrow[r, "f^{-1}t"] \arrow[rr, bend left, "f^*t"] & f^{-1}\mathcal{O}_X \arrow[r, "f^\sharp"] & \mathcal{O}_{X'} \end{tikzcd}$$

Note that $\mathrm{d}_{X/Y}(t) \in \Omega_{X/Y}(U)$. Hence we may think of $\mathrm{d}_{X/Y}(t)$ as a map $\mathrm{d}_{X/Y}(t) : h_U^\# \to \Omega_{X/Y}$. Then $f^{-1}\mathrm{d}_{X/Y}(t) : f^{-1}h_U^\# \to f^{-1}\Omega_{X/Y}$. By $f^*\mathrm{d}_{X/Y}(t)$ we mean the composition

$$\begin{tikzcd} f^{-1}h_U^\# \arrow[r, "f^{-1}\mathrm{d}_{X/Y}(t)"] \arrow[rr, bend left, "f^*\mathrm{d}_{X/Y}(t)"] & f^{-1}\Omega_{X/Y} \arrow[r, "1\otimes\mathrm{id}"] & f^*\Omega_{X/Y} \end{tikzcd}$$

OK, and now the statement of the lemma means that we have

$$c_f \circ f^*t = f^*\mathrm{d}_{X/Y}(t)$$

as maps from $f^{-1}h_U^\#$ to $\Omega_{X'/Y'}$. We omit the verification that this property holds for c_f as defined in the lemma. (Hint: The first map $c'_f : \Omega_{X/Y} \to f_*\Omega_{X'/Y'}$ satisfies $c'_f(\mathrm{d}_{X/Y}(t)) = f_*\mathrm{d}_{X'/Y'}(f^\sharp(t))$ as sections of $f_*\Omega_{X'/Y'}$ over U, and you have to turn this into the equality above by using adjunction.) The reason that this uniquely characterizes c_f is that the images of $f^*\mathrm{d}_{X/Y}(t)$ generate the $\mathcal{O}_{X'}$-module $f^*\Omega_{X/Y}$ simply because the local sections $\mathrm{d}_{X/Y}(t)$ generate the $\mathcal{O}_X$-module $\Omega_{X/Y}$. □

16.30. Stalks of modules

We have to be a bit careful when taking stalks at points, since the colimit defining a stalk (see Sites, Equation 9.28.1.1) may not be filtered[3]. On the other hand, by definition of a point of a site the stalk functor is exact and commutes with arbitrary colimits. In other words, it behaves exactly as if the colimit were filtered.

Lemma 16.30.1. *Let $\mathcal{C}$ be a site. Let p be a point of $\mathcal{C}$.*

(1) *We have $(\mathcal{F}^\#)_p = \mathcal{F}_p$ for any presheaf of sets on $\mathcal{C}$.*
(2) *The stalk functor $Sh(\mathcal{C}) \to Sets$, $\mathcal{F} \mapsto \mathcal{F}_p$ is exact (see Categories, Definition 4.21.1) and commutes with arbitrary colimits.*
(3) *The stalk functor $PSh(\mathcal{C}) \to Sets$, $\mathcal{F} \mapsto \mathcal{F}_p$ is exact (see Categories, Definition 4.21.1) and commutes with arbitrary colimits.*

Proof. By Sites, Lemma 9.28.5 we have (1). By Sites, Lemmas 9.28.4 we see that $PSh(\mathcal{C}) \to Sets$, $\mathcal{F} \mapsto \mathcal{F}_p$ is a left adjoint, and by Sites, Lemma 9.28.5 we see the same thing for $PSh(\mathcal{C}) \to Sets$, $\mathcal{F} \mapsto \mathcal{F}_p$. Hence the stalk functor commutes with arbitrary colimits (see Categories, Lemma 4.22.2). It follows from the definition of a point of a site, see Sites, Definition 9.28.2 that $Sh(S_{\acute{e}tale}) \to Sets$, $\mathcal{F} \mapsto \mathcal{F}_p$ is exact. Since sheafification is exact (Sites, Lemma 9.10.14) it follows that $PSh(S_{\acute{e}tale}) \to Sets$, $\mathcal{F} \mapsto \mathcal{F}_p$ is exact. □

[3]Of course in almost any naturally occurring case the colimit is filtered and some of the discussion in this section may be simplified.

In particular, since the stalk functor $\mathcal{F} \mapsto \mathcal{F}_p$ on presheaves commutes with all finite limits and colimits we may apply the reasoning of the proof of Sites, Proposition 9.38.3. The result of such an argument is that if $\mathcal{F}$ is a (pre)sheaf of algebraic structures listed in Sites, Proposition 9.38.3 then the stalk $\mathcal{F}_p$ is naturally an algebraic structure of the same kind. Let us explain this in detail when $\mathcal{F}$ is an abelian presheaf. In this case the addition map $+ : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ induces a map

$$+ : \mathcal{F}_p \times \mathcal{F}_p = (\mathcal{F} \times \mathcal{F})_p \longrightarrow \mathcal{F}_p$$

where the equal sign uses that stalk functor on presheaves of sets commutes with finite limits. This defines a group structure on the stalk $\mathcal{F}_p$. In this way we obtain our stalk functor

$$PAb(\mathcal{C}) \longrightarrow Ab, \quad \mathcal{F} \longmapsto \mathcal{F}_p$$

By construction the underlying set of $\mathcal{F}_p$ is the stalk of the underlying presheaf of sets. This also defines our stalk functor for sheaves of abelian groups by precomposing with the inclusion $Ab(\mathcal{C}) \subset PAb(\mathcal{C})$.

Lemma 16.30.2. *Let $\mathcal{C}$ be a site. Let p be a point of $\mathcal{C}$.*

1. *The functor $Ab(\mathcal{C}) \to Ab$, $\mathcal{F} \mapsto \mathcal{F}_p$ is exact.*
2. *The stalk functor $PAb(\mathcal{C}) \to Ab$, $\mathcal{F} \mapsto \mathcal{F}_p$ is exact.*
3. *For $\mathcal{F} \in Ob(PAb(\mathcal{C}))$ we have $\mathcal{F}_p = \mathcal{F}^\#_p$.*

Proof. This is formal from the results of Lemma 16.30.1 and the construction of the stalk functor above. $\square$

Next, we turn to the case of sheaves of modules. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. (It suffices for the discussion that $\mathcal{O}$ be a presheaf of rings.) Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules. Let p be a point of $\mathcal{C}$. In this case we get a map

$$\cdot : \mathcal{O}_p \times \mathcal{O}_p = (\mathcal{O} \times \mathcal{O})_p \longrightarrow \mathcal{O}_p$$

which is the stalk of the multiplication map and

$$\cdot : \mathcal{O}_p \times \mathcal{F}_p = (\mathcal{O} \times \mathcal{F})_p \longrightarrow \mathcal{F}_p$$

which is the stalk of the multiplication map. We omit the verification that this defines a ring structure on $\mathcal{O}_p$ and an $\mathcal{O}_p$-module structure on $\mathcal{F}_p$. In this way we obtain a functor

$$PMod(\mathcal{O}) \longrightarrow Mod(\mathcal{O}_p), \quad \mathcal{F} \longmapsto \mathcal{F}_p$$

By construction the underlying set of $\mathcal{F}_p$ is the stalk of the underlying presheaf of sets. This also defines our stalk functor for sheaves of $\mathcal{O}$-modules by precomposing with the inclusion $Mod(\mathcal{O}) \subset PMod(\mathcal{O})$.

Lemma 16.30.3. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let p be a point of $\mathcal{C}$.*

1. *The functor $Mod(\mathcal{O}) \to Mod(\mathcal{O}_p)$, $\mathcal{F} \mapsto \mathcal{F}_p$ is exact.*
2. *The stalk functor $PMod(\mathcal{O}) \to Mod(\mathcal{O}_p)$, $\mathcal{F} \mapsto \mathcal{F}_p$ is exact.*
3. *For $\mathcal{F} \in Ob(PMod(\mathcal{O}))$ we have $\mathcal{F}_p = \mathcal{F}^\#_p$.*

Proof. This is formal from the results of Lemma 16.30.2, the construction of the stalk functor above, and Lemma 16.14.1. $\square$

Lemma 16.30.4. *Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi or ringed sites. Let p be a point of $\mathcal{C}$ or $Sh(\mathcal{C})$ and set $q = f \circ p$. Then*

$$(f^*\mathcal{F})_p = \mathcal{F}_q \otimes_{\mathcal{O}_{\mathcal{D},q}} \mathcal{O}_{\mathcal{C},p}$$

for any $\mathcal{O}_\mathcal{D}$-module $\mathcal{F}$.

Proof. We have

$$f^*\mathcal{F} = f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_\mathcal{D}} \mathcal{O}_\mathcal{C}$$

by definition. Since taking stalks at p (i.e., applying p^{-1}) commutes with $\otimes$ by Lemma 16.24.1 we win by the relation between the stalk of pullbacks at p and stalks at q explained in Sites, Lemma 9.30.1 or Sites, Lemma 9.30.2. $\square$

16.31. Skyscraper sheaves

Let p be a point of a site $\mathcal{C}$ or a topos $Sh(\mathcal{C})$. In this section we study the exactness properties of the functor which associates to an abelian group A the skyscraper sheaf p_*A. First, recall that $p_* : Sets \to Sh(\mathcal{C})$ has a lot of exactness properties, see Sites, Lemmas 9.28.9 and 9.28.10.

Lemma 16.31.1. *Let $\mathcal{C}$ be a site. Let p be a point of $\mathcal{C}$ or of its associated topos.*

(1) *The functor $p_* : Ab \to Ab(\mathcal{C})$, $A \mapsto p_*A$ is exact.*
(2) *There is a functorial direct sum decomposition*

$$p^{-1}p_*A = A \oplus I(A)$$

for $A \in Ob(Ab)$.

Proof. By Sites, Lemma 9.28.9 there are functorial maps $A \to p^{-1}p_*A \to A$ whose composition equals id_A. Hence a functorial direct sum decomposition as in (2) with $I(A)$ the kernel of the adjunction map $p^{-1}p_*A \to A$. The functor p_* is left exact by Lemma 16.14.3. The functor p_* transforms surjections into surjections by Sites, Lemma 9.28.10. Hence (1) holds. $\square$

To do the same thing for sheaves of modules, suppose given a point p of a ringed topos $(Sh(\mathcal{C}), \mathcal{O})$. Recall that p^{-1} is just the stalk functor. Hence we can think of p as a morphism of ringed topoi

$$(p, \mathrm{id}_{\mathcal{O}_p}) : (Sh(pt), \mathcal{O}_p) \longrightarrow (Sh(\mathcal{C}), \mathcal{O}).$$

Thus we get a pullback functor $p^* : Mod(\mathcal{O}) \to Mod(\mathcal{O}_p)$ which equals the stalk functor, and which we discussed in Lemma 16.30.3. In this section we consider the functor $p_* : Mod(\mathcal{O}_p) \to Mod(\mathcal{O})$.

Lemma 16.31.2. *Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. Let p be a point of the topos $Sh(\mathcal{C})$.*

(1) *The functor $p_* : Mod(\mathcal{O}_p) \to Mod(\mathcal{O})$, $M \mapsto p_*M$ is exact.*
(2) *There is a functorial direct sum decomposition of $\mathcal{O}_p$-modules*

$$p^{-1}p_*M = M \oplus I(M)$$

for M a $\mathcal{O}_p$-module.

Proof. This follows immediately from the corresponding result for abelian sheaves in Lemma 16.31.1. $\square$

Example 16.31.3. Let G be a group. Consider the site $\mathcal{T}_G$ and its point p, see Sites, Example 9.29.6. Let R be a ring with a G-action which corresponds to a sheaf of rings $\mathcal{O}$ on $\mathcal{T}_G$. Then $\mathcal{O}_p = R$ where we forget the G-action. In this case $p^{-1}p_*M = \mathrm{Map}(G, M)$ and $I(M) = \{f : G \to M \mid f(1_G) = 0\}$ and $M \to \mathrm{Map}(G, M)$ assigns to $m \in M$ the constant function with value m.

16.32. Localization and points

Lemma 16.32.1. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let p be a point of $\mathcal{C}$. Let U be an object of $\mathcal{C}$. For $\mathcal{G}$ in $Mod(\mathcal{O}_U)$ we have*

$$(j_{U!}\mathcal{G})_p = \bigoplus\nolimits_q \mathcal{G}_q$$

where the coproduct is over the points q of $\mathcal{C}/U$ lying over p, see Sites, Lemma 9.31.2.

Proof. We use the description of $j_{U!}\mathcal{G}$ as the sheaf associated to the presheaf $V \mapsto \bigoplus_{\varphi \in Mor_{\mathcal{C}}(V,U)} \mathcal{G}(V/_\varphi U)$ of Lemma 16.19.2. The stalk of $j_{U!}\mathcal{G}$ at p is equal to the stalk of this presheaf, see Lemma 16.30.3. Let $u : \mathcal{C} \to Sets$ be the functor corresponding to p (see Sites, Section 9.28). Hence we see that

$$(j_{U!}\mathcal{G})_p = colim_{(V,y)} \bigoplus\nolimits_{\varphi : V \to U} \mathcal{G}(V/_\varphi U)$$

where the colimit is taken in the category of abelian groups. To a quadruple (V, y, φ, s) occuring in this colimit, we can assign $x = u(\varphi)(y) \in u(U)$. Hence we obtain

$$(j_{U!}\mathcal{G})_p = \bigoplus\nolimits_{x \in u(U)} colim_{(\varphi : V \to U, y),\ u(\varphi)(y) = x} \mathcal{G}(V/_\varphi U).$$

This is equal to the expression of the lemma by the description of the points q lying over x in Sites, Lemma 9.31.2. □

Remark 16.32.2. Warning: The result of Lemma 16.32.1 has no analogue for $j_{U,*}$.

16.33. Pullbacks of flat modules

The pullback of a flat module along a morphism of ringed topoi is flat. This is quite tricky to prove, except when there are enough points. Here we prove it only in this case and we will add the general case if we ever need it.

Lemma 16.33.1. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let p be a point of $\mathcal{C}$. If $\mathcal{F}$ is a flat $\mathcal{O}$-module, then $\mathcal{F}_p$ is a flat $\mathcal{O}_p$-module.*

Proof. Let M be an $\mathcal{O}_p$-module. Then

$$\begin{aligned}(p_*M \otimes_{\mathcal{O}} \mathcal{F})_p &= p^{-1}(p_*M \otimes_{\mathcal{O}} \mathcal{F}) \\ &= p^{-1}p_*M \otimes_{\mathcal{O}_p} \mathcal{F}_p \\ &= M \otimes_{\mathcal{O}_p} \mathcal{F}_p \oplus I(M) \otimes_{\mathcal{O}_p} \mathcal{F}_p\end{aligned}$$

where we have used the description of the stalk functor as a pullback, Lemma 16.24.1, and Lemma 16.31.2. Since p_* is exact by Lemma 16.31.2, it is clear that if $\mathcal{F}$ is exact, then also the functor $M \mapsto M \otimes_{\mathcal{O}_p} \mathcal{F}_p$ is exact, i.e., $\mathcal{F}_p$ is flat. □

Lemma 16.33.2. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. If $\mathcal{C}$ has enough points, then $\mathcal{F}$ is flat if and only if $\mathcal{F}_p$ is a flat $\mathcal{O}_p$-module for all points p of $\mathcal{C}$.*

Proof. By Lemma 16.33.1 we see one of the implications. For the converse, use that $(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G})_p = \mathcal{F}_p \otimes_{\mathcal{O}_p} \mathcal{G}_p$ by Lemma 16.24.1 and Lemma 16.14.4. □

Lemma 16.33.3. *Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \to (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$ be a morphism of ringed topoi or ringed sites. Assume $\mathcal{C}$ has enough points[4]. Then $f^*\mathcal{F}$ is a flat $\mathcal{O}_{\mathcal{C}}$-module whenever $\mathcal{F}$ is a flat $\mathcal{O}_{\mathcal{D}}$-module.*

[4]This assumption is not necessary, see introduction to this section.

Proof. Let p be a point of $\mathcal{C}$ and set $q = f \circ p$. Then

$$(f^*\mathcal{F})_p = \mathcal{F}_q \otimes_{\mathcal{O}_{\mathcal{D},q}} \mathcal{O}_{\mathcal{C},p}$$

by Lemma 16.30.4. Hence if $\mathcal{F}$ is flat, then $\mathcal{F}_q$ is a flat $\mathcal{O}_{\mathcal{D},q}$-module by Lemma 16.33.1 and hence by Algebra, Lemma 7.35.6 we see that $(f^*\mathcal{F})_p$ is a flat $\mathcal{O}_{\mathcal{C},p}$-module. This implies that $f^*\mathcal{F}$ is a flat $\mathcal{O}_{\mathcal{C}}$-module by Lemma 16.33.2. □

16.34. Locally ringed topoi

A reference for this section is [**MA71**, Exposé IV, Exercice 13.9].

Lemma 16.34.1. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. The following are equivalent*

(1) *For every object U of $\mathcal{C}$ and $f \in \mathcal{O}(U)$ there exists a covering $\{U_j \to U\}$ such that for each j either $f|_{U_j}$ is invertible or $(1-f)|_{U_j}$ is invertible.*
(2) *For every object U of $\mathcal{C}$ and $f_1, \ldots, f_n \in \mathcal{O}(U)$ which generate the unit ideal in $\mathcal{O}(U)$ there exists a covering $\{U_j \to U\}$ such that for each j there exists an i such that $f_i|_{U_j}$ is invertible.*
(3) *The map of sheaves of sets*

$$(\mathcal{O} \times \mathcal{O}) \amalg (\mathcal{O} \times \mathcal{O}) \longrightarrow \mathcal{O} \times \mathcal{O}$$

which maps (f, a) in the first component to (f, af) and (f, b) in the second component to $(f, b(1-f))$ is surjective.

Proof. It is clear that (2) implies (1). To show that (1) implies (2) we argue by induction on n. The first case is $n = 2$ (since $n = 1$ is trivial). In this case we have $a_1 f_1 + a_2 f_2 = 1$ for some $a_1, a_2 \in \mathcal{O}(U)$. By assumption we can find a covering $\{U_j \to U\}$ such that for each j either $a_1 f_1|_{U_j}$ is invertible or $a_2 f_2|_{U_j}$ is invertible. Hence either $f_1|_{U_j}$ is invertible or $f_2|_{U_j}$ is invertible as desired. For $n > 2$ we have $a_1 f_1 + \ldots + a_n f_n = 1$ for some $a_1, \ldots, a_n \in \mathcal{O}(U)$. By the case $n = 2$ we see that we have some covering $\{U_j \to U\}_{j \in J}$ such that for each j either $f_n|_{U_j}$ is invertible or $a_1 f_1 + \ldots + a_{n-1} f_{n-1}|_{U_j}$ is invertible. Say the first case happens for $j \in J_n$. Set $J' = J \setminus J_n$. By induction hypothesis, for each $j \in J'$ we can find a covering $\{U_{jk} \to U_j\}_{k \in K_j}$ such that for each $k \in K_j$ there exists an $i \in \{1, \ldots, n-1\}$ such that $f_i|_{U_{jk}}$ is invertible. By the axioms of a site the family of morphisms $\{U_j \to U\}_{j \in J_n} \cup \{U_{jk} \to U\}_{j \in J', k \in K_j}$ is a covering which has the desired property.

Assume (1). To see that the map in (3) is surjective, let (f, c) be a section of $\mathcal{O} \times \mathcal{O}$ over U. By assumption there exists a covering $\{U_j \to U\}$ such that for each j either f or $1-f$ restricts to an invertible section. In the first case we can take $a = c|_{U_j}(f|_{U_j})^{-1}$, and in the second case we can take $b = c|_{U_j}(1 - f|_{U_j})^{-1}$. Hence (f, c) is in the image of the map on each of the members. Conversely, assume (3) holds. For any U and $f \in \mathcal{O}(U)$ there exists a covering $\{U_j \to U\}$ of U such that the section $(f, 1)|_{U_j}$ is in the image of the map in (3) on sections over U_j. This means precisely that either f or $1-f$ restricts to an invertible section over U_j, and we see that (1) holds. □

Lemma 16.34.2. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Consider the following conditions*

(1) *For every object U of $\mathcal{C}$ and $f \in \mathcal{O}(U)$ there exists a covering $\{U_j \to U\}$ such that for each j either $f|_{U_j}$ is invertible or $(1-f)|_{U_j}$ is invertible.*
(2) *For every point p of $\mathcal{C}$ the stalk $\mathcal{O}_p$ is either the zero ring or a local ring.*

We always have (1) ⇒ (2). If $\mathcal{C}$ has enough points then (1) and (2) are equivalent.

Proof. Assume (1). Let p be a point of $\mathcal{C}$ given by a functor $u : \mathcal{C} \to \mathit{Sets}$. Let $f_p \in \mathcal{O}_p$. Since $\mathcal{O}_p$ is computed by Sites, Equation (9.28.1.1) we may represent f_p by a triple (U, x, f) where $x \in U(U)$ and $f \in \mathcal{O}(U)$. By assumption there exists a covering $\{U_i \to U\}$ such that for each i either f or $1 - f$ is invertible on U_i. Because u defines a point of the site we see that for some i there exists an $x_i \in u(U_i)$ which maps to $x \in u(U)$. By the discussion surrounding Sites, Equation (9.28.1.1) we see that (U, x, f) and $(U_i, x_i, f|_{U_i})$ define the same element of $\mathcal{O}_p$. Hence we conclude that either f_p or $1 - f_p$ is invertible. Thus $\mathcal{O}_p$ is a ring such that for every element a either a or $1 - a$ is invertible. This means that $\mathcal{O}_p$ is either zero or a local ring, see Algebra, Lemma 7.17.2.

Assume (2) and assume that $\mathcal{C}$ has enough points. Consider the map of sheaves of sets

$$\mathcal{O} \times \mathcal{O} \amalg \mathcal{O} \times \mathcal{O} \longrightarrow \mathcal{O} \times \mathcal{O}$$

of Lemma 16.34.1 part (3). For any local ring R the corresponding map $(R\times R)\amalg(R\times R) \to R \times R$ is surjective, see for example Algebra, Lemma 7.17.2. Since each $\mathcal{O}_p$ is a local ring or zero the map is surjective on stalks. Hence, by our assumption that $\mathcal{C}$ has enough points it is surjective and we win. $\square$

In Modules, Section 15.2 we pointed out how in a ringed space $(X, \mathcal{O}_X)$ there can be an open subspace over which the structure sheaf is zero. To prevent this we can require the sections 1 and 0 to have different values in every stalk of the space X. In the setting of ringed topoi and ringed sites the condition is that

$$\emptyset^\# \longrightarrow \mathrm{Equalizer}(0, 1 : * \longrightarrow \mathcal{O}) \tag{16.34.2.1}$$

is an isomorphism of sheaves. Here $*$ is the singleton sheaf, resp. $\emptyset^\#$ is the ``empty sheaf'', i.e., the final, resp. initial object in the category of sheaves, see Sites, Example 9.10.2, resp. Section 9.37. In other words, the condition is that whenever $U \in \mathit{Ob}(\mathcal{C})$ is not sheaf theoretically empty, then $1, 0 \in \mathcal{O}(U)$ are not equal. Let us state the obligatory lemma.

Lemma 16.34.3. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Consider the statements*

1. *(16.34.2.1) is an isomorphism, and*
2. *for every point p of $\mathcal{C}$ the stalk $\mathcal{O}_p$ is not the zero ring.*

We always have (1) $\Rightarrow$ (2) and if $\mathcal{C}$ has enough points then (1) $\Leftrightarrow$ (2).

Proof. Omitted. $\square$

Lemmas 16.34.1, 16.34.2, and 16.34.3 motivate the following definition.

Definition 16.34.4. A ringed site $(\mathcal{C}, \mathcal{O})$ is said to be *locally ringed site* if (16.34.2.1) is an isomorphism, and the equivalent properties of Lemma 16.34.1 are satisfied.

In [**MA71**, Exposé IV, Exercice 13.9] the condition that (16.34.2.1) be an isomorphism is missing leading to a slightly different notion of a locally ringed site and locally ringed topos. As we are motivated by the notion of a locally ringed space we decided to add this condition (see explanation above).

Lemma 16.34.5. *Being a locally ringed site is an intrinsic property. More precisely,*

1. *if $f : Sh(\mathcal{C}') \to Sh(\mathcal{C})$ is a morphism of topoi and $(\mathcal{C}, \mathcal{O})$ is a locally ringed site, then $(\mathcal{C}', f^{-1}\mathcal{O})$ is a locally ringed site, and*
2. *if $(f, f^\sharp) : (Sh(\mathcal{C}'), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ is an equivalence of ringed topoi, then $(\mathcal{C}, \mathcal{O})$ is locally ringed if and only if $(\mathcal{C}', \mathcal{O}')$ is locally ringed.*

Proof. It is clear that (2) follows from (1). To prove (1) note that as f^{-1} is exact we have $f^{-1}* = *$, $f^{-1}\emptyset^\# = \emptyset^\#$, and f^{-1} commutes with products, equalizers and transforms isomorphisms and surjections into isomorphisms and surjections. Thus f^{-1} transforms the isomorphism (16.34.2.1) into its analogue for $f^{-1}\mathcal{O}$ and transforms the surjection of Lemma 16.34.1 part (3) into the corresponding surjection for $f^{-1}\mathcal{O}$. □

In fact Lemma 16.34.5 part (2) is the analogue of Schemes, Lemma 21.2.2. It assures us that the following definition makes sense.

Definition 16.34.6. A ringed topos $(Sh(\mathcal{C}), \mathcal{O})$ is said to be *locally ringed* if the underlying ringed site $(\mathcal{C}, \mathcal{O})$ is locally ringed.

Next, we want to work out what it means to have a morphism of locally ringed spaces. In order to do this we have the following lemma.

Lemma 16.34.7. *Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Consider the following conditions*

(1) *The diagram of sheaves*

$$\begin{tikzcd} f^{-1}(\mathcal{O}^*_\mathcal{D}) \arrow[r, "f^\sharp"'] \arrow[d] & \mathcal{O}^*_\mathcal{C} \arrow[d] \\ f^{-1}(\mathcal{O}_\mathcal{D}) \arrow[r, "f^\sharp"] & \mathcal{O}_\mathcal{C} \end{tikzcd}$$

is cartesian.

(2) *For any point p of $\mathcal{C}$, setting $q = f \circ p$, the diagram*

$$\begin{tikzcd} \mathcal{O}^*_{\mathcal{D},q} \arrow[r] \arrow[d] & \mathcal{O}^*_{\mathcal{C},p} \arrow[d] \\ \mathcal{O}_{\mathcal{D},q} \arrow[r] & \mathcal{O}_{\mathcal{C},p} \end{tikzcd}$$

of sets is cartesian.

We always have (1) ⇒ (2). If $\mathcal{C}$ has enough points then (1) and (2) are equivalent. If $(Sh(\mathcal{C}), \mathcal{O}_\mathcal{C})$ and $(Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ are locally ringed topoi then (2) is equivalent to

(3) *For any point p of $\mathcal{C}$, setting $q = f \circ p$, the ring map $\mathcal{O}_{\mathcal{D},q} \to \mathcal{O}_{\mathcal{C},p}$ is a local ring map.*

In fact, properties (2), or (3) for a conservative family of points implies (1).

Proof. This lemma proves itself, in other words, it follows by unwinding the definitions. □

Definition 16.34.8. Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Assume $(Sh(\mathcal{C}), \mathcal{O}_\mathcal{C})$ and $(Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ are locally ringed topoi. We say that $(f, f^\sharp)$ is a *morphism of locally ringed topoi* if and only if the diagram of sheaves

$$\begin{tikzcd} f^{-1}(\mathcal{O}^*_\mathcal{D}) \arrow[r, "f^\sharp"'] \arrow[d] & \mathcal{O}^*_\mathcal{C} \arrow[d] \\ f^{-1}(\mathcal{O}_\mathcal{D}) \arrow[r, "f^\sharp"] & \mathcal{O}_\mathcal{C} \end{tikzcd}$$

(see Lemma 16.34.7) is cartesian. If $(f, f^\sharp)$ is a morphism of ringed sites, then we say that it is a *morphism of locally ringed sites* if the associated morphism of ringed topoi is a morphism of locally ringed topoi.

It is clear that an isomorphism of ringed topoi between locally ringed topoi is automatically an isomorphism of locally ringed topoi.

Lemma 16.34.9. *Let $(f, f^\sharp) : (Sh(\mathcal{C}_1), \mathcal{O}_1) \to (Sh(\mathcal{C}_2), \mathcal{O}_2)$ and $(g, g^\sharp) : (Sh(\mathcal{C}_2), \mathcal{O}_2) \to (Sh(\mathcal{C}_3), \mathcal{O}_3)$ be morphisms of locally ringed topoi. Then the composition $(g, g^\sharp) \circ (f, f^\sharp)$ (see Definition 16.7.1) is also a morphism of locally ringed topoi.*

Proof. Omitted. □

Lemma 16.34.10. *If $f : Sh(\mathcal{C}') \to Sh(\mathcal{C})$ is a morphism of topoi. If $\mathcal{O}$ is a sheaf of rings on $\mathcal{C}$, then*

$$f^{-1}(\mathcal{O}^*) = (f^{-1}\mathcal{O})^*.$$

In particular, if $\mathcal{O}$ turns $\mathcal{C}$ into a locally ringed site, then setting $f^\sharp = id$ the morphism of ringed topoi

$$(f, f^\sharp) : (Sh(\mathcal{C}'), f^{-1}\mathcal{O}) \to (Sh(\mathcal{C}, \mathcal{O})$$

is a morphism of locally ringed topoi.

Proof. Note that the diagram

$$\begin{tikzcd}
\mathcal{O}^* \arrow[r] \arrow[d, "u \mapsto (u, u^{-1})"'] & * \arrow[d, "1"] \\
\mathcal{O} \times \mathcal{O} \arrow[r, "(a,b) \mapsto ab"] & \mathcal{O}
\end{tikzcd}$$

is cartesian. Since f^{-1} is exact we conclude that

$$\begin{tikzcd}
f^{-1}(\mathcal{O}^*) \arrow[r] \arrow[d, "u \mapsto (u, u^{-1})"'] & * \arrow[d, "1"] \\
f^{-1}\mathcal{O} \times f^{-1}\mathcal{O} \arrow[r, "(a,b) \mapsto ab"] & f^{-1}\mathcal{O}
\end{tikzcd}$$

is cartesian which implies the first assertion. For the second, note that $(\mathcal{C}', f^{-1}\mathcal{O})$ is a locally ringed site by Lemma 16.34.5 so that the assertion makes sense. Now the first part implies that the morphism is a morphism of locally ringed topoi. □

Lemma 16.34.11. *Localization of locally ringed sites and topoi.*

(1) *Let $(\mathcal{C}, \mathcal{O})$ be a locally ringed site. Let U be an object of $\mathcal{C}$. Then the localization $(\mathcal{C}/U, \mathcal{O}_U)$ is a locally ringed site, and the localization morphism*

$$(j_U, j_U^\sharp) : (Sh(\mathcal{C}/U), \mathcal{O}_U) \to (Sh(\mathcal{C}), \mathcal{O})$$

is a morphism of locally ringed topoi.

(2) *Let $(\mathcal{C}, \mathcal{O})$ be a locally ringed site. Let $f : V \to U$ be a morphism of $\mathcal{C}$. Then the morphism*

$$(j, j^\sharp) : (Sh(\mathcal{C}/V), \mathcal{O}_V) \to (Sh(\mathcal{C}/U), \mathcal{O}_U)$$

of Lemma 16.19.4 is a morphism of locally ringed topoi.

(3) *Let $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}) \longrightarrow (\mathcal{D}, \mathcal{O}')$ be a morphism of locally ringed sites where f is given by the continuous functor $u : \mathcal{D} \to \mathcal{C}$. Let V be an object of $\mathcal{D}$ and let $U = u(V)$. Then the morphism*

$$(f', (f')^\sharp) : (Sh(\mathcal{C}/U), \mathcal{O}_U) \to (Sh(\mathcal{D}/V), \mathcal{O}'_V)$$

of Lemma 16.20.1 is a morphism of locally ringed sites.

(4) *Let $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}) \longrightarrow (\mathcal{D}, \mathcal{O}')$ be a morphism of locally ringed sites where f is given by the continuous functor $u : \mathcal{D} \to \mathcal{C}$. Let $V \in Ob(\mathcal{D})$, $U \in Ob(\mathcal{C})$, and $c : U \to u(V)$. Then the morphism*

$$(f_c, (f_c)^\sharp) : (Sh(\mathcal{C}/U), \mathcal{O}_U) \to (Sh(\mathcal{D}/V), \mathcal{O}'_V)$$

of Lemma 16.20.2 is a morphism of locally ringed topoi.

(5) *Let $(Sh(\mathcal{C}), \mathcal{O})$ be a locally ringed topos. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$. Then the localization $(Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F})$ is a locally ringed topos and the localization morphism*

$$(j_\mathcal{F}, j_\mathcal{F}^\sharp) : (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F}) \to (Sh(\mathcal{C}), \mathcal{O})$$

is a morphism of locally ringed topoi.

(6) *Let $(Sh(\mathcal{C}), \mathcal{O})$ be a locally ringed topos. Let $s : \mathcal{G} \to \mathcal{F}$ be a map of sheaves on $\mathcal{C}$. Then the morphism*

$$(j, j^\sharp) : (Sh(\mathcal{C})/\mathcal{G}, \mathcal{O}_\mathcal{G}) \longrightarrow (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F})$$

of Lemma 16.21.4 is a morphism of locally ringed topoi.

(7) *Let $f : (Sh(\mathcal{C}), \mathcal{O}) \longrightarrow (Sh(\mathcal{D}), \mathcal{O}')$ be a morphism of locally ringed topoi. Let $\mathcal{G}$ be a sheaf on $\mathcal{D}$. Set $\mathcal{F} = f^{-1}\mathcal{G}$. Then the morphism*

$$(f', (f')^\sharp) : (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F}) \longrightarrow (Sh(\mathcal{D})/\mathcal{G}, \mathcal{O}'_\mathcal{G})$$

of Lemma 16.22.1 is a morphism of locally ringed topoi.

(8) *Let $f : (Sh(\mathcal{C}), \mathcal{O}) \longrightarrow (Sh(\mathcal{D}), \mathcal{O}')$ be a morphism of locally ringed topoi. Let $\mathcal{G}$ be a sheaf on $\mathcal{D}$, let $\mathcal{F}$ be a sheaf on $\mathcal{C}$, and let $s : \mathcal{F} \to f^{-1}\mathcal{G}$ be a morphism of sheaves. Then the morphism*

$$(f_s, (f_s)^\sharp) : (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F}) \longrightarrow (Sh(\mathcal{D})/\mathcal{G}, \mathcal{O}'_\mathcal{G})$$

of Lemma 16.22.3 is a morphism of locally ringed topoi.

Proof. Part (1) is clear since $\mathcal{O}_U$ is just the restriction of $\mathcal{O}$, so Lemmas 16.34.5 and 16.34.10 apply. Part (2) is clear as the morphism $(j, j^\sharp)$ is actually a localization of a locally ringed site so (1) applies. Part (3) is clear also since $(f')^\sharp$ is just the restriction of $f^\sharp$ to the topos $Sh(\mathcal{C})/\mathcal{F}$, see proof of Lemma 16.22.1 (hence the diagram of Definition 16.34.8 for the morphism f' is just the restriction of the corresponding diagram for f, and restriction is an exact functor). Part (4) follows formally on combining (2) and (3). Parts (5), (6), (7), and (8) follow from their counterparts (1), (2), (3), and (4) by enlarging the sites as in Lemma 16.7.2 and translating everything in terms of sites and morphisms of sites using the comparisons of Lemmas 16.21.3, 16.21.5, 16.22.2, and 16.22.4. (Alternatively one could use the same arguments as in the proofs of (1), (2), (3), and (4) to prove (5), (6), (7), and (8) directly.) □

16.35. Lower shriek for modules

In this section we extend the construction of $g_!$ discussed in Section 16.16 to the case of sheaves of modules.

Lemma 16.35.1. *Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous and concontinuous functor between sites. Denote $g : Sh(\mathcal{C}) \to Sh(\mathcal{O}_\mathcal{D})$ the associated morphism of topoi. Let $\mathcal{O}_\mathcal{D}$ be a sheaf of rings on $\mathcal{D}$. Set $\mathcal{O}_\mathcal{C} = g^{-1}\mathcal{O}_\mathcal{D}$. Hence g becomes a morphism of ringed topoi with $g^* = g^{-1}$. In this case there exists a functor*

$$g_! : Mod(\mathcal{O}_\mathcal{C}) \longrightarrow Mod(\mathcal{O}_\mathcal{D})$$

which is left adjoint to g^*.

Proof. Let U be an object of $\mathcal{C}$. For any $\mathcal{O}_\mathcal{D}$-module $\mathcal{G}$ we have

$$\begin{aligned} Hom_{\mathcal{O}_\mathcal{C}}(j_{U!}\mathcal{O}_U, g^{-1}\mathcal{G}) &= g^{-1}\mathcal{G}(U) \\ &= \mathcal{G}(u(U)) \\ &= Hom_{\mathcal{O}_\mathcal{C}}(j_{u(U)!}\mathcal{O}_{u(U)}, \mathcal{G}) \end{aligned}$$

because g^{-1} is described by restriction, see Sites, Lemma 9.19.5. Of course a similar formula holds a direct sum of modules of the form $j_{U!}\mathcal{O}_U$. By Homology, Lemma 10.22.6 and Lemma 16.26.6 we see that $g_!$ exists. □

Remark 16.35.2. Warning! Let $u : \mathcal{C} \to \mathcal{D}$, g, $\mathcal{O}_\mathcal{D}$, and $\mathcal{O}_\mathcal{C}$ be as in Lemma 16.35.1. In general it is **not** the case that the diagram

$$\begin{tikzcd} Mod(\mathcal{O}_\mathcal{C}) \arrow[r, "g_!"'] \arrow[d, "forget"'] & Mod(\mathcal{O}_\mathcal{D}) \arrow[d, "forget"] \\ Ab(\mathcal{C}) \arrow[r, "g_!^{Ab}"] & Ab(\mathcal{D}) \end{tikzcd}$$

commutes (here $g_!^{Ab}$ is the one from Lemma 16.16.2). There is a transformation of functors

$$g_!^{Ab} \circ forget \longrightarrow forget \circ g_!$$

From the proof of Lemma 16.35.1 we see that this is an isomorphism if and only if $g_! j_{U!}\mathcal{O}_U = g_!^{Ab} j_{U!}\mathcal{O}_U$ for all objects U of $\mathcal{C}$, in other words, if and only if

$$g_!^{Ab} j_{U!}\mathcal{O}_U = j_{u(U)!}\mathcal{O}_{u(U)}$$

for all objects U of $\mathcal{C}$. Note that for such a U we obtain a commutative diagram

$$\begin{tikzcd} \mathcal{C}/U \arrow[r, "u'"'] \arrow[d, "j_U"'] & \mathcal{D}/u(U) \arrow[d, "j_{u(U)}"] \\ \mathcal{C} \arrow[r, "u"] & \mathcal{D} \end{tikzcd}$$

of cocontinuous functors of sites, see Sites, Lemma 9.24.4. Hence we see that $g_! = g_!^{Ab}$ if the canonical map

$$(g')_!^{Ab}\mathcal{O}_U \longrightarrow \mathcal{O}_{u(U)} \tag{16.35.2.1}$$

is an isomorphism for all objects U of $\mathcal{C}$. Here $g' : Sh(\mathcal{C}/U) \to Sh(\mathcal{D}/u(U))$ is the morphism of topoi induced by the cocontinuous functor u'.

16.36. Other chapters

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CHAPTER 17

Injectives

17.1. Introduction

We will use the existence of sufficiently many injectives to do cohomology of abelian sheaves on a site. So we briefly explain why there are enough injectives. At the end we explain the more general story.

17.2. Abelian groups

In this section we show the category of abelian groups has enough injectives. Recall that an abelian group M is *divisible* if and only if for every $x \in M$ and every $n \in \mathbf{N}$ there exists a $y \in M$ such that $ny = x$.

Lemma 17.2.1. *An abelian group J is an injective object in the category of abelian groups if and only if J is divisible.*

Proof. Suppose that J is not divisible. Then there exists an $x \in J$ and $n \in \mathbf{N}$ such that there is no $y \in J$ with $ny = x$. Then the morphism $\mathbf{Z} \to J$, $m \mapsto mx$ does not extend to $\frac{1}{n}\mathbf{Z} \supset \mathbf{Z}$. Hence J is not injective.

Let $A \subset B$ be abelian groups. Assume that J is a divisible abelian group. Let $\varphi : A \to J$ be a morphism. Consider the set of homomorphisms $\varphi' : A' \to J$ with $A \subset A' \subset B$ and $\varphi'|_A = \varphi$. Define $(A', \varphi') \geq (A'', \varphi'')$ if and only if $A' \supset A''$ and $\varphi'|_{A''} = \varphi''$. If $(A_i, \varphi_i)_{i\in I}$ is a totally ordered collection of such pairs, then we obtain a map $\bigcup_{i\in I} A_i \to J$ defined by $a \in A_i$ maps to $\varphi_i(a)$. Thus Zorn's lemma applies. To conclude we have to show that if the pair (A', φ') is maximal then $A' = B$. In other words, it suffices to show, given any subgroup $A \subset B$, $A \neq B$ and any $\varphi : A \to J$, then we can find $\varphi' : A' \to J$ with $A \subset A' \subset B$ such that (a) the inclusion $A \subset A'$ is strict, and (b) the morphism φ' extends φ.

To prove this, pick $x \in B$, $x \notin A$. If there exists no $n \in \mathbf{N}$ such that $nx \in A$, then $A \oplus \mathbf{Z} \cong A + \mathbf{Z}x$. Hence we can extend φ to $A' = A + \mathbf{Z}x$ by using φ on A and mapping x to zero for example. If there does exist an $n \in \mathbf{N}$ such that $nx \in A$, then let n be the minimal such integer. Let $z \in J$ be an element such that $nz = \varphi(nx)$. Define a morphism $\tilde{\varphi} : A \oplus \mathbf{Z} \to J$ by $(a, m) \mapsto \varphi(a) + mz$. By our choice of z the kernel of $\tilde{\varphi}$ contains the kernel of the map $A \oplus \mathbf{Z} \to B$, $(a, m) \mapsto a + mx$. Hence $\tilde{\varphi}$ factors through the image $A' = A + \mathbf{Z}x$, and this extends the morphism φ. $\square$

We can use this lemma to show that every abelian group can be embbeded in a injective abelian group. But this is a special case of the result of the following section.

17.3. Modules

As an example theorem let us try to prove that there are enough injective modules over a ring R. We start with the fact that $\mathbf{Q}/\mathbf{Z}$ is an injective abelian group. This follows from Lemma 17.2.1 above.

Definition 17.3.1. Let R be a ring.

(1) For any R-module M over R we denote $M^\vee = Hom(M, \mathbf{Q}/\mathbf{Z})$ with its natural R-module structure. We think of $M \mapsto M^\vee$ as a contravariant functor from the category of R-modules to itself.
(2) For any R-module M we denote
$$F(M) = \bigoplus\nolimits_{m \in M} R[m]$$
the *free module* with basis given by the elements $[m]$ with $m \in M$. We let $F(M) \to M$, $\sum f_i[m_i] \mapsto \sum f_i m_i$ be the natural surjection of R-modules. We think of $M \mapsto (F(M) \to M)$ as a functor from the category of R-modules to the category of arrows in R-modules.

Lemma 17.3.2. *Let R be a ring. The functor $M \mapsto M^\vee$ is exact.*

Proof. This because $\mathbf{Q}/\mathbf{Z}$ is an injective abelian group. $\square$

There is a canonical map $ev : M \to (M^\vee)^\vee$ given by evaluation: given $x \in M$ we let $ev(x) \in (M^\vee)^\vee = Hom(M^\vee, \mathbf{Q}/\mathbf{Z})$ be the map $\varphi \mapsto \varphi(x)$.

Lemma 17.3.3. *For any R-module M the evaluation map $ev : M \to (M^\vee)^\vee$ is injective.*

Proof. You can check this using that $\mathbf{Q}/\mathbf{Z}$ is an injective abelian group. Namely, if $x \in M$ is not zero, then let $M' \subset M$ be the cyclic group it generates. There exists a nonzero map $M' \to \mathbf{Q}/\mathbf{Z}$ which necessarily does not annihilate x. This extends to a map $\varphi : M \to \mathbf{Q}/\mathbf{Z}$ And then $ev(x)(\varphi) = \varphi(x) \neq 0$. $\square$

The canonical surjection $F(M) \to M$ of R-modules turns into a a canonical injection, see above, of R-modules
$$(M^\vee)^\vee \longrightarrow (F(M^\vee))^\vee.$$
Set $J(M) = (F(M^\vee))^\vee$. The composition of ev with this the displayed map gives $M \to J(M)$ functorially in M.

Lemma 17.3.4. *Let R be a ring. For every R-module M the R-module $J(M)$ is injective.*

Proof. Note that $J(M) \cong \prod_{m \in M} R^\vee$ as an R-module. As the product of injective modules is injective, it suffices to show that $R^\vee$ is injective. For this we use that
$$Hom_R(N, R^\vee) = Hom_R(N, Hom_{\mathbf{Z}}(R, \mathbf{Q}/\mathbf{Z})) = N^\vee$$
and the fact that $(-)^\vee$ is an exact functor by Lemma 17.3.2. $\square$

Lemma 17.3.5. *Let R be a ring. The construction above defines a covariant functor $M \mapsto (M \to J(M))$ from the category of R-modules to the category of arrows of R-modules such that for every module M the output $M \to J(M)$ is an injective map of M into an injective R-module $J(M)$.*

Proof. Follows from the above. $\square$

In particular, for any map of R-modules $M \to N$ there is an associated morphism $J(M) \to J(N)$ making the following diagram commute:

$$\begin{tikzcd} M \arrow[r] \arrow[d] & N \arrow[d] \\ J(M) \arrow[r] & J(N) \end{tikzcd}$$

This the kind of construction we would like to have in general. In Homology, Section 10.20 we introduced terminology to express this. Namely, we say this means that the category of R-modules has functorial injective embeddings.

17.4. Projective resolutions

Totally unimportant. Skip this section.

For any set S we let $F(S)$ denote the free R-module on S. Then any left R-module has the following two step resolution

$$F(M \times M) \oplus F(R \times M) \to F(M) \to M \to 0.$$

The first map is given by the rule

$$[m_1, m_2] \oplus [r, m] \mapsto [m_1 + m_2] - [m_1] - [m_2] + [rm] - r[m].$$

17.5. Modules over noncommutative rings

In the stacks project a ring is always commutative with 1. The material of Section 17.3 continues to work when R is only a noncommutative ring, except that if M is a right R-module, then $M^\wedge$ is a left R-module and vice-versa. The conclusion is that the category of right R-modules and the category of left R-modules have functorial injective embeddings.

Precise statements and proofs omitted.

17.6. Baer's argument for modules

There is another, more set-theoretic approach to showing that any R-module M can be imbedded in an injective module. This approach constructs the injective module by a transfinite colimit of push-outs. While this method is somewhat abstract and more complicated than the one of Section 17.3, it is also more general. Apparently this method originates with Baer, and was revisited by Cartan and Eilenberg in [**CE56**] and by Grothendieck in [**Gro57**]. There Grothendieck uses it to show that many other abelian categories have enough injectives. We will get back to the general case later (insert future reference here).

We begin with a few set theoretic remarks. Let $\{B_\beta\}_{\beta\in\alpha}$ be an inductive system of objects in some category $\mathcal{C}$, indexed by an ordinal α. Assume that $colim_{\beta\in\alpha} B_\beta$ exists in $\mathcal{C}$. If A is an object of $\mathcal{C}$, then there is a natural map

(17.6.0.1) $$colim_{\beta\in\alpha} Mor_{\mathcal{C}}(A, B_\beta) \longrightarrow Mor_{\mathcal{C}}(A, colim_{\beta\in\alpha} B_\beta).$$

because if one is given a map $A \to B_\beta$ for some β, one naturally gets a map from A into the colimit by composing with $B_\beta \to colim_{\beta\in\alpha} B_\alpha$. Note that the left colimit is one of sets! In general, (17.6.0.1) is neither injective or surjective.

Example 17.6.1. Consider the category of sets. Let $A = \mathbf{N}$ and $B_n = \{1, \ldots, n\}$ be the inductive system indexed by the natural numbers where $B_n \to B_m$ for $n \leq m$ is the obvious map. Then $colim\, B_n = \mathbf{N}$, so there is a map $A \to \varinjlim B_n$, which does not factor as $A \to B_m$ for any m. Consequently, $colim\, Mor(A, B_n) \to Mor(A, colim\, B_n)$ is not surjective.

Example 17.6.2. Next we give an example where the map fails to be injective. Let $B_n = \mathbf{N}/\{1, 2, \ldots, n\}$, that is, the quotient set of $\mathbf{N}$ with the first n elements collapsed to one element. There are natural maps $B_n \to B_m$ for $n \leq m$, so the $\{B_n\}$ form a system of sets over $\mathbf{N}$. It is easy to see that $colim\, B_n = \{*\}$: it is the one-point set. So it follows that $Mor(A, colim\, B_n)$ is a one-element set for every set A. However, $colim\, Mor(A, B_n)$ is **not** a one-element set. Consider the family of maps $A \to B_n$ which are just the natural projections $\mathbf{N} \to \mathbf{N}/\{1, 2, \ldots, n\}$ and the family of maps $A \to B_n$ which map the whole of A to the class of 1. These two families of maps are distinct at each step and thus are distinct in $colim\, Mor(A, B_n)$, but they induce the same map $A \to colim\, B_n$.

Nonetheless, if we map out of a finite set then (17.6.0.1) is an isomorphism always.

Lemma 17.6.3. *Suppose that, in (17.6.0.1), $\mathcal{C}$ is the category of sets and A is a finite set, then the map is a bijection.*

Proof. Let $f : A \to colim\, B_\beta$. The range of f is finite, containing say elements $c_1, \ldots, c_r \in colim\, B_\beta$. These all come from some elements in B_β for $\beta \in \alpha$ large by definition of the colimit. Thus we can define $\tilde{f} : A \to B_\beta$ lifting f at a finite stage. This proves that (17.6.0.1) is surjective. Next, suppose two maps $f : A \to B_\gamma, f' : A \to B_{\gamma'}$ define the same map $A \to colim\, B_\beta$. Then each of the finitely many elements of A gets sent to the same point in the colimit. By definition of the colimit for sets, there is $\beta \geq \gamma, \gamma'$ such that the finitely many elements of A get sent to the same points in B_β under f and f'. This proves that (17.6.0.1) is injective. □

The most interesting case of the lemma is when $\alpha = \omega$, i.e., when the system $\{B_\beta\}$ is a system $\{B_n\}_{n \in \mathbf{N}}$ over the natural numbers as in Examples 17.6.1 and 17.6.2. The essential idea is that A is ``small'' relative to the long chain of compositions $B_1 \to B_2 \to \ldots$, so that it has to factor through a finite step. A more general version of this lemma can be found in Sets, Lemma 3.7.1. Next, we generalize this to the category of modules.

Definition 17.6.4. Let $\mathcal{C}$ be a category, let $I \subset \text{Arrow}(\mathcal{C})$, and let α be an ordinal. An object A of $\mathcal{C}$ is said to be *α-small with respect to I* if whenever $\{B_\beta\}$ is a system over α with transition maps in I, then the map (17.6.0.1) is an isomorphism.

In the rest of this section we shall restrict ourselves to the category of R-modules for a fixed commutative ring R. We shall also take I to be the collection of injective maps, i.e., the *monomorphisms* in the category of modules over R. In this case, for any system $\{B_\beta\}$ as in the definition each of the maps

$$B_\beta \to colim_{\beta \in \alpha} B_\beta$$

is an injection. It follows that the map (17.6.0.1) is an *injection*. We can in fact interpret the B_β's as submodules of the module $B = colim_{\beta \in \alpha} B_\beta$, and then we have $B = \bigcup_{\beta \in \alpha} B_\beta$. This is not an abuse of notation if we identify B_α with the image in the colimit. We now want to show that modules are always small for ``large'' ordinals α.

Proposition 17.6.5. *Let R be a ring. Let M be an R-module. Let κ the cardinality of the set of submodules of M. If α is an ordinal whose cofinality is bigger than κ, then M is α-small with respect to injections.*

Proof. The proof is straightforward, but let us first think about a special case. If M is finite, then the claim is that for any inductive system $\{B_\beta\}$ with injections between them, parametrized by a limit ordinal, any map $M \to colim\, B_\beta$ factors through one of the B_β. And this we proved in Lemma 17.6.3.

Now we start the proof in the general case. We need only show that the map (17.6.0.1) is a surjection. Let $f : M \to colim\, B_\beta$ be a map. Consider the subobjects $\{f^{-1}(B_\beta)\}$ of M, where B_β is considered as a subobject of the colimit $B = \bigcup_\beta B_\beta$. If one of these, say $f^{-1}(B_\beta)$, fills M, then the map factors through B_β.

So suppose to the contrary that all of the $f^{-1}(B_\beta)$ were proper subobjects of M. However, we know that

$$\bigcup f^{-1}(B_\beta) = f^{-1}\left(\bigcup B_\beta\right) = M.$$

Now there are at most κ different subobjects of M that occur among the $f^{-1}(B_\alpha)$, by hypothesis. Thus we can find a subset $S \subset \alpha$ of cardinality at most κ such that as β' ranges over S, the $f^{-1}(B_{\beta'})$ range over *all* the $f^{-1}(B_\alpha)$.

However, S has an upper bound $\tilde{\alpha} < \alpha$ as α has cofinality bigger than κ. In particular, all the $f^{-1}(B_{\beta'})$, $\beta' \in S$ are contained in $f^{-1}(B_{\tilde{\alpha}})$. It follows that $f^{-1}(B_{\tilde{\alpha}}) = M$. In particular, the map f factors through $B_{\tilde{\alpha}}$. $\square$

From this lemma we will be able to deduce the existence of lots of injectives. Let us recall the criterion of Baer.

Lemma 17.6.6. *Let R be a ring. An R-module Q is injective if and only if in every commutative diagram*

$$\begin{tikzcd} \mathfrak{a} \arrow[r] \arrow[d] & Q \\ R \arrow[ur, dashed] & \end{tikzcd}$$

for $\mathfrak{a} \subset R$ an ideal, the dotted arrow exists.

Proof. Assume Q satisfies the assumption of the lemma. Let $M \subset N$ be R-modules, and let $\varphi : M \to Q$ be an R-module map. Arguing as in the proof of Lemma 17.2.1 we see that it suffices to prove that if $M \neq N$, then we can find an R-module M', $M \subset M' \subset N$ such that (a) the inclusion $M \subset M'$ is strict, and (b) φ can be extended to M'. To find M', let $x \in N$, $x \notin M$. Let $\psi : R \to N$, $r \mapsto rx$. Set $\mathfrak{a} = \psi^{-1}(M)$. By assumption the morphism

$$\mathfrak{a} \xrightarrow{\psi} M \xrightarrow{\varphi} Q$$

can be extended to a morphism $\varphi' : R \to Q$. Note that φ' annihilates the kernel of ψ (as this is true for φ). Thus φ' gives rise to a morphism $\varphi'' : \mathrm{Im}(\psi) \to Q$ which agrees with φ on the intersection $M \cap \mathrm{Im}(\psi)$ by construction. Thus φ and φ'' glue to give an extension of φ to the strictly bigger module $M' = \mathcal{F} + \mathrm{Im}(\psi)$. $\square$

If M is an R-module, then in general we may have a semi-complete diagram as in Lemma 17.6.6. In it, we can form the *push-out*

$$\begin{tikzcd} \mathfrak{a} \arrow[r] \arrow[d] & Q \arrow[d] \\ R \arrow[r] & R \oplus_{\mathfrak{a}} Q. \end{tikzcd}$$

Here the vertical map is injective, and the diagram commutes. The point is that we can extend $\mathfrak{a} \to Q$ to R *if* we extend Q to the larger module $R \oplus_{\mathfrak{a}} Q$.

The key point of Baer's argument is to repeat this procedure transfinitely many times. To do this we first define, given an R-module M the following (huge) pushout

(17.6.6.1)
$$\begin{tikzcd}
\bigoplus_{\mathfrak a} \bigoplus_{\varphi \in Hom_R(\mathfrak a, M)} \mathfrak a \arrow[r] \arrow[d] & M \arrow[d] \\
\bigoplus_{\mathfrak a} \bigoplus_{\varphi \in Hom_R(\mathfrak a, M)} R \arrow[r] & \mathbf{M}(M).
\end{tikzcd}$$

Here the top horizontal arrow maps the element $a \in \mathfrak a$ in the summand corresponding to φ to the element $\varphi(a) \in M$. The left vertical arrow maps $a \in \mathfrak a$ in the summand corresponding to φ simply to the element $a \in R$ in the summand corresponding to φ. The fundamental properties of this construction are formulated in the following lemma.

Lemma 17.6.7. *Let R be a ring.*

(1) *The construction $M \mapsto (M \to \mathbf{M}(M))$ is functorial in M.*
(2) *The map $M \to \mathbf{M}(M)$ is injective.*
(3) *For any ideal $\mathfrak a$ and any R-module map $\varphi : \mathfrak a \to M$ there is an R-module map $\varphi' : R \to \mathbf{M}(M)$ such that*

$$\begin{tikzcd}
\mathfrak a \arrow[r, "\varphi"'] \arrow[d] & M \arrow[d] \\
R \arrow[r, "\varphi'"] & \mathbf{M}(M)
\end{tikzcd}$$

commutes.

Proof. Parts (2) and (3) are immediate from the construction. To see (1), let $\chi : M \to N$ be an R-module map. We claim there exists a canonical commutative diagram

$$\begin{tikzcd}
\bigoplus_{\mathfrak a} \bigoplus_{\varphi \in Hom_R(\mathfrak a, M)} \mathfrak a \arrow[r] \arrow[d] \arrow[drr] & M \arrow[drr, "\chi"] & & \\
\bigoplus_{\mathfrak a} \bigoplus_{\varphi \in Hom_R(\mathfrak a, M)} R \arrow[drr] & & \bigoplus_{\mathfrak a} \bigoplus_{\psi \in Hom_R(\mathfrak a, N)} \mathfrak a \arrow[r] \arrow[d] & N \\
& & \bigoplus_{\mathfrak a} \bigoplus_{\psi \in Hom_R(\mathfrak a, N)} R &
\end{tikzcd}$$

which induces the desired map $\mathbf{M}(M) \to \mathbf{M}(N)$. The middle east-south-east arrow maps the summand $\mathfrak a$ corresponding to φ via $\mathrm{id}_{\mathfrak a}$ to the summand $\mathfrak a$ corresponding to $\psi = \chi \circ \varphi$. Similarly for the lower east-south-east arrow. Details omitted. $\square$

The idea will now be to apply the functor $\mathbf{M}$ a transfinite number of times. We define for each ordinal α a functor $\mathbf{M}_\alpha$ on the category of R-modules, together with a natural injection $N \to \mathbf{M}_\alpha(N)$. We do this by transfinite induction. First, $\mathbf{M}_1 = \mathbf{M}$ is the functor defined above. Now, suppose given an ordinal α, and suppose $\mathbf{M}_{\alpha'}$ is defined for $\alpha' < \alpha$. If α has an immediate predecessor $\tilde\alpha$, we let

$$\mathbf{M}_\alpha = \mathbf{M} \circ \mathbf{M}_{\tilde\alpha}.$$

If not, i.e., if α is a limit ordinal, we let

$$\mathbf{M}_\alpha(N) = colim_{\alpha' < \alpha} \mathbf{M}_{\alpha'}(N).$$

It is clear (e.g., inductively) that the $\mathbf{M}_\alpha(N)$ form an inductive system over ordinals, so this is reasonable.

Theorem 17.6.8. *Let κ be the cardinality of the set of ideals in R, and let α be an ordinal whose cofinality is greater than κ. Then $\mathbf{M}_\alpha(N)$ is an injective R-module, and $N \to \mathbf{M}_\alpha(N)$ is a functorial injective embedding.*

Proof. By Baer's criterion Lemma 17.6.6, it suffices to show that if $\mathfrak{a} \subset R$ is an ideal, then any map $f : \mathfrak{a} \to \mathbf{M}_\alpha(N)$ extends to $R \to \mathbf{M}_\alpha(N)$. However, we know since α is a limit ordinal that

$$\mathbf{M}_\alpha(N) = colim_{\beta<\alpha}\mathbf{M}_\beta(N),$$

so by Proposition 17.6.5, we find that

$$Hom_R(\mathfrak{a}, \mathbf{M}_\alpha(N)) = colim_{\beta<\alpha} Hom_R(\mathfrak{a}, \mathbf{M}_\beta(N)).$$

This means in particular that there is some $\beta' < \alpha$ such that f factors through the submodule $\mathbf{M}_{\beta'}(N)$, as

$$f : \mathfrak{a} \to \mathbf{M}_{\beta'}(N) \to \mathbf{M}_\alpha(N).$$

However, by the fundamental property of the functor $\mathbf{M}$, see Lemma 17.6.7 part (3), we know that the map $\mathfrak{a} \to \mathbf{M}_{\beta'}(N)$ can be extended to

$$R \to \mathbf{M}(\mathbf{M}_{\beta'}(N)) = \mathbf{M}_{\beta'+1}(N),$$

and the last object imbeds in $\mathbf{M}_\alpha(N)$ (as $\beta' + 1 < \alpha$ since α is a limit ordinal). In particular, f can be extended to $\mathbf{M}_\alpha(N)$. $\square$

17.7. G-modules

Lemma 17.7.1. *Let G be a topological group. The category Mod_G of discrete G-modules, see Étale Cohomology, Definition 38.57.1 has functorial injective hulls.*

Proof. By Section 17.5 the category $\text{Mod}_{\mathbf{Z}[G]}$ has functorial injective embeddings. Consider the forgetful functor $v : Mod_G \to \text{Mod}_{\mathbf{Z}[G]}$. This functor is fully faithful, transforms injective maps into injective maps and has a right adjoint, namely

$$u : M \mapsto u(M) = \{x \in M \mid \text{stabilizer of } x \text{ is open}\}$$

Since it is true that $v(M) = 0 \Rightarrow M = 0$ we conclude by Homology, Lemma 10.22.5. $\square$

17.8. Abelian sheaves on a space

Lemma 17.8.1. *Let X be a topological space. The category of abelian sheaves on X has enough injectives. In fact it has functorial injective embeddings.*

Proof. For an abelian group A we denote $j : A \to J(A)$ the functorial injective embedding constructed in Section 17.3. Let $\mathcal{F}$ be an abelian sheaf on X. By Sheaves, Example 6.7.5 the assignment

$$\mathcal{I} : U \mapsto \mathcal{I}(U) = \prod\nolimits_{x\in U} J(\mathcal{F}_x)$$

is an abelian sheaf. There is a canonical map $\mathcal{F} \to \mathcal{I}$ given by mapping $s \in \mathcal{F}(U)$ to $\prod_{x\in U} j(s_x)$ where $s_x \in \mathcal{F}_x$ denotes the germ of s at x. This map is injective, see Sheaves, Lemma 6.11.1 for example.

It remains to prove the following: Given a rule $x \mapsto I_x$ which assigns to each point $x \in X$ an injective abelian group the sheaf $\mathcal{I} : U \mapsto \prod_{x\in U} I_x$ is injective. Note that

$$\mathcal{I} = \prod\nolimits_{x\in X} i_{x,*}I_x$$

is the product of the skyscraper sheaves $i_{x,*}I_x$ (see Sheaves, Section 6.27 for notation.) We have

$$Mor_{Ab}(\mathcal{F}_x, I_x) = Mor_{Ab(X)}(\mathcal{F}, i_{x,*}I_x).$$

see Sheaves, Lemma 6.27.3. Hence it is clear that each $i_{x,*}I_x$ is injective. Hence the injectivity of $\mathcal{I}$ follows from Homology, Lemma 10.20.3. □

17.9. Sheaves of modules on a ringed space

Lemma 17.9.1. *Let $(X, \mathcal{O}_X)$ be a ringed space, see Sheaves, Section 6.25. The category of sheaves of $\mathcal{O}_X$-modules on X has enough injectives. In fact it has functorial injective embeddings.*

Proof. For any ring R and any R-module M we denote $j : M \to J_R(M)$ the functorial injective embedding constructed in Section 17.3. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules on X. By Sheaves, Examples 6.7.5 and 6.15.6 the assignment

$$\mathcal{J} : U \mapsto \mathcal{J}(U) = \prod\nolimits_{x \in U} J_{\mathcal{O}_{X,x}}(\mathcal{F}_x)$$

is an abelian sheaf. There is a canonical map $\mathcal{F} \to \mathcal{J}$ given by mapping $s \in \mathcal{F}(U)$ to $\prod_{x \in U} j(s_x)$ where $s_x \in \mathcal{F}_x$ denotes the germ of s at x. This map is injective, see Sheaves, Lemma 6.11.1 for example.

It remains to prove the following: Given a rule $x \mapsto I_x$ which assigns to each point $x \in X$ an injective $\mathcal{O}_{X,x}$-module the sheaf $\mathcal{I} : U \mapsto \prod_{x \in U} I_x$ is injective. Note that

$$\mathcal{I} = \prod\nolimits_{x \in X} i_{x,*}I_x$$

is the product of the skyscraper sheaves $i_{x,*}I_x$ (see Sheaves, Section 6.27 for notation.) We have

$$Hom_{\mathcal{O}_{X,x}}(\mathcal{F}_x, I_x) = Hom_{\mathcal{O}_X}(\mathcal{F}, i_{x,*}I_x).$$

see Sheaves, Lemma 6.27.3. Hence it is clear that each $i_{x,*}I_x$ is an injective $\mathcal{O}_X$-module (see Homology, Lemma 10.22.1 or argue directly). Hence the injectivity of $\mathcal{I}$ follows from Homology, Lemma 10.20.3. □

17.10. Abelian presheaves on a category

Let $\mathcal{C}$ be a category. Recall that this means that $Ob(\mathcal{C})$ is a set. On the one hand, consider abelian presheaves on $\mathcal{C}$, see Sites, Section 9.2. On the other hand, consider families of abelian groups indexed by elements of $Ob(\mathcal{C})$; in other words presheaves on the discrete category with underlying set of objects $Ob(\mathcal{C})$. Let us denote this discrete category simply $Ob(\mathcal{C})$. There is a natural functor

$$i : Ob(\mathcal{C}) \longrightarrow \mathcal{C}$$

and hence there is a natural restriction or forgetful functor

$$v = i^p : PAb(\mathcal{C}) \longrightarrow PAb(Ob(\mathcal{C}))$$

compare Sites, Section 9.5. We will denote presheaves on $\mathcal{C}$ by B and presheaves on $Ob(\mathcal{C})$ by A.

There are also two functors, namely i_p and ${}_pi$ which assign an abelian presheaf on $\mathcal{C}$ to an abelian presheaf on $Ob(\mathcal{C})$, see Sites, Sections 9.5 and 9.17. Here we will use $u = {}_pi$ which is defined (in the case at hand) as follows:

$$uA(U) = \prod\nolimits_{U' \to U} A(U').$$

So an element is a family $(a_\phi)_\phi$ with ϕ ranging through all morphisms in $\mathcal{C}$ with target U. The restriction map on uA corresponding to $g : V \to U$ maps our element $(a_\phi)_\phi$ to the element $(a_{g\circ\psi})_\psi$.

There is a canonical surjective map $vuA \to A$ and a canonical injective map $B \to uvB$. We leave it to the reader to show that

$$Mor_{PAb(Ob(\mathcal{C}))}(B, uA) = Mor_{PAb(\mathcal{C})}(vB, A).$$

in this simple case; the general case is in Sites, Section 9.5. Thus the pair (u, v) is an example of a pair of adjoint functors, see Categories, Section 4.22.

At this point we can list the following facts about the situation above.

(1) The functors u and v are exact. This follows from the explicit description of these functors given above.
(2) In particular the functor v transforms injective maps into injective maps.
(3) The category $PAb(Ob(\mathcal{C}))$ has enough injectives.
(4) In fact there is a functorial injective embedding $A \mapsto \big(A \to J(A)\big)$ as in Homology, Definition 10.20.5. Namely, we can take $J(A)$ to be the presheaf $U \mapsto J(A(U))$, where $J(-)$ is the functor constructed in Section 17.3 for the ring $\mathbf{Z}$.

Putting all of this together gives us the following procedure for embedding objects B of $PAb(\mathcal{C})$) into an injective object: $B \to uJ(vB)$. See Homology, Lemma 10.22.5.

Proposition 17.10.1. *For abelian presheaves on a category there is a functorial injective embedding.*

Proof. See discussion above. □

17.11. Abelian Sheaves on a site

Let $\mathcal{C}$ be a site. In this section we prove that there are enough injectives for abelian sheaves on $\mathcal{C}$.

Denote $i : Ab(\mathcal{C}) \longrightarrow PAb(\mathcal{C})$ the forgetful functor from abelian sheaves to abelian presheaves. Let $^\#$: $PAb(\mathcal{C}) \longrightarrow Ab(\mathcal{C})$ denote the sheafification functor. Recall that $^\#$ is a left adjoint to i, that $^\#$ is exact, and that $i\mathcal{F}^\# = \mathcal{F}$ for any abelian sheaf $\mathcal{F}$. Finally, let $\mathcal{G} \to J(\mathcal{G})$ denote the canonical embedding into an injective presheaf we found in Section 17.10.

For any sheaf $\mathcal{F}$ in $Ab(\mathcal{C})$ and any ordinal β we define a sheaf $J_\beta(\mathcal{F})$ by transfinite induction. We set $J_0(\mathcal{F}) = \mathcal{F}$. We define $J_1(\mathcal{F}) = J(i\mathcal{F})^\#$. Sheafification of the canonical map $i\mathcal{F} \to J(i\mathcal{F})$ gives a functorial map

$$\mathcal{F} \longrightarrow J_1(\mathcal{F})$$

which is injective as # is exact. We set $J_{\alpha+1}(\mathcal{F}) = J_1(J_\alpha(\mathcal{F}))$. So that there are canonical injective maps $J_\alpha(\mathcal{F}) \to J_{\alpha+1}(\mathcal{F})$. For a limit ordinal β, we define

$$J_\beta(\mathcal{F}) = colim_{\alpha<\beta}\, J_\alpha(\mathcal{F}).$$

Note that this is a directed colimit. Hence for any ordinals $\alpha < \beta$ we have an injective map $J_\alpha(\mathcal{F}) \to J_\beta(\mathcal{F})$.

Lemma 17.11.1. *With notation as above. Suppose that $\mathcal{G}_1 \to \mathcal{G}_2$ is an injective map of abelian sheaves on $\mathcal{C}$. Let α be an ordinal and let $\mathcal{G}_1 \to J_\alpha(\mathcal{F})$ be a morphism of sheaves. There exists a morphism $\mathcal{G}_2 \to J_{\alpha+1}(\mathcal{F})$ such that the following diagram commutes*

$$\begin{tikzcd} \mathcal{G}_1 \arrow[r] \arrow[d] & \mathcal{G}_2 \arrow[d] \\ J_\alpha(\mathcal{F}) \arrow[r] & J_{\alpha+1}(\mathcal{F}) \end{tikzcd}$$

Proof. This is because the map $i\mathcal{G}_1 \to i\mathcal{G}_2$ is injective and hence $i\mathcal{G}_1 \to iJ_\alpha(\mathcal{F})$ extends to $i\mathcal{G}_2 \to J(iJ_\alpha(\mathcal{F}))$ which gives the desired map after applying the sheafification functor. □

This lemma says that somehow the system $\{J_\alpha(\mathcal{F})\}$ is an injective embedding of $\mathcal{F}$. Of course we cannot take the limit over all α because they form a class and not a set. However, the idea is now that you don't have to check injectivity on all injections $\mathcal{G}_1 \to \mathcal{G}_2$, plus the following lemma.

Lemma 17.11.2. *Suppose that $\mathcal{G}_i$, $i \in I$ is set of abelian sheaves on $\mathcal{C}$. There exists an ordinal β such that for any sheaf $\mathcal{F}$, any $i \in I$, and any map $\varphi : \mathcal{G}_i \to J_\beta(\mathcal{F})$ there exists an $\alpha < \beta$ such that φ factors through $J_\alpha(\mathcal{F})$.*

Proof. This reduces to the case of a single sheaf $\mathcal{G}$ by taking the direct sum of all the $\mathcal{G}_i$.

Consider the sets

$$S = \coprod\nolimits_{U \in Ob(\mathcal{C})} \mathcal{G}(U).$$

and

$$T_\beta = \coprod\nolimits_{U \in Ob(\mathcal{C})} J_\beta(\mathcal{F})(U)$$

Then $T_\beta = colim_{\alpha<\beta} T_\alpha$ with injective transition maps. A morphism $\mathcal{G} \to J_\beta(\mathcal{F})$ factors through $J_\alpha(\mathcal{F})$ if and only if the associated map $S \to T_\beta$ factors through T_α. By Sets, Lemma 3.7.1 the cofinality of β is bigger than the cardinality of S, then the result of the lemma is true. Hence the lemma follows from the fact that there are ordinals with arbitrarily large cofinality, see Sets, Proposition 3.7.2. □

Recall that for an object X of $\mathcal{C}$ we denote $\mathbf{Z}_X$ the presheaf of abelian groups $\Gamma(U, \mathbf{Z}_X) = \oplus_{U \to X} \mathbf{Z}$, see Modules on Sites, Section 16.4. The sheaf associated to this presheaf is denoted $\mathbf{Z}_X^\#$, see Modules on Sites, Section 16.5. It can be characterized by the property

$$Mor_{Ab(\mathcal{C})}(\mathbf{Z}_X^\#, \mathcal{G}) = \mathcal{G}(X) \tag{17.11.2.1}$$

where the element φ of the left hand side is mapped to $\varphi(1 \cdot \mathrm{id}_X)$ in the right hand side. We can use these sheaves to characterize injective abelian sheaves.

Lemma 17.11.3. *Suppose $\mathcal{J}$ is a sheaf of abelian groups with the following property: For all $X \in Ob(\mathcal{C})$, for any abelian subsheaf $\mathcal{S} \subset \mathbf{Z}_X^\#$ and any morphism $\varphi : \mathcal{S} \to \mathcal{J}$, there exists a morphism $\mathbf{Z}_X^\# \to \mathcal{J}$ extending φ. Then $\mathcal{J}$ is an injective sheaf of abelian groups.*

Proof. Let $\mathcal{F} \to \mathcal{G}$ be an injective map of abelian sheaves. Suppose $\varphi : \mathcal{F} \to \mathcal{J}$ is a morphism. Arguing as in the proof of Lemma 17.2.1 we see that it suffices to prove that if $\mathcal{F} \neq \mathcal{G}$, then we can find an abelian sheaf $\mathcal{F}'$, $\mathcal{F} \subset \mathcal{F}' \subset \mathcal{G}$ such that (a) the inclusion $\mathcal{F} \subset \mathcal{F}'$ is strict, and (b) φ can be extended to $\mathcal{F}'$. To find $\mathcal{F}'$, let X be an object of $\mathcal{C}$ such that the inclusion $\mathcal{F}(X) \subset \mathcal{G}(X)$ is strict. Pick $s \in \mathcal{G}(X)$, $s \notin \mathcal{F}(X)$. Let $\psi : \mathbf{Z}_X^\# \to \mathcal{G}$ be the morphism corresponding to the section s via (17.11.2.1). Set $\mathcal{S} = \psi^{-1}(\mathcal{F})$. By assumption the morphism

$$\mathcal{S} \xrightarrow{\psi} \mathcal{F} \xrightarrow{\varphi} \mathcal{J}$$

can be extended to a morphism $\varphi' : \mathbf{Z}_X^\# \to \mathcal{J}$. Note that φ' annihilates the kernel of ψ (as this is true for φ). Thus φ' gives rise to a morphism $\varphi'' : \mathrm{Im}(\psi) \to \mathcal{J}$ which agrees with φ on the intersection $\mathcal{F} \cap \mathrm{Im}(\psi)$ by construction. Thus φ and φ'' glue to give an extension of φ to the strictly bigger subsheaf $\mathcal{F}' = \mathcal{F} + \mathrm{Im}(\psi)$. □

Theorem 17.11.4. *The category of sheaves of abelian groups on a site has enough injectives. In fact there exists a functorial injective embedding, see Homology, Definition 10.20.5.*

Proof. Let $\mathcal{G}_i$, $i \in I$ be a set of abelian sheaves such that every subsheaf of every $\mathbf{Z}_X^\#$ occurs as one of the $\mathcal{G}_i$. Apply Lemma 17.11.2 to this collection to get an ordinal β. We claim that for any sheaf of abelian groups $\mathcal{F}$ the map $\mathcal{F} \to J_\beta(\mathcal{F})$ is an injection of $\mathcal{F}$ into an injective. Note that by construction the assigment $\mathcal{F} \mapsto \left(\mathcal{F} \to J_\beta(\mathcal{F})\right)$ is indeed functorial.

The proof of the claim comes from the fact that by Lemma 17.11.3 it suffices to extend any morphism $\gamma : \mathcal{G} \to J_\beta(\mathcal{F})$ from a subsheaf $\mathcal{G}$ of some $\mathbf{Z}_X^\#$ to all of $\mathbf{Z}_X^\#$. Then by Lemma 17.11.2 the map γ lifts into $J_\alpha(\mathcal{F})$ for some $\alpha < \beta$. Finally, we apply Lemma 17.11.1 to get the desired extension of γ to a morphism into $J_{\alpha+1}(\mathcal{F}) \to J_\beta(\mathcal{F})$. □

17.12. Modules on a ringed site

Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. By analogy with Section 17.3 let us try to prove that there are enough injective $\mathcal{O}$-modules. First of all, we pick an injective embedding

$$\bigoplus\nolimits_{U,\mathcal{I}} j_{U!}\mathcal{O}_U/\mathcal{I} \longrightarrow \mathcal{J}$$

where $\mathcal{J}$ is an injective abelian sheaf (which exists by the previous section). Here the direct sum is over all objects U of $\mathcal{C}$ and over all $\mathcal{O}$-submodules $\mathcal{I} \subset j_{U!}\mathcal{O}_U$. Please see Modules on Sites, Section 16.19 to read about the functors restriction and extension by 0 for the localization functor $j_U : \mathcal{C}/U \to \mathcal{C}$.

For any sheaf of $\mathcal{O}$-modules $\mathcal{F}$ denote

$$\mathcal{F}^\vee = \mathcal{H}om(\mathcal{F}, \mathcal{J})$$

with its natural $\mathcal{O}$-module structure. Insert here future reference to internal hom. We will also need a canonical flat resolution of a sheaf of $\mathcal{O}$-modules. This we can do as follows: For any $\mathcal{O}$-module $\mathcal{F}$ we denote

$$F(\mathcal{F}) = \bigoplus\nolimits_{U \in Ob(\mathcal{C}), s \in \mathcal{F}(U)} j_{U!}\mathcal{O}_U.$$

This is a flat sheaf of $\mathcal{O}$-modules which comes equipped with a canonical surjection $F(\mathcal{F}) \to \mathcal{F}$, see Modules on Sites, Lemma 16.26.6. Moreover the construction $\mathcal{F} \mapsto F(\mathcal{F})$ is functorial in $\mathcal{F}$.

Lemma 17.12.1. *The functor $\mathcal{F} \mapsto \mathcal{F}^\vee$ is exact.*

Proof. This because $\mathcal{J}$ is an injective abelian sheaf. □

There is a canonical map $ev : \mathcal{F} \to (\mathcal{F}^\vee)^\vee$ given by evaluation: given $x \in \mathcal{F}(U)$ we let $ev(x) \in (\mathcal{F}^\vee)^\vee = \mathcal{H}om(\mathcal{F}^\vee, \mathcal{J})$ be the map $\varphi \mapsto \varphi(x)$.

Lemma 17.12.2. *For any $\mathcal{O}$-module $\mathcal{F}$ the evaluation map $ev : \mathcal{F} \to (\mathcal{F}^\vee)^\vee$ is injective.*

Proof. You can check this using the definition of $\mathcal{J}$. Namely, if $s \in \mathcal{F}(U)$ is not zero, then let $j_{U!}\mathcal{O}_U \to \mathcal{F}$ be the map of $\mathcal{O}$-modules it corresponds to via adjunction. Let $\mathcal{I}$ be the kernel of this map. There exists a nonzero map $\mathcal{F} \supset j_{U!}\mathcal{O}_U/\mathcal{I} \to \mathcal{J}$ which does not annihilate s. As $\mathcal{J}$ is an injective $\mathcal{O}$-module, this extends to a map $\varphi : \mathcal{F} \to \mathcal{J}$. Then $ev(s)(\varphi) = \varphi(s) \neq 0$ which is what we had to prove. $\square$

The canonical surjection $F(\mathcal{F}) \to \mathcal{F}$ of $\mathcal{O}$-modules turns into a a canonical injection, see above, of $\mathcal{O}$-modules

$$(\mathcal{F}^\vee)^\vee \longrightarrow (F(\mathcal{F}^\vee))^\vee.$$

Set $J(\mathcal{F}) = (F(\mathcal{F}^\vee))^\vee$. The composition of ev with this the displayed map gives $\mathcal{F} \to J(\mathcal{F})$ functorially in $\mathcal{F}$.

Lemma 17.12.3. *Let $\mathcal{O}$ be a sheaf of rings. For every $\mathcal{O}$-module $\mathcal{F}$ the $\mathcal{O}$-module $J(\mathcal{F})$ is injective.*

Proof. We have to show that the functor $Hom_{\mathcal{O}}(\mathcal{G}, J(\mathcal{F}))$ is exact. Note that

$$\begin{aligned} Hom_{\mathcal{O}}(\mathcal{G}, J(\mathcal{F})) &= Hom_{\mathcal{O}}(\mathcal{G}, (F(\mathcal{F}^\vee))^\vee) \\ &= Hom_{\mathcal{O}}(\mathcal{G}, \mathcal{H}om(F(\mathcal{F}^\vee), \mathcal{J})) \\ &= Hom(\mathcal{G} \otimes_{\mathcal{O}} F(\mathcal{F}^\vee), \mathcal{J}) \end{aligned}$$

Thus what we want follows from the fact that $F(\mathcal{F}^\vee)$ is flat and $\mathcal{J}$ is injective. $\square$

Theorem 17.12.4. *Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. The category of sheaves of $\mathcal{O}$-modules on a site has enough injectives. In fact there exists a functorial injective embedding, see Homology, Definition 10.20.5.*

Proof. From the discussion in this section. $\square$

Proposition 17.12.5. *Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings on $\mathcal{C}$. The category $PMod(\mathcal{O})$ of presheaves of $\mathcal{O}$-modules has functorial injective embeddings.*

Proof. We could prove this along the lines of the discussion in Section 17.10. But instead we argue using the theorem above. Endow $\mathcal{C}$ with the structure of a site by letting the set of coverings of an object U consist of all singletons $\{f : V \to U\}$ where f is an isomorphism. We omit the verification that this defines a site. A sheaf for this topology is the same as a presheaf (proof omitted). Hence the theorem applies. $\square$

17.13. Embedding abelian categories

In this section we show that an abelian category embeds in the category of abelian sheaves on a site having enough points. The site will be the one described in the following lemma.

Lemma 17.13.1. *Let $\mathcal{A}$ be an abelian category. Let*

$$Cov = \{\{f : V \to U\} \mid f \textit{ is surjective}\}.$$

Then $(\mathcal{A}, Cov)$ is a site, see Sites, Definition 9.6.2.

Proof. Note that $Ob(\mathcal{A})$ is a set by our conventions about categories. An isomorphism is a surjective morphism. The composition of surjective morphisms is surjective. And the base change of a surjective morphism in $\mathcal{A}$ is surjective, see Homology, Lemma 10.3.22. $\square$

Let $\mathcal{A}$ be a pre-additive category. In this case the Yoneda embedding $\mathcal{A} \to PSh(\mathcal{A})$, $X \mapsto h_X$ factors through a functor $\mathcal{A} \to PAb(\mathcal{A})$.

Lemma 17.13.2. *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{C} = (\mathcal{A}, Cov)$ be the site defined in Lemma 17.13.1. Then $X \mapsto h_X$ defines a fully faithful, exact functor*

$$\mathcal{A} \longrightarrow Ab(\mathcal{C}).$$

Moreover, the site $\mathcal{C}$ has enough points.

Proof. Suppose that $f : V \to U$ is a surjective morphism of $\mathcal{A}$. Let $K = \mathrm{Ker}(f)$. Recall that $V \times_U V = \mathrm{Ker}((f, -f) : V \oplus V \to U)$, see Homology, Example 10.3.17. In particular there exists an injection $K \oplus K \to V \times_U V$. Let $p, q : V \times_U V \to V$ be the two projection morphisms. Note that $p - q : V \times_U V \to V$ is a morphism such that $f \circ (p - q) = 0$. Hence $p - q$ factors through $K \to V$. Let us denote this morphism by $c : V \times_K V \to K$. And since the composition $K \oplus K \to V \times_U V \to K$ is surjective, we conclude that c is surjective. It follows that

$$V \times_K V \xrightarrow{p-q} V \to U \to 0$$

is an exact sequence of $\mathcal{A}$. Hence for an object X of $\mathcal{A}$ the sequence

$$0 \to Hom_{\mathcal{A}}(U, X) \to Hom_{\mathcal{A}}(V, X) \to Hom_{\mathcal{A}}(V \times_U V, X)$$

is an exact sequence of abelian groups, see Homology, Lemma 10.3.19. This means that h_X satisfies the sheaf condition on $\mathcal{C}$.

The functor is fully faithful by Categories, Lemma 4.3.5. The functor is a left exact functor between abelian categories by Homology, Lemma 10.3.19. To show that it is right exact, let $X \to Y$ be a surjective morphism of $\mathcal{A}$. Let U be an object of $\mathcal{A}$, and let $s \in h_Y(U) = Mor_{\mathcal{A}}(U, Y)$ be a section of h_Y over U. By Homology, Lemma 10.3.22 the projection $U \times_Y X \to U$ is surjective. Hence $\{V = U \times_Y X \to U\}$ is a covering of U such that $s|_V$ lifts to a section of h_X. This proves that $h_X \to h_Y$ is a surjection of abelian sheaves, see Sites, Lemma 9.11.2.

The site $\mathcal{C}$ has enough points by Sites, Proposition 9.35.3. □

Remark 17.13.3. The Freyd-Mitchell embedding theorem says there exists a fully faithful exact functor from any abelian category $\mathcal{A}$ to the category of modules over a ring. Lemma 17.13.2 is not quite as strong. But the result is suitable for the stacks project as we have to understand sheaves of abelian groups on sites in detail anyway. Moreover, ``diagram chasing'' works in the category of abelian sheaves on $\mathcal{C}$, for example by working with sections over objects, or by working on the level of stalks using that $\mathcal{C}$ has enough points. To see how to deduce the Freyd-Mitchell embedding theorem from Lemma 17.13.2 see Remark 17.13.5.

Remark 17.13.4. If $\mathcal{A}$ is a ``big'' abelian category, i.e., if $\mathcal{A}$ has a class of objects, then Lemma 17.13.2 does not work. In this case, given any set of objects $E \subset Ob(\mathcal{A})$ there exists an abelian full subcategory $\mathcal{A}' \subset \mathcal{A}$ such that $Ob(\mathcal{A}')$ is a set and $E \subset Ob(\mathcal{A}')$. Then one can apply Lemma 17.13.2 to $\mathcal{A}'$. One can use this to prove that results depending on a diagram chase hold in $\mathcal{A}$.

Remark 17.13.5. Let $\mathcal{C}$ be a site. Note that $Ab(\mathcal{C})$ has enough injectives, see Theorem 17.11.4. (In the case that $\mathcal{C}$ has enough points this is straightforward because p_*I is an injective sheaf if I is an injective $\mathbf{Z}$-module and p is a point.) Also, $Ab(\mathcal{C})$ has a cogenerator (details omitted). Hence Lemma 17.13.2 proves that we have a fully faithful, exact embedding $\mathcal{A} \to \mathcal{B}$ where $\mathcal{B}$ has a cogenerator and enough injectives. We can apply this to $\mathcal{A}^{opp}$ and we get a fully faithful exact functor $i : \mathcal{A} \to \mathcal{D} = \mathcal{B}^{opp}$ where $\mathcal{D}$ has enough projectives and a generator. Hence $\mathscr{D}$ has a projective generator P. Set $R = Mor_{\mathscr{D}}(P, P)$. Then

$$\mathscr{A} \longrightarrow \text{Mod}_R, \quad X \longmapsto Hom_{\mathscr{D}}(P, X).$$

One can check this is a fully faithful, exact functor. In other words, one retrieves the Freyd-Mitchell theorem mentioned in Remark 17.13.3 above.

Remark 17.13.6. The arguments proving Lemmas 17.13.1 and 17.13.2 work also for *exact categories*, see [**Büh10**, Appendix A] and [**BBD82**, 1.1.4]. We quickly review this here and we add more details if we ever need it in the stacks project.

Let $\mathscr{A}$ be an additive category. A *kernel-cokernel* pair is a pair (i, p) of morphisms of $\mathscr{A}$ with $i : A \to B$, $p : B \to C$ such that i is the kernel of p and p is the cokernel of i. Given a set $\mathscr{E}$ of kernel-cokernel pairs we say $i : A \to B$ is an *admissible monomorphism* if $(i, p) \in \mathscr{E}$ for some morphism p. Similarly we say a morphism $p : B \to C$ is an *admissible epimorphism* if $(i, p) \in \mathscr{E}$ for some morphism i. The pair $(\mathscr{A}, \mathscr{E})$ is said to be an *exact category* if the following axioms hold

(1) $\mathscr{E}$ is closed under isomorphisms of kernel-cokernel pairs,
(2) for any object A the pairs 1_A is both an admissible epimorphism and an admissible monomorphism,
(3) admissible monomorphisms are stable under composition,
(4) admissible epimorphisms are stable under composition,
(5) the push-out of an admissible monomorphism $i : A \to B$ via any morphism $A \to A'$ exist and the induced morphism $i' : A' \to B'$ is an admissible monomorphism, and
(6) the base change of an admissible epimorphism $p : B \to C$ via any morphism $C' \to C$ exist and the induced morphism $p' : B' \to C'$ is an admissible epimorphism.

Given such a structure let $\mathscr{C} = (\mathscr{A}, \text{Cov})$ where coverings (i.e., elements of Cov) are given by admissible epimorphisms. The axioms listed above immediately imply that this is a site. Consider the functor

$$F : \mathscr{A} \longrightarrow Ab(\mathscr{C}), \quad X \longmapsto h_X$$

exactly as in Lemma 17.13.2. It turns out that this functor is fully faithful, exact, and reflects exactness. Moreover, any extension of objects in the essential image of F is in the essential image of F.

17.14. Grothendieck's AB conditions

This and the next few sections are mostly interesting for ``big'' abelian categories, i.e., those categories listed in Categories, Remark 4.2.2. A good case to keep in mind is the category of sheaves of modules on a ringed site.

Grothendieck proved the existence of injectives in great generality in the paper [**Gro57**]. He used the following conditions to single out abelian categories with special properties.

Definition 17.14.1. Let $\mathscr{A}$ be an abelian category. We name some conditions

AB3 $\mathscr{A}$ has direct sums,
AB4 $\mathscr{A}$ has AB3 and direct sums are exact,
AB5 $\mathscr{A}$ has AB3 and filtered colimits are exact.

Here are the dual notions

AB3* $\mathscr{A}$ has products,

AB4* $\mathcal{A}$ has AB3* and products are exact,

AB5* $\mathcal{A}$ has AB3* and filtered limits are exact.

We say an object U of $\mathcal{A}$ is a *generator* if for every $N \subset M$, $N \neq M$ in $\mathcal{A}$ there exists a morphism $U \to M$ which does not factor through N. We say $\mathcal{A}$ is a *Grothendieck abelian category* if it has AB5 and a generator.

Discussion: A direct sum in an abelian category is a coproduct. If an abelian category has direct sums (i.e., AB3), then it has colimits, see Categories, Lemma 4.13.11. Similarly if $\mathcal{A}$ has AB3* then it has limits, see Categories, Lemma 4.13.10. Exactness of direct sums means the following: given an index set I and short exact sequences

$$0 \to A_i \to B_i \to C_i \to 0, \quad i \in I$$

in $\mathcal{A}$ then the sequence

$$0 \to \bigoplus\nolimits_{i \in I} A_i \to \bigoplus\nolimits_{i \in I} B_i \to \bigoplus\nolimits_{i \in I} C_i \to 0$$

is exact as well. Without assuming AB4 it is only true in general that the sequence is exact on the right (i.e., taking direct sums is a right exact functor if direct sums exist). Similarly, exactness of filtered colimits means the following: given a directed partially ordered set I and a system of short exact sequences

$$0 \to A_i \to B_i \to C_i \to 0$$

over I in $\mathcal{A}$ then the sequence

$$0 \to colim_{i \in I} A_i \to colim_{i \in I} B_i \to colim_{i \in I} C_i \to 0$$

is exact as well. Without assuming AB5 it is only true in general that the sequence is exact on the right (i.e., taking colimits is a right exact functor if colimits exist). A similar explanation holds for AB4* and AB5*.

17.15. Injectives in Grothendieck categories

The existence of a generator implies that given an object M of a Grothendieck abelian category $\mathcal{A}$ there is a set of subobjects. (This may not be true for a general ``big'' abelian category.)

Definition 17.15.1. Let $\mathcal{A}$ be a Grothendieck abelian category. Let M be an object of $\mathcal{A}$. The *size* $|M|$ of M is the cardinality of the set of subobjects of M.

Lemma 17.15.2. *Let $\mathcal{A}$ be a Grothendieck abelian category. If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of $\mathcal{A}$, then $|M'|, |M''| \leq |M|$.*

Proof. Immediate from the definitions. □

Lemma 17.15.3. *Let $\mathcal{A}$ be a Grothendieck abelian category with generator U.*

(1) *If $|M| \leq \kappa$, then M is the quotient of a direct sum of at most κ copies of U.*
(2) *For every cardinal κ there exists a set of isomorphism classes of objects M with $|M| \leq \kappa$.*

Proof. For (1) choose for every proper subobject $M' \subset M$ a morphism $\varphi_{M'} : U \to M$ whose image is not contained in M'. Then $\bigoplus_{M' \subset M} \varphi_{M'} : \bigoplus_{M' \subset N} U \to M$ is surjective. It is clear that (1) implies (2). □

Proposition 17.15.4. *Let $\mathcal{A}$ be a Grothendieck abelian category. Let M be an object of $\mathcal{A}$. Let $\kappa = |M|$. If α is an ordinal whose cofinality is bigger than κ, then M is α-small with respect to injections.*

Proof. Please compare with Proposition 17.6.5. We need only show that the map (17.6.0.1) is a surjection. Let $f : M \to colim\, B_\beta$ be a map. Consider the subobjects $\{f^{-1}(B_\beta)\}$ of M, where B_β is considered as a subobject of the colimit $B = \bigcup_\beta B_\beta$. If one of these, say $f^{-1}(B_\beta)$, fills M, then the map factors through B_β.

So suppose to the contrary that all of the $f^{-1}(B_\beta)$ were proper subobjects of M. However, because $\mathscr{A}$ has AB5 we have

$$colim\, f^{-1}(B_\beta) = f^{-1}\left(colim\, B_\beta\right) = M.$$

Now there are at most κ different subobjects of M that occur among the $f^{-1}(B_\alpha)$, by hypothesis. Thus we can find a subset $S \subset \alpha$ of cardinality at most κ such that as β' ranges over S, the $f^{-1}(B_{\beta'})$ range over *all* the $f^{-1}(B_\alpha)$.

However, S has an upper bound $\tilde{\alpha} < \alpha$ as α has cofinality bigger than κ. In particular, all the $f^{-1}(B_{\beta'})$, $\beta' \in S$ are contained in $f^{-1}(B_{\tilde{\alpha}})$. It follows that $f^{-1}(B_{\tilde{\alpha}}) = M$. In particular, the map f factors through $B_{\tilde{\alpha}}$. □

Lemma 17.15.5. *Let $\mathscr{A}$ be a Grothendieck abelian category with generator U. An object I of $\mathscr{A}$ is injective if and only if in every commutative diagram*

$$\begin{tikzcd} M \arrow[r] \arrow[d] & I \\ U \arrow[ur, dashed] & \end{tikzcd}$$

for $M \subset U$ a subobject, the dotted arrow exists.

Proof. Please see Lemma 17.6.6 for the case of modules. Choose an injection $A \subset B$ and a morphism $\varphi : A \to I$. Consider the set S of pairs (A', φ') consisting of subobjects $A \subset A' \subset B$ and a morphism $\varphi' : A' \to I$ extending φ. Define a partial ordering on this set in the obvious manner. Choose a totally ordered subset $T \subset S$. Then

$$A' = colim_{t \in T} A_t \xrightarrow{colim_{t\in T} \varphi_t} I$$

is an upper bound. Hence by Zorn's lemma the set S has a maximal element (A', φ'). We claim that $A' = B$. If not, then choose a morphism $\psi : U \to B$ which does not factor through A'. Set $N = A' \cap \psi(U)$. Set $M = \psi^{-1}(N)$. Then the map

$$M \to N \to A' \xrightarrow{\varphi'} I$$

can be extended to a morphism $\chi : U \to I$. Since $\chi|_{\mathrm{Ker}(\psi)} = 0$ we see that χ factors as

$$U \to \mathrm{Im}(\psi) \xrightarrow{\varphi''} I$$

Since φ' and φ'' agree on $N = A' \cap \mathrm{Im}(\psi)$ we see that combined the define a morphism $A' + \mathrm{Im}(\psi) \to I$ contradicting the assumed maximality of A'. □

Theorem 17.15.6. *Let $\mathscr{A}$ be a Grothendieck abelian category. Then $\mathscr{A}$ has functorial injective embeddings.*

Proof. Please compare with the proof of Theorem 17.6.8. Choose a generator U of $\mathcal{A}$. For an object M we define $\mathbf{M}(M)$ by the following pushout diagram

$$\begin{tikzcd} \bigoplus_{N \subset U} \bigoplus_{\varphi \in Hom(N,M)} N \arrow[r] \arrow[d] & M \arrow[d] \\ \bigoplus_{N \subset U} \bigoplus_{\varphi \in Hom(U,M)} U \arrow[r] & \mathbf{M}(M). \end{tikzcd}$$

Note that $M \to \mathbf{M}(N)$ is a functor and that there exist functorial injective maps $M \to \mathbf{M}(M)$. By transfinite induction we define functors $\mathbf{M}_\alpha(M)$ for every ordinal α. Namely, set $\mathbf{M}_0(M) = M$. Given $\mathbf{M}_\alpha(M)$ set $\mathbf{M}_{\alpha+1}(M) = \mathbf{M}(\mathbf{M}_\alpha(M))$. For a limit ordinal β set

$$\mathbf{M}_\beta(M) = colim_{\alpha < \beta} \mathbf{M}_\alpha(M).$$

Finally, choose an ordinal α whose cofinality is greater than $|U|$, see Sets, Proposition 3.7.2. We claim that $M \to \mathbf{M}_\alpha(M)$ is the desired functorial injective embedding. Namely, if $N \subset U$ is a subobject and $\varphi : N \to \mathbf{M}_\alpha(M)$ is a morphism, then we see that φ factors through $\mathbf{M}_{\alpha'}(M)$ for some $\alpha' < \alpha$ by Proposition 17.15.4. By construction of $\mathbf{M}(-)$ we see that φ extends to a morphism from U into $\mathbf{M}_{\alpha'+1}(M)$ and hence into $\mathbf{M}_\alpha(M)$. By Lemma 17.15.5 we conclude that $\mathbf{M}_\alpha(M)$ is injective. □

17.16. K-injectives in Grothendieck categories

Most of the material in this section is taken from the paper [**Ser03**] which generalizes some of the results in the paper [**Spa88**] by Spaltenstein to general Grothendieck abelian categories. The exposition is also simplified somewhat: Lemma 17.16.3 gives a new characterization of K-injective complexes, and we have consistently tried to mimick Grothendieck's approach (see proof of Theorem 17.15.6).

Lemma 17.16.1. *Let $\mathcal{A}$ be a Grothendieck abelian category with generator U. Let c be the function on cardinals defined by $c(\kappa) = |\bigoplus_{\alpha \in \kappa} U|$. If $\pi : M \to N$ is a surjection then there exists a subobject $M' \subset M$ which surjects onto N with $|N'| \leq c(|N|)$.*

Proof. For every proper subobject $N' \subset N$ choose a morphism $\varphi_{N'} : U \to M$ such that $U \to M \to N$ does not factor through N'. Set

$$N' = \operatorname{Im}\left(\bigoplus\nolimits_{N' \subset N} \varphi_{N'} : \bigoplus\nolimits_{N' \subset N} U \longrightarrow M\right)$$

Then N' works. □

Lemma 17.16.2. *Let $\mathcal{A}$ be a Grothendieck abelian category. The exists a cardinal κ such that given any acyclic complex $M^\bullet$ there exists a surjection of complexes*

$$\bigoplus\nolimits_{i \in I} M_i^\bullet \longrightarrow M^\bullet$$

where $M_i^\bullet$ is bounded above, acyclic and $|M_i^n| \leq \kappa$.

Proof. Choose a generator U of $\mathcal{A}$. Denote c the function of Lemma 17.16.1. Set $\kappa = \sup\{c^n(|U|), n = 1, 2, 3, \ldots\}$. It suffices to prove that for every $\varphi : U \to M^n$ there exists a morphism of complexes $\varphi^\bullet : N^\bullet \to M^\bullet$ with $N^n = U$, $\varphi^n = \varphi$, $N^\bullet$ bounded above, acyclic and $|N^m| \leq \kappa$ for all m. To do this set $N^n = U$, $\varphi^n = \varphi$, $N^{n+1} = \operatorname{Im}(U \to M^n \to M^{n+1})$

and $\varphi^{n+1} : N^{n+1} \to M^{n+1}$ the inclusion map. Moreover, we set $N^m = 0$ for $m \geq n+2$. Suppose we have constructed $\varphi^m : N^m \to M^m$ and $N^m \to M^m$ for all $m \geq k$ such that

$$\begin{tikzcd} & N^k \arrow[r] \arrow[d] & N^{k+1} \arrow[r] \arrow[d] & \ldots \\ M^{k-1} \arrow[r] & M^k \arrow[r] & M^{k+1} \arrow[r] & \ldots \end{tikzcd}$$

commutes, such that $|N^m| \leq c^{\max\{n-m,1\}}(|U|)$, and such that $\mathrm{Im}(N^{m-1} \to N^m) = \mathrm{Ker}(N^m \to N^{m+1})$ for all $m \geq k+1$. Then we can choose

$$N^{k-1} \longrightarrow \mathrm{Ker}(N^k \to N^{k+1}) \times_{M^k} M^{k-1}$$

surjecting onto $\mathrm{Ker}(N^k \to N^{k+1})$ as in Lemma 17.16.1. This is possible as the complex $M^\bullet$ is exact by assumption. Hence we win by induction. □

Lemma 17.16.3. *Let $\mathcal{A}$ be a Grothendieck abelian category. Let κ be a cardinal as in Lemma 17.16.2. Suppose that $I^\bullet$ is a complex such that*

1. *each I^j is injective, and*
2. *for every bounded above acyclic complex $M^\bullet$ such that $|M^n| \leq \kappa$ we have $Hom_{K(\mathcal{A})}(M^\bullet, I^\bullet) = 0$.*

Then $I^\bullet$ is an K-injective complex.

Proof. Let $M^\bullet$ be an acyclic complex. By Lemma 17.16.2 and Derived Categories, Lemma 11.15.5 we can find a resolution

$$\ldots \to M_2^\bullet \to M_1^\bullet \to M_0^\bullet \to M^\bullet \to 0$$

(in the abelian category of complexes) where each $M_k^\bullet$ is a direct sum of complexes which satisfy the hypotheses of (2). Then we obtain a short exact sequence of complexes

$$0 \to \bigoplus\nolimits_{i \geq 1} M_i^\bullet \to \bigoplus\nolimits_{i \geq 0} M_i^\bullet \to M^\bullet \to 0$$

In other words we have constructed a short exact sequence of complexes $0 \to K^\bullet \to L^\bullet \to M^\bullet \to 0$ where $K^\bullet$ and $L^\bullet$ is a direct sum of complexes as in (2). Since each component of $I^\bullet$ is injective we see that we obtain an exact sequence

$$\mathrm{Hom}_{K(\mathcal{A})}(K^\bullet[1], I^\bullet) \to \mathrm{Hom}_{K(\mathcal{A})}(M^\bullet, I^\bullet) \to \mathrm{Hom}_{K(\mathcal{A})}(L^\bullet, I^\bullet)$$

Hence we see that it suffices to prove $\mathrm{Hom}_{K(\mathcal{A})}(M^\bullet, I^\bullet)$ is zero when $M^\bullet$ is a direct sum of complexes as in (2). This follows from assumption (2). □

Lemma 17.16.4. *Let $\mathcal{A}$ be a Grothendieck abelian category. Let $(K_i^\bullet)_{i \in I}$ be a set of acyclic complexes. There exists a functor $M^\bullet \mapsto \mathbf{M}^\bullet(M^\bullet)$ and a natural transformation $j_{M^\bullet} : M^\bullet \to \mathbf{M}^\bullet(M^\bullet)$ such*

1. *$j_{M^\bullet}$ is a (termwise) injective quasi-isomorphism, and*
2. *for every $i \in I$ and $w : K_i^\bullet \to M^\bullet$ the morphism $j_{M^\bullet} \circ w$ is homotopic to zero.*

Proof. For every $i \in I$ choose a (termwise) injective map of complexes $K_i^\bullet \to L_i^\bullet$ which is homotopic to zero with $L_i^\bullet$ quasi-isomorphic to zero. For example, take $L_i^\bullet$ to be the cone on the identity of $K_i^\bullet$. We define $\mathbf{M}^\bullet(M^\bullet)$ by the following pushout diagram

$$\begin{tikzcd}
\bigoplus_{i\in I}\bigoplus_{w:K_i^\bullet\to M^\bullet} K_i^\bullet \arrow[r] \arrow[d] & M^\bullet \arrow[d] \\
\bigoplus_{i\in I}\bigoplus_{w:K_i^\bullet\to M^\bullet} L_i^\bullet \arrow[r] & \mathbf{M}^\bullet(M^\bullet).
\end{tikzcd}$$

Then $M^\bullet \to \mathbf{M}^\bullet(M^\bullet)$ is a functor. The right vertical arrow defines the functorial injective map $j_{M^\bullet}$. The cokernel of $j_{M^\bullet}$ is isomorphic to the direct sum of the cokernels of the maps $K_i^\bullet \to L_i^\bullet$ hence acyclic. Thus $j_{M^\bullet}$ is a quasi-isomorphism. Part (2) holds by construction. □

Lemma 17.16.5. *Let $\mathcal{A}$ be a Grothendieck abelian category. There exists a functor $M^\bullet \mapsto \mathbf{N}^\bullet(M^\bullet)$ and a natural transformation $j_{M^\bullet} : M^\bullet \to \mathbf{N}^\bullet(M^\bullet)$ such*

1. *$j_{M^\bullet}$ is a (termwise) injective quasi-isomorphism, and*
2. *for every $n \in \mathbf{Z}$ the map $M^n \to \mathbf{N}^n(M^\bullet)$ factors through a subobject $I^n \subset \mathbf{N}^n(M^\bullet)$ where I^n is an injective object of $\mathcal{A}$.*

Proof. Choose a functorial injective embeddings $i_M : M \to I(M)$, see Theorem 17.15.6. For every complex $M^\bullet$ denote $J^\bullet(M^\bullet)$ the complex with terms $J^n(M^\bullet) = I(M^n)\oplus I(M^{n+1})$ and differential

$$d_{J^\bullet(M^\bullet)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

There exists a canonical injective map of complexes $u_{M^\bullet} : M^\bullet \to J^\bullet(M^\bullet)$ by mapping M^n to $I(M^n) \oplus I(M^{n+1})$ via the maps $i_{M^n} : M^n \to I(M^n)$ and $i_{M^{n+1}} \circ d : M^n \to M^{n+1} \to I(M^{n+1})$. Hence a short exact sequence of complexes

$$0 \to M^\bullet \xrightarrow{u_{M^\bullet}} J^\bullet(M^\bullet) \xrightarrow{v_{M^\bullet}} Q^\bullet(M^\bullet) \to 0$$

functorial in $M^\bullet$. Set

$$\mathbf{N}^\bullet(M^\bullet) = C(v_{M^\bullet})^\bullet[-1].$$

Note that

$$\mathbf{N}^n(M^\bullet) = Q^{n-1}(M^\bullet) \oplus J^n(M^\bullet)$$

with differential

$$\begin{pmatrix} -d^{n-1}_{Q^\bullet(M^\bullet)} & -v^n_{M^\bullet} \\ 0 & d^n_{J^\bullet(M)} \end{pmatrix}$$

Hence we see that there is a map of complexes $j_{M^\bullet} : M^\bullet \to \mathbf{N}^\bullet(M^\bullet)$ induced by u. It is injective and factors through an injective subobject by construction. The map $j_{M^\bullet}$ is a quasi-isomorphism as one can prove by looking at the long exact sequence of cohomology associated to the short exact sequences of complexes above. □

Theorem 17.16.6. *Let $\mathcal{A}$ be a Grothendieck abelian category. For every complex $M^\bullet$ there exists a quasi-isomorphism $M^\bullet \to I^\bullet$ where $I^\bullet$ is a K-injective complex. In fact, we may also assume that I^n is an injective object of $\mathcal{A}$ for all n. Moreover, there exists a functorial injective quasi-isomorphism into such a K-injective complex.*

Proof. Please compare with the proof of Theorem 17.6.8 and Theorem 17.15.6. Choose a cardinal κ as in Lemmas 17.16.2 and 17.16.3. Choose a set $(K_i^\bullet)_{i\in I}$ of bounded above, acyclic complexes such that every bounded above acyclic complex $K^\bullet$ such that $|K^n| \le \kappa$ is isomorphic to $K_i^\bullet$ for some $i \in I$. This is possible by Lemma 17.15.3. Denote $\mathbf{M}^\bullet(-)$ the functor constructed in Lemma 17.16.4. Denote $\mathbf{N}^\bullet(-)$ the functor constructed in Lemma 17.16.5. Both of these functors come with injective transformations $\text{id} \to \mathbf{M}$ and $\text{id} \to \mathbf{N}$.

By transfinite induction we define a sequence of functors $\mathbf{T}_\alpha(-)$ and corresponding transformations $\text{id} \to \mathbf{T}_\alpha$. Namely we set $\mathbf{T}_0(M^\bullet) = M^\bullet$. If $\mathbf{T}_\alpha$ is given then we set

$$\mathbf{T}_{\alpha+1}(M^\bullet) = \mathbf{N}^\bullet(\mathbf{M}^\bullet(\mathbf{T}_\alpha(M^\bullet)))$$

If β is a limit ordinal we set

$$\mathbf{T}_\beta(M^\bullet) = colim_{\alpha<\beta} \mathbf{T}_\alpha(M^\bullet)$$

The transition maps of the system are injective quasi-isomorphisms. By AB5 we see that the colimit is still quasi-isomorphic to $M^\bullet$. We claim that $M^\bullet \to \mathbf{T}_\alpha(M^\bullet)$ does the job if the cofinality of α is larger than $\max(\kappa, |U|)$ where U is a generator of $\mathcal{A}$. Namely, it suffices to check conditions (1) and (2) of Lemma 17.16.3.

For (1) we use the criterion of Lemma 17.15.5. Suppose that $M \subset U$ and $\varphi : M \to \mathbf{T}^n_\alpha(M^\bullet)$ is a morphism for some $n \in \mathbf{Z}$. By Proposition 17.15.4 we see that φ factor through $\mathbf{T}^n_{\alpha'}(M^\bullet)$ for some $\alpha' < \alpha$. In particular, by the construction of the functor $\mathbf{N}^\bullet(-)$ we see that φ factors through an injective object of $\mathcal{A}$ which shows that φ lifst to a morphism on U.

For (2) let $w : K^\bullet \to \mathbf{T}_\alpha(M^\bullet)$ be a morphism of complexes where $K^\bullet$ is a bounded above acyclic complex such that $|K^n| \leq \kappa$. Then $K^\bullet \cong K_i^\bullet$ for some $i \in I$. Moreover, by Proposition 17.15.4 once again we see that w factor through $\mathbf{T}^n_{\alpha'}(M^\bullet)$ for some $\alpha' < \alpha$. In particular, by the construction of the functor $\mathbf{M}^\bullet(-)$ we see that w is homotopic to zero. This finishes the proof. □

17.17. Additional remarks on Grothendieck abelian categories

In this section we put some results on Grothendieck abelian categories which are folklore.

Lemma 17.17.1. *Let $\mathcal{A}$ be a Grothendieck abelian category. Let $F : \mathcal{A}^{opp} \to Sets$ be a functor. Then F is representable if and only if F commutes with colimits, i.e.,*

$$F(colim_i\, N_i) = lim\, F(N_i)$$

for any diagram $\mathcal{I} \to \mathcal{A}$, $i \in \mathcal{I}$.

Proof. If F is representable, then it commutes with colimits by definition of colimits.

Assume that F commutes with colimits. Then $F(M \oplus N) = F(M) \prod F(N)$ and we can use this to define a group structure on $F(M)$. Hence we get $F : \mathcal{A} \to Ab$ which is additive and right exact, i.e., transforms a short exact sequence $0 \to K \to L \to M \to 0$ into an exact sequence $F(K) \leftarrow F(L) \leftarrow F(M) \leftarrow 0$ (compare with Homology, Section 10.5).

Let U be a generator for $\mathcal{A}$. Set $A = \bigoplus_{s\in F(U)} U$. Let $s_{univ} = (s)_{s\in F(U)} \in F(A) = \prod_{s\in F(U)} F(U)$. Let $A' \subset A$ be the largest subobject such that s_{univ} restricts to zero on A'. This exists because $\mathcal{A}$ is a grothendieck category and because F commutes with colimits. Because F commutes with colimits there exists a unique element $\overline{s}_{univ} \in F(A/A')$ which maps to s_{univ} in $F(A)$. We claim that A/A' represents F, in other words, the Yoneda map

$$\overline{s}_{univ} : h_{A/A'} \longrightarrow F$$

is an isomorphism. Let $M \in Ob(\mathcal{A})$ and $s \in F(M)$. Consider the surjection

$$c_M : A_M = \bigoplus\nolimits_{\varphi\in Hom_{\mathcal{A}}(U,M)} U \longrightarrow M.$$

This gives $F(c_M)(s) = (s_\varphi) \in \prod_\varphi F(U)$. Consider the map

$$\psi : A_M = \bigoplus\nolimits_{\varphi \in Hom_{\mathscr{A}}(U,M)} U \longrightarrow \bigoplus\nolimits_{s \in F(U)} U = A$$

which maps the summand corresponding to φ to the summand corresponding to s_φ by the identity map on U. Then s_{univ} maps to $(s_\varphi)_\varphi$ by construction. in other words the right square in the diagram

$$\begin{tikzcd} A' \arrow[r] & A \arrow[r, dotted, "s_{univ}"'] & F \\ K \arrow[u, "?"] \arrow[r] & A_M \arrow[u, "\psi"] \arrow[r] & M \arrow[u, dotted, "s"'] \end{tikzcd}$$

commutes. Let $K = \mathrm{Ker}(A_M \to M)$. Since s restricts to zero on K we see that $\psi(K) \subset A'$ by definition of A'. Hence there is an induced morphism $M \to A/A'$. This construction gives an inverse to the map $h_{A/A'}(M) \to F(M)$ (details omitted). $\square$

Lemma 17.17.2. *A Grothendieck abelian category has Ab3*.*

Proof. Let M_i, $i \in I$ be a family of objects of $\mathscr{A}$ indexed by a set I. The functor $F = \prod_{i \in I} h_{M_i}$ commutes with colimits. Hence Lemma 17.17.1 applies. $\square$

Remark 17.17.3. In the chapter on derived categories we consistently work with ``small'' abelian categories (as is the convention in the stacks project). For a ``big'' abelian category $\mathscr{A}$ it isn't clear that the derived category $D(\mathscr{A})$ exists because it isn't clear that morphisms in the derived category are sets (in general they aren't, insert future reference here). However, if $\mathscr{A}$ is a Grothendieck abelian category, then we can use Theorem 17.16.6 to see that $\mathrm{Hom}_{D(\mathscr{A})}(K^\bullet, L^\bullet) = \mathrm{Hom}_{K(\mathscr{A})}(K^\bullet, I^\bullet)$ where $L^\bullet \to I^\bullet$ is a quasi-isomorphism into a K-injective complex. And $\mathrm{Hom}_{K(\mathscr{A})}(K^\bullet, I^\bullet)$ is a set.

Lemma 17.17.4. *Let $\mathscr{A}$ be a Grothendieck abelian category. Then $D(\mathscr{A})$ has both direct sums and products.*

Proof. Let $K_i^\bullet$, $i \in I$ be a family of objects of $D(\mathscr{A})$ indexed by a set I. We claim that the termwise direct sum $\bigoplus_{i \in I} K_i^\bullet$ is a direct sum in $D(\mathscr{A})$. Namely, let $I^\bullet$ be a K-injective complex. Then we have

$$\begin{aligned} Hom_{D(\mathscr{A})}(\bigoplus\nolimits_{i \in I} K_i^\bullet, I^\bullet) &= Hom_{K(\mathscr{A})}(\bigoplus\nolimits_{i \in I} K_i^\bullet, I^\bullet) \\ &= \prod\nolimits_{i \in I} Hom_{K(\mathscr{A})}(K_i^\bullet, I^\bullet) \\ &= \prod\nolimits_{i \in I} Hom_{D(\mathscr{A})}(K_i^\bullet, I^\bullet) \end{aligned}$$

as desired. This is sufficient since any complex can be represented by a K-injective complex by Theorem 17.16.6. To construct the product, choose a K-injective resolution $K_i^\bullet \to I_i^\bullet$ for each i. Then we claim that $\prod_{i \in I} I_i^\bullet$ is a product in $D(\mathscr{A})$. Namely, let $K^\bullet$ be an complex. Note that a product of K-injective complexes is K-injective (follows immediately from the definition). Thus we have

$$\begin{aligned} Hom_{D(\mathscr{A})}(K^\bullet, \prod\nolimits_{i \in I} I_i^\bullet) &= Hom_{K(\mathscr{A})}(K^\bullet, \prod\nolimits_{i \in I} I_i^\bullet) \\ &= \prod\nolimits_{i \in I} Hom_{K(\mathscr{A})}(K^\bullet, I_i^\bullet) \\ &= \prod\nolimits_{i \in I} Hom_{D(\mathscr{A})}(K^\bullet, I_i^\bullet) \end{aligned}$$

which proves the result. $\square$

Remark 17.17.5. Let R be a ring. Suppose that M_n, $n \in \mathbf{Z}$ are R-modules. Denote $E_n = M_n[-n] \in D(R)$. We claim that $E = \bigoplus M_n[-n]$ is *both* the direct sum and the product of the objects E_n in $D(R)$. To see that it is the direct sum, take a look at the proof of Lemma 17.17.4. To see that it is the direct product, take injective resolutions $M_n \to I_n^\bullet$. By the proof of Lemma 17.17.4 we have

$$\prod E_n = \prod I_n^\bullet[-n]$$

in $D(R)$. Since products in Mod_R are exact, we see that $\prod I_n^\bullet$ is quasi-isomorphic to E. This works more generally in $D(\mathcal{A})$ where $\mathcal{A}$ is a Grothendieck abelian category with Ab4*.

Remark 17.17.6. Let $\mathcal{A}$ be a Grothendieck abelian category with Ab4*. Let $K_e^\bullet$ be a collection of complexes indexed by $e \in \mathbf{N}$. Then the termwise product $\prod_e K_e^\bullet$ represents the product in $D(\mathcal{A})$. Namely, choose injective resolutions $K_e^\bullet \to I_e^\bullet$. The proof of Lemma 17.17.4 shows that the product in $D(\mathcal{A})$ is equal to $\prod_e I_e^\bullet$ and Ab4* shows that this is isomorphic to $\prod_e K_e^\bullet$. This works more generally in $D(\mathcal{A})$ where $\mathcal{A}$ is a Grothendieck abelian category with exact countable products.

17.18. Other chapters

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CHAPTER 18

Cohomology of Sheaves

18.1. Introduction

In this document we work out some topics on cohomology of sheaves on topological spaces. We mostly work in the generality of modules over a sheaf of rings and we work with morphisms of ringed spaces. To see what happens for sheaves on sites as well, see the chapter Cohomology on Sites, Section 19. Basic references are [**God73**] and [**Ive86**].

18.2. Topics

Here are some topics that should be discussed in this chapter, and have not yet been written.

(1) Ext-groups.
(2) Ext sheaves.
(3) Tor functors.
(4) Derived pullback for morphisms between ringed spaces.
(5) Cup-product.
(6) Etc, etc, etc.

18.3. Cohomology of sheaves

Let X be a topological space. Let $\mathcal{F}$ be a abelian sheaf. We know that the category of abelian sheaves on X has enough injectives, see Injectives, Lemma 17.8.1. Hence we can choose an injective resolution $\mathcal{F}[0] \to \mathcal{I}^\bullet$. As is customary we define

(18.3.0.1) $$H^i(X, \mathcal{F}) = H^i(\Gamma(X, \mathcal{I}^\bullet))$$

to be the *ith cohomology group of the abelian sheaf $\mathcal{F}$*. The family of functors $H^i((X, -)$ forms a universal δ-functor from $Ab(X) \to Ab$.

Let $f : X \to Y$ be a continuous map of topological spaces. With $\mathcal{F}[0] \to \mathcal{I}^\bullet$ as above we define

(18.3.0.2) $$R^i f_* \mathcal{F} = H^i(f_* \mathcal{I}^\bullet)$$

to be the *ith higher direct image of $\mathcal{F}$*. The family of functors $R^i f_*$ forms a universal δ-functor from $Ab(X) \to Ab(Y)$.

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We know that the category of $\mathcal{O}_X$-modules on X has enough injectives, see Injectives, Lemma 17.9.1. Hence we can choose an injective resolution $\mathcal{F}[0] \to \mathcal{I}^\bullet$. As is customary we define

(18.3.0.3) $$H^i(X, \mathcal{F}) = H^i(\Gamma(X, \mathcal{I}^\bullet))$$

to be the *ith cohomology group of $\mathcal{F}$*. The family of functors $H^i((X, -)$ forms a universal δ-functor from $Mod(\mathcal{O}_X) \to \mathrm{Mod}(\mathcal{O}_X(X))$.

Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. With $\mathcal{F}[0] \to \mathcal{I}^\bullet$ as above we define

$$R^i f_* \mathcal{F} = H^i(f_* \mathcal{I}^\bullet) \tag{18.3.0.4}$$

to be the *ith higher direct image of $\mathcal{F}$*. The family of functors $R^i f_*$ forms a universal δ-functor from $Mod(\mathcal{O}_X) \to Mod(\mathcal{O}_Y)$.

18.4. Derived functors

We briefly explain an approach to right derived functors using resolution functors. Let $(X, \mathcal{O}_X)$ be a ringed space. The category $Mod(\mathcal{O}_X)$ is abelian, see Modules, Lemma 15.3.1. In this chapter we will write

$$K(X) = K(\mathcal{O}_X) = K(Mod(\mathcal{O}_X)) \quad \text{and} \quad D(X) = D(\mathcal{O}_X) = D(Mod(\mathcal{O}_X)).$$

and similarly for the bounded versions for the triangulated categories introduced in Derived Categories, Definition 11.7.1 and Definition 11.10.3. By Derived Categories, Remark 11.23.3 there exists a resolution functor

$$j = j_X : K^+(Mod(\mathcal{O}_X)) \longrightarrow K^+(\mathcal{I})$$

where $\mathcal{I}$ is the strictly full additive subcategory of $Mod(\mathcal{O}_X)$ consisting of injective sheaves. For any left exact functor $F : Mod(\mathcal{O}_X) \to \mathcal{B}$ into any abelian category $\mathcal{B}$ we will denote RF the right derived functor described in Derived Categories, Section 11.19 and constructed using the resolution functor j_X just described:

$$RF = F \circ j'_X : D^+(X) \longrightarrow D^+(\mathcal{B}) \tag{18.4.0.5}$$

see Derived Categories, Lemma 11.24.1 for notation. Note that we may think of RF as defined on $Mod(\mathcal{O}_X)$, $\text{Comp}^+(Mod(\mathcal{O}_X))$, $K^+(X)$, or $D^+(X)$ depending on the situation. According to Derived Categories, Definition 11.16.2 we obtain the $i$the right derived functor

$$R^i F = H^i \circ RF : Mod(\mathcal{O}_X) \longrightarrow \mathcal{B} \tag{18.4.0.6}$$

so that $R^0 F = F$ and $\{R^i F, \delta\}_{i \geq 0}$ is universal δ-functor, see Derived Categories, Lemma 11.19.4.

Here are two special cases of this construction. Given a ring R we write $K(R) = K(\text{Mod}_R)$ and $D(R) = D(\text{Mod}_R)$ and similarly for bounded versions. For any open $U \subset X$ we have a left exact functor $\Gamma(U, -) : Mod(\mathcal{O}_X) \longrightarrow \text{Mod}(\mathcal{O}_X(U))$ which gives rise to

$$R\Gamma(U, -) : D^+(X) \longrightarrow D^+(\mathcal{O}_X(U)) \tag{18.4.0.7}$$

by the discussion above. We set $H^i(U, -) = R^i\Gamma(U, -)$. If $U = X$ we recover (18.3.0.3). If $f : X \to Y$ is a morphism of ringed spaces, then we have the left exact functor $f_* : Mod(\mathcal{O}_X) \longrightarrow Mod(\mathcal{O}_Y)$ which gives rise to the *derived pushforward*

$$Rf_* : D^+(X) \longrightarrow D^+(Y) \tag{18.4.0.8}$$

The ith cohomology sheaf of $Rf_*\mathcal{F}^\bullet$ is denoted $R^i f_* \mathcal{F}^\bullet$ and called the ith *higher direct image* in accordance with (18.3.0.4). The two displayed functors above are exact functor of derived categories.

Abuse of notation: When the functor Rf_*, or any other derived functor, is applied to a sheaf $\mathcal{F}$ on X or a complex of sheaves it is understood that $\mathcal{F}$ has been replaced by a suitable resolution of $\mathcal{F}$. To facilitate this kind of operation we will say, given an object $\mathcal{F}^\bullet \in D(X)$, that a bounded below complex $\mathcal{I}^\bullet$ of injectives of $Mod(\mathcal{O}_X)$ *represents $\mathcal{F}^\bullet$ in* *the derived category* if there exists a quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$. In the same vein the phrase ``let $\alpha : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ be a morphism of $D(X)$" does not mean that α is represented by a morphism of complexes. If we have an actual morphism of complexes we will say so.

18.5. First cohomology and torsors

Definition 18.5.1. Let X be a topological space. Let $\mathcal{G}$ be a sheaf of (possibly non-commutative) groups on X. A *torsor*, or more precisely a *$\mathcal{G}$-torsor*, is a sheaf of sets $\mathcal{F}$ on X endowed with an action $\mathcal{G} \times \mathcal{F} \to \mathcal{F}$ such that

(1) whenever $\mathcal{F}(U)$ is nonempty the action $\mathcal{G}(U)\times\mathcal{F}(U) \to \mathcal{F}(U)$ is simply transitive, and
(2) for every $x \in X$ the stalk $\mathcal{F}_x$ is nonempty.

A *morphism of $\mathcal{G}$-torsors* $\mathcal{F} \to \mathcal{F}'$ is simply a morphism of sheaves of sets compatible with the $\mathcal{G}$-actions. The *trivial $\mathcal{G}$-torsor* is the sheaf $\mathcal{G}$ endowed with the obvious left $\mathcal{G}$-action.

It is clear that a morphism of torsors is automatically an isomorphism.

Lemma 18.5.2. *Let X be a topological space. Let $\mathcal{G}$ be a sheaf of (possibly non-commutative) groups on X. A $\mathcal{G}$-torsor $\mathcal{F}$ is trivial if and only if $\mathcal{F}(X)\neq\emptyset$.*

Proof. Omitted. □

Lemma 18.5.3. *Let X be a topological space. Let $\mathcal{H}$ be an abelian sheaf on X. There is a canonical bijection between the set of isomorphism classes of $\mathcal{H}$-torsors and $H^1(X, \mathcal{H})$.*

Proof. Let $\mathcal{F}$ be a $\mathcal{H}$-torsor. Consider the free abelian sheaf $\mathbf{Z}[\mathcal{F}]$ on $\mathcal{F}$. It is the sheafification of the rule which associates to $U \subset X$ open the collection of finite formal sums $\sum n_i[s_i]$ with $n_i \in \mathbf{Z}$ and $s_i \in \mathcal{F}(U)$. There is a natural map

$$\sigma : \mathbf{Z}[\mathcal{F}] \longrightarrow \underline{\mathbf{Z}}$$

which to a local section $\sum n_i[s_i]$ associates $\sum n_i$. The kernel of σ is generated by the local section of the form $[s] - [s']$. There is a canonical map $a : \mathrm{Ker}(\sigma) \to \mathcal{H}$ which maps $[s] - [s'] \mapsto h$ where h is the local section of $\mathcal{H}$ such that $h \cdot s = s'$. Consider the push out diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(\sigma) \arrow[r] \arrow[d, "a"] & \mathbf{Z}[\mathcal{F}] \arrow[r] \arrow[d] & \underline{\mathbf{Z}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{H} \arrow[r] & \mathcal{E} \arrow[r] & \underline{\mathbf{Z}} \arrow[r] & 0
\end{tikzcd}$$

Here $\mathcal{E}$ is the extension obtained by push out. From the long exact cohomology sequence associated to the lower short exact sequence we obtain an element $\xi = \xi_\mathcal{F} \in H^1(X, \mathcal{H})$ by applying the boundary operator to $1 \in H^0(X, \underline{\mathbf{Z}})$.

Conversely, given $\xi \in H^1(X, \mathcal{H})$ we can associate to ξ a torsor as follows. Choose an embedding $\mathcal{H} \to \mathcal{I}$ of $\mathcal{H}$ into an injective abelian sheaf $\mathcal{I}$. We set $\mathcal{Q} = \mathcal{I}/\mathcal{H}$ so that we have a short exact sequence

$$0 \longrightarrow \mathcal{H} \longrightarrow \mathcal{I} \longrightarrow \mathcal{Q} \longrightarrow 0$$

The element ξ is the image of a global section $q \in H^0(X, \mathcal{Q})$ because $H^1(X, \mathcal{I}) = 0$ (see Derived Categories, Lemma 11.19.4). Let $\mathcal{F} \subset \mathcal{I}$ be the subsheaf (of sets) of sections that map to q in the sheaf $\mathcal{Q}$. It is easy to verify that $\mathcal{F}$ is a torsor.

We omit the verification that the two constructions given above are mutually inverse. □

18.6. Locality of cohomology

The following lemma says there is no ambguity in defining the cohomology of a sheaf $\mathcal{F}$ over an open.

Lemma 18.6.1. *Let X be a ringed space. Let $U \subset X$ be an open subspace.*

(1) *If $\mathcal{I}$ is an injective $\mathcal{O}_X$-module then $\mathcal{I}|_U$ is an injective $\mathcal{O}_U$-module.*
(2) *For any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ we have $H^p(U, \mathcal{F}) = H^p(U, \mathcal{F}|_U)$.*

Proof. Denote $j : U \to X$ the open immersion. Recall that the functor j^{-1} of restriction to U is a right adjoint to the functor $j_!$ of extension by 0, see Sheaves, Lemma 6.31.8. Moreover, $j_!$ is exact. Hence (1) follows from Homology, Lemma 10.22.1.

By definition $H^p(U, \mathcal{F}) = H^p(\Gamma(U, \mathcal{I}^\bullet))$ where $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution in $\textit{Mod}(\mathcal{O}_X)$. By the above we see that $\mathcal{F}|_U \to \mathcal{I}^\bullet|_U$ is an injective resolution in $\textit{Mod}(\mathcal{O}_U)$. Hence $H^p(U, \mathcal{F}|_U)$ is equal to $H^p(\Gamma(U, \mathcal{I}^\bullet|_U))$. Of course $\Gamma(U, \mathcal{F}) = \Gamma(U, \mathcal{F}|_U)$ for any sheaf $\mathcal{F}$ on X. Hence the equality in (2). □

Let X be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $U \subset V \subset X$ be open subsets. Then there is a canonical *restriction mapping*

$$H^n(V, \mathcal{F}) \longrightarrow H^n(U, \mathcal{F}), \quad \xi \longmapsto \xi|_U \tag{18.6.1.1}$$

functorial in $\mathcal{F}$. Namely, choose any injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. The restriction mappings of the sheaves $\mathcal{I}^p$ give a morphism of complexes

$$\Gamma(V, \mathcal{I}^\bullet) \longrightarrow \Gamma(U, \mathcal{I}^\bullet)$$

The LHS is a complex representing $R\Gamma(V, \mathcal{F})$ and the RHS is a complex representing $R\Gamma(U, \mathcal{F})$. We get the map on cohomology groups by applying the functor H^n. As indicated we will use the notation $\xi \mapsto \xi|_U$ to denote this map. Thus the rule $U \mapsto H^n(U, \mathcal{F})$ is a presheaf of $\mathcal{O}_X$-modules. This presheaf is customarily denoted $\underline{H}^n(\mathcal{F})$. We will give another interpretation of this presheaf in Lemma 18.11.3.

Lemma 18.6.2. *Let X be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $U \subset X$ be an open subspace. Let $n > 0$ and let $\xi \in H^n(U, \mathcal{F})$. Then there exists an open covering $U = \bigcup_{i \in I} U_i$ such that $\xi|_{U_i} = 0$ for all $i \in I$.*

Proof. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution. Then

$$H^n(U, \mathcal{F}) = \frac{\text{Ker}(\mathcal{I}^n(U) \to \mathcal{I}^{n+1}(U))}{\text{Im}(\mathcal{I}^{n-1}(U) \to \mathcal{I}^n(U))}.$$

Pick an element $\tilde{\xi} \in \mathcal{I}^n(U)$ representing the cohomology class in the presentation above. Since $\mathcal{I}^\bullet$ is an injective resolution of $\mathcal{F}$ and $n > 0$ we see that the complex $\mathcal{I}^\bullet$ is exact in degree n. Hence $\text{Im}(\mathcal{I}^{n-1} \to \mathcal{I}^n) = \text{Ker}(\mathcal{I}^n \to \mathcal{I}^{n+1})$ as sheaves. Since $\tilde{\xi}$ is a section of the kernel sheaf over U we conclude there exists an open covering $U = \bigcup_{i \in I} U_i$ such that $\tilde{\xi}|_{U_i}$ is the image under d of a section $\xi_i \in \mathcal{I}^{n-1}(U_i)$. By our definition of the restriction $\xi|_{U_i}$ as corresponding to the class of $\tilde{\xi}|_{U_i}$ we conclude. □

Lemma 18.6.3. *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}$ be a $\mathcal{O}_X$-module. The sheaves $R^i f_* \mathcal{F}$ are the sheaves associated to the presheaves*

$$V \longmapsto H^i(f^{-1}(V), \mathcal{F})$$

with restriction mappings as in Equation (18.6.1.1). There is a similar statement for $R^i f_*$ applied to a bounded below complex $\mathcal{F}^\bullet$.

Proof. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution. Then $R^i f_*\mathcal{F}$ is by definition the ith cohomology sheaf of the complex

$$f_*\mathcal{I}^0 \to f_*\mathcal{I}^1 \to f_*\mathcal{I}^2 \to \ldots$$

By definition of the abelian category structure on $\mathcal{O}_Y$-modules this cohomology sheaf is the sheaf associated to the presheaf

$$V \longmapsto \frac{\mathrm{Ker}(f_*\mathcal{I}^i(V) \to f_*\mathcal{I}^{i+1}(V))}{\mathrm{Im}(f_*\mathcal{I}^{i-1}(V) \to f_*\mathcal{I}^i(V))}$$

and this is obviously equal to

$$\frac{\mathrm{Ker}(\mathcal{I}^i(f^{-1}(V)) \to \mathcal{I}^{i+1}(f^{-1}(V)))}{\mathrm{Im}(\mathcal{I}^{i-1}(f^{-1}(V)) \to \mathcal{I}^i(f^{-1}(V)))}$$

which is equal to $H^i(f^{-1}(V), \mathcal{F})$ and we win. □

Lemma 18.6.4. *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let $V \subset Y$ be an open subspace. Denote $g : f^{-1}(V) \to V$ the restriction of f. Then we have*

$$R^p g_*(\mathcal{F}|_{f^{-1}(V)}) = (R^p f_*\mathcal{F})|_V$$

There is a similar statement for the derived image $Rf_*\mathcal{F}^\bullet$ where $\mathcal{F}^\bullet$ is a bounded below complex of $\mathcal{O}_X$-modules.

Proof. First proof. Apply Lemmas 18.6.3 and 18.6.1 to see the displayed equality. Second proof. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ and use that $\mathcal{F}|_{f^{-1}(V)} \to \mathcal{I}^\bullet|_{f^{-1}(V)}$ is an injective resolution also. □

Remark 18.6.5. Here is a different approach to the proofs of Lemmas 18.6.2 and 18.6.3 above. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $i_X : Mod(\mathcal{O}_X) \to Mod(\mathcal{O}_X)$ be the inclusion functor and let # be the sheafification functor. Recall that i_X is left exact and # is exact.

(1) First prove Lemma 18.11.3 below which says that the right derived functors of i_X are given by $R^p i_X\mathcal{F} = \underline{H}^p(\mathcal{F})$. Here is another proof: The equality is clear for $p = 0$. Both $(R^p i_X)_{p\geq 0}$ and $(\underline{H}^p)_{p\geq 0}$ are delta functors vanishing on injectives, hence both are universal, hence they are isomorphic. See Homology, Section 10.9.
(2) A restatement of Lemma 18.6.2 is that $(\underline{H}^p(\mathcal{F}))^\# = 0$, $p > 0$ for any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$. To see this is true, use that $^\#$ is exact so

$$(\underline{H}^p(\mathcal{F}))^\# = (R^p i_X\mathcal{F})^\# = R^p(\# \circ i_X)(\mathcal{F}) = 0$$

because $\# \circ i_X$ is the identity functor.
(3) Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. The presheaf $V \mapsto H^p(f^{-1}V, \mathcal{F})$ is equal to $R^p(i_Y \circ f_*)\mathcal{F}$. You can prove this by noticing that both give universal delta functors as in the argument of (1) above. Hence Lemma 18.6.3 says that $R^p f_*\mathcal{F} = (R^p(i_Y \circ f_*)\mathcal{F})^\#$. Again using that # is exact a that $\# \circ i_Y$ is the identity functor we see that

$$R^p f_*\mathcal{F} = R^p(\# \circ i_Y \circ f_*)\mathcal{F} = (R^p(i_Y \circ f_*)\mathcal{F})^\#$$

as desired.

18.7. Projection formula

In this section we collect variants of the projection formula. The most basic version is Lemma 18.7.2.

Lemma 18.7.1. *Let X be a ringed space. Let $\mathcal{I}$ be an injective $\mathcal{O}_X$-module. Let $\mathcal{E}$ be an $\mathcal{O}_X$-module. Assume $\mathcal{E}$ is finite locally free on X, see Modules, Definition 15.14.1. Then $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{I}$ is an injective $\mathcal{O}_X$-module.*

Proof. This is true because under the assumptions of the lemma we have

$$Hom_{\mathcal{O}_X}(\mathcal{F}, \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{I}) = Hom_{\mathcal{O}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}^\wedge, \mathcal{I})$$

where $\mathcal{E}^\wedge = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$ is the dual of $\mathcal{E}$ which is finite locally free also. Since tensoring with a finite locally free sheaf is an exact functor we win by Homology, Lemma 10.20.2. $\square$

Lemma 18.7.2. *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let $\mathcal{E}$ be an $\mathcal{O}_Y$-module. Assume $\mathcal{E}$ is finite locally free on Y, see Modules, Definition 15.14.1. Then there exist isomorphisms*

$$\mathcal{E} \otimes_{\mathcal{O}_Y} R^q f_* \mathcal{F} \longrightarrow R^q f_*(f^*\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F})$$

for all $q \geq 0$. In fact there exists an isomorphism

$$\mathcal{E} \otimes_{\mathcal{O}_Y} Rf_* \mathcal{F} \longrightarrow Rf_*(f^*\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F})$$

in $D^+(Y)$ functorial in $\mathcal{F}$.

Proof. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ on X. Note that $f^*\mathcal{E}$ is finite locally free also, hence we get a resolution

$$f^*\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow f^*\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{I}^\bullet$$

which is an injective resolution by Lemma 18.7.1. Apply f_* to see that

$$Rf_*(f^*\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}) = f_*(f^*\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{I}^\bullet).$$

Hence the lemma follows if we can show that $f_*(f^*\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}) = \mathcal{E} \otimes_{\mathcal{O}_Y} f_*(\mathcal{F})$ functorially in the $\mathcal{O}_X$-module $\mathcal{F}$. This is clear when $\mathcal{E} = \mathcal{O}_Y^{\oplus n}$, and follows in general by working locally on Y. Details omitted. $\square$

18.8. Mayer-Vietoris

Below will construct the Čech-to-cohomology spectral sequence, see Lemma 18.11.4. A special case of that spectral sequence is the Mayer-Vietoris long exact sequence. Since it is such a basic, useful and easy to understand variant of the spectral sequence we treat it here separately.

Lemma 18.8.1. *Let X be a ringed space. Let $U' \subset U \subset X$ be open subspaces. For any injective $\mathcal{O}_X$-module $\mathcal{I}$ the restriction mapping $\mathcal{I}(U) \to \mathcal{I}(U')$ is surjective.*

Proof. Let $j : U \to X$ and $j' : U' \to X$ be the open immersions. Recall that $j_!\mathcal{O}_U$ is the extension by zero of $\mathcal{O}_U = \mathcal{O}_X|_U$, see Sheaves, Section 6.31. Since $j_!$ is a left adjoint to restriction we see that for any sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules

$$Hom_{\mathcal{O}_X}(j_!\mathcal{O}_U, \mathcal{F}) = Hom_{\mathcal{O}_U}(\mathcal{O}_U, \mathcal{F}|_U) = \mathcal{F}(U)$$

see Sheaves, Lemma 6.31.8. Similarly, the sheaf $j'_!\mathcal{O}_{U'}$ represents the functor $\mathcal{F} \mapsto \mathcal{F}(U')$. Moreover there is an obvious canonical map of $\mathcal{O}_X$-modules

$$j'_!\mathcal{O}_{U'} \longrightarrow j_!\mathcal{O}_U$$

which corresponds to the restriction mapping $\mathcal{F}(U) \to \mathcal{F}(U')$ via Yoneda's lemma (Categories, Lemma 4.3.5). By the description of the stalks of the sheaves $j'_!\mathcal{O}_{U'}$, $j_!\mathcal{O}_U$ we see that the displayed map above is injective (see lemma cited above). Hence if $\mathcal{I}$ is an injective $\mathcal{O}_X$-module, then the map

$$Hom_{\mathcal{O}_X}(j_!\mathcal{O}_U, \mathcal{I}) \longrightarrow Hom_{\mathcal{O}_X}(j'_!\mathcal{O}_{U'}, \mathcal{I})$$

is surjective, see Homology, Lemma 10.20.2. Putting everything together we obtain the lemma. $\square$

Lemma 18.8.2. *(Mayer-Vietoris.) Let X be a ringed space. Suppose that $X = U \cup V$ is a union of two open subsets. For every $\mathcal{O}_X$-module $\mathcal{F}$ there exists a long exact cohomology sequence*

$$0 \to H^0(X, \mathcal{F}) \to H^0(U, \mathcal{F}) \oplus H^0(V, \mathcal{F}) \to H^0(U \cap V, \mathcal{F}) \to H^1(X, \mathcal{F}) \to \ldots$$

This long exact sequence is functorial in $\mathcal{F}$.

Proof. The sheaf condition says that the kernel of $(1, -1) : \mathcal{F}(U) \oplus \mathcal{F}(V) \to \mathcal{F}(U \cap V)$ is equal to the image of $\mathcal{F}(X)$ by the first map for any abelian sheaf $\mathcal{F}$. Lemma 18.8.1 above implies that the map $(1, -1) : \mathcal{I}(U) \oplus \mathcal{I}(V) \to \mathcal{I}(U \cap V)$ is surjective whenever $\mathcal{I}$ is an injective $\mathcal{O}_X$-module. Hence if $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution of $\mathcal{F}$, then we get a short exact sequence of complexes

$$0 \to \mathcal{I}^\bullet(X) \to \mathcal{I}^\bullet(V) \oplus \mathcal{I}^\bullet(V) \to \mathcal{I}^\bullet(U \cap V) \to 0.$$

Taking cohomology gives the result (use Homology, Lemma 10.10.12). We omit the proof of the functoriality of the sequence. $\square$

Lemma 18.8.3. *(Relative Mayer-Vietoris.) Let $f : X \to Y$ be a morphism of ringed spaces. Suppose that $X = U \cup V$ is a union of two open subsets. Denote $a = f|_U : U \to Y$, $b = f|_V : V \to Y$, and $c = f|_{U \cap V} : U \cap V \to Y$. For every $\mathcal{O}_X$-module $\mathcal{F}$ there exists a long exact sequence*

$$0 \to f_*\mathcal{F} \to a_*(\mathcal{F}|_U) \oplus b_*(\mathcal{F}|_V) \to c_*(\mathcal{F}|_{U \cap V}) \to R^1 f_*\mathcal{F} \to \ldots$$

This long exact sequence is functorial in $\mathcal{F}$.

Proof. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution of $\mathcal{F}$. We claim that we get a short exact sequence of complexes

$$0 \to f_*\mathcal{I}^\bullet \to a_*\mathcal{I}^\bullet|_V \oplus b_*\mathcal{I}^\bullet|_V \to c_*\mathcal{I}^\bullet|_{U \cap V} \to 0.$$

Namely, for any open $W \subset Y$, and for any $n \geq 0$ the corresponding sequence of groups of sections over W

$$0 \to \mathcal{I}^n(f^{-1}(W)) \to \mathcal{I}^n(V \cap f^{-1}(W)) \oplus \mathcal{I}^n(V \cap f^{-1}(W)) \to \mathcal{I}^n(U \cap V \cap f^{-1}(W)) \to 0$$

was shown to be short exact in the proof of Lemma 18.8.2. The lemma follows by taking cohomology sheaves and using the fact that $\mathcal{I}^\bullet|_U$ is an injective resolution of $\mathcal{F}|_U$ and similarly for $\mathcal{I}^\bullet|_V$, $\mathcal{I}^\bullet|_{U \cap V}$ see Lemma 18.6.1. $\square$

18.9. The Čech complex and Čech cohomology

Let X be a topological space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering, see Topology, Basic notion (6). As is customary we denote $U_{i_0 \ldots i_p} = U_{i_0} \cap \ldots \cap U_{i_p}$ for the $(p+1)$-fold intersection of members of $\mathcal{U}$. Let $\mathcal{F}$ be an abelian presheaf on X. Set

$$\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) = \prod\nolimits_{(i_0, \ldots, i_p) \in I^{p+1}} \mathcal{F}(U_{i_0 \ldots i_p}).$$

This is an abelian group. For $s \in \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})$ we denote $s_{i_0 \ldots i_p}$ its value in $\mathcal{F}(U_{i_0 \ldots i_p})$. Note that if $s \in \check{\mathcal{C}}^2(\mathcal{U}, \mathcal{F})$ and $i, j \in I$ then s_{ij} and s_{ji} are both elements of $\mathcal{F}(U_i \cap U_j)$ but there is no imposed relation between s_{ij} and s_{ji}. In other words, we are *not* working with alternating cochains (these will be defined in Section 18.17). We define

$$d : \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^{p+1}(\mathcal{U}, \mathcal{F})$$

by the formula

$$d(s)_{i_0 \ldots i_{p+1}} = \sum\nolimits_{j=0}^{p+1} (-1)^j s_{i_0 \ldots \hat{i}_j \ldots i_p}|_{U_{i_0 \ldots i_{p+1}}} \tag{18.9.0.1}$$

It is straightforward to see that $d \circ d = 0$. In other words $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is a complex.

Definition 18.9.1. Let X be a topological space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. Let $\mathcal{F}$ be an abelian presheaf on X. The complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is the *Čech complex* associated to $\mathcal{F}$ and the open covering $\mathcal{U}$. Its cohomology groups $H^i(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}))$ are called the *Čech cohomology groups* associated to $\mathcal{F}$ and the covering $\mathcal{U}$. They are denoted $\check{H}^i(\mathcal{U}, \mathcal{F})$.

Lemma 18.9.2. *Let X be a topological space. Let $\mathcal{F}$ be an abelian presheaf on X. The following are equivalent*

(1) *$\mathcal{F}$ is an abelian sheaf and*
(2) *for every open covering $\mathcal{U} : U = \bigcup_{i \in I} U_i$ the natural map*

$$\mathcal{F}(U) \to \check{H}^0(\mathcal{U}, \mathcal{F})$$

is bijective.

Proof. This is true since the sheaf condition is exactly that $\mathcal{F}(U) \to \check{H}^0(\mathcal{U}, \mathcal{F})$ is bijective for every open covering. □

18.10. Cech cohomology as a functor on presheaves

Warning: In this section we work almost exclusively with presheaves and categories of presheaves and the results are completely wrong in the setting of sheaves and categories of sheaves!

Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_X$-modules. We have the Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ of $\mathcal{F}$ just by thinking of $\mathcal{F}$ as a presheaf of abelian groups. However, each term $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})$ has a natural structure of a $\mathcal{O}_X(U)$-module and the differential is given by $\mathcal{O}_X(U)$-module maps. Moreover, it is clear that the construction

$$\mathcal{F} \longmapsto \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

is functorial in $\mathcal{F}$. In fact, it is a functor

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, -) : PMod(\mathcal{O}_X) \longrightarrow \text{Comp}^+(\text{Mod}(\mathcal{O}_X(U))) \tag{18.10.0.1}$$

see Derived Categories, Definition 11.7.1 for notation. Recall that the category of bounded below complexes in an abelian category is an abelian category, see Homology, Lemma 10.10.9.

Lemma 18.10.1. *The functor given by Equation (18.10.0.1) is an exact functor (see Homology, Lemma 10.5.1).*

Proof. For any open $W \subset U$ the functor $\mathcal{F} \mapsto \mathcal{F}(W)$ is an additive exact functor from $PMod(\mathcal{O}_X)$ to $\text{Mod}(\mathcal{O}_X(U))$. The terms $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})$ of the complex are products of these exact functors and hence exact. Moreover a sequence of complexes is exact if and only if the sequence of terms in a given degree is exact. Hence the lemma follows. $\square$

Lemma 18.10.2. *Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. The functors $\mathcal{F} \mapsto \check{H}^n(\mathcal{U}, \mathcal{F})$ form a δ-functor from the abelian category of presheaves of $\mathcal{O}_X$-modules to the category of $\mathcal{O}_X(U)$-modules (see Homology, Definition 10.9.1).*

Proof. By Lemma 18.10.1 a short exact sequence of presheaves of $\mathcal{O}_X$-modules $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is turned into a short exact sequence of complexes of $\mathcal{O}_X(U)$-modules. Hence we can use Homology, Lemma 10.10.12 to get the boundary maps $\delta_{\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3} : \check{H}^n(\mathcal{U}, \mathcal{F}_3) \to \check{H}^{n+1}(\mathcal{U}, \mathcal{F}_1)$ and a corresponding long exact sequence. We omit the verification that these maps are compatible with maps between short exact sequences of presheaves. $\square$

In the formulation of the following lemma we use the functor $j_{p!}$ of extension by 0 for presheaves of modules relative to an open immersion $j : U \to X$. See Sheaves, Section 6.31. For any open $W \subset X$ and any presheaf $\mathcal{G}$ of $\mathcal{O}_X|_U$-modules we have

$$(j_{p!}\mathcal{G})(W) = \begin{cases} \mathcal{G}(W) & \text{if } W \subset U \\ 0 & \text{else.} \end{cases}$$

Moreover, the functor $j_{p!}$ is a left adjoint to the restriction functor see Sheaves, Lemma 6.31.8. In particular we have the following formula

$$Hom_{\mathcal{O}_X}(j_{p!}\mathcal{O}_U, \mathcal{F}) = Hom_{\mathcal{O}_U}(\mathcal{O}_U, \mathcal{F}|_U) = \mathcal{F}(U).$$

Since the functor $\mathcal{F} \mapsto \mathcal{F}(U)$ is an exact functor on the category of presheaves we conclude that the presheaf $j_{p!}\mathcal{O}_U$ is a projective object in the category $PMod(\mathcal{O}_X)$, see Homology, Lemma 10.21.2.

Note that if we are given open subsets $U \subset V \subset X$ with associated open immersions j_U, j_V, then we have a canonical map $(j_U)_{p!}\mathcal{O}_U \to (j_V)_{p!}\mathcal{O}_V$. It is the identity on sections over any open $W \subset U$ and 0 else. In terms of the identification $Hom_{\mathcal{O}_X}((j_U)_{p!}\mathcal{O}_U, (j_V)_{p!}\mathcal{O}_V) = (j_V)_{p!}\mathcal{O}_V(U) = \mathcal{O}_V(U)$ it corresponds to the element $1 \in \mathcal{O}_V(U)$.

Lemma 18.10.3. *Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be a covering. Denote $j_{i_0 \ldots i_p} : U_{i_0 \ldots i_p} \to X$ the open immersion. Consider the chain complex $K(\mathcal{U})_\bullet$ of presheaves of $\mathcal{O}_X$-modules*

$$\ldots \to \bigoplus_{i_0 i_1 i_2} (j_{i_0 i_1 i_2})_{p!}\mathcal{O}_{U_{i_0 i_1 i_2}} \to \bigoplus_{i_0 i_1} (j_{i_0 i_1})_{p!}\mathcal{O}_{U_{i_0 i_1}} \to \bigoplus_{i_0} (j_{i_0})_{p!}\mathcal{O}_{U_{i_0}} \to 0 \to \ldots$$

where the last nonzero term is placed in degree 0 and where the map

$$(j_{i_0 \ldots i_{p+1}})_{p!}\mathcal{O}_{U_{i_0 \ldots i_{p+1}}} \longrightarrow (j_{i_0 \ldots \hat{i}_j \ldots i_{p+1}})_{p!}\mathcal{O}_{U_{i_0 \ldots \hat{i}_j \ldots i_{p+1}}}$$

is given by $(-1)^j$ *times the canonical map. Then there is an isomorphism*

$$Hom_{\mathcal{O}_X}(K(\mathcal{U})_\bullet, \mathcal{F}) = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

functorial in $\mathcal{F} \in Ob(PMod(\mathcal{O}_X))$.

Proof. We saw in the discussion just above the lemma that

$$Hom_{\mathcal{O}_X}((j_{i_0\ldots i_p})_{p!}\mathcal{O}_{U_{i_0\ldots i_p}}, \mathcal{F}) = \mathcal{F}(U_{i_0\ldots i_p}).$$

Hence we see that it is indeed the case that the direct sum

$$\bigoplus\nolimits_{i_0\ldots i_p}(j_{i_0\ldots i_p})_{p!}\mathcal{O}_{U_{i_0\ldots i_p}}$$

represents the functor

$$\mathcal{F} \longmapsto \prod\nolimits_{i_0\ldots i_p}\mathcal{F}(U_{i_0\ldots i_p}).$$

Hence by Categories, Yoneda Lemma 4.3.5 we see that there is a complex $K(\mathcal{U})_\bullet$ with terms as given. It is a simple matter to see that the maps are as given in the lemma. □

Lemma 18.10.4. *Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i\in I} U_i$ be a covering. Let $\mathcal{O}_\mathcal{U} \subset \mathcal{O}_X$ be the image presheaf of the map $\bigoplus j_{p!}\mathcal{O}_{U_i} \to \mathcal{O}_X$. The chain complex $K(\mathcal{U})_\bullet$ of presheaves of Lemma 18.10.3 above has homology presheaves*

$$H_i(K(\mathcal{U})_\bullet) = \begin{cases} 0 & if \quad i \neq 0 \\ \mathcal{O}_\mathcal{U} & if \quad i = 0 \end{cases}$$

Proof. Consider the extended complex $K^{ext}_\bullet$ one gets by putting $\mathcal{O}_\mathcal{U}$ in degree -1 with the obvious map $K(\mathcal{U})_0 = \bigoplus_{i_0}(j_{i_0})_{p!}\mathcal{O}_{U_{i_0}} \to \mathcal{O}_\mathcal{U}$. It suffices to show that taking sections of this extended complex over any open $W \subset X$ leads to an acyclic complex. In fact, we claim that for every $W \subset X$ the complex $K^{ext}_\bullet(W)$ is homotopy equivalent to the zero complex. Write $I = I_1 \amalg I_2$ where $W \subset U_i$ if and only if $i \in I_1$.

If $I_1 = \emptyset$, then the complex $K^{ext}_\bullet(W) = 0$ so there is nothing to prove.

If $I_1 \neq \emptyset$, then $\mathcal{O}_\mathcal{U}(W) = \mathcal{O}_X(W)$ and

$$K^{ext}_p(W) = \bigoplus\nolimits_{i_0\ldots i_p \in I_1} \mathcal{O}_X(W).$$

This is true because of the simple description of the presheaves $(j_{i_0\ldots i_p})_{p!}\mathcal{O}_{U_{i_0\ldots i_p}}$. Moreover, the differential of the complex $K^{ext}_\bullet(W)$ is given by

$$d(s)_{i_0\ldots i_p} = \sum\nolimits_{j=0,\ldots,p+1}\sum\nolimits_{i\in I_1}(-1)^j s_{i_0\ldots i_{j-1} i i_j \ldots i_p}.$$

The sum is finite as the element s has finite support. Fix an element $i_{\text{fix}} \in I_1$. Define a map

$$h : K^{ext}_p(W) \longrightarrow K^{ext}_{p+1}(W)$$

by the rule

$$h(s)_{i_0\ldots i_{p+1}} = \begin{cases} 0 & \text{if} \quad i_0 \neq i \\ s_{i_1\ldots i_{p+1}} & \text{if} \quad i_0 = i_{\text{fix}} \end{cases}$$

We will use the shorthand $h(s)_{i_0\ldots i_{p+1}} = (i_0 = i_{\text{fix}})s_{i_1\ldots i_p}$ for this. Then we compute

$$\begin{aligned} & (dh + hd)(s)_{i_0\ldots i_p} \\ = \quad & \sum_j \sum_{i\in I_1}(-1)^j h(s)_{i_0\ldots i_{j-1} i i_j \ldots i_p} + (i = i_0)d(s)_{i_1\ldots i_p} \\ = \quad & s_{i_0\ldots i_p} + \sum_{j\geq 1}\sum_{i\in I_1}(-1)^j(i_0 = i_{\text{fix}})s_{i_1\ldots i_{j-1} i i_j \ldots i_p} + (i_0 = i_{\text{fix}})d(s)_{i_1\ldots i_p} \end{aligned}$$

which is equal to $s_{i_0\ldots i_p}$ as desired. $\square$

Lemma 18.10.5. *Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i\in I} U_i$ be an open covering of $U \subset X$. The Čech cohomology functors $\check{H}^p(\mathcal{U}, -)$ are canonically isomomorphic as a δ-functor to the right derived functors of the functor*

$$\check{H}^0(\mathcal{U}, -) : PMod(\mathcal{O}_X) \longrightarrow Mod(\mathcal{O}_X(U)).$$

Moreover, there is a functorial quasi-isomorphism

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow R\check{H}^0(\mathcal{U}, \mathcal{F})$$

where the right hand side indicates the right derived functor

$$R\check{H}^0(\mathcal{U}, -) : D^+(PMod(\mathcal{O}_X)) \longrightarrow D^+(\mathcal{O}_X(U))$$

of the left exact functor $\check{H}^0(\mathcal{U}, -)$.

Proof. Note that the category of presheaves of $\mathcal{O}_X$-modules has enough injectives, see Injectives, Proposition 17.12.5. Note that $\check{H}^0(\mathcal{U}, -)$ is a left exact functor from the category of presheaves of $\mathcal{O}_X$-modules to the category of $\mathcal{O}_X(U)$-modules. Hence the derived functor and the right derived fuctor exist, see Derived Categories, Section 11.19.

Let $\mathcal{I}$ be a injective presheaf of $\mathcal{O}_X$-modules. In this case the functor $Hom_{\mathcal{O}_X}(-, \mathcal{I})$ is exact on $PMod(\mathcal{O}_X)$. By Lemma 18.10.3 we have

$$Hom_{\mathcal{O}_X}(K(\mathcal{U})_\bullet, \mathcal{I}) = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}).$$

By Lemma 18.10.4 we have that $K(\mathcal{U})_\bullet$ is quasi-isomorphic to $\mathcal{O}_{\mathcal{U}}[0]$. Hence by the exactness of Hom into $\mathcal{I}$ mentioned above we see that $\check{H}^i(\mathcal{U}, \mathcal{I}) = 0$ for all $i > 0$. Thus the δ-functor $(\check{H}^n, \delta)$ (see Lemma 18.10.2) satisfies the assumptions of Homology, Lemma 10.9.4, and hence is a universal δ-functor.

By Derived Categories, Lemma 11.19.4 also the sequence $R^i\check{H}^0(\mathcal{U}, -)$ forms a universal δ-functor. By the uniqueness of universal δ-functors, see Homology, Lemma 10.9.5 we conclude that $R^i\check{H}^0(\mathcal{U}, -) = \check{H}^i(\mathcal{U}, -)$. This is enough for most applications and the reader is suggested to skip the rest of the proof.

Let $\mathcal{F}$ be any presheaf of $\mathcal{O}_X$-modules. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in the category $PMod(\mathcal{O}_X)$. Consider the double complex $A^{\bullet,\bullet}$ with terms

$$A^{p,q} = \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{I}^q).$$

Consider the simple complex $sA^\bullet$ associated to this double complex. There is a map of complexes

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow sA^\bullet$$

coming from the maps $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) \to A^{p,0} = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^0)$ and there is a map of complexes

$$\check{H}^0(\mathcal{U}, \mathcal{I}^\bullet) \longrightarrow sA^\bullet$$

coming from the maps $\check{H}^0(\mathcal{U}, \mathcal{I}^q) \to A^{0,q} = \check{\mathcal{C}}^0(\mathcal{U}, \mathcal{I}^q)$. Both of these maps are quasi-isomorphisms by an application of Homology, Lemma 10.19.6. Namely, the columns of the double complex are exact in positive degrees because the Čech complex as a functor is exact (Lemma 18.10.1) and the rows of the double complex are exact in positive degrees since as we just saw the higher Čech cohomology groups of the injective presheaves $\mathcal{I}^q$ are zero. Since quasi-isomorphisms become invertible in $D^+(\mathcal{O}_X(U))$ this gives the last displayed morphism of the lemma. We omit the verification that this morphism is functorial. $\square$

18.11. Cech cohomology and cohomology

Lemma 18.11.1. *Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i\in I} U_i$ be a covering. Let $\mathcal{I}$ be an injective $\mathcal{O}_X$-module. Then*

$$\check{H}^p(\mathcal{U}, \mathcal{I}) = \begin{cases} \mathcal{I}(U) & if \quad p = 0 \\ 0 & if \quad p > 0 \end{cases}$$

Proof. An injective $\mathcal{O}_X$-module is also injective as an object in the category $PMod(\mathcal{O}_X)$ (for example since sheafification is an exact left adjoint to the inclusion functor, using Homology, Lemma 10.22.1). Hence we can apply Lemma 18.10.5 (or its proof) to see the result. □

Lemma 18.11.2. *Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i\in I} U_i$ be a covering. There is a transformation*

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, -) \longrightarrow R\Gamma(U, -)$$

of functors $Mod(\mathcal{O}_X) \to D^+(\mathcal{O}_X(U))$. In particular this provides canonical maps $\check{H}^p(\mathcal{U}, \mathcal{F}) \to H^p(U, \mathcal{F})$ for $\mathcal{F}$ ranging over $Mod(\mathcal{O}_X)$.

Proof. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. Consider the double complex $A^{\bullet,\bullet}$ with terms $A^{p,q} = \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{I}^q)$. Moreover, consider the associated simple complex $sA^\bullet$, see Homology, Definition 10.19.2. There is a map of complexes

$$\alpha : \Gamma(U, \mathcal{I}^\bullet) \longrightarrow sA^\bullet$$

coming from the maps $\mathcal{I}^q(U) \to \check{H}^0(\mathcal{U}, \mathcal{I}^q)$ and a map of complexes

$$\beta : \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow sA^\bullet$$

coming from the map $\mathcal{F} \to \mathcal{I}^0$. We can apply Homology, Lemma 10.19.6 to see that α is a quasi-isomorphism. Namely, Lemma 18.11.1 implies that the qth row of the double complex $A^{\bullet,\bullet}$ is a resolution of $\Gamma(U, \mathcal{I}^q)$. Hence α becomes invertible in $D^+(\mathcal{O}_X(U))$ and the transformation of the lemma is the composition of β followed by the inverse of α. We omit the verification that this is functorial. □

Lemma 18.11.3. *Let X be a ringed space. Consider the functor $i : Mod(\mathcal{O}_X) \to PMod(\mathcal{O}_X)$. It is a left exact functor with right derived functors given by*

$$R^p i(\mathcal{F}) = \underline{H}^p(\mathcal{F}) : U \longmapsto H^p(U, \mathcal{F})$$

see discussion in Section 18.6.

Proof. It is clear that i is left exact. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. By definition $R^p i$ is the pth cohomology *presheaf* of the complex $\mathcal{I}^\bullet$. In other words, the sections of $R^p i(\mathcal{F})$ over an open U are given by

$$\frac{\mathrm{Ker}(\mathcal{I}^n(U) \to \mathcal{I}^{n+1}(U))}{\mathrm{Im}(\mathcal{I}^{n-1}(U) \to \mathcal{I}^n(U))}.$$

which is the definition of $H^p(U, \mathcal{F})$. □

Lemma 18.11.4. *Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i\in I} U_i$ be a covering. For any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ there is a spectral sequence $(E_r, d_r)_{r\geq 0}$ with*

$$E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F}))$$

converging to $H^{p+q}(U, \mathcal{F})$. This spectral sequence is functorial in $\mathcal{F}$.

Proof. This is a Grothendieck spectral sequence (see Derived Categories, Lemma 11.21.2) for the functors

$$i : Mod(\mathcal{O}_X) \to PMod(\mathcal{O}_X) \quad \text{and} \quad \check{H}^0(\mathcal{U}, -) : PMod(\mathcal{O}_X) \to \text{Mod}(\mathcal{O}_X(U)).$$

Namely, we have $\check{H}^0(\mathcal{U}, i(\mathcal{F})) = \mathcal{F}(U)$ by Lemma 18.9.2. We have that $i(\mathcal{I})$ is Cech acyclic by Lemma 18.11.1. And we have that $\check{H}^p(\mathcal{U}, -) = R^p\check{H}^0(\mathcal{U}, -)$ as functors on $PMod(\mathcal{O}_X)$ by Lemma 18.10.5. Putting everything together gives the lemma. □

Lemma 18.11.5. *Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i\in I} U_i$ be a covering. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Assume that $H^i(U_{i_0\ldots i_p}, \mathcal{F}) = 0$ for all $i > 0$, all $p \geq 0$ and all $i_0, \ldots, i_p \in I$. Then $\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(U, \mathcal{F})$ as $\mathcal{O}_X(U)$-modules.*

Proof. We will use the spectral sequence of Lemma 18.11.4. The assumptions mean that $E_2^{p,q} = 0$ for all (p, q) with $q \neq 0$. Hence the spectral sequence degenerates at E_2 and the result follows. □

Lemma 18.11.6. *Let X be a ringed space. Let*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

be a short exact sequence of $\mathcal{O}_X$-modules. Let $U \subset X$ be an open subset. If there exists a cofinal system of open coverings $\mathcal{U}$ of U such that $\check{H}^1(\mathcal{U}, \mathcal{F}) = 0$, then the map $\mathcal{G}(U) \to \mathcal{H}(U)$ is surjective.

Proof. Take an element $s \in \mathcal{H}(U)$. Choose an open covering $\mathcal{U} : U = \bigcup_{i\in I} U_i$ such that (a) $\check{H}^1(\mathcal{U}, \mathcal{F}) = 0$ and (b) $s|_{U_i}$ is the image of a section $s_i \in \mathcal{G}(U_i)$. Since we can certainly find a covering such that (b) holds it follows from the assumptions of the lemma that we can find a covering such that (a) and (b) both hold. Consider the sections

$$s_{i_0i_1} = s_{i_1}|_{U_{i_0i_1}} - s_{i_0}|_{U_{i_0i_1}}.$$

Since s_i lifts s we see that $s_{i_0i_1} \in \mathcal{F}(U_{i_0i_1})$. By the vanishing of $\check{H}^1(\mathcal{U}, \mathcal{F})$ we can find sections $t_i \in \mathcal{F}(U_i)$ such that

$$s_{i_0i_1} = t_{i_1}|_{U_{i_0i_1}} - t_{i_0}|_{U_{i_0i_1}}.$$

Then clearly the sections $s_i - t_i$ satsify the sheaf condition and glue to a section of $\mathcal{G}$ over U which maps to s. Hence we win. □

Lemma 18.11.7. *Let X be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module such that*

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$$

for all $p > 0$ and any open covering $\mathcal{U} : U = \bigcup_{i\in I} U_i$ of X. Then $H^p(U, \mathcal{F}) = 0$ for all $p > 0$ and any open $U \subset X$.

Proof. Let $\mathcal{F}$ be a sheaf satisfying the assumption of the lemma. We will indicate this by saying ``$\mathcal{F}$ has vanishing higher Cech cohomology for any open covering". Choose an embedding $\mathcal{F} \to \mathcal{I}$ into an injective $\mathcal{O}_X$-module. By Lemma 18.11.1 $\mathcal{I}$ has vanishing higher Cech cohomology for any open covering. Let $\mathcal{Q} = \mathcal{I}/\mathcal{F}$ so that we have a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{Q} \to 0.$$

By Lemma 18.11.6 and our assumptions this sequence is actually exact as a sequence of presheaves! In particular we have a long exact sequence of Čech cohomology groups for any open covering $\mathcal{U}$, see Lemma 18.10.2 for example. This implies that $\mathcal{Q}$ is also an $\mathcal{O}_X$-module with vanishing higher Čech cohomology for all open coverings.

Next, we look at the long exact cohomology sequence

$$\begin{tikzcd} 0 \arrow[r] & H^0(U, \mathcal{F}) \arrow[r] & H^0(U, \mathcal{I}) \arrow[r] & H^0(U, \mathcal{Q}) \arrow[dll] \\ & H^1(U, \mathcal{F}) \arrow[r] & H^1(U, \mathcal{I}) \arrow[r] & H^1(U, \mathcal{Q}) \arrow[dll] \\ & \ldots & \ldots & \ldots \end{tikzcd}$$

for any open $U \subset X$. Since $\mathcal{I}$ is injective we have $H^n(U, \mathcal{I}) = 0$ for $n > 0$ (see Derived Categories, Lemma 11.19.4). By the above we see that $H^0(U, \mathcal{I}) \to H^0(U, \mathcal{Q})$ is surjective and hence $H^1(U, \mathcal{F}) = 0$. Since $\mathcal{F}$ was an arbitrary $\mathcal{O}_X$-module with vanishing higher Čech cohomology we conclude that also $H^1(U, \mathcal{Q}) = 0$ since $\mathcal{Q}$ is another of these sheaves (see above). By the long exact sequence this in turn implies that $H^2(U, \mathcal{F}) = 0$. And so on and so forth. □

Lemma 18.11.8. *(Variant of Lemma 18.11.7.) Let X be a ringed space. Let $\mathcal{B}$ be a basis for the topology on X. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Assume there exists a set of open coverings Cov with the following properties:*

1. *For every $\mathcal{U} \in$ Cov with $\mathcal{U} : U = \bigcup_{i \in I} U_i$ we have $U, U_i \in \mathcal{B}$ and every $U_{i_0 \ldots i_p} \in \mathcal{B}$.*
2. *For every $U \in \mathcal{B}$ the open coverings of U occuring in Cov is a cofinal system of open coverings of U.*
3. *For every $\mathcal{U} \in$ Cov we have $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for all $p > 0$.*

Then $H^p(U, \mathcal{F}) = 0$ for all $p > 0$ and any $U \in \mathcal{B}$.

Proof. Let $\mathcal{F}$ and Cov be as in the lemma. We will indicate this by saying ``$\mathcal{F}$ has vanishing higher Cech cohomology for any $\mathcal{U} \in$ Cov". Choose an embedding $\mathcal{F} \to \mathcal{I}$ into an injective $\mathcal{O}_X$-module. By Lemma 18.11.1 $\mathcal{I}$ has vanishing higher Čech cohomology for any $\mathcal{U} \in$ Cov. Let $\mathcal{Q} = \mathcal{I}/\mathcal{F}$ so that we have a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{Q} \to 0.$$

By Lemma 18.11.6 and our assumption (2) this sequence gives rise to an exact sequence

$$0 \to \mathcal{F}(U) \to \mathcal{I}(U) \to \mathcal{Q}(U) \to 0.$$

for every $U \in \mathcal{B}$. Hence for any $\mathcal{U} \in$ Cov we get a short exact sequence of Čech complexes

$$0 \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{Q}) \to 0$$

since each term in the Čech complex is made up out of a product of values over elements of $\mathcal{B}$ by assumption (1). In particular we have a long exact sequence of Čech cohomology groups for any open covering $\mathcal{U} \in$ Cov. This implies that $\mathcal{Q}$ is also an $\mathcal{O}_X$-module with vanishing higher Čech cohomology for all $\mathcal{U} \in$ Cov.

Next, we look at the long exact cohomology sequence

$$\begin{tikzcd} 0 \arrow[r] & H^0(U, \mathcal{F}) \arrow[r] & H^0(U, \mathcal{I}) \arrow[r] & H^0(U, \mathcal{Q}) \arrow[dll] \\ & H^1(U, \mathcal{F}) \arrow[r] & H^1(U, \mathcal{I}) \arrow[r] & H^1(U, \mathcal{Q}) \arrow[dll] \\ & \ldots & \ldots & \ldots \end{tikzcd}$$

for any $U \in \mathcal{B}$. Since $\mathcal{I}$ is injective we have $H^n(U, \mathcal{I}) = 0$ for $n > 0$ (see Derived Categories, Lemma 11.19.4). By the above we see that $H^0(U, \mathcal{I}) \to H^0(U, \mathcal{Q})$ is surjective and hence $H^1(U, \mathcal{F}) = 0$. Since $\mathcal{F}$ was an arbitrary $\mathcal{O}_X$-module with vanishing higher Čech cohomology for all $\mathcal{U} \in \text{Cov}$ we conclude that also $H^1(U, \mathcal{Q}) = 0$ since $\mathcal{Q}$ is another of these sheaves (see above). By the long exact sequence this in turn implies that $H^2(U, \mathcal{F}) = 0$. And so on and so forth. □

Lemma 18.11.9. *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{I}$ be an injective $\mathcal{O}_X$-module. Then*

(1) $\check{H}^p(\mathcal{V}, f_*\mathcal{I}) = 0$ *for all $p > 0$ and any open covering $\mathcal{V} : V = \bigcup_{j \in J} V_j$ of Y.*
(2) $H^p(V, f_*\mathcal{I}) = 0$ *for all $p > 0$ and every open $V \subset Y$.*

In other words, $f_*\mathcal{I}$ is right acyclic for $\Gamma(U, -)$ (see Derived Categories, Definition 11.15.3) for any $U \subset X$ open.

Proof. Set $\mathcal{U} : f^{-1}(V) = \bigcup_{j \in J} f^{-1}(V_j)$. It is an open covering of X and

$$\check{\mathcal{C}}^\bullet(\mathcal{V}, f_*\mathcal{I}) = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}).$$

This is true because

$$f_*\mathcal{I}(V_{j_0 \ldots j_p}) = \mathcal{I}(f^{-1}(V_{j_0 \ldots j_p})) = \mathcal{I}(f^{-1}(V_{j_0}) \cap \ldots \cap f^{-1}(V_{j_p})) = \mathcal{I}(U_{j_0 \ldots j_p}).$$

Thus the first statement of the lemma follows from Lemma 18.11.1. The second statement follows from the first and Lemma 18.11.7. □

The following lemma implies in particular that $f_* : Ab(X) \to Ab(Y)$ transforms injective abelian sheaves into injective abelian sheaves.

Lemma 18.11.10. *Let $f : X \to Y$ be a morphism of ringed spaces. Assume f is flat. Then $f_*\mathcal{I}$ is an injective $\mathcal{O}_Y$-module for any injective $\mathcal{O}_X$-module $\mathcal{I}$.*

Proof. In this case the functor f^* transforms injections into injections. Hence the result follows from Modules, Lemma 15.17.2 and Homology, Lemma 10.22.1 □

18.12. The Leray spectral sequence

Lemma 18.12.1. *Let $f : X \to Y$ be a morphism of ringed spaces. There is a commutative diagram*

$$\begin{tikzcd} D^+(X) \arrow[r, "R\Gamma(X,-)"'] \arrow[d, "Rf_*"'] & D^+(\mathcal{O}_X(X)) \arrow[d, "restriction"] \\ D^+(Y) \arrow[r, "R\Gamma(Y,-)"] & D^+(\mathcal{O}_Y(Y)) \end{tikzcd}$$

More generally for any $V \subset Y$ open and $U = f^{-1}(V)$ there is a commutative diagram

$$\begin{tikzcd} D^+(X) \arrow[r, "R\Gamma(U,-)"'] \arrow[d, "Rf_*"'] & D^+(\mathcal{O}_X(U)) \arrow[d, "restriction"] \\ D^+(Y) \arrow[r, "R\Gamma(V,-)"] & D^+(\mathcal{O}_Y(V)) \end{tikzcd}$$

See also Remark 18.12.2 for more explanation.

Proof. Let $\Gamma_{res} : Mod(\mathcal{O}_X) \to \text{Mod}(\mathcal{O}_Y(Y))$ be the functor which associates to an $\mathcal{O}_X$-module $\mathcal{F}$ the global sections of $\mathcal{F}$ viewed as a $\mathcal{O}_Y(Y)$-module via the map $f^\sharp : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$. Let $restriction : \text{Mod}(\mathcal{O}_X(X)) \to \text{Mod}(\mathcal{O}_Y(Y))$ be the restriction functor induced by $f^\sharp : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$. Note that $restriction$ is exact so that its right derived functor is computed by simply applying the restriction functor, see Derived Categories, Lemma 11.16.8. It is clear that

$$\Gamma_{res} = restriction \circ \Gamma(X, -) = \Gamma(Y, -) \circ f_*$$

We claim that Derived Categories, Lemma 11.21.1 applies to both compositions. For the first this is clear by our remarks above. For the second, it follows from Lemma 18.11.9 which implies that injective $\mathcal{O}_X$-modules are mapped to $\Gamma(Y, -)$-acyclic sheaves on Y. □

Remark 18.12.2. Here is a down-to-earth explanation of the meaning of Lemma 18.12.1. It says that given $f : X \to Y$ and $\mathcal{F} \in Mod(\mathcal{O}_X)$ and given an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ we have

$R\Gamma(X, \mathcal{F})$	is represented by	$\Gamma(X, \mathcal{I}^\bullet)$
$Rf_*\mathcal{F}$	is represented by	$f_*\mathcal{I}^\bullet$
$R\Gamma(Y, Rf_*\mathcal{F})$	is represented by	$\Gamma(Y, f_*\mathcal{I}^\bullet)$

the last fact coming from Leray's acyclicity lemma (Derived Categories, Lemma 11.16.7) and Lemma 18.11.9. Finally, it combines this with the trivial observation that

$$\Gamma(X, \mathcal{I}^\bullet) = \Gamma(Y, f_*\mathcal{I}^\bullet).$$

to arrive at the commutativity of the diagram of the lemma.

Lemma 18.12.3. *Let X be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module.*

(1) *The cohomology groups $H^i(U, \mathcal{F})$ for $U \subset X$ open of $\mathcal{F}$ computed as an $\mathcal{O}_X$-module, or computed as an abelian sheaf are identical.*
(2) *Let $f : X \to Y$ be a morphism of ringed spaces. The higher direct images $R^if_*\mathcal{F}$ of $\mathcal{F}$ computed as an $\mathcal{O}_X$-module, or computed as an abelian sheaf are identical.*

There are similar statements in the case of bounded below complexes of $\mathcal{O}_X$-modules.

Proof. Consider the morphism of ringed spaces $(X, \mathcal{O}_X) \to (X, \underline{\mathbf{Z}}_X)$ given by the identity on the underlying topological space and by the unique map of sheaves of rings $\underline{\mathbf{Z}}_X \to \mathcal{O}_X$. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Denote $\mathcal{F}_{ab}$ the same sheaf seens as an $\underline{\mathbf{Z}}_X$-module, i.e., seens as a sheaf of abelian groups. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution. By Remark 18.12.2 we see that $\Gamma(X, \mathcal{I}^\bullet)$ computes both $R\Gamma(X, \mathcal{F})$ and $R\Gamma(X, \mathcal{F}_{ab})$. This proves (1).

To prove (2) we use (1) and Lemma 18.6.3. The result follows immediately. □

Lemma 18.12.4. *(Leray spectral sequence.) Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}^\bullet$ be a bounded below complex of $\mathcal{O}_X$-modules. There is a spectral sequence*

$$E_2^{p,q} = H^p(Y, R^qf_*(\mathcal{F}^\bullet))$$

converging to $H^{p+q}(X, \mathcal{F}^\bullet)$.

Proof. This is just the Grothendieck spectral sequence Derived Categories, Lemma 11.21.2 coming from the composition of functors $\Gamma_{res} = \Gamma(Y, -) \circ f_*$ where Γ_{res} is as in the proof of Lemma 18.12.1. To see that the assumptions of Derived Categories, Lemma 11.21.2 are satisfied, see the proof of Lemma 18.12.1 or Remark 18.12.2. □

Remark 18.12.5. The Leray spectral sequence, the way we proved it in Lemma 18.12.4 is a spectral sequence of $\Gamma(Y, \mathcal{O}_Y)$-modules. However, it is quite easy to see that it is in fact a spectral sequence of $\Gamma(X, \mathcal{O}_X)$-modules. For example f gives rise to a morphism of ringed spaces $f' : (X, \mathcal{O}_X) \to (Y, f_*\mathcal{O}_X)$. By Lemma 18.12.3 the terms $E_r^{p,q}$ of the Leray spectral sequence for an $\mathcal{O}_X$-module $\mathcal{F}$ and f are identical with those for $\mathcal{F}$ and f' at least for $r \geq 2$. Namely, they both agree with the terms of the Leray spectral sequence for $\mathcal{F}$ as an abelian sheaf. And since $(f_*\mathcal{O}_X)(Y) = \mathcal{O}_X(X)$ we see the result. It is often the case that the Leray spectral sequence carries additional structure.

Lemma 18.12.6. *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module.*

(1) *If $R^q f_*\mathcal{F} = 0$ for $q > 0$, then $H^p(X, \mathcal{F}) = H^p(Y, f_*\mathcal{F})$ for all p.*
(2) *If $H^p(Y, R^q f_*\mathcal{F}) = 0$ for all q and $p > 0$, then $H^q(X, \mathcal{F}) = H^0(Y, R^q f_*\mathcal{F})$ for all q.*

Proof. These are two simple conditions that force the Leray spectral sequence to converge. You can also prove these facts directly (without using the spectral sequence) which is a good exercise in cohomology of sheaves. □

Lemma 18.12.7. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of ringed spaces. In this case $Rg_* \circ Rf_* = R(g \circ f)_*$ as functors from $D^+(X) \to D^+(Z)$.*

Proof. We are going to apply Derived Categories, Lemma 11.21.1. It is clear that $g_* \circ f_* = (g \circ f)_*$, see Sheaves, Lemma 6.21.2. It remains to show that $f_*\mathcal{I}$ is g_*-acyclic. This follows from Lemma 18.11.9 and the description of the higher direct images $R^i g_*$ in Lemma 18.6.3. □

Lemma 18.12.8. *(Relative Leray spectral sequence.) Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of ringed spaces. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. There is a spectral sequence with*

$$E_2^{p,q} = R^p g_*(R^q f_*\mathcal{F})$$

converging to $R^{p+q}(g \circ f)_*\mathcal{F}$. This spectral sequence is functorial in $\mathcal{F}$, and there is a version for bounded below complexes of $\mathcal{O}_X$-modules.

Proof. This is a Grothendieck spectral sequence for composition of functors and follows from Lemma 18.12.7 and Derived Categories, Lemma 11.21.2. □

18.13. Functoriality of cohomology

Lemma 18.13.1. *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{G}^\bullet$, resp. $\mathcal{F}^\bullet$ be a bounded below complex of $\mathcal{O}_Y$-modules, resp. $\mathcal{O}_X$-modules. Let $\varphi : \mathcal{G}^\bullet \to f_*\mathcal{F}^\bullet$ be a morphism of complexes. There is a canonical morphism*

$$\mathcal{G}^\bullet \longrightarrow Rf_*(\mathcal{F}^\bullet)$$

in $D^+(Y)$. Moreover this construction is functorial in the triple $(\mathcal{G}^\bullet, \mathcal{F}^\bullet, \varphi)$.

Proof. Choose an injective resolution $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$. By definition $Rf_*(\mathcal{F}^\bullet)$ is represented by $f_*\mathcal{I}^\bullet$ in $K^+(\mathcal{O}_Y)$. The composition

$$\mathcal{G}^\bullet \to f_*\mathcal{F}^\bullet \to f_*\mathcal{I}^\bullet$$

is a morphism in $K^+(Y)$ which turns into the morphism of the lemma upon applying the localization functor $j_Y : K^+(Y) \to D^+(Y)$. □

Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{G}$ be an $\mathcal{O}_Y$-module and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Recall that an f-map φ from $\mathcal{G}$ to $\mathcal{F}$ is a map $\varphi : \mathcal{G} \to f_*\mathcal{F}$, or what is the same thing, a map $\varphi : f^*\mathcal{G} \to \mathcal{F}$. See Sheaves, Definition 6.21.7. Such an f-map gives rise to a morphism of complexes

$$\varphi : R\Gamma(Y, \mathcal{G}) \longrightarrow R\Gamma(X, \mathcal{F}) \tag{18.13.1.1}$$

in $D^+(\mathcal{O}_Y(Y))$. Namely, we use the morphism $\mathcal{G} \to Rf_*\mathcal{F}$ in $D^+(Y)$ of Lemma 18.13.1, and we apply $R\Gamma(Y, -)$. By Lemma 18.12.1 we see that $R\Gamma(X, \mathcal{F}) = R\Gamma(Y, Rf_*\mathcal{F})$ and we get the displayed arrow. We spell this out completely in Remark 18.13.2 below. In particular it gives rise to maps on cohomology

$$\varphi : H^i(Y, \mathcal{G}) \longrightarrow H^i(X, \mathcal{F}). \tag{18.13.1.2}$$

Remark 18.13.2. Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{G}$ be an $\mathcal{O}_Y$-module. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let φ be an f-map from $\mathcal{G}$ to $\mathcal{F}$. Choose a resolution $\mathcal{F} \to \mathcal{I}^\bullet$ by a complex of injective $\mathcal{O}_X$-modules. Choose resolutions $\mathcal{G} \to \mathcal{J}^\bullet$ and $f_*\mathcal{I} \to (\mathcal{J}')^\bullet$ by complexes of injective $\mathcal{O}_Y$-modules. By Derived Categories, Lemma 11.17.6 there exists a map of complexes β such that the diagram

$$\begin{tikzcd} \mathcal{G} \arrow[r] \arrow[d] & f_*\mathcal{F} \arrow[r] & f_*\mathcal{I}^\bullet \arrow[d] \\ \mathcal{J}^\bullet \arrow[rr, "\beta"] & & (\mathcal{J}')^\bullet \end{tikzcd} \tag{18.13.2.1}$$

commutes. Applying global section functors we see that we get a diagram

$$\begin{tikzcd} & \Gamma(Y, f_*\mathcal{I}^\bullet) \arrow[r, equal] \arrow[d, "qis"'] & \Gamma(X, \mathcal{I}^\bullet) \\ \Gamma(Y, \mathcal{J}^\bullet) \arrow[r, "\beta"] & \Gamma(Y, (\mathcal{J}')^\bullet) & \end{tikzcd}$$

The complex on the bottom left represents $R\Gamma(Y, \mathcal{G})$ and the complex on the top right represents $R\Gamma(X, \mathcal{F})$. The vertical arrow is a quasi-isomorphism by Lemma 18.12.1 which becomes invertible after applying the localization functor $K^+(\mathcal{O}_Y(Y)) \to D^+(\mathcal{O}_Y(Y))$. The arrow (18.13.1.1) is given by the composition of the horizontal map by the inverse of the vertical map.

Lemma 18.13.3. *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let $\mathcal{G}$ be an $\mathcal{O}_Y$-module. Let $\varphi : f^*\mathcal{G} \to \mathcal{F}$ be an f-map. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be an open covering. Let $\mathcal{V} : Y = \bigcup_{j \in J} V_j$ be an open covering. Assume that $\mathcal{U}$ is a refinement of $f^{-1}\mathcal{V} : X = \bigcup_{j \in J} f^{-1}(V_j)$. In this case there exists a commutative diagram*

$$\begin{tikzcd} \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \arrow[r] & R\Gamma(X, \mathcal{F}) \\ \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G}) \arrow[r] \arrow[u, "\gamma"] & R\Gamma(Y, \mathcal{G}) \arrow[u] \end{tikzcd}$$

in $D^+(\mathcal{O}_X(X))$ where the horizontal arrows come from Lemma 18.11.2 and the right vertical arrow is Equation (18.13.1.1). In particular we get commutative diagrams of cohomology *groups*

$$\begin{tikzcd}
\check{H}^p(\mathcal{U},\mathcal{F}) \arrow[r] & H^p(X,\mathcal{F}) \\
\check{H}^p(\mathcal{V},\mathcal{G}) \arrow[r] \arrow[u, "\gamma"] & H^p(Y,\mathcal{G}) \arrow[u]
\end{tikzcd}$$

where the right vertical arrow is (18.13.1.2)

Proof. We first define the left vertical arrow. Namely, choose a map $c : I \to J$ such that $U_i \subset f^{-1}(V_{c(i)})$ for all $i \in I$. In degree p we define the map by the rule

$$\gamma(s)_{i_0\ldots i_p} = \varphi(s)_{c(i_0)\ldots c(i_p)}$$

This makes sense because φ does indeed induce maps $\mathcal{G}(V_{c(i_0)\ldots c(i_p)}) \to \mathcal{F}(U_{i_0\ldots i_p})$ by assumption. It is also clear that this defines a morphism of complexes. Choose injective resolutions $\mathcal{F} \to \mathcal{I}^\bullet$ on X and $\mathcal{G} \to \mathcal{J}^\bullet$ on Y. According to the proof of Lemma 18.11.2 we introduce the double complexes $A^{\bullet,\bullet}$ and $B^{\bullet,\bullet}$ with terms

$$B^{p,q} = \check{\mathcal{C}}^p(\mathcal{V},\mathcal{J}^q) \quad \text{and} \quad A^{p,q} = \check{\mathcal{C}}^p(\mathcal{U},\mathcal{I}^q).$$

As in Remark 18.13.2 above we also choose an injective resolution $f_*\mathcal{I} \to (\mathcal{J}')^\bullet$ on Y and a morphism of complexes $\beta : \mathcal{J} \to (\mathcal{J}')^\bullet$ making (18.13.2.1) commutes. We introduce some more double complexes, namely $(B')^{\bullet,\bullet}$ and $(B'')\bullet, \bullet$ with

$$(B')^{p,q} = \check{\mathcal{C}}^p(\mathcal{V},(\mathcal{J}')^q) \quad \text{and} \quad (B'')^{p,q} = \check{\mathcal{C}}^p(\mathcal{V},f_*\mathcal{I}^q).$$

Note that there is an f-map of complexes from $f_*\mathcal{I}^\bullet$ to $\mathcal{I}^\bullet$. Hence it is clear that the same rule as above defines a morphism of double complexes

$$\gamma : (B'')^{\bullet,\bullet} \longrightarrow A^{\bullet,\bullet}.$$

Consider the diagram of complexes

$$\begin{tikzcd}
\check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{F}) \arrow[r] & sA^\bullet & & & \Gamma(X,\mathcal{I}^\bullet) \arrow[llll, "qis"'] \\
\check{\mathcal{C}}^\bullet(\mathcal{V},\mathcal{G}) \arrow[r] \arrow[u, "\gamma"] & sB^\bullet \arrow[r, "\beta"] & s(B')^\bullet & s(B'')^\bullet \arrow[l] \arrow[ull, "s\gamma"'] & \\
& \Gamma(Y,\mathcal{J}^\bullet) \arrow[r, "\beta"] \arrow[u, "qis"] & \Gamma(Y,(\mathcal{J}')^\bullet) \arrow[u] & \Gamma(Y,f_*\mathcal{I}^\bullet) \arrow[l, "qis"'] \arrow[u] \arrow[uur, equal] &
\end{tikzcd}$$

The two horizontal arrows with targets $sA^\bullet$ and $sB^\bullet$ are the ones explained in Lemma 18.11.2. The left upper shape (a pentagon) is commutative simply because (18.13.2.1) is commutative. The two lower squares are trivially commutative. It is also immediate from the definitions that the right upper shape (a square) is commutative. The result of the lemma now follows from the definitions and the fact that going around the diagram on the outer sides from $\check{\mathcal{C}}^\bullet(\mathcal{V},\mathcal{G})$ to $\Gamma(X,\mathcal{I}^\bullet)$ either on top or on bottom is the same (where you have to invert any quasi-isomorphisms along the way). □

18.14. The base change map

We will need to know how to construct the base change map in some cases. Since we have not yet discussed derived pullback we only discuss this in the case of a base change by a flat morphism of ringed spaces. Before we state the result, let us discuss flat pullback on the derived category. Namely, suppose that $g : X \to Y$ is a flat morphism of ringed spaces.

By Modules, Lemma 15.17.2 the functor $g^* : Mod(\mathcal{O}_Y) \to Mod(\mathcal{O}_X)$ is exact. Hence it has a derived functor
$$g^* : D^+(Y) \to D^+(X)$$
which is computed by simply pulling back an representative of a given object in $D^+(Y)$, see Derived Categories, Lemma 11.16.8. Hence as indicated we indicate this functor by g^* rather than Lg^*.

Lemma 18.14.1. *Let*
$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$
be a commutative diagram of ringed spaces. Let $\mathcal{F}^\bullet$ be a bounded below complex of $\mathcal{O}_X$-modules. Assume both g and g' are flat. Then there exists a canonical base change map
$$g^* Rf_*\mathcal{F}^\bullet \longrightarrow R(f')_*(g')^*\mathcal{F}^\bullet$$
in $D^+(S')$.

Proof. Choose injective resolutions $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ and $(g')^*\mathcal{F}^\bullet \to \mathcal{J}^\bullet$. By Lemma 18.11.10 we see that $(g')_*\mathcal{J}^\bullet$ is a complex of injectives representing $R(g')_*(g')^*\mathcal{F}^\bullet$. Hence by Derived Categories, Lemmas 11.17.6 and 11.17.7 the arrow β in the diagram
$$\begin{tikzcd} (g')_*(g')^*\mathcal{F}^\bullet \arrow[r] & (g')_*\mathcal{J}^\bullet \\ \mathcal{F}^\bullet \arrow[r] \arrow[u, "adjunction"] & \mathcal{I}^\bullet \arrow[u, "\beta"'] \end{tikzcd}$$
exists and is unique up to homotopy. Pushing down to S we get
$$f_*\beta : f_*\mathcal{I}^\bullet \longrightarrow f_*(g')_*\mathcal{J}^\bullet = g_*(f')_*\mathcal{J}^\bullet$$
By adjunction of g^* and g_* we get a map of complexes $g^*f_*\mathcal{I}^\bullet \to (f')_*\mathcal{J}^\bullet$. Note that this map is unique up to homotopy since the only choice in the whole process was the choice of the map β and everything was done on the level of complexes. $\square$

Remark 18.14.2. The ``correct" version of the base change map is map
$$Lg^*Rf_*\mathcal{F}^\bullet \longrightarrow R(f')_*L(g')^*\mathcal{F}^\bullet.$$
The construction of this map really involves dealing with unbounded complexes and having adjoint functors Lj^*, Rj_* on unbounded complexes. We will deal with this later (insert future reference here).

18.15. Cohomology and colimits

Let X be a ringed space. Let $(\mathcal{F}_i, \varphi_{ii'})$ be a directed system of sheaves of $\mathcal{O}_X$-modules over the partially ordered set I, see Categories, Section 4.19. Since for each i there is a canonical map $\mathcal{F}_i \to colim_i\, \mathcal{F}_i$ we get a canonical map
$$colim_i\, H^p(X, \mathcal{F}_i) \longrightarrow H^p(X, colim_i\, \mathcal{F}_i)$$
for every $p \geq 0$. Of course there is a similar map for every open $U \subset X$. These maps are in general not isomorphisms, even for $p = 0$. In this section we generalize the results of Sheaves, Lemma 6.29.1. See also Modules, Lemma 15.11.6 (in the special case $\mathcal{G} = \mathcal{O}_X$).

Lemma 18.15.1. *Let X be a ringed space. Assume that the underlying topological space of X has the following properties:*

(1) *there exists a basis of quasi-compact open subsets, and*
(2) *the intersection of any two quasi-compact opens is quasi-compact.*

Then for any directed system $(\mathcal{F}_i, \varphi_{ii'})$ of sheaves of $\mathcal{O}_X$-modules and for any quasi-compact open $U \subset X$ the canonical map

$$colim_i\, H^q(U, \mathcal{F}_i) \longrightarrow H^q(U, colim_i\, \mathcal{F}_i)$$

is an isomorphism for every $q \geq 0$.

Proof. It is important in this proof to argue for all quasi-compact opens $U \subset X$ at the same time. The result is true for $i = 0$ and any quasi-compact open $U \subset X$ by Sheaves, Lemma 6.29.1 (combined with Topology, Lemma 5.18.2). Assume that we have proved the result for all $q \leq q_0$ and let us prove the result for $q = q_0 + 1$.

By our conventions on directed systems the index set I is directed, and any system of $\mathcal{O}_X$-modules $(\mathcal{F}_i, \varphi_{ii'})$ over I is directed. By Injectives, Lemma 17.9.1 the category of $\mathcal{O}_X$-modules has functorial injective embeddings. Thus for any system $(\mathcal{F}_i, \varphi_{ii'})$ there exists a system $(\mathcal{I}_i, \varphi_{ii'})$ with each $\mathcal{I}_i$ an injective $\mathcal{O}_X$-module and a morphism of systems given by injective $\mathcal{O}_X$-module maps $\mathcal{F}_i \to \mathcal{I}_i$. Denote $\mathcal{Q}_i$ the cokernel so that we have short exact sequences

$$0 \to \mathcal{F}_i \to \mathcal{I}_i \to \mathcal{Q}_i \to 0.$$

We claim that the sequence

$$0 \to colim_i\, \mathcal{F}_i \to colim_i\, \mathcal{I}_i \to colim_i\, \mathcal{Q}_i \to 0.$$

is also a short exact sequence of $\mathcal{O}_X$-modules. We may check this on stalks. By Sheaves, Sections 6.28 and 6.29 taking stalks commutes with colimits. Since a directed colimit of short exact sequences of abelian groups is short exact (see Algebra, Lemma 7.8.9) we deduce the result. We claim that $H^q(U, colim_i\, \mathcal{I}_i) = 0$ for all quasi-compact open $U \subset X$ and all $q \geq 1$. Accepting this claim for the moment consider the diagram

$$\begin{tikzcd}
colim_i\, H^{q_0}(U, \mathcal{I}_i) \arrow[r] \arrow[d] & colim_i\, H^{q_0}(U, \mathcal{Q}_i) \arrow[r] \arrow[d] & colim_i\, H^{q_0+1}(U, \mathcal{F}_i) \arrow[r] \arrow[d] & 0 \arrow[d] \\
H^{q_0}(U, colim_i\, \mathcal{I}_i) \arrow[r] & H^{q_0}(U, colim_i\, \mathcal{Q}_i) \arrow[r] & H^{q_0+1}(U, colim_i\, \mathcal{F}_i) \arrow[r] & 0
\end{tikzcd}$$

The zero at the lower right corner comes from the claim and the zero at the upper right corner comes from the fact that the sheaves $\mathcal{I}_i$ are injective. The top row is exact by an application of Algebra, Lemma 7.8.9. Hence by the snake lemma we deduce the result for $q = q_0 + 1$.

It remains to show that the claim is true. We will use Lemma 18.11.8. Let $\mathcal{B}$ be the collection of all quasi-compact open subsets of X. This is a basis for the topology on X by assumption. Let Cov be the collection of finite open coverings $\mathcal{U} : U = \bigcup_{j=1,\ldots,m} U_j$ with each of U, U_j quasi-compact open in X. By the result for $q = 0$ we see that for $\mathcal{U} \in$ Cov we have

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, colim_i\, \mathcal{I}_i) = colim_i\, \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}_i)$$

because all the multiple intersections $U_{j_0\ldots j_p}$ are quasi-compact. By Lemma 18.11.1 each of the complexes in the colimit of Čech complexes is acyclic in degree ≥ 1. Hence by Algebra, Lemma 7.8.9 we see that also the Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, colim_i\, \mathcal{I}_i)$ is acyclic in degrees ≥ 1. In other words we see that $\check{H}^p(\mathcal{U}, colim_i \mathcal{I}_i) = 0$ for all $p \geq 1$. Thus the assumptions of Lemma 18.11.8 are satisfied and the claim follows. □

18.16. Vanishing on Noetherian topological spaces

The aim is to prove a theorem of Grothendieck namely Lemma 18.16.5. See [**Gro57**].

Lemma 18.16.1. *Let $i : Z \to X$ be a closed immersion of topological spaces. For any abelian sheaf $\mathcal{F}$ on Z we have $H^p(Z, \mathcal{F}) = H^p(X, i_*\mathcal{F})$.*

Proof. This is true because i_* is exact (see Modules, Lemma 15.6.1), and hence $R^p i_* = 0$ as a functor (Derived Categories, Lemma 11.16.8). Thus we may apply Lemma 18.12.6. □

Lemma 18.16.2. *Let X be an irreducible topological space. Then $H^p(X, \underline{A}) = 0$ for all $p > 0$ and any abelian group A.*

Proof. Recall that $\underline{A}$ is the constant sheaf as defined in Sheaves, Definition 6.7.4. It is clear that for any nonempty open $U \subset X$ we have $\underline{A}(U) = A$ as X is irreducible (and hence U is connected). We will show that the higher Čech cohomology groups $\check{H}^p(\mathcal{U}, \underline{A})$ are zero for any open covering $\mathcal{U} : U = \bigcup_{i \in I} U_i$ of an open $U \subset X$. Then the lemma will follow from Lemma 18.11.7.

Recall that the value of an abelian sheaf on the empty open set is 0. Hence we may clearly assume $U_i \neq \emptyset$ for all $i \in I$. In this case we see that $U_i \cap U_{i'} \neq \emptyset$ for all $i, i' \in I$. Hence we see that the Čech complex is simply the complex

$$\prod_{i_0 \in I} A \to \prod_{(i_0, i_1) \in I^2} A \to \prod_{(i_0, i_1, i_2) \in I^3} A \to \ldots$$

We have to see this has trivial higher cohomology groups. We can see this for example because this is the cech complex for the covering of a 1-point space and Čech cohomology agrees with cohomology on such a space. (You can also directly verify it by writing an explicit homotopy.) □

Lemma 18.16.3. *Let X be a topological space. Let $n \geq 0$ be an integer. Assume*

1. *there exists a basis of quasi-compact open subsets, and*
2. *the intersection of any two quasi-compact opens is quasi-compact.*
3. *$H^p(X, \mathcal{F}) = 0$ for any abelian sheaf $\mathcal{F}$ which is a quotient of $j_!\underline{\mathbf{Z}}_U$ for some open $j : U \to X$.*

Then $H^p(X, \mathcal{F}) = 0$ for all $p \geq n$ and any abelian sheaf $\mathcal{F}$ on X.

Proof. Let $S = \coprod_{U \subset X} \mathcal{F}(U)$. For any finite subset $A = \{s_1, \ldots, s_n\} \subset S$, let $\mathcal{F}_A$ be the subsheaf of $\mathcal{F}$ generated by all s_i (see Modules, Definition 15.4.5). Note that if $A \subset A'$, then $\mathcal{F}_A \subset \mathcal{F}_{A'}$. Hence $\{\mathcal{F}_A\}$ forms a system over the partially ordered set of finite subsets of S. By Modules, Lemma 15.4.6 it is clear that

$$colim_A \mathcal{F}_A = \mathcal{F}$$

by looking at stalks. By Lemma 18.15.1 we have

$$H^p(X, \mathcal{F}) = colim_A H^p(X, \mathcal{F}_A)$$

Hence it suffices to prove the vanishing for the abelian sheaves $\mathcal{F}_A$. In other words, it suffices to prove the result when $\mathcal{F}$ is generated by finitely many local sections.

Suppose that $\mathcal{F}$ is gerated by the local sections $s_1, \ldots, s_n$. Let $\mathcal{F}' \subset \mathcal{F}$ be the subsheaf generated by $s_1, \ldots, s_{n-1}$. Then we have a short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}/\mathcal{F}' \to 0$$

From the long exact sequence of cohomology we see that it suffices to prove the vanishing for the abelian sheaves $\mathcal{F}'$ and $\mathcal{F}/\mathcal{F}'$ which are generated by fewer than n local sections. Hence it suffices to prove the vanishing for sheaves generated by at most one local section. These sheaves are exactly the quotients of the sheaves $j_!\underline{\mathbf{Z}}_U$ mentioned in the lemma. □

Lemma 18.16.4. *Let X be an irreducible topological space. Let $\mathcal{H} \subset \underline{\mathbf{Z}}$ be an abelian subsheaf of the constant sheaf. Then there exists a nonempty open $U \subset X$ such that $\mathcal{H}|_U = \underline{d\mathbf{Z}}_U$ for some $d \in \mathbf{Z}$.*

Proof. Recall that $\underline{\mathbf{Z}}(V) = \mathbf{Z}$ for any nonempty open V of X (see proof of Lemma 18.16.2). If $\mathcal{H} = 0$, then the lemma holds with $d = 0$. If $\mathcal{H} \neq 0$, then there exists a nonempty open $U \subset X$ such that $\mathcal{H}(U) \neq 0$. Say $\mathcal{H}(U) = n\mathbf{Z}$ for some $n \geq 1$. Hence we see that $\underline{n\mathbf{Z}}_U \subset \mathcal{H}|_U \subset \underline{\mathbf{Z}}_U$. If the first inclusion is strict we can find a nonempty $U' \subset U$ and an integer $1 \leq n' < n$ such that $\underline{n'\mathbf{Z}}_{U'} \subset \mathcal{H}|_{U'} \subset \underline{\mathbf{Z}}_{U'}$. This process has to stop after a finite number of steps, and hence we get the lemma. □

Lemma 18.16.5. *Let X be a Noetherian topological space. If $\dim(X) \leq n$, then $H^p(X, \mathcal{F}) = 0$ for all $p > n$ and any abelian sheaf $\mathcal{F}$ on X.*

Proof. We prove this lemma by induction on n. So fix n and assume the lemma holds for all Noetherian topological spaces of dimension $< n$.

Let $\mathcal{F}$ be an abelian sheaf on X. Suppose $U \subset X$ is an open. Let $Z \subset X$ denote the closed complement. Denote $j : U \to X$ and $i : Z \to X$ the inclusion maps. Then there is a short exact sequence

$$0 \to j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \to 0$$

see Modules, Lemma 15.7.1. Note that $j_!j^*\mathcal{F}$ is supported on the topological closure Z' of U, i.e., it is of the form $i'_*\mathcal{F}'$ for some abelian sheaf $\mathcal{F}'$ on Z', where $i' : Z' \to X$ is the inclusion.

We can use this to reduce to the case where X is irreducible. Namely, according to Topology, Lemma 5.6.2 X has finitely many irreducible components. If X has more than one irreducible component, then let $Z \subset X$ be an irreducible component of X and set $U = X \setminus Z$. By the above, and the long exact sequence of cohomology, it suffices to prove the vanishing of $H^p(X, i_*i^*\mathcal{F})$ and $H^p(X, i'_*\mathcal{F}')$ for $p > n$. By Lemma 18.16.1 it suffices to prove $H^p(Z, i^*\mathcal{F})$ and $H^p(Z', \mathcal{F}')$ for $p > n$. Since Z' and Z have fewer irreducible components we indeed reduce to the case of an irreducible X.

If $n = 0$ and $X = \{*\}$, then every sheaf is constant and higher cohomology groups vanish (for example by Lemma 18.16.2).

Suppose X is irreducible of dimension n. By Lemma 18.16.3 we reduce to the case where $\mathcal{F}$ is generated by a single local section, i.e., to the case where there is an exact sequence

$$0 \to \mathcal{H} \to j'_!\mathbf{Z}_V \to \mathcal{F} \to 0$$

for some open $j' : V \to X$. By Lemma 18.16.4 (applied to the restriction of $\mathcal{H}$ to V) there exists a nonempty open $U \subset V$, and $d \in \mathbf{Z}$ such that $\mathcal{H}|_U = \underline{d\mathbf{Z}}_U$. Hence we see that

$\mathcal{F}|_U \cong \underline{\mathbf{Z}/d\mathbf{Z}}_U$. Let Z be the complement of U in X. Denote $j : U \to X$ and $i : Z \to X$ the inclusion maps. As in the first paragraph of the proof we obtain a short exact sequence

$$0 \to j_!\underline{\mathbf{Z}/d\mathbf{Z}} \to \mathcal{F} \to i_*i^*\mathcal{F} \to 0$$

OK, and now $\dim(Z) < n$ so by induction we have $H^p(X, i_*i^*\mathcal{F}) = H^p(Z, i^*\mathcal{F}) = 0$ for all $p \geq n$. Hence it suffices to prove the vanishing for sheaves of the form $j_!(\underline{A}_U)$ where $j : U \to X$ is an open immersion and A is an abelian group.

In this case we again look at the short exact sequence

$$0 \to j_!(\underline{A}_U) \to \underline{A} \to i_*\underline{A}_Z \to 0$$

By Lemma 18.16.2 we have the vanishing of $H^p(X, \underline{A})$ for all $p \geq 1$. By induction we have $H^p(X, i_*\underline{A}_Z) = H^p(Z, \underline{A}_Z) = 0$ for $p \leq n$. Hence we win by the long exact cohomology sequence. $\square$

18.17. The alternating Čech complex

This section compares the Čech complex with the alternating Čech complex and some related complexes.

Let X be a topological space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. For $p \geq 0$ set

$$\check{\mathcal{C}}^p_{alt}(\mathcal{U}, \mathcal{F}) = \left\{ \begin{matrix} s \in \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) \text{ such that } s_{i_0\ldots i_p} = 0 \text{ if } i_n = i_m \text{ for some } n \neq m \\ \text{and } s_{i_0\ldots i_n \ldots i_m \ldots i_p} = -s_{i_0\ldots i_m \ldots i_n \ldots i_p} \text{ in any case.} \end{matrix} \right\}$$

We omit the verification that the differential d of Equation (18.9.0.1) maps $\check{\mathcal{C}}^p_{alt}(\mathcal{U}, \mathcal{F})$ into $\check{\mathcal{C}}^{p+1}_{alt}(\mathcal{U}, \mathcal{F})$.

Definition 18.17.1. Let X be a topological space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. Let $\mathcal{F}$ be an abelian presheaf on X. The complex $\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F})$ is the *alternating Čech complex* associated to $\mathcal{F}$ and the open covering $\mathcal{U}$.

Hence there is a canonical morphism of complexes

$$\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

namely the inclusion of the alternating Čech complex into the usual Čech complex.

Suppose our covering $\mathcal{U} : U = \bigcup_{i \in I} U_i$ comes equipped with a total ordering $<$ on I. In this case, set

$$\check{\mathcal{C}}^p_{ord}(\mathcal{U}, \mathcal{F}) = \prod\nolimits_{(i_0, \ldots, i_p) \in I^{p+1}, i_0 < \ldots < i_p} \mathcal{F}(U_{i_0\ldots i_p}).$$

This is an abelian group. For $s \in \check{\mathcal{C}}^p_{ord}(\mathcal{U}, \mathcal{F})$ we denote $s_{i_0\ldots i_p}$ its value in $\mathcal{F}(U_{i_0\ldots i_p})$. We define

$$d : \check{\mathcal{C}}^p_{ord}(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^{p+1}_{ord}(\mathcal{U}, \mathcal{F})$$

by the formula

$$d(s)_{i_0\ldots i_{p+1}} = \sum\nolimits_{j=0}^{p+1} (-1)^j s_{i_0\ldots \hat{i}_j \ldots i_p}|_{U_{i_0\ldots i_{p+1}}}$$

for any $i_0 < \ldots < i_{p+1}$. Note that this formula is identical to Equation (18.9.0.1). It is straightforward to see that $d \circ d = 0$. In other words $\check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{F})$ is a complex.

Definition 18.17.2. Let X be a topological space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. Assume given a total ordering on I. Let $\mathcal{F}$ be an abelian presheaf on X. The complex $\check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{F})$ is the *ordered Čech complex* associated to $\mathcal{F}$, the open covering $\mathcal{U}$ and the given total ordering on I.

This complex is sometimes called the alternating Čech complex. The reason is that there is an obvious comparison map between the ordered Čech complex and the alternating Čech complex. Namely, consider the map

$$c : \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

given by the rule

$$c(s)_{i_0\ldots i_p} = \begin{cases} 0 & \text{if} \quad i_n = i_m \text{ for some } n \neq m \\ \text{sgn}(\sigma) s_{i_{\sigma(0)}\ldots i_{\sigma(p)}} & \text{if} \quad i_{\sigma(0)} < i_{\sigma(1)} < \ldots < i_{\sigma(p)} \end{cases}$$

Here σ denotes a permutation of $\{0, \ldots, p\}$ and $\text{sgn}(\sigma)$ denotes its sign. The alternating and ordered Cech complexes are often identified in the literature via the map c. Namely we have the following easy lemma.

Lemma 18.17.3. *Let X be a topological space. Let $\mathcal{U} : U = \bigcup_{i\in I} U_i$ be an open covering. Assume I comes equipped with a total ordering. The map c is a morphism of complexes. In fact it induces an isomorphism*

$$c : \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F})$$

of complexes.

Proof. Omitted. □

There is also a map

$$\pi : \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{F})$$

which is described by the rule

$$\pi(s)_{i_0\ldots i_p} = s_{i_0\ldots i_p}$$

whenever $i_0 < i_1 < \ldots < i_p$.

Lemma 18.17.4. *Let X be a topological space. Let $\mathcal{U} : U = \bigcup_{i\in I} U_i$ be an open covering. Assume I comes equipped with a total ordering. The map $\pi : \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{F})$ is a morphism of complexes. It induces an isomorphism*

$$\pi : \check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{F})$$

of complexes which is a left inverse to the morphism c.

Remark 18.17.5. This means that if we have two total orderings $<_1$ and $<_2$ on the index set I, then we get an isomorphism of complexes $\tau = \pi_2 \circ c_1 : \check{\mathcal{C}}_{ord\text{-}1}(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}_{ord\text{-}2}(\mathcal{U}, \mathcal{F})$. It is clear that

$$\tau(s)_{i_0\ldots i_p} = \text{sign}(\sigma) s_{i_{\sigma(0)}\ldots i_{\sigma(p)}}$$

where $i_0 <_1 i_1 <_1 \ldots <_1 i_p$ and $i_{\sigma(0)} <_2 i_{\sigma(1)} <_2 \ldots <_2 i_{\sigma(p)}$. This is the sense in which the ordered Čech complex is independent of the chosen total ordering.

Lemma 18.17.6. *Let X be a topological space. Let $\mathcal{U} : U = \bigcup_{i\in I} U_i$ be an open covering. Assume I comes equipped with a total ordering. The map $c \circ \pi$ is homotopic to the identity on $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$. In particular the inclusion map $\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is a homotopy equivalence.*

Proof. For any multi-index $(i_0, \ldots, i_p) \in I^{p+1}$ there exists a unique permutation $\sigma : \{0, \ldots, p\} \to \{0, \ldots, p\}$ such that

$$i_{\sigma(0)} \leq i_{\sigma(1)} \leq \ldots \leq i_{\sigma(p)} \quad \text{and} \quad \sigma(j) < \sigma(j+1) \quad \text{if} \quad i_{\sigma(j)} = i_{\sigma(j+1)}.$$

We denote this permutation $\sigma = \sigma^{i_0\ldots i_p}$.

For any permutation $\sigma : \{0, \ldots, p\} \to \{0, \ldots, p\}$ and any a, $0 \leq a \leq p$ we denote σ_a the permutation of $\{0, \ldots, p\}$ such that

$$\sigma_a(j) = \begin{cases} \sigma(j) & \text{if} \quad 0 \leq j < a, \\ \min\{j' \mid j' > \sigma_a(j-1), j' \neq \sigma(k), \forall k < a\} & \text{if} \quad a \leq j \end{cases}$$

So if $p = 3$ and σ, τ are given by

$$\begin{matrix} \text{id} & 0 & 1 & 2 & 3 \\ \sigma & 3 & 2 & 1 & 0 \end{matrix} \quad \text{and} \quad \begin{matrix} \text{id} & 0 & 1 & 2 & 3 \\ \tau & 3 & 0 & 2 & 1 \end{matrix}$$

then we have

$$\begin{matrix} \text{id} & 0 & 1 & 2 & 3 \\ \sigma_0 & 0 & 1 & 2 & 3 \\ \sigma_1 & 3 & 0 & 1 & 2 \\ \sigma_2 & 3 & 2 & 0 & 1 \\ \sigma_3 & 3 & 2 & 1 & 0 \end{matrix} \quad \text{and} \quad \begin{matrix} \text{id} & 0 & 1 & 2 & 3 \\ \tau_0 & 0 & 1 & 2 & 3 \\ \tau_1 & 3 & 0 & 1 & 2 \\ \tau_2 & 3 & 0 & 1 & 2 \\ \tau_3 & 3 & 0 & 2 & 1 \end{matrix}$$

It is clear that always $\sigma_0 = \text{id}$ and $\sigma_p = \sigma$.

Having introduced this notation we define for $s \in \check{\mathcal{C}}^{p+1}(\mathcal{U}, \mathcal{F})$ the element $h(s) \in \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})$ to be the element with components

$$(18.17.6.1) \qquad h(s)_{i_0 \ldots i_p} = \sum\nolimits_{0 \leq a \leq p} (-1)^a \text{sign}(\sigma_a) s_{i_{\sigma(0)} \ldots i_{\sigma(a)} i_{\sigma_a(a)} \ldots i_{\sigma_a(p)}}$$

where $\sigma = \sigma^{i_0 \ldots i_p}$. The index $i_{\sigma(a)}$ occurs twice in $i_{\sigma(0)} \ldots i_{\sigma(a)} i_{\sigma_a(a)} \ldots i_{\sigma_a(p)}$ once in the first group of $a + 1$ indices and once in the second group of $p - a + 1$ indices since $\sigma_a(j) = \sigma(a)$ for some $j \geq a$ by definition of σ_a. Hence the sum makes sense since each of the elements $s_{i_{\sigma(0)} \ldots i_{\sigma(a)} i_{\sigma_a(a)} \ldots i_{\sigma_a(p)}}$ is defined over the open $U_{i_0 \ldots i_p}$. Note also that for $a = 0$ we get $s_{i_0 \ldots i_p}$ and for $a = p$ we get $(-1)^p \text{sign}(\sigma) s_{i_{\sigma(0)} \ldots i_{\sigma(p)}}$.

We claim that

$$(dh + hd)(s)_{i_0 \ldots i_p} = s_{i_0 \ldots i_p} - \text{sign}(\sigma) s_{i_{\sigma(0)} \ldots i_{\sigma(p)}}$$

where $\sigma = \sigma^{i_0 \ldots i_p}$. We omit the verification of this claim. (There is a PARI/gp script called first-homotopy.gp in the stacks-project subdirectory scripts which can be used to check finitely many instances of this claim. We wrote this script to make sure the signs are correct.) Write

$$\kappa : \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

for the operator given by the rule

$$\kappa(s)_{i_0 \ldots i_p} = \text{sign}(\sigma^{i_0 \ldots i_p}) s_{i_{\sigma(0)} \ldots i_{\sigma(p)}}.$$

The claim above implies that κ is a morphism of complexes and that κ is homotopic to the identity map of the Čech complex. This does not immediately imply the lemma since the image of the operator κ is not the alternating subcomplex. Namely, the image of κ is the ``semi-alternating'' complex $\check{\mathcal{C}}^p_{semi\text{-}alt}(\mathcal{U}, \mathcal{F})$ where s is a p-cochain of this complex if and only if

$$s_{i_0 \ldots i_p} = \text{sign}(\sigma) s_{i_{\sigma(0)} \ldots i_{\sigma(p)}}$$

for any $(i_0, \ldots, i_p) \in I^{p+1}$ with $\sigma = \sigma^{i_0 \ldots i_p}$. We introduce yet another variant Čech complex, namely the semi-ordered Cech complex defined by

$$\check{\mathcal{C}}^p_{semi\text{-}ord}(\mathcal{U}, \mathcal{F}) = \prod\nolimits_{i_0 \leq i_1 \leq \ldots \leq i_p} \mathcal{F}(U_{i_0 \ldots i_p})$$

It is easy to see that Equation (18.9.0.1) also defines a differential and hence that we get a complex. It is also clear (analogous to Lemma 18.17.4) that the projection map

$$\check{\mathcal{C}}^\bullet_{semi\text{-}alt}(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet_{semi\text{-}ord}(\mathcal{U}, \mathcal{F})$$

is an isomorphism of complexes.

Hence the Lemma follows if we can show that the obvious inclusion map

$$\check{\mathcal{C}}^p_{ord}(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^p_{semi\text{-}ord}(\mathcal{U}, \mathcal{F})$$

is a homotopy equivalence. To see this we use the homotopy

$$(18.17.6.2) \quad h(s)_{i_0\ldots i_p} = \begin{cases} 0 & \text{if} \quad i_0 < i_1 < \ldots < i_p \\ (-1)^a s_{i_0\ldots i_{a-1} i_a i_a i_{a+1}\ldots i_p} & \text{if} \quad i_0 < i_1 < \ldots < i_{a-1} < i_a = i_{a+1} \end{cases}$$

We claim that

$$(dh + hd)(s)_{i_0\ldots i_p} = \begin{cases} 0 & \text{if} \quad i_0 < i_1 < \ldots < i_p \\ s_{i_0\ldots i_p} & \text{else} \end{cases}$$

We omit the verification. (There is a PARI/gp script called second-homotopy.gp in the stacks-project subdirectory scripts which can be used to check finitely many instances of this claim. We wrote this script to make sure the signs are correct.) The claim clearly shows that the composition

$$\check{\mathcal{C}}^\bullet_{semi\text{-}ord}(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet_{semi\text{-}ord}(\mathcal{U}, \mathcal{F})$$

of the projection with the natural inclusion is homotopic to the identity map as desired. □

18.18. Locally finite coverings and the Čech complex

In this section we discuss an alternative way to esthablish the relationship between the Čech complex and cohomology in case the covering is locally finite.

Definition 18.18.1. Let X be a topological space. An open covering $X = \bigcup_{i\in I} U_i$ is said to be *locally finite* if for every $x \in X$ there exists an open neighbourhood W of x such that $\{i \in I \mid W \cap U_i \neq \emptyset\}$ is finite.

Remark 18.18.2. Let $X = \bigcup_{i\in I} U_i$ be a locally finite open covering. Denote $j_i : U_i \to X$ the inclusion map. Suppose that for each i we are given an abelian sheaf $\mathcal{F}_i$ on U_i. Consider the abelian sheaf $\mathcal{G} = \bigoplus_{i\in I} (j_i)_*\mathcal{F}_i$. Then actually

$$\Gamma(X, \mathcal{G}) = \prod\nolimits_{i\in I} \mathcal{F}_i(U_i).$$

This seems strange until you realize that the direct sum of a collection of sheaves is the sheafification of what you think it should be. See discussion in Modules, Section 15.3.

Lemma 18.18.3. *Let X be a ringed space. Let $\mathcal{U} : X = \bigcup_{i\in I} U_i$ be a locally finite open covering of X. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Denote $\mathcal{F}_{i_0\ldots i_p}$ the restriction of $\mathcal{F}$ to $U_{i_0\ldots i_p}$. There exists a complex $\mathfrak{C}^\bullet(\mathcal{U}, \mathcal{F})$ of $\mathcal{O}_X$-modules with*

$$\mathfrak{C}^p(\mathcal{U}, \mathcal{F}) = \bigoplus\nolimits_{i_0\ldots i_p} (j_{i_0\ldots i_p})_*\mathcal{F}_{i_0\ldots i_p}$$

and differential $d : \mathfrak{C}^p(\mathcal{U}, \mathcal{F}) \to \mathfrak{C}^{p+1}(\mathcal{U}, \mathcal{F})$ as in Equation (18.9.0.1). Moreover, there exists a canonical map

$$\mathcal{F} \to \mathfrak{C}^\bullet(\mathcal{U}, \mathcal{F})$$

which is a quasi-isomorphism, i.e., $\mathfrak{C}^\bullet(\mathcal{U}, \mathcal{F})$ is a resolution of $\mathcal{F}$.

Proof. Omitted. □

With this lemma it is easy to reprove the Čech to cohomology spectral sequence of Lemma 18.11.4 in the special case of locally finite coverings. Namely, let X, $\mathcal{U}$, $\mathcal{F}$ as in Lemma 18.18.3 and let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution. Then we may consider the double complex

$$A^{\bullet,\bullet} = \Gamma(X, \mathfrak{C}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)).$$

Note that

$$A^{p,q} = \prod\nolimits_{i_0\ldots i_p} \mathcal{I}^q(U_{i_0\ldots i_p})$$

see Remark 18.18.2. Consider the two spectral sequences of Homology, Section 10.19 associated to this double complex. See especially Homology, Lemma 10.19.3. For the spectral sequence $({}'E_r, {}'d_r)_{r\geq 0}$ we get ${}'E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F}))$ and for the spectral sequence $({}''E_r, {}''d_r)_{r\geq 0}$ we get ${}''E_2^{p,q} = 0$ if $p > 0$ and ${}''E_2^{0,q} = H^q(X, \mathcal{F})$. Whence the result since both spectral sequences converge to the cohomology of the simple complex $sA^\bullet$.

18.19. Čech cohomology of complexes

In general for sheaves of abelian groups $\mathcal{F}$ and $\mathcal{G}$ on X there is a cupproduct map

$$H^i(X, \mathcal{F}) \times H^j(X, \mathcal{G}) \longrightarrow H^{i+j}(X, \mathcal{F} \otimes_{\mathbf{Z}} \mathcal{G}).$$

In this section we define it using Čech cocyles by an explicit formula for the cup product. If you are worried about the fact that cohomology may not equal Čech cohomology, then you can use hypercoverings and still use the cocycle notation. This also has the advantage that it works to define the cup product for hypercohomology on any site.

Let $\mathcal{F}^\bullet$ be a bounded below complex of sheaves of abelian groups on X. We can often compute $H^n(X, \mathcal{F}^\bullet)$ using Čech cocycles. Namely, let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open covering of X. Consider the associated total complex to $\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)$ with degree n term

$$\text{Tot}^n(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) = \prod\nolimits_{p+q=n} \mathcal{F}^q(U_{i_0\ldots i_p})$$

A typical element in Tot^n will be denoted $\alpha = \{\alpha_{i_0\ldots i_p}\}$ where $\alpha_{i_0\ldots i_p} \in \mathcal{F}^q(U_{i_0\ldots i_p})$. In other words the $\mathcal{F}$-degree of $\alpha_{i_0\ldots i_p}$ is $q = n-p$. This notation requires us to be aware of the degree α lives in at all times. We indicate this situation by the formula $\deg_{\mathcal{F}}(\alpha_{i_0\ldots i_p}) = q$. According to our conventions in Homology, Definition 10.19.2 the differential of an element α of degree n is given by

$$d(\alpha)_{i_0\ldots i_{p+1}} = \sum\nolimits_{j=0}^{p+1} (-1)^j \alpha_{i_0\ldots\hat{i}_j\ldots i_{p+1}} + (-1)^{p+1} d_{\mathcal{F}}(\alpha_{i_0\ldots i_{p+1}})$$

where $d_{\mathcal{F}}$ denotes the differential on the complex $\mathcal{F}$. The expression $\alpha_{i_0\ldots\hat{i}_j\ldots i_{p+1}}$ means the restriction of $\alpha_{i_0\ldots\hat{i}_j\ldots i_{p+1}} \in \mathcal{F}(U_{i_0\ldots\hat{i}_j\ldots i_{p+1}})$ to $U_{i_0\ldots i_{p+1}}$. To check this is a complex, let α be an element of degree n in $\mathrm{Tot}(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$. We compute:

$$\begin{aligned} d^2(\alpha)_{i_0\ldots i_{p+2}} &= \sum\nolimits_{j=0}^{p+2}(-1)^j d(\alpha)_{i_0\ldots\hat{i}_j\ldots i_{p+2}} + (-1)^{p+2}d_{\mathcal{F}}(d(\alpha)_{i_0\ldots i_{p+2}}) \\ &= \sum\nolimits_{j=0}^{p+2}(-1)^j \sum\nolimits_{j'=0\ldots j-1}(-1)^{j'}\alpha_{i_0\ldots\hat{i}_{j'}\ldots\hat{i}_j\ldots i_{p+2}} + \\ &\quad \sum\nolimits_{j=0}^{p+2}(-1)^j \sum\nolimits_{j'=j+1\ldots p+2}(-1)^{j'-1}\alpha_{i_0\ldots\hat{i}_j\ldots\hat{i}_{j'}\ldots i_{p+2}} + \\ &\quad \sum\nolimits_{j=0}^{p+2}(-1)^{j+(p+1)}d_{\mathcal{F}}(\alpha_{i_0\ldots\hat{i}_j\ldots i_{p+2}}) + \\ &\quad (-1)^{p+2}d_{\mathcal{F}}\Big(\sum\nolimits_{j=0}^{p+2}(-1)^j\,\alpha_{i_0\ldots\hat{i}_j\ldots i_{p+2}}\Big) + \\ &\quad (-1)^{(p+2)(p+1)}d_{\mathcal{F}}(d_{\mathcal{F}}(\alpha_{i_0\ldots i_{p+2}})) \end{aligned}$$

which equals zero by the nullity of $d_{\mathcal{F}}^2$, a trivial sign change between the third and fourth terms, and the usual argument for the first two double Čech terms.

The construction of $\mathrm{Tot}(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$ is functorial in $\mathcal{F}^\bullet$. As well there is a functorial transformation

$$\Gamma(X, \mathcal{F}^\bullet) \longrightarrow \mathrm{Tot}(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \tag{18.19.0.1}$$

of complexes defined by the following rule: The section $s \in \Gamma(X, \mathcal{F}^n)$ is mapped to the element $\alpha = \{\alpha_{i_0\ldots i_p}\}$ with $\alpha_{i_0} = s|_{U_{i_0}}$ and $\alpha_{i_0\ldots i_p} = 0$ for $p > 0$.

Refinements. Let $\mathcal{V} = \{V_j\}_{j\in J}$ be a refinement of $\mathcal{U}$. This means there is a map $t : J \to I$ such that $V_j \subset U_{t(j)}$ for all $j \in J$. This gives rise to a functorial transformation

$$T_t : \mathrm{Tot}(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \longrightarrow \mathrm{Tot}(\mathcal{C}^\bullet(\mathcal{V}, \mathcal{F}^\bullet)).$$

defined by the rule

$$T_t(\alpha)_{j_0\ldots j_p} = \alpha_{t(j_0)\ldots t(j_p)}|_{V_{j_0\ldots j_p}}.$$

Given two maps $t, t' : J \to I$ as above the maps T_t and $T_{t'}$ constructed above are homotopic. The homotopy is given by

$$h(\alpha)_{j_0\ldots j_p} = \sum\nolimits_{a=0}^{p}(-1)^a\alpha_{t(j_0)\ldots t(j_a)t'(j_a)\ldots t'(j_p)}$$

for an element α of degree n. This works because of the following computation, again with α an elemement of degree n (so $d(\alpha)$ has degree $n+1$ and $h(\alpha)$ has degree $n-1$):

$$
\begin{aligned}
(d(h(\alpha)) + h(d(\alpha)))_{j_0\ldots j_p} = & \sum\nolimits_{k=0}^{p} (-1)^k h(\alpha)_{j_0\ldots \hat{j}_k \ldots j_p} + \\
& (-1)^p d_{\mathcal{F}}(h(\alpha)_{j_0\ldots j_p}) + \\
& \sum\nolimits_{a=0}^{p} (-1)^a d(\alpha)_{t(j_0)\ldots t(j_a) t'(j_a) \ldots t'(j_p)} \\
= & \sum\nolimits_{k=0}^{p} \sum\nolimits_{a=0}^{k-1} (-1)^{k+a} \alpha_{t(j_0)\ldots t(j_a) t'(j_a) \ldots t'(\hat{j_k}) \ldots t'(j_p)} + \\
& \sum\nolimits_{k=0}^{p} \sum\nolimits_{a=k+1}^{p} (-1)^{k+a-1} \alpha_{t(j_0)\ldots t(\hat{j_k}) \ldots t(j_a) t'(j_a) \ldots t'(j_p)} + \\
& \sum\nolimits_{a=0}^{p} (-1)^{p+a} d_{\mathcal{F}}(\alpha_{t(j_0)\ldots t(j_a) t'(j_a) \ldots t'(j_p)}) + \\
& \sum\nolimits_{a=0}^{p} \sum\nolimits_{k=0}^{a} (-1)^{a+k} \alpha_{t(j_0)\ldots t(\hat{j_k}) \ldots t(j_a) t'(j_a) \ldots t'(j_p)} + \\
& \sum\nolimits_{a=0}^{p} \sum\nolimits_{k=a}^{p} (-1)^{a+k+1} \alpha_{t(j_0)\ldots t(j_a) t'(j_a) \ldots t'(\hat{j_k}) \ldots t'(j_p)} + \\
& \sum\nolimits_{a=0}^{p} (-1)^{a+p+1} d_{\mathcal{F}}(\alpha_{t(j_0)\ldots t(j_a) t'(j_a) \ldots t'(j_p)}) \\
= & \alpha_{t'(j_0)\ldots t'(j_p)} + (-1)^{2p+1} \alpha_{t(j_0)\ldots t(j_p)} \\
= & T_{t'}(\alpha)_{j_0\ldots j_p} - T_t(\alpha)_{j_0\ldots j_p}
\end{aligned}
$$

We leave it to the reader to verify the cancellations. (Note that the terms having both k and a in the 1st, 2nd and 4th, 5th summands cancel, except the ones where $a = k$ which only occur in the 4th and 5th and these cancel against each other except for the two desired terms.) It follows that the induced map

$$
H^n(T_t) : H^n(\text{Tot}(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))) \to H^n(\text{Tot}(\mathcal{C}^\bullet(\mathcal{V}, \mathcal{F}^\bullet)))
$$

is independend of the choice of t. We define Cech hypercohomology as the limit of the Čech cohomology groups over all refinements via the maps $H^\bullet(T_t)$.

Let $\mathcal{I}^\bullet$ be a bounded below complex of injectives. Consider the map (18.19.0.1) for the $\mathcal{I}^\bullet$ which is a map $\Gamma(X, \mathcal{I}^\bullet) \to \text{Tot}(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))$. This is a quasi-isomorphism of complexes of abelian groups as follows from a spectral sequence argument on the double complex $\mathcal{C}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)$ using Lemma 18.11.1. Suppose $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ is a quasi-isomorphism of $\mathcal{F}^\bullet$ into a bounded below complex of injectives. The cohomology $H^n(X, \mathcal{F}^\bullet)$ is defined to be $H^n(\Gamma(X, \mathcal{I}^\bullet))$. Thus the map

$$
\text{Tot}(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \longrightarrow \text{Tot}(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))
$$

induces maps $H^n(\text{Tot}(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))) \to H^n(X, \mathcal{F}^\bullet)$. In the limit (over all open coverings of X) this induces a map of Čech cohomology into the cohomology, which is often an isomorphism and is always an isomorphism if we use hypercoverings.

Consider the map $\tau : \text{Tot}(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \to \text{Tot}(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$ defined by

$$
\tau(\alpha)_{i_0\ldots i_p} = (-1)^{p(p+1)/2} \alpha_{i_p\ldots i_0}.
$$

Then we have for an element α of degree n that

$$
\begin{aligned}
& d(\tau(\alpha))_{i_0\ldots i_{p+1}} \\
& = \sum\nolimits_{j=0}^{p+1} (-1)^j \tau(\alpha)_{i_0\ldots \hat{i}_j \ldots i_{p+1}} + (-1)^{p+1} d_{\mathcal{F}}(\tau(\alpha)_{i_0\ldots i_{p+1}}) \\
& = \sum\nolimits_{j=0}^{p+1} (-1)^{j+\frac{p(p+1)}{2}} \alpha_{i_{p+1}\ldots \hat{i}_j \ldots i_0} + (-1)^{p+1+\frac{(p+1)(p+2)}{2}} d_{\mathcal{F}}(\alpha_{i_{p+1}\ldots i_0})
\end{aligned}
$$

On the other hand we have

$$
\begin{aligned}
&\tau(d(\alpha))_{i_0\ldots i_{p+1}}\\
&= (-1)^{\frac{(p+1)(p+2)}{2}} d(\alpha)_{i_{p+1}\ldots i_0}\\
&= (-1)^{\frac{(p+1)(p+2)}{2}} \left(\sum\nolimits_{j=0}^{p+1} (-1)^j \alpha_{i_{p+1}\ldots \hat{i}_{p+1-j}\ldots i_0} + (-1)^{p+1} d_{\mathcal{F}}(\alpha_{i_{p+1}\ldots i_0})\right)
\end{aligned}
$$

Thus we conclude that $d(\tau(\alpha)) = \tau(d(\alpha))$ because $p(p+1)/2 \equiv (p+1)(p+2)/2 + p + 1 \bmod 2$. In other words τ is an endomorphism of the complex $\text{Tot}(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$. Note that the diagram

$$
\begin{matrix}
\Gamma(X, \mathcal{F}^\bullet) & \longrightarrow & \text{Tot}(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \\
\downarrow \text{id} & & \downarrow \tau \\
\Gamma(X, \mathcal{F}^\bullet) & \longrightarrow & \text{Tot}(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))
\end{matrix}
$$

commutes. In addition τ is clearly compatible with refinements. This suggests that τ acts as the identity on Čech cohomology (i.e., in the limit -- provided Čech hypercohomology agrees with hypercohomology, which is always the case if we use hypercoverings). We claim that τ actually is homotopic to the identity on the total Čech complex $\text{Tot}(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$. To prove this, we use as homotopy

$$
h(\alpha)_{i_0\ldots i_p} = \sum\nolimits_{a=0}^{p} \epsilon_p(a) \alpha_{i_0\ldots i_a i_p \ldots i_a} \quad \text{with} \quad \epsilon_p(a) = (-1)^{\frac{(p-a)(p-a-1)}{2} + p}
$$

for α of degree n. As usual we omit writing $|_{U_{i_0\ldots i_p}}$. This works because of the following computation, again with α an elemement of degree n:

$$
\begin{aligned}
(d(h(\alpha)) + h(d(\alpha)))_{i_0\ldots i_p} &= \sum\nolimits_{k=0}^{p} (-1)^k h(\alpha)_{i_0\ldots \hat{i}_k \ldots i_p} + \\
&\quad (-1)^p d_{\mathcal{F}}(h(\alpha)_{i_0\ldots i_p}) + \\
&\quad \sum\nolimits_{a=0}^{p} \epsilon_p(a) d(\alpha)_{i_0\ldots i_a i_p \ldots i_a} \\
&= \sum\nolimits_{k=0}^{p} \sum\nolimits_{a=0}^{k-1} (-1)^k \epsilon_{p-1}(a) \alpha_{i_0\ldots i_a i_p \ldots \hat{i_k} \ldots i_a} + \\
&\quad \sum\nolimits_{k=0}^{p} \sum\nolimits_{a=k+1}^{p} (-1)^k \epsilon_{p-1}(a-1) \alpha_{i_0\ldots \hat{i_k} \ldots i_a i_p \ldots i_a} + \\
&\quad \sum\nolimits_{a=0}^{p} (-1)^p \epsilon_p(a) d_{\mathcal{F}}(\alpha_{i_0\ldots i_a i_p \ldots i_a}) + \\
&\quad \sum\nolimits_{a=0}^{p} \sum\nolimits_{k=0}^{a} \epsilon_p(a) (-1)^k \alpha_{i_0\ldots \hat{i_k} \ldots i_a i_p \ldots i_a} + \\
&\quad \sum\nolimits_{a=0}^{p} \sum\nolimits_{k=a}^{p} \epsilon_p(a) (-1)^{p+a+1-k} \alpha_{i_0\ldots i_a i_p \ldots \hat{i_k} \ldots i_a} + \\
&\quad \sum\nolimits_{a=0}^{p} \epsilon_p(a) (-1)^{p+1} d_{\mathcal{F}}(\alpha_{i_0\ldots i_a i_p \ldots i_a}) \\
&= \epsilon_p(0) \alpha_{i_p\ldots i_0} + \epsilon_p(p)(-1)^{p+1} \alpha_{i_0\ldots i_p} \\
&= (-1)^{\frac{p(p+1)}{2}} \alpha_{i_p\ldots i_0} - \alpha_{i_0\ldots i_p}
\end{aligned}
$$

The cancellations follow because

$$
(-1)^k \epsilon_{p-1}(a) + \epsilon_p(a)(-1)^{p+a+1-k} = 0 \quad \text{and} \quad (-1)^k \epsilon_{p-1}(a-1) + \epsilon_p(a)(-1)^k = 0
$$

We leave it to the reader to verify the cancellations.

Suppose we have two bounded below complexes complexes of abelian sheaves $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$. We define the complex $\text{Tot}(\mathcal{F}^\bullet \otimes_Z \mathcal{G}^\bullet)$ to be to complex with terms $\bigoplus_{p+q=n} \mathcal{F}^p \otimes \mathcal{G}^q$ and differential according to the rule

$$
d(\alpha \otimes \beta) = d(\alpha) \otimes \beta + (-1)^{\deg(\alpha)} \alpha \otimes d(\beta) \tag{18.19.0.2}
$$

when α and β are homogenous, see Homology, Definition 10.19.2.

Suppose that $M^\bullet$ and $N^\bullet$ are two bounded below complexes of abelian groups. Then if m, resp. n is a cocycle for $M^\bullet$, resp. $N^\bullet$, it is immediate that $m \otimes n$ is a cocycle for $\text{Tot}(M^\bullet \otimes N^\bullet)$. Hence a cupproduct

$$H^i(M^\bullet) \times H^j(N^\bullet) \longrightarrow H^{i+j}(Tot(M^\bullet \otimes N^\bullet)).$$

This is discussed also in More on Algebra, Section 12.7.

So the construction of the cup product in hypercohomology of complexes rests on a construction of a map of complexes

$$\text{Tot}\left(\text{Tot}(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \otimes_{\mathbf{Z}} \text{Tot}(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{G}^\bullet))\right) \longrightarrow \text{Tot}(\mathcal{C}^\bullet(\mathcal{U}, \text{Tot}(\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet))) \tag{18.19.0.3}$$

This map is denoted $\cup$ and is given by the rule

$$(\alpha \cup \beta)_{i_0 \ldots i_p} = \sum\nolimits_{r=0}^{p} \epsilon(n, m, p, r) \alpha_{i_0 \ldots i_r} \otimes \beta_{i_r \ldots i_p}.$$

where α has degree n and β has degree m and with

$$\epsilon(n, m, p, r) = (-1)^{(p+r)n + rp + r}.$$

Note that $\epsilon(n, m, p, n) = 1$. Hence if $\mathcal{F}^\bullet = \mathcal{F}[0]$ is the complex consisting in a single abelian sheaf $\mathcal{F}$ placed in degree 0, then there no signs in the formula for $\cup$ (as in that case $\alpha_{i_0 \ldots i_r} = 0$ unless $r = n$). For an explanation of why there has to be a sign and how to compute it see [**MA71**, Exposee XVII] by Deligne. To check (18.19.0.3) is a map of complexes we have to show that

$$d(\alpha \cup \beta) = d(\alpha) \cup \beta + (-1)^{\deg(\alpha)} \alpha \cup d(\beta)$$

by the definition of the differential on $\text{Tot}(\text{Tot}(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \otimes_{\mathbf{Z}} \text{Tot}(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{G}^\bullet)))$ as given in Homology, Definition 10.19.2. We compute first

$$\begin{aligned} d(\alpha \cup \beta)_{i_0 \ldots i_{p+1}} &= \sum\nolimits_{j=0}^{p+1} (-1)^j (\alpha \cup \beta)_{i_0 \ldots \hat{i}_j \ldots i_{p+1}} + (-1)^{p+1} d_{\mathcal{F} \otimes \mathcal{G}}((\alpha \cup \beta)_{i_0 \ldots i_{p+1}}) \\ &= \sum\nolimits_{j=0}^{p+1} \sum\nolimits_{r=0}^{j-1} (-1)^j \epsilon(n, m, p, r) \alpha_{i_0 \ldots i_r} \otimes \beta_{i_r \ldots \hat{i}_j \ldots i_{p+1}} + \\ &\quad \sum\nolimits_{j=0}^{p+1} \sum\nolimits_{r=j+1}^{p+1} (-1)^j \epsilon(n, m, p, r-1) \alpha_{i_0 \ldots \hat{i}_j \ldots i_r} \otimes \beta_{i_r \ldots i_{p+1}} + \\ &\quad \sum\nolimits_{r=0}^{p+1} (-1)^{p+1} \epsilon(n, m, p+1, r) d_{\mathcal{F} \otimes \mathcal{G}}(\alpha_{i_0 \ldots i_r} \otimes \beta_{i_r \ldots i_{p+1}}) \end{aligned}$$

and note that the summands in the last term equal

$$(-1)^{p+1} \epsilon(n, m, p+1, r) \left(d_{\mathcal{F}}(\alpha_{i_0 \ldots i_r}) \otimes \beta_{i_r \ldots i_{p+1}} + (-1)^{n-r} \alpha_{i_0 \ldots i_r} \otimes d_{\mathcal{G}}(\beta_{i_r \ldots i_{p+1}}) \right).$$

because $\deg_{\mathcal{F}}(\alpha_{i_0 \ldots i_r}) = n - r$. On the other hand

$$\begin{aligned} (d(\alpha) \cup \beta)_{i_0 \ldots i_{p+1}} &= \sum\nolimits_{r=0}^{p+1} \epsilon(n+1, m, p+1, r) d(\alpha)_{i_0 \ldots i_r} \otimes \beta_{i_r \ldots i_{p+1}} \\ &= \sum\nolimits_{r=0}^{p+1} \sum\nolimits_{j=0}^{r} \epsilon(n+1, m, p+1, r)(-1)^j \alpha_{i_0 \ldots \hat{i}_j \ldots i_r} \otimes \beta_{i_r \ldots i_{p+1}} + \\ &\quad \sum\nolimits_{r=0}^{p+1} \epsilon(n+1, m, p+1, r)(-1)^r d_{\mathcal{F}}(\alpha_{i_0 \ldots i_r}) \otimes \beta_{i_r \ldots i_{p+1}} \end{aligned}$$

and

$$\begin{aligned}(\alpha \cup d(\beta))_{i_0\ldots i_{p+1}} &= \sum\nolimits_{r=0}^{p+1} \epsilon(n, m+1, p+1, r)\alpha_{i_0\ldots i_r} \otimes d(\beta)_{i_r\ldots i_{p+1}} \\ &= \sum\nolimits_{r=0}^{p+1} \sum\nolimits_{j=r}^{p+1} \epsilon(n, m+1, p+1, r)(-1)^{j-r}\alpha_{i_0\ldots i_r} \otimes \beta_{i_r\ldots \hat{i_j}\ldots i_{p+1}} + \\ &\quad \sum\nolimits_{r=0}^{p+1} \epsilon(n, m+1, p+1, r)(-1)^{p+1-r}\alpha_{i_0\ldots i_r} \otimes d_{\mathcal{G}}(\beta_{i_r\ldots i_{p+1}})\end{aligned}$$

The desired equality holds if we have

$$\begin{aligned}(-1)^{p+1}\epsilon(n, m, p+1, r) &= \epsilon(n+1, m, p+1, r)(-1)^r \\ (-1)^{p+1}\epsilon(n, m, p+1, r)(-1)^{n-r} &= (-1)^n\epsilon(n, m+1, p+1, r)(-1)^{p+1-r} \\ \epsilon(n+1, m, p+1, r)(-1)^r &= (-1)^{1+n}\epsilon(n, m+1, p+1, r-1) \\ (-1)^j\epsilon(n, m, p, r) &= (-1)^n\epsilon(n, m+1, p+1, r)(-1)^{j-r} \\ (-1)^j\epsilon(n, m, p, r-1) &= \epsilon(n+1, m, p+1, r)(-1)^j\end{aligned}$$

(The third equality is necessary to get the terms with $r = j$ from $d(\alpha) \cup \beta$ and $(-1)^n\alpha \cup d(\beta)$ to cancel each other.) We leave the verifications to the reader. (Alternatively, check the script signs.gp in the scripts subdirectory of the stacks project.)

Associativity of the cupproduct. Suppose that $\mathcal{F}^\bullet$, $\mathcal{G}^\bullet$ and $\mathcal{H}^\bullet$ are bounded below complexes of abelian groups on X. The obvious map (without the intervention of signs) is an isomorphism of complexes

$$\mathrm{Tot}(\mathrm{Tot}(\mathcal{F}^\bullet \otimes_Z \mathcal{G}^\bullet) \otimes_Z \mathcal{H}^\bullet) \longrightarrow \mathrm{Tot}(\mathcal{F}^\bullet \otimes_Z \mathrm{Tot}(\mathcal{G}^\bullet \otimes_Z \mathcal{H}^\bullet)).$$

Another way to say this is that the triple complex $\mathcal{F}^\bullet \otimes_Z \mathcal{G}^\bullet \otimes_Z \mathcal{H}^\bullet$ gives rise to a well defined total complex with differential satisfying

$$d(\alpha \otimes \beta \otimes \gamma) = d(\alpha) \otimes \beta \otimes \gamma + (-1)^{\deg(\alpha)}\alpha \otimes d(\beta) \otimes \gamma + (-1)^{\deg(\alpha)+\deg(\beta)}\alpha \otimes \beta \otimes d(\gamma)$$

for homogeneous elements. Using this map it is easy to verify that

$$(\alpha \cup \beta) \cup \gamma = \alpha \cup (\beta \cup \gamma)$$

namely, if α has degree a, β has degree b and γ has degree c, then

$$\begin{aligned}((\alpha \cup \beta) \cup \gamma)_{i_0\ldots i_p} &= \sum\nolimits_{r=0}^{p} \epsilon(a+b, c, p, r)(\alpha \cup \beta)_{i_0\ldots i_r} \otimes \gamma_{i_r\ldots i_p} \\ &= \sum\nolimits_{r=0}^{p} \sum\nolimits_{s=0}^{r} \epsilon(a+b, c, p, r)\epsilon(a, b, r, s)\alpha_{i_0\ldots i_s} \otimes \beta_{i_s\ldots i_r} \otimes \gamma_{i_r\ldots i_p}\end{aligned}$$

and

$$\begin{aligned}(\alpha \cup (\beta \cup \gamma))_{i_0\ldots i_p} &= \sum\nolimits_{s=0}^{p} \epsilon(a, b+c, p, s)\alpha_{i_0\ldots i_s} \otimes (\beta \cup \gamma)_{i_s\ldots i_p} \\ &= \sum\nolimits_{s=0}^{p} \sum\nolimits_{r=s}^{p} \epsilon(a, b+c, p, s)\epsilon(b, c, p-s, r-s)\alpha_{i_0\ldots i_s} \otimes \beta_{i_s\ldots i_r} \otimes \gamma_{i_r\ldots i_p}\end{aligned}$$

and a trivial mod 2 calculation shows the signs match up. (Alternatively, check the script signs.gp in the scripts subdirectory of the stacks project.)

Finally, we indicate why the cup product preserves a graded commutative structure, at least on a cohomological level. For this we use the operator τ introduced above. Let $\mathcal{F}^\bullet$ be a bounded below complexes of abelian groups, and assume we are given a graded commutative multiplication

$$\wedge^\bullet : \mathrm{Tot}(\mathcal{F}^\bullet \otimes \mathcal{F}^\bullet) \longrightarrow \mathcal{F}^\bullet.$$

This means the following: For s a local section of $\mathcal{F}^a$, and t a local section of $\mathcal{F}^b$ we have $s \wedge t$ a local section of $\mathcal{F}^{a+b}$. Graded commutative means we have $s \wedge t = (-1)^{ab} t \wedge s$. Since $\wedge$ is a map of complexes we have $d(s \wedge t) = d(s) \wedge t + (-1)^a s \wedge t$. The composition

$$\text{Tot}(\text{Tot}(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \otimes \text{Tot}(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))) \to \text{Tot}(\mathcal{C}^\bullet(\mathcal{U}, \text{Tot}(\mathcal{F}^\bullet \otimes_{Z} \mathcal{F}^\bullet))) \to \text{Tot}(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$$

induces a cup product on cohomology

$$H^n(\text{Tot}(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))) \times H^m(\text{Tot}(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))) \longrightarrow H^{n+m}(\text{Tot}(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)))$$

and so in the limit also a product on Čech cohomology and therefore (using hypercoverings if needed) a product in cohomology of $\mathcal{F}^\bullet$. We claim this product (on cohomology) is graded commutative as well. To prove this we first consider an element α of degree n in $\text{Tot}(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$ and an element β of degree m in $\text{Tot}(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$ and we compute

$$\begin{aligned}
\wedge^\bullet(\alpha \cup \beta)_{i_0 \ldots i_p} &= \sum\nolimits_{r=0}^{p} \epsilon(n, m, p, r) \alpha_{i_0 \ldots i_r} \wedge \beta_{i_r \ldots i_p} \\
&= \sum\nolimits_{r=0}^{p} \epsilon(n, m, p, r) (-1)^{\deg(\alpha_{i_0 \ldots i_r}) \deg(\beta_{i_r \ldots i_p})} \beta_{i_r \ldots i_p} \wedge \alpha_{i_0 \ldots i_r}
\end{aligned}$$

because $\wedge$ is graded commutative. On the other hand we have

$$\begin{aligned}
\tau(\wedge^\bullet(\tau(\beta) \cup \tau(\alpha)))_{i_0 \ldots i_p} &= \chi(p) \sum\nolimits_{r=0}^{p} \epsilon(m, n, p, r) \tau(\beta)_{i_p \ldots i_{p-r}} \wedge \tau(\alpha)_{i_{p-r} \ldots i_0} \\
&= \chi(p) \sum\nolimits_{r=0}^{p} \epsilon(m, n, p, r) \chi(r) \chi(p-r) \beta_{i_{p-r} \ldots i_p} \wedge \alpha_{i_0 \ldots i_{p-r}} \\
&= \chi(p) \sum\nolimits_{r=0}^{p} \epsilon(m, n, p, p-r) \chi(r) \chi(p-r) \beta_{i_r \ldots i_p} \wedge \alpha_{i_0 \ldots i_r}
\end{aligned}$$

where $\chi(t) = (-1)^{\frac{t(t+1)}{2}}$. Since we proved earlier that τ acts as the identity on cohomology we have to verify that

$$\epsilon(n, m, p, r)(-1)^{(n-r)(m-(p-r))} = (-1)^{nm} \chi(p) \epsilon(m, n, p, p-r) \chi(r) \chi(p-r)$$

A trivial mod 2 calculation shows these signs match up. (Alternatively, check the script signs.gp in the scripts subdirectory of the stacks project.)

Finally, we study the compatibility of cup product with boundary maps. Suppose that

$$0 \to \mathcal{F}_1^\bullet \to \mathcal{F}_2^\bullet \to \mathcal{F}_3^\bullet \to 0 \quad \text{and} \quad 0 \leftarrow \mathcal{G}_1^\bullet \leftarrow \mathcal{G}_2^\bullet \leftarrow \mathcal{G}_3^\bullet \leftarrow 0$$

are short exact sequences of bounded below complexes of abelian sheaves on X. Let $\mathcal{H}^\bullet$ be another bounded below complex of abelian sheaves, and suppose we have maps of complexes

$$\gamma_i : \text{Tot}(\mathcal{F}_i^\bullet \otimes_Z \mathcal{G}_i^\bullet) \longrightarrow \mathcal{H}^\bullet$$

which are compatible with the maps between the complexes, namely such that the diagrams

$$\begin{tikzcd}
\text{Tot}(\mathcal{F}_1^\bullet \otimes_Z \mathcal{G}_1^\bullet) \arrow[d, "\gamma_1"'] & \text{Tot}(\mathcal{F}_1^\bullet \otimes_Z \mathcal{G}_2^\bullet) \arrow[l] \arrow[d] \\
\mathcal{H}^\bullet & \text{Tot}(\mathcal{F}_2^\bullet \otimes_Z \mathcal{G}_2^\bullet) \arrow[l, "\gamma_2"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
\text{Tot}(\mathcal{F}_2^\bullet \otimes_Z \mathcal{G}_2^\bullet) \arrow[d, "\gamma_2"'] & \text{Tot}(\mathcal{F}_2^\bullet \otimes_Z \mathcal{G}_3^\bullet) \arrow[l] \arrow[d] \\
\mathcal{H}^\bullet & \text{Tot}(\mathcal{F}_3^\bullet \otimes_Z \mathcal{G}_3^\bullet) \arrow[l, "\gamma_3"']
\end{tikzcd}$$

are commutative.

Lemma 18.19.1. *In the situation above, assume Čech cohomology agrees with cohomology for the sheaves $\mathcal{F}_i^p$ and $\mathcal{G}_j^q$. Let $a_3 \in H^n(X, \mathcal{F}_3^\bullet)$ and $b_1 \in H^m(X, \mathcal{G}_1^\bullet)$. Then we have*

$$\gamma_1(\partial a_3 \cup b_1) = (-1)^{n+1}\gamma_3(a_3 \cup \partial b_1)$$

in $H^{n+m}(X, \mathcal{H}^\bullet)$ where ∂ indicates the boundary map on cohomology associated to the short exact sequences of complexes above.

Proof. We will use the following conventions and notation. We think of $\mathcal{F}_1^p$ as a subsheaf of $\mathcal{F}_2^p$ and we think of $\mathcal{G}_3^q$ as a subsheaf of $\mathcal{G}_2^q$. Hence if s is a local section of $\mathcal{F}_1^p$ we use s to denote the corresponding section of $\mathcal{F}_2^p$ as well. Similarly for local sections of $\mathcal{G}_3^q$. Furthermore, if s is a local section of $\mathcal{F}_2^p$ then we denote $\bar{s}$ its image in $\mathcal{F}_3^p$. Similarly for the map $\mathcal{G}_2^q \to \mathcal{G}_1^q$. In particular if s is a local section of $\mathcal{F}_2^p$ and $\bar{s} = 0$ then s is a local section of $\mathcal{F}_1^p$. The commutativity of the diagrams above implies, for local sections s of $\mathcal{F}_2^p$ and t of $\mathcal{G}_3^q$ that $\gamma_2(s \otimes t) = \gamma_3(\bar{s} \otimes t)$ as sections of $\mathcal{H}^{p+q}$.

Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be an open covering of X. Suppose that α_3, resp. β_1 is a degree n, resp. m cocycle of $\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}_3^\bullet))$, resp. $\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{G}_1^\bullet))$ representing a_3, resp. b_1. After refining $\mathcal{U}$ if necessary, we can find cochains α_2, resp. β_2 of degree n, resp. m in $\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}_2^\bullet))$, resp. $\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{G}_2^\bullet))$ mapping to α_3, resp. β_1. Then we see that

$$\overline{d(\alpha_2)} = d(\bar{\alpha}_2) = 0 \quad \text{and} \quad \overline{d(\beta_2)} = d(\bar{\beta}_2) = 0.$$

This means that $\alpha_1 = d(\alpha_2)$ is a degree $n+1$ cocycle in $\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}_1^\bullet))$ repesenting ∂a_3. Similarly, $\beta_3 = d(\beta_2)$ is a degree $m+1$ cocycle in $\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{G}_3^\bullet))$ representing ∂b_1. Thus we may compute

$$\begin{aligned} d(\gamma_2(\alpha_2 \cup \beta_2)) &= \gamma_2(d(\alpha_2 \cup \beta_2)) \\ &= \gamma_2(d(\alpha_2) \cup \beta_2 + (-1)^n \alpha_2 \cup d(\beta_2)) \\ &= \gamma_2(\alpha_1 \cup \beta_2) + (-1)^n \gamma_2(\alpha_2 \cup \beta_3) \\ &= \gamma_1(\alpha_1 \cup \beta_1) + (-1)^n \gamma_3(\alpha_3 \cup \beta_3) \end{aligned}$$

So this even tells us that the sign is $(-1)^{n+1}$ as indicated in the lemma[1]. □

18.20. Flat resolutions

A reference for the material in this section is [**Spa88**]. Let $(X, \mathcal{O}_X)$ be a ringed space. By Modules, Lemma 15.16.6 any $\mathcal{O}_X$-module is a quotient of a flat $\mathcal{O}_X$-module. By Derived Categories, Lemma 11.15.5 any bounded above complex of $\mathcal{O}_X$-modules has a left resolution by a bounded above complex of flat $\mathcal{O}_X$-modules. However, for unbounded complexes, it turns out that flat resolutions aren't good enough.

Lemma 18.20.1. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{G}^\bullet$ be a complex of $\mathcal{O}_X$-modules. The functor*

$$K(Mod(\mathcal{O}_X)) \longrightarrow K(Mod(\mathcal{O}_X)), \quad \mathcal{F}^\bullet \longmapsto Tot(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{G}^\bullet)$$

is an exact functor of triangulated categories.

Proof. Omitted. Hint: See More on Algebra, Lemmas 12.3.1 and 12.3.2. □

[1]The sign depends on the convention for the signs in the long exact sequence in cohomology associated to a triangle in $D(X)$. The conventions in the stacks project are (a) distinguished triangles correspond to termwise split exact sequences and (b) the boundary maps in the long exact sequence are given by the maps in the snake lemma without the intervention of signs. See Derived Categories, Section 11.9.

Definition 18.20.2. Let $(X, \mathcal{O}_X)$ be a ringed space. A complex $\mathcal{K}^\bullet$ of $\mathcal{O}_X$-modules is called *K-flat* if for every acyclic complex $\mathcal{F}^\bullet$ of $\mathcal{O}_X$-modules the complex

$$\mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{K}^\bullet)$$

is acyclic.

Lemma 18.20.3. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{K}^\bullet$ be a K-flat complex. Then the functor*

$$K(Mod(\mathcal{O}_X)) \longrightarrow K(Mod(\mathcal{O}_X)), \quad \mathcal{F}^\bullet \longmapsto Tot(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{K}^\bullet)$$

transforms quasi-isomorphisms into quasi-isomorphisms.

Proof. Follows from Lemma 18.20.1 and the fact that quasi-isomorphisms are characterized by having acyclic cones. □

Lemma 18.20.4. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{K}^\bullet$ be a complex of $\mathcal{O}_X$-modules. Then $\mathcal{K}^\bullet$ is K-flat if and only if for all $x \in X$ the complex $\mathcal{K}_x^\bullet$ of $\mathcal{O}_{X,x}$ is K-flat (More on Algebra, Definition 12.3.3).*

Proof. If $\mathcal{K}_x^\bullet$ is K-flat for all $x \in X$ then we see that $\mathcal{K}^\bullet$ is K-flat because $\otimes$ and direct sums commute with taking stalks and because we can check exactness at stalks, see Modules, Lemma 15.3.1. Conversely, assume $\mathcal{K}^\bullet$ is K-flat. Pick $x \in X$ $M^\bullet$ be an acyclic complex of $\mathcal{O}_{X,x}$-modules. Then $i_{x,*}M^\bullet$ is an acyclic complex of $\mathcal{O}_X$-modules. Thus $\mathrm{Tot}(i_{x,*}M^\bullet \otimes_{\mathcal{O}_X} \mathcal{K}^\bullet)$ is acyclic. Taking stalks at x shows that $\mathrm{Tot}(M^\bullet \otimes_{\mathcal{O}_{X,x}} \mathcal{K}_x^\bullet)$ is acyclic. □

Lemma 18.20.5. *Let $(X, \mathcal{O}_X)$ be a ringed space. If $\mathcal{K}^\bullet$, $\mathcal{L}^\bullet$ are K-flat complexes of $\mathcal{O}_X$-modules, then $Tot(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet)$ is a K-flat complex of $\mathcal{O}_X$-modules.*

Proof. Follows from the isomorphism

$$\mathrm{Tot}(\mathcal{M}^\bullet \otimes_{\mathcal{O}_X} \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet)) = \mathrm{Tot}(\mathrm{Tot}(\mathcal{M}^\bullet \otimes_{\mathcal{O}_X} \mathcal{K}^\bullet) \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet)$$

and the definition. □

Lemma 18.20.6. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $(\mathcal{K}_1^\bullet, \mathcal{K}_2^\bullet, \mathcal{K}_3^\bullet)$ be a distinguished triangle in $K(Mod(\mathcal{O}_X))$. If two out of three of $\mathcal{K}_i^\bullet$ are K-flat, so is the third.*

Proof. Follows from Lemma 18.20.1 and the fact that in a distinguished triangle in $K(Mod(\mathcal{O}_X))$ if two out of three are acyclic, so is the third. □

Lemma 18.20.7. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The pullback of a K-flat complex of $\mathcal{O}_Y$-modules is a K-flat complex of $\mathcal{O}_X$-modules.*

Proof. We can check this on stalks, see Lemma 18.20.4. Hence this follows from Sheaves, Lemma 6.26.4 and More on Algebra, Lemma 12.3.5. □

Lemma 18.20.8. *Let $(X, \mathcal{O}_X)$ be a ringed space. A bounded above complex of flat $\mathcal{O}_X$-modules is K-flat.*

Proof. We can check this on stalks, see Lemma 18.20.4. Thus this lemma follows from Modules, Lemma 15.16.2 and More on Algebra, Lemma 12.3.8. □

In the following lemma by a colimit of a system of complexes we mean the termwise colimit.

Lemma 18.20.9. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{K}_1^\bullet \to \mathcal{K}_2^\bullet \to \ldots$ be a system of K-flat complexes. Then $colim_i \mathcal{K}_i^\bullet$ is K-flat.*

Proof. Because we are taking termwise colimits it is clear that

$$colim_i \operatorname{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{K}_i^\bullet) = \operatorname{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} colim_i \mathcal{K}_i^\bullet)$$

Hence the lemma follows from the fact that filtered colimits are exact. □

Lemma 18.20.10. *Let $(X, \mathcal{O}_X)$ be a ringed space. For any complex $\mathcal{G}^\bullet$ of $\mathcal{O}_X$-modules there exists a commutative diagram of complexes of $\mathcal{O}_X$-modules*

$$\begin{tikzcd} \mathcal{K}_1^\bullet \arrow[r] \arrow[d] & \mathcal{K}_2^\bullet \arrow[r] \arrow[d] & \ldots \\ \tau_{\leq 1}\mathcal{G}^\bullet \arrow[r] & \tau_{\leq 2}\mathcal{G}^\bullet \arrow[r] & \ldots \end{tikzcd}$$

with the following properties: (1) the vertical arrows are quasi-isomorphisms, (2) each $\mathcal{K}_n^\bullet$ is a bounded above complex whose terms are direct sums of $\mathcal{O}_X$-modules of the form $j_{U!}\mathcal{O}_U$, and (3) the maps $\mathcal{K}_n^\bullet \to \mathcal{K}_{n+1}^\bullet$ are termwise split injections whose cokernels are direct sums of $\mathcal{O}_X$-modules of the form $j_{U!}\mathcal{O}_U$. Moreover, the map $colim\, \mathcal{K}_n^\bullet \to \mathcal{G}^\bullet$ is a quasi-isomorphism.

Proof. The existence of the diagram and properties (1), (2), (3) follows immediately from Modules, Lemma 15.16.6 and Derived Categories, Lemma 11.27.1. The induced map $colim\, \mathcal{K}_n^\bullet \to \mathcal{G}^\bullet$ is a quasi-isomorphism because filtered colimits are exact. □

Lemma 18.20.11. *Let $(X, \mathcal{O}_X)$ be a ringed space. For any complex $\mathcal{G}^\bullet$ there exists a K-flat complex $\mathcal{K}^\bullet$ and a quasi-isomorphism $\mathcal{K}^\bullet \to \mathcal{G}^\bullet$.*

Proof. Choose a diagram as in Lemma 18.20.10. Each complex $\mathcal{K}_n^\bullet$ is a bounded above complex of flat modules, see Modules on Sites, Lemma 16.26.5. Hence $\mathcal{K}_n^\bullet$ is K-flat by Lemma 18.20.8. The induced map $colim\, \mathcal{K}_n^\bullet \to \mathcal{G}^\bullet$ is a quasi-isomorphism by construction. Since $colim\, \mathcal{K}_n^\bullet$ is K-flat by Lemma 18.20.9 we win. □

Lemma 18.20.12. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\alpha : \mathcal{P}^\bullet \to \mathcal{Q}^\bullet$ be a quasi-isomorphism of K-flat complexes of $\mathcal{O}_X$-modules. For every complex $\mathcal{F}^\bullet$ of $\mathcal{O}_X$-modules the induced map*

$$Tot(id_{\mathcal{F}^\bullet} \otimes \alpha) : Tot(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{P}^\bullet) \longrightarrow Tot(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{Q}^\bullet)$$

is a quasi-isomorphism.

Proof. Choose a quasi-isomorphism $\mathcal{K}^\bullet \to \mathcal{F}^\bullet$ with $\mathcal{K}^\bullet$ a K-flat complex, see Lemma 18.20.11. Consider the commutative diagram

$$\begin{tikzcd} \operatorname{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{P}^\bullet) \arrow[r] \arrow[d] & \operatorname{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{Q}^\bullet) \arrow[d] \\ \operatorname{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{P}^\bullet) \arrow[r] & \operatorname{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{Q}^\bullet) \end{tikzcd}$$

The result follows as by Lemma 18.20.3 the vertical arrows and the top horizontal arrow are quasi-isomorphisms. □

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}^\bullet$ be an object of $D(\mathcal{O}_X)$. Choose a K-flat resolution $\mathcal{K}^\bullet \to \mathcal{F}^\bullet$, see Lemma 18.20.11. By Lemma 18.20.1 we obtain an exact functor of triangulated categories

$$K(\mathcal{O}_X) \longrightarrow K(\mathcal{O}_X), \quad \mathcal{G}^\bullet \longmapsto \operatorname{Tot}(\mathcal{G}^\bullet \otimes_{\mathcal{O}_X} \mathcal{K}^\bullet)$$

By Lemma 18.20.3 this functor induces a functor $D(\mathcal{O}_X) \to D(\mathcal{O}_X)$ simply because $D(\mathcal{O}_X)$ is the localization of $K(\mathcal{O}_X)$ at quasi-isomorphisms. By Lemma 18.20.12 the resulting functor (up to isomorphism) does not depend on the choice of the K-flat resolution.

Definition 18.20.13. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}^\bullet$ be an object of $D(\mathcal{O}_X)$. The *derived tensor product*

$$- \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}^\bullet : D(\mathcal{O}_X) \longrightarrow D(\mathcal{O}_X)$$

is the exact functor of triangulated categories described above.

It is clear from our explicit constructions that there is a canonical isomorphism

$$\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet \cong \mathcal{G}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}^\bullet$$

for $\mathcal{G}^\bullet$ and $\mathcal{F}^\bullet$ in $D(\mathcal{O}_X)$. Hence when we write $\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet$ we will usually be agnostic about which variable we are using to define the derived tensor product with.

18.21. Derived pullback

Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. We can use K-flat resolutions to define a derived pullback functor

$$Lf^* : D(\mathcal{O}_Y) \to D(\mathcal{O}_X)$$

Namely, for every complex of $\mathcal{O}_Y$-modules $\mathcal{G}^\bullet$ we can choose a K-flat resolution $\mathcal{K}^\bullet \to \mathcal{G}^\bullet$ and set $Lf^*\mathcal{G}^\bullet = f^*\mathcal{K}^\bullet$. You can use Lemmas 18.20.7, 18.20.11, and 18.20.12 to see that this is well defined. However, to cross all the t's and dot all the i's it is perhaps more convenient to use some general theory.

Lemma 18.21.1. *The construction above is independent of choices and defines an exact functor of triangulated categories $Lf^* : D(\mathcal{O}_Y) \to D(\mathcal{O}_X)$.*

Proof. To see this we use the general theory developed in Derived Categories, Section 11.14. Set $\mathcal{D} = K(\mathcal{O}_Y)$ and $\mathcal{D}' = D(\mathcal{O}_X)$. Let us write $F : \mathcal{D} \to \mathcal{D}'$ the exact functor of triangulated categories defined by the rule $F(\mathcal{G}^\bullet) = f^*\mathcal{G}^\bullet$. We let S be the set of quasi-isomorphisms in $\mathcal{D} = K(\mathcal{O}_Y)$. This gives a situation as in Derived Categories, Situation 11.14.1 so that Derived Categories, Definition 11.14.2 applies. We claim that LF is everywhere defined. This follows from Derived Categories, Lemma 11.14.15 with $\mathcal{P} \subset Ob(\mathcal{D})$ the collection of K-flat complexes: (1) follows from Lemma 18.20.11 and to see (2) we have to show that for a quasi-isomorphism $\mathcal{K}_1^\bullet \to \mathcal{K}_2^\bullet$ between K-flat complexes of $\mathcal{O}_Y$-modules the map $f^*\mathcal{K}_1^\bullet \to f^*\mathcal{K}_2^\bullet$ is a quasi-isomorphism. To see this write this as

$$f^{-1}\mathcal{K}_1^\bullet \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \longrightarrow f^{-1}\mathcal{K}_2^\bullet \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

The functor f^{-1} is exact, hence the map $f^{-1}\mathcal{K}_1^\bullet \to f^{-1}\mathcal{K}_2^\bullet$ is a quasi-isomorphism. By Lemma 18.20.7 applied to the morphsm $(X, f^{-1}\mathcal{O}_Y) \to (Y, \mathcal{O}_Y)$ the complexes $f^{-1}\mathcal{K}_1^\bullet$ and $f^{-1}\mathcal{K}_2^\bullet$ are K-flat complexes of $f^{-1}\mathcal{O}_Y$-modules. Hence Lemma 18.20.12 guarantees that the displayed map is a quasi-isomorphism. Thus we obtain a derived functor

$$LF : D(\mathcal{O}_Y) = S^{-1}\mathcal{D} \longrightarrow \mathcal{D}' = D(\mathcal{O}_X)$$

see Derived Categories, Equation (11.14.9.1). Finally, Derived Categories, Lemma 11.14.15 also guarantees that $LF(\mathcal{K}^\bullet) = F(\mathcal{K}^\bullet) = f^*\mathcal{K}^\bullet$ when $\mathcal{K}^\bullet$ is K-flat, i.e., $Lf^* = LF$ is indeed computed in the way described above. $\square$

Lemma 18.21.2. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. There is a canonical bifunctorial isomorphism*

$$Lf^*(\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathcal{G}^\bullet) = Lf^*\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{G}^\bullet$$

for $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in Ob(D(X))$.

Proof. We may assume that $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ are K-flat complexes. In this case $\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathcal{G}^\bullet$ is just the total complex associated to the double complex $\mathcal{F}^\bullet \otimes_{\mathcal{O}_Y} \mathcal{G}^\bullet$. By Lemma 18.20.5 $\text{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_Y} \mathcal{G}^\bullet)$ is K-flat also. Hence the isomorphism of the lemma comes from the isomorphism

$$\text{Tot}(f^*\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{G}^\bullet) \longrightarrow f^*\text{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_Y} \mathcal{G}^\bullet)$$

whose constituents are the isomorphisms $f^*\mathcal{F}^p \otimes_{\mathcal{O}_X} f^*\mathcal{G}^q \to f^*(\mathcal{F}^p \otimes_{\mathcal{O}_Y} \mathcal{G}^q)$ of Modules, Lemma 15.15.4. □

18.22. Cohomology of unbounded complexes

Let $(X, \mathcal{O}_X)$ be a ringed space. The category $Mod(\mathcal{O}_X)$ is a Grothendieck abelian category: it has all colimits, filtered colimits are exact, and it has a generator, namely

$$\bigoplus\nolimits_{U \subset X \text{ open}} j_{U!}\mathcal{O}_U,$$

see Modules, Section 15.3 and Lemmas 15.16.5 and 15.16.6. By Injectives, Theorem 17.16.6 for every complex $\mathcal{F}^\bullet$ of $\mathcal{O}_X$-modules there exists an injective quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ to a K-injective complex of $\mathcal{O}_X$-modules. Hence we can define

$$R\Gamma(X, \mathcal{F}^\bullet) = \Gamma(X, \mathcal{I}^\bullet)$$

and similarly for any left exact functor, see Derived Categories, Lemma 11.28.5. For any morphism of ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ we obtain

$$Rf_* : D(X) \longrightarrow D(Y)$$

on the unbounded derived categories.

Lemma 18.22.1. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The functor Rf_* defined above and the functor Lf^* defined in Lemma 18.21.1 are adjoint:*

$$Hom_{D(X)}(Lf^*\mathcal{G}^\bullet, \mathcal{F}^\bullet) = Hom_{D(Y)}(\mathcal{G}^\bullet, Rf_*\mathcal{F}^\bullet)$$

bifunctorially in $\mathcal{F}^\bullet \in Ob(D(X))$ and $\mathcal{G}^\bullet \in Ob(D(Y))$.

Proof. This is formal from the results obtained above. Choose a K-flat resolution $\mathcal{K}^\bullet \to \mathcal{G}^\bullet$ and a K-injective resolution $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$. Then

$$\text{Hom}_{D(X)}(Lf^*\mathcal{G}^\bullet, \mathcal{F}^\bullet) = \text{Hom}_{D(X)}(f^*\mathcal{K}^\bullet, \mathcal{I}^\bullet) = \text{Hom}_{K(Mod(\mathcal{O}_X))}(f^*\mathcal{K}^\bullet, \mathcal{I}^\bullet)$$

by our definition of Lf^* and because $\mathcal{I}^\bullet$ is K-injective, see Derived Categories, Lemma 11.28.2. On the other hand

$$\text{Hom}_{D(Y)}(\mathcal{G}^\bullet, Rf_*\mathcal{F}^\bullet) = \text{Hom}_{D(Y)}(\mathcal{K}^\bullet, f_*\mathcal{I}^\bullet)$$

by our definition of Rf_*. By definition of morphisms in $D(Y)$ this is equal to

$$colim_{s : \mathcal{H}^\bullet \to \mathcal{K}^\bullet} \text{Hom}_{K(Mod(\mathcal{O}_Y))}(\mathcal{H}^\bullet, f_*\mathcal{I}^\bullet)$$

where the colimit is over all quasi-isomorphisms $s : \mathcal{H}^\bullet \to \mathcal{K}^\bullet$ of complexes of $\mathcal{O}_Y$-modules. Since every complex has a left K-flat resolution it suffices to look at quasi-isomorphisms $s : (\mathcal{K}')^\bullet \to \mathcal{K}^\bullet$ where $(\mathcal{K}')^\bullet$ is K-flat as well. In this case we have

$$\begin{aligned}\mathrm{Hom}_{K(Mod(\mathcal{O}_Y))}((\mathcal{K}')^\bullet, f_*\mathcal{I}^\bullet) &= \mathrm{Hom}_{K(Mod(\mathcal{O}_Y))}(f^*(\mathcal{K}')^\bullet, \mathcal{I}^\bullet)\\ &= \mathrm{Hom}_{K(Mod(\mathcal{O}_Y))}(f^*\mathcal{K}^\bullet, \mathcal{I}^\bullet)\end{aligned}$$

The first equality because f^* and f_* are adjoint functors and the second because $\mathcal{I}^\bullet$ is K-injective and because $f^*(\mathcal{K}')^\bullet \to f^*\mathcal{K}^\bullet$ is a quasi-isomorphism (by virtue of the fact that Lf^* is well defined). □

18.23. Producing K-injective resolutions

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}_X$-modules. The category $Mod(\mathcal{O}_X)$ has enough injectives, hence we can use Derived Categories, Lemma 11.27.3 produce a diagram

$$\begin{tikzcd} \ldots \arrow[r] & \tau_{\geq -2}\mathcal{F}^\bullet \arrow[r] \arrow[d] & \tau_{\geq -1}\mathcal{F}^\bullet \arrow[d] \\ \ldots \arrow[r] & \mathcal{I}_2^\bullet \arrow[r] & \mathcal{I}_1^\bullet \end{tikzcd}$$

in the category of complexes of $\mathcal{O}_X$-modules such that

(1) the vertical arrows are quasi-isomorphisms,
(2) $\mathcal{I}_n^\bullet$ is a bounded above complex of injectives,
(3) the arrows $\mathcal{I}_{n+1}^\bullet \to \mathcal{I}_n^\bullet$ are termwise split surjections.

The category of $\mathcal{O}_X$-modules has limits (they are computed on the level of presheaves), hence we can form the termwise limit $\mathcal{I}^\bullet = lim_n \mathcal{I}_n^\bullet$. By Derived Categories, Lemmas 11.28.3 and 11.28.6 this is a K-injective complex. In general the canonical map

$$\mathcal{F}^\bullet \to \mathcal{I}^\bullet \tag{18.23.0.1}$$

may not be a quasi-isomorphism. In the following lemma we describe some conditions under which it is.

Lemma 18.23.1. *In the situation described above. Denote $\mathcal{H}^i = H^i(\mathcal{F}^\bullet)$ the ith cohomology sheaf. Let $\mathcal{B}$ be a set of open subsets of X. Let $d \in \mathbf{N}$. Assume*

(1) *every open in X has a covering whose members are elements of $\mathcal{B}$,*
(2) *for every $U \in \mathcal{B}$ we have $H^p(U, \mathcal{H}^q) = 0$ for $p > d$*[2].

Then (18.23.0.1) is a quasi-isomorphism.

Proof. Let $U \in \mathcal{B}$. Note that $H^m(\mathcal{I}^\bullet(U))$ is the cohomology of

$$lim_n \mathcal{I}_n^{m-2}(U) \to lim_n \mathcal{I}_n^{m-1}(U) \to lim_n \mathcal{I}_n^{m}(U) \to lim_n \mathcal{I}_n^{m+1}(U)$$

in the third spot from the left. Note that the transition maps $\mathcal{I}_{n+1}^m(U) \to \mathcal{I}_n^m(U)$ are always surjective by our construction of the inverse system. By construction there are distingushed triangles in $D(\mathcal{O}_X)$

$$\mathcal{H}^{-n}[n] \to \mathcal{I}_n^\bullet \to \mathcal{I}_{n-1}^\bullet \to \mathcal{H}^{-n}[n+1]$$

By assumption (2) we see that if $m > d - n$ then

$$H^m(U, \mathcal{H}^{-n}[n]) = H^{n+m}(U, \mathcal{H}^{-n}) = 0$$

[2]In fact, analyzing the proof we see that it suffices if there exists a function $d : \mathbf{Z} \to \mathbf{Z} \cup \{+\infty\}$ such that $H^p(U, \mathcal{H}^q) = 0$ for $p > d(q)$ where $q + d(q) \to -\infty$ as $q \to -\infty$

and similarly $H^m(U, \mathcal{H}^{-n}[n+1]) = 0$ which implies that $H^m(\mathcal{I}_n^\bullet(U)) \to H^m(\mathcal{I}_{n-1}^\bullet(U))$ is an isomorphism. Thus the cohomologies of the complexes $\mathcal{I}_n^\bullet(U)$ are eventually constant in every cohomological degree. Thus we may apply Homology, Lemma 10.23.7 to conclude that

$$H^m(\mathcal{I}^\bullet(U)) = lim\, H^m(\mathcal{I}_n^\bullet(U)).$$

Using the stabilization above once again we see that

$$H^m(\mathcal{I}^\bullet(U)) = H^m(\mathcal{I}^\bullet_{\max\{1,-m+d\}}(U))$$

for every $U \in \mathcal{B}$.

Since every object of $\mathcal{C}$ has a covering whose members are elements of $\mathcal{B}$ we see that it suffices to show that the sheafification of

$$U \longmapsto H^m(\mathcal{I}^\bullet_{\max\{1,-m+d\}}(U))$$

is $\mathcal{H}^m$. But since for any complex $\mathcal{J}^\bullet$ of abelian sheaves the sheafification of $U \mapsto H^m(\mathcal{J}^\bullet(U))$ is isomorphic to the mth cohomology sheaf of $\mathcal{J}^\bullet$ this is clear from the fact that $\tau_{\leq -n}\mathcal{F}^\bullet \to \mathcal{I}_n^\bullet$ is a quasi-isomorphism. □

18.24. Other chapters

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CHAPTER 19

Cohomology on Sites

19.1. Introduction

In this document we work out some topics on cohomology of sheaves. We work out what happens for sheaves on sites, allthough often we will simply duplicate the discussion, the constructions, and the proofs from the topological case in the case. Basic references are [**MA71**], [**God73**] and [**Ive86**].

19.2. Topics

Here are some topics that should be discussed in this chapter, and have not yet been written.

(1) Cohomology of a sheaf of modules on a site is the same as the cohomology of the underlying abelian sheaf.
(2) Hypercohomology on a site.
(3) Ext-groups.
(4) Ext sheaves.
(5) Tor functors.
(6) Higher direct images for a morphism of sites.
(7) Derived pullback for morphisms between ringed sites.
(8) Cup-product.
(9) Group cohomology.
(10) Comparison of group cohomology and cohomology on $\mathcal{T}_G$.
(11) Cech cohomology on sites.
(12) Cech to cohomology spectral sequence on sites.
(13) Leray Spectral sequence for a morphism between ringed sites.
(14) Etc, etc, etc.

19.3. Cohomology of sheaves

Let $\mathcal{C}$ be a site, see Sites, Definition 9.6.2. Let $\mathcal{F}$ be a abelian sheaf on $\mathcal{C}$. We know that the category of abelian sheaves on $\mathcal{C}$ has enough injectives, see Injectives, Theorem 17.11.4. Hence we can choose an injective resolution $\mathcal{F}[0] \to \mathcal{I}^\bullet$. For any object U of the site $\mathcal{C}$ we define

$$H^i(U, \mathcal{F}) = H^i(\Gamma(U, \mathcal{I}^\bullet)) \tag{19.3.0.1}$$

to be the *ith cohomology group of the abelian sheaf $\mathcal{F}$ over the object U*. In other words, these are the right derived functors of the functor $\mathcal{F} \mapsto \mathcal{F}(U)$. The family of functors $H^i(U, -)$ forms a universal δ-functor $Ab(\mathcal{C}) \to Ab$.

It sometimes happens that the site $\mathcal{C}$ does not have a final object. In this case we define the *global sections* of a presheaf of sets $\mathcal{F}$ over $\mathcal{C}$ to be the set

$$\Gamma(\mathcal{C}, \mathcal{F}) = Mor_{PSh(\mathcal{C})}(e, \mathcal{F}) \tag{19.3.0.2}$$

where e is a final object in the category of presheaves on $\mathcal{C}$. In this case, given an abelian sheaf $\mathcal{F}$ on $\mathcal{C}$, we define the *ith cohomology group of $\mathcal{F}$ on $\mathcal{C}$* as follows

$$H^i(\mathcal{C}, \mathcal{F}) = H^i(\Gamma(\mathcal{C}, \mathcal{I}^\bullet)) \tag{19.3.0.3}$$

in other words, it is the ith right derived functor of the global sections functor. The family of functors $H^i(\mathcal{C}, -)$ forms a universal δ-functor $Ab(\mathcal{C}) \to Ab$.

Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi, see Sites, Definition 9.15.1. With $\mathcal{F}[0] \to \mathcal{I}^\bullet$ as above we define

$$R^i f_* \mathcal{F} = H^i(f_* \mathcal{I}^\bullet) \tag{19.3.0.4}$$

to be the *ith higher direct image of $\mathcal{F}$*. These are the right derived functors of f_*. The family of functors $R^i f_*$ forms a universal δ-functor from $Ab(\mathcal{C}) \to Ab(\mathcal{D})$.

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site, see Modules on Sites, Definition 16.6.1. Let $\mathcal{F}$ be an $\mathcal{O}$-module. We know that the category of $\mathcal{O}$-modules has enough injectives, see Injectives, Theorem 17.12.4. Hence we can choose an injective resolution $\mathcal{F}[0] \to \mathcal{I}^\bullet$. For any object U of the site $\mathcal{C}$ we define

$$H^i(U, \mathcal{F}) = H^i(\Gamma(U, \mathcal{I}^\bullet)) \tag{19.3.0.5}$$

to be the *the ith cohomology group of $\mathcal{F}$ over U*. The family of functors $H^i(U, -)$ forms a universal δ-functor $Mod(\mathcal{O}) \to \text{Mod}(\mathcal{O}(U))$. Similarly

$$H^i(\mathcal{C}, \mathcal{F}) = H^i(\Gamma(\mathcal{C}, \mathcal{I}^\bullet)) \tag{19.3.0.6}$$

it the *ith cohomology group of $\mathcal{F}$ on $\mathcal{C}$*. The family of functors $H^i(\mathcal{C}, -)$ forms a universal δ-functor $Mod(\mathcal{C}) \to \text{Mod}(\Gamma(\mathcal{C}, \mathcal{O}))$.

Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a morphism of ringed topoi, see Modules on Sites, Definition 16.7.1. With $\mathcal{F}[0] \to \mathcal{I}^\bullet$ as above we define

$$R^i f_* \mathcal{F} = H^i(f_* \mathcal{I}^\bullet) \tag{19.3.0.7}$$

to be the *ith higher direct image of $\mathcal{F}$*. These are the right derived functors of f_*. The family of functors $R^i f_*$ forms a universal δ-functor from $Mod(\mathcal{O}) \to Mod(\mathcal{O}')$.

19.4. Derived functors

We briefly explain an approach to right derived functors using resolution functors. Namely, suppose that $(\mathcal{C}, \mathcal{O})$ is a ringed site. In this chapter we will write

$$K(\mathcal{O}) = K(Mod(\mathcal{O})) \quad \text{and} \quad D(\mathcal{O}) = D(Mod(\mathcal{O}))$$

and similarly for the bounded versions for the triangulated categories introduced in Derived Categories, Definition 11.7.1 and Definition 11.10.3. By Derived Categories, Remark 11.23.3 there exists a resolution functor

$$j = j_{(\mathcal{C}, \mathcal{O})} : K^+(Mod(\mathcal{O})) \longrightarrow K^+(\mathcal{I})$$

where $\mathcal{I}$ is the strictly full additive subcategory of $Mod(\mathcal{O})$ which consists of injective $\mathcal{O}$-modules. For any left exact functor $F : Mod(\mathcal{O}) \to \mathcal{B}$ into any abelian category $\mathcal{B}$ we will denote RF the right derived functor of Derived Categories, Section 11.19 constructed using the resolution functor j just described:

$$RF = F \circ j' : D^+(\mathcal{O}) \longrightarrow D^+(\mathcal{B}) \tag{19.4.0.8}$$

see Derived Categories, Lemma 11.24.1 for notation. Note that we may think of RF as defined on $\mathit{Mod}(\mathcal{O})$, $\text{Comp}^+(\mathit{Mod}(\mathcal{O}))$, or $K^+(\mathcal{O})$ depending on the situation. According to Derived Categories, Definition 11.16.2 we obtain the $i$the right derived functor

$$R^iF = H^i \circ RF : \mathit{Mod}(\mathcal{O}) \longrightarrow \mathcal{B} \tag{19.4.0.9}$$

so that $R^0F = F$ and $\{R^iF, \delta\}_{i\geq 0}$ is universal δ-functor, see Derived Categories, Lemma 11.19.4.

Here are two special cases of this construction. Given a ring R we write $K(R) = K(\text{Mod}_R)$ and $D(R) = D(\text{Mod}_R)$ and similarly for the bounded versions. For any object U of $\mathcal{C}$ have a left exact functor $\Gamma(U, -) : \mathit{Mod}(\mathcal{O}) \longrightarrow \text{Mod}(\mathcal{O}(U))$ which gives rise to

$$R\Gamma(U, -) : D^+(\mathcal{O}) \longrightarrow D^+(\mathcal{O}(U))$$

by the discussion above. Note that $H^i(U, -) = R^i\Gamma(U, -)$ is compatible with (19.3.0.5) above. We similarly have

$$R\Gamma(\mathcal{C}, -) : D^+(\mathcal{O}) \longrightarrow D^+(\Gamma(\mathcal{C}, \mathcal{O}))$$

compatible with (19.3.0.6). If $f : (\mathit{Sh}(\mathcal{C}), \mathcal{O}) \to (\mathit{Sh}(\mathcal{D}), \mathcal{O}')$ is a morphism of ringed topoi then we get a left exact functor $f_* : \mathit{Mod}(\mathcal{O}) \to \mathit{Mod}(\mathcal{O}')$ which gives rise to *derived pushforward*

$$Rf_* : D^+(\mathcal{O}) \to D^+(\mathcal{O}')$$

The ith cohomology sheaf of $Rf_*\mathcal{F}^\bullet$ is denoted $R^if_*\mathcal{F}^\bullet$ and called the ith *higher direct image* in accordance with (19.3.0.7). The displayed functors above are exact functor of derived categories.

19.5. First cohomology and torsors

Definition 19.5.1. Let $\mathcal{C}$ be a site. Let $\mathcal{G}$ be a sheaf of (possibly non-commutative) groups on $\mathcal{C}$. A *pseudo torsor*, or more precisely a *pseudo $\mathcal{G}$-torsor*, is a sheaf of sets $\mathcal{F}$ on $\mathcal{C}$ endowed with an action $\mathcal{G} \times \mathcal{F} \to \mathcal{F}$ such that

(1) whenever $\mathcal{F}(U)$ is nonempty the action $\mathcal{G}(U)\times\mathcal{F}(U) \to \mathcal{F}(U)$ is simply transitive.

A *morphism of pseudo $\mathcal{G}$-torsors $\mathcal{F} \to \mathcal{F}'$* is simply a morphism of sheaves of sets compatible with the $\mathcal{G}$-actions. A *torsor*, or more precisely a *$\mathcal{G}$-torsor*, is a pseudo G-torsor such that in addition

(2) for every $U \in \mathit{Ob}(\mathcal{C})$ there exists a covering $\{U_i \to U\}_{i\in I}$ of U such that $\mathcal{F}(U_i)$ is nonempty for all $i \in I$.

A *morphism of G-torsors* is simply a morphism of pseudo G-torsors. The *trivial $\mathcal{G}$-torsor* is the sheaf $\mathcal{G}$ endowed with the obvious left $\mathcal{G}$-action.

It is clear that a morphism of torsors is automatically an isomorphism.

Lemma 19.5.2. *Let $\mathcal{C}$ be a site. Let $\mathcal{G}$ be a sheaf of (possibly non-commutative) groups on $\mathcal{C}$. A $\mathcal{G}$-torsor $\mathcal{F}$ is trivial if and only if $\Gamma(\mathcal{C}, \mathcal{F})\neq\emptyset$.*

Proof. Omitted. □

Lemma 19.5.3. *Let $\mathcal{C}$ be a site. Let $\mathcal{H}$ be an abelian sheaf on $\mathcal{C}$. There is a canonical bijection between the set of isomorphism classes of $\mathcal{H}$-torsors and $H^1(\mathcal{C}, \mathcal{H})$.*

Proof. Let $\mathcal{F}$ be a $\mathcal{H}$-torsor. Consider the free abelian sheaf $\mathbf{Z}[\mathcal{F}]$ on $\mathcal{F}$. It is the sheafification of the rule which associates to $U \in Ob(\mathcal{C})$ the collection of finite formal sums $\sum n_i[s_i]$ with $n_i \in \mathbf{Z}$ and $s_i \in \mathcal{F}(U)$. There is a natural map

$$\sigma : \mathbf{Z}[\mathcal{F}] \longrightarrow \underline{\mathbf{Z}}$$

which to a local section $\sum n_i[s_i]$ associates $\sum n_i$. The kernel of σ is generated by sections of the form $[s]-[s']$. There is a canonical map $a : \mathrm{Ker}(\sigma) \to \mathcal{H}$ which maps $[s]-[s'] \mapsto h$ where h is the local section of $\mathcal{H}$ such that $h \cdot s = s'$. Consider the push out diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ker}(\sigma) \arrow[r] \arrow[d, "a"] & \mathbf{Z}[\mathcal{F}] \arrow[r] \arrow[d] & \underline{\mathbf{Z}} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{H} \arrow[r] & \mathcal{E} \arrow[r] & \underline{\mathbf{Z}} \arrow[r] & 0 \end{tikzcd}$$

Here $\mathcal{E}$ is the extension obtained by push out. From the long exact cohomology sequence associated to the lower short exact sequence we obtain an element $\xi = \xi_{\mathcal{F}} \in H^1(\mathcal{C}, \mathcal{H})$ by applying the boundary operator to $1 \in H^0(\mathcal{C}, \underline{\mathbf{Z}})$.

Conversely, given $\xi \in H^1(\mathcal{C}, \mathcal{H})$ we can associate to ξ a torsor as follows. Choose an embedding $\mathcal{H} \to \mathcal{I}$ of $\mathcal{H}$ into an injective abelian sheaf $\mathcal{I}$. We set $\mathcal{Q} = \mathcal{I}/\mathcal{H}$ so that we have a short exact sequence

$$0 \longrightarrow \mathcal{H} \longrightarrow \mathcal{I} \longrightarrow \mathcal{Q} \longrightarrow 0$$

The element ξ is the image of a global section $q \in H^0(\mathcal{C}, \mathcal{Q})$ because $H^1(\mathcal{C}, \mathcal{I}) = 0$ (see Derived Categories, Lemma 11.19.4). Let $\mathcal{F} \subset \mathcal{I}$ be the subsheaf (of sets) of sections that map to q in the sheaf $\mathcal{Q}$. It is easy to verify that $\mathcal{F}$ is a $\mathcal{H}$-torsor.

We omit the verification that the two constructions given above are mutually inverse. □

19.6. First cohomology and extensions

Lemma 19.6.1. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules on $\mathcal{C}$. There is a canonical bijection*

$$Ext^1_{Mod(\mathcal{O})}(\mathcal{O}, \mathcal{F}) \longrightarrow H^1(\mathcal{C}, \mathcal{F})$$

which associates to the extension

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{O} \to 0$$

the image of $1 \in \Gamma(\mathcal{C}, \mathcal{O})$ in $H^1(\mathcal{C}, \mathcal{F})$.

Proof. Let us construct the inverse of the map given in the lemma. Let $\xi \in H^1(\mathcal{C}, \mathcal{F})$. Choose an injection $\mathcal{F} \subset \mathcal{I}$ with $\mathcal{I}$ injective in $Mod(\mathcal{O})$. Set $\mathcal{Q} = \mathcal{I}/\mathcal{F}$. By the long exact sequence of cohomology, we see that ξ is the image of of a section $\tilde{\xi} \in \Gamma(\mathcal{C}, \mathcal{Q}) = Hom_{\mathcal{O}}(\mathcal{O}, \mathcal{Q})$. Now, we just form the pullback

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d, equal] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{O} \arrow[r] \arrow[d, "\tilde{\xi}"] & 0 \\ 0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{I} \arrow[r] & \mathcal{Q} \arrow[r] & 0 \end{tikzcd}$$

see Homology, Section 10.4. □

The following lemma will be superceded by the more general Lemma 19.12.4.

Lemma 19.6.2. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules on $\mathcal{C}$. Let $\mathcal{F}_{ab}$ denote the underlying sheaf of abelian groups. Then there is a functorial isomorphism*

$$H^1(\mathcal{C}, \mathcal{F}_{ab}) = H^1(\mathcal{C}, \mathcal{F})$$

where the left hand side is cohomology computed in $Ab(\mathcal{C})$ and the right hand side is cohomology computed in $Mod(\mathcal{O})$.

Proof. Let $\underline{\mathbf{Z}}$ denote the constant sheaf $\mathbf{Z}$. As $Ab(\mathcal{C}) = Mod(\underline{\mathbf{Z}})$ we may apply Lemma 19.6.1 twice, and it follows that we have to show

$$\mathrm{Ext}^1_{Mod(\mathcal{O})}(\mathcal{O}, \mathcal{F}) = \mathrm{Ext}^1_{Mod(\underline{\mathbf{Z}})}(\underline{\mathbf{Z}}, \mathcal{F}_{ab}).$$

Suppose that $0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{O} \to 0$ is an extension in $Mod(\mathcal{O})$. Then we can use the obvious map of abelian sheaves $1 : \underline{\mathbf{Z}} \to \mathcal{O}$ and pullback to obtain an extension $\mathcal{E}_{ab}$, like so:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}_{ab} \arrow[r] \arrow[d, equal] & \mathcal{E}_{ab} \arrow[r] \arrow[d] & \underline{\mathbf{Z}} \arrow[r] \arrow[d, "1"] & 0 \\
0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{O} \arrow[r] & 0
\end{tikzcd}$$

The converse is a little more fun. Suppose that $0 \to \mathcal{F}_{ab} \to \mathcal{E}_{ab} \to \underline{\mathbf{Z}} \to 0$ is an extension in $Mod(\underline{\mathbf{Z}})$. Since $\underline{\mathbf{Z}}$ is a flat $\underline{\mathbf{Z}}$-module we see that the sequence

$$0 \to \mathcal{F}_{ab} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} \to \mathcal{E}_{ab} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} \to \underline{\mathbf{Z}} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} \to 0$$

is exact, see Modules on Sites, Lemma 16.26.7. Of course $\underline{\mathbf{Z}} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} = \mathcal{O}$. Hence we can push out this via the ($\mathcal{O}$-linear) multiplication map $\mu : \mathcal{F} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} \to \mathcal{F}$ to get an extension of $\mathcal{O}$ by $\mathcal{F}$, like this

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}_{ab} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} \arrow[r] \arrow[d, "\mu"] & \mathcal{E}_{ab} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} \arrow[r] \arrow[d] & \mathcal{O} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{O} \arrow[r] & 0
\end{tikzcd}$$

which is the desired extension. We omit the verification that these constructions are mutually inverse. ☐

19.7. First cohomology and invertible sheaves

The Picard group of a ringed site is defined in Modules on Sites, Section 16.28.

Lemma 19.7.1. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. There is a canonical isomorphism*

$$H^1(\mathcal{C}, \mathcal{O}^*) = Pic(\mathcal{O}).$$

of abelian groups.

Proof. Let $\mathcal{L}$ be an invertible $\mathcal{O}$-module. Consider the presheaf $\mathcal{L}^*$ defined by the rule

$$U \longmapsto \{s \in \mathcal{L}(U) \text{ such that } \mathcal{O}_U \xrightarrow{s\cdot -} \mathcal{L}_U \text{ is an isomorphism}\}$$

This presheaf satisfies the sheaf condition. Moreover, if $f \in \mathcal{O}^*(U)$ and $s \in \mathcal{L}^*(U)$, then clearly $fs \in \mathcal{L}^*(U)$. By the same token, if $s, s' \in \mathcal{L}^*(U)$ then there exists a unique $f \in \mathcal{O}^*(U)$ such that $fs = s'$. Moreover, the sheaf $\mathcal{L}^*$ has sections locally by the very definition of an invertible sheaf. In other words we see that $\mathcal{L}^*$ is a $\mathcal{O}^*$-torsor. Thus we get a map

$$\begin{matrix} \text{set of invertible sheaves on } (\mathcal{C}, \mathcal{O}) \\ \text{up to isomorphism} \end{matrix} \longrightarrow \begin{matrix} \text{set of } \mathcal{O}^*\text{-torsors} \\ \text{up to isomorphism} \end{matrix}$$

We omit the verification that this is a homomorphism of abelian groups. By Lemma 19.5.3 the right hand side is canonically bijective to $H^1(\mathcal{C}, \mathcal{O}^*)$. Thus we have to show this map is injective and surjective.

Injective. If the torsor $\mathcal{L}^*$ is trivial, this means by Lemma 19.5.2 that $\mathcal{L}^*$ has a global section. Hence this means exactly that $\mathcal{L} \cong \mathcal{O}$ is the neutral element in $\mathrm{Pic}(\mathcal{O})$.

Surjective. Let $\mathcal{F}$ be an $\mathcal{O}^*$-torsor. Consider the presheaf of sets

$$\mathcal{L}_1 : U \longmapsto (\mathcal{F}(U) \times \mathcal{O}(U))/\mathcal{O}^*(U)$$

where the action of $f \in \mathcal{O}^*(U)$ on (s, g) is $(fs, f^{-1}g)$. Then $\mathcal{L}_1$ is a presheaf of $\mathcal{O}$-modules by setting $(s, g) + (s', g') = (s, g + (s'/s)g')$ where s'/s is the local section f of $\mathcal{O}^*$ such that $fs = s'$, and $h(s, g) = (s, hg)$ for h a local section of $\mathcal{O}$. We omit the verification that the sheafification $\mathcal{L} = \mathcal{L}_1^\#$ is an invertible $\mathcal{O}$-module whose associated $\mathcal{O}^*$-torsor $\mathcal{L}^*$ is isomorphic to $\mathcal{F}$. $\square$

19.8. Locality of cohomology

The following lemma says there is no ambguity in defining the cohomology of a sheaf $\mathcal{F}$ over an object of the site.

Lemma 19.8.1. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$.*

(1) *If $\mathcal{I}$ is an injective $\mathcal{O}$-module then $\mathcal{I}|_U$ is an injective $\mathcal{O}_U$-module.*
(2) *For any sheaf of $\mathcal{O}$-modules $\mathcal{F}$ we have $H^p(U, \mathcal{F}) = H^p(\mathcal{C}/U, \mathcal{F}|_U)$.*

Proof. Recall that the functor j_U^{-1} of restriction to U is a right adjoint to the functor $j_{U!}$ of extension by 0, see Modules on Sites, Section 16.19. Moreover, $j_{U!}$ is exact. Hence (1) follows from Homology, Lemma 10.22.1.

By definition $H^p(U, \mathcal{F}) = H^p(\mathcal{I}^\bullet(U))$ where $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution in $Mod(\mathcal{O})$. By the above we see that $\mathcal{F}|_U \to \mathcal{I}^\bullet|_U$ is an injective resolution in $Mod(\mathcal{O}_U)$. Hence $H^p(U, \mathcal{F}|_U)$ is equal to $H^p(\mathcal{I}^\bullet|_U(U))$. Of course $\mathcal{F}(U) = \mathcal{F}|_U(U)$ for any sheaf $\mathcal{F}$ on $\mathcal{C}$. Hence the equality in (2). $\square$

The following lemma will be use to see what happens if we change a partial universe, or to compare cohomology of the small and big étale sites.

Lemma 19.8.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume u satisfies the hypotheses of Sites, Lemma 9.19.8. Let $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be the associated morphism of topoi. For any abelian sheaf $\mathcal{F}$ on $\mathcal{D}$ we have isomorphisms*

$$R\Gamma(\mathcal{C}, g^{-1}\mathcal{F}) = R\Gamma(\mathcal{D}, \mathcal{F}),$$

in particular $H^p(\mathcal{C}, g^{-1}\mathcal{F}) = H^p(\mathcal{D}, \mathcal{F})$ and for any $U \in Ob(\mathcal{C})$ we have isomorphisms

$$R\Gamma(U, g^{-1}\mathcal{F}) = R\Gamma(u(U), \mathcal{F}),$$

in particular $H^p(U, g^{-1}\mathcal{F}) = H^p(u(U), \mathcal{F})$. All of these isomorphisms are functorial in $\mathcal{F}$.

Proof. Since it is clear that $\Gamma(\mathcal{C}, g^{-1}\mathcal{F}) = \Gamma(\mathcal{D}, \mathcal{F})$ by hypothesis (e), it suffices to show that g^{-1} transforms injective abelian sheaves into injective abelian sheaves. As usual we use Homology, Lemma 10.22.1 to see this. The left adjoint to g^{-1} is $g_! = f^{-1}$ with the notation of Sites, Lemma 9.19.8 which is an exact functor. Hence the lemma does indeed apply. $\square$

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. Let $\varphi : U \to V$ be a morphism of $\mathcal{O}$. Then there is a canonical *restriction mapping*

$$H^n(V, \mathcal{F}) \longrightarrow H^n(U, \mathcal{F}), \quad \xi \longmapsto \xi|_U \tag{19.8.2.1}$$

functorial in $\mathcal{F}$. Namely, choose any injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. The restriction mappings of the sheaves $\mathcal{I}^p$ give a morphism of complexes

$$\Gamma(V, \mathcal{I}^\bullet) \longrightarrow \Gamma(U, \mathcal{I}^\bullet)$$

The LHS is a complex representing $R\Gamma(V, \mathcal{F})$ and the RHS is a complex representing $R\Gamma(U, \mathcal{F})$. We get the map on cohomology groups by applying the functor H^n. As indicated we will use the notation $\xi \mapsto \xi|_U$ to denote this map. Thus the rule $U \mapsto H^n(U, \mathcal{F})$ is a presheaf of $\mathcal{O}$-modules. This presheaf is customarily denoted $\underline{H}^n(\mathcal{F})$. We will give another interpretation of this presheaf in Lemma 19.11.4.

The following lemma says that it is possible to kill higher cohomology classes by going to a covering.

Lemma 19.8.3. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. Let U be an object of $\mathcal{C}$. Let $n > 0$ and let $\xi \in H^n(U, \mathcal{F})$. Then there exists a covering $\{U_i \to U\}$ of $\mathcal{C}$ such that $\xi|_{U_i} = 0$ for all $i \in I$.*

Proof. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution. Then

$$H^n(U, \mathcal{F}) = \frac{\mathrm{Ker}(\mathcal{I}^n(U) \to \mathcal{I}^{n+1}(U))}{\mathrm{Im}(\mathcal{I}^{n-1}(U) \to \mathcal{I}^n(U))}.$$

Pick an element $\tilde{\xi} \in \mathcal{I}^n(U)$ representing the cohomology class in the presentation above. Since $\mathcal{I}^\bullet$ is an injective resolution of $\mathcal{F}$ and $n > 0$ we see that the complex $\mathcal{I}^\bullet$ is exact in degree n. Hence $\mathrm{Im}(\mathcal{I}^{n-1} \to \mathcal{I}^n) = \mathrm{Ker}(\mathcal{I}^n \to \mathcal{I}^{n+1})$ as sheaves. Since $\tilde{\xi}$ is a section of the kernel sheaf over U we conclude there exists a covering $\{U_i \to U\}$ of the site such that $\tilde{\xi}|_{U_i}$ is the image under d of a section $\xi_i \in \mathcal{I}^{n-1}(U_i)$. By our definition of the restriction $\xi|_{U_i}$ as corresponding to the class of $\tilde{\xi}|_{U_i}$ we conclude. □

Lemma 19.8.4. *Let $f : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a morphism of ringed sites corresponding to the continuous functor $u : \mathcal{D} \to \mathcal{C}$. For any $\mathcal{F} \in Ob(Mod(\mathcal{O}_\mathcal{C}))$ the sheaf $R^i f_* \mathcal{F}$ is the sheaf associated to the presheaf*

$$V \longmapsto H^i(u(V), \mathcal{F})$$

Proof. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution. Then $R^i f_* \mathcal{F}$ is by definition the ith cohomology sheaf of the complex

$$f_*\mathcal{I}^0 \to f_*\mathcal{I}^1 \to f_*\mathcal{I}^2 \to \ldots$$

By definition of the abelian category structure on $\mathcal{O}_\mathcal{D}$-modules this cohomology sheaf is the sheaf associated to the presheaf

$$V \longmapsto \frac{\mathrm{Ker}(f_*\mathcal{I}^i(V) \to f_*\mathcal{I}^{i+1}(V))}{\mathrm{Im}(f_*\mathcal{I}^{i-1}(V) \to f_*\mathcal{I}^i(V))}$$

and this is obviously equal to

$$\frac{\mathrm{Ker}(\mathcal{I}^i(u(V)) \to \mathcal{I}^{i+1}(u(V)))}{\mathrm{Im}(\mathcal{I}^{i-1}(u(V)) \to \mathcal{I}^i(u(V)))}$$

which is equal to $H^i(u(V), \mathcal{F})$ and we win. □

19.9. The Cech complex and Cech cohomology

Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a family of morphisms with fixed target, see Sites, Definition 9.6.1. Let $\mathcal{F}$ be an abelian presheaf on $\mathcal{C}$. Set

$$\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) = \prod\nolimits_{(i_0, \ldots, i_p) \in I^{p+1}} \mathcal{F}(U_{i_0} \times_U \ldots \times_U U_{i_p}).$$

This is an abelian group. For $s \in \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})$ we denote $s_{i_0 \ldots i_p}$ its value in the factor $\mathcal{F}(U_{i_0} \times_U \ldots \times_U U_{i_p})$. We define

$$d : \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^{p+1}(\mathcal{U}, \mathcal{F})$$

by the formula

(19.9.0.1) $$d(s)_{i_0 \ldots i_{p+1}} = \sum\nolimits_{j=0}^{p+1} (-1)^j (U_{i_0} \times_U \ldots \times_U U_{i_{p+1}} \longrightarrow U_{i_0} \times_U \ldots \widehat{U_{i_j}} \ldots \times_U U_{i_{p+1}})^* s_{i_0 \ldots \hat{i}_j \ldots i_p}$$

It is straightforward to see that $d \circ d = 0$. In other words $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is a complex.

Definition 19.9.1. Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a family of morphisms with fixed target. Let $\mathcal{F}$ be an abelian presheaf on $\mathcal{C}$. The complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is the *Cech complex* associated to $\mathcal{F}$ and the family $\mathcal{U}$. Its cohomology groups $H^i(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}))$ are called the *Cech cohomology groups* associated to $\mathcal{F}$ and $\mathcal{U}$. They are denoted $\check{H}^i(\mathcal{U}, \mathcal{F})$.

Lemma 19.9.2. *Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be an abelian presheaf on $\mathcal{C}$. The following are equivalent*

(1) *$\mathcal{F}$ is an abelian sheaf on $\mathcal{C}$ and*
(2) *for every covering $\mathcal{U} = \{U_i \to U\}_{i \in I}$ of the site $\mathcal{C}$ the natural map*
$$\mathcal{F}(U) \to \check{H}^0(\mathcal{U}, \mathcal{F})$$
(see Sites, Section 9.10) is bijective.

Proof. This is true since the sheaf condition is exactly that $\mathcal{F}(U) \to \check{H}^0(\mathcal{U}, \mathcal{F})$ is bijective for every open covering. □

Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a family of morphisms of $\mathcal{C}$ with fixed target. Let $\mathcal{V} = \{V_j \to V\}_{j \in J}$ be another. Let $f : U \to V$, $\alpha : I \to J$ and $f_i : U_i \to V_{\alpha(i)}$ be a morphism of families of morphisms with fixed target, see Sites, Section 9.8. In this case we get a map of Cech complexes

(19.9.2.1) $$\varphi : \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

which in degree p is given by

$$\varphi(s)_{i_0 \ldots i_p} = (f_{i_0} \times \ldots \times f_{i_p})^* s_{\alpha(i_0) \ldots \alpha(i_p)}$$

19.10. Cech cohomology as a functor on presheaves

Warning: In this section we work exclusively with abelian presheaves on a category. The results are completely wrong in the setting of sheaves and categories of sheaves!

Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a family of morphisms with fixed target. Let $\mathcal{F}$ be an abelian presheaf on $\mathcal{C}$. The construction

$$\mathcal{F} \longmapsto \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

is functorial in $\mathcal{F}$. In fact, it is a functor

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, -) : PAb(\mathcal{C}) \longrightarrow \text{Comp}^+(Ab) \tag{19.10.0.2}$$

see Derived Categories, Definition 11.7.1 for notation. Recall that the category of bounded below complexes in an abelian category is an abelian category, see Homology, Lemma 10.10.9.

Lemma 19.10.1. *The functor given by Equation (19.10.0.2) is an exact functor (see Homology, Lemma 10.5.1).*

Proof. For any object W of $\mathcal{C}$ the functor $\mathcal{F} \mapsto \mathcal{F}(W)$ is an additive exact functor from $PAb(\mathcal{C})$ to Ab. The terms $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})$ of the complex are products of these exact functors and hence exact. Moreover a sequence of complexes is exact if and only if the sequence of terms in a given degree is exact. Hence the lemma follows. □

Lemma 19.10.2. *Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a family of morphisms with fixed target. The functors $\mathcal{F} \mapsto \check{H}^n(\mathcal{U}, \mathcal{F})$ form a δ-functor from the abelian category $PAb(\mathcal{C})$ to the category of $\mathbf{Z}$-modules (see Homology, Definition 10.9.1).*

Proof. By Lemma 19.10.1 a short exact sequence of abelian presheaves $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is turned into a short exact sequence of complexes of $\mathbf{Z}$-modules. Hence we can use Homology, Lemma 10.10.12 to get the boundary maps $\delta_{\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3} : \check{H}^n(\mathcal{U}, \mathcal{F}_3) \to \check{H}^{n+1}(\mathcal{U}, \mathcal{F}_1)$ and a corresponding long exact sequence. We omit the verification that these maps are compatible with maps between short exact sequences of presheaves. □

Lemma 19.10.3. *Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a family of morphisms with fixed target. Consider the chain complex $\mathbf{Z}_{\mathcal{U}, \bullet}$ of abelian presheaves*

$$\ldots \to \bigoplus_{i_0 i_1 i_2} \mathbf{Z}_{U_{i_0} \times_U U_{i_1} \times_U U_{i_2}} \to \bigoplus_{i_0 i_1} \mathbf{Z}_{U_{i_0} \times_U U_{i_1}} \to \bigoplus_{i_0} \mathbf{Z}_{U_{i_0}} \to 0 \to \ldots$$

where the last nonzero term is placed in degree 0 and where the map

$$\mathbf{Z}_{U_{i_0} \times_U \ldots \times_u U_{i_{p+1}}} \longrightarrow \mathbf{Z}_{U_{i_0} \times_U \ldots \widehat{U_{i_j}} \ldots \times_U U_{i_{p+1}}}$$

is given by $(-1)^j$ times the canonical map. Then there is an isomorphism

$$Hom_{PAb(\mathcal{C})}(\mathbf{Z}_{U, \bullet}, \mathcal{F}) = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

functorial in $\mathcal{F} \in Ob(PAb(\mathcal{C}))$.

Proof. This is a tautology based on the fact that

$$\begin{aligned} Hom_{PAb(\mathcal{C})}(\bigoplus_{i_0 \ldots i_p} \mathbf{Z}_{U_{i_0} \times_U \ldots \times_U U_{i_p}}, \mathcal{F}) &= \prod_{i_0 \ldots i_p} Hom_{PAb(\mathcal{C})}(\mathbf{Z}_{U_{i_0} \times_U \ldots \times_U U_{i_p}}, \mathcal{F}) \\ &= \prod_{i_0 \ldots i_p} \mathcal{F}(U_{i_0} \times_U \ldots \times_U U_{i_p}) \end{aligned}$$

see Modules on Sites, Lemma 16.4.2. □

Lemma 19.10.4. *Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{f_i : U_i \to U\}_{i \in I}$ be a family of morphisms with fixed target. The chain complex $\mathbf{Z}_{\mathcal{U}, \bullet}$ of presheaves of Lemma 19.10.3 above is exact in positive degrees, i.e., the homology presheaves $H_i(\mathbf{Z}_{\mathcal{U}, \bullet})$ are zero for $i > 0$.*

Proof. Let V be an object of $\mathcal{C}$. We have to show that the chain complex of abelian groups $\mathbf{Z}_{\mathcal{U},\bullet}(V)$ is exact in degrees > 0. This is the complex

$$\begin{tikzcd}
\ldots \arrow[d] \\
\bigoplus_{i_0 i_1 i_2} \mathbf{Z}[Mor_\mathcal{C}(V, U_{i_0} \times_U U_{i_1} \times_U U_{i_2})] \arrow[d] \\
\bigoplus_{i_0 i_1} \mathbf{Z}[Mor_\mathcal{C}(V, U_{i_0} \times_U U_{i_1})] \arrow[d] \\
\bigoplus_{i_0} \mathbf{Z}[Mor_\mathcal{C}(V, U_{i_0})] \arrow[d] \\
0
\end{tikzcd}$$

For any morphism $\varphi : V \to U$ denote $Mor_\varphi(V, U_i) = \{\varphi_i : V \to U_i \mid f_i \circ \varphi_i = \varphi\}$. We will use a similar notation for $Mor_\varphi(V, U_{i_0} \times_U \ldots \times_U U_{i_p})$. Note that composing with the various projection maps between the fibred products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ preserves these morphism sets. Hence we see that the complex above is the same as the complex

$$\begin{tikzcd}
\ldots \arrow[d] \\
\bigoplus_\varphi \bigoplus_{i_0 i_1 i_2} \mathbf{Z}[Mor_\varphi(V, U_{i_0} \times_U U_{i_1} \times_U U_{i_2})] \arrow[d] \\
\bigoplus_\varphi \bigoplus_{i_0 i_1} \mathbf{Z}[Mor_\varphi(V, U_{i_0} \times_U U_{i_1})] \arrow[d] \\
\bigoplus_\varphi \bigoplus_{i_0} \mathbf{Z}[Mor_\varphi(V, U_{i_0})] \arrow[d] \\
0
\end{tikzcd}$$

Next, we make the remark that we have

$$Mor_\varphi(V, U_{i_0} \times_U \ldots \times_U U_{i_p}) = Mor_\varphi(V, U_{i_0}) \times \ldots \times Mor_\varphi(V, U_{i_p})$$

Using this and the fact that $\mathbf{Z}[A] \oplus \mathbf{Z}[B] = \mathbf{Z}[A \coprod B]$ we see that the complex becomes

$$\begin{array}{c} \ldots \\ \downarrow \\ \bigoplus_\varphi \mathbf{Z}\left[\coprod_{i_0 i_1 i_2} Mor_\varphi(V, U_{i_0}) \times Mor_\varphi(V, U_{i_2})\right] \\ \downarrow \\ \bigoplus_\varphi \mathbf{Z}\left[\coprod_{i_0 i_1} Mor_\varphi(V, U_{i_0}) \times Mor_\varphi(V, U_{i_1})\right] \\ \downarrow \\ \bigoplus_\varphi \mathbf{Z}\left[\coprod_{i_0} Mor_\varphi(V, U_{i_0})\right] \\ \downarrow \\ 0 \end{array}$$

Finally, on setting $S_\varphi = \coprod_{i \in I} Mor_\varphi(V, U_i)$ we see that we get

$$\bigoplus\nolimits_\varphi \left(\ldots \to \mathbf{Z}[S_\varphi \times S_\varphi \times S_\varphi] \to \mathbf{Z}[S_\varphi \times S_\varphi] \to \mathbf{Z}[S_\varphi] \to 0 \to \ldots\right)$$

Thus we have simplified our task. Namely, it suffices to show that for any nonempty set S the (extended) complex of free abelian groups

$$\ldots \to \mathbf{Z}[S \times S \times S] \to \mathbf{Z}[S \times S] \to \mathbf{Z}[S] \xrightarrow{\Sigma} \mathbf{Z} \to 0 \to \ldots$$

is exact in all degrees. To see this fix an element $s \in S$, and use the homotopy

$$n_{(s_0, \ldots, s_p)} \longmapsto n_{(s, s_0, \ldots, s_p)}$$

with obvious notations. □

Lemma 19.10.5. *Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{f_i : U_i \to U\}_{i \in I}$ be a family of morphisms with fixed target. Let $\mathcal{O}$ be a presheaf of rings on $\mathcal{C}$. The chain complex*

$$\mathbf{Z}_{\mathcal{U}, \bullet} \otimes_{p, \mathbf{Z}} \mathcal{O}$$

is exact in positive degrees. Here $\mathbf{Z}_{\mathcal{U}, \bullet}$ is the cochain complex of Lemma 19.10.3, and the tensor product is over the constant presheaf of rings with value $\mathbf{Z}$.

Proof. Let V be an object of $\mathcal{C}$. In the proof of Lemma 19.10.4 we saw that $\mathbf{Z}_{\mathcal{U}, \bullet}(V)$ is isomorphic as a complex to a direct sum of complexes which are homotopic to $\mathbf{Z}$ placed in degree zero. Hence also $\mathbf{Z}_{\mathcal{U}, \bullet}(V) \otimes_{\mathbf{Z}} \mathcal{O}(V)$ is isomorphic as a complex to a direct sum of complexes which are homotopic to $\mathcal{O}(V)$ placed in degree zero. Or you can use Modules on Sites, Lemma 16.26.9, which applies since the presheaves $\mathbf{Z}_{\mathcal{U}, i}$ are flat, and the proof of Lemma 19.10.4 shows that $H_0(\mathbf{Z}_{\mathcal{U}, \bullet})$ is a flat presheaf also. □

Lemma 19.10.6. *Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{f_i : U_i \to U\}_{i \in I}$ be a family of morphisms with fixed target. The Cech cohomology functors $\check{H}^p(\mathcal{U}, -)$ are canonically isomomorphic as a δ-functor to the right derived functors of the functor*

$$\check{H}^0(\mathcal{U}, -) : PAb(\mathcal{C}) \longrightarrow Ab.$$

Moreover, there is a functorial quasi-isomorphism

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow R\check{H}^0(\mathcal{U}, \mathcal{F})$$

where the right hand side indicates the derived functor

$$R\check{H}^0(\mathcal{U}, -) : D^+(PAb(\mathcal{C})) \longrightarrow D^+(\mathbf{Z})$$

of the left exact functor $\check{H}^0(\mathcal{U}, -)$.

Proof. Note that the category of abelian presheaves has enough injectives, see Injectives, Proposition 17.10.1. Note that $\check{H}^0(\mathcal{U}, -)$ is a left exact functor from the category of abelian presheaves to the category of $\mathbf{Z}$-modules. Hence the derived functor and the right derived fuctor exist, see Derived Categories, Section 11.19.

Let $\mathcal{I}$ be a injective abelian presheaf. In this case the functor $Hom_{PAb(\mathcal{C})}(-, \mathcal{I})$ is exact on $PAb(\mathcal{C})$. By Lemma 19.10.3 we have

$$Hom_{PAb(\mathcal{C})}(\mathbf{Z}_{\mathcal{U},\bullet}, \mathcal{I}) = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}).$$

By Lemma 19.10.4 we have that $\mathbf{Z}_{\mathcal{U},\bullet}$ is exact in positive degrees. Hence by the exactness of Hom into $\mathcal{I}$ mentioned above we see that $\check{H}^i(\mathcal{U}, \mathcal{I}) = 0$ for all $i > 0$. Thus the δ-functor $(\check{H}^n, \delta)$ (see Lemma 19.10.2) satisfies the assumptions of Homology, Lemma 10.9.4, and hence is a universal δ-functor.

By Derived Categories, Lemma 11.19.4 also the sequence $R^i\check{H}^0(\mathcal{U}, -)$ forms a universal δ-functor. By the uniqueness of universal δ-functors, see Homology, Lemma 10.9.5 we conclude that $R^i\check{H}^0(\mathcal{U}, -) = \check{H}^i(\mathcal{U}, -)$. This is enough for most applications and the reader is suggested to skip the rest of the proof.

Let $\mathcal{F}$ be any abelian presheaf on $\mathcal{C}$. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in the category $PAb(\mathcal{C})$. Consider the double complex $A^{\bullet,\bullet}$ with terms

$$A^{p,q} = \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{I}^q).$$

Consider the simple complex $sA^\bullet$ associated to this double complex. There is a map of complexes

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow sA^\bullet$$

coming from the maps $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) \to A^{p,0} = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^0)$ and there is a map of complexes

$$\check{H}^0(\mathcal{U}, \mathcal{I}^\bullet) \longrightarrow sA^\bullet$$

coming from the maps $\check{H}^0(\mathcal{U}, \mathcal{I}^q) \to A^{0,q} = \check{\mathcal{C}}^0(\mathcal{U}, \mathcal{I}^q)$. Both of these maps are quasi-isomorphisms by an application of Homology, Lemma 10.19.6. Namely, the columns of the double complex are exact in positive degrees because the Cech complex as a functor is exact (Lemma 19.10.1) and the rows of the double complex are exact in positive degrees since as we just saw the higher Cech cohomology groups of the injective presheaves $\mathcal{I}^q$ are zero. Since quasi-isomorphisms become invertible in $D^+(\mathbf{Z})$ this gives the last displayed morphism of the lemma. We omit the verification that this morphism is functorial. □

19.11. Cech cohomology and cohomology

The relationship between cohomology and Cech cohomology comes from the fact that the Cech cohomology of an injective abelian sheaf is zero. To see this we note that an injective abelian sheaf is an injective abelian presheaf and then we apply results in Cech cohomology in the preceding section.

Lemma 19.11.1. *Let $\mathcal{C}$ be a site. An injective abelian sheaf is also injective as an object in the category $PAb(\mathcal{C})$.*

Proof. Apply Homology, Lemma 10.22.1 to the categories $\mathcal{A} = Ab(\mathcal{C})$, $\mathcal{B} = PAb(\mathcal{C})$, the inclusion functor and sheafification. (See Modules on Sites, Section 16.3 to see that all assumptions of the lemma are satisfied.) □

Lemma 19.11.2. *Let $\mathcal{C}$ be a site. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a covering of $\mathcal{C}$. Let $\mathcal{I}$ be an injective abelian sheaf, i.e., an injective object of $Ab(\mathcal{C})$. Then*

$$\check{H}^p(\mathcal{U}, \mathcal{I}) = \begin{cases} \mathcal{I}(U) & \text{if} \quad p = 0 \\ 0 & \text{if} \quad p > 0 \end{cases}$$

Proof. By Lemma 19.11.1 we see that $\mathcal{I}$ is an injective object in $PAb(\mathcal{C})$. Hence we can apply Lemma 19.10.6 (or its proof) to see the vanishing of higher Cech cohomology group. For the zeroth see Lemma 19.9.2. □

Lemma 19.11.3. *Let $\mathcal{C}$ be a site. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a covering of $\mathcal{C}$. There is a transformation*

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, -) \longrightarrow R\Gamma(U, -)$$

of functors $Ab(\mathcal{C}) \to D^+(\mathbf{Z})$. In particular this gives a transformation of functors $\check{H}^p(U, \mathcal{F}) \to H^p(U, \mathcal{F})$ for $\mathcal{F}$ ranging over $Ab(\mathcal{C})$.

Proof. Let $\mathcal{F}$ be an abelian sheaf. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. Consider the double complex $A^{\bullet,\bullet}$ with terms $A^{p,q} = \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{I}^q)$. Moreover, consider the associated simple complex $sA^\bullet$, see Homology, Definition 10.19.2. There is a map of complexes

$$\alpha : \Gamma(U, \mathcal{I}^\bullet) \longrightarrow sA^\bullet$$

coming from the maps $\mathcal{I}^q(U) \to \check{H}^0(\mathcal{U}, \mathcal{I}^q)$ and a map of complexes

$$\beta : \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow sA^\bullet$$

coming from the map $\mathcal{F} \to \mathcal{I}^0$. We can apply Homology, Lemma 10.19.6 to see that α is a quasi-isomorphism. Namely, Lemma 19.11.2 implies that the qth row of the double complex $A^{\bullet,\bullet}$ is a resolution of $\Gamma(U, \mathcal{I}^q)$. Hence α becomes invertible in $D^+(\mathbf{Z})$ and the transformation of the lemma is the composition of β followed by the inverse of α. We omit the verification that this is functorial. □

Lemma 19.11.4. *Let $\mathcal{C}$ be a site. Consider the functor $i : Ab(\mathcal{C}) \to PAb(\mathcal{C})$. It is a left exact functor with right derived functors given by*

$$R^p i(\mathcal{F}) = \underline{H}^p(\mathcal{F}) : U \longmapsto H^p(U, \mathcal{F})$$

see discussion in Section 19.8.

Proof. It is clear that i is left exact. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. By definition $R^p i$ is the pth cohomology *presheaf* of the complex $\mathcal{I}^\bullet$. In other words, the sections of $R^p i(\mathcal{F})$ over an open U are given by

$$\frac{\mathrm{Ker}(\mathcal{I}^n(U) \to \mathcal{I}^{n+1}(U))}{\mathrm{Im}(\mathcal{I}^{n-1}(U) \to \mathcal{I}^n(U))}.$$

which is the definition of $H^p(U, \mathcal{F})$. □

Lemma 19.11.5. *Let $\mathcal{C}$ be a site. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a covering of $\mathcal{C}$. For any abelian sheaf $\mathcal{F}$ there is a spectral sequence $(E_r, d_r)_{r \geq 0}$ with*

$$E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F}))$$

converging to $H^{p+q}(U, \mathcal{F})$. This spectral sequence is functorial in $\mathcal{F}$.

Proof. This is a Grothendieck spectral sequence (see Derived Categories, Lemma 11.21.2) for the functors

$$i : Ab(\mathcal{C}) \to PAb(\mathcal{C}) \quad \text{and} \quad \check{H}^0(\mathcal{U}, -) : PAb(\mathcal{C}) \to Ab.$$

Namely, we have $\check{H}^0(\mathcal{U}, i(\mathcal{F})) = \mathcal{F}(U)$ by Lemma 19.9.2. We have that $i(\mathcal{I})$ is Cech acyclic by Lemma 19.11.2. And we have that $\check{H}^p(\mathcal{U}, -) = R^p\check{H}^0(\mathcal{U}, -)$ as functors on $PAb(\mathcal{C})$ by Lemma 19.10.6. Putting everything together gives the lemma. □

Lemma 19.11.6. *Let $\mathcal{C}$ be a site. Let $\mathcal{U} = \{U_i \to U\}_{i\in I}$ be a covering. Let $\mathcal{F} \in Ob(Ab(\mathcal{C}))$. Assume that $H^i(U_{i_0} \times_U \ldots \times_U U_{i_p}, \mathcal{F}) = 0$ for all $i > 0$, all $p \geq 0$ and all $i_0, \ldots, i_p \in I$. Then $\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(U, \mathcal{F})$.*

Proof. We will use the spectral sequence of Lemma 19.11.5. The assumptions mean that $E_2^{p,q} = 0$ for all (p, q) with $q \neq 0$. Hence the spectral sequence degenerates at E_2 and the result follows. □

Lemma 19.11.7. *Let $\mathcal{C}$ be a site. Let*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

be a short exact sequence of abelian sheaves on $\mathcal{C}$. Let U be an object of $\mathcal{C}$. If there exists a cofinal system of coverings $\mathcal{U}$ of U such that $\check{H}^1(\mathcal{U}, \mathcal{F}) = 0$, then the map $\mathcal{G}(U) \to \mathcal{H}(U)$ is surjective.

Proof. Take an element $s \in \mathcal{H}(U)$. Choose a covering $\mathcal{U} = \{U_i \to U\}_{i\in I}$ such that (a) $\check{H}^1(\mathcal{U}, \mathcal{F}) = 0$ and (b) $s|_{U_i}$ is the image of a section $s_i \in \mathcal{G}(U_i)$. Since we can certainly find a covering such that (b) holds it follows from the assumptions of the lemma that we can find a covering such that (a) and (b) both hold. Consider the sections

$$s_{i_0i_1} = s_{i_1}|_{U_{i_0}\times_U U_{i_1}} - s_{i_0}|_{U_{i_0}\times_U U_{i_1}}.$$

Since s_i lifts s we see that $s_{i_0i_1} \in \mathcal{F}(U_{i_0} \times_U U_{i_1})$. By the vanishing of $\check{H}^1(\mathcal{U}, \mathcal{F})$ we can find sections $t_i \in \mathcal{F}(U_i)$ such that

$$s_{i_0i_1} = t_{i_1}|_{U_{i_0}\times_U U_{i_1}} - t_{i_0}|_{U_{i_0}\times_U U_{i_1}}.$$

Then clearly the sections $s_i - t_i$ satsify the sheaf condition and glue to a section of $\mathcal{G}$ over U which maps to s. Hence we win. □

Lemma 19.11.8. *(Variant of Cohomology, Lemma 18.11.7.) Let $\mathcal{C}$ be a site. Let $Cov_{\mathcal{C}}$ be the set of coverings of $\mathcal{C}$ (see Sites, Definition 9.6.2). Let $\mathcal{B} \subset Ob(\mathcal{C})$, and $Cov \subset Cov_{\mathcal{C}}$ be subsets. Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}$. Assume that*

1. *For every $\mathcal{U} \in Cov$, $\mathcal{U} = \{U_i \to U\}_{i\in I}$ we have $U, U_i \in \mathcal{B}$ and every $U_{i_0} \times_U \ldots \times_U U_{i_p} \in \mathcal{B}$.*
2. *For every $U \in \mathcal{B}$ the coverings of U occuring in Cov is a cofinal system of coverings of U.*
3. *For every $\mathcal{U} \in Cov$ we have $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for all $p > 0$.*

Then $H^p(U, \mathcal{F}) = 0$ for all $p > 0$ and any $U \in \mathcal{B}$.

Proof. Let $\mathcal{F}$ and Cov be as in the lemma. We will indicate this by saying ``$\mathcal{F}$ has vanishing higher Cech cohomology for any $\mathcal{U} \in$ Cov". Choose an embedding $\mathcal{F} \to \mathcal{I}$ into an injective abelian sheaf. By Lemma 19.11.2 $\mathcal{I}$ has vanishing higher Cech cohomology for any $\mathcal{U} \in$ Cov. Let $\mathcal{Q} = \mathcal{I}/\mathcal{F}$ so that we have a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{Q} \to 0.$$

By Lemma 19.11.7 and our assumption (2) this sequence gives rise to an exact sequence

$$0 \to \mathcal{F}(U) \to \mathcal{I}(U) \to \mathcal{Q}(U) \to 0.$$

for every $U \in \mathcal{B}$. Hence for any $\mathcal{U} \in \text{Cov}$ we get a short exact sequence of Cech complexes

$$0 \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{Q}) \to 0$$

since each term in the Cech complex is made up out of a product of values over elements of $\mathcal{B}$ by assumption (1). In particular we have a long exact sequence of Cech cohomology groups for any covering $\mathcal{U} \in \text{Cov}$. This implies that $\mathcal{Q}$ is also an abelian sheaf with vanishing higher Cech cohomology for all $\mathcal{U} \in \text{Cov}$.

Next, we look at the long exact cohomology sequence

$$\begin{tikzcd}
0 \arrow[r] & H^0(U, \mathcal{F}) \arrow[r] & H^0(U, \mathcal{I}) \arrow[r] & H^0(U, \mathcal{Q}) \arrow[dll] \\
 & H^1(U, \mathcal{F}) \arrow[r] & H^1(U, \mathcal{I}) \arrow[r] & H^1(U, \mathcal{Q}) \arrow[dll] \\
 & \ldots & \ldots & \ldots
\end{tikzcd}$$

for any $U \in \mathcal{B}$. Since $\mathcal{I}$ is injective we have $H^n(U, \mathcal{I}) = 0$ for $n > 0$ (see Derived Categories, Lemma 11.19.4). By the above we see that $H^0(U, \mathcal{I}) \to H^0(U, \mathcal{Q})$ is surjective and hence $H^1(U, \mathcal{F}) = 0$. Since $\mathcal{F}$ was an arbitrary abelian sheaf with vanishing higher Cech cohomology for all $\mathcal{U} \in \text{Cov}$ we conclude that also $H^1(U, \mathcal{Q}) = 0$ since $\mathcal{Q}$ is another of these sheaves (see above). By the long exact sequence this in turn implies that $H^2(U, \mathcal{F}) = 0$. And so on and so forth. □

19.12. Cohomology of modules

Everything that was said for cohomology of abelian sheaves goes for cohomology of modules, since the two agree.

Lemma 19.12.1. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. An injective sheaf of modules is also injective as an object in the category $PMod(\mathcal{O})$.*

Proof. Apply Homology, Lemma 10.22.1 to the categories $\mathcal{A} = Mod(\mathcal{O})$, $\mathcal{B} = PMod(\mathcal{O})$, the inclusion functor and sheafification. (See Modules on Sites, Section 16.11 to see that all assumptions of the lemma are satisfied.) □

Lemma 19.12.2. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Consider the functor $i : Mod(\mathcal{C}) \to PMod(\mathcal{C})$. It is a left exact functor with right derived functors given by*

$$R^p i(\mathcal{F}) = \underline{H}^p(\mathcal{F}) : U \longmapsto H^p(U, \mathcal{F})$$

see discussion in Section 19.8.

Proof. It is clear that i is left exact. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in $Mod(\mathcal{O})$. By definition $R^p i$ is the pth cohomology *presheaf* of the complex $\mathcal{I}^\bullet$. In other words, the sections of $R^p i(\mathcal{F})$ over an open U are given by

$$\frac{\text{Ker}(\mathcal{I}^n(U) \to \mathcal{I}^{n+1}(U))}{\text{Im}(\mathcal{I}^{n-1}(U) \to \mathcal{I}^n(U))}.$$

which is the definition of $H^p(U, \mathcal{F})$. □

Lemma 19.12.3. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a covering of $\mathcal{C}$. Let $\mathcal{I}$ be an injective $\mathcal{O}$-module, i.e., an injective object of $Mod(\mathcal{O})$. Then*

$$\check{H}^p(\mathcal{U}, \mathcal{I}) = \begin{cases} \mathcal{I}(U) & \text{if} \quad p = 0 \\ 0 & \text{if} \quad p > 0 \end{cases}$$

Proof. Lemma 19.10.3 gives the first equality in the following sequence of equalities

$$\begin{aligned} \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}) &= Mor_{PAb(\mathcal{C})}(\mathbf{Z}_{\mathcal{U},\bullet}, \mathcal{I}) \\ &= Mor_{PMod(\mathbf{Z})}(\mathbf{Z}_{\mathcal{U},\bullet}, \mathcal{I}) \\ &= Mor_{PMod(\mathcal{O})}(\mathbf{Z}_{\mathcal{U},\bullet} \otimes_{p,\mathbf{Z}} \mathcal{O}, \mathcal{I}) \end{aligned}$$

The third equality by Modules on Sites, Lemma 16.9.2. By Lemma 19.12.1 we see that $\mathcal{I}$ is an injective object in $PMod(\mathcal{O})$. Hence $Hom_{PMod(\mathcal{O})}(-, \mathcal{I})$ is an exact functor. By Lemma 19.10.5 we see the vanishing of higher Cech cohomology groups. For the zeroth see Lemma 19.9.2. □

Lemma 19.12.4. *Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. Let $\mathcal{F}$ be an $\mathcal{O}$-module, and denote $\mathcal{F}_{ab}$ the underlying sheaf of abelian groups. Then we have*

$$H^i(\mathcal{C}, \mathcal{F}_{ab}) = H^i(\mathcal{C}, \mathcal{F})$$

and for any object U of $\mathcal{C}$ we also have

$$H^i(U, \mathcal{F}_{ab}) = H^i(U, \mathcal{F}).$$

Here the left hand side is cohomology computed in $Ab(\mathcal{C})$ and the right hand side is cohomology computed in $Mod(\mathcal{O})$.

Proof. By Derived Categories, Lemma 11.19.4 the δ-funcor $(\mathcal{F} \mapsto H^p(U, \mathcal{F}))_{p \geq 0}$ is universal. The functor $Mod(\mathcal{O}) \to Ab(\mathcal{C})$, $\mathcal{F} \mapsto \mathcal{F}_{ab}$ is exact. Hence $(\mathcal{F} \mapsto H^p(U, \mathcal{F}_{ab}))_{p \geq 0}$ is a δ-functor also. Suppose we show that $(\mathcal{F} \mapsto H^p(U, \mathcal{F}_{ab}))_{p \geq 0}$ is also universal. This will imply the second statement of the lemma by uniqueness of universal δ-functors, see Homology, Lemma 10.9.5. Since $Mod(\mathcal{O})$ has enough injectives, it suffices to show that $H^i(U, \mathcal{I}_{ab}) = 0$ for any injective object $\mathcal{I}$ in $Mod(\mathcal{O})$, see Homology, Lemma 10.9.4.

Let $\mathcal{I}$ be an injective object of $Mod(\mathcal{O})$. Apply Lemma 19.11.8 with $\mathcal{F} = \mathcal{I}$, $\mathcal{B} = \mathcal{C}$ and $\text{Cov} = \text{Cov}_{\mathcal{C}}$. Assumption (3) of that lemma holds by Lemma 19.12.3. Hence we see that $H^i(U, \mathcal{I}_{ab}) = 0$ for every object U of $\mathcal{C}$.

If $\mathcal{C}$ has a final object then this also implies the first equality. If not, then according to Sites, Lemma 9.25.5 we see that the ringed topos $(Sh(\mathcal{C}), \mathcal{O})$ is equivalent to a ringed topos where the underlying site does have a final object. Hence the lemma follows. □

Lemma 19.12.5. *Cohomology and products. Let $\mathcal{F}_i$ be a family of abelian sheaves on a site $\mathcal{C}$. Then there are canonical maps*

$$H^p(U, \prod\nolimits_{i \in I} \mathcal{F}_i) \longrightarrow \prod\nolimits_{i \in I} H^p(U, \mathcal{F}_i)$$

for any object U of $\mathcal{C}$. For $p = 0$ this map is an isomorphism and for $p = 1$ this map is injective.

Proof. Choose injective resolutions $\mathcal{F}_i \to \mathcal{I}_i^\bullet$. Then $\mathcal{F} = \prod \mathcal{F}_i$ maps to the complex $(\prod \mathcal{I}_i)^\bullet$ which consists of injectives, see Homology, Lemma 10.20.3. Choose an injective resolution $\mathcal{F} \to \mathcal{J}^\bullet$. There exists a map of complexes $\beta : \mathcal{J}^\bullet \to (\prod \mathcal{I}_i)^\bullet$ which induces the identity on $\prod \mathcal{F}_i$, see Derived Categories, Lemma 11.17.6. Since $\Gamma(U, \prod \mathcal{I}_i^p) =$

$\prod \Gamma(U, \mathcal{I}_i^p)$ and since H^p commutes with products (see Homology, Lemma 10.24.1) we obtain a canonical map

$$H^p(U, \prod \mathcal{F}_i) = H^p(\Gamma(U, \mathcal{I}^\bullet)) \longrightarrow H^p(\Gamma(U, (\prod \mathcal{I}_i)^\bullet)) = \prod H^p(U, \mathcal{F}_i).$$

To prove the assertion for H^1, pick an element $\xi \in H^1(U, \mathcal{F})$ which maps to zero in $\prod H^1(U, \mathcal{F}_i)$. By locality of cohomology, see Lemma 19.8.3, there exists a covering $\mathcal{U} = \{U_j \to U\}$ such that $\xi|_{U_j} = 0$ for all j. Hence ξ comes from an element of $\check{\xi} \in \check{H}^1(\mathcal{U}, \mathcal{F})$ in the spectral sequence of Lemma 19.11.5. Since the edge maps $\check{H}^1(\mathcal{U}, \mathcal{F}_i) \to H^1(U, \mathcal{F}_i)$ are injective for all i, and since the image of ξ is zero in $\prod H^1(U, \mathcal{F}_i)$ we see that the image $\check{\xi}_i = 0$ in $\check{H}^1(\mathcal{U}, \mathcal{F}_i)$. However, since $\mathcal{F} = \prod \mathcal{F}_i$ we see that $\check{\mathcal{C}}(\mathcal{U}, \mathcal{F})$ is the product of the complexes $\check{\mathcal{C}}(\mathcal{U}, \mathcal{F}_i)$, hence by Homology, Lemma 10.24.1 we conclude that $\check{\xi} = 0$ as desired. $\square$

19.13. Limp sheaves

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K be a sheaf of sets on $\mathcal{C}$ (we intentionally use a roman capital here to distinguish from abelian sheaves). Given an abelian sheaf $\mathcal{F}$ we denote $\mathcal{F}(K) = Mor_{Sh(\mathcal{C})}(K, \mathcal{F})$. The functor $\mathcal{F} \mapsto \mathcal{F}(K)$ is a left exact functor $Mod(\mathcal{O}) \to Ab$ hence we have its right derived functors. We will denote these $H^p(K, \mathcal{F})$ so that $H^0(K, \mathcal{F}) = \mathcal{F}(K)$.

We mention two special cases. The first is the case where $K = h_U^\#$ for some object U of $\mathcal{C}$. In this case $H^p(K, \mathcal{F}) = H^p(U, \mathcal{F})$, because $Mor_{Sh(\mathcal{C})}(h_U^\#, \mathcal{F}) = \mathcal{F}(U)$, see Sites, Section 9.12. The second is the case $\mathcal{O} = \mathbf{Z}$ (the constant sheaf). In this case the cohomology groups are functors $H^p(K, -) : Ab(\mathcal{C}) \to Ab$. Here is the analogue of Lemma 19.12.4.

Lemma 19.13.1. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K be a sheaf of sets on $\mathcal{C}$. Let $\mathcal{F}$ be an $\mathcal{O}$-module and denote $\mathcal{F}_{ab}$ the underlying sheaf of abelian groups. Then $H^p(K, \mathcal{F}) = H^p(K, \mathcal{F}_{ab})$.*

Proof. Note that both $H^p(K, \mathcal{F})$ and $H^p(K, \mathcal{F}_{ab})$ depend only on the topos, not on the underlying site. Hence by Sites, Lemma 9.25.5 we may replace $\mathcal{C}$ by a ``larger'' site such that $K = h_U$ for some object U of $\mathcal{C}$. In this case the result follows from Lemma 19.12.4. $\square$

Lemma 19.13.2. *Let $\mathcal{C}$ be a site. Let $K' \to K$ be a surjective map of sheaves of sets on $\mathcal{C}$. Set $K'_p = K' \times_K \ldots \times_K K'$ ($p + 1$-factors). For every abelian sheaf $\mathcal{F}$ there is a spectral sequence with $E_1^{p,q} = H^q(K'_p, \mathcal{F})$ converging to $H^{p+q}(K, \mathcal{F})$.*

Proof. After replacing $\mathcal{C}$ by a ``larger'' site as in Sites, Lemma 9.25.5 we may assume that K, K' are objects of $\mathcal{C}$ and that $\mathcal{U} = \{K' \to K\}$ is a covering. Then we have the Čech to cohomology spectral sequence of Lemma 19.11.5 whose E_1 page is as indicated in the statement of the lemma. $\square$

Lemma 19.13.3. *Let $\mathcal{C}$ be a site. Let K be a sheaf of sets on $\mathcal{C}$. Consider the morphism of topoi $j : Sh(\mathcal{C}/K) \to Sh(\mathcal{C})$, see Sites, Lemma 9.26.3. Then j^{-1} preserves injectives and $H^p(K, \mathcal{F}) = H^p(\mathcal{C}/K, j^{-1}\mathcal{F})$ for any abelian sheaf $\mathcal{F}$ on $\mathcal{C}$.*

Proof. By Sites, Lemmas 9.26.1 and 9.26.3 the morphism of topoi j is equivalent to a localization. Hence this follows from Lemma 19.8.1. $\square$

Keeping in mind Lemma 19.13.1 we see that the following definition is the ``correct one'' also for sheaves of modules on ringed sites.

Definition 19.13.4. Let $\mathcal{C}$ be a site. We say an abelian sheaf $\mathcal{F}$ is *limp*[1] if for every sheaf of sets K we have $H^p(K, \mathcal{F}) = 0$ for all $p \geq 1$.

It is clear that being limp is an intrinsic property, i.e., preserved under equivalences of topoi. A limp sheaf has vanishing higher cohomology on all objects of the site, but in general the condition of being limp is strictly stronger. Here is a characterization of limp sheaves which is sometimes useful.

Lemma 19.13.5. *Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be an abelian sheaf. If*

(1) $H^p(U, \mathcal{F}) = 0$ *for* $p > 0$, *and*
(2) *for every surjection $K' \to K$ of sheaves of sets the extended Čech complex*

$$0 \to H^0(K, \mathcal{F}) \to H^0(K', \mathcal{F}) \to H^0(K' \times_K K', \mathcal{F}) \to \ldots$$

is exact,

then $\mathcal{F}$ is limp (and the converse holds too).

Proof. By assumption (1) we have $H^p(h_U^\#, g^{-1}\mathcal{I}) = 0$ for all $p > 0$ and all objects U of $\mathcal{C}$. Note that if $K = \coprod K_i$ is a coproduct of sheaves of sets on $\mathcal{C}$ then $H^p(K, g^{-1}\mathcal{I}) = \prod H^p(K_i, g^{-1}\mathcal{I})$. For any sheaf of sets K there exists a surjection

$$K' = \coprod h_{U_i}^\# \longrightarrow K$$

see Sites, Lemma 9.12.4. Thus we conclude that: (*) for every sheaf of sets K there exists a surjection $K' \to K$ of sheaves of sets such that $H^p(K', \mathcal{F}) = 0$ for $p > 0$. We claim that (*) and condition (2) imply that $\mathcal{F}$ is limp. Note that conditions (*) and (2) only depend on $\mathcal{F}$ as an object of the topos $Sh(\mathcal{C})$ and not on the underlying site. (We will not use property (1) in the rest of the proof.)

We are going to prove by induction on $n \geq 0$ that (*) and (2) imply the following induction hypothesis IH_n: $H^p(K, \mathcal{F}) = 0$ for all $0 < p \leq n$ and all sheaves of sets K. Note that IH_0 holds. Assume IH_n. Pick a sheaf of sets K. Pick a surjection $K' \to K$ such that $H^p(K', \mathcal{F}) = 0$ for all $p > 0$. We have a spectral sequence with

$$E_1^{p,q} = H^q(K'_p, \mathcal{F})$$

convering to $H^{p+q}(K, \mathcal{F})$, see Lemma 19.13.2. By IH_n we see that $E_1^{p,q} = 0$ for $0 < q \leq n$ and by assumption (2) we see that $E_2^{p,0} = 0$ for $p > 0$. Finally, we have $E_1^{0,q} = 0$ for $q > 0$ because $H^q(K', \mathcal{F}) = 0$ by choice of K'. Hence we conclude that $H^{n+1}(K, \mathcal{F}) = 0$ because all the terms $E_2^{p,q}$ with $p + q = n + 1$ are zero. $\square$

19.14. The Leray spectral sequence

The key to proving the existence of the Leray spectral sequence is the following lemma.

Lemma 19.14.1. *Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Then for any injective object $\mathcal{I}$ in $Mod(\mathcal{O}_\mathcal{C})$ the pushforward $f_*\mathcal{I}$ is limp.*

Proof. Let K be a sheaf of sets on $\mathcal{D}$. By Modules on Sites, Lemma 16.7.2 we may replace $\mathcal{C}$, $\mathcal{D}$ by ``larger'' sites such that f comes from a morphism of ringed sites induced by a continuous functor $u : \mathcal{D} \to \mathcal{C}$ such that $K = h_V$ for some object V of $\mathcal{D}$.

[1]This is probably nonstandard notation. Please email stacks.project@gmail.com if you know the correct terminology.

Thus we have to show that $H^q(V, f_*\mathcal{I})$ is zero for $q > 0$ and all objects V of $\mathcal{D}$ when f is given by a morphism of ringed sites. Let $\mathcal{V} = \{V_j \to V\}$ be any covering of $\mathcal{D}$. Since u is continuous we see that $\mathcal{U} = \{u(V_j) \to u(v)\}$ is a covering of $\mathcal{C}$. Then we have an equality of Čech complexes

$$\check{\mathcal{C}}^\bullet(\mathcal{V}, f_*\mathcal{I}) = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I})$$

by the definition of f_*. By Lemma 19.12.3 we see that the cohomology of this complex is zero in positive degrees. We win by Lemma 19.11.8. □

For flat morphisms the functor f_* preserves injective modules. In particular the functor $f_* : Ab(\mathcal{C}) \to Ab(\mathcal{D})$ always transforms injective abelian sheaves into injective abelian sheaves.

Lemma 19.14.2. *Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. If f is flat, then $f_*\mathcal{I}$ is an injective $\mathcal{O}_\mathcal{D}$-module for any injective $\mathcal{O}_\mathcal{C}$-module $\mathcal{I}$.*

Proof. In this case the functor f^* is exact, see Modules on Sites, Lemma 16.27.2. Hence the result follows from Homology, Lemma 10.22.1. □

Lemma 19.14.3. *Let $(Sh(\mathcal{C}), \mathcal{O}_\mathcal{C})$ be a ringed topos. A limp sheaf is right acyclic for the following functors:*

1. *the functor $H^0(U, -)$ for any object U of $\mathcal{C}$,*
2. *the functor $\mathcal{F} \mapsto \mathcal{F}(K)$ for any presheaf of sets K,*
3. *the functor $\Gamma(\mathcal{C}, -)$ of global sections,*
4. *the functor f_* for any morphism $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ of ringed topoi.*

Proof. Part (2) is the definition of a limp sheaf. Part (1) is a consequence of (2) as pointed out in the discussion following the definition of limp sheaves. Part (3) is a special case of (2) where $K = e$ is the final object of $Sh(\mathcal{C})$.

To prove (4) we may assume, by Modules on Sites, Lemma 16.7.2 that f is given by a morphism of sites. In this case we see that $R^i f_*$, $i > 0$ of a limp sheaf are zero by the description of higher direct images in Lemma 19.8.4. □

Lemma 19.14.4. *(Leray spectral sequence.) Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let $\mathcal{F}^\bullet$ be a bounded below complex of $\mathcal{O}_\mathcal{C}$-modules. There is a spectral sequence*

$$E_2^{p,q} = H^p(\mathcal{D}, R^q f_*(\mathcal{F}^\bullet))$$

converging to $H^{p+q}(\mathcal{C}, \mathcal{F}^\bullet)$.

Proof. This is just the Grothendieck spectral sequence Derived Categories, Lemma 11.21.2 coming from the composition of functors $\Gamma(\mathcal{C}, -) = \Gamma(\mathcal{D}, -) \circ f_*$. To see that the assumptions of Derived Categories, Lemma 11.21.2 are satisfied, see Lemmas 19.14.1 and 19.14.3. □

Lemma 19.14.5. *Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let $\mathcal{F}$ be an $\mathcal{O}_\mathcal{C}$-module.*

1. *If $R^q f_*\mathcal{F} = 0$ for $q > 0$, then $H^p(\mathcal{C}, \mathcal{F}) = H^p(\mathcal{D}, f_*\mathcal{F})$ for all p.*
2. *If $H^p(\mathcal{D}, R^q f_*\mathcal{F}) = 0$ for all q and $p > 0$, then $H^q(\mathcal{C}, \mathcal{F}) = H^0(\mathcal{D}, R^q f_*\mathcal{F})$ for all q.*

Proof. These are two simple conditions that force the Leray spectral sequence to converge. You can also prove these facts directly (without using the spectral sequence) which is a good exercise in cohomology of sheaves. □

Lemma 19.14.6. *(Relative Leray spectral sequence.) Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ and $g : (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D}) \to (Sh(\mathcal{E}), \mathcal{O}_\mathcal{E})$ be morphisms of ringed topoi. Let $\mathcal{F}$ be an $\mathcal{O}_\mathcal{C}$-module. There is a spectral sequence with*

$$E_2^{p,q} = R^p g_*(R^q f_* \mathcal{F})$$

converging to $R^{p+q}(g \circ f)_* \mathcal{F}$. This spectral sequence is functorial in $\mathcal{F}$, and there is a version for bounded below complexes of $\mathcal{O}_\mathcal{C}$-modules.

Proof. This is a Grothendieck spectral sequence for composition of functors, see Derived Categories, Lemma 11.21.2 and Lemmas 19.14.1 and 19.14.3. ☐

19.15. The base change map

In this section we construct the base change map in some cases; the general case is treated in Remark 19.19.2. The discussion in this section avoids using derived pullback by restricting to the case of a base change by a flat morphism of ringed sites. Before we state the result, let us discuss flat pullback on the derived category. Suppose $g : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ is a flat morphism of ringed topoi. By Modules on Sites, Lemma 16.27.2 the functor $g^* : Mod(\mathcal{O}_\mathcal{D}) \to Mod(\mathcal{O}_\mathcal{C})$ is exact. Hence it has a derived functor

$$g^* : D(\mathcal{O}_\mathcal{C}) \to D(\mathcal{O}_\mathcal{D})$$

which is computed by simply pulling back an representative of a given object in $D(\mathcal{O}_\mathcal{D})$, see Derived Categories, Lemma 11.16.8. It preserved the bounded (above, below) subcategories. Hence as indicated we indicate this functor by g^* rather than Lg^*.

Lemma 19.15.1. *Let*

$$\begin{tikzcd} (Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) \arrow[r, "g'"'] \arrow[d, "f'"'] & (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \arrow[d, "f"] \\ (Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'}) \arrow[r, "g"] & (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D}) \end{tikzcd}$$

be a commutative diagram of ringed topoi. Let $\mathcal{F}^\bullet$ be a bounded below complex of $\mathcal{O}_\mathcal{C}$-modules. Assume both g and g' are flat. Then there exists a canonical base change map

$$g^* Rf_* \mathcal{F}^\bullet \longrightarrow R(f')_*(g')^* \mathcal{F}^\bullet$$

in $D^+(\mathcal{O}_{\mathcal{D}'})$.

Proof. Choose injective resolutions $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ and $(g')^*\mathcal{F}^\bullet \to \mathcal{J}^\bullet$. By Lemma 19.14.2 we see that $(g')_*\mathcal{J}^\bullet$ is a complex of injectives representing $R(g')_*(g')^*\mathcal{F}^\bullet$. Hence by Derived Categories, Lemmas 11.17.6 and 11.17.7 the arrow β in the diagram

$$\begin{tikzcd} (g')_*(g')^*\mathcal{F}^\bullet \arrow[r] & (g')_*\mathcal{J}^\bullet \\ \mathcal{F}^\bullet \arrow[r] \arrow[u, "adjunction"] & \mathcal{I}^\bullet \arrow[u, "\beta"'] \end{tikzcd}$$

exists and is unique up to homotopy. Pushing down to $\mathcal{D}$ we get

$$f_*\beta : f_*\mathcal{I}^\bullet \longrightarrow f_*(g')_*\mathcal{J}^\bullet = g_*(f')_*\mathcal{J}^\bullet$$

By adjunction of g^* and g_* we get a map of complexes $g^*f_*\mathcal{I}^\bullet \to (f')_*\mathcal{J}^\bullet$. Note that this map is unique up to homotopy since the only choice in the whole process was the choice of the map β and everything was done on the level of complexes. ☐

19.16. Cohomology and colimits

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \to Mod(\mathcal{O})$, $i \mapsto \mathcal{F}_i$ be a diagram over the index category $\mathcal{I}$, see Categories, Section 4.13. For each i there is a canonical map $\mathcal{F}_i \to colim_i \mathcal{F}_i$ which induces a map on cohomology. Hence we get a canonical map

$$colim_i H^p(U, \mathcal{F}_i) \longrightarrow H^p(U, colim_i \mathcal{F}_i)$$

for every $p \geq 0$ and every object U of $\mathcal{C}$. These maps are in general not isomorphisms, even for $p = 0$.

To repeat the arguments given in the case of topological spaces we will say that an object U of a site $\mathcal{C}$ is *quasi-compact* if every covering of U in $\mathcal{C}$ can be refined by a finite covering.

Lemma 19.16.1. *Let $\mathcal{C}$ be a site. Let $\mathcal{I} \to Sh(\mathcal{C})$, $i \mapsto \mathcal{F}_i$ be a filtered diagram of sheaves of sets. Let $U \in Ob(\mathcal{C})$. Consider the canonical map*

$$\Psi : colim_i \mathcal{F}_i(U) \longrightarrow \left(colim_i \mathcal{F}_i\right)(U)$$

With the terminology introduced above:

(1) *If all the transition maps are injective then Ψ is injective for any U.*
(2) *If U is quasi-compact, then Ψ is injective.*
(3) *If U is quasi-compact and all the transition maps are injective then Ψ is an isomorphism.*
(4) *If U has a cofinal system of coverings $\{U_j \to U\}_{j \in J}$ with J finite and $U_j \times_U U_{j'}$ quasi-compact for all $j, j' \in J$, then Ψ is bijective.*

Proof. Assume all the transition maps are injective. In this case the presheaf $\mathcal{F}' : V \mapsto colim_i \mathcal{F}_i(V)$ is separated (see Sites, Definition 9.10.9). By Sites, Lemma 9.10.13 we have $(\mathcal{F}')^\# = colim_i \mathcal{F}_i$. By Sites, Theorem 9.10.10 we see that $\mathcal{F}' \to (\mathcal{F}')^\#$ is injective. This proves (1).

Assume U is quasi-compact. Suppose that $s \in \mathcal{F}_i(U)$ and $s' \in \mathcal{F}_{i'}(U)$ give rise to elements on the left hand side which have the same image under Ψ. Since U is quasi-compact this means there exists a finite covering $\{U_j \to U\}_{j=1,\ldots,m}$ and for each j an index $i_j \in I$, $i_j \geq i$, $i_j \geq i'$ such that $\varphi_{ii_j}(s) = \varphi_{i'i_j}(s')$. Let $i'' \in I$ be $\geq$ than all of the i_j. We conclude that $\varphi_{ii''}(s)$ and $\varphi_{i'i''}(s)$ agree on U_j for all j and hence that $\varphi_{ii''}(s) = \varphi_{i'i''}(s)$. This proves (2).

Assume U is quasi-compact and all transition maps injective. Let s be an element of the target of Ψ. Since U is quasi-compact there exists a finite covering $\{U_j \to U\}_{j=1,\ldots,m}$, for each j an index $i_j \in I$ and $s_j \in \mathcal{F}_{i_j}(U_j)$ such that $s|_{U_j}$ comes from s_j for all j. Pick $i \in I$ which is $\geq$ than all of the i_j. By (1) the sections $\varphi_{i_j i}(s_j)$ agree over $U_j \times_U U_{j'}$. Hence they glue to a section $s' \in \mathcal{F}_i(U)$ which maps to s under Ψ. This proves (3).

Assume the hypothesis of (4). Let s be an element of the target of Ψ. By assumption there exists a finite covering $\{U_j \to U\}_{j=1,\ldots,m} U_j$, with $U_j \times_U U_{j'}$ quasi-compact for all $j, j' \in J$ and for each j an index $i_j \in I$ and $s_j \in \mathcal{F}_{i_j}(U_j)$ such that $s|_{U_j}$ is the image of s_j for all j. Since $U_j \times_U U_{j'}$ is quasi-compact we can apply (2) and we see that there exists an $i_{jj'} \in I$, $i_{jj'} \geq i_j$, $i_{jj'} \geq i_{j'}$ such that $\varphi_{i_j i_{jj'}}(s_j)$ and $\varphi_{i_{j'} i_{jj'}}(s_{j'})$ agree over $U_j \times_U U_{j'}$. Choose an index $i \in I$ wich is bigger or equal than all the $i_{jj'}$. Then we see that the sections $\varphi_{i_j i}(s_j)$ of $\mathcal{F}_i$ glue to a section of $\mathcal{F}_i$ over U. This section is mapped to the element s as desired. $\square$

The following lemma is the analogue of the previous lemma for cohomology.

Lemma 19.16.2. *Let $\mathcal{C}$ be a site. Let $Cov_{\mathcal{C}}$ be the set of coverings of $\mathcal{C}$ (see Sites, Definition 9.6.2). Let $\mathcal{B} \subset Ob(\mathcal{C})$, and $Cov \subset Cov_{\mathcal{C}}$ be subsets. Assume that*

(1) *For every $\mathcal{U} \in Cov$ we have $\mathcal{U} = \{U_i \to U\}_{i \in I}$ with I finite, $U, U_i \in \mathcal{B}$ and every $U_{i_0} \times_U \ldots \times_U U_{i_p} \in \mathcal{B}$.*
(2) *For every $U \in \mathcal{B}$ the coverings of U occuring in Cov is a cofinal system of coverings of U.*

Then the map

$$colim_i\, H^p(U, \mathcal{F}_i) \longrightarrow H^p(U, colim_i\, \mathcal{F}_i)$$

is an isomorphism for every $p \geq 0$, every $U \in \mathcal{B}$, and every filtered diagram $\mathcal{I} \to Ab(\mathcal{C})$.

Proof. To prove the lemma we will argue by induction on p. Note that we require in (1) the coverings $\mathcal{U} \in$ Cov to be finite, so that all the elements of $\mathcal{B}$ are quasi-compact. Hence (2) and (1) imply that any $U \in \mathcal{B}$ satsifies the hypothesis of Lemma 19.16.1 (4). Thus we see that the result holds for $p = 0$. Now we assume the lemma holds for p and prove it for $p + 1$.

Choose a filtered diagram $\mathcal{F} : \mathcal{I} \to Ab(\mathcal{C})$, $i \mapsto \mathcal{F}_i$. Since $Ab(\mathcal{C})$ has functorial injective embeddings, see Injectives, Theorem 17.11.4, we can find a morphism of filtered diagrams $\mathcal{F} \to \mathcal{I}$ such that each $\mathcal{F}_i \to \mathcal{I}_i$ is an injective map of abelian sheaves into an injective abelian sheaf. Denote $\mathcal{Q}_i$ the cokernel so that we have short exact sequences

$$0 \to \mathcal{F}_i \to \mathcal{I}_i \to \mathcal{Q}_i \to 0.$$

Since colimits of sheaves are the sheafification of colimits on the level of preshease, since sheafification is exact, and since filtered colimits of abelian groups are exact (see Algebra, Lemma 7.8.9), we see the sequence

$$0 \to colim_i\, \mathcal{F}_i \to colim_i\, \mathcal{I}_i \to colim_i\, \mathcal{Q}_i \to 0.$$

is also a short exact sequence. We claim that $H^q(U, colim_i\, \mathcal{I}_i) = 0$ for all $U \in \mathcal{B}$ and all $q \geq 1$. Accepting this claim for the moment consider the diagram

$$\begin{tikzcd}
colim_i\, H^p(U, \mathcal{I}_i) \arrow[r] \arrow[d] & colim_i\, H^p(U, \mathcal{Q}_i) \arrow[r] \arrow[d] & colim_i\, H^{p+1}(U, \mathcal{F}_i) \arrow[r] \arrow[d] & 0 \arrow[d] \\
H^p(U, colim_i\, \mathcal{I}_i) \arrow[r] & H^p(U, colim_i\, \mathcal{Q}_i) \arrow[r] & H^{p+1}(U, colim_i\, \mathcal{F}_i) \arrow[r] & 0
\end{tikzcd}$$

The zero at the lower right corner comes from the claim and the zero at the upper right corner comes from the fact that the sheaves $\mathcal{I}_i$ are injective. The top row is exact by an application of Algebra, Lemma 7.8.9. Hence by the snake lemma we deduce the result for $p + 1$.

It remains to show that the claim is true. We will use Lemma 19.11.8. By the result for $p = 0$ we see that for $\mathcal{U} \in$ Cov we have

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, colim_i\, \mathcal{I}_i) = colim_i\, \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}_i)$$

because all the $U_{j_0} \times_U \ldots \times_U U_{j_p}$ are in $\mathcal{B}$. By Lemma 19.11.2 each of the complexes in the colimit of Cech complexes is acyclic in degree ≥ 1. Hence by Algebra, Lemma 7.8.9 we see that also the Cech complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, colim_i\, \mathcal{I}_i)$ is acyclic in degrees ≥ 1. In other words we see that $\check{H}^p(\mathcal{U}, colim_i\, \mathcal{I}_i) = 0$ for all $p \geq 1$. Thus the assumptions of Lemma 19.11.8. are satisfied and the claim follows. $\square$

19.17. Flat resolutions

In this section we redo the arguments of Cohomology, Section 18.20 in the setting of ringed sites and ringed topoi.

Lemma 19.17.1. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{G}^\bullet$ be a complex of $\mathcal{O}$-modules. The functor*

$$K(Mod(\mathcal{O})) \longrightarrow K(Mod(\mathcal{O})), \quad \mathcal{F}^\bullet \longmapsto Tot(\mathcal{F}^\bullet \otimes_\mathcal{O} \mathcal{G}^\bullet)$$

is an exact functor of triangulated categories.

Proof. Omitted. Hint: See More on Algebra, Lemmas 12.3.1 and 12.3.2. □

Definition 19.17.2. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. A complex $\mathcal{K}^\bullet$ of $\mathcal{O}$-modules is called *K-flat* if for every acyclic complex $\mathcal{F}^\bullet$ of $\mathcal{O}$-modules the complex

$$\text{Tot}(\mathcal{F}^\bullet \otimes_\mathcal{O} \mathcal{K}^\bullet)$$

is acyclic.

Lemma 19.17.3. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{K}^\bullet$ be a K-flat complex. Then the functor*

$$K(Mod(\mathcal{O})) \longrightarrow K(Mod(\mathcal{O})), \quad \mathcal{F}^\bullet \longmapsto Tot(\mathcal{F}^\bullet \otimes_\mathcal{O} \mathcal{K}^\bullet)$$

transforms quasi-isomorphisms into quasi-isomorphisms.

Proof. Follows from Lemma 19.17.1 and the fact that quasi-isomorphisms are characterized by having acyclic cones. □

Lemma 19.17.4. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. If $\mathcal{K}^\bullet$, $\mathcal{L}^\bullet$ are K-flat complexes of $\mathcal{O}$-modules, then $Tot(\mathcal{K}^\bullet \otimes_\mathcal{O} \mathcal{L}^\bullet)$ is a K-flat complex of $\mathcal{O}$-modules.*

Proof. Follows from the isomorphism

$$\text{Tot}(\mathcal{M}^\bullet \otimes_\mathcal{O} \text{Tot}(\mathcal{K}^\bullet \otimes_\mathcal{O} \mathcal{L}^\bullet)) = \text{Tot}(\text{Tot}(\mathcal{M}^\bullet \otimes_\mathcal{O} \mathcal{K}^\bullet) \otimes_\mathcal{O} \mathcal{L}^\bullet)$$

and the definition. □

Lemma 19.17.5. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{K}_1^\bullet, \mathcal{K}_2^\bullet, \mathcal{K}_3^\bullet)$ be a distinguished triangle in $K(Mod(\mathcal{O}))$. If two out of three of $\mathcal{K}_i^\bullet$ are K-flat, so is the third.*

Proof. Follows from Lemma 19.17.1 and the fact that in a distinguished triangle in $K(Mod(\mathcal{O}))$ if two out of three are acyclic, so is the third. □

Lemma 19.17.6. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed space. A bounded above complex of flat $\mathcal{O}$-modules is K-flat.*

Proof. Let $\mathcal{K}^\bullet$ be a bounded above complex of flat $\mathcal{O}$-modules. Let $\mathcal{L}^\bullet$ be an acyclic complex of $\mathcal{O}$-modules. Note that $\mathcal{L}^\bullet = colim_m \tau_{\leq m}\mathcal{L}^\bullet$ where we take termwise colimits. Hence also

$$\text{Tot}(\mathcal{K}^\bullet \otimes_\mathcal{O} \mathcal{L}^\bullet) = colim_m \text{Tot}(\mathcal{K}^\bullet \otimes_\mathcal{O} \tau_{\leq m}\mathcal{L}^\bullet)$$

termwise. Hence to prove the complex on the left is acyclic it suffices to show each of the complexes on the right is acyclic. Since $\tau_{\leq m}\mathcal{L}^\bullet$ is acyclic this reduces us to the case where $\mathcal{L}^\bullet$ is bounded above. In this case the spectral sequence of Homology, Lemma 10.19.5 has

$${}'E_1^{p,q} = H^p(\mathcal{L}^\bullet \otimes_R \mathcal{K}^q)$$

which is zero as $\mathcal{K}^q$ is flat and $\mathcal{L}^\bullet$ acyclic. Hence we win. □

Lemma 19.17.7. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{K}_1^\bullet \to \mathcal{K}_2^\bullet \to \ldots$ be a system of K-flat complexes. Then $colim_i \mathcal{K}_i^\bullet$ is K-flat.*

Proof. Because we are taking termwise colimits it is clear that

$$colim_i \operatorname{Tot}(\mathcal{F}^\bullet \otimes_\mathcal{O} \mathcal{K}_i^\bullet) = \operatorname{Tot}(\mathcal{F}^\bullet \otimes_\mathcal{O} colim_i \mathcal{K}_i^\bullet)$$

Hence the lemma follows from the fact that filtered colimits are exact. □

Lemma 19.17.8. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. For any complex $\mathcal{G}^\bullet$ of $\mathcal{O}$-modules there exists a commutative diagram of complexes of $\mathcal{O}$-modules*

$$\begin{tikzcd} \mathcal{K}_1^\bullet \arrow[r] \arrow[d] & \mathcal{K}_2^\bullet \arrow[r] \arrow[d] & \ldots \\ \tau_{\leq 1}\mathcal{G}^\bullet \arrow[r] & \tau_{\leq 2}\mathcal{G}^\bullet \arrow[r] & \ldots \end{tikzcd}$$

with the following properties: (1) the vertical arrows are quasi-isomorphisms, (2) each $\mathcal{K}_n^\bullet$ is a bounded above complex whose terms are direct sums of $\mathcal{O}$-modules of the form $j_{U!}\mathcal{O}_U$, and (3) the maps $\mathcal{K}_n^\bullet \to \mathcal{K}_{n+1}^\bullet$ are termwise split injections whose cokernels are direct sums of $\mathcal{O}$-modules of the form $j_{U!}\mathcal{O}_U$. Moreover, the map $colim\, \mathcal{K}_n^\bullet \to \mathcal{G}^\bullet$ is a quasi-isomorphism.

Proof. The existence of the diagram and properties (1), (2), (3) follows immediately from Modules on Sites, Lemma 16.26.6 and Derived Categories, Lemma 11.27.1. The induced map $colim\, \mathcal{K}_n^\bullet \to \mathcal{G}^\bullet$ is a quasi-isomorphism because filtered colimits are exact. □

Lemma 19.17.9. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. For any complex $\mathcal{G}^\bullet$ of $\mathcal{O}$-modules there exists a K-flat complex $\mathcal{K}^\bullet$ and a quasi-isomorphism $\mathcal{K}^\bullet \to \mathcal{G}^\bullet$.*

Proof. Choose a diagram as in Lemma 19.17.8. Each complex $\mathcal{K}_n^\bullet$ is a bounded above complex of flat modules, see Modules on Sites, Lemma 16.26.5. Hence $\mathcal{K}_n^\bullet$ is K-flat by Lemma 19.17.6. The induced map $colim\, \mathcal{K}_n^\bullet \to \mathcal{G}^\bullet$ is a quasi-isomorphism by construction. Since $colim\, \mathcal{K}_n^\bullet$ is K-flat by Lemma 19.17.7 we win. □

Lemma 19.17.10. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\alpha : \mathcal{P}^\bullet \to \mathcal{Q}^\bullet$ be a quasi-isomorphism of K-flat complexes of $\mathcal{O}$-modules. For every complex $\mathcal{F}^\bullet$ of $\mathcal{O}$-modules the induced map*

$$Tot(id_{\mathcal{F}^\bullet} \otimes \alpha) : Tot(\mathcal{F}^\bullet \otimes_\mathcal{O} \mathcal{P}^\bullet) \longrightarrow Tot(\mathcal{F}^\bullet \otimes_\mathcal{O} \mathcal{Q}^\bullet)$$

is a quasi-isomorphism.

Proof. Choose a quasi-isomorphism $\mathcal{K}^\bullet \to \mathcal{F}^\bullet$ with $\mathcal{K}^\bullet$ a K-flat complex, see Lemma 19.17.9. Consider the commutative diagram

$$\begin{tikzcd} \operatorname{Tot}(\mathcal{K}^\bullet \otimes_\mathcal{O} \mathcal{P}^\bullet) \arrow[r] \arrow[d] & \operatorname{Tot}(\mathcal{K}^\bullet \otimes_\mathcal{O} \mathcal{Q}^\bullet) \arrow[d] \\ \operatorname{Tot}(\mathcal{F}^\bullet \otimes_\mathcal{O} \mathcal{P}^\bullet) \arrow[r] & \operatorname{Tot}(\mathcal{F}^\bullet \otimes_\mathcal{O} \mathcal{Q}^\bullet) \end{tikzcd}$$

The result follows as by Lemma 19.17.3 the vertical arrows and the top horizontal arrow are quasi-isomorphisms. □

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}^\bullet$ be an object of $D(\mathcal{O})$. Choose a K-flat resolution $\mathcal{K}^\bullet \to \mathcal{F}^\bullet$, see Lemma 19.17.9. By Lemma 19.17.1 we obtain an exact functor of triangulated categories

$$K(\mathcal{O}) \longrightarrow K(\mathcal{O}), \quad \mathcal{G}^\bullet \longmapsto \operatorname{Tot}(\mathcal{G}^\bullet \otimes_\mathcal{O} \mathcal{K}^\bullet)$$

By Lemma 19.17.3 this functor induces a functor $D(\mathcal{O}) \to D(\mathcal{O})$ simply because $D(\mathcal{O})$ is the localization of $K(\mathcal{O})$ at quasi-isomorphisms. By Lemma 19.17.10 the resulting functor (up to isomorphism) does not depend on the choice of the K-flat resolution.

Definition 19.17.11. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}^\bullet$ be an object of $D(\mathcal{O})$. The *derived tensor product*

$$- \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{F}^\bullet : D(\mathcal{O}) \longrightarrow D(\mathcal{O})$$

is the exact functor of triangulated categories described above.

It is clear from our explicit constructions that there is a canonical isomorphism

$$\mathcal{F}^\bullet \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{G}^\bullet \cong \mathcal{G}^\bullet \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{F}^\bullet$$

for $\mathcal{G}^\bullet$ and $\mathcal{F}^\bullet$ in $D(\mathcal{O})$. Hence when we write $\mathcal{F}^\bullet \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{G}^\bullet$ we will usually be agnostic about which variable we are using to define the derived tensor product with.

19.18. Derived pullback

Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. We can use K-flat resolutions to define a derived pullback functor

$$Lf^* : D(\mathcal{O}') \to D(\mathcal{O})$$

However, we have to be a little careful since we haven't yet proved the pullback of a flat module is flat in complete generality, see Modules on Sites, Section 16.33. In this section, we will use the hypothesis that our sites have enough points, but once we improve the result of the aforementioned section, all of the results in this section will hold without the assumption on the existence of points.

Lemma 19.18.1. *Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{C}')$ be a morphism of topoi. Let $\mathcal{O}'$ be a sheaf of rings on $\mathcal{C}'$. Assume $\mathcal{C}$ has enough points. For any complex of $\mathcal{O}'$-modules $\mathcal{G}^\bullet$, there exists a quasi-isomorphism $\mathcal{K}^\bullet \to \mathcal{G}^\bullet$ such that $\mathcal{K}^\bullet$ is a K-flat complex of $\mathcal{O}'$-modules and $f^{-1}\mathcal{K}^\bullet$ is a K-flat complex of $f^{-1}\mathcal{O}'$-modules.*

Proof. In the proof of Lemma 19.17.9 we find a quasi-isomorphism $\mathcal{K}^\bullet = colim_i \mathcal{K}_i^\bullet \to \mathcal{G}^\bullet$ where each $\mathcal{K}_i^\bullet$ is a bounded above complex of flat $\mathcal{O}'$-modules. By Modules on Sites, Lemma 16.33.3 applied to the morphism of ringed topoi $(Sh(\mathcal{C}), f^{-1}\mathcal{O}') \to (Sh(\mathcal{C}'), \mathcal{O}')$ we see that $f^{-1}\mathcal{F}_i^\bullet$ is a bounded above complex of flat $f^{-1}\mathcal{O}'$-modules. Hence $f^{-1}\mathcal{K}^\bullet = colim_i f^{-1}\mathcal{K}_i^\bullet$ is K-flat by Lemmas 19.17.6 and 19.17.7. □

Remark 19.18.2. It is straightforward to show that the pullback of a K-flat complex is K-flat for a morphism of ringed topoi with enough points; this slightly improves the result of Lemma 19.18.1. However, in applications it seems rather that the explicit form of the K-flat complexes constructed in Lemma 19.17.9 is what is useful (as in the proof above) and not the plain fact that they are K-flat. Note for example that the terms of the complex constructed are each direct sums of modules of the form $j_{U!}\mathcal{O}_U$, see Lemma 19.17.8.

Lemma 19.18.3. *Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. Assume $\mathcal{C}$ has enough points. There exists an exact functor*

$$Lf^* : D(\mathcal{O}') \longrightarrow D(\mathcal{O})$$

of triangulated categories so that $Lf^*\mathcal{K}^\bullet = f^*\mathcal{K}^\bullet$ for any complex as in Lemma 19.18.1 in particular for any bounded above complex of flat $\mathcal{O}'$-modules.

Proof. To see this we use the general theory developed in Derived Categories, Section 11.14. Set $\mathcal{D} = K(\mathcal{O}')$ and $\mathcal{D}' = D(\mathcal{O})$. Let us write $F : \mathcal{D} \to \mathcal{D}'$ the exact functor of triangulated categories defined by the rule $F(\mathcal{G}^\bullet) = f^*\mathcal{G}^\bullet$. We let S be the set of quasi-isomorphisms in $\mathcal{D} = K(\mathcal{O}')$. This gives a situation as in Derived Categories, Situation 11.14.1 so that Derived Categories, Definition 11.14.2 applies. We claim that LF is everywhere defined. This follows from Derived Categories, Lemma 11.14.15 with $\mathcal{P} \subset Ob(\mathcal{D})$ the collection of complexes $\mathcal{K}^\bullet$ such that $f^{-1}\mathcal{K}^\bullet$ is a K-flat complex of $f^{-1}\mathcal{O}'$-modules: (1) follows from Lemma 19.18.1 and to see (2) we have to show that for a quasi-isomorphism $\mathcal{K}_1^\bullet \to \mathcal{K}_2^\bullet$ between elements of $\mathcal{P}$ the map $f^*\mathcal{K}_1^\bullet \to f^*\mathcal{K}_2^\bullet$ is a quasi-isomorphism. To see this write this as

$$f^{-1}\mathcal{K}_1^\bullet \otimes_{f^{-1}\mathcal{O}'} \mathcal{O} \longrightarrow f^{-1}\mathcal{K}_2^\bullet \otimes_{f^{-1}\mathcal{O}'} \mathcal{O}$$

The functor f^{-1} is exact, hence the map $f^{-1}\mathcal{K}_1^\bullet \to f^{-1}\mathcal{K}_2^\bullet$ is a quasi-isomorphism. The complexes $f^{-1}\mathcal{K}_1^\bullet$ and $f^{-1}\mathcal{K}_2^\bullet$ are K-flat complexes of $f^{-1}\mathcal{O}'$-modules by our choice of $\mathcal{P}$. Hence Lemma 19.17.10 guarantees that the displayed map is a quasi-isomorphism. Thus we obtain a derived functor

$$LF : D(\mathcal{O}') = S^{-1}\mathcal{D} \longrightarrow \mathcal{D}' = D(\mathcal{O})$$

see Derived Categories, Equation (11.14.9.1). Finally, Derived Categories, Lemma 11.14.15 also guarantees that $LF(\mathcal{K}^\bullet) = F(\mathcal{K}^\bullet) = f^*\mathcal{K}^\bullet$ when $\mathcal{K}^\bullet$ is in $\mathcal{P}$. Since the proof of Lemma 19.18.1 shows that bounded above complexes of flat modules are in $\mathcal{P}$ we win. ☐

Lemma 19.18.4. *Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a morphism of ringed topoi. Assume $\mathcal{C}$ has enough points. There is a canonical bifunctorial isomorphism*

$$Lf^*(\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}'} \mathcal{G}^\bullet) = Lf^*\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}} Lf^*\mathcal{G}^\bullet$$

for $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in Ob(D(\mathcal{O}'))$.

Proof. We may assume that $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ are K-flat complexes of $\mathcal{O}'$-modules. In this case $\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}'} \mathcal{G}^\bullet$ is just the total complex associated to the double complex $\mathcal{F}^\bullet \otimes_{\mathcal{O}'} \mathcal{G}^\bullet$. By Lemma 19.17.4 $\text{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}'} \mathcal{G}^\bullet)$ is K-flat also. Hence the isomorphism of the lemma comes from the isomorphism

$$\text{Tot}(f^*\mathcal{F}^\bullet \otimes_{\mathcal{O}} f^*\mathcal{G}^\bullet) \longrightarrow f^*\text{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}'} \mathcal{G}^\bullet)$$

whose constituents are the isomorphisms $f^*\mathcal{F}^p \otimes_{\mathcal{O}} f^*\mathcal{G}^q \to f^*(\mathcal{F}^p \otimes_{\mathcal{O}'} \mathcal{G}^q)$ of Modules on Sites, Lemma 16.24.1. ☐

19.19. Cohomology of unbounded complexes

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. The category $Mod(\mathcal{O})$ is a Grothendieck abelian category: it has all colimits, filtered colimits are exact, and it has a generator, namely

$$\bigoplus\nolimits_{U \in Ob(\mathcal{C})} j_{U!}\mathcal{O}_U,$$

see Modules on Sites, Section 16.14 and Lemmas 16.26.5 and 16.26.6. By Injectives, Theorem 17.16.6 for every complex $\mathcal{F}^\bullet$ of $\mathcal{O}$-modules there exists an injective quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ to a K-injective complex of $\mathcal{O}$-modules. Hence we can define

$$R\Gamma(\mathcal{C}, \mathcal{F}^\bullet) = \Gamma(\mathcal{C}, \mathcal{I}^\bullet)$$

and similarly for any left exact functor, see Derived Categories, Lemma 11.28.5. For any morphism of ringed topoi $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ we obtain

$$Rf_* : D(\mathcal{O}) \longrightarrow D(\mathcal{O}')$$

on the unbounded derived categories.

Lemma 19.19.1. *Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a morphism of ringed topoi. Assume $\mathcal{C}$ has enough points. The functor Rf_* defined above and the functor Lf^* defined in Lemma 19.18.3 are adjoint:*

$$Hom_{D(\mathcal{O})}(Lf^*\mathcal{G}^\bullet, \mathcal{F}^\bullet) = Hom_{D(\mathcal{O}')}(\mathcal{G}^\bullet, Rf_*\mathcal{F}^\bullet)$$

bifunctorially in $\mathcal{F}^\bullet \in Ob(D(\mathcal{O}))$ and $\mathcal{G}^\bullet \in Ob(D(\mathcal{O}'))$.

Proof. This is formal from the results obtained above. Choose a K-flat resolution $\mathcal{K}^\bullet \to \mathcal{G}^\bullet$ and a K-injective resolution $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$. Then

$$\mathrm{Hom}_{D(\mathcal{O})}(Lf^*\mathcal{G}^\bullet, \mathcal{F}^\bullet) = \mathrm{Hom}_{D(\mathcal{O})}(f^*\mathcal{K}^\bullet, \mathcal{I}^\bullet) = \mathrm{Hom}_{K(Mod(\mathcal{O}))}(f^*\mathcal{K}^\bullet, \mathcal{I}^\bullet)$$

by our definition of Lf^* and because $\mathcal{I}^\bullet$ is K-injective, see Derived Categories, Lemma 11.28.2. On the other hand

$$\mathrm{Hom}_{D(\mathcal{O}')}(\mathcal{G}^\bullet, Rf_*\mathcal{F}^\bullet) = \mathrm{Hom}_{D(\mathcal{O}')}(\mathcal{K}^\bullet, f_*\mathcal{I}^\bullet)$$

by our definition of Rf_*. By definition of morphisms in $D(\mathcal{O}')$ this is equal to

$$colim_{s : \mathcal{H}^\bullet \to \mathcal{K}^\bullet} \mathrm{Hom}_{K(Mod(\mathcal{O}'))}(\mathcal{H}^\bullet, f_*\mathcal{I}^\bullet)$$

where the colimit is over all quasi-isomorphisms $s : \mathcal{H}^\bullet \to \mathcal{K}^\bullet$ of complexes of $\mathcal{O}'$-modules. Since every complex has a left K-flat resolution it suffices to look at quasi-isomorphisms $s : (\mathcal{K}')^\bullet \to \mathcal{K}^\bullet$ where $(\mathcal{K}')^\bullet$ is K-flat as well. In this case we have

$$\begin{aligned} \mathrm{Hom}_{K(Mod(\mathcal{O}'))}((\mathcal{K}')^\bullet, f_*\mathcal{I}^\bullet) &= \mathrm{Hom}_{K(Mod(\mathcal{O}'))}(f^*(\mathcal{K}')^\bullet, \mathcal{I}^\bullet) \\ &= \mathrm{Hom}_{K(Mod(\mathcal{O}'))}(f^*\mathcal{K}^\bullet, \mathcal{I}^\bullet) \end{aligned}$$

The first equality because f^* and f_* are adjoint functors and the second because $\mathcal{I}^\bullet$ is K-injective and because $f^*(\mathcal{K}')^\bullet \to f^*\mathcal{K}^\bullet$ is a quasi-isomorphism (by virtue of the fact that Lf^* is well defined). $\square$

Remark 19.19.2. The construction of unbounded derived functor Lf^* and Rf_* allows one to construct the base change map in full generality. Namely, suppose that

$$\begin{tikzcd} (Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) \arrow[r, "g'"'] \arrow[d, "f'"'] & (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \arrow[d, "f"] \\ (Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'}) \arrow[r, "g"] & (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}}) \end{tikzcd}$$

is a commutative diagram of ringed topoi. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}_{\mathcal{C}}$-modules. Then there exists a canonical base change map

$$Lg^*Rf_*\mathcal{F}^\bullet \longrightarrow R(f')_*L(g')^*\mathcal{F}^\bullet$$

in $D(\mathcal{O}_{\mathcal{D}'})$. Namely, this map is adjoint to a map $L(f')^*Lg^*Rf_*\mathcal{F}^\bullet \to L(g')^*\mathcal{F}^\bullet$ Since $L(f')^*Lg^* = L(g')^*Lf^*$ we see this is the same as a map $L(g')^*Lf^*Rf_*\mathcal{F}^\bullet \to L(g')^*\mathcal{F}^\bullet$ which we can take to be $L(g')^*$ of the adjunction map $Lf^*Rf_*\mathcal{F}^\bullet \to \mathcal{F}^\bullet$.

19.20. Producing K-injective resolutions

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}$-modules. The category $Mod(\mathcal{O})$ has enough injectives, hence we can use Derived Categories, Lemma 11.27.3 produce a diagram

$$\begin{tikzcd} \ldots \arrow[r] & \tau_{\geq -2}\mathcal{F}^\bullet \arrow[r] \arrow[d] & \tau_{\geq -1}\mathcal{F}^\bullet \arrow[d] \\ \ldots \arrow[r] & \mathcal{I}_2^\bullet \arrow[r] & \mathcal{I}_1^\bullet \end{tikzcd}$$

in the category of complexes of $\mathcal{O}$-modules such that

(1) the vertical arrows are quasi-isomorphisms,
(2) $\mathcal{I}_n^\bullet$ is a bounded below complex of injectives,
(3) the arrows $\mathcal{I}_{n+1}^\bullet \to \mathcal{I}_n^\bullet$ are termwise split surjections.

The category of $\mathcal{O}$-modules has limits (they are computed on the level of presheaves), hence we can form the termwise limit $\mathcal{I}^\bullet = lim_n \mathcal{I}_n^\bullet$. By Derived Categories, Lemmas 11.28.3 and 11.28.6 this is a K-injective complex. In general the canonical map

$$\mathcal{F}^\bullet \to \mathcal{I}^\bullet \tag{19.20.0.1}$$

may not be a quasi-isomorphism. In the following lemma we describe some conditions under which it is.

Lemma 19.20.1. *In the situation described above. Denote $\mathcal{H}^i = H^i(\mathcal{F}^\bullet)$ the ith cohomology sheaf. Let $\mathcal{B} \subset Ob(\mathcal{C})$ be a subset. Let $d \in \mathbf{N}$. Assume*

(1) *every object of $\mathcal{C}$ has a covering whose members are elements of $\mathcal{B}$,*
(2) *for every $U \in \mathcal{B}$ we have $H^p(U, \mathcal{H}^q) = 0$ for $p > d$*[2]*.*

Then (19.20.0.1) is a quasi-isomorphism.

Proof. Let $U \in \mathcal{B}$. Note that $H^m(\mathcal{I}^\bullet(U))$ is the cohomology of

$$lim_n \mathcal{I}_n^{m-2}(U) \to lim_n \mathcal{I}_n^{m-1}(U) \to lim_n \mathcal{I}_n^m(U) \to lim_n \mathcal{I}_n^{m+1}(U)$$

in the third spot from the left. Note that the transition maps $\mathcal{I}_{n+1}^m(U) \to \mathcal{I}_n^m(U)$ are always surjective because the maps $\mathcal{I}_{n+1}^\bullet \to \mathcal{I}_n^\bullet$ are termwise split surjections. By construction there are distingushed triangles

$$\mathcal{H}^{-n}[n] \to \mathcal{I}_n^\bullet \to \mathcal{I}_{n-1}^\bullet \to \mathcal{H}^{-n}[n+1]$$

in $D(\mathcal{O})$. As $\mathcal{I}_n^\bullet$ is a bounded below complex of injectives we have $H^m(U, \mathcal{I}_n^\bullet) = H^m(\mathcal{I}_n^\bullet(U))$. By assumption (2) we see that if $m > d - n$ then

$$H^m(U, \mathcal{H}^{-n}[n]) = H^{n+m}(U, \mathcal{H}^{-n}) = 0$$

and similarly $H^m(U, \mathcal{H}^{-n}[n+1]) = 0$. This implies that $H^m(\mathcal{I}_n^\bullet(U)) \to H^m(\mathcal{I}_{n-1}^\bullet(U))$ is an isomorphism for $m > d - n$. In other words, the cohomologies of the complexes $\mathcal{I}_n^\bullet(U)$ are eventually constant in every cohomological degree. Thus we may apply Homology, Lemma 10.23.7 to conclude that

$$H^m(\mathcal{I}^\bullet(U)) = lim\, H^m(\mathcal{I}_n^\bullet(U)).$$

Using the eventual stabilization once again we see that

$$H^m(\mathcal{I}^\bullet(U)) = H^m(\mathcal{I}_{\max(1,-m+d)}^\bullet(U))$$

[2] In fact, analyzing the proof we see that it suffices if there exists a function $d : \mathbf{Z} \to \mathbf{Z} \cup \{+\infty\}$ such that $H^p(U, \mathcal{H}^q) = 0$ for $p > d(q)$ where $q + d(q) \to -\infty$ as $q \to -\infty$

for every $U \in \mathcal{B}$.

We want to show that the map $\mathcal{H}^m \to H^m(\mathcal{J}^\bullet)$ is an isomorphism for all m. The sheaf $H^m(\mathcal{J}^\bullet)$ is the sheafification of the presheaf $U \mapsto H^m(\mathcal{J}^\bullet(U))$. We have seen above that this presheaf equals the presheaf

$$U \longmapsto H^m(\mathcal{J}^\bullet_{\max(1,-m+d)}(U))$$

when U runs through the elements of $\mathcal{B}$. Since every object of $\mathcal{C}$ has a covering whose members are elements of $\mathcal{B}$ we see that it suffices to prove the sheafification of $U \longmapsto H^m(\mathcal{J}^\bullet_{\max(1,-m+d)}(U))$ is $\mathcal{H}^m$. On the other hand, this sheafification is equal to $H^m(\mathcal{J}^\bullet_{\max(1,-m+d)})$. Since $\tau_{\geq -\max(1,-m+d)}\mathcal{F}^\bullet \to \mathcal{J}^\bullet_{\max(1,-m+d)}$ is a quasi-isomorphism we win. $\square$

The construction above can be used in the following setting. Let $\mathcal{C}$ be a category. Let $\text{Cov}(\mathcal{C}) \supset \text{Cov}'(\mathcal{C})$ be two ways to endow $\mathcal{C}$ with the structure of a site. Denote τ the topology corresponding to $\text{Cov}(\mathcal{C})$ and τ' the topology corresponding to $\text{Cov}'(\mathcal{C})$. Then the identity functor on $\mathcal{C}$ defines a morphism of sites

$$\epsilon : \mathcal{C}_\tau \longrightarrow \mathcal{C}_{\tau'}$$

where ϵ_* is the identity functor on underlying presheaves and where ϵ^{-1} is the τ-sheafification of a τ'-sheaf (hence clearly exact). Let $\mathcal{O}$ be a sheaf of rings for the τ-topology. Then $\mathcal{O}$ is also a sheaf for the τ'-topology and ϵ becomes a morphism of ringed sites

$$\epsilon : (\mathcal{C}_\tau, \mathcal{O}_\tau) \longrightarrow (\mathcal{C}_{\tau'}, \mathcal{O}_{\tau'})$$

In this situation we can sometimes point out subcategories of $D(\mathcal{O}_\tau)$ and $D(\mathcal{O}_{\tau'})$ which are identified by the functors ϵ^* and $R\epsilon_*$.

Lemma 19.20.2. *With $\epsilon : (\mathcal{C}_\tau, \mathcal{O}_\tau) \longrightarrow (\mathcal{C}_{\tau'}, \mathcal{O}_{\tau'})$ as above. Let $\mathcal{B} \subset Ob(\mathcal{C})$ be a subset. Let $\mathcal{A} \subset PMod(\mathcal{O})$ be a full subcategory. Assume*

1. *every object of $\mathcal{A}$ is a sheaf for the τ-topology,*
2. *$\mathcal{A}$ is a weak Serre subcategory of $Mod(\mathcal{O}_\tau)$,*
3. *every object of $\mathcal{C}$ has a τ'-covering whose members are elements of $\mathcal{B}$, and*
4. *for every $U \in \mathcal{B}$ we have $H^p_\tau(U, \mathcal{F}) = 0$, $p > 0$ for all $\mathcal{F} \in \mathcal{A}$.*

Then $\mathcal{A}$ is a weak Serre subcategory of $Mod(\mathcal{O}_{\tau'})$ and there is an equivalence of triangulated categories $D_\mathcal{A}(\mathcal{O}_\tau) = D_\mathcal{A}(\mathcal{O}_{\tau'})$ given by ϵ^* and $R\epsilon_*$.

Proof. Note that for $A \in \mathcal{A}$ we can think of A as a sheaf in either topology and (abusing notation) that $\epsilon_* A = A$ and $\epsilon^* A = A$. Consider an exact sequence

$$A_0 \to A_1 \to A_2 \to A_3 \to A_4$$

in $Mod(\mathcal{O}_{\tau'})$ with A_0, A_1, A_3, A_4 in $\mathcal{A}$. We have to show that A_2 is an element of $\mathcal{A}$, see Homology, Definition 10.7.1. Apply the exact functor $\epsilon^* = \epsilon^{-1}$ to conclude that $\epsilon^* A_2$ is an object of $\mathcal{A}$. Consider the map of sequences

$$\begin{tikzcd} A_0 \arrow[r] \arrow[d] & A_1 \arrow[r] \arrow[d] & A_2 \arrow[r] \arrow[d] & A_3 \arrow[r] \arrow[d] & A_4 \arrow[d] \\ A_0 \arrow[r] & A_1 \arrow[r] & \epsilon_*\epsilon^* A_2 \arrow[r] & A_3 \arrow[r] & A_4 \end{tikzcd}$$

to conclude that $A_2 = \epsilon_*\epsilon^* A_2$ is an object of $\mathcal{A}$. At this point it makes sense to talk about the derived categories $D_\mathcal{A}(\mathcal{O}_\tau)$ and $D_\mathcal{A}(\mathcal{O}_{\tau'})$, see Derived Categories, Section 11.12.

Since ϵ^* is exact and preserves $\mathcal{A}$, it is clear that we obtain a functor $\epsilon^* : D_\mathcal{A}(\mathcal{O}_{\tau'}) \to D_\mathcal{A}(\mathcal{O}_\tau)$. We claim that $R\epsilon_*$ is a quasi-inverse. Namely, let $\mathcal{F}^\bullet$ be an object of $D_\mathcal{A}(\mathcal{O}_\tau)$.

Construct a map $\mathscr{F}^\bullet \to \mathscr{I}^\bullet = lim\, \mathscr{I}_n^\bullet$ as in (19.20.0.1). By Lemma 19.20.1 and assumption (4) we see that $\mathscr{F}^\bullet \to \mathscr{I}^\bullet$ is a quasi-isomorphism. Then

$$R\epsilon_*\mathscr{F}^\bullet = \epsilon_*\mathscr{I}^\bullet = lim_n\, \epsilon_*\mathscr{I}_n^\bullet$$

For every $U \in \mathscr{B}$ we have

$$H^m(\epsilon_*\mathscr{I}_n^\bullet(U)) = H^m(\mathscr{I}_n^\bullet(U)) = \begin{cases} H^m(\mathscr{F}^\bullet)(U) & \text{if} \quad m \geq -n \\ 0 & \text{if} \quad m < n \end{cases}$$

by the assumed vanishing of (4), the spectral sequence Derived Categories, Lemma 11.20.3, and the fact that $\tau_{\geq -n}\mathscr{F}^\bullet \to \mathscr{I}_n^\bullet$ is a quasi-isomorphism. The maps $\epsilon_*\mathscr{I}_{n+1}^\bullet \to \epsilon_*\mathscr{I}_n^\bullet$ are termwise split surjections as ϵ_* is a functor. Hence we can apply Homology, Lemma 10.23.7 to the sequence of complexes

$$lim_n\, \epsilon_*\mathscr{I}_n^{m-2}(U) \to lim_n\, \epsilon_*\mathscr{I}_n^{m-1}(U) \to lim_n\, \epsilon_*\mathscr{I}_n^{m}(U) \to lim_n\, \epsilon_*\mathscr{I}_n^{m+1}(U)$$

to conclude that $H^m(\epsilon_*\mathscr{I}^\bullet(U)) = H^m(\mathscr{F}^\bullet)(U)$ for $U \in \mathscr{B}$. Sheafifying and using property (3) this proves that $H^m(\epsilon_*\mathscr{I}^\bullet)$ is isomorphic to $\epsilon_* H^m(\mathscr{F}^\bullet)$, i.e., is an object of $\mathscr{A}$. Thus $R\epsilon_*$ indeed gives rise to a functor

$$R\epsilon_* : D_{\mathscr{A}}(\mathcal{O}_\tau) \longrightarrow D_{\mathscr{A}}(\mathcal{O}_{\tau'})$$

For $\mathscr{F}^\bullet \in D_{\mathscr{A}}(\mathcal{O}_\tau)$ the adjunction map $\epsilon^* R\epsilon_*\mathscr{F}^\bullet \to \mathscr{F}^\bullet$ is a quasi-isomorphism as we've seen above that the cohomology sheaves of $R\epsilon_*\mathscr{F}^\bullet$ are $\epsilon_* H^m(\mathscr{F}^\bullet)$. For $\mathscr{G}^\bullet \in D_{\mathscr{A}}(\mathcal{O}_{\tau'})$ the adjunction map $\mathscr{G}^\bullet \to R\epsilon_*\epsilon^*\mathscr{G}^\bullet$ is a quasi-isomorphism for the same reason, i.e., because the cohomology sheaves of $R\epsilon_*\epsilon^*\mathscr{G}^\bullet$ are isomorphic to $\epsilon_* H^m(\epsilon^*\mathscr{G}) = H^m(\mathscr{G}^\bullet)$. □

19.21. Spectral sequences for Ext

In this section we collect various spectral sequences that come up when considering the Ext functors. For any pair of complexes $\mathscr{G}^\bullet, \mathscr{F}^\bullet$ of complexes of modules on a ringed site $(\mathscr{C}, \mathcal{O})$ we denote

$$\mathrm{Ext}^n_{\mathcal{O}}(\mathscr{G}^\bullet, \mathscr{F}^\bullet) = \mathrm{Hom}_{D(\mathcal{O})}(\mathscr{G}^\bullet, \mathscr{F}^\bullet[n])$$

according to our general conventions in Derived Categories, Section 11.26.

Example 19.21.1. Let $(\mathscr{C}, \mathcal{O})$ be a ringed site. Let $\mathscr{K}^\bullet$ be a bounded above complex of $\mathcal{O}$-modules. Let $\mathscr{F}$ be an $\mathcal{O}$-module. Then there is a spectral sequence with E_2-page

$$E_2^{i,j} = \mathrm{Ext}^i_{\mathcal{O}}(H^{-j}(\mathscr{K}^\bullet), \mathscr{F}) \Rightarrow \mathrm{Ext}^{i+j}_{\mathcal{O}}(\mathscr{K}^\bullet, \mathscr{F})$$

and another spectral sequence with E_1-page

$$E_1^{i,j} = \mathrm{Ext}^j_{\mathcal{O}}(\mathscr{K}^{-i}, \mathscr{F}) \Rightarrow \mathrm{Ext}^{i+j}_{\mathcal{O}}(\mathscr{K}^\bullet, \mathscr{F}).$$

To construct these spectral sequences choose an injective resolution $\mathscr{F} \to \mathscr{I}^\bullet$ and consider the two spectral sequences coming from the double complex $\mathrm{Hom}_{\mathcal{O}}(\mathscr{K}^\bullet, \mathscr{I}^\bullet)$, see Homology, Section 10.19.

19.22. Derived lower shriek

In this section we study some situations where besides Lf^* and Rf_* there also a derived functor $Lf_!$.

Lemma 19.22.1. *Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous and cocontinuous functor of sites which induces a morphism of topoi $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$. Let $\mathcal{O}_\mathcal{D}$ be a sheaf of rings and set $\mathcal{O}_\mathcal{C} = g^{-1}\mathcal{O}_\mathcal{D}$. The functor $g_! : Mod(\mathcal{O}_\mathcal{C}) \to Mod(\mathcal{O}_\mathcal{D})$ (see Modules on Sites, Lemma 16.35.1) has a left derived functor*

$$Lg_! : D(\mathcal{O}_\mathcal{C}) \longrightarrow D(\mathcal{O}_\mathcal{D})$$

which is left adjoint to g^*.

Proof. We are going to use Derived Categories, Proposition 11.27.2 to construct $Lg_!$. To do this we have to verify assumptions (1), (2), (3), (4), and (5) of that proposition. First, since $g_!$ is a left adjoint we see that it is right exact and commutes with all colimits, so (5) holds. Conditions (3) and (4) hold because the category of modules on a ringed site is a Grothendieck abelian category. Let $\mathcal{P} \subset Ob(Mod(\mathcal{O}_\mathcal{C}))$ be the collection of $\mathcal{O}_\mathcal{C}$-modules which are direct sums of modules of the form $j_{U!}\mathcal{O}_U$. Here $U \in Ob(\mathcal{C})$ and $j_{U!}$ is the extension by zero associated to the localization morphism $j_U : \mathcal{C}/U \to \mathcal{C}$. Every $\mathcal{O}_\mathcal{C}$-module is a quotient of an object of $\mathcal{P}$, see Modules on Sites, Lemma 16.26.6. Thus (1) holds. Finally, we have to prove (2). Let $\mathcal{K}^\bullet$ be a bounded above acyclic complex of $\mathcal{O}_\mathcal{C}$-modules with $\mathcal{K}^n \in \mathcal{P}$ for all n. We have to show that $g_!\mathcal{K}^\bullet$ is exact. To do this it suffices to show, for every injective $\mathcal{O}_\mathcal{D}$-module $\mathcal{I}$ that

$$Hom_{D(\mathcal{O}_\mathcal{D})}(g_!\mathcal{K}^\bullet, \mathcal{I}[n]) = 0$$

for all $n \in \mathbf{Z}$. Since $\mathcal{I}$ is injective we have

$$\begin{aligned} Hom_{D(\mathcal{O}_\mathcal{D})}(g_!\mathcal{K}^\bullet, \mathcal{I}[n]) &= \mathrm{Hom}_{K(\mathcal{O}_\mathcal{D})}(g_!\mathcal{K}^\bullet, \mathcal{I}[n]) \\ &= H^n(\mathrm{Hom}_{\mathcal{O}_\mathcal{D}}(g_!\mathcal{K}^\bullet, \mathcal{I})) \\ &= H^n(\mathrm{Hom}_{\mathcal{O}_\mathcal{C}}(\mathcal{K}^\bullet, g^{-1}\mathcal{I})) \end{aligned}$$

the last equality by the adjointness of $g_!$ and g^{-1}.

The vanishing of this group would be clear if $g^{-1}\mathcal{I}$ were an injective $\mathcal{O}_\mathcal{C}$-module. But $g^{-1}\mathcal{I}$ isn't necessarily an injective $\mathcal{O}_\mathcal{C}$-module as $g_!$ isn't exact in general. We do know that

$$\mathrm{Ext}^p_{\mathcal{O}_\mathcal{C}}(j_{U!}\mathcal{O}_U, g^{-1}\mathcal{I}) = H^p(U, g^{-1}\mathcal{I}) = 0 \text{ for } p \geq 1$$

Namely, the first equality follows from $Hom_{\mathcal{O}_\mathcal{C}}(j_{U!}\mathcal{O}_U, \mathcal{H}) = \mathcal{H}(U)$ and taking derived functors. The vanishing of $H^p(U, g^{-1}\mathcal{I})$ for all $U \in Ob(\mathcal{C})$ comes from the vanishing of all higher Čech cohomology groups $\check{H}^p(\mathcal{U}, g^{-1}\mathcal{I})$ via Lemma 19.11.8. Namely, for a covering $\mathcal{U} = \{U_i \to U\}_{i\in I}$ in $\mathcal{C}$ we have $\check{H}^p(\mathcal{U}, g^{-1}\mathcal{I}) = \check{H}^p(u(\mathcal{U}), \mathcal{I})$. Since $\mathcal{I}$ is an injective $\mathcal{O}$-module these Čech cohomology groups vanish, see Lemma 19.12.3. Since each $\mathcal{K}^{-q}$ is a direct sum of modules of the form $j_{U!}\mathcal{O}_U$ we see that

$$\mathrm{Ext}^p_{\mathcal{O}_\mathcal{C}}(\mathcal{K}^{-q}, g^{-1}\mathcal{I}) = 0 \text{ for } p \geq 1 \text{ and all } q$$

Let us use the spectral sequence (see Example 19.21.1)

$$E_1^{p,q} = \mathrm{Ext}^p_{\mathcal{O}_\mathcal{C}}(\mathcal{K}^{-q}, g^{-1}\mathcal{I}) \Rightarrow \mathrm{Ext}^{p+q}_{\mathcal{O}_\mathcal{C}}(\mathcal{K}^\bullet, g^{-1}\mathcal{I}) = 0.$$

Note that the spectral sequence abuts to zero as $\mathcal{K}^\bullet$ is acyclic (hence vanishes in the derived category, hence produces vanishing ext groups). By the vanishing of higher exts proved above the only nonzero terms on the E_1 page are the terms $E_1^{0,q} = \mathrm{Hom}_{\mathcal{O}_\mathcal{C}}(\mathcal{K}^{-q}, g^{-1}\mathcal{I})$. We conclude that the complex $\mathrm{Hom}_{\mathcal{O}_\mathcal{C}}(\mathcal{K}^\bullet, g^{-1}\mathcal{I})$ is acyclic as desired.

Thus the left derived functor $Lg_!$ exists. We still have to show that it is left adjoint to $g^{-1} = g^* = Rg^* = Lg^*$, i.e., that we have

$$Hom_{D(\mathcal{O}_\mathcal{C})}(\mathcal{H}^\bullet, g^{-1}\mathcal{E}^\bullet) = Hom_{D(\mathcal{O}_\mathcal{D})}(Lg_!\mathcal{H}^\bullet, \mathcal{E}^\bullet) \tag{19.22.1.1}$$

This is actually a formal consequence of the discussion above. Choose a quasi-isomorphism $\mathcal{K}^\bullet \to \mathcal{H}^\bullet$ such that $\mathcal{K}^\bullet$ computes $Lg_!$. Moreover, choose a quasi-isomorphism $\mathcal{E}^\bullet \to \mathcal{I}^\bullet$ into a K-injective complex of $\mathcal{O}_\mathcal{D}$-modules $\mathcal{I}^\bullet$. Then the RHS of (19.22.1.1) is

$$Hom_{K(\mathcal{O}_\mathcal{D})}(g_!\mathcal{K}^\bullet, \mathcal{I}^\bullet)$$

On the other hand, by the definition of morphisms in the derived category the LHS of (19.22.1.1) is

$$\begin{aligned} Hom_{D(\mathcal{O}_\mathcal{C})}(\mathcal{K}^\bullet, g^{-1}\mathcal{I}^\bullet) &= colim_{s : \mathcal{L}^\bullet \to \mathcal{K}^\bullet} Hom_{K(\mathcal{O}_\mathcal{C})}(\mathcal{L}^\bullet, g^{-1}\mathcal{I}^\bullet) \\ &= colim_{s : \mathcal{L}^\bullet \to \mathcal{K}^\bullet} Hom_{K(\mathcal{O}_\mathcal{D})}(g_!\mathcal{L}^\bullet, \mathcal{I}^\bullet) \end{aligned}$$

by the adjointness of $g_!$ and g^* on the level of sheaves of modules. The colimit is over all quasi-isomorphisms with target $\mathcal{K}^\bullet$. Since for every complex $\mathcal{L}^\bullet$ there exists a quasi-isomorphism $(\mathcal{K}')^\bullet \to \mathcal{L}^\bullet$ such that $(\mathcal{K}')^\bullet$ computes $Lg_!$ we see that we may as well take the colimit over quasi-isomorphisms of the form $s : (\mathcal{K}')^\bullet \to \mathcal{K}^\bullet$ where $(\mathcal{K}')^\bullet$ computes $Lg_!$. In this case

$$Hom_{K(\mathcal{O}_\mathcal{D})}(g_!\mathcal{K}^\bullet, \mathcal{I}^\bullet) \longrightarrow Hom_{K(\mathcal{O}_\mathcal{D})}(g_!(\mathcal{K}')^\bullet, \mathcal{I}^\bullet)$$

is an isomorphism as $g_!(\mathcal{K}')^\bullet \to g_!\mathcal{K}^\bullet$ is a quasi-isomorphism and $\mathcal{I}^\bullet$ is K-injective. This finishes the proof. ☐

Remark 19.22.2. Warning! Let $u : \mathcal{C} \to \mathcal{D}$, g, $\mathcal{O}_\mathcal{D}$, and $\mathcal{O}_\mathcal{C}$ be as in Lemma 19.22.1. In general it is **not** the case that the diagram

$$\begin{tikzcd} D(\mathcal{O}_\mathcal{C}) \arrow[r, "Lg_!"'] \arrow[d, "forget"'] & D(\mathcal{O}_\mathcal{D}) \arrow[d, "forget"] \\ D(\mathcal{C}) \arrow[r, "Lg_!^{Ab}"] & D(\mathcal{D}) \end{tikzcd}$$

commutes where the functor $Lg_!^{Ab}$ is the one constructed in Lemma 19.22.1 but using the constant sheaf $\mathbf{Z}$ as the structure sheaf on both $\mathcal{C}$ and $\mathcal{D}$. In general it isn't even the case that $g_! = g_!^{Ab}$ (see Modules on Sites, Remark 16.35.2), but this phenomenon **can occur even if** $g_! = g_!^{Ab}$! In general all we can say is that there exists a natural transformation

$$Lg_!^{Ab} \circ forget \longrightarrow forget \circ Lg_!$$

19.23. Other chapters

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CHAPTER 20

Hypercoverings

20.1. Introduction

Let $\mathscr{C}$ be a site, see Sites, Definition 9.6.2. Let X be an object of $\mathscr{C}$. Given an abelian sheaf $\mathscr{F}$ on $\mathscr{C}$ we would like to compute its cohomology groups

$$H^i(X, \mathscr{F}).$$

According to our general definitions (insert future reference here) this cohomology group is computed by choosing an injective resolution

$$0 \to \mathscr{F} \to \mathscr{I}^0 \to \mathscr{I}^1 \to \ldots$$

and setting

$$H^i(X, \mathscr{F}) = H^i(\Gamma(X, \mathscr{I}^0) \to \Gamma(X, \mathscr{I}^1) \to \Gamma(X, \mathscr{I}^2) \to \ldots)$$

We will have to do quite a bit of work to prove that we may also compute these cohomology groups without choosing an injective resolution. Also, we will only do this in case the site $\mathscr{C}$ has fibre products.

A hypercovering in a site is a generalization of a covering. See [**MA71**, Exposé V, Sec. 7]. A hypercovering is a special case of a simplicial augmentation where one has cohomological descent, see [**MA71**, Exposé Vbis]. A nice manuscript on cohomological descent is the text by Brian Conrad, see `http://math.stanford.edu/~conrad/papers/hypercover.pdf`. Brian's text follows the exposition in [**MA71**, Exposé Vbis], and in particular discusses a more general kind of hypercoverings, such as proper hypercoverings of schemes used to compute étale cohomology for example. A proper hypercovering can be seen as a hypercovering in the category of schemes endowed with a different topology than the étale topology, but still they can be used to compute the étale cohomology.

20.2. Hypercoverings

In order to start we make the following definition. The letters ``SR'' stand for Semi-Representable.

Definition 20.2.1. Let $\mathscr{C}$ be a site with fibre products. Let $X \in Ob(\mathscr{C})$ be an object of $\mathscr{C}$. We denote $\text{SR}(\mathscr{C}, X)$ the category of *semi-representable objects* defined as follows

(1) objects are families of morphisms $\{U_i \to X\}_{i \in I}$, and
(2) morphisms $\{U_i \to X\}_{i \in I} \to \{V_j \to X\}_{j \in J}$ are given by a map $\alpha : I \to J$ and for each $i \in I$ a morphism $f_i : U_i \to V_{\alpha(i)}$ over X.

This definition is different from the one in [**MA71**, Exposé V, Sec. 7], but it seems flexible enough to do all the required arguments. Note that this is a ``big'' category. We will later ``bound'' the size of the index sets I that we need and we can then redefine $\text{SR}(\mathscr{C}, X)$ to become a category.

Definition 20.2.2. Let $\mathcal{C}$ be a site with fibre products. Let $X \in Ob(\mathcal{C})$ be an object of $\mathcal{C}$. We denote F the functor *which associates a sheaf to a semi-representable object*. In a formula

$$\begin{array}{rcl} F : \mathrm{SR}(\mathcal{C}, X) & \longrightarrow & PSh(\mathcal{C}) \\ \{U_i \to X\}_{i\in I} & \longmapsto & \amalg_{i\in I} h_{U_i} \end{array}$$

where h_U denotes the representable presheaf associated to the object U.

Given a morphism $U \to X$ we obtain a morphism $h_U \to h_X$ of representable presheaves. Thus it makes more sense to think of F as a functor into the category of presheaves of sets over h_X, namely $PSh(\mathcal{C})/h_X$.

Lemma 20.2.3. *Let $\mathcal{C}$ be a site with fibre products. Let $X \in Ob(\mathcal{C})$ be an object of $\mathcal{C}$. The category $SR(\mathcal{C}, X)$ has coproducts and finite limits. Moreover, the functor F commutes with coproducts and fibre products, and transforms products into fibre products over h_X. In other words, it commutes with finite limits as a functor into $PSh(\mathcal{C})/h_X$.*

Proof. It is clear that the coproduct of $\{U_i \to X\}_{i\in I}$ and $\{V_j \to X\}_{j\in J}$ is $\{U_i \to X\}_{i\in I} \amalg \{V_j \to X\}_{j\in J}$ and similarly for coproducts of families of families of morphisms with target X. The object $\{X \to X\}$ is a final object of $\mathrm{SR}(\mathcal{C}, X)$. Suppose given a morphism $(\alpha, f_i) : \{U_i \to X\}_{i\in I} \to \{V_j \to X\}_{j\in J}$ and a morphism $(\beta, g_k) : \{W_k \to X\}_{k\in K} \to \{V_j \to X\}_{j\in J}$. The fibred product of these morphisms is given by

$$\{U_i \times_{f_i, V_j, g_k} W_k \to X\}_{(i,j,k)\in I\times J\times K \text{ such that } k=\alpha(i)=\beta(j)}$$

The fibre products exist by the assumption that $\mathcal{C}$ has fibre products. Thus $\mathrm{SR}(\mathcal{C}, X)$ has finite limits, see Categories, Lemma 4.16.4. The statements on the functor F are clear from the constructions above. $\square$

Definition 20.2.4. Let $\mathcal{C}$ be a site with fibred products. Let X be an object of $\mathcal{C}$. Let $f = (\alpha, f_i) : \{U_i \to X\}_{i\in I} \to \{V_j \to X\}_{j\in J}$ be a morphism in the category $\mathrm{SR}(\mathcal{C}, X)$. We say that f is a *covering* if for every $j \in J$ the family of morphisms $\{U_i \to V_j\}_{i\in I, \alpha(i)=j}$ is a covering for the site $\mathcal{C}$.

Lemma 20.2.5. *Let $\mathcal{C}$ be a site with fibred products. Let $X \in Ob(\mathcal{C})$.*

(1) *A composition of coverings in $SR(\mathcal{C}, X)$ is a covering.*
(2) *A base change of coverings is a covering.*
(3) *If $A \to B$ and $K \to L$ are coverings, then $A \times K \to B \times L$ is a covering.*

Proof. Immediate from the axioms of a site. (Number (3) is the composition $A \times K \to B \times K \to B \times L$ and hence a composition of basechanges of coverings.) $\square$

According to the results in the chapter on simplicial methods the coskelet of a truncated simplicial object of $\mathrm{SR}(\mathcal{C}, X)$ exists. Hence the following definition makes sense.

Definition 20.2.6. Let $\mathcal{C}$ be a site. Let $X \in Ob(\mathcal{C})$ be an object of $\mathcal{C}$. A *hypercovering* of X is a simplicial object K in the category $\mathrm{SR}(\mathcal{C}, X)$ such that

(1) The object K_0 is a covering of X for the site $\mathcal{C}$.
(2) For every $n \geq 0$ the canonical morphism

$$K_{n+1} \longrightarrow (\mathrm{cosk}_n \mathrm{sk}_n K)_{n+1}$$

is a covering in the sense defined above.

Condition (1) makes sense since each object of $\mathrm{SR}(\mathcal{C}, X)$ is after all a family of morphisms with target X. It could also be formulated as saying that the morphism of K_0 to the final object of $\mathrm{SR}(\mathcal{C}, X)$ is a covering.

Example 20.2.7. Let $\{U_i \to X\}_{i\in I}$ be a covering of the site $\mathcal{C}$. Set $K_0 = \{U_i \to X\}_{i\in I}$. Then K_0 is a 0-truncated simplicial object of $\mathrm{SR}(\mathcal{C}, X)$. Hence we may form

$$K = \mathrm{cosk}_0 K_0.$$

Clearly K passes condition (1) of Definition 20.2.6. Since all the morphisms $K_{n+1} \to (\mathrm{cosk}_n \mathrm{sk}_n K)_{n+1}$ are isomorphisms it also passes condition (2). Note that the terms K_n are the usual

$$K_n = \{U_{i_0} \times_X U_{i_1} \times_X \ldots \times_X U_{i_n} \to X\}_{(i_0,i_1,\ldots,i_n)\in I^{n+1}}$$

Lemma 20.2.8. *Let $\mathcal{C}$ be a site with fibre products. Let $X \in Ob(\mathcal{C})$ be an object of $\mathcal{C}$. The collection of all hypercoverings of X forms a set.*

Proof. Since $\mathcal{C}$ is a site, the set of all coverings of S forms a set. Thus we see that the collection of possible K_0 forms a set. Suppose we have shown that the collection of all possible $K_0, \ldots, K_n$ form a set. Then it is enough to show that given $K_0, \ldots, K_n$ the collection of all possible K_{n+1} forms a set. And this is clearly true since we have to choose K_{n+1} among all possible coverings of $(\mathrm{cosk}_n \mathrm{sk}_n K)_{n+1}$. □

Remark 20.2.9. The lemma does not just say that there is a cofinal system of choices of hypercoverings that is a set, but that really the hypercoverings form a set.

The category of presheaves on $\mathcal{C}$ has finite (co)limits. Hence the functors cosk_n exists for presheaves of sets.

Lemma 20.2.10. *Let $\mathcal{C}$ be a site with fibre products. Let $X \in Ob(\mathcal{C})$ be an object of $\mathcal{C}$. Let K be a hypercovering of X. Consider the simplicial object $F(K)$ of $PSh(\mathcal{C})$, endowed with its augmentation to the constant simplicial presheaf h_X.*

(1) *The morphism of presheaves $F(K)_0 \to h_X$ becomes a surjection after sheafification.*
(2) *The morphism*

$$(d^1_0, d^1_1) : F(K)_1 \longrightarrow F(K)_0 \times_{h_X} F(K)_0$$

becomes a surjection after sheafification.
(3) *For every $n \geq 1$ the morphism*

$$F(K)_{n+1} \longrightarrow (cosk_n sk_n F(K))_{n+1}$$

turns into a surjection after sheafification.

Proof. We will use the fact that if $\{U_i \to U\}_{i\in I}$ is a covering of the site $\mathcal{C}$, then the morphism

$$\amalg_{i\in I} h_{U_i} \to h_U$$

becomes surjective after sheafification, see Sites, Lemma 9.12.5. Thus the first assertion follows immediately.

For the second assertion, note that according to Simplicial, Example 14.17.2 the simplicial object $\mathrm{cosk}_0 \mathrm{sk}_0 K$ has terms $K_0 \times \ldots \times K_0$. Thus according to the definition of a hypercovering we see that $(d^1_0, d^1_1) : K_1 \to K_0 \times K_0$ is a covering. Hence (2) follows from the claim above and the fact that F transforms products into fibred products over h_X.

For the third, we claim that $\text{cosk}_n \text{sk}_n F(K) = F(\text{cosk}_n \text{sk}_n K)$ for $n \geq 1$. To prove this, denote temporarily F' the functor $\text{SR}(\mathcal{C}, X) \to PSh(\mathcal{C})/h_X$. By Lemma 20.2.3 the functor F' commutes with finite limits. By our description of the cosk_n functor in Simplicial, Section 14.17 we see that $\text{cosk}_n \text{sk}_n F'(K) = F'(\text{cosk}_n \text{sk}_n K)$. Recall that the category used in the description of $(\text{cosk}_n U)_m$ in Simplicial, Lemma 14.17.3 is the category $(\Delta/[m])^{opp}_{\leq n}$. It is an amusing exercise to show that $(\Delta/[m])_{\leq n}$ is a nonempty connected category (see Categories, Definition 4.15.1) as soon as $n \geq 1$. Hence, Categories, Lemma 4.15.2 shows that $\text{cosk}_n \text{sk}_n F'(K) = \text{cosk}_n \text{sk}_n F(K)$. Whence the claim. Property (2) follows from this, because now we see that the morphism in (2) is the result of applying the functor F to a covering as in Definition 20.2.4, and the result follows from the first fact mentioned in this proof. □

20.3. Acyclicity

Let $\mathcal{C}$ be a site. For a presheaf of sets $\mathcal{F}$ we denote $\mathbf{Z}_\mathcal{F}$ the presheaf of abelian groups defined by the rule

$$\mathbf{Z}_\mathcal{F}(U) = \text{free abelian group on } \mathcal{F}(U).$$

We will sometimes call this the *free abelian presheaf on* $\mathcal{F}$. Of course the construction $\mathcal{F} \mapsto \mathbf{Z}_\mathcal{F}$ is a functor and it is left adjoint to the forgetful functor $PAb(\mathcal{C}) \to PSh(\mathcal{C})$. Of course the sheafification $\mathbf{Z}^\#_\mathcal{F}$ is a sheaf of abelian groups, and the functor $\mathcal{F} \mapsto \mathbf{Z}^\#_\mathcal{F}$ is a left adjoint as well. We sometimes call $\mathbf{Z}^\#_\mathcal{F}$ the *free abelian sheaf on* $\mathcal{F}$.

For an object X of the site $\mathcal{C}$ we denote $\mathbf{Z}_X$ the free abelian presheaf on h_X, and we denote $\mathbf{Z}^\#_X$ its sheafification.

Definition 20.3.1. Let $\mathcal{C}$ be a site. Let K be a simplicial object of $PSh(\mathcal{C})$. By the above we get a simplicial object $\mathbf{Z}^\#_K$ of $Ab(\mathcal{C})$. We can take its associated complex of abelian presheaves $s(\mathbf{Z}^\#_K)$, see Simplicial, Section 14.21. The *homology of K* is the homology of the complex of abelian sheaves $s(\mathbf{Z}^\#_K)$.

In other words, the *ith homology* $H_i(K)$ *of* K is the sheaf of abelian groups $H_i(K) = H_i(s(\mathbf{Z}^\#_K))$. In this section we worry about the homology in case K is a hypercovering of an object X of $\mathcal{C}$.

Lemma 20.3.2. *Let $\mathcal{C}$ be a site. Let $\mathcal{F} \to \mathcal{G}$ be a morphism of presheaves of sets. Denote K the simplicial object of $PSh(\mathcal{C})$ whose nth term is the $(n+1)$st fibre product of $\mathcal{F}$ over $\mathcal{G}$, see Simplicial, Example 14.3.5. Then, if $\mathcal{F} \to \mathcal{G}$ is surjective after sheafification, we have*

$$H_i(K) = \begin{cases} 0 & \text{if} \quad i > 0 \\ \mathbf{Z}^\#_\mathcal{G} & \text{if} \quad i = 0 \end{cases}$$

The isomorphism in degree 0 *is given by the morphsm $H_0(K) \to \mathbf{Z}^\#_\mathcal{G}$ coming from the map $(\mathbf{Z}^\#_K)_0 = \mathbf{Z}^\#_\mathcal{F} \to \mathbf{Z}^\#_\mathcal{G}$.*

Proof. Let $\mathcal{G}' \subset \mathcal{G}$ be the image of the morphism $\mathcal{F} \to \mathcal{G}$. Let $U \in Ob(\mathcal{C})$. Set $A = \mathcal{F}(U)$ and $B = \mathcal{G}'(U)$. Then the simplicial set $K(U)$ is equal to the simplicial set with n-simplices given by

$$A \times_B A \times_B \ldots \times_B A \ (n + 1 \text{ factors}).$$

By Simplicial, Lemma 14.28.4 the morphism $K(U) \to B$ is a homotopy equivalence. Hence applying the functor ``free abelian group on'' to this we deduce that

$$\mathbf{Z}_K(U) \longrightarrow \mathbf{Z}_B$$

is a homotopy equivalence. Note that $s(\mathbf{Z}_B)$ is the complex

$$\ldots \to \bigoplus\nolimits_{b\in B}\mathbf{Z} \xrightarrow{0} \bigoplus\nolimits_{b\in B}\mathbf{Z} \xrightarrow{1} \bigoplus\nolimits_{b\in B}\mathbf{Z} \xrightarrow{0} \bigoplus\nolimits_{b\in B}\mathbf{Z} \to 0$$

see Simplicial, Lemma 14.21.3. Thus we see that $H_i(s(\mathbf{Z}_K(U))) = 0$ for $i > 0$, and $H_0(s(\mathbf{Z}_K(U))) = \bigoplus_{b\in B}\mathbf{Z} = \bigoplus_{s\in\mathcal{G}'(U)}\mathbf{Z}$. These identifications are compatible with restriction maps.

We conclude that $H_i(s(\mathbf{Z}_K)) = 0$ for $i > 0$ and $H_0(s(\mathbf{Z}_K)) = \mathbf{Z}_{\mathcal{G}'}$, where here we compute homology groups in $PAb(\mathcal{C})$. Since sheafification is an exact functor we deduce the result of the lemma. Namely, the exactness implies that $H_0(s(\mathbf{Z}_K))^\# = H_0(s(\mathbf{Z}_K^\#))$, and similarly for other indices. $\square$

Lemma 20.3.3. *Let $\mathcal{C}$ be a site. Let $f : L \to K$ be a morphism of simplicial objects of $PSh(\mathcal{C})$. Let $n \geq 0$ be an integer. Assume that*

(1) *For $i < n$ the morphism $L_i \to K_i$ is an isomorphism.*
(2) *The morphism $L_n \to K_n$ is surjective after sheafification.*
(3) *The canonical map $L \to cosk_n sk_n L$ is an isomorphism.*
(4) *The canonical map $K \to cosk_n sk_n K$ is an isomorphism.*

Then $H_i(f) : H_i(L) \to H_i(K)$ is an isomorphism.

Proof. This proof is exactly the same as the proof of Lemma 20.3.2 above. Namely, we first let $K'_n \subset K_n$ be the sub presheaf which is the image of the map $L_n \to K_n$. Assumption (2) means that the sheafification of K'_n is equal to the sheafification of K_n. Moreover, since $L_i = K_i$ for all $i < n$ we see that get an n-truncated simplicial presheaf U by taking $U_0 = L_0 = K_0, \ldots, U_{n-1} = L_{n-1} = K_{n-1}, U_n = K'_n$. Denote $K' = \text{cosk}_n U$, a simplicial presheaf. Because we can construct K'_m as a finite limit, and since sheafification is exact, we see that $(K'_m)^\# = K_m$. In other words, $(K')^\# = K^\#$. We conclude, by exactness of sheafification once more, that $H_i(K) = H_i(K')$. Thus it suffices to prove the lemma for the morphism $L \to K'$, in other words, we may assume that $L_n \to K_n$ is a surjective morphism of *presheaves*!

In this case, for any object U of $\mathcal{C}$ we see that the morphism of simplicial sets

$$L(U) \longrightarrow K(U)$$

satisfies all the assumptions of Simplicial, Lemma 14.28.3. Hence it is a homotopy equivalence, and thus

$$\mathbf{Z}_L(U) \longrightarrow \mathbf{Z}_K(U)$$

is a homotopy equivalence too. This for all U. The result follows. $\square$

Lemma 20.3.4. *Let $\mathcal{C}$ be a site. Let K be a simplicial presheaf. Let $\mathcal{G}$ be a presheaf. Let $K \to \mathcal{G}$ be an augmentation of K towards $\mathcal{G}$. Assume that*

(1) *The morphism of presheaves $K_0 \to \mathcal{G}$ becomes a surjection after sheafification.*
(2) *The morphism*
$$(d^1_0, d^1_1) : K_1 \longrightarrow K_0 \times_{\mathcal{G}} K_0$$
becomes a surjection after sheafification.
(3) *For every $n \geq 1$ the morphism*
$$K_{n+1} \longrightarrow (cosk_n sk_n K)_{n+1}$$
turns into a surjection after sheafification.

Then $H_i(K) = 0$ for $i > 0$ and $H_0(K) = \mathbf{Z}_{\mathcal{G}}^\#$.

Proof. Denote $K^n = \text{cosk}_n \text{sk}_n K$ for $n \geq 1$. Define K^0 as the simplicial object with terms $(K^0)_n$ equal to the $(n+1)$-fold fibred product $K_0 \times_{\mathcal{G}} \ldots \times_{\mathcal{G}} K_0$, see Simplicial, Example 14.3.5. We have morphisms

$$K \longrightarrow \ldots \to K^n \to K^{n-1} \to \ldots \to K^1 \to K^0.$$

The morphisms $K \to K^i$, $K^j \to K^i$ for $j \geq i \geq 1$ come from the universal properties of the cosk_n functors. The morphism $K^1 \to K^0$ is the canonical morphism from Simplicial, Remark 14.18.4. We also recall that $K^0 \to \text{cosk}_1 \text{sk}_1 K^0$ is an isomorphism, see Simplicial, Lemma 14.18.3.

By Lemma 20.3.2 we see that $H_i(K^0) = 0$ for $i > 0$ and $H_0(K^0) = \mathbf{Z}^\#_{\mathcal{G}}$.

Pick $n \geq 1$. Consider the morphism $K^n \to K^{n-1}$. It is an isomorphism on terms of degree $< n$. Note that $K^n \to \text{cosk}_n \text{sk}_n K^n$ and $K^{n-1} \to \text{cosk}_n \text{sk}_n K^{n-1}$ are isomorphisms. Note that $(K^n)_n = K_n$ and that $(K^{n-1})_n = (\text{cosk}_{n-1} \text{sk}_{n-1} K)_n$. Hence by assumption, we have that $(K^n)_n \to (K^{n-1})_n$ is a morphism of presheaves which becomes surjective after sheafification. By Lemma 20.3.3 we conclude that $H_i(K^n) = H_i(K^{n-1})$. Combined with the above this proves the lemma. $\square$

Lemma 20.3.5. *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of of $\mathcal{C}$. Let K be a hypercovering of X. The homology of the simplicial presheaf $F(K)$ is 0 in degrees > 0 and equal to $\mathbf{Z}^\#_X$ in degree 0.*

Proof. Combine Lemmas 20.3.4 and 20.2.10. $\square$

20.4. Covering hypercoverings

Here are some ways to construct hypercoverings. We note that since the category $\text{SR}(\mathcal{C}, X)$ has fibre products the category of simplicial objects of $\text{SR}(\mathcal{C}, X)$ has fibre products as well, see Simplicial, Lemma 14.7.2.

Lemma 20.4.1. *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K, L, M be simplicial objects of $SR(\mathcal{C}, X)$. Let $a : K \to L$, $b : M \to L$ be morphisms. Assume*

1. *K is a hypercovering of X,*
2. *the morphism $M_0 \to L_0$ is a covering, and*
3. *for all $n \geq 0$ in the diagram*

$$\begin{tikzcd}
M_{n+1} \arrow[rr] \arrow[dr, "\gamma"] \arrow[dd] & & (cosk_n sk_n M)_{n+1} \arrow[dd] \\
& L_{n+1} \times_{(cosk_n sk_n L)_{n+1}} (cosk_n sk_n M)_{n+1} \arrow[ur] \arrow[dl] & \\
L_{n+1} \arrow[rr] & & (cosk_n sk_n L)_{n+1}
\end{tikzcd}$$

the arrow γ is a covering.

Then the fibre product $K \times_L M$ is a hypercovering of X.

Proof. The morphism $(K \times_L M)_0 = K_0 \times_{L_0} M_0 \to K_0$ is a base change of a covering by (2), hence a covering, see Lemma 20.2.5. And $K_0 \to \{X \to X\}$ is a covering by (1). Thus $(K \times_L M)_0 \to \{X \to X\}$ is a covering by Lemma 20.2.5. Hence $K \times_L M$ satisfies the first condition of Definition 20.2.6.

We still have to check that

$$K_{n+1} \times_{L_{n+1}} M_{n+1} = (K \times_L M)_{n+1} \longrightarrow (\mathrm{cosk}_n \mathrm{sk}_n (K \times_L M))_{n+1}$$

is a covering for all $n \geq 0$. We abbreviate as follows: $A = (\mathrm{cosk}_n \mathrm{sk}_n K)_{n+1}$, $B = (\mathrm{cosk}_n \mathrm{sk}_n L)_{n+1}$, and $C = (\mathrm{cosk}_n \mathrm{sk}_n M)_{n+1}$. The functor $\mathrm{cosk}_n \mathrm{sk}_n$ commutes with fibre products, see Simplicial, Lemma 14.17.13. Thus the right hand side above is equal to $A \times_B C$. Consider the following commutative diagram

$$\begin{tikzcd}
K_{n+1} \times_{L_{n+1}} M_{n+1} \arrow[r] \arrow[d] & M_{n+1} \arrow[d] \arrow[dr, "\gamma"] \arrow[drr] & & \\
K_{n+1} \arrow[r] \arrow[dr] & L_{n+1} \arrow[drr] & L_{n+1} \times_B C \arrow[l] \arrow[r] & C \arrow[d] \\
& A \arrow[rr] & & B
\end{tikzcd}$$

This diagram shows that

$$K_{n+1} \times_{L_{n+1}} M_{n+1} = (K_{n+1} \times_B C) \times_{(L_{n+1} \times_B C), \gamma} M_{n+1}$$

Now, $K_{n+1} \times_B C \to A \times_B C$ is a base change of the covering $K_{n+1} \to A$ via the morphism $A \times_B C \to A$, hence is a covering. By assumption (3) the morphism γ is a covering. Hence the morphism

$$(K_{n+1} \times_B C) \times_{(L_{n+1} \times_B C), \gamma} M_{n+1} \longrightarrow K_{n+1} \times_B C$$

is a covering as a base change of a covering. The lemma follows as a composition of coverings is a covering. □

Lemma 20.4.2. *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. If K, L are hypercoverings of X, then $K \times L$ is a hypercovering of X.*

Proof. You can either verify this directly, or use Lemma 20.4.1 above and check that $L \to \{X \to X\}$ has property (3). □

Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Since the category $\mathrm{SR}(\mathcal{C}, X)$ has coproducts and finite limits, it is permissible to speak about the objects $U \times K$ and $Hom(U, K)$ for certain simplicial sets U (for example those with finitely many nondegenerate simplices) and any simplicial object K of $\mathrm{SR}(\mathcal{C}, X)$. See Simplicial, Sections 14.12 and 14.15.

Lemma 20.4.3. *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K be a hypercovering of X. Let $k \geq 0$ be an integer. Let $u : Z \to K_k$ be a covering in in $SR(\mathcal{C}, X)$. Then there exists a morphism of hypercoverings $f : L \to K$ such that $L_k \to K_k$ factors through u.*

Proof. Denote $Y = K_k$. There is a canonical morphism $K \to Hom(\Delta[k], Y)$ corresponding to id_Y via Simplicial, Lemma 14.15.5. We will use the description of $Hom(\Delta[k], Y)$ and $Hom(\Delta[k], Z)$ given in that lemma. In particular there is a morphism $Hom(\Delta[k], Y) \to Hom(\Delta[k], Z)$ which on degree n terms is the morphism

$$\prod\nolimits_{\alpha : [k] \to [n]} Y \longrightarrow \prod\nolimits_{\alpha : [k] \to [n]} Z.$$

Set

$$L = K \times_{Hom(\Delta[n], Y)} Hom(\Delta[n], Z).$$

The morphism $L_k \to K_k$ sits in to a commutative diagram

$$\begin{tikzcd} L_k \arrow[r] \arrow[d] & \prod_{\alpha : [k]\to[n]} Y \arrow[r, "\mathrm{pr}_{\mathrm{id}_{[k]}}"] \arrow[d] & Y \arrow[d] \\ K_k \arrow[r] & \prod_{\alpha : [k]\to[n]} Z \arrow[r, "\mathrm{pr}_{\mathrm{id}_{[k]}}"] & Z \end{tikzcd}$$

Since the composition of the two bottom arrows is the identity we conclude that we have the desired factorization.

We still have to show that L is a hypercovering of X. To see this we will use Lemma 20.4.1. Condition (1) is satisfied by assumption. For (2), the morphism

$$Hom(\Delta[k], Y)_0 \to Hom(\Delta[k], Z)_0$$

is a covering because it is a product of coverings, see Lemma 20.2.5. For (3) suppose first that $n \geq 1$. In this case by Simplicial, Lemma 14.19.12 we have $Hom(\Delta[k], Y) = \mathrm{cosk}_n \mathrm{sk}_n\, Hom(\Delta[k], Y)$ and similarly for Z. Thus condition (3) for $n > 0$ is clear. For $n = 0$, the diagram of condition (3) of Lemma 20.4.1 is, according to Simplicial, Lemma 14.19.13, the diagram

$$\begin{tikzcd} \prod_{\alpha : [k]\to[1]} Z \arrow[r] \arrow[d] & Z \times Z \arrow[d] \\ \prod_{\alpha : [k]\to[1]} Y \arrow[r] & Y \times Y \end{tikzcd}$$

with obvious horizontal arrows. Thus the morphism γ is the morphism

$$\prod\nolimits_{\alpha : [k]\to[1]} Z \longrightarrow \prod\nolimits_{\alpha : [k]\to[1] \text{ not onto}} Z \times \prod\nolimits_{\alpha : [k]\to[1] \text{ onto}} Y$$

which is a product of coverings and hence a covering according to Lemma 20.4.1 once again. $\square$

Lemma 20.4.4. *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K be a hypercovering of X. Let $n \geq 0$ be an integer. Let $u : \mathcal{F} \to F(K_n)$ be a morphism of presheaves which becomes surjective on sheafification. Then there exists a morphism of hypercoverings $f : L \to K$ such that $F(f_n) : F(L_n) \to F(K_n)$ factors through u.*

Proof. Write $K_n = \{U_i \to X\}_{i\in I}$. Thus the map u is a morphism of presheaves of sets $u : \mathcal{F} \to \amalg h_{u_i}$. The assumption on u means that for every $i \in I$ there exists a covering $\{U_{ij} \to U_i\}_{j\in I_i}$ of the site $\mathcal{C}$ and a morphism of presheaves $t_{ij} : h_{U_{ij}} \to \mathcal{F}$ such that $u \circ t_{ij}$ is the map $h_{U_{ij}} \to h_{U_i}$ coming from the morphism $U_{ij} \to U_i$. Set $J = \amalg_{i\in I} I_i$, and let $\alpha : J \to I$ be the obvious map. For $j \in J$ denote $V_j = U_{\alpha(j)j}$. Set $Z = \{V_j \to X\}_{j\in J}$. Finally, consider the morphism $u' : Z \to K_n$ given by $\alpha : J \to I$ and the morphisms $V_j = U_{\alpha(j)j} \to U_{\alpha(j)}$ above. Clearly, this is a covering in the category $\mathrm{SR}(\mathcal{C}, X)$, and by construction $F(u') : F(Z) \to F(K_n)$ factors through u. Thus the result follows from Lemma 20.4.3 above. $\square$

20.5. Adding simplices

In this section we prove some technical lemmas which we will need later. Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. As we pointed out in Section 20.4 above, the objects $U \times K$ and $Hom(U, K)$ for certain simplicial sets U and any simplicial object K of $\mathrm{SR}(\mathcal{C}, X)$ are defined. See Simplicial, Sections 14.12 and 14.15.

Lemma 20.5.1. *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K be a hypercovering of X. Let $U \subset V$ be simplicial sets, with U_n, V_n finite nonempty for all n. Assume that U has finitely many nondegenerate simplices. Suppose $n \geq 0$ and $x \in V_n$, $x \notin U_n$ are such that*

(1) $V_i = U_i$ *for* $i < n$,
(2) $V_n = U_n \cup \{x\}$,
(3) *any* $z \in V_j$, $z \notin U_j$ *for* $j > n$ *is degenerate.*

Then the morphism

$$Hom(V, K)_0 \longrightarrow Hom(U, K)_0$$

of $SR(\mathcal{C}, X)$ is a covering.

Proof. If $n = 0$, then it follows easily that $V = U \amalg \Delta[0]$ (see below). In this case $Hom(V, K)_0 = Hom(U, K)_0 \times K_0$. The result, in this case, then follows from Lemma 20.2.5.

Let $a : \Delta[n] \to V$ be the morphism associated to x as in Simplicial, Lemma 14.11.3. Let us write $\partial\Delta[n] = i_{(n-1)!}\text{sk}_{n-1}\Delta[n]$ for the $(n-1)$-skeleton of $\Delta[n]$. Let $b : \partial\Delta[n] \to U$ be the restriction of a to the $(n-1)$ skeleton of $\Delta[n]$. By Simplicial, Lemma 14.19.7 we have $V = U \amalg_{\partial\Delta[n]} \Delta[n]$. By Simplicial, Lemma 14.15.6 we get that

$$\begin{tikzcd}
Hom(V, K)_0 \arrow[r] \arrow[d] & Hom(U, K)_0 \arrow[d] \\
Hom(\Delta[n], K)_0 \arrow[r] & Hom(\partial\Delta[n], K)_0
\end{tikzcd}$$

is a fibre product square. Thus it suffices to show that the bottom horizontal arrow is a covering. By Simplicial, Lemma 14.19.11 this arrow is identified with

$$K_n \to (\text{cosk}_{n-1}\text{sk}_{n-1}K)_n$$

and hence is a covering by definition of a hypercovering. □

Lemma 20.5.2. *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K be a hypercovering of X. Let $U \subset V$ be simplicial sets, with U_n, V_n finite nonempty for all n. Assume that U and V have finitely many nondegenerate simplices. Then the morphism*

$$Hom(V, K)_0 \longrightarrow Hom(U, K)_0$$

of $SR(\mathcal{C}, X)$ is a covering.

Proof. By Lemma 20.5.1 above, it suffices to prove a simple lemma about inclusions of simplicial sets $U \subset V$ as in the lemma. And this is exactly the result of Simplicial, Lemma 14.19.8. □

20.6. Homotopies

Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let L be a simplicial object of $\text{SR}(\mathcal{C}, X)$. According to Simplicial, Lemma 14.15.4 there exists an object $Hom(\Delta[1], L)$ in the category $\text{Simp}(\text{SR}(\mathcal{C}, X))$ which represents the functor

$$T \longmapsto Mor_{\text{Simp}(\text{SR}(\mathcal{C},X))}(\Delta[1] \times T, L)$$

There is a canonical morphism

$$Hom(\Delta[1], L) \to L \times L$$

coming from $e_i : \Delta[0] \to \Delta[1]$ and the identification $Hom(\Delta[0], L) = L$.

Lemma 20.6.1. *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let L be a simplicial object of $SR(\mathcal{C}, X)$. Let $n \geq 0$. Consider the commutative diagram*

$$\begin{tikzcd} Hom(\Delta[1], L)_{n+1} \arrow[r] \arrow[d] & (cosk_n sk_n\, Hom(\Delta[1], L))_{n+1} \arrow[d] \\ (L \times L)_{n+1} \arrow[r] & (cosk_n sk_n(L \times L))_{n+1} \end{tikzcd} \tag{20.6.1.1}$$

coming from the morphism defined above. We can identify the terms in this diagram as follows, where $\partial\Delta[n+1] = i_{n!}sk_n\Delta[n+1]$ is the n-skeleton of the $(n+1)$-simplex:

$$\begin{aligned} Hom(\Delta[1], L)_{n+1} &= Hom(\Delta[1] \times \Delta[n+1], L)_0 \\ (cosk_n sk_n\, Hom(\Delta[1], L))_{n+1} &= Hom(\Delta[1] \times \partial\Delta[n+1], L)_0 \\ (L \times L)_{n+1} &= Hom((\Delta[n+1] \amalg \Delta[n+1], L)_0 \\ (cosk_n sk_n(L \times L))_{n+1} &= Hom(\partial\Delta[n+1] \amalg \partial\Delta[n+1], L)_0 \end{aligned}$$

and the morphism between these objects of $SR(\mathcal{C}, X)$ come from the commutative diagram of simplicial sets

$$\begin{tikzcd} \Delta[1] \times \Delta[n+1] & \Delta[1] \times \partial\Delta[n+1] \arrow[l] \\ \Delta[n+1] \amalg \Delta[n+1] \arrow[u] & \partial\Delta[n+1] \amalg \partial\Delta[n+1] \arrow[l] \arrow[u] \end{tikzcd} \tag{20.6.1.2}$$

Moreover the fibre product of the bottom arrow and the right arrow in (20.6.1.1) is equal to

$$Hom(U, L)_0$$

where $U \subset \Delta[1] \times \Delta[n+1]$ is the smallest simplicial subset such that both $\Delta[n+1] \amalg \Delta[n+1]$ and $\Delta[1] \times \partial\Delta[n+1]$ map into it.

Proof. The first and third equalities are Simplicial, Lemma 14.15.4. The second and fourth follow from the cited lemma combined with Simplicial, Lemma 14.19.11. The last assertion follows from the fact that U is the push-out of the bottom and right arrow of the diagram (20.6.1.2), via Simplicial, Lemma 14.15.6. To see that U is equal to this push-out it suffices to see that the intersection of $\Delta[n+1] \amalg \Delta[n+1]$ and $\Delta[1] \times \partial\Delta[n+1]$ in $\Delta[1] \times \Delta[n+1]$ is equal to $\partial\Delta[n+1] \amalg \partial\Delta[n+1]$. This we leave to the reader. $\square$

Lemma 20.6.2. *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K, L be hypercoverings of X. Let $a, b : K \to L$ be morphisms of hypercoverings. There exists a morphism of hypercoverings $c : K' \to K$ such that $a \circ c$ is homotopic to $b \circ c$.*

Proof. Consider the following commutative diagram

$$\begin{tikzcd} K' \arrow[r, equal, "def"] \arrow[dr, "c"'] & K \times_{(L \times L)} Hom(\Delta[1], L) \arrow[r] \arrow[d] & Hom(\Delta[1], L) \arrow[d] \\ & K \arrow[r, "(a,b)"] & L \times L \end{tikzcd}$$

By the functorial property of $Hom(\Delta[1], L)$ the composition of the horizontal morphisms corresponds to a morphism $K'\Delta[1] \to L$ which defines a homotopy between $c \circ a$ and $c \circ b$. Thus if we can show that K' is a hypercovering of X, then we obtain the lemma. To see this we will apply Lemma 20.4.1 to the pair of morphisms $K \to L \times L$ and $Hom(\Delta[1], L) \to L \times L$. Condition (1) of Lemma 20.4.1 is statisfied. Condition (2) of Lemma 20.4.1 is true because $Hom(\Delta[1], L)_0 = L_1$, and the morphism $(d^1_0, d^1_1) : L_1 \to L_0 \times L_0$ is a covering of $\text{SR}(\mathcal{C}, X)$ by our assumption that L is a hypercovering. To prove condition (3) of Lemma 20.4.1 we use Lemma 20.6.1 above. According to this lemma the morphism γ of condition (3) of Lemma 20.4.1 is the morphism

$$Hom(\Delta[1] \times \Delta[n+1], L)_0 \longrightarrow Hom(U, L)_0$$

where $U \subset \Delta[1] \times \Delta[n+1]$. According to Lemma 20.5.2 this is a covering and hence the claim has been proven. □

Remark 20.6.3. Note that the crux of the proof is to use Lemma 20.5.2. This lemma is completely general and does not care about the exact shape of the simplicial sets (as long as they have only finitely many nondegenerate simplices). It seems altogether reasonable to expect a result of the following kind: Given any morphism $a : K \times \partial\Delta[k] \to L$, with K and L hypercoverings, there exists a morphism of hypercoverings $c : K' \to K$ and a morphism $g : K' \times \Delta[k] \to L$ such that $g|_{K' \times \partial\Delta[k]} = a \circ (c \times \text{id}_{\partial\Delta[k]})$. In other words, the category of hypercoverings is in a suitable sense contractible.

20.7. Cech cohomology associated to hypercoverings

Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Consider a presheaf of abelian groups $\mathcal{F}$ on the site $\mathcal{C}$. It defines a functor

$$\begin{array}{rcl} \mathcal{F} : \text{SR}(\mathcal{C}, X)^{opp} & \longrightarrow & Ab \\ \{U_i \to X\}_{i \in I} & \longmapsto & \prod_{i \in I} \mathcal{F}(U_i) \end{array}$$

Thus a simplicial object K of $\text{SR}(\mathcal{C}, X)$ is turned into a cosimplicial object $\mathcal{F}(K)$ of Ab. In this situation we define

$$\check{H}^i(K, \mathcal{F}) = H^i(s(\mathcal{F}(K))).$$

Recall that $s(\mathcal{F}(K))$ is the cochain complex associated to the cosimplicial abelian group $\mathcal{F}(K)$, see Simplicial, Section 14.23. In this section we prove analogues of some of the results for Cech cohomology of open coverings proved in Cohomology, Sections 18.9, 18.10 and 18.11.

Lemma 20.7.1. *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K be a hypercovering of X. Let $\mathcal{F}$ be a sheaf of abelian groups on $\mathcal{C}$. Then $\check{H}^0(K, \mathcal{F}) = \mathcal{F}(X)$.*

Proof. We have

$$\check{H}^0(K, \mathcal{F}) = \text{Ker}(\mathcal{F}(K_0) \longrightarrow \mathcal{F}(K_1))$$

Write $K_0 = \{U_i \to X\}$. It is a covering in the site $\mathcal{C}$. As well, we have that $K_1 \to K_0 \times K_0$ is a covering in $\text{SR}(\mathcal{C}, X)$. Hence we may write $K_1 = \amalg_{i_0, i_1 \in I}\{V_{i_0 i_1 j} \to X\}$ so that the morphism $K_1 \to K_0 \times K_0$ is given by coverings $\{V_{i_0 i_1 j} \to U_{i_0} \times_X U_{i_1}\}$ of the site $\mathcal{C}$. Thus we can further identify

$$\check{H}^0(K, \mathcal{F}) = \text{Ker}(\prod\nolimits_i \mathcal{F}(U_i) \longrightarrow \prod\nolimits_{i_0 i_1 j} \mathcal{F}(V_{i_0 i_1 j}))$$

with obvious map. The sheaf property of $\mathcal{F}$ implies that $\check{H}^0(K, \mathcal{F}) = H^0(X, \mathcal{F})$. □

In fact this property characterizes the abelian sheaves among all abelian presheaves on $\mathcal{C}$ of course. The analogue of Cohomology, Lemma 20.7.2 in this case is the following.

Lemma 20.7.2. *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K be a hypercovering of X. Let $\mathcal{I}$ be an injective sheaf of abelian groups on $\mathcal{C}$. Then*

$$\check{H}^p(K, \mathcal{I}) = \begin{cases} \mathcal{I}(X) & \text{if} \quad p = 0 \\ 0 & \text{if} \quad p > 0 \end{cases}$$

Proof. Observe that for any object $Z = \{U_i \to X\}$ of $\text{SR}(\mathcal{C}, X)$ and any abelian sheaf $\mathcal{F}$ on $\mathcal{C}$ we have

$$\begin{aligned} \mathcal{F}(Z) &= \prod \mathcal{F}(U_i) \\ &= \prod Mor_{PSh(\mathcal{C})}(h_{U_i}, \mathcal{F}) \\ &= Mor_{PSh(\mathcal{C})}(F(Z), \mathcal{F}) \\ &= Mor_{PAb(\mathcal{C})}(\mathbf{Z}_{F(Z)}, \mathcal{F}) \\ &= Mor_{Ab(\mathcal{C})}(\mathbf{Z}^\#_{F(Z)}, \mathcal{F}) \end{aligned}$$

Thus we see, for any simplicial object K of $\text{SR}(\mathcal{C}, X)$ that we have

$$s(\mathcal{F}(K)) = Hom_{Ab(\mathcal{C})}(s(\mathbf{Z}^\#_K), \mathcal{F}) \tag{20.7.2.1}$$

see Definition 20.3.1 for notation. Now, we know that $s(\mathbf{Z}^\#_K)$ is quasi-isomorphic to $\mathbf{Z}^\#_X$ if K is a hypercovering, see Lemma 20.3.5. We conclude that if $\mathcal{I}$ is an injective abelian sheaf, and K a hypercovering, then the complex $s(\mathcal{I}(K))$ is acyclic except possibly in degree 0. In other words, we have

$$\check{H}^i(K, \mathcal{I}) = 0$$

for $i > 0$. Combined with Lemma 20.7.1 the lemma is proved. □

Next we come to the analogue of Cohomology, Lemma 20.7.3. To state it we need to introduce a little more notation. Let $\mathcal{C}$ be a site with fibre products. Let $\mathcal{F}$ be a sheaf of abelian groups on $\mathcal{C}$. The symbol $\underline{H}^i(\mathcal{F})$ indicates the presheaf of abelian groups on $\mathcal{C}$ which is defined by the rule

$$\underline{H}^i(\mathcal{F}) : U \longmapsto H^i(U, \mathcal{F})$$

where U ranges over the objects of $\mathcal{C}$.

Lemma 20.7.3. *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K be a hypercovering of X. Let $\mathcal{F}$ be a sheaf of abelian groups on $\mathcal{C}$. There is a map*

$$s(\mathcal{F}(K)) \longrightarrow R\Gamma(X, \mathcal{F})$$

in $D^+(Ab)$ functorial in $\mathcal{F}$, which induces natural transformations

$$\check{H}^i(K, -) \longrightarrow H^i(X, -)$$

as functors $Ab(\mathcal{C}) \to Ab$. Moreover, there is a spectral sequence $(E_r, d_r)_{r \geq 0}$ with

$$E_2^{p,q} = \check{H}^p(K, \underline{H}^q(\mathcal{F}))$$

converging to $H^{p+q}(X, \mathcal{F})$. This spectral sequence is functorial in $\mathcal{F}$ and in the hypercovering K.

Proof. We could prove this by the same method as employed in the corresponding lemma in the chapter on cohomology. Instead let us prove this by a double complex argument.

Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in the category of abelian sheaves on $\mathcal{C}$. Consider the double complex $A^{\bullet,\bullet}$ with terms

$$A^{p,q} = \mathcal{I}^q(K_p)$$

where the differential $d_1^{p,q} : A^{p,q} \to A^{p+1,q}$ is the one coming from the differential $\mathcal{I}^p \to \mathcal{I}^{p+1}$ and the differential $d_2^{p,q} : A^{p,q} \to A^{p,q+1}$ is the one coming from the differential on the complex $s(\mathcal{I}^p(K))$ associated to the cosimplicial abelian group $\mathcal{I}^p(K)$ as explained above. As usual we denote $sA^\bullet$ the simple complex associated to the double complex $A^{\bullet,\bullet}$. We will use the two spectral sequences $({}'E_r, {}'d_r)$ and $({}''E_r, {}''d_r)$ associated to this double complex, see Homology, Section 10.19.

By Lemma 20.7.2 the complexes $s(\mathcal{I}^p(K))$ are acyclic in positive degrees and have H^0 equal to $\mathcal{I}^p(X)$. Hence by Homology, Lemma 10.19.6 and its proof the spectral sequence $({}'E_r, {}'d_r)$ degenerates, and the natural map

$$\mathcal{I}^\bullet(X) \longrightarrow sA^\bullet$$

is a quasi-isomorphism of complexes of abelian groups. In particular we conclude that $H^n(sA^\bullet) = H^n(X, \mathcal{F})$.

The map $s(\mathcal{F}(K)) \longrightarrow R\Gamma(X, \mathcal{F})$ of the lemma is the composition of the natural map $s(\mathcal{F}(K)) \to sA^\bullet$ followed by the inverse of the displayed quasi-isomorphism above. This works because $\mathcal{I}^\bullet(X)$ is a representative of $R\Gamma(X, \mathcal{F})$.

Consider the spectral sequence $({}''E_r, {}''d_r)_{r\geq 0}$. By Homology, Lemma 10.19.3 we see that

$${}''E_2^{p,q} = H^p_{II}(H^q_I(A^{\bullet,\bullet}))$$

In other words, we first take cohomology with respect to d_1 which gives the groups ${}''E_1^{p,q} = \underline{H}^p(\mathcal{F})(K_q)$. Hence it is indeed the case (by the description of the differential ${}''d_1$) that ${}''E_2^{p,q} = \check{H}^p(K, \underline{H}^q(\mathcal{F}))$. And by the other spectral sequence above we see that this one converges to $H^n(X, \mathcal{F})$ as desired.

We omit the proof of the statements regarding the functoriality of the above constructions in the abelian sheaf $\mathcal{F}$ and the hypercovering K. $\square$

20.8. Cohomology and hypercoverings

Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let $\mathcal{F}$ be a sheaf of abelian groups on $\mathcal{C}$. Let K, L be hypercoverings of X. If $a, b : K \to L$ are homotopic maps, then $\mathcal{F}(a), \mathcal{F}(b) : \mathcal{F}(K) \to \mathcal{F}(L)$ are homotopic maps, see Simplicial, Lemma 14.26.3. Hence have the same effect on cohomology groups of the associated cochain complexes, see Simplicial, Lemma 14.26.5. We are going to use this to define the colimit over all hypercoverings.

Let us temporarily denote $\text{HC}(\mathcal{C}, X)$ the category of hypercoverings of X. We have seen that this is a category and not a ``big'' category, see Lemma 20.2.8. This will be the index category for our diagram, see Categories, Section 4.13 for notation. Consider the diagram

$$\check{H}^i(-, \mathcal{F}) : \text{HC}(\mathcal{C}, X) \longrightarrow Ab.$$

By Lemma 20.4.2 and Lemma 20.6.2, and the remark on homotopies above, this diagram is directed, see Categories, Definition 4.17.1. Thus the colimit

$$\check{H}^i_{\text{HC}}(X, \mathcal{F}) = colim_{K\in \text{HC}(\mathcal{C},X)} \check{H}^i(K, \mathcal{F})$$

has a particularly simple discription (see location cited).

Theorem 20.8.1. *Let $\mathscr{C}$ be a site with fibre products. Let X be an object of $\mathscr{C}$. Let $i \geq 0$. The functors*

$$\begin{array}{ccc} Ab(\mathscr{C}) & \longrightarrow & Ab \\ \mathscr{F} & \longmapsto & H^i(X, \mathscr{F}) \\ \mathscr{F} & \longmapsto & \check{H}^i_{HC}(X, \mathscr{F}) \end{array}$$

are canonically isomorphic.

Proof using spectral sequences. Suppose that $\xi \in H^p(X, \mathscr{F})$ for some $p \geq 0$. Let us show that ξ is in the image of the map $\check{H}^p(X, \mathscr{F}) \to H^p(X, \mathscr{F})$ of Lemma 20.7.3 for some hypercovering K of X.

This is true if $p = 0$ by Lemma 20.7.1. If $p = 1$, choose a Cech hypercovering K of X as in Example 20.2.7 starting with a covering $K_0 = \{U_i \to X\}$ in the site $\mathscr{C}$ such that $\xi|_{U_i} = 0$, see Cohomology on Sites, Lemma 19.8.3. It follows immediately from the spectral sequence in Lemma 20.7.3 that ξ comes from an element of $\check{H}^1(K, \mathscr{F})$ in this case. In general, choose any hypercovering K of X such that ξ maps to zero in $\underline{H}^p(\mathscr{F})(K_0)$ (using Example 20.2.7 and Cohomology on Sites, Lemma 19.8.3 again). By the spectral sequence of Lemma 20.7.3 the obstruction for ξ to come from an element of $\check{H}^p(K, \mathscr{F})$ is a sequence of elements $\xi_1, \ldots, \xi_{p-1}$ with $\xi_q \in \check{H}^{p-q}(K, \underline{H}^q(\mathscr{F}))$ (more precisely the images of the ξ_q in certain subquotients of these groups).

We can inductively replace the hypercovering K by refinements such that the obstructions $\xi_1, \ldots, \xi_{p-1}$ restrict to zero (and not just the images in the subquotients -- so no subtlety here). Indeed, suppose we have already managed to reach the situation where $\xi_{q+1}, \ldots, \xi_{p-1}$ are zero. Note that $\xi_q \in \check{H}^{p-q}(K, \underline{H}^q(\mathscr{F}))$ is the class of some element

$$\tilde{\xi}_q \in \underline{H}^q(\mathscr{F})(K_{p-q}) = \prod H^q(U_i, \mathscr{F})$$

if $K_{p-q} = \{U_i \to X\}_{i \in I}$. Let $\xi_{q,i}$ be the component of $\tilde{\xi}_q$ in $H^q(U_i, \mathscr{F})$. As $q \geq 1$ we can use Cohomology on Sites, Lemma 19.8.3 yet again to choose coverings $\{U_{i,j} \to U_i\}$ of the site such that each restriction $\xi_{q,i}|_{U_{i,j}} = 0$. Consider the object $Z = \{U_{i,j} \to X\}$ of the category $\mathrm{SR}(\mathscr{C}, X)$ and its obvious morphism $u : Z \to K_{p-q}$. It is clear that u is a covering, see Definition 20.2.4. By Lemma 20.4.3 there exists a morphism $L \to K$ of hypercoverings of X such that $L_{p-q} \to K_{p-q}$ factors through u. Then clearly the image of ξ_q in $\underline{H}^q(\mathscr{F})(L_{p-q})$. is zero. Since the spectral sequence of Lemma 20.7.3 is functorial this means that after replacing K by L we reach the situation where $\xi_q, \ldots, \xi_{p-1}$ are all zero. Continuing like this we end up with a hypercovering where they are all zero and hence ξ is in the image of the map $\check{H}^p(X, \mathscr{F}) \to H^p(X, \mathscr{F})$.

Suppose that K is a hypercovering of X, that $\xi \in \check{H}^p(K, \mathscr{F})$ and that the image of ξ under the map $\check{H}^p(X, \mathscr{F}) \to H^p(X, \mathscr{F})$ of Lemma 20.7.3 is zero. To finish the proof of the theorem we have to show that there exists a morphism of hypercoverings $L \to K$ such that ξ restricts to zero in $\check{H}^p(L, \mathscr{F})$. By the spectral sequence of Lemma 20.7.3 the vanishing of the image of ξ in $H^p(X, \mathscr{F})$ means that there exist elements $\xi_1, \ldots, \xi_{p-2}$ with $\xi_q \in \check{H}^{p-1-q}(K, \underline{H}^q(\mathscr{F}))$ (more precisely the images of these in certain subquotients) such that the images $d_{q+1}^{p-1-q,q}\xi_q$ (in the spectral sequence) add up to ξ. Hence by exacly the same mechanism as above we can find a morphism of hypercoverings $L \to K$ such that the restrictions of the elements ξ_q, $q = 1, \ldots, p-2$ in $\check{H}^{p-1-q}(L, \underline{H}^q(\mathscr{F}))$ are zero. Then it follows that ξ is zero since the morphism $L \to K$ induces a morphism of spectral sequences according to Lemma 20.7.3. $\square$

Proof without using spectral sequences. We have seen the result for $i = 0$, see Lemma 20.7.1. We know that the functors $H^i(X, -)$ form a universal δ-functor, see Derived Categories, Lemma 11.19.4. In order to prove the theorem it suffices to show that the sequence of functors $\check{H}^i_{HC}(X, -)$ forms a δ-functor. Namely we know that Cech cohomology is zero on injective sheaves (Lemma 20.7.2) and then we can apply Homology, Lemma 10.9.4.

Let

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

be a short exact sequence of abelian sheaves on $\mathcal{C}$. Let $\xi \in \check{H}^p_{HC}(X, \mathcal{H})$. Choose a hypercovering K of X and an element $\sigma \in \mathcal{H}(K_p)$ representing ξ in cohomology. There is a corresponding exact sequence of complexes

$$0 \to s(\mathcal{F}(K)) \to s(\mathcal{G}(K)) \to s(\mathcal{H}(K))$$

but we are not assured that there is a zero on the right also and this is the only thing that prevents us from defining $\delta(\xi)$ by a simple application of the snake lemma. Recall that

$$\mathcal{H}(K_p) = \prod \mathcal{H}(U_i)$$

if $K_p = \{U_i \to X\}$. Let $\sigma = \prod \sigma_i$ with $\sigma_i \in \mathcal{H}(U_i)$. Since $\mathcal{G} \to \mathcal{H}$ is a surjection of sheaves we see that there exist coverings $\{U_{i,j} \to U_i\}$ such that $\sigma_i|_{U_{i,j}}$ is the image of some element $\tau_{i,j} \in \mathcal{G}(U_{i,j})$. Consider the object $Z = \{U_{i,j} \to X\}$ of the category $\text{SR}(\mathcal{C}, X)$ and its obvious morphism $u : Z \to K_p$. It is clear that u is a covering, see Definition 20.2.4. By Lemma 20.4.3 there exists a morphism $L \to K$ of hypercoverings of X such that $L_p \to K_p$ factors through u. After replacing K by L we may therefore assume that σ is the image of an element $\tau \in \mathcal{G}(K_p)$. Note that $d(\sigma) = 0$, but not necessarily $d(\tau) = 0$. Thus $d(\tau) \in \mathcal{F}(K_{p+1})$ is a cocycle. In this situation we define $\delta(\xi)$ as the class of the cocycle $d(\tau)$ in $\check{H}^{p+1}_{HC}(X, \mathcal{F})$.

At this point there are several things to verify: (a) $\delta(\xi)$ does not depend on the choice of τ, (b) $\delta(\xi)$ does not depend on the choice of the hypercovering $L \to K$ such that σ lifts, and (c) $\delta(\xi)$ does not depend on the initial hypercovering and σ chosen to represent ξ. We omit the verification of (a), (b), and (c); the independence of the choices of the hypercoverings really comes down to Lemmas 20.4.2 and 20.6.2. We also omit the verification that δ is functorial with respect to morphisms of short exact sequences of abelian sheaves on $\mathcal{C}$.

Finally, we have to verify that with this definition of δ our short exact sequence of abelian sheaves above leads to a long exact sequence of Cech cohomology groups. First we show that if $\delta(\xi) = 0$ (with ξ as above) then ξ is the image of some element $\xi' \in \check{H}^p_{HC}(X, \mathcal{G})$. Namely, if $\delta(\xi) = 0$, then, with notation as above, we see that the class of $d(\tau)$ is zero in $\check{H}^{p+1}_{HC}(X, \mathcal{F})$. Hence there exists a morphism of hypercoverings $L \to K$ such that the restriction of $d(\tau)$ to an element of $\mathcal{F}(L_{p+1})$ is equal to $d(v)$ for some $v \in \mathcal{F}(L_p)$. This implies that $\tau|_{L_p} + v$ form a cocycle, and determine a class $\xi' \in \check{H}^p(L, \mathcal{G})$ which maps to ξ as desired.

We omit the proof that if $\xi' \in \check{H}^{p+1}_{HC}(X, \mathcal{F})$ maps to zero in $\check{H}^{p+1}_{HC}(X, \mathcal{G})$, then it is equal to $\delta(\xi)$ for some $\xi \in \check{H}^p_{HC}(X, \mathcal{H})$. □

20.9. Hypercoverings of spaces

The theory above is mildly interesting even in the case of topological spaces. In this case we can work out what is a hypercovering and see what the result actually says.

Let X be a topological space. Consider the site $\mathcal{T}_X$ of Sites, Example 9.6.4. Recall that an object of $\mathcal{T}_X$ is simply an open of X and that morphisms of $\mathcal{T}_X$ correspond simply to inclusions. So what is a hypercovering of X for the site $\mathcal{T}_X$?

Let us first unwind Definition 20.2.1. An object of $\text{SR}(\mathcal{C}, X)$ is simply given by a set I and for each $i \in I$ an open $U_i \subset X$. Let us denote this by $\{U_i\}_{i \in I}$ since there can be no confusion about the morphism $U_i \to X$. A morphism $\{U_i\}_{i \in I} \to \{V_j\}_{j \in J}$ between two such objects is given by a map of sets $\alpha : I \to J$ such that $U_i \subset V_{\alpha(i)}$ for all $i \in I$. When is such a morphism a covering? This is the case if and only if for every $j \in J$ we have $V_j = \bigcup_{i \in I,\ \alpha(i) = j} U_i$ (and is a covering in the site $\mathcal{T}_X$).

Using the above we get the following description of a hypercovering in the site $\mathcal{T}_X$. A hypercovering of X in $\mathcal{T}_X$ is given by the following data

(1) a simplicial set I (see Simplicial, Section 14.11), and
(2) for each $n \geq 0$ and every $i \in I_n$ an open set $U_i \subset X$.

We will denote such a collection of data by the notation $(I, \{U_i\})$. In order for this to be a hypercovering of X we require the following properties

- for $i \in I_n$ and $0 \leq a \leq n + 1$ we have $U_i \subset U_{d^n_a(i)}$,
- for $i \in I_n$ and $0 \leq a \leq n$ we have $U_i = U_{s^n_a(i)}$,
- we have

$$X = \bigcup\nolimits_{i \in I_0} U_i, \tag{20.9.0.1}$$

- for every $i_0, i_1 \in I_0$, we have

$$U_{i_0} \cap U_{i_1} = \bigcup\nolimits_{i \in I_1,\ d^1_0(i) = i_0,\ d^1_1(i) = i_1} U_i, \tag{20.9.0.2}$$

- for every $n \geq 1$ and every $(i_0, \ldots, i_{n+1}) \in (I_n)^{n+2}$ such that $d^n_{b-1}(i_a) = d^n_a(i_b)$ for all $0 \leq a < b \leq n + 1$ we have

$$U_{i_0} \cap \ldots \cap U_{i_{n+1}} = \bigcup\nolimits_{i \in I_{n+1},\ d^{n+1}_a(i) = i_a,\ a = 0, \ldots, n+1} U_i, \tag{20.9.0.3}$$

- each of the open coverings (20.9.0.1), (20.9.0.2), and (20.9.0.3) is an element of $\text{Cov}(\mathcal{T}_X)$ (this is a set theoretic condition, bounding the size of the index sets of the coverings).

Condititions (20.9.0.1) and (20.9.0.2) should be familiar from the chapter on sheaves on spaces for example, and condition (20.9.0.3) is the natural generalization.

Remark 20.9.1. One feature of this description is that if one of the multiple intersections $U_{i_0} \cap \ldots \cap U_{i_{n+1}}$ is empty then the covering on the right hand side may be the empty covering. Thus it is not automatically the case that the maps $I_{n+1} \to (\text{cosk}_n \text{sk}_n I)_{n+1}$ are surjective. This means that the geometric realization of I may be an interesting (non-contractible) space.

In fact, let $I'_n \subset I_n$ be the subset consisting of those simplices $i \in I_n$ such that $U_i \neq \emptyset$. It is easy to see that $I' \subset I$ is a subsimplicial set, and that $(I', \{U_i\})$ is a hypercovering. Hence we can always refine a hypercovering to a hypercovering where none of the opens U_i is empty.

Remark 20.9.2. Let us repackage this information in yet another way. Namely, suppose that $(I, \{U_i\})$ is a hypercovering of the topological space X. Given this data we can construct a simplicial toplogical space $U_\bullet$ by setting

$$U_n = \coprod\nolimits_{i \in I_n} U_i,$$

and where for given $\varphi : [n] \to [m]$ we let morphisms $U(\varphi) : U_n \to U_m$ be the morphism coming from the inclusions $U_i \subset U_{\varphi(i)}$ for $i \in I_n$. This simplicial topological space comes with an augmentation $\epsilon : U_\bullet \to X$. With this morphism the simplicial space $U_\bullet$ becomes a hypercovering of X along which one has cohomological descent in the sense of [**MA71**, Exposé Vbis]. In other words, $H^n(U_\bullet, \epsilon^*\mathcal{F}) = H^n(X, \mathcal{F})$. (Insert future reference here to cohomology over simplicial spaces and cohomological descent formulated in those terms.) Suppose that $\mathcal{F}$ is an abelian sheaf on X. In this case the spectral sequence of Lemma 20.7.3 becomes the spectral sequence with E_1-term

$$E_1^{p,q} = H^q(U_p, \epsilon_q^*\mathcal{F}) \Rightarrow H^{p+q}(U_\bullet, \epsilon^*\mathcal{F}) = H^{p+q}(X, \mathcal{F})$$

comparing the total cohomology of $\epsilon^*\mathcal{F}$ to the cohomology groups of $\mathcal{F}$ over the pieces of $U_\bullet$. (Insert future reference to this spectral sequence here.)

In topology we often want to find hypercoverings of X which have the property that all the U_i come from a given basis for the topology of X and that all the coverings (20.9.0.2) and (20.9.0.3) are from a given cofinal collection of coverings. Here are two example lemmas.

Lemma 20.9.3. *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology of X. There exists a hypercovering $(I, \{U_i\})$ of X such that each U_i is an element of $\mathcal{B}$.*

Proof. Let $n \geq 0$. Let us say that an *n-truncated hypercovering* of X is given by an n-truncated simplicial set I and for each $i \in I_a$, $0 \leq a \leq n$ an open U_i of X such that the conditions defining a hypercovering hold whenever they make sense. In other words we require the inclusion relations and covering conditions only when all simplices that occur in them are a-simplices with $a \leq n$. The lemma follows if we can prove that given a n-truncated hypercovering $(I, \{U_i\})$ with all $U_i \in \mathcal{B}$ we can extend it to an $(n+1)$-truncated hypercovering without adding any a-simplices for $a \leq n$. This we do as follows. First we consider the $(n+1)$-truncated simplicial set I' defined by $I' = \text{sk}_{n+1}(\text{cosk}_n I)$. Recall that

$$I'_{n+1} = \left\{ \begin{matrix} (i_0, \ldots, i_{n+1}) \in (I_n)^{n+2} \text{ such that} \\ d^n_{b-1}(i_a) = d^n_a(i_b) \text{ for all } 0 \leq a < b \leq n+1 \end{matrix} \right\}$$

If $i' \in I'_{n+1}$ is degenerate, say $i' = s^n_a(i)$ then we set $U_{i'} = U_i$ (this is forced on us anyway by the second condition). We also set $J_{i'} = \{i'\}$ in this case. If $i' \in I'_{n+1}$ is nondegerate, say $i' = (i_0, \ldots, i_{n+1})$, then we choose a set $J_{i'}$ and an open covering

$$U_{i_0} \cap \ldots \cap U_{i_{n+1}} = \bigcup\nolimits_{i \in J_{i'}} U_i, \tag{20.9.3.1}$$

with $U_i \in \mathcal{B}$ for $i \in J_{i'}$. Set

$$I_{n+1} = \coprod\nolimits_{i' \in I'_{n+1}} J_{i'}$$

There is a canonical map $\pi : I_{n+1} \to I'_{n+1}$ which is a bijection over the set of degenerate simplices in I'_{n+1} by construction. For $i \in I_{n+1}$ we define $d^{n+1}_a(i) = d^{n+1}_a(\pi(i))$. For $i \in I_n$ we define $s^n_a(i) \in I_{n+1}$ as the unique simplex lying over the degenerate simplex $s^n_a(i) \in I'_{n+1}$. We omit the verification that this defines an $(n+1)$-truncated hypercovering of X. □

Lemma 20.9.4. *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology of X. Assume that*

1. *X is quasi-compact,*
2. *each $U \in \mathcal{B}$ is quasi-compact open, and*
3. *the intersection of any two quasi-compact opens in X is quasi-compact.*

Then there exists a hypercovering $(I, \{U_i\})$ of X with the following properties

(1) *each U_i is an element of the basis $\mathcal{B}$,*
(2) *each of the I_n is a finite set, and in particular*
(3) *each of the coverings (20.9.0.1), (20.9.0.2), and (20.9.0.3) is finite.*

Proof. This follows directly from the construction in the proof of Lemma 20.9.3 if we choose finite coverings by elements of $\mathcal{B}$ in (20.9.3.1). Details omitted. $\square$

20.10. Other chapters

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CHAPTER 21

Schemes

21.1. Introduction

In this document we define schemes. A basic reference is [**DG67**].

21.2. Locally ringed spaces

Recall that we defined ringed spaces in Sheaves, Section 6.25. Briefly, a ringed space is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$. A morphism of ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is given by a continuous map $f : X \to Y$ and an f-map of sheaves of rings $f^\sharp : \mathcal{O}_Y \to \mathcal{O}_X$. You can think of $f^\sharp$ as a map $\mathcal{O}_Y \to f_*\mathcal{O}_X$, see Sheaves, Definition 6.21.7 and Lemma 6.21.8.

A good geometric example of this to keep in mind is $\mathcal{C}^\infty$-manifolds and morphisms of $\mathcal{C}^\infty$-manifolds. Namely, if M is a $\mathcal{C}^\infty$-manifold, then the sheaf $\mathcal{C}^\infty_M$ of smooth functions is a sheaf of rings on M. And any map $f : M \to N$ of manifolds is smooth if and only if for every local section h of $\mathcal{C}^\infty_N$ the composition $h \circ f$ is a local section of $\mathcal{C}^\infty_M$. Thus a smooth map f gives rise in a natural way to a morphism of ringed spaces

$$f : (M, \mathcal{C}^\infty_M) \longrightarrow (N, \mathcal{C}^\infty_N)$$

see Sheaves, Example 6.25.2. It is instructive to consider what happens to stalks. Namely, let $m \in M$ with image $f(m) = n \in N$. Recall that the stalk $\mathcal{C}^\infty_{M,m}$ is the ring of germs of smooth functions at m, see Sheaves, Example 6.11.4. The algebra of germs of functions on (M, m) is a local ring with maximal ideal the functions which vanish at m. Similarly for $\mathcal{C}^\infty_{N,n}$. The map on stalks $f^\sharp : \mathcal{C}^\infty_{N,n} \to \mathcal{C}^\infty_{M,m}$ maps the maximal ideal into the maximal ideal, simply because $f(m) = n$.

In algebraic geometry we study schemes. On a scheme the sheaf of rings is not determined by an intrinsic property of the space. The spectrum of a ring R (see Algebra, Section 7.16) endowed with a sheaf of rings constructed out of R (see below), will be our basic building block. It will turn out that the stalks of $\mathcal{O}$ on $Spec(R)$ are the local rings of R at its primes. There are two reasons to introduce locally ringed spaces in this setting: (1) There is in general no mechanism that assigns to a continuous map of spectra a map of the corresponding rings. This is why we add as an extra datum the map $f^\sharp$. (2) If we consider morphisms of these spectra in the category of ringed spaces, then the maps on stalks may not be local homomorphisms. Since our geometric intuition says it should we introduce locally ringed spaces as follows.

Definition 21.2.1. Locally ringed spaces.

(1) A *locally ringed space* $(X, \mathcal{O}_X)$ is a pair consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ all of whose stalks are local rings.

(2) Given a locally ringed space $(X, \mathcal{O}_X)$ we say that $\mathcal{O}_{X,x}$ is the *local ring of X at x*. We denote $\mathfrak{m}_{X,x}$ or simply $\mathfrak{m}_x$ the maximal ideal of $\mathcal{O}_{X,x}$. Moreover, the *residue field of X at x* is the residue field $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$.
(3) A *morphism of locally ringed spaces* $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces such that for all $x \in X$ the induced ring map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local ring map.

We will usually suppress the sheaf of rings $\mathcal{O}_X$ in the notation when discussing locally ringed spaces. We will simply refer to ``the locally ringed space X''. We will by abuse of notation think of X also as the underlying topological space. Finally we will denote the corresponding sheaf of rings $\mathcal{O}_X$ as the *structure sheaf of X*. In addition, it is customary to denote the maximal ideal of the local ring $\mathcal{O}_{X,x}$ by $\mathfrak{m}_{X,x}$ or simply $\mathfrak{m}_x$. We will say ``let $f : X \to Y$ be a morphism of locally ringed spaces'' thereby surpressing the structure sheaves even further. In this case, we will by abuse of notation think of $f : X \to Y$ also as the underlying continuous map of topological spaces. The f-map corresponding to f will customarily be denoted $f^\sharp$. The condition that f is a morphism of locally ringed spaces can then be expressed by saying that for every $x \in X$ the map on stalks

$$f^\sharp_x : \mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$$

maps the maximal ideal $\mathfrak{m}_{Y,f(x)}$ into $\mathfrak{m}_{X,x}$.

Let us use these notational conventions to show that the collection of locally ringed spaces and morphisms of locally ringed spaces forms a category. In order to see this we have to show that the composition of morphisms of locally ringed spaces is a morphism of locally ringed spaces. OK, so let $f : X \to Y$ and $g : Y \to Z$ be morphism of locally ringed spaces. The composition of f and g is defined in Sheaves, Definition 6.25.3. Let $x \in X$. By Sheaves, Lemma 6.21.10 the composition

$$\mathcal{O}_{Z,g(f(x))} \xrightarrow{g^\sharp} \mathcal{O}_{Y,f(x)} \xrightarrow{f^\sharp} \mathcal{O}_{X,x}$$

is the associated map on stalks for the morphism $g \circ f$. The result follows since a composition of local ring homomorphisms is a local ring homomorphism.

A pleasing feature of the definition is the fact that the functor

$$\textit{Locally ringed spaces} \longrightarrow \textit{Ringed spaces}$$

reflects isomorphisms (plus more). Here is a less abstract statement.

Lemma 21.2.2. *Let X, Y be locally ringed spaces. If $f : X \to Y$ is an isomorphism of ringed spaces, then f is an isomorphism of locally ringed spaces.*

Proof. This follows trivially from the corresponding fact in algebra: Suppose A, B are local rings. Any isomorphism of rings $A \to B$ is a local ring homomorphism. □

21.3. Open immersions of locally ringed spaces

Definition 21.3.1. Let $f : X \to Y$ be a morphism of locally ringed spaces. We say that f is an *open immersion* if f is a homeomorphism of X onto an open subset of Y, and the map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ is an isomorphism.

The following construction is parallel to Sheaves, Definition 6.31.2 (3).

Example 21.3.2. Let X be a locally ringed space. Let $U \subset X$ be an open subset. Let $\mathcal{O}_U = \mathcal{O}_X|_U$ be the restriction of $\mathcal{O}_X$ to U. For $u \in U$ the stalk $\mathcal{O}_{U,u}$ is equal to the stalk $\mathcal{O}_{X,u}$, and hence is a local ring. Thus $(U, \mathcal{O}_U)$ is a locally ringed space and the morphism $j : (U, \mathcal{O}_U) \to (X, \mathcal{O}_X)$ is an open immersion.

Definition 21.3.3. Let X be a locally ringed space. Let $U \subset X$ be an open subset. The locally ringed space $(U, \mathcal{O}_U)$ of Example 21.3.2 above is the *open subspace of X associated to U*.

Lemma 21.3.4. *Let $f : X \to Y$ be an open immersion of locally ringed spaces. Let $j : V = f(X) \to Y$ be the open subspace of Y associated to the image of f. There is a unique isomorphism $f' : X \cong V$ of locally ringed spaces such that $f = j \circ f'$.*

Proof. Omitted. ☐

From now on we do not distinguish between open subsets and their associated subspaces.

Lemma 21.3.5. *Let $f : X \to Y$ be a morphism of locally ringed spaces. Let $U \subset X$, and $V \subset Y$ be open subsets. Suppose that $f(U) \subset V$. There exists a unique morphism of locally ringed spaces $f|_U : U \to V$ such that the following diagram is a commutative square of locally ringed spaces*

$$\begin{tikzcd} U \arrow[r] \arrow[d, "f|_U"'] & X \arrow[d, "f"] \\ V \arrow[r] & Y \end{tikzcd}$$

Proof. Omitted. ☐

In the following we will use without further mention the following fact which follows from the lemma above. Given any morphism $f : Y \to X$ of locally ringed spaces, and any open subset $U \subset X$ such that $f(Y) \subset U$, then there exists a unique morphism of locally ringed spaces $Y \to U$ such that the composition $Y \to U \to X$ is equal to f. In fact, we will even by abuse of notation write $f : Y \to U$ since this rarely gives rise to confusion.

21.4. Closed immersions of locally ringed spaces

We follow our conventions introduced in Modules, Definition 15.13.1.

Definition 21.4.1. Let $i : Z \to X$ be a morphism of locally ringed spaces. We say that i is an *closed immersion* if:

(1) The map i is a homeomorphism of Z onto a closed subset of X.
(2) The map $\mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective; let $\mathcal{I}$ denote the kernel.
(3) The $\mathcal{O}_X$-module $\mathcal{I}$ is locally generated by sections.

Lemma 21.4.2. *Let $f : Z \to X$ be a morphism of locally ringed spaces. In order for f to be a closed immersion it suffices if there exists an open covering $X = \bigcup U_i$ such that each $f : f^{-1}U_i \to U_i$ is a closed immersion.*

Proof. Omitted. ☐

Example 21.4.3. Let X be a locally ringed space. Let $\mathcal{I} \subset \mathcal{O}_X$ be a sheaf of ideals which is locally generated by sections as a sheaf of $\mathcal{O}_X$-modules. Let Z be the support of the sheaf of rings $\mathcal{O}_X/\mathcal{I}$. This is a closed subset of X, by Modules, Lemma 15.5.3. Denote $i : Z \to X$ the inclusion map. By Modules, Lemma 15.6.1 there is a unique sheaf of rings $\mathcal{O}_Z$ on Z with $i_*\mathcal{O}_Z = \mathcal{O}_X/\mathcal{I}$. For any $z \in Z$ the local ring $\mathcal{O}_{Z,z}$ is equal to the quotient ring $\mathcal{O}_{X,x}/\mathcal{I}_x$

and nonzero, hence a local ring. Thus $i : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ is a closed immersion of locally ringed spaces.

Definition 21.4.4. Let X be a locally ringed space. Let $\mathcal{I}$ be a sheaf of ideals on X which is locally generated by sections. The locally ringed space $(Z, \mathcal{O}_Z)$ of Example 21.4.3 above is the *closed subspace of X associated to the sheaf of ideals $\mathcal{I}$.*

Lemma 21.4.5. *Let $f : X \to Y$ be a closed immersion of locally ringed spaces. Let $\mathcal{I}$ be the kernel of the map $\mathcal{O}_Y \to f_*\mathcal{O}_X$. Let $i : Z \to Y$ be the closed subspace of Y associated to $\mathcal{I}$. There is a unique isomorphism $f' : X \cong Z$ of locally ringed spaces such that $f = i \circ f'$.*

Proof. Omitted. $\square$

Lemma 21.4.6. *Let X, Y be a locally ringed spaces. Let $\mathcal{I} \subset \mathcal{O}_X$ be a locally generated sheaf of ideals. Let $i : Z \to X$ be the associated closed subspace. A morphism $f : Y \to X$ factors through Z if and only if the map $f^*\mathcal{I} \to f^*\mathcal{O}_X = \mathcal{O}_Y$ is zero. If this is the case the morphism $g : Y \to Z$ such that $f = i \circ g$ is unique.*

Proof. Clearly if f factors as $Y \to Z \to X$ then the map $f^*\mathcal{I} \to \mathcal{O}_Y$ is zero. Conversely suppose that $f^*\mathcal{I} \to \mathcal{O}_Y$ is zero. Pick any $y \in Y$, and consider the ring map $f^\sharp_y : \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$. Since the composition $\mathcal{I}_y \to \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is zero by assumption and since $f^\sharp_y(1) = 1$ we see that $1 \notin \mathcal{I}_y$, i.e., $\mathcal{I}_y \neq \mathcal{O}_{X,f(y)}$. We conclude that $f(Y) \subset Z = \text{Supp}(\mathcal{O}_X/\mathcal{I})$. Hence $f = i \circ g$ where $g : Y \to Z$ is continuous. Consider the map $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_Y$. The assumption $f^*\mathcal{I} \to \mathcal{O}_Y$ is zero implies that the composition $\mathcal{I} \to \mathcal{O}_X \to f_*\mathcal{O}_Y$ is zero by adjointness of f_* and f^*. In other words, we obtain a morphism of sheaves of rings $\overline{f^\sharp} : \mathcal{O}_X/\mathcal{I} \to f_*\mathcal{O}_Y$. Note that $f_*\mathcal{O}_Y = i_*g_*\mathcal{O}_Y$ and that $\mathcal{O}_X/\mathcal{I} = i_*\mathcal{O}_Z$. By Sheaves, Lemma 6.32.4 we obtain a unique morphism of sheaves of rings $g^\sharp : \mathcal{O}_Z \to g_*\mathcal{O}_Y$ whose pushforward under i is $\overline{f^\sharp}$. We omit the verification that $(g, g^\sharp)$ defines a morphism of locally ringed spaces and that $f = i \circ g$ as a morphism of locally ringed spaces. The uniqueness of $(g, g^\sharp)$ was pointed out above. $\square$

Lemma 21.4.7. *Let $f : X \to Y$ be a morphism of locally ringed spaces. Let $\mathcal{I} \subset \mathcal{O}_Y$ be a sheaf of ideals which is locally generated by functions. Let $i : Z \to Y$ be the closed subspace associated to the sheaf of ideals $\mathcal{I}$. Let $\mathcal{J}$ be the image of the map $f^*\mathcal{I} \to f^*\mathcal{O}_Y = \mathcal{O}_X$. Then this ideal is locally generated by sections. Moreover, let $i' : Z' \to X$ be the associated closed subspace of X. There exists a unique morphism of locally ringed spaces $f' : Z' \to Z$ such that the following diagram is a commutative square of locally ringed spaces*

$$\begin{tikzcd}
Z' \arrow[r, "i'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Z \arrow[r, "i"] & Y
\end{tikzcd}$$

Moreover, this diagram is a fibre square in the category of locally ringed spaces.

Proof. The ideal $\mathcal{J}$ is locally generated by sections by Modules, Lemma 15.8.2. The rest of the lemma follows from the characterization, in Lemma 21.4.6 above, of what it means for a morphism to factor through a closed subscheme. $\square$

21.5. Affine schemes

Let R be a ring. Consider the topological space $Spec(R)$ associated to R, see Algebra, Section 7.16. We will endow this space with a sheaf of rings $\mathcal{O}_{Spec(R)}$ and the resulting pair $(Spec(R), \mathcal{O}_{Spec(R)})$ will be an affine scheme.

Recall that $Spec(R)$ has a basis of open sets $D(f)$, $f \in R$ which we call standard opens, see Algebra, Definition 7.16.3. In addition, the intersection of two standard opens is another: $D(f) \cap D(g) = D(fg)$, $f, g \in R$.

Lemma 21.5.1. *Let R be a ring. Let $f \in R$.*

1. *If $g \in R$ and $D(g) \subset D(f)$, then*
   (a) *f is invertible in R_g,*
   (b) *$g^e = af$ for some $e \geq 1$ and $a \in R$,*
   (c) *there is a canonical ring map $R_f \to R_g$, and*
   (d) *there is a canonical R_f-module map $M_f \to M_g$ for any R-module M.*
2. *Any open covering of $D(f)$ can be refined to a finite open covering of the form $D(f) = \bigcup_{i=1}^{n} D(g_i)$.*
3. *If $g_1, \ldots, g_n \in R$, then $D(f) \subset \bigcup D(g_i)$ if and only if $g_1, \ldots, g_n$ generate the unit ideal in R_f.*

Proof. Recall that $D(g) = Spec(R_g)$ (see Algebra, Lemma 7.16.6). Thus (a) holds because f maps to an element of R_g which is not contained in any prime ideal, and hence invertible, see Algebra, Lemma 7.16.2. Write the inverse of f in R_g as a/g^d. This means $g^d - af$ is annihilated by a power of g, whence (b). For (c), the map $R_f \to R_g$ exists by (a) from the universal property of localization, or we can define it by mapping b/f^n to $a^n b/g^{ne}$. The equality $M_f = M \otimes_R R_f$ can be used to obtain the map on modules, or we can define $M_f \to M_g$ by mapping x/f^n to $a^n x/g^{ne}$.

Recall that $D(f)$ is quasi-compact, see Algebra, Lemma 7.26.1. Hence the second statement follows directly from the fact that the standard opens form a basis for the topology.

The third statement follows directly from Algebra, Lemma 7.16.2. $\square$

In Sheaves, Section 6.30 we defined the notion of a sheaf on a basis, and we showed that it is essentially equivalent to the notion of a sheaf on the space, see Sheaves, Lemmas 6.30.6 and 6.30.9. Moreover, we showed in Sheaves, Lemma 6.30.4 that it is sufficient to check the sheaf condition on a cofinal system of open coverings for each standard open. By the lemma above it suffices to check on the finite coverings by standard opens.

Definition 21.5.2. Let R be a ring.

1. A *standard open covering* of $Spec(R)$ is a covering $Spec(R) = \bigcup_{i=1}^{n} D(f_i)$, where $f_1, \ldots, f_n \in R$.
2. Suppose that $D(f) \subset Spec(R)$ is a standard open. A *standard open covering* of $D(f)$ is a covering $D(f) = \bigcup_{i=1}^{n} D(g_i)$, where $g_1, \ldots, g_n \in R$.

Let R be a ring. Let M be an R-module. We will define a presheaf $\widetilde{M}$ on the basis of standard opens. Suppose that $U \subset Spec(R)$ is a standard open. If $f, g \in R$ are such that $D(f) = D(g)$, then by Lemma 21.5.1 above there are canonical maps $M_f \to M_g$ and $M_g \to M_f$ which are mutually inverse. Hence we may choose any f such that $U = D(f)$ and define

$$\widetilde{M}(U) = M_f.$$

Note that if $D(g) \subset D(f)$, then by Lemma 21.5.1 above we have a canonical map

$$\widetilde{M}(D(f)) = M_f \longrightarrow M_g = \widetilde{M}(D(g)).$$

Clearly, this defines a presheaf of abelian groups on the basis of standard opens. If $M = R$, then $\widetilde{R}$ is a presheaf of rings on the basis of standard opens.

Let us compute the stalk of $\widetilde{M}$ at a point $x \in Spec(R)$. Suppose that x corresponds to the prime $\mathfrak{p} \subset R$. By definition of the stalk we see that

$$\widetilde{M}_x = colim_{f \in R, f \notin \mathfrak{p}} M_f$$

Here the set $\{f \in R, f \notin \mathfrak{p}\}$ is partially ordered by the rule $f \geq f' \Leftrightarrow D(f) \subset D(f')$. If $f_1, f_2 \in R \setminus \mathfrak{p}$, then we have $f_1 f_2 \geq f_1$ in this ordering. Hence by Algebra, Lemma 7.9.9 we conclude that

$$\widetilde{M}_x = M_{\mathfrak{p}}.$$

Next, we check the sheaf condition for the standard open coverings. If $D(f) = \bigcup_{i=1}^n D(g_i)$, then the sheaf condition for this covering is equivalent with the exactness of the sequence

$$0 \to M_f \to \bigoplus M_{g_i} \to \bigoplus M_{g_i g_j}.$$

Note that $D(g_i) = D(fg_i)$, and hence we can rewrite this sequence as the sequence

$$0 \to M_f \to \bigoplus M_{fg_i} \to \bigoplus M_{fg_i g_j}.$$

In addition, by Lemma 21.5.1 above we see that $g_1, \ldots, g_n$ generate the unit ideal in R_f. Thus we may apply Algebra, Lemma 7.20.2 to the module M_f over R_f and the elements $g_1, \ldots, g_n$. We conclude that the sequence is exact. By the remarks made above, we see that $\widetilde{M}$ is a sheaf on the basis of standard opens.

Thus we conclude from the material in Sheaves, Section 6.30 that there exists a unique sheaf of rings $\mathcal{O}_{Spec(R)}$ which agrees with $\widetilde{R}$ on the standard opens. Note that by our computation of stalks above, the stalks of this sheaf of rings are all local rings.

Similarly, for any R-module M there exists a unique sheaf of $\mathcal{O}_{Spec(R)}$-modules $\mathcal{F}$ which agrees with $\widetilde{M}$ on the standard opens, see Sheaves, Lemma 6.30.12.

Definition 21.5.3. Let R be a ring.

(1) The *structure sheaf* $\mathcal{O}_{Spec(R)}$ *of the spectrum of* R is the unique sheaf of rings $\mathcal{O}_{Spec(R)}$ which agrees with $\widetilde{R}$ on the basis of standard opens.
(2) The locally ringed space $(Spec(R), \mathcal{O}_{Spec(R)})$ is called the *spectrum* of R and denoted $Spec(R)$.
(3) The sheaf of $\mathcal{O}_{Spec(R)}$-modules extending $\widetilde{M}$ to all opens of $Spec(R)$ is called the sheaf of $\mathcal{O}_{Spec(R)}$-modules associated to M. This sheaf is denoted $\widetilde{M}$ as well.

We summarize the results obtained so far.

Lemma 21.5.4. *Let R be a ring. Let M be an R-module. Let $\widetilde{M}$ be the sheaf of $\mathcal{O}_{Spec(R)}$-modules associated to M.*

(1) *We have $\Gamma(Spec(R), \mathcal{O}_{Spec(R)}) = R$.*
(2) *We have $\Gamma(Spec(R), \widetilde{M}) = M$ as an R-module.*
(3) *For every $f \in R$ we have $\Gamma(D(f), \mathcal{O}_{Spec(R)}) = R_f$.*
(4) *For every $f \in R$ we have $\Gamma(D(f), \widetilde{M}) = M_f$ as an R_f-module.*

(5) *Whenever $D(g) \subset D(f)$ the restriction mappings on $\mathcal{O}_{Spec(R)}$ and $\widetilde{M}$ are the maps $R_f \to R_g$ and $M_f \to M_g$ from Lemma 21.5.1.*
(6) *Let $\mathfrak{p}$ be a prime of R, and let $x \in Spec(R)$ be the corresponding point. We have $\mathcal{O}_{Spec(R),x} = R_{\mathfrak{p}}$.*
(7) *Let $\mathfrak{p}$ be a prime of R, and let $x \in Spec(R)$ be the corresponding point. We have $\mathcal{F}_x = M_{\mathfrak{p}}$ as an $R_{\mathfrak{p}}$-module.*

Moreover, all these identifications are functorial in the R module M. In particular, the functor $M \mapsto \widetilde{M}$ is an exact functor from the category of R-modules to the category of $\mathcal{O}_{Spec(R)}$-modules.

Proof. Assertions (1) - (7) are clear from the discussion above. The exactness of the functor $M \mapsto \widetilde{M}$ follows from the fact that the functor $M \mapsto M_{\mathfrak{p}}$ is exact and the fact that exactness of short exact sequences may be checked on stalks, see Modules, Lemma 15.3.1. □

Definition 21.5.5. An *affine scheme* is a locally ringed space isomorphic as a locally ringed space to $Spec(R)$ for some ring R. A *morphism of affine schemes* is a morphism in the category of locally ringed spaces.

It turns out that affine schemes play a special role among all locally ringed spaces, which is what the next section is about.

21.6. The category of affine schemes

Note that if Y is an affine scheme, then its points are in canonical $1-1$ bijection with prime ideals in $\Gamma(Y, \mathcal{O}_Y)$.

Lemma 21.6.1. *Let X be a locally ringed space. Let Y be an affine scheme. Let $f \in Mor(X, Y)$ be a morphism of locally ringed spaces. Given a point $x \in X$ consider the ring maps*

$$\Gamma(Y, \mathcal{O}_Y) \xrightarrow{f^\sharp} \Gamma(X, \mathcal{O}_X) \to \mathcal{O}_{X,x}$$

Let $\mathfrak{p} \subset \Gamma(Y, \mathcal{O}_Y)$ denote the inverse image of $\mathfrak{m}_x$. Let $y \in Y$ be the corresponding point. Then $f(x) = y$.

Proof. Consider the commutative diagram

$$\begin{tikzcd} \Gamma(X, \mathcal{O}_X) \arrow[r] & \mathcal{O}_{X,x} \\ \Gamma(Y, \mathcal{O}_Y) \arrow[r] \arrow[u] & \mathcal{O}_{Y,f(x)} \arrow[u] \end{tikzcd}$$

(see the discussion of f-maps below Sheaves, Definition 6.21.7). Since the right vertical arrow is local we see that $\mathfrak{m}_{f(x)}$ is the inverse image of $\mathfrak{m}_x$. The result follows. □

Lemma 21.6.2. *Let X be a locally ringed space. Let $f \in \Gamma(X, \mathcal{O}_X)$. The set*

$$D(f) = \{x \in X \mid image\ f \not\in \mathfrak{m}_x\}$$

is open. Moreover $f|_{D(f)}$ has an inverse.

Proof. This is a special case of Modules, Lemma 15.21.7, but we also give a direct proof. Suppose that $U \subset X$ and $V \subset X$ are two open subsets such that $f|_U$ has an inverse g and $f|_V$ has an inverse h. Then clearly $g|_{U \cap V} = h|_{U \cap V}$. Thus it suffices to show that f is invertible in an open neighbourhood of any $x \in D(f)$. This is clear because $f \not\in \mathfrak{m}_x$ implies that $f \in \mathcal{O}_{X,x}$

has an inverse $g \in \mathcal{O}_{X,x}$ which means there is some open neighbourhood $x \in U \subset X$ so that $g \in \mathcal{O}_X(U)$ and $g \cdot f|_U = 1$. □

Lemma 21.6.3. *In Lemma 21.6.2 above, if X is an affine scheme, then the open $D(f)$ agrees with the standard open $D(f)$ defined previously (in Algebra, Definition 7.16.1).*

Proof. Omitted. □

Lemma 21.6.4. *Let X be a locally ringed space. Let Y be an affine scheme. The map*

$$Mor(X, Y) \longrightarrow Hom(\Gamma(Y, \mathcal{O}_Y), \Gamma(X, \mathcal{O}_X))$$

which maps f to $f^\sharp$ (on global sections) is bijective.

Proof. Since Y is affine we have $(Y, \mathcal{O}_Y) \cong (Spec(R), \mathcal{O}_{Spec(R)})$ for some ring R. During the proof we will use facts about Y and its structure sheaf which are direct consequences of things we know about the spectrum of a ring, see e.g. Lemma 21.5.4.

Motivated by the lemmas above we construct the inverse map. Let $\psi_Y : \Gamma(Y, \mathcal{O}_Y) \to \Gamma(X, \mathcal{O}_X)$ be a ring map. First, we define the corresponding map of spaces

$$\Psi : X \longrightarrow Y$$

by the rule of Lemma 21.6.1. In other words, given $x \in X$ we define $\Psi(x)$ to be the point of Y corresponding to the prime in $\Gamma(Y, \mathcal{O}_Y)$ which is the inverse image of $\mathfrak{m}_x$ under the composition $\Gamma(Y, \mathcal{O}_Y) \xrightarrow{\psi_Y} \Gamma(X, \mathcal{O}_X) \to \mathcal{O}_{X,x}$.

We claim that the map $\Psi : X \to Y$ is continuous. The standard opens $D(g)$, for $g \in \Gamma(Y, \mathcal{O}_Y)$ are a basis for the toppology of Y. Thus it suffices to prove that $\Psi^{-1}(D(g))$ is open. By construction of Ψ the inverse image $\Psi^{-1}(D(g))$ is exactly the set $D(\psi_Y(g)) \subset X$ which is open by Lemma 21.6.2. Hence Ψ is continuous.

Next we construct a Ψ-map of sheaves from $\mathcal{O}_Y$ to $\mathcal{O}_X$. By Sheaves, Lemma 6.30.14 it suffices to define ring maps $\psi_{D(g)} : \Gamma(D(g), \mathcal{O}_Y) \to \Gamma(\Psi^{-1}(D(g)), \mathcal{O}_X)$ compatible with restriction maps. We have a canonical isomorphism $\Gamma(D(g), \mathcal{O}_Y) = \Gamma(Y, \mathcal{O}_Y)_g$, because Y is an affine scheme. Because $\psi_Y(g)$ is invertible on $D(\psi_Y(g))$ we see that there is a canonical map

$$\Gamma(Y, \mathcal{O}_Y)_g \longrightarrow \Gamma(\Psi^{-1}(D(g)), \mathcal{O}_X) = \Gamma(D(\psi_Y(g)), \mathcal{O}_X)$$

extending the map ψ_Y by the universal property of localization. Note that there is no choice but to take the canonical map here! And we take this, combined with the canonical identification $\Gamma(D(g), \mathcal{O}_Y) = \Gamma(Y, \mathcal{O}_Y)_g$, to be $\psi_{D(g)}$. This is compatible with localization since the restriction mapping on the affine schemes are defined in terms of the universal properties of localization also, see Lemmas 21.5.4 and 21.5.1.

Thus we have defined a morphism of ringed spaces $(\Psi, \psi) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ recovering ψ_Y on global sections. To see that it is a morphism of locally ringed spaces we have to show that the induced maps on local rings

$$\psi_x : \mathcal{O}_{Y,\Psi(x)} \longrightarrow \mathcal{O}_{X,x}$$

are local. This follows immediately from the commutative diagram of the proof of Lemma 21.6.1 and the definition of Ψ.

Finally, we have to show that the constructions $(\Psi, \psi) \mapsto \psi_Y$ and the construction $\psi_Y \mapsto (\Psi, \psi)$ are inverse to each other. Clearly, $\psi_Y \mapsto (\Psi, \psi) \mapsto \psi_Y$. Hence the only thing to prove is that given ψ_Y there is at most one pair (Ψ, ψ) giving rise to it. The uniqueness of Ψ was shown in Lemma 21.6.1 and given the uniqueness of Ψ the uniqueness of the map ψ was pointed out during the course of the proof above. □

Lemma 21.6.5. *The category of affine schemes is equivalent to the opposite of the category of rings. The equivalence is given by the functor that associates to an affine scheme the global sections of its structure sheaf.*

Proof. This is now clear from Definition 21.5.5 and Lemma 21.6.4. □

Lemma 21.6.6. *Let Y be an affine scheme. Let $f \in \Gamma(Y, \mathcal{O}_Y)$. The open subspace $D(f)$ is an affine scheme.*

Proof. We may assume that $Y = Spec(R)$ and $f \in R$. Consider the morphism of affine schemes $\phi : U = Spec(R_f) \to Spec(R) = Y$ induced by the ring map $R \to R_f$. By Algebra, Lemma 7.16.6 we know that it is a homeomorphism onto $D(f)$. On the other hand, the map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_U$ is an isomorphism on stalks, hence an isomorphism. Thus we see that ϕ is an open immersion. We conclude that $D(f)$ is isomorphic to U by Lemma 21.3.4. □

Lemma 21.6.7. *The category of affine schemes has finite products, and fibre products. In other words, it has finite limits. Moreover, the products and fibre products in the category of affine schemes are the same as in the category of locally ringed spaces. In a formula, we have (in the category of locally ringed spaces)*

$$Spec(R) \times Spec(S) = Spec(R \otimes_{\mathbf{Z}} S)$$

and given ring maps $R \to A$, $R \to B$ we have

$$Spec(A) \times_{Spec(R)} Spec(B) = Spec(A \otimes_R B).$$

Proof. This is just an application of Lemma 21.6.4. First of all, by that lemma, the affine scheme $Spec(\mathbf{Z})$ is the final object in the category of locally ringed spaces. Thus the first displayed formula follows from the second. To prove the second note that for any locally ringed space X we have

$$\begin{aligned} Mor(X, Spec(A \otimes_R B)) &= Hom(A \otimes_R B, \mathcal{O}_X(X)) \\ &= Hom(A, \mathcal{O}_X(X)) \times_{Hom(R, \mathcal{O}_X(X))} Hom(B, \mathcal{O}_X(X)) \\ &= Mor(X, Spec(A)) \times_{Mor(X, Spec(R))} Mor(X, Spec(B)) \end{aligned}$$

which proves the formula. See Categories, Section 4.6 for the relevant definitions. □

Lemma 21.6.8. *Let X be a locally ringed space. Assume $X = U \amalg V$ with U and V open and such that U, V are affine schemes. Then X is an affine scheme.*

Proof. Set $R = \Gamma(X, \mathcal{O}_X)$. Note that $R = \mathcal{O}_X(U) \times \mathcal{O}_X(V)$ by the sheaf property. By Lemma 21.6.4 there is a canonical morphism of locally ringed spaces $X \to Spec(R)$. By Algebra, Lemma 7.18.2 we see that as a topological space $Spec(\mathcal{O}_X(U)) \amalg Spec(\mathcal{O}_X(V)) = Spec(R)$ with the maps coming from the ring homomorphisms $R \to \mathcal{O}_X(U)$ and $R \to \mathcal{O}_X(V)$. This of course means that $Spec(R)$ is the coproduct in the category of locally ringed spaces as well. By assumption the morphism $X \to Spec(R)$ induces an isomorphism of $Spec(\mathcal{O}_X(U))$ with U and similarly for V. Hence $X \to Spec(R)$ is an isomorphism. □

21.7. Quasi-Coherent sheaves on affines

Recall that we have defined the abstract notion of a quasi-coherent sheaf in Modules, Definition 15.10.1. In this section we show that any quasi-coherent sheaf on an affine scheme $Spec(R)$ corresponds the the sheaf $\widetilde{M}$ associated to an R-module M.

Lemma 21.7.1. *Let $(X, \mathcal{O}_X) = (Spec(R), \mathcal{O}_{Spec(R)})$ be an affine scheme. Let M be an R-module. There exists a canonical isomorphism between the sheaf $\widetilde{M}$ associated to the R-module M (Definition 21.5.3) and the sheaf $\mathcal{F}_M$ associated to the R-module M (Modules, Definition 15.10.6). This isomorphism is functorial in M. In particular, the sheaves $\widetilde{M}$ are quasi-coherent. Moreover, they are characterized by the following mapping property*

$$Hom_{\mathcal{O}_X}(\widetilde{M}, \mathcal{F}) = Hom_R(M, \Gamma(X, \mathcal{F}))$$

for any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$. Here a map $\alpha : \widetilde{M} \to \mathcal{F}$ corresponds to its effect on global sections.

Proof. By Modules, Lemma 15.10.5 we have a morphism $\mathcal{F}_M \to \widetilde{M}$ corresponding to the map $M \to \Gamma(X, \widetilde{M}) = M$. Let $x \in X$ correspond to the prime $\mathfrak{p} \subset R$. The induced map on stalks are the maps $\mathcal{O}_{X,x} \otimes_R M \to M_{\mathfrak{p}}$ which are isomorphisms because $R_{\mathfrak{p}} \otimes_R M = M_{\mathfrak{p}}$. Hence the map $\mathcal{F}_M \to \widetilde{M}$ is an isomorphism. The mapping property follows from the mapping property of the sheaves $\mathcal{F}_M$. □

Lemma 21.7.2. *Let $(X, \mathcal{O}_X) = (Spec(R), \mathcal{O}_{Spec(R)})$ be an affine scheme. There are canonical isomorphisms*

1. *$\widetilde{M \otimes_R N} \cong \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$, see Modules, Section 15.15.*
2. *$\widetilde{T^n(M)} \cong T^n(\widetilde{M})$, $\widetilde{Sym^n(M)} \cong Sym^n(\widetilde{M})$, and $\widetilde{\wedge^n(M)} \cong \wedge^n(\widetilde{M})$, see Modules, Section 15.18.*
3. *if M is a finitely presented R-module, then $\mathcal{H}om_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}) \cong \widetilde{Hom_R(M, N)}$, see Modules, Section 15.19.*

Proof. To give a map $\widetilde{M \otimes_R N}$ into $\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$ we have to give a map on global sections $M \otimes_R N \to \Gamma(X, \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N})$ which exists by definition of the tensor product of sheaves of modules. To see that this map is an isomorphism it suffices to check that it is an isomorphism on stalks. And this follows from the description of the stalks of $\widetilde{M}$ (as a functor) and Modules, Lemma 15.15.1.

The proof of (2) is similar, using Modules, Lemma 15.18.2.

For (3) note that if M is finitely presented as an R-module then $\widetilde{M}$ has a global finite presentation as an $\mathcal{O}_X$-module. Hence Modules, Lemma 15.19.3 applies. □

Lemma 21.7.3. *Let $(X, \mathcal{O}_X) = (Spec(S), \mathcal{O}_{Spec(S)})$, $(Y, \mathcal{O}_Y) = (Spec(R), \mathcal{O}_{Spec(R)})$ be affine schemes. Let $\psi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of affine schemes, corresponding to the ring map $\psi^\sharp : R \to S$ (see Lemma 21.6.5).*

1. *We have $\psi^* \widetilde{M} = \widetilde{S \otimes_R M}$ functorially in the R-module M.*
2. *We have $\psi_* \widetilde{N} = \widetilde{N_R}$ functorially in the S-module N.*

Proof. The first assertion follows from the identification in Lemma 21.7.1 and the result of Modules, Lemma 15.10.7. The second assertion follows from the fact that $\psi^{-1}(D(f)) = D(\psi^\sharp(f))$ and hence

$$\psi_* \widetilde{N}(D(f)) = \widetilde{N}(D(\psi^\sharp(f))) = N_{\psi^\sharp(f)} = (N_R)_f = \widetilde{N_R}(D(f))$$

as desired. □

Lemma 21.7.3 above says in particular that if you restrict the sheaf $\widetilde{M}$ to a standard affine open subspace $D(f)$, then you get $\widetilde{M_f}$. We will use this from now on without further mention.

Lemma 21.7.4. *Let $(X, \mathcal{O}_X) = (Spec(R), \mathcal{O}_{Spec(R)})$ be an affine scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is isomorphic to the sheaf associated to the R-module $\Gamma(X, \mathcal{F})$.*

Proof. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Since every standard open $D(f)$ is quasi-compact we see that X is a locally quasi-compact, i.e., every point has a fundamental system of quasi-compact neighbourhoods, see Topology, Definition 5.18.1. Hence by Modules, Lemma 15.10.8 for every prime $\mathfrak{p} \subset R$ corresponding to $x \in X$ there exists an open neighbourhood $x \in U \subset X$ such that $\mathcal{F}|_U$ is isomorphic to the quasi-coherent sheaf associated to some $\mathcal{O}_X(U)$-module M. In other words, we get an open covering by U's with this property. By Lemma 21.5.1 for example we can refine this covering to a standard open covering. Thus we get a covering $Spec(R) = \bigcup D(f_i)$ and R_{f_i}-modules M_i and isomorphisms $\varphi_i : \mathcal{F}|_{D(f_i)} \to \mathcal{F}_{M_i}$ for some R_{f_i}-module M_i. On the overlaps we get isomorphisms

$$\mathcal{F}_{M_i}|_{D(f_i f_j)} \xrightarrow{\varphi_i^{-1}|_{D(f_i f_j)}} \mathcal{F}|_{D(f_i f_j)} \xrightarrow{\varphi_j|_{D(f_i f_j)}} \mathcal{F}_{M_j}|_{D(f_i f_j)}.$$

Let us denote these ψ_{ij}. It is clear that we have the cocycle condition

$$\psi_{jk}|_{D(f_i f_j f_k)} \circ \psi_{ij}|_{D(f_i f_j f_k)} = \psi_{ik}|_{D(f_i f_j f_k)}$$

on triple overlaps.

Recall that each of the open subspaces $D(f_i)$, $D(f_i f_j)$, $D(f_i f_j f_k)$ is an affine scheme. Hence the sheaves $\mathcal{F}_{M_i}$ are isomorphic to the sheaves $\widetilde{M_i}$ by Lemma 21.7.1 above. In particular we see that $\mathcal{F}_{M_i}(D(f_i f_j)) = (M_i)_{f_j}$, etc. Also by Lemma 21.7.1 above we see that ψ_{ij} corresponds to a unique $R_{f_i f_j}$-module isomorphism

$$\psi_{ij} : (M_i)_{f_j} \longrightarrow (M_j)_{f_i}$$

namely, the effect of ψ_{ij} on sections over $D(f_i f_j)$. Moreover these then satisfy the cocycle condition that

$$\begin{tikzcd}
(M_i)_{f_j f_k} \arrow[rr, "\psi_{ik}"] \arrow[dr, "\psi_{ij}"'] & & (M_k)_{f_i f_j} \\
& (M_j)_{f_i f_k} \arrow[ur, "\psi_{jk}"'] &
\end{tikzcd}$$

commutes (for any triple i, j, k).

Now Algebra, Lemma 7.21.4 shows that there exist an R-module M such that $M_i = M_{f_i}$ compatible with the morphisms ψ_{ij}. Consider $\mathcal{F}_M = \widetilde{M}$. At this point it is a formality to show that $\widetilde{M}$ is isomorphic to the quasi-coherent sheaf $\mathcal{F}$ we started out with. Namely, the sheaves $\mathcal{F}$ and $\widetilde{M}$ give rise to isomorphic sets of glueing data of sheaves of $\mathcal{O}_X$-modules with respect to the covering $X = \bigcup D(f_i)$, see Sheaves, Section 6.33 and in particular Lemma 6.33.4. Explicitly, in the current situation, this boils down to the following argument: Let us construct an R-module map

$$M \longrightarrow \Gamma(X, \mathcal{F}).$$

Namely, given $m \in M$ we get $m_i = m/1 \in M_{f_i} = M_i$ by construction of M. By construction of M_i this corresponds to a section $s_i \in \mathcal{F}(U_i)$. (Namely, $\varphi_i^{-1}(m_i)$.) We claim that $s_i|_{D(f_i f_j)} = s_j|_{D(f_i f_j)}$. This is true because, by construction of M, we have $\psi_{ij}(m_i) = m_j$, and by the construction of the ψ_{ij}. By the sheaf condition of $\mathcal{F}$ this collection of sections gives rise to a unique section s of $\mathcal{F}$ over X. We leave it to the reader to show that $m \mapsto s$ is a R-module map. By Lemma 21.7.1 we obtain an associated $\mathcal{O}_X$-module map

$$\widetilde{M} \longrightarrow \mathcal{F}.$$

By construction this map reduces to the isomorphisms φ_i^{-1} on each $D(f_i)$ and hence is an isomorphism. □

Lemma 21.7.5. *Let $(X, \mathcal{O}_X) = (Spec(R), \mathcal{O}_{Spec(R)})$ be an affine scheme. The functors $M \mapsto \widetilde{M}$ and $\mathcal{F} \mapsto \Gamma(X, \mathcal{F})$ define quasi-inverse equivalences of categories*

$$QCoh(\mathcal{O}_X) \rightleftarrows Mod\text{-}R$$

between the category of quasi-coherent $\mathcal{O}_X$-modules and the category of R-modules.

Proof. See Lemmas 21.7.1 and 21.7.4 above. □

From now on we will not distinghuish between quasi-coherent sheaves on affine schemes and sheaves of the form $\widetilde{M}$.

Lemma 21.7.6. *Let $X = Spec(R)$ be an affine scheme. Kernels and cokernels of maps of quasi-coherent $\mathcal{O}_X$-modules are quasi-coherent.*

Proof. This follows from the exactness of the functor $\tilde{\ }$ since by Lemma 21.7.1 we know that any map $\psi : \widetilde{M} \to \widetilde{N}$ comes from an R-module map $\varphi : M \to N$. (So we have $\mathrm{Ker}(\psi) = \widetilde{\mathrm{Ker}(\varphi)}$ and $\mathrm{Coker}(\psi) = \widetilde{\mathrm{Coker}(\varphi)}$.) □

Lemma 21.7.7. *Let $X = Spec(R)$ be an affine scheme. The direct sum of an arbitrary collection of quasi-coherent sheaves on X is quasi-coherent. The same holds for colimits.*

Proof. Suppose $\mathcal{F}_i$, $i \in I$ is a collection of quasi-coherent sheaves on X. By Lemma 21.7.5 above we can write $\mathcal{F}_i = \widetilde{M_i}$ for some R-module M_i. Set $M = \bigoplus M_i$. Consider the sheaf $\widetilde{M}$. For each standard open $D(f)$ we have

$$\widetilde{M}(D(f)) = M_f = \left(\bigoplus M_i\right)_f = \bigoplus M_{i,f}.$$

Hence we see that the quasi-coherent $\mathcal{O}_X$-module $\widetilde{M}$ is the direct sum of the sheaves $\mathcal{F}_i$. A similar argument works for general colimits. □

Lemma 21.7.8. *Let $(X, \mathcal{O}_X) = (Spec(R), \mathcal{O}_{Spec(R)})$ be an affine scheme. Suppose that*

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

is a short exact sequence of sheaves $\mathcal{O}_X$-modules. If two out of three are quasi-coherent then so is the third.

Proof. This is clear in case both $\mathcal{F}_1$ and $\mathcal{F}_2$ are quasi-coherent because the functor $M \mapsto \widetilde{M}$ is exact, see Lemma 21.5.4. Similarly in case both $\mathcal{F}_2$ and $\mathcal{F}_3$ are quasi-coherent. Now, suppose that $\mathcal{F}_1 = \widetilde{M_1}$ and $\mathcal{F}_3 = \widetilde{M_3}$ are quasi-coherent. Set $M_2 = \Gamma(X, \mathcal{F}_2)$. We claim it suffices to show that the sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is exact. Namely, if this is the case, then (by using the mapping property of Lemma 21.7.1) we get a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{M}_1 \arrow[r] \arrow[d] & \widetilde{M}_2 \arrow[r] \arrow[d] & \widetilde{M}_3 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{F}_1 \arrow[r] & \mathcal{F}_2 \arrow[r] & \mathcal{F}_3 \arrow[r] & 0 \end{tikzcd}$$

and we win by the snake lemma.

The ``correct'' argument here would be to show first that $H^1(X, \mathcal{F}) = 0$ for any quasi-coherent sheaf $\mathcal{F}$. This is actually not all that hard, but it is perhaps better to postpone this till later. Instead we use a small trick.

Pick $m \in M_3 = \Gamma(X, \mathcal{F}_3)$. Consider the following set

$$I = \{f \in R \mid \text{the element } fm \text{ comes from } M_2\}.$$

Clearly this is an ideal. It suffices to show $1 \in I$. Hence it suffices to show that for any prime $\mathfrak{p}$ there exists an $f \in I$, $f \not\in \mathfrak{p}$. Let $x \in X$ be the point corresponding to $\mathfrak{p}$. Because surjectivity can be checked on stalks there exists an open neighbourhood U of x such that $m|_U$ comes from a local section $s \in \mathcal{F}_2(U)$. In fact we may assume that $U = D(f)$ is a standard open, i.e., $f \in R$, $f \not\in \mathfrak{p}$. We will show that for some $N \gg 0$ we have $f^N \in I$, which will finish the proof.

Take any point $z \in V(f)$, say corresponding to the prime $\mathfrak{q} \subset R$. We can also find a $g \in R$, $g \not\in \mathfrak{q}$ such that $m|_{D(g)}$ lifts to some $s' \in \mathcal{F}_2(D(g))$. Consider the difference $s|_{D(fg)} - s'|_{D(fg)}$. This is an element m' of $\mathcal{F}_1(D(fg)) = (M_1)_{fg}$. For some integer $n = n(z)$ the element $f^n m'$ comes from some $m'_1 \in (M_1)_g$. We see that $f^n s$ extends to a section σ of $\mathcal{F}_2$ on $D(f) \cup D(g)$ because it agrees with the restriction of $f^n s' + m'_1$ on $D(f) \cap D(g) = D(fg)$. Moreover, σ maps to the restriction of $f^n m$ to $D(f) \cup D(g)$.

Since $V(f)$ is quasi-compact, there exists a finite list of elements $g_1, \ldots, g_m \in R$ such that $V(f) \subset \bigcup D(g_j)$, an integer $n > 0$ and sections $\sigma_j \in \mathcal{F}_2(D(f) \cup D(g_j))$ such that $\sigma_j|_{D(f)} = f^n s$ and σ_j maps to the section $f^n m|_{D(f) \cup D(g_j)}$ of $\mathcal{F}_3$. Consider the differences

$$\sigma_j|_{D(f) \cup D(g_j g_k)} - \sigma_k|_{D(f) \cup D(g_j g_k)}.$$

These correspond to sections of $\mathcal{F}_1$ over $D(f) \cup D(g_j g_k)$ which are zero on $D(f)$. In particular their images in $\mathcal{F}_1(D(g_j g_k)) = (M_1)_{g_j g_k}$ are zero in $(M_1)_{g_j g_k f}$. Thus some high power of f kills each and every one of these. In other words, the elements $f^N \sigma_j$, for some $N \gg 0$ satisfy the glueing condition of the sheaf property and give rise to a section σ of $\mathcal{F}_2$ over $\bigcup (D(f) \cup D(g_j)) = X$ as desired. □

21.8. Closed subspaces of affine schemes

Example 21.8.1. Let R be a ring. Let $I \subset R$ be an ideal. Consider the morphism of affine schemes $i : Z = Spec(R/I) \to Spec(R) = X$. By Algebra, Lemma 7.16.7 this is a homeomorphism of Z onto a closed subset of X. Moreover, if $I \subset \mathfrak{p} \subset R$ is a prime corresponding to a point $x = i(z)$, $x \in X$, $z \in Z$, then on stalks we get the map

$$\mathcal{O}_{X,x} = R_{\mathfrak{p}} \longrightarrow R_{\mathfrak{p}}/IR_{\mathfrak{p}} = \mathcal{O}_{Z,z}$$

Thus we see that i is a closed immersion of locally ringed spaces, see Definition 21.4.1. Clearly, this is (isomorphic) to the closed subspace associated to the quasi-coherent sheaf of ideals $\tilde{I}$, as in Example 21.4.3.

Lemma 21.8.2. *Let $(X, \mathcal{O}_X) = (Spec(R), \mathcal{O}_{Spec(R)})$ be an affine scheme. Let $i : Z \to X$ be any closed immersion of locally ringed spaces. Then there exists an unique ideal $I \subset R$ such that the morphism $i : Z \to X$ can be identified with the closed immersion $Spec(R/I) \to Spec(R)$ constructed in Example 21.8.1 above.*

Proof. This is kind of silly! Namely, by Lemma 21.4.5 we can identify $Z \to X$ with the closed subspace associated to a sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ as in Definition 21.4.4 and Example 21.4.3. By our conventions this sheaf of ideals is locally generated by sections as a sheaf of $\mathcal{O}_X$-modules. Hence the quotient sheaf $\mathcal{O}_X/\mathcal{I}$ is locally on X the cokernel of a map $\bigoplus_{j \in J} \mathcal{O}_U \to \mathcal{O}_U$. Thus by definition, $\mathcal{O}_X/\mathcal{I}$ is quasi-coherent. By our results in Section 21.7 it is of the form $\widetilde{S}$ for some R-module S. Moreover, since $\mathcal{O}_X = \widetilde{R} \to \widetilde{S}$ is surjective we see by Lemma 21.7.8 that also $\mathcal{I}$ is quasi-coherent, say $\mathcal{I} = \tilde{I}$. Of course $I \subset R$ and $S = R/I$ and everything is clear. □

21.9. Schemes

Definition 21.9.1. A *scheme* is a locally ringed space with the property that every point has an open neighbourhood which is an affine scheme. A *morphism of schemes* is a morphism of locally ringed spaces. The category of schemes will be denoted *Sch*.

Let X be a scheme. We will use the following (very slight) abuse of language. We will say $U \subset X$ is an *affine open*, or an *open affine* if the open subspace U is an affine scheme. We will often write $U = Spec(R)$ to indicate that U is isomorphic to $Spec(R)$ and moreover that we will identify (temporarily) U and $Spec(R)$.

Lemma 21.9.2. *Let X be a scheme. Let $j : U \to X$ be an open immersion of locally ringed spaces. Then U is a scheme. In particular, any open subspace of X is a scheme.*

Proof. Let $U \subset X$. Let $u \in U$. Pick an affine open neighbourhood $u \in V \subset X$. Because standard opens of V form a basis of the topology on V we see that there exists a $f \in \mathcal{O}_V(V)$ such that $D(f) \subset U$. And $D(f)$ is an affine scheme by Lemma 21.6.6. This proves that every point of U has an open neighbourhood which is affine. □

Clearly the lemma (or its proof) shows that any scheme X has a basis (see Topology, Section 5.3) for the topology consisting of affine opens.

Example 21.9.3. Let k be a field. An example of a scheme which is not affine is given by the open subspace $U = Spec(k[x, y]) \setminus \{(x, y)\}$ of the affine scheme $X = Spec(k[x, y])$. It is covered by two affines, namely $D(x) = Spec(k[x, y, 1/x])$ and $D(y) = Spec(k[x, y, 1/y])$ whose intersection is $D(xy) = Spec(k[x, y, 1/xy])$. By the sheaf property for $\mathcal{O}_U$ there is an exact sequence

$$0 \to \Gamma(U, \mathcal{O}_U) \to k[x, y, 1/x] \times k[x, y, 1/y] \to k[x, y, 1/xy]$$

We conclude that the map $k[x, y] \to \Gamma(U, \mathcal{O}_U)$ (coming from the morphism $U \to X$) is an isomorphism. Therefore U cannot be affine since if it was then by Lemma 21.6.5 we would have $U \cong X$.

21.10. Immersions of schemes

In Lemma 21.9.2 we saw that any open subspace of a scheme is a scheme. Below we will prove that the same holds for a closed subspace of a scheme.

Note that the notion of a quasi-coherent sheaf of $\mathcal{O}_X$-modules is defined for any ringed space X in particular when X is a scheme. By our efforts in Section 21.7 we know that such a sheaf is on any affine open $U \subset X$ of the form $\widetilde{M}$ for some $\mathcal{O}_X(U)$-module M.

Lemma 21.10.1. *Let X be a scheme. Let $i : Z \to X$ be a closed immersion of locally ringed spaces.*

1. *The locally ringed space Z is a scheme,*
2. *the kernel $\mathcal{I}$ of the map $\mathcal{O}_X \to i_*\mathcal{O}_Z$ is a quasi-coherent sheaf of ideals,*
3. *for any affine open $U = Spec(R)$ of X the morphism $i^{-1}(U) \to U$ can be identified with $Spec(R/I) \to Spec(R)$ for some ideal $I \subset R$, and*
4. *we have $\mathcal{I}|_U = \tilde{I}$.*

In particular, any sheaf of ideals locally generated by sections is a quasi-coherent sheaf of ideals (and vice versa), and any closed subspace of X is a scheme.

Proof. Let $i : Z \to X$ be a closed immersion. Let $z \in Z$ be a point. Choose any affine open neighbourhood $i(z) \in U \subset X$. Say $U = Spec(R)$. By Lemma 21.8.2 we know that $i^{-1}(U) \to U$ can be identified with the morphism of affine schemes $Spec(R/I) \to Spec(R)$. First of all this implies that $z \in i^{-1}(U) \subset Z$ is an affine neighbourhood of z. Thus Z is a scheme. Second this implies that $\mathcal{I}|_U$ is $\tilde{I}$. In other words for every point $x \in i(Z)$ there exists an open neighbourhood such that $\mathcal{I}$ is quasi-coherent in that neighbourhood. Note that $\mathcal{I}|_{X \setminus i(Z)} \cong \mathcal{O}_{X \setminus i(Z)}$. Thus the restriction of the sheaf of ideals is quasi-coherent on $X \setminus i(Z)$ also. We conclude that $\mathcal{I}$ is quasi-coherent. □

Definition 21.10.2. Let X be a scheme.

1. A morphism of schemes is called an *open immersion* if it is an open immersion of locally ringed spaces (see Definition 21.3.1).
2. An *open subscheme* of X is an open subspace of X which is a scheme by Lemma 21.9.2 above.
3. A morphism of schemes is called a *closed immersion* if it is a closed immersion of locally ringed spaces (see Definition 21.4.1).
4. A *closed subscheme* of X is a closed subspace of X which is a scheme by Lemma 21.10.1 above.
5. A morphism of schemes $f : X \to Y$ is called an *immersion*, or a *locally closed immersion* if it can be factored as $j \circ i$ where i is a closed immersion and j is an open immersion.

It follows from the lemmas in Sections 21.3 and 21.4 that any open (resp. closed) immersion of schemes is isomorphic to the inclusion of an open (resp. closed) subscheme of the target. We will define locally closed subschemes below.

Remark 21.10.3. If $f : X \to Y$ is an immersion of schemes, then it is in general not possible to factor f as an open immersion followed by a closed immersion. See Morphisms, Example 24.2.10.

Lemma 21.10.4. *Let $f : Y \to X$ be an immersion of schemes. Then f is a closed immersion if and only if $f(Y) \subset X$ is a closed subset.*

Proof. If f is a closed immersion then $f(Y)$ is closed by definition. Conversely, suppose that $f(Y)$ is closed. By definition there exists an open subscheme $U \subset X$ such that f is the composition of a closed immersion $i : Y \to U$ and the open immersion $j : U \to X$. Let $\mathcal{I} \subset \mathcal{O}_U$ be the quasi-coherent sheaf of ideals associated to the closed immerion i. Note that $\mathcal{I}|_{U \setminus i(Y)} = \mathcal{O}_{U \setminus i(Y)} = \mathcal{O}_{X \setminus i(Y)}|_{U \setminus i(Y)}$. Thus we may glue (see Sheaves, Section 6.33) $\mathcal{I}$ and $\mathcal{O}_{X \setminus i(Y)}$ to a sheaf of ideals $\mathcal{J} \subset \mathcal{O}_X$. Since every point of X has a neighbourhood where $\mathcal{J}$ is quasi-coherent, we see that $\mathcal{J}$ is quasi-coherent (in particular locally generated by sections). By construction $\mathcal{O}_X/\mathcal{J}$ is supported on U and equal to $\mathcal{O}_U/\mathcal{I}$. Thus we see that the closed subspaces associated to $\mathcal{I}$ and $\mathcal{J}$ are canonically isomorphic, see Example 21.4.3. In particular the closed subspace of U associated to $\mathcal{I}$ is isomorphic to a closed subspace of X. Since $Y \to U$ is identified with the closed subspace associated to $\mathcal{I}$, see Lemma 21.4.5, we conclude that $Y \to U \to X$ is a closed immersion. $\square$

Let $f : Y \to X$ be an immersion. Let $Z = \overline{f(Y)} \setminus f(Y)$ which is a closed subset of X. Let $U = X \setminus Z$. The lemma implies that U is the biggest open subspace of X such that $f : Y \to X$ factors through a closed immersion into U. If we define a *locally closed subscheme of* X as a pair (Z, U) consisting of a closed subscheme Z of an open subscheme U of X such that in addition $\overline{Z} \cup U = X$. We usually just say ``let Z be a locally closed subscheme of X" since we may recover U from the morphism $Z \to X$. The above then shows that any immersion $f : Y \to X$ factors uniquely as $Y \to Z \to X$ where Z is a locally closed subspace of X and $Y \to Z$ is an isomorphism.

The interest of this is that the collection of locally closed subschemes of X forms a set. We may define a partial ordering on this set, which we call inclusion for obvious reasons. To be explicit, if $Z \to X$ and $Z' \to X$ are two locally closed subschemes of X, then we say that Z *is contained in* Z' simply if the morphism $Z \to X$ factors through Z'. If it does, then of course Z is identified with a unique locally closed subscheme of Z', and so on.

21.11. Zariski topology of schemes

See Topology, Section 5.1 for some basic material in topology adapted to the Zariski topology of schemes.

Lemma 21.11.1. *Let X be a scheme. Any irreducible closed subset of X has a unique generic point. In other words, X is a sober topological space, see Topology, Definition 5.5.4.*

Proof. Let $Z \subset X$ be an irreducible closed subset. For every affine open $U \subset X$, $U = Spec(R)$ we know that $Z \cap U = V(I)$ for a unique radical ideal $I \subset R$. Note that $Z \cap I$ is either empty or irreducible. In the second case (which occurs for at least one U) we see that $I = \mathfrak{p}$ is a prime ideal, which is a generic point ξ of $Z \cap U$. It follows that $Z = \overline{\{\xi\}}$, in other words ξ is a generic point of Z. If ξ' was a second generic point, then $\xi' \in Z \cap U$ and it follows immediately that $\xi' = \xi$. $\square$

Lemma 21.11.2. *Let X be a scheme. The collection of affine opens of X forms a basis for the topology on X.*

Proof. This follows from the discussion on open subschemes in Section 21.9. $\square$

Remark 21.11.3. In general the intersection of two affine opens in X is not affine open. See Example 21.14.3.

Lemma 21.11.4. *The underlying topological space of any scheme is locally quasi-compact, see Topology, Definition 5.18.1.*

Proof. This follows from Lemma 21.11.2 above and the fact that the spectrum of ring is quasi-compact, see Algebra, Lemma 7.16.10. □

Lemma 21.11.5. *Let X be a scheme. Let U, V be affine opens of X, and let $x \in U \cap V$. There exists an affine open neighbourhood W of x such that W is a standard open of both U and V.*

Proof. Write $U = Spec(A)$ and $V = Spec(B)$. Say x corresponds to the prime $\mathfrak{p} \subset A$ and the prime $\mathfrak{q} \subset B$. We may choose a $f \in A$, $f \not\in \mathfrak{p}$ such that $D(f) \subset U \cap V$. Note that any standard open of $D(f)$ is a standard open of $Spec(A) = U$. Hence we may assume that $U \subset V$. In other words, now we may think of U as an affine open of V. Next we choose a $g \in B$, $g \not\in \mathfrak{q}$ such that $D(g) \subset U$. In this case we see that $D(g) = D(g_A)$ where $g_A \in A$ denotes the image of $g \in A$. Thus the lemma is proved. □

Lemma 21.11.6. *Let X be a scheme. Let $X = \bigcup_i U_i$ be an affine open covering. Let $V \subset X$ be an affine open. There exists a standard open covering $V = \bigcup_{j=1,\ldots,m} V_j$ (see Definition 21.5.2) such that each V_j is a standard open in one of the U_i.*

Proof. Pick $v \in V$. Then $v \in U_i$ for some i. By Lemma 21.11.5 above there exists an open $v \in W_v \subset V \cap U_i$ such that W_v is a standard open in both V and U_i. Since V is quasi-compact the lemma follows. □

Lemma 21.11.7. *Let X be a scheme whose underlying topological space is a finite discrete set. Then X is affine.*

Proof. Say $X = \{x_1, \ldots, x_n\}$. Then $U_i = \{x_i\}$ is an open neighbourhood of x_i. By Lemma 21.11.2 it is affine. Hence X is a finite disjoint union of affine schemes, and hence is affine by Lemma 21.6.8. □

Example 21.11.8. There exists a scheme without closed points. Namely, let R be a local domain whose spectrum looks like $(0) = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \ldots \subset \mathfrak{m}$. Then the open subscheme $Spec(R) \setminus \{\mathfrak{m}\}$ does not have a closed point. To see that such a ring R exists, we use that given any totally ordered group $(\Gamma, \geq)$ there exists a valuation ring A with valuation group $(\Gamma, \geq)$, see [**Kru32**]. See Algebra, Section 7.46 for notation. We take $\Gamma = \mathbf{Z}x_1 \oplus \mathbf{Z}x_2 \oplus \mathbf{Z}x_3 \oplus \ldots$ and we define $\sum_i a_i x_i \geq 0$ if and only if the first nonzero a_i is > 0, or all $a_i = 0$. So $x_1 \geq x_2 \geq x_3 \geq \ldots \geq 0$. The subsets $x_i + \Gamma_{\geq 0}$ are prime ideals of $(\Gamma, \geq)$, see Algebra, notation above Lemma 7.46.11. These together with $\emptyset$ and $\Gamma_{\geq 0}$ are the only prime ideals. Hence A is an example of a ring with the given structure of its spectrum, by Algebra, Lemma 7.46.11.

21.12. Reduced schemes

Definition 21.12.1. Let X be a scheme. We say X is *reduced* if every local ring $\mathcal{O}_{X,x}$ is reduced.

Lemma 21.12.2. *A scheme X is reduced if and only if $\mathcal{O}_X(U)$ is a reduced ring for all $U \subset X$ open.*

Proof. Assume that X is reduced. Let $f \in \mathcal{O}_X(U)$ be a section such that $f^n = 0$. Then the image of f in $\mathcal{O}_{U,u}$ is zero for all $u \in U$. Hence f is zero, see Sheaves, Lemma 6.11.1. Conversely, assume that $\mathcal{O}_X(U)$ is reduced for all opens U. Pick any nonzero element $f \in \mathcal{O}_{X,x}$. Any representative $(U, f \in \mathcal{O}(U))$ of f is nonzero and hence not nilpotent. Hence f is not nilpotent in $\mathcal{O}_{X,x}$. □

Lemma 21.12.3. *An affine scheme $Spec(R)$ is reduced if and only if R is reduced.*

Proof. The direct implication follows immediately from Lemma 21.12.2 above. In the other direction it follows since any localization of a reduced ring is reduced, and in particular the local rings of a reduced ring are reduced. □

Lemma 21.12.4. *Let X be a scheme. Let $T \subset X$ be a closed subset. There exists a unique closed subscheme $Z \subset X$ with the following properties: (a) the underlying topological space of Z is equal to T, and (b) Z is reduced.*

Proof. Let $\mathcal{I} \subset \mathcal{O}_X$ be the sub presheaf defined by the rule

$$\mathcal{I}(U) = \{ f \in \mathcal{O}_X(U) \mid f(t) = 0 \text{ for all } t \in T \cap U\}$$

Here we use $f(t)$ to indicate the image of f in the residue field $\kappa(t)$ of X at t. Because of the local nature of the condition it is clear that $\mathcal{I}$ is a sheaf of ideals. Moreover, let $U = Spec(R)$ be an affine open. We may write $T \cap U = V(I)$ for a unique radical ideal $I \subset R$. Given a prime $\mathfrak{p} \in V(I)$ corresponding to $t \in T \cap U$ and an element $f \in R$ we have $f(t) = 0 \Leftrightarrow f \in \mathfrak{p}$. Hence $\mathcal{I}(U) = \cap_{\mathfrak{p} \in V(I)} \mathfrak{p} = I$ by Algebra, Lemma 7.16.2. Moreover, for any standard open $D(g) \subset Spec(R) = U$ we have $\mathcal{I}(D(g)) = I_g$ by the same reasoning. Thus $\tilde{I}$ and $\mathcal{I}|_U$ agree (as ideals) on a basis of opens and hence are equal. Therefore $\mathcal{I}$ is a quasi-coherent sheaf of ideals.

At this point we may define Z as the closed subspace associated to the sheaf of ideals $\mathcal{I}$. For every affine open $U = Spec(R)$ of X we see that $Z \cap U = Spec(R/I)$ where I is a radical ideal and hence Z is reduced (by Lemma 21.12.3 above). By construction the underlying closed subset of Z is T. Hence we have found a closed subscheme with properties (a) and (b).

Let $Z' \subset X$ be a second closed subscheme with properties (a) and (b). For every affine open $U = Spec(R)$ of X we see that $Z' \cap U = Spec(R/I')$ for some ideal $I' \subset R$. By Lemma 21.12.3 the ring R/I' is reduced and hence I' is radical. Since $V(I') = T \cap U = V(I)$ we deduced that $I = I'$ by Algebra, Lemma 7.16.2. Hence Z' and Z are defined by the same sheaf of ideals and hence are equal. □

Definition 21.12.5. Let X be a scheme. Let $i : Z \to X$ be the inclusion of a closed subset. A *scheme structure on Z* is given by a closed subscheme Z' of X whose underlying closed is equal to Z. We often say ``let $(Z, \mathcal{O}_Z)$ be a scheme structure on Z" to indicate this. The *reduced induced scheme structure* on Z is the one constructed in Lemma 21.12.4. The *reduction X_{red} of X* is the reduced induced scheme structure on X itself.

Often when we say ``let $Z \subset X$ be an irreducible component of X" we think of Z as a reduced closed subscheme of X using the reduced induced scheme structure.

Lemma 21.12.6. *Let X be a scheme. Let $Z \subset X$ be a closed subscheme. Let Y be a reduced scheme. A morphism $f : Y \to X$ factors through Z if and only if $f(Y) \subset Z$ (set theoretically). In particular, any morphism $Y \to X$ factors as $Y \to X_{red} \to X$.*

Proof. Assume $f(Y) \subset Z$ (set theoretically). Let $I \subset \mathcal{O}_X$ be the ideal sheaf of Z. For any affine opens $U \subset X$, $Spec(B) = V \subset Y$ with $f(V) \subset U$ and any $g \in \mathcal{I}(U)$ the pullback $b = f^\sharp(g) \in \Gamma(V, \mathcal{O}_Y) = B$ maps to zero in the residue field of any $y \in V$. In other words $g \in \bigcap_{\mathfrak{p} \subset B} \mathfrak{p}$. This implies $b = 0$ as B is reduced (Lemma 21.12.2, and Algebra, Lemma 7.16.2). Hence f factors through Z by Lemma 21.4.6. □

21.13. Points of schemes

Given a scheme X we can define a functor

$$h_X : Sch^{opp} \longrightarrow Sets, \quad T \longmapsto Mor(T, X).$$

See Categories, Example 4.3.4. This is called the *functor of points of X*. A fun part of scheme theory is to find descriptions of the internal geometry of X in terms of this functor h_X. In this section we find a simple way to describe points of X.

Let X be a scheme. Let R be a local ring with maximal ideal $\mathfrak{m} \subset R$. Suppose that $f : Spec(R) \to X$ is a morphism of schemes. Let $x \in X$ be the image of the closed point $\mathfrak{m} \in Spec(R)$. Then we obtain a local homomorphism of local rings

$$f^\sharp : \mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{Spec(R),\mathfrak{m}} = R.$$

Lemma 21.13.1. *Let X be a scheme. Let R be a local ring. The construction above gives a bijective correspondence between morphisms $Spec(R) \to X$ and pairs (x, φ) consisting of a point $x \in X$ and a local homomorphism of local rings $\varphi : \mathcal{O}_{X,x} \to R$.*

Proof. Let A be a ring. For any ring homomorphism $\psi : A \to R$ there exists a unique prime ideal $\mathfrak{p} \subset A$ and a factorization $A \to A_{\mathfrak{p}} \to R$ where the last map is a local homomorphism of local rings. Namely, $\mathfrak{p} = \psi^{-1}(\mathfrak{m})$. Via Lemma 21.6.4 this proves that the lemma holds if X is an affine scheme.

Let X be a general scheme. Any $x \in X$ is contained in an open affine $U \subset X$. By the affine case we conclude that every pair (x, φ) occurs as the end product of the construction above the lemma.

To finish the proof it suffices to show that any morphism $f : Spec(R) \to X$ has image contained in any affine open containing the image x of the closed point of $Spec(R)$. In fact, let $x \in V \subset X$ be any open neighbourhood containing x. Then $f^{-1}(V) \subset Spec(R)$ is an open containing the unique closed point and hence equal to $Spec(R)$. □

As a special case of the lemma above we obtain for every point x of a scheme X a canonical morphism

$$Spec(\mathcal{O}_{X,x}) \longrightarrow X \tag{21.13.1.1}$$

corresponding to the identity map on the local ring of X at x. We may reformulate the lemma above as saying that for any morphism $f : Spec(R) \to X$ there exists a unique point $x \in X$ such that f factors as $Spec(R) \to Spec(\mathcal{O}_{X,x}) \to X$ where the first map comes from a local homomorphism $\mathcal{O}_{X,x} \to R$.

In case we have a morphism of schemes $f : X \to S$, and a point x mapping to a point $s \in S$ we obtain a commutative diagram

$$\begin{tikzcd} Spec(\mathcal{O}_{X,x}) \arrow[r] \arrow[d] & X \arrow[d] \\ Spec(\mathcal{O}_{S,s}) \arrow[r] & S \end{tikzcd}$$

where the left vertical map corresponds to the local ring map $f^\sharp_x : \mathcal{O}_{X,x} \to \mathcal{O}_{S,s}$.

Lemma 21.13.2. *Let X be a scheme. Let $x, x' \in X$ be points of X. Then $x' \in X$ is a generalization of x if and only if x' is in the image of the canonical morphism $Spec(\mathcal{O}_{X,x}) \to X$.*

Proof. A continuous map preserves the relation of specialization/generalization. Since every point of $Spec(\mathcal{O}_{X,x})$ is a generalization of the closed point we see every point in the image of $Spec(\mathcal{O}_{X,x}) \to X$ is a generalization of x. Conversely, suppose that x' is a generalization of x. Choose an affine open neighbourhood $U = Spec(R)$ of x. Then $x' \in U$. Say $\mathfrak{p} \subset R$ and $\mathfrak{p}' \subset R$ are the primes corresponding to x and x'. Since x' is a generalization of x we see that $\mathfrak{p}' \subset \mathfrak{p}$. This means that $\mathfrak{p}'$ is in the image of the morphism $Spec(\mathcal{O}_{X,x}) = Spec(R_{\mathfrak{p}}) \to Spec(R) = U \subset X$ as desired. □

Now, let us discuss morphisms from spectra of fields. Let $(R, \mathfrak{m}, \kappa)$ be a local ring with maximal ideal $\mathfrak{m}$ and residue field κ. Let K be a field. A local homomorphism $R \to K$ by definition factors as $R \to \kappa \to K$, i.e., is the same thing as a morphism $\kappa \to K$. Thus we see that morphisms

$$Spec(K) \longrightarrow X$$

correspond to pairs $(x, \kappa(x) \to K)$. We may define a partial ordering on morphisms of spectra of fields to X by saying that $Spec(K) \to X$ dominates $Spec(L) \to X$ if $Spec(K) \to X$ factors through $Spec(L) \to X$. This suggests the following notion: Let us temporarily say that two morphisms $p : Spec(K) \to X$ and $q : Spec(L) \to X$ are *equivalent* if there exists a third field Ω and a commutative diagram

$$\begin{tikzcd} Spec(\Omega) \arrow[r] \arrow[d] & Spec(L) \arrow[d, "q"] \\ Spec(K) \arrow[r, "p"] & X \end{tikzcd}$$

Of course this immediately implies that the unique points of all three of the schemes $Spec(K)$, $Spec(L)$, and $Spec(\Omega)$ map to the same $x \in X$. Thus a diagram (by the remarks above) corresponds to a point $x \in X$ and a commutative diagram

$$\begin{tikzcd} \Omega & L \arrow[l] \\ K \arrow[u] & \kappa(x) \arrow[l] \arrow[u] \end{tikzcd}$$

of fields. This defines an equivalence relation, because given any set of extensions $\kappa \subset K_i$ there exists some field extension $\kappa \subset \Omega$ such that all the field extensions K_i are contained in the extension Ω.

Lemma 21.13.3. *Let X be a scheme. Points of X correspond bijectively to equivalence classes of morphisms from spectra of fields into X. Moreover, each equivalence class contains a (unique up to unique isomorphism) smallest element $Spec(\kappa(x)) \to X$.*

Proof. Follows from the discussion above. □

Of course the morphisms $Spec(\kappa(x)) \to X$ factor through the canonical morphisms $Spec(\mathcal{O}_{X,x}) \to X$. And the content of Lemma 21.13.2 is in this setting that the morphism $Spec(\kappa(x')) \to X$ factors as $Spec(\kappa(x')) \to Spec(\mathcal{O}_{X,x}) \to X$ whenever x' is a generalization of x. In case we have a morphism of schemes $f : X \to S$, and a point x mapping to a point $s \in S$ we obtain a commutative diagram

$$\begin{tikzcd} Spec(\kappa(x)) \arrow[r] \arrow[d] & Spec(\mathcal{O}_{X,x}) \arrow[r] \arrow[d] & X \arrow[d] \\ Spec(\kappa(s)) \arrow[r] & Spec(\mathcal{O}_{S,s}) \arrow[r] & S. \end{tikzcd}$$

21.14. Glueing schemes

Let I be a set. For each $i \in I$ let $(X_i, \mathcal{O}_i)$ be a locally ringed space. (Actually the construction that follows works equally well for ringed spaces.) For each pair $i, j \in I$ let $U_{ij} \subset X_i$ be an open subspace. For each pair $i, j \in I$, let

$$\varphi_{ij} : U_{ij} \to U_{ji}$$

be an isomorphism of locally ringed spaces. For convenience we assume that $U_{ii} = X_i$ and $\varphi_{ii} = \text{id}_{X_i}$. For each triple $i, j, k \in I$ assume that

(1) we have $\varphi_{ij}^{-1}(U_{ji} \cap U_{jk}) = U_{ij} \cap U_{ik}$, and
(2) the diagram

$$\begin{tikzcd} U_{ij} \cap U_{ik} \arrow[rr, "\varphi_{ik}"] \arrow[dr, "\varphi_{ij}"'] & & U_{ki} \cap U_{kj} \\ & U_{ji} \cap U_{jk} \arrow[ur, "\varphi_{jk}"'] & \end{tikzcd}$$

is commutative.

Let us call a collection $(I, (X_i)_{i\in I}, (U_{ij})_{i,j\in I}, (\varphi_{ij})_{i,j\in I})$ satisfying the conditions above a glueing data.

Lemma 21.14.1. *Given any glueing data of locally ringed spaces there exists a locally ringed space X and open subspaces $U_i \subset X$ together with isomorphisms $\varphi_i : X_i \to U_i$ of locally ringed spaces such that*

(1) $\varphi_i(U_{ij}) = U_i \cap U_j$, *and*
(2) $\varphi_{ij} = \varphi_j^{-1}|_{U_i \cap U_j} \circ \varphi_i|_{U_{ij}}$.

The locally ringed space X is characterized by the following mapping properties: Given a locally ringed space Y we have

$$\begin{array}{rcl} Mor(X, Y) & = & \{(f_i)_{i\in I} \mid f_i : X_i \to Y,\ f_j \circ \varphi_{ij} = f_i|_{U_{ij}}\} \\ f & \mapsto & (f|_{U_i} \circ \varphi_i)_{i\in I} \\ Mor(Y, X) & = & \left\{ \begin{array}{c} \text{open covering } Y = \bigcup_{i\in I} V_i \text{ and } (g_i : V_i \to X_i)_{i\in I} \text{ such that} \\ g_i^{-1}(U_{ij}) = V_i \cap V_j \text{ and } g_j|_{V_i \cap V_j} = \varphi_{ij} \circ g_i|_{V_i \cap V_j} \end{array} \right\} \\ g & \mapsto & V_i = g^{-1}(U_i),\ g_i = g|_{V_i} \end{array}$$

Proof. We construct X in stages. As a set we take

$$X = (\coprod X_i)/\sim .$$

Here given $x \in X_i$ and $x' \in X_j$ we say $x \sim x'$ if and only if $x \in U_{ij}$, $x' \in U_{ji}$ and $\varphi_{ij}(x) = x'$. This is an equivalence relation since if $x \in X_i$, $x' \in X_j$, $x'' \in X_k$, and $x \sim x'$ and $x' \sim x''$, then $x' \in U_{ji} \cap U_{jk}$, hence by condition (1) of a glueing data also $x \in U_{ij} \cap U_{ik}$ and $x'' \in U_{ki} \cap U_{kj}$ and by condition (2) we see that $\varphi_{ik}(x) = x''$. (Reflexivity and symmetry follows from our assumptions that $U_{ii} = X_i$ and $\varphi_{ii} = \text{id}_{X_i}$.) Denote $\varphi_i : X_i \to X$ the natural maps. Denote $U_i = \varphi_i(X_i) \subset X$. Note that $\varphi_i : X_i \to U_i$ is a bijection.

The topology on X is defined by the rule that $U \subset X$ is open if and only if $\varphi_i^{-1}(U)$ is open for all i. We leave it to the reader to verify that this does indeed define a topology. Note that in particular U_i is open since $\varphi_j^{-1}(U_i) = U_{ji}$ which is open in X_j for all j. Moreover, for any open set $W \subset X_i$ the image $\varphi_i(W) \subset U_i$ is open because $\varphi_j^{-1}(\varphi_i(W)) = \varphi_{ji}^{-1}(W \cap U_{ij})$. Therefore $\varphi_i : X_i \to U_i$ is a homeomorphism.

To obtain a locally ringed space we have to construct the sheaf of rings $\mathcal{O}_X$. We do this by glueing the sheaves of rings $\mathcal{O}_{U_i} := \varphi_{i,*}\mathcal{O}_{X_i}$. Namely, in the commutative diagram

$$\begin{tikzcd} U_{ij} \arrow[rr, "\varphi_{ij}"] \arrow[dr, "\varphi_i|_{U_{ij}}"'] & & U_{ji} \arrow[dl, "\varphi_j|_{U_{ji}}"] \\ & U_i \cap U_j & \end{tikzcd}$$

the arrow on top is an isomorphism of ringed spaces, and hence we get unique isomorphisms of sheaves of rings

$$\mathcal{O}_{U_i}|_{U_i \cap U_j} \longrightarrow \mathcal{O}_{U_j}|_{U_i \cap U_j}.$$

These satisfy a cocycle condition as in Sheaves, Section 6.33. By the results of that section we obtain a sheaf of rings $\mathcal{O}_X$ on X such that $\mathcal{O}_X|_{U_i}$ is isomorphic to $\mathcal{O}_{U_i}$ compatibly with the glueing maps displayed above. In particular $(X, \mathcal{O}_X)$ is a locally ringed space since the stalks of $\mathcal{O}_X$ are equal to the stalks of $\mathcal{O}_{X_i}$ at corresponding points.

The proof of the mapping properties is omitted. □

Lemma 21.14.2. *In Lemma 21.14.1 above, assume that all X_i are schemes. Then the resulting locally ringed space X is a scheme.*

Proof. This is clear since each of the U_i is a scheme and hence every $x \in X$ has an affine neighbourhood. □

It is customary to think of X_i as an open subspace of X via the isomorphisms φ_i. We will do this in the next two examples.

Example 21.14.3. (Affine space with zero doubled.) Let k be a field. Let $n \geq 1$. Let $X_1 = Spec(k[x_1, \ldots, x_n])$, let $X_2 = Spec(k[y_1, \ldots, y_n])$. Let $0_1 \in X_1$ be the point corresponding to the maximal ideal $(x_1, \ldots, x_n) \subset k[x_1, \ldots, x_n]$. Let $0_2 \in X_2$ be the point corresponding to the maximal ideal $(y_1, \ldots, y_n) \subset k[y_1, \ldots, y_n]$. Let $U_{12} = X_1 \setminus \{0_1\}$ and let $U_{21} = X_2 \setminus \{0_2\}$. Let $\varphi_{12} : U_{12} \to U_{21}$ be the isomorphism coming from the isomorphism of k-algebras $k[y_1, \ldots, y_n] \to k[x_1, \ldots, x_n]$ mapping y_i to x_i (which induces $X_1 \cong X_2$ mapping 0_1 to 0_2). Let X be the scheme obtained from the glueing data $(X_1, X_2, U_{12}, U_{21}, \varphi_{12}, \varphi_{21} = \varphi_{12}^{-1})$. Via the slight abuse of notation introduced above the example we think of $X_i \subset X$ as open subschemes. There is a morphism $f : X \to Spec(k[t_1, \ldots, t_n])$ which on X_i corresponds to k algebra map $k[t_1, \ldots, t_n] \to k[x_1, \ldots, x_n]$ (resp. $k[t_1, \ldots, t_n] \to k[y_1, \ldots, y_n]$) mapping t_i to x_i (resp. t_i to y_i). It is easy to see that this morphism identifies $k[t_1, \ldots, t_n]$ with $\Gamma(X, \mathcal{O}_X)$. Since $f(0_1) = f(0_2)$ we see that X is not affine.

Note that X_1 and X_2 are affine opens of X. But, if $n = 2$, then $X_1 \cap X_2$ is the scheme described in Example 21.9.3 and hence not affine. Thus in general the intersection of affine opens of a scheme is not affine. (This fact holds more generally for any $n > 1$.)

Another curious feature of this example is the following. If $n > 1$ there are many irreducible closed subsets $T \subset X$ (take the closure of any non closed point in X_1 for example). But unless $T = \{0_1\}$, or $T = \{0_2\}$ we have $0_1 \in T \Leftrightarrow 0_2 \in T$. Proof omitted.

Example 21.14.4. (Projective line.) Let k be a field. Let $X_1 = Spec(k[x])$, let $X_2 = Spec(k[y])$. Let $0 \in X_1$ be the point corresponding to the maximal ideal $(x) \subset k[x]$. Let $\infty \in X_2$ be the point corresponding to the maximal ideal $(y) \subset k[y]$. Let $U_{12} = X_1 \setminus \{0\} = D(x) = Spec(k[x, 1/x])$ and let $U_{21} = X_2 \setminus \{\infty\} = D(y) = Spec(k[y, 1/y])$. Let $\varphi_{12} : U_{12} \to U_{21}$ be the isomorphism coming from the isomorphism of k-algebras $k[y, 1/y] \to k[x, 1/x]$ mapping y to $1/x$. Let $\mathbf{P}^1_k$ be the scheme obtained from the glueing data $(X_1, X_2, U_{12}, U_{21}, \varphi_{12}, \varphi_{21} = \varphi_{12}^{-1})$. Via the slight abuse of notation introduced above the example we think of $X_i \subset \mathbf{P}^1_k$ as open subschemes. In this case we see that $\Gamma(\mathbf{P}^1_k, \mathcal{O}) = k$ because the only polynomials $g(x)$ in x such that $g(1/y)$ is also a polynomial in y are constant polynomials. Since $\mathbf{P}^1_k$ is infinite we see that $\mathbf{P}^1_k$ is not affine.

We claim that there exists an affine open $U \subset \mathbf{P}^1_k$ which contains both 0 and ∞. Namely, let $U = \mathbf{P}^1_k \setminus \{1\}$, where 1 is the point of X_1 corresponding to the maximal ideal $(x - 1)$ and also the point of X_2 corresponding to the maximal ideal $(y - 1)$. Then it is easy to see that $s = 1/(x - 1) = y/(1 - y) \in \Gamma(U, \mathcal{O}_U)$. In fact you can show that $\Gamma(U, \mathcal{O}_U)$ is equal to the polynomial ring $k[s]$ and that the corresponding morphism $U \to Spec(k[s])$ is an isomorphism of schemes. Details omitted.

21.15. A representability criterion

In this section we reformulate the glueing lemma of Section 21.14 in terms of functors. We recall some of the material from Categories, Section 4.3. Recall that given a scheme X we can define a functor

$$h_X : Sch^{opp} \longrightarrow Sets, \quad T \longmapsto Mor(T, X).$$

This is called the *functor of points of X*.

Let F be a contravariant functor from the category of schemes to the category of sets. In a formula

$$F : Sch^{opp} \longrightarrow Sets.$$

We will use the same terminology as in Sites, Section 9.2. Namely, given a scheme T, an element $\xi \in F(T)$, and a morphism $f : T' \to T$ we will denote $f^*\xi$ the element $F(f)(\xi)$, and sometimes we will even use the notation $\xi|_{T'}$

Definition 21.15.1. (See Categories, Definition 4.3.6.) Let F be a contravariant functor from the category of schemes to the category of sets (as above). We say that F is *representable by a scheme* or *representable* if there exists a scheme X such that $h_X \cong F$.

Suppose that F is representable by the scheme X and that $s : h_X \to F$ is an isomorphism. By Categories, Yoneda Lemma 4.3.5 the pair $(X, s : h_X \to F)$ is unique up to unique isomorphism if it exists. Moreover, the Yoneda lemma says that given any contravariant functor F as above and any scheme Y, we have a bijection

$$Mor_{\mathrm{Fun}(Sch^{opp}, Sets)}(h_Y, F) \longrightarrow F(Y), \quad s \longmapsto s(\mathrm{id}_Y).$$

Here is the reverse construction. Given any $\xi \in F(Y)$ the transformation of functors $s_\xi : h_Y \to F$ associates to any morphism $f : T \to Y$ the element $f^*\xi \in F(T)$.

In particular, in the case that F is representable, there exists a scheme X and an element $\xi \in F(X)$ such that the corresponding morphism $h_X \to F$ is an isomorphism. In this case we also say *the pair (X, ξ) represents F*. The element $\xi \in F(X)$ is often called the *``universal family''* for reasons that will become more clear when we talk about algebraic stacks (insert future reference here). For the moment we simply observe that the fact that if the pair (X, ξ) represents F, then every element $\xi' \in F(T)$ for any T is of the form $\xi' = f^*\xi$ for a unique morphism $f : T \to X$.

Example 21.15.2. Consider the rule which associates to every scheme T the set $F(T) = \Gamma(T, \mathcal{O}_T)$. We can turn this into a contravariant functor by using for a morphism $f : T' \to T$ the pullback map $f^\sharp : \Gamma(T, \mathcal{O}_T) \to \Gamma(T', \mathcal{O}_{T'})$. Given a ring R and an element $t \in R$ there exists a unique ring homomorphism $\mathbf{Z}[x] \to R$ which maps x to t. Thus, using Lemma 21.6.4, we see that

$$Mor(T, Spec(\mathbf{Z}[x])) = Hom(\mathbf{Z}[x], \Gamma(T, \mathcal{O}_T)) = \Gamma(T, \mathcal{O}_T).$$

This does indeed give an isomorphism $h_{Spec(\mathbf{Z}[x])} \to F$. What is the ``universal family'' ξ? To get it we have to apply the identifications above to $\mathrm{id}_{Spec(\mathbf{Z}[x])}$. Clearly under the identifications above this gives that $\xi = x \in \Gamma(Spec(\mathbf{Z}[x]), \mathcal{O}_{Spec(\mathbf{Z}[x])}) = \mathbf{Z}[x]$ as expected.

Definition 21.15.3. Let F be a contravariant functor on the category of schemes with values in sets.

(1) We say that F *satisfies the sheaf property for the Zariski topology* if for every scheme T and every open covering $T = \bigcup_{i \in I} U_i$, and for any collection of elements $\xi_i \in F(U_i)$ such that $\xi_i|_{U_i \cap U_j} = \xi_j|_{U_i \cap U_j}$ there exists a unique element $\xi \in F(T)$ such that $\xi_i = \xi|_{U_i}$ in $F(U_i)$.
(2) A *subfunctor* $H \subset F$ is a rule that associates to every scheme T a subset $H(T) \subset F(T)$ such that the maps $F(f) : F(T) \to F(T')$ maps $H(T)$ into $H(T')$ for all morphisms of schemes $f : T' \to T$.
(3) Let $H \subset F$ be a subfunctor. We say that $H \subset F$ is *representable by open immersions* if for all pairs (T, ξ), where T is a scheme and $\xi \in F(T)$ there exists an open subscheme $U_\xi \subset T$ with the following property:
    (∗) A morphism $f : T' \to T$ factors through U_ξ if and only if $f^*\xi \in H(T')$.
(4) Let I be a set. For each $i \in I$ let $H_i \subset F$ be a subfunctor. We say that the collection $(H_i)_{i\in I}$ *covers* F if and only if for every $\xi \in F(T)$ there exists an open covering $T = \bigcup U_i$ such that $\xi|_{U_i} \in H_i(U_i)$.

Lemma 21.15.4. *Let F be a contravariant functor on the category of schemes with values in the category of sets. Suppose that*

(1) *F satisfies the sheaf property for the Zariski topology,*
(2) *there exists a set I and a collection of subfunctors $F_i \subset F$ such that*
    (a) *each F_i is representable,*
    (b) *each $F_i \subset F$ is representable by open immersions, and*
    (c) *the collection $(F_i)_{i\in I}$ covers F.*

Then F is representable.

Proof. Let X_i be a scheme representing F_i and let $\xi_i \in F_i(X_i) \subset F(X_i)$ be the ``universal family''. Because $F_j \subset F$ is representable by open immersions, there exists an open $U_{ij} \subset X_i$ such that $T \to X_i$ factors through U_{ij} if and only if $\xi_i|_T \in F_j(T)$. In particular $\xi_i|_{U_{ij}} \in F_j(U_{ij})$ and therefore we obtain a canonical morphism $\varphi_{ij} : U_{ij} \to X_j$ such that $\varphi_{ij}^*\xi_j = \xi_i|_{U_{ij}}$. By defintion of U_{ji} this implies that φ_{ij} factors through U_{ji}. Since $(\varphi_{ij} \circ \varphi_{ji})^*\xi_j = \varphi_{ji}^*(\varphi_{ij}^*\xi_j) = \varphi_{ji}^*\xi_i = \xi_j$ we conclude that $\varphi_{ij} \circ \varphi_{ji} = \mathrm{id}_{U_{ji}}$ because the pair (X_j, ξ_j) represents F_j. In particular the maps $\varphi_{ij} : U_{ij} \to U_{ji}$ are isomorphisms of schemes. Next we have to show that $\varphi_{ij}^{-1}(U_{ji} \cap U_{jk}) = U_{ij} \cap U_{ik}$. This is true because (a) $U_{ji} \cap U_{jk}$ is the largest open of U_{ji} such that ξ_j restricts to an element of F_k, (b) $U_{ij} \cap U_{ik}$ is the largest open of U_{ij} such that ξ_i restricts to an element of F_k, and (c) $\varphi_{ij}^*\xi_j = \xi_i$. Moreover, the cocycle condition in Section 21.14 follows because both $\varphi_{jk}|_{U_{ji}\cap U_{jk}} \circ \varphi_{ij}|_{U_{ij}\cap U_{ik}}$ and $\varphi_{ik}|_{U_{ij}\cap U_{ik}}$ pullback ξ_k to the element ξ_i. Thus we may apply Lemma 21.14.2 to obtain a scheme X with an open covering $X = \bigcup U_i$ and isomorphisms $\varphi_i : X_i \to U_i$ with properties as in Lemma 21.14.1. Let $\xi_i' = (\varphi_i^{-1})^*\xi_i$. The conditions of Lemma 21.14.1 imply that $\xi_i'|_{U_i\cap U_j} = \xi_j'|_{U_i\cap U_j}$. Therefore, by the condition that F satisfies the sheaf condition in the Zariski topology we see that there exists an element $\xi' \in F(X)$ such that $\xi_i = \varphi_i^*\xi'|_{U_i}$ for all i. Since φ_i is an isomorphism we also get that $(U_i, \xi'|_{U_i})$ represents the functor F_i.

We claim that the pair (X, ξ') represents the functor F. To show this, let T be a scheme and let $\xi \in F(T)$. We will construct a unqiue morphism $g : T \to X$ such that $g^*\xi' = \xi$. Namely, by the condition that the subfunctors F_i cover T there exists an open covering $T = \bigcup V_i$ such that for each i the restriction $\xi|_{V_i} \in F_i(V_i)$. Moreover, since each of the inclusions $F_i \subset F$ are representable by open immersions we may assume that each $V_i \subset T$ is maximal open with this property. Because, (U_i, ξ'_{U_i}) represents the functor F_i we get a unique morphism $g_i : V_i \to U_i$ such that $g_i^*\xi'|_{U_i} = \xi|_{V_i}$. On the overlaps $V_i \cap V_j$ the morphisms g_i and g_j agree, for example because they both pull back $\xi'|_{U_i\cap U_j} \in F_i(U_i \cap U_j)$ to the same element. Thus the morphisms g_i glue to a unique morphism from $T \to X$ as desired. □

Remark 21.15.5. Suppose the functor F is defined on all locally ringed spaces, and if conditions of Lemma 21.15.4 are replaced by the following:

(1) F satisfies the sheaf property on the category of locally ringed spaces,
(2) there exists a set I and a collection of subfunctors $F_i \subset F$ such that
    (a) each F_i is representable by a scheme,
    (b) each $F_i \subset F$ is representable by open immersions on the category of locally ringed spaces, and
    (c) the collection $(F_i)_{i\in I}$ covers F as a functor on the category of locally ringed spaces.

We leave it to the reader to spell this out further. Then the end result is that the functor F is representable in the category of locally ringed spaces and that the representing object is a scheme.

21.16. Existence of fibre products of schemes

A very basic question is whether or not products and fibre products exist on the category of schemes. We first prove abstractly that products and fibre products exist, and in the next section we show how we may think in a reasonable way about fibre products of schemes.

Lemma 21.16.1. *The category of schemes has a final object, products and fibre products. In other words, the category of schemes has finite limits, see Categories, Lemma 4.16.4.*

Proof. Please skip this proof. It is more important to learn how to work with the fibre product which is explained in the next section.

By Lemma 21.6.4 the scheme $Spec(\mathbf{Z})$ is a final object in the category of locally ringed spaces. Thus it suffices to prove that fibred products exist.

Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes. We have to show that the functor

$$\begin{array}{rcl} F : Sch^{opp} & \longrightarrow & Sets \\ T & \longmapsto & Mor(T, X) \times_{Mor(T,S)} Mor(T, Y) \end{array}$$

is representable. We claim that Lemma 21.15.4 applies to the functor F. If we prove this then the lemma is proved.

First we show that F satisfies the sheaf property in the Zariski topology. Namely, suppose that T is a scheme, $T = \bigcup_{i \in I} U_i$ is an open covering, and $\xi_i \in F(U_i)$ such that $\xi_i|_{U_i \cap U_j} = \xi_j|_{U_i \cap U_j}$ for all pairs i, j. By definition ξ_i corresponds to a pair (a_i, b_i) where $a_i : U_i \to X$ and $b_i : U_i \to Y$ are morphisms of schemes such that $f \circ a_i = g \circ b_i$. The glueing condition says that $a_i|_{U_i \cap U_j} = a_j|_{U_i \cap U_j}$ and $b_i|_{U_i \cap U_j} = b_j|_{U_i \cap U_j}$. Thus by glueing the morphisms a_i we obtain a morphism of locally ringed spaces (i.e., a morphism of schemes) $a : T \to X$ and similarly $b : T \to Y$ (see for example the mapping property of Lemma 21.14.1). Moreover, on the members of an open covering the compositions $f \circ a$ and $g \circ b$ agree. Therefore $f \circ a = g \circ b$ and the pair (a, b) defines an element of $F(T)$ which restricts to the pairs (a_i, b_i) on each U_i. The sheaf condition is verified.

Next, we construct the family of subfunctors. Choose an open covering by open affines $S = \bigcup_{i \in I} U_i$. For every $i \in I$ choose open coverings by open affines $f^{-1}(U_i) = \bigcup_{j \in J_i} V_j$ and $g^{-1}(U_i) = \bigcup_{k \in K_i} W_k$. Note that $X = \bigcup_{i \in I} \bigcup_{j \in J_i} V_j$ is an open covering and similarly for Y. For any $i \in I$ and each pair $(j, k) \in J_i \times K_i$ we have a commutative diagram

$$
\begin{tikzcd}
 & W_k \arrow[d] \arrow[dr] & \\
V_j \arrow[r] \arrow[dr] & U_i \arrow[dr] & Y \arrow[d] \\
 & X \arrow[r] & S
\end{tikzcd}
$$

where all the skew arrows are open immersions. For such a triple we get a functor

$$
\begin{array}{rcl}
F_{i,j,k} : Sch^{opp} & \longrightarrow & Sets \\
T & \longmapsto & Mor(T, V_j) \times_{Mor(T, U_i)} Mor(T, W_j).
\end{array}
$$

There is an obvious transformation of functors $F_{i,j,k} \to F$ (coming from the huge commutative diagram above) which is injective, so we may think of $F_{i,j,k}$ as a subfunctor of F.

We check condition (2)(a) of Lemma 21.15.4. This follows directly from Lemma 21.6.7. (Note that we use here that the fibre products in the category of affine schemes are also fibre products in the whole category of locally ringed spaces.)

We check condition (2)(b) of Lemma 21.15.4. Let T be a scheme and let $\xi \in F(T)$. In other words, $\xi = (a, b)$ where $a : T \to X$ and $b : T \to Y$ are morphisms of schemes such that $f \circ a = g \circ b$. Set $V_{i,j,k} = a^{-1}(V_j) \cap b^{-1}(W_k)$. For any further morphism $h : T' \to T$ we have $h^*\xi = (a \circ h, b \circ h)$. Hence we see that $h^*\xi \in F_{i,j,k}(T')$ if and only if $a(h(T')) \subset V_j$ and $b(h(T')) \subset W_k$. In other words, if and only if $h(T') \subset V_{i,j,k}$. This proves condition (2)(b).

We check condition (2)(c) of Lemma 21.15.4. Let T be a scheme and let $\xi = (a, b) \in F(T)$ as above. Set $V_{i,j,k} = a^{-1}(V_j) \cap b^{-1}(W_k)$ as above. Condition (2)(c) just means that $T = \bigcup V_{i,j,k}$ which is evident. Thus the lemma is proved and fibre products exist. $\square$

Remark 21.16.2. Using Remark 21.15.5 you can show that the fibre product of morphisms of schemes exists in the category of locally ringed spaces and is a scheme.

21.17. Fibre products of schemes

Here is a review of the general definition, even though we have already shown that fibre products of schemes exist.

Definition 21.17.1. Given morphisms of schemes $f : X \to S$ and $g : Y \to S$ the *fibre product* is a scheme $X \times_S Y$ together with projection morphisms $p : X \times_S Y \to X$ and $q : X \times_S Y \to Y$ sitting into the following commutative diagram

$$\begin{tikzcd} X \times_S Y \arrow[r, "q"'] \arrow[d, "p"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & S \end{tikzcd}$$

which is universal among all diagrams of this sort, see Categories, Definition 4.6.1.

In other words, given any solid commutative diagram of morphisms of schemes

$$\begin{tikzcd} T \arrow[rrd, bend left] \arrow[rdd, bend right] \arrow[rd, dashed] & & \\ & X \times_S Y \arrow[r] \arrow[d] & Y \arrow[d] \\ & X \arrow[r] & S \end{tikzcd}$$

there exists a unique dotted arrow making the diagram commute. We will prove some lemmas which will tell us how to think about fibre products.

Lemma 21.17.2. *Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes with the same target. If X, Y, S are all affine then $X \times_S Y$ is affine.*

Proof. Suppose that $X = Spec(A)$, $Y = Spec(B)$ and $S = Spec(R)$. By Lemma 21.6.7 the affine scheme $Spec(A \otimes_R B)$ is the fibre product $X \times_S Y$ in the category of locally ringed spaces. Hence it is a fortiori the fibre product in the category of schemes. $\square$

Lemma 21.17.3. *Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes with the same target. Let $X \times_S Y$, p, q be the fibre product. Suppose that $U \subset S$, $V \subset X$, $W \subset Y$ are open subschemes such that $f(V) \subset U$ and $g(W) \subset U$. Then the canonical morphism $V \times_U W \to X \times_S Y$ is an open immersion which identifies $V \times_U W$ with $p^{-1}(V) \cap q^{-1}(W)$.*

Proof. Let T be a scheme Suppose $a : T \to V$ and $b : T \to W$ are morphisms such that $f \circ a = g \circ b$ as morphisms into U. Then they agree as morphisms into S. By the universal property of the fibre product we get a unique morphism $T \to X \times_S Y$. Of course this morphism has image contained in the open $p^{-1}(V) \cap q^{-1}(W)$. Thus $p^{-1}(V) \cap q^{-1}(W)$ is a fibre product of V and W over U. The result follows from the uniqueness of fibre products, see Categories, Section 4.6. $\square$

In particular this shows that $V \times_U W = V \times_S W$ in the situation of the lemma. Moreover, if U, V, W are all affine, then we know that $V \times_U W$ is affine. And of course we may cover $X \times_S Y$ by such affine opens $V \times_U W$. We formulate this as a lemma.

Lemma 21.17.4. *Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes with the same target. Let $S = \bigcup U_i$ be any affine open covering of S. For each $i \in I$, let $f^{-1}(U_i) =$*

$\bigcup_{j \in J_i} V_j$ be an affine open covering of $f^{-1}(U_i)$ and let $g^{-1}(U_i) = \bigcup_{k \in K_i} W_k$ be an affine open covering of $f^{-1}(U_i)$. Then

$$X \times_S Y = \bigcup_{i \in I} \bigcup_{j \in J_i,\ k \in K_i} V_j \times_{U_i} W_k$$

is an affine open covering of $X \times_S Y$.

Proof. See discussion above the lemma. □

In other words, we might have used the previous lemma as a way of construction the fibre product directly by glueing the affine schemes. (Which is of course exactly what we did in the proof of Lemma 21.16.1 anyway.) Here is a way to describe the set of points of a fibre product of schemes.

Lemma 21.17.5. *Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes with the same target. Points z of $X \times_S Y$ are in bijective correspondence to quadruples*

$$(x, y, s, \mathfrak{p})$$

where $x \in X$, $y \in Y$, $s \in S$ are points with $f(x) = s$, $g(y) = s$ and $\mathfrak{p}$ is a prime ideal of the ring $\kappa(x) \otimes_{\kappa(s)} \kappa(y)$. The residue field of z corresponds to the residue field of the prime $\mathfrak{p}$.

Proof. Let z be a point of $X \times_S Y$ and let us construct a triple as above. Recall that we may think of z as a morphism $Spec(\kappa(z)) \to X \times_S Y$, see Lemma 21.13.3. This morphism corresponds to morphisms $a : Spec(\kappa(z)) \to X$ and $b : Spec(\kappa(z)) \to Y$ such that $f \circ a = g \circ b$. By the same lemma again we get points $x \in X$, $y \in Y$ lying over the same point $s \in S$ as well as field maps $\kappa(x) \to \kappa(z)$, $\kappa(y) \to \kappa(z)$ such that the compositions $\kappa(s) \to \kappa(x) \to \kappa(z)$ and $\kappa(s) \to \kappa(y) \to \kappa(z)$ are the same. In other words we get a ring map $\kappa(x) \otimes_{\kappa(s)} \kappa(y) \to \kappa(z)$. We let $\mathfrak{p}$ be the kernel of this map.

Conversely, given a quadruple $(x, y, s, \mathfrak{p})$ we get a commutative solid diagram

$$\begin{tikzcd}
X \times_S Y \arrow[rrrd, bend left] \arrow[rdd, bend right] & & & \\
& Spec(\kappa(x) \otimes_{\kappa(s)} \kappa(y)/\mathfrak{p}) \arrow[ul, dashed] \arrow[r] \arrow[d] & Spec(\kappa(y)) \arrow[r] \arrow[d] & Y \arrow[dd] \\
& Spec(\kappa(x)) \arrow[r] \arrow[d] & Spec(\kappa(s)) \arrow[rd] & \\
& X \arrow[rr] & & S
\end{tikzcd}$$

see the discussion in Section 21.13. Thus we get the dotted arrow. The corrsponding point z of $X \times_S Y$ is the image of the generic point of $Spec(\kappa(x) \otimes_{\kappa(s)} \kappa(y)/\mathfrak{p})$. We omit the verification that the two constructions are inverse to each other. □

Lemma 21.17.6. *Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes with the same target.*

1. *If $f : X \to S$ is a closed immersion, then $X \times_S Y \to Y$ is a closed immersion. Moreover, if $X \to S$ corresponds to the quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_S$, then $X \times_S Y \to Y$ corresponds to the sheaf of ideals $Im(g^*\mathcal{I} \to \mathcal{O}_Y)$.*
2. *If $f : X \to S$ is an open immersion, then $X \times_S Y \to Y$ is an open immersion.*
3. *If $f : X \to S$ is an immersion, then $X \times_S Y \to Y$ is an immersion.*

Proof. Assume that $X \to S$ is a closed immersion corresponding to the quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_S$. By Lemma 21.4.7 the closed subspace $Z \subset Y$ defined by the sheaf of ideals $\mathrm{Im}(g^*\mathcal{I} \to \mathcal{O}_Y)$ is the fibre product in the category of locally ringed spaces. By Lemma 21.10.1 Z is a scheme. Hence $Z = X \times_S Y$ and the first statement follows. The second follows from Lemma 21.17.3 for example. The third is a combination of the first two. $\square$

Definition 21.17.7. Let $f : X \to Y$ be a morphism of schemes. Let $Z \subset Y$ be a closed subscheme of Y. The *inverse image $f^{-1}(Z)$ of the closed subscheme Z* is the closed subscheme $Z \times_Y X$ of X. See Lemma 21.17.6 above.

We may occasionally also use this terminology with locally closed and open subschemes.

21.18. Base change in algebraic geometry

One motivation for the introduction of the language of schemes is that it gives a very precise notion of what it means to define a variety over a particular field. For example a variety X over $\mathbf{Q}$ is synonymous (insert future reference here) with $X \to Spec(\mathbf{Q})$ which is of finite type, separated, irreducible and reduced[1]. In any case, the idea is more generally to work with schemes over a given *base scheme*, often denoted S. We use the language: ``let X be a scheme over S'' to mean simply that X comes equipped with a morphism $X \to S$. In diagrams we will try to picture the *structure morphism* $X \to S$ as a downward arrow from X to S. We are often more interested in the properties of X relative to S rather than the internal geometry of X. For example, we would like to know things about the fibres of $X \to S$, what happens to X after base change, etc, etc.

We introduce some of the language that is customarily used. Of course this language is just a special case of thinking about the category of objects over a given object in a category, see Categories, Example 4.2.13.

Definition 21.18.1. Let S be a scheme.

(1) We say X is a *scheme over S* to mean that X comes equipped with a morphism of schemes $X \to S$. The morphism $X \to S$ is sometimes called the *structure morphism*.
(2) If R is a ring we say X is a *scheme over R* instead of X is a scheme over $Spec(R)$.
(3) A *morphism $f : X \to Y$ of schemes over S* is a morphism of schemes such that the composition $X \to Y \to S$ of f with the structure morphism of Y is equal to the structure morphism of X.
(4) We denote $Mor_S(X, Y)$ the set of all morphisms from X to Y over S.
(5) Let X be a scheme over S. Let $S' \to S$ be a morphism of schemes. The *base change* of X is the scheme $X_{S'} = S' \times_S X$ over S'.
(6) Let $f : X \to Y$ be a morphism of schemes over S. Let $S' \to S$ be a morphism of schemes. The *base change* of f is the induced morphism $f' : X_{S'} \to Y_{S'}$ (namely the morphsm $\mathrm{id}_{S'} \times_{\mathrm{id}_S} f$).
(7) Let R be a ring. Let X be a scheme over R. Let $R \to R'$ be a ring map. The *base change* $X_{R'}$ is the scheme $Spec(R') \times_{Spec(R)} X$ over R'.

Here is a typical result.

[1]Of course algebraic geometers still quibble over whether one should require X to be geometrically irreducible over $\mathbf{Q}$.

Lemma 21.18.2. *Let S be a scheme. Let $f : X \to Y$ be an immersion (resp. closed immersion, resp. open immersion) of schemes over S. Then any base change of f is an immersion (resp. closed immersion, resp. open immersion).*

Proof. We can think of the base change of f via the morphism $S' \to S$ as the top left vertical arrow in the following commutative diagram:

$$\begin{tikzcd} X_{S'} \arrow[r] \arrow[d] & X \arrow[d] \arrow[dd, bend left] \\ Y_{S'} \arrow[r] \arrow[d] & Y \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

The diagram implies $X_{S'} \cong Y_{S'} \times_Y X$, and the lemma follows from Lemma 21.17.6. □

In fact this type of result is so typical that there is a piece of language to express it. Here it is.

Definition 21.18.3. Properties and base change.

(1) Let $\mathcal{P}$ be a property of schemes over a base. We say that $\mathcal{P}$ is *preserved under arbitrary base change*, or simply that *preserved under base change* if whenever X/S has $\mathcal{P}$, any base change $X_{S'}/S'$ has $\mathcal{P}$.
(2) Let $\mathcal{P}$ be a property of morphisms of schemes over a base. We say that $\mathcal{P}$ is *preserved under arbitrary base change*, or simply that *preserved under base change* if whenever $f : X \to Y$ over S has $\mathcal{P}$, any base change $f' : X_{S'} \to Y_{S'}$ over S' has $\mathcal{P}$.

At this point we can say that ``being a closed immersion" is preserved under arbitrary base change.

Definition 21.18.4. Let $f : X \to S$ be a morphism of schemes. Let $s \in S$ be a point. The *scheme theoretic fibre X_s of f over s*, or simply the *fibre of f over s* is the scheme fitting in the following fibre product diagram

$$\begin{tikzcd} X_s = Spec(\kappa(s)) \times_S X \arrow[r] \arrow[d] & X \arrow[d] \\ Spec(\kappa(s)) \arrow[r] & S \end{tikzcd}$$

We think of the fibre X_s always as a scheme over $\kappa(s)$.

Lemma 21.18.5. *Let $f : X \to S$ be a morphism of schemes. Consider the diagrams*

$$\begin{tikzcd} X_s \arrow[r] \arrow[d] & X \arrow[d] \\ Spec(\kappa(s)) \arrow[r] & S \end{tikzcd} \qquad \begin{tikzcd} Spec(\mathcal{O}_{S,s}) \times_S X \arrow[r] \arrow[d] & X \arrow[d] \\ Spec(\mathcal{O}_{S,s}) \arrow[r] & S \end{tikzcd}$$

In both cases the top horizontal arrow is a homeomorphism onto its image.

Proof. Choose an open affine $U \subset S$ that contains s. The bottom horizontal morphisms factor through U, see Lemma 21.13.1 for example. Thus we may assume that S is affine. If

X is also affine, then the result follows from Algebra, Remark 7.16.8. In the general case the result follows by covering X by open affines. □

21.19. Quasi-compact morphisms

A scheme is *quasi-compact* if its underlying topological space is quasi-compact. There is a relative notion which is defined as follows.

Definition 21.19.1. A morphism of schemes is called *quasi-compact* if the underlying map of topological spaces is quasi-compact, see Topology, Definition 5.9.1.

Lemma 21.19.2. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *$f : X \to S$ is quasi-compact,*
(2) *the inverse image of every affine open is quasi-compact, and*
(3) *there exists some affine open covering $S = \bigcup_{i \in I} U_i$ such that $f^{-1}(U_i)$ is quasi-compact for all i.*

Proof. Suppose we are given a covering $X = \bigcup_{i \in I} U_i$ as in (3). First, let $U \subset S$ be any affine open. For any $u \in U$ we can find an index $i(u) \in I$ such that $u \in U_{i(u)}$. By Lemma 21.11.5 we can find an affine open $W_u \subset U \cap U_{i(u)}$ which is standard open in both U and $U_{i(u)}$. By compactness we can find finitely many points $u_1, \ldots, u_n \in U$ such that $U = \bigcup_{j=1}^n W_{u_j}$. For each j write $f^{-1} U_{i(u_j)} = \bigcup_{k \in K_j} V_{jk}$ as a finite union of affine opens. Since $W_{u_j} \subset U_{i(u)}$ is a standard open we see that $f^{-1}(W_{u_j}) \cap V_{jk}$ is a standard open of V_{jk}, see Algebra, Lemma 7.16.4. Hence $f^{-1}(W_{u_j}) \cap V_{jk}$ is affine, and so $f^{-1}(W_{u_j})$ is a finite union of affines. This proves that the inverse image of any affine open is a finite union of affine opens.

Next, assume that the inverse image of every affine open is a finite union of affine opens. Let $K \subset X$ be any quasi-compact open. Since X has a basis of the topology consisting of affine opens we see that K is a finite union of affine opens. Hence the inverse image of K is a finite union of affine opens. Hence f is quasi-compact.

Finally, assume that f is quasi-compact. In this case the argument of the previous paragraph shows that the inverse image of any affine is a finite union of affine opens. □

Lemma 21.19.3. *Being quasi-compact is a property of morphisms of schemes over a base which is preserved under arbitrary base change.*

Proof. Omitted. □

Lemma 21.19.4. *The composition of quasi-compact morphisms is quasi-compact.*

Proof. Omitted. □

Lemma 21.19.5. *A closed immersion is quasi-compact.*

Proof. Follows from the definitions and Topology, Lemma 5.9.3. □

Example 21.19.6. An open immersion is in general not quasi-compact. The standard example of this is the open subspace $U \subset X$, where $X = Spec(k[x_1, x_2, x_3, \ldots])$, where U is $X \setminus \{0\}$, and where 0 is the point of X corresponding to the maximal ideal $(x_1, x_2, x_3, \ldots)$.

Lemma 21.19.7. *Let $f : X \to S$ be a quasi-compact morphism of schemes. The following are equivalent*

(1) *$f(X) \subset S$ is closed, and*
(2) *$f(X) \subset S$ is stable under specialization.*

Proof. We have (1) $\Rightarrow$ (2) by Topology, Lemma 5.14.2. Assume (2). Let $U \subset S$ be an affine open. It suffices to prove that $f(X) \cap U$ is closed. Since $U \cap f(X)$ is stable under specializations, we have reduced to the case where S is affine. Because f is quasi-compact we deduce that $X = f^{-1}(S)$ is quasi-compact as S is affine. Thus we may write $X = \bigcup_{i=1}^n U_i$ with $U_i \subset X$ open affine. Say $S = Spec(R)$ and $U_i = Spec(A_i)$ for some R-algebra A_i. Then $f(X) = \text{Im}(Spec(A_1 \times \ldots \times A_n) \to Spec(R))$. Thus the lemma follows from Algebra, Lemma 7.36.5. □

Lemma 21.19.8. *Let $f : X \to S$ be a quasi-compact morphism of schemes. Then f is closed if and only if specializations lift along f, see Topology, Definition 5.14.3.*

Proof. According to Topology, Lemma 5.14.6 if f is closed then specializations lift along f. Conversely, suppose that specializations lift along f. Let $Z \subset X$ be a closed subset. We may think of Z as a scheme with the reduced induced scheme structure, see Definition 21.12.5. Since $Z \subset X$ is closed the restriction of f to Z is still quasi-compact. Moreover specializations lift along $Z \to S$ as well, see Topology, Lemma 5.14.4. Hence it suffices to prove $f(X)$ is closed if specializations lift along f. In particular $f(X)$ is stable under specializations, see Topology, Lemma 5.14.5. Thus $f(X)$ is closed by Lemma 21.19.7. □

21.20. Valuative criterion for universal closedness

In Topology, Section 5.12 there is a discussion of proper maps as closed maps of topological spaces all of whose fibres are quasi-compact, or as maps such that all base changes are closed maps. Here is the corresponding notion in algebraic geometry.

Definition 21.20.1. A morphism of schemes $f : X \to S$ is said to be *universally closed* if every base change $f' : X_{S'} \to S'$ is closed.

In fact the adjective ``universally'' is often used in this way. In other words, given a property $\mathcal{P}$of morphisms the we say that ``$X \to S$ is universally $\mathcal{P}$'' if and only if every base change $X_{S'} \to S'$ has $\mathcal{P}$.

Please take a look at Morphisms, Section 24.40 for a more detailed discussion of the properties of universally closed morphisms. In this section we restrict the discussion to the relationship between universal closed morphisms and morphisms satisfying the existence part of the valuative criterion.

Lemma 21.20.2. *Let $f : X \to S$ be a morphism of schemes.*

(1) *If f is universally closed then specializations lift along any base change of f, see Topology, Definition 5.14.3.*
(2) *If f is quasi-compact and specializations lift along any base change of f, then f is universally closed.*

Proof. Part (1) is a direct consequence of Topology, Lemma 5.14.6. Part (2) follows from Lemmas 21.19.8 and 21.19.3. □

Definition 21.20.3. Let $f : X \to S$ be a morphism of schemes. We say *f satisfies the existence part of the valuative criterion* if given any commutative solid diagram

$$\begin{tikzcd} Spec(K) \arrow[r] \arrow[d] & X \arrow[d] \\ Spec(A) \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

where A is a valuation ring with field of fractions K, the dotted arrow exists. We say f *satisfies the uniqueness part of the valuative criterion* if there is at most one dotted arrow given any diagram as above (without requiring existence of course).

A *valuation ring* is a local domain maximal among the relation of domination in its fraction field, see Algebra, Definition 7.46.1. Hence the spectrum of a valuation ring has a unique generic point η and a unique closed point 0, and of course we have the specialization $\eta \rightsquigarrow 0$. The significance of valuation rings is that any specialization of points in any scheme is the image of $\eta \rightsquigarrow 0$ under some morphism from the spectrum of some valuation ring. Here is the precise result.

Lemma 21.20.4. *Let S be a scheme. Let $s' \rightsquigarrow s$ be a specialization of points of S. Then*

(1) *there exists a valuation ring A and a morphism $Spec(A) \to S$ such that the generic point η of $Spec(A)$ maps to s' and the special point maps to s, and*
(2) *given a field extension $\kappa(s') \subset K$ we may arrange it so that the extension $\kappa(s') \subset \kappa(\eta)$ induced by f is isomorphic to the given extension.*

Proof. Let $s' \rightsquigarrow s$ be a specialization in S, and let $\kappa(s') \subset K$ be an extension of fields. By Lemma 21.13.2 and the discussion following Lemma 21.13.3 this leads to ring maps $\mathcal{O}_{S,s} \to \kappa(s') \to K$. Let $A \subset K$ be any valuation ring whose field of fractions is K and which dominates the image of $\mathcal{O}_{S,s} \to K$, see Algebra, Lemma 7.46.2. The ring map $\mathcal{O}_{S,s} \to A$ induces the morphism $f : Spec(A) \to S$, see Lemma 21.13.1. This morphism has all the desired properties by construction. $\square$

Lemma 21.20.5. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *Specializations lift along any base change of f*
(2) *The morphism f satisfies the existence part of the valuative criterion.*

Proof. Assume (1) holds. Let a solid diagram as in Definition 21.20.3 be given. In order to find the dotted arrow we may replace $X \to S$ by $X_{Spec(A)} \to Spec(A)$ since after all the assumption is stable under base change. Thus we may assume $S = Spec(A)$. Let $x' \in X$ be the image of $Spec(K) \to X$, so that we have $\kappa(x') \subset K$, see Lemma 21.13.3. By assumption there exists a specialization $x' \rightsquigarrow x$ in X such that x maps to the closed point of $S = Spec(A)$. We get a local ring map $A \to \mathcal{O}_{X,x}$ and a ring map $\mathcal{O}_{X,x} \to \kappa(x')$, see Lemma 21.13.2 and the discussion following Lemma 21.13.3. The composition $A \to \mathcal{O}_{X,x} \to \kappa(x') \to K$ is the given injection $A \to K$. Since $A \to \mathcal{O}_{X,x}$ is local, the image of $\mathcal{O}_{X,x} \to K$ dominates A and hence is equal to A, by Algebra, Definition 7.46.1. Thus we obtain a ring map $\mathcal{O}_{X,x} \to A$ and hence a morphism $Spec(A) \to X$ (see Lemma 21.13.1 and discussion following it). This proves (2).

Conversely, assume (2) holds. It is immediate that the existence part of the valuative criterion holds for any base change $X_{S'} \to S'$ of f by considering the following commutative diagram

$$\begin{tikzcd}
Spec(K) \arrow[r] \arrow[d] & X_{S'} \arrow[r] \arrow[d] & X \arrow[d] \\
Spec(A) \arrow[r] \arrow[ru, dashed] \arrow[rru, dashed] & S' \arrow[r] & S
\end{tikzcd}$$

Namely, the more horizontal dotted arrow will lead to the other one by definition of the fibre product. OK, so it clearly suffices to show that specializations lift along f. Let $s' \rightsquigarrow s$

be a specialization in S, and let $x' \in X$ be a point lying over s'. Apply Lemma 21.20.4 to $s' \rightsquigarrow s$ and the extension of fields $\kappa(s') \subset \kappa(x') = K$. We get a commutative diagram

$$\begin{tikzcd} Spec(K) \arrow[rr] \arrow[d] & & X \arrow[d] \\ Spec(A) \arrow[r] \arrow[urr, dashed] & Spec(\mathcal{O}_{S,s}) \arrow[r] & S \end{tikzcd}$$

and by condition (2) we get the dotted arrow. The image x of the closed point of $Spec(A)$ in X will be a solution to our problem, i.e., x is a specialization of x' and maps to s. □

Proposition 21.20.6. *(Valuative criterion of universal closedness.) Let f be a quasi-compact morphism of schemes. Then f is universally closed if and only if f satisfies the existence part of the valuative criterion.*

Proof. This is a formal consequence of Lemmas 21.20.2 and 21.20.5 above. □

Example 21.20.7. Let k be a field. Consider the structure morphism $p : \mathbf{P}^1_k \to Spec(k)$ of the projective line over k, see Example 21.14.4. Let us use the valuative criterion above to prove that p is universally closed. By construction $\mathbf{P}^1_k$ is covered by two affine opens and hence p is quasi-compact. Let a commutative diagram

$$\begin{tikzcd} Spec(K) \arrow[r, "\xi"'] \arrow[d] & \mathbf{P}^1_k \arrow[d] \\ Spec(A) \arrow[r, "\varphi"] & Spec(k) \end{tikzcd}$$

be given, where A is a valuation ring and K is its field of fractions. Recall that $\mathbf{P}^1_k$ is gotten by glueing $Spec(k[x])$ to $Spec(k[y])$ by glueing $D(x)$ to $D(y)$ via $x = y^{-1}$ (or more symmetrically $xy = 1$). To show there is a morphism $Spec(A) \to \mathbf{P}^1_k$ fitting diagonally into the diagram above we may assume that ξ maps into the open $Spec(k[x])$ (by symmetry). This gives the following commutative diagram of rings

$$\begin{tikzcd} K & k[x] \arrow[l, "\xi^\sharp"'] \\ A \arrow[u] & k \arrow[l, "\varphi^\sharp"] \arrow[u] \end{tikzcd}$$

By Algebra, Lemma 7.46.3 we see that either $\xi^\sharp(x) \in A$ or $\xi^\sharp(x)^{-1} \in A$. In the first case we get a ring map

$$k[x] \to A,\ \lambda \mapsto \varphi^\sharp(\lambda),\ x \mapsto \xi^\sharp(x)$$

fitting into the diagram of rings above, and we win. In the second case we see that we get a ring map

$$k[y] \to A,\ \lambda \mapsto \varphi^\sharp(\lambda),\ y \mapsto \xi^\sharp(x)^{-1}.$$

This gives a morphism $Spec(A) \to Spec(k[y]) \to \mathbf{P}^1_k$ which fits diagonally into the initial commutative diagram of this example (check omitted).

21.21. Separation axioms

A topological space X is Hausdorff if and only if the diagonal $\Delta \subset X \times X$ is a closed subset. The analogue in algebraic geometry is, given a scheme X over a base scheme S, to consider the diagonal morphism

$$\Delta_{X/S} : X \longrightarrow X \times_S X.$$

This is the unique morphism of schemes such that $\mathrm{pr}_1 \circ \Delta_{X/S} = \mathrm{id}_X$ and $\mathrm{pr}_2 \circ \Delta_{X/S} = \mathrm{id}_X$ (it exists in any category with fibre products).

Lemma 21.21.1. *The diagonal morphism of a morphism between affines is closed.*

Proof. The diagonal morphism associated to the morphism $Spec(S) \to Spec(R)$ is the morphism on spectra corresponding to the ring map $S \otimes_R S \to S$, $a \otimes b \mapsto ab$. This map is clearly surjective, so $S \cong S \otimes_R S/J$ for some ideal $J \subset S \otimes_R S$. Hence Δ is a closed immersion according to Example 21.8.1 $\square$

Lemma 21.21.2. *Let X be a scheme over S. The diagonal morphism $\Delta_{X/S}$ is an immersion.*

Proof. Recall that if $V \subset X$ is affine open and maps into $U \subset S$ affine open, then $V \times_U V$ is affine open in $X \times_S X$, see Lemmas 21.17.2 and 21.17.3. Consider the open subscheme W of $X \times_S X$ which is the union of these affine opens $V \times_U V$. By Lemma 21.4.2 it is enough to show that each morphism $\Delta_{X/S}^{-1}(V \times_U V) \to V \times_U V$ is a closed immersion. Since $V = \Delta_{X/S}^{-1}(V \times_U V)$ we are just checking that $\Delta_{V/U}$ is a closed immersion, which is Lemma 21.21.1. $\square$

Definition 21.21.3. Let $f : X \to S$ be a morphism of schemes.

(1) We say f is *separated* if the diagonal morphism $\Delta_{X/S}$ is a closed immersion.
(2) We say f is *quasi-separated* if the diagonal morphism $\Delta_{X/S}$ is a quasi-compact morphism.
(3) We say a scheme Y is *separated* if the morphism $Y \to Spec(\mathbf{Z})$ is separated.
(4) We say a scheme Y is *quasi-separated* if the morphism $Y \to Spec(\mathbf{Z})$ is quasi-separated.

By Lemmas 21.21.2 and 21.10.4 we see that $\Delta_{X/S}$ is a closed immersion if an only if $\Delta_{X/S}(X) \subset X \times_S X$ is a closed subset. Moreover, by Lemma 21.19.5 we see that a separated morphism is quasi-separated. The reason for introducing quasi-separated morphisms is that nonseparated morphisms come up naturally in studying algebraic varieties (especially when doing moduli, algebraic stacks, etc). But most often they are still quasi-separated.

Example 21.21.4. Here is an example of a non-quasi-separated morphism. Suppose $X = X_1 \cup X_2 \to S = Spec(k)$ with $X_1 = X_2 = Spec(k[t_1, t_2, t_3, \ldots])$ glued along the complement of $\{0\} = \{(t_1, t_2, t_3, \ldots)\}$ (glued as in Example 21.14.3). In this case the inverse image of the affine scheme $X_1 \times_S X_2$ under $\Delta_{X/S}$ is the scheme $Spec(k[t_1, t_2, t_3, \ldots]) \setminus \{0\}$ which is not quasi-compact.

Lemma 21.21.5. *Let X, Y be schemes over S. Let $a, b : X \to Y$ be morphisms of schemes over S. There exists a largest locally closed subscheme $Z \subset X$ such that $a|_Z = b|_Z$. In fact Z is the equalizer of (a, b). Moreover, if Y is separated over S, then Z is a closed subscheme.*

Proof. The equalizer of (a, b) is for categorical reasons the fibre product Z in the following diagram

$$\begin{tikzcd} Z = Y \times_{(Y\times_S Y)} X \arrow[r] \arrow[d] & X \arrow[d, "{(a,b)}"] \\ Y \arrow[r, "\Delta_{Y/S}"] & Y \times_S Y \end{tikzcd}$$

Thus the lemma follows from Lemmas 21.18.2, 21.21.2 and Definition 21.21.3. □

Lemma 21.21.6. *An affine scheme is separated. A morphism of affine schemes is separated.*

Proof. See Lemma 21.21.1. □

Lemma 21.21.7. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *The morphism f is quasi-separated.*
(2) *For every pair of affine opens $U, V \subset X$ which map into a common affine open of S the intersection $U \cap V$ is a finite union of affine opens of X.*
(3) *There exists an affine open covering $S = \bigcup_{i\in I} U_i$ and for each i an affine open covering $f^{-1}U_i = \bigcup_{j\in I_i} V_j$ such that for each i and each pair $j, j' \in I_i$ the intersection $V_j \cap V_{j'}$ is a finite union of affine opens of X.*

Proof. Let us prove that (3) implies (1). By Lemma 21.17.4 the covering $X \times_S X = \bigcup_i \bigcup_{j,j'} V_j \times_{U_i} V_{j'}$ is an affine open covering of $X \times_S X$. Moreover, $\Delta^{-1}_{X/S}(V_j \times_{U_i} V_{j'}) = V_j \cap V_{j'}$. Hence the implication follows from Lemma 21.19.2.

The implication (1) ⇒ (2) follows from the fact that under the hypotheses of (1) the fibre product $U \times_S V$ is an affine open of $X \times_S X$. The implication (2) ⇒ (3) is trivial. □

Lemma 21.21.8. *Let $f : X \to S$ be a morphism of schemes.*

(1) *If f is separated then for every pair of affine opens (U, V) of X which map into a common affine open of S we have*
   (a) *the intersection $U \cap V$ is affine.*
   (b) *the ring map $\mathcal{O}_X(U) \otimes_{\mathbf{Z}} \mathcal{O}_X(V) \to \mathcal{O}_X(U \cap V)$ is surjective.*
(2) *If any pair of points $x_1, x_2 \in X$ lying over a common point $s \in S$ are contained in affine opens $x_1 \in U$, $x_2 \in V$ which map into a common affine open of S such that (a), (b) hold, then f is separated.*

Proof. Assume f separated. Suppose (U, V) is a pair as in (1). Let $W = Spec(R)$ be an affine open of S containing both $f(U)$ and $g(V)$. Write $U = Spec(A)$ and $V = Spec(B)$ for R-algebras A and B. By Lemma 21.17.3 we see that $U \times_S V = U \times_W V = Spec(A \otimes_R B)$ is an affine open of $X \times_S X$. Hence, by Lemma 21.10.1 we see that $\Delta^{-1}(U \times_S V) \to U \times_S V$ can be identified with $Spec(A \otimes_R B/J)$ for some ideal $J \subset A \otimes_R B$. Thus $U \cap V = \Delta^{-1}(U \times_S V)$ is affine. Assertion (1)(b) holds because $A \otimes_{\mathbf{Z}} B \to (A \otimes_R B)/J$ is surjective.

Assume the hypothesis formulated in (2) holds. Clearly the collection of affine opens $U \times_S V$ for pairs (U, V) as in (2) form an affine open covering of $X \times_S X$ (see e.g. Lemma 21.17.4). Hence it suffices to show that each morphism $U \cap V = \Delta^{-1}_{X/S}(U \times_S V) \to U \times_S V$ is a closed immersion, see Lemma 21.4.2. By assumption (a) we have $U \cap V = Spec(C)$ for some ring C. After choosing an affine open $W = Spec(R)$ of S into which both U and V map and writing $U = Spec(A)$, $V = Spec(B)$ we see that the assumption (b) means that the composition

$$A \otimes_{\mathbf{Z}} B \to A \otimes_R B \to C$$

is surjective. Hence $A \otimes_R B \to C$ is surjective and we conclude that $Spec(C) \to Spec(A \otimes_R B)$ is a closed immersion. $\square$

Example 21.21.9. Let k be a field. Consider the structure morphism $p : \mathbf{P}^1_k \to Spec(k)$ of the projective line over k, see Example 21.14.4. Let us use the lemma above to prove that p is separated. By construction $\mathbf{P}^1_k$ is covered by two affine opens $U = Spec(k[x])$ and $V = Spec(k[y])$ with intersection $U \cap V = Spec(k[x, y]/(xy - 1))$ (using obvious notation). Thus it suffices to check that conditions (2)(a) and (2)(b) of Lemma 21.21.8 hold for the pairs of affine opens (U, U), (U, V), (V, U) and (V, V). For the pairs (U, U) and (V, V) this is trivial. For the pair (U, V) this amounts to proving that $U \cap V$ is affine, which is true, and that the ring map

$$k[x] \otimes_{\mathbf{Z}} k[y] \longrightarrow k[x, y]/(xy - 1)$$

is surjective. This is clear because any element in the right hand side can be written as a sum of a polynomial in x and a polynomial in y.

Lemma 21.21.10. *Let $f : X \to T$ and $g : Y \to T$ be morphisms of schemes with the same target. Let $h : T \to S$ be a morphism of schemes. Then the induced morphism $i : X \times_T Y \to X \times_S Y$ is an immersion. If $T \to S$ is separated, then i is a closed immersion. If $T \to S$ is quasi-separated, then i is a quasi-compact morphism.*

Proof. By general category theory the following diagram

$$\begin{tikzcd} X \times_T Y \arrow[r] \arrow[d] & X \times_S Y \arrow[d] \\ T \arrow[r, "\Delta_{T/S}"] & T \times_S T \end{tikzcd}$$

is a fibre product diagram. The lemma follows from Lemmas 21.21.2, 21.17.6 and 21.19.3. $\square$

Lemma 21.21.11. *Let $g : X \to Y$ be a morphism of schemes over S. The morphism $i : X \to X \times_S Y$ is an immersion. If Y is separated over S it is a closed immersion. If Y is quasi-separated over S it is quasi-compact.*

Proof. This is a special case of Lemma 21.21.10 applied to the morphism $X = X \times_Y Y \to X \times_S Y$. $\square$

Lemma 21.21.12. *Let $f : X \to S$ be a morphism of schemes. Let $s : S \to X$ be a section of f (in a formula $f \circ s = id_S$). Then s is an immersion. If f is separated then s is a closed immersion. If f is quasi-separated, then s is quasi-compact.*

Proof. This is a special case of Lemma 21.21.11 applied to $g = s$ so the morphism $i = s : S \to S \times_S X$. $\square$

Lemma 21.21.13. *Permanence properties.*

1. *A composition of separated morphisms is separated.*
2. *A composition of quasi-separated morphisms is quasi-separated.*
3. *The base change of a separated morphism is separated.*
4. *The base change of a quasi-separated morphism is quasi-separated.*
5. *A (fibre) product of separated morphisms is separated.*
6. *A (fibre) product of quasi-separated morphisms is quasi-separated.*

Proof. Let $X \to Y \to Z$ be morphisms. Assume that $X \to Y$ and $Y \to Z$ are separated. The composition

$$X \to X \times_Y X \to X \times_Z X$$

is closed because the first one is by assumption and the second one by Lemma 21.21.10. The same argument works for ``quasi-separated'' (with the same references).

Let $f : X \to Y$ be a morphism of schemes over a base S. Let $S' \to S$ be a morphism of schemes. Let $f' : X_{S'} \to Y_{S'}$ be the base change of f. Then the diagonal morphism of f' is a morphism

$$\Delta_{f'} : X_{S'} = S' \times_S X \longrightarrow X_{S'} \times_{Y_{S'}} X_{S'} = S' \times_S (X \times_Y X)$$

which is easily seen to be the base change of Δ_f. Thus (3) and (4) follow from the fact that closed immersions and quasi-compact morphisms are preserved under arbitrary base change (Lemmas 21.17.6 and 21.19.3).

If $f : X \to Y$ and $g : U \to V$ are morphisms of schemes over a base S, then $f \times g$ is the composition of $X \times_S U \to X \times_S V$ (a base change of g) and $X \times_S V \to Y \times_S V$ (a base change of f). Hence (5) and (6) follow from (1) -- (4). □

Lemma 21.21.14. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. If $g \circ f$ is separated then so is f. If $g \circ f$ is quasi-separated then so is f.*

Proof. Assume that $g \circ f$ is separated. Consider the factorization $X \to X \times_Y X \to X \times_Z X$ of the diagonal morphism of $g \circ f$. By Lemma 21.21.10 the last morphism is an immersion. By assumption the image of X in $X \times_Z X$ is closed. Hence it is also closed in $X \times_Y X$. Thus we see that $X \to X \times_Y X$ is a closed immersion by Lemma 21.10.4.

Assume that $g \circ f$ is quasi-separated. Let $V \subset Y$ be an affine open which maps into an affine open of Z. Let $U_1, U_2 \subset X$ be affine opens which map into V. Then $U_1 \cap U_2$ is a finite union of affine opens because U_1, U_2 map into a common affine open of Z. Since we may cover Y by affine opens like V we deduce the lemma from Lemma 21.21.7. □

Lemma 21.21.15. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. If $g \circ f$ is quasi-compact and g is quasi-separated then f is quasi-compact.*

Proof. This is true because f equals the composition $(1, f) : X \to X \times_Z Y \to Y$. The first map is quasi-compact by Lemma 21.21.12 because it is a section of the quasi-separated morphism $X \times_Z Y \to X$ (a base change of g, see Lemma 21.21.13). The second map is quasi-compact as it is the base change of f, see Lemma 21.19.3. And compositions of quasi-compact morphisms are quasi-compact, see Lemma 21.19.4. □

You may have been wondering whether the condition of only considering pairs of affine opens whose image is contained in an affine open is really necessary to be able to conclude that their intersection is affine. Often it isn't!

Lemma 21.21.16. *Let $f : X \to S$ be a morphism. Assume f is separated and S is a separated scheme. Suppose $U \subset X$ and $V \subset X$ are affine. Then $U \cap V$ is affine (and a closed subscheme of $U \times V$).*

Proof. In this case X is separated by Lemma 21.21.13. Hence $U \cap V$ is affine by applying Lemma 21.21.8 to the morphism $X \to Spec(\mathbf{Z})$. □

On the other hand, the following example shows that we cannot expect the image of an affine to be contained in an affine.

Example 21.21.17. Consider the nonaffine scheme $U = Spec(k[x, y]) \setminus \{(x, y)\}$ of Example 21.9.3. On the other hand, consider the scheme

$$\mathbf{GL}_{2,k} = Spec(k[a, b, c, d, 1/ad - bc]).$$

There is a morphism $\mathbf{GL}_{2,k} \to U$ corresponding to the ring map $x \mapsto a$, $y \mapsto b$. It is easy to see that this is a surjective morphism, and hence the image is not contained in any affine open of U. In fact, the affine scheme $\mathbf{GL}_{2,k}$ also surjects onto $\mathbf{P}^1_k$, and $\mathbf{P}^1_k$ does not even have an immersion into *any* affine scheme.

21.22. Valuative criterion of separatedness

Lemma 21.22.1. *Let $f : X \to S$ be a morphism of schemes. If f is separated, then f satisfies the uniqueness part of the valuative criterion.*

Proof. Let a diagram as in Definition 21.20.3 be given. Suppose there are two morphisms $a, b : Spec(A) \to X$ fitting into the diagram. Let $Z \subset Spec(A)$ be the equalizer of a and b. By Lemma 21.21.5 this is a closed subscheme of $Spec(A)$. By assumption it contains the generic point of $Spec(A)$. Since A is a domain this implies $Z = Spec(A)$. Hence $a = b$ as desired. □

Lemma 21.22.2. *(Valuative criterion separatedness.) Let $f : X \to S$ be a morphism. Assume*

(1) *the morphism f is quasi-separated, and*
(2) *the morphism f satisfies the uniqueness part of the valuative criterion.*

Then f is separated.

Proof. By assumption (1) and Proposition 21.20.6 we see that it suffices to prove the morphism $\Delta_{X/S} : X \to X \times_S X$ satisfies the existence part of the valuative criterion. Let a solid commutative diagram

$$\begin{tikzcd} Spec(K) \arrow[r] \arrow[d] & X \arrow[d] \\ Spec(A) \arrow[r] \arrow[ur, dashed] & X \times_S X \end{tikzcd}$$

be given. The lower right arrow corresponds to a pair of morphisms $a, b : Spec(A) \to X$ over S. By (2) we see that $a = b$. Hence using a as the dotted arrow works. □

21.23. Monomorphisms

Definition 21.23.1. A morphism of schemes is called a *monomorphism* if it is a monomorphism in the category of schemes, see Categories, Definition 4.23.1.

Lemma 21.23.2. *Let $j : X \to Y$ be a morphism of schemes. Then j is a monomorphism if and only if the diagonal morphism $\Delta_{X/Y} : X \to X \times_Y X$ is an isomorphism.*

Proof. This is true in any category with fibre products. □

Lemma 21.23.3. *A monomorphism of schemes is separated.*

Proof. This is true because an isomorphism is a closed immersion, and Lemma 21.23.2 above. □

Lemma 21.23.4. *A composition of monomorphisms is a monomorpism.*

Proof. True in any category. □

Lemma 21.23.5. *The base change of a monomorphism is a monomorphism.*

Proof. True in any category with fibre products. □

Lemma 21.23.6. *Let $j : X \to Y$ be a morphism of schemes. If*

(1) *j is injective on points, and*
(2) *for any $x \in X$ the ring map $j_x^\sharp : \mathcal{O}_{Y,j(x)} \to \mathcal{O}_{X,x}$ is surjective,*

then j is a monomorphism.

Proof. Let $a, b : Z \to X$ be two morphisms of schemes such that $j \circ a = j \circ b$. Then (1) implies $a = b$ as underlying maps of topological spaces. For any $z \in Z$ we have $a_z^\sharp \circ j_{a(z)}^\sharp = b_z^\sharp \circ j_{b(z)}^\sharp$ as maps $\mathcal{O}_{Y,j(a(z))} \to \mathcal{O}_{Z,z}$. The surjectivity of the maps $j_x^\sharp$ forces $a_z^\sharp = b_z^\sharp$, $\forall z \in Z$. This implies that $a^\sharp = b^\sharp$. Hence we conclude $a = b$ as morphisms of schemes as desired. □

Lemma 21.23.7. *An immersion of schemes is a monomorphism. In particular, any immersion is separated.*

Proof. We can see this by checking that the criterion of Lemma 21.23.6 applies. More elegantly perhaps, we can use that Lemmas 21.3.5 and 21.4.6 imply that open and closed immersions are monomorphisms and hence any immersion (which is a composition of such) is a monomorphism. □

Lemma 21.23.8. *Let $f : X \to S$ be a separated morphism. Any locally closed subscheme $Z \subset X$ is separated over S.*

Proof. Follows from Lemma 21.23.7 and the fact that a composition of separated morphisms is separated (Lemma 21.21.13). □

Example 21.23.9. The morphism $Spec(\mathbf{Q}) \to Spec(\mathbf{Z})$ is a monomorphism. This is true because $\mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{Q} = \mathbf{Q}$. More generally, for any scheme S and any point $s \in S$ the canonical morphism

$$Spec(\mathcal{O}_{S,s}) \longrightarrow S$$

is a monomorphism.

Lemma 21.23.10. *Let $k_1, \ldots, k_n$ be fields. For any monomorphism of schemes $X \to Spec(k_1 \times \ldots \times k_n)$ there exists a subset $I \subset \{1, \ldots, n\}$ such that $X \cong Spec(\prod_{i \in I} k_i)$ as schemes over $Spec(k_1 \times \ldots \times k_n)$. More generally, if $X = \coprod_{i \in I} Spec(k_i)$ is a disjoint union of spectra of fields and $Y \to X$ is a monomorphism, then there exists a subset $J \subset I$ such that $Y = \coprod_{i \in J} Spec(k_i)$.*

Proof. First reduce to the case $n = 1$ (or $\#I = 1$) by taking the inverse images of the open and closed subschemes $Spec(k_i)$. In this case X has only one point hence is affine. The corresponding algebra problem is this: If $k \to R$ is an algebra map with $R \otimes_k R \cong R$, then $R \cong k$. This holds for dimension reasons. See also Algebra, Lemma 7.99.8 □

21.24. Functoriality for quasi-coherent modules

Let X be a scheme. We denote $QCoh(\mathcal{O}_X)$ or $QCoh(X)$ the category of quasi-coherent $\mathcal{O}_X$-modules as defined in Modules, Definition 15.10.1. We have seen in Section 21.7 that the category $QCoh(\mathcal{O}_X)$ has a lot of good properties when X is affine. Since the property of being quasi-coherent is local on X, these properties are inherited by the category of quasi-coherent sheaves on any scheme X. We enumerate them here.

(1) A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is quasi-coherent if and only if the restriction of $\mathcal{F}$ to each affine open $U = Spec(R)$ is of the form $\widetilde{M}$ for some R-module M.
(2) A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is quasi-coherent if and only if the restriction of $\mathcal{F}$ to each of the members of an affine open covering is quasi-coherent.
(3) Any direct sum of quasi-coherent sheaves is quasi-coherent.
(4) Any colimit of quasi-coherent sheaves is quasi-coherent.
(5) The kernel and cokernel of a morphism of quasi-coherent sheaves is quasi-coherent.
(6) Given a short exact sequence of $\mathcal{O}_X$-modules $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ if two out of three are quasi-coherent so is the third.
(7) Given a morphism of schemes $f : Y \to X$ the pullback of a quasi-coherent $\mathcal{O}_X$-module is a quasi-coherent $\mathcal{O}_Y$-module. See Modules, Lemma 15.10.4.
(8) Given two quasi-coherent $\mathcal{O}_X$-modules the tensor product is quasi-coherent, see Modules, Lemma 15.15.5.
(9) Given a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the tensor, symmetric and exterior algebras on $\mathcal{F}$ are quasi-coherent, see Modules, Lemma 15.18.6.
(10) Given two quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}$, $\mathcal{G}$ such that $\mathcal{F}$ is of finite presentation, then the internal hom $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is quasi-coherent, see Modules, Lemma 15.19.4 and (5) above.

On the other hand, it is in general not the case that the push forward of a quasi-coherent module is quasi-coherent. Here is a case where it this does hold.

Lemma 21.24.1. *Let $f : X \to S$ be a morphism of schemes. If f is quasi-compact and quasi-separated then f_* transforms quasi-coherent $\mathcal{O}_X$-modules into quasi-coherent $\mathcal{O}_S$-modules.*

Proof. The question is local on S and hence we may assume that S is affine. Because X is quasi-compact we may write $X = \bigcup_{i=1}^{n} U_i$ with each U_i open affine. Because f is quasi-separated we may write $U_i \cap U_j = \bigcup_{k=1}^{n_{ij}} U_{ijk}$ for some affine open U_{ijk}, see Lemma 21.21.7. Denote $f_i : U_i \to S$ and $f_{ijk} : U_{ijk} \to S$ the restrictions of f. For any open V of S and any sheaf $\mathcal{F}$ on X we have

$$\begin{aligned} f_*\mathcal{F}(V) &= \mathcal{F}(f^{-1}V) \\ &= \mathrm{Ker}\left(\bigoplus\nolimits_i \mathcal{F}(f^{-1}V \cap U_i) \to \bigoplus\nolimits_{i,j,k} \mathcal{F}(f^{-1}V \cap U_{ijk})\right) \\ &= \mathrm{Ker}\left(\bigoplus\nolimits_i f_{i,*}(\mathcal{F}|_{U_i})(V) \to \bigoplus\nolimits_{i,j,k} f_{ijk,*}(\mathcal{F}|_{U_{ijk}})\right)(V) \\ &= \mathrm{Ker}\left(\bigoplus\nolimits_i f_{i,*}(\mathcal{F}|_{U_i}) \to \bigoplus\nolimits_{i,j,k} f_{ijk,*}(\mathcal{F}|_{U_{ijk}})\right)(V) \end{aligned}$$

In other words there is a short exact sequence of sheaves

$$0 \to f_*\mathcal{F} \to \bigoplus f_{i,*}\mathcal{F}_i \to \bigoplus f_{ijk,*}\mathcal{F}_{ijk}$$

where $\mathcal{F}_i, \mathcal{F}_{ijk}$ denotes the restriction of $\mathcal{F}$ to the corresponding open. If $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-modules then $\mathcal{F}_i$, $\mathcal{F}_{ijk}$ is a quasi-coherent $\mathcal{O}_{U_i}$, $\mathcal{O}_{U_{ijk}}$-module. Hence by Lemma 21.7.3 we see that the second and third term of the exact sequence are quasi-coherent $\mathcal{O}_S$-modules. Thus we conclude that $f_*\mathcal{F}$ is a quasi-coherent $\mathcal{O}_S$-module. □

Using this we can characterize (closed) immersions of schemes as follows.

Lemma 21.24.2. *Let $f : X \to Y$ be a morphism of schemes. Suppose that*

(1) *f induces a homeomorphism of X with a closed subset of Y, and*

(2) $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ *is surjective.*

Then f is a closed immersion of schemes.

Proof. Assume (1) and (2). By (1) the morphism f is quasi-compact (see Topology, Lemma 5.9.3). Conditions (1) and (2) imply conditions (1) and (2) of Lemma 21.23.6. Hence $f : X \to Y$ is a monomorphism. In particular, f is separated, see Lemma 21.23.3. Hence Lemma 21.24.1 above applies and we conclude that $f_*\mathcal{O}_X$ is a quasi-coherent $\mathcal{O}_Y$-module. Therefore the kernel of $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is quasi-coherent by Lemma 21.7.8. Since a quasi-coherent sheaf is locally generated by sections (see Modules, Definition 15.10.1) this implies that f is a closed immersion, see Definition 21.4.1. □

We can use this lemma to prove the following lemma.

Lemma 21.24.3. *A composition of immersions of schemes is an immersion, a composition of closed immersions of schemes is a closed immersion, and a composition of open immersions of schemes is an open immersion.*

Proof. This is clear for the case of open immersions since an open subspace of an open subspace is also an open subspace.

Suppose $a : Z \to Y$ and $b : Y \to X$ are closed immersions of schemes. We will verify that $c = b \circ a$ is also a closed immersion. The assumption implies that a and b are homeomorphisms onto closed subsets, and hence also $c = b \circ a$ is a homeomorphism onto a closed subset. Moreover, the map $\mathcal{O}_X \to c_*\mathcal{O}_Z$ is surjective since it factors as the composition of the surjective maps $\mathcal{O}_X \to b_*\mathcal{O}_Y$ and $b_*\mathcal{O}_Y \to b_*a_*\mathcal{O}_Z$ (surjective as b_* is exact, see Modules, Lemma 15.6.1). Hence by Lemma 21.24.2 above c is a closed immersion.

Finally, we come to the case of immersions. Suppose $a : Z \to Y$ and $b : Y \to X$ are immersions of schemes. This means there exist open subschemes $V \subset Y$ and $U \subset X$ such that $a(Z) \subset V$, $b(Y) \subset U$ and $a : Z \to V$ and $b : Y \to U$ are closed immersions. Since the topology on Y is induced from the topology on U we can find an open $U' \subset U$ such that $V = b^{-1}(U')$. Then we see that $Z \to V = b^{-1}(U') \to U'$ is a composition of closed immersions and hence a closed immersion. This proves that $Z \to X$ is an immersion and we win. □

21.25. Other chapters

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(7) Commutative Algebra
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CHAPTER 22

Constructions of Schemes

22.1. Introduction

In this chapter we introduce ways of constructing schemes out of others. A basic reference is [**DG67**].

22.2. Relative glueing

The following lemma is relevant in case we are trying to construct a scheme X over S, and we already know how to construct the restriction of X to the affine opens of S. The actual result is completely general and works in the setting of (locally) ringed spaces, allthough our proof is written in the language of schemes.

Lemma 22.2.1. *Let S be a scheme. Let $\mathcal{B}$ be a basis for the topology of S. Suppose given the following data:*

1. *For every $U \in \mathcal{B}$ a scheme $f_U : X_U \to U$ over U.*
2. *For every pair $U, V \in \mathcal{B}$ such that $V \subset U$ a morphism $\rho^U_V : X_V \to X_U$.*

Assume that

- (a) *each ρ^U_V induces an isomorphism $X_V \to f_U^{-1}(V)$ of schemes over V,*
- (b) *whenever $W, V, U \in \mathcal{B}$, with $W \subset V \subset U$ we have $\rho^U_W = \rho^U_V \circ \rho^V_W$.*

Then there exists a unique scheme $f : X \to S$ over S and isomorphisms $i_U : f^{-1}(U) \to X_U$ over U such that for $V \subset U \subset S$ affine open the composition

$$X_V \xrightarrow{i_V^{-1}} f^{-1}(V) \xrightarrow{\text{inclusion}} f^{-1}(U) \xrightarrow{i_U} X_U$$

is the morphism ρ^U_V.

Proof. To prove this we will use Schemes, Lemma 21.15.4. First we define a contravariant functor F from the category of schemes to the category of sets. Namely, for a scheme T we set

$$F(T) = \left\{ \begin{matrix} (g, \{h_U\}_{U\in\mathcal{B}}),\ g : T \to S,\ h_U : g^{-1}(U) \to X_U, \\ f_U \circ h_U = g|_{g^{-1}(U)},\ h_U|_{g^{-1}(V)} = \rho^U_V \circ h_V \,\forall\, V, U \in \mathcal{B}, V \subset U \end{matrix} \right\}.$$

The restriction mapping $F(T) \to F(T')$ given a morphism $T' \to T$ is just gotten by composition. For any $W \in \mathcal{B}$ we consider the subfunctor $F_W \subset F$ consisting of those systems $(g, \{h_U\})$ such that $g(T) \subset W$.

First we show F satisfies the sheaf property for the Zariski topology. Suppose that T is a scheme, $T = \bigcup V_i$ is an open covering, and $\xi_i \in F(V_i)$ is an element such that $\xi_i|_{V_i \cap V_j} = \xi_j|_{V_i \cap V_j}$. Say $\xi_i = (g_i, \{f_{i,U}\})$. Then we immediately see that the morphisms g_i glue to a unique global morphism $g : T \to S$. Moreover, it is clear that $g^{-1}(U) = \bigcup g_i^{-1}(U)$. Hence the morphisms $h_{i,U} : g_i^{-1}(U) \to X_U$ glue to a unique morphism $h_U : U \to X_U$. It is easy to verify that the system $(g, \{f_U\})$ is an element of $F(T)$. Hence F satisfies the sheaf property for the Zariski topology.

Next we verify that each F_W, $W \in \mathcal{B}$ is representable. Namely, we claim that the transformation of functors

$$F_W \longrightarrow Mor(-, X_W),\ (g, \{h_U\}) \longmapsto h_W$$

is an isomorphism. To see this suppose that T is a scheme and $\alpha : T \to X_W$ is a morphism. Set $g = f_W \circ \alpha$. For any $U \in \mathcal{B}$ such that $U \subset W$ we can define $h_U : g^{-1}(U) \to X_U$ be the composition $(\rho^W_U)^{-1} \circ \alpha|_{g^{-1}(U)}$. This works because the image $\alpha(g^{-1}(U))$ is contained in $f_W^{-1}(U)$ and condition (a) of the lemma. It is clear that $f_U \circ h_U = g|_{g^{-1}(U)}$ for such a U. Moreover, if also $V \in \mathcal{B}$ and $V \subset U \subset W$, then $\rho^U_V \circ h_V = h_U|_{g^{-1}(V)}$ by property (b) of the lemma. We still have to define h_U for an arbitrary element $U \in \mathcal{B}$. Since $\mathcal{B}$ is a basis for the topology on S we can find an open covering $U \cap W = \bigcup U_i$ with $U_i \in \mathcal{B}$. Since g maps into W we have $g^{-1}(U) = g^{-1}(U \cap W) = \bigcup g^{-1}(U_i)$. Consider the morphisms $h_i = \rho^U_{U_i} \circ h_{U_i} : g^{-1}(U_i) \to X_U$. It is a simple matter to use condition (b) of the lemma to prove that $h_i|_{g^{-1}(U_i)\cap g^{-1}(U_j)} = h_j|_{g^{-1}(U_i)\cap g^{-1}(U_j)}$. Hence these morphisms glue to give the desired morphism $h_U : g^{-1}(U) \to X_U$. We omit the (easy) verification that the system $(g, \{h_U\})$ is an element of $F_W(T)$ which maps to α under the displayed arrow above.

Next, we verify each $F_W \subset F$ is representable by open immersions. This is clear from the definitions.

Finally we have to verify the collection $(F_W)_{W\in\mathcal{B}}$ covers F. This is clear by construction and the fact that $\mathcal{B}$ is a basis for the topology of S.

Let X be a scheme representing the functor F. Let $(f, \{i_U\}) \in F(X)$ be a ``universal family". Since each F_W is representable by X_W (via the morphism of functors displayed above) we see that $i_W : f^{-1}(W) \to X_W$ is an isomorphism as desired. The lemma is proved. □

Lemma 22.2.2. *Let S be a scheme. Let $\mathcal{B}$ be a basis for the topology of S. Suppose given the following data:*

1. *For every $U \in \mathcal{B}$ a scheme $f_U : X_U \to U$ over U.*
2. *For every $U \in \mathcal{B}$ a quasi-coherent sheaf $\mathcal{F}_U$ over X_U.*
3. *For every pair $U, V \in \mathcal{B}$ such that $V \subset U$ a morphism $\rho^U_V : X_V \to X_U$.*
4. *For every pair $U, V \in \mathcal{B}$ such that $V \subset U$ a morphism $\theta^U_V : (\rho^U_V)^*\mathcal{F}_U \to \mathcal{F}_V$.*

Assume that

(a) *each ρ^U_V induces an isomorphism $X_V \to f_U^{-1}(V)$ of schemes over V,*
(b) *each θ^U_V is an isomorphism,*
(c) *whenever $W, V, U \in \mathcal{B}$, with $W \subset V \subset U$ we have $\rho^U_W = \rho^U_V \circ \rho^V_W$,*
(d) *whenever $W, V, U \in \mathcal{B}$, with $W \subset V \subset U$ we have $\theta^U_W = \theta^V_W \circ (\rho^V_W)^*\theta^U_V$.*

Then there exists a unique scheme $f : X \to S$ over S together with a unique quasi-coherent sheaf $\mathcal{F}$ on X and isomorphisms $i_U : f^{-1}(U) \to X_U$ and $\theta_U : i_U^*\mathcal{F}_U \to \mathcal{F}|_{f^{-1}(U)}$ over U such that for $V \subset U \subset S$ affine open the composition

$$X_V \xrightarrow{i_V^{-1}} f^{-1}(V) \xrightarrow{inclusion} f^{-1}(U) \xrightarrow{i_U} X_U$$

is the morphism ρ^U_V, and the composition

$$(\rho^U_V)^*\mathcal{F}_U = (i_V^{-1})^*((i_U^*\mathcal{F}_U)|_{f^{-1}(V)}) \xrightarrow{\theta_U|_{f^{-1}(V)}} (i_V^{-1})^*(\mathcal{F}|_{f^{-1}(V)}) \xrightarrow{\theta_V^{-1}} \mathcal{F}_V \tag{22.2.2.1}$$

is equal to θ_V^U.

Proof. By Lemma 22.2.1 we get the scheme X over S and the isomorphisms i_U. Set $\mathcal{F}'_U = i_U^*\mathcal{F}_U$ for $U \in \mathcal{B}$. This is a quasi-coherent $\mathcal{O}_{f^{-1}(U)}$-module. The maps

$$\mathcal{F}'_U|_{f^{-1}(V)} = i_U^*\mathcal{F}_U|_{f^{-1}(V)} = i_V^*(\rho_V^U)^*\mathcal{F}_U \xrightarrow{i_V^*\theta_V^U} i_V^*\mathcal{F}_V = \mathcal{F}'_V$$

define isomorphisms $(\theta')_V^U : \mathcal{F}'_U|_{f^{-1}(V)} \to \mathcal{F}'_V$ whenever $V \subset U$ are elements of $\mathcal{B}$. Condition (d) says exactly that this is compatible in case we have a triple of elements $W \subset V \subset U$ of $\mathcal{B}$. This allows us to get well defined isomorphisms

$$\varphi_{12} : \mathcal{F}'_{U_1}|_{f^{-1}(U_1 \cap U_2)} \longrightarrow \mathcal{F}'_{U_2}|_{f^{-1}(U_1 \cap U_2)}$$

whenever $U_1, U_2 \in \mathcal{B}$ by covering the intersection $U_1 \cap U_2 = \bigcup V_j$ by elements V_j of $\mathcal{B}$ and taking

$$\varphi_{12}|_{V_j} = \left((\theta')_{V_j}^{U_2}\right)^{-1} \circ (\theta')_{V_j}^{U_1}.$$

We omit the verification that these maps do indeed glue to a φ_{12} and we omit the verification of the cocycle condition of a glueing datum for sheaves (as in Sheaves, Section 6.33). By Sheaves, Lemma 6.33.2 we get our $\mathcal{F}$ on X. We omit the verification of (22.2.2.1). □

Remark 22.2.3. There is a functoriality property for the constructions explained in Lemmas 22.2.1 and 22.2.2. Namely, suppose given two collections of data $(f_U : X_U \to U, \rho_V^U)$ and $(g_U : Y_U \to U, \sigma_V^U)$ as in Lemma 22.2.1. Suppose for every $U \in \mathcal{B}$ given a morphism $h_U : X_U \to Y_U$ over U compatible with the restrictions ρ_V^U and σ_V^U. Functoriality means that this gives rise to a morphism of schemes $h : X \to Y$ over S restricting back to the morphisms h_U, where $f : X \to S$ is obtained from the datum $(f_U : X_U \to U, \rho_V^U)$ and $g : Y \to S$ is obtained from the datum $(g_U : Y_U \to U, \sigma_V^U)$.

Similarly, suppose given two collections of data $(f_U : X_U \to U, \mathcal{F}_U, \rho_V^U, \theta_V^U)$ and $(g_U : Y_U \to U, \mathcal{G}_U, \sigma_V^U, \eta_V^U)$ as in Lemma 22.2.2. Suppose for every $U \in \mathcal{B}$ given a morphism $h_U : X_U \to Y_U$ over U compatible with the restrictions ρ_V^U and σ_V^U, and a morphism $\tau_U : h_U^*\mathcal{G}_U \to \mathcal{F}_U$ compatible with the maps θ_V^U and η_V^U. Functoriality means that these give rise to a morphism of schemes $h : X \to Y$ over S restricting back to the morphisms h_U, and a morphism $h^*\mathcal{G} \to \mathcal{F}$ restricting back to the maps h_U where $(f : X \to S, \mathcal{F})$ is obtained from the datum $(f_U : X_U \to U, \mathcal{F}_U, \rho_V^U, \theta_V^U)$ and where $(g : Y \to S, \mathcal{G})$ is obtained from the datum $(g_U : Y_U \to U, \mathcal{G}_U, \sigma_V^U, \eta_V^U)$.

We omit the verifications and we omit a suitable formulation of ``equivalence of categories'' between relative glueing data and relative objects.

22.3. Relative spectrum via glueing

Situation 22.3.1. Here S is a scheme, and $\mathcal{A}$ is a quasi-coherent $\mathcal{O}_S$-algebra.

In this section we outline how to construct a morphism of schemes

$$\underline{Spec}_S(\mathcal{A}) \longrightarrow S$$

by glueing the spectra $Spec(\Gamma(U, \mathcal{A}))$ where U ranges over the affine opens of S. We first show that the spectra of the values of $\mathcal{A}$ over affines form a suitable collection of schemes, as in Lemma 22.2.1.

Lemma 22.3.2. *In Situation 22.3.1. Suppose $U \subset U' \subset S$ are affine opens. Let $A = \mathcal{A}(U)$ and $A' = \mathcal{A}(U')$. The map of rings $A' \to A$ induces a morphism $Spec(A) \to Spec(A')$, and the diagram*

$$\begin{tikzcd} Spec(A) \arrow[r] \arrow[d] & Spec(A') \arrow[d] \\ U \arrow[r] & U' \end{tikzcd}$$

is cartesian.

Proof. Let $R = \mathcal{O}_S(U)$ and $R' = \mathcal{O}_S(U')$. Note that the map $R \otimes_{R'} A' \to A$ is an isomorphism as $\mathcal{A}$ is quasi-coherent (see Schemes, Lemma 21.7.3 for example). The result follows from the description of the fibre product of affine schemes in Schemes, Lemma 21.6.7. □

In particular the morphism $Spec(A) \to Spec(A')$ of the lemma is an open immersion.

Lemma 22.3.3. *In Situation 22.3.1. Suppose $U \subset U' \subset U'' \subset S$ are affine opens. Let $A = \mathcal{A}(U)$, $A' = \mathcal{A}(U')$ and $A'' = \mathcal{A}(U'')$. The composition of the morphisms $Spec(A) \to Spec(A')$, and $Spec(A') \to Spec(A'')$ of Lemma 22.3.2 gives the morphism $Spec(A) \to Spec(A'')$ of Lemma 22.3.2.*

Proof. This follows as the map $A'' \to A$ is the composition of $A'' \to A'$ and $A' \to A$ (because $\mathcal{A}$ is a sheaf). □

Lemma 22.3.4. *In Situation 22.3.1. There exists a morphism of schemes*

$$\pi : \underline{Spec}_S(\mathcal{A}) \longrightarrow S$$

with the following properties:

(1) *for every affine open $U \subset S$ there exists an isomorphism $i_U : \pi^{-1}(U) \to Spec(\mathcal{A}(U))$, and*
(2) *for $U \subset U' \subset S$ affine open the composition*

$$Spec(\mathcal{A}(U)) \xrightarrow{i_U^{-1}} \pi^{-1}(U) \xrightarrow{inclusion} \pi^{-1}(U') \xrightarrow{i_{U'}} Spec(\mathcal{A}(U'))$$

is the open immersion of Lemma 22.3.2 above.

Proof. Follows immediately from Lemmas 22.2.1, 22.3.2, and 22.3.3. □

22.4. Relative spectrum as a functor

We place ourselves in Situation 22.3.1. So S is a scheme and $\mathcal{A}$ is a quasi-coherent sheaf of $\mathcal{O}_S$-algebras. (This means that $\mathcal{A}$ is a sheaf of $\mathcal{O}_S$-algebras which is quasi-coherent as an $\mathcal{O}_S$-module.)

For any $f : T \to S$ the pullback $f^*\mathcal{A}$ is a quasi-coherent sheaf of $\mathcal{O}_T$-algebras. We are going to consider pairs $(f : T \to S, \varphi)$ where f is a morphism of schemes and $\varphi : f^*\mathcal{A} \to \mathcal{O}_T$ is a morphism of $\mathcal{O}_T$-algebras. Note that this is the same as giving a $f^{-1}\mathcal{O}_S$-algebra homomorphism $\varphi : f^{-1}\mathcal{A} \to \mathcal{O}_T$, see Sheaves, Lemma 6.20.2. This is also the same as giving a $\mathcal{O}_S$-algebra map $\varphi : \mathcal{A} \to f_*\mathcal{O}_T$, see Sheaves, Lemma 6.24.7. We will use all three ways of thinking about φ, without further mention.

Given such a pair $(f : T \to S, \varphi)$ and a morphism $a : T' \to T$ we get a second pair $(f' = f \circ a, \varphi' = a^*\varphi)$ which we call the pull back of (f, φ). One way to describe $\varphi' = a^*\varphi$ is as the composition $\mathcal{A} \to f_*\mathcal{O}_T \to f'_*\mathcal{O}_{T'}$ where the second map is $f_*a^\sharp$ with $a^\sharp : \mathcal{O}_T \to a_*\mathcal{O}_{T'}$. In this way we have defined a functor

$$\begin{array}{rcl} F : Sch^{opp} & \longrightarrow & Sets \\ T & \longmapsto & F(T) = \{\text{pairs } (f, \varphi) \text{ as above}\} \end{array} \tag{22.4.0.1}$$

Lemma 22.4.1. *In Situation 22.3.1. Let F be the functor associated to $(S, \mathcal{A})$ above. Let $g : S' \to S$ be a morphism of schemes. Set $\mathcal{A}' = g^*\mathcal{A}$. Let F' be the functor associated to $(S', \mathcal{A}')$ above. Then there is a canonical isomorphism*

$$F' \cong h_{S'} \times_{h_S} F$$

of functors.

Proof. A pair $(f' : T \to S', \varphi' : (f')^*\mathcal{A}' \to \mathcal{O}_T)$ is the same as a pair $(f, \varphi : f^*\mathcal{A} \to \mathcal{O}_T)$ together with a factorization of f as $f = g \circ f'$. Namely with this notation we have $(f')^*\mathcal{A}' = (f')^*g^*\mathcal{A} = f^*\mathcal{A}$. Hence the lemma. □

Lemma 22.4.2. *In Situation 22.3.1. Let F be the functor associated to $(S, \mathcal{A})$ above. If S is affine, then F is representable by the affine scheme $Spec(\Gamma(S, \mathcal{A}))$.*

Proof. Write $S = Spec(R)$ and $A = \Gamma(S, \mathcal{A})$. Then A is an R-algebra and $\mathcal{A} = \widetilde{A}$. The ring map $R \to A$ gives rise to a canonical map

$$f_{univ} : Spec(A) \longrightarrow S = Spec(R).$$

We have $f_{univ}^*\mathcal{A} = \widetilde{A \otimes_R A}$ by Schemes, Lemma 21.7.3. Hence there is a canonical map

$$\varphi_{univ} : f_{univ}^*\mathcal{A} = \widetilde{A \otimes_R A} \longrightarrow \widetilde{A} = \mathcal{O}_{Spec(A)}$$

coming from the A-module map $A \otimes_R A \to A$, $a \otimes a' \mapsto aa'$. We claim that the pair $(f_{univ}, \varphi_{univ})$ represents F in this case. In other words we claim that for any scheme T the map

$$Mor(T, Spec(A)) \longrightarrow \{\text{pairs } (f, \varphi)\}, \quad a \longmapsto (a^* f_{univ}, a^*\varphi)$$

is bijective.

Let us construct the inverse map. For any pair $(f : T \to S, \varphi)$ we get the induced ring map

$$A = \Gamma(S, \mathcal{A}) \xrightarrow{f^*} \Gamma(T, f^*\mathcal{A}) \xrightarrow{\varphi} \Gamma(T, \mathcal{O}_T)$$

This induces a morphism of schemes $T \to Spec(A)$ by Schemes, Lemma 21.6.4.

The verification that this map is inverse to the map displayed above is omitted. □

Lemma 22.4.3. *In Situation 22.3.1. The functor F is representable by a scheme.*

Proof. We are going to use Schemes, Lemma 21.15.4.

First we check that F satisfies the sheaf property for the Zariski topology. Namely, suppose that T is a scheme, that $T = \bigcup_{i \in I} U_i$ is an open covering, and that $(f_i, \varphi_i) \in F(U_i)$ such that $(f_i, \varphi_i)|_{U_i \cap U_j} = (f_j, \varphi_j)|_{U_i \cap U_j}$. This implies that the morphisms $f_i : U_i \to S$ glue to a morphism of schemes $f : T \to S$ such that $f|_{I_i} = f_i$, see Schemes, Section 21.14. Thus $f_i^*\mathcal{A} = f^*\mathcal{A}|_{U_i}$ and by assumption the morphisms φ_i agree on $U_i \cap U_j$. Hence by Sheaves, Section 6.33 these glue to a morphism of $\mathcal{O}_T$-algebras $f^*\mathcal{A} \to \mathcal{O}_T$. This proves that F satisfies the sheaf condition with respect to the Zariski topology.

Let $S = \bigcup_{i \in I} U_i$ be an affine open covering. Let $F_i \subset F$ be the subfunctor consisting of those pairs $(f : T \to S, \varphi)$ such that $f(T) \subset U_i$.

We have to show each F_i is representable. This is the case because F_i is identified with the functor associated to U_i equipped with the quasi-coherent $\mathcal{O}_{U_i}$-algebra $\mathcal{A}|_{U_i}$, by Lemma 22.4.1. Thus the result follows from Lemma 22.4.2.

Next we show that $F_i \subset F$ is representable by open immersions. Let $(f : T \to S, \varphi) \in F(T)$. Consider $V_i = f^{-1}(U_i)$. It follows from the definition of F_i that given $a : T' \to T$ we gave $a^*(f, \varphi) \in F_i(T')$ if and only if $a(T') \subset V_i$. This is what we were required to show.

Finally, we have to show that the collection $(F_i)_{i \in I}$ covers F. Let $(f : T \to S, \varphi) \in F(T)$. Consider $V_i = f^{-1}(U_i)$. Since $S = \bigcup_{i \in I} U_i$ is an open covering of S we see that $T = \bigcup_{i \in I} V_i$ is an open covering of T. Moreover $(f, \varphi)|_{V_i} \in F_i(V_i)$. This finishes the proof of the lemma. $\square$

Lemma 22.4.4. *In Situation 22.3.1. The scheme $\pi : \underline{Spec}_S(\mathcal{A}) \to S$ constructed in Lemma 22.3.4 and the scheme representing the functor F are canonically isomorphic as schemes over S.*

Proof. Let $X \to S$ be the scheme representing the functor F. Consider the sheaf of $\mathcal{O}_S$-algebras $\mathcal{R} = \pi_* \mathcal{O}_{\underline{Spec}_S(\mathcal{A})}$. By construction of $\underline{Spec}_S(\mathcal{A})$ we have isomorphisms $\mathcal{A}(U) \to \mathcal{R}(U)$ for every affine open $U \subset S$; this follows from Lemma 22.3.4 part (1). For $U \subset U' \subset S$ open these isomorphisms are compatible with the restriction mappings; this follows from Lemma 22.3.4 part (2). Hence by Sheaves, Lemma 6.30.13 these isomorphisms result from an isomorphism of $\mathcal{O}_S$-algebras $\varphi : \mathcal{A} \to \mathcal{R}$. Hence this gives an element $(\underline{Spec}_S(\mathcal{A}), \varphi) \in F(\underline{Spec}_S(\mathcal{A}))$. Since X represents the functor F we get a corresponding morphism of schemes $can : \underline{Spec}_S(\mathcal{A}) \to X$ over S.

Let $U \subset S$ be any affine open. Let $F_U \subset F$ be the subfunctor of F corresponding to pairs (f, φ) over schemes T with $f(T) \subset U$. Clearly the base change X_U represents F_U. Moreover, F_U is represented by $Spec(\mathcal{A}(U)) = \pi^{-1}(U)$ according to Lemma 22.4.2. In other words $X_U \cong \pi^{-1}(U)$. We omit the verification that this identification is brought about by the base change of the morphism can to U. $\square$

Definition 22.4.5. Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_S$-algebras. The *relative spectrum of $\mathcal{A}$ over S*, or simply the *spectrum of $\mathcal{A}$ over S* is the scheme constructed in Lemma 22.3.4 which represents the functor F (22.4.0.1), see Lemma 22.4.4. We denote it $\pi : \underline{Spec}_S(\mathcal{A}) \to S$. The ``universal family'' is a morphism of $\mathcal{O}_S$-algebras

$$\mathcal{A} \longrightarrow \pi_* \mathcal{O}_{\underline{Spec}_S(\mathcal{A})}$$

The following lemma says among other things that forming the relative spectrum commutes with base change.

Lemma 22.4.6. *Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_S$-algebras. Let $\pi : \underline{Spec}_S(\mathcal{A}) \to S$ be the relative spectrum of $\mathcal{A}$ over S.*

(1) *For every affine open $U \subset S$ the inverse image $f^{-1}(U)$ is affine.*
(2) *For every morphism $g : S' \to S$ we have $S' \times_S \underline{Spec}_S(\mathcal{A}) = \underline{Spec}_{S'}(g^*\mathcal{A})$.*
(3) *The universal map*

$$\mathcal{A} \longrightarrow \pi_* \mathcal{O}_{\underline{Spec}_S(\mathcal{A})}$$

is an isomorphism of $\mathcal{O}_S$-algebras.

Proof. Part (1) comes from the description of the relative spectrum by glueing, see Lemma 22.3.4. Part (2) follows immediately from Lemma 22.4.1. Part (3) follows because it is local on S and it is clear in case S is affine by Lemma 22.4.2 for example. $\square$

Lemma 22.4.7. *Let $f : X \to S$ be a quasi-compact and quasi-separated morphism of schemes. By Schemes, Lemma 21.24.1 the sheaf $f_*\mathcal{O}_X$ is a quasi-coherent sheaf of $\mathcal{O}_S$-algebras. There is a canonical morphism*

$$can : X \longrightarrow \underline{Spec}_S(f_*\mathcal{O}_X)$$

of schemes over S. For any affine open $U \subset S$ the restriction $can|_{f^{-1}(U)}$ is identified with the canonical morphism

$$f^{-1}(U) \longrightarrow Spec(\Gamma(f^{-1}(U), \mathcal{O}_X))$$

coming from Schemes, Lemma 21.6.4.

Proof. The morphism comes, via the definition of $\underline{Spec}$ as the scheme representing the functor F, from the canonical map $\varphi : f^*f_*\mathcal{O}_X \to \mathcal{O}_X$ (which by adjointness of push and pull corresponds to $\mathrm{id} : f_*\mathcal{O}_X \to f_*\mathcal{O}_X$). The statement on the restriction to $f^{-1}(U)$ follows from the description of the relative spectrum over affines, see Lemma 22.4.2. $\square$

22.5. Affine n-space

As an application of the relative spectrum we define affine n-space over a base scheme S as follows. For any integer $n \geq 0$ we can consider the quasi-coherent sheaf of $\mathcal{O}_S$-algebras $\mathcal{O}_S[T_1, \ldots, T_n]$. It is quasi-coherent because as a sheaf of $\mathcal{O}_S$-modules it is just the direct sum of copies of $\mathcal{O}_S$ indexed by multi-indices.

Definition 22.5.1. Let S be a scheme and $n \geq 0$. The scheme

$$\mathbf{A}^n_S = \underline{Spec}_S(\mathcal{O}_S[T_1, \ldots, T_n])$$

over S is called *affine n-space over S*. If $S = Spec(R)$ is affine then we also call this *affine n-space over R* and we denote it $\mathbf{A}^n_R$.

Note that $\mathbf{A}^n_R = Spec(R[T_1, \ldots, T_n])$. For any morphism $g : S' \to S$ of schemes we have $g^*\mathcal{O}_S[T_1, \ldots, T_n] = \mathcal{O}_{S'}[T_1, \ldots, T_n]$ and hence $\mathbf{A}^n_{S'} = S' \times_S \mathbf{A}^n_S$ is the base change. Therefore an alternative definition of affine n-space is the formula

$$\mathbf{A}^n_S = S \times_{Spec(\mathbf{Z})} \mathbf{A}^n_{\mathbf{Z}}.$$

Also, a morphism from an S-scheme $f : X \to S$ to $\mathbf{A}^n_S$ is given by a homomorphism of $\mathcal{O}_S$-algebras $\mathcal{O}_S[T_1, \ldots, T_n] \to f_*\mathcal{O}_X$. This is clearly the same thing as giving the images of the T_i. In other words, a morphism from X to $\mathbf{A}^n_S$ over S is the same as giving n elements $h_1, \ldots, h_n \in \Gamma(X, \mathcal{O}_X)$.

22.6. Vector bundles

Let S be a scheme. Let $\mathcal{E}$ be a quasi-coherent sheaf of $\mathcal{O}_S$-modules. By Modules, Lemma 15.18.6 the symmetric algebra $\mathrm{Sym}(\mathcal{E})$ of $\mathcal{E}$ over $\mathcal{O}_S$ is a quasi-coherent sheaf of $\mathcal{O}_S$-algebras. Hence it makes sense to apply the construction of the previous section to it.

Definition 22.6.1. Let S be a scheme. Let $\mathcal{E}$ be a quasi-coherent $\mathcal{O}_S$-module[1]. The *vector bundle associated to* $\mathcal{E}$ is

$$\mathbf{V}(\mathcal{E}) = \underline{Spec}_S(\mathrm{Sym}(\mathcal{E})).$$

The vector bundle associated to $\mathcal{E}$ comes with a bit of extra structure. Namely, we have a grading

$$\pi_*\mathcal{O}_{\mathbf{V}(\mathcal{E})} = \bigoplus\nolimits_{n\geq 0} \mathrm{Sym}^n(\mathcal{E}).$$

which turns $\pi_*\mathcal{O}_{\mathbf{V}(\mathcal{E})}$ into a graded $\mathcal{O}_S$-algebra. Conversely, we can recover $\mathcal{E}$ from the degree 1 part of this. Thus we define an abstract vector bundle as follows.

Definition 22.6.2. Let S be a scheme. A *vector bundle* $\pi : V \to S$ *over* S is an affine morphism of schemes such that $\pi_*\mathcal{O}_V$ is endowed with the structure of a graded $\mathcal{O}_S$-algebra $\pi_*\mathcal{O}_V = \bigoplus_{n\geq 0} \mathcal{E}_n$ such that $\mathcal{E}_0 = \mathcal{O}_S$ and such that the maps

$$\mathrm{Sym}^n(\mathcal{E}_1) \longrightarrow \mathcal{E}_n$$

are isomorphisms for all $n \geq 0$. A *morphism of vector bundles over* S is a morphism $f : V \to V'$ such that the induced map

$$f^* : \pi'_*\mathcal{O}_{V'} \longrightarrow \pi_*\mathcal{O}_V$$

is compatible with the given gradings.

An example of a vector bundle over S is affine n-space $\mathbf{A}^n_S$ over S, see Definition 22.5.1. This is true because $\mathcal{O}_S[T_1, \ldots, T_n] = \mathrm{Sym}(\mathcal{O}_S^{\oplus n})$.

Lemma 22.6.3. *The category of vector bundles over a scheme S is anti-equivalent to the category of quasi-coherent $\mathcal{O}_S$-modules.*

Proof. Omitted. Hint: In one direction one uses the functor $\underline{Spec}_S(-)$ and in the other the functor $(\pi : V \to S) \leadsto (\pi_*\mathcal{O}_V)_1$ (degree 1 part). $\square$

22.7. Cones

In algebraic geometry cones correspond to graded algebras. By our conventions a graded ring or algebra A comes with a grading $A = \bigoplus_{d\geq 0} A_d$ by the nonnegative integers, see Algebra, Section 7.52.

Definition 22.7.1. Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Assume that $\mathcal{O}_S \to \mathcal{A}_0$ is an isomorphism[2]. The *cone associated to* $\mathcal{A}$ or the *affine cone associated to* $\mathcal{A}$ is

$$C(\mathcal{A}) = \underline{Spec}_S(\mathcal{A}).$$

The cone associated to a graded sheaf of $\mathcal{O}_S$-algebras comes with a bit of extra structure. Namely, we obtain a grading

$$\pi_*\mathcal{O}_{C(\mathcal{A})} = \bigoplus\nolimits_{n\geq 0} \mathcal{A}_n$$

Thus we can define an abstract cone as follows.

[1]The reader may expect here the condition that $\mathcal{E}$ is finite locally free. We do not do so in order to be consistent with [**DG67**, II, Definition 1.7.8].

[2]Often one imposes the assumption that $\mathcal{A}$ is generated by $\mathcal{A}_1$ over $\mathcal{O}_S$. We do not assume this in order to be consisten with [**DG67**, II, (8.3.1)].

Definition 22.7.2. Let S be a scheme. A *cone* $\pi : C \to S$ *over* S is an affine morphism of schemes such that $\pi_*\mathcal{O}_C$ is endowed with the structure of a graded $\mathcal{O}_S$-algebra $\pi_*\mathcal{O}_C = \bigoplus_{n\geq 0} \mathcal{A}_n$ such that $\mathcal{A}_0 = \mathcal{O}_S$. A *morphism of cones* from $\pi : C \to S$ to $\pi' : C' \to S$ is a morphism $f : C \to C'$ such that the induced map

$$f^* : \pi'_*\mathcal{O}_{C'} \longrightarrow \pi_*\mathcal{O}_C$$

is compatible with the given gradings.

Any vector bundle is an example of a cone. In fact the category of vector bundles over S is a full subcategory of the category of cones over S.

22.8. Proj of a graded ring

Let S be a graded ring. Consider the topological space $\text{Proj}(S)$ associated to S, see Algebra, Section 7.53. We will endow this space with a sheaf of rings $\mathcal{O}_{\text{Proj}(S)}$ such that the resulting pair $(\text{Proj}(R), \mathcal{O}_{\text{Proj}(R)})$ will be a scheme.

Recall that $\text{Proj}(S)$ has a basis of open sets $D_+(f)$, $f \in S_d$, $d \geq 1$ which we call *standard opens*, see Algebra, Section 7.53. This terminology will always imply that f is homogeneous of positive degree even if we forget to mention it. In addition, the intersection of two standard opens is another: $D_+(f) \cap D_+(g) = D_+(fg)$, for $f, g \in S$ homogeneous of positive degree.

Lemma 22.8.1. *Let S be a graded ring. Let $f \in S$ homogeneous of positive degree.*

(1) *If $g \in S$ homogeneous of positive degree and $D_+(g) \subset D_+(f)$, then*
  (a) *f is invertible in S_g, and $f^{\deg(g)}/g^{\deg(f)}$ is invertible in $S_{(g)}$,*
  (b) *$g^e = af$ for some $e \geq 1$ and $a \in S$ homogeneous,*
  (c) *there is a canonical S-algebra map $S_f \to S_g$,*
  (d) *there is a canonical S_0-algebra map $S_{(f)} \to S_{(g)}$ compatible with the map $S_f \to S_g$,*
  (e) *the map $S_{(f)} \to S_{(g)}$ induces an isomorphism*
$$(S_{(f)})_{g^{\deg(f)}/f^{\deg(g)}} \cong S_{(g)},$$
  (f) *these maps induce a commutative diagram of topological spaces*
$$\begin{tikzcd} D_+(g) \arrow[d] & \{\mathbf{Z}\text{-graded primes of } S_g\} \arrow[l] \arrow[r] \arrow[d] & Spec(S_{(g)}) \arrow[d] \\ D_+(f) & \{\mathbf{Z}\text{-graded primes of } S_f\} \arrow[l] \arrow[r] & Spec(S_{(f)}) \end{tikzcd}$$
  where the horizontal maps are homeomorphisms and the vertical maps are open immersions,
  (g) *there are a compatible canonical S_f and $S_{(f)}$-module maps $M_f \to M_g$ and $M_{(f)} \to M_{(g)}$ for any graded S-module M, and*
  (h) *the map $M_{(f)} \to M_{(g)}$ induces an isomorphism*
$$(M_{(f)})_{g^{\deg(f)}/f^{\deg(g)}} \cong M_{(g)}.$$

(2) *Any open covering of $D_+(f)$ can be refined to a finite open covering of the form $D_+(f) = \bigcup_{i=1}^n D_+(g_i)$.*

(3) *Let $g_1, \ldots, g_n \in S$ be homogeneous of positive degree. Then $D_+(f) \subset \bigcup D_+(g_i)$ if and only if $g_1^{\deg(f)}/f^{\deg(g_1)}, \ldots, g_n^{\deg(f)}/f^{\deg(g_n)}$ generate the unit ideal in $S_{(f)}$.*

Proof. Recall that $D_+(g) = Spec(S_{(g)})$ with identification given by the ring maps $S \to S_g \leftarrow S_{(g)}$, see Algebra, Lemma 7.53.3. Thus $f^{\deg(g)}/g^{\deg(f)}$ is an element of $S_{(g)}$ which is not contained in any prime ideal, and hence invertible, see Algebra, Lemma 7.16.2. We conclude that (a) holds. Write the inverse of f in S_g as a/g^d. We may replace a by its homogeneous part of degree $d\deg(g) - \deg(f)$. This means $g^d - af$ is annihilated by a power of g, whence $g^e = af$ for some $a \in S$ homogeneous of degree $e\deg(g) - \deg(f)$. This proves (b). For (c), the map $S_f \to S_g$ exists by (a) from the universal property of localization, or we can define it by mapping b/f^n to $a^n b/g^{ne}$. This clearly induces a map of the subrings $S_{(f)} \to S_{(g)}$ of degree zero elements as well. We can similarly define $M_f \to M_g$ and $M_{(f)} \to M_{(g)}$ by mapping x/f^n to $a^n x/g^{ne}$. The statements writing $S_{(g)}$ resp. $M_{(g)}$ as principal localizations of $S_{(f)}$ resp. $M_{(f)}$ are clear from the formulas above. The maps in the commutative diagram of topological spaces correspond to the ring maps given above. The horizontal arrows are homeomorphisms by Algebra, Lemma 7.53.3. The vertical arrows are open immersions since the left one is the inclusion of an open subset.

The open $D_+(f)$ is quasi-compact because it is homeomorphic to $Spec(S_{(f)})$, see Algebra, Lemma 7.26.1. Hence the second statement follows directly from the fact that the standard opens form a basis for the topology.

The third statement follows directly from Algebra, Lemma 7.16.2. □

In Sheaves, Section 6.30 we defined the notion of a sheaf on a basis, and we showed that it is essentially equivalent to the notion of a sheaf on the space, see Sheaves, Lemmas 6.30.6 and 6.30.9. Moreover, we showed in Sheaves, Lemma 6.30.4 that it is sufficient to check the sheaf condition on a cofinal system of open coverings for each standard open. By the lemma above it suffices to check on the finite coverings by standard opens.

Definition 22.8.2. Let S be a graded ring. Suppose that $D_+(f) \subset \text{Proj}(S)$ is a standard open. A *standard open covering* of $D_+(f)$ is a covering $D_+(f) = \bigcup_{i=1}^n D_+(g_i)$, where $g_1, \ldots, g_n \in S$ are homogeneous of positive degree.

Let S be a graded ring. Let M be a graded S-module. We will define a presheaf $\widetilde{M}$ on the basis of standard opens. Suppose that $U \subset \text{Proj}(S)$ is a standard open. If $f, g \in S$ are homogeneous of positive degree such that $D_+(f) = D_+(g)$, then by Lemma 22.8.1 above there are canonical maps $M_{(f)} \to M_{(g)}$ and $M_{(g)} \to M_{(f)}$ which are mutually inverse. Hence we may choose any f such that $U = D_+(f)$ and define

$$\widetilde{M}(U) = M_{(f)}.$$

Note that if $D_+(g) \subset D_+(f)$, then by Lemma 22.8.1 above we have a canonical map

$$\widetilde{M}(D_+(f)) = M_{(f)} \longrightarrow M_{(g)} = \widetilde{M}(D_+(g)).$$

Clearly, this defines a presheaf of abelian groups on the basis of standard opens. If $M = S$, then $\widetilde{S}$ is a presheaf of rings on the basis of standard opens. And for general M we see that $\widetilde{M}$ is a presheaf of $\widetilde{S}$-modules on the basis of standard opens.

Let us compute the stalk of $\widetilde{M}$ at a point $x \in \text{Proj}(S)$. Suppose that x corresponds to the homogeneous prime ideal $\mathfrak{p} \subset S$. By definition of the stalk we see that

$$\widetilde{M}_x = colim_{f \in S_d, d > 0, f \notin \mathfrak{p}} M_{(f)}$$

Here the set $\{f \in S_d, d > 0, f \notin \mathfrak{p}\}$ is partially ordered by the rule $f \geq f' \Leftrightarrow D_+(f) \subset D_+(f')$. If $f_1, f_2 \in S \setminus \mathfrak{p}$ are homogeneous of positive degree, then we have $f_1 f_2 \geq f_1$ in this ordering. In Algebra, Section 7.53 we defined $M_{(\mathfrak{p})}$ as the ring whose elements are fractions x/f with x, f homogeneous, $\deg(x) = \deg(f)$, $f \not\in \mathfrak{p}$. Since $\mathfrak{p} \in \text{Proj}(S)$ there exists at least one $f_0 \in S$ homogeneous of positive degree with $f_0 \not\in \mathfrak{p}$. Hence $x/f = f_0 x / f f_0$ and we see that we may always assume the denominator of an element in $M_{(\mathfrak{p})}$ has positive degree. From these remarks it follows easily that

$$\widetilde{M}_x = M_{(\mathfrak{p})}.$$

Next, we check the sheaf condition for the standard open coverings. If $D_+(f) = \bigcup_{i=1}^n D_+(g_i)$, then the sheaf condition for this covering is equivalent with the exactness of the sequence

$$0 \to M_{(f)} \to \bigoplus M_{(g_i)} \to \bigoplus M_{(g_i g_j)}.$$

Note that $D_+(g_i) = D_+(f g_i)$, and hence we can rewrite this sequence as the sequence

$$0 \to M_{(f)} \to \bigoplus M_{(f g_i)} \to \bigoplus M_{(f g_i g_j)}.$$

By Lemma 22.8.1 we see that $g_1^{\deg(f)}/f^{\deg(g_1)}, \ldots, g_n^{\deg(f)}/f^{\deg(g_n)}$ generate the unit ideal in $S_{(f)}$, and that the modules $M_{(f g_i)}$, $M_{(f g_i g_j)}$ are the principal localizations of the $S_{(f)}$-module $M_{(f)}$ at these elements and their products. Thus we may apply Algebra, Lemma 7.20.2 to the module $M_{(f)}$ over $S_{(f)}$ and the elements $g_1^{\deg(f)}/f^{\deg(g_1)}, \ldots, g_n^{\deg(f)}/f^{\deg(g_n)}$. We conclude that the sequence is exact. By the remarks made above, we see that $\widetilde{M}$ is a sheaf on the basis of standard opens.

Thus we conclude from the material in Sheaves, Section 6.30 that there exists a unique sheaf of rings $\mathcal{O}_{\text{Proj}(S)}$ which agrees with $\widetilde{S}$ on the standard opens. Note that by our computation of stalks above and Algebra, Lemma 7.53.5 the stalks of this sheaf of rings are all local rings.

Similarly, for any graded S-module M there exists a unique sheaf of $\mathcal{O}_{\text{Proj}(S)}$-modules $\mathcal{F}$ which agrees with $\widetilde{M}$ on the standard opens, see Sheaves, Lemma 6.30.12.

Definition 22.8.3. Let S be a graded ring.

(1) The *structure sheaf $\mathcal{O}_{Proj(S)}$ of the homogeneous spectrum of S* is the unique sheaf of rings $\mathcal{O}_{\text{Proj}(S)}$ which agrees with $\widetilde{S}$ on the basis of standard opens.
(2) The locally ringed space $(\text{Proj}(S), \mathcal{O}_{\text{Proj}(S)})$ is called the *homogeneous spectrum* of S and denoted $\text{Proj}(S)$.
(3) The sheaf of $\mathcal{O}_{\text{Proj}(S)}$-modules extending $\widetilde{M}$ to all opens of $\text{Proj}(S)$ is called the sheaf of $\mathcal{O}_{\text{Proj}(S)}$-modules associated to M. This sheaf is denoted $\widetilde{M}$ as well.

We summarize the results obtained so far.

Lemma 22.8.4. *Let S be a graded ring. Let M be a graded S-module. Let $\widetilde{M}$ be the sheaf of $\mathcal{O}_{Proj(S)}$-modules associated to M.*

(1) *For every $f \in S$ homogeneous of positive degree we have*
$$\Gamma(D_+(f), \mathcal{O}_{Proj(S)}) = S_{(f)}.$$
(2) *For every $f \in S$ homogeneous of positive degree we have $\Gamma(D_+(f), \widetilde{M}) = M_{(f)}$ as an $S_{(f)}$-module.*
(3) *Whenever $D_+(g) \subset D_+(f)$ the restriction mappings on $\mathcal{O}_{Proj(S)}$ and $\widetilde{M}$ are the maps $S_{(f)} \to S_{(g)}$ and $M_{(f)} \to M_{(g)}$ from Lemma 22.8.1.*
(4) *Let $\mathfrak{p}$ be a homogeneous prime of S not containing S_+, and let $x \in Proj(S)$ be the corresponding point. We have $\mathcal{O}_{Proj(S),x} = S_{(\mathfrak{p})}$.*

(5) *Let $\mathfrak{p}$ be a homogeneous prime of S not containing S_+, and let $x \in Proj(S)$ be the corresponding point. We have $\mathcal{F}_x = M_{(\mathfrak{p})}$ as an $S_{(\mathfrak{p})}$-module.*

(6) *There is a canonical ring map $S_0 \longrightarrow \Gamma(Proj(S), \widetilde{S})$ and a canonical S_0-module map $M_0 \longrightarrow \Gamma(Proj(S), \widetilde{M})$ compatible with the descriptions of sections over standard opens above and stalks above.*

Moreover, all these identifications are functorial in the graded S-module M. In particular, the functor $M \mapsto \widetilde{M}$ is an exact functor from the category of graded S-modules to the category of $\mathcal{O}_{Proj(S)}$-modules.

Proof. Assertions (1) - (5) are clear from the discussion above. We see (6) since there are canonical maps $M_0 \to M_{(f)}$, $x \mapsto x/1$ compatible with the restriction maps described in (3). The exactness of the functor $M \mapsto \widetilde{M}$ follows from the fact that the functor $M \mapsto M_{(\mathfrak{p})}$ is exact (see Algebra, Lemma 7.53.5) and the fact that exactness of short exact sequences may be checked on stalks, see Modules, Lemma 15.3.1. □

Remark 22.8.5. The map from M_0 to the global sections of $\widetilde{M}$ is generally far from being an isomorphism. A trivial example is to take $S = k[x, y, z]$ with $1 = \deg(x) = \deg(y) = \deg(z)$ (or any number of variables) and to take $M = S/(x^{100}, y^{100}, z^{100})$. It is easy to see that $\widetilde{M} = 0$, but $M_0 = k$.

Lemma 22.8.6. *Let S be a graded ring. Let $f \in S$ be homogeneous of positive degree. Suppose that $D(g) \subset Spec(S_{(f)})$ is a standard open. Then there exists a $h \in S$ homogeneous of positive degree such that $D(g)$ corresponds to $D_+(h) \subset D_+(f)$ via the homeomorphism of Algebra, Lemma 7.53.3. In fact we can take h such that $g = h/f^n$ for some n.*

Proof. Write $g = h/f^n$ for some h homogeneous of positive degree and some $n \geq 1$. If $D_+(h)$ is not contained in $D_+(f)$ then we replace h by hf and n by $n + 1$. Then h has the required shape and $D_+(h) \subset D_+(f)$ corresponds to $D(g) \subset Spec(S_{(f)})$. □

Lemma 22.8.7. *Let S be a graded ring. The locally ringed space $Proj(S)$ is a scheme. The standard opens $D_+(f)$ are affine opens. For any graded S-module M the sheaf $\widetilde{M}$ is a quasi-coherent sheaf of $\mathcal{O}_{Proj(S)}$-modules.*

Proof. Consider a standard open $D_+(f) \subset \text{Proj}(S)$. By Lemmas 22.8.1 and 22.8.4 we have $\Gamma(D_+(f), \mathcal{O}_{\text{Proj}(S)}) = S_{(f)}$, and we have a homeomorphism $\varphi : D_+(f) \to Spec(S_{(f)})$. For any standard open $D(g) \subset Spec(S_{(f)})$ we may pick a $h \in S_+$ as in Lemma 22.8.6. Then $\varphi^{-1}(D(g)) = D_+(h)$, and by Lemmas 22.8.4 and 22.8.1 we see

$$\Gamma(D_+(h), \mathcal{O}_{\text{Proj}(S)}) = S_{(h)} = (S_{(f)})_{h^{\deg(f)}/f^{\deg(h)}} = (S_{(f)})_g = \Gamma(D(g), \mathcal{O}_{Spec(S_{(f)})}).$$

Thus the restriction of $\mathcal{O}_{\text{Proj}(S)}$ to $D_+(f)$ corresponds via the homeomorphism φ exactly to the sheaf $\mathcal{O}_{Spec(S_{(f)})}$ as defined in Schemes, Section 21.5. We conclude that $D_+(f)$ is an affine scheme isomorphic to $Spec(S_{(f)})$ via φ and hence that $\text{Proj}(S)$ is a scheme.

In exactly the same way we show that $\widetilde{M}$ is a quasi-coherent sheaf of $\mathcal{O}_{\text{Proj}(S)}$-modules. Namely, the argument above will show that

$$\widetilde{M}|_{D_+(f)} \cong \varphi^*\left(\widetilde{M_{(f)}}\right)$$

which shows that $\widetilde{M}$ is quasi-coherent. □

Lemma 22.8.8. *Let S be a graded ring. The scheme $Proj(S)$ is separated.*

Proof. We have to show that the canonical morphism $\mathrm{Proj}(S) \to \mathit{Spec}(\mathbf{Z})$ is separated. We will use Schemes, Lemma 21.21.8. Thus it suffices to show given any pair of standard opens $D_+(f)$ and $D_+(g)$ that $D_+(f) \cap D_+(g) = D_+(fg)$ is affine (clear) and that the ring map

$$S_{(f)} \otimes_{\mathbf{Z}} S_{(g)} \longrightarrow S_{(fg)}$$

is surjective. Any element s in $S_{(fg)}$ is of the form $s = h/(f^n g^m)$ with $h \in S$ homogeneous of degree $n \deg(f) + m \deg(g)$. We may multiply h by a suitable monomial $f^i g^j$ and assume that $n = n' \deg(g)$, and $m = m' \deg(f)$. Then we can rewrite s as $s = h/f^{(n'+m')\deg(g)} \cdot f^{m' \deg(g)}/g^{m' \deg(f)}$. So s is indeed in the image of the displayed arrow. □

Lemma 22.8.9. *Let S be a graded ring. The scheme $\mathrm{Proj}(S)$ is quasi-compact if and only if there exist finitely many homogeneous elements $f_1, \ldots, f_n \in S_+$ such that $S_+ \subset \sqrt{(f_1, \ldots, f_n)}$.*

Proof. Given such a collection of elements the standard affine opens $D_+(f_i)$ cover $\mathrm{Proj}(S)$ by Algebra, Lemma 7.53.3. Conversely, if $\mathrm{Proj}(S)$ is quasi-compact, then we may cover it by finitely many standard opens $D_+(f_i)$, $i = 1, \ldots, n$ and we see that $S_+ \subset \sqrt{(f_1, \ldots, f_n)}$ by the lemma referenced above. □

Lemma 22.8.10. *Let S be a graded ring. The scheme $\mathrm{Proj}(S)$ has a canonical morphism towards the affine scheme $\mathit{Spec}(S_0)$, agreeing with the map on topological spaces coming from Algebra, Definition 7.53.1.*

Proof. We saw above that our construction of $\widetilde{S}$, resp. $\widetilde{M}$ gives a sheaf of S_0-algebras, resp. S_0-modules. Hence we get a morphism by Schemes, Lemma 21.6.4. This morphism, when restricted to $D_+(f)$ comes from the canonical ring map $S_0 \to S_{(f)}$. The maps $S \to S_f$, $S_{(f)} \to S_f$ are S_0-algebra maps, see Lemma 22.8.1. Hence if the homogeneous prime $\mathfrak{p} \subset S$ corresponds to the $\mathbf{Z}$-graded prime $\mathfrak{p}' \subset S_f$ and the (usual) prime $\mathfrak{p}'' \subset S_{(f)}$, then each of these has the same inverse image in S_0. □

Lemma 22.8.11. *Let S be a graded ring. If S is finitely generated as an algebra over S_0, then the morphism $\mathrm{Proj}(S) \to \mathit{Spec}(S_0)$ satisfies the existence and uniqueness parts of the valuative criterion, see Schemes, Definition 21.20.3.*

Proof. The uniqueness part follows from the fact that $\mathrm{Proj}(S)$ is separated (Lemma 22.8.8 and Schemes, Lemma 21.22.1). Choose $x_i \in S_+$ homogeneous, $i = 1, \ldots, n$ which generate S over S_0. Let $d_i = \deg(x_i)$ and set $d = \mathrm{lcm}\{d_i\}$. Suppose we are given a diagram

$$\begin{tikzcd} \mathit{Spec}(K) \arrow[r] \arrow[d] & \mathrm{Proj}(S) \arrow[d] \\ \mathit{Spec}(A) \arrow[r] & \mathit{Spec}(S_0) \end{tikzcd}$$

as in Schemes, Definition 21.20.3. Denote $v : K^* \to \Gamma$ the valuation of A, see Algebra, Definition 7.46.8. We may choose an $f \in S_+$ homogeneous such that $\mathit{Spec}(K)$ maps into $D_+(f)$. Then we get a commutative diagram of ring maps

$$\begin{tikzcd} K & S_{(f)} \arrow[l, "\varphi"'] \\ A \arrow[u] & S_0 \arrow[l] \arrow[u] \end{tikzcd}$$

Let $i_0 \in \{1, \ldots, n\}$ be an index minimizing the valuation $(d/d_i)v(\varphi(x_i^{\deg(f)}/f^{d_i}))$ where we temporarily use the convention that the valuation of zero is bigger than any element of the value group. For convenience set $x_0 = x_{i_0}$ and $d_0 = d_{i_0}$. Since the open sets $D_+(x_i)$ cover $\text{Proj}(S)$ we see that $\varphi(x_0) \neq 0$. This means that the ring map φ factors though a map $\varphi' : S_{(fx_0)} \to K$. We see that

$$\deg(f)v(\varphi'(x_i^{d_0}/x_0^{d_i})) = d_0 v(\varphi(x_i^{\deg(f)}/f^{d_i})) - d_i v(\varphi(x_0^{\deg(f)}/f^{d_0})) \geq 0$$

by our choice of i_0. This implies that the S_0-algebra $S_{(x_0)}$, which is generated by the elements $x_i^{d_0}/x_0^{d_i}$ over S_0, maps into A via φ'. The corresponding morphism of schemes $Spec(A) \to Spec(S_{(x_0)}) = D_+(x_0) \subset \text{Proj}(S)$ provides the morphism fitting into the first commutative diagram of this proof. $\square$

We saw in the proof of Lemma 22.8.11 that, under the hypotheses of that lemma, the morphism $\text{Proj}(S) \to Spec(S_0)$ is quasi-compact as well. Hence (by Schemes, Proposition 21.20.6) we see that $\text{Proj}(S) \to Spec(S_0)$ is universally closed in the situation of the lemma. We give two examples showing these results do not hold without some assumption on the graded ring S.

Example 22.8.12. Let $\mathbf{C}[X_1, X_2, X_3, \ldots]$ be the graded $\mathbf{C}$-algebra with each X_i in degree 0. Consider the ring map

$$\mathbf{C}[X_1, X_2, X_3, \ldots] \longrightarrow \mathbf{C}[t^\alpha; \alpha \in \mathbf{Q}_{\geq 0}]$$

which maps X_i to $t^{1/i}$. The right hand side becomes a valuation ring A upon localization at the ideal $\mathfrak{m} = (t^\alpha; \alpha > 0)$. This gives a morphism from $Spec(f.f.(A))$ to $\text{Proj}(\mathbf{C}[X_1, X_2, X_3, \ldots])$ which does not extend to a morphism defined on all of $Spec(A)$. The reason is that the image of $Spec(A)$ would be contained in one of the $D_+(X_i)$ but then X_{i+1}/X_i would map to an element of A which it doesn't since it maps to $t^{1/(i+1)-1/i}$.

Example 22.8.13. Let $R = \mathbf{C}[t]$ and

$$S = R[X_1, X_2, X_3, \ldots]/(X_i^2 - tX_{i+1}).$$

The grading is such that $R = S_0$ and $\deg(X_i) = 2^{i-1}$. Note that if $\mathfrak{p} \in \text{Proj}(S)$ then $t \notin \mathfrak{p}$ (otherwise $\mathfrak{p}$ has to contain all of the X_i which is not allowed for an element of the homogeneous spectrum). Thus we see that $D_+(X_i) = D_+(X_{i+1})$ for all i. Hence $\text{Proj}(S)$ is quasi-compact; in fact it is affine since it is equal to $D_+(X_1)$. It is easy to see that the image of $\text{Proj}(S) \to Spec(R)$ is $D(t)$. Hence the morphism $\text{Proj}(S) \to Spec(R)$ is not closed. Thus the valuative criterion cannot apply because it would imply that the morphism is closed (see Schemes, Proposition 21.20.6).

Example 22.8.14. Let A be a ring. Let $S = A[T]$ as a graded A algebra with T in degree 1. Then the canonical morphism $\text{Proj}(S) \to Spec(A)$ (see Lemma 22.8.10) is an isomorphism.

22.9. Quasi-coherent sheaves on Proj

Let S be a graded ring. Let M be a graded S-module. We saw in the previous section how to construct a quasi-coherent sheaf of modules $\widetilde{M}$ on $\text{Proj}(S)$ and a map

$$M_0 \longrightarrow \Gamma(\text{Proj}(S), \widetilde{M})$$

of the degree 0 part of M to the global sections of $\widetilde{M}$. The degree 0 part of the nth twist $M(n)$ of the graded module M (see Algebra, Section 7.52) is equal to M_n. Hence we can get maps

$$M_n \longrightarrow \Gamma(\text{Proj}(S), \widetilde{M(n)}).$$

We would like to be able to perform this operation for any quasi-coherent sheaf $\mathcal{F}$ on Proj(S). We will do this by tensoring with the nth twist of the structure sheaf, see Definition 22.10.1. In order to relate the two notions we will use the following lemma.

Lemma 22.9.1. *Let S be a graded ring. Let $(X, \mathcal{O}_X) = (Proj(S), \mathcal{O}_{Proj(S)})$ be the scheme of Lemma 22.8.7. Let $f \in S_+$ be homogeneous. Let $x \in X$ be a point corresponding to the homogeneous prime $\mathfrak{p} \subset S$. Let M, N be graded S-modules. There is a canonical map of $\mathcal{O}_{Proj(S)}$-modules*

$$\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \longrightarrow \widetilde{M \otimes_S N}$$

which induces the canonical map $M_{(f)} \otimes_{S_{(f)}} N_{(f)} \to (M \otimes_S N)_{(f)}$ on sections over $D_+(f)$ and the canonical map $M_{(\mathfrak{p})} \otimes_{S_{(\mathfrak{p})}} N_{(\mathfrak{p})} \to (M \otimes_S N)_{(\mathfrak{p})}$ on stalks at x. Moreover, the following diagram

$$\begin{tikzcd}
M_0 \otimes_{S_0} N_0 \arrow[r] \arrow[d] & (M \otimes_S N)_0 \arrow[d] \\
\Gamma(X, \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}) \arrow[r] & \Gamma(X, \widetilde{M \otimes_R N})
\end{tikzcd}$$

is commutative.

Proof. To construct a morphism as displayed is the same as constructing a $\mathcal{O}_X$-bilinear map

$$\widetilde{M} \times \widetilde{N} \longrightarrow \widetilde{M \otimes_R N}$$

see Modules, Section 15.15. It suffices to define this on sections over the opens $D_+(f)$ compatible with restriction mappings. On $D_+(f)$ we use the $S_{(f)}$-bilinear map $M_{(f)} \times N_{(f)} \to (M \otimes_S N)_{(f)}$, $(x/f^n, y/f^m) \mapsto (x \otimes y)/f^{n+m}$. Details omitted. □

Remark 22.9.2. In general the map constructed in Lemma 22.9.1 above is not an isomorphism. Here is an example. Let k be a field. Let $S = k[x, y, z]$ with k in degree 0 and $\deg(x) = 1$, $\deg(y) = 2$, $\deg(z) = 3$. Let $\mathfrak{p} = (x, y) \in \text{Proj}(S)$. Let $M = S(1)$ and $N = S(2)$, see Algebra, Section 7.52 for notation. Then $M \otimes_S N = S(3)$. Note that

$$\begin{aligned}
S_z &= k[x, y, z, 1/z] \\
S_{(z)} &= k[x^3/z, xy/z, y^3/z^2] \cong k[u, v, w]/(uw - v^3) \\
M_{(z)} &= S_{(z)} \cdot x + S_{(z)} \cdot y^2/z \subset S_z \\
N_{(z)} &= S_{(z)} \cdot y + S_{(z)} \cdot x^2 \subset S_z \\
S(3)_{(z)} &= S_{(z)} \cdot z \subset S_z
\end{aligned}$$

Consider the maximal ideal $\mathfrak{m} = (u, v, w) \subset S_{(z)}$. It is not hard to see that both $M_{(z)}/\mathfrak{m}M_{(z)}$ and $N_{(z)}/\mathfrak{m}N_{(z)}$ have dimension 2 over $\kappa(\mathfrak{m})$. But $S(3)_{(z)}/\mathfrak{m}S(3)_{(z)}$ has dimension 1. Thus the map $M_{(z)} \otimes N_{(z)} \to S(3)_{(z)}$ is not an isomorphism.

22.10. Invertible sheaves on Proj

Recall from Algebra, Section 7.52 the construction of the twisted module $M(n)$ associated to a graded module over a graded ring.

Definition 22.10.1. Let S be a graded ring. Let $X = \text{Proj}(S)$.

(1) We define $\mathcal{O}_X(n) = \widetilde{S(n)}$. This is called the nth *twist of the structure sheaf of Proj(S)*.
(2) For any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ we set $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$.

We are going to use Lemma 22.9.1 to construct some canonical maps. Since $S(n) \otimes_S S(m) = S(n+m)$ we see that there are canonical maps

$$\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{O}_X(m) \longrightarrow \mathcal{O}_X(n+m). \tag{22.10.1.1}$$

These maps are not isomorphisms in general, see the example in Remark 22.9.2. The same example shows that $\mathcal{O}_X(n)$ is *not* an invertible sheaf on X in general. Tensoring with an arbitrary $\mathcal{O}_X$-module $\mathcal{F}$ we get maps

$$\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{F}(m) \longrightarrow \mathcal{F}(n+m). \tag{22.10.1.2}$$

The maps (22.10.1.1) on global sections give a map of graded rings

$$S \longrightarrow \bigoplus\nolimits_{n \geq 0} \Gamma(X, \mathcal{O}_X(n)). \tag{22.10.1.3}$$

And for an arbitrary $\mathcal{O}_X$-module $\mathcal{F}$ the maps (22.10.1.2) give a graded module structure

$$\bigoplus\nolimits_{n \geq 0} \Gamma(X, \mathcal{O}_X(n)) \times \bigoplus\nolimits_{m \in \mathbf{Z}} \Gamma(X, \mathcal{F}(m)) \longrightarrow \bigoplus\nolimits_{m \in \mathbf{Z}} \Gamma(X, \mathcal{F}(m)) \tag{22.10.1.4}$$

and via (22.10.1.3) also a S-module structure. More generally, given any graded S-module M we have $M(n) = M \otimes_S S(n)$. Hence we get maps

$$\widetilde{M}(n) = \widetilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) \longrightarrow \widetilde{M(n)}. \tag{22.10.1.5}$$

On global sections we get a map of graded S-modules

$$M \longrightarrow \bigoplus\nolimits_{n \in \mathbf{Z}} \Gamma(X, \widetilde{M(n)}). \tag{22.10.1.6}$$

Here is an important fact which follows basically immediately from the definitions.

Lemma 22.10.2. *Let S be a graded ring. Set $X = Proj(S)$. Let $f \in S$ be homogeneous of degree $d > 0$. The sheaves $\mathcal{O}_X(nd)|_{D_+(f)}$ are invertible, and in fact trivial for all $n \in \mathbf{Z}$ (see Modules, Definition 15.21.1). The maps (22.10.1.1) restricted to $D_+(f)$*

$$\mathcal{O}_X(nd)|_{D_+(f)} \otimes_{\mathcal{O}_{D_+(f)}} \mathcal{O}_X(m)|_{D_+(f)} \longrightarrow \mathcal{O}_X(nd+m)|_{D_+(f)}$$

and the maps (22.10.1.5) restricted to $D_+(f)$

$$\widetilde{M}(nd)|_{D_+(f)} = \widetilde{M}|_{D_+(f)} \otimes_{\mathcal{O}_{D_+(f)}} \mathcal{O}_X(nd)|_{D_+(f)} \longrightarrow \widetilde{M(nd)}|_{D_+(f)}$$

are isomorphisms for all $n, m \in \mathbf{Z}$.

Proof. The (not graded) S-module maps $S \to S(n)$, and $M \to M(n)$, given by $x \mapsto f^{n/d}x$ become isomorphisms after inverting f. The first shows that $S_{(f)} \cong S(n)_{(f)}$ which gives an isomorphism $\mathcal{O}_{D_+(f)} \cong \mathcal{O}_X(n)|_{D_+(f)}$. The second shows that the map $S(n)_{(f)} \otimes_{S_{(f)}} M_{(f)} \to M(n)_{(f)}$ is an isomorphism. $\square$

Lemma 22.10.3. *Let S be a graded ring generated as an S_0-algebra by the elements of S_1. Set $X = Proj(S)$. In this case the sheaves $\mathcal{O}_X(n)$ are all invertible, and all the maps (22.10.1.1) and (22.10.1.5) are isomorphisms. In particular, these maps induce isomorphisms*

$$\mathcal{O}_X(n) \cong \mathcal{O}_X(1)^{\otimes n} \quad \textit{and} \quad \widetilde{M(n)} = \widetilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(1)^{\otimes n}.$$

In fact the lemma holds more generally if X is covered by the standard opens $D_+(f)$ with $f \in S_1$.

Proof. Under the assumptions of the lemma X is covered by the open subsets $D_+(f)$ with $f \in S_1$ and the lemma is a consequence of Lemma 22.10.2 above. □

Lemma 22.10.4. *Let S be a graded ring. Set $X = Proj(S)$. Fix $d \geq 1$ an integer. The following open subsets of X are equal:*

1. *The largest open subset $W = W_d \subset X$ such that each $\mathcal{O}_X(dn)|_W$ is invertible and all the multiplication maps $\mathcal{O}_X(nd)|_W \otimes_{\mathcal{O}_W} \mathcal{O}_X(md)|_W \to \mathcal{O}_X(nd + md)|_W$ (see 22.10.1.1) are isomorphisms.*
2. *The union of the open subsets $D_+(fg)$ with $f, g \in S$ homogeneous and $\deg(f) = \deg(g) + d$.*

Moreover, all the maps $\widetilde{M}(nd)|_W = \widetilde{M}|_W \otimes_{\mathcal{O}_W} \mathcal{O}_X(nd)|_W \to \widetilde{M(nd)}|_W$ (see 22.10.1.5) are isomorphisms.

Proof. If $x \in D_+(fg)$ with $\deg(f) = \deg(g) + d$ then on $D_+(fg)$ the sheaves $\mathcal{O}_X(dn)$ are generated by the element $(f/g)^n = f^{2n}/(fg)^n$. This implies x is in the open subset W defined in (1) by arguing as in the proof of Lemma 22.10.2.

Conversely, suppose that $\mathcal{O}_X(d)$ is free of rank 1 in an open neighbourhood V of $x \in X$ and all the multiplication maps $\mathcal{O}_X(nd)|_V \otimes_{\mathcal{O}_V} \mathcal{O}_X(md)|_V \to \mathcal{O}_X(nd + md)|_V$ are isomorphisms. We may choose $h \in S_+$ homogeneous such that $D_+(h) \subset V$. By the definition of the twists of the structure sheaf we conclude there exists an element s of $(S_h)_d$ such that s^n is a basis of $(S_h)_{nd}$ as a module over $S_{(h)}$ for all $n \in \mathbf{Z}$. We may write $s = f/h^m$ for some $m \geq 1$ and $f \in S_{d+m\deg(h)}$. Set $g = h^m$ so $s = f/g$. Note that $x \in D(g)$ by construction. Note that $g^d \in (S_h)_{-d\deg(g)}$. By assumption we can write this as a multiple of $s^{\deg(g)} = f^{\deg(g)}/g^{\deg(g)}$, say $g^d = a/g^e \cdot f^{\deg(g)}/g^{\deg(g)}$. Then we conclude that $g^{d+e+\deg(g)} = af^{\deg(g)}$ and hence also $x \in D_+(f)$. So x is an element of the set defined in (2).

The existence of the generating section $s = f/g$ over the affine open $D_+(fg)$ whose powers freely generate the sheaves of modules $\mathcal{O}_X(nd)$ easily implies that the multiplication maps $\widetilde{M}(nd)|_W = \widetilde{M}|_W \otimes_{\mathcal{O}_W} \mathcal{O}_X(nd)|_W \to \widetilde{M(nd)}|_W$ (see 22.10.1.5) are isomorphisms. Compare with the proof of Lemma 22.10.2. □

Recall from Modules, Lemma 15.21.7 that given an invertible sheaf $\mathcal{L}$ on a locally ringed space X, and given a global section s of $\mathcal{L}$ the set $X_s = \{x \in X \mid s \notin \mathfrak{m}_x \mathcal{L}_x\}$ is open.

Lemma 22.10.5. *Let S be a graded ring. Set $X = Proj(S)$. Fix $d \geq 1$ an integer. Let $W = W_d \subset X$ be the open subscheme defined in Lemma 22.10.4. Let $n \geq 1$ and $f \in S_{nd}$. Denote $s \in \Gamma(W, \mathcal{O}_W(nd))$ the section which is the image of f via (22.10.1.3) restricted to W. Then*

$$W_s = D_+(f) \cap W.$$

Proof. Let $D_+(ab) \subset W$ be a standard affine open with $a, b \in S$ homogeneous and $\deg(a) = \deg(b) + d$. Note that $D_+(ab) \cap D_+(f) = D_+(abf)$. On the other hand the restriction of s to $D_+(ab)$ corresponds to the element $f/1 = b^n f/a^n (a/b)^n \in (S_{ab})_{nd}$. We have seen in the proof of Lemma 22.10.4 that $(a/b)^n$ is a generator for $\mathcal{O}_W(nd)$ over $D_+(ab)$. We conclude that $W_s \cap D_+(ab)$ is the principal open associated to $b^n f/a^n \in \mathcal{O}_X(D_+(ab))$. Thus the result of the lemma is clear. □

The following lemma states the properties that we will later use to characterize schemes with an ample invertible sheaf.

Lemma 22.10.6. *Let S be a graded ring. Let $X = Proj(S)$. Let $Y \subset X$ be a quasi-compact open subscheme. Denote $\mathcal{O}_Y(n)$ the restriction of $\mathcal{O}_X(n)$ to Y. There exists an integer $d \geq 1$ such that*

1. *the subscheme Y is contained in the open W_d defined in Lemma 22.10.4,*
2. *the sheaf $\mathcal{O}_Y(dn)$ is invertible for all $n \in \mathbf{Z}$,*
3. *all the maps $\mathcal{O}_Y(nd) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(m) \longrightarrow \mathcal{O}_Y(nd + m)$ of Equation (22.10.1.1) are isomorphisms,*
4. *all the maps $\widetilde{M}(nd)|_Y = \widetilde{M}|_Y \otimes_{\mathcal{O}_Y} \mathcal{O}_X(nd)|_Y \to \widetilde{M(n)}|_Y$ (see 22.10.1.5) are isomorphisms,*
5. *given $f \in S_{nd}$ denote $s \in \Gamma(Y, \mathcal{O}_Y(nd))$ the image of f via (22.10.1.3) restricted to Y, then $D_+(f) \cap Y = Y_s$,*
6. *a basis for the topology on Y is given by the collection of opens Y_s, where $s \in \Gamma(Y, \mathcal{O}_Y(nd))$, $n \geq 1$, and*
7. *a basis for the topology of Y is given by those opens $Y_s \subset Y$, for $s \in \Gamma(Y, \mathcal{O}_Y(nd))$, $n \geq 1$ which are affine.*

Proof. Since X is quasi-compact there exist finitely many homogeneous $f_i \in S_+$, $i = 1, \ldots, n$ such that the standard opens $D_+(f_i)$ give an open covering of X. Let $d_i = \deg(f_i)$ and set $d = d_1 \ldots d_n$. Note that $D_+(f_i) = D_+(f_i^{d/d_i})$ and hence we see immediately that $Y \subset W_d$, by characterization (2) in Lemma 22.10.4 or by (1) using Lemma 22.10.2. Note that (1) implies (2), (3) and (4) by Lemma 22.10.4. (Note that (3) is a special case of (4).) Assertion (5) follows from Lemma 22.10.5. Assertions (6) and (7) follow because the open subsets $D_+(f)$ form a basis for the topology of X and are affine. $\square$

22.11. Functoriality of Proj

A graded ring map $\psi : A \to B$ does not always give rise to a morphism of associated projective homogeneous spectra. The reason is that the inverse image $\psi^{-1}(\mathfrak{q})$ of a homogeneous prime $\mathfrak{q} \subset B$ may contain the irrelevant prime A_+ even if $\mathfrak{q}$ does not contain B_+. The correct result is stated as follows.

Lemma 22.11.1. *Let A, B be two graded rings. Set $X = Proj(A)$ and $Y = Proj(B)$. Let $\psi : A \to B$ be a graded ring map. Set*

$$U(\psi) = \bigcup\nolimits_{f \in A_+ \ homogeneous} D_+(\psi(f)) \subset Y.$$

Then there is a canonical morphism of schemes

$$r_\psi : U(\psi) \longrightarrow X$$

and a map of $\mathbf{Z}$-graded $\mathcal{O}_{U(\psi)}$-algebras

$$\theta = \theta_\psi : r_\psi^*\left(\bigoplus\nolimits_{d \in \mathbf{Z}} \mathcal{O}_X(d)\right) \longrightarrow \bigoplus\nolimits_{d \in \mathbf{Z}} \mathcal{O}_{U(\psi)}(d).$$

The triple $(U(\psi), r_\psi, \theta)$ is characterized by the following properties:

1. *For every $d \geq 0$ the diagram*

$$\begin{tikzcd} A_d \arrow[rr, "\psi"] \arrow[d] & & B_d \arrow[d] \\ \Gamma(X, \mathcal{O}_X(d)) \arrow[r, "\theta"] & \Gamma(U(\psi), \mathcal{O}_Y(d)) & \Gamma(Y, \mathcal{O}_Y(d)) \arrow[l] \end{tikzcd}$$

   is commutative.

(2) *For any* $f \in A_+$ *homogeneous we have* $r_\psi^{-1}(D_+(f)) = D_+(\psi(f))$ *and the restriction of* r_ψ *to* $D_+(\psi(f))$ *corresponds to the ring map* $A_{(f)} \to B_{(\psi(f))}$ *induced by* ψ.

Proof. Clearly condition (2) uniquely determines the morphism of schemes and the open subset $U(\psi)$. Pick $f \in A_d$ with $d \geq 1$. Note that $\mathcal{O}_X(n)|_{D_+(f)}$ corresponds to the $A_{(f)}$-module $(A_f)_n$ and that $\mathcal{O}_Y(n)|_{D_+(\psi(f))}$ corresponds to the $B_{(\psi(f))}$-module $(B_{\psi(f)})_n$. In other words θ when restricted to $D_+(\psi(f))$ corresponds to a map of $\mathbf{Z}$-graded $B_{(\psi(f))}$-algebras

$$A_f \otimes_{A_{(f)}} B_{(\psi(f))} \longrightarrow B_{\psi(f)}$$

Condition (1) determines the images of all elements of A. Since f is an invertible element which is mapped to $\psi(f)$ we see that $1/f^m$ is mapped to $1/\psi(f)^m$. It easily follows from this that θ is uniquely determined, namely it is given by the rule

$$a/f^m \otimes b/\psi(f)^e \longmapsto \psi(a)b/\psi(f)^{m+e}.$$

To show existence we remark that the proof of uniqueness above gave a well defined prescription for the morphism r and the map θ when restricted to every standard open of the form $D_+(\psi(f)) \subset U(\psi)$ into $D_+(f)$. Call these r_f and θ_f. Hence we only need to verify that if $D_+(f) \subset D_+(g)$ for some $f, g \in A_+$ homogeneous, then the restriction of r_g to $D_+(\psi(f))$ matches r_f. This is clear from the formulas given for r and θ above. $\square$

Lemma 22.11.2. *Let A, B, and C be graded rings. Set $X = Proj(A)$, $Y = Proj(B)$ and $Z = Proj(C)$. Let $\varphi : A \to B$, $\psi : B \to C$ be graded ring maps. Then we have*

$$U(\psi \circ \varphi) = r_\varphi^{-1}(U(\psi)) \quad \textit{and} \quad r_{\psi\circ\varphi} = r_\varphi \circ r_\psi|_{U(\psi\circ\varphi)}.$$

In addition we have

$$\theta_\psi \circ r_\psi^* \theta_\varphi = \theta_{\psi\circ\varphi}$$

with obvious notation.

Proof. Omitted. $\square$

Lemma 22.11.3. *With hypotheses and notation as in Lemma 22.11.1 above. Assume $A \to B$ is surjective. Then*

(1) $U(\psi) = Y$,
(2) $r_\psi : Y \to X$ *is a closed immersion, and*
(3) *the maps* $\theta : r_\psi^* \mathcal{O}_X(n) \to \mathcal{O}_Y(n)$ *are surjective but not isomorphisms in general.*

Proof. Write $B = A/I$ for some graded ideal $I \subset A$. Part (1) is obvious from the definition of $U(\psi)$. For $f \in A_+$ homogeneous we see that $A_{(f)} \to B_{(f)} = A_{(f)}/I_{(f)}$ is surjective. This proves (2). Also the map

$$A_f \otimes_{A_{(f)}} A_{(f)}/I_{(f)} \to (A/I)_f$$

is surjective which proves the surjectivity of θ. For an example where this map is not an isomorphism consider the graded ring $A = k[x, y]$ where k is a field and $\deg(x) = 1$, $\deg(y) = 2$. Set $I = (x)$, so that $B = k[y]$. Note that $\mathcal{O}_Y(1) = 0$ in this case. But it is easy to see that $r_\psi^* \mathcal{O}_Y(1)$ is not zero. (There are less silly examples.) $\square$

Lemma 22.11.4. *With hypotheses and notation as in Lemma 22.11.1 above. Assume $A \to B$ is surjective, and assume that A is generated by A_1 over A_0. Then*

(1) $U(\psi) = Y$,
(2) $r_\psi : Y \to X$ *is a closed immersion, and*
(3) *the maps* $\theta : r_\psi^* \mathcal{O}_X(n) \to \mathcal{O}_Y(n)$ *are isomorphisms.*

Proof. We apply Lemma 22.11.3. By Lemma 22.10.3 we see that both $\mathcal{O}_X(n)$ and $\mathcal{O}_Y(n)$ are invertible. Hence θ is an isomorphism. $\square$

Lemma 22.11.5. *With hypotheses and notation as in Lemma 22.11.1 above. Assume there exists a ring map $R \to A_0$ and a ring map $R \to R'$ such that $B = R' \otimes_R A$. Then*

(1) *$U(\psi) = Y$,*
(2) *the diagram*
$$\begin{tikzcd} Y = Proj(B) \arrow[r, "r_\psi"'] \arrow[d] & Proj(A) = X \arrow[d] \\ Spec(R') \arrow[r] & Spec(R) \end{tikzcd}$$
is a fibre product square, and
(3) *the maps $\theta : r_\psi^*\mathcal{O}_X(n) \to \mathcal{O}_Y(n)$ are isomorphisms.*

Proof. This follows immediately by looking at what happens over the standard opens $D_+(f)$ for $f \in A_+$. $\square$

Lemma 22.11.6. *With hypotheses and notation as in Lemma 22.11.1 above. Assume there exists a $g \in A_0$ such that ψ induces an isomorphism $A_g \to B$. Then $U(\psi) = Y$, $r_\psi : Y \to X$ is an open immersion which induces an isomorphism of Y with the inverse image of $D(g) \subset Spec(A_0)$. Moreover the map θ is an isomorphism.*

Proof. This is a special case of Lemma 22.11.5 above. $\square$

22.12. Morphisms into Proj

Let S be a graded ring. Let $X = \text{Proj}(S)$ be the homogeneous spectrum of S. Let $d \geq 1$ be an integer. Consider the open subscheme

(22.12.0.1) $$U_d = \bigcup\nolimits_{f \in S_d} D_+(f) \quad \subset \quad X = \text{Proj}(S)$$

Note that $d | d' \Rightarrow U_d \subset U_{d'}$ and $X = \bigcup_d U_d$. Neither X nor U_d need be quasi-compact, see Algebra, Lemma 7.53.3. Let us write $\mathcal{O}_{U_d}(n) = \mathcal{O}_X(n)|_{U_d}$. By Lemma 22.10.2 we know that $\mathcal{O}_{U_d}(nd)$, $n \in \mathbf{Z}$ is an invertible $\mathcal{O}_{U_d}$-module and that all the multiplication maps $\mathcal{O}_{U_d}(nd) \otimes_{\mathcal{O}_{U_d}} \mathcal{O}_X(m) \to \mathcal{O}_{U_d}(nd + m)$ of (22.10.1.1) are isomorphisms. In particular we have $\mathcal{O}_{U_d}(nd) \cong \mathcal{O}_{U_d}(d)^{\otimes n}$. The graded ring map (22.10.1.3) on global sections combined with restriction to U_d give a homomorphism of graded rings

(22.12.0.2) $$\psi^d : S^{(d)} \longrightarrow \Gamma_*(U_d, \mathcal{O}_{U_d}(d)).$$

For the notation $S^{(d)}$, see Algebra, Section 7.52. For the notation Γ_* see Modules, Definition 15.21.4. Moreover, since U_d is covered by the opens $D_+(f)$, $f \in S_d$ we see that $\mathcal{O}_{U_d}(d)$ is globally generated by the sections in the image of $\psi^d_1 : S^{(d)}_1 = S_d \to \Gamma(U_d, \mathcal{O}_{U_d}(d))$, see Modules, Definition 15.4.1.

Let Y be a scheme, and let $\varphi : Y \to X$ be a morphism of schemes. Assume the image $\varphi(Y)$ is contained in the open subscheme U_d of X. By the discussion following Modules, Definition 15.21.4 we obtain a homomorphism of graded rings

$$\Gamma_*(U_d, \mathcal{O}_{U_d}(d)) \longrightarrow \Gamma_*(Y, \varphi^*\mathcal{O}_X(d)).$$

The composition of this and ψ^d gives a graded ring homomorphism

(22.12.0.3) $$\psi^d_\varphi : S^{(d)} \longrightarrow \Gamma_*(Y, \varphi^*\mathcal{O}_X(d))$$

which has the property that the invertible sheaf $\varphi^*\mathcal{O}_X(d)$ is globally generated by the sections in the image of $(S^{(d)})_1 = S_d \to \Gamma(Y, \varphi^*\mathcal{O}_X(d))$.

Lemma 22.12.1. *Let S be a graded ring, and $X = Proj(S)$. Let $d \geq 1$ and $U_d \subset X$ as above. Let Y be a scheme. Let $\mathcal{L}$ be an invertible sheaf on Y. Let $\psi : S^{(d)} \to \Gamma_*(Y, \mathcal{L})$ be a graded ring homomorphism such that $\mathcal{L}$ is generated by the sections in the image of $\psi|_{S_d} : S_d \to \Gamma(Y, \mathcal{L})$. Then there exists a morphism $\varphi : Y \to X$ such that $\varphi(Y) \subset U_d$ and an isomorphism $\alpha : \varphi^*\mathcal{O}_{U_d}(d) \to \mathcal{L}$ such that ψ^d_φ agrees with ψ via α:*

$$\begin{tikzcd}
\Gamma_*(Y, \mathcal{L}) & \Gamma_*(Y, \varphi^*\mathcal{O}_{U_d}(d)) \arrow[l, "\alpha"'] & \Gamma_*(U_d, \mathcal{O}_{U_d}(d)) \arrow[l, "\varphi^*"'] \\
S^{(d)} \arrow[u, "\psi"] & & S^{(d)} \arrow[ll, "id"'] \arrow[u, "\psi^d"'] \arrow[ul, "\psi^d_\varphi"]
\end{tikzcd}$$

commutes. Moreover, the pair (φ, α) is unique.

Proof. Pick $f \in S_d$. Denote $s = \psi(f) \in \Gamma(Y, \mathcal{L})$. On the open set Y_s where s does not vanish multiplication by s induces an isomorphism $\mathcal{O}_{Y_s} \to \mathcal{L}|_{Y_s}$, see Modules, Lemma 15.21.7. We will denote the inverse of this map $x \mapsto x/s$, and similarly for powers of $\mathcal{L}$. Using this we define a ring map $\psi_{(f)} : S_{(f)} \to \Gamma(Y_s, \mathcal{O})$ by mapping the fraction a/f^n to $\psi(a)/s^n$. By Schemes, Lemma 21.6.4 this corresponds to a morphism $\varphi_f : Y_s \to Spec(S_{(f)}) = D_+(f)$. We also introduce the isomorphism $\alpha_f : \varphi_f^*\mathcal{O}_{D_+(f)}(d) \to \mathcal{L}|_{Y_s}$ which maps the pullback of the trivializing section f over $D_+(f)$ to the trivializing section s over Y_s. With this choice the commutativity of the diagram in the lemma holds with Y replace by Y_s, φ replaced by φ_f, and α replaced by α_f; verification omitted.

Suppose that $f' \in S_d$ is a second element, and denote $s' = \psi(f') \in \Gamma(Y, \mathcal{L})$. Then $Y_s \cap Y_{s'} = Y_{ss'}$ and similarly $D_+(f) \cap D_+(f') = D_+(ff')$. In Lemma 22.10.6 we saw that $D_+(f') \cap D_+(f)$ is the same as the set of points of $D_+(f)$ where the section of $\mathcal{O}_X(d)$ defined by f' does not vanish. Hence $\varphi_f^{-1}(D_+(f') \cap D_+(f)) = Y_s \cap Y_{s'} = \varphi_{f'}^{-1}(D_+(f') \cap D_+(f))$. On $D_+(f) \cap D_+(f')$ the fraction f/f' is an invertible section of the structure sheaf with inverse f'/f. Note that $\psi_{(f')}(f/f') = \psi(f)/s' = s/s'$ and $\psi_{(f)}(f'/f) = \psi(f')/s = s'/s$. We claim there is a unique ring map $S_{(ff')} \to \Gamma(Y_{ss'}, \mathcal{O})$ making the following diagram commute

$$\begin{tikzcd}
\Gamma(Y_s, \mathcal{O}) \arrow[r] & \Gamma(Y_{ss'}, \mathcal{O}) & \Gamma(Y_{s'}, \mathcal{O}) \arrow[l] \\
S_{(f)} \arrow[u, "\psi_{(f)}"] \arrow[r] & S_{(ff')} \arrow[u] & S_{(f')} \arrow[l] \arrow[u, "\psi_{(f')}"']
\end{tikzcd}$$

It exists because we may use the rule $x/(ff')^n \mapsto \psi(x)/(ss')^n$, which ``works" by the formulas above. Uniqueness follows as Proj(S) is separated, see Lemma 22.8.8 and its proof. This shows that the morphisms φ_f and $\varphi_{f'}$ agree over $Y_s \cap Y_{s'}$. The restrictions of α_f and $\alpha_{f'}$ agree over $Y_s \cap Y_{s'}$ because the regular functions s/s' and $\psi_{(f')}(f)$ agree. This proves that the morphisms ψ_f glue to a global morphism from Y into $U_d \subset X$, and that the maps α_f glue to an isomorphism satsifying the conditions of the lemma.

We still have to show the pair (φ, α) is unique. Suppose (φ', α') is a second such pair. Let $f \in S_d$. By the commutativity of the diagrams in the lemma we have that the inverse images of $D_+(f)$ under both φ and φ' are equal to $Y_{\psi(f)}$. Since the opens $D_+(f)$ are a basis for the topology on X, and since X is a sober topological space (see Schemes, Lemma 21.11.1) this means the maps φ and φ' are the same on underlying topological spaces. Let us use $s = \psi(f)$ to trivialize the invertible sheaf $\mathcal{L}$ over $Y_{\psi(f)}$. By the commutativity of the diagrams we have that $\alpha^{\otimes n}(\psi^d_\varphi(x)) = \psi(x) = (\alpha')^{\otimes n}(\psi^d_{\varphi'}(x))$ for all $x \in S_{nd}$. By construction of ψ^d_φ and $\psi^d_{\varphi'}$ we have $\psi^d_\varphi(x) = \varphi^\sharp(x/f^n)\psi^d_\varphi(f^n)$ over $Y_{\psi(f)}$, and similarly for $\psi^d_{\varphi'}$. by the commutativity of the diagrams of the lemma we deduce that $\varphi^\sharp(x/f^n) = (\varphi')^\sharp(x/f^n)$. This proves that φ and φ' induce the same morphism from $Y_{\psi(f)}$ into the affine scheme $D_+(f) = Spec(S_{(f)})$. Hence φ and φ' are the same as morphisms. Finally, it remains to show that the commutativity of the diagram of the lemma singles out, given φ, a unique α. We omit the verification. □

We continue the discussion from above the lemma. Let S be a graded ring. Let Y be a scheme. We will consider *triples* $(d, \mathcal{L}, \psi)$ where

(1) $d \geq 1$ is an integer,
(2) $\mathcal{L}$ is an invertible $\mathcal{O}_Y$-module, and
(3) $\psi : S^{(d)} \to \Gamma_*(Y, \mathcal{L})$ is a graded ring homomorphism such that $\mathcal{L}$ is generated by the global sections $\psi(f)$, with $f \in S_d$.

Given a morphism $h : Y' \to Y$ and a triple $(d, \mathcal{L}, \psi)$ over Y we can pull it back to the triple $(d, h^*\mathcal{L}, h^* \circ \psi)$. Given two triples $(d, \mathcal{L}, \psi)$ and $(d, \mathcal{L}', \psi')$ with the same integer d we say they are *strictly equivalent* if there exists an isomorphism $\beta : \mathcal{L} \to \mathcal{L}'$ such that $\beta \circ \psi = \psi'$ as graded ring maps $S^{(d)} \to \Gamma_*(Y, \mathcal{L}')$.

For each integer $d \geq 1$ we define

$$\begin{aligned} F_d : Sch^{opp} &\longrightarrow Sets, \\ Y &\longmapsto \{\text{strict equivalence classes of triples } (d, \mathcal{L}, \psi) \text{ as above}\} \end{aligned}$$

with pullbacks as defined above.

Lemma 22.12.2. *Let S be a graded ring. Let $X = Proj(S)$. The open subscheme $U_d \subset X$ (22.12.0.1) represents the functor F_d and the triple $(d, \mathcal{O}_{U_d}(d), \psi^d)$ defined above is the universal family (see Schemes, Section 21.15).*

Proof. This is a reformulation of Lemma 22.12.1 □

Lemma 22.12.3. *Let S be a graded ring generated as an S_0-algebra by the elements of S_1. In this case the scheme $X = Proj(S)$ represents the functor which associates to a scheme Y the set of pairs $(\mathcal{L}, \psi)$, where*

(1) *$\mathcal{L}$ is an invertible $\mathcal{O}_Y$-module, and*
(2) *$\psi : S \to \Gamma_*(Y, \mathcal{L})$ is a graded ring homomorphism such that $\mathcal{L}$ is generated by the global sections $\psi(f)$, with $f \in S_1$*

up to strict equivalence as above.

Proof. Under the assumptions of the lemma we have $X = U_1$ and the lemma is a reformulation of Lemma 22.12.2 above. □

We end this section with a discussion of a functor corresponding to Proj(S) for a general graded ring S. We advise the reader to skip the rest of this section.

Fix an arbitrary graded ring S. Let T be a scheme. We will say two triples $(d, \mathcal{L}, \psi)$ and $(d', \mathcal{L}', \psi')$ over T with possibly different integers d, d' are *equivalent* if there exists an isomorphism $\beta : \mathcal{L}^{\otimes d'} \to (\mathcal{L}')^{\otimes d}$ of invertible sheaves over T such that $\beta \circ \psi|_{S^{(dd')}}$ and $\psi'|_{S^{(dd')}}$ agree as graded ring maps $S^{(dd')} \to \Gamma_*(Y, (\mathcal{L}')^{\otimes dd'})$.

Lemma 22.12.4. *Let S be a graded ring. Set $X = Proj(S)$. Let T be a scheme. Let $(d, \mathcal{L}, \psi)$ and $(d', \mathcal{L}', \psi')$ be two triples over T. The following are equivalent:*

(1) *Let $n = lcm(d, d')$. Write $n = ad = a'd'$. There exists an isomorphism $\beta : \mathcal{L}^{\otimes a} \to (\mathcal{L}')^{\otimes a'}$ with the property that $\beta \circ \psi|_{S^{(n)}}$ and $\psi'|_{S^{(n)}}$ agree as graded ring maps $S^{(n)} \to \Gamma_*(Y, (\mathcal{L}')^{\otimes n})$.*
(2) *The triples $(d, \mathcal{L}, \psi)$ and $(d', \mathcal{L}', \psi')$ are equivalent.*
(3) *For some positive integer $n = ad = a'd'$ there exists an isomorphism $\beta : \mathcal{L}^{\otimes a} \to (\mathcal{L}')^{\otimes a'}$ with the property that $\beta \circ \psi|_{S^{(n)}}$ and $\psi'|_{S^{(n)}}$ agree as graded ring maps $S^{(n)} \to \Gamma_*(Y, (\mathcal{L}')^{\otimes n})$.*
(4) *The morphisms $\varphi : T \to X$ and $\varphi' : T \to X$ assocated to $(d, \mathcal{L}, \psi)$ and $(d', \mathcal{L}', \psi')$ are equal.*

Proof. Clearly (1) implies (2) and (2) implies (3) by restricting to more divisible degrees and powers of invertible sheaves. Also (3) implies (4) by the uniqueness statement in Lemma 22.12.1. Thus we have to prove that (4) implies (1). Assume (4), in other words $\varphi = \varphi'$. Note that this implies that we may write $\mathcal{L} = \varphi^*\mathcal{O}_X(d)$ and $\mathcal{L}' = \varphi^*\mathcal{O}_X(d')$. Moreover, via these identifications we have that the graded ring maps ψ and ψ' correspond to the restriction of the canonical graded ring map

$$S \longrightarrow \bigoplus\nolimits_{n\geq 0} \Gamma(X, \mathcal{O}_X(n))$$

to $S^{(d)}$ and $S^{(d')}$ composed with pullback by φ (by Lemma 22.12.1 again). Hence taking β to be the isomorphism

$$(\varphi^*\mathcal{O}_X(d))^{\otimes a} = \varphi^*\mathcal{O}_X(n) = (\varphi^*\mathcal{O}_X(d'))^{\otimes a'}$$

works. □

Let S be a graded ring. Let $X = \text{Proj}(S)$. Over the open subscheme scheme $U_d \subset X = \text{Proj}(S)$ (22.12.0.1) we have the triple $(d, \mathcal{O}_{U_d}(d), \psi^d)$. Clearly, if $d|d'$ the triples $(d, \mathcal{O}_{U_d}(d), \psi^d)$ and $(d', \mathcal{O}_{U_{d'}}(d'), \psi^{d'})$ are equivalent when restricted to the open U_d (which is a subset of $U_{d'}$). This, combined with Lemma 22.12.1 shows that morphisms $Y \to X$ correspond roughly to equivalence classes of triples over Y. This is not quite true since if Y is not quasi-compact, then there may not be a single triple which works. Thus we have to be slightly careful in defining the corresponding functor.

Here is one possible way to do this. Suppose $d' = ad$. Consider the transformation of functors $F_d \to F_{d'}$ which assigns to the triple $(d, \mathcal{L}, \psi)$ over T the triple $(d', \mathcal{L}^{\otimes a}, \psi|_{S^{(d')}})$. One of the implications of Lemma 22.12.4 is that the transformation $F_d \to F_{d'}$ is injective! For a quasi-compact scheme T we define

$$F(T) = \bigcup\nolimits_{d\in\mathbf{N}} F_d(T)$$

with transition maps as explained above. This clearly defines a contravariant functor on the category of quasi-compact schemes with values in sets. For a general scheme T we define

$$F(T) = lim_{V\subset T \text{ quasi-compact open}} F(V).$$

In other words, an element ξ of $F(T)$ corresponds to a compatible system of choices of elements $\xi_V \in F(V)$ where V ranges over the quasi-compact opens of T. We omit the definition of the pullback map $F(T) \to F(T')$ for a morphism $T' \to T$ of schemes. Thus we have defined our functor

$$F : Sch^{opp} \longrightarrow Sets$$

Lemma 22.12.5. *Let S be a graded ring. Let $X = Proj(S)$. The functor F defined above is representable by the scheme X.*

Proof. We have seen above that the functor F_d corresponds to the open subscheme $U_d \subset X$. Moreover the transformation of functors $F_d \to F_{d'}$ (if $d|d'$) defined above corresponds to the inclusion morphism $U_d \to U_{d'}$ (see discussion above). Hence to show that F is represented by X it suffices to show that $T \to X$ for a quasi-compact scheme T ends up in some U_d, and that for a general scheme T we have

$$Mor(T, X) = lim_{V \subset T \text{ quasi-compact open}} Mor(V, X).$$

These verifications are omitted. □

22.13. Projective space

Projective space is one of the fundamental objects studied in algebraic geometry. In this section we just give its construction as Proj of a polynomial ring. Later we will discover many of its beautiful properties.

Lemma 22.13.1. *Let $S = \mathbf{Z}[T_0, \ldots, T_n]$ with $\deg(T_i) = 1$. The scheme*

$$\mathbf{P}^n_{\mathbf{Z}} = Proj(S)$$

represents the functor which associates to a scheme Y the pairs $(\mathcal{L}, (s_0, \ldots, s_n))$ where

(1) *$\mathcal{L}$ is an invertible $\mathcal{O}_Y$-module, and*
(2) *$s_0, \ldots, s_n$ are global sections of $\mathcal{L}$ which generate $\mathcal{L}$*

up to the following equivalence: $(\mathcal{L}, (s_0, \ldots, s_n)) \sim (\mathcal{N}, (t_0, \ldots, t_n)) \Leftrightarrow$ there exists an isomorphism $\beta : \mathcal{L} \to \mathcal{N}$ with $\beta(s_i) = t_i$ for $i = 0, \ldots, n$.

Proof. This is a special case of Lemma 22.12.3 above. Namely, for any graded ring A we have

$$\begin{array}{rcl} Mor_{gradedrings}(\mathbf{Z}[T_0, \ldots, T_n], A) & = & A_1 \times \ldots \times A_1 \\ \psi & \mapsto & (\psi(T_0), \ldots, \psi(T_n)) \end{array}$$

and the degree 1 part of $\Gamma_*(Y, \mathcal{L})$ is just $\Gamma(Y, \mathcal{L})$. □

Definition 22.13.2. The scheme $\mathbf{P}^n_{\mathbf{Z}} = \text{Proj}(\mathbf{Z}[T_0, \ldots, T_n])$ is called *projective n-space over* $\mathbf{Z}$. Its base change $\mathbf{P}^n_S$ to a scheme S is called *projective n-space over S*. If R is a ring the base change to $Spec(R)$ is denoted $\mathbf{P}^n_R$ and called *projective n-space over R*.

Given a scheme Y over S and a pair $(\mathcal{L}, (s_0, \ldots, s_n))$ as in Lemma 22.13.1 the induced morphism to $\mathbf{P}^n_S$ is denoted

$$\varphi_{(\mathcal{L}, (s_0, \ldots, s_n))} : Y \longrightarrow \mathbf{P}^n_S$$

This makes sense since the pair defines a morphism into $\mathbf{P}^n_{\mathbf{Z}}$ and we already have the structure morphism into S so combined we get a morphism into $\mathbf{P}^n_S = \mathbf{P}^n_{\mathbf{Z}} \times S$. Note that this is the S-morphism characterized by

$$\mathcal{L} = \varphi^*_{(\mathcal{L}, (s_0, \ldots, s_n))} \mathcal{O}_{\mathbf{P}^n_R}(1) \quad \text{and} \quad s_i = \varphi^*_{(\mathcal{L}, (s_0, \ldots, s_n))} T_i$$

where we think of T_i as a global section of $\mathcal{O}_{\mathbf{P}^n_S}(1)$ via (22.10.1.3).

Lemma 22.13.3. *Projective n-space over $\mathbf{Z}$ is covered by $n + 1$ standard opens*

$$\mathbf{P}^n_{\mathbf{Z}} = \bigcup\nolimits_{i=0, \ldots, n} D_+(T_i)$$

where each $D_+(T_i)$ is isomorphic to $\mathbf{A}^n_{\mathbf{Z}}$ affine n-space over $\mathbf{Z}$.

Proof. This is true because $\mathbf{Z}[T_0, \ldots, T_n]_+ = (T_0, \ldots, T_n)$ and since

$$Spec\left(\mathbf{Z}\left[\frac{T_0}{T_i}, \ldots, \frac{T_n}{T_i}\right]\right) \cong \mathbf{A}^n_{\mathbf{Z}}$$

in an obvious way. □

Lemma 22.13.4. *Let S be a scheme. The structure morphism $\mathbf{P}^n_S \to S$ is*

(1) *separated,*
(2) *quasi-compact,*
(3) *satisfies the existence and uniqueness parts of the valuative criterion, and*
(4) *universally closed.*

Proof. All these properties are stable under base change (this is clear for the last two and for the other two see Schemes, Lemmas 21.21.13 and 21.19.3). Hence it suffices to prove them for the morphism $\mathbf{P}^n_{\mathbf{Z}} \to Spec(\mathbf{Z})$. Separatedness is Lemma 22.8.8. Quasi-compactness follows from Lemma 22.13.3. Existence and uniqueness of the valuative criterion follow from Lemma 22.8.11. Universally closed follows from the above and Schemes, Proposition 21.20.6. □

Remark 22.13.5. What's missing in the list of properties above? Well to be sure the property of being of finite type. The reason we do not list this here is that we have not yet defined the notion of finite type at this point. (Another property which is missing is ``smoothness''. And I'm sure there are many more you can think of.)

We finish this section with two simple lemmas. These lemmas are special cases of more general results later, but perhaps it makes sense to prove these directly here now.

Lemma 22.13.6. *Let R be a ring. Let $Z \subset \mathbf{P}^n_R$ be a closed subscheme. Let*

$$I_d = Ker\left(R[T_0, \ldots, T_n]_d \longrightarrow \Gamma(Z, \mathcal{O}_{\mathbf{P}^n_R}(d)|_Z)\right)$$

Then $I = \bigoplus I_d \subset R[T_0, \ldots, T_n]$ is a graded ideal and $Z = Proj(R[T_0, \ldots, T_n]/I)$.

Proof. It is clear that I is a graded ideal. Set $Z' = \text{Proj}(R[T_0, \ldots, T_n]/I)$. By Lemma 22.11.4 we see that Z' is a closed subscheme of $\mathbf{P}^n_R$. To see the equality $Z = Z'$ it suffices to check on an standard affine open $D_+(T_i)$. By renumbering the homogeneous coordinates we may assume $i = 0$. Say $Z \cap D_+(T_0)$, resp. $Z' \cap D_+(T_0)$ is cut out by the ideal J, resp. J' of $R[T_1/T_0, \ldots, T_n/T_0]$. Then J' is the ideal generated by the elements $F/T_0^{\deg(F)}$ where $F \in I$ is homogeneous. Suppose the degree of $F \in I$ is d. Since F vanishes as a section of $\mathcal{O}_{\mathbf{P}^n_R}(d)$ restricted to Z we see that F/T_0^d is an element of J. Thus $J' \subset J$.

Conversely, suppose that $f \in J$. If f has total degree d in $T_1/T_0, \ldots, T_n/T_0$, then we can write $f = F/T_0^d$ for some $F \in R[T_0, \ldots, T_n]_d$. Pick $i \in \{1, \ldots, n\}$. Then $Z \cap D_+(T_i)$ is cut out by some ideal $J_i \subset R[T_0/T_i, \ldots, T_n/T_i]$. Moreover,

$$J \cdot R\left[\frac{T_1}{T_0}, \ldots, \frac{T_n}{T_0}, \frac{T_0}{T_i}, \ldots, \frac{T_n}{T_i}\right] = J_i \cdot R\left[\frac{T_1}{T_0}, \ldots, \frac{T_n}{T_0}, \frac{T_0}{T_i}, \ldots, \frac{T_n}{T_i}\right]$$

The left hand side is the localization of J with respect to the element T_i/T_0 and the right hand side is the localization of J_i with respect to the element T_0/T_i. It follows that $T_0^{d_i} F/T_i^{d+d_i}$ is an element of J_i for some d_i sufficiently large. This proves that $T_0^{\max(d_i)} F$ is an element of I, because its restriction to each standard affine open $D_+(T_i)$ vanishes on the closed subscheme $Z \cap D_+(T_i)$. Hence $f \in J'$ and we conclude $J \subset J'$ as desired. □

The following lemma is a special case of the more general Properties, Lemma 23.25.3.

Lemma 22.13.7. *Let R be a ring. Let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathbf{P}^n_R$. For $d \geq 0$ set*

$$M_d = \Gamma(\mathbf{P}^n_R, \mathcal{F} \otimes_{\mathcal{O}_{\mathbf{P}^n_R}} \mathcal{O}_{\mathbf{P}^n_R}(d)) = \Gamma(\mathbf{P}^n_R, \mathcal{F}(d))$$

Then $M = \bigoplus_{d \geq 0} M_d$ is a graded $R[T_0, \ldots, R_n]$-module and there is a canonical isomorphism $\mathcal{F} = \widetilde{M}$.

Proof. The multiplication maps

$$R[T_0, \ldots, R_n]_e \times M_d \longrightarrow M_{d+e}$$

come from the natural isomorphisms

$$\mathcal{O}_{\mathbf{P}^n_R}(e) \otimes_{\mathcal{O}_{\mathbf{P}^n_R}} \mathcal{F}(d) \longrightarrow \mathcal{F}(e + d)$$

see Equation (22.10.1.4). Let us construct the map $c : \widetilde{M} \to \mathcal{F}$. On each of the standard affines $U_i = D_+(T_i)$ we see that $\Gamma(U_i, \widetilde{M}) = (M[1/T_i])_0$ where the subscript $_0$ means degree 0 part. An element of this can be written as m/T_i^d with $m \in M_d$. Since T_i is a generator of $\mathcal{O}(1)$ over U_i we can always write $m|_{U_i} = m_i \otimes T_i^d$ where $m_i \in \Gamma(U_i, \mathcal{F})$ is a unique section. Thus a natural guess is $c(m/T_i^d) = m_i$. A small argument, which is omitted here, shows that this gives a well defined map $c : \widetilde{M} \to \mathcal{F}$ if we can show that

$$(T_i/T_j)^d m_i|_{U_i \cap U_j} = m_j|_{U_i \cap U_j}$$

in $M[1/T_iT_j]$. But this is clear since on the overlap the generators T_i and T_j of $\mathcal{O}(1)$ differ by the invertible function T_i/T_j.

Injectivity of c. We may check for injectivity over the affine opens U_i. Let $i \in \{0, \ldots, n\}$ and let s be an element $s = m/T_i^d \in \Gamma(U_i, \widetilde{M})$ such that $c(m/T_i^d) = 0$. By the description of c above this means that $m_i = 0$, hence $m|_{U_i} = 0$. Hence $T_i^e m = 0$ in M for some e. Hence $s = m/T_i^d = T_i^e/T_i^{e+d} = 0$ as desired.

Surjectivity of c. We may check for surjectivity over the affine opens U_i. By renumbering it suffices to check it over U_0. Let $s \in \mathcal{F}(U_0)$. Let us write $\mathcal{F}|_{U_i} = \widetilde{N_i}$ for some $R[T_0/T_i, \ldots, T_0/T_i]$-module N_i, which is possible because $\mathcal{F}$ is quasi-coherent. So s corresponds to an element $x \in N_0$. Then we have that

$$(N_i)_{T_j/T_i} \cong (N_j)_{T_i/T_j}$$

(where the subscripts mean ``principal localization at") as modules over the ring

$$R\left[\frac{T_0}{T_i}, \ldots, \frac{T_n}{T_i}, \frac{T_0}{T_j}, \ldots, \frac{T_n}{T_j}\right].$$

This means that for some large integer d there exist elements $s_i \in N_i$, $i = 1, \ldots, n$ such that

$$s = (T_i/T_0)^d s_i$$

on $U_0 \cap U_i$. Next, we look at the difference

$$t_{ij} = s_i - (T_j/T_i)^d s_j$$

on $U_i \cap U_j$, $0 < i < j$. By our choice of s_i we know that $t_{ij}|_{U_0 \cap U_i \cap U_j} = 0$. Hence there exists a large integer e such that $(T_0/T_i)^e t_{ij} = 0$. Set $s_i' = (T_0/T_i)^e s_i$, and $s_0' = s$. Then we will have

$$s_a' = (T_b/T_a)^{e+d} s_b'$$

on $U_a \cap U_b$ for all a, b. This is exactly the condition that the elements s'_a glue to a global section $m \in \Gamma(\mathbf{P}^n_R, \mathcal{F}(e+d))$. And moreover $c(m/T_0^{e+d}) = s$ by construction. Hence c is surjective and we win. $\square$

22.14. Invertible sheaves and morphisms into Proj

Let T be a scheme and let $\mathcal{L}$ be an invertible sheaf on T. For a section $s \in \Gamma(T, \mathcal{L})$ we denote T_s the open subset of points where s does not vanish. See Modules, Lemma 15.21.7. We can view the following lemma as a slight generalization of Lemma 22.12.3. It also is a generalization of Lemma 22.11.1.

Lemma 22.14.1. *Let A be a graded ring. Set $X = Proj(A)$. Let T be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_T$-module. Let $\psi : A \to \Gamma_*(T, \mathcal{L})$ be a homomorphism of graded rings. Set*

$$U(\psi) = \bigcup\nolimits_{f \in A_+ \ homogeneous} T_{\psi(f)}$$

The morphism ψ induces a canonical morphism of schemes

$$r_{\mathcal{L},\psi} : U(\psi) \longrightarrow X$$

together with a map of $\mathbf{Z}$-graded $\mathcal{O}_T$-algebras

$$\theta : r^*_{\mathcal{L},\psi}\left(\bigoplus\nolimits_{d \in \mathbf{Z}} \mathcal{O}_X(d)\right) \longrightarrow \bigoplus\nolimits_{d \in \mathbf{Z}} \mathcal{L}^{\otimes d}|_{U(\psi)}.$$

The triple $(U(\psi), r_{\mathcal{L},\psi}, \theta)$ is characterized by the following properties:

1. *For $f \in A_+$ homogeneous we have $r^{-1}_{\mathcal{L},\psi}(D_+(f)) = T_{\psi(f)}$.*
2. *For every $d \geq 0$ the diagram*

$$\begin{tikzcd} A_d \arrow[r, "\psi"] \arrow[d, "(22.10.1.3)"'] & \Gamma(T, \mathcal{L}^{\otimes d}) \arrow[d, "restrict"] \\ \Gamma(X, \mathcal{O}_X(d)) \arrow[r, "\theta"] & \Gamma(U(\psi), \mathcal{L}^{\otimes d}) \end{tikzcd}$$

   is commutative.

Moreover, for any $d \geq 1$ and any open subscheme $V \subset T$ such that the sections in $\psi(A_d)$ generate $\mathcal{L}^{\otimes d}|_V$ the morphism $r_{\mathcal{L},\psi}|_V$ agrees with the morphism $\varphi : V \to Proj(A)$ and the map $\theta|_V$ agrees with the map $\alpha : \varphi^*\mathcal{O}_X(d) \to \mathcal{L}^{\otimes d}|_V$ where (φ, α) is the pair of Lemma 22.12.1 associated to $\psi|_{A^{(d)}} : A^{(d)} \to \Gamma_*(V, \mathcal{L}^{\otimes d})$.

Proof. Suppose that we have two triples $(U, r : U \to X, \theta)$ and $(U', r' : U' \to X, \theta')$ satisfying (1) and (2). Property (1) implies that $U = U' = U(\psi)$ and that $r = r'$ as maps of underlying topological spaces, since the opens $D_+(f)$ form a basis for the topology on X, and since X is a sober topological space (see Algebra, Section 7.53 and Schemes, Lemma 21.11.1). Let $f \in A_+$ be homogeneous. Note that $\Gamma(D_+(f), \bigoplus_{n \in \mathbf{Z}} \mathcal{O}_X(n)) = A_f$ as a $\mathbf{Z}$-graded algebra. Consider the two $\mathbf{Z}$-graded ring maps

$$\theta, \theta' : A_f \longrightarrow \Gamma(T_{\psi(f)}, \bigoplus \mathcal{L}^{\otimes n}).$$

We know that multiplication by f (resp. $\psi(f)$) is an isomorphism on the left (resp. right) hand side. We also know that $\theta(x/1) = \theta'(x/1) = \psi(x)|_{T_{\psi(f)}}$ by (2) for all $x \in A$. Hence we deduce easily that $\theta = \theta'$ as desired. Considering the degree 0 parts we deduce that $r^\sharp = (r')^\sharp$, i.e., that $r = r'$ as morphisms of schemes. This proves the uniqueness.

Now we come to existence. By the uniqueness just proved, it is enought to construct the pair (r, θ) locally on T. Hence we may assume that $T = Spec(R)$ is affine, that $\mathcal{L} = \mathcal{O}_T$ and that for some $f \in A_+$ homogeneous we have $\psi(f)$ generates $\mathcal{O}_T = \mathcal{O}_T^{\otimes \deg(f)}$. In other words, $\psi(f) = u \in R^*$ is a unit. In this case the map ψ is a graded ring map

$$A \longrightarrow R[x] = \Gamma_*(T, \mathcal{O}_T)$$

which maps f to $ux^{\deg(f)}$. Clearly this extends (uniquely) to a $\mathbf{Z}$-graded ring map $\theta : A_f \to R[x, x^{-1}]$ by mapping $1/f$ to $u^{-1}x^{-\deg(f)}$. This map in degree zero gives the ring map $A_{(f)} \to R$ which gives the morphism $r : T = Spec(R) \to Spec(A_{(f)}) = D_+(f) \subset X$. Hence we have constructed (r, θ) in this special case.

Let us show the last statement of the lemma. According to Lemma 22.12.1 the morphism constructed there is the unique one such that the displayed diagram in its statement commutes. The commutativity of the diagram in the lemma implies the commutativity when restricted to V and $A^{(d)}$. Whence the result. □

Remark 22.14.2. Assumptions as in Lemma 22.14.1 above. The image of the morphism $r_{\mathcal{L},\psi}$ need not be contained in the locus where the sheaf $\mathcal{O}_X(1)$ is invertible. Here is an example. Let k be a field. Let $S = k[A, B, C]$ graded by $\deg(A) = 1$, $\deg(B) = 2$, $\deg(C) = 3$. Set $X = \text{Proj}(S)$. Let $T = \mathbf{P}^2_k = \text{Proj}(k[X_0, X_1, X_2])$. Recall that $\mathcal{L} = \mathcal{O}_T(1)$ is invertible and that $\mathcal{O}_T(n) = \mathcal{L}^{\otimes n}$. Consider the composition ψ of the maps

$$S \to k[X_0, X_1, X_2] \to \Gamma_*(T, \mathcal{L}).$$

Here the first map is $A \mapsto X_0^6$, $B \mapsto X_1^3$, $C \mapsto X_2^3$ and the second map is (22.10.1.3). By the lemma this corresponds to a morphism $r_{\mathcal{L},\psi} : T \to X = \text{Proj}(S)$ which is easily seen to be surjective. On the other hand, in Remark 22.9.2 we showed that the sheaf $\mathcal{O}_X(1)$ is not invertible at all points of X.

22.15. Relative Proj via glueing

Situation 22.15.1. Here S is a scheme, and $\mathcal{A}$ is a quasi-coherent graded $\mathcal{O}_S$-algebra.

In this section we outline how to construct a morphism of schemes

$$\underline{\text{Proj}}_S(\mathcal{A}) \longrightarrow S$$

by glueing the homogeneous spectra $\text{Proj}(\Gamma(U, \mathcal{A}))$ where U ranges over the affine opens of S. We first show that the homogeneous spectra of the values of $\mathcal{A}$ over affines form a suitable collection of schemes, as in Lemma 22.2.1.

Lemma 22.15.2. *In Situation 22.15.1. Suppose $U \subset U' \subset S$ are affine opens. Let $A = \mathcal{A}(U)$ and $A' = \mathcal{A}(U')$. The map of graded rings $A' \to A$ induces a morphism $r : Proj(A) \to Proj(A')$, and the diagram*

$$\begin{array}{ccc} Proj(A) & \longrightarrow & Proj(A') \\ \downarrow & & \downarrow \\ U & \longrightarrow & U' \end{array}$$

is cartesian. Moreover there are canonical isomorphisms $\theta : r^*\mathcal{O}_{Proj(A')}(n) \to \mathcal{O}_{Proj(A)}(n)$ compatible with multiplication maps.

Proof. Let $R = \mathcal{O}_S(U)$ and $R' = \mathcal{O}_S(U')$. Note that the map $R \otimes_{R'} A' \to A$ is an isomorphism as $\mathcal{A}$ is quasi-coherent (see Schemes, Lemma 21.7.3 for example). Hence the lemma follows from Lemma 22.11.5. $\square$

In particular the morphism $\text{Proj}(A) \to \text{Proj}(A')$ of the lemma is an open immersion.

Lemma 22.15.3. *In Situation 22.15.1. Suppose $U \subset U' \subset U'' \subset S$ are affine opens. Let $A = \mathcal{A}(U)$, $A' = \mathcal{A}(U')$ and $A'' = \mathcal{A}(U'')$. The composition of the morphisms $r : Proj(A) \to Proj(A')$, and $r' : Proj(A') \to Proj(A'')$ of Lemma 22.15.2 gives the morphism $r'' : Proj(A) \to Proj(A'')$ of Lemma 22.15.2. A similar statement holds for the isomorphisms θ.*

Proof. This follows from Lemma 22.11.2 since the map $A'' \to A$ is the composition of $A'' \to A'$ and $A' \to A$. $\square$

Lemma 22.15.4. *In Situation 22.15.1. There exists a morphism of schemes*

$$\pi : \underline{Proj}_S(\mathcal{A}) \longrightarrow S$$

with the following properties:

(1) *for every affine open $U \subset S$ there exists an isomorphism $i_U : \pi^{-1}(U) \to Proj(A)$ with $A = \mathcal{A}(U)$, and*
(2) *for $U \subset U' \subset S$ affine open the composition*

$$Proj(A) \xrightarrow{i_U^{-1}} \pi^{-1}(U) \xrightarrow{inclusion} \pi^{-1}(U') \xrightarrow{i_{U'}} Proj(A')$$

with $A = \mathcal{A}(U)$, $A' = \mathcal{A}(U')$ is the open immersion of Lemma 22.15.2 above.

Proof. Follows immediately from Lemmas 22.2.1, 22.15.2, and 22.15.3. $\square$

Lemma 22.15.5. *In Situation 22.15.1. The morphism $\pi : \underline{Proj}_S(\mathcal{A}) \to S$ of Lemma 22.15.4 comes with the following additional structure. There exists a quasi-coherent $\mathbf{Z}$-graded sheaf of $\mathcal{O}_{\underline{Proj}_S(\mathcal{A})}$-algebras $\bigoplus_{n\in\mathbf{Z}} \mathcal{O}_{\underline{Proj}_S(\mathcal{A})}(n)$, and a morphism of graded $\mathcal{O}_S$-algebras*

$$\psi : \mathcal{A} \longrightarrow \bigoplus\nolimits_{n\geq 0} \pi_*\left(\mathcal{O}_{\underline{Proj}_S(\mathcal{A})}(n)\right)$$

uniquely determined by the following property: For every affine open $U \subset S$ with $A = \mathcal{A}(U)$ there is an isomorphism

$$\theta_U : i_U^*\left(\bigoplus\nolimits_{n\in\mathbf{Z}} \mathcal{O}_{Proj(A)}(n)\right) \longrightarrow \left(\bigoplus\nolimits_{n\in\mathbf{Z}} \mathcal{O}_{\underline{Proj}_S(\mathcal{A})}(n)\right)|_{\pi^{-1}(U)}$$

of $\mathbf{Z}$-graded $\mathcal{O}_{\pi^{-1}(U)}$-algebras such that

$$\begin{tikzcd} A_n \arrow[rr, "\psi"] \arrow[dr, "(22.10.1.3)"'] & & \Gamma(\pi^{-1}(U), \mathcal{O}_{\underline{Proj}_S(\mathcal{A})}(n)) \\ & \Gamma(Proj(A), \mathcal{O}_{Proj(A)}(n)) \arrow[ur, "\theta_U"'] & \end{tikzcd}$$

is commutative.

Proof. We are going to use Lemma 22.2.2 to glue the sheaves of $\mathbf{Z}$-graded algebras $\bigoplus_{n\in\mathbf{Z}} \mathcal{O}_{\text{Proj}(A)}(n)$ for $A = \mathcal{A}(U)$, $U \subset S$ affine open over the scheme $\underline{\text{Proj}}_S(\mathcal{A})$. We have constructed the data necessary for this in Lemma 22.15.2 and we have checked condition (d) of Lemma 22.2.2 in Lemma 22.15.3. Hence we get the sheaf of $\mathbf{Z}$-graded $\mathcal{O}_{\underline{\text{Proj}}_S(\mathcal{A})}$-algebras $\bigoplus_{n\in\mathbf{Z}} \mathcal{O}_{\underline{\text{Proj}}_S(\mathcal{A})}(n)$

together with the isomorphisms θ_U for all $U \subset S$ affine open and all $n \in \mathbf{Z}$. For every affine open $U \subset S$ with $A = \mathcal{A}(U)$ we have a map $A \to \Gamma(\text{Proj}(A), \bigoplus_{n \geq 0} \mathcal{O}_{\text{Proj}(A)}(n))$. Hence the map ψ exists by functoriality of relative glueing, see Remark 22.2.3. The diagram of the lemma commutes by construction. This characterizes the sheaf of $\mathbf{Z}$-graded $\mathcal{O}_{\underline{\text{Proj}}_S(\mathcal{A})}$-algebras $\bigoplus \mathcal{O}_{\underline{\text{Proj}}_S(\mathcal{A})}(n)$ because the proof of Lemma 22.11.1 shows that having these diagrams commute uniquely determines the maps θ_U. Some details omitted. $\square$

22.16. Relative Proj as a functor

We place ourselves in Situation 22.15.1. So S is a scheme and $\mathcal{A} = \bigoplus_{d \geq 0} \mathcal{A}_d$ is a quasi-coherent graded $\mathcal{O}_S$-algebra. In this section we relativize the construction of Proj by constructing a functor which the relative homogeneous spectrum will represent. As a result we will construct a morphism of schemes

$$\underline{\text{Proj}}_S(\mathcal{A}) \longrightarrow S$$

which above affine opens of S will look like the homogeneous spectrum of a graded ring. The discussion will be modeled after our discussion of the relative spectrum in Section 22.4. The easier method using glueing schemes of the form Proj(A), $A = \Gamma(U, \mathcal{A})$, $U \subset S$ affine open, is explained in Section 22.15, and the result in this section will be shown to be isomorphic to that one.

Fix for the moment an integer $d \geq 1$. We denote $\mathcal{A}^{(d)} = \bigoplus_{n \geq 0} \mathcal{A}_{nd}$ similarly to the notation in Algebra, Section 7.52. Let T be a scheme. Let us consider *quadruples* $(d, f : T \to S, \mathcal{L}, \psi)$ *over* T where

(1) d is the integer we fixed above,
(2) $f : T \to S$ is a morphism of schemes,
(3) $\mathcal{L}$ is an invertible $\mathcal{O}_T$-module, and
(4) $\psi : f^*\mathcal{A}^{(d)} \to \bigoplus_{n \geq 0} \mathcal{L}^{\otimes n}$ is a homomorphism of graded $\mathcal{O}_T$-algebras such that $f^*\mathcal{A}_d \to \mathcal{L}$ is surjective.

Given a morphism $h : T' \to T$ and a quadruple $(d, f, \mathcal{L}, \psi)$ over T we can pull it back to the quadruple $(d, f \circ h, h^*\mathcal{L}, h^*\psi)$ over T'. Given two quadruples $(d, f, \mathcal{L}, \psi)$ and $(d, f', \mathcal{L}', \psi')$ over T with the same integer d we say they are *strictly equivalent* if $f = f'$ and there exists an isomorphism $\beta : \mathcal{L} \to \mathcal{L}'$ such that $\beta \circ \psi = \psi'$ as graded $\mathcal{O}_T$-algebra maps $f^*\mathcal{A}^{(d)} \to \bigoplus_{n \geq 0} (\mathcal{L}')^{\otimes n}$.

For each integer $d \geq 1$ we define

$$\begin{array}{rcl} F_d : Sch^{opp} & \longrightarrow & Sets, \\ T & \longmapsto & \{\text{strict equivalence classes of } (d, f : T \to S, \mathcal{L}, \psi) \text{ as above}\} \end{array}$$

with pullbacks as defined above.

Lemma 22.16.1. *In Situation 22.15.1. Let $d \geq 1$. Let F_d be the functor associated to $(S, \mathcal{A})$ above. Let $g : S' \to S$ be a morphism of schemes. Set $\mathcal{A}' = g^*\mathcal{A}$. Let F'_d be the functor associated to $(S', \mathcal{A}')$ above. Then there is a canonical isomorphism*

$$F'_d \cong h_{S'} \times_{h_S} F_d$$

of functors.

Proof. A quadruple $(d, f' : T \to S', \mathcal{L}', \psi' : (f')^*(\mathcal{A}')^{(d)} \to \bigoplus_{n \geq 0} (\mathcal{L}')^{\otimes n})$ is the same as a quadruple $(d, f, \mathcal{L}, \psi : f^*\mathcal{A}^{(d)} \to \bigoplus_{n \geq 0} \mathcal{L}^{\otimes n})$ together with a factorization of f as

$f = g \circ f'$. Namely, the correspondence is $f = g \circ f'$, $\mathcal{L} = \mathcal{L}'$ and $\psi = \psi'$ via the identifications $(f')^*(\mathcal{A}')^{(d)} = (f')^* g^*(\mathcal{A}^{(d)}) = f^*\mathcal{A}^{(d)}$. Hence the lemma. $\square$

Lemma 22.16.2. *In Situation 22.15.1. Let F_d be the functor associated to $(d, S, \mathcal{A})$ above. If S is affine, then F_d is representable by the open subscheme U_d (22.12.0.1) of the scheme $Proj(\Gamma(S, \mathcal{A}))$.*

Proof. Write $S = Spec(R)$ and $A = \Gamma(S, \mathcal{A})$. Then A is a graded R-algebra and $\mathcal{A} = \widetilde{A}$. To prove the lemma we have to identify the functor F_d with the functor $F_d^{triples}$ of triples defined in Section 22.12.

Let $(d, f : T \to S, \mathcal{L}, \psi)$ be a quadruple. We may think of ψ as a $\mathcal{O}_S$-module map $\mathcal{A}^{(d)} \to \bigoplus_{n \geq 0} f_*\mathcal{L}^{\otimes n}$. Since $\mathcal{A}^{(d)}$ is quasi-coherent this is the same thing as an R-linear homomorphism of graded rings $A^{(d)} \to \Gamma(S, \bigoplus_{n \geq 0} f_*\mathcal{L}^{\otimes n})$. Clearly, $\Gamma(S, \bigoplus_{n \geq 0} f_*\mathcal{L}^{\otimes n}) = \Gamma_*(T, \mathcal{L})$. Thus we may associate to the quadruple the triple $(d, \mathcal{L}, \psi)$.

Conversely, let $(d, \mathcal{L}, \psi)$ be a triple. The composition $R \to A_0 \to \Gamma(T, \mathcal{O}_T)$ determines a morphism $f : T \to S = Spec(R)$, see Schemes, Lemma 21.6.4. With this choice of f the map $A^{(d)} \to \Gamma(S, \bigoplus_{n \geq 0} f_*\mathcal{L}^{\otimes n})$ is R-linear, and hence corresponds to a ψ which we can use for a quadruple $(d, f : T \to S, \mathcal{L}, \psi)$. We omit the verification that this establishes an isomorphism of functors $F_d = F_d^{triples}$. $\square$

Lemma 22.16.3. *In Situation 22.15.1. The functor F_d is representable by a scheme.*

Proof. We are going to use Schemes, Lemma 21.15.4.

First we check that F_d satisfies the sheaf property for the Zariski topology. Namely, suppose that T is a scheme, that $T = \bigcup_{i \in I} U_i$ is an open covering, and that $(d, f_i, \mathcal{L}_i, \psi_i) \in F_d(U_i)$ such that $(d, f_i, \mathcal{L}_i, \psi_i)|_{U_i \cap U_j}$ and $(d, f_j, \mathcal{L}_j, \psi_j)|_{U_i \cap U_j}$ are strictly equivalent. This implies that the morphisms $f_i : U_i \to S$ glue to a morphism of schemes $f : T \to S$ such that $f|_{I_i} = f_i$, see Schemes, Section 21.14. Thus $f_i^*\mathcal{A}^{(d)} = f^*\mathcal{A}^{(d)}|_{U_i}$. It also implies there exist isomorphisms $\beta_{ij} : \mathcal{L}_i|_{U_i \cap U_j} \to \mathcal{L}_j|_{U_i \cap U_j}$ such that $\beta_{ij} \circ \psi_i = \psi_j$ on $U_i \cap U_j$. Note that the isomorphisms β_{ij} are uniquely determined by this requirement because the maps $f_i^*\mathcal{A}_d \to \mathcal{L}_i$ are surjective. In particular we see that $\beta_{jk} \circ \beta_{ij} = \beta_{ik}$ on $U_i \cap U_j \cap U_k$. Hence by Sheaves, Section 6.33 the invertible sheaves $\mathcal{L}_i$ glue to an invertible $\mathcal{O}_T$-module $\mathcal{L}$ and the morphisms ψ_i glue to morphism of $\mathcal{O}_T$-algebras $\psi : f^*\mathcal{A}^{(d)} \to \bigoplus_{n \geq 0} \mathcal{L}^{\otimes n}$. This proves that F_d satisfies the sheaf condition with respect to the Zariski topology.

Let $S = \bigcup_{i \in I} U_i$ be an affine open covering. Let $F_{d,i} \subset F_d$ be the subfunctor consisting of those pairs $(f : T \to S, \varphi)$ such that $f(T) \subset U_i$.

We have to show each $F_{d,i}$ is representable. This is the case because $F_{d,i}$ is identified with the functor associated to U_i equipped with the quasi-coherent graded $\mathcal{O}_{U_i}$-algebra $\mathcal{A}|_{U_i}$) by Lemma 22.16.1. Thus the result follows from Lemma 22.16.2.

Next we show that $F_{d,i} \subset F_d$ is representable by open immersions. Let $(f : T \to S, \varphi) \in F_d(T)$. Consider $V_i = f^{-1}(U_i)$. It follows from the definition of $F_{d,i}$ that given $a : T' \to T$ we gave $a^*(f, \varphi) \in F_{d,i}(T')$ if and only if $a(T') \subset V_i$. This is what we were required to show.

Finally, we have to show that the collection $(F_{d,i})_{i \in I}$ covers F_d. Let $(f : T \to S, \varphi) \in F_d(T)$. Consider $V_i = f^{-1}(U_i)$. Since $S = \bigcup_{i \in I} U_i$ is an open covering of S we see that $T = \bigcup_{i \in I} V_i$ is an open covering of T. Moreover $(f, \varphi)|_{V_i} \in F_{d,i}(V_i)$. This finishes the proof of the lemma. $\square$

At this point we can redo the material at the end of Section 22.12 in the current relative setting and define a functor which is representable by $\underline{\text{Proj}}_S(\mathcal{A})$. To do this we introduce the notion of equivalence between two quadruples $(d, f : T \to S, \mathcal{L}, \psi)$ and $(d', f' : T \to S, \mathcal{L}', \psi')$ with possibly different values of the integers d, d'. Namely, we say these are *equivalent* if $f = f'$, and there exists an isomorphism $\beta : \mathcal{L}^{\otimes d'} \to (\mathcal{L}')^{\otimes d}$ such that $\beta \circ \psi|_{f^*\mathcal{A}^{(dd')}} = \psi'|_{f^*\mathcal{A}^{(dd')}}$. The following lemma implies that this defines an equivalence relation. (This is not a complete triviality.)

Lemma 22.16.4. *In Situation 22.15.1. Let T be a scheme. Let $(d, f, \mathcal{L}, \psi)$, $(d', f', \mathcal{L}', \psi')$ be two quadruples over T. The following are equivalent:*

(1) *Let $m = lcm(d, d')$. Write $m = ad = a'd'$. We have $f = f'$ and there exists an isomorphism $\beta : \mathcal{L}^{\otimes a} \to (\mathcal{L}')^{\otimes a'}$ with the property that $\beta \circ \psi|_{f^*\mathcal{A}^{(m)}}$ and $\psi'|_{f^*\mathcal{A}^{(m)}}$ agree as graded ring maps $f^*\mathcal{A}^{(m)} \to \bigoplus_{n \geq 0} (\mathcal{L}')^{\otimes mn}$.*
(2) *The quadruples $(d, f, \mathcal{L}, \psi)$ and $(d', f', \mathcal{L}', \psi')$ are equivalent.*
(3) *We have $f = f'$ and for some positive integer $m = ad = a'd'$ there exists an isomorphism $\beta : \mathcal{L}^{\otimes a} \to (\mathcal{L}')^{\otimes a'}$ with the property that $\beta \circ \psi|_{f^*\mathcal{A}^{(m)}}$ and $\psi'|_{f^*\mathcal{A}^{(m)}}$ agree as graded ring maps $f^*\mathcal{A}^{(m)} \to \bigoplus_{n \geq 0} (\mathcal{L}')^{\otimes mn}$.*

Proof. Clearly (1) implies (2) and (2) implies (3) by restricting to more divisible degrees and powers of invertible sheaves. Assume (3) for some integer $m = ad = a'd'$. Let $m_0 = \text{lcm}(d, d')$ and write it as $m_0 = a_0 d = a_0' d'$. We are given an isomorphism $\beta : \mathcal{L}^{\otimes a} \to (\mathcal{L}')^{\otimes a'}$ with the property described in (3). We want to find an isomorphism $\beta_0 : \mathcal{L}^{\otimes a_0} \to (\mathcal{L}')^{\otimes a_0'}$ having that property as well. Since by assumption the maps $\psi : f^*\mathcal{A}_d \to \mathcal{L}$ and $\psi' : (f')^*\mathcal{A}_{d'} \to \mathcal{L}'$ are surjective the same is true for the maps $\psi : f^*\mathcal{A}_{m_0} \to \mathcal{L}^{\otimes a_0}$ and $\psi' : (f')^*\mathcal{A}_{m_0} \to (\mathcal{L}')^{\otimes a_0}$. Hence if β_0 exists it is uniquely determined by the condition that $\beta_0 \circ \psi = \psi'$. This means that we may work locally on T. Hence we may assume that $f = f' : T \to S$ maps into an affine open, in other words we may assume that S is affine. In this case the result follows from the corresponding result for triples (see Lemma 22.12.4) and the fact that triples and quadruples correspond in the affine base case (see proof of Lemma 22.16.2). $\square$

Suppose $d' = ad$. Consider the transformation of functors $F_d \to F_{d'}$ which assigns to the quadruple $(d, f, \mathcal{L}, \psi)$ over T the quadruple $(d', f, \mathcal{L}^{\otimes a}, \psi|_{f^*\mathcal{A}^{(d')}})$. One of the implications of Lemma 22.16.4 is that the transformation $F_d \to F_{d'}$ is injective! For a quasi-compact scheme T we define

$$F(T) = \bigcup\nolimits_{d \in \mathbf{N}} F_d(T)$$

with transition maps as explained above. This clearly defines a contravariant functor on the category of quasi-compact schemes with values in sets. For a general scheme T we define

$$F(T) = lim_{V \subset T \text{ quasi-compact open}} F(V).$$

In other words, an element ξ of $F(T)$ corresponds to a compatible system of choices of elements $\xi_V \in F(V)$ where V ranges over the quasi-compact opens of T. We omit the definition of the pullback map $F(T) \to F(T')$ for a morphism $T' \to T$ of schemes. Thus we have defined our functor

$$F : Sch^{opp} \longrightarrow Sets \qquad (22.16.4.1)$$

Lemma 22.16.5. *In Situation 22.15.1. The functor F above is representable by a scheme.*

Proof. Let $U_d \to S$ be the scheme representing the functor F_d defined above. Let $\mathcal{L}_d, \psi^d : \pi_d^*\mathcal{A}^{(d)} \to \bigoplus_{n \geq 0} \mathcal{L}_d^{\otimes n}$ be the universal object. If $d|d'$, then we may consider the quadruple $(d', \pi_d, \mathcal{L}_d^{\otimes d'/d}, \psi^d|_{\mathcal{A}^{(d')}})$ which determines a canonical morphism $U_d \to U_{d'}$ over S. By construction this morphism corresponds to the transformation of functors $F_d \to F_{d'}$ defined above.

For every affine open $Spec(R) = V \subset S$ setting $A = \Gamma(V, \mathcal{A})$ we have a canonical identification of the base change $U_{d,V}$ with the corresponding open subscheme of $\text{Proj}(A)$, see Lemma 22.16.2. Moreover, the morphisms $U_{d,V} \to U_{d',V}$ constructed above correspond to the inclusions of opens in $\text{Proj}(A)$. Thus we conclude that $U_d \to U_{d'}$ is an open immersion.

This allows us to construct X by glueing the schemes U_d along the open immersions $U_d \to U_{d'}$. Technically, it is convenient to choose a sequence $d_1|d_2|d_3|\ldots$ such that every positive integer divides one of the d_i and to simply take $X = \bigcup U_{d_i}$ using the open immersions above. It is then a simple matter to prove that X represents the functor F. □

Lemma 22.16.6. *In Situation 22.15.1. The scheme $\pi : \underline{Proj}_S(\mathcal{A}) \to S$ constructed in Lemma 22.15.4 and the scheme representing the functor F are canonically isomorphic as schemes over S.*

Proof. Let X be the scheme representing the functor F. Note that X is a scheme over S since the functor F comes equipped with a natural transformation $F \to h_S$. Write $Y = \underline{\text{Proj}}_S(\mathcal{A})$. We have to show that $X \cong Y$ as S-schemes. We give two arguments.

The first argument uses the construction of X as the union of the schemes U_d representing F_d in the proof of Lemma 22.16.5. Over each affine open of S we can identify X with the homogeneous spectrum of the sections of $\mathcal{A}$ over that open, since this was true for the opens U_d. Moreover, these identifications are compatible with further restrictions to smaller affine opens. On the other hand, Y was constructed by glueing these homogeneous spectra. Hence we can glue these isomorphisms to an isomorphism between X and $\underline{\text{Proj}}_S(\mathcal{A})$ as desired. Details omitted.

Here is the second argument. Lemma 22.15.5 shows that there exists a morphism of graded algebras

$$\psi : \pi^*\mathcal{A} \longrightarrow \bigoplus\nolimits_{n \geq 0} \mathcal{O}_Y(n)$$

over Y which on sections over affine opens of S agrees with (22.10.1.3). Hence for every $y \in Y$ there exists an open neighbourhood $V \subset Y$ of y and an integer $d \geq 1$ such that for $d|n$ the sheaf $\mathcal{O}_Y(n)|_V$ is invertible and the multiplication maps $\mathcal{O}_Y(n)|_V \otimes_{\mathcal{O}_V} \mathcal{O}_Y(m)|_V \to \mathcal{O}_Y(n+m)|_V$ are isomorphisms. Thus ψ restricted to the sheaf $\pi^*\mathcal{A}^{(d)}|_V$ gives an element of $F_d(V)$. Since the opens V cover Y we see ``ψ'' gives rise to an element of $F(Y)$. Hence a canonical morphism $Y \to X$ over S. Because this construction is completely canonical to see that it is an isomorphism we may work locally on S. Hence we reduce to the case S affine where the result is clear. □

Definition 22.16.7. Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent sheaf of graded $\mathcal{O}_S$-algebras. The *relative homogeneous spectrum of $\mathcal{A}$ over S*, or the *homogeneous spectrum of $\mathcal{A}$ over S*, or the *relative Proj of $\mathcal{A}$ over S* is the scheme constructed in Lemma 22.15.4 which represents the functor F (22.16.4.1), see Lemma 22.16.6. We denote it $\pi : \underline{\text{Proj}}_S(\mathcal{A}) \to S$.

The relative Proj comes equipped with a quasi-coherent sheaf of $\mathbf{Z}$-graded algebras $\bigoplus_{n\in\mathbf{Z}} \mathcal{O}_{\underline{\text{Proj}}_S(\mathcal{A})}(n)$ (the twists of the structure sheaf) and a ``universal'' homomorphism of graded algebras

$$\psi_{univ} : \mathcal{A} \longrightarrow \pi_*\left(\bigoplus\nolimits_{n\geq 0} \mathcal{O}_{\underline{\text{Proj}}_S(\mathcal{A})}(n)\right)$$

see Lemma 22.15.5. We may also think of this as a homomorphism

$$\psi_{univ} : \pi^*\mathcal{A} \longrightarrow \bigoplus\nolimits_{n\geq 0} \mathcal{O}_{\underline{\text{Proj}}_S(\mathcal{A})}(n)$$

if we like. The following lemma is a formulation of the universality of this object.

Lemma 22.16.8. *In Situation 22.15.1. Let $(f : T \to S, d, \mathcal{L}, \psi)$ be a quadruple. Let $r_{d,\mathcal{L},\psi} : T \to \underline{Proj}_S(\mathcal{A})$ be the associated S-morphism. There exists an isomorphism of $\mathbf{Z}$-graded $\mathcal{O}_T$-algebras*

$$\theta : r^*_{d,\mathcal{L},\psi}\left(\bigoplus\nolimits_{n\in\mathbf{Z}} \mathcal{O}_{\underline{Proj}_S(\mathcal{A})}(nd)\right) \longrightarrow \bigoplus\nolimits_{n\in\mathbf{Z}} \mathcal{L}^{\otimes n}$$

such that the following diagram commutes

$$\begin{tikzcd}
\mathcal{A}^{(d)} \arrow[rr, "\psi"] \arrow[dr, "\psi_{univ}"'] & & f_*\left(\bigoplus_{n\in\mathbf{Z}} \mathcal{L}^{\otimes n}\right) \\
& \pi_*\left(\bigoplus_{n\geq 0} \mathcal{O}_{\underline{Proj}_S(\mathcal{A})}(nd)\right) \arrow[ur, "\theta"'] &
\end{tikzcd}$$

The commutativity of this diagram uniquely determines θ.

Proof. Note that the quadruple $(f : T \to S, d, \mathcal{L}, \psi)$ defines an element of $F_d(T)$. Let $U_d \subset \underline{\text{Proj}}_S(\mathcal{A})$ be the locus where the sheaf $\mathcal{O}_{\underline{\text{Proj}}_S(\mathcal{A})}(d)$ is invertible and generated by the image of $\psi_{univ} : \pi^*\mathcal{A}_d \to \mathcal{O}_{\underline{\text{Proj}}_S(\mathcal{A})}(d)$. Recall that U_d represents the functor F_d, see the proof of Lemma 22.16.5. Hence the result will follow if we can show the quadruple $(U_d \to S, d, \mathcal{O}_{U_d}(d), \psi_{univ}|_{\mathcal{A}^{(d)}})$ is the universal family, i.e., the representing object in $F_d(U_d)$. We may do this after restricting to an affine open of S because (a) the formation of the functors F_d commutes with base change (see Lemma 22.16.1), and (b) the pair $(\bigoplus_{n\in\mathbf{Z}} \mathcal{O}_{\underline{\text{Proj}}_S(\mathcal{A})}(n), \psi_{univ})$ is constructed by glueing over affine opens in S (see Lemma 22.15.5). Hence we may assume that S is affine. In this case the functor of quadruples F_d and the functor of triples F_d agree (see proof of Lemma 22.16.2) and moreover Lemma 22.12.2 shows that $(d, \mathcal{O}_{U_d}(d), \psi^d)$ is the universal triple over U_d. Going backwards through the identifications in the proof of Lemma 22.16.2 shows that $(U_d \to S, d, \mathcal{O}_{U_d}(d), \psi_{univ}|_{\mathcal{A}^{(d)}})$ is the universal quadruple as desired. $\square$

Lemma 22.16.9. *Let S be a scheme and $\mathcal{A}$ be a quasi-coherent sheaf of graded $\mathcal{O}_S$-algebras. The morphism $\pi : \underline{Proj}_S(\mathcal{A}) \to S$ is separated.*

Proof. To prove a morphism is separated we may work locally on the base, see Schemes, Section 21.21. By construction $\underline{\text{Proj}}_S(\mathcal{A})$ is over any affine $U \subset S$ isomorphic to $\text{Proj}(A)$ with $A = \mathcal{A}(U)$. By Lemma 22.8.8 we see that $\text{Proj}(A)$ is separated. Hence $\text{Proj}(A) \to U$ is separated (see Schemes, Lemma 21.21.14) as desired. $\square$

Lemma 22.16.10. *Let S be a scheme and $\mathcal{A}$ be a quasi-coherent sheaf of graded $\mathcal{O}_S$-algebras. Let $g : S' \to S$ be any morphism of schemes. Then there is a canonical isomorphism*

$$\underline{Proj}_{S'}(g^*\mathcal{A}) \longrightarrow S' \times_S \underline{Proj}_S(\mathcal{A})$$

Proof. This follows from Lemma 22.16.1 and the construction of $\underline{\text{Proj}}_S(\mathcal{A})$ in Lemma 22.16.5 as the union of the schemes U_d representing the functors F_d. $\square$

Lemma 22.16.11. *Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent sheaf of graded $\mathcal{O}_S$-modules generated as an $\mathcal{A}_0$-algebra by $\mathcal{A}_1$. In this case the scheme $X = \underline{Proj}_S(\mathcal{A})$ represents the functor F_1 which associates to a scheme $f : T \to S$ over S the set of pairs $(\mathcal{L}, \psi)$, where*

1. *$\mathcal{L}$ is an invertible $\mathcal{O}_T$-module, and*
2. *$\psi : f^*\mathcal{A} \to \bigoplus_{n\geq 0} \mathcal{L}^{\otimes n}$ is a graded $\mathcal{O}_T$-algebra homomorphism such that $f^*\mathcal{A}_1 \to \mathcal{L}$ is surjective*

up to strict equivalence as above. Moreover, in this case all the quasi-coherent sheaves $\mathcal{O}_{\underline{Proj}(\mathcal{A})}(n)$ are invertible $\mathcal{O}_{\underline{Proj}(\mathcal{A})}$-modules and the multiplication maps induce isomorphsms $\mathcal{O}_{\underline{Proj}(\mathcal{A})}(n) \otimes_{\mathcal{O}_{\underline{Proj}(\mathcal{A})}} \mathcal{O}_{\underline{Proj}(\mathcal{A})}(m) = \mathcal{O}_{\underline{Proj}(\mathcal{A})}(n+m)$.

Proof. Under the assumptions of the lemma the sheaves $\mathcal{O}_{\underline{\text{Proj}}(\mathcal{A})}(n)$ are invertible and the multiplication maps isomorphisms by Lemma 22.16.5 and Lemma 22.12.3 over affine opens of S. Thus X actually represents the functor F_1, see proof of Lemma 22.16.5. $\square$

22.17. Quasi-coherent sheaves on relative Proj

We briefly discuss how to deal with graded modules in the relative setting.

We plave ourselves in Situation 22.15.1. So S is a scheme, and $\mathcal{A}$ is a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $\mathcal{M} = \bigoplus_{n\in\mathbf{Z}} \mathcal{M}_n$ be a graded $\mathcal{A}$-module, quasi-coherent as an $\mathcal{O}_S$-module. We are going to describe the associated quasi-coherent sheaf of modules on $\underline{\text{Proj}}_S(\mathcal{A})$. We first describe the value of this sheaf schemes T mapping into the relative Proj.

Let T be a scheme. Let $(d, f : T \to S, \mathcal{L}, \psi)$ be a quadruple over T, as in Section 22.16. We define a quasi-coherent sheaf $\widetilde{\mathcal{M}}_T$ of $\mathcal{O}_T$-modules as follows

$$\widetilde{\mathcal{M}}_T = \left(f^*\mathcal{M}^{(d)} \otimes_{f^*\mathcal{A}^{(d)}} \left(\bigoplus\nolimits_{n\in\mathbf{Z}} \mathcal{L}^{\otimes n}\right)\right)_0 \quad (22.17.0.1)$$

So $\widetilde{\mathcal{M}}_T$ is the degree 0 part of the tensor product of the graded $f^*\mathcal{A}^{(d)}$-modules $\mathcal{M}^{(d)}$ and $\bigoplus_{n\in\mathbf{Z}} \mathcal{L}^{\otimes n}$. Note that the sheaf $\widetilde{\mathcal{M}}_T$ depends on the quadruple even though we suppressed this in the notation. This construction has the pleasing property that given any morphism $g : T' \to T$ we have $\widetilde{\mathcal{M}}_{T'} = g^*\widetilde{\mathcal{M}}_T$ where $\widetilde{\mathcal{M}}_{T'}$ denotes the quasi-coherent sheaf associated to the pullback quadruple $(d, f \circ g, g^*\mathcal{L}, g^*\psi)$.

Since all sheaves in (22.17.0.1) are quasi-coherent we can spell out the construction over an affine open $Spec(C) = V \subset T$ which maps into an affine open $Spec(R) = U \subset S$. Namely, suppose that $\mathcal{A}|_U$ corresponds to the graded R-algebra A, that $\mathcal{M}|_U$ corresponds to the graded A-module M, and that $\mathcal{L}|_V$ corresponds to the invertible C-module L. The map ψ gives rise to a graded R-algebra map $\gamma : A^{(d)} \to \bigoplus_{n\geq 0} L^{\otimes n}$. (Tensor powers of L over C.) Then $(\widetilde{\mathcal{M}}_T)|_V$ is the quasi-coherent sheaf associated to the C-module

$$N_{R,C,A,M,\gamma} = \left(M^{(d)} \otimes_{A^{(d)},\gamma} \left(\bigoplus\nolimits_{n\in\mathbf{Z}} L^{\otimes n}\right)\right)_0$$

By assumption we may even cover T by affine opens V such that there exists some $a \in A_d$ such that $\gamma(a) \in L$ is a C-basis for the module L. In that case any element of $N_{R,C,A,M,\gamma}$ is a sum of pure tensors $\sum m_i \otimes \gamma(a)^{-n_i}$ with $m \in M_{n_i d}$. In fact we may multiply each m_i with a suitable positive power of a and collect terms to see that each element of $N_{R,C,A,M,\gamma}$ can be written as $m \otimes \gamma(a)^{-n}$ with $m \in M_{nd}$ and $n \gg 0$. In other words we see that in this case

$$N_{R,C,A,M,\gamma} = M_{(a)} \otimes_{A_{(a)}} C$$

where the map $A_{(a)} \to C$ is the map $x/a^n \mapsto \gamma(x)/\gamma(a)^n$. In other words, this is the value of $\widetilde{M}$ on $D_+(a) \subset \text{Proj}(A)$ pulled back to $Spec(C)$ via the morphism $Spec(C) \to D_+(a)$ coming from γ.

Lemma 22.17.1. *In Situation 22.15.1. For any quasi-coherent sheaf of graded $\mathcal{A}$-modules $\mathcal{M}$ on S, there exists a canonical associated sheaf of $\mathcal{O}_{\underline{Proj}_S(\mathcal{A})}$-modules $\widetilde{\mathcal{M}}$ with the following properties:*

(1) *Given a scheme T and a quadruple $(T \to S, d, \mathcal{L}, \psi)$ over T corresponding to a morphism $h : T \to \underline{Proj}_S(\mathcal{A})$ there is a canonical isomorphism $\widetilde{\mathcal{M}}_T = h^*\widetilde{\mathcal{M}}$ where $\widetilde{\mathcal{M}}_T$ is defined by (22.17.0.1).*
(2) *The isomorphisms of (1) are compatible with pullbacks.*
(3) *There is a canonical map*

$$\pi^*\mathcal{M}_0 \longrightarrow \widetilde{\mathcal{M}}.$$

(4) *The construction $\mathcal{M} \mapsto \widetilde{\mathcal{M}}$ is functorial in $\mathcal{M}$.*
(5) *The construction $\mathcal{M} \mapsto \widetilde{\mathcal{M}}$ is exact.*
(6) *There are canonical maps*

$$\widetilde{\mathcal{M}} \otimes_{\mathcal{O}_{\underline{Proj}_S(\mathcal{A})}} \widetilde{\mathcal{N}} \longrightarrow \widetilde{\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}}$$

as in Lemma 22.9.1.
(7) *There exist canonical maps*

$$\pi^*\mathcal{M} \longrightarrow \bigoplus\nolimits_{n \in \mathbf{Z}} \widetilde{\mathcal{M}(n)}$$

generalizing (22.10.1.6).
(8) *The formation of $\widetilde{\mathcal{M}}$ commutes with base change.*

Proof. Omitted. We should split this lemma into parts and prove the parts separately. □

22.18. Invertible sheaves and morphisms into relative Proj

It seems that we may need the following lemma somewhere. The situation is the following:

(1) Let S be a scheme.
(2) Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra.
(3) Denote $\pi : \underline{\text{Proj}}_S(\mathcal{A}) \to S$ the relative homogeneous spectrum over S.
(4) Let $f : X \to S$ be a morphism of schemes.
(5) Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module.
(6) Let $\psi : f^*\mathcal{A} \to \bigoplus_{d \geq 0} \mathcal{L}^{\otimes d}$ be a homomorphism of graded $\mathcal{O}_X$-algebras.

Given this data set

$$U(\psi) = \bigcup\nolimits_{(U,V,a)} U_{\psi(a)}$$

where (U, V, a) satisfies:

(1) $V \subset S$ affine open,
(2) $U = f^{-1}(V)$, and
(3) $a \in \mathcal{A}(V)_+$ is homogeneous.

Namely, then $\psi(a) \in \Gamma(U, \mathcal{L}^{\otimes \deg(a)})$ and $U_{\psi(a)}$ is the corresponding open (see Modules, Lemma 15.21.7).

Lemma 22.18.1. *With assumptions and notation as above. The morphism ψ induces a canonical morphism of schemes over S*

$$r_{\mathcal{L},\psi} : U(\psi) \longrightarrow \underline{Proj}_S(\mathcal{A})$$

together with a map of graded $\mathcal{O}_{U(\psi)}$-algebras

$$\theta : r^*_{\mathcal{L},\psi}\left(\bigoplus\nolimits_{d\geq 0} \mathcal{O}_{\underline{Proj}_S(\mathcal{A})}(d)\right) \longrightarrow \bigoplus\nolimits_{d\geq 0} \mathcal{L}^{\otimes d}|_{U(\psi)}$$

characterized by the following properties:

(1) *For every open $V \subset S$ and every $d \geq 0$ the diagram*

$$\begin{CD} \mathcal{A}_d(V) @>{\psi}>> \Gamma(f^{-1}(V), \mathcal{L}^{\otimes d}) \\ @V{\psi}VV @VV{restrict}V \\ \Gamma(\pi^{-1}(V), \mathcal{O}_{\underline{Proj}_S(\mathcal{A})}(d)) @>{\theta}>> \Gamma(f^{-1}(V) \cap U(\psi), \mathcal{L}^{\otimes d}) \end{CD}$$

is commutative.

(2) *For any $d \geq 1$ and any open subscheme $W \subset X$ such that $\psi|_W : f^*\mathcal{A}_d|_W \to \mathcal{L}^{\otimes d}|_W$ is surjective the restriction of the morphism $r_{\mathcal{L},\psi}$ agrees with the morphism $W \to \underline{Proj}_S(\mathcal{A})$ which exists by the construction of the relative homogeneous spectrum, see Definition 22.16.7.*

(3) *For any affine open $V \subset S$, the restriction*

$$(U(\psi) \cap f^{-1}(V), r_{\mathcal{L},\psi}|_{U(\psi)\cap f^{-1}(V)}, \theta|_{U(\psi)\cap f^{-1}(V)})$$

agrees via i_V (see Lemma 22.15.4) with the triple $(U(\psi'), r_{\mathcal{L},\psi'}, \theta')$ of Lemma 22.14.1 associated to the map $\psi' : A = \mathcal{A}(V) \to \Gamma_*(f^{-1}(V), \mathcal{L}|_{f^{-1}(V)})$ induced by ψ.

Proof. Use characterization (3) to construct the morphism $r_{\mathcal{L},\psi}$ and θ locally over S. Use the uniqueness of Lemma 22.14.1 to show that the construction glues. Details omitted. $\square$

22.19. Twisting by invertible sheaves and relative Proj

Let S be a scheme. Let $\mathcal{A} = \bigoplus_{d\geq 0} \mathcal{A}_d$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $\mathcal{L}$ be an invertible sheaf on S. In this situation we obtain another quasi-coherent graded $\mathcal{O}_S$-algebra, namely

$$\mathcal{B} = \bigoplus\nolimits_{d\geq 0} \mathcal{A}_d \otimes_{\mathcal{O}_S} \mathcal{L}^{\otimes d}$$

It turns out that $\mathcal{A}$ and $\mathcal{B}$ have isomorphic relative homogeneous spectra.

Lemma 22.19.1. *With notation S, $\mathcal{A}$, $\mathcal{L}$ and $\mathcal{B}$ as above. There is a canonical isomorphism*

$$\begin{tikzcd} P = \underline{Proj}_S(\mathcal{A}) \arrow[rr, "g"] \arrow[dr, "\pi"'] & & \underline{Proj}_S(\mathcal{B}) = P' \arrow[dl, "\pi'"] \\ & S & \end{tikzcd}$$

with the following properties

(1) *There are isomorphisms* $\theta_n : g^*\mathcal{O}_{P'}(n) \to \mathcal{O}_P(n) \otimes \pi^*\mathcal{L}^{\otimes n}$ *which fit together to give an isomorphism of* $\mathbf{Z}$*-graded algebras*

$$\theta : g^*\left(\bigoplus\nolimits_{n\in\mathbf{Z}} \mathcal{O}_{P'}(n)\right) \longrightarrow \bigoplus\nolimits_{n\in\mathbf{Z}} \mathcal{O}_P(n) \otimes \pi^*\mathcal{L}^{\otimes n}$$

(2) *For every open* $V \subset S$ *the diagrams*

$$\begin{tikzcd} \mathcal{A}_n(V) \otimes \mathcal{L}^{\otimes n}(V) \arrow[rr, "multiply"'] \arrow[d, "\psi\otimes\pi^*"] & & \mathcal{B}_n(V) \arrow[dd, "\psi"] \\ \Gamma(\pi^{-1}V, \mathcal{O}_P(n)) \otimes \Gamma(\pi^{-1}V, \pi^*\mathcal{L}^{\otimes n}) \arrow[d, "multiply"] & & \\ \Gamma(\pi^{-1}V, \mathcal{O}_P(n) \otimes \pi^*\mathcal{L}^{\otimes n}) & & \Gamma(\pi'^{-1}V, \mathcal{O}_{P'}(n)) \arrow[ll, "\theta_n"'] \end{tikzcd}$$

are commutative.

(3) *Add more here as necessary.*

Proof. This is the identity map when $\mathcal{L} \cong \mathcal{O}_S$. In general choose an open covering of S such that $\mathcal{L}$ is trivialized over the pieces and glue the corresponding maps. Details omitted. □

22.20. Projective bundles

Let S be a scheme. Let $\mathcal{E}$ be a quasi-coherent sheaf of $\mathcal{O}_S$-modules. By Modules, Lemma 15.18.6 the symmetric algebra $\mathrm{Sym}(\mathcal{E})$ of $\mathcal{E}$ over $\mathcal{O}_S$ is a quasi-coherent sheaf of $\mathcal{O}_S$-algebras. Note that it is generated in degree 1 over $\mathcal{O}_S$. Hence it makes sense to apply the construction of the previous section to it, specifically Lemmas 22.16.5 and 22.16.11.

Definition 22.20.1. Let S be a scheme. Let $\mathcal{E}$ be a quasi-coherent $\mathcal{O}_S$-module[3]. We denote

$$\pi : \mathbf{P}(\mathcal{E}) = \underline{\mathrm{Proj}}_S(\mathrm{Sym}(\mathcal{E})) \longrightarrow S$$

and we call it the *projective bundle associated to* $\mathcal{E}$. The symbol $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(n)$ indicates the invertible $\mathcal{O}_{\mathbf{P}(\mathcal{E})}$-modules introduced in Lemma 22.16.5 and is called the nth *twist of the structure sheaf*.

Note that according to Lemma 22.16.5 there are canonical $\mathcal{O}_S$-module homomorphisms

$$\mathrm{Sym}^n(\mathcal{E}) \longrightarrow \pi_*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(n))$$

for all $n \geq 0$. This, combined with the fact that $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ is the canonical relatively ample invertible sheaf on $\mathbf{P}(\mathcal{E})$, is a good way to remember how we have normalized our construction of $\mathbf{P}(\mathcal{E})$. Namely, in some references the space $\mathbf{P}(\mathcal{E})$ is only defined for $\mathcal{E}$ finite locally free on S, and sometimes $\mathbf{P}(\mathcal{E})$ is actually defined as our $\mathbf{P}(\mathcal{E}^\wedge)$ where $\mathcal{E}^\wedge$ is the dual of the sheaf $\mathcal{E}$.

Example 22.20.2. The map $\mathrm{Sym}^n(\mathcal{E}) \to \pi_*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(n))$ is an isomorphism if $\mathcal{E}$ is locally free, but in general need not be an isomorphism. In fact we will give an example where this map is not injective for $n = 1$. Set $S = Spec(A)$ with

$$A = k[u, v, s_1, s_2, t_1, t_2]/I$$

[3]The reader may expect here the condition that $\mathcal{E}$ is finite locally free. We do not do so in order to be consistent with [**DG67**, II, Definition 4.1.1].

where k is a field and

$$I = (-us_1 + vt_1 + ut_2, vs_1 + us_2 - vt_2, vs_2, ut_1).$$

Denote $\overline{u}$ the class of u in A and similarly for the other variables. Let $M = (Ax \oplus Ay)/A(\overline{u}x + \overline{v}y)$ so that

$$\mathrm{Sym}(M) = A[x, y]/(\overline{u}x + \overline{v}y) = k[x, y, u, v, s_1, s_2, t_1, t_2]/J$$

where

$$J = (-us_1 + vt_1 + ut_2, vs_1 + us_2 - vt_2, vs_2, ut_1, ux + vy).$$

In this case the projective bundle associated to the quasi-coherent sheaf $\mathcal{E} = \widetilde{M}$ on $S = Spec(A)$ is the scheme

$$P = \mathrm{Proj}(\mathrm{Sym}(M)).$$

Note that this scheme as an affine open covering $P = D_+(x) \cup D_+(y)$. Consider the element $m \in M$ which is the image of the element $us_1x + vt_2y$. Note that

$$x(us_1x + vt_2y) = (s_1x + s_2y)(ux + vy) \bmod I$$

and

$$y(us_1x + vt_2y) = (t_1x + t_2y)(ux + vy) \bmod I.$$

The first equation implies that m maps to zero as a section of $\mathcal{O}_P(1)$ on $D_+(x)$ and the second that it maps to zero as a section of $\mathcal{O}_P(1)$ on $D_+(y)$. This shows that m maps to zero in $\Gamma(P, \mathcal{O}_P(1))$. On the other hand we claim that $m \neq 0$, so that m gives an example of a nonzero global section of $\mathcal{E}$ mapping to zero in $\Gamma(P, \mathcal{O}_P(1))$. Assume $m = 0$ to get a contradiction. In this case there exists an element $f \in k[u, v, s_1, s_2, t_1, t_2]$ such that

$$us_1x + vt_2y = f(ux + vy) \bmod I$$

Since I is generated by homogeneous polynomials of degree 2 we may decompose f into its homogeneous components and take the degree 1 component. In other words we may assume that

$$f = au + bv + \alpha_1 s_1 + \alpha_2 s_2 + \beta_1 t_1 + \beta_2 t_2$$

for some $a, b, \alpha_1, \alpha_2, \beta_1, \beta_2 \in k$. The resulting conditions are that

$$\begin{aligned} us_1 - u(au + bv + \alpha_1 s_1 + \alpha_2 s_2 + \beta_1 t_1 + \beta_2 t_2) &\in I \\ vt_2 - v(au + bv + \alpha_1 s_1 + \alpha_2 s_2 + \beta_1 t_1 + \beta_2 t_2) &\in I \end{aligned}$$

There are no terms u^2, uv, v^2 in the generators of I and hence we see $a = b = 0$. Thus we get the relations

$$\begin{aligned} us_1 - u(\alpha_1 s_1 + \alpha_2 s_2 + \beta_1 t_1 + \beta_2 t_2) &\in I \\ vt_2 - v(\alpha_1 s_1 + \alpha_2 s_2 + \beta_1 t_1 + \beta_2 t_2) &\in I \end{aligned}$$

We may use the first generator of I to replace any occurence of us_1 by $vt_1 + ut_2$, the second generator of I to replace any occurence of vs_1 by $-us_2 + vt_2$, the third generator to remove occurences of vs_2 and the third to remove occurences of ut_1. Then we get the relations

$$\begin{aligned} (1 - \alpha_1)vt_1 + (1 - \alpha_1)ut_2 - \alpha_2 us_2 - \beta_2 ut_2 &= 0 \\ (1 - \alpha_1)vt_2 + \alpha_1 us_2 - \beta_1 vt_1 - \beta_2 vt_2 &= 0 \end{aligned}$$

This implies that α_1 should be both 0 and 1 which is a contradiction as desired.

Lemma 22.20.3. *Let S be a scheme. The structure morphism $\mathbf{P}(\mathcal{E}) \to S$ of a projective bundle over S is separated.*

Proof. Immediate from Lemma 22.16.9. $\square$

Lemma 22.20.4. *Let S be a scheme. Let $n \geq 0$. Then $\mathbf{P}^n_S$ is a projective bundle over S.*

Proof. Note that

$$\mathbf{P}^n_{\mathbf{Z}} = \text{Proj}(\mathbf{Z}[T_0, \ldots, T_n]) = \underline{\text{Proj}}_{Spec(\mathbf{Z})}\left(\widetilde{\mathbf{Z}[T_0, \ldots, T_n]}\right)$$

where the grading on the ring $\mathbf{Z}[T_0, \ldots, T_n]$ is given by $\deg(T_i) = 1$ and the elements of $\mathbf{Z}$ are in degree 0. Recall that $\mathbf{P}^n_S$ is defined as $\mathbf{P}^n_{\mathbf{Z}} \times_{Spec(\mathbf{Z})} S$. Moreover, forming the relative homogeneous spectrum commutes with base change, see Lemma 22.16.10. For any scheme $g : S \to Spec(\mathbf{Z})$ we have $g^*\mathcal{O}_{Spec(\mathbf{Z})}[T_0, \ldots, T_n] = \mathcal{O}_S[T_0, \ldots, T_n]$. Combining the above we see that

$$\mathbf{P}^n_S = \underline{\text{Proj}}_S(\mathcal{O}_S[T_0, \ldots, T_n]).$$

Finally, note that $\mathcal{O}_S[T_0, \ldots, T_n] = \text{Sym}(\mathcal{O}_S^{\oplus n+1})$. Hence we see that $\mathbf{P}^n_S$ is a projective bundle over S. □

22.21. Blowing up

Definition 22.21.1. Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals, and let $Z \subset X$ be the closed subscheme corresponding to $\mathcal{I}$, see Schemes, Definition 21.10.2. The *blowing up of X along Z*, or the *blowing up of X in the ideal sheaf $\mathcal{I}$* is the morphism

$$b : \underline{\text{Proj}}_X\left(\bigoplus\nolimits_{n \geq 0} \mathcal{I}^n\right) \longrightarrow X$$

We will see later, that blowing up turns any closed subscheme into an effective Cartier divisor, see Divisors, Lemma 26.9.18.

Lemma 22.21.2. *Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. If X is integral, then the blow up X' of X in $\mathcal{I}$ is integral.*

Proof. If A is a domain, and $I \subset A$ an ideal, then $\bigoplus_{n \geq 0} I^n$ is a domain. Details omitted. □

22.22. Other chapters

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CHAPTER 23

Properties of Schemes

23.1. Introduction

In this chapter we introduce some absolute properties of schemes. A foundational reference is [**DG67**].

23.2. Constructible sets

Constructible and locally construcible sets are introduced in Topology, Section 5.10. We may characterize locally constructible subsets of schemes as follows.

Lemma 23.2.1. *Let X be a scheme. A subset E of X is locally constructible in X if and only if $E \cap U$ is constructible in U for every affine open U of X.*

Proof. Assume E is locally constructible. Then there exists an open covering $X = \bigcup U_i$ such that $E \cap U_i$ is constructible in U_i for each i. Let $V \subset X$ be any affine open. We can find a finite open affine covering $V = V_1 \cup \ldots \cup V_m$ such that for each j we have $V_j \subset U_i$ for some $i = i(j)$. By Topology, Lemma 5.10.4 we see that each $E \cap V_j$ is construcible in V_j. Since the inclusions $V_j \to V$ are quasi-compact (see Schemes, Lemma 21.19.2) we conclude that $E \cap V$ is constructible in V by Topology, Lemma 5.10.5. The converse implication is immediate. □

Lemma 23.2.2. *Let X be a quasi-separated scheme. The intersection of any two quasi-compact opens of X is a quasi-compact open of X. Every quasi-compact open of X is retrocompact in X.*

Proof. If U and V are quasi-compact open then $U \cap V = \Delta^{-1}(U \times V)$, where $\Delta : X \to X \times X$ is the diagonal. As X is quasi-separated we see that Δ is quasi-compact. Hence we see that $U \cap V$ is quasi-compact as $U \times V$ is quasi-compact (details omitted; use Schemes, Lemma 21.17.4 to see $U \times V$ is a finite union of affines). The second assertion follows from the first and the definitions. □

Lemma 23.2.3. *Let X be a quasi-compact and quasi-separated scheme. Any locally constructible subset of X is constructible.*

Proof. As X is quasi-compact we can choose a finite affine open covering $X = V_1 \cup \ldots \cup V_m$. As X is quasi-separated each V_i is retrocompact in X by Lemma 23.2.2. Hence by Topology, Lemma 5.10.5 we see that $E \subset X$ is constructible in X if and only if $E \cap V_j$ is constructible in V_j. Thus we win by Lemma 23.2.1. □

23.3. Integral, irreducible, and reduced schemes

Definition 23.3.1. Let X be a scheme. We say X is *integral* if it is nonempty and for every nonempty affine open $Spec(R) = U \subset X$ the ring R is an integral domain.

Lemma 23.3.2. *Let X be a scheme. The following are equivalent.*

(1) *The scheme X is reduced, see Schemes, Definition 21.12.1.*
(2) *There exists an affine open covering $X = \bigcup U_i$ such that each $\Gamma(U_i, \mathcal{O}_X)$ is reduced.*
(3) *For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is reduced.*
(4) *For every open $U \subset X$ the ring $\mathcal{O}_X(U)$ is reduced.*

Proof. See Schemes, Lemmas 21.12.2 and 21.12.3. □

Lemma 23.3.3. *Let X be a scheme. The following are equivalent.*
(1) *The scheme X is irreducible.*
(2) *There exists an affine open covering $X = \bigcup_{i \in I} U_i$ such that I is not empty, U_i is irreducible for all $i \in I$, and $U_i \cap U_j \neq \emptyset$ for all $i, j \in I$.*
(3) *The scheme X is nonempty and every nonempty affine open $U \subset X$ is irreducible.*

Proof. Assume (1). By Schemes, Lemma 21.11.1 we see that X has a unique generic point η. Then $X = \overline{\{\eta\}}$. Hence η is an element of every nonempty affine open $U \subset X$. This implies that $U = \overline{\{\eta\}}$ and that any two nonempty affines meet. Thus (1) implies both (2) and (3).

Assume (2). Suppose $X = Z_1 \cup Z_2$ is a union of two closed subsets. For every i we see that either $U_i \subset Z_1$ or $U_i \subset Z_2$. Pick some $i \in I$ and assume $U_i \subset Z_1$ (possibly after renumbering Z_1, Z_2). For any $j \in I$ the open subset $U_i \cap U_j$ is dense in U_j and contained in the closed subset $Z_1 \cap U_j$. We conclude that also $U_j \subset Z_1$. Thus $X = Z_1$ as desired.

Assume (3). Choose an affine open covering $X = \bigcup_{i \in I} U_i$. We may assume that each U_i is nonempty. Since X is nonempty we see that I is not empty. By assumption each U_i is irreducible. Suppose $U_i \cap U_j = \emptyset$ for some pair $i, j \in I$. Then the open $U_i \coprod U_j = U_i \cup U_j$ is affine, see Schemes, Lemma 21.6.8. Hence it is irreducible by assumption which is absurd. We conclude that (3) implies (2). The lemma is proved. □

Lemma 23.3.4. *A scheme X is integral if and only if it is reduced and irreducible.*

Proof. If X is irreducible, then every affine open $Spec(R) = U \subset X$ is irreducible. If X is reduced, then R is reduced, by Lemma 23.3.2 above. Hence R is reduced and (0) is a prime ideal, i.e., R is an integral domain.

If X is integral, then for every nonempty affine open $Spec(R) = U \subset X$ the ring R is reduced and hence X is reduced by Lemma 23.3.2. Moreover, every nonempty affine open is irreducible. Hence X is irreducible, see Lemma 23.3.3. □

Example 23.3.5. We give an example of an affine scheme $X = Spec(A)$ which is connected, all of whose local rings are domains, but which is not integral. Connectedness for A means A has no nontrivial idempotents, see Algebra, Lemma 7.18.3. Integrality means A is a domain (see above). Local rings being domains means that whenever $fg = 0$ in A, every point of X has a neighborhood where either f or g vanishes.

Roughly speaking, the construction is as follows: let X_0 be the cross (the union of coordinate axes) on the affine plane. Then let X_1 be the (reduced) full preimage of X_0 on the blow-up of the plane (X_1 has three rational components forming a chain). Then blow up the resulting surface at the two singularities of X_1, and let X_2 be the reduced preimage of X_1 (which has five rational components), etc. Take X to be the inverse limit. The only problem with this construction is that blow-ups glue in a projective line, so X_1 is not affine. Let us correct this by glueing in an affine line instead (so our scheme will be an open subset in what was described above).

Here is a completely algebraic construction: For every $k \geq 0$, let A_k be the following ring: its elements are collections of polynomials $p_i \in \mathbf{C}[x]$ where $i = 0, \ldots, 2^k$ such that $p_i(1) = p_{i+1}(0)$. Set $X_k = Spec(A_k)$. Observe that X_k is a union of $2^k + 1$ affine lines that meet transversally in a chain. Define a ring homomorphism $A_k \to A_{k+1}$ by

$$(p_0, \ldots, p_{2^k}) \longmapsto (p_0, p_0(1), p_1, p_1(1), \ldots, p_{2^k}),$$

in other words, every other polynomial is constant. This identifies A_k with a subring of A_{k+1}. Let A be the direct limit of A_k (basically, their union). Set $X = Spec(A)$. For every k, we have a natural embedding $A_k \to A$, that is, a map $X \to X_k$. Each A_k is connected but not integral; this implies that A is connected but not integral. It remains to show that the local rings of A are domains.

Take $f, g \in A$ with $fg = 0$ and $x \in X$. Let us construct a neighborhood of x on which one of f and g vanishes. Choose k such that $f, g \in A_{k-1}$ (note the $k - 1$ index). Let y be the image of x in X_k. It suffices to prove that y has a neighborhood on which either f or g viewed as sections of $\mathcal{O}_{X_k}$ vanishes. If y is a smooth point of X_k, that is, it lies on only one of the $2^k + 1$ lines, this is obvious. We can therefore assume that y is one of the 2^k singular points, so two components of X_k pass through y. However, on one of these two components (the one with odd index), both f and g are constant, since they are pullbacks of functions on X_{k-1}. Since $fg = 0$ everywhere, either f or g (say, f) vanishes on the other component. This implies that f vanishes on both components, as required.

23.4. Types of schemes defined by properties of rings

In this section we study what properties of rings allow one to define local properties of schemes.

Definition 23.4.1. Let P be a property of rings. We say that P is *local* if the following hold:

(1) For any ring R, and any $f \in R$ we have $P(R) \Rightarrow P(R_f)$.
(2) For any ring R, and $f_i \in R$ such that $(f_1, \ldots, f_n) = R$ then $\forall i, P(R_{f_i}) \Rightarrow P(R)$.

Definition 23.4.2. Let P be a property of rings. Let X be a scheme. We say X is *locally P* if for any $x \in X$ there exists an affine open neighbourhood U of x in X such that $\mathcal{O}_X(U)$ has property P.

This is only a good notion if the property is local. Even if P is a local property we will not automatically use this definition to say that a scheme is ``locally P'' unless we also explicitly state the definition elsewhere.

Lemma 23.4.3. *Let X be a scheme. Let P be a local property of rings. The following are equivalent:*

(1) *The scheme X is locally P.*
(2) *For every affine open $U \subset X$ the property $P(\mathcal{O}_X(U))$ holds.*
(3) *There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ satisfies P.*
(4) *There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is locally P.*

Moreover, if X is locally P then every open subscheme is locally P.

Proof. Of course (1) $\Leftrightarrow$ (3) and (2) $\Rightarrow$ (1). If (3) $\Rightarrow$ (2), then the final statement of the lemma holds and it follows easily that (4) is also equivalent to (1). Thus we show (3) $\Rightarrow$ (2).

Let $X = \bigcup U_i$ be an affine open covering, say $U_i = Spec(R_i)$. Assume $P(R_i)$. Let $Spec(R) = U \subset X$ be an arbitrary affine open. By Schemes, Lemma 21.11.6 there exists a standard covering of $U = Spec(R)$ by standard opens $D(f_j)$ such that each ring R_{f_j} is a principal localization of one of the rings R_i. By Definition 23.4.1 (1) we get $P(R_{f_j})$. Whereupon $P(R)$ by Definition 23.4.1 (2). □

Here is a sample application.

Lemma 23.4.4. *Let X be a scheme. Then X is reduced if and only if X is ``locally reduced" in the sense of Definition 23.4.2.*

Proof. This is clear from Lemma 23.3.2. □

Lemma 23.4.5. *The following properties of a ring R are local.*

(1) *(Cohen-Macauley.) The ring R is Noetherian and CM, see Algebra, Definition 7.96.6.*
(2) *(Regular.) The ring R is Noetherian and regular, see Algebra, Definition 7.102.6.*
(3) *(Absolutely Noetherian.) The ring R is of finite type over Z.*
(4) *Add more here as needed.*[1]

Proof. Omitted. □

23.5. Noetherian schemes

Recall that a ring R is *Noetherian* if it satsifies the ascending chain condition of ideals. Equivalently every ideal of R is finitely generated.

Definition 23.5.1. Let X be a scheme.

(1) We say X is *locally Noetherian* if every $x \in X$ has an affine open neighbourhood $Spec(R) = U \subset X$ such that the ring R is Noetherian.
(2) We say X is *Noetherian* if X is Noetherian and quasi-compact.

Here is the standard result characterizing locally Noetherian schemes.

Lemma 23.5.2. *Let X be a scheme. The following are equivalent:*

(1) *The scheme X is locally Noetherian.*
(2) *For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is Noetherian.*
(3) *There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is Noetherian.*
(4) *There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is locally Noetherian.*

Moreover, if X is locally Noetherian then every open subscheme is locally Noetherian.

Proof. To show this it suffices to show that being Noetherian is a local property of rings, see Lemma 23.4.3. Any localization of a Noetherian ring is Noetherian, see Algebra, Lemma 7.28.1. By Algebra, Lemma 7.21.2 we see the second property to Definition 23.4.1. □

Lemma 23.5.3. *Any immersion $Z \to X$ with X locally Noetherian is quasi-compact.*

[1] But we only list those properties here which we have not already dealt with separately somewhere else.

Proof. A closed immersion is clearly quasi-compact. A composition of quasi-compact morphisms is quasi-compact, see Topology, Lemma 5.9.2. Hence it suffices to show that an open immersion into a locally Noetherian scheme is quasi-compact. Using Schemes, Lemma 21.19.2 we reduce to the case where X is affine. Any open subset of the spectrum of a Noetherian ring is quasi-compact (for example combine Algebra, Lemma 7.28.5 and Topology, Lemmas 5.6.2 and 5.9.9). ☐

Lemma 23.5.4. *A locally Noetherian scheme is quasi-separated.*

Proof. By Schemes, Lemma 21.21.7 we have to show that the intersection $U \cap V$ of two affine opens of X is quasi-compact. This follows from Lemma 23.5.3 above on considering the open immersion $U \cap V \to U$ for example. (But really it is just because any open of the spectrum of a Noetherian ring is quasi-compact.) ☐

Lemma 23.5.5. *A (locally) Noetherian scheme has a (locally) Noetherian underlying topological space, see Topology, Definition 5.6.1.*

Proof. This is because a Noetherian scheme is a finite union of spectra of Noetherian rings and Algebra, Lemma 7.28.5 and Topology, Lemma 5.6.4. ☐

Lemma 23.5.6. *Any morphism of schemes $f : X \to Y$ with X Noetherian is quasi-compact.*

Proof. Use Lemma 23.5.5 and use that any subset of a Noetherian topological space is quasi-compact (see Topology, Lemmas Lemmas 5.6.2 and 5.9.9). ☐

Lemma 23.5.7. *Any locally closed subscheme of a (locally) Noetherian scheme is (locally) Noetherian.*

Proof. Omitted. Hint: Any quotient, and any localization of a Noetherian ring is Noetherian. For the Noetherian case use again that any subset of a Noetherian space is a Noetherian space (with induced topology). ☐

Here is a fun lemma. It says that every locally Noetherian scheme has plenty of closed points (at least one in every closed subset).

Lemma 23.5.8. *Any locally Noetherian scheme has a closed point. Any closed subset of a locally Noetherian scheme has a closed point. Equivalently, any point of a locally Noetherian scheme specializes to a closed point.*

Proof. The second assertion follows from the first (using Schemes, Lemma 21.12.4 and Lemma 23.5.7). Consider any nonempty affine open $U \subset X$. Let $x \in U$ be a closed point. If x is a closed point of X then we are done. If not, let $y \in \overline{\{x\}}$ be a specialization of x. Note that $y \in X \setminus U$. Consider the local ring $R = \mathcal{O}_{X,y}$. This is a Noetherian local ring. Denote $V \subset Spec(R)$ the inverse image of U in $Spec(R)$ by the canonical morphism $Spec(R) \to X$ (see Schemes, Section 21.13.) By construction V is a singleton with unique point corresponding to x (use Schemes, Lemma 21.13.2). Say $V = \{\mathfrak{q}\}$. Consider the Noetherian local domain $R/\mathfrak{q}$. By Algebra, Lemma 7.58.1 we see that $\dim(R/\mathfrak{q}) = 1$. In other words, we see that y is an immediate specialization of x (see Topology, Definition 5.16.1). In other words, any point $y \neq x$ such that $x \rightsquigarrow y$ is an immediate specialization of x. Clearly each of these points is a closed point, and we win. ☐

Lemma 23.5.9. *Let X be a locally Noetherian scheme. Let $x' \rightsquigarrow x$ be a specialization of points of X. Then*

(1) *there exists a discrete valuation ring R and a morphism $f : Spec(R) \to X$ such that the generic point η of $Spec(R)$ maps to x' and the special point maps to x, and*
(2) *given a finitely generated field extension $\kappa(x') \subset K$ we may arrange it so that the extension $\kappa(x') \subset \kappa(\eta)$ induced by f is isomorphic to the given one.*

Proof. Let $x' \rightsquigarrow x$ be a specialization in X, and let $\kappa(x') \subset K$ be a finitely generated extension of fields. By Schemes, Lemma 21.13.2 and the discussion following Schemes, Lemma 21.13.3 this leads to ring maps $\mathcal{O}_{X,x} \to \kappa(x') \to K$. Let $R \subset K$ be any discrete valuation ring whose field of fractions is K and which dominates the image of $\mathcal{O}_{X,x} \to K$, see Algebra, Lemma 7.110.11. The ring map $\mathcal{O}_{X,x} \to R$ induces the morphism $f : Spec(R) \to X$, see Schemes, Lemma 21.13.1. This morphism has all the desired properties by construction. □

23.6. Jacobson schemes

Recall that a space is said to be *Jacobson* if the closed points are dense in every closed subset, see Topology, Section 5.13.

Definition 23.6.1. A scheme S is said to be *Jacobson* if its underlying topological space is Jacobson.

Recall that a ring R is Jacobson if every radical ideal of R is the intersection of maximal ideals, see Algebra, Definition 7.31.1.

Lemma 23.6.2. *An affine scheme $Spec(R)$ is Jacobson if and only if the ring R is Jacobson.*

Proof. This is Algebra, Lemma 7.31.4. □

Here is the standard result characterizing Jacobson schemes. Intuitively it claims that Jacobson ⇔ locally Jacobson.

Lemma 23.6.3. *Let X be a scheme. The following are equivalent:*

(1) *The scheme X is Jacobson.*
(2) *The scheme X is ``locally Jacobson'' in the sense of Definition 23.4.2.*
(3) *For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is Jacobson.*
(4) *There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is Jacobson.*
(5) *There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is Jacobson.*

Moreover, if X is Jacobson then every open subscheme is Jacobson.

Proof. The final assertion of the lemma holds by Topology, Lemma 5.13.5. The equivalence of (5) and (1) is Topology, Lemma 5.13.4. Hence, using Lemma 23.6.2, we see that (1) ⇔ (2). To finish proving the lemma it suffices to show that ``Jacobson'' is a local property of rings, see Lemma 23.4.3. Any localization of a Jacobson ring at an element is Jacobson, see Algebra, Lemma 7.31.14. Suppose R is a ring, $f_1, \ldots, f_n \in R$ generate the unit ideal and each R_{f_i} is Jacobson. Then we see that $Spec(R) = \bigcup D(f_i)$ is a union of open subsets which are all Jacobson, and hence $Spec(R)$ is Jacobson by Topology, Lemma 5.13.4 again. This proves the second property of Definition 23.4.1. □

Many schemes used commonly in algebraic geometry are Jacobson, see Morphisms, Lemma 24.15.10. We mention here the following interesting case.

Lemma 23.6.4. *Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. In this case the scheme $S = Spec(R) \setminus \{\mathfrak{m}\}$ is Jacobson.*

Proof. Since $Spec(R)$ is a Noetherian scheme, hence S is a Noetherian scheme (Lemma 23.5.7). Hence S is a sober, Noetherian topological space (use Schemes, Lemma 21.11.1). Assume S is not Jacobson to get a contradiction. By Topology, Lemma 5.13.3 there exists some non-closed point $\xi \in S$ such that $\{\xi\}$ is locally closed. This corresponds to a prime $\mathfrak{p} \subset R$ such that (1) there exists a prime $\mathfrak{q}$, $\mathfrak{p} \subset \mathfrak{q} \subset \mathfrak{m}$ with both inclusions strict, and (2) $\{\mathfrak{p}\}$ is open in $Spec(R/\mathfrak{p})$. This is impossible by Algebra, Lemma 7.58.1. □

23.7. Normal schemes

Recall that a ring R is said to be normal if all its local rings are normal domains, see Algebra, Definition 7.33.10. A normal domain is a domain which is integrally closed in its field of fractions, see Algebra, Definition 7.33.1. Thus it makes sense to define a normal scheme as follows.

Definition 23.7.1. A scheme X is *normal* if and only if for all $x \in X$ the local ring $\mathcal{O}_{X,x}$ is a normal domain.

This seems to be the definition used in EGA, see [**DG67**, 0, 4.1.4]. Suppose $X = Spec(A)$, and A is reduced. Then saying that X is normal is not equivalent to saying that A is integrally closed in its total ring of fractions. However, if A is Noetherian then this is the case (see Algebra, Lemma 7.33.14).

Lemma 23.7.2. *Let X be a scheme. The following are equivalent:*

(1) *The scheme X is normal.*
(2) *For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is normal.*
(3) *There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is normal.*
(4) *There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is normal.*

Moreover, if X is normal then every open subscheme is normal.

Proof. This is clear from the definitions. □

Lemma 23.7.3. *A normal scheme is reduced.*

Proof. Immediate from the defintions. □

Lemma 23.7.4. *Let X be an integral scheme. Then X is normal if and only if for every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is a normal domain.*

Proof. This follows from Algebra, Lemma 7.33.9. □

Lemma 23.7.5. *Let X be a scheme with a finite number of irreducible components. The following are equivalent:*

(1) *X is normal, and*
(2) *X is a finite disjoint union of normal integral schemes.*

Proof. It is immediate from the definitions that (2) implies (1). Let X be a normal scheme with a finite number of irreducible components. If X is affine then X satisfies (2) by Algebra, Lemma 7.33.14. For a general X, let $X = \bigcup X_i$ be an affine open covering. Note that also each X_i has but a finite number of irreducible components, and the lemma holds for each X_i. Let $T \subset X$ be an irreducible component. By the affine case each intersection $T \cap X_i$

is open in X_i and an integral normal scheme. Hence $T \subset X$ is open, and an integral normal scheme. This proves that X is the disjoint union of its irreducible components, which are integral normal schemes. There are only finitely many by assumption. ☐

Lemma 23.7.6. *Let X be a Noetherian scheme. The following are equivalent:*

(1) *X is normal, and*
(2) *X is a finite disjoint union of normal integral schemes.*

Proof. This is a special case of Lemma 23.7.5 because a Noetherian scheme has a Noetherian underlying topological space (Lemma 23.5.5 and Topology, Lemma 5.6.2. ☐

Lemma 23.7.7. *Let X be a locally Noetherian normal scheme. The following are equivalent:*

(1) *X is normal, and*
(2) *X is a disjoint union of integral normal schemes.*

Proof. Omitted. Hint: This is purely topological from Lemma 23.7.6. ☐

Remark 23.7.8. Let X be a normal scheme. If X is locally Noetherian then we see that X is integral if and only if X is connected, see Lemma 23.7.7. But there exists a connected affine scheme X such that $\mathcal{O}_{X,x}$ is a domain for all $x \in X$, but X is not irreducible, see Example 23.3.5. This example is even a normal scheme (proof omitted), so beware!

Lemma 23.7.9. *Let X be an integral normal scheme. Then $\Gamma(X, \mathcal{O}_X)$ is a normal domain.*

Proof. Set $R = \Gamma(X, \mathcal{O}_X)$. It is clear that R is a domain. Suppose $f = a/b$ is an element of its fraction field which is integral over R. Say we have $f^d + \sum_{i=1,\ldots,d} a_i f^i = 0$ with $a_i \in R$. Let $U \subset X$ be affine open. Since $b \in R$ is not zero and since X is integral we see that also $b|_U \in \mathcal{O}_X(U)$ is not zero. Hence a/b is an element of the fraction field of $\mathcal{O}_X(U)$ which is integral over $\mathcal{O}_X(U)$ (because we can use the same polynomial $f^d + \sum_{i=1,\ldots,d} a_i|_U f^i = 0$ on U). Since $\mathcal{O}_X(U)$ is a normal domain (Lemma 23.7.2), we see that $f_U = (a|_U)/(b|_U) \in \mathcal{O}_X(U)$. It is easy to see that $f_U|_V = f_V$ whenever $V \subset U \subset X$ are affine open. Hence the local sections f_U glue to a global section f as desired. ☐

23.8. Cohen-Macaulay schemes

Recall, see Algebra, Definition 7.96.1, that a local Noetherian ring $(R, \mathfrak{m})$ is said to be Cohen-Macaulay if $\text{depth}_{\mathfrak{m}}(R) = \dim(R)$. Recall that a Noetherian ring R is said to be Cohen-Macaulay if every local ring $R_{\mathfrak{p}}$ of R is Cohen-Macaulay, see Algebra, Definition 7.96.6.

Definition 23.8.1. Let X be a scheme. We say X is *Cohen-Macaulay* if for every $x \in X$ there exists an affine open neighbourhood $U \subset X$ of x such that the ring $\mathcal{O}_X(U)$ is Noetherian and Cohen-Macaulay.

Lemma 23.8.2. *Let X be a scheme. The following are equivalent:*

(1) *X is Cohen-Macaulay,*
(2) *X is locally Noetherian and all of its local rings are Cohen-Macaulay, and*
(3) *X is locally Noetherian and for any closed point $x \in X$ the local ring $\mathcal{O}_{X,x}$ is Cohen-Macaulay.*

Proof. Algebra, Lemma 7.96.5 says that the localization of a Cohen-Macaulay local ring is Cohen-Macaulay. The lemma follows by combining this with Lemma 23.5.2, with the existence of closed points on locally Noetherian schemes (Lemma 23.5.8), and the definitions. ☐

Lemma 23.8.3. *Let X be a scheme. The following are equivalent:*

1. *The scheme X is Cohen-Macaulay.*
2. *For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is Noetherian and Cohen-Macaulay.*
3. *There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is Noetherian and Cohen-Macaulay.*
4. *There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is Cohen-Macaulay.*

Moreover, if X is Cohen-Macaulay then every open subscheme is Cohen-Macaulay.

Proof. Combine Lemmas 23.5.2 and 23.8.2. □

More information on Cohen-Macaulay schemes and depth can be found in Coherent, Section 25.13.

23.9. Regular schemes

Recall, see Algebra, Definition 7.57.9, that a local Noetherian ring $(R, \mathfrak{m})$ is said to be *regular* if $\mathfrak{m}$ can be generated by $\dim(R)$ elements. Recall that a Noetherian ring R is said to be *regular* if every local ring $R_{\mathfrak{p}}$ of R is regular, see Algebra, Definition 7.102.6.

Definition 23.9.1. Let X be a scheme. We say X is *regular*, or *nonsingular* if for every $x \in X$ there exists an affine open neighbourhood $U \subset X$ of x such that the ring $\mathcal{O}_X(U)$ is Noetherian and regular.

Lemma 23.9.2. *Let X be a scheme. The following are equivalent:*

1. *X is regular,*
2. *X is locally Noetherian and all of its local rings are regular, and*
3. *X is locally Noetherian and for any closed point $x \in X$ the local ring $\mathcal{O}_{X,x}$ is regular.*

Proof. By the discussion in Algebra preceding Algebra, Definition 7.102.6 we know that the localization of a regular local ring is regular. The lemma follows by combining this with Lemma 23.5.2, with the existence of closed points on locally Noetherian schemes (Lemma 23.5.8), and the definitions. □

Lemma 23.9.3. *Let X be a scheme. The following are equivalent:*

1. *The scheme X is regular.*
2. *For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is Noetherian and regular.*
3. *There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is Noetherian and regular.*
4. *There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is regular.*

Moreover, if X is regular then every open subscheme is regular.

Proof. Combine Lemmas 23.5.2 and 23.9.2. □

Lemma 23.9.4. *A regular scheme is normal.*

Proof. See Algebra, Lemma 7.140.5. □

23.10. Dimension

The dimension of a scheme is just the dimension of its underlying topological space.

Definition 23.10.1. Let X be a scheme.

(1) The *dimension* of X is just the dimension of X as a topological spaces, see Topology, Definition 5.7.1.
(2) For $x \in X$ we denote $\dim_x(X)$ the dimension of the underlying topological space of X at x as in Topology, Definition 5.7.1. We say $\dim_x(X)$ is the *dimension of X at x*.

As a scheme has a sober underlying topological space (Schemes, Lemma 21.11.1) we may compute the dimension of X as the supremum of the lengths n of chains

$$T_0 \subset T_1 \subset \ldots \subset T_n$$

of irreducible closed subsets of X, or as the supremum of the lengths n of chains of specializations

$$\xi_n \leadsto \xi_{n-1} \leadsto \ldots \leadsto \xi_0$$

of points of X.

Lemma 23.10.2. *Let X be a scheme. The following are equal*

(1) *The dimension of X.*
(2) *The supremum of the dimensions of the local rings of X.*
(3) *The supremum of* $\dim_x(X)$ *for $x \in X$.*

Proof. Note that given a chain of specializations

$$\xi_n \leadsto \xi_{n-1} \leadsto \ldots \leadsto \xi_0$$

of points of X all of the points ξ_i correspond to prime ideals of the local ring of X at ξ_0 by Schemes, Lemma 21.13.2. Hence we see that the dimension of X is the supremum of the dimensions of its local rings. In particular $\dim_x(X) \geq \dim(\mathcal{O}_{X,x})$ as $\dim_x(X)$ is the minimum of the dimensions of open neighbourhoods of x. Thus $\sup_{x \in X} \dim_x(X) \geq \dim(X)$. On the other hand, it is clear that $\sup_{x\in X} \dim_x(X) \leq \dim(X)$ as $\dim(U) \leq \dim(X)$ for any open subset of X. $\square$

23.11. Catenary schemes

Recall that a topological space X is called *catenary* if for every pair of irreducible closed subsets $T \subset T'$ there exist a maximal chain of irreducible closed subsets

$$T = T_0 \subset T_1 \subset \ldots \subset T_e = T'$$

and every such chain has the same length. See Topology, Definition 5.8.1.

Definition 23.11.1. Let S be a scheme. We say S is *catenary* if the underlying topological space of S is catenary.

Recall that a ring A is called *catenary* if for any pair of prime ideals $\mathfrak{p} \subset \mathfrak{q}$ there exists a maximal chain of primes

$$\mathfrak{p} = \mathfrak{p}_0 \subset \ldots \subset \mathfrak{p}_e = \mathfrak{q}$$

and all of these have the same length. See Algebra, Definition 7.97.1.

Lemma 23.11.2. *Let S be a scheme. The following are equivalent*

(1) *S is catenary,*

(2) *there exists an open covering of S all of whose members are catenary schemes,*
(3) *for every affine open $Spec(R) = U \subset S$ the ring R is catenary, and*
(4) *there exists an affine open covering $S = \bigcup U_i$ such that each U_i is the spectrum of a catenary ring.*

Moreover, in this case any locally closed subscheme of S is catenary as well.

Proof. Combine Topology, Lemma 5.8.2, and Algebra, Lemma 7.97.2. $\square$

Lemma 23.11.3. *Let S be a locally Noetherian scheme. The following are equivalent:*

(1) *S is catenary, and*
(2) *locally in the Zariski topology there exists a dimension function on S (see Topology, Definition 5.16.1).*

Proof. This follows from Topology, Lemmas 5.8.2, 5.16.2, and 5.16.4, Schemes, Lemma 21.11.1 and finally Lemma 23.5.5. $\square$

Lemma 23.11.4. *Let X be a scheme. Let $Y \subset X$ be an irreducible closed subset. Let $\xi \in Y$ be the generic point. Then*

$$codim(Y, X) = \dim(\mathcal{O}_{X,\xi})$$

where the codimension is as defined in Topology, Definition 5.8.3.

Proof. By Topology, Lemma 5.8.4 we may replace X by an affine open neighbourhood of ξ. In this case the result follows easily from Algebra, Lemma 7.23.2. $\square$

In particular the dimension of a scheme is the supremum of the dimensions of all of its local rings. It turns out that we can use this lemma to characterize a catenary scheme as a scheme all of whose local rings are catenary.

Lemma 23.11.5. *Let X be a scheme. The following are equivalent*

(1) *X is catenary, and*
(2) *for any $x \in X$ the local ring $\mathcal{O}_{X,x}$ is catenary.*

Proof. Assume X is catenary. Let $x \in X$. By Lemma 23.11.2 we may replace X by an affine open neighbourhood of x, and then $\Gamma(X, \mathcal{O}_X)$ is a catenary ring. By Algebra, Lemma 7.97.3 any localization of a catenary ring is catenary. Whence $\mathcal{O}_{X,x}$ is catenary.

Conversely assume all local rings of X are catenary. Let $Y \subset Y'$ be an inclusion of irreducible closed subsets of X. Let $\xi \in Y$ be the generic point. Let $\mathfrak{p} \subset \mathcal{O}_{X,\xi}$ be the prime corresponding to the generic point of Y', see Schemes, Lemma 21.13.2. By that same lemma the irreducible closed subsets of X in between Y and Y' correspond to primes $\mathfrak{q} \subset \mathcal{O}_{X,\xi}$ with $\mathfrak{p} \subset \mathfrak{q} \subset \mathfrak{m}_\xi$. Hence we see all maximal chains of these are finite and have the same length as $\mathcal{O}_{X,\xi}$ is a catenary ring. $\square$

23.12. Serre's conditions

Here are two technical notions that are often useful. See also Coherent, Section 25.13.

Definition 23.12.1. Let X be a locally Noetherian scheme. Let $k \geq 0$.

(1) We say X is *regular in codimension k*, or we say X has property (R_k) if for every $x \in X$ we have
$$\dim(\mathcal{O}_{X,x}) \leq k \Rightarrow \mathcal{O}_{X,x} \text{ is regular}$$
(2) We say X has property (S_k) if for every $x \in X$ we have $\text{depth}(\mathcal{O}_{X,x}) \geq \min(k, \dim(\mathcal{O}_{X,x}))$.

The phrase ``regular in codimension k'' makes sense since we have seen in Section 23.11 that if $Y \subset X$ is irreducible closed with generic point x, then $\dim(\mathcal{O}_{X,x}) = \text{codim}(Y, X)$. For example condition (R_0) means that for every generic point $\eta \in X$ of an irreducible component of X the local ring $\mathcal{O}_{X,\eta}$ is a field. But for general Noetherian schemes it can happen that the regular locus of X is badly behaved, so care has to be taken.

Lemma 23.12.2. *Let X be a locally Noetherian scheme. Then X is Cohen-Macaulay if and only if X has (S_k) for all $k \geq 0$.*

Proof. By Lemma 23.8.2 we reduce to looking at local rings. Hence the lemma is true because a Noetherian local ring is Cohen-Macaulay if and only if it has depth equal to its dimension. □

Lemma 23.12.3. *Let X be a locally Noetherian scheme. Then X is reduced if and only if X has properties (S_1) and (R_0).*

Proof. This is Algebra, Lemma 7.140.3. □

Lemma 23.12.4. *Let X be a locally Noetherian scheme. Then X is normal if and only if X has properties (S_2) and (R_1).*

Proof. This is Algebra, Lemma 7.140.4. □

23.13. Japanese and Nagata schemes

The notions considered in this section are not prominently defined in EGA. A ``universally Japanese scheme'' is mentioned and defined in [**DG67**, IV Corollary 5.11.4]. A ``Japanese scheme'' is mentioned in [**DG67**, IV Remark 10.4.14 (ii)] but no definition is given. A Nagata scheme (as given below) occurs in a few places in the literature (see for example [**Liu02**, Definition 8.2.30] and [**Gre76**, Page 142]).

We briefly recall that a domain R is called *Japanese* if the integral closure of R in any finite extension of its fraction field is finite over R. A ring R is called *universally Japanese* if for any finite type ring map $R \to S$ with S a domain S is Japanese. A ring R is called *Nagata* if it is Noetherian and $R/\mathfrak{p}$ is Japanese for every prime $\mathfrak{p}$ of R.

Definition 23.13.1. Let X be a scheme.

(1) Assume X integral. We say X is *Japanese* if for every $x \in X$ there exists an affine open neighbourhood $x \in U \subset X$ such that the ring $\mathcal{O}_X(U)$ is Japanese (see Algebra, Definition 7.144.1).
(2) We say X is *universally Japanese* if for every $x \in X$ there exists an affine open neighbourhood $x \in U \subset X$ such that the ring $\mathcal{O}_X(U)$ is universally Japanese (see Algebra, Definition 7.144.15).
(3) We say X is *Nagata* if for every $x \in X$ there exists an affine open neighbourhood $x \in U \subset X$ such that the ring $\mathcal{O}_X(U)$ is Nagata (see Algebra, Definition 7.144.15).

Being Nagata is the same thing as being locally Noetherian and universally Japanese, see Lemma 23.13.8.

Remark 23.13.2. In [**Hoo72**] a (locally Noetherian) scheme X is called Japanese if for every $x \in X$ and every associated prime $\mathfrak{p}$ of $\mathcal{O}_{X,x}$ the ring $\mathcal{O}_{X,x}/\mathfrak{p}$ is Japanese. We do not use this definition since it is not clear that this gives the same notion as above for Noetherian integral schemes. In other words, we do not know whether a Noetherian domain all of whose local rings are Japanese is Japanese. If you do please email stacks.project@gmail.com. On the other hand, we could circumvent this problem by calling a scheme X Japanese if for every affine open $Spec(A) \subset X$ the ring $A/\mathfrak{p}$ is Japanese for every associated prime $\mathfrak{p}$ of A.

Lemma 23.13.3. *A Nagata scheme is locally Noetherian.*

Proof. This is true because a Nagata ring is Noetherian by definition. □

Lemma 23.13.4. *Let X be an integral scheme. The following are equivalent:*

(1) *The scheme X is Japanese.*
(2) *For every affine open $U \subset X$ the domain $\mathcal{O}_X(U)$ is Japanese.*
(3) *There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is Japanese.*
(4) *There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is Japanese.*

Moreover, if X is Japanese then every open subscheme is Japanese.

Proof. This follows from Lemma 23.4.3 and Algebra, Lemmas 7.144.3 and 7.144.4. □

Lemma 23.13.5. *Let X be a scheme. The following are equivalent:*

(1) *The scheme X is universally Japanese.*
(2) *For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is universally Japanese.*
(3) *There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is universally Japanese.*
(4) *There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is universally Japanese.*

Moreover, if X is universally Japanese then every open subscheme is universally Japanese.

Proof. This follows from Lemma 23.4.3 and Algebra, Lemmas 7.144.18 and 7.144.21. □

Lemma 23.13.6. *Let X be a scheme. The following are equivalent:*

(1) *The scheme X is Nagata.*
(2) *For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is Nagata.*
(3) *There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is Nagata.*
(4) *There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is Nagata.*

Moreover, if X is Nagata then every open subscheme is Nagata.

Proof. This follows from Lemma 23.4.3 and Algebra, Lemmas 7.144.20 and 7.144.21. □

Lemma 23.13.7. *Let X be a locally Noetherian scheme. Then X is Nagata if and only if every integral closed subscheme $Z \subset X$ is Japanese.*

Proof. Assume X is Nagata. Let $Z \subset X$ be an integral closed subscheme. Let $z \in Z$. Let $Spec(A) = U \subset X$ be an affine open containing z such that A is Nagata. Then $Z \cap U \cong Spec(A/\mathfrak{p})$ for some prime $\mathfrak{p}$, see Schemes, Lemma 21.10.1 (and Definition 23.3.1). By Algebra, Definition 7.144.15 we see that $A/\mathfrak{p}$ is Japanese. Hence Z is Japanese by definition.

Assume every integral closed subscheme of X is Japanese. Let $Spec(A) = U \subset X$ be any affine open. As X is locally Noetherian we see that A is Noetherian (Lemma 23.5.2). Let $\mathfrak{p} \subset A$ be a prime ideal. We have to show that $A/\mathfrak{p}$ is Japanese. Let $T \subset U$ be the closed subset $V(\mathfrak{p}) \subset Spec(A)$. Let $\overline{T} \subset X$ be the closure. Then $\overline{T}$ is irreducible as the closure of an irreducible subset. Hence the reduced closed subscheme defined by $\overline{T}$ is an integral closed subscheme (called $\overline{T}$ again), see Schemes, Lemma 21.12.4. In other words, $Spec(A/\mathfrak{p})$

is an affine open of an integral closed subscheme of X. This subscheme is Japanese by assumption and by Lemma 23.13.4 we see that $A/\mathfrak{p}$ is Japanese. □

Lemma 23.13.8. *Let X be a scheme. The following are equivalent:*

(1) *X is Nagata, and*
(2) *X is locally Noetherian and universally Japanese.*

Proof. This is Algebra, Proposition 7.144.30. □

This discussion will be continued in Morphisms, Section 24.17.

23.14. The singular locus

Here is the definition.

Definition 23.14.1. Let X be a locally Noetherian scheme. The *regular locus* $\text{Reg}(X)$ of X is the set of $x \in X$ such that $\mathcal{O}_{X,x}$ is a regular local ring. The *singular locus* $\text{Sing}(X)$ is the complement $X \setminus \text{Reg}(X)$, i.e., the set of points $x \in X$ such that $\mathcal{O}_{X,x}$ is not a regular local ring.

The regular locus of a locally Noetherian scheme is stable under generalizations, see the discussion preceding Algebra, Definition 7.102.6. However, for general locally Noetherian schemes the regular locus need not be open. In More on Algebra, Section 12.35 the reader can find some criteria for when this is the case. We will discuss this further in Morphisms, Section 24.18.

23.15. Quasi-affine schemes

Definition 23.15.1. A scheme X is called *quasi-affine* if it is quasi-compact and isomorphic to an open subscheme of an affine scheme.

Lemma 23.15.2. *Let X be a scheme. Let $f \in \Gamma(X, \mathcal{O}_X)$. Denote X_f the maximal open subscheme of X where f is invertible, see Schemes, Lemma 21.6.2 or Modules, Lemma 15.21.7. If X is quasi-compact and quasi-separated, the canonical map*

$$\Gamma(X, \mathcal{O}_X)_f \longrightarrow \Gamma(X_f, \mathcal{O}_X)$$

is an isomorphism. Moreover, if $\mathcal{F}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules the map

$$\Gamma(X, \mathcal{F})_f \longrightarrow \Gamma(X_f, \mathcal{F})$$

is an isomorphism.

Proof. Write $R = \Gamma(X, \mathcal{O}_X)$. Consider the canonical morphism

$$\varphi : X \longrightarrow Spec(R)$$

of schemes, see Schemes, Lemma 21.6.4. Then the inverse image of the standard open $D(f)$ on the right hand side is X_f on the left hand side. Moreover, since X is assumed quasi-compact and quasi-separated the morphism φ is quasi-compact and quasi-separated, see Schemes, Lemma 21.19.2 and 21.21.14. Hence by Schemes, Lemma 21.24.1 we see that $\varphi_*\mathcal{F}$ is quasi-coherent. Hence we see that $\varphi_*\mathcal{F} = \widetilde{M}$ with $M = \Gamma(X, \mathcal{F})$ as an R-module. Thus we see that

$$\Gamma(X_f, \mathcal{F}) = \Gamma(D(f), \varphi_*\mathcal{F}) = \Gamma(D(f), \widetilde{M}) = M_f$$

which is exactly the content of the lemma. The case of $\mathcal{F} = \mathcal{O}_X$ will given the first displayed isomorphism of the lemma. □

Lemma 23.15.3. *Let X be a scheme. Let $f \in \Gamma(X, \mathcal{O}_X)$. Assume X is quasi-compact and quasi-separated and assume that X_f is affine. Then the canonical morphism*

$$j : X \longrightarrow Spec(\Gamma(X, \mathcal{O}_X))$$

from Schemes, Lemma 21.6.4 induces an isomorphism of $X_f = j^{-1}(D(f))$ onto the standard affine open $D(f) \subset Spec(\Gamma(X, \mathcal{O}_X))$.

Proof. This is clear as j induces an isomorphism of rings $\Gamma(X, \mathcal{O}_X)_f \to \mathcal{O}_X(X_f)$ by Lemma 23.15.2 above. □

Lemma 23.15.4. *Let X be a scheme. Then X is quasi-affine if and only if the canonical morphism*

$$X \longrightarrow Spec(\Gamma(X, \mathcal{O}_X))$$

from Schemes, Lemma 21.6.4 is a quasi-compact open immersion.

Proof. If the displayed morphism is a quasi-compact open immersion then X is ismorphic to a quasi-compact open subscheme of $Spec(\Gamma(X, \mathcal{O}_X))$ and clearly X is quasi-affine.

Assume X is quasi-affine, say $X \subset Spec(R)$ is quasi-compact open. This in particular implies that X is separated, see Schemes, Lemma 21.23.8. Let $A = \Gamma(X, \mathcal{O}_X)$. Consider the ring map $R \to A$ coming from $R = \Gamma(Spec(R), \mathcal{O}_{Spec(R)})$ and the restriction mapping of the sheaf $\mathcal{O}_{Spec(R)}$. By Schemes, Lemma 21.6.4 we obtain a factorization:

$$X \longrightarrow Spec(A) \longrightarrow Spec(R)$$

of the inclusion morphism. Let $x \in X$. Choose $r \in R$ such that $x \in D(r)$ and $D(r) \subset X$. Denote $f \in A$ the image of r in A. The open X_f of Lemma 23.15.2 above is equal to $D(r) \subset X$ and hence $A_f \cong R_r$ by the conclusion of that lemma. Hence $D(r) \to Spec(A)$ is an isomorphism onto the standard affine open $D(f)$ of $Spec(A)$. Since X can be covered by such affine opens $D(f)$ we win. □

23.16. Characterizing modules of finite type and finite presentation

Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The following lemma implies that $\mathcal{F}$ is of finite type (see Modules, Definition 15.9.1) if and only if $\mathcal{F}$ is on each open affine $Spec(A) = U \subset X$ of the form $\widetilde{M}$ for some finite type A-module M. Similarly, $\mathcal{F}$ is of finite presentation (see Modules, Definition 15.11.1) if and only if $\mathcal{F}$ is on each open affine $Spec(A) = U \subset X$ of the form $\widetilde{M}$ for some finitely presented A-module M.

Lemma 23.16.1. *Let $X = Spec(R)$ be an affine scheme. The quasi-coherent sheaf of $\mathcal{O}_X$-modules $\widetilde{M}$ is a finite type $\mathcal{O}_X$-module if and only if M is a finite R-module.*

Proof. Assume $\widetilde{M}$ is a finite type $\mathcal{O}_X$-module. This means there exists an open covering of X such that $\widetilde{M}$ restricted to the members of this covering is globally generated by finitely many sections. Thus there also exists a standard open covering $X = \bigcup_{i=1,\ldots,n} D(f_i)$ such that $\widetilde{M}|_{D(f_i)}$ is generated by finitely many sections. Thus M_{f_i} is finitely generated for each i. Hence we conclude by Algebra, Lemma 7.21.2. □

Lemma 23.16.2. *Let $X = Spec(R)$ be an affine scheme. The quasi-coherent sheaf of $\mathcal{O}_X$-modules $\widetilde{M}$ is an $\mathcal{O}_X$-module of finite presentation if and only if M is an R-module of finite presentation.*

Proof. Assume $\widetilde{M}$ is an $\mathcal{O}_X$-module of finite presentation. By Lemma 23.16.1 we see that M is a finite R-module. Choose a surjection $R^n \to M$ with kernel K. By Schemes, Lemma 21.5.4 there is a short exact sequence

$$0 \to \widetilde{K} \to \bigoplus \mathcal{O}_X^{\oplus n} \to \widetilde{M} \to 0$$

By Modules, Lemma 15.11.3 we see that $\widetilde{K}$ is a finite type $\mathcal{O}_X$-module. Hence by Lemma 23.16.1 again we see that K is a finite R-module. Hence M is an R-module of finite presentation. □

23.17. Flat modules

On any ringed space $(X, \mathcal{O}_X)$ we know what it means for an $\mathcal{O}_X$-module to be flat (at a point), see Modules, Definition 15.16.1 (Definition 15.16.3). On an affine scheme this matches the notion defined in the algebra chapter.

Lemma 23.17.1. *Let $X = Spec(R)$ be an affine scheme. Let $\mathcal{F} = \widetilde{M}$ for some R-module M. The quasi-coherent sheaf $\mathcal{F}$ is a flat $\mathcal{O}_X$-module of if and only if M is a flat R-module.*

Proof. Flatness of $\mathcal{F}$ may be checked on the stalks, see Modules, Lemma 15.16.2. The same is true in the case of modules over a ring, see Algebra, Lemma 7.35.19. And since $\mathcal{F}_x = M_{\mathfrak{p}}$ if x corresponds to $\mathfrak{p}$ the lemma is true. □

23.18. Locally free modules

On any ringed space we know what it means for an $\mathcal{O}_X$-module to be (finite) locally free. On an affine scheme this matches the notion defined in the algebra chapter.

Lemma 23.18.1. *Let $X = Spec(R)$ be an affine scheme. Let $\mathcal{F} = \widetilde{M}$ for some R-module M. The quasi-coherent sheaf $\mathcal{F}$ is a (finite) locally free $\mathcal{O}_X$-module of if and only if M is a (finite) locally free R-module.*

Proof. Follows from the definitions, see Modules, Definition 15.14.1 and Algebra, Definition 7.72.1. □

We can characterize finite locally free modules in many different ways.

Lemma 23.18.2. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The following are equivalent:*

(1) *$\mathcal{F}$ is a flat $\mathcal{O}_X$-module of finite presentation,*
(2) *$\mathcal{F}$ is $\mathcal{O}_X$-module of finite presentation and for all $x \in X$ the stalk $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module,*
(3) *$\mathcal{F}$ is a locally free, finite type $\mathcal{O}_X$-module,*
(4) *$\mathcal{F}$ is a finite locally free $\mathcal{O}_X$-module, and*
(5) *$\mathcal{F}$ is an $\mathcal{O}_X$-module of finite type, for every $x \in X$ the the stalk $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module, and the function*

$$\rho_{\mathcal{F}} : X \to \mathbf{Z}, \quad x \longmapsto \dim_{\kappa(x)} \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$$

is locally constant in the Zariski topology on X.

Proof. This lemma immediately reduces to the affine case. In this case the lemma is a reformulation of Algebra, Lemma 7.72.2. The translation uses Lemmas 23.16.1, 23.16.2, 23.17.1, and 23.18.1. □

23.19. Locally projective modules

A consequence of the work done in the algebra chapter is that it makes sense to define a locally projective module as follows.

Definition 23.19.1. Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. We say $\mathcal{F}$ is *locally projective* if for every affine open $U \subset X$ the $\mathcal{O}_X(U)$-module $\mathcal{F}(U)$ is projective.

Lemma 23.19.2. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The following are equivalent*

1. *$\mathcal{F}$ is locally projective, and*
2. *there exists an affine open covering $X = \bigcup U_i$ such that the $\mathcal{O}_X(U_i)$-module $\mathcal{F}(U_i)$ is projective for every i.*

In particular, if $X = Spec(A)$ and $\mathcal{F} = \widetilde{M}$ then $\mathcal{F}$ is locally projective if and only if M is a projective A-module.

Proof. First, note that if M is a projective A-module and $A \to B$ is a ring map, then $M \otimes_A B$ is a projective B-module, see Algebra, Lemma 7.88.1. Hence if U is an affine open such that $\mathcal{F}(U)$ is a projective $\mathcal{O}_X(U)$-module, then the standard open $D(f)$ is an affine open such that $\mathcal{F}(D(f))$ is a projective $\mathcal{O}_X(D(f))$-module for all $f \in \mathcal{O}_X(U)$. Assume (2) holds. Let $U \subset X$ be an arbitrary affine open. We can find an open covering $U = \bigcup_{j=1,\ldots,m} D(f_j)$ by finitely many standard opens $D(f_j)$ such that for each j the open $D(f_j)$ is a standard open of some U_i, see Schemes, Lemma 21.11.5. Hence, if we set $A = \mathcal{O}_X(U)$ and if M is an A-module such that $\mathcal{F}|_U$ corresponds to M, then we see that M_{f_j} is a projective A_{f_j}-module. It follows that $A \to B = \prod A_{f_j}$ is a faithfully flat ring map such that $M \times_A B$ is a projective B-module. Hence M is projective by Algebra, Theorem 7.89.5. $\square$

Lemma 23.19.3. *Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module. If $\mathcal{G}$ is locally projective on Y, then $f^*\mathcal{G}$ is locally projective on X.*

Proof. See Algebra, Lemma 7.88.1. $\square$

23.20. Extending quasi-coherent sheaves

It is sometimes useful to be able to show that a given quasi-coherent sheaf on an open subscheme extends to the whole scheme.

Lemma 23.20.1. *Let $j : U \to X$ be a quasi-compact open immersion of schemes.*

1. *Any quasi-coherent sheaf on U extends to a quasi-coherent sheaf on X.*
2. *Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $\mathcal{G} \subset \mathcal{F}|_U$ be a quasi-coherent subsheaf. There exists a quasi-coherent subsheaf $\mathcal{H}$ of $\mathcal{F}$ such that $\mathcal{H}|_U = \mathcal{G}$ as subsheaves of $\mathcal{F}|_U$.*
3. *Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $\mathcal{G}$ be a quasi-coherent sheaf on U. Let $\varphi : \mathcal{G} \to \mathcal{F}|_U$ be a morphism of $\mathcal{O}_U$-modules. There exists a quasi-coherent sheaf $\mathcal{H}$ of $\mathcal{O}_X$-modules and a map $\psi : \mathcal{H} \to \mathcal{F}$ such that $\mathcal{H}|_U = \mathcal{G}$ and that $\psi|_U = \varphi$.*

Proof. An immersion is separated (see Schemes, Lemma 21.23.7) and j is quasi-compact by assumption. Hence for any quasi-coherent sheaf $\mathcal{G}$ on U the sheaf $j_*\mathcal{G}$ is an extension to X. See Schemes, Lemma 21.24.1 and Sheaves, Section 6.31.

Assume $\mathcal{F}$, $\mathcal{G}$ are as in (2). Then $j_*\mathcal{G}$ is a quasi-coherent sheaf on X (see above). It is a subsheaf of $j_*j^*\mathcal{F}$. Hence the kernel

$$\mathcal{H} = \ker(\mathcal{F} \oplus j_*\mathcal{G} \longrightarrow j_*j^*\mathcal{F})$$

is quasi-coherent as well, see Schemes, Section 21.24. It is formal to check that $\mathcal{H} \subset \mathcal{F}$ and that $\mathcal{H}|_U = \mathcal{G}$ (using the material in Sheaves, Section 6.31 again).

The same proof as above works. Just take $\mathcal{H} = \ker(\mathcal{F} \oplus j_*\mathcal{G} \to j_*j^*\mathcal{F})$ with its obvious map to $\mathcal{F}$ and its obvious identification with $\mathcal{G}$ over U. $\square$

Lemma 23.20.2. *Let X be a quasi-compact and quasi-separated scheme. Let $U \subset X$ be a quasi-compact open. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{G} \subset \mathcal{F}|_U$ be a quasi-coherent $\mathcal{O}_U$-submodule which is of finite type. Then there exists a quasi-coherent submodule $\mathcal{G}' \subset \mathcal{F}$ which is of finite type such that $\mathcal{G}'|_U = \mathcal{G}$.*

Proof. Let n be the minimal number of affine opens $U_i \subset X$, $i = 1, \ldots, n$ such that $X = U \cup \bigcup U_i$. (Here we use that X is quasi-compact.) Suppose we can prove the lemma for the case $n = 1$. Then we can successively extend $\mathcal{G}$ to a $\mathcal{G}_1$ over $U \cup U_1$ to a $\mathcal{G}_2$ over $U \cup U_1 \cup U_2$ to a $\mathcal{G}_3$ over $U \cup U_1 \cup U_2 \cup U_3$, and so on. Thus we reduce to the case $n = 1$.

Thus we may assume that $X = U \cup V$ with V affine. Since X is quasi-separated and U, V are quasi-compact open, we see that $U \cap V$ is a quasi-compact open. It suffices to prove the lemma for the system $(V, U \cap V, \mathcal{F}|_V, \mathcal{G}|_{U\cap V})$ since we can glue the resulting sheaf $\mathcal{G}'$ over V to the given sheaf $\mathcal{G}$ over U along the common value over $U \cap V$. Thus we reduce to the case where X is affine.

Assume $X = Spec(R)$. Write $\mathcal{F} = \widetilde{M}$ for some R-module M. By Lemma 23.20.1 above we may find a quasi-coherent subsheaf $\mathcal{H} \subset \mathcal{F}$ which restricts to $\mathcal{G}$ over U. Write $\mathcal{H} = \widetilde{N}$ for some R-module N. For every $u \in U$ there exists an $f \in R$ such that $u \in D(f) \subset U$ and such that N_f is finitely generated, see Lemma 23.16.1. Since U is quasi-compact we can cover it by finitely many $D(f_i)$ such that N_{f_i} is generated by finitely many elements, say $x_{i,1}/f_i^N, \ldots, x_{i,r_i}/f_i^N$. Let $N' \subset N$ be the submodule generated by the elements $x_{i,j}$. Then the subsheaf $\mathcal{G} := \widetilde{N'} \subset \mathcal{H} \subset \mathcal{F}$ works. $\square$

Lemma 23.20.3. *Let X be a quasi-compact and quasi-separated scheme. Any quasi-coherent sheaf of $\mathcal{O}_X$-modules is the directed colimit of its quasi-coherent $\mathcal{O}_X$-submodules which are of finite type.*

Proof. The colimit is direct because if $\mathcal{G}_1, \mathcal{G}_2$ are quasi-coherent subsheaves of finite type, then $\mathcal{G}_1 + \mathcal{G}_2 \subset \mathcal{F}$ is a quasi-coherent subsheaf of finite type. Let $U \subset X$ be any affine open, and let $s \in \Gamma(U, \mathcal{F})$ be any section. Let $\mathcal{G} \subset \mathcal{F}|_U$ be the subsheaf generated by s. Then clearly $\mathcal{G}$ is quasi-coherent and has finite type as an $\mathcal{O}_U$-module. By Lemma 23.20.2 we see that $\mathcal{G}$ is the restriction of a quasi-coherent subsheaf $\mathcal{G}' \subset \mathcal{F}$ which has finite type. Since X has a basis for the topology consisting of affine opens we conclude that every local section of $\mathcal{F}$ is locally contained in a quasi-coherent submodule of finite type. Thus we win. $\square$

Lemma 23.20.4. *Let X be a scheme. Assume X is quasi-compact and quasi-separated. For any quasi-compact open $U \subset X$ there exists a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ which is of finite type such that the corresponding closed subscheme $Z \subset X$ has the property $X = U \coprod Z$ (set theoretically).*

Proof. Let $T = X \setminus U$. By Schemes, Lemma 21.12.4 there exists a unique quasi-coherent sheaf of ideals $\mathcal{J}$ cutting out the reduced induced closed subscheme structure on T. Note that $\mathcal{J}|_U = \mathcal{O}_U$ which is an $\mathcal{O}_U$-modules of finite type. By Lemma 23.20.2 there exists a quasi-coherent subsheaf $\mathcal{I} \subset \mathcal{J}$ which is of finite type and has the property that $\mathcal{I}|_U = \mathcal{J}|_U$. It is easy to see that $\mathcal{I}$ has the required properties. $\square$

Lemma 23.20.5. *(Variant of Lemma 23.20.2 dealing with modules of finite presentation.) Let X be a quasi-compact and quasi-separated scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $U \subset X$ be a quasi-compact open. Let $\mathcal{G}$ be an $\mathcal{O}_U$-module which of finite presentation. Let $\varphi : \mathcal{G} \to \mathcal{F}|_U$ be a morphism of $\mathcal{O}_U$-modules. Then there exists an $\mathcal{O}_X$-module $\mathcal{G}'$ of finite presentation, and a morphism of $\mathcal{O}_X$-modules $\varphi' : \mathcal{G}' \to \mathcal{F}$ such that $\mathcal{G}'|_U = \mathcal{G}$ and such that $\varphi'|_U = \varphi$.*

Proof. The beginning of the proof is a repeat of the beginning of the proof of Lemma 23.20.2. We write it out carefully anyway.

Let n be the minimal number of affine opens $U_i \subset X$, $i = 1, \ldots, n$ such that $X = U \cup \bigcup U_i$. (Here we use that X is quasi-compact.) Suppose we can prove the lemma for the case $n = 1$. Then we can successively extend the pair $(\mathcal{G}, \varphi)$ to a pair $(\mathcal{G}_1, \varphi_1)$ over $U \cup U_1$ to a pair $(\mathcal{G}_2, \varphi_2)$ over $U \cup U_1 \cup U_2$ to a pair $(\mathcal{G}_3, \varphi_3)$ over $U \cup U_1 \cup U_2 \cup U_3$, and so on. Thus we reduce to the case $n = 1$.

Thus we may assume that $X = U \cup V$ with V affine. Since X is quasi-separated and U quasi-compact, we see that $U \cap V \subset V$ is quasi-compact. Suppose we prove the lemma for the system $(V, U \cap V, \mathcal{F}|_V, \mathcal{G}|_{U \cap V}, \varphi|_{U \cap V})$ thereby producing $(\mathcal{G}', \varphi')$ over V. Then we can glue $\mathcal{G}'$ over V to the given sheaf $\mathcal{G}$ over U along the common value over $U \cap V$, and similarly we can glue the map φ' to the map φ along the common value over $U \cap V$. Thus we reduce to the case where X is affine.

Assume $X = Spec(R)$. By Lemma 23.20.1 above we may find a quasi-coherent sheaf $\mathcal{H}$ with a map $\psi : \mathcal{H} \to \mathcal{F}$ over X which restricts to $\mathcal{G}$ and φ over U. By Lemma 23.20.2 we can find a finite type quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{H}' \subset \mathcal{H}$ such that $\mathcal{H}'|_U = \mathcal{G}$. Thus after replacing $\mathcal{H}$ by $\mathcal{H}'$ and ψ by the restriction of ψ to $\mathcal{H}'$ we may assume that $\mathcal{H}$ is of finite type. By Lemma 23.16.2 we conclude that $\mathcal{H} = \widetilde{N}$ with N a finitely generated R-module. Hence there exists a surjection as in the following short exact sequence of quasi-coherent $\mathcal{O}_X$-modules

$$0 \to \mathcal{K} \to \mathcal{O}_X^{\oplus n} \to \mathcal{H} \to 0$$

where $\mathcal{K}$ is defined as the kernel. Since $\mathcal{G}$ is of finite presentation and $\mathcal{H}|_U = \mathcal{G}$ by Modules, Lemma 15.11.3 the restriction $\mathcal{K}|_U$ is an $\mathcal{O}_U$-module of finite type. Hence by Lemma 23.20.2 again we see that there exists a finite type quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{K}' \subset \mathcal{K}$ such that $\mathcal{K}'|_U = \mathcal{K}|_U$. The solution to the problem posed in the lemma is to set

$$\mathcal{G}' = \mathcal{O}_X^{\oplus n}/\mathcal{K}'$$

which is clearly of finite presentation and restricts to give $\mathcal{G}$ on U with φ' equal to the composition

$$\mathcal{G}' = \mathcal{O}_X^{\oplus n}/\mathcal{K}' \to \mathcal{O}_X^{\oplus n}/\mathcal{K} = \mathcal{H} \xrightarrow{\psi} \mathcal{F}.$$

This finishes the proof of the lemma. □

The following lemma says that every quasi-coherent sheaf on a quasi-compact and quasi-separated scheme is a filtered colimit of $\mathcal{O}$-modules of finite presentation. Actually, we reformulate this in (perhaps more familiar) terms of directed colimits over posets in the next lemma.

Lemma 23.20.6. *Let X be a scheme. Assume X is quasi-compact and quasi-separated. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. There exist*

(1) *a filtered index category $\mathcal{I}$ (see Categories, Definition 4.17.1),*
(2) *a diagram $\mathcal{I} \to Mod(\mathcal{O}_X)$ (see Categories, Section 4.13), $i \mapsto \mathcal{F}_i$,*

(3) *morphisms of $\mathcal{O}_X$-modules $\varphi_i : \mathcal{F}_i \to \mathcal{F}$*

such that each $\mathcal{F}_i$ is of finite presentation and such that the morphisms φ_i induce an isomorphism

$$colim_i \mathcal{F}_i = \mathcal{F}.$$

Proof. Choose a set I and for each $i \in I$ an $\mathcal{O}_X$-module of finite presentation and a homomorphism of $\mathcal{O}_X$-modules $\varphi_i : \mathcal{F}_i \to \mathcal{F}$ with the following property: For any $\psi : \mathcal{G} \to \mathcal{F}$ with $\mathcal{G}$ of finite presentation there is an $i \in I$ such that there exists an isomorphism $\alpha : \mathcal{F}_i \to \mathcal{G}$ with $\varphi_i = \psi \circ \alpha$. It is clear from Modules, Lemma 15.9.8 that such a set exists (see also its proof). We denote $\mathcal{I}$ the category with $Ob(\mathcal{I}) = I$ and given $i, i' \in I$ we set

$$Mor_{\mathcal{I}}(i, i') = \{\alpha : \mathcal{F}_i \to \mathcal{F}_{i'} \mid \alpha \circ \varphi_{i'} = \varphi_i\}.$$

We claim that $\mathcal{I}$ filtered category and that $\mathcal{F} = colim_i \mathcal{F}_i$.

Let $i, i' \in I$. Then we can consider the morphism

$$\mathcal{F}_i \oplus \mathcal{F}_{i'} \longrightarrow \mathcal{F}$$

which is the direct sum of φ_i and $\varphi_{i'}$. Since a direct sum of finitely presented $\mathcal{O}_X$-modules is finitely presented we see that there exists some $i'' \in I$ such that $\varphi_{i''} : \mathcal{F}_{i''} \to \mathcal{F}$ is isomorphic to the displayed arrow towards $\mathcal{F}$ above. Since there are commutative diagrams

$$\begin{tikzcd} \mathcal{F}_i \arrow[r] \arrow[d] & \mathcal{F} \arrow[d, equal] \\ \mathcal{F}_i \oplus \mathcal{F}_{i'} \arrow[r] & \mathcal{F} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{F}_{i'} \arrow[r] \arrow[d] & \mathcal{F} \arrow[d, equal] \\ \mathcal{F}_i \oplus \mathcal{F}_{i'} \arrow[r] & \mathcal{F} \end{tikzcd}$$

we see that there are morphisms $i \to i''$ and $i' \to i''$ in $\mathcal{I}$. Next, suppose that we have $i, i' \in I$ and morphisms $\alpha, \beta : i \to i'$ (corresponding to $\mathcal{O}_X$-module maps $\alpha, \beta : \mathcal{F}_i \to \mathcal{F}_{i'}$). In this case consider the coequalizer

$$\mathcal{G} = \text{Coker}(\mathcal{F}_i \xrightarrow{\alpha-\beta} \mathcal{F}_{i'})$$

Note that $\mathcal{G}$ is an $\mathcal{O}_X$-module of finite presentation. Since by definition of morphisms in the category $\mathcal{I}$ we have $\varphi_{i'} \circ \alpha = \varphi_{i'} \circ \beta$ we see that we get an induced map $\psi : \mathcal{G} \to \mathcal{F}$. Hence again the pair $(\mathcal{G}, \psi)$ is isomorphic to the pair $(\mathcal{F}_{i''}, \varphi_{i''})$ for some i''. Hence we see that there exists a morphism $i' \to i''$ in $\mathcal{I}$ which equalizes α and β. Thus we have shown that the category $\mathcal{I}$ is filtered.

We still have to show that the colimit of the diagram is $\mathcal{F}$. By definition of the colimit, and by our definition of the category $\mathcal{I}$ there is a canonical map

$$\varphi : colim_i \mathcal{F}_i \longrightarrow \mathcal{F}.$$

Pick $x \in X$. Let us show that φ_x is an isomorphism. Recall that

$$(colim_i \mathcal{F}_i)_x = colim_i \mathcal{F}_{i,x},$$

see Sheaves, Section 6.29. First we show that the map φ_x is injective. Suppose that $s \in \mathcal{F}_{i,x}$ is an element such that s maps to zero in $\mathcal{F}_x$. Then there exists a quasi-compact open U such that s comes from $s \in \mathcal{F}_i(U)$ and such that $\varphi_i(s) = 0$ in $\mathcal{F}(U)$. By Lemma 23.20.2 we can find a finite type quasi-coherent subsheaf $\mathcal{K} \subset \text{Ker}(\varphi_i)$ which restricts to the quasi-coherent $\mathcal{O}_U$-submodule of $\mathcal{F}_i$ generated by s: $\mathcal{K}|_U = \mathcal{O}_U \cdot s \subset \mathcal{F}_i|_U$. Clearly, $\mathcal{F}_i/\mathcal{K}$ is of finite presentation and the map φ_i factors through the quotient map $\mathcal{F}_i \to \mathcal{F}_i/\mathcal{K}$. Hence we can find an $i' \in I$ and a morphism $\alpha : \mathcal{F}_i \to \mathcal{F}_{i'}$ in $\mathcal{I}$ which can be identified with the quotient map $\mathcal{F}_i \to \mathcal{F}_i/\mathcal{K}$. Then it follows that the section s maps to zero in $\mathcal{F}_{i'}(U)$ and in particular in $(colim_i \mathcal{F}_i)_x = colim_i \mathcal{F}_{i,x}$. The injectivity follows. Finally, we show that the map φ_x is surjective. Pick $s \in \mathcal{F}_x$. Choose a quasi-compact open neighbourhood $U \subset X$ of x such that s corresponds to a section $s \in \mathcal{F}(U)$. Consider the map $s : \mathcal{O}_U \to \mathcal{F}$ (multiplication by s). By Lemma 23.20.5 there exists an $\mathcal{O}_X$-module $\mathcal{G}$ of finite presentation and an $\mathcal{O}_X$-module map $\mathcal{G} \to \mathcal{F}$ such that $\mathcal{G}|_U \to \mathcal{F}|_U$ is identified with $s : \mathcal{O}_U \to \mathcal{F}$. Again by definition of $\mathcal{I}$ there exists an $i \in I$ such that $\mathcal{G} \to \mathcal{F}$ is isomorphic to $\varphi_i : \mathcal{F}_i \to \mathcal{F}$. Clearly there exists a section $s' \in \mathcal{F}_i(U)$ mapping to $s \in \mathcal{F}(U)$. This proves surjectivity and the proof of the lemma is complete. □

Lemma 23.20.7. *Let X be a scheme. Assume X is quasi-compact and quasi-separated. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. There exist*

(1) *a directed partially ordered set I (see Categories, Definition 4.19.2),*
(2) *a system $(\mathcal{F}_i, \varphi_{ii'})$ over I in $Mod(\mathcal{O}_X)$ (see Categories, Definition 4.19.1)*
(3) *morphisms of $\mathcal{O}_X$-modules $\varphi_i : \mathcal{F}_i \to \mathcal{F}$*

such that each $\mathcal{F}_i$ is of finite presentation and such that the morphisms φ_i induce an isomorphism

$$colim_i \mathcal{F}_i = \mathcal{F}.$$

Proof. This is a direct consequence of Lemma 23.20.6 and Categories, Lemma 4.19.3 (combined with the fact that colimits exist in the category of sheaves of $\mathcal{O}_X$-modules, see Sheaves, Section 6.29). □

Lemma 23.20.8. *Let X be a scheme. Assume X is quasi-compact and quasi-separated. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is the directed colimit of its finite type quasi-coherent submodules.*

Proof. If $\mathcal{G}, \mathcal{H} \subset \mathcal{F}$ are finite type quasi-coherent $\mathcal{O}_X$-submodules then the image of $\mathcal{G} \oplus \mathcal{H} \to \mathcal{F}$ is another finite type quasi-coherent $\mathcal{O}_X$-submodule which contains both of them. In this way we see that the system is directed. To show that $\mathcal{F}$ is the colimit of this system, write $\mathcal{F} = colim_i \mathcal{F}_i$ as a directed colimit of finitely presented quasi-coherent sheaves as in Lemma 23.20.7. Then the images $\mathcal{G}_i = \text{Im}(\mathcal{F}_i \to \mathcal{F})$ are finite type quasi-coherent subsheaves of $\mathcal{F}$. Since $\mathcal{F}$ is the colimit of these the result follows. □

Let X be a scheme. In the following lemma we use the notion of a *quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{A}$ of finite presentation*. This means that for every affine open $Spec(R) \subset X$ we have $\mathcal{A} = \widetilde{A}$ where A is a (commutative) R-algebra which is of finite presentation as an R-algebra.

Lemma 23.20.9. *Let X be a scheme. Assume X is quasi-compact and quasi-separated. Let $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_X$-algebra. There exist*

(1) *a directed partially ordered set I (see Categories, Definition 4.19.2),*
(2) *a system $(\mathcal{A}_i, \varphi_{ii'})$ over I in the category of $\mathcal{O}_X$-algebras,*
(3) *morphisms of $\mathcal{O}_X$-algebras $\varphi_i : \mathcal{A}_i \to \mathcal{A}$*

such that each $\mathcal{A}_i$ is a quasi-coherent $\mathcal{O}_X$-algebra of finite presentation and such that the morphisms φ_i induce an isomorphism

$$colim_i \mathcal{A}_i = \mathcal{A}.$$

Proof. First we write $\mathcal{A} = colim_i \mathcal{F}_i$ as a directed colimit of finitely presented quasi-coherent sheaves as in Lemma 23.20.7. For each i let $\mathcal{B}_i = \text{Sym}(\mathcal{F}_i)$ be the symmetric algebra on $\mathcal{F}_i$ over $\mathcal{O}_X$. Write $\text{I}_i = \ker(\mathcal{B}_i \to \mathcal{A})$. Write $\mathcal{I}_i = colim_j \mathcal{F}_{i,j}$ where $\mathcal{F}_{i,j}$ is a finite type quasi-coherent submodule of $\mathcal{I}_i$, see Lemma 23.20.8. Set $\mathcal{I}_{i,j} \subset \mathcal{I}_i$ equal to the $\mathcal{B}_i$-ideal generated by $\mathcal{F}_{i,j}$. Set $\mathcal{A}_{i,j} = \mathcal{B}_i/\mathcal{I}_{i,j}$. Then $\mathcal{A}_{i,j}$ is a quasi-coherent finitely presented $\mathcal{O}_X$-algebra. Define $(i,j) \leq (i',j')$ if $i \leq i'$ and the map $\mathcal{B}_i \to \mathcal{B}_{i'}$ maps the ideal $\mathcal{I}_{i,j}$ into the ideal $\mathcal{I}_{i',j'}$. Then it is clear that $\mathcal{A} = colim_{i,j}\mathcal{A}_{i,j}$. $\square$

Let X be a scheme. In the following lemma we use the notion of a *quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{A}$ of finite type*. This means that for every affine open $Spec(R) \subset X$ we have $\mathcal{A} = \widetilde{A}$ where A is a (commutative) R-algebra which is of finite type as an R-algebra.

Lemma 23.20.10. *Let X be a scheme. Assume X is quasi-compact and quasi-separated. Let $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_X$-algebra. Then $\mathcal{A}$ is the directed colimit of its finite type quasi-coherent $\mathcal{O}_X$-subalgebras.*

Proof. Omitted. Hint: Compare with the proof of Lemma 23.20.8. $\square$

23.21. Gabber's result

In this section we prove a result of Gabber which guarantees that on every scheme there exists a cardinal κ such that every quasi-coherent module $\mathcal{F}$ is the union of its quasi-coherent κ-generated subsheaves. It follows that the category of quasi-coherent sheaves on a scheme is a Grothendieck abelian category having limits and enough injectives[2].

Definition 23.21.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let κ be an infinite cardinal. We say a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is *κ-generated* if there exists an open covering $X = \bigcup U_i$ such that $\mathcal{F}|_{U_i}$ is generated by a subset $R_i \subset \mathcal{F}(U_i)$ whose cardinality is at most κ.

Note that a direct sum of at most κ κ-generated modules is again κ-generated because $\kappa \otimes \kappa = \kappa$, see Sets, Section 3.6. In particular this holds for the direct sum of two κ-generated modules. Moreover, a quotient of a κ-generated sheaf is κ-generated. (But the same needn't be true for submodules.)

Lemma 23.21.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let κ be a cardinal. There exists a set T and a family $(\mathcal{F}_t)_{t\in T}$ of κ-generated $\mathcal{O}_X$-modules such that every κ-generated $\mathcal{O}_X$-module is isomorphic to one of the $\mathcal{F}_t$.*

Proof. There is a set of coverings of X (provided we disallow repeats). Suppose $X = \bigcup U_i$ is a covering and suppose $\mathcal{F}_i$ is an $\mathcal{O}_{U_i}$-module. Then there is a set of isomorphism classes of $\mathcal{O}_X$-modules $\mathcal{F}$ with the property that $\mathcal{F}|_{U_i} \cong \mathcal{F}_i$ since there is a set of glueing maps. This reduces us to proving there is a set of (isomorphism classes of) quotients $\bigoplus_{k\in\kappa}\mathcal{O}_X \to \mathcal{F}$ for any ringed space X. This is clear. $\square$

Here is the result the title of this section refers to.

Lemma 23.21.3. *Let X be a scheme. There exists a cardinal κ such that every quasi-coherent module $\mathcal{F}$ is the directed colimit of its quasi-coherent κ-generated quasi-coherent subsheaves.*

Proof. Choose an affine open covering $X = \bigcup_{i\in I} U_i$. For each pair i, j choose an affine open covering $U_i \cap U_j = \bigcup_{k\in I_{ij}} U_{ijk}$. Write $U_i = \mathrm{Spec}(A_i)$ and $U_{ijk} = \mathrm{Spec}(A_{ijk})$. Let κ be any infinite cardinal $\geq$ than the cardinality of any of the sets I, I_{ij}.

Let $\mathcal{F}$ be a quasi-coherent sheaf. Set $M_i = \mathcal{F}(U_i)$ and $M_{ijk} = \mathcal{F}(U_{ijk})$. Note that

$$M_i \otimes_{A_i} A_{ijk} = M_{ijk} = M_j \otimes_{A_j} A_{ijk}.$$

[2]Nicely explained in a blog post by Akhil Mathew.

see Schemes, Lemma 21.7.3. Using the axiom of choice we choose a map

$$(i, j, k, m) \mapsto S(i, j, k, m)$$

which associates to every $i, j \in I$, $k \in I_{ij}$ and $m \in M_i$ a finite subset $S(i, j, k, m) \subset M_j$ such that we have

$$m \otimes 1 = \sum\nolimits_{m' \in S(i,j,k,m)} m' \otimes a_{m'}$$

in M_{ijk} for some $a_{m'} \in A_{ijk}$. Moreover, let's agree that $S(i, i, k, m) = \{m\}$ for all $i, j = i, k, m$ as above. Fix such a map.

Given a family $\mathcal{S} = (S_i)_{i\in I}$ of subsets $S_i \subset M_i$ of cardinality at most κ we set $\mathcal{S}' = (S'_i)$ where

$$S'_j = \bigcup\nolimits_{(i,j,k,m) \text{ such that } m\in S_i} S(i, j, k, m)$$

Note that $S_i \subset S'_i$. Note that S'_i has cardinality at most κ because it is a union over a set of cardinality at most κ of finite sets. Set $\mathcal{S}^{(0)} = \mathcal{S}$, $\mathcal{S}^{(1)} = \mathcal{S}'$ and by induction $\mathcal{S}^{(n+1)} = (\mathcal{S}^{(n)})'$. Then set $\mathcal{S}^{(\infty)} = \bigcup_{n\geq 0} \mathcal{S}^{(n)}$. Writing $\mathcal{S}^{(\infty)} = (S_i^{(\infty)})$ we see that for any element $m \in S_i^{(\infty)}$ the image of m in M_{ijk} can be written as a finite sum $\sum m' \otimes a_{m'}$ with $m' \in S_j^{(\infty)}$. In this way we see that setting

$$N_i = A_i\text{-submodule of } M_i \text{ generated by } S_i^{(\infty)}$$

we have

$$N_i \otimes_{A_i} A_{ijk} = N_j \otimes_{A_j} A_{ijk}.$$

as submodules of M_{ijk}. Thus there exists a quasi-coherent subsheaf $\mathcal{G} \subset \mathcal{F}$ with $\mathcal{G}(U_i) = N_i$. Moreover, by construction the sheaf $\mathcal{G}$ is κ-generated.

Let $\{\mathcal{G}_t\}_{t\in T}$ be the set of κ-generated quasi-coherent subsheaves. If $t, t' \in T$ then $\mathcal{G}_t + \mathcal{G}_{t'}$ is also a κ-generated quasi-coherent subsheaf as it is the image of the map $\mathcal{G}_t \oplus \mathcal{G}_{t'} \to \mathcal{F}$. Hence the system (ordered by inclusion) is directed. The arguments above show that every section of $\mathcal{F}$ over U_i is in one of the $\mathcal{G}_t$ (because we can start with $\mathcal{S}$ such that the given section is an element of S_i). Hence $colim_t \mathcal{G}_t \to \mathcal{F}$ is both injective and surjective as desired. $\square$

Proposition 23.21.4. *Let X be a scheme. The inclusion functor $QCoh(\mathcal{O}_X) \to Mod(\mathcal{O}_X)$ has a right adjoint*

$$Q^3 : Mod(\mathcal{O}_X) \longrightarrow QCoh(\mathcal{O}_X)$$

such that for every quasi-coherent sheaf $\mathcal{F}$ the adjunction mapping $Q(\mathcal{F}) \to \mathcal{F}$ is an isomorphism. Moreover, the category $QCoh(\mathcal{O}_X)$ has limits and enough injectives.

Proof. The two assertions about $Q(\mathcal{F}) \to \mathcal{F}$ and limits in $QCoh(\mathcal{O}_X)$ are formal consequences of the existence of Q, the fact that the inclusion is fully faithful, and the fact that $Mod(\mathcal{O}_X)$ has limits (see Modules, Section 15.3). The existence of injectives follows from the existence of injectives in $Mod(\mathcal{O}_X)$ (see Injectives, Lemma 17.9.1) and Homology, Lemma 10.22.3. Thus it suffices to construct Q.

Pick a cardinal κ as in Lemma 23.21.3. Pick a collection $(\mathcal{F}_t)_{t\in T}$ of κ-generated quasi-coherent sheaves as in Lemma 23.21.2. Given an object $\mathcal{G}$ of $QCoh(\mathcal{O}_X)$ we set

$$Q(\mathcal{G}) = colim_{(t,\alpha)} \mathcal{F}_t$$

The colimit is over the category of pairs (t, α) where $t \in T$ and $\alpha : \mathcal{F}_t \to \mathcal{G}$ is a morphism of $\mathcal{O}_X$-modules. A morphism $(t, \alpha) \to (t', \alpha')$ is given by a morphism $\beta : \mathcal{F}_t \to \mathcal{F}_{t'}$ such

[3] This functor is sometimes called the *coherator*.

that $\alpha' \circ \beta = \alpha$. By Schemes, Section 21.24 the colimit is quasi-coherent. Note that there is a canonical map $Q(\mathcal{G}) \to \mathcal{G}$ by definition of the colimit. The formula

$$Hom(\mathcal{H}, Q(\mathcal{G})) = Hom(\mathcal{H}, \mathcal{G})$$

holds for κ-generated quasi-coherent modules $\mathcal{H}$ by choice of the system $(\mathcal{F}_t)_{t\in T}$. It follows formally from Lemma 23.21.3 that this equality continuous to hold for any quasi-coherent module $\mathcal{H}$ on X. This finishes the proof. □

23.22. Sections of quasi-coherent sheaves

Here is a computation of sections of a quasi-coherent sheaf on a quasi-compact open of an affine spectrum.

Lemma 23.22.1. *Let A be a ring. Let $I \subset A$ be a finitely generated ideal. Let M be an A-module. Then there is a canonical map*

$$colim_n\, Hom_A(I^n, M) \longrightarrow \Gamma(Spec(A) \setminus V(I), \widetilde{M}).$$

This map is always injective. If for all $x \in M$ we have $Ix = 0 \Rightarrow x = 0$ then this map is an isomorphism. In general, set $M_n = \{x \in M \mid I^n x = 0\}$, then there is an isomorphism

$$colim_n\, Hom_A(I^n, M/M_n) \longrightarrow \Gamma(Spec(A) \setminus V(I), \widetilde{M}).$$

Proof. Since $I^n \subset I^{n+1}$ and $M_n \subset M_{n+1}$ we can use composition via these maps to get canonical maps of A-modules

$$Hom_A(I^n, M) \longrightarrow Hom_A(I^{n+1}, M)$$

and

$$Hom_A(I^n, M/M_n) \longrightarrow Hom_A(I^{n+1}, M/M_{n+1})$$

which we will use as the transition maps in the systems. Given an A-module map $\varphi : I^n \to M$, then we get a map of sheaves $\widetilde{\varphi} : \tilde{I} \to \widetilde{M}$ which we can restrict to the open $Spec(A) \setminus V(I)$. Since $\tilde{I}$ restricted to this open gives the structure sheaf we get an element of $\Gamma(Spec(A) \setminus V(I), \widetilde{M})$. We omit the verification that this is compatible with the transition maps in the system $Hom_A(I^n, M)$. This gives the first arrow. To get the second arrow we note that $\widetilde{M}$ and $\widetilde{M/M_n}$ agree over the open $Spec(A) \setminus V(I)$ since the sheaf $\widetilde{M_n}$ is clearly supported on $V(I)$. Hence we can use the same mechanism as before.

Next, we work out how to define this arrow in terms of algebra. Say $I = (f_1, \ldots, f_t)$. Then $Spec(A) \setminus V(I) = \bigcup_{i=1,\ldots,t} D(f_i)$. Hence

$$0 \to \Gamma(Spec(A) \setminus V(I), \widetilde{M}) \to \bigoplus\nolimits_i M_{f_i} \to \bigoplus\nolimits_{i,j} M_{f_i f_j}$$

is exact. Suppose that $\varphi : I^n \to M$ is an A-module map. Consider the vector of elements $\varphi(f_i^n)/f_i^n \in M_{f_i}$. It is easy to see that this vector maps to zero in the second direct sum of the exact sequence above. Whence an element of $\Gamma(Spec(A) \setminus V(I), \widetilde{M})$. We omit the verification that this description agrees with the one given above.

Let us show that the first arrow is injective using this description. Namely, if φ maps to zero, then for each i the element $\varphi(f_i^n)/f_i^n$ is zero in M_{f_i}. In other words we see that for each i we have $f_i^m \varphi(f_i^n) = 0$ for some $m \geq 0$. We may choose a single m which works for all i. Then we see that $\varphi(f_i^{n+m}) = 0$ for all i. It is easy to see that this means that $\varphi|_{I^{t(n+m-1)+1}} = 0$ in other words that φ maps to zero in the $t(n + m - 1) + 1$st term of the colimit. Hence injectivity follows.

Note that each $M_n = 0$ in case we have $Ix = 0 \Rightarrow x = 0$ for $x \in M$. Thus to finish the proof of the lemma it suffices to show that the second arrow is an isomorphism.

Let us attempt to construct an inverse of the second map of the lemma. Let $s \in \Gamma(Spec(A) \setminus V(I), \widetilde{M})$. This corresponds to a vector x_i/f_i^n with $x_i \in M$ of the first direct sum of the exact sequence above. Hence for each i, j there exists $m \geq 0$ such that $f_i^m f_j^m (f_j^n x_i - f_i^n x_j) = 0$ in M. We may choose a single m which works for all pairs i, j. After replacing x_i by $f_i^m x_i$ and n by $n + m$ we see that we get $f_j^n x_i = f_i^n x_j$ in M for all i, j. Let us introduce

$$K_n = \{x \in M \mid f_1^n x = \ldots = f_t^n x = 0\}$$

We claim there is an A-module map

$$\varphi : I^{t(n-1)+1} \longrightarrow M/K_n$$

which maps the monomial $f_1^{e_1} \ldots f_t^{e_t}$ with $\sum e_i = t(n-1) + 1$ to the class modulo K_n of the expression $f_1^{e_1} \ldots f_i^{e_i - n} \ldots f_t^{e_t} x_i$ where i is chosen such that $e_i \geq n$ (note that there is at least one such i). To see that this is indeed the case suppose that

$$\sum\nolimits_{E=(e_1,\ldots,e_t),|E|=t(n-1)+1} a_E f_1^{e_1} \ldots f_t^{e_t} = 0$$

is a relation between the monomials with coefficients a_E in A. Then we would map this to

$$z = \sum\nolimits_{E=(e_1,\ldots,e_t),|E|=t(n-1)+1} a_E f_1^{e_1} \ldots f_{i(E)}^{e_{i(E)}-n} \ldots f_t^{e_t} x_i$$

where for each multiindex E we have chosen a particular $i(E)$ such that $e_{i(E)} \geq n$. Note that if we multiply this by f_j^n for any j, then we get zero, since by the relations $f_j^n x_i = f_i^n x_j$ above we get

$$\begin{aligned} f_j^n z &= \sum\nolimits_{E=(e_1,\ldots,e_t),|E|=t(n-1)+1} a_E f_1^{e_1} \ldots f_j^{e_j+n} \ldots f_{i(E)}^{e_{i(E)}-n} \ldots f_t^{e_t} x_i \\ &= \sum\nolimits_{E=(e_1,\ldots,e_t),|E|=t(n-1)+1} a_E f_1^{e_1} \ldots f_t^{e_t} x_j = 0. \end{aligned}$$

Hence $z \in K_n$ and we see that every relation gets mapped to zero in M/K_n. This proves the claim.

Note that $K_n \subset M_{t(n-1)+1}$. Hence the map φ in particular gives rise to a A-module map $I^{t(n-1)+1} \to M/M_{t(n-1)+1}$. This proves the second arrow of the lemma is surjective. We omit the proof of injectivity. $\square$

Example 23.22.2. Let k be a field. Consider the ring

$$A = k[f, g, x, y, \{a_n, b_n\}_{n \geq 1}]/(fy - gx, \{a_n f^n + b_n g^n\}_{n \geq 1}).$$

Then $x/f \in A_f$ and $y/g \in A_g$ map to the same element of A_{fg}. Hence these define a section s of the structure sheaf of $Spec(A)$ over $D(f) \cup D(g) = Spec(A) \setminus V(I)$. Here $I = (f, g) \subset A$. However, there is no $n \geq 0$ such that s comes from an A-module map $\varphi : I^n \to A$ as in the source of the first displayed arrow of Lemma 23.22.1. Namely, given such a module map set $x_n = \varphi(f^n)$ and $y_n = \varphi(g^n)$. Then $f^m x_n = f^{n+m-1} x$ and $g^m y_n = g^{n+m-1} y$ for some $m \geq 0$ (see proof of the lemma). But then we would have $0 = \varphi(0) = \varphi(a_{n+m} f^{n+m} + b_{n+m} g^{n+m}) = a_{n+m} f^{n+m-1} x + b_{n+m} g^{n+m-1} y$ which is not the case in the ring A.

Lemma 23.22.3. *Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Consider the sheaf of $\mathcal{O}_X$-modules $\mathcal{F}'$ which associates to every open $U \subset X$*

$$\mathcal{F}'(U) = \{s \in \mathcal{F}(U) \mid \mathcal{I}s = 0\}$$

Assume $\mathcal{I}$ is of finite type. Then

(1) *$\mathcal{F}'$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules,*
(2) *on any affine open $U \subset X$ we have $\mathcal{F}'(U) = \{s \in \mathcal{F}(U) \mid \mathcal{I}(U)s = 0\}$, and*
(3) *$\mathcal{F}'_x = \{s \in \mathcal{F}_x \mid \mathcal{I}_x s = 0\}$.*

Proof. It is clear that the rule defining $\mathcal{F}'$ gives a subsheaf of $\mathcal{F}$ (the sheaf condition is easy to verify). Hence we may work locally on X to verify the other statements. In other words we may assume that $X = Spec(A)$, $\mathcal{F} = \widetilde{M}$ and $\mathcal{I} = \tilde{I}$. It is clear that in this case $\mathcal{F}'(U) = \{x \in M \mid Ix = 0\} =: M'$ because $\tilde{I}$ is generated by its global sections I which proves (2). To show $\mathcal{F}'$ is quasi-coherent it suffices to show that for every $f \in A$ we have $\{x \in M_f \mid I_f x = 0\} = (M')_f$. Write $I = (g_1, \ldots, g_t)$, which is possible because $\mathcal{I}$ is of finite type, see Lemma 23.16.1. If $x = y/f^n$ and $I_f x = 0$, then that means that for every i there exists an $m \geq 0$ such that $f^m g_i x = 0$. We may choose one m which works for all i (and this is where we use that I is finitely generated). Then we see that $f^m x \in M'$ and $x/f^n = f^m x/f^{n+m}$ in $(M')_f$ as desired. The proof of (3) is similar and omitted. □

Definition 23.22.4. Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The subsheaf $\mathcal{F}' \subset \mathcal{F}$ defined in Lemma 23.22.3 above is called the *subsheaf of sections annihilated by $\mathcal{I}$*.

Lemma 23.22.5. *Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $Z \subset X$ be the closed subscheme defined by $\mathcal{I}$ and set $U = X \setminus Z$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume that X is quasi-compact and quasi-separated and that $\mathcal{I}$ is of finite type. Let $\mathcal{F}_n \subset \mathcal{F}$ be subsheaf of sections annihilated by $\mathcal{I}^n$. The canonical map*

$$colim_n\, Hom_{\mathcal{O}_X}(\mathcal{I}^n, \mathcal{F}) \longrightarrow \Gamma(U, \mathcal{F})$$

is injective and the canonical map

$$colim_n\, Hom_{\mathcal{O}_X}(\mathcal{I}^n, \mathcal{F}/\mathcal{F}_n) \longrightarrow \Gamma(U, \mathcal{F})$$

is an isomorphism.

Proof. Let $Spec(A) = W \subset X$ be an affine open. Write $\mathcal{F}|_W = \widetilde{M}$ for some A-module M and $\mathcal{I}|_W = \tilde{I}$ for some ideal $I \subset A$. We omit the verification that $\mathcal{F}_n = \widetilde{M_n}$ where $M_n \subset M$ is defined as in Lemma 23.22.1. This proves (1). It also follows from Lemma 23.22.1 that we have an injection

$$colim_n\, Hom_{\mathcal{O}_W}(\mathcal{I}^n|_W, \mathcal{F}|_W) \longrightarrow \Gamma(U \cap W, \mathcal{F})$$

and a bijection

$$colim_n\, Hom_{\mathcal{O}_W}(\mathcal{I}^n|_W, (\mathcal{F}/\mathcal{F}_n)|_W) \longrightarrow \Gamma(U \cap W, \mathcal{F})$$

for any such affine open W.

To see (2) we choose a finite affine open covering $X = \bigcup_{j=1,\ldots,m} W_j$. The injectivity of the first arrow of (2) follows immediately from the above and the finiteness of the covering. Moreover for each pair j, j' we choose a finite affine open covering

$$W_j \cap W_{j'} = \bigcup\nolimits_{k=1,\ldots,m_{jj'}} W_{jj'k}.$$

Let $s \in \Gamma(U, \mathcal{F})$. As seen above for each j there exists an n_j and a map $\varphi_j : \mathcal{I}^{n_j}|_{W_j} \to (\mathcal{F}/\mathcal{F}_{n_j})|_{W_j}$ which corresponds to $s|_{W_j}$. By the same token for each triple (j, j', k) there exists an integer $n_{jj'k}$ such that the restriction of φ_j and $\varphi_{j'}$ as maps $\mathcal{I}^{n_{jj'k}} \to \mathcal{F}/\mathcal{F}_{n_{jj'k}}$

agree over $W_{jj'l}$. Let $n = \max\{n_j, n_{jj'k}\}$ and we see that the φ_j glue as maps $\mathcal{I}^n \to \mathcal{F}/\mathcal{F}_n$ over X. This proves surjectivity of the map. We omit the proof of injectivity. $\square$

23.23. Ample invertible sheaves

Recall from Modules, Lemma 15.21.7 that given an invertible sheaf $\mathcal{L}$ on a locally ringed space X, and given a global section s of $\mathcal{L}$ the set $X_s = \{x \in X \mid s \not\in \mathfrak{m}_x\mathcal{L}_x\}$ is open. A general remark is that $X_s \cap X_{s'} = X_{ss'}$, where ss' denote the section $s \otimes s' \in \Gamma(X, \mathcal{L} \otimes \mathcal{L}')$.

Definition 23.23.1. Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. We say $\mathcal{L}$ is *ample* if

(1) X is quasi-compact, and
(2) for every $x \in X$ there exists an $n \geq 1$ and $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that $x \in X_s$ and X_s is affine.

Lemma 23.23.2. *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $n \geq 1$. Then $\mathcal{L}$ is ample if and only if $\mathcal{L}^{\otimes n}$ is ample.*

Proof. This follows from the fact that $X_{s^n} = X_s$. $\square$

Lemma 23.23.3. *Let X be a scheme. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. For any closed subscheme $Z \subset X$ the restriction of $\mathcal{L}$ to Z is ample.*

Proof. This is clear since a closed subset of a quasi-compact space is quasi-compact and a closed subscheme of an affine scheme is affine (see Schemes, Lemma 21.8.2). $\square$

Lemma 23.23.4. *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$. For any affine $U \subset X$ the intersection $U \cap X_s$ is affine.*

Proof. This translates into the following algebra problem. Let R be a ring. Let N be an invertible R-module (i.e., locally free of rank 1). Let $s \in N$ be an element. Then $U = \{\mathfrak{p} \mid s \not\in \mathfrak{p}N\}$ is an affine open subset of $Spec(R)$. This you can see as follows. Think of s as an R-module map $R \to N$. This gives rise to R-module maps $N^{\otimes k} \to N^{\otimes k+1}$. Consider

$$R' = colim_n \, N^{\otimes n}$$

with transition maps as above. Define an R-algebra structure on R' by the rule $x \cdot y = x \otimes y \in N^{\otimes n+m}$ if $x \in N^{\otimes n}$ and $y \in N^{\otimes m}$. We claim that $Spec(R') \to Spec(R)$ is an open immersion with image U.

To prove this is a local question on $Spec(R)$. Let $\mathfrak{p} \in Spec(R)$. Pick $f \in R$, $f \not\in \mathfrak{p}$ such that $N_f \cong R_f$ as a module. Replacing R by R_f, N by N_f and R' by $R'_f = colim\, N_f^{\otimes n}$ we may assume that $N \cong R$. Say $N = R$. In this case s is an element of R and it is easy to see that $R' \cong R_s$. Thus the lemma follows. $\square$

Recall that given a scheme X and an invertible sheaf $\mathcal{L}$ on X we get a graded ring $\Gamma_*(X, \mathcal{L}) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$, see Modules, Definition 15.21.4. Also, given a sheaf of $\mathcal{O}_X$-modules we have the graded $\Gamma_*(X, \mathcal{L})$-module $\Gamma_*(X, \mathcal{F}) = \Gamma_*(X, \mathcal{L}, \mathcal{F})$.

Lemma 23.23.5. *Let X be a scheme. Let $\mathcal{L}$ be an invertible sheaf on X. Let $s \in \Gamma(X, \mathcal{L})$. If X is quasi-compact and quasi-separated, the canonical map*

$$\Gamma_*(X, \mathcal{L})_{(s)} \longrightarrow \Gamma(X_s, \mathcal{O})$$

which maps a/s^n to $a \otimes s^{-n}$ is an isomorphism. Moreover, if $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module then the map

$$\Gamma_*(X, \mathcal{L}, \mathcal{F})_{(s)} \longrightarrow \Gamma(X_s, \mathcal{F})$$

is an isomorphism.

Proof. Consider the scheme

$$\pi : L^* = \underline{Spec}_X \left(\bigoplus\nolimits_{n \in \mathbf{Z}} \mathcal{L}^{\otimes n} \right) \longrightarrow X$$

see Constructions, Section 22.4. Since the inverse image $\pi^{-1}(U)$ of every affine open $U \subset X$ is affine (see Constructions, Lemma 22.4.6), it follows that L^* quasi-compact and separated, since X is assumed quasi-compact and separated (use Schemes, Lemma 21.21.7). Note that s gives rise to an element $f \in \Gamma(L^*, \mathcal{O})$, via $\pi_*\mathcal{O}_{L^*} = \bigoplus_{n\in\mathbf{Z}} \mathcal{L}^{\otimes n}$. Note that $(L^*)_f = \pi^{-1}(X_s)$. Hence we have

$$\begin{aligned} \left(\bigoplus\nolimits_{n \in \mathbf{Z}} \Gamma(X, \mathcal{L}^{\otimes n}) \right)_s &= \Gamma(L^*, \mathcal{O}_{L^*})_f \\ &= \Gamma((L^*)_f, \mathcal{O}_{L^*}) \\ &= \bigoplus\nolimits_{n \in \mathbf{Z}} \Gamma(X_s, \mathcal{L}^{\otimes n}) \end{aligned}$$

where the middle "=" is Lemma 23.15.2. The first statement of the lemma follows from this equality by looking at degree zero terms. The second statement also follows from Lemma 23.15.2 applied to the quasi-coherent sheaf of $\mathcal{O}_{L^*}$-modules $\pi^*\mathcal{F}$ using that

$$\pi_*\pi^*\mathcal{F} = \mathcal{F} \otimes_{\mathcal{O}_X} \left(\bigoplus\nolimits_{n \in \mathbf{Z}} \mathcal{L}^{\otimes n} \right) = \bigoplus\nolimits_{n \in \mathbf{Z}} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$$

which is proved by computing both sides on affine opens of X. □

Lemma 23.23.6. *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume the open sets X_s, where $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ and $n \geq 1$, form a basis for the topology on X. Then among those opens, the open sets X_s which are affine form a basis for the topology on X.*

Proof. Let $x \in X$. Choose an affine open neighbourhood $Spec(R) = U \subset X$ of x. By assumption, there exists a $n \geq 1$ and a $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that $X_s \subset U$. By Lemma 23.23.4 above the intersection $X_s = U \cap X_s$ is affine. Since U can be chosen arbitrarily small we win. □

Lemma 23.23.7. *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume for every point x of X there exists $n \geq 1$ and $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that $x \in X_s$ and X_s is affine. Then X is quasi-separated.*

Proof. By assumption we can find a covering of X by affine opens of the form X_s. By Schemes, Lemma 21.21.7 it suffices to show that $X_s \cap X_{s'}$ is quasi-compact whenever X_s is affine. This is true by Lemma 23.23.4. □

Lemma 23.23.8. *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Set $S = \Gamma_*(X, \mathcal{L})$ as a graded ring. If every point of X is contained in one of the open subschemes X_s, for some $s \in S_+$ homogeneous, then there is a canonical morphism of schemes*

$$f : X \longrightarrow Y = Proj(S),$$

to the homogeneous spectrum of S (see Constructions, Section 22.8). This morphism has the following properties

1. *$f^{-1}(D_+(s)) = X_s$ for any $s \in S_+$ homogeneous,*
2. *there are $\mathcal{O}_Y$-module maps $f^*\mathcal{O}_Y(n) \to \mathcal{L}^{\otimes n}$ compatible with multiplication maps, see Constructions, Equation (22.10.1.1),*
3. *the compositions $S_n \to \Gamma(Y, \mathcal{O}_Y(n)) \to \Gamma(X, \mathcal{L}^{\otimes n})$ are equal to the identity maps, and*

(4) *for every $x \in X$ there is an integer $d \geq 1$ and an open neighbourhood $U \subset X$ of x such that $f^*\mathcal{O}_Y(dn)|_U \to \mathcal{L}^{\otimes dn}|_U$ is an isomorphism for all $n \in \mathbf{Z}$.*

Proof. Denote $\psi : S \to \Gamma_*(X, \mathcal{L})$ the identity map. We are going to use the triple $(U(\psi), r_{\mathcal{L},\psi}, \theta)$ of Constructions, Lemma 22.14.1. By assumption the open subscheme $U(\psi)$ of equals X. Hence $r_{\mathcal{L},\psi} : U(\psi) \to Y$ is defined on all of X. We set $f = r_{\mathcal{L},\psi}$. The maps in part (2) are the components of θ. Part (3) follows from condition (2) in the lemma cited above. Part (1) follows from (3) combined with condition (1) in the lemma cited above. Part (4) follows from the last statement in Constructions, Lemma 22.14.1 since the map α mentioned there is an isomorphism. □

Lemma 23.23.9. *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Set $S = \Gamma_*(X, \mathcal{L})$. Assume (a) every point of X is contained in one of the open subschemes X_s, for some $s \in S_+$ homogeneous, and (b) X is quasi-compact. Then the canonical morphism of schemes $f : X \longrightarrow Proj(S)$ of Lemma 23.23.8 above is quasi-compact.*

Proof. It suffices to show that $f^{-1}(D_+(s))$ is quasi-compact for any $s \in S_+$ homogeneous. Write $X = \bigcup_{i=1,\ldots,n} X_i$ as a finite union of affine opens. By Lemma 23.23.4 each intersection $X_s \cap X_i$ is affine. Hence $X_s = \bigcup_{i=1,\ldots,n} X_s \cap X_i$ is quasi-compact. □

Lemma 23.23.10. *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Set $S = \Gamma_*(X, \mathcal{L})$. Assume $\mathcal{L}$ is ample. Then the canonical morphism of schemes $f : X \longrightarrow Proj(S)$ of Lemma 23.23.8 is an open immersion.*

Proof. By Lemma 23.23.7 we see that X is quasi-separated. Choose finitely many $s_1, \ldots, s_n \in S_+$ homogeneous such that X_{s_i} are affine, and $X = \bigcup X_{s_i}$. Say s_i has degree d_i. The inverse image of $D_+(s_i)$ under f is X_{s_i}, see Lemma 23.23.8. By Lemma 23.23.5 the ring map

$$(S^{(d_i)})_{(s_i)} = \Gamma(D_+(s_i), \mathcal{O}_{\mathrm{Proj}(S)}) \longrightarrow \Gamma(X_{s_i}, \mathcal{O}_X)$$

is an isomorphism. Hence f induces an isomorphism $X_{s_i} \to D_+(s_i)$. Thus f is an isomorphism of X onto the open subscheme $\bigcup_{i=1,\ldots,n} D_+(s_i)$ of $\mathrm{Proj}(S)$. □

Lemma 23.23.11. *Let X be a scheme. Let S be a graded ring. Assume X is quasi-compact, and assume there exists an open immersion*

$$j : X \longrightarrow Y = Proj(S).$$

Then $j^*\mathcal{O}_Y(d)$ is an invertible ample sheaf for some $d > 0$.

Proof. This is Constructions, Lemma 22.10.6. □

Proposition 23.23.12. *Let X be a quasi-compact scheme. Let $\mathcal{L}$ be an invertible sheaf on X. Set $S = \Gamma_*(X, \mathcal{L})$. The following are equivalent:*

(1) *$\mathcal{L}$ is ample,*
(2) *the open sets X_s, with $s \in S_+$ homogeneous, cover X and the associated morphism $X \to Proj(S)$ is an open immersion,*
(3) *the open sets X_s, with $s \in S_+$ homogeneous, form a basis for the topology of X,*
(4) *the open sets X_s, with $s \in S_+$ homogeneous, which are affine form a basis for the topology of X,*
(5) *for every quasi-coherent sheaf $\mathcal{F}$ on X the sum of the images of the canonical maps*

$$\Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) \otimes_{\mathbf{Z}} \mathcal{L}^{\otimes -n} \longrightarrow \mathcal{F}$$

with $n \geq 1$ equals $\mathcal{F}$,

(6) *same property as (5) with $\mathcal{F}$ ranging over all quasi-coherent sheaves of ideals,*
(7) *X is quasi-separated and for every quasi-coherent sheaf $\mathcal{F}$ of finite type on X there exists an integer n_0 such that $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ is globally generated for all $n \geq n_0$,*
(8) *X is quasi-separated and for every quasi-coherent sheaf $\mathcal{F}$ of finite type on X there exist integers $n > 0$, $k \geq 0$ such that $\mathcal{F}$ is a quotient of a direct sum of k copies of $\mathcal{L}^{\otimes -n}$, and*
(9) *same as in (8) with $\mathcal{F}$ ranging over all sheaves of ideals of finite type on X.*

Proof. Lemma 23.23.10 is (1) $\Rightarrow$ (2). Lemmas 23.23.2 and 23.23.11 provide the implication (1) $\Leftarrow$ (2). The implications (2) $\Rightarrow$ (4) $\Rightarrow$ (3) are clear from Constructions, Section 22.8. Lemma 23.23.6 is (3) $\Rightarrow$ (1). Thus we see that the first 4 conditions are all equivalent.

Assume the equivalent conditions (1) -- (4). Note that in particular X is separated (as an open subscheme of the separated scheme $\text{Proj}(S)$). Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Choose $s \in S_+$ homogeneous such that X_s is affine. We claim that any section $m \in \Gamma(X_s, \mathcal{F})$ is in the image of one of the maps displayed in (5) above. This will imply (5) since these affines X_s cover X. Namely, by Lemma 23.23.5 we may write m as the image of $m' \otimes s^{-n}$ for some $n \geq 1$, some $m' \in \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$. This proves the claim.

Clearly (5) $\Rightarrow$ (6). Let us assume (6) and prove $\mathcal{L}$ is ample. Pick $x \in X$. Let $U \subset X$ be an affine open which contains x. Set $Z = X \setminus U$. We may think of Z as a reduced closed subscheme, see Schemes, Section 21.12. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaf of ideals corresponding to the closed subscheme Z. By assumption (6), there exists an $n \geq 1$ and a section $s \in \Gamma(X, \mathcal{I} \otimes \mathcal{L}^{\otimes n})$ such that s does not vanish at x (more precisely such that $s \not\in \mathfrak{m}_x \mathcal{I}_x \otimes \mathcal{L}_x^{\otimes n}$). We may think of s as a section of $\mathcal{L}^{\otimes n}$. Since it clearly vanishes along Z we see that $X_s \subset U$. Hence X_s is affine, see Lemma 23.23.4. This proves that $\mathcal{L}$ is ample. At this point we have proved that (1) -- (6) are equivalent.

Assume the equivalent conditions (1) -- (6). In the following we will use the fact that the tensor product of two sheaves of modules which are globally generated is globally generated without further mention (see Modules, Lemma 15.4.3). By (1) we can find elements $s_i \in S_{d_i}$ with $d_i \geq 1$ such that $X = \bigcup_{i=1,\ldots,n} X_{s_i}$. Set $d = d_1 \ldots d_n$. It follows that $\mathcal{L}^{\otimes d}$ is globally generated by

$$s_1^{d/d_1}, \ldots, s_n^{d/d_n}.$$

This means that if $\mathcal{L}^{\otimes j}$ is globally generated then so is $\mathcal{L}^{\otimes j+dn}$ for all $n \geq 0$. Fix a $j \in \{0, \ldots, d-1\}$. For any point $x \in X$ there exists an $n \geq 1$ and a global section s of $\mathcal{L}^{j+dn}$ which does not vanish at x, as follows from (5) applied to $\mathcal{F} = \mathcal{L}^{\otimes j}$ and ample invertible sheaf $\mathcal{L}^{\otimes d}$. Since X is quasi-compact there we may find a finite list of integers n_i and global sections s_i of $\mathcal{L}^{\otimes j+dn_i}$ which do not vanish at any point of X. Since $\mathcal{L}^{\otimes d}$ is globally generated this means that $\mathcal{L}^{\otimes j+dn}$ is globally generated where $n = \max\{n_i\}$. Since we proved this for every congruence class mod d we conclude that there exists an $n_0 = n_0(\mathcal{L})$ such that $\mathcal{L}^{\otimes n}$ is globally generated for all $n \geq n_0$. At this point we see that if $\mathcal{F}$ is globally generated then so is $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ for all $n \geq n_0$.

We continue to assume the equivalent conditions (1) -- (6). Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules of finite type. Denote $\mathcal{F}_n \subset \mathcal{F}$ the image of the canonical map of (5). By construction $\mathcal{F}_n \otimes \mathcal{L}^{\otimes n}$ is globally generated. By (5) we see $\mathcal{F}$ is the sum of the subsheaves $\mathcal{F}_n$, $n \geq 1$. By Modules, Lemma 15.9.7 we see that $\mathcal{F} = \sum_{n=1,\ldots,N} \mathcal{F}_n$ for some $N \geq 1$. It follows that $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated whenever $n \geq N + n_0(\mathcal{L})$ with $n_0(\mathcal{L})$ as above. We conclude that (1) -- (6) implies (7).

Assume (7). Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules of finite type. By (7) there exists an integer $n \geq 1$ such that the canonical map

$$\Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) \otimes_{\mathbf{Z}} \mathcal{L}^{\otimes -n} \longrightarrow \mathcal{F}$$

is surjective. Let I be the set of finite subsets of $\Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$ partially ordered by inclusion. Then I is a directed partially ordered set. For $i = \{s_1, \ldots, s_{r(i)}\}$ let $\mathcal{F}_i \subset \mathcal{F}$ be the image of the map

$$\bigoplus\nolimits_{j=1,\ldots,r(i)} \mathcal{L}^{\otimes -n} \longrightarrow \mathcal{F}$$

which is multiplication by s_j on the jth factor. The surjectivity above implies that $\mathcal{F} = \operatorname{colim}_{i \in I} \mathcal{F}_i$. Hence Modules, Lemma 15.9.7 applies and we conclude that $\mathcal{F} = \mathcal{F}_i$ for some i. Hence we have proved (8). In other words, (7) $\Rightarrow$ (8).

The implication (8) $\Rightarrow$ (9) is trivial.

Finally, assume (9). Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. By Lemma 23.20.3 (this is where we use the condition that X be quasi-separated) we see that $\mathcal{I} = \operatorname{colim}_\alpha I_\alpha$ with each I_α quasi-coherent of finite type. Since by assumption each of the I_α is a quotient of negative tensor powers of $\mathcal{L}$ we conclude the same for $\mathcal{I}$ (but of course without the finiteness or boundedness of the powers). Hence we conclude that (9) implies (6). This ends the proof of the proposition. □

23.24. Affine and quasi-affine schemes

Lemma 23.24.1. *Let X be a scheme. Then X is quasi-affine if and only if $\mathcal{O}_X$ is ample.*

Proof. Suppose that X is quasi-affine. Consider the open immersion

$$j : X \longrightarrow Spec(\Gamma(X, \mathcal{O}_X))$$

from Lemma 23.15.4. Note that $Spec(A) = \text{Proj}(A[T])$, see Constructions, Example 22.8.14. Hence we can apply Lemma 23.23.11 to deduce that $\mathcal{O}_X$ is ample.

Suppose that $\mathcal{O}_X$ is ample. Note that $\Gamma_*(X, \mathcal{O}_X) \cong \Gamma(X, \mathcal{O}_X)[T]$ as graded rings. Hence the result follows from Lemmas 23.23.10 and 23.15.4 taking into account that $Spec(A) = \text{Proj}(A[T])$ for any ring A as seen above. □

Lemma 23.24.2. *Let X be a scheme. Suppose that there exist finitely many elements $f_1, \ldots, f_n \in \Gamma(X, \mathcal{O}_X)$ such that*

1. *each X_{f_i} is an affine open of X, and*
2. *the ideal generated by $f_1, \ldots, f_n$ in $\Gamma(X, \mathcal{O}_X)$ is equal to the unit ideal.*

Then X is affine.

Proof. Assume we have $f_1, \ldots, f_n$ as in the lemma. We may write $1 = \sum g_i f_i$ for some $g_j \in \Gamma(X, \mathcal{O}_X)$ and hence it is clear that $X = \bigcup X_{f_i}$. (The f_i's cannot all vanish at a point.) Since each X_{f_i} is quasi-compact (being affine) it follows that X is quasi-compact. Hence we see that X is quasi-affine by Lemma 23.24.1 above. Consider the open immersion

$$j : X \to Spec(\Gamma(X, \mathcal{O}_X)),$$

see Lemma 23.15.4. The inverse image of the standard open $D(f_i)$ on the right hand side is equal to X_{f_i} on the left hand side and the morphism j induces an isomorphism $X_{f_i} \cong D(f_i)$, see Lemma 23.15.3. Since the f_i generate the unit ideal we see that $Spec(\Gamma(X, \mathcal{O}_X)) = \bigcup_{i=1,\ldots,n} D(f_i)$. Thus j is an isomorphism. □

23.25. Quasi-coherent sheaves and ample invertible sheaves

Situation 23.25.1. Let X be a scheme. Let $\mathcal{L}$ be an invertible sheaf on X. Assume $\mathcal{L}$ is ample. Set $S = \Gamma_*(X, \mathcal{L})$ as a graded ring. Set $Y = \text{Proj}(S)$. Let $f : X \to Y$ be the canonical morphism of Lemma 23.23.8. It comes equipped with a $\mathbf{Z}$-graded $\mathcal{O}_X$-algebra map $\bigoplus f^*\mathcal{O}_Y(n) \to \bigoplus \mathcal{L}^{\otimes n}$.

The following lemma is really a special case of the next lemma but it seems like a good idea to point out its validity first.

Lemma 23.25.2. *In Situation 23.25.1. The canonical morphism $f : X \to Y$ maps X into the open subscheme $W = W_1 \subset Y$ where $\mathcal{O}_Y(1)$ is invertible and where all multiplication maps $\mathcal{O}_Y(n) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(m) \to \mathcal{O}_Y(n+m)$ are isomorphisms (see Constructions, Lemma 22.10.4). Moreover, the maps $f^*\mathcal{O}_Y(n) \to \mathcal{L}^{\otimes n}$ are all isomorphisms.*

Proof. By Proposition 23.23.12 there exists an integer n_0 such that $\mathcal{L}^{\otimes n}$ is globally generated for all $n \geq n_0$. Let $x \in X$ be a point. By the above we can find $a \in S_{n_0}$ and $b \in S_{n_0+1}$ such that a and b do not vanish at x. Hence $f(x) \in D_+(a) \cap D_+(b) = D_+(ab)$. By Constructions, Lemma 22.10.4 we see that $f(x) \in W_1$ as desired. By Constructions, Lemma 22.14.1 which was used in the construction of the map f the maps $f^*\mathcal{O}_Y(n_0) \to \mathcal{L}^{\otimes n_0}$ and $f^*\mathcal{O}_Y(n_0+1) \to \mathcal{L}^{\otimes n_0+1}$ are isomorphisms in a neighbourhood of x. By compatibility with the algebra structure and the fact that f maps into W we conclude all the maps $f^*\mathcal{O}_Y(n) \to \mathcal{L}^{\otimes n}$ are isomorphisms in a neighbourhood of x. Hence we win. $\square$

Recall from Modules, Definition 15.21.4 that given a locally ringed space X, an invertible sheaf $\mathcal{L}$, and a $\mathcal{O}_X$-module $\mathcal{F}$ we have the graded $\Gamma_*(X, \mathcal{L})$-module

$$\Gamma(X, \mathcal{L}, \mathcal{F}) = \bigoplus\nolimits_{n \in \mathbf{Z}} \Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}).$$

The following lemma says that, in Situation 23.25.1, we can recover a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ from this graded module. Take a look also at Constructions, Lemma 22.13.7 where we prove this lemma in the special case $X = \mathbf{P}^n_R$.

Lemma 23.25.3. *In Situation 23.25.1. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Set $M = \Gamma_*(X, \mathcal{L}, \mathcal{F})$ as a graded S-module. There are isomorphisms*

$$f^*\widetilde{M} \longrightarrow \mathcal{F}$$

functorial in $\mathcal{F}$ such that $M_0 \to \Gamma(Proj(S), \widetilde{M}) \to \Gamma(X, \mathcal{F})$ is the identity map.

Proof. Let $s \in S_+$ be homogeneous such that X_s is affine open in X. Recall that $\widetilde{M}|_{D_+(s)}$ corresponds to the $S_{(s)}$-module $M_{(s)}$, see Constructions, Lemma 22.8.4. Recall that $f^{-1}(D_+(s)) = X_s$. As X carries an ample invertible sheaf it is quasi-compact and quasi-separated, see Section 23.23. By Lemma 23.23.5 there is a canonical isomorphism $M_{(s)} = \Gamma_*(X, \mathcal{L}, \mathcal{F})_{(s)} \to \Gamma(X_s, \mathcal{F})$. Since $\mathcal{F}$ is quasi-coherent this leads to a canonical isomorphism

$$f^*\widetilde{M}|_{X_s} \to \mathcal{F}|_{X_s}$$

Since $\mathcal{L}$ is ample on X we know that X is covered by the affine opens of the form X_s. Hence it suffices to prove that the displayed maps glue on overlaps. Proof of this is omitted. $\square$

Remark 23.25.4. With assumptions and notation of Lemma 23.25.3. Denote the displayed map of the lemma by $\theta_{\mathcal{F}}$. Note that the isomorphism $f^*\mathcal{O}_Y(n) \to \mathcal{L}^{\otimes n}$ of Lemma 23.25.2 is just $\theta_{\mathcal{L}^{\otimes n}}$. Consider the multiplication maps

$$\widetilde{M} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(n) \longrightarrow \widetilde{M(n)}$$

see Constructions, Equation (22.10.1.5). Pull this back to X and consider

$$\begin{tikzcd} f^*\widetilde{M} \otimes_{\mathcal{O}_X} f^*\mathcal{O}_Y(n) \arrow[r] \arrow[d, "\theta_{\mathcal{F}} \otimes \theta_{\mathcal{L}^{\otimes n}}"'] & f^*\widetilde{M(n)} \arrow[d, "\theta_{\mathcal{F}\otimes\mathcal{L}^{\otimes n}}"] \\ \mathcal{F} \otimes \mathcal{L}^{\otimes n} \arrow[r, "\mathrm{id}"] & \mathcal{F} \otimes \mathcal{L}^{\otimes n} \end{tikzcd}$$

Here we have used the obvious identification $M(n) = \Gamma_*(X, \mathcal{L}, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$. This diagram commutes. Proof omitted.

23.26. Finding suitable affine opens

In this section we collect some results on the existence of affine opens in more and less general situations.

Lemma 23.26.1. *Let X be a quasi-separated scheme. Let $Z_1, \ldots, Z_n$ be pairwise distinct irreducible components of X, see Topology, Section 5.5. Let $\eta_i \in Z_i$ be their generic points, see Schemes, Lemma 21.11.1. There exist affine open neighbourhoods $\eta_i \in U_i$ such that $U_i \cap U_j = \emptyset$ for all $i \neq j$. In particular, $U = U_1 \cup \ldots \cup U_n$ is an affine open containing all of the points $\eta_1, \ldots, \eta_n$.*

Proof. Let V_i be any affine open containing η_i and disjoint from the closed set $Z_1 \cup \ldots \hat{Z}_i \ldots \cup Z_n$. Since X is quasi-separated for each i the union $W_i = \bigcup_{j, j \neq i} V_i \cap V_j$ is a quasi-compact open of V_i not containing η_i. We can find open neighbourhoods $U_i \subset V_i$ containing η_i and disjoint from W_i by Algebra, Lemma 7.23.4. Finally, U is affine since it is the spectrum of the ring $R_1 \times \ldots \times R_n$ where $R_i = \mathcal{O}_X(U_i)$, see Schemes, Lemma 21.6.8. □

Remark 23.26.2. Lemma 23.26.1 above is false if X is not quasi-separated. Here is an example. Take $R = \mathbf{Q}[x, y_1, y_2, \ldots]/((x-i)y_i)$. Consider the minimal prime ideal $\mathfrak{p} = (y_1, y_2, \ldots)$ of R. Glue two copies of $Spec(R)$ along the (not quasi-compact) open $Spec(R) \setminus V(\mathfrak{p})$ to get a scheme X (glueing as in Schemes, Example 21.14.3). Then the two maximal points of X corresponding to $\mathfrak{p}$ are not contained in a common affine open. The reason is that any open of $Spec(R)$ containing $\mathfrak{p}$ contains infinitely many of the ``lines'' $x = i$, $y_j = 0$, $j \neq i$ with parameter y_i. Details omitted.

Notwithstanding the example above, for ``most'' finite sets of irreducible closed subsets one can apply Lemma 23.26.1 above, at least if X is quasi-compact. This is true because X contains a dense open which is separated.

Lemma 23.26.3. *Let X be a quasi-compact scheme. There exists a dense open $V \subset X$ which is separated.*

Proof. Say $X = \bigcup_{i=1,\ldots,n} U_i$ is a union of n affine open subschemes. We will prove the lemma by induction on n. It is trivial for $n = 1$. Let $V' \subset \bigcup_{i=1,\ldots,n-1} U_i$ be a separated dense open subscheme, which exists by induction hypothesis. Consider

$$V = V' \coprod (U_n \setminus \overline{V'}).$$

It is clear that V is separated and a dense open subscheme of X. □

Here is a slight refinement of Lemma 23.26.1 above.

Lemma 23.26.4. *Let X be a quasi-separated scheme. Let $Z_1, \ldots, Z_n$ be pairwise distinct irreducible components of X. Let $\eta_i \in Z_i$ be their generic points. Let $x \in X$ be arbitrary. There exists an affine open $U \subset X$ containing x and all the η_i.*

Proof. Suppose that $x \in Z_1 \cap \ldots \cap Z_r$ and $x \not\in Z_{r+1}, \ldots, Z_n$. Then we may choose an affine open $W \subset X$ such that $x \in W$ and $W \cap Z_i = \emptyset$ for $i = r+1, \ldots, n$. Note that clearly $\eta_i \in W$ for $i = 1, \ldots, r$. By Lemma 23.26.1 we may choose affine opens $U_i \subset X$ which are pairwise disjoint such that $\eta_i \in U_i$ for $i = r+1, \ldots, n$. Since X is quasi-separated the opens $W \cap U_i$ are quasi-compact and do not contain η_i for $i = r+1, \ldots, n$. Hence by Algebra, Lemma 7.23.4 we may shrink U_i such that $W \cap U_i = \emptyset$ for $i = r+1, \ldots, n$. Then the union $U = W \cup \bigcup_{i=r+1,\ldots,n} U_i$ is disjoint and hence (by Schemes, Lemma 21.6.8) a suitable affine open. $\square$

Lemma 23.26.5. *Let X be a scheme. Assume either*

1. *The scheme X is quasi-affine.*
2. *The scheme X is isomorphic to an open subscheme of an affine scheme.*
3. *There exists an ample invertible sheaf on X.*
4. *The scheme X is isomorphic to an open subscheme of $Proj(S)$ for some graded ring S.*

Then for any finite subset $E \subset X$ there exists an affine open $U \subset X$ with $E \subset U$.

Proof. By Properties, Definition 23.15.1 a quasi-affine scheme is a quasi-compact open subscheme of an affine scheme. Any affine scheme $Spec(R)$ is isomorphic to $\text{Proj}(R[X])$ where $R[X]$ is graded by setting $\deg(X) = 1$. By Properties, Proposition 23.23.12 if X has an ample invertible sheaf then X is isomorphic to an open subscheme of $\text{Proj}(S)$ for some graded ring S. Hence, it suffices to prove the lemma in case (4). Thus assume $X \subset \text{Proj}(S)$ is an open subscheme where S is some graded ring. Since E is finite we may assume $E \subset D_+(f_1) \cup \ldots \cup D_+(f_n) \subset X$ for some finite number of homogeneous elements $f_i \in S_+$. Suppose that $E = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}$ as a subset of $\text{Proj}(S)$. Consider the ideal $I = (f_1, \ldots, f_n) \subset S$. Since $I \not\subset \mathfrak{p}_j$ for all $j = 1, \ldots, m$ we see from Algebra, Lemma 7.53.6 that there exists a homogeneous element $f \in I$, $f \not\in \mathfrak{p}_j$ for all $j = 1, \ldots, m$. Then $E \subset D_+(f) \subset D_+(f_1) \cup \ldots \cup D_+(f_n)$ is an affine open as desired. $\square$

23.27. Other chapters

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CHAPTER 24

Morphisms of Schemes

24.1. Introduction

In this chapter we introduce some types of morphisms of schemes. A basic reference is [**DG67**].

24.2. Closed immersions

In this section we elucidate some of the results obtained previously on closed immersions of schemes. Recall that a morphism of schemes $i : Z \to X$ is defined to be a closed immersion if (a) i induces a homeomorphism onto a closed subset of X, (b) $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective, and (c) the kernel of $i^\sharp$ is locally generated by sections, see Schemes, Definitions 21.10.2 and 21.4.1. It turns out that, given that Z and X are schemes, there are many different ways of characterizing a closed immersion.

Lemma 24.2.1. *Let $i : Z \to X$ be a morphism of schemes. The following are equivalent:*

(1) *The morphism i is a closed immersion.*
(2) *For every affine open $Spec(R) = U \subset X$, there exists an ideal $I \subset R$ such that $i^{-1}(U) = Spec(R/I)$ as schemes over $U = Spec(R)$.*
(3) *There exists an affine open covering $X = \bigcup_{j\in J} U_j$, $U_j = Spec(R_j)$ and for every $j \in J$ there exists an ideal $I_j \subset R_j$ such that $i^{-1}(U_j) = Spec(R_j/I_j)$ as schemes over $U_j = Spec(R_j)$.*
(4) *The morphism i induces a homeomorphism of Z with a closed subset of X and $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective.*
(5) *The morphism i induces a homeomorphism of Z with a closed subset of X, the map $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective, and the kernel $Ker(i^\sharp) \subset \mathcal{O}_X$ is a quasi-coherent sheaf of ideals.*
(6) *The morphism i induces a homeomorphism of Z with a closed subset of X, the map $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective, and the kernel $Ker(i^\sharp) \subset \mathcal{O}_X$ is a sheaf of ideals which is locally generated by sections.*

Proof. Condition (6) is our definition of a closed immersion, see Schemes, Definitions 21.4.1 and 21.10.2. So (6) $\Leftrightarrow$ (1). We have (1) $\Rightarrow$ (2) by Schemes, Lemma 21.10.1. Trivially (2) $\Rightarrow$ (3).

Assume (3). Each of the morphisms $Spec(R_j/I_j) \to Spec(R_j)$ is a closed immersion, see Schemes, Example 21.8.1. Hence $i^{-1}(U_j) \to U_j$ is a homeomorphism onto its image and $i^\sharp|_{U_j}$ is surjective. Hence i is a homeomorphism onto its image and $i^\sharp$ is surjective since this may be checked locally. We conclude that (3) $\Rightarrow$ (4).

The implication (4) $\Rightarrow$ (1) is Schemes, Lemma 21.24.2. The implication (5) $\Rightarrow$ (6) is trivial. And the implication (6) $\Rightarrow$ (5) follows from Schemes, Lemma 21.10.1. $\square$

Lemma 24.2.2. *Let X be a scheme. Suppose $i : Z \to X$ and $i' : Z' \to X$ are closed immersions corresponding to the quasi-coherent ideal sheaves $\mathcal{I} = Ker(i^\sharp)$ and $\mathcal{I}' = Ker((i')^\sharp)$ of $\mathcal{O}_X$.*

1. *The morphism $i : Z \to X$ factors as $Z \to Z' \to X$ for some $a : Z \to Z'$ if and only if $\mathcal{I}' \subset \mathcal{I}$. If this happens, then a is a closed immersion.*
2. *We have $Z \cong Z'$ as schemes over X if and only if $\mathcal{I} = \mathcal{I}'$.*

Proof. This follows from our discussion of closed subspaces in Schemes, Section 21.4 especially Schemes, Lemma 21.4.6. It also follows in a straightforward way from characterization (3) in Lemma 24.2.1 above. $\square$

Lemma 24.2.3. *Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a sheaf of ideals. The following are equivalent:*

1. *The sheaf of ideals $\mathcal{I}$ is locally generated by sections as a sheaf of $\mathcal{O}_X$ modules.*
2. *The sheaf of ideals $\mathcal{I}$ is quasi-coherent as a sheaf of $\mathcal{O}_X$-modules.*
3. *There exists a closed immersion $i : Z \to X$ whose corresponding sheaf of ideals $Ker(i^\sharp)$ is equal to $\mathcal{I}$.*

Proof. In Schemes, Section 21.4 we constructed the closed subspace associated to a sheaf of ideals locally generated by sections. This closed subspace is a scheme by Schemes, Lemma 21.10.1. Hence we see that (1) $\Rightarrow$ (3) by our definition of a closed immersion of schemes. By Lemma 24.2.1 above we see that (3) $\Rightarrow$ (2). And of course (2) $\Rightarrow$ (1). $\square$

Lemma 24.2.4. *The base change of a closed immersion is a closed immersion.*

Proof. See Schemes, Lemma 21.18.2. $\square$

Lemma 24.2.5. *A composition of closed immersions is a closed immersion.*

Proof. We have seen this in Schemes, Lemma 21.24.3, but here is another proof. Namely, it follows from the characterization (3) of closed immersions in Lemma 24.2.1. Since if $I \subset R$ is an ideal, and $\overline{J} \subset R/I$ is an ideal, then $\overline{J} = J/I$ for some ideal $J \subset R$ which contains I and $(R/I)/\overline{J} = R/J$. $\square$

Lemma 24.2.6. *A closed immersion is quasi-compact.*

Proof. This lemma is a duplicate of Schemes, Lemma 21.19.5. $\square$

Lemma 24.2.7. *A closed immersion is separated.*

Proof. This lemma is a special case of Schemes, Lemma 21.23.7. $\square$

Lemma 24.2.8. *Let $h : Z \to X$ be an immersion. If h is quasi-compact, then we can factor $h = i \circ j$ with $j : Z \to \overline{Z}$ an open immersion and $i : \overline{Z} \to X$ a closed immersion.*

Proof. Note that h is quasi-compact and quasi-separated (see Schemes, Lemma 21.23.7). Hence $h_*\mathcal{O}_Z$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules by Schemes, Lemma 21.24.1. This implies that $\mathcal{I} = \text{Ker}(\mathcal{O}_X \to h_*\mathcal{O}_Z)$ is a quasi-coherent sheaf of ideals, see Schemes, Section 21.24. Let $\overline{Z} \subset X$ be the closed subscheme corresponding to $\mathcal{I}$, see Lemma 24.2.3. By Schemes, Lemma 21.4.6 the morphism h factors as $h = i \circ j$ where $i : \overline{Z} \to X$ is the inclusion morphism. To see that j is an open immersion, choose an open subscheme $U \subset X$ such that h induces a closed immersion of Z into U. Then it is clear that $\mathcal{I}|_U$ is the sheaf of ideals corresponding to the closed immersion $Z \to U$. Hence we see that $Z = \overline{Z} \cap U$. $\square$

Lemma 24.2.9. *Let $h : Z \to X$ be an immersion. If Z is reduced, then we can factor $h = i \circ j$ with $j : Z \to \overline{Z}$ an open immersion and $i : \overline{Z} \to X$ a closed immersion.*

Proof. Let $\overline{Z} \subset X$ be the closure of $h(Z)$ with the reduced induced closed subscheme structure, see Schemes, Definition 21.12.5. By Schemes, Lemma 21.12.6 the morphism h factors as $h = i \circ j$ with $i : \overline{Z} \to X$ the inclusion morphism and $j : Z \to \overline{Z}$. From the definition of an immersion we see there exists an open subscheme $U \subset X$ such that h factors through a closed immersion into U. Hence $\overline{Z} \cap U$ and $h(Z)$ are reduced closed subschemes of U with the same underlying closed set. Hence by the uniqueness in Schemes, Lemma 21.12.4 we see that $h(Z) \cong \overline{Z} \cap U$. So j induces an isomorphism of Z with $\overline{Z} \cap U$. In other words j is an open immersion. □

Example 24.2.10. Here is an example of an immersion which is not a composition of an open immersion followed by a closed immersion. Let k be a field. Let $X = Spec(k[x_1, x_2, x_3, \ldots])$. Let $U = \bigcup_{n=1}^{\infty} D(x_n)$. Then $U \to X$ is an open immersion. Consider the ideals

$$I_n = (x_1^n, x_2^n, \ldots, x_{n-1}^n, x_n - 1, x_{n+1}, x_{n+2}, \ldots) \subset k[x_1, x_2, x_3, \ldots][1/x_n].$$

Note that $I_n k[x_1, x_2, x_3, \ldots][1/x_n x_m] = (1)$ for any $m \neq n$. Hence the quasi-coherent ideals $\tilde{I}_n$ on $D(x_n)$ agree on $D(x_n x_m)$, namely $\tilde{I}_n|_{D(x_n x_m)} = \mathcal{O}_{D(x_n x_m)}$ if $n \neq m$. Hence these ideals glue to a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_U$. Let $Z \subset U$ be the closed subscheme corresponding to $\mathcal{I}$. Thus $Z \to X$ is an immersion.

We claim that we cannot factor $Z \to X$ as $Z \to \overline{Z} \to X$, where $\overline{Z} \to X$ is closed and $Z \to \overline{Z}$ is open. Namely, $\overline{Z}$ would have to be defined by an ideal $I \subset k[x_1, x_2, x_3, \ldots]$ such that $I_n = Ik[x_1, x_2, x_3, \ldots][1/x_n]$. But the only element $f \in k[x_1, x_2, x_3, \ldots]$ which ends up in all I_n is 0! Hence I does not exist.

24.3. Closed immersions and quasi-coherent sheaves

The following lemma finally does for quasi-coherent sheaves on schemes what Modules, Lemma 15.6.1 does for abelian sheaves. See also the discussion in Modules, Section 15.13.

Lemma 24.3.1. *Let $i : Z \to X$ be a closed immersion of schemes. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaf of ideals cutting out Z. The functor*

$$i_* : QCoh(\mathcal{O}_Z) \longrightarrow QCoh(\mathcal{O}_X)$$

is exact, fully faithful, with essential image those quasi-coherent $\mathcal{O}_X$-modules $\mathcal{G}$ such that $\mathcal{I}\mathcal{G} = 0$.

Proof. A closed immersion is quasi-compact and separated, see Lemmas 24.2.6 and 24.2.7. Hence Schemes, Lemma 21.24.1 applies and the pushforward of a quasi-coherent sheaf on Z is indeed a quasi-coherent sheaf on X.

By Modules, Lemma 15.6.1 the functor i_* is faithful. We claim that for any quasi-coherent sheaf $\mathcal{F}$ on Z the canonical map

$$i^* i_* \mathcal{F} \longrightarrow \mathcal{F}$$

is an isomorphism. This claim implies in particular that i_* is fully faithful. To prove the claim let $U = Spec(R)$ be any affine open of X, and write $Z \cap U = Spec(R/I)$, see Lemma 24.2.1 above. We may write $\mathcal{F}|_{U \cap Z} = \widetilde{M}$ where M is an R/I-module (see Schemes, Section 21.24). By Schemes, Lemma 21.7.3 we see that $i_*\mathcal{F}|_U$ corresponds to M_R and then $i^*i_*\mathcal{F}|_{Z\cap U}$ corresponds to $M_R \otimes_R R/I$. In other words, we have to see that for any R/I-module M the canonical map

$$M_R \otimes_R R/I \longrightarrow M, \quad m \otimes f \longmapsto fm$$

is an isomorphism. Proof of this easy algebra fact is omitted.

Now we turn to the description of the essential image of the functor i_*. It is clear that $\mathcal{I}(i_*\mathcal{F}) = 0$ for any quasi-coherent $\mathcal{O}_Z$-module, for example by our local description above. Next, suppose that $\mathcal{G}$ is any quasi-coherent $\mathcal{O}_X$-module such that $\mathcal{I}\mathcal{G} = 0$. It suffices to show that the canonical map

$$\mathcal{G} \longrightarrow i_*i^*\mathcal{G}$$

is an isomorphism. By exactly the same arguments as above we see that it suffices to prove the following algebraic statement: Given a ring R, an ideal I and an R-module N such that $IN = 0$ the canonical map

$$N \longrightarrow N \otimes_R R/I, \quad n \longmapsto n \otimes 1$$

is an isomorphism of R-modules. Proof of this easy algebra fact is omitted. □

Let $i : Z \to X$ be a closed immersion. Because of the lemma above we often, by abuse of notation, denote $\mathcal{F}$ the sheaf $i_*\mathcal{F}$ on X.

Lemma 24.3.2. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{G} \subset \mathcal{F}$ be a $\mathcal{O}_X$-submodule. There exists a unique quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{G}' \subset \mathcal{G}$ with the following property: For every quasi-coherent $\mathcal{O}_X$-module $\mathcal{H}$ the map*

$$Hom_{\mathcal{O}_X}(\mathcal{H}, \mathcal{G}') \longrightarrow Hom_{\mathcal{O}_X}(\mathcal{H}, \mathcal{G})$$

is bijective. In particular $\mathcal{G}'$ is the largest quasi-coherent $\mathcal{O}_X$-submodule of $\mathcal{F}$ contained in $\mathcal{G}$.

Proof. Let $\mathcal{G}_a$, $a \in A$ be the set of quasi-coherent $\mathcal{O}_X$-submodules contained in $\mathcal{G}$. Then the image $\mathcal{G}'$ of

$$\bigoplus\nolimits_{a\in A} \mathcal{G}_a \longrightarrow \mathcal{F}$$

is quasi-coherent as the image of a map of quasi-coherent sheaves on X is quasi-coherent and since a direct sum of quasi-coherent sheaves is quasi-coherent, see Schemes, Section 21.24. The module $\mathcal{G}'$ is contained in $\mathcal{G}$. Hence this is the largest quasi-coherent $\mathcal{O}_X$-module contained in $\mathcal{G}$.

To prove the formula, let $\mathcal{H}$ be a quasi-coherent $\mathcal{O}_X$-module and let $\alpha : \mathcal{H} \to \mathcal{G}$ be an $\mathcal{O}_X$-module map. The image of the composition $\mathcal{H} \to \mathcal{G} \to \mathcal{F}$ is quasi-coherent as the image of a map of quasi-coherent sheaves. Hence it is contained in $\mathcal{G}'$. Hence α factors through $\mathcal{G}'$ as desired. □

Lemma 24.3.3. *Let $i : Z \to X$ be a closed immersion of schemes. There is a functor*[1] *$i^! : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Z)$ which is a right adjoint to i_*. (Compare Modules, Lemma 15.6.3.)*

Proof. Given quasi-coherent $\mathcal{O}_X$-module $\mathcal{G}$ we consider the subsheaf $\mathcal{H}_Z(\mathcal{G})$ of $\mathcal{G}$ of local sections annihilated by $\mathcal{I}$. By Lemma 24.3.2 there is a canonical largest quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{H}_Z(\mathcal{G})'$. By construction we have

$$Hom_{\mathcal{O}_X}(i_*\mathcal{F}, \mathcal{H}_Z(\mathcal{G})') = Hom_{\mathcal{O}_X}(i_*\mathcal{F}, \mathcal{G})$$

[1] This is likely nonstandard notation.

for any quasi-coherent $\mathcal{O}_Z$-module $\mathcal{F}$. Hence we can set $i^!\mathcal{G} = i^*(\mathcal{H}_Z(\mathcal{G})')$. Details omitted. ☐

24.4. Scheme theoretic image

Lemma 24.4.1. *Let $f : X \to Y$ be a morphism of schemes. There exists a closed subscheme $Z \subset Y$ such that f factors through Z and such that for any other closed subscheme $Z' \subset Y$ such that f factors through Z' we have $Z \subset Z'$.*

Proof. Let $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_Y \to f_*\mathcal{O}_X)$. If $\mathcal{I}$ is quasi-coherent then we just take Z to be the closed subscheme determined by $\mathcal{I}$, see Lemma 24.2.3. This works by Schemes, Lemma 21.4.6. In general the same lemma requires us to show that there exists a largest quasi-coherent sheaf of ideals $\mathcal{I}'$ contained in $\mathcal{I}$. This follows from Lemma 24.3.2. ☐

Definition 24.4.2. Let $f : X \to Y$ be a morphism of schemes. The *scheme theoretic image* of f is the smallest closed subscheme $Z \subset Y$ through which f factors, see Lemma 24.4.1 above.

We often just denote $f : X \to Z$ the factorization of f. If the morphism f is not quasi-compact, then (in general) the construction of the scheme theoretic image does not commute with restriction to open subschemes to Y. Namely, if f is the immersion $Z \to X$ of Example 24.2.10 above then the scheme theoretic image of $Z \to X$ is X. But clearly the scheme theoretic image of $Z = Z \cap U \to U$ is just Z.

Lemma 24.4.3. *Let $f : X \to Y$ be a morphism of schemes. Let $Z \subset Y$ be the scheme theoretic image of f. If f is quasi-compact then*

1. *the sheaf of ideals $\mathcal{I} = Ker(\mathcal{O}_Y \to f_*\mathcal{O}_X)$ is quasi-coherent,*
2. *the scheme theoretic image Z is the closed subscheme determined by $\mathcal{I}$,*
3. *for any open $U \subset Y$ the scheme theoretic image of $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is equal to $Z \cap U$, and*
4. *the image $f(X) \subset Z$ is a dense subset of Z, in other words the morphism $X \to Z$ is dominant (see Definition 24.6.1).*

Proof. Part (4) follows from part (3). To show (3) it suffices to prove (1) since the formation of $\mathcal{I}$ commutes with restriction to open subschemes of Y. And if (1) holds then in the proof of Lemma 24.4.1 we showed (2). Thus it suffices to prove that $\mathcal{I}$ is quasi-coherent. Since the property of being quasi-coherent is local we may assume Y is affine. As f is quasi-compact, we can find a finite affine open covering $X = \bigcup_{i=1,\ldots,n} U_i$. Denote f' the composition

$$X' = \coprod U_i \longrightarrow X \longrightarrow Y.$$

Then $f_*\mathcal{O}_X$ is a subsheaf of $f'_*\mathcal{O}_{X'}$, and hence $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_Y \to \mathcal{O}_{X'})$. By Schemes, Lemma 21.24.1 the sheaf $f'_*\mathcal{O}_{X'}$ is quasi-coherent on Y. Hence we win. ☐

Example 24.4.4. If $A \to B$ is a ring map with kernel I, then the scheme theoretic image of $Spec(B) \to Spec(A)$ is the closed subscheme $Spec(A/I)$ of $Spec(A)$. This follows from Lemma 24.4.3.

If the morphism is quasi-compact, then the scheme theoretic image only adds points which are specializations of points in the image.

Lemma 24.4.5. *Let $f : X \to Y$ be a quasi-compact morphism. Let Z be the scheme theoretic image of f. Let $z \in Z$. There exists a valuation ring A with fraction field K and a commutative diagram*

$$\begin{tikzcd} Spec(K) \arrow[rr] \arrow[d] & & X \arrow[dl] \arrow[d] \\ Spec(A) \arrow[r] & Z \arrow[r] & Y \end{tikzcd}$$

such that the closed point of $Spec(A)$ maps to z. In particular any point of Z is the specialization of a point of $f(X)$.

Proof. Let $z \in Spec(R) = V \subset Y$ be an affine open neighbourhood of z. By Lemma 24.4.3 we have $Z \cap V$ is the scheme theoretic closure of $f^{-1}(V) \to V$, and hence we may replace Y by V and assume $Y = Spec(R)$ is affine. In this case X is quasi-compact as f is quasi-compact. Say $X = U_1 \cup \ldots \cup U_n$ is a finite affine open covering. Write $U_i = Spec(A_i)$. Let $I = \text{Ker}(R \to A_1 \times \ldots \times A_n)$. By Lemma 24.4.3 again we see that Z corresponds to the closed subscheme $Spec(R/I)$ of Y. If $\mathfrak{p} \subset R$ is the prime corresponding to z, then we see that $I_\mathfrak{p} \subset R_\mathfrak{p}$ is not an equality. Hence (as localization is exact, see Algebra, Proposition 7.9.12) we see that $R_\mathfrak{p} \to (A_1)_\mathfrak{p} \times \ldots \times (A_1)_\mathfrak{p}$ is not zero. Hence one of the rings $(A_i)_\mathfrak{p}$ is not zero. Hence there exists an i and a prime $\mathfrak{q}_i \subset A_i$ lying over a prime $\mathfrak{p}_i \subset \mathfrak{p}$. By Algebra, Lemma 7.46.2 we can choose a valuation ring $A \subset K = f.f.(A_i/\mathfrak{q}_i)$ dominating the local ring $R_\mathfrak{p}/\mathfrak{p}_1 R_\mathfrak{p} \subset f.f.(A_i/\mathfrak{q}_i)$. This gives the desired diagram. Some details omitted. ☐

Lemma 24.4.6. *Let $f_1 : X \to Y_1$ and $Y_1 \to Y_2$ be morphisms of schemes. Let $f_2 : X \to Y_2$ be the composition. Let $Z_i \subset Y_i$, $i = 1, 2$ be the scheme theoretic image of f_i. Then the morphism $Y_1 \to Y_2$ induces a morphism $Z_1 \to Z_2$ and a commutative diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & Z_1 \arrow[r] \arrow[d] & Y_1 \arrow[d] \\ & Z_2 \arrow[r] & Y_2 \end{tikzcd}$$

Proof. See Schemes, Lemma 21.4.6. ☐

Lemma 24.4.7. *Let $f : X \to Y$ be a morphism of schemes. If X is reduced, then the scheme theoretic image of f is the reduced induced scheme structure on $\overline{f(X)}$.*

Proof. This is true because the reduced induced scheme structure on $\overline{f(X)}$ is clearly the smallest closed subscheme of Y through which f factors, see Schemes, Lemma 21.12.6. ☐

24.5. Scheme theoretic closure and density

Definition 24.5.1. Let X be a scheme. Let $U \subset X$ be an open subscheme.

(1) The scheme theoretic image of the morphism $U \to X$ is called the *scheme theoretic closure of U in X*.
(2) We say U is *scheme theoretically dense in X* if for every open $V \subset X$ the scheme theoretic closure of $U \cap V$ in V is equal to V.

This is [**DG67**, IV, Definition 11.10.2]. With this definition it is **not** the case that U is scheme theoretically dense in X if and only if the scheme theoretic closure of U is X, see Example 24.5.2. This is somewhat inelegant; but see Lemmas 24.5.3 and 24.5.8 below. On the other hand, with this definition U is scheme theoretically dense in X if and only if for every $V \subset X$ open the ring map $\mathcal{O}_X(V) \to \mathcal{O}_X(U \cap V)$ is injective, see Lemma 24.5.5 below. In particular we see that scheme theoretically dense implies dense which is pleasing.

Example 24.5.2. Here is an example where scheme theoretic closure being X does not imply dense for the underlying topological spaces. Let k be a field. Set $A = k[x, z_1, z_2, \ldots]/(x^n z_n)$ Set $I = (z_1, z_2, \ldots) \subset A$. Consider the affine scheme $X = Spec(A)$ and the open subscheme $U = X \setminus V(I)$. Since $A \to \prod_n A_{z_n}$ is injective we see that the scheme theoretic closure of U is X. Consider the morphism $X \to Spec(k[x])$. This morphism is surjective (set all $z_n = 0$ to see this). But the restriction of this morphism to U is not surjective because it maps to the point $x = 0$. Hence U cannot be topologically dense in X.

Lemma 24.5.3. *Let X be a scheme. Let $U \subset X$ be an open subscheme. If the inclusion morphism $U \to X$ is quasi-compact, then U is scheme theoretically dense in X if and only if the scheme theoretic closure of U in X is X.*

Proof. Follows from Lemma 24.4.3 part (3). $\square$

Example 24.5.4. Let A be a ring and $X = Spec(A)$. Let $f_1, \ldots, f_n \in A$ and let $U = D(f_1) \cup \ldots \cup D(f_n)$. Let $I = \text{Ker}(A \to \prod A_{f_i})$. Then the scheme theoretic closure of U in X is the closed subscheme $Spec(A/I)$ of X. Note that $U \to X$ is quasi-compact. Hence by Lemma 24.5.3 we see U is scheme theoretically dense in X if and only if $I = 0$.

Lemma 24.5.5. *Let $j : U \to X$ be an open immersion of schemes. Then U is scheme theoretically dense in X if and only if $\mathcal{O}_X \to j_*\mathcal{O}_U$ is injective.*

Proof. If $\mathcal{O}_X \to j_*\mathcal{O}_U$ is injective, then the same is true when restricted to any open V of X. Hence the scheme theoretic closure of $U \cap V$ in V is equal to V, see Lemma 24.4.3 for example. Conversely, suppose that the scheme theoretic closure of $U \cap V$ is equal to V for all opens V. Suppose that $\mathcal{O}_X \to j_*\mathcal{O}_U$ is not injective. Then we can find an affine open, say $Spec(A) = V \subset X$ and a nonzero element $f \in A$ such that f maps to zero in $\Gamma(V \cap U, \mathcal{O}_X)$. In this case the scheme theoretic closure of $V \cap U$ in V is clearly contained in $Spec(A/(f))$ a contradiction. $\square$

Lemma 24.5.6. *Let X be a scheme. If U, V are scheme theoretically dense open subschemes of X, then so is $U \cap V$.*

Proof. Let $W \subset X$ be any open. Consider the map $\mathcal{O}_X(W) \to \mathcal{O}_X(W \cap V) \to \mathcal{O}_X(W \cap V \cap U)$. By Lemma 24.5.5 both maps are injective. Hence the composite is injective. Hence by Lemma 24.5.5 $U \cap V$ is scheme theoretically dense in X. $\square$

Lemma 24.5.7. *Let $Z \to X$ be an immersion. Assume either $Z \to X$ is quasi-compact or Z is reduced. Let $\overline{Z} \subset X$ be the scheme theoretic image of h. Then the morphism $Z \to \overline{Z}$ is an open immersion which identifies Z with a scheme theoretically dense open subscheme of $\overline{Z}$. Moreover, Z is topologically dense in $\overline{Z}$.*

Proof. By Lemma 24.2.8 or Lemma 24.2.9 we can factor $Z \to X$ as $Z \to \overline{Z}_1 \to X$ with $Z \to \overline{Z}_1$ open and $\overline{Z}_1 \to X$ closed. On the other hand, let $Z \to \overline{Z} \subset X$ be the scheme theoretic closure of $Z \to X$. We conclude that $\overline{Z} \subset \overline{Z}_1$. Since Z is an open subscheme of $\overline{Z}_1$ it follows that Z is an open subscheme of $\overline{Z}$ as well. In the case that Z is reduced we know that $Z \subset \overline{Z}_1$ is topologically dense by the construction of $\overline{Z}_1$ in the proof of Lemma 24.2.9. Hence $\overline{Z}_1$ and $\overline{Z}$ have the same underlying topological spaces. Thus $\overline{Z} \subset \overline{Z}_1$ is a closed immersion into a reduced scheme which induces a bijection on underlying topological spaces, and hence it is an isomorphism. In the case that $Z \to X$ is quasi-compact we argue as follows: The assertion that Z is scheme theoretically dense in $\overline{Z}$ follows from Lemma 24.4.3 part (3). The last assertion follows from Lemma 24.4.3 part (4). $\square$

Lemma 24.5.8. *Let X be a reduced scheme and let $U \subset X$ be an open subscheme. Then the following are equivalent*

(1) *U is topologically dense in X,*
(2) *the scheme theoretic closure of U in X is X, and*
(3) *U is scheme theoretically dense in X.*

Proof. This follows from Lemma 24.5.7 and the fact that the a closed subscheme Z of X whose underlying topological space equals X must be equal to X as a scheme. $\square$

Lemma 24.5.9. *Let X be a scheme and let $U \subset X$ be a reduced open subscheme. Then the following are equivalent*

(1) *the scheme theoretic closure of U in X is X, and*
(2) *U is scheme theoretically dense in X.*

If this holds then X is a reduced scheme.

Proof. This follows from Lemma 24.5.7 and the fact that the scheme theoretic closure of U in X is reduced by Lemma 24.4.7. $\square$

Lemma 24.5.10. *Let S be a scheme. Let X, Y be schemes over S. Let $f, g : X \to Y$ be morphisms of schemes over S. Let $U \subset X$ be an open subscheme such that $f|_U = g|_U$. If the scheme theoretic closure of U in X is X and $Y \to S$ is separated, then $f = g$.*

Proof. Follows from the definitions and Schemes, Lemma 21.21.5. $\square$

24.6. Dominant morphisms

The definition of a morphism of schemes being dominant is a little different from what you might expect if you are used to the notion of a dominant morphism of varieties.

Definition 24.6.1. A morphism $f : X \to S$ of schemes is called *dominant* if the image of f is a dense subset of S.

So for example, if k is an infinite field and $\lambda_1, \lambda_2, \ldots$ is a countable collection of elements of k, then the morphism

$$\coprod\nolimits_{i=1,2,\ldots} Spec(k) \longrightarrow Spec(k[x])$$

with ith factor mapping to the point $x = \lambda_i$ is dominant.

Lemma 24.6.2. *Let $f : X \to S$ be a morphism of schemes. If every generic point of every irreducible component of S is in the image of f, then f is dominant.*

Proof. This is a topological fact which follows directly from the fact that the topological space underlying a scheme is sober, see Schemes, Lemma 21.11.1, and that every point of S is contained in an irreducible component of S, see Topology, Lemma 5.5.3. $\square$

The expectation that morphisms are dominant only if generic points of the target are in the image does hold if the morphism is quasi-compact.

Lemma 24.6.3. *Let $f : X \to S$ be a quasi-compact morphism of schemes. Then f is dominant (if and) only if for every irreducible component $Z \subset S$ the generic point of Z is in the image of f.*

Proof. Let $V \subset S$ be an affine open. Because f is quasi-compact we may choose finitely many affine opens $U_i \subset f^{-1}(V)$, $i = 1, \ldots, n$ covering $f^{-1}(V)$. Consider the morphism of affines

$$f' : \coprod_{i=1,\ldots,n} U_i \longrightarrow V.$$

A disjoint union of affines is affine, see Schemes, Lemma 21.6.8. Generic points of irreducible components of V are exactly the generic points of the irreducible components of S that meet V. Also, f is dominant if and only f' is dominant no matter what choices of V, n, U_i we make above. Thus we have reduced the lemma to the case of a morphism of affine schemes. The affine case is Algebra, Lemma 7.27.6. □

Here is a slightly more useful variant of the lemma above.

Lemma 24.6.4. *Let $f : X \to S$ be a quasi-compact morphism of schemes. Let $\eta \in S$ be a generic point of an irreducible component of S. If $\eta \notin f(X)$ then there exists an open neighbourhood $V \subset S$ of η such that $f^{-1}(V) = \emptyset$.*

Proof. Let $Z \subset S$ be the scheme theoretic image of f. We have to show that $\eta \notin Z$. This follows from Lemma 24.4.5 but can also be seen as follows. By Lemma 24.4.3 the morphism $X \to Z$ is dominant, which by Lemma 24.6.3 means all the generic points of all irreducible components of Z are in the image of $X \to Z$. By assumption we see that $\eta \notin Z$ since η would be the generic point of some irreducible component of Z if it were in Z. □

There is another case where dominant is the same as having all generic points of irreducible components in the image.

Lemma 24.6.5. *Let $f : X \to S$ be a morphism of schemes. Suppose that X has finitely many irreducible components. Then f is dominant (if and) only if for every irreducible component $Z \subset S$ the generic point of Z is in the image of f. If so, then S has finitely many irreducible components as well.*

Proof. Assume f is dominant. Say $X = Z_1 \cup Z_2 \cup \ldots \cup Z_n$ is the decomposition of X into irreducible components. Let $\xi_i \in Z_i$ be its generic point, so $Z_i = \overline{\{\xi_i\}}$. Note that $f(Z_i)$ is an irreducible subset of S. Hence

$$S = \overline{f(X)} = \bigcup \overline{f(Z_i)} = \bigcup \overline{\{f(\xi_i)\}}$$

is a finite union of irreducible subsets whose generic points are in the image of f. The lemma follows. □

24.7. Birational morphisms

You may be used to the notion of a birational map of varieties having the property that it is an isomorphism over an open subset of the target. However, in general a birational morphism may not be an isomorphism over any nonempty open, see Example 24.7.3. Here is the formal definition.

Definition 24.7.1. Let X, Y be schemes. Assume X and Y have finitely many irreducible components. We say a morphism $f : X \to Y$ is *birational* if

(1) f induces a bijection between the set of generic points of irreducible components of X and the set of generic points of the irreducible components of Y, and
(2) for every generic point $\eta \in X$ of an irreducible component of X the local ring map $\mathcal{O}_{Y,f(\eta)} \to \mathcal{O}_{X,\eta}$ is an isomorphism.

Lemma 24.7.2. *Let $f : X \to Y$ be a morphism of schemes having finitely many irreducible components. If f is birational then f is dominant.*

Proof. Follows immediately from the definitions. □

Example 24.7.3. Here is an example of a birational morphism which is not an isomorphism over any open of the target. Let k be an infinite field. Let $A = k[x]$. Let $B = k[x, \{y_\alpha\}_{\alpha \in k}]/((x - \alpha)y_\alpha, y_\alpha y_\beta)$. There is an inclusion $A \subset B$ and a retraction $B \to A$ setting all y_α equal to zero. Both the morphism $Spec(A) \to Spec(B)$ and the morphism $Spec(B) \to Spec(A)$ are birational but not an isomorphism over any open.

24.8. Rational maps

Let X be a scheme. Note that if U, V are dense open in X, then so is $U \cap V$.

Definition 24.8.1. Let X, Y be schemes.

(1) Let $f : U \to Y$, $g : V \to Y$ be morphisms of schemes defined on dense open subsets U, V of X. We say that f is *equivalent* to g if $f|_W = g|_W$ for some $W \subset U \cap V$ dense open in X.
(2) A *rational map from X to Y* is an equivalence class for the equivalence relation defined in (1).
(3) If X, Y are schemes over a base scheme S we say that a rational map from X to Y is an *S-rational map from X to Y* if there exists a representative $f : U \to Y$ of the equivalence class which is an S-morphism.

We say that two morphisms f, g as in (1) of the definition define the same rational map instead of saying that they are equivalent.

Definition 24.8.2. Let X be a scheme. A *rational function on X* is a rational morphism from X to $\mathbf{A}^1_{\mathbf{Z}}$.

See Constructions, Definition 22.5.1 for the definition of the affine line $\mathbf{A}^1$. Let X be a scheme over S. For any open $U \subset X$ a morphism $U \to \mathbf{A}^1_{\mathbf{Z}}$ is the same as a morphism $U \to \mathbf{A}^1_S$ over S. Hence a rational function is also the same as a S-rational map from X into $\mathbf{A}^1_S$.

Recall that we have the canonical identification $Mor(T, \mathbf{A}^1_{\mathbf{Z}}) = \Gamma(T, \mathcal{O}_T)$ for any scheme T, see Schemes, Example 21.15.2. Hence $\mathbf{A}^1_{\mathbf{Z}}$ is a ring-object in the category of schemes. More precisely, the morphisms

$$\begin{array}{rcl} + : \mathbf{A}^1_{\mathbf{Z}} \times \mathbf{A}^1_{\mathbf{Z}} & \longrightarrow & \mathbf{A}^1_{\mathbf{Z}} \\ (f, g) & \longmapsto & f + g \\ * : \mathbf{A}^1_{\mathbf{Z}} \times \mathbf{A}^1_{\mathbf{Z}} & \longrightarrow & \mathbf{A}^1_{\mathbf{Z}} \\ (f, g) & \longmapsto & fg \end{array}$$

satisfy all the axioms of the addition and multiplication in a ring (commutative with 1 as always). Hence also the set of rational maps into $\mathbf{A}^1_{\mathbf{Z}}$ has a natural ring structure.

Definition 24.8.3. Let X be a scheme. The *ring of rational functions on X* is the ring $R(X)$ whose elements are rational functions with addition and multiplication as just described.

Lemma 24.8.4. *Let X be an irreducible scheme. Let $\eta \in X$ be the generic point of X. There is a canonical identification $R(X) \cong \mathcal{O}_{X,\eta}$. If X is integral then $R(X) = \kappa(\eta) = \mathcal{O}_{X,\eta}$ is a field.*

Proof. Omitted. □

Definition 24.8.5. Let X be an integral scheme. The *function field*, or the *field of rational functions* of X is the field $R(X)$.

We may occasionally indicate this field $k(X)$ instead of $R(X)$.

Remark 24.8.6. There is a variant of Definition 24.8.1 where we consider only those morphism $U \to Y$ defined on scheme theoretically dense open subschemes $U \subset X$. We use Lemma 24.5.6 to see that we obtain an equivalence relation. An equivalence class of these is called a *pseudo-morphism from X to Y*. If X is reduced the two notions coincide.

Here is a fun application of these notions. Note that by Lemma 24.8.4 on an integral scheme every local ring $\mathcal{O}_{X,x}$ may be viewed as a local subring of $R(X)$.

Lemma 24.8.7. *Let X be an integral separated scheme. Let Z_1, Z_2 be distinct irreducible closed subsets of X. Let η_i be the generic point of Z_i. If $Z_1 \not\subset Z_2$, then $\mathcal{O}_{X,\eta_1} \not\subset \mathcal{O}_{X,\eta_2}$ as subrings of $R(X)$. In particular, if $Z_1 = \{x\}$ consists of one closed point x, there exists a function regular in a neighborhood of x which is not in $\mathcal{O}_{X,\eta_2}$.*

Proof. First observe that under the assumption of X being seperated, there is a unique map of schemes $Spec(\mathcal{O}_{X,\eta_2}) \to X$ over X such that the composition

$$Spec(R(X)) \longrightarrow Spec(\mathcal{O}_{X,\eta_2}) \longrightarrow X$$

is the canonical map $Spec(R(X)) \to X$. Namely, there is the canonical map $can : Spec(\mathcal{O}_{X,\eta_2}) \to X$, see Schemes, Equation (21.13.1.1). Given a second morphism a to X, we have that a agrees with can on the generic point of $Spec(\mathcal{O}_{X,\eta_2})$ by assumption. Now being X being seperated guarantees that the subset in $Spec(\mathcal{O}_{X,\eta_2})$ where these two maps agree is closed, see Schemes, Lemma 21.21.5. Hence $a = can$ on all of $Spec(\mathcal{O}_{X,\eta_2})$.

Assume $Z_1 \not\subset Z_2$ and assume on the contrary that $\mathcal{O}_{X,\eta_1} \subset \mathcal{O}_{X,\eta_2}$ as subrings of $R(X)$. Then we would obtain a second morphism

$$Spec(\mathcal{O}_{X,\eta_2}) \longrightarrow Spec(\mathcal{O}_{X,\eta_1}) \longrightarrow X.$$

By the above this composition would have to be equal to can. This implies that η_2 specializes to η_1 (see Schemes, Lemma 21.13.2). But this contradicts our assumption $Z_1 \not\subset Z_2$. □

24.9. Surjective morphisms

Definition 24.9.1. A morphism of schemes is said to be *surjective* if it is surjective on underlying topological spaces.

Lemma 24.9.2. *The composition of surjective morphisms is surjective.*

Proof. Omitted. □

Lemma 24.9.3. *Let X and Y be schemes over a base scheme S. Given points $x \in X$ and $y \in Y$, there is a point of $X \times_S Y$ mapping to x and y under the projections if and only if x and y lie above the same point of S.*

Proof. The condition is obviously necessary, and the converse follows from the proof of Schemes, Lemma 21.17.5. □

Lemma 24.9.4. *The base change of a surjective morphism is surjective.*

Proof. Let $f : X \to Y$ be a morphism of schemes over a base scheme S. If $S' \to S$ is a morphism of schemes, let $p : X_{S'} \to X$ and $q : Y_{S'} \to Y$ be the canonical projections. The commutative square

$$\begin{tikzcd} X_{S'} \arrow[r, "p"] \arrow[d, "f_{S'}"'] & X \arrow[d, "f"] \\ Y_{S'} \arrow[r, "q"] & Y. \end{tikzcd}$$

identifies $X_{S'}$ as a fibre product of $X \to Y$ and $Y_{S'} \to Y$. Let Z be a subset of the underlying topological space of X. Then $q^{-1}(f(Z)) = f_{S'}(p^{-1}(Z))$, because $y' \in q^{-1}(f(Z))$ if and only if $q(y') = f(x)$ for some $x \in Z$, if and only if, by Lemma 24.9.3, there exists $x' \in X_{S'}$ such that $f_{S'}(x') = y'$ and $p(x') = x$. In particular taking $Z = X$ we see that if f is surjective so is the base change $f_{S'} : X_{S'} \to Y_{S'}$. □

Example 24.9.5. Bijectivity is not stable under base change, and so neither is injectivity. For example consider the bijection $Spec(\mathbf{C}) \to Spec(\mathbf{R})$. The base change $Spec(\mathbf{C}\otimes_{\mathbf{R}}\mathbf{C}) \to Spec(\mathbf{C})$ is not injective, since there is an isomorphism $\mathbf{C}\otimes_{\mathbf{R}}\mathbf{C} \cong \mathbf{C}\times\mathbf{C}$ (the decomposition comes from the idempotent $\frac{1\otimes 1+i\otimes i}{2}$) and hence $Spec(\mathbf{C}\otimes_{\mathbf{R}}\mathbf{C})$ has two points.

Lemma 24.9.6. *Let*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. If f is surjective and p is quasi-compact, then q is quasi-compact.

Proof. Let $W \subset Z$ be a quasi-compact open. By assumption $p^{-1}(W)$ is quasi-compact. Hence by Topology, Lemma 5.9.5 the inverse image $q^{-1}(W) = f(p^{-1}(W))$ is quasi-compact too. This proves the lemma. □

24.10. Radicial and universally injective morphisms

In this section we define what it means for a morphism of schemes to be *radicial* and what it means for a morphism of schemes to be *universally injective*. We then show that these notions agree. The reason for introducing both is that in the case of algebraic spaces there are corresponding notions which may not always agree.

Definition 24.10.1. Let $f : X \to S$ be a morphism.

(1) We say that f is *universally injective* if and only if for any morphism of schemes $S' \to S$ the base change $f' : X_{S'} \to S'$ is injective (on underlying topological spaces).
(2) We say f is *radicial* if f is injective as a map of topological spaces, and for every $x \in X$ the field extension $\kappa(x) \supset \kappa(f(x))$ is purely inseparable.

Lemma 24.10.2. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *For every field K the induced map $Mor(Spec(K), X) \to Mor(Spec(K), S)$ is injective.*
(2) *The morphism f is universally injective.*
(3) *The morphism f is radicial.*
(4) *The diagonal morphism $\Delta_{X/S} : X \longrightarrow X \times_S X$ is surjective.*

Proof. Let K be a field, and let $s : Spec(K) \to S$ be a morphism. Giving a morphism $x : Spec(K) \to X$ such that $f \circ x = s$ is the same as giving a section of the projection $X_K = Spec(K) \times_S X \to Spec(K)$, which in turn is the same as giving a point $x \in X_K$ whose residue field is K. Hence we see that (2) implies (1).

Conversely, suppose that (1) holds. Assume that $x, x' \in X_{S'}$ map to the same point $s' \in S'$. Choose a commutative diagram

$$\begin{tikzcd} K & \kappa(x) \arrow[l] \\ \kappa(x') \arrow[u] & \kappa(s') \arrow[l] \arrow[u] \end{tikzcd}$$

of fields. By Schemes, Lemma 21.13.3 we get two morphisms $a, a' : Spec(K) \to X_{S'}$. One corresponding to the point x and the embedding $\kappa(x) \subset K$ and the other corresponding to the point x' and the embedding $\kappa(x') \subset K$. Also we have $f' \circ a = f' \circ a'$. Condition (1) now implies that the compositions of a and a' with $X_{S'} \to X$ are equal. Since $X_{S'}$ is the fibre product of S' and X over S we see that $a = a'$. Hence $x = x'$. Thus (1) implies (2).

If there are two different points $x, x' \in X$ mapping to the same point of s then (2) is violated. If for some $s = f(x)$, $x \in X$ the field extension $\kappa(s) \subset \kappa(x)$ is not purely inseparable, then we may find a field extension $\kappa(s) \subset K$ such that $\kappa(x)$ has two $\kappa(s)$-homomorphisms into K. By Schemes, Lemma 21.13.3 this implies that the map $Mor(Spec(K), X) \to Mor(Spec(K), S)$ is not injective, and hence (1) is violated. Thus we see that the equivalent conditions (1) and (2) imply f is radicial, i.e., they imply (3).

Assume (3). By Schemes, Lemma 21.13.3 a morphism $Spec(K) \to X$ is given by a pair $(x, \kappa(x) \to K)$. Property (3) says exactly that associating to the pair $(x, \kappa(x) \to K)$ the pair $(s, \kappa(s) \to \kappa(x) \to K)$ is injective. In other words (1) holds. At this point we know that (1), (2) and (3) are all equivalent.

Finally, we prove the equivalence of (4) with (1), (2) and (3). A point of $X \times_S X$ is given by a quadruple $(x_1, x_2, s, \mathfrak{p})$, where $x_1, x_2 \in X$, $f(x_1) = f(x_2) = s$ and $\mathfrak{p} \subset \kappa(x_1) \otimes_{\kappa(s)} \kappa(x_2)$ is a prime ideal, see Schemes, Lemma 21.17.5. If f is universally injective, then by taking $S' = X$ in the definition of universally injective, $\Delta_{X/S}$ must be surjective since it is a section of the injective morphism $X \times_S X \longrightarrow X$. Conversely, if $\Delta_{X/S}$ is surjective, then always $x_1 = x_2 = x$ and there is exactly one such prime ideal $\mathfrak{p}$, which means that $\kappa(s) \subset \kappa(x)$ is purely inseparable. Hence f is radicial. Alternatively, if $\Delta_{X/S}$ is surjective, then for any $S' \to S$ the base change $\Delta_{X_{S'}/S'}$ is surjective which implies that f is universally injective. This finishes the proof of the lemma. □

Lemma 24.10.3. *A universally injective morphism is separated.*

Proof. Combine Lemma 24.10.2 with the remark that $X \to S$ is separated if and only if the image of $\Delta_{X/S}$ is closed in $X \times_S X$, see Schemes, Definition 21.21.3 and the discussion following it. □

Lemma 24.10.4. *A base change of a universally injective morphism is universally injective.*

Proof. This is formal. □

Lemma 24.10.5. *A composition of radicial morphisms is radicial, and so the same holds for the equivalent condition of being universally injective.*

Proof. Omitted. □

24.11. Affine morphisms

Definition 24.11.1. A morphism of schemes $f : X \to S$ is called *affine* if the inverse image of every affine open of S is an affine open of X.

Lemma 24.11.2. *An affine morphism is separated and quasi-compact.*

Proof. Let $f : X \to S$ be affine. Quasi-compactness is immediate from Schemes, Lemma 21.19.2. We will show f is separated using Schemes, Lemma 21.21.8. Let $x_1, x_2 \in X$ be points of X which map to the same point $s \in S$. Choose any affine open $W \subset S$ containing s. By assumption $f^{-1}(W)$ is affine. Apply the lemma cited with $U = V = f^{-1}(W)$. $\square$

Lemma 24.11.3. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *The morphism f is affine.*
(2) *There exists an affine open covering $S = \bigcup W_j$ such that each $f^{-1}(W_j)$ is affine.*
(3) *There exists a quasi-coherent sheaf of $\mathcal{O}_S$-algebras $\mathcal{A}$ and an isomorphism $X \cong \underline{Spec}_S(\mathcal{A})$ of schemes over S. See Constructions, Section 22.4 for notation.*

Moreover, in this case $X = \underline{Spec}_S(f_*\mathcal{O}_X)$.

Proof. It is obvious that (1) implies (2).

Assume $S = \bigcup_{j\in J} W_j$ is an affine open covering such that each $f^{-1}(W_j)$ is affine. By Schemes, Lemma 21.19.2 we see that f is quasi-compact. By Schemes, Lemma 21.21.7 we see the morphism f is quasi-separated. Hence by Schemes, Lemma 21.24.1 the sheaf $\mathcal{A} = f_*\mathcal{O}_X$ is a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. Thus we have the scheme $g : Y = \underline{Spec}_S(\mathcal{A}) \to S$ over S. The identity map $\mathrm{id} : \mathcal{A} = f_*\mathcal{O}_X \to f_*\mathcal{O}_X$ provides, via the definition of the relative spectrum, a morphism $can : X \to Y$ over S, see Constructions, Lemma 22.4.7. By assumption and the lemma just cited the restriction $can|_{f^{-1}(W_j)} : f^{-1}(W_j) \to g^{-1}(W_j)$ is an isomorphism. Thus can is an isomorphism. We have shown that (2) implies (3).

Assume (3). By Constructions, Lemma 22.4.6 we see that the inverse image of every affine open is affine, and hence the morphism is affine by definition. $\square$

Remark 24.11.4. We can also argue directly that (2) implies (1) in Lemma 24.11.3 above as follows. Assume $S = \bigcup W_j$ is an affine open covering such that each $f^{-1}(W_j)$ is affine. First argue that $\mathcal{A} = f_*\mathcal{O}_X$ is quasi-coherent as in the proof above. Let $Spec(R) = V \subset S$ be affine open. We have to show that $f^{-1}(V)$ is affine. Set $A = \mathcal{A}(V) = f_*\mathcal{O}_X(V) = \mathcal{O}_X(f^{-1}(V))$. By Schemes, Lemma 21.6.4 there is a canonical morphism $\psi : f^{-1}(V) \to Spec(A)$ over $Spec(R) = V$. By Schemes, Lemma 21.11.6 there exists an integer $n \geq 0$, a standard open covering $V = \bigcup_{i=1,\ldots,n} D(h_i)$, $h_i \in R$, and a map $a : \{1, \ldots, n\} \to J$ such that each $D(h_i)$ is also a standard open of the affine scheme $W_{a(i)}$. The inverse image of a standard open under a morphism of affine schemes is standard open, see Algebra, Lemma 7.16.4. Hence we see that $f^{-1}(D(h_i))$ is a standard open of $f^{-1}(W_{a(i)})$, in particular that $f^{-1}(D(h_i))$ is affine. Because $\mathcal{A}$ is quasi-coherent we have $A_{h_i} = \mathcal{A}(D(h_i)) = \mathcal{O}_X(f^{-1}(D(h_i)))$, so $f^{-1}(D(h_i))$ is the spectrum of A_{h_i}. It follows that the morphism ψ induces an isomorphism of the open $f^{-1}(D(h_i))$ with the open $Spec(A_{h_i})$ of $Spec(A)$. Since $f^{-1}(V) = \bigcup f^{-1}(D(h_i))$ and $Spec(A) = \bigcup Spec(A_{h_i})$ we win.

Lemma 24.11.5. *Let S be a scheme. There is an anti-equivalence of categories*

$$\begin{matrix} \textit{Schemes affine} \\ \textit{over } S \end{matrix} \longleftrightarrow \begin{matrix} \textit{quasi-coherent sheaves} \\ \textit{of } \mathcal{O}_S\textit{-algebras} \end{matrix}$$

which associates to $f : X \to S$ the sheaf $f_*\mathcal{O}_X$.

Proof. Omitted. $\square$

Lemma 24.11.6. *Let $f : X \to S$ be an affine morphism of schemes. Let $\mathcal{A} = f_*\mathcal{O}_X$. The functor $\mathcal{F} \mapsto f_*\mathcal{F}$ induces an equivalence of categories*

$$\left\{\begin{matrix} \textit{category of quasi-coherent} \\ \mathcal{O}_X\textit{-modules}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix} \textit{category of quasi-coherent} \\ \mathcal{A}\textit{-modules}\end{matrix}\right\}$$

Moreover, an $\mathcal{A}$-module is quasi-coherent as an $\mathcal{O}_S$-module if and only if it is quasi-coherent as an $\mathcal{A}$-module.

Proof. Omitted. $\square$

Lemma 24.11.7. *The composition of affine morphisms is affine.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be affine morphisms. Let $U \subset Z$ be affine open. Then $g^{-1}(U)$ is affine by assumption on g. Whereupon $f^{-1}(g^{-1}(U))$ is affine by assumption on f. Hence $(g \circ f)^{-1}(U)$ is affine. $\square$

Lemma 24.11.8. *The base change of an affine morphism is affine.*

Proof. Let $f : X \to S$ be an affine morphism. Let $S' \to S$ be any morphism. Denote $f' : X_{S'} = S' \times_S X \to S'$ the base change of f. For every $s' \in S'$ there exists an open affine neighbourhood $s' \in V \subset S'$ which maps into some open affine $U \subset S$. By assumption $f^{-1}(U)$ is affine. By the material in Schemes, Section 21.17 we see that $f^{-1}(U)_V = V \times_U f^{-1}(U)$ is affine and equal to $(f')^{-1}(V)$. This proves that S' has an open covering by affines whose inverse image under f' is affine. We conclude by Lemma 24.11.3 above. $\square$

Lemma 24.11.9. *A closed immersion is affine.*

Proof. The first indication of this is Schemes, Lemma 21.8.2. See Schemes, Lemma 21.10.1 for a complete statement. $\square$

Lemma 24.11.10. *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$. The inclusion morphism $j : X_s \to X$ is affine.*

Proof. This follows from Properties, Lemma 23.23.4 and the definition. $\square$

Lemma 24.11.11. *Suppose $g : X \to Y$ is a morphism of schemes over S. If X is affine over S and Y is separated over S, then g is affine. In particular, any morphism from an affine scheme to a separated scheme is affine.*

Proof. The base change $X \times_S Y \to Y$ is affine by Lemma 24.11.8. The morphism $X \to X \times_S Y$ is a closed immersion as $Y \to S$ is separated, see Schemes, Lemma 21.21.12. A closed immersion is affine (see Lemma 24.11.9) and the composition of affine morphisms is affine (see Lemma 24.11.7). Thus we win. $\square$

Lemma 24.11.12. *A morphism between affine schemes is affine.*

Proof. Immediate from Lemma 24.11.11 with $S = Spec(\mathbf{Z})$. It also follows directly from the equivalence of (1) and (2) in Lemma 24.11.3. $\square$

Lemma 24.11.13. *Let S be a scheme. Let A be an Artinian ring. Any morphism $Spec(A) \to S$ is affine.*

Proof. Omitted. $\square$

24.12. Quasi-affine morphisms

Recall that a scheme X is called *quasi-affine* if it is quasi-compact and isomorphic to an open subscheme of an affine scheme, see Properties, Definition 23.15.1.

Definition 24.12.1. A morphism of schemes $f : X \to S$ is called *quasi-affine* if the inverse image of every affine open of S is a quasi-affine scheme.

Lemma 24.12.2. *A quasi-affine morphism is separated and quasi-compact.*

Proof. Let $f : X \to S$ be quasi-affine. Quasi-compactness is immediate from Schemes, Lemma 21.19.2. We will show f is separated using Schemes, Lemma 21.21.8. Let $x_1, x_2 \in X$ be points of X which map to the same point $s \in S$. Choose any affine open $W \subset S$ containing s. By assumption $f^{-1}(W)$ is isomorphic to an open subscheme of an affine scheme, say $f^{-1}(W) \to Y$ is such an open immersion. Choose affine open neighbourhoods $x_1 \in U \subset f^{-1}(W)$ and $x_2 \in V \subset f^{-1}(W)$. We may think of U and V as open subschemes of Y and hence we see that $U \cap V$ is affine and that $\mathcal{O}(U) \otimes_{\mathbf{Z}} \mathcal{O}(V) \to \mathcal{O}(U \cap V)$ is surjective (by the lemma cited above applied to U, V in Y). Hence by the lemma cited we conclude that f is separated. $\square$

Lemma 24.12.3. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *The morphism f is quasi-affine.*
(2) *There exists an affine open covering $S = \bigcup W_j$ such that each $f^{-1}(W_j)$ is quasi-affine.*
(3) *There exists a quasi-coherent sheaf of $\mathcal{O}_S$-algebras $\mathcal{A}$ and a quasi-compact open immersion*

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & \underline{Spec}_S(\mathcal{A}) \arrow[dl] \\ & S & \end{tikzcd}$$

over S.
(4) *Same as in (3) but with $\mathcal{A} = f_*\mathcal{O}_X$ and the horizontal arrow the canonical morphism of Constructions, Lemma 22.4.7.*

Proof. It is obvious that (1) implies (2) and that (4) implies (3).

Assume $S = \bigcup_{j \in J} W_j$ is an affine open covering such that each $f^{-1}(W_j)$ is quasi-affine. By Schemes, Lemma 21.19.2 we see that f is quasi-compact. By Schemes, Lemma 21.21.7 we see the morphism f is quasi-separated. Hence by Schemes, Lemma 21.24.1 the sheaf $\mathcal{A} = f_*\mathcal{O}_X$ is a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. Thus we have the scheme $g : Y = \underline{Spec}_S(\mathcal{A}) \to S$ over S. The identity map $\text{id} : \mathcal{A} = f_*\mathcal{O}_X \to f_*\mathcal{O}_X$ provides, via the definition of the relative spectrum, a morphism $can : X \to Y$ over S, see Constructions, Lemma 22.4.7. By assumption, the lemma just cited, and Properties, Lemma 23.15.4 the restriction $can|_{f^{-1}(W_j)} : f^{-1}(W_j) \to g^{-1}(W_j)$ is a quasi-compact open immersion. Thus can is a quasi-compact open immersion. We have shown that (2) implies (4).

Assume (3). Choose any affine open $U \subset S$. By Constructions, Lemma 22.4.6 we see that the inverse image of U in the relative spectrum is affine. Hence we conclude that $f^{-1}(U)$ is quasi-affine (note that quasi-compactness is encoded in (3) as well). Thus (3) implies (1). $\square$

Lemma 24.12.4. *The composition of quasi-affine morphisms is quasi-affine.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be quasi-affine morphisms. Let $U \subset Z$ be affine open. Then $g^{-1}(U)$ is quasi-affine by assumption on g. Let $j : g^{-1}(U) \to V$ be a quasi-compact open immersion into an affine scheme V. By Lemma 24.12.3 above we see that $f^{-1}(g^{-1}(U))$ is a quasi-compact open subscheme of the relative spectrum $\underline{Spec}_{g^{-1}(U)}(\mathscr{A})$ for some quasi-coherent sheaf of $\mathcal{O}_{g^{-1}(U)}$-algebras $\mathscr{A}$. By Schemes, Lemma 21.24.1 the sheaf $\mathscr{A}' = j_*\mathscr{A}$ is a quasi-coherent sheaf of $\mathcal{O}_V$-algebras with the property that $j^*\mathscr{A}' = \mathscr{A}$. Hence we get a commutative diagram

$$\begin{tikzcd}
f^{-1}(g^{-1}(U)) \arrow[r] & \underline{Spec}_{g^{-1}(U)}(\mathscr{A}) \arrow[r] \arrow[d] & \underline{Spec}_V(\mathscr{A}') \arrow[d] \\
 & g^{-1}(U) \arrow[r, "j"] & V
\end{tikzcd}$$

with the square being a fibre square, see Constructions, Lemma 22.4.6. Note that the upper right corner is an affine scheme. Hence $(g \circ f)^{-1}(U)$ is quasi-affine. $\square$

Lemma 24.12.5. *The base change of a quasi-affine morphism is quasi-affine.*

Proof. Let $f : X \to S$ be a quasi-affine morphism. By Lemma 24.12.3 above we can find a quasi-coherent sheaf of $\mathcal{O}_S$-algebras $\mathscr{A}$ and a quasi-compact open immersion $X \to \underline{Spec}_S(\mathscr{A})$ over S. Let $g : S' \to S$ be any morphism. Denote $f' : X_{S'} = S' \times_S X \to S'$ the base change of f. Since the base change of a quasi-compact open immersion is a quasi-compact open immersion we see that $X_{S'} \to \underline{Spec}_{S'}(g^*\mathscr{A})$ is a quasi-compact open immersion (we have used Schemes, Lemmas 21.19.3 and 21.18.2 and Constructions, Lemma 22.4.6). By Lemma 24.12.3 again we conclude that $X_{S'} \to S'$ is quasi-affine. $\square$

Lemma 24.12.6. *A quasi-compact immersion is quasi-affine.*

Proof. Let $X \to S$ be a quasi-compact immersion. We have to show the inverse image of every affine open is quasi-affine. Hence, assuming S is an affine scheme, we have to show X is quasi-affine. By Lemma 24.5.7 the morphism $X \to S$ factors as $X \to Z \to S$ where Z is a closed subscheme of S and $X \subset Z$ is a quasi-compact open. Since S is affine Lemma 24.2.1 implies Z is affine. Hence we win. $\square$

Lemma 24.12.7. *Let S be a scheme. Let X be an affine scheme. A morphism $f : X \to S$ is quasi-affine if and only if it is quasi-compact. In particular any morphism from an affine scheme to a quasi-separated scheme is quasi-affine.*

Proof. Let $V \subset S$ be an affine open. Then $f^{-1}(V)$ is an open subscheme of the affine scheme X, hence quasi-affine if and only if it is quasi-compact. This proves the first assertion. The quasi-compactness of any $f : X \to S$ where X is affine and S quasi-separated follows from Schemes, Lemma 21.21.15 applied to $X \to S \to Spec(\mathbf{Z})$. $\square$

Lemma 24.12.8. *Suppose $g : X \to Y$ is a morphism of schemes over S. If X is quasi-affine over S and Y is quasi-separated over S, then g is quasi-affine. In particular, any morphism from a quasi-affine scheme to a quasi-separated scheme is quasi-affine.*

Proof. The base change $X \times_S Y \to Y$ is quasi-affine by Lemma 24.12.5. The morphism $X \to X \times_S Y$ is a quasi-compact immersion as $Y \to S$ is quasi-separated, see Schemes, Lemma 21.21.12. A quasi-compact immersion is quasi-affine by Lemma 24.12.6 and the composition of quasi-affine morphisms is quasi-affine (see Lemma 24.12.4). Thus we win. $\square$

24.13. Types of morphisms defined by properties of ring maps

In this section we study what properties of ring maps allow one to define local properties of morphisms of schemes.

Definition 24.13.1. Let P be a property of ring maps.

1. We say that P is *local* if the following hold:
   (a) For any ring map $R \to A$, and any $f \in R$ we have $P(R \to A) \Rightarrow P(R_f \to A_f)$.
   (b) For any rings R, A, any $f \in R$, $a \in A$, and any ring map $R_f \to A$ we have $P(R_f \to A) \Rightarrow P(R \to A_a)$.
   (c) For any ring map $R \to A$, and $a_i \in A$ such that $(a_1, \ldots, a_n) = A$ then $\forall i, P(R \to A_{a_i}) \Rightarrow P(R \to A)$.
2. We say that P is *stable under base change* if for any ring maps $R \to A$, $R \to R'$ we have $P(R \to A) \Rightarrow P(R' \to R' \otimes_R A)$.
3. We say that P is *stable under composition* if for any ring maps $A \to B$, $B \to C$ we have $P(A \to B) \wedge P(B \to C) \Rightarrow P(A \to C)$.

Definition 24.13.2. Let P be a property of ring maps. Let $f : X \to S$ be a morphisms of schemes. We say f is *locally of type P* if for any $x \in X$ there exists an affine open neighbourhood U of x in X which maps into an affine open $V \subset S$ such that the induced ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ has property P.

This is not a ``good'' definition unless the property $P$ is a local property. Even if $P$ is a local property we will not automatically use this definition to say that a morphism is ``locally of type P' unless we also explicitly state the definition elsewhere.

Lemma 24.13.3. *Let $f : X \to S$ be a morphism of schemes. Let P be a property of ring maps. Let U be an affine open of X, and V an affine open of S such that $f(U) \subset V$. If f is locally of type P and P is local, then $P(\mathcal{O}_S(V) \to \mathcal{O}_X(U))$ holds.*

Proof. As f is locally of type P for every $u \in U$ there exists an affine open $U_u \subset X$ mapping into an affine open $V_u \subset S$ such that $P(\mathcal{O}_S(V_u) \to \mathcal{O}_X(U_u))$ holds. Choose an open neighbourhood $U'_u \subset U \cap U_u$ of u which is standard affine open in both U and U_u, see Schemes, Lemma 21.11.5. By Definition 24.13.1 (1)(b) we see that $P(\mathcal{O}_S(V_u) \to \mathcal{O}_X(U'_u))$ holds. Hence we may assume that $U_u \subset U$ is a standard affine open. Choose an open neighbourhood $V'_u \subset V \cap V_u$ of $f(u)$ which is standard affine open in both V and V_u, see Schemes, Lemma 21.11.5. Then $U'_u = f^{-1}(V'_u) \cap U_u$ is a standard affine open of U_u (hence of U) and we have $P(\mathcal{O}_S(V'_u) \to \mathcal{O}_X(U'_u))$ by Definition 24.13.1 (1)(a). Hence we may assume both $U_u \subset U$ and $V_u \subset V$ are standard affine open. Applying Definition 24.13.1 (1)(b) one more time we conclude that $P(\mathcal{O}_S(V) \to \mathcal{O}_X(U_u))$ holds. Because U is quasi-compact we may choose a finite number of points $u_1, \ldots, u_n \in U$ such that

$$U = U_{u_1} \cup \ldots \cup U_{u_n}.$$

By Definition 24.13.1 (1)(c) we conclude that $P(\mathcal{O}_S(V) \to \mathcal{O}_X(U))$ holds. □

Lemma 24.13.4. *Let P be a local property of ring maps. Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

1. *The morphism f is locally of type P.*
2. *For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ we have $P(\mathcal{O}_S(V) \to \mathcal{O}_X(U))$.*

(3) *There exists an open covering* $S = \bigcup_{j \in J} V_j$ *and open coverings* $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ *such that each of the morphisms* $U_i \to V_j$, $j \in J, i \in I_j$ *is locally of type* P.
(4) *There exists an affine open covering* $S = \bigcup_{j \in J} V_j$ *and affine open coverings* $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ *such that* $P(\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i))$ *holds, for all* $j \in J, i \in I_j$.

Moreover, if f *is locally of type* P *then for any open subschemes* $U \subset X$, $V \subset S$ *with* $f(U) \subset V$ *the restriction* $f|_U : U \to V$ *is locally of type* P.

Proof. This follows from Lemma 24.13.3 above. □

Lemma 24.13.5. *Let* P *be a property of ring maps. Assume* P *is local and stable under composition. The composition of morphisms locally of type* P *is locally of type* P.

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms locally of type P. Let $x \in X$. Choose an affine open neighbourhood $W \subset Z$ of $g(f(x))$. Choose an affine open neighbourhood $V \subset g^{-1}(W)$ of $f(x)$. Choose an affine open neighbourhood $U \subset f^{-1}(V)$ of x. By Lemma 24.13.4 the ring maps $\mathcal{O}_Z(W) \to \mathcal{O}_Y(V)$ and $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ satisfy P. Hence $\mathcal{O}_Z(W) \to \mathcal{O}_X(U)$ satisfies P as P is assumed stable under composition. □

Lemma 24.13.6. *Let* P *be a property of ring maps. Assume* P *is local and stable under base change. The base change of a morphism locally of type* P *is locally of type* P.

Proof. Let $f : X \to S$ be a morphism locally of type P. Let $S' \to S$ be any morphism. Denote $f' : X_{S'} = S' \times_S X \to S'$ the base change of f. For every $s' \in S'$ there exists an open affine neighbourhood $s' \in V' \subset S'$ which maps into some open affine $V \subset S$. By Lemma 24.13.4 the open $f^{-1}(V)$ is a union of affines U_i such that the ring maps $\mathcal{O}_S(V) \to \mathcal{O}_X(U_i)$ all satisfy P. By the material in Schemes, Section 21.17 we see that $f^{-1}(U)_{V'} = V' \times_V f^{-1}(V)$ is the union of the affine opens $V' \times_V U_i$. Since $\mathcal{O}_{X_{S'}}(V' \times_V U_i) = \mathcal{O}_{S'}(V') \otimes_{\mathcal{O}_S(V)} \mathcal{O}_X(U_i)$ we see that the ring maps $\mathcal{O}_{S'}(V') \to \mathcal{O}_{X_{S'}}(V' \times_V U_i)$ satisfy P as P is assumed stable under base change. □

Lemma 24.13.7. *The following properties of a ring map* $R \to A$ *are local.*

(1) *(Isomorphism on local rings.) For every prime* $\mathfrak{q}$ *of* A *lying over* $\mathfrak{p} \subset R$ *the ring map* $R \to A$ *induces an isomorphism* $R_{\mathfrak{p}} \to A_{\mathfrak{q}}$.
(2) *(Open immersion.) For every prime* $\mathfrak{q}$ *of* A *there exists an* $f \in R$, $\varphi(f) \not\in \mathfrak{q}$ *such that the ring map* $\varphi : R \to A$ *induces an isomorphism* $R_f \to A_f$.
(3) *(Reduced fibres.) For every prime* $\mathfrak{p}$ *of* R *the fibre ring* $A \otimes_R \kappa(\mathfrak{p})$ *is reduced.*
(4) *(Fibres of dimension at most n.) For every prime* $\mathfrak{p}$ *of* R *the fibre ring* $A \otimes_R \kappa(\mathfrak{p})$ *has Krull dimension at most n.*
(5) *(Locally Noetherian on the target.) The ring map* $R \to A$ *has the property that* A *is Noetherian.*
(6) *Add more here as needed*[2].

Proof. Omitted. □

Lemma 24.13.8. *The following properties of ring maps are stable under base change.*

(1) *(Isomorphism on local rings.) For every prime* $\mathfrak{q}$ *of* A *lying over* $\mathfrak{p} \subset R$ *the ring map* $R \to A$ *induces an isomorphism* $R_{\mathfrak{p}} \to A_{\mathfrak{q}}$.

[2] But only those properties that are not already dealt with separately elsewhere.

(2) *(Open immersion.) For every prime $\mathfrak{q}$ of A there exists an $f \in R$, $\varphi(f) \notin \mathfrak{q}$ such that the ring map $\varphi : R \to A$ induces an isomorphism $R_f \to A_f$.*
(3) *(Reduced fibres.) For every prime $\mathfrak{p}$ of R the fibre ring $A \otimes_R \kappa(\mathfrak{p})$ is reduced.*
(4) *(Fibres of dimension at most n.) For every prime $\mathfrak{p}$ of R the fibre ring $A \otimes_R \kappa(\mathfrak{p})$ has Krull dimension at most n.*
(5) *Add more here as needed*[3].

Proof. Omitted. □

Lemma 24.13.9. *The following properties of ring maps are stable under composition.*
(1) *(Isomorphism on local rings.) For every prime $\mathfrak{q}$ of A lying over $\mathfrak{p} \subset R$ the ring map $R \to A$ induces an isomorphism $R_{\mathfrak{p}} \to A_{\mathfrak{q}}$.*
(2) *(Open immersion.) For every prime $\mathfrak{q}$ of A there exists an $f \in R$, $\varphi(f) \notin \mathfrak{q}$ such that the ring map $\varphi : R \to A$ induces an isomorphism $R_f \to A_f$.*
(3) *(Locally Noetherian on the target.) The ring map $R \to A$ has the property that A is Noetherian.*
(4) *Add more here as needed*[4].

Proof. Omitted. □

24.14. Morphisms of finite type

Recall that a ring map $R \to A$ is said to be of finite type if A is isomorphic to a quotient of $R[x_1, \ldots, x_n]$ as an R-algebra, see Algebra, Definition 7.6.1.

Definition 24.14.1. Let $f : X \to S$ be a morphism of schemes.
(1) We say that f is of *finite type at $x \in X$* if there exists an affine open neighbourhood $Spec(A) = U \subset X$ of x and and an affine open $Spec(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is of finite type.
(2) We say that f is *locally of finite type* if it is of finite type at every point of X.
(3) We say that f is of *finite type* if it is locally of finite type and quasi-compact.

Lemma 24.14.2. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*
(1) *The morphism f is locally of finite type.*
(2) *For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is of finite type.*
(3) *There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$, $j \in J, i \in I_j$ is locally of finite type.*
(4) *There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring map $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i)$ is of finite type, for all $j \in J, i \in I_j$.*

Moreover, if f is locally of finite type then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \to V$ is locally of finite type.

Proof. This follows from Lemma 24.13.3 if we show that the property "$R \to A$ is of finite type" is local. We check conditions (a), (b) and (c) of Definition 24.13.1. By Algebra, Lemma 7.13.2 being of finite type is stable under base change and hence we conclude (a)

[3] But only those properties that are not already dealt with separately elsewhere.
[4] But only those properties that are not already dealt with separately elsewhere.

holds. By the same lemma being of finite type is stable under composition and trivially for any ring R the ring map $R \to R_f$ is of finite type. We conclude (b) holds. Finally, property (c) is true according to Algebra, Lemma 7.21.3. $\square$

Lemma 24.14.3. *The composition of two morphisms which locally of finite type is locally of finite type. The same is true for morphisms of finite type.*

Proof. In the proof of Lemma 24.14.2 we saw that being of finite type is a local property of ring maps. Hence the first statement of the lemma follows from Lemma 24.13.5 combined with the fact that being of finite type is a property of ring maps that is stable under composition, see Algebra, Lemma 7.6.2. By the above and the fact that compositions of quasi-compact morphisms are quasi-compact, see Schemes, Lemma 21.19.4 we see that the composition of morphisms of finite type is of finite type. $\square$

Lemma 24.14.4. *The base change of a morphism which is locally of finite type is locally of finite type. The same is true for morphisms of finite type.*

Proof. In the proof of Lemma 24.14.2 we saw that being of finite type is a local property of ring maps. Hence the first statement of the lemma follows from Lemma 24.13.5 combined with the fact that being of finite type is a property of ring maps that is stable under base change, see Algebra, Lemma 7.13.2. By the above and the fact that a base change of a quasi-compact morphism is quasi-compact, see Schemes, Lemma 21.19.3 we see that the base change of a morphism of finite type is a morphism of finite type. $\square$

Lemma 24.14.5. *A closed immersion is of finite type. An immersion is locally of finite type.*

Proof. This is true because an open immersion is a local isomorphism, and a closed immersion is obviously of finite type. $\square$

Lemma 24.14.6. *Let $f : X \to S$ be a morphism. If S is (locally) Noetherian and f (locally) of finite type then X is (locally) Noetherian.*

Proof. This follows immediately from the fact that a ring of finite type over a Noetherian ring is Noetherian, see Algebra, Lemma 7.28.1. (Also: use the fact that the source of a quasi-compact morphism with quasi-compact target is quasi-compact.) $\square$

Lemma 24.14.7. *Let $f : X \to S$ be locally of finite type with S locally Noetherian. Then f is quasi-separated.*

Proof. In fact, it is true that X is quasi-separated, see Properties, Lemma 23.5.4 and Lemma 24.14.6 above. Then apply Schemes, Lemma 21.21.14 to conclude that f is quasi-separated. $\square$

Lemma 24.14.8. *Let $X \to Y$ be a morphism of schemes over a base scheme S. If X is locally of finite type over S, then $X \to Y$ is locally of finite type.*

Proof. Via Lemma 24.14.2 this translates into the following algebra fact: Given ring maps $A \to B \to C$ such that $A \to C$ is of finite type, then $B \to C$ is of finite type. (See Algebra, Lemma 7.6.2). $\square$

24.15. Points of finite type and Jacobson schemes

Let S be a scheme. A finite type point s of S is a point such that the morphism $Spec(\kappa(s)) \to S$ is of finite type. The reason for studying this is that finite type points can replace closed points in a certain sense and in certain situations. There are always enough of them for example. Moreover, a scheme is Jacobson if and only if all finite type points are closed points.

Lemma 24.15.1. *Let S be a scheme. Let k be a field. Let $f : Spec(k) \to S$ be a morphism. The following are equivalent:*

(1) *The morphism f is of finite type.*
(2) *The morphism f is locally of finite type.*
(3) *There exists an affine open $U = Spec(R)$ of S such that f corresponds to a finite ring map $R \to k$.*
(4) *There exists an affine open $U = Spec(R)$ of S such that the image of f consists of a closed point u in U and the field extension $\kappa(u) \subset k$ is finite.*

Proof. The equivalence of (1) and (2) is obvious as $Spec(k)$ is a singleton and hence any morphism from it is quasi-compact.

Suppose f is locally of finite type. Choose any affine open $Spec(R) = U \subset S$ such that the image of f is contained in U, and the ring map $R \to k$ is of finite type. Let $\mathfrak{p} \subset R$ be the kernel. Then $R/\mathfrak{p} \subset k$ is of finite type. By Algebra, Lemma 7.30.2 there exist a $\overline{f} \in R/\mathfrak{p}$ such that $(R/\mathfrak{p})_{\overline{f}}$ is a field and $(R/\mathfrak{p})_{\overline{f}} \to k$ is a finite field extension. If $f \in R$ is a lift of $\overline{f}$, then we see that k is a finite R_f-module. Thus (2) $\Rightarrow$ (3).

Suppose that $Spec(R) = U \subset S$ is an affine open such that f corresponds to a finite ring map $R \to k$. Then f is locally of finite type by Lemma 24.14.2. Thus (3) $\Rightarrow$ (2).

Suppose $R \to k$ is finite. The image of $R \to k$ is a field over which k is finite by Algebra, Lemma 7.32.16. Hence the kernel of $R \to k$ is a maximal ideal. Thus (3) $\Rightarrow$ (4).

The implication (4) $\Rightarrow$ (3) is immediate. $\square$

Lemma 24.15.2. *Let S be a scheme. Let A be an Artinian local ring with residue field κ. Let $f : Spec(A) \to S$ be a morphism of schemes. Then f is of finite type if and only if the composition $Spec(\kappa) \to Spec(A) \to S$ is of finite type.*

Proof. Since the morphism $Spec(\kappa) \to Spec(A)$ is of finite type it is clear that if f is of finite type so is the composition $Spec(\kappa) \to S$ (see Lemma 24.14.3). For the converse, note that $Spec(A) \to S$ maps into some affine open $U = Spec(B)$ of S as $Spec(A)$ has only one point. To finish apply Algebra, Lemma 7.50.3 to $B \to A$. $\square$

Recall that given a point s of a scheme S there is a canonical morphism $Spec(\kappa(s)) \to S$, see Schemes, Section 21.13.

Definition 24.15.3. Let S be a scheme. Let us say that a point s of S is a *finite type point* if the canonical morphism $Spec(\kappa(s)) \to S$ is of finite type. We denote $S_{\text{ft-pts}}$ the set of finite type points of S.

We can describe the set of finite type points as follows.

Lemma 24.15.4. *Let S be a scheme. We have*

$$S_{ft\text{-}pts} = \bigcup\nolimits_{U \subset S \ open} U_0$$

where U_0 is the set of closed points of U. Here we may let U range over all opens or over all affine opens of S.

Proof. Immediate from Lemma 24.15.1. □

Lemma 24.15.5. *Let $f : T \to S$ be a morphism of schemes. If f is locally of finite type, then $f(T_{ft\text{-}pts}) \subset S_{ft\text{-}pts}$.*

Proof. If T is the spectrum of a field this is Lemma 24.15.1. In general it follows since the composition of morphisms locally of finite type is locally of finite type (Lemma 24.14.3). □

Lemma 24.15.6. *Let $f : T \to S$ be a morphism of schemes. If f is locally of finite type and surjective, then $f(T_{ft\text{-}pts}) = S_{ft\text{-}pts}$.*

Proof. We have $f(T_{\text{ft-pts}}) \subset S_{\text{ft-pts}}$ by Lemma 24.15.5. Let $s \in S$ be a finite type point. As f is surjective the scheme $T_s = Spec(\kappa(s)) \times_S T$ is nonempty, therefore has a finite type point $t \in T_s$ by Lemma 24.15.4. Now $T_s \to T$ is a morphism of finite type as a base change of $s \to S$ (Lemma 24.14.4). Hence the image of t in T is a finite type point by Lemma 24.15.5 which maps to s by construction. □

Lemma 24.15.7. *Let S be a scheme. For any locally closed subset $T \subset S$ we have*

$$T \neq \emptyset \Rightarrow T \cap S_{ft\text{-}pts} \neq \emptyset.$$

In particular, for any closed subset $T \subset S$ we see that $T \cap S_{ft\text{-}pts}$ is dense in T.

Proof. Note that T carries a scheme structure (see Schemes, Lemma 21.12.4) such that $T \to S$ is a locally closed immersion. Any locally closed immersion is locally of finite type, see Lemma 24.14.5. Hence by Lemma 24.15.5 we see $T_{\text{ft-pts}} \subset S_{\text{ft-pts}}$. Finally, any nonempty affine open of T has at least one closed point which is a finite type point of T by Lemma 24.15.4. □

It follows that most of the material from Topology, Section 5.13 goes through with the set of closed points replaced by the set of points of finite type. In fact, if S is Jacobson then we recover the closed points as the finite type points.

Lemma 24.15.8. *Let S be a scheme. The following are equivalent:*

(1) *For every finite type morphism $f : Spec(k) \to S$ with k a field the image consists of a closed point of S. In the terminology introduced above: finite type points of S are closed points of S.*
(2) *For every locally finite type morphism $T \to S$ closed points map to closed points.*
(3) *For every locally finite type morphism $f : T \to S$ any closed point $t \in T$ maps to a closed point $s \in S$ and $\kappa(s) \subset \kappa(t)$ is finite.*
(4) *The scheme S is Jacobson.*

Proof. We have trivially (3) ⇒ (2) ⇒ (1). The discussion above shows that (1) implies (4). Hence it suffices to show that (4) implies (3). Suppose that $T \to S$ is locally of finite type. Choose $t \in T$ with $s = f(t)$ as in (3). Choose affine open neighbourhoods $Spec(R) = U \subset S$ of s and $Spec(A) = V \subset T$ of t with $f(V) \subset U$. The induced ring map $R \to A$ is of finite type (see Lemma 24.14.2) and R is Jacobson by Properties, Lemma 23.6.3. Thus the result follows from Algebra, Proposition 7.31.18. □

Lemma 24.15.9. *Let S be a Jacobson scheme. Any scheme locally of finite type over S is Jacobson.*

Proof. This is clear from Algebra, Proposition 7.31.18 (and Properties, Lemma 23.6.3 and Lemma 24.14.2). ☐

Lemma 24.15.10. *The following types of schemes are Jacobson.*

(1) *Any scheme locally of finite type over a field.*
(2) *Any scheme locally of finite type over* $\mathbf{Z}$*.*
(3) *Any scheme locally of finite type over a* 1*-dimensional Noetherian domain with infinitely many primes.*
(4) *A scheme of the form* $Spec(R) \setminus \{\mathfrak m\}$ *where* $(R, \mathfrak m)$ *is a Noetherian local ring. Also any scheme locally of finite type over it.*

Proof. We will use Lemma 24.15.9 without mention. The spectrum of a field is clearly Jacobson. The spectrum of $\mathbf{Z}$ is Jacobson, see Algebra, Lemma 7.31.6. For (3) see Algebra, Lemma 7.58.2. For (4) see Properties, Lemma 23.6.4. ☐

24.16. Universally catenary schemes

Recall that a topological space X is called *catenary* if for every pair of irreducible closed subsets $T \subset T'$ there exist a maximal chain of irreducible closed subsets

$$T = T_0 \subset T_1 \subset \ldots \subset T_e = T'$$

and every such chain has the same length. See Topology, Definition 5.8.1. Recall that a scheme is catenary if its underlying topological space is catenary. See Properties, Definition 23.11.1.

Definition 24.16.1. Let S be a scheme. Assume S is locally Noetherian. We say S is *universally catenary* if for every morphism $X \to S$ locally of finite type the scheme X is catenary.

This is a ``better'' notion than catenary as there exist Noetherian schemes which are catenary but not universally catenary. See Examples, Section 64.9. Many schemes are universally catenary, see Lemma 24.16.4 below.

Recall that a ring A is called *catenary* if for any pair of prime ideals $\mathfrak p \subset \mathfrak q$ there exists a maximal chain of primes

$$\mathfrak p = \mathfrak p_0 \subset \ldots \subset \mathfrak p_e = \mathfrak q$$

and all of these have the same length. See Algebra, Definition 7.97.1. We have seen the relationship between catenary schemes and catenary rings in Properties, Section 23.11. Recall that a ring A is called *universally catenary* if A is Noetherian and for every finite type ring map $A \to B$ the ring B is catenary. See Algebra, Definition 7.97.5. Many interesting rings which come up in algebraic geometry satisfy this property.

Lemma 24.16.2. *Let S be a locally Noetherian scheme. The following are equivalent*

(1) *S is universally catenary,*
(2) *there exists an open covering of S all of whose members are universally catenary schemes,*
(3) *for every affine open $Spec(R) = U \subset S$ the ring R is universally catenary, and*
(4) *there exists an affine open covering $S = \bigcup U_i$ such that each U_i is the spectrum of a universally catenary ring.*

Moreover, in this case any scheme locally of finite type over S is universally catenary as well.

Proof. By Lemma 24.14.5 an open immersion is locally of finite type. A composition of morphisms locally of finite type is locally of finite type (Lemma 24.14.3). Thus it is clear that if S is universally catenary then any open and any scheme locally of finite type over S is universally catenary as well. This proves the final statement of the lemma and that (1) implies (2).

If $Spec(R)$ is a universally catenary scheme, then every scheme $Spec(A)$ with A a finite type R-algebra is catenary. Hence all these rings A are catenary by Algebra, Lemma 7.97.2. Thus R is universally catenary. Combined with the remarks above we conclude that (1) implies (3), and (2) implies (4). Of course (3) implies (4) trivially.

To finish the proof we show that (4) implies (1). Assume (4) and let $X \to S$ be a morphism locally of finite type. We can find an affine open covering $X = \bigcup V_j$ such that each $V_j \to S$ maps into one of the U_i. By Lemma 24.14.2 the induced ring map $\mathcal{O}(U_i) \to \mathcal{O}(V_j)$ is of finite type. Hence $\mathcal{O}(V_j)$ is catenary. Hence X is catenary by Properties, Lemma 23.11.2. □

Lemma 24.16.3. *Let S be a locally Noetherian scheme. The following are equivalent:*

1. *S is universally catenary, and*
2. *all local rings $\mathcal{O}_{S,s}$ of S are universally catenary.*

Proof. Assume that all local rings of S are universally catenary. Let $f : X \to S$ be locally of finite type. We know that X is catenary if and only if $\mathcal{O}_{X,x}$ is catenary for all $x \in X$. If $f(x) = s$, then $\mathcal{O}_{X,x}$ is essentially of finite type over $\mathcal{O}_{S,s}$. Hence $\mathcal{O}_{X,x}$ is catenary by the assumption that $\mathcal{O}_{S,s}$ is universally catenary.

Conversly, assume that S is universally catenary. Let $s \in S$. We may replace S by an affine open neighbourhood of s by Lemma 24.16.2. Say $S = Spec(R)$ and s corresponds to the prime ideal $\mathfrak{p}$. Any finite type $R_{\mathfrak{p}}$-algebra A' is of the form $A_{\mathfrak{p}}$ for some finite type R-algebra A. By assumption (and Lemma 24.16.2 if you like) the ring A is catenary, and hence A' (a localization of A) is catenary. Thus $R_{\mathfrak{p}}$ is universally catenary. □

Lemma 24.16.4. *The following types of schemes are universally catenary.*

1. *Any scheme locally of finite type over a field.*
2. *Any scheme locally of finite type over a Cohen-Macaulay scheme.*
3. *Any scheme locally of finite type over $\mathbf{Z}$.*
4. *Any scheme locally of finite type over a 1-dimensional Noetherian domain.*
5. *And so on.*

Proof. All of these follow from the fact that a Cohen-Macaulay ring is universally catenary, see Algebra, Lemma 7.97.6. Also, use the last assertion of Lemma 24.16.2. Some details omitted. □

24.17. Nagata schemes, reprise

See Properties, Section 23.13 for the definitions and basic properties of Nagata and universally Japanese schemes.

Lemma 24.17.1. *Let $f : X \to S$ be a morphism. If S is Nagata and f locally of finite type then X is Nagata. If S is universally Japanese and f locally of finite type then X is universally Japanese.*

Proof. For ``universally Japanese'' this follows from Algebra, Lemma 7.144.18. For ``Nagata'' this follows from Algebra, Proposition 7.144.30. □

Lemma 24.17.2. *The following types of schemes are Nagata.*

1. *Any scheme locally of finite type over a field.*
2. *Any scheme locally of finite type over a Noetherian complete local ring.*
3. *Any scheme locally of finite type over* $\mathbf{Z}$*.*
4. *Any scheme locally of finite type over a Dedeking ring of characteristic zero.*
5. *And so on.*

Proof. By Lemma 24.17.1 we only need to show that the rings mentioned above are Nagata rings. For this see Algebra, Proposition 7.144.31. $\square$

24.18. The singular locus, reprise

We look for a criterion that implies openness of the regular locus for any scheme locally of finite type over the base. Here is the definition.

Definition 24.18.1. Let X be a locally Noetherian scheme. We say X is *J-2* if for every morphism $Y \to X$ which is locally of finite type the regular locus $\text{Reg}(Y)$ is open in Y.

This is the analogue of the corresponding notion for Noetherian rings, see More on Algebra, Definition 12.35.1.

Lemma 24.18.2. *Let X be a locally Noetherian scheme. The following are equivalent*

1. *X is J-2,*
2. *there exists an open covering of X all of whose members are J-2 schemes,*
3. *for every affine open $Spec(R) = U \subset X$ the ring R is J-2, and*
4. *there exists an affine open covering $S = \bigcup U_i$ such that each $\mathcal{O}(U_i)$ is J-2 for all i.*

Moreover, in this case any scheme locally of finite type over X is J-2 as well.

Proof. By Lemma 24.14.5 an open immersion is locally of finite type. A composition of morphisms locally of finite type is locally of finite type (Lemma 24.14.3). Thus it is clear that if X is J-2 then any open and any scheme locally of finite type over X is J-2 as well. This proves the final statement of the lemma.

If $Spec(R)$ is J-2, then for every finite type R-algebra A the regular locus of the scheme $Spec(A)$ is open. Hence R is J-2, by definition (see More on Algebra, Definition 12.35.1). Combined with the remarks above we conclude that (1) implies (3), and (2) implies (4). Of course (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) trivially.

To finish the proof we show that (4) implies (1). Assume (4) and let $Y \to X$ be a morphism locally of finite type. We can find an affine open covering $Y = \bigcup V_j$ such that each $V_j \to X$ maps into one of the U_i. By Lemma 24.14.2 the induced ring map $\mathcal{O}(U_i) \to \mathcal{O}(V_j)$ is of finite type. Hence the regular locus of $V_j = Spec(\mathcal{O}(V_j))$ is open. Since $\text{Reg}(Y) \cap V_j = \text{Reg}(V_j)$ we conclude that $\text{Reg}(Y)$ is open as desired. $\square$

Lemma 24.18.3. *The following types of schemes are J-2.*

1. *Any scheme locally of finite type over a field.*
2. *Any scheme locally of finite type over a Noetherian complete local ring.*
3. *Any scheme locally of finite type over* $\mathbf{Z}$*.*
4. *Any scheme locally of finite type over a Dedeking ring of characteristic zero.*
5. *And so on.*

Proof. By Lemma 24.18.2 we only need to show that the rings mentioned above are J-2. For this see More on Algebra, Proposition 12.36.6. $\square$

24.19. Quasi-finite morphisms

A solid treatment of quasi-finite morphisms is the basis of many developments further down the road. It will lead to various versions of Zariski's Main Theorem, behaviour of dimensions of fibres, descent for étale morphisms, etc, etc. Before reading this section it may be a good idea to take a look at the algebra results in Algebra, Section 7.113.

Recall that a finite type ring map $R \to A$ is quasi-finite at a prime $\mathfrak{q}$ if $\mathfrak{q}$ defines an isolated point of its fibre, see Algebra, Definition 7.113.3.

Definition 24.19.1. Let $f : X \to S$ be a morphism of schemes.

(1) We say that f is *quasi-finite at a point* $x \in X$ if there exist an affine neighbourhood $Spec(A) = U \subset X$ of x and an affine open $Spec(R) = V \subset S$ such that $f(U) \subset V$, the ring map $R \to A$ is of finite type, and $R \to A$ is quasi-finite at the prime of A corresponding to x (see above).
(2) We say f is *locally quasi-finite* if f is quasi-finite at every point x of X.
(3) We say that f is *quasi-finite* if f is of finite type and every point x is an isolated point of its fibre.

Trivially, a locally quasi-finite morphism is locally of finite type. We will see below that a morphism f which is locally of finite type is quasi-finite at x if and only if x is isolated in its fibre. Moreover, the set of points at which a morphism is quasi-finite is open; we will see this in Section 24.47 on Zariski's Main Theorem.

Lemma 24.19.2. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point. Set $s = f(x)$. If $\kappa(s) \supset \kappa(x)$ is an algebraic field extension, then*

(1) *x is a closed point of its fibre, and*
(2) *if in addition s is a closed point of S, then x is a closed point of X.*

Proof. The second statement follows from the first by elementary topology. According to Schemes, Lemma 21.18.5 to prove the first statement we may replace X by X_s and S by $Spec(\kappa(s))$. Thus we may assume that $S = Spec(k)$ is the spectrum of a field. In this case, let $Spec(A) = U \subset X$ be any affine open containing x. The point x corresponds to a prime ideal $\mathfrak{q} \subset A$ such that $k \subset \kappa(\mathfrak{q})$ is an algebraic field extension. By Algebra, Lemma 7.31.9 we see that $\mathfrak{q}$ is a maximal ideal, i.e., $x \in U$ is a closed point. Since the affine opens form a basis of the topology of X we conclude that $\{x\}$ is closed. $\square$

The following lemma is a version of the Hilbert Nullstellensatz.

Lemma 24.19.3. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point. Set $s = f(x)$. Assume f is locally of finite type. Then x is a closed point of its fibre if and only if $\kappa(s) \subset \kappa(x)$ is a finite field extension.*

Proof. If the extension is finite, then x is a closed point of the fibre by Lemma 24.19.2 above. For the converse, assume that x is a closed point of its fibre. Choose affine opens $Spec(A) = U \subset X$ and $Spec(R) = V \subset S$ such that $f(U) \subset V$. By Lemma 24.14.2 the ring map $R \to A$ is of finite type. Let $\mathfrak{q} \subset A$, resp. $\mathfrak{p} \subset R$ be the prime ideal corresponding to x, resp. s. Consider the fibre ring $\overline{A} = A \otimes_R \kappa(\mathfrak{p})$. Let $\overline{\mathfrak{q}}$ be the prime of $\overline{A}$ corresponding to $\mathfrak{q}$. The assumption that x is a closed point of its fibre implies that $\overline{\mathfrak{q}}$ is a maximal ideal of $\overline{A}$. Since $\overline{A}$ is an algebra of finite type over the field $\kappa(\mathfrak{p})$ we see by the Hilbert Nullstellensatz, see Algebra, Theorem 7.30.1, that $\kappa(\overline{\mathfrak{q}})$ is a finite extension of $\kappa(\mathfrak{p})$. Since $\kappa(s) = \kappa(\mathfrak{p})$ and $\kappa(x) = \kappa(\mathfrak{q}) = \kappa(\overline{\mathfrak{q}})$ we win. $\square$

Lemma 24.19.4. *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $g : S' \to S$ be any morphism. Denote $f' : X' \to S'$ the base change. If $x' \in X'$ maps to a point $x \in X$ which is closed in $X_{f(s)}$ then x' is closed in $X'_{f'(x')}$.*

Proof. The residue field $\kappa(x')$ is a quotient of $\kappa(f'(x')) \otimes_{\kappa(f(x))} \kappa(x)$, see Schemes, Lemma 21.17.5. Hence it is a finite extension of $\kappa(f'(x'))$ as $\kappa(x)$ is a finite extension of $\kappa(f(x))$ by Lemma 24.19.3. Thus we see that x' is closed in its fibre by applying that lemma one more time. □

Lemma 24.19.5. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point. Set $s = f(x)$. If f is quasi-finite at x, then the residue field extension $\kappa(s) \subset \kappa(x)$ is finite.*

Proof. This is clear from Algebra, Definition 7.113.3. □

Lemma 24.19.6. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point. Set $s = f(x)$. Let X_s be the fibre of f at s. Assume f is locally of finite type. The following are equivalent:*

(1) *The morphism f is quasi-finite at x.*
(2) *The point x is isolated in X_s.*
(3) *The point x is closed in X_s and there is no point $x' \in X_s$, $x' \neq x$ which specializes to x.*
(4) *For any pair of affine opens $Spec(A) = U \subset X$, $Spec(R) = V \subset S$ with $f(U) \subset V$ and $x \in U$ corresponding to $\mathfrak{q} \subset A$ the ring map $R \to A$ is quasi-finite at $\mathfrak{q}$.*

Proof. Assume f is quasi-finite at x. By assumption there exist opens $U \subset X$, $V \subset S$ such that $f(U) \subset V$, $x \in U$ and x an isolated point of U_s. Hence $\{x\} \subset U_s$ is an open subset. Since $U_s = U \cap X_s \subset X_s$ is also open we conclude that $\{x\} \subset X_s$ is an open subset also. Thus we conclude that x is an isolated point of X_s.

Note that X_s is a Jacobson scheme by Lemma 24.15.10 (and Lemma 24.14.4). If x is isolated in X_s, i.e., $\{x\} \subset X_s$ is open, then $\{x\}$ contains a closed point (by the Jacobson property), hence x is closed in X_s. It is clear that there is no point $x' \in X_s$, distinct from x, specializing to x.

Assume that x is closed in X_s and that there is no point $x' \in X_s$, distinct from x, specializing to x. Consider a pair of affine opens $Spec(A) = U \subset X$, $Spec(R) = V \subset S$ with $f(U) \subset V$ and $x \in U$. Let $\mathfrak{q} \subset A$ correspond to x and $\mathfrak{p} \subset R$ correspond to s. By Lemma 24.14.2 the ring map $R \to A$ is of finite type. Consider the fibre ring $\overline{A} = A \otimes_R \kappa(\mathfrak{p})$. Let $\overline{\mathfrak{q}}$ be the prime of $\overline{A}$ corresponding to $\mathfrak{q}$. Since $Spec(\overline{A})$ is an open subscheme of the fibre X_s we see that $\overline{q}$ is a maximal ideal of $\overline{A}$ and that there is no point of $Spec(\overline{A})$ specializing to $\overline{\mathfrak{q}}$. This implies that $\dim(\overline{A}_{\overline{q}}) = 0$. Hence by Algebra, Definition 7.113.3 we see that $R \to A$ is quasi-finite at $\mathfrak{q}$, i.e., $X \to S$ is quasi-finite at x by definition.

At this point we have shown conditions (1) -- (3) are all equivalent. It is clear that (4) implies (1). And it is also clear that (2) implies (4) since if x is an isolated point of X_s then it is also an isolated point of U_s for any open U which contains it. □

Lemma 24.19.7. *Let $f : X \to S$ be a morphism of schemes. Let $s \in S$. Assume that*

(1) *f is locally of finite type, and*
(2) *$f^{-1}(\{s\})$ is a finite set.*

Then X_s is a finite discrete topological space, and f is quasi-finite at each point of X lying over s.

Proof. Suppose T is a scheme which (a) is locally of finite type over a field k, and (b) has finitely many points. Then Lemma 24.15.10 shows T is a Jacobson scheme. A finite sober Jacobson space is discrete, see Topology, Lemma 5.13.6. Apply this remark to the fibre X_s which is locally of finite type over $Spec(\kappa(s))$ to see the first statement. Finally, apply Lemma 24.19.6 to see the second. □

Lemma 24.19.8. *Let $f : X \to S$ be a morphism of schemes. Assume f is locally of finite type. Then the following are equivalent*

(1) *f is locally quasi-finite,*
(2) *for every $s \in S$ the fibre X_s is a discrete topological space, and*
(3) *for every morphism $Spec(k) \to S$ where k is a field the base change X_k has an underlying discrete topological space.*

Proof. It is immediate that (3) implies (2). Lemma 24.19.6 shows that (2) is equivalent to (1). Assume (2) and let $Spec(k) \to S$ be as in (3). Denote $s \in S$ the image of $Spec(k) \to S$. Then X_k is the base change of X_s via $Spec(k) \to Spec(\kappa(s))$. Hence every point of X_k is closed by Lemma 24.19.4. As $X_k \to Spec(k)$ is locally of finite type (by Lemma 24.14.4), we may apply Lemma 24.19.6 to conclude that every point of X_k is isolated, i.e., X_k has a discrete underlying topological space. □

Lemma 24.19.9. *Let $f : X \to S$ be a morphism of schemes. Then f is quasi-finite if and only if f is locally quasi-finite and quasi-compact.*

Proof. Assume f is quasi-finite. It is quasi-compact by Definition 24.14.1. Let $x \in X$. We see that f is quasi-finite at x by Lemma 24.19.6. Hence f is quasi-compact and locally quasi-finite.

Assume f is quasi-compact and locally quasi-finite. Then f is of finite type. Let $x \in X$ be a point. By Lemma 24.19.6 we see that x is an isolated point of its fibre. The lemma is proved. □

Lemma 24.19.10. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *f is quasi-finite, and*
(2) *f is locally of finite type, quasi-compact, and has finite fibres.*

Proof. Assume f is quasi-finite. In particular f is locally of finite type and quasi-compact (since it is of finite type). Let $s \in S$. Since every $x \in X_s$ is isolated in X_s we see that $X_s = \bigcup_{x \in X_s} \{x\}$ is an open covering. As f is quasi-compact, the fibre X_s is quasi-compact. Hence we see that X_s is finite.

Conversely, assume f is locally of finite type, quasi-compact and has finite fibres. Then it is locally quasi-finite by Lemma 24.19.7. Hence it is quasi-finite by Lemma 24.19.9. □

Recall that a ring map $R \to A$ is quasi-finite if it is of finite type and quasi-finite at *all* primes of A, see Algebra, Definition 7.113.3.

Lemma 24.19.11. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *The morphism f is locally quasi-finite.*
(2) *For every pair of affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is quasi-finite.*

(3) *There exists an open covering $S = \bigcup_{j\in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i\in I_j} U_i$ such that each of the morphisms $U_i \to V_j$, $j \in J, i \in I_j$ is locally quasi-finite.*
(4) *There exists an affine open covering $S = \bigcup_{j\in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i\in I_j} U_i$ such that the ring map $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i)$ is quasi-finite, for all $j \in J, i \in I_j$.*

Moreover, if f is locally quasi-finite then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \to V$ is locally quasi-finite.

Proof. For a ring map $R \to A$ let us define $P(R \to A)$ to mean ``$R \to A$ is quasi-finite'' (see remark above lemma). We claim that $P$ is a local property of ring maps. We check conditions (a), (b) and (c) of Definition 24.13.1. In the proof of Lemma 24.14.2 we have seen that (a), (b) and (c) hold for the property of being ``of finite type''. Note that, for a finite type ring map $R \to A$, the property $R \to A$ is quasi-finite at $\mathfrak{q}$ depends only on the local ring $A_\mathfrak{q}$ as an algebra over $R_\mathfrak{p}$ where $\mathfrak{p} = R \cap \mathfrak{q}$ (usual abuse of notation). Using these remarks (a), (b) and (c) of Definition 24.13.1 follow immediately. For example, suppose $R \to A$ is a ring map such that all of the ring maps $R \to A_{a_i}$ are quasi-finite for $a_1, \ldots, a_n \in A$ generating the unit ideal. We conclude that $R \to A$ is of finite type. Also, for any prime $\mathfrak{q} \subset A$ the local ring $A_\mathfrak{q}$ is isomorphic as an R-algebra to the local ring $(A_{a_i})_{\mathfrak{q}_i}$ for some i and some $\mathfrak{q}_i \subset A_{a_i}$. Hence we conclude that $R \to A$ is quasi-finite at $\mathfrak{q}$.

We conclude that Lemma 24.13.3 applies with P as in the previous paragraph. Hence it suffices to prove that f is locally quasi-finite is equivalent to f is locally of type P. Since $P(R \to A)$ is ``$R \to A$ is quasi-finite'' which means $R \to A$ is quasi-finite at every prime of A, this follows from Lemma 24.19.6. $\square$

Lemma 24.19.12. *The composition of two morphisms which are locally quasi-finite is locally quasi-finite. The same is true for quasi-finite morphisms.*

Proof. In the proof of Lemma 24.19.11 we saw that P =``quasi-finite'' is a local property of ring maps, and that a morphism of schemes is locally quasi-finite if and only if it is locally of type P as in Definition 24.13.2. Hence the first statement of the lemma follows from Lemma 24.13.5 combined with the fact that being quasi-finite is a property of ring maps that is stable under composition, see Algebra, Lemma 7.113.7. By the above, Lemma 24.19.9 and the fact that compositions of quasi-compact morphisms are quasi-compact, see Schemes, Lemma 21.19.4 we see that the composition of quasi-finite morphisms is quasi-finite. $\square$

Lemma 24.19.13. *Let $f : X \to S$ be a morphism of schemes. Let $g : S' \to S$ be a morphism of schemes. Denote $f' : X_{S'} \to S'$ the base change of f by g and denote $g' : X_{S'} \to X$ the projection. Assume X is locally of finite type over S.*

(1) *Let $U \subset X$ (resp. $U' \subset X'$) be the set of points where f (resp. f') is quasi-finite. Then $U' = U_{S'} = (g')^{-1}(U)$.*
(2) *The base change of a locally quasi-finite morphism is locally quasi-finite.*
(3) *The base change of a quasi-finite morphism is quasi-finite.*

Proof. The first and second assertion follow from the corresponding algebra result, see Algebra, Lemma 7.113.8 (combined with the fact that f' is also locally of finite type by Lemma 24.14.4). By the above, Lemma 24.19.9 and the fact that a base change of a quasi-compact morphism is quasi-compact, see Schemes, Lemma 21.19.3 we see that the base change of a quasi-finite morphism is quasi-finite. $\square$

Lemma 24.19.14. *Any immersion is locally quasi-finite.*

Proof. This is true because an open immersion is a local isomorphism and a closed immersion is clearly quasi-finite. □

Lemma 24.19.15. *Let $X \to Y$ be a morphism of schemes over a base scheme S. Let $x \in X$. If $X \to S$ is quasi-finite at x, then $X \to Y$ is quasi-finite at x. If X is locally quasi-finite over S, then $X \to Y$ is locally quasi-finite.*

Proof. Via Lemma 24.19.11 this translates into the following algebra fact: Given ring maps $A \to B \to C$ such that $A \to C$ is quasi-finite, then $B \to C$ is quasi-finite. This follows from Algebra, Lemma 7.113.6 with $R = A$, $S = S' = C$ and $R' = B$. □

24.20. Morphisms of finite presentation

Recall that a ring map $R \to A$ is of finite presentation if A is isomorphic to $R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ as an R-algebra for some n, m and some polynomials f_j, see Algebra, Definition 7.6.1.

Definition 24.20.1. Let $f : X \to S$ be a morphism of schemes.

1. We say that f is of *finite presentation at $x \in X$* if there exists a affine open neighbourhood $Spec(A) = U \subset X$ of x and and affine open $Spec(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is of finite presentation.
2. We say that f is *locally of finite presentation* if it is of finite presentation at every point of X.
3. We say that f is of *finite presentation* if it is locally of finite presentation, quasi-compact and quasi-separated.

Note that a morphism of finite presentation is **not** just a quasi-compact morphism which is locally of finite presentation. Later we will characterize morphisms which are locally of finite presentation as those morphisms such that

$$colim\, Mor_S(T_i, X) = Mor_S(lim\, T_i, X)$$

for any directed system of affine schemes T_i over S. See Limits, Proposition 27.4.1. In Limits, Section 27.6 we show that, if $S = lim_i S_i$ is a limit of affine schemes, any scheme X of finite presentation over S descends to a scheme X_i over S_i for some i.

Lemma 24.20.2. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

1. *The morphism f is locally of finite presentation.*
2. *For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is of finite presentation.*
3. *There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$, $j \in J, i \in I_j$ is locally of finite presentation.*
4. *There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring map $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i)$ is of finite presentation, for all $j \in J, i \in I_j$.*

Moreover, if f is locally of finite presentation then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \to V$ is locally of finite presentation.

Proof. This follows from Lemma 24.13.3 if we show that the property ``$R \to A$ is of finite presentation" is local. We check conditions (a), (b) and (c) of Definition 24.13.1. By Algebra, Lemma 7.13.2 being of finite presentation is stable under base change and hence we conclude (a) holds. By the same lemma being of finite presentation is stable under composition and trivially for any ring R the ring map $R \to R_f$ is of finite presentation. We conclude (b) holds. Finally, property (c) is true according to Algebra, Lemma 7.21.3. $\square$

Lemma 24.20.3. *The composition of two morphisms which locally of finite presentation is locally of finite presentation. The same is true for morphisms of finite presentation.*

Proof. In the proof of Lemma 24.20.2 we saw that being of finite presentation is a local property of ring maps. Hence the first statement of the lemma follows from Lemma 24.13.5 combined with the fact that being of finite presentation is a property of ring maps that is stable under composition, see Algebra, Lemma 7.6.2. By the above and the fact that compositions of quasi-compact, quasi-separated morphisms are quasi-compact and quasi-separated, see Schemes, Lemmas 21.19.4 and 21.21.13 we see that the composition of morphisms of finite presentation is of finite presentation. $\square$

Lemma 24.20.4. *The base change of a morphism which is locally of finite presentation is locally of finite presentation. The same is true for morphisms of finite presentation.*

Proof. In the proof of Lemma 24.20.2 we saw that being of finite presentation is a local property of ring maps. Hence the first statement of the lemma follows from Lemma 24.13.5 combined with the fact that being of finite presentation is a property of ring maps that is stable under base change, see Algebra, Lemma 7.13.2. By the above and the fact that a base change of a quasi-compact, quasi-separated morphism is quasi-compact and quasi-separated, see Schemes, Lemmas 21.19.3 and 21.21.13 we see that the base change of a morphism of finite presentation is a morphism of finite presentation. $\square$

Lemma 24.20.5. *Any open immersion is locally of finite presentation.*

Proof. This is true because an open immersion is a local isomorphism. $\square$

Lemma 24.20.6. *Any open immersion is of finite presentation if and only if it is quasi-compact.*

Proof. We have seen (Lemma 24.20.5) that an open immersion is locally of finite presentation. We have see (Schemes, Lemma 21.23.7) that an immersion is separated and hence quasi-separated. From this and Definition 24.20.1 the lemma follows. $\square$

Lemma 24.20.7. *Any closed immersion $i : Z \to X$ is of finite presentation if and only if the associated quasi-coherent sheaf of ideals $\mathcal{I} = Ker(\mathcal{O}_X \to i_*\mathcal{O}_Z)$ is of finite type (as an $\mathcal{O}_X$-module).*

Proof. On any affine open $Spec(R) \subset X$ we have $i^{-1}(Spec(R)) = Spec(R/I)$ and $\mathcal{I} = \tilde{I}$. Moreover, $\mathcal{I}$ is of finite type if and only if I is a finite R-module for every such affine open (see Properties, Lemma 23.16.1). And R/I is of finite presentation over R if and only if I is a finite R-module. Hence we win. $\square$

Lemma 24.20.8. *A morphism which is locally of finite presentation is locally of finite type. A morphism of finite presentation is of finite type.*

Proof. Omitted. $\square$

Lemma 24.20.9. *Let $f : X \to S$ be a morphism.*

(1) *If S is locally Noetherian and f locally of finite type then f is locally of finite presentation.*
(2) *If S is locally Noetherian and f of finite type then f is of finite presentation.*

Proof. The first statement follows from the fact that a ring of finite type over a Noetherian ring is of finite presentation, see Algebra, Lemma 7.28.4. Suppose that f is of finite type and S is locally Noetherian. Then f is quasi-compact and locally of finite presentation by (1). Hence it suffices to prove that f is quasi-separated. This follows from Lemma 24.14.7 (and Lemma 24.20.8). ☐

Lemma 24.20.10. *Let S be a scheme which is quasi-compact and quasi-separated. If X is of finite presentation over S, then X is quasi-compact and quasi-separated.*

Proof. Omitted. ☐

Lemma 24.20.11. *Let $f : X \to Y$ be a morphism of schemes over S. If X is locally of finite presentation over S, and Y is locally of finite type over S, then f is locally of finite presentation.*

Proof. Via Lemma 24.20.2 this translates into the following algebra fact: Given ring maps $A \to B \to C$ such that $A \to C$ is of finite presentation and $A \to B$ is of finite type, then $B \to C$ is of finite type. (See Algebra, Lemma 7.6.2). ☐

24.21. Constructible sets

Constructible and locally construcible sets of schemes have been discussed in Properties, Section 23.2. In this section we prove some results concerning images and inverse images of (locally) constructible sets. The main result is Chevalley's theorem which states that the image of a locally constructible set under a morphism of finite presentation is locally constructible.

Lemma 24.21.1. *Let $f : X \to Y$ be a morphism of schemes. Let $E \subset Y$ be a subset. If E is (locally) construcible in Y, then $f^{-1}(E)$ is (locally) constructible in X.*

Proof. To show that the inverse image of every construcible subset is constructible it suffices to show that the inverse image of every retrocompact open V of Y is retrocompact in X, see Topology, Lemma 5.10.3. The significance of V being retrocompact in Y is just that the open immersion $V \to Y$ is quasi-compact. Hence the base change $f^{-1}(V) = X \times_Y V \to X$ is quasi-compact too, see Schemes, Lemma 21.19.3. Hence we see $f^{-1}(V)$ is retrocompact in X. Suppose E is locally constructible in Y. Choose $x \in X$. Choose an affine neighbourhood V of $f(x)$ and an affine neighbourhood $U \subset X$ of x such that $f(U) \subset V$. Thus we think of $f|_U : U \to V$ as a morphism into V. By Properties, Lemma 23.2.1 we see that $E \cap V$ is constructible in V. By the constructible case we see that $(f|_U)^{-1}(E \cap V)$ is constructible in U. Since $(f|_U)^{-1}(E \cap V) = f^{-1}(E) \cap U$ we win. ☐

Lemma 24.21.2. *Let $f : X \to Y$ be a morphism of schemes. Assume*

(1) *f is quasi-compact and locally of finite presentation, and*
(2) *Y is quasi-compact and quasi-separated.*

Then the image of every constructible subset of X is constructible in Y.

Proof. By Properties, Lemma 23.2.3 it suffices to prove this lemma in case Y is affine. In this case X is quasi-compact. Hence we can write $X = U_1 \cup \ldots \cup U_n$ with each U_i affine open in X. If $E \subset X$ is constructible, then each $E \cap U_i$ is constructible too, see Topology, Lemma 5.10.4. Hence, since $f(E) = \bigcup f(E \cap U_i)$ and since finite unions of constructible sets are constructible, this reduces us to the case where X is affine. In this case the result is Algebra, Theorem 7.26.9. ☐

Theorem 24.21.3. *(Chevalley's Theorem.) Let $f : X \to Y$ be a morphism of schemes. Assume f is quasi-compact and locally of finite presentation. Then the image of every locally constructible subset is locally constructible.*

Proof. Let $E \subset X$ be locally constructible. We have to show that $f(E)$ is locally constructible too. We will show that $f(E) \cap V$ is constructible for any affine open $V \subset Y$. Thus we reduce to the case where Y is affine. In this case X is quasi-compact. Hence we can write $X = U_1 \cup \ldots \cup U_n$ with each U_i affine open in X. If $E \subset X$ is locally constructible, then each $E \cap U_i$ is constructible, see Properties, Lemma 23.2.1. Hence, since $f(E) = \bigcup f(E \cap U_i)$ and since finite unions of constructible sets are constructible, this reduces us to the case where X is affine. In this case the result is Algebra, Theorem 7.26.9. □

Lemma 24.21.4. *Let X be a scheme. Let $x \in X$. Let $E \subset X$ be a locally constructible subset. If $\{x' \mid x' \leadsto x\} \subset E$, then E contains an open neighbourhood of x.*

Proof. Assume $\{x' \mid x' \leadsto x\} \subset E$. We may assume X is affine. In this case E is constructible, see Properties, Lemma 23.2.1. In particular, also the complement E^c is constructible. By Algebra, Lemma 7.26.3 we can find a morphism of affine schemes $f : Y \to X$ such that $E^c = f(Y)$. Let $Z \subset X$ be the scheme theoretic image of f. By Lemma 24.4.5 and the assumption $\{x' \mid x' \leadsto x\} \subset E$ we see that $x \notin Z$. Hence $X \setminus Z \subset E$ is an open neighbourhood of x contained in E. □

24.22. Open morphisms

Definition 24.22.1. Let $f : X \to S$ be a morphism.

(1) We say f is *open* if the map on underlying topological spaces is open.
(2) We say f is *universally open* if for any morphism of schemes $S' \to S$ the base change $f' : X_{S'} \to S'$ is open.

According to Topology, Lemma 5.14.6 generalizations lift along certain types of open maps of topological spaces. In fact generalizations lift along any open morphism of schemes (see Lemma 24.22.5). Also, we will see that generalizations lift along flat morphisms of schemes (Lemma 24.24.8). This sometimes in turn implies that the morphism is open.

Lemma 24.22.2. *Let $f : X \to S$ be a morphism.*

(1) *If f is locally of finite presentation and generalizations lift along f, then f is open.*
(2) *If f is locally of finite presentation and generalizations lift along every base change of f, then f is universally open.*

Proof. It suffices to prove the first assertion. This reduces to the case where both X and S are affine. In this case the result follows from Algebra, Lemma 7.36.3 and Proposition 7.36.8. □

See also Lemma 24.24.9 for the case of a morphism flat of finite presentation.

Lemma 24.22.3. *A composition of (universally) open morphisms is (universally) open.*

Proof. Omitted. □

Lemma 24.22.4. *Let k be a field. Let X be a scheme over k. The structure morphism $X \to Spec(k)$ is universally open.*

Proof. Let $S \to Spec(k)$ be a morphism. We have to show that the base change $X_S \to S$ is open. The question is local on S and X, hence we may assume that S and X are affine. In this case the result is Algebra, Lemma 7.36.10. □

Lemma 24.22.5. *Let $\varphi : X \to Y$ be a morphism of schemes. If φ is open, then φ is generizing (i.e., generalizations lift along φ). If φ is universally open, then φ is universally generizing.*

Proof. Assume φ is open. Let $y' \rightsquigarrow y$ be a specialization of points of Y. Let $x \in X$ with $\varphi(x) = y$. Choose affine opens $U \subset X$ and $V \subset Y$ such that $\varphi(U) \subset V$ and $x \in U$. Then also $y' \in V$. Hence we may replace X by U and Y by V and assume X, Y affine. The affine case is Algebra, Lemma 7.36.2 (combined with Algebra, Lemma 7.36.3). $\square$

Lemma 24.22.6. *Let $f : X \to Y$ be a morphism of schemes. Let $g : Y' \to Y$ be open and surjective such that the base change $f' : X' \to Y'$ is quasi-compact. Then f is quasi-compact.*

Proof. Let $V \subset Y$ be a quasi-compact open. As g is open and surjective we can find a quasi-compact open $W' \subset W$ such that $g(W') = V$. By assumption $(f')^{-1}(W')$ is quasi-compact. The image of $(f')^{-1}(W')$ in X is equal to $f^{-1}(V)$, see Lemma 24.9.3. Hence $f^{-1}(V)$ is quasi-compact as the image of a quasi-compact space, see Topology, Lemma 5.9.5. Thus f is quasi-compact. $\square$

24.23. Submersive morphisms

Definition 24.23.1. Let $f : X \to Y$ be a morphism of schemes.

(1) We say f is *submersive*[5] if the continuous map of underlying topological spaces is submersive, see Topology, Definition 5.15.1.
(2) We say f is *universally submersive* if for every morphism of schemes $Y' \to Y$ the base change $Y' \times_Y X \to Y'$ is submersive.

We note that a submersive morphism is in particular surjective.

24.24. Flat morphisms

Flatness is one of the most important technical tools in algebraic geometry. In this section we introduce this notion. We intentionally limit the discussion to straightforward observations, apart from Lemma 24.24.9. A very important class of results, namely criteria for flatness will be discussed (insert future reference here).

Recall that a module M over a ring R is *flat* if the functor $- \otimes_R M : \text{Mod}_R \to \text{Mod}_R$ is exact. A ring map $R \to A$ is said to be *flat* if A is flat as an R-module. See Algebra, Definition 7.35.1.

Definition 24.24.1. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules.

(1) We say f is *flat at a point* $x \in X$ if the local ring $\mathcal{O}_{X,x}$ is flat over the local ring $\mathcal{O}_{S,f(x)}$.
(2) We say that $\mathcal{F}$ is *flat over S at a point* $x \in X$ if the stalk $\mathcal{F}_x$ is a flat $\mathcal{O}_{S,f(x)}$-module.
(3) We say f is *flat* if f is flat at every point of X.
(4) We say that $\mathcal{F}$ is *flat over S* if $\mathcal{F}$ is flat over S at every point x of X.

Thus we see that f is flat if and only if the structure sheaf $\mathcal{O}_X$ is flat over S.

Lemma 24.24.2. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. The following are equivalent*

[5]This is very different from the notion of a submersion of differential manifolds.

(1) *The sheaf* $\mathcal{F}$ *is flat over* S.
(2) *For every affine opens* $U \subset X$, $V \subset S$ *with* $f(U) \subset V$ *the* $\mathcal{O}_S(V)$*-module* $\mathcal{F}(U)$ *is flat.*
(3) *There exists an open covering* $S = \bigcup_{j \in J} V_j$ *and open coverings* $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ *such that each of the modules* $\mathcal{F}|_{U_i}$ *is flat over* V_j*, for all* $j \in J, i \in I_j$.
(4) *There exists an affine open covering* $S = \bigcup_{j \in J} V_j$ *and affine open coverings* $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ *such that* $\mathcal{F}(U_i)$ *is a flat* $\mathcal{O}_S(V_j)$*-module, for all* $j \in J, i \in I_j$.

Moreover, if $\mathcal{F}$ *is flat over* S *then for any open subschemes* $U \subset X$, $V \subset S$ *with* $f(U) \subset V$ *the restriction* $\mathcal{F}|_U$ *is flat over* V.

Proof. Let $R \to A$ be a ring map. Let M be an A-module. If M is R-flat, then for all primes $\mathfrak{q}$ the module $M_\mathfrak{q}$ is flat over $R_\mathfrak{p}$ with $\mathfrak{p}$ the prime of R lying under $\mathfrak{q}$. Conversely, if $M_\mathfrak{q}$ is flat over $R_\mathfrak{p}$ for all primes $\mathfrak{q}$ of A, then M is flat over R. See Algebra, Lemma 7.35.19. This equivalence easily implies the statements of the lemma. $\square$

Lemma 24.24.3. *Let* $f : X \to S$ *be a morphism of schemes. The following are equivalent*

(1) *The morphism* f *is flat.*
(2) *For every affine opens* $U \subset X$, $V \subset S$ *with* $f(U) \subset V$ *the ring map* $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ *is flat.*
(3) *There exists an open covering* $S = \bigcup_{j \in J} V_j$ *and open coverings* $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ *such that each of the morphisms* $U_i \to V_j$, $j \in J, i \in I_j$ *is flat.*
(4) *There exists an affine open covering* $S = \bigcup_{j \in J} V_j$ *and affine open coverings* $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ *such that* $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i)$ *is flat, for all* $j \in J, i \in I_j$.

Moreover, if f *is flat then for any open subschemes* $U \subset X$, $V \subset S$ *with* $f(U) \subset V$ *the restriction* $f|_U : U \to V$ *is flat.*

Proof. This is a special case of Lemma 24.24.2 above. $\square$

Lemma 24.24.4. *Let* $X \to Y \to Z$ *be morphisms of schemes. Let* $\mathcal{F}$ *be a quasi-coherent* $\mathcal{O}_X$*-module. If* $\mathcal{F}$ *is flat over* Y*, and* Y *is flat over* Z*, then* $\mathcal{F}$ *is flat over* Z.

Proof. See Algebra, Lemma 7.35.3. $\square$

Lemma 24.24.5. *The composition of flat morphisms is flat.*

Proof. This is a special case of Lemma 24.24.4. $\square$

Lemma 24.24.6. *Let* $f : X \to S$ *be a morphism of schemes. Let* $\mathcal{F}$ *be a quasi-coherent sheaf of* $\mathcal{O}_X$*-modules. Let* $g : S' \to S$ *be a morphism of schemes. Denote* $g' : X' = X_{S'} \to X$ *the projection. Let* $x' \in X'$ *be a point with image* $x = g(x') \in X$. *If* $\mathcal{F}$ *is flat over* S *at* x*, then* $(g')^*\mathcal{F}$ *is flat over* S' *at* x'. *In particular, if* $\mathcal{F}$ *is flat over* S*, then* $(g')^*\mathcal{F}$ *is flat over* S'.

Proof. See Algebra, Lemma 7.35.6. $\square$

Lemma 24.24.7. *The base change of a flat morphism is flat.*

Proof. This is a special case of Lemma 24.24.6. $\square$

Lemma 24.24.8. *Let* $f : X \to S$ *be a flat morphism of schemes. Then generalizations lift along* f*, see Topology, Definition 5.14.3.*

Proof. See Algebra, Section 7.36. $\square$

Lemma 24.24.9. *A flat morphism locally of finite presentation is universally open.*

Proof. This follows from Lemmas 24.24.8 and Lemma 24.22.2 above. We can also argue directly as follows.

Let $f : X \to S$ be flat locally of finite presentation. To show f is open it suffices to show that we may cover X by open affines $X = \bigcup U_i$ such that $U_i \to S$ is open. By definition we may cover X by affine opens $U_i \subset X$ such that each U_i maps into an affine open $V_i \subset S$ and such that the induced ring map $\mathcal{O}_S(V_i) \to \mathcal{O}_X(U_i)$ is of finite presentation. Thus $U_i \to V_i$ is open by Algebra, Proposition 7.36.8. The lemma follows. □

Lemma 24.24.10. *Let $f : X \to Y$ be a quasi-compact, surjective, flat morphism. A subset $T \subset Y$ is open (resp. closed) if and only $f^{-1}(T)$ is open (resp. closed). In other words, f is a submersive morphism.*

Proof. The question is local on Y, hence we may assume that Y is affine. In this case X is quasi-compact as f is quasi-compact. Write $X = X_1 \cup \ldots \cup X_n$ as a finite union of affine opens. Then $f' : X' = X_1 \coprod \ldots \coprod X_n \to Y$ is a surjective flat morphism of affine schemes. Note that for $T \subset Y$ we have $(f')^{-1}(T) = f^{-1}(T) \cap X_1 \coprod \ldots \coprod f^{-1}(T) \cap X_n$. Hence, $f^{-1}(T)$ is open if and only if $(f')^{-1}(T)$ is open. Thus we may assume both X and Y are affine.

Let $f : Spec(B) \to Spec(A)$ be a surjective morphism of affine schemes corresponding to a flat ring map $A \to B$. Suppose that $f^{-1}(T)$ is closed, say $f^{-1}(T) = V(I)$ for $I \subset A$ an ideal. Then $T = f(f^{-1}(T)) = f(V(I))$ is the image of $Spec(A/I) \to Spec(B)$ (here we use that f is surjective). On the other hand, generalizations lift along f (Lemma 24.24.8). Hence by Topology, Lemma 5.14.5 we see that $Y \setminus T = f(X \setminus f^{-1}(T))$ is stable under generalization. Hence T is stable under specialization (Topology, Lemma 5.14.2). Thus T is closed by Algebra, Lemma 7.36.5. □

Lemma 24.24.11. *Let $h : X \to Y$ be a morphism of schemes over S. Let $\mathcal{G}$ be a quasi-coherent sheaf on Y. Let $x \in X$ with $y = h(x) \in Y$. If h is flat at x, then*

$$\mathcal{G} \text{ flat over } S \text{ at } y \Leftrightarrow h^*\mathcal{G} \text{ flat over } S \text{ at } x.$$

In particular: If h is surjective and flat, then $\mathcal{G}$ is flat over S, if and only if $h^*\mathcal{G}$ is flat over S. If h is surjective and flat, and X is flat over S, then Y is flat over S.

Proof. You can prove this by applying Algebra, Lemma 7.35.8. Here is a direct proof. Let $s \in S$ be the image of y. Consider the local ring maps $\mathcal{O}_{S,s} \to \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$. By assumption the ring map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is faithfully flat, see Algebra, Lemma 7.35.16. Let $N = \mathcal{G}_y$. Note that $h^*\mathcal{G}_x = N \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x}$, see Sheaves, Lemma 6.26.4. Let $M' \to M$ be an injection of $\mathcal{O}_{S,s}$-modules. By the faithful flatness mentioned above we have

$$\begin{aligned} & \mathrm{Ker}(M' \otimes_{\mathcal{O}_{S,s}} N \to M \otimes_{\mathcal{O}_{S,s}} N) \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \\ = {} & \mathrm{Ker}(M' \otimes_{\mathcal{O}_{S,s}} N \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \to M \otimes_{\mathcal{O}_{S,s}} N \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x}) \end{aligned}$$

Hence the equivalence of the lemma follows from the second characterization of flatness in Algebra, Lemma 7.35.4. □

24.25. Flat closed immersions

Connected components of schemes are not always open. But they do always have a canonical scheme structure. We explain this in this section.

Lemma 24.25.1. *Let X be a scheme. The rule which associates to a closed subscheme of X its underlying closed subset defines a bijection*

$$\left\{ \begin{array}{c} \text{closed subschemes } Z \subset X \\ \text{such that } Z \to X \text{ is flat} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{closed subsets } Z \subset X \\ \text{closed under generalizations} \end{array} \right\}$$

Proof. The affine case is Algebra, Lemma 7.100.4. In general the lemma follows by covering X by affines and glueing. Details omitted. $\square$

Note that a connected component T of a scheme X is a closed subset stable under generalization. Hence the following definition makes sense.

Definition 24.25.2. Let X be a scheme. Let $T \subset X$ be a connected component. The *canonical scheme structure on T* is the unique scheme structure on T such that the closed immersion $T \to X$ is flat, see Lemma 24.25.1.

It turns out that we can determine when every finite flat $\mathcal{O}_X$-module is finite locally free using the previous lemma.

Lemma 24.25.3. *Let X be a scheme. The following are equivalent*

(1) *every finite flat quasi-coherent $\mathcal{O}_X$-module is finite locally free, and*
(2) *every closed subset $Z \subset X$ which is closed under generalizations is open.*

Proof. In the affine case this is Algebra, Lemma 7.100.6. The scheme case does not follow directly from the affine case, so we simply repeat the arguments.

Assume (1). Consider a closed immersion $i : Z \to X$ such that i is flat. Then $i_*\mathcal{O}_Z$ is quasi-coherent and flat, hence finite locally free by (1). Thus $Z = \text{Supp}(i_*\mathcal{O}_Z)$ is also open and we see that (2) holds. Hence the implication (1) $\Rightarrow$ (2) follows from the characterization of flat closed immersions in Lemma 24.25.1.

For the converse assume that X satisfies (2). Let $\mathcal{F}$ be a finite flat quasi-coherent $\mathcal{O}_X$-module. The support $Z = \text{Supp}(\mathcal{F})$ of $\mathcal{F}$ is closed, see Modules, Lemma 15.9.6. On the other hand, if $x \rightsquigarrow x'$ is a specialization, then by Algebra, Lemma 7.72.4 the module $\mathcal{F}_{x'}$ is free over $\mathcal{O}_{X,x'}$, and

$$\mathcal{F}_x = \mathcal{F}_{x'} \otimes_{\mathcal{O}_{X,x'}} \mathcal{O}_{X,x}.$$

Hence $x' \in \text{Supp}(\mathcal{F}) \Rightarrow x \in \text{Supp}(\mathcal{F})$, in other words, the support is closed under generalization. As X satisfies (2) we see that the support of $\mathcal{F}$ is open and closed. The modules $\wedge^i(\mathcal{F})$, $i = 1, 2, 3, \ldots$ are finite flat quasi-coherent $\mathcal{O}_X$-modules also, see Modules, Section 15.18. Note that $\text{Supp}(\wedge^{i+1}(\mathcal{F})) \subset \text{Supp}(\wedge^i(\mathcal{F}))$. Thus we see that there exists a decomposition

$$X = U_0 \coprod U_1 \coprod U_2 \coprod \ldots$$

by open and closed subsets such that the support of $\wedge^i(\mathcal{F})$ is $U_i \cup U_{i+1} \cup \ldots$ for all i. Let x be a point of X, and say $x \in U_r$. Note that $\wedge^i(\mathcal{F})_x \otimes \kappa(x) = \wedge^i(\mathcal{F}_x \otimes \kappa(x))$. Hence, $x \in U_r$ implies that $\mathcal{F}_x \otimes \kappa(x)$ is a vector space of dimension r. By Nakayama's lemma, see Algebra, Lemma 7.14.5 we can choose an affine open neighbourhood $U \subset U_r \subset X$ of x and sections $s_1, \ldots, s_r \in \mathcal{F}(U)$ such that the induced map

$$\mathcal{O}_U^{\oplus r} \longrightarrow \mathcal{F}|_U, \quad (f_1, \ldots, f_r) \longmapsto \sum f_i s_i$$

is surjective. This means that $\wedge^r(\mathcal{F}|_U)$ is a finite flat quasi-coherent $\mathcal{O}_U$-module whose support is all of U. By the above it is generated by a single element, namely $s_1 \wedge \ldots \wedge s_r$. Hence $\wedge^r(\mathcal{F}|_U) \cong \mathcal{O}_U/\mathcal{I}$ for some quasi-coherent sheaf of ideals $\mathcal{I}$ such that $\mathcal{O}_U/\mathcal{I}$ is flat over $\mathcal{O}_U$ and such that $V(\mathcal{I}) = U$. It follows that $\mathcal{I} = 0$ by applying Lemma 24.25.1. Thus

$s_1 \wedge \ldots \wedge s_r$ is a basis for $\wedge^r(\mathcal{F}|_U)$ and it follows that the displayed map is injective as well as surjective. This proves that $\mathcal{F}$ is finite locally free as desired. □

24.26. Generic flatness

A scheme of finite type over an integral base is flat over a dense open of the base. In Algebra, Section 24.26 we proved a Noetherian version, a version for morphisms of finite presentation, and a general version. We only state and prove the general version here. However, it turns out that this will be superseded by Proposition 24.26.2 which shows the result holds if we only assume the base is reduced.

Proposition 24.26.1. *(Generic flatness) Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. Assume*

1. *S is integral,*
2. *f is of finite type, and*
3. *$\mathcal{F}$ is a finite type $\mathcal{O}_X$-module.*

Then there exists an open dense subscheme $U \subset S$ such that $X_U \to U$ is flat and of finite presentation and such that $\mathcal{F}|_{X_U}$ is flat over U and of finite presentation over $\mathcal{O}_{X_U}$.

Proof. As S is integral it is irreducible (see Properties, Lemma 23.3.4) and any nonempty open is dense. Hence we may replace S by an affine open of S and assume that $S = Spec(A)$ is affine. As S is integral we see that A is a domain. As f is of finite type, it is quasi-compact, so X is quasi-compact. Hence we can find a finite affine open cover $X = \bigcup_{i=1,\ldots,n} X_i$. Write $X_i = Spec(B_i)$. Then B_i is a finite type A-algebra, see Lemma 24.14.2. Moreover there are finite type B_i-modules M_i such that $\mathcal{F}|_{X_i}$ is the quasi-coherent sheaf associated to the B_i-module M_i, see Properties, Lemma 23.16.1. Next, for each pair of indices i, j choose an ideal $I_{ij} \subset B_i$ such that $X_i \setminus X_i \cap X_j = V(I_{ij})$ inside $X_i = Spec(B_i)$. Set $M_{ij} = B_i/I_{ij}$ and think of it as a B_i-module. Then $V(I_{ij}) = \text{Supp}(M_{ij})$ and M_{ij} is a finite B_i-module.

At this point we apply Algebra, Lemma 7.109.3 the pairs $(A \to B_i, M_{ij})$ and to the pairs $(A \to B_i, M_i)$. Thus we obtain nonzero $f_{ij}, f_i \in A$ such that (a) $A_{f_{ij}} \to B_{i,f_{ij}}$ is flat and of finite presentation and $M_{ij,f_{ij}}$ is flat over $A_{f_{ij}}$ and of finite presentation over $B_{i,f_{ij}}$, and (b) B_{i,f_i} is flat and of finite presentation over A_f and M_{i,f_i} is flat and of finite presentation over B_{i,f_i}. Set $f = (\prod f_i)(\prod f_{ij})$. We claim that taking $U = D(f)$ works.

To prove our claim we may replace A by A_f, i.e., perform the base change by $U = Spec(A_f) \to S$. After this base change we see that each of $A \to B_i$ is flat and of finite presentation and that M_i, M_{ij} are flat over A and of finite presentation over B_i. This already proves that $X \to S$ is quasi-compact, locally of finite presentation, flat, and that $\mathcal{F}$ is flat over S and of finite presentation over $\mathcal{O}_X$, see Lemma 24.20.2 and Properties, Lemma 23.16.2. Since M_{ij} is of finite presentation over B_i we see that $X_i \cap X_j = X_i \setminus \text{Supp}(M_{ij})$ is a quasi-compact open of X_i, see Algebra, Lemma 7.59.5. Hence we see that $X \to S$ is quasi-separated by Schemes, Lemma 21.21.7. This proves the proposition. □

It actually turns out that there is also a version of generic flatness over an arbitrary reduced base. Here it is.

Proposition 24.26.2. *(Generic flatness, reduced case) Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. Assume*

1. *S is reduced,*
2. *f is of finite type, and*
3. *$\mathcal{F}$ is a finite type $\mathcal{O}_X$-module.*

Then there exists an open dense subscheme $U \subset S$ such that $X_U \to U$ is flat and of finite presentation and such that $\mathcal{F}|_{X_U}$ is flat over U and of finite presentation over $\mathcal{O}_{X_U}$.

Proof. For the impatient reader: This proof is a repeat of the proof of Proposition 24.26.1 using Algebra, Lemma 7.109.7 instead of Algebra, Lemma 7.109.3.

Since being flat and being of finite presentation is local on the base, see Lemmas 24.24.2 and 24.20.2, we may work affine locally on S. Thus we may assume that $S = Spec(A)$, where A is a reduced ring (see Properties, Lemma 23.3.2). As f is of finite type, it is quasi-compact, so X is quasi-compact. Hence we can find a finite affine open cover $X = \bigcup_{i=1,\ldots,n} X_i$. Write $X_i = Spec(B_i)$. Then B_i is a finite type A-algebra, see Lemma 24.14.2. Moreover there are finite type B_i-modules M_i such that $\mathcal{F}|_{X_i}$ is the quasi-coherent sheaf associated to the B_i-module M_i, see Properties, Lemma 23.16.1. Next, for each pair of indices i, j choose an ideal $I_{ij} \subset B_i$ such that $X_i \setminus X_i \cap X_j = V(I_{ij})$ inside $X_i = Spec(B_i)$. Set $M_{ij} = B_i/I_{ij}$ and think of it as a B_i-module. Then $V(I_{ij}) = \text{Supp}(M_{ij})$ and M_{ij} is a finite B_i-module.

At this point we apply Algebra, Lemma 7.109.7 the pairs $(A \to B_i, M_{ij})$ and to the pairs $(A \to B_i, M_i)$. Thus we obtain dense opens $U(A \to B_i, M_{ij}) \subset S$ and dense opens $U(A \to B_i, M_i) \subset S$ with notation as in Algebra, Equation (7.109.3.2). Since a finite intersection of dense opens is dense open, we see that

$$U = \bigcap\nolimits_{i,j} U(A \to B_i, M_{ij}) \quad \cap \quad \bigcap\nolimits_i U(A \to B_i, M_i)$$

is open and dense in S. We claim that U is the desired open.

Pick $u \in U$. By definition of the loci $U(A \to B_i, M_{ij})$ and $U(A \to B, M_i)$ there exist $f_{ij}, f_i \in A$ such that (a) $u \in D(f_i)$ and $u \in D(f_{ij})$, (b) $A_{f_{ij}} \to B_{i,f_{ij}}$ is flat and of finite presentation and $M_{ij,f_{ij}}$ is flat over $A_{f_{ij}}$ and of finite presentation over $B_{i,f_{ij}}$, and (c) B_{i,f_i} is flat and of finite presentation over A_f and M_{i,f_i} is flat and of finite presentation over B_{i,f_i}. Set $f = (\prod f_i)(\prod f_{ij})$. Now it suffices to prove that $X \to S$ is flat and of finite presentation over $D(f)$ and that $\mathcal{F}$ restricted to $X_{D(f)}$ is flat over $D(f)$ and of finite presentation over the structure sheaf of $X_{D(f)}$.

Hence we may replace A by A_f, i.e., perform the base change by $Spec(A_f) \to S$. After this base change we see that each of $A \to B_i$ is flat and of finite presentation and that M_i, M_{ij} are flat over A and of finite presentation over B_i. This already proves that $X \to S$ is quasi-compact, locally of finite presentation, flat, and that $\mathcal{F}$ is flat over S and of finite presentation over $\mathcal{O}_X$, see Lemma 24.20.2 and Properties, Lemma 23.16.2. Since M_{ij} is of finite presentation over B_i we see that $X_i \cap X_j = X_i \setminus \text{Supp}(M_{ij})$ is a quasi-compact open of X_i, see Algebra, Lemma 7.59.5. Hence we see that $X \to S$ is quasi-separated by Schemes, Lemma 21.21.7. This proves the proposition. □

Remark 24.26.3. The results above are a first step towards more refined flattening techniques for morphisms of schemes. The article [**GR71**] by Raynaud and Gruson contains many wonderful results in this direction.

24.27. Morphisms and dimensions of fibres

Let X be a topological space, and $x \in X$. Recall that we have defined $\dim_x(X)$ as the minimum of the dimensions of the open neighbourhoods of x in X. See Topology, Definition 5.7.1.

Lemma 24.27.1. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ and set $s = f(x)$. Assume f is locally of finite type. Then*

$$\dim_x(X_s) = \dim(\mathcal{O}_{X_s,x}) + trdeg_{\kappa(s)}(\kappa(x)).$$

Proof. This immediately reduces to the case $S = s$, and X affine. In this case the result follows from Algebra, Lemma 7.107.3. □

Lemma 24.27.2. *Let $f : X \to Y$ and $g : Y \to S$ be morphisms of schemes. Let $x \in X$ and set $y = f(x)$, $s = g(y)$. Assume f and g locally of finite type. Then*

$$\dim_x(X_s) \leq \dim_x(X_y) + \dim_y(Y_s).$$

Moreover, equality holds if $\mathcal{O}_{X_s,x}$ is flat over $\mathcal{O}_{Y_s,y}$, which holds for example if $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,y}$.

Proof. Note that $\text{trdeg}_{\kappa(s)}(\kappa(x)) = \text{trdeg}_{\kappa(y)}(\kappa(x)) + \text{trdeg}_{\kappa(s)}(\kappa(y))$. Thus by Lemma 24.27.1 the statement is equivalent to

$$\dim(\mathcal{O}_{X_s,x}) \leq \dim(\mathcal{O}_{X_y,x}) + \dim(\mathcal{O}_{Y_s,y}).$$

For this see Algebra, Lemma 7.103.6. For the flat case see Algebra, Lemma 7.103.7. □

Lemma 24.27.3. *Let*

$$\begin{tikzcd} X' \arrow[r, "g'"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

be a fibre product diagram of schemes. Assume f locally of finite type. Suppose that $x' \in X'$, $x = g'(x')$, $s' = f'(x')$ and $s = g(s') = f(x)$. Then $\dim_x(X_s) = \dim_{x'}(X'_{s'})$.

Proof. Follows immediately from Algebra, Lemma 7.107.6. □

The following lemma follows from a nontrivial algebraic result. Namely, the algebraic version of Zariski's main theorem.

Lemma 24.27.4. *Let $f : X \to S$ be a morphism of schemes. Let $n \geq 0$. Assume f is locally of finite type. The set*

$$U_n = \{x \in X \mid \dim_x X_{f(x)} \leq n\}$$

is open in X.

Proof. This is immediate from Algebra, Lemma 7.116.6 □

Lemma 24.27.5. *Let $f : X \to S$ be a morphism of schemes. Let $n \geq 0$. Assume f is locally of finite presentation. The open*

$$U_n = \{x \in X \mid \dim_x X_{f(x)} \leq n\}$$

of Lemma 24.27.4 is retrocompact in X. (See Topology, Definition 5.9.1.)

Proof. The topological space X has a basis for its topology consisting of affine opens $U \subset X$ such that the infuced morphism $f|_U : U \to S$ factors through an affine open $V \subset S$. Hence it is enough to show that $U \cap U_n$ is quasi-compact for such a U. Note that $U_n \cap U$ is the same as the open $\{x \in U \mid \dim_x U_{f(x)} \leq n\}$. This reduces us to the case where X and S are affine. In this case the lemma follows from Algebra, Lemma 7.116.8 (and Lemma 24.20.2). □

Lemma 24.27.6. *Let $f : X \to S$ be a morphism of schemes. Let $x \rightsquigarrow x'$ be a nontrivial specialization of points in X lying over the same point $s \in S$. Assume f is locally of finite type. Then*

(1) $\dim_x(X_s) \leq \dim_{x'}(X_s)$,
(2) $trdeg_{\kappa(s)}(\kappa(x)) > trdeg_{\kappa(s)}(\kappa(x'))$, *and*
(3) $\dim(\mathcal{O}_{X_s,x}) < \dim(\mathcal{O}_{X_s,x'})$.

Proof. The first inequality follows from Lemma 24.27.4. The third inequality follows since $\mathcal{O}_{X_s,x}$ is a localization of $\mathcal{O}_{X_s,x}$ in a prime ideal, hence any chain of prime ideals in $\mathcal{O}_{X_s,x}$ is part of a strictly longer chain of primes in $\mathcal{O}_{X_s,x'}$. The second inequality follows from Algebra, Lemma 7.107.2. □

24.28. Morphisms of given relative dimension

In order to be able to speak comfortably about morphisms of a given relative dimension we introduce the following notion.

Definition 24.28.1. Let $f : X \to S$ be a morphism of schemes. Assume f is locally of finite type.

(1) We say f is of *relative dimension $\leq d$ at x* if $\dim_x(X_{f(x)}) \leq d$.
(2) We say f is of *relative dimension $\leq d$* if $\dim_x(X_{f(x)}) \leq d$ for all $x \in X$.
(3) We say f is of *relative dimension d* if all nonempty fibres X_s are equidimensional of dimension d.

This is not a particularly well behaved notion, but it works well in a number of situations.

Lemma 24.28.2. *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. If f has relative dimension d, then so does any base change of f. Same for relative dimension $\leq d$.*

Proof. This is immediate from Lemma 24.27.3. □

Lemma 24.28.3. *Let $f : X \to Y$, $g : Y \to Z$ be locally of finite type. If f has relative dimension $\leq d$ and g has relative dimension $\leq e$ then $g \circ f$ has relative dimension $\leq d + e$. If*

(1) *f has relative dimension d,*
(2) *g has relative dimension e, and*
(3) *f is flat,*

then $g \circ f$ has relative dimension $d + e$.

Proof. This is immediate from Lemma 24.27.2. □

In general it is not possible to decompose a morphism into its pieces where the relative dimension is a given one. However, it is possible if the morphism has Cohen-Macaulay fibres and is flat of finite presentation.

Lemma 24.28.4. *Let $f : X \to S$ be a morphism of schemes. Assume that*

(1) *f is flat,*
(2) *f is locally of finite presentation, and*
(3) *for all $s \in S$ the fibre X_s is Cohen-Macaulay (Properties, Definition 23.8.1)*

Then there exist open and closed subschemes $X_d \subset X$ such that $X = \coprod_{d \geq 0} X_d$ and $f|_{X_d} : X_d \to S$ has relative dimension d.

Proof. This is immediate from Algebra, Lemma 7.121.8. □

Lemma 24.28.5. *Let $f : X \to S$ be a morphism of schemes. Assume f is locally of finite type. Let $x \in X$ with $s = f(x)$. Then f is quasi-finite at x if and only if $\dim_x(X_s) = 0$. In particular, f is locally quasi-finite if and only if f has relative dimension 0.*

Proof. If f is quasi-finite at x then $\kappa(x)$ is a finite extension of $\kappa(s)$ (by Lemma 24.19.5) and x is isolated in X_s (by Lemma 24.19.6), hence $\dim_x(X_s) = 0$ by Lemma 24.27.1. Conversely, if $\dim_x(X_s) = 0$ then by Lemma 24.27.1 we see $\kappa(s) \subset \kappa(x)$ is algebraic and there are no other points of X_s specializing to x. Hence x is closed in its fibre by Lemma 24.19.2 and by Lemma 24.19.6 (3) we conclude that f is quasi-finite at x. □

24.29. The dimension formula

For morphisms between Noetherian schemes we can say a little more about dimensions of local rings. Here is an important (and not so hard to prove) result. Recall that $R(X)$ denotes the function field of an integral scheme X.

Lemma 24.29.1. *Let S be a scheme. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$, and set $s = f(x)$. Assume*

1. *S is locally Noetherian,*
2. *f is locally of finite type,*
3. *X and S integral, and*
4. *f dominant.*

We have

$$\dim(\mathcal{O}_{X,x}) \leq \dim(\mathcal{O}_{S,s}) + trdeg_{R(S)} R(X) - trdeg_{\kappa(s)} \kappa(x). \tag{24.29.1.1}$$

Moreover, equality holds if S is universally catenary.

Proof. The corresponding algebra statement is Algebra, Lemma 7.104.1. □

An application is the construction of a dimension function on any scheme of finite type over a universally catenary scheme endowed with a dimension function. For the definition of dimension functions, see Topology, Definition 5.16.1.

Lemma 24.29.2. *Let S be a universally catenary scheme. Let $\delta : S \to \mathbf{Z}$ be a dimension function. Let $f : X \to S$ be a morphism of schemes. Assume f locally of finite type. Then the map*

$$\begin{aligned} \delta = \delta_{X/S} : X &\longrightarrow \mathbf{Z} \\ x &\longmapsto \delta(f(x)) + trdeg_{\kappa(f(x))} \kappa(x) \end{aligned}$$

is a dimension function on X.

Proof. Let $f : X \to S$ be locally of finite type. Let $x \rightsquigarrow y$, $x \neq y$ be a specialization in X. We have to show that $\delta_{X/S}(x) > \delta_{X/S}(y)$ and that $\delta_{X/S}(x) = \delta_{X/S}(y) + 1$ if y is an immediate specialization of x.

Choose an affine open $V \subset S$ containing the image of y and choose an affine open $U \subset X$ mapping into V and containing y. We may clearly replace X by U and S by V. Thus we may assume that $X = Spec(A)$ and $S = Spec(R)$ and that f is given by a ring map $R \to A$. The ring R is universally catenary (Lemma 24.16.2) and the map $R \to A$ is of finite type (Lemma 24.14.2).

Let $\mathfrak{q} \subset A$ be the prime ideal corresponding to the point x and let $\mathfrak{p} \subset R$ be the prime ideal corresponding to $f(x)$. The restriction δ' of δ to $S' = Spec(R/\mathfrak{p}) \subset S$ is a dimension function. The ring $R/\mathfrak{p}$ is universally catenary. The restriction of $\delta_{X/S}$ to $X' = Spec(A/\mathfrak{q})$ is clearly equal to the function $\delta_{X'/S'}$ constructed using the dimension function δ'. Hence we may assume in addition to the above that $R \subset A$ are domains, in other words that X and S are integral schemes.

Note that $\mathcal{O}_{X,x}$ is a localization of $\mathcal{O}_{X,y}$ at a non-maximal prime (Schemes, Lemma 21.13.2). Hence $\dim(\mathcal{O}_{X,x}) < \dim(\mathcal{O}_{X,y})$ and $\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{X,y}) - 1$ if y is an immediate specialization of x.

Write $s = f(x) \neq f(y) = s'$. We see, using equality in (24.29.1.1), that

$$\begin{aligned}\delta_{X/S}(x) - \delta_{X/S}(y) = \delta(s) - \delta(s') & \\ + \dim(\mathcal{O}_{S,s}) - \dim(\mathcal{O}_{S,s'}) & \\ - \dim(\mathcal{O}_{X,x}) + \dim(\mathcal{O}_{X,y}). &\end{aligned}$$

Since δ is a dimension function on the scheme S the difference $\delta(s) - \delta(s')$ is equal to $\text{codim}(\overline{\{s'\}}, \overline{\{s\}})$ by Topology, Lemma 5.16.2. As S is integral, catenary this is equal to $\text{codim}(\overline{\{s'\}}, S) - \text{codim}(\overline{\{s\}}, S)$ (Topology, Lemma 5.8.6). And this in turn is equal to $\dim(\mathcal{O}_{S,s'}) - \dim(\mathcal{O}_{S,s})$ by Properties, Lemma 23.11.4. Hence we conclude that

$$\delta_{X/S}(x) - \delta_{X/S}(y) = -\dim(\mathcal{O}_{X,x}) + \dim(\mathcal{O}_{X,y})$$

and hence the lemma follows from our remarks about the dimensions of these local rings above. □

Another application of the dimension formula is that the dimension does not change under ``alterations'' (to be defined later).

Lemma 24.29.3. *Let $f : X \to Y$ be a morphism of schemes. Assume that*

(1) *Y is locally Noetherian,*
(2) *X and Y are integral schemes,*
(3) *f is dominant, and*
(4) *f is locally of finite type.*

Then we have

$$\dim(X) \leq \dim(Y) + trdeg_{R(Y)} R(X).$$

If f is closed[6] *then equality holds.*

Proof. Let $f : X \to Y$ be as in the lemma. Let $\xi_0 \rightsquigarrow \xi_1 \rightsquigarrow \ldots \rightsquigarrow \xi_e$ be a sequence of specializations in X. We may assume that $x = \xi_e$ is a closed point of X, see Properties, Lemma 23.5.8. In particular, setting $y = f(x)$, we see x is a closed point of its fibre X_y. By the Hilbert Nullstellensatz we see that $\kappa(x)$ is a finite extension of $\kappa(y)$, see Lemma 24.19.3. By the dimension formula, Lemma 24.29.1, we see that

$$\dim(\mathcal{O}_{X,x}) \leq \dim(\mathcal{O}_{Y,y}) + \text{trdeg}_{R(Y)} R(X)$$

Hence we conclude that $e \leq \dim(Y) + \text{trdeg}_{R(Y)} R(X)$ as desired.

Next, assume f is also closed. Say $\overline{\xi}_0 \rightsquigarrow \overline{\xi}_1 \rightsquigarrow \ldots \rightsquigarrow \overline{\xi}_d$ is a sequence of specializations in Y. We want to show that $\dim(X) \geq d + r$. We may assume that $\overline{\xi}_0 = \eta$ is the generic point of Y. The generic fibre X_η is a scheme locally of finite type over $\kappa(\eta) = R(Y)$. It is

[6] For example if f is proper, see Definition 24.40.1.

nonempty as f is dominant. Hence by Lemma 24.15.10 it is a Jacobson scheme. Thus by Lemma 24.15.8 we can find a closed point $\xi_0 \in X_\eta$ and the extension $\kappa(\eta) \subset \kappa(\xi_0)$ is a finite extension. Note that $\mathcal{O}_{X,\xi_0} = \mathcal{O}_{X_\eta,\xi_0}$ because η is the generic point of Y. Hence we see that $\dim(\mathcal{O}_{X,\xi_0}) = r$ by Lemma 24.29.1 applied to the scheme X_η over the universally catenary scheme $Spec(\kappa(\eta))$ (see Lemma 24.16.4) and the point ξ_0. This means that we can find $\xi_{-r} \rightsquigarrow \ldots \rightsquigarrow \xi_{-1} \rightsquigarrow \xi_0$ in X. On the other hand, as f is closed specializations lift along f, see Topology, Lemma 5.14.6. Thus, as ξ_0 lies over $\eta = \overline{\xi}_0$ we can find specializations $\xi_0 \rightsquigarrow \xi_1 \rightsquigarrow \ldots \rightsquigarrow \xi_d$ lying over $\overline{\xi}_0 \rightsquigarrow \overline{\xi}_1 \rightsquigarrow \ldots \rightsquigarrow \overline{\xi}_d$. In other words we have

$$\xi_{-r} \rightsquigarrow \ldots \rightsquigarrow \xi_{-1} \rightsquigarrow \xi_0 \rightsquigarrow \xi_1 \rightsquigarrow \ldots \rightsquigarrow \xi_d$$

which means that $\dim(X) \geq d + r$ as desired. □

24.30. Syntomic morphisms

An algebra A over a field k is called a *global complete intersection over* k if $A \cong k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ and $\dim(A) = n - c$. An algebra A over a field k is called a *local complete intersection* if $Spec(A)$ can be covered by standard opens each of which are global complete intersections over k. See Algebra, Section 7.124. Recall that a ring map $R \to A$ is *syntomic* if it is of finite presentation, flat with local complete intersection rings as fibres, see Algebra, Definition 7.125.1.

Definition 24.30.1. Let $f : X \to S$ be a morphism of schemes.

(1) We say that f is *syntomic at* $x \in X$ if there exists a affine open neighbourhood $Spec(A) = U \subset X$ of x and and affine open $Spec(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is syntomic.
(2) We say that f is *syntomic* if it is syntomic at every point of X.
(3) If $S = Spec(k)$ and f is syntomic, then we say that X is a *local complete intersection over* k.
(4) A morphism of affine schemes $f : X \to S$ is called *standard syntomic* if there exists a global relative complete intersection $R \to R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ (see Algebra, Definition 7.125.5) such that $X \to S$ is isomorphic to

$$Spec(R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)) \to Spec(R).$$

In the literature a syntomic morphism is sometimes referred to as a *flat local complete intersection morphism*. It turns out this is a convenient class of morphisms. For example one can define a syntomic topology using these, which is finer than the smooth and étale topologies, but has many of the same formal properties.

A global relative complete intersection (which we used to define standard syntomic ring maps) is in particular flat. In More on Morphisms, Section 33.38 we will consider morphisms $X \to S$ which locally are of the form

$$Spec(R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)) \to Spec(R).$$

for some Koszul-regular sequence $f_1, \ldots, f_r$ in $R[x_1, \ldots, x_n]$. Such a morphism will be called a *local complete intersection morphism*. One we have this definition in place it will be the case that a morphism is syntomic if and only if it is a flat, local complete intersection morphism.

Note that there is no separation or quasi-compactness hypotheses in the definition of a syntomic morphism. Hence the question of being syntomic is local in nature on the source. Here is the precise result.

Lemma 24.30.2. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *The morphism f is syntomic.*
(2) *For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is syntomic.*
(3) *There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$, $j \in J, i \in I_j$ is syntomic.*
(4) *There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring map $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i)$ is syntomic, for all $j \in J, i \in I_j$.*

Moreover, if f is syntomic then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \to V$ is syntomic.

Proof. This follows from Lemma 24.13.3 if we show that the property "$R \to A$ is syntomic" is local. We check conditions (a), (b) and (c) of Definition 24.13.1. By Algebra, Lemma 7.125.3 being syntomic is stable under base change and hence we conclude (a) holds. By Algebra, Lemma 7.125.18 being syntomic is stable under composition and trivially for any ring R the ring map $R \to R_f$ is syntomic. We conclude (b) holds. Finally, property (c) is true according to Algebra, Lemma 7.125.4. □

Lemma 24.30.3. *The composition of two morphisms which are syntomic is syntomic.*

Proof. In the proof of Lemma 24.30.2 we saw that being syntomic is a local property of ring maps. Hence the first statement of the lemma follows from Lemma 24.13.5 combined with the fact that being syntomic is a property of ring maps that is stable under composition, see Algebra, Lemma 7.125.18. □

Lemma 24.30.4. *The base change of a morphism which is syntomic is syntomic.*

Proof. In the proof of Lemma 24.30.2 we saw that being syntomic is a local property of ring maps. Hence the lemma follows from Lemma 24.13.5 combined with the fact that being syntomic is a property of ring maps that is stable under base change, see Algebra, Lemma 7.125.3. □

Lemma 24.30.5. *Any open immersion is syntomic.*

Proof. This is true because an open immersion is a local isomorphism. □

Lemma 24.30.6. *A syntomic morphism is locally of finite presentation.*

Proof. True because a syntomic ring map is of finite presentation by definition. □

Lemma 24.30.7. *A syntomic morphism is flat.*

Proof. True because a syntomic ring map is flat by definition. □

Lemma 24.30.8. *A syntomic morphism is universally open.*

Proof. Combine Lemmas 24.30.6, 24.30.7, and 24.24.9. □

Let k be a field. Let A be a local k-algebra essentially of finite type over k. Recall that A is called a *complete intersection over k* if we can write $A \cong R/(f_1, \ldots, f_c)$ where R is a regular local ring essentially of finite type over k, and $f_1, \ldots, f_c$ is a regular sequence in R, see Algebra, Definition 7.124.5.

Lemma 24.30.9. *Let k be a field. Let X be a scheme locally of finite type over k. The following are equivalent:*

1. *X is a local complete intersection over k,*
2. *for every $x \in X$ there exists an affine open $U = Spec(R) \subset X$ neighbourhood of x such that $R \cong k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a global complete intersection over k, and*
3. *for every $x \in X$ the local ring $\mathcal{O}_{X,x}$ is a complete intersection over k.*

Proof. The corresponding algebra results can be found in Algebra, Lemmas 7.124.8 and 7.124.9. □

The following lemma says locally any syntomic morphism is standard syntomic. Hence we can use standard syntomic morphisms as a *local model* for a syntomic morphism. Moreover, it says that a flat morphism of finite presentation is syntomic if and only if the fibres are local complete intersection schemes.

Lemma 24.30.10. *Let $f : X \to S$ be a morphism of schemes. Assume f locally of finite presentation. Let $x \in X$ be a point. Set $s = f(x)$. The following are equivalent*

1. *The morphism f is syntomic at x.*
2. *There exist affine opens $U \subset X$, and $V \subset S$ such that $x \in U$, $f(U) \subset V$ and the induced morphism $f|_U : U \to V$ is standard syntomic.*
3. *The local ring map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat and $\mathcal{O}_{X,x}/\mathfrak{m}_s\mathcal{O}_{X,x}$ is a complete intersection over $\kappa(s)$ (see Algebra, Definition 7.124.5).*

Proof. Follows from the definitions and Algebra, Lemma 7.125.16. □

Lemma 24.30.11. *Let $f : X \to S$ be a morphism of schemes. If f is flat, locally of finite presentation, and all fibres X_s are local complete intersections, then f is syntomic.*

Proof. Clear from Lemmas 24.30.9 and 24.30.10 and the isomorphisms of local rings $\mathcal{O}_{X,x}/\mathfrak{m}_s\mathcal{O}_{X,x} \cong \mathcal{O}_{X_s,x}$. □

Lemma 24.30.12. *Let $f : X \to S$ be a morphism of schemes. Assume f locally of finite type. Formation of the set*

$$T = \{x \in X \mid \mathcal{O}_{X_{f(x)},x} \text{ is a complete intersection over } \kappa(f(x))\}$$

commutes with arbitrary base change: For any morphism $g : S' \to S$, consider the base change $f' : X' \to S'$ of f and the projection $g' : X' \to X$. Then the corresponding set T' for the morphism f' is equal to $T' = (g')^{-1}(T)$. In particular, if f is assumed flat, and locally of finite presentation then the same holds for the open set of points where f is syntomic.

Proof. Let $s' \in S'$ be a point, and let $s = g(s')$. Then we have

$$X'_{s'} = Spec(\kappa(s')) \times_{Spec(\kappa(s))} X_s$$

In other words the fibres of the base change are the base changes of the fibres. Hence the first part is equivalent to Algebra, Lemma 7.124.10. The second part follows from the first because in that case T is the set of points where f is syntomic according to Lemma 24.30.10. □

Lemma 24.30.13. *Let R be a ring. Let $R \to A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ be a relative global complete intersection. Set $S = Spec(R)$ and $X = Spec(A)$. Consider the morphism $f : X \to S$ associated to the ring map $R \to A$. The function $x \mapsto \dim_x(X_{f(x)})$ is constant with value $n - c$.*

Proof. By Algebra, Definition 7.125.5 $R \to A$ being a relative global complete intersection means all nonzero fibre rings have dimension $n - c$. Thus for a prime $\mathfrak{p}$ of R the fibre ring $\kappa(\mathfrak{p})[x_1, \ldots, x_n]/(\overline{f}_1, \ldots, \overline{f}_c)$ is either zero or a global complete intersection ring of dimension $n - c$. By the discussion following Algebra, Definition 7.124.1 this implies it is equidimensional of dimension $n - c$. Whence the lemma. $\square$

Lemma 24.30.14. *Let $f : X \to S$ be a syntomic morphism. The function $x \mapsto \dim_x(X_{f(x)})$ is locally constant on X.*

Proof. By Lemma 24.30.10 the morphism f locally looks like a standard syntomic morphism of affines. Hence the result follows from Lemma 24.30.13. $\square$

Lemma 24.30.14 says that the following definition makes sense.

Definition 24.30.15. Let $d \geq 0$ be an integer. We say a morphism of schemes $f : X \to S$ is *syntomic of relative dimension d* if f is syntomic and the function $\dim_x(X_{f(x)}) = d$ for all $x \in X$.

In other words, f is syntomic and the nonempty fibres are equidimensional of dimension d.

Lemma 24.30.16. *Let*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & S & \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume that

(1) *f is surjective and syntomic,*
(2) *p is syntomic, and*
(3) *q is locally of finite presentation*[7].

Then q is syntomic.

Proof. By Lemma 24.24.11 we see that q is flat. Hence it suffices to show that the fibres of $Y \to S$ are local complete intersections, see Lemma 24.30.11. Let $s \in S$. Consider the morphism $X_s \to Y_s$. This is a base change of the morphism $X \to Y$ and hence surjective, and syntomic (Lemma 24.30.4). For the same reason X_s is syntomic over $\kappa(s)$. Moreover, Y_s is locally of finite type over $\kappa(s)$ (Lemma 24.14.4). In this way we reduce to the case where S is the spectrum of a field.

Assume $S = Spec(k)$. Let $y \in Y$. Choose an affine open $Spec(A) \subset Y$ neighbourhood of y. Let $Spec(B) \subset X$ be an affine open such that $f(Spec(B)) \subset Spec(A)$, containing a point $x \in X$ such that $f(x) = y$. Choose a surjection $k[x_1, \ldots, x_n] \to A$ with kernel I. Choose a surjection $A[y_1, \ldots, y_m] \to B$, which gives rise in turn to a surjection $k[x_i, y_j] \to B$ with kernel J. Let $\mathfrak{q} \subset k[x_i, y_j]$ be the prime corresponding to $y \in Spec(B)$ and let $\mathfrak{p} \subset k[x_i]$ the prime corresponding to $x \in Spec(A)$. Since x maps to y we have $\mathfrak{p} = \mathfrak{q} \cap k[x_i]$. Consider the following commutative diagram of local rings:

$$\begin{tikzcd} \mathcal{O}_{X,x} \arrow[r, equal] & B_{\mathfrak{q}} & k[x_1, \ldots, x_n, y_1, \ldots, y_m]_{\mathfrak{q}} \arrow[l] \\ \mathcal{O}_{Y,y} \arrow[r, equal] \arrow[u] & A_{\mathfrak{p}} \arrow[u] & k[x_1, \ldots, x_n]_{\mathfrak{p}} \arrow[l] \arrow[u] \end{tikzcd}$$

[7] In fact this is implied by (1) and (2), see Descent, Lemma 31.10.3. See also Descent, Remark 31.10.7 for further discussion.

We claim that the hypotheses of Algebra, Lemma 7.124.12 are satisfied. Conditions (1) and (2) are trivial. Condition (4) follows as $X \to Y$ is flat. Condition (3) follows as the rings $\mathcal{O}_{Y,y}$ and $\mathcal{O}_{X_y,x} = \mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$ are complete intersection rings by our assumptions that f and p are syntomic, see Lemma 24.30.10. The output of Algebra, Lemma 7.124.12 is exactly that $\mathcal{O}_{Y,y}$ is a complete intersection ring! Hence by Lemma 24.30.10 again we see that Y is syntomic over k at y as desired. □

24.31. Conormal sheaf of an immersion

Let $i : Z \to X$ be a closed immersion. Let $\mathcal{I} \subset \mathcal{O}_X$ be the corresponding quasi-coherent sheaf of ideals. Consider the short exact sequence

$$0 \to \mathcal{I}^2 \to \mathcal{I} \to \mathcal{I}/\mathcal{I}^2 \to 0$$

of quasi-coherent sheaves on X. Since the sheaf $\mathcal{I}/\mathcal{I}^2$ is annihilated by $\mathcal{I}$ it corresponds to a sheaf on Z by Lemma 24.3.1. This quasi-coherent $\mathcal{O}_Z$-module is called the *conormal sheaf of Z in X* and is often simply denoted $\mathcal{I}/\mathcal{I}^2$ by the abuse of notation mentioned in Section 24.3.

In case $i : Z \to X$ is a (locally closed) immersion we define the conormal sheaf of i as the conormal sheaf of the closed immersion $i : Z \to X \setminus \partial Z$, where $\partial Z = \overline{Z} \setminus Z$. It is often denoted $\mathcal{I}/\mathcal{I}^2$ where $\mathcal{I}$ is the ideal sheaf of the closed immersion $i : Z \to X \setminus \partial Z$.

Definition 24.31.1. Let $i : Z \to X$ be an immersion. The *conormal sheaf $\mathcal{C}_{Z/X}$ of Z in X* or the *conormal sheaf of i* is the quasi-coherent $\mathcal{O}_Z$-module $\mathcal{I}/\mathcal{I}^2$ described above.

In [**DG67**, IV Definition 16.1.2] this sheaf is denoted $\mathcal{N}_{Z/X}$. We will not follow this convention since we would like to reserve the notation $\mathcal{N}_{Z/X}$ for the *normal sheaf of the immersion*. It is defined as

$$\mathcal{N}_{Z/X} = \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{C}_{Z/X}, \mathcal{O}_Z) = \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Z)$$

provided the conormal sheaf is of finite presentation (otherwise the normal sheaf may not even be quasi-coherent). We will come back to the normal sheaf later (insert future reference here).

Lemma 24.31.2. *Let $i : Z \to X$ be an immersion. The conormal sheaf of i has the following properties:*

(1) *Let $U \subset X$ be any open such that $i(Z)$ is a closed subset of U. Let $\mathcal{I} \subset \mathcal{O}_U$ be the sheaf of ideals corresponding to the closed subscheme $i(Z) \subset U$. Then*
$$\mathcal{C}_{Z/X} = i^*\mathcal{I} = i^{-1}(\mathcal{I}/\mathcal{I}^2)$$
(2) *For any affine open $Spec(R) = U \subset X$ such that $Z \cap U = Spec(R/I)$ there is a canonical isomorphism $\Gamma(Z \cap U, \mathcal{C}_{Z/X}) = I/I^2$.*

Proof. Mostly clear from the definitions. Note that given a ring R and an ideal I of R we have $I/I^2 = I \otimes_R R/I$. Details omitted. □

Lemma 24.31.3. *Let*

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ Z' \arrow[r, "i'"] & X' \end{tikzcd}$$

be a commutative diagram in the category of schemes. Assume i, i' immersions. There is a canonical map of $\mathcal{O}_Z$-modules

$$f^*\mathcal{C}_{Z'/X'} \longrightarrow \mathcal{C}_{Z/X}$$

characterized by the following property: For every pair of affine opens $(Spec(R) = U \subset X, Spec(R') = U' \subset X')$ with $f(U) \subset U'$ such that $Z \cap U = Spec(R/I)$ and $Z' \cap U' = Spec(R'/I')$ the induced map

$$\Gamma(Z' \cap U', \mathcal{C}_{Z'/X'}) = I'/I'^2 \longrightarrow I/I^2 = \Gamma(Z \cap U, \mathcal{C}_{Z/X})$$

is the one induced by the ring map $f^\sharp : R' \to R$ which has the property $f^\sharp(I') \subset I$.

Proof. Let $\partial Z' = \overline{Z'} \setminus Z'$ and $\partial Z = \overline{Z} \setminus Z$. These are closed subsets of X' and of X. Replacing X' by $X' \setminus \partial Z'$ and X by $X \setminus \big(g^{-1}(\partial Z') \cup \partial Z\big)$ we see that we may assume that i and i' are closed immersions.

The fact that $g \circ i$ factors through i' implies that $g^*\mathcal{I}'$ maps into $\mathcal{I}$ under the canonical map $g^*\mathcal{I}' \to \mathcal{O}_X$, see Schemes, Lemmas 21.4.6 and 21.4.7. Hence we get an induced map of quasi-coherent sheaves $g^*(\mathcal{I}'/(\mathcal{I}')^2) \to \mathcal{I}/\mathcal{I}^2$. Pulling back by i gives $i^*g^*(\mathcal{I}'/(\mathcal{I}')^2) \to i^*(\mathcal{I}/\mathcal{I}^2)$. Note that $i^*(\mathcal{I}/\mathcal{I}^2) = \mathcal{C}_{Z/X}$. On the other hand, $i^*g^*(\mathcal{I}'/(\mathcal{I}')^2) = f^*(i')^*(\mathcal{I}'/(\mathcal{I}')^2) = f^*\mathcal{C}_{Z'/X'}$. This gives the desired map.

Checking that the map is locally described as the given map $I'/(I')^2 \to I/I^2$ is a matter of unwinding the definitions and is omitted. Another observation is that given any $x \in i(Z)$ there do exist affine open neighbourhoods U, U' with $f(U) \subset U'$ and $Z \cap U$ as well as $U' \cap Z'$ closed such that $x \in U$. Proof omitted. Hence the requirement of the lemma indeed characterizes the map (and could have been used to define it). $\square$

Lemma 24.31.4. *Let*

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ Z' \arrow[r, "i'"] & X' \end{tikzcd}$$

be a fibre product diagram in the category of schemes with i, i' immersions. Then the canonical map $f^*\mathcal{C}_{Z'/X'} \to \mathcal{C}_{Z/X}$ of Lemma 24.31.3 is surjective. If g is flat, then it is an isomorphism.

Proof. Let $R' \to R$ be a ring map, and $I' \subset R'$ an ideal. Set $I = I'R$. Then $I'/(I')^2 \otimes_{R'} R \to I/I^2$ is surjective. If $R' \to R$ is flat, then $I = I' \otimes_{R'} R$ and $I^2 = (I')^2 \otimes_{R'} R$ and we see the map is an isomorphism. $\square$

Lemma 24.31.5. *Let $Z \to Y \to X$ be immersions of schemes. Then there is a canonical exact sequence*

$$i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

where the maps come from Lemma 24.31.3 and $i : Z \to Y$ is the first morphism.

Proof. Via Lemma 24.31.3 this translates into the following algebra fact. Suppose that $C \to B \to A$ are surjective ring maps. Let $I = \mathrm{Ker}(B \to A)$, $J = \mathrm{Ker}(C \to A)$ and $K = \mathrm{Ker}(C \to B)$. Then there is an exact sequence

$$K/K^2 \otimes_B A \to J/J^2 \to I/I^2 \to 0.$$

This follows immediately from the observation that $I = J/K$. $\square$

24.32. Sheaf of differentials of a morphism

We suggest the reader take a look at the corresponding section in the chapter on commutative algebra (Algebra, Section 7.122).

Definition 24.32.1. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. A *derivation* or more precisely an *S-derivation* into $\mathcal{F}$ is a map $D : \mathcal{O}_X \to \mathcal{F}$ which is additive, annihilates the image of $f^{-1}\mathcal{O}_S \to \mathcal{O}_X$, and satisfies the *Leibniz rule*

$$D(ab) = aD(b) + D(a)b$$

for all a, b local sections of $\mathcal{O}_X$ (wherever they are both defined). We denote

$$\mathrm{Der}_S(\mathcal{O}_X, \mathcal{F})$$

the set of S-derivations into $\mathcal{F}$.

This is the sheaf theoretic analogue of Algebra, Definition 24.32.1. Given a derivation $D : \mathcal{O}_X \to \mathcal{F}$ as in the definition the map on global sections

$$D : \Gamma(X, \mathcal{O}_X) \longrightarrow \Gamma(X, \mathcal{F})$$

clearly is a $\Gamma(S, \mathcal{O}_S)$-derivation as in the algebra definition.

Lemma 24.32.2. *Let $R \to A$ be a ring map. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules on $X = Spec(A)$. Set $S = Spec(R)$. The rule which associates to an S-derivation on $\mathcal{F}$ its action on global sections defines a bijection between the set of S-derivations of $\mathcal{F}$ and the set of R-derivations on $M = \Gamma(X, \mathcal{F})$.*

Proof. Let $D : A \to M$ be an R-derivation. We have to show there exists a unique S-derivation on $\mathcal{F}$ which gives rise to D on global sections. Let $U = D(f) \subset X$ be a standard affine open. Any element of $\Gamma(U, \mathcal{O}_X)$ is of the form a/f^n for some $a \in A$ and $n \geq 0$. By the Leibniz rule we have

$$D(a)|_U = a/f^n D(f^n)|_U + f^n D(a/f^n)$$

in $\Gamma(U, \mathcal{F})$. Since f acts invertibly on $\Gamma(U, \mathcal{F})$ this completely determines the value of $D(a/f^n) \in \Gamma(U, \mathcal{F})$. This proves uniqueness. Existence follows by simply defining

$$D(a/f^n) := (1/f^n)D(a)|_U - a/f^{2n} D(f^n)|_U$$

and proving this has all the desired properties (on the basis of standard opens of X). Details omitted. $\square$

Here is a particular situation where derivations come up naturally.

Lemma 24.32.3. *Let $f : X \to S$ be a morphism of schemes. Consider a short exact sequence*

$$0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{O}_X \to 0$$

Here $\mathcal{A}$ is a sheaf of $f^{-1}\mathcal{O}_S$-algebras, $\pi : \mathcal{A} \to \mathcal{O}_X$ is a surjection of sheaves of $f^{-1}\mathcal{O}_S$-algebras, and $\mathcal{I} = Ker(\pi)$ is its kernel. Assume $\mathcal{I}$ an ideal sheaf with square zero in $\mathcal{A}$. So $\mathcal{I}$ has a natural structure of an $\mathcal{O}_X$-module. A section $s : \mathcal{O}_X \to \mathcal{A}$ of π is a $f^{-1}\mathcal{O}_S$-algebra map such that $\pi \circ s = id$. Given any section $s : \mathcal{O}_X \to \mathcal{I}$ of π and any S-derivation $D : \mathcal{O}_X \to \mathcal{I}$ the map

$$s + D : \mathcal{O}_X \to \mathcal{A}$$

is a section of π and every section s' is of the form $s + D$ for a unique S-derivation D.

Proof. Recall that the $\mathcal{O}_X$-module structure on $\mathcal{I}$ is given by $h\tau = \tilde{h}\tau$ (multiplication in $\mathcal{A}$) where h is a local section of $\mathcal{O}_X$, and $\tilde{h}$ is a local lift of h to a local section of $\mathcal{A}$, and τ is a local section of $\mathcal{I}$. In particular, given s, we may use $\tilde{h} = s(h)$. To verify that $s + D$ is a homomorphism of sheaves of rings we compute

$$\begin{aligned}(s+D)(ab) &= s(ab) + D(ab)\\ &= s(a)s(b) + aD(b) + D(a)b\\ &= s(a)s(b) + s(a)D(b) + D(a)s(b)\\ &= (s(a)+D(a))(s(b)+D(b))\end{aligned}$$

by the Leibniz rule. In the same manner one shows $s + D$ is a $f^{-1}\mathcal{O}_S$-algebra map because D is an S-derivation. Conversely, given s' we set $D = s' - s$. Details omitted. □

Let $f : X \to S$ be a morphism of schemes. We now esthablish the existence of a couple of ``global'' sheaves and maps of sheaves, and in the next paragraph we describe the constructions over some affine opens.

Recall that $\Delta = \Delta_{X/S} : X \to X \times_S X$ is an immersion, see Schemes, Lemma 21.21.2. Let $\mathcal{J}$ be the ideal sheaf of the immersion. It lives over any open subscheme U of $X \times_S X$ such that $\Delta(X) \subset U$ is closed. For example the one from the proof of the lemma just cited; if f is separated then we can take $U = X \times_S X$. Note that the sheaf of rings $\mathcal{O}_U/\mathcal{J}^2$ is supported on $\Delta(X)$. Moreover it sits in a short exact sequence of sheaves

$$0 \to \mathcal{J}/\mathcal{J}^2 \to \mathcal{O}_U/\mathcal{J}^2 \to \Delta_*\mathcal{O}_X \to 0.$$

Using Δ^{-1} we can think of this as a surjection of sheaves of $f^{-1}\mathcal{O}_S$-algebras with kernel the conormal sheaf of Δ (see Definition 24.31.1 and Lemma 24.31.2).

$$0 \to \mathcal{C}_{X/X\times_S X} \to \Delta^{-1}(\mathcal{O}_U/\mathcal{J}^2) \to \mathcal{O}_X \to 0$$

This places us in the sitation of Lemma 24.32.3. The projection morphisms $p_i : X \times_S X \to X$, $i = 1, 2$ induce maps of sheaves of rings $p_i^\sharp : p_i^{-1}\mathcal{O}_X \to \mathcal{O}_{X\times_S X}$. We may restrict to U and divide by $\mathcal{J}^2$ to get $p_i^{-1}\mathcal{O}_X \to \mathcal{O}_U/\mathcal{J}^2$. Since $\Delta^{-1}p_i^{-1}\mathcal{O}_X = \mathcal{O}_X$ we get maps

$$s_i : \mathcal{O}_X \to \Delta^{-1}(\mathcal{O}_U/\mathcal{J}^2).$$

Both s_1 and s_2 are sections to the map $\Delta^{-1}(\mathcal{O}_U/\mathcal{J}^2) \to \mathcal{O}_X$, as in Lemma 24.32.3. Thus we get an S-derivation $\mathrm{d} = s_2 - s_1 : \mathcal{O}_X \to \mathcal{C}_{X/X\times_S X}$.

Let us work this out on a suitable affine open. We can cover X by affine opens $Spec(A) = W \subset X$ whose image is contained in an affine open $Spec(R) = V \subset S$. According to the proof of Schemes, Lemma 21.21.2 $W \times_V W \subset X \times_S X$ is an affine open contained in the open U mentioned above. Also $W \times_V W = Spec(A \otimes_R A)$. The sheaf $\mathcal{J}$ corresponds to the ideal $J = \mathrm{Ker}(A \otimes_R A \to A)$. The short exact sequence to the short exact sequence of $A \otimes_R A$-modules

$$0 \to J/J^2 \to (A \otimes_R A)/J^2 \to A \to 0$$

The sections s_i correspond to the ring maps

$$A \longrightarrow (A \otimes_R A)/J^2,\ \ s_1 : a \mapsto a \otimes 1,\ \ s_2 : a \mapsto 1 \otimes a.$$

By Lemma 24.31.2 the conormal sheaf of $\Delta_{X/S}$ restricted to $U \times_V U$ is the quasi-coherent sheaf associated to the A-module J/J^2. Comparing with Algebra, Lemma 7.122.13 (or by a direct computation) we see that the induced map $\mathrm{d} : A \to J/J^2$ is isomorphic to the universal R-derivation on A. Thus the following definition makes sense.

Definition 24.32.4. Let $f : X \to S$ be a morphism of schemes.

(1) The *sheaf of differentials* $\Omega_{X/S}$ *of* X *over* S is the conormal sheaf of the immersion $\Delta_{X/S} : X \to X \times_S X$, see Definition 24.31.1.
(2) The *universal* S*-derivation* is the S-derivation
$$\mathrm{d}_{X/S} : \mathcal{O}_X \longrightarrow \Omega_{X/S}$$
which maps a local section f of $\mathcal{O}_X$ to the class of the local section $\mathrm{d}(f) = \mathrm{d}_{X/S}(f) = s_2(f) - s_1(f)$ with s_2 and s_1 as described above.

Here is the universal property of the universal derivation. If you have any other construction of the sheaf of relative differentials which satisfies this universal property then, by the Yoneda lemma, it will be canonically isomorphic to the one defined above.

Lemma 24.32.5. *Let* $f : X \to S$ *be a morphism of schemes. The map*
$$Hom_{\mathcal{O}_X}(\Omega_{X/S}, \mathcal{F}) \longrightarrow Der_S(\mathcal{O}_X, \mathcal{F}), \quad \alpha \longmapsto \alpha \circ d_{X/S}$$
is an isomorphism of functors from the category of $\mathcal{O}_X$*-modules to the category of sets.*

Proof. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let $D \in \mathrm{Der}_S(\mathcal{O}_X, \mathcal{F})$. We have to show there exists a unique $\mathcal{O}_X$-linear map $\alpha : \Omega_{X/S} \to \mathcal{F}$ such that $D = \alpha \circ \mathrm{d}_{X/S}$.

We claim that the image of $\mathrm{d}_{X/S} : \mathcal{O}_X \to \Omega_{X/S}$ generates $\Omega_{X/S}$ as an $\mathcal{O}_X$-module. To see this it suffices to prove this is true on suitable affine opens. We can cover X by affine opens $Spec(A) = W \subset X$ whose image is contained in an affine open $Spec(R) = V \subset S$. As seen in the discussion leading up to Definition 24.32.4 we have
$$\Omega_{X/S}|_W = \widetilde{J/J^2}$$
with $J = \mathrm{Ker}(A \otimes_R A \to A)$. Now clearly J is generated by the elements $1 \otimes f - f \otimes 1$. Hence the claim follows. This claim implies immediately that α, if it exists, is unique.

Next, we come to existence of α. Note that the construction of the pair $(\Omega_{X/S}, \mathrm{d}_{X/S})$ commutes with restriction to open subschemes (in both X and S). Proof omitted. By the uniqueness just shown, it therefore suffices to prove existence in case both X and S are affine. Thus we may write $X = Spec(A)$, $S = Spec(S)$ and $M = \Gamma(X, \mathcal{F})$. Set as usual $J = \mathrm{Ker}(A \otimes_R A \to A)$ so that $\Omega_{X/S} = \widetilde{J/J^2}$. According to Algebra, Lemmas 7.122.3 and 7.122.13 there exists a unique A-linear map $\alpha' : J/J^2 \to M$ such that the composition $\mathrm{d} \circ \alpha' : A \to J/J^2 \to M$ is equal to the action of D on global sections over X. By Schemes, Lemma 21.7.1 the A-linear map α' corresponds to a map $\alpha : \Omega_{X/S} = \widetilde{J/J^2} \to \mathcal{F}$. Then the derivations $\alpha \circ \mathrm{d}_{X/S}$ and D have the same effect on global sections and hence agree by Lemma 24.32.2. This proves existence and we win. □

Lemma 24.32.6. *Let* $f : X \to S$ *be a morphism of schemes. Let* $U \subset X$*,* $V \subset S$ *be open subschemes such that* $f(U) \subset V$*. Then there is a unique isomorphism* $\Omega_{X/S}|_U = \Omega_{U/V}$ *of* $\mathcal{O}_U$*-modules such that* $d_{X/S}|_U = d_{U/V}$*.*

Proof. The existence of the isomorphism is clear from the construction of $\Omega_{X/S}$. Uniqueness comes from the fact, seen in the proof of Lemma 24.32.5, that the image of $\mathrm{d}_{X/S} : \mathcal{O}_X \to \Omega_{X/S}$ generates $\Omega_{X/S}$ as an $\mathcal{O}_X$-module. □

From now on we will use these canonical identifications and simply write $\Omega_{U/S}$ or $\Omega_{U/V}$ for the restriction of $\Omega_{X/S}$ to U.

Lemma 24.32.7. *Let $f : X \to S$ be a morphism of schemes. For any pair of affine opens $Spec(A) = U \subset X$, $Spec(R) = V \subset S$ with $f(U) \subset V$ there is a unique isomorphism*

$$\Gamma(U, \mathcal{O}_{X/S}) = \Omega_{A/R}.$$

compatible with $d_{X/S}$ and $d : A \to \Omega_{A/R}$.

Proof. During the construction of $\Omega_{X/S}$ we have seen that the restriction of $\Omega_{X/S}$ to U is isomorphic to the quasi-coherent sheaf associated to the A-module J/J^2 where $J = \mathrm{Ker}(A \otimes_R A \to A)$. Hence the result follows from Algebra, Lemma 7.122.13.

An alternative proof is to show that the A-module $M = \Gamma(U, \Omega_{X/S}) = \Gamma(U, \Omega_{U/V})$ together with $\mathrm{d}_{X/S} = \mathrm{d}_{U/V} : A \to M$ is a universal R-derivation of A. This follows by combining Lemmas 24.32.2 and 24.32.5 above. The universal property of $\mathrm{d} : A \to \Omega_{A/R}$ (see Algebra, Lemma 7.122.3) and the Yoneda lemma (Categories, Lemma 4.3.5) imply there is a unique isomorphism of A-modules $M \cong \Omega_{A/R}$ compatible with derivations. This gives the second proof. □

Remark 24.32.8. The lemma above gives a second way of constructing the module of differentials. Namely, let $f : X \to S$ be a morphism of schemes. Consider the collection of all affine opens $U \subset X$ which map into an affine open of S. These form a basis for the topology on X. Thus it suffices to define $\Gamma(U, \Omega_{X/S})$ for such U. We simply set $\Gamma(U, \Omega_{X/S}) = \Omega_{A/R}$ if A, R are as in Lemma 24.32.7 above. This works, but it takes somewhat more algebraic preliminaries to construct the restriction mappings and to verify the sheaf condition with this ansatz.

Lemma 24.32.9. *Let*

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

be a commutative diagram of schemes. The canonical map $\mathcal{O}_X \to f_*\mathcal{O}_{X'}$ composed with the map $f_*d_{X'/S'} : f_*\mathcal{O}_{X'} \to f_*\Omega_{X'/S'}$ is a S-derivation. Hence we obtain a canonical map of $\mathcal{O}_X$-modules $\Omega_{X/S} \to f_*\Omega_{X'/S'}$, and by adjointness of f_* and f^* a canonical $\mathcal{O}_{X'}$-module homomorphism

$$c_f : f^*\Omega_{X/S} \longrightarrow \Omega_{X'/S'}.$$

It is uniquely characterized by the property that $f^*d_{X/S}(h)$ mapsto $d_{X'/S'}(f^*h)$ for any local section h of $\mathcal{O}_X$.

Proof. Everything but the last assertion of the lemma is proven in the lemma; the universal property of $\Omega_{X/S}$ is Lemma 24.32.5. The last assertion means that c_f is the unique $\mathcal{O}_{X'}$-linear map such that whenever $U \subset X$ is open and $h \in \mathcal{O}_X(U)$, then the pullback $f^*h \in \mathcal{O}_{X'}(f^{-1}U)$ of h satisfies $\mathrm{d}_{X'/S'}(f^*h) = c_f(f^*\mathrm{d}_{X/S}(h))$. We omit the proof. We can also use the functoriality of the conormal sheaves (see Lemma 24.31.3) to define c_f. Or we can use the characterization in the last line of the lemma to glue maps defined on affine patches (see Algebra, Equation (7.122.5.1)). □

Lemma 24.32.10. *Let*

$$\begin{tikzcd} X'' \arrow[r, "g"] \arrow[d] & X' \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ S'' \arrow[r] & S' \arrow[r] & S \end{tikzcd}$$

be a commutative diagram of schemes. Then we have

$$c_{f\circ g} = c_g \circ g^* c_f$$

as maps $(f \circ g)^*\Omega_{X/S} \to \Omega_{X''/S''}$.

Proof. Omitted. One way to see this is to restrict to affine opens. □

Lemma 24.32.11. *Let* $f : X \to Y$, $g : Y \to S$ *be morphisms of schemes. Then there is a canonical exact sequence*

$$f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0$$

where the maps come from applications of Lemma 24.32.9.

Proof. This is the sheafified version of Algebra, Lemma 7.122.7. □

Lemma 24.32.12. *Let* $X \to S$ *be a morphism of schemes. Let* $g : S' \to S$ *be a morphism of schemes. Let* $X' = X_{S'}$ *be the base change of* X*. Denote* $g' : X' \to X$ *the projection. Then the map*

$$(g')^*\Omega_{X/S} \to \Omega_{X'/S'}$$

of Lemma 24.32.9 is an isomorphism.

Proof. This is the sheafified version of Algebra, Lemma 7.122.12. □

Lemma 24.32.13. *Let* $f : X \to S$ *and* $g : Y \to S$ *be morphisms of schemes with the same target. Let* $p : X \times_S Y \to X$ *and* $q : X \times_S Y \to Y$ *be the projection morphisms. The maps from Lemma 24.32.9*

$$p^*\Omega_{X/S} \oplus q^*\Omega_{Y/S} \longrightarrow \Omega_{X\times_S Y/S}$$

give an isomorphism.

Proof. By Lemma 24.32.12 the composition $p^*\Omega_{X/S} \to \Omega_{X\times_S Y/S} \to \Omega_{X\times_S Y/Y}$ is an isomorphism, and similarly for q. Moreover, the cokernel of $p^*\Omega_{X/S} \to \Omega_{X\times_S Y/S}$ is $\Omega_{X\times_S Y/X}$ by Lemma 24.32.11. The result follows. □

Lemma 24.32.14. *Let* $f : X \to S$ *be a morphism of schemes. If* f *is locally of finite type, then* $\Omega_{X/S}$ *is a finite type* $\mathcal{O}_X$*-module.*

Proof. Immediate from Algebra, Lemma 7.122.16, Lemma 24.32.7, Lemma 24.14.2, and Properties, Lemma 23.16.1. □

Lemma 24.32.15. *Let* $f : X \to S$ *be a morphism of schemes. If* f *is locally of finite type, then* $\Omega_{X/S}$ *is an* $\mathcal{O}_X$*-module of finite presentation.*

Proof. Immediate from Algebra, Lemma 7.122.15, Lemma 24.32.7, Lemma 24.20.2, and Properties, Lemma 23.16.2. □

Lemma 24.32.16. *If* $X \to S$ *is an immersion, or more generally a monomorphism, then* $\Omega_{X/S}$ *is zero.*

Proof. This is true because $\Delta_{X/S}$ is an isomorphism in this case and hence has trivial conormal sheaf. The algebraic version is Algebra, Lemma 7.122.5. □

Lemma 24.32.17. *Let* $i : Z \to X$ *be an immersion of schemes over* S*. There is a canonical exact sequence*

$$\mathcal{C}_{Z/X} \to i^*\Omega_{X/S} \to \Omega_{Z/S} \to 0$$

where the first arrow is induced by $\mathrm{d}_{X/S}$ *and the second arrow comes from Lemma 24.32.9.*

Proof. This is the sheafified version of Algebra, Lemma 7.122.9. However we should make sure we can define the first arrow globally. Hence we explain the meaning of ``induced by $\text{d}_{X/S}$'' here. Namely, we may assume that i is a closed immersion by shrinking X. Let $\mathcal{I} \subset \mathcal{O}_X$ be the sheaf of ideals corresponding to $Z \subset X$. Then $\text{d}_{X/S} : \mathcal{I} \to \Omega_{X/S}$ maps the subsheaf $\mathcal{I}^2 \subset \mathcal{I}$ to $\mathcal{I}\Omega_{X/S}$. Hence it induces a map $\mathcal{I}/\mathcal{I}^2 \to \Omega_{X/S}/\mathcal{I}\Omega_{X/S}$ which is $\mathcal{O}_X/\mathcal{I}$-linear. By Lemma 24.3.1 this corresponds to a map $\mathcal{C}_{Z/X} \to i^*\Omega_{X/S}$ as desired. $\square$

Lemma 24.32.18. *Let $i : Z \to X$ be an immersion of schemes over S, and assume i (locally) has a left inverse. Then the canonical sequence*

$$0 \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/S} \to \Omega_{Z/S} \to 0$$

of Lemma 24.32.17 is (locally) split exact. In particular, if $s : S \to X$ is a section of the structure morphism $X \to S$ then the map $\mathcal{C}_{S/X} \to s^*\Omega_{X/S}$ induced by $d_{X/S}$ is an isomorphism.

Proof. Follows from Algebra, Lemma 7.122.10. Clarification: if $g : X \to Z$ is a left inverse of i, then i^*c_g is a right inverse of the map $i^*\Omega_{X/S} \to \Omega_{Z/S}$. Also, if s is a section, then it is an immersion $s : Z = S \to X$ over S (see Schemes, Lemma 21.21.12) and in that case $\Omega_{Z/S} = 0$. $\square$

Remark 24.32.19. Let $X \to S$ be a morphism of schemes. According to Lemma 24.32.13 we have

$$\Omega_{X \times_S X/S} = \text{pr}_1^*\Omega_{X/S} \oplus \text{pr}_2^*\Omega_{X/S}$$

On the other hand, the diagonal morphism $\Delta : X \to X \times_S X$ is an immersion, which locally has a left inverse. Hence by Lemma 24.32.18 we obtain a canonical short exact sequence

$$0 \to \mathcal{C}_{X/X \times_S X} \to \Omega_{X/S} \oplus \Omega_{X/S} \to \Omega_{X/S} \to 0$$

Note that the right arrow is $(1, 1)$ which is indeed a split surjection. On the other hand, by our very definition we have $\Omega_{X/S} = \mathcal{C}_{X/X \times_S X}$. Because we chose $\text{d}_{X/S}(f) = s_2(f) - s_1(f)$ in Definition 24.32.4 it turns out that the left arrow is the map $(-1, 1)$[8].

Lemma 24.32.20. *Let*

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[dr, "j"'] & X \arrow[d] \\ & Y \end{tikzcd}$$

be a commutative diagram of schemes where i and j are immersions. Then there is a canonical exact sequence

$$\mathcal{C}_{Z/Y} \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/Y} \to 0$$

where the first arrow comes from Lemma 24.31.3 and the second from Lemma 24.32.17.

Proof. The algebraic version of this is Algebra, Lemma 7.123.5. $\square$

[8]Namely, the local section $\text{d}_{X/S}(f) = 1 \otimes f - f \otimes 1$ of the ideal sheaf of Δ maps via $\text{d}_{X \times_S X/X}$ to the local section $1 \otimes 1 \otimes 1 \otimes f - 1 \otimes f \otimes 1 \otimes 1 - 1 \otimes 1 \otimes f \otimes 1 + f \otimes 1 \otimes 1 \otimes 1 = \text{pr}_2^*\text{d}_{X/S}(f) - \text{pr}_1^*\text{d}_{X/S}(f)$.

24.33. Smooth morphisms

Let $f : X \to Y$ be a map of topological spaces. Consider the following condition:

(*) For every $x \in X$ there exist open neighbourhoods $x \in U \subset X$ and $f(x) \in V \subset Y$, and an integer d such that $f(U) = V$ and such that there is an isomorphism

$$\begin{tikzcd} V \times B_d(0,1) \arrow[r, "\cong"] \arrow[d] & U \arrow[r] \arrow[d] & X \arrow[d] \\ V \arrow[r, equal] & V \arrow[r] & Y \end{tikzcd}$$

where $B_d(0,1) \subset \mathbf{R}^d$ is a ball of radius 1 around 0.

Smooth morphisms are the analogue of such morphisms in the category of schemes. See Lemma 24.33.11 and Lemma 24.35.20.

Contrary to expectations (perhaps) the notion of a smooth ring map is not defined solely in terms of the module of differentials. Namely, recall that $R \to A$ is a *smooth ring map* if A is of finite presentation over R and if the naive cotangent complex of A over R is quasi-isomorphic to a projective module placed in degree 0, see Algebra, Definition 7.126.1.

Definition 24.33.1. Let $f : X \to S$ be a morphism of schemes.

(1) We say that f is *smooth at* $x \in X$ if there exists a affine open neighbourhood $Spec(A) = U \subset X$ of x and and affine open $Spec(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is smooth.
(2) We say that f is *smooth* if it is smooth at every point of X.
(3) A morphism of affine schemes $f : X \to S$ is called *standard smooth* there exists a standard smooth ring map $R \to R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ (see Algebra, Definition 7.126.6) such that $X \to S$ is isomorphic to

$$Spec(R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)) \to Spec(R).$$

A pleasing feature of this definition is that the set of points where a morphism is smooth is automatically open.

Note that there is no separation or quasi-compactness hypotheses in the definition. Hence the question of being smooth is local in nature on the source. Here is the precise result.

Lemma 24.33.2. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *The morphism f is smooth.*
(2) *For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is smooth.*
(3) *There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$, $j \in J, i \in I_j$ is smooth.*
(4) *There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring map $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i)$ is smooth, for all $j \in J, i \in I_j$.*

Moreover, if f is smooth then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \to V$ is smooth.

Proof. This follows from Lemma 24.13.3 if we show that the property "$R \to A$ is smooth" is local. We check conditions (a), (b) and (c) of Definition 24.13.1. By Algebra, Lemma 7.126.4 being smooth is stable under base change and hence we conclude (a) holds. By

Algebra, Lemma 7.126.14 being smooth is stable under composition and for any ring R the ring map $R \to R_f$ is (standard) smooth. We conclude (b) holds. Finally, property (c) is true according to Algebra, Lemma 7.126.13. $\square$

The following lemma characterizes a smooth morphism as a flat, finitely presented morphism with smooth fibres. Note that schemes smooth over a field are discussed in more detail in Varieties, Section 28.15.

Lemma 24.33.3. *Let $f : X \to S$ be a morphism of schemes. If f is flat, locally of finite presentation, and all fibres X_s are smooth, then f is smooth.*

Proof. Follows from Algebra, Lemma 7.126.16. $\square$

Lemma 24.33.4. *The composition of two morphisms which are smooth is smooth.*

Proof. In the proof of Lemma 24.33.2 we saw that being smooth is a local property of ring maps. Hence the first statement of the lemma follows from Lemma 24.13.5 combined with the fact that being smooth is a property of ring maps that is stable under composition, see Algebra, Lemma 7.126.14. $\square$

Lemma 24.33.5. *The base change of a morphism which is smooth is smooth.*

Proof. In the proof of Lemma 24.33.2 we saw that being smooth is a local property of ring maps. Hence the lemma follows from Lemma 24.13.5 combined with the fact that being smooth is a property of ring maps that is stable under base change, see Algebra, Lemma 7.126.4. $\square$

Lemma 24.33.6. *Any open immersion is smooth.*

Proof. This is true because an open immersion is a local isomorphism. $\square$

Lemma 24.33.7. *A smooth morphism is syntomic.*

Proof. See Algebra, Lemma 7.126.10. $\square$

Lemma 24.33.8. *A smooth morphism is locally of finite presentation.*

Proof. True because a smooth ring map is of finite presentation by definition. $\square$

Lemma 24.33.9. *A smooth morphism is flat.*

Proof. Combine Lemmas 24.30.7 and 24.33.7. $\square$

Lemma 24.33.10. *A smooth morphism is universally open.*

Proof. Combine Lemmas 24.33.9, 24.33.8, and 24.24.9. Or alternatively, combine Lemmas 24.33.7, 24.30.8. $\square$

The following lemma says locally any smooth morphism is standard smooth. Hence we can use standard smooth morphisms as a *local model* for a smooth morphism.

Lemma 24.33.11. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point. Set $s = f(x)$. The following are equivalent*

(1) *The morphism f is smooth at x.*
(2) *There exist affine opens $U \subset X$, and $V \subset S$ such that $x \in U$, $f(U) \subset V$ and the induced morphism $f|_U : U \to V$ is standard smooth.*

Proof. Follows from the definitions and Algebra, Lemmas 7.126.7 and 7.126.10. $\square$

Lemma 24.33.12. *Let $f : X \to S$ be a morphism of schemes. Assume f is smooth. Then the module of differentials $\Omega_{X/S}$ of X over S is finite locally free and*

$$rank_x(\Omega_{X/S}) = \dim_x(X_{f(x)})$$

for every $x \in X$.

Proof. The statement is local on X and S. By Lemma 24.33.11 above we may assume that f is a standard smooth morphism of affines. In this case the result follows from Algebra, Lemma 7.126.7 (and the definition of a relative global complete intersection, see Algebra, Definition 7.125.5). □

Lemma 24.33.12 says that the following definition makes sense.

Definition 24.33.13. Let $d \geq 0$ be an integer. We say a morphism of schemes $f : X \to S$ is *smooth of relative dimension d* if f is smooth and $\Omega_{X/S}$ is finite locally free of constant rank d.

In other words, f is smooth and the nonempty fibres are equidimensional of dimension d. By Lemma 24.33.14 below this is also the same as requiring: (a) f is locally of finite presentation, (b) f is flat, (c) all nonempty fibres equidimensional of dimension d, and (d) $\Omega_{X/S}$ finite locally free of rank d. It is not enough to simply assume that f is flat, of finite presentation, and $\Omega_{X/S}$ is finite locally free of rank d. A counter example is given by $Spec(\mathbf{F}_p[t]) \to Spec(\mathbf{F}_p[t^p])$.

Here is a differential criterion of smoothness at a point. There are many variants of this result all of which may be useful at some point. We will just add them here as needed.

Lemma 24.33.14. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. Set $s = f(x)$. Assume f is locally of finite presentation. The following are equivalent:*

(1) *The morphism f is smooth at x.*
(2) *The local ring map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat and the $\mathcal{O}_{X,x}$-module $\Omega_{X/S,x}$ can be generated by at most $\dim_x(X_{f(x)})$ elements.*
(3) *The local ring map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat and the $\kappa(x)$-vector space*
$$\Omega_{X_s/s,x} \otimes_{\mathcal{O}_{X_s,x}} \kappa(x) = \Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$$
can be generated by at most $\dim_x(X_{f(x)})$ elements.
(4) *There exist affine opens $U \subset X$, and $V \subset S$ such that $x \in U$, $f(U) \subset V$ and the induced morphism $f|_U : U \to V$ is standard smooth.*
(5) *There exist affine opens $Spec(A) = U \subset X$ and $Spec(R) = V \subset S$ with $x \in U$ corresponding to $\mathfrak{q} \subset A$, and $f(U) \subset V$ such that there exists a presentation*
$$A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$$
with
$$g = \det \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_2/\partial x_1 & \ldots & \partial f_c/\partial x_1 \\ \partial f_1/\partial x_2 & \partial f_2/\partial x_2 & \ldots & \partial f_c/\partial x_2 \\ \ldots & \ldots & \ldots & \ldots \\ \partial f_1/\partial x_c & \partial f_2/\partial x_c & \ldots & \partial f_c/\partial x_c \end{pmatrix}$$
mapping to an element of A not in $\mathfrak{q}$.

Proof. Note that if f is smooth at x, then we see from Lemma 24.33.11 that (4) holds, and (5) is a slightly weakened version of (4). Moreover, this implies that the ring map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat (see Lemma 24.33.9) and that $\Omega_{X/S}$ is finite locally free of rank equal to $\dim_x(X_s)$ (see Lemma 24.33.12). This implies (2) and (3).

By Lemma 24.32.12 the module of differentials $\Omega_{X_s/s}$ of the fibre X_s over $\kappa(s)$ is the pullback of the module of differentials $\Omega_{X/S}$ of X over S. Hence the displayed equality in part (3) of the lemma. By Lemma 24.32.14 these modules are of finite type. Hence the mimimal number of generators of the modules $\Omega_{X/S,x}$ and $\Omega_{X_s/s,x}$ is the same and equal to the dimension of this $\kappa(x)$-vector space by Nakayama's Lemma (Algebra, Lemma 7.14.5). This in particular shows that (2) and (3) are equivalent.

Combining Algebra, Lemmas 7.126.16 and 7.129.3 shows that (2) and (3) imply (1). Finally, (5) implies (4) see for example Algebra, Example 7.126.8. □

Lemma 24.33.15. *Let $f : X \to S$ be a morphism of schemes. Assume f locally of finite type. Formation of the set*

$$T = \{x \in X \mid X_{f(x)} \text{ is smooth over } \kappa(f(x)) \text{ at } x\}$$

commutes with arbitrary base change: For any morphism $g : S' \to S$, consider the base change $f' : X' \to S'$ of f and the projection $g' : X' \to X$. Then the corresponding set T' for the morphism f' is equal to $T' = (g')^{-1}(T)$. In particular, if f is assumed flat, and locally of finite presentation then the same holds for the open set of points where f is smooth.

Proof. Let $s' \in S'$ be a point, and let $s = g(s')$. Then we have

$$X'_{s'} = Spec(\kappa(s')) \times_{Spec(\kappa(s))} X_s$$

In other words the fibres of the base change are the base changes of the fibres. Hence the first part is equivalent to Algebra, Lemma 7.126.18. The second part follows from the first because in that case T is the (open) set of points where f is smooth according to Lemma 24.33.3. □

Here is a lemma that actually uses the vanishing of H^{-1} of the naive cotangent complex for a smooth ring map.

Lemma 24.33.16. *Let $f : X \to Y$, $g : Y \to S$ be morphisms of schemes. Assume f is smooth. Then*

$$0 \to f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0$$

(see Lemma 24.32.11) is short exact.

Proof. The algebraic version of this lemma is the following: Given ring maps $A \to B \to C$ with $B \to C$ smooth, then the sequence

$$0 \to C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

of Algebra, Lemma 7.122.7 is exact. This is Algebra, Lemma 7.128.1. □

Lemma 24.33.17. *Let $i : Z \to X$ be an immersion of schemes over S. Assume that Z is smooth over S. Then the canonical exact sequence*

$$0 \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/S} \to \Omega_{Z/S} \to 0$$

of Lemma 24.32.17 is short exact.

Proof. The algebraic version of this lemma is the following: Given ring maps $A \to B \to C$ with $A \to C$ smooth and $B \to C$ surjective with kernel J, then the sequence

$$0 \to J/J^2 \to C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to 0$$

of Algebra, Lemma 7.122.9 is exact. This is Algebra, Lemma 7.128.2. □

Lemma 24.33.18. *Let*

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[rd, "j"'] & X \arrow[d] \\ & Y \end{tikzcd}$$

be a commutative diagram of schemes where i and j are immersions and $X \to Y$ is smooth. Then the canonical exact sequence

$$0 \to \mathcal{C}_{Z/Y} \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/Y} \to 0$$

of Lemma 24.32.20 is exact.

Proof. The algebraic version of this lemma is the following: Given ring maps $A \to B \to C$ with $A \to C$ surjective and $A \to B$ smooth, then the sequence

$$0 \to I/I^2 \to J/J^2 \to C \otimes_B \Omega_{B/A} \to 0$$

of Algebra, Lemma 7.123.5 is exact. This is Algebra, Lemma 7.128.3. □

Lemma 24.33.19. *Let*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[rd, "p"'] & & Y \arrow[ld, "q"] \\ & S & \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume that

(1) *f is surjective, and smooth,*
(2) *p is smooth, and*
(3) *q is locally of finite presentation*[9].

Then q is smooth.

Proof. By Lemma 24.24.11 we see that q is flat. Pick a point $y \in Y$. Pick a point $x \in X$ mapping to y. Suppose f has relative dimension a at x and p has relative dimension b at x. By Lemma 24.33.12 this means that $\Omega_{X/S,x}$ is free of rank b and $\Omega_{X/Y,x}$ is free of rank a. By the short exact sequence of Lemma 24.33.16 this means that $(f^*\Omega_{Y/S})_x$ is free of rank $b - a$. By Nakayama's Lemma this implies that $\Omega_{Y/S,y}$ can be generated by $b - a$ elements. Also, by Lemma 24.27.2 we see that $\dim_y(Y_s) = b - a$. Hence we conclude that $Y \to S$ is smooth at y by Lemma 24.33.14 part (2). □

In the situation of the following lemma the image of σ is locally on X cut out by a regular sequence, see Divisors, Lemma 26.14.7.

Lemma 24.33.20. *Let $f : X \to S$ be a morphism of schemes. Let $\sigma : S \to X$ be a section of f. Let $s \in S$ be a point such that f is smooth at $x = \sigma(s)$. Then there exist affine open neighbourhoods $Spec(A) = U \subset S$ of s and $Spec(B) = V \subset X$ of x such that*

(1) *$f(V) \subset U$ and $\sigma(U) \subset V$,*
(2) *with $I = Ker(\sigma^\# : B \to A)$ the module I/I^2 is a free A-module, and*
(3) *$B^\wedge \cong A[[x_1, \ldots, x_d]]$ as A-algebras where $B^\wedge$ denotes the completion of B with respect to I.*

---

[9]In fact this is implied by (1) and (2), see Descent, Lemma 31.10.3. Moreover, it suffices to assume f is surjective, flat and locally of finite presentation, see Descent, Lemma 31.10.5.

Proof. Pick an affine open $U \subset S$ containing s Pick an affine open $V \subset f^{-1}(U)$ containing x. Pick an affine open $U' \subset \sigma^{-1}(V)$ containing s. Note that $V' = f^{-1}(U') \cap V$ is affine as it is equal to the fibre product $V' = U' \times_U V$. Then U' and V' satisfy (1). Write $U' = Spec(A')$ and $V' = Spec(B')$. By Algebra, Lemma 7.128.4 the module $I'/(I')^2$ is finite locally free as a A'-module. Hence after replacing U' by a smaller affine open $U'' \subset U'$ and V' by $V'' = V' \cap f^{-1}(U'')$ we obtain the situation where $I''/(I'')^2$ is free, i.e., (2) holds. In this case (3) holds also by Algebra, Lemma 7.128.4. □

24.34. Unramified morphisms

We briefly discuss unramified morphisms before the (perhaps) more interesting class of étale morphisms. Recall that a ring map $R \to A$ is *unramified* if it is of finite type and $\Omega_{A/R} = 0$ (this is the definition of [**Ray70**]). A ring map $R \to A$ is called *G-unramified* if it is of finite presentation and $\Omega_{A/R} = 0$ (this is the definition of [**DG67**]). See Algebra, Definition 7.138.1.

Definition 24.34.1. Let $f : X \to S$ be a morphism of schemes.

(1) We say that f is *unramified at $x \in X$* if there exists a affine open neighbourhood $Spec(A) = U \subset X$ of x and and affine open $Spec(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is unramified.
(2) We say that f is *G-unramified at $x \in X$* if there exists a affine open neighbourhood $Spec(A) = U \subset X$ of x and and affine open $Spec(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is G-unramified.
(3) We say that f is *unramified* if it is unramified at every point of X.
(4) We say that f is *G-unramified* if it is G-unramified at every point of X.

Note that a G-unramified morphism is unramified. Hence any result for unramified morphisms implies the corresponding result for G-unramified morphisms. Moreover, if S is locally Noetherian then there is no difference between G-unramified and unramified morphisms, see Lemma 24.34.6. A pleasing feature of this definition is that the set of points where a morphism is unramified (resp. G-unramified) is automatically open.

Lemma 24.34.2. *Let $f : X \to S$ be a morphism of schemes. Then*

(1) *f is unramified if and only if f is locally of finite type and $\Omega_{X/S} = 0$, and*
(2) *f is G-unramified if and only if f is locally of finite presentation and $\Omega_{X/S} = 0$.*

Proof. By definition a ring map $R \to A$ is unramified (resp. G-unramified) if and only if it is of finite type (resp. finite presentation) and $\Omega_{A/R} = 0$. Hence the lemma follows directly from the definitions and Lemma 24.32.7. □

Note that there is no separation or quasi-compactness hypotheses in the definition. Hence the question of being unramified is local in nature on the source. Here is the precise result.

Lemma 24.34.3. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *The morphism f is unramified (resp. G-unramified).*
(2) *For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is unramified (resp. G-unramified).*
(3) *There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$, $j \in J, i \in I_j$ is unramified (resp. G-unramified).*

(4) *There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring map $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i)$ is unramified (resp. G-unramified), for all $j \in J, i \in I_j$.*

Moreover, if f is unramified (resp. G-unramified) then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \to V$ is unramified (resp. G-unramified).

Proof. This follows from Lemma 24.13.3 if we show that the property "$R \to A$ is unramified" is local. We check conditions (a), (b) and (c) of Definition 24.13.1. These properties are proved in Algebra, Lemma 7.138.3. □

Lemma 24.34.4. *The composition of two morphisms which are unramified is unramified. The same holds for G-unramified morphisms.*

Proof. The proof of Lemma 24.34.3 shows that being unramified (resp. G-unramified) is a local property of ring maps. Hence the first statement of the lemma follows from Lemma 24.13.5 combined with the fact that being unramified (resp. G-unramified) is a property of ring maps that is stable under composition, see Algebra, Lemma 7.138.3. □

Lemma 24.34.5. *The base change of a morphism which is unramified is unramified. The same holds for G-unramified morphisms.*

Proof. The proof of Lemma 24.34.3 shows that being unramified (resp. G-unramified) is a local property of ring maps. Hence the lemma follows from Lemma 24.13.5 combined with the fact that being unramified (resp. G-unramified) is a property of ring maps that is stable under base change, see Algebra, Lemma 7.138.3. □

Lemma 24.34.6. *Let $f : X \to S$ be a morphism of schemes. Assume S is locally Noetherian. Then f is unramified if and only if f is G-unramified.*

Proof. Follows from the definitions and Lemma 24.20.9. □

Lemma 24.34.7. *Any open immersion is G-unramified.*

Proof. This is true because an open immersion is a local isomorphism. □

Lemma 24.34.8. *A closed immersion $i : Z \to X$ is unramified. It is G-unramified if and only if the associated quasi-coherent sheaf of ideals $\mathcal{I} = Ker(\mathcal{O}_X \to i_*\mathcal{O}_Z)$ is of finite type (as an $\mathcal{O}_X$-module).*

Proof. Follows from Lemma 24.20.7 and Algebra, Lemma 7.138.3. □

Lemma 24.34.9. *An unramified morphism is locally of finite type. A G-unramified morphism is locally of finite presentation.*

Proof. An unramified ring map is of finite type by definition. A G-unramified ring map is of finite presentation by definition. □

Lemma 24.34.10. *Let $f : X \to S$ be a morphism of schemes. If f is unramified at x then f is quasi-finite at x. In particular, an unramified morphism is locally quasi-finite.*

Proof. See Algebra, Lemma 7.138.6. □

Lemma 24.34.11. *Fibres of unramified morphisms.*

(1) *Let X be a scheme over a field k. The structure morphism $X \to Spec(k)$ is unramified if and only if X is a disjoint union of spectra of finite separable field extensions of k.*

(2) *If $f : X \to S$ is an unramified morphism then for every $s \in S$ the fibre X_s is a disjoint union of spectra of finite separable field extensions of $\kappa(s)$.*

Proof. Part (2) follows from part (1) and Lemma 24.34.5. Let us prove part (1). We first use Algebra, Lemma 7.138.7. This lemma implies that if X is a disjoint union of spectra of finite separable field extensions of k then $X \to Spec(k)$ is unramified. Conversely, suppose that $X \to Spec(k)$ is unramified. By Algebra, Lemma 7.138.5 for every $x \in X$ the residue field extension $k \subset \kappa(x)$ is finite separable. Hence all points of X are closed points (see Lemma 24.19.2 for example). Thus X is a discrete space, in particular the disjoint union of the spectra of its local rings. By Algebra, Lemma 7.138.5 again these local rings are fields, and we win. □

The following lemma characterizes an unramified morphisms as morphisms locally of finite type with unramified fibres.

Lemma 24.34.12. *Let $f : X \to S$ be a morphism of schemes.*

(1) *If f is unramified then for any $x \in X$ the field extension $\kappa(f(x)) \subset \kappa(x)$ is finite separable.*
(2) *If f is locally of finite type, and for every $s \in S$ the fibre X_s is a disjoint union of spectra of finite separable field extensions of $\kappa(s)$ then f is unramified.*
(3) *If f is locally of finite presentation, and for every $s \in S$ the fibre X_s is a disjoint union of spectra of finite separable field extensions of $\kappa(s)$ then f is G-unramified.*

Proof. Follows from Algebra, Lemmas 7.138.5 and 7.138.7. □

Here is a characterization of unramified morphisms in terms of the diagonal morphism.

Lemma 24.34.13. *Let $f : X \to S$ be a morphism.*

(1) *If f is unramified, then the diagonal morphism $\Delta : X \to X \times_S X$ is an open immersion.*
(2) *If f is locally of finite type and Δ is an open immersion, then f is unramified.*
(3) *If f is locally of finite presentation and Δ is an open immersion, then f is G-unramified.*

Proof. The first statement follows from Algebra, Lemma 7.138.4. The second statement from the fact that $\Omega_{X/S}$ (see Definition 24.32.4) is the conormal sheaf of the diagonal morphism and hence clearly zero if Δ is an open immersion. □

Lemma 24.34.14. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. Set $s = f(x)$. Assume f is locally of finite type (resp. locally of finite presentation). The following are equivalent:*

(1) *The morphism f is unramified (resp. G-unramified) at x.*
(2) *The fibre X_s is unramified over $\kappa(s)$ at x.*
(3) *The $\mathcal{O}_{X,x}$-module $\Omega_{X/S,x}$ is zero.*
(4) *The $\mathcal{O}_{X_s,x}$-module $\Omega_{X_s/s,x}$ is zero.*
(5) *The $\kappa(x)$-vector space*

$$\Omega_{X_s/s,x} \otimes_{\mathcal{O}_{X_s,x}} \kappa(x) = \Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$$

is zero.
(6) *We have $\mathfrak{m}_s \mathcal{O}_{X,x} = \mathfrak{m}_x$ and the field extension $\kappa(s) \subset \kappa(x)$ is finite separable.*

Proof. Note that if f is unramified at x, then we see that $\Omega_{X/S} = 0$ in a neighbourhood of x by the definitions and the results on modules of differentials in Section 24.32. Hence (1) implies (3) and the vanishing of the right hand vector space in (5). It also implies (2) because by Lemma 24.32.12 the module of differentials $\Omega_{X_s/s}$ of the fibre X_s over $\kappa(s)$ is the pullback of the module of differentials $\Omega_{X/S}$ of X over S. This fact on modules of differentials also implies the displayed equality of vector spaces in part (4). By Lemma 24.32.14 the modules $\Omega_{X/S,x}$ and $\Omega_{X_s/s,x}$ are of finite type. Hence he modules $\Omega_{X/S,x}$ and $\Omega_{X_s/s,x}$ are zero if and only if the corresponding $\kappa(x)$-vector space in (4) is zero by Nakayama's Lemma (Algebra, Lemma 7.14.5). This in particular shows that (3), (4) and (5) are equivalent. The support of $\Omega_{X/S}$ is closed in X, see Modules, Lemma 15.9.6. Assumption (3) implies that x is not in the support. Hence $\Omega_{X/S}$ is zero in a neighbourhood of x, which implies (1). The equivalence of (1) and (3) applied to $X_s \to s$ implies the equivalence of (2) and (4). At this point we have seen that (1) -- (5) are equivalent.

Alternatively you can use Algebra, Lemma 7.138.3 to see the equivalence of (1) -- (5) more directly.

The equivalence of (1) and (6) follows from Lemma 24.34.12. It also follows more directly from Algebra, Lemmas 7.138.5 and 7.138.7. ☐

Lemma 24.34.15. *Let $f : X \to S$ be a morphism of schemes. Assume f locally of finite type. Formation of the open set*

$$\begin{aligned} T &= \{x \in X \mid X_{f(x)} \text{ is unramified over } \kappa(f(x)) \text{ at } x\} \\ &= \{x \in X \mid X \text{ is unramified over } S \text{ at } x\} \end{aligned}$$

commutes with arbitrary base change: For any morphism $g : S' \to S$, consider the base change $f' : X' \to S'$ of f and the projection $g' : X' \to X$. Then the corresponding set T' for the morphism f' is equal to $T' = (g')^{-1}(T)$. If f is assumed locally of finite presentation then the same holds for the open set of points where f is G-unramified.

Proof. Let $s' \in S'$ be a point, and let $s = g(s')$. Then we have

$$X'_{s'} = Spec(\kappa(s')) \times_{Spec(\kappa(s))} X_s$$

In other words the fibres of the base change are the base changes of the fibres. In particular

$$\Omega_{X_s/s,x} \otimes_{\mathcal{O}_{X_s,x}} \kappa(x') = \Omega_{X'_{s'}/s',x'} \otimes_{\mathcal{O}_{X'_{s'},x'}} \kappa(x')$$

see Lemma 24.32.12. Whence $x' \in T'$ if and only if $x \in T$ by Lemma 24.34.14. The second part follows from the first because in that case T is the (open) set of points where f is G-unramified according to Lemma 24.34.14. ☐

Lemma 24.34.16. *Let $f : X \to Y$ be a morphism of schemes over S.*

(1) *If X is unramified over S, then f is unramified.*
(2) *If X is G-unramified over S and Y of finite type over S, then f is G-unramified.*

Proof. Assume that X is unramified over S. By Lemma 24.14.8 we see that f is locally of finite type. By assumption we have $\Omega_{X/S} = 0$. Hence $\Omega_{X/Y} = 0$ by Lemma 24.32.11. Thus f is unramified. If X is G-unramified over S and Y of finite type over S, then by Lemma 24.20.11 we see that f is locally of finite presentation and we conclude that f is G-unramified. ☐

Lemma 24.34.17. *Let S be a scheme. Let X, Y be schemes over S. Let $f, g : X \to Y$ be morphisms over S. Let $x \in X$. Assume that*

(1) *the structure morphism $Y \to S$ is unramified,*
(2) *$f(x) = g(x)$ in Y, say $y = f(x) = g(x)$, and*
(3) *the induced maps $f^\sharp, g^\sharp : \kappa(y) \to \kappa(x)$ are equal.*

Then there exists an open neighbourhood of x in X on which f and g are equal.

Proof. Consider the morphism $(f, g) : X \to Y \times_S Y$. By assumption (1) and Lemma 24.34.13 the inverse image of $\Delta_{Y/S}(Y)$ is open in X. And assumptions (2) and (3) imply that x is in this open subset. □

24.35. Étale morphisms

The Zariski topology of a scheme is a very coarse topology. This is particularly clear when looking at varieties over $\mathbf{C}$. It turns out that declaring an étale morphism to be the analogue of a local isomorphism in topology introduces a much finer topology. On varieties over $\mathbf{C}$ this topology gives rise to the ``correct'' betti numbers when computing cohomology with finite coefficients. Another observable is that if $f : X \to Y$ is an étale morphism of varieties over $\mathbf{C}$, and if x is a closed point of X, then f induces an isomorphism $\mathcal{O}^\wedge_{Y,f(x)} \to \mathcal{O}^\wedge_{X,x}$ of complete local rings.

In this section we start our study of these matters. In fact we deliberately restrict our discussion to a minimum since we will discuss more interesting results elsewhere. Recall that a ring map $R \to A$ is said to be *étale* if it is smooth and $\Omega_{A/R} = 0$, see Algebra, Definition 7.132.1.

Definition 24.35.1. Let $f : X \to S$ be a morphism of schemes.

(1) We say that f is *étale at $x \in X$* if there exists a affine open neighbourhood $Spec(A) = U \subset X$ of x and and affine open $Spec(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is étale.
(2) We say that f is *étale* if it is étale at every point of X.
(3) A morphism of affine schemes $f : X \to S$ is called *standard étale* if $X \to S$ is isomorphic to
$$Spec(R[x]_g/(f)) \to Spec(R)$$
where $R \to R[x]_g/(f)$ is a standard étale ring map, see Algebra, Definition 7.132.13, i.e., f is monic and f' invertible in $R[x]_g$.

A morphism is étale if and only if it is smooth of relative dimension 0 (see Definition 24.33.13). A pleasing feature of the definition is that the set of points where a morphism is étale is automatically open.

Note that there is no separation or quasi-compactness hypotheses in the definition. Hence the question of being étale is local in nature on the source. Here is the precise result.

Lemma 24.35.2. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *The morphism f is étale.*
(2) *For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is étale.*
(3) *There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$, $j \in J, i \in I_j$ is étale.*
(4) *There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring map $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i)$ is étale, for all $j \in J, i \in I_j$.*

Moreover, if f is étale then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \to V$ is étale.

Proof. This follows from Lemma 24.13.3 if we show that the property ``$R \to A$ is étale'' is local. We check conditions (a), (b) and (c) of Definition 24.13.1. These all follow from Algebra, Lemma 7.132.3. □

Lemma 24.35.3. *The composition of two morphisms which are étale is étale.*

Proof. In the proof of Lemma 24.35.2 we saw that being étale is a local property of ring maps. Hence the first statement of the lemma follows from Lemma 24.13.5 combined with the fact that being étale is a property of ring maps that is stable under composition, see Algebra, Lemma 7.132.3. □

Lemma 24.35.4. *The base change of a morphism which is étale is étale.*

Proof. In the proof of Lemma 24.35.2 we saw that being étale is a local property of ring maps. Hence the lemma follows from Lemma 24.13.5 combined with the fact that being étale is a property of ring maps that is stable under base change, see Algebra, Lemma 7.132.3. □

Lemma 24.35.5. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. Then f is étale at x if and only if f is smooth and unramified at x.*

Proof. This follows immediately from the definitions. □

Lemma 24.35.6. *An étale morphism is locally quasi-finite.*

Proof. By Lemma 24.35.5 an étale morphism is unramified. By Lemma 24.34.10 an unramified morphism is locally quasi-finite. □

Lemma 24.35.7. *Fibres of étale morphisms.*

(1) *Let X be a scheme over a field k. The structure morphism $X \to Spec(k)$ is étale if and only if X is a disjoint union of spectra of finite separable field extensions of k.*
(2) *If $f : X \to S$ is an étale morphism, then for every $s \in S$ the fibre X_s is a disjoint union of spectra of finite separable field extensions of $\kappa(s)$.*

Proof. You can deduce this from Lemma 24.34.11 via Lemma 24.35.5 above. Here is a direct proof.

We will use Algebra, Lemma 7.132.4. Hence it is clear that if X is a disjoint union of spectra of finite separable field extensions of k then $X \to Spec(k)$ is étale. Conversely, suppose that $X \to Spec(k)$ is étale. Then for any affine open $U \subset X$ we see that U is a finite disjoint union of spectra of finite separable field extensions of k. Hence all points of X are closed points (see Lemma 24.19.2 for example). Thus X is a discrete space and we win. □

The following lemma characterizes an étale morphism as a flat, finitely presented morphism with ``étale fibres''.

Lemma 24.35.8. *Let $f : X \to S$ be a morphism of schemes. If f is flat, locally of finite presentation, and for every $s \in S$ the fibre X_s is a disjoint union of spectra of finite separable field extensions of $\kappa(s)$, then f is étale.*

Proof. You can deduce this from Algebra, Lemma 7.132.7. Here is another proof.

By Lemma 24.35.7 a fibre X_s is étale and hence smooth over s. By Lemma 24.33.3 we see that $X \to S$ is smooth. By Lemma 24.34.12 we see that f is unramified. We conclude by Lemma 24.35.5. ☐

Lemma 24.35.9. *Any open immersion is étale.*

Proof. This is true because an open immersion is a local isomorphism. ☐

Lemma 24.35.10. *An étale morphism is syntomic.*

Proof. See Algebra, Lemma 7.126.10 and use that an étale morphism is the same as a smooth morphism of relative dimension 0. ☐

Lemma 24.35.11. *An étale morphism is locally of finite presentation.*

Proof. True because an étale ring map is of finite presentation by definition. ☐

Lemma 24.35.12. *An étale morphism is flat.*

Proof. Combine Lemmas 24.30.7 and 24.35.10. ☐

Lemma 24.35.13. *An étale morphism is open.*

Proof. Combine Lemmas 24.35.12, 24.35.11, and 24.24.9. ☐

The following lemma says locally any étale morphism is standard étale. This is actually kind of a tricky result to prove in complete generality. The tricky parts are hidden in the chapter on commutative algebra. Hence a standard étale morphism is a *local model* for a general étale morphism.

Lemma 24.35.14. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point. Set $s = f(x)$. The following are equivalent*

1. *The morphism f is étale at x.*
2. *There exist affine opens $U \subset X$, and $V \subset S$ such that $x \in U$, $f(U) \subset V$ and the induced morphism $f|_U : U \to V$ is standard étale (see Definition 24.35.1).*

Proof. Follows from the definitions and Algebra, Proposition 7.132.16. ☐

Here is a differential criterion of étaleness at a point. There are many variants of this result all of which may be useful at some point. We will just add them here as needed.

Lemma 24.35.15. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. Set $s = f(x)$. Assume f is locally of finite presentation. The following are equivalent:*

1. *The morphism f is étale at x.*
2. *The local ring map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat and the $\mathcal{O}_{X,x}$-module $\Omega_{X/S,x}$ is zero.*
3. *The local ring map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat and the $\kappa(x)$-vector space*
$$\Omega_{X_s/s,x} \otimes_{\mathcal{O}_{X_s,x}} \kappa(x) = \Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$$
is zero.
4. *The local ring map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat, we have $\mathfrak{m}_s\mathcal{O}_{X,x} = \mathfrak{m}_x$ and the field extension $\kappa(s) \subset \kappa(x)$ is finite separable.*
5. *There exist affine opens $U \subset X$, and $V \subset S$ such that $x \in U$, $f(U) \subset V$ and the induced morphism $f|_U : U \to V$ is standard smooth of relative dimension 0.*

(6) *There exist affine opens $Spec(A) = U \subset X$ and $Spec(R) = V \subset S$ with $x \in U$ corresponding to $\mathfrak{q} \subset A$, and $f(U) \subset V$ such that there exists a presentation*

$$A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$$

with

$$g = \det \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_2/\partial x_1 & \ldots & \partial f_n/\partial x_1 \\ \partial f_1/\partial x_2 & \partial f_2/\partial x_2 & \ldots & \partial f_n/\partial x_2 \\ \ldots & \ldots & \ldots & \ldots \\ \partial f_1/\partial x_n & \partial f_2/\partial x_n & \ldots & \partial f_n/\partial x_n \end{pmatrix}$$

mapping to an element of A not in $\mathfrak{q}$.

(7) *There exist affine opens $U \subset X$, and $V \subset S$ such that $x \in U$, $f(U) \subset V$ and the induced morphism $f|_U : U \to V$ is standard étale.*

(8) *There exist affine opens $Spec(A) = U \subset X$ and $Spec(R) = V \subset S$ with $x \in U$ corresponding to $\mathfrak{q} \subset A$, and $f(U) \subset V$ such that there exists a presentation*

$$A = R[x]_Q/(P) = R[x, 1/Q]/(P)$$

with $P, Q \in R[x]$, P monic and $P' = dP/dx$ mapping to an element of A not in $\mathfrak{q}$.

Proof. Use Lemma 24.35.14 and the definitions to see that (1) implies all of the other conditions. For each of the conditions (2) -- (7) combine Lemmas 24.33.14 and 24.34.14 to see that (1) holds by showing f is both smooth and unramified at x and applying Lemma 24.35.5. Some details omitted. □

Lemma 24.35.16. *A morphism is étale at a point if and only if it is flat and G-unramified at that point. A morphism is étale if and only if it is flat and G-unramified.*

Proof. This is clear from Lemmas 24.35.15 and 24.34.14. □

Lemma 24.35.17. *Let $f : X \to S$ be a morphism of schemes. Assume f locally of finite type. Formation of the set*

$$T = \{x \in X \mid X_{f(x)} \text{ is étale over } \kappa(f(x)) \text{ at } x\}$$

commutes with arbitrary base change: For any morphism $g : S' \to S$, consider the base change $f' : X' \to S'$ of f and the projection $g' : X' \to X$. Then the corresponding set T' for the morphism f' is equal to $T' = (g')^{-1}(T)$. In particular, if f is assumed locally of finite presentation and flat then the same holds for the open set of points where f is étale.

Proof. Combine Lemmas 24.35.16 and 24.34.15. □

Our proof of the following lemma is somewhat complicated. It uses the ``Critère de platitude par fibres" to see that a morphism $X \to Y$ over S between schemes étale over S is automatically flat. The details are in the chapter on commutative algebra.

Lemma 24.35.18. *Let $f : X \to Y$ be a morphism of schemes over S. If X and Y are étale over S, then f is étale.*

Proof. See Algebra, Lemma 7.132.8. □

Lemma 24.35.19. *Let*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & S & \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume that

(1) *f is surjective, and étale,*
(2) *p is étale, and*
(3) *q is locally of finite presentation*[10].

Then q is étale.

Proof. By Lemma 24.33.19 we see that q is smooth. Thus we only need to see that q has relative dimension 0. This follows from Lemma 24.27.2 and the fact that f and p have relative dimension 0. $\square$

A final characterization of smooth morphisms is that a smooth morphism $f : X \to S$ is locally the composition of an étale morphism by a projection $\mathbf{A}^d_S \to S$.

Lemma 24.35.20. *Let $\varphi : X \to Y$ be a morphism of schemes. Let $x \in X$. If φ is smooth at x, then there exist exist an integer $d \geq 0$ and affine opens $V \subset Y$ and $U \subset X$ with $x \in U$ and $\varphi(U) \subset V$ such that there exists a commutative diagram*

$$\begin{tikzcd} X \arrow[d] & U \arrow[l] \arrow[r, "\pi"'] \arrow[d] & \mathbf{A}^d_V \arrow[dl] \\ Y & V \arrow[l] & \end{tikzcd}$$

where π is étale.

Proof. By Lemma 24.33.11 we can find affine opens U and V as in the lemma such that $\varphi|_U : U \to V$ is standard smooth. Write $U = Spec(A)$ and $V = Spec(R)$ so that we can write

$$A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$$

with

$$g = \det \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_2/\partial x_1 & \ldots & \partial f_c/\partial x_1 \\ \partial f_1/\partial x_2 & \partial f_2/\partial x_2 & \ldots & \partial f_c/\partial x_2 \\ \ldots & \ldots & \ldots & \ldots \\ \partial f_1/\partial x_c & \partial f_2/\partial x_c & \ldots & \partial f_c/\partial x_c \end{pmatrix}$$

mapping to an invertible element of A. Then it is clear that $R[x_{c+1}, \ldots, x_n] \to A$ is standard smooth of relative dimension 0. Hence it is smooth of relative dimension 0. In other words the ring map $R[x_{c+1}, \ldots, x_n] \to A$ is étale. As $\mathbf{A}^{n-c}_V = Spec(R[x_{c+1}, \ldots, x_n])$ the lemma with $d = n - c$. $\square$

24.36. Relatively ample sheaves

Let X be a scheme and $\mathcal{L}$ an invertible sheaf on X. Then $\mathcal{L}$ is ample on X if X is quasi-compact and every point of X is contained in an affine open of the form X_s, where $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ and $n \geq 1$, see Properties, Definition 23.23.1. We relativize this as follows.

Definition 24.36.1. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. We say $\mathcal{L}$ is *relatively ample*, or *f-relatively ample*, or *ample on X/S*, or *f-ample* if $f : X \to S$ is quasi-compact, and if for every affine open $V \subset S$ the restriction of $\mathcal{L}$ to the open subscheme $f^{-1}(V)$ of X is ample.

We note that the existence of a relatively ample sheaf on X does not force the morphism $X \to S$ to be of finite type.

[10] In fact this is implied by (1) and (2), see Descent, Lemma 31.10.3. Moreover, it suffices to assume that f is surjective, flat and locally of finite presentation, see Descent, Lemma 31.10.5.

Lemma 24.36.2. *Let $X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $n \geq 1$. Then $\mathcal{L}$ is f-ample if and only if $\mathcal{L}^{\otimes n}$ is f-ample.*

Proof. This follows from Properties, Lemma 23.23.2. □

Lemma 24.36.3. *Let $f : X \to S$ be a morphism of schemes. If there exists an f-ample invertible sheaf, then f is separated.*

Proof. Being separated is local on the base (see Schemes, Lemma 21.21.8 for example; it also follows easily from the definition). Hence we may assume S is affine and X has an ample invertible sheaf. In this case the result follows from Properties, Lemma 23.23.10 and Constructions, Lemma 22.8.8. □

There are many ways to charactarize relatively ample invertible sheaves, by relativizing any of the list of equivalent conditions in Properties, Proposition 23.23.12. We will add these here as needed.

Lemma 24.36.4. *Let $f : X \to S$ be a quasi-compact morphism of schemes. Let $\mathcal{L}$ be an invertible sheaf on X. The following are equivalent:*

(1) *The invertible sheaf $\mathcal{L}$ is f-ample.*
(2) *There exists an open covering $S = \bigcup V_i$ such that each $\mathcal{L}|_{f^{-1}(V_i)}$ is ample relative to $f^{-1}(V_i) \to V_i$.*
(3) *There exists an affine open covering $S = \bigcup V_i$ such that each $\mathcal{L}|_{f^{-1}(V_i)}$ is ample.*
(4) *There exists a quasi-coherent graded $\mathcal{O}_S$-algebra $\mathcal{A}$ and a map of graded $\mathcal{O}_X$-algebras $\psi : f^*\mathcal{A} \to \bigoplus_{d \geq 0} \mathcal{L}^{\otimes d}$ such that $U(\psi) = X$ and*
$$r_{\mathcal{L},\psi} : X \longrightarrow \underline{Proj}_S(\mathcal{A})$$
is an open immersion (see Constructions, Lemma 22.18.1 for notation).
(5) *The morphism f is quasi-separated and part (4) above holds with $\mathcal{A} = f_*(\bigoplus_{d \geq 0} \mathcal{L}^{\otimes d})$ and ψ the adjunction mapping.*
(6) *Same as (4) but just requiring $r_{\mathcal{L},\psi}$ to be an immersion.*

Proof. It is immediate from the definition that (1) implies (2) and (2) implies (3). It is clear that (5) implies (4).

Assume (3) holds for the affine open covering $S = \bigcup V_i$. We are going to show (5) holds. Since each $f^{-1}(V_i)$ has an ample invertible sheaf we see that $f^{-1}(V_i)$ is separated (see Properties, Lemma 23.23.10 and Constructions, Lemma 22.8.8). Hence f is separated. By Schemes, Lemma 21.24.1 we see that $\mathcal{A} = f_*(\bigoplus_{d \geq 0} \mathcal{L}^{\otimes d})$ is a quasi-coherent graded $\mathcal{O}_S$-algebra. Denote $\psi : f^*\mathcal{A} \to \bigoplus_{d \geq 0} \mathcal{L}^{\otimes d}$ the adjunction mapping. The description of the open $U(\psi)$ in Constructions, Section 22.18 and the definition of ampleness of $\mathcal{L}|_{f^{-1}(V_i)}$ show that $U(\psi) = X$. Moreover, Constructions, Lemma 22.18.1 part (3) shows that the restriction of $r_{\mathcal{L},\psi}$ to $f^{-1}(V_i)$ is the same as the morphism from Properties, Lemma 23.23.8 which is an open immersion according to Properties, Lemma 23.23.10. Hence (5) holds.

Let us show that (4) implies (1). Assume (4). Denote $\pi : \underline{\mathrm{Proj}}_S(\mathcal{A}) \to S$ the structure morphism. Choose $V \subset S$ affine open. By Constructions, Definition 22.16.7 we see that $\pi^{-1}(V) \subset \underline{\mathrm{Proj}}_S(\mathcal{A})$ is equal to $\mathrm{Proj}(A)$ where $A = \mathcal{A}(V)$ as a graded ring. Hence $r_{\mathcal{L},\psi}$ maps $f^{-1}(V)$ isomorphically onto a quasi-compact open of $\mathrm{Proj}(A)$. Moreover, $\mathcal{L}^{\otimes d}$ is isomorphic to the pullback of $\mathcal{O}_{\mathrm{Proj}(A)}(d)$ for some $d \geq 1$. (See part (3) of Constructions, Lemma 22.18.1 and the final statement of Constructions, Lemma 22.14.1.) This implies that $\mathcal{L}|_{f^{-1}(V)}$ is ample by Properties, Lemmas 23.23.11 and 23.23.2.

Assume (6). By the equivalence of (1) - (5) above we see that the property of being relatively ample on X/S is local on S. Hence we may assume that S is affine, and we have to show that $\mathcal{L}$ is ample on X. In this case the morphism $r_{\mathcal{L},\psi}$ is identified with the morphism, also denoted $r_{\mathcal{L},\psi} : X \to \text{Proj}(A)$ associated to the map $\psi : A = \mathcal{A}(V) \to \Gamma_*(X, \mathcal{L})$. (See references above.) As above we also see that $\mathcal{L}^{\otimes d}$ is the pullback of the sheaf $\mathcal{O}_{\text{Proj}(A)}(d)$ for some $d \geq 1$. Moreover, since X is quasi-compact we see that X gets identified with a closed subscheme of a quasi-compact open subscheme $Y \subset \text{Proj}(A)$. By Constructions, Lemma 22.10.6 (see also Properties, Lemma 23.23.11) we see that $\mathcal{O}_Y(d')$ is an ample invertible sheaf on Y for some $d' \geq 1$. Since the restriction of an ample sheaf to a closed subscheme is ample, see Properties, Lemma 23.23.3 we conclude that the pullback of $\mathcal{O}_Y^{d'}$ is ample. Combining these results with Properties, Lemma 23.23.2 we conclude that $\mathcal{L}$ is ample as desired. $\square$

Lemma 24.36.5. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume S affine. Then $\mathcal{L}$ is f-relatively ample if and only if $\mathcal{L}$ is ample on X.*

Proof. Immediate from Lemma 24.36.4 and the definitions. $\square$

24.37. Very ample sheaves

Recall that given a quasi-coherent sheaf $\mathcal{E}$ on a scheme S the *projective bundle* associated to $\mathcal{E}$ is the morphism $\mathbf{P}(\mathcal{E}) \to S$, where $\mathbf{P}(\mathcal{E}) = \underline{\text{Proj}}_S(\text{Sym}(\mathcal{E}))$, see Constructions, Definition 22.20.1.

Definition 24.37.1. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. We say $\mathcal{L}$ is *relatively very ample* or more precisely *f-relatively very ample*, or *very ample on X/S*, or *f-very ample* if there exist a quasi-coherent $\mathcal{O}_S$-module $\mathcal{E}$ and an immersion $i : X \to \mathbf{P}(\mathcal{E})$ over S such that $\mathcal{L} \cong i^*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$.

Since there is no assumption of quasi-compactness in this definition it is not true in general that a relatively very ample invertible sheaf is a relatively ample invertible sheaf.

Lemma 24.37.2. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. If f is quasi-compact and $\mathcal{L}$ is a relatively very ample invertible sheaf, then $\mathcal{L}$ is a relatively ample invertible sheaf.*

Proof. By definition there exists quasi-coherent $\mathcal{O}_S$-module $\mathcal{E}$ and an immersion $i : X \to \mathbf{P}(\mathcal{E})$ over S such that $\mathcal{L} \cong i^*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$. Set $\mathcal{A} = \text{Sym}(\mathcal{E})$, so $\mathbf{P}(\mathcal{E}) = \underline{\text{Proj}}_S(\mathcal{A})$ by definition. The graded $\mathcal{O}_S$-algebra $\mathcal{A}$ comes equipped with a map

$$\psi : \mathcal{A} \to \bigoplus\nolimits_{n \geq 0} \pi_*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(n) \to \bigoplus\nolimits_{n \geq 0} f_*\mathcal{L}^{\otimes n}$$

where the second arrow uses the identification $\mathcal{L} \cong i^*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$. By adjointness of f_* and f^* we get a morphism $\psi : f^*\mathcal{A} \to \bigoplus_{n \geq 0} \mathcal{L}^{\otimes n}$. We omit the verification that the morphism $r_{\mathcal{L},\psi}$ associated to this map is exactly the immersion i. Hence the result follows from part (6) of Lemma 24.36.4. $\square$

To arrive at the correct converse of this lemma we ask whether given a relatively ample invertible sheaf $\mathcal{L}$ there exists an integer $n \geq 1$ such that $\mathcal{L}^{\otimes n}$ is relatively very ample? In general this is false. There are several things that prevent this from being true:

1. Even if S is affine, it can happen that no finite integer n works because $X \to S$ is not of finite type, see Example 24.37.3.

(2) The base not being quasi-compact means the result can be prevented from being true even with f finite type. Namely, given a field k there exists a scheme X_d of finite type over k with an ample invertible sheaf $\mathcal{O}_{X_d}(1)$ so that the smallest tensor power of $\mathcal{O}_{X_d}(1)$ which is very ample is the dth power. See Example 24.37.4. Taking f to be the disjoint union of the schemes X_d mapping to the disjoint union of copies of $Spec(k)$ gives an example.

To see our version of the converse take a look at Lemma 24.38.5 below. We will do some preliminary work before proving it.

Example 24.37.3. Let k be a field. Consider the graded k-algebra

$$A = k[U, V, Z_1, Z_2, Z_3, \ldots]/I \quad \text{with} \quad I = (U^2 - Z_1^2, U^4 - Z_2^2, U^6 - Z_3^2, \ldots)$$

with grading given by $\deg(U) = \deg(V) = \deg(Z_1) = 1$ and $\deg(Z_d) = d$. Note that $X = \text{Proj}(A)$ is covered by $D_+(U)$ and $D_+(V)$. Hence the sheaves $\mathcal{O}_X(n)$ are all invertible and isomorphic to $\mathcal{O}_X(1)^{\otimes n}$. In particular $\mathcal{O}_X(1)$ is ample and f-ample for the morphism $f : X \to Spec(k)$. We claim that no power of $\mathcal{O}_X(1)$ is f-relatively very ample. Namely, it is easy to see that $\Gamma(X, \mathcal{O}_X(n))$ is the degree n summand of the algebra A. Hence if $\mathcal{O}_X(n)$ were very ample, then X would be a closed subscheme of a projective space over k and hence of finite type over k. On the other hand $D_+(V)$ is the spectrum of $k[t, t_1, t_2, \ldots]/(t^2 - t_1^2, t^4 - t_2^2, t^6 - t_3^2, \ldots)$ which is not of finite type over k.

Example 24.37.4. Let k be an infinite field. Let $\lambda_1, \lambda_2, \lambda_3, \ldots$ be pairwise distinct elements of k^*. (This is not strictly necessary, and in fact the example works perfectly well even if all λ_i are equal to 1.) Consider the graded k-algebra

$$A_d = k[U, V, Z]/I_d \quad \text{with} \quad I_d = (Z^2 - \prod\nolimits_{i=1}^{2d}(U - \lambda_i V)).$$

with grading given by $\deg(U) = \deg(V) = 1$ and $\deg(Z) = d$. Then $X_d = \text{Proj}(A_d)$ has ample invertible sheaf $\mathcal{O}_{X_d}(1)$. We claim that if $\mathcal{O}_{X_d}(n)$ is very ample, then $n \geq d$. The reason for this is that Z has degree d, and hence $\Gamma(X_d, \mathcal{O}_{X_d}(n)) = k[U, V]_n$ for $n < d$. Details omitted.

Lemma 24.37.5. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible sheaf on X. If $\mathcal{L}$ is relatively very ample on X/S then f is separated.*

Proof. Being separated is local on the base (see Schemes, Section 21.21). An immersion is separated (see Schemes, Lemma 21.23.7). Hence the lemma follows since locally X has an immersion into the homogeneous spectrum of a graded ring which is separated, see Constructions, Lemma 22.8.8. □

Lemma 24.37.6. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible sheaf on X. Assume f is quasi-compact. The following are equivalent*

(1) *$\mathcal{L}$ is relatively very ample on X/S,*
(2) *there exists an open covering $S = \bigcup V_j$ such that $\mathcal{L}|_{f^{-1}(V_j)}$ is relatively very ample on $f^{-1}(V_j)/V_j$ for all j,*
(3) *there exists a quasi-coherent sheaf of graded $\mathcal{O}_S$-algebras $\mathcal{A}$ generated in degree 1 over $\mathcal{O}_S$ and a map of graded $\mathcal{O}_X$-algebras $\psi : f^*\mathcal{A} \to \bigoplus_{n\geq 0} \mathcal{L}^{\otimes n}$ such that $f^*\mathcal{A}_1 \to \mathcal{L}$ is surjective and the associated morphism $r_{\mathcal{L},\psi} : X \to \underline{Proj}_S(\mathcal{A})$ is an immersion, and*
(4) *f is quasi-separated, the canonical map $\psi : f^*f_*\mathcal{L} \to \mathcal{L}$ is surjective, and the associated map $r_{\mathcal{L},\psi} : X \to \mathbf{P}(f_*\mathcal{L})$ is an immersion.*

Proof. It is clear that (1) implies (2). It is also clear that (4) implies (1); the hypothesis of quasi-spearation in (4) is used to garantee that $f_*\mathcal{L}$ is quasi-coherent via Schemes, Lemma 21.24.1.

Assume (2). We will prove (4). Let $S = \bigcup V_j$ be an open covering as in (2). Set $X_j = f^{-1}(V_j)$ and $f_j : X_j \to V_j$ the restriction of f. We see that f is separated by Lemma 24.37.5 (as being separated is local on the base). Consider the map $\psi : f^*f_*\mathcal{L} \to \mathcal{L}$. On each V_j there exists a quasi-coherent sheaf $\mathcal{E}_j$ and an embedding $i : X_j \to \mathbf{P}(\mathcal{E}_j)$ with $\mathcal{L}_{X_j} \cong i^*\mathcal{O}_{\mathbf{P}(\mathcal{E}_j)}(1)$. In other words there is a map $\mathcal{E}_j \to (f_*\mathcal{L})|_{X_j}$ such that the composition

$$f_j^*\mathcal{E}_j \to (f^*f_*\mathcal{L})|_{X_j} \to \mathcal{L}|_{X_j}$$

is surjective. Hence we conclude that ψ is surjective. Let $r_{\mathcal{L},\psi} : X \to \mathbf{P}(f_*\mathcal{L})$ be the associated morphism. Using the maps $\mathcal{E}_j \to (f_*\mathcal{L})|_{X_j}$ we see that there is a factorization

$$X_j \xrightarrow{r_{\mathcal{L},\psi}} \mathbf{P}(f_*\mathcal{L})|_{V_j} \longrightarrow \mathbf{P}(\mathcal{E}_j)$$

which shows that $r_{\mathcal{L},\psi}$ is an immersion.

At this point we see that (1), (2) and (4) are equivalent. Clearly (4) implies (3). Assume (3). We will prove (1). Let $\mathcal{A}$ be a quasi-coherent sheaf of graded $\mathcal{O}_S$-algebras generated in degree 1 over $\mathcal{O}_S$. Consider the map of graded $\mathcal{O}_S$-algebras $\text{Sym}(\mathcal{A}_1) \to \mathcal{A}$. This is surjective by hypothesis and hence induces a closed immersion

$$\underline{\text{Proj}}_S(\mathcal{A}) \longrightarrow \mathbf{P}(\mathcal{A}_1)$$

which pulls back $\mathcal{O}(1)$ to $\mathcal{O}(1)$, see (insert future reference here -- but see Constructions, Lemma 22.11.3 for the case where S is affine). Hence it is clear that (3) implies (1). □

24.38. Ample and very ample sheaves relative to finite type morphisms

In fact most of the material in this section is about the notion of a (quasi-)projective morphism which we have not defined yet.

Lemma 24.38.1. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible sheaf on X. Assume that*

1. *the invertible sheaf $\mathcal{L}$ is very ample on X/S,*
2. *the morphism $X \to S$ is of finite type, and*
3. *S is affine.*

Then there exists an $n \geq 0$ and an immersion $i : X \to \mathbf{P}^n_S$ over S such that $\mathcal{L} \cong i^*\mathcal{O}_{\mathbf{P}^n_S}(1)$.

Proof. Assume (1), (2) and (3). Condition (3) means $S = Spec(R)$ for some ring R. Condition (1) means by definition there exists a quasi-coherent $\mathcal{O}_S$-module $\mathcal{E}$ and an immersion $\alpha : X \to \mathbf{P}(\mathcal{E})$ such that $\mathcal{L} = \alpha^*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$. Write $\mathcal{E} = \widetilde{M}$ for some R-module M. Thus we have

$$\mathbf{P}(\mathcal{E}) = \text{Proj}(\text{Sym}_R(M)).$$

Since α is an immersion, and since the topology of $\text{Proj}(\text{Sym}_R(M))$ is generated by the standard opens $D_+(f)$, $f \in \text{Sym}^d_R(M)$, $d \geq 1$, we can find for each $x \in X$ an $f \in \text{Sym}^d_R(M)$, $d \geq 1$, with $\alpha(x) \in D_+(f)$ such that

$$\alpha|_{\alpha^{-1}(D_+(f))} : \alpha^{-1}(D_+(f)) \to D_+(f)$$

is a closed immersion. Condition (2) implies X is quasi-compact. Hence we can find a finite collection of elements $f_j \in \text{Sym}^{d_j}_R(M)$, $d_j \geq 1$ such that for each $f = f_j$ the displayed map above is a closed immersion and such that $\alpha(X) \subset \bigcup D_+(f_j)$. Write $U_j = \alpha^{-1}(D_+(f_j))$. Note that U_j is affine as a closed subscheme of the affine scheme $D_+(f_j)$. Write $U_j = Spec(A_j)$. Condition (2) also implies that A_j is of finite type over R, see Lemma 24.14.2. Choose finitely many $x_{j,k} \in A_j$ which generate A_j as a R-algebra. Since $\alpha|_{U_j}$ is a closed immersion we see that $x_{j,k}$ is the image of an element

$$f_{j,k}/f_j^{e_{j,k}} \in \mathrm{Sym}_R(M)_{(f_j)} = \Gamma(D_+(f_j), \mathcal{O}_{\mathrm{Proj}(\mathrm{Sym}_R(M))}).$$

Finally, choose $n \geq 1$ and elements $y_0, \ldots, y_n \in M$ such that each of the polynomials $f_j, f_{j,k} \in \mathrm{Sym}_R(M)$ is a polynomial in the elements y_t with coefficients in R. Consider the graded ring map

$$\psi : R[Y_0, \ldots, Y_n] \longrightarrow \mathrm{Sym}_R(M), \quad Y_i \longmapsto y_i.$$

Denote F_j, $F_{j,k}$ the elements of $R[Y_0, \ldots, Y_n]$ such that $\psi(F_j) = f_j$ and $\psi(F_{j,k}) = f_{j,k}$. By Constructions, Lemma 22.11.1 we obtain an open subscheme

$$U(\psi) \subset \mathrm{Proj}(\mathrm{Sym}_R(M))$$

and a morphism $r_\psi : U(\psi) \to \mathbf{P}^n_R$. This morphism satisfies $r_\psi^{-1}(D_+(F_j)) = D_+(f_j)$, and hence we see that $\alpha(X) \subset U(\psi)$. Moreover, it is clear that

$$i = r_\psi \circ \alpha : X \longrightarrow \mathbf{P}^n_R$$

is still an immersion since $i^\sharp(F_{j,k}/F_j^{e_{j,k}}) = x_{j,k} \in A_j = \Gamma(U_j, \mathcal{O}_X)$ by construction. Moreover, the morphism r_ψ comes equipped with a map $\theta : r_\psi^*\mathcal{O}_{\mathbf{P}^n_R}(1) \to \mathcal{O}_{\mathrm{Proj}(\mathrm{Sym}_R(M))}(1)|_{U(\psi)}$ which is an isomorphism in this case (for construction θ see lemma cited above; some details omitted). Since the original map α was assumed to have the property that $\mathcal{L} = \alpha^*\mathcal{O}_{\mathrm{Proj}(\mathrm{Sym}_R(M))}(1)$ we win. □

Lemma 24.38.2. *Let $\pi : X \to S$ be a morphism of schemes. Assume that X is quasi-affine and that π is locally of finite type. Then there exist $n \geq 0$ and an immersion $i : X \to \mathbf{A}^n_S$ over S.*

Proof. Let $A = \Gamma(X, \mathcal{O}_X)$. By assumption X is quasi-compact and is identified with an open subscheme of $Spec(A)$, see Properties, Lemma 23.15.4. Moreover, the set of opens X_f, for those $f \in A$ such that X_f is affine, forms a basis for the topology of X, see the proof of Properties, Lemma 23.15.4. Hence we can find a finite number of $f_j \in A$, $j = 1, \ldots, m$ such that $X = \bigcup X_{f_j}$, and such that $\pi(X_{f_j}) \subset V_j$ for some affine open $V_j \subset S$. By Lemma 24.14.2 the ring maps $\mathcal{O}(V_j) \to \mathcal{O}(X_{f_j}) = A_{f_j}$ are of finite type. Thus we may choose $a_1, \ldots, a_N \in A$ such that the elements $a_1, \ldots, a_N, f_1, \ldots, f_m, 1/f_j$ generate A_{f_j} over $\mathcal{O}(V_j)$ for each j. Take $n = N + m$ and let

$$i : X \longrightarrow \mathbf{A}^n_S$$

be the morphism given by the global sections $a_1, \ldots, a_n, f_1, \ldots, f_n$ of the structure sheaf of X. Let $D(x_j) \subset \mathbf{A}^n_S$ be the open subscheme where the jth coordinate function is nonzero. Then it is clear that $i^{-1}(D(x_j))$ is X_{f_j} and that the induced morphism $X_{f_j} \to D(x_j)$ factors through the affine open $Spec(\mathcal{O}(V_j)[x_1, \ldots, x_n, 1/x_j])$ of $D(x_j)$. Since the ring map $\mathcal{O}(V_j)[x_1, \ldots, x_n, 1/x_j] \to A_{f_j}$ is surjective by construction we conclude that the restriction of i to X_{f_j} is an immersion as desired. □

Lemma 24.38.3. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible sheaf on X. Assume that*

(1) *the invertible sheaf $\mathcal{L}$ is ample on X, and*

(2) *the morphism $X \to S$ is locally of finite type.*

Then there exists a $d_0 \geq 1$ such that for every $d \geq d_0$ there exists an $n \geq 0$ and an immersion $i : X \to \mathbf{P}^n_S$ over S such that $\mathcal{L}^{\otimes d} \cong i^*\mathcal{O}_{\mathbf{P}^n_S}(1)$.

Proof. Let $A = \Gamma_*(X, \mathcal{L}) = \bigoplus_{d \geq 0} \Gamma(X, \mathcal{L}^{\otimes d})$. By Properties, Proposition 23.23.12 the set of affine opens X_a with $a \in A_+$ homogeneous forms a basis for the topology of X. Hence we can find finitely many such elements $a_0, \ldots, a_n \in A_+$ such that

(1) we have $X = \bigcup_{i=0,\ldots,n} X_{a_i}$,
(2) each X_{a_i} is affine, and
(3) each X_{a_i} maps into an affine open $V_i \subset S$.

By Lemma 24.14.2 we see that the ring maps $\mathcal{O}_S(V_i) \to \mathcal{O}_X(X_{a_i})$ are of finite type. Hence we can find finitely many elements $f_{ij} \in \mathcal{O}_X(X_{a_i})$, $j = 1, \ldots, n_i$ which generate $\mathcal{O}_X(X_{a_i})$ as an $\mathcal{O}_S(V_i)$-algebra. By Properties, Lemma 23.23.5 we may write each f_{ij} as $a_{ij}/a_i^{e_{ij}}$ for some $a_{ij} \in A_+$ homogeneous. Let N be a positive integer which is a common multiple of all the degrees of the elements a_i, a_{ij}. Consider the elements

$$a_i^{N/\deg(a_i)},\ a_{ij}a_i^{(N/\deg(a_i))-e_{ij}} \in A_N.$$

By construction these generate the invertible sheaf $\mathcal{L}^{\otimes N}$ over X. Hence they give rise to a morphism

$$j : X \longrightarrow \mathbf{P}^m_S \quad \text{with } m = n + \sum n_i$$

over S, see Constructions, Lemma 22.13.1 and Definition 22.13.2. Moreover, $j^*\mathcal{O}_{\mathbf{P}_S}(1) = \mathcal{L}^{\otimes N}$. We name the homogeneous coordinates $T_0, \ldots, T_n, T_{ij}$ instead of $T_0, \ldots, T_m$. For $i = 0, \ldots, n$ we have $i^{-1}(D_+(T_i)) = X_{a_i}$. Moreover, pulling back the element T_{ij}/T_i via $j^\sharp$ we get the element $f_{ij} \in \mathcal{O}_X(X_{a_i})$. Hence the morphism j restricted to X_{a_i} gives a closed immersion of X_{a_i} into the affine open $D_+(T_i) \cap \mathbf{P}^m_{V_i}$ of $\mathbf{P}^N_S$. Hence we conclude that the morphism j is an immersion. This implies the lemma holds for some d and n which is enough in virtually all applications.

This proves that for one $d_2 \geq 1$ (namely $d_2 = N$ above), some $m \geq 0$ there exists some immersion $j : X \to \mathbf{P}^m_S$ given by global sections $s'_0, \ldots, s'_m \in \Gamma(X, \mathcal{L}^{\otimes d_2})$. By Properties, Proposition 23.23.12 we know there exists an integer d_1 such that $\mathcal{L}^{\otimes d}$ is globally generated for all $d \geq d_1$. Set $d_0 = d_1 + d_2$. We claim that the lemma holds with this value of d_0. Namely, given an integer $d \geq d_0$ we may choose $s''_1, \ldots, s''_t \in \Gamma(X, \mathcal{L}^{\otimes d - d_2})$ which generate $\mathcal{L}^{\otimes d - d_2}$ over X. Set $n = (m + 1)t$ and denote $s_0, \ldots, s_n$ the collection of sections $s'_\alpha s''_\beta$, $\alpha = 0, \ldots, m$, $\beta = 1, \ldots, t$. These generate $\mathcal{L}^{\otimes d}$ over X and therefore define a morphism

$$i : X \longrightarrow \mathbf{P}^n_S$$

such that $i^*\mathcal{O}_{\mathbf{P}^n_S}(1) \cong \mathcal{L}^{\otimes d}$. We omit the verification that since j was an immersion also the morphism i so obtained is an immersion also. (Hint: Segre embedding.) □

Lemma 24.38.4. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume S affine and f of finite type. The following are equivalent*

(1) *$\mathcal{L}$ is ample on X,*
(2) *$\mathcal{L}$ is f-ample,*
(3) *$\mathcal{L}^{\otimes d}$ is f-very ample for some $d \geq 1$,*
(4) *$\mathcal{L}^{\otimes d}$ is f-very ample for all $d \gg 1$,*

(5) *for some* $d \geq 1$ *there exist* $n \geq 1$ *and an immersion* $i : X \to \mathbf{P}^n_S$ *such that* $\mathcal{L}^{\otimes d} \cong i^*\mathcal{O}_{\mathbf{P}^n_S}(1)$, *and*
(6) *for all* $d \gg 1$ *there exist* $n \geq 1$ *and an immersion* $i : X \to \mathbf{P}^n_S$ *such that* $\mathcal{L}^{\otimes d} \cong i^*\mathcal{O}_{\mathbf{P}^n_S}(1)$.

Proof. The equivalence of (1) and (2) is Lemma 24.36.5. The implication (2) ⇒ (6) is Lemma 24.38.3. Trivially (6) implies (5). As $\mathbf{P}^n_S$ is a projective bundle over S (see Constructions, Lemma 22.20.4) we see that (5) implies (3) and (6) implies (4) from the definition of a relatively very ample sheaf. Trivially (4) implies (3). To finish we have to show that (3) implies (2) which follows from Lemma 24.37.2 and Lemma 24.36.2. □

Lemma 24.38.5. *Let* $f : X \to S$ *be a morphism of schemes. Let* $\mathcal{L}$ *be an invertible* $\mathcal{O}_X$*-module. Assume* S *quasi-compact and* f *of finite type. The following are equivalent*

(1) $\mathcal{L}$ *is* f*-ample,*
(2) $\mathcal{L}^{\otimes d}$ *is* f*-very ample for some* $d \geq 1$,
(3) $\mathcal{L}^{\otimes d}$ *is* f*-very ample for all* $d \gg 1$.

Proof. Trivially (3) implies (2). Lemma 24.37.2 garantees that (2) implies (1) since a morphism of finite type is quasi-compact by definition. Assume that $\mathcal{L}$ is f-ample. Choose a finite affine open covering $S = V_1 \cup \ldots \cup V_m$. Write $X_i = f^{-1}(V_i)$. By Lemma 24.38.4 above we see there exists a d_0 such that $\mathcal{L}^{\otimes d}$ is relatively very ample on X_i/V_i for all $d \geq d_0$. Hence we conclude (1) implies (3) by Lemma 24.37.6. □

The following two lemmas provide the most used and most useful characterizations of relatively very ample and relatively ample invertible sheaves when the morphism is of finite type.

Lemma 24.38.6. *Let* $f : X \to S$ *be a morphism of schemes. Let* $\mathcal{L}$ *be an invertible sheaf on* X. *Assume* f *is of finite type. The following are equivalent:*

(1) $\mathcal{L}$ *is* f*-relatively very ample, and*
(2) *there exist an open covering* $S = \bigcup V_j$, *for each* j *an integer* n_j, *and immersions*

$$i_j : X_j = f^{-1}(V_j) = V_j \times_S X \longrightarrow \mathbf{P}^{n_j}_{V_j}$$

over V_j *such that* $\mathcal{L}|_{X_j} \cong i_j^*\mathcal{O}_{\mathbf{P}^{n_j}_{V_j}}(1)$.

Proof. We see that (1) implies (2) by taking an affine open covering of S and applying Lemma 24.38.1 to each of the restrictions of f and $\mathcal{L}$. We see that (2) implies (1) by Lemma 24.37.6. □

Lemma 24.38.7. *Let* $f : X \to S$ *be a morphism of schemes. Let* $\mathcal{L}$ *be an invertible sheaf on* X. *Assume* f *is of finite type. The following are equivalent:*

(1) $\mathcal{L}$ *is* f*-relatively ample, and*
(2) *there exist an open covering* $S = \bigcup V_j$, *for each* j *an integers* $d_j \geq 1$, $n_j \geq 0$, *and immersions*

$$i_j : X_j = f^{-1}(V_j) = V_j \times_S X \longrightarrow \mathbf{P}^{n_j}_{V_j}$$

over V_j *such that* $\mathcal{L}^{\otimes d_j}|_{X_j} \cong i_j^*\mathcal{O}_{\mathbf{P}^{n_j}_{V_j}}(1)$.

Proof. We see that (1) implies (2) by taking an affine open covering of S and applying Lemma 24.38.4 to each of the restrictions of f and $\mathcal{L}$. We see that (2) implies (1) by Lemma 24.36.4. □

24.39. Quasi-projective morphisms

The discussion in the previous section suggests the following definitions. We take our definition of quasi-projective from [**DG67**]. The version with the letter ``H'' is the definition in [**Har77**].

Definition 24.39.1. Let $f : X \to S$ be a morphism of schemes.

1. We say f is *quasi-projective* if f is of finite type and there exists an f-relatively ample invertible $\mathcal{O}_X$-module.
2. We say f is *H-quasi-projective* if f if there exists a quasi-compact immersion $X \to \mathbf{P}^n_S$ over S for some n.[11]
3. We say f is *locally quasi-projective* if there exists an open covering $S = \bigcup V_j$ such that each $f^{-1}(V_j) \to V_j$ is quasi-projective.

As this definition suggests the property of being quasi-projective is not local on S.

Lemma 24.39.2. *Let $f : X \to S$ be a morphism of schemes. If f is quasi-projective, or H-quasi-projective or locally quasi-projective, then f is separated of finite type.*

Proof. Omitted. □

Lemma 24.39.3. *A H-quasi-projective morphism is quasi-projective.*

Proof. Omitted. □

Lemma 24.39.4. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

1. *The morphism f is locally quasi-projective.*
2. *There exists an open covering $S = \bigcup V_j$ such that each $f^{-1}(V_j) \to V_j$ is H-quasi-projective.*

Proof. By Lemma 24.39.3 we see that (2) implies (1). Assume (1). The question is local on S and hence we may assume S is affine, X of finite type over S and $\mathcal{L}$ is a relatively ample invertible sheaf on X/S. By Lemma 24.38.4 we may assume $\mathcal{L}$ is ample on X. By Lemma 24.38.3 we see that there exists an immersion of X into a projective space over S, i.e., X is H-quasi-projective over S as desired. □

24.40. Proper morphisms

The notion of a proper morphism plays an important role in algebraic geometry. An important example of a proper morphism will be the structure morphism $\mathbf{P}^n_S \to S$ of projective n-space, and this is in fact the motivating example leading to the definition.

Definition 24.40.1. Let $f : X \to S$ be a morphism of schemes. We say f is *proper* if f is separated, finite type, and universally closed.

The morphism from the affine line with zero doubled to the affine line is of finite type and universally closed, so the separation condition is necessary in the definition above. In the rest of this section we prove some of the basic properties of proper morphisms and of universally closed morphisms.

Lemma 24.40.2. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

[11] This is not exactly the same as the definition in Hartshorne. Namely, the definition in Hartshorne (8th corrected printing, 1997) is that f should be the composition of an open immersion followed by a H-projective morphism (see Definition 24.41.1), which does not imply f is quasi-compact. See Lemma 24.41.3 for the implication in the other direction.

(1) *The morphism f is universally closed.*
(2) *There exists an open covering $S = \bigcup V_j$ such that $f^{-1}(V_j) \to V_j$ is universally closed for all indices j.*

Proof. This is clear from the definition. □

Lemma 24.40.3. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *The morphism f is proper.*
(2) *There exists an open covering $S = \bigcup V_j$ such that $f^{-1}(V_j) \to V_j$ is proper for all indices j.*

Proof. Omitted. □

Lemma 24.40.4. *The composition of proper morphisms is proper. The same is true for universally closed morphisms.*

Proof. A composition of closed morphisms is closed. If $X \to Y \to Z$ are universally closed morphisms and $Z' \to Z$ is any morphism, then we see that $Z' \times_Z X = (Z' \times_Z Y) \times_Y X \to Z' \times_Z Y$ is closed and $Z' \times_Z Y \to Z'$ is closed. Hence the result for universally closed morphisms. We have seen that ``separated'' and ``finite type'' are preserved under compositions (Schemes, Lemma 21.21.13 and Lemma 24.14.3). Hence the result for proper morphisms. □

Lemma 24.40.5. *The base change of a proper morphism is proper. The same is true for universally closed morphisms.*

Proof. This is true by definition for universally closed morphisms. It is true for separated morphisms (Schemes, Lemma 21.21.13). It is true for morphisms of finite type (Lemma 24.14.4). Hence it is true for proper morphisms. □

Lemma 24.40.6. *A closed immersion is proper, hence a fortiori universally closed.*

Proof. The base change of a closed immersion is a closed immersion (Schemes, Lemma 21.18.2). Hence it is universally closed. A closed immersion is separated (Schemes, Lemma 21.23.7). A closed immersion is of finite type (Lemma 24.14.5). Hence a closed immersion is proper. □

Lemma 24.40.7. *Suppose given a commutative diagram of schemes*

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

with Y separated over S.

(1) *If $X \to S$ is universally closed, then the morphism $X \to Y$ is universally closed.*
(2) *If X proper over S, then the morphism $X \to Y$ is proper.*

In particular, in both cases the image of X in Y is closed.

Proof. Assume that $X \to S$ is universally closed (resp. proper). We factor the morphism as $X \to X \times_S Y \to Y$. The first morphism is a closed immersion, see Schemes, Lemma 21.21.11. Hence the first morphism is proper (Lemma 24.40.6). The projection $X \times_S Y \to Y$ is the base change of a unviversally closed (resp. proper) morphism and hence universally closed (resp. proper), see Lemma 24.40.5. Thus $X \to Y$ is universally closed (resp. proper) as the composition of universally closed (resp. proper) morphisms (Lemma 24.40.4). □

The following lemma says that the image of a proper scheme (in a separated scheme of finite type over the base) is proper.

Lemma 24.40.8. *Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes over S. If X is universally closed over S and f is surjective then Y is universally closed over S. In particular, if also Y is separated and of finite type over S, then Y is proper over S.*

Proof. Assume X is universally closed and f surjective. Denote $p : X \to S$, $q : Y \to S$ the structure morphisms. Let $S' \to S$ be a morphism of schemes. The base change $f' : X_{S'} \to Y_{S'}$ is surjective (Lemma 24.9.4), and the base change $p' : X_{S'} \to S'$ is closed. If $T \subset Y_{S'}$ is closed, then $(f')^{-1}(T) \subset X_{S'}$ is closed, hence $p'((f')^{-1}(T)) = q'(T)$ is closed. So q' is closed. □

The proof of the following lemma is due to Bjorn Poonen, see this location.

Lemma 24.40.9. *A universally closed morphism of schemes is quasi-compact.*

Proof. Let $f : X \to S$ be a morphism. Assume that f is not quasi-compact. Our goal is to show that f is not universally closed. By Schemes, Lemma 21.19.2 there exists an affine open $V \subset S$ such that $f^{-1}(V)$ is not quasi-compact. To achieve our goal it suffices to show that $f^{-1}(V) \to V$ is not universally closed, hence we may assume that $S = Spec(A)$ for some ring A.

Write $X = \bigcup_{i \in I} X_i$ where the X_i are affine open subschemes of X. Let $T = Spec(A[y_i; i \in I])$. Let $T_i = D(y_i) \subset T$. Let Z be the closed set $(X \times_S T) - \bigcup_{i \in I}(X_i \times_S T_i)$. It suffices to prove that the image $f_T(Z)$ of Z under $f_T : X \times_S T \to T$ is not closed.

There exists a point $s \in S$ such that there is no neighborhood U of s in S such that X_U is quasi-compact. Otherwise we could cover S with finitely many such U and Schemes, Lemma 21.19.2 would imply f quasi-compact. Fix such an $s \in S$.

First we check that $f_T(Z_s) \neq T_s$. Let $t \in T$ be the point lying over s with $\kappa(t) = \kappa(s)$ such that $y_i = 1$ in $\kappa(t)$ for all i. Then $t \in T_i$ for all i, and the fiber of $Z_s \to T_s$ above t is isomorphic to $(X - \bigcup_{i \in I} X_i)_s$, which is empty. Thus $t \in T_s - f_T(Z_s)$.

Assume $f_T(Z)$ is closed in T. Then there exists an element $g \in A[y_i; i \in I]$ with $f_T(Z) \subset V(g)$ but $t \notin V(g)$. Hence the image of g in $\kappa(t)$ is nonzero. In particular some coefficient of g has nonzero image in $\kappa(s)$. Hence this coefficient is invertible on some neighborhood U of s. Let J be the finite set of $j \in I$ such that y_j appears in g. Since X_U is not quasi-compact, we may choose a point $x \in X - \bigcup_{j \in J} X_j$ lying above some $u \in U$. Since g has a coefficient that is invertible on U, we can find a point $t' \in T$ lying above u such that $t' \notin V(g)$ and $t' \in V(y_i)$ for all $i \notin J$. This is true because $V(y_i; i \in I, i \notin J) = Spec(A[t_j; j \in J])$ and the set of points of this scheme lying over u is bijective with $Spec(\kappa(u)[t_j; j \in J])$. In other words $t' \notin T_i$ for each $i \notin J$. By Schemes, Lemma 21.17.5 we can find a point z of $X \times_S T$ mapping to $x \in X$ and to $t' \in T$. Since $x \notin X_j$ for $j \in J$ and $t' \notin T_i$ for $i \in I \setminus J$ we see that $z \in Z$. On the other hand $f_T(z) = t' \notin V(g)$ which contradicts $f_T(Z) \subset V(g)$. Thus the assumption ``$f_T(Z)$ closed'' is wrong and we conclude indeed that f_T is not closed, as desired. □

24.41. Projective morphisms

We will use the definition of a projective morphism from [**DG67**]. The version of the definition with the ``H'' is the one from [**Har77**]. The resulting definitions are different. Both are useful.

Definition 24.41.1. Let $f : X \to S$ be a morphism of schemes.

(1) We say f is *projective* if X is isomorphic as an S-scheme to a closed subscheme of a projective bundle $\mathbf{P}(\mathcal{E})$ for some quasi-coherent, finite type $\mathcal{O}_S$-module $\mathcal{E}$.
(2) We say f is *H-projective* if there exists and integer n and a closed immersion $X \to \mathbf{P}^n_S$ over S.
(3) We say f is *locally projective* if there exists an open covering $S = \bigcup U_i$ such that each $f^{-1}(U_i) \to U_i$ is projective.

Lemma 24.41.2. *An H-projective morphism is projective.*

Proof. This is true as $\mathbf{P}^n_S$ is a projective bundle over S, see Constructions, Lemma 22.20.4. $\square$

Lemma 24.41.3. *Let $f : X \to S$ be a H-quasi-projective morphism. Then f factors as $X \to X' \to S$ where $X \to X'$ is an open immersion and $X' \to S$ is H-projective.*

Proof. By definition we can factor f as a quasi-compact immersion $i : X \to \mathbf{P}^n_S$ followed by the projection $\mathbf{P}^n_S \to S$. By Lemma 24.5.7 there exists a closed subscheme $X' \subset \mathbf{P}^n_S$ such that i factors through an open immersion $X \to X'$. The lemma follows. $\square$

Lemma 24.41.4. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *The morphism f is locally projective.*
(2) *There exists an open covering $S = \bigcup U_i$ such that each $f^{-1}(U_i) \to U_i$ is H-projective.*

Proof. By Lemma 24.41.2 we see that (2) implies (1). Assume (1). For every point $s \in S$ we can find $Spec(R) = U \subset S$ an affine open neighbourhood of s such that X_U is isomorphic to a closed subscheme of $\mathbf{P}(\mathcal{E})$ for some finite type, quasi-coherent sheaf of $\mathcal{O}_U$-modules $\mathcal{E}$. Write $\mathcal{E} = \widetilde{M}$ for some finite type R-module M (see Properties, Lemma 23.16.1). Choose generators $x_0, \ldots, x_n \in M$ of M as an R-module. Consider the surjective graded R-algebra map

$$R[X_0, \ldots, X_n] \longrightarrow \text{Sym}_R(M).$$

According to Constructions, Lemma 22.11.3 the corresponding morphism

$$\mathbf{P}(\mathcal{E}) \to \mathbf{P}^n_R$$

is a closed immersion. Hence we conclude that $f^{-1}(U)$ is isomorphic to a closed subscheme of $\mathbf{P}^n_U$ (as a scheme over U). In other words: (2) holds. $\square$

Lemma 24.41.5. *A locally projective morphism is proper.*

Proof. Let $f : X \to S$ be locally projective. In order to show that f is proper we may work locally on the base, see Lemma 24.40.3. Hence, by Lemma 24.41.4 above we may assume there exists a closed immersion $X \to \mathbf{P}^n_S$. By Lemmas 24.40.4 and 24.40.6 it suffices to prove that $\mathbf{P}^n_S \to S$ is proper. Since $\mathbf{P}^n_S \to S$ is the base change of $\mathbf{P}^n_{\mathbf{Z}} \to Spec(\mathbf{Z})$ it suffices to show that $\mathbf{P}^n_{\mathbf{Z}} \to Spec(\mathbf{Z})$ is proper, see Lemma 24.40.5. By Constructions, Lemma 22.8.8 the scheme $\mathbf{P}^n_{\mathbf{Z}}$ is separated. By Constructions, Lemma 22.8.9 the scheme $\mathbf{P}^n_{\mathbf{Z}}$ is quasi-compact. It is clear that $\mathbf{P}^n_{\mathbf{Z}} \to Spec(\mathbf{Z})$ is locally of finite type since $\mathbf{P}^n_{\mathbf{Z}}$ is covered by the affine opens $D_+(X_i)$ each of which is the spectrum of the finite type $\mathbf{Z}$-algebra

$$\mathbf{Z}[X_0/X_i, \ldots, X_n/X_i].$$

Finally, we have to show that $\mathbf{P}^n_{\mathbf{Z}} \to Spec(\mathbf{Z})$ is universally closed. This follows from Constructions, Lemma 22.8.11 and the valuative criterion (see Schemes, Proposition 21.20.6). $\square$

Lemma 24.41.6. *Let S be a scheme. There exists a closed immersion*

$$\mathbf{P}^n_S \times_S \mathbf{P}^m_S \longrightarrow \mathbf{P}^{nm+n+m}_S$$

called the Segre embedding.

Proof. It suffices to prove this when $S = Spec(\mathbf{Z})$. Hence we will drop the index S and work in the absolute setting. Write $\mathbf{P}^n = \text{Proj}(\mathbf{Z}[X_0, \ldots, X_n])$, $\mathbf{P}^m = \text{Proj}(\mathbf{Z}[Y_0, \ldots, Y_m])$, and $\mathbf{P}^{nm+n+m} = \text{Proj}(\mathbf{Z}[Z_0, \ldots, Z_{nm+n+m}])$. In order to map into $\mathbf{P}^{nm+n+m}$ we have to write down an invertible sheaf $\mathcal{L}$ on the left hand side and $(n+1)(m+1)$ sections s_i which generate it. See Constructions, Lemma 22.13.1. The invertible sheaf we take is

$$\mathcal{L} = \text{pr}_1^*\mathcal{O}_{\mathbf{P}^n}(1) \otimes \text{pr}_2^*\mathcal{O}_{\mathbf{P}^m}(1)$$

The sections we take are

$$s_0 = X_0Y_0,\ s_1 = X_1Y_0, \ldots,\ s_n = X_nY_0,\ s_{n+1} = X_0Y_1, \ldots,\ s_{nm+n+m} = X_nY_m.$$

These generate $\mathcal{L}$ since the sections X_i generate $\mathcal{O}_{\mathbf{P}^n}(1)$ and the sections Y_j generate $\mathcal{O}_{\mathbf{P}^m}(1)$. The induced morphism φ has the property that

$$\varphi^{-1}(D_+(Z_{i+(n+1)j})) = D_+(X_i) \times D_+(Y_j).$$

Hence it is an affine morphism. The corresponding ring map in case $(i,j) = (0,0)$ is the map

$$\mathbf{Z}[Z_1/Z_0, \ldots, Z_{nm+n+m}/Z_0] \longrightarrow \mathbf{Z}[X_1/X_0, \ldots, X_n/X_0, Y_1/Y_0, \ldots, Y_n/Y_0]$$

which maps Z_i/Z_0 to the element X_i/X_0 for $i \leq n$ and the element $Z_{(n+1)j}/Z_0$ to the element Y_j/Y_0. Hence it is surjective. A similar argument works for the other affine open subsets. Hence the morphism φ is a closed immersion. □

Lemma 24.41.7. *A composition of H-projective morphisms is H-projective.*

Proof. Suppose $X \to Y$ and $Y \to Z$ are H-projective. Then there exist closed immersions $X \to \mathbf{P}^n_Y$ over Y, and $Y \to \mathbf{P}^m_Z$ over Z. Consider the following diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & \mathbf{P}^n_Y \arrow[r] \arrow[dl] & \mathbf{P}^n_{\mathbf{P}^m_Z} \arrow[r, equal] \arrow[dl] & \mathbf{P}^n_Z \times_Z \mathbf{P}^m_Z \arrow[r] & \mathbf{P}^{nm+n+m}_Z \arrow[ddllll] \\
Y \arrow[r] \arrow[d] & \mathbf{P}^m_Z \arrow[dl] & & & \\
Z & & & &
\end{tikzcd}$$

Here the rightmost top horizontal arrow is the Segre embedding, see Lemma 24.41.6. The diagram identifies X as a closed subscheme of $\mathbf{P}^{nm+n+m}_Z$ as desired. □

Lemma 24.41.8. *A base change of a H-projective morphism is H-projective.*

Proof. This is true because the base change of projective space over a scheme is projective space, and the fact that the base change of a closed immersion is a closed immersion, see Schemes, Lemma 21.18.2. □

Lemma 24.41.9. *A base change of a (locally) projective morphism is (locally) projective.*

Proof. This is true because the base change of a projective bundle over a scheme is a projective bundle, the pullback of a finite type $\mathcal{O}$-module is of finite type (Modules, Lemma 15.9.2) and the fact that the base change of a closed immersion is a closed immersion, see Schemes, Lemma 21.18.2. Some details omitted. □

Lemma 24.41.10. *Le X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. If $\mathcal{I}$ is of finite type, then the blowing up of X in the ideal sheaf $\mathcal{I}$ is a projective morphism $b : \underline{Proj}_X(\bigoplus_{n \geq 0} \mathcal{I}^n) \to X$.*

Proof. Omitted. Hint: Use $\mathcal{I}$ as the sheaf $\mathcal{E}$ of the definition of a projective morphism. □

24.42. Integral and finite morphisms

Recall that a ring map $R \to A$ is said to be *integral* if every element of A satisfies a monic equation with coefficients in R. Recall that a ring map $R \to A$ is said to be finite if A is *finite* as an R-module. See Algebra, Definition 7.32.1.

Definition 24.42.1. Let $f : X \to S$ be a morphism of schemes.

1. We say that f is *integral* if f is affine and if for every affine open $Spec(R) = V \subset S$ with inverse image $Spec(A) = f^{-1}(V) \subset X$ the associated ring map $R \to A$ is integral.
2. We say that f is *finite* if f is affine and if for every affine open $Spec(R) = V \subset S$ with inverse image $Spec(A) = f^{-1}(V) \subset X$ the associated ring map $R \to A$ is finite.

It is clear that integral/finite morphisms are separated and quasi-compact. It is also clear that a finite morphism is a morphism of finite type. Most of the lemmas in this section are completely standard. But note the fun Lemma 24.42.7 at the end of the section.

Lemma 24.42.2. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

1. *The morphism f is integral.*
2. *There exists an affine open covering $S = \bigcup U_i$ such that each $f^{-1}(U_i)$ is affine and $\mathcal{O}_S(U_i) \to \mathcal{O}_X(f^{-1}(U_i))$ is integral.*
3. *There exists an open covering $S = \bigcup S_j$ such that each $f^{-1}(U_i) \to U_i$ is integral.*

Moreover, if f is integral then for every open subscheme $U \subset S$ the morphism $f : f^{-1}(U) \to U$ is integral.

Proof. See Algebra, Lemma 7.32.12. Some details omitted. □

Lemma 24.42.3. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

1. *The morphism f is finite.*
2. *There exists an affine open covering $S = \bigcup U_i$ such that each $f^{-1}(U_i)$ is affine and $\mathcal{O}_S(U_i) \to \mathcal{O}_X(f^{-1}(U_i))$ is finite.*
3. *There exists an open covering $S = \bigcup S_j$ such that each $f^{-1}(U_i) \to U_i$ is finite.*

Moreover, if f is finite then for every open subscheme $U \subset S$ the morphim $f : f^{-1}(U) \to U$ is finite.

Proof. See Algebra, Lemma 7.32.12. Some details omitted. □

Lemma 24.42.4. *A finite morphism is integral. An integral morphism which is locally of finite type is finite.*

Proof. See Algebra, Lemma 7.32.3 and Lemma 7.32.5. □

Lemma 24.42.5. *A composition of finite morphisms is finite. Same is true for integral morphisms.*

Proof. See Algebra, Lemmas 7.7.3 and 7.32.6. ☐

Lemma 24.42.6. *A base change of a finite morphism is finite. Same is true for integral morphisms.*

Proof. See Algebra, Lemma 7.32.11. ☐

Lemma 24.42.7. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *f is integral, and*
(2) *f is affine and universally closed.*

Proof. Assume (1). An integral morphism is affine by definition. A base change of an integral morphism is integral so in order to prove (2) it suffices to show that an integral morphism is closed. This follows from Algebra, Lemmas 7.32.20 and 7.36.6.

Assume (2). We may assume f is the morphism $f : Spec(A) \to Spec(R)$ coming from a ring map $R \to A$. Let a be an element of A. We have to show that a is integral over R, i.e. that in the kernel I of the map $R[x] \to A$ sending x to a there is a monic polynomial. Consider the ring $B = A[x]/(ax - 1)$ and let J be the kernel of the composition $R[x] \to A[x] \to B$. If $f \in J$ there exists $q \in A[x]$ such that $f = (ax - 1)q$ in $A[x]$ so if $f = \sum_i f_i x^i$ and $q = \sum_i q_i x^i$, for all $i \geq 0$ we have $f_i = aq_{i-1} - q_i$. For $n \geq \deg q + 1$ the polynomial

$$\sum_{i \geq 0} f_i x^{n-i} = \sum_{i \geq 0} (aq_{i-1} - q_i) x^{n-i} = (a - x) \sum_{i \geq 0} q_i x^{n-i-1}$$

is clearly in I; if $f_0 = 1$ this polynomial is also monic, so we are reduced to prove that J contains a polynomial with constant term 1. We do it by proving $Spec(R[x]/(J + (x))$ is empty.

Since f is universally closed the base change $Spec(A[x]) \to Spec(R[x])$ is closed. Hence the image of the closed subset $Spec(B) \subset Spec(A[x])$ is the closed subset $Spec(R[x]/J) \subset Spec(R[x])$, see Example 24.4.4 and Lemma 24.4.3. In particular $Spec(B) \to Spec(R[x]/J)$ is surjective. Consider the following diagram where every square is a pullback:

$$\begin{tikzcd} Spec(B) \arrow[r, two heads, "g"] & Spec(R[x]/J) \arrow[r] & Spec(R[x]) \\ \emptyset \arrow[u] \arrow[r] & Spec(R[x]/(J + (x))) \arrow[u] \arrow[r] & Spec(R) \arrow[u, "0"'] \end{tikzcd}$$

The bottom left corner is empty because it is the spectrum of $R \otimes_{R[x]} B$ where the map $R[x] \to B$ sends x to an invertible element and $R[x] \to R$ sends x to 0. Since g is surjective this implies $Spec(R[x]/(J + (x)))$ is empty, as we wanted to show. ☐

Lemma 24.42.8. *Let $f : X \to S$ be an integral morphism. Then every point of X is closed in its fibre.*

Proof. See Algebra, Lemma 7.32.18. ☐

Lemma 24.42.9. *A finite morphism is quasi-finite.*

Proof. This is implied by Algebra, Lemma 7.113.4 and Lemma 24.19.9. Alternatively, all points in fibres are closed points by Lemma 24.42.8 (and the fact that a finite morphism is integral) and use Lemma 24.19.6 (3) to see that f is quasi-finite at x for all $x \in X$. ☐

Lemma 24.42.10. *A finite morphism is proper.*

Proof. A finite morphism is integral and hence universally closed by Lemma 24.42.7. It is also clearly separated and of finite type. Hence it is proper by definition. □

Lemma 24.42.11. *A closed immersion is finite (and a fortiori integral).*

Proof. True because a closed immersion is affine (Lemma 24.11.9) and a surjective ring map is finite and integral. □

Lemma 24.42.12. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms.*

(1) *If $g \circ f$ is finite and g separated then f is finite.*
(2) *If $g \circ f$ is integral and g separated then f is integral.*

Proof. Assume $g \circ f$ is finite (resp. integral) and g separated. The base change $X \times_Z Y \to Y$ is finite (resp. integral) by Lemma 24.42.6. The morphism $X \to X \times_Z Y$ is a closed immersion as $Y \to Z$ is separated, see Schemes, Lemma 21.21.12. A closed immersion is finite (resp. integral), see Lemma 24.42.11. The composition of finite (resp. integral) morphisms is finite (resp. integral), see Lemma 24.42.5. Thus we win. □

Lemma 24.42.13. *Let $f : X \to Y$ be a morphism of schemes. If f is finite and a monomorphism, then f is a closed immersion.*

Proof. This reduces to Algebra, Lemma 7.99.6. □

24.43. Universal homeomorphisms

The following definition is really superfluous since a universal homeomorphism is really just an integral, universally injective and surjective morphism, see Lemma 24.43.3.

Definition 24.43.1. A morphisms $f : X \to Y$ of schemes is called a *universal homeomorphism* if the base change $f' : Y' \times_Y X \to Y'$ is a homeomorphism for every morphism $Y' \to Y$.

Lemma 24.43.2. *Let $f : X \to Y$ be a morphism of schemes. If f is a homeomorphism then f is affine.*

Proof. Let $y \in Y$ be a point. Let $y \in V$ be an affine open neighbourhood. let $x \in X$ be the unique point of X mapping to y. Let $U \subset X$ be an affine open neighbourhood of x which maps into V. Since $f(U) \subset V$ is open we may choose a $h \in \Gamma(V, \mathcal{O}_Y)$ such that $y \in D(h) \subset f(U)$. Denote $h' \in \Gamma(U, \mathcal{O}_X)$ the restriction of $f^\sharp(h)$ to U. Then we see that $D(h') \subset U$ is equal to $f^{-1}(D(h))$. In other words, every point of Y has an open neighbourhood whose inverse image is affine. Thus f is affine, see Lemma 24.11.3. □

Lemma 24.43.3. *Let $f : X \to Y$ be a morphism of schemes. The following are equivalent:*

(1) *f is a universal homeomorphism, and*
(2) *f is integral, universally injective and surjective.*

Proof. Assume f is a universal homeomorphism. By Lemma 24.43.2 we see that f is affine. Since f is clearly universally closed we see that f is integral by Lemma 24.42.7. It is also clear that f is universally injective and surjective.

Assume f is integral, universally injective and surjective. By Lemma 24.42.7 f is universally closed. Since it is also universally bijective (see Lemma 24.9.4) we see that it is a universal homeomorphism. □

Lemma 24.43.4. *Let X be a scheme. The canonical closed immersion $X_{red} \to X$ (see Schemes, Definition 21.12.5) is a universal homeomorphism.*

Proof. Omitted. □

24.44. Finite locally free morphisms

In many papers the authors use finite flat morphisms when they really mean finite locally free morphisms. The reason is that if the base is locally Noetherian then this is the same thing. But in general it is not, see Exercises, Exercise 65.4.3.

Definition 24.44.1. Let $f : X \to S$ be a morphism of schemes. We say f is *finite locally free* if f is affine and $f_*\mathcal{O}_X$ is a finite locally free $\mathcal{O}_S$-module. In this case we say f is has *rank* or *degree* d if the sheaf $f_*\mathcal{O}_X$ is finite locally free of degree d.

Note that if $f : X \to S$ is finite locally free then S is the disjoint union of open and closed subschemes S_d such that $f^{-1}(S_d) \to S_d$ is finite locally free of degree d.

Lemma 24.44.2. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *f is finite locally free,*
(2) *f is finite, flat, and locally of finite presentation.*

If S is locally Noetherian these are also equivalent to

(3) *f is finite and flat.*

Proof. See Algebra, Lemma 7.72.2. The Noetherian case follows as a finite module over a Noetherian ring is a finitely presented module, see Algebra, Lemma 7.28.4. □

Lemma 24.44.3. *A composition of finite locally free morphisms is finite locally free.*

Proof. Omitted. □

Lemma 24.44.4. *A base change of a finite locally free morphism is finite locally free.*

Proof. Omitted. □

Lemma 24.44.5. *Let $f : X \to S$ be a finite locally free morphism of schemes. There exists a disjoint union decomposition $S = \coprod_{d \geq 0} S_d$ by open and closed subschemes such that setting $X_d = f^{-1}(S_d)$ the restrictions $f|_{X_d}$ are finite locally free morphisms $X_d \to S_d$ of degree d.*

Proof. This is true because a finite locally free sheaf locally has a well defined rank. Details omittted. □

Lemma 24.44.6. *Let $f : Y \to X$ be a finite morphism with X affine. There exists a diagram*

$$\begin{tikzcd} Z' \arrow[rd] & Y' \arrow[l, "i"'] \arrow[r] \arrow[d] & Y \arrow[d] \\ & X' \arrow[r] & X \end{tikzcd}$$

where

(1) *$Y' \to Y$ and $X' \to X$ are surjective finite locally free,*
(2) *$Y' = X' \times_X Y$,*
(3) *$i : Y' \to Z'$ is a closed immersion,*
(4) *$Z' \to X'$ is finite locally free, and*

(5) $Z' = \bigcup_{j=1,\ldots,m} Z'_j$ *is a (set theoretic) finite union of closed subschemes, each of which maps isomorphically to* X'.

Proof. Write $X = Spec(A)$ and $Y = Spec(B)$. See also More on Algebra, Section 12.16. Let $x_1, \ldots, x_n \in B$ be generators of B over A. For each i we can choose a monic polynomial $P_i(T) \in A[T]$ such that $P(x_i) = 0$ in B. By Algebra, Lemma 7.125.9 (applied n times) there exists a finite locally free ring extension $A \subset A'$ such that each P_i splits completely:

$$P_i(T) = \prod\nolimits_{k=1,\ldots,d_i}(T - \alpha_{ik})$$

for certain $\alpha_{ik} \in A'$. Set

$$C = A'[T_1, \ldots, T_n]/(P_1(T_1), \ldots, P_n(T_n))$$

and $B' = A' \otimes_A B$. The map $C \to B'$, $T_i \mapsto 1 \otimes x_i$ is an A'-algebra surjection. Setting $X' = Spec(A')$, $Y' = Spec(B')$ and $Z' = Spec(C)$ we see that (1) -- (4) hold. Part (5) holds because set theoretically $Spec(C)$ is the union of the closed subschemes cut out by the ideals

$$(T_1 - \alpha_{1k_1}, T_2 - \alpha_{2k_2}, \ldots, T_n - \alpha_{nk_n})$$

for any $1 \leq k_i \leq d_i$. □

The following lemma is stated in the correct generality in Lemma 24.47.4 below.

Lemma 24.44.7. *Let $f : Y \to X$ be a finite morphism of schemes. Let $T \subset Y$ be a closed nowhere dense subset of Y. Then $f(T) \subset X$ is a closed nowhere dense subset of X.*

Proof. By Lemma 24.42.10 we know that $f(T) \subset X$ is closed. Let $X = \bigcup X_i$ be an affine covering. Since T is nowhere dense in Y, we see that also $T \cap f^{-1}(X_i)$ is nowhere dense in $f^{-1}(X_i)$. Hence if we can prove the theorem in the affine case, then we see that $f(T) \cap X_i$ is nowhere dense. This then implies that T is nowhere dense in X by Topology, Lemma 5.17.4.

Assume X is affine. Choose a diagram

$$\begin{tikzcd} Z' \arrow[dr] & Y' \arrow[l, "i"'] \arrow[r, "a"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ & X' \arrow[r, "b"] & X \end{tikzcd}$$

as in Lemma 24.44.6. The morphisms a, b are open since they are finite locally free (Lemmas 24.44.2 and 24.24.9). Hence $T' = a^{-1}(T)$ is nowhere dense, see Topology, Lemma 5.17.6. The morphism b is surjective and open. Hence, if we can prove $f'(T') = b^{-1}(f(T))$ is nowhere dense, then $f(T)$ is nowhere dense, see Topology, Lemma 5.17.6. As i is a closed immersion, by Topology, Lemma 5.17.5 we see that $i(T') \subset Z'$ is closed and nowhere dense. Thus we have reduced the problem to the case discussed in the following paragraph.

Assume that $Y = \bigcup_{i=1,\ldots,n} Y_i$ is a finite union of closed subsets, each mapping isomorphically to X. Consider $T_i = Y_i \cap T$. If each of the T_i is nowhere dense in Y_i, then each $f(T_i)$ is nowhere dense in X as $Y_i \to X$ is an isomorphism. Hence $f(T) = f(T_i)$ is a finite union of nowhere dense closed subsets of X and we win, see Topology, Lemma 5.17.2. Suppose not, say T_1 contains a nonempty open $V \subset Y_1$. We are going to show this leads to a contradiction. Consider $Y_2 \cap V \subset V$. This is either a proper closed subset, or equal to V. In the first case we replace V by $V \setminus V \cap Y_2$, so $V \subset T_1$ is open in Y_1 and does not meet Y_2. In the second case we have $V \subset Y_1 \cap Y_2$ is open in both Y_1 and Y_2. Repeat sequentially with $i = 3, \ldots, n$. The result is a disjoint union decomposition

$$\{1, \ldots, n\} = I_1 \coprod I_2, \quad 1 \in I_1$$

and an open V of Y_1 contained in T_1 such that $V \subset Y_i$ for $i \in I_1$ and $V \cap Y_i = \emptyset$ for $i \in I_2$. Set $U = f(V)$. This is an open of X since $f|_{Y_1} : Y_1 \to X$ is an isomorphism. Then

$$f^{-1}(U) = V \coprod \bigcup\nolimits_{i \in I_2} (Y_i \cap f^{-1}(U))$$

As $\bigcup_{i \in I_2} Y_i$ is closed, this implies that $V \subset f^{-1}(U)$ is open, hence $V \subset Y$ is open. This contradicts the assumption that T is nowhere dense in Y, as desired. □

24.45. Generically finite morphisms

In this section we characterize maps between schemes which are locally of finite type and which are ``generically finite'' in some sense.

Lemma 24.45.1. *Let X, Y be schemes. Let $f : X \to Y$ be locally of finite type. Let $\eta \in Y$ be a generic point of an irreducible component of Y. The following are equivalent:*

1. *the set $f^{-1}(\{\eta\})$ is finite,*
2. *there exist affine opens $U_i \subset X$, $i = 1, \ldots, n$ and $V \subset Y$ with $f(U_i) \subset V$, $\eta \in V$ and $f^{-1}(\{\eta\}) \subset \bigcup U_i$ such that each $f|_{U_i} : U_i \to V$ is finite.*

If f is quasi-separated, then these are also equivalent to

3. *there exist affine opens $V \subset Y$, and $U \subset X$ with $f(U) \subset V$, $\eta \in V$ and $f^{-1}(\{\eta\}) \subset U$ such that $f|_U : U \to V$ is finite.*

If f is quasi-compact and quasi-separated, then these are also equivalent to

4. *there exists an affine open $V \subset Y$, $\eta \in V$ such that $f^{-1}(V) \to V$ is finite.*

Proof. The question is local on the base. Hence we may replace Y by an affine neighbourhood of η, and we may and do assume throughout the proof below that Y is affine, say $Y = Spec(R)$.

It is clear that (2) implies (1). Assume that $f^{-1}(\{\eta\}) = \{\xi_1, \ldots, \xi_n\}$ is finite. Choose affine opens $U_i \subset X$ with $\xi_i \in U_i$. By Algebra, Lemma 7.113.9 we see that after replacing Y by a standard open in Y each of the morphisms $U_i \to Y$ is finite. In other words (2) holds.

It is clear that (3) implies (1). Assume $f^{-1}(\{\eta\}) = \{\xi_1, \ldots, \xi_n\}$ and assume that f is quasi-separated. Since Y is affine this implies that X is quasi-separated. Since each ξ_i maps to a generic point of an irreducible component of Y, we see that each ξ_i is a generic point of an irreducible component of X. By Properties, Lemma 23.26.1 we can find an affine open $U \subset X$ containing each ξ_i. By Algebra, Lemma 7.113.9 we see that after replacing Y by a standard open in Y the morphisms $U \to Y$ is finite. In other words (3) holds.

It is clear that (4) implies all of (1) -- (3) with no further assumptions on f. Suppose that f is quasi-compact and quasi-separated. We have to show that the equivalent conditions (1) -- (3) imply (4). Let U, V be as in (3). Replace Y by V. Since f is quasi-compact and Y is quasi-compact (being affine) we see that X is quasi-compact. Hence $Z = X \setminus U$ is quasi-compact, hence the morphism $f|_Z : Z \to Y$ is quasi-compact. By construction of Z we see that $\eta \notin f(Z)$. Hence by Lemma 24.6.4 we see that there exists an affine open neighbourhood V' of η in Y such that $f^{-1}(V') \cap Z = \emptyset$. Then we have $f^{-1}(V') \subset U$ and this means that $f^{-1}(V') \to V'$ is finite. □

Example 24.45.2. Let $A = \prod_{n \in \mathbf{N}} \mathbf{F}_2$. Every element of A is an idempotent. Hence every prime ideal is maximal with residue field $\mathbf{F}_2$. Thus the topology on $X = Spec(A)$ is totally disconnected and quasi-compact. The projection maps $A \to \mathbf{F}_2$ define open points of $Spec(A)$. It cannot be the case that all the points of X are open since X is quasi-compact. Let $x \in X$ be a closed point which is not open. Then we can form a scheme Y which is two copies of X glued along $X \setminus \{x\}$. In other words, this is X with x doubled, compare Schemes, Example 21.14.3. The morphism $f : Y \to X$ is quasi-compact, finite type and has finite fibres but is not quasi-separated. The point $x \in X$ is a generic point of an irreducible component of X (since X is totally disconnected). But properties (3) and (4) of Lemma 24.45.1 do not hold. The reason is that for any open neighbourhood $x \in U \subset X$ the inverse image $f^{-1}(U)$ is not affine because functions on $f^{-1}(U)$ cannot separated the two points lying over x (proof omitted; this is a nice exercise). Hence the condition that f is quasi-separated is necessary in parts (3) and (4) of the lemma.

Remark 24.45.3. An alternative to Lemma 24.45.1 is the statement that a quasi-finite morphism is finite over a dense open of the target. This will be shown in More on Morphisms, Section 33.29.

Lemma 24.45.4. *Let X, Y be integral schemes. Let $f : X \to Y$ be locally of finite type. Assume f is dominant. The following are equivalent:*

1. *the extension $R(Y) \subset R(X)$ has transcendence degree* 0*,*
2. *the extension $R(Y) \subset R(X)$ is finite,*
3. *there exist nonempty affine opens $U \subset X$ and $V \subset Y$ such that $f(U) \subset V$ and $f|_U : U \to V$ is finite, and*
4. *the generic point of X is the only point of X mapping to the generic point of Y.*

If f is separated, or if f is quasi-compact, then these are also equivalent to

5. *there exists a nonempty affine open $V \subset Y$ such that $f^{-1}(V) \to V$ is finite.*

Proof. Choose any affine opens $Spec(A) = U \subset X$ and $Spec(R) = V \subset Y$ such that $f(U) \subset V$. Then R and A are domains by definition. The ring map $R \to A$ is of finite type Lemma 24.14.2). Let $K = f.f.(R) = R(Y)$ and $L = f.f.(A) = R(X)$. Then $K \subset L$ is a finitely generated field extension. Hence we see that (1) is equivalent to (2).

Suppose (2) holds. Let $x_1, \ldots, x_n \in A$ be generators of A over R. By assumption there exist nonzero polynomials $P_i(X) \in R[X]$ such that $P_i(x_i) = 0$. Let $f_i \in R$ be the leading coefficient of P_i. Then we conclude that $R_{f_1 \ldots f_n} \to A_{f_1 \ldots f_n}$ is finite, i.e., (3) holds. Note that (3) implies (2). So now we see that (1), (2) and (3) are all equivalent.

Let η be the generic point of X, and let $\eta' \in Y$ be the generic point of Y. Assume (4). Then $\dim_\eta(X_{\eta'}) = 0$ and we see that $R(X) = \kappa(\eta)$ has transcendence degree 0 over $R(Y) = \kappa(\eta')$ by Lemma 24.27.1. In other words (1) holds. Assume the equivalent conditions (1), (2) and (3). Suppose that $x \in X$ is a point mapping to η'. As x is a specialization of η, this gives inclusions $R(Y) \subset \mathcal{O}_{X,x} \subset R(X)$, which implies $\mathcal{O}_{X,x}$ is a field, see Algebra, Lemma 7.32.17. Hence $x = \eta$. Thus we see that (1) -- (4) are all equivalent.

It is clear that (5) implies (3) with no additional assumptions on f. What remains is to prove that if f is either separated or quasi-compact, then the equivalent conditions (1) -- (4) imply (5).

Assume U, V as in (3) and assume f is separated. Then $U \to f^{-1}(V)$ is a morphism from a scheme proper over VLemma 24.42.10) into a scheme separated over V. Hence $U \subset f^{-1}(V)$ is closed Lemma 24.40.7. Since X is irreducible we conclude $U = f^{-1}(V)$. This proves (5).

Assume f is quasi-compact. Let U, V be as in (3). Then $f^{-1}(V)$ is quasi-compact. Consider the closed subset $Z = f^{-1}(V) \setminus U$. Since Z does not contain the generic point of X we see that the quasi-compact morphism $f : Z \to V$ is not dominant by Lemma 24.6.3. Hence after shrinking V we may assume that $f^{-1}(V) = U$ which implies that (5) holds. $\square$

Definition 24.45.5. Let X and Y be integral schemes. Let $f : X \to Y$ be locally of finite type and dominant. Assume $[R(X) : R(Y)] < \infty$, or any other of the equivalent conditions (1) -- (4) of Lemma 24.45.4. Then the positive integer

$$\deg(X/Y) = [R(X) : R(Y)]$$

is called the *degree of X over Y*.

It is possible to extend this notion to a morphism $f : X \to Y$ if (a) Y is integral with generic point η, (b) f is locally of finite type, and (c) $f^{-1}(\{\eta\})$ is finite. Namely, in this case we can define

$$\deg(X/Y) = \sum\nolimits_{\xi \in X,\ f(\xi)=\eta} \dim_{R(Y)}(\mathcal{O}_{X,\xi}).$$

Namely, given that $R(Y) = \kappa(\eta) = \mathcal{O}_{Y,\eta}$ (Lemma 24.8.4) the dimensions above are finite by Lemma 24.45.1 above. However, for most applications the definition given above is the right one.

Lemma 24.45.6. *Let X, Y, Z be integral schemes. Let $f : X \to Y$ and $g : Y \to Z$ be dominant morphisms locally of finite type. Assume that $[R(X) : R(Y)] < \infty$ and $[R(Y) : R(Z)] < \infty$. Then*

$$\deg(X/Z) = \deg(X/Y)\deg(Y/Z).$$

Proof. This comes from the multiplicativity of degrees in towers of finite extensions of fields. $\square$

Remark 24.45.7. Let $f : X \to Y$ be a morphism of schemes which is locally of finite type. There are (at least) two properties that we could use to define *generically finite* morphisms. These correspond to whether you want the property to be local on the source or local on the target:

(1) (Local on the target; suggested by Ravi Vakil.) Assume every quasi-compact open of Y has finitely many irreducible components (for example if Y is locally Noetherian). The requirement is that the inverse image of each generic point is finite, see Lemma 24.45.1.
(2) (Local on the source.) The requirement is that there exists a dense open $U \subset X$ such that $U \to Y$ is locally quasi-finite.

In case (1) the requirement can be formulated without the auxiliary condition on Y, but probably doesn't give the right notion for general schemes. Property (2) as formulated doesn't imply that the fibres over generic points are finite; however, if f is quasi-compact and Y is as in (1) then it does.

24.46. Normalization

In this section we construct the *normalization*, and the *normalization of one scheme in another*.

Lemma 24.46.1. *Let X be a scheme. Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. The subsheaf $\mathcal{A}' \subset \mathcal{A}$ defined by the rule*

$$U \longmapsto \{f \in \mathcal{A}(U) \mid f_x \in \mathcal{A}_x \text{ integral over } \mathcal{O}_{X,x} \text{ for all } x \in U\}$$

is a quasi-coherent $\mathcal{O}_X$-algebra, and for any affine open $U \subset X$ the ring $\mathcal{A}'(U) \subset \mathcal{A}(U)$ is the integral closure of $\mathcal{O}_X(U)$ in $\mathcal{A}(U)$.

Proof. This is a subsheaf by the local nature of the conditions. It is an $\mathcal{O}_X$-algebra by Algebra, Lemma 7.32.7. Let $U \subset X$ be an affine open. Say $U = Spec(R)$ and say $\mathcal{A}$ is the quasi-coherent sheaf associated to the R-algebra A. Then according to Algebra, Lemma 7.32.10 the value of $\mathcal{A}'$ over U is given by the integral closure A' of R in A. This proves the last assertion of the lemma. To prove that $\mathcal{A}'$ is quasi-coherent, it suffices to show that $\mathcal{A}'(D(f)) = A'_f$. This follows from the fact that integral closure and localization commute, see Algebra, Lemma 7.32.9. $\square$

Definition 24.46.2. Let X be a scheme. Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. The *integral closure of $\mathcal{O}_X$ in $\mathcal{A}$* is the quasi-coherent $\mathcal{O}_X$-subalgebra $\mathcal{A}' \subset \mathcal{A}$ constructed in Lemma 24.46.1 above.

In the setting of the definition above we can consider the morphism of relative spectra

$$\begin{tikzcd} Y = \underline{Spec}_X(\mathcal{A}) \arrow[rr] \arrow[dr] & & X' = \underline{Spec}_X(\mathcal{A}') \arrow[dl] \\ & X & \end{tikzcd}$$

see Lemma 24.11.5. The scheme $X' \to X$ will be the normalization of X in the scheme Y. Here is a slightly more general setting. Suppose we have a quasi-compact and quasi-separated morphism $f : Y \to X$ of schemes. In this case the sheaf of $\mathcal{O}_X$-algebras $f_*\mathcal{O}_Y$ is quasi-coherent, see Schemes, Lemma 21.24.1. Taking the integral closure $\mathcal{O}' \subset f_*\mathcal{O}_Y$ we obtain a quasi-coherent sheaf of $\mathcal{O}_X$-algebras whose relative spectrum is the normalization of X in Y. Here is the formal definition.

Definition 24.46.3. Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of schemes. Let $\mathcal{O}'$ be the integral closure of $\mathcal{O}_X$ in $f_*\mathcal{O}_Y$. The *normalization of X in Y* is the scheme[12]

$$\nu : X' = \underline{Spec}_X(\mathcal{O}') \to X$$

over X. It comes equipped with a natural factorization

$$Y \xrightarrow{f'} X' \xrightarrow{\nu} X$$

of the initial morphism f.

The factorization is the composition of the canonical morphism $Y \to \underline{Spec}(f_*\mathcal{O}_Y)$ (see Constructions, Lemma 22.4.7) and the morphism of relative spectra coming from the inclusion map $\mathcal{O}' \to f_*\mathcal{O}_Y$. We can characterize the normalization as follows.

Lemma 24.46.4. *Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of schemes. The factorization $f = \nu \circ f'$, where $\nu : X' \to X$ is the normalization of X in Y is characterized by the following two properties:*

(1) *the morphism ν is integral, and*

[12]The scheme X' need not be normal, for example if $Y = X$ and $f = \mathrm{id}_X$, then $X' = X$.

(2) *for any factorization* $f = \pi \circ g$*, with* $\pi : Z \to X$ *integral, there exists a commutative diagram*

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "f'"'] & Z \arrow[d, "\pi"] \\ X' \arrow[r, "\nu"] \arrow[ur, "h"] & X \end{tikzcd}$$

for some unique morphism $h : X' \to Z$.

Moreover, in (2) the morphism $h : X' \to Z$ *is the normalization of* Z *in* Y.

Proof. Let $\mathcal{O}' \subset f_*\mathcal{O}_Y$ be the integral closure of $\mathcal{O}_X$ as in Definition 24.46.3. The morphism ν is integral by construction, which proves (1). Assume given a factorization $f = \pi \circ g$ with $\pi : Z \to X$ integral as in (2). By Definition 24.42.1 Then π is affine, and hence Z is the relative spectrum of a quasi-coherent sheaf of $\mathcal{O}_X$-algebras $\mathcal{B}$. The morphism $g : X \to Z$ corresponds to a map of $\mathcal{O}_X$-algebras $\chi : \mathcal{B} \to f_*\mathcal{O}_Y$. Since $\mathcal{B}(U)$ is integral over $\mathcal{O}_X(U)$ for every affine open $U \subset X$ (by Definition 24.42.1) we see from Lemma 24.46.1 that $\chi(\mathcal{B}) \subset \mathcal{O}'$. By the functoriality of the relative spectrum Lemma 24.11.5 this provides us with a unique morphism $h : X' \to Z$. We omit the verification that the diagram commutes.

It is clear that (1) and (2) characterize the factorization $f = \nu \circ f'$ since it characterizes it as an initial object in a category. The morphism h in (2) is integral by Lemma 24.42.12. Given a factorization $g = \pi' \circ g'$ with $\pi' : Z' \to Z$ integral, we get a factorization $f = (\pi \circ \pi') \circ g'$ and we get a morphism $h' : X' \to Z'$. Uniqueness implies that $\pi' \circ h' = h$. Hence the characterization (1), (2) applies to the morphism $h : X' \to Z$ which gives the last statement of the lemma. □

Lemma 24.46.5. *Let*

$$\begin{tikzcd} Y_2 \arrow[r] \arrow[d, "f_2"'] & Y_1 \arrow[d, "f_1"] \\ X_2 \arrow[r] & X_1 \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume f_1, f_2 *quasi-compact and quasi-separated. Let* $f_i = \nu_i \circ f'_i$, $i = 1, 2$ *be the canonical factorizations, where* $\nu_i : X'_i \to X_i$ *is the normalization of* X_i *in* Y_i. *Then there exists a canonical commutative diagram*

$$\begin{tikzcd} Y_2 \arrow[r] \arrow[d, "f'_2"'] & Y_1 \arrow[d, "f'_1"] \\ X'_2 \arrow[r] \arrow[d, "\nu_2"'] & X'_1 \arrow[d, "\nu_1"] \\ X_2 \arrow[r] & X_1 \end{tikzcd}$$

Proof. By Lemmas 24.46.4 (1) and 24.42.6 the base change $X_2 \times_{X_1} X'_1 \to X_2$ is integral. Note that f_2 factors through this morphism. Hence we get a canonical morphism $X'_2 \to X_2 \times_{X_1} X'_1$ from Lemma 24.46.4 (2). This gives the middle horizontal arrow in the last diagram. □

Lemma 24.46.6. *Let* $f : Y \to X$ *be a quasi-compact and quasi-separated morphism of schemes. Let* $U \subset X$ *be an open subscheme and set* $V = f^{-1}(U)$. *Then the normalization of* U *in* V *is the inverse image of* U *in the normalization of* X *in* Y.

Proof. Clear from the construction. □

Lemma 24.46.7. *Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of schemes. Suppose that $Y = Y_1 \coprod Y_2$ is a disjoint union of two schemes. Write $f_i = f_{Y_i}$. Let X_i' be the normalization of X in Y_i. Then $Y_1' \coprod Y_2'$ is the normalization of X in Y.*

Proof. In terms of integral closures this corresponds to the following fact: Let $A \to B$ be a ring map. Suppose that $B = B_1 \times B_2$. Let A_i' be the integral closure of A in B_i. Then $A_1' \times A_2'$ is the integral closure of A in B. The reason this works is that the elements $(1, 0)$ and $(0, 1)$ of B are idempotents and hence integral over A. Thus the integral closure A' of A in B is a product and it is not hard to see that the factors are the integral closures A_i' as described above (some details omitted). □

Lemma 24.46.8. *Let $f : Y \to X$ be an integral morphism. Then the integral closure of X in Y is equal to Y.*

Proof. Omitted. □

The following lemma is a generalization of the preceding one.

Lemma 24.46.9. *Let $f : X \to S$ be a quasi-compact, quasi-separated and universally closed morphisms of schemes. Then $f_*\mathcal{O}_X$ is integral over $\mathcal{O}_S$. In other words, the normalization of S in X is equal to the factorization*

$$X \longrightarrow \underline{Spec}_S(f_*\mathcal{O}_X) \longrightarrow S$$

of Constructions, Lemma 22.4.7.

Proof. The question is local on S, hence we may assume $S = Spec(R)$ is affine. Let $h \in \Gamma(X, \mathcal{O}_X)$. We have to show that h satisfies a monic equation over R. Think of h as a morphism as in the following commutative diagram

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & \mathbf{A}^1_S \arrow[dl] \\ & S & \end{tikzcd}$$

Let $Z \subset \mathbf{A}^1_S$ be the scheme theoretic image of h, see Definition 24.4.2. The morphism h is quasi-compact as f is quasi-compact and $\mathbf{A}^1_S \to S$ is separated, see Schemes, Lemma 21.21.15. By Lemma 24.4.3 the morphism $X \to Z$ is dominant. By Lemma 24.40.7 the morphism $X \to Z$ is closed. Hence $h(X) = Z$ (set theoretically). Thus we can use Lemma 24.40.8 to conclude that $Z \to S$ is universally closed (and even proper). Since $Z \subset \mathbf{A}^1_S$, we see that $Z \to S$ is affine and proper, hence integral by Lemma 24.42.7. Writing $\mathbf{A}^1_S = Spec(R[T])$ we conclude that the ideal $I \subset R[T]$ of Z contains a monic polynomial $P(T) \in R[T]$. Hence $P(h) = 0$ and we win. □

Lemma 24.46.10. *Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of schemes. Assume*

(1) *Y is a normal scheme,*
(2) *any quasi-compact open $V \subset Y$ has a finite number of irreducible components.*

Then the normalization X' of X in Y is a normal scheme. Moreover, the morphism $Y \to X'$ is dominant and induces a bijection of irreducible components.

Proof. We first prove that X' is normal. Let $U \subset X$ be an affine open. It suffices to prove that the inverse image of U in X' is normal (see Properties, Lemma 23.7.2). By Lemma 24.46.6 we may replace X by U, and hence we may assume $X = Spec(A)$ affine. In this case Y is quasi-compact, and hence has a finite number of irreducible components by assumption. Hence $Y = \coprod_{i=1,\ldots n} Y_i$ is a finite disjoint union of normal integral schemes by Properties, Lemma 23.7.5. By Lemma 24.46.7 we see that $X' = \coprod_{i=1,\ldots,n} X'_i$, where X'_i is the normalization of X in Y_i. By Properties, Lemma 23.7.9 we see that $B_i = \Gamma(Y_i, \mathcal{O}_{Y_i})$ is a normal domain. Note that $X'_i = Spec(A'_i)$, where $A'_i \subset B_i$ is the integral closure of A in B_i, see Lemma 24.46.1. By Algebra, Lemma 7.33.2 we see that $A'_i \subset B_i$ is a normal domain. Hence $X' = \coprod X'_i$ is a finite union of normal schemes and hence is normal.

It is clear from the description of X' above that $Y \to X'$ is dominant and induces a bijection on irreducible components if X is affine. The result in general follows from this by a topological argument (omitted). $\square$

Lemma 24.46.11. *Let $f : X \to S$ be a morphism. Assume that*

(1) *S is a Nagata scheme,*
(2) *f is of finite type*[13]*, and*
(3) *X is reduced.*

Then the normalization $\nu : S' \to S$ of S in X is finite.

Proof. There is an immediate reduction to the case $S = Spec(R)$ where R is a Nagata ring. In this case we have to show that the integral closure A of R in $\Gamma(X, \mathcal{O}_X)$ is finite over R. Since f is of finite type we can write $X = \bigcup_{i=1,\ldots,n} U_i$ with each U_i affine. Say $U_i = Spec(B_i)$. Each B_i is a reduced ring of finite type over R (Lemma 24.14.2). Moreover, $\Gamma(X, \mathcal{O}_X) \subset B = B_1 \times \ldots \times B_n$. So A is contained in the integral closure A' of R in B. Note that B is a reduced finite type R-algebra. Since R is Noetherian it suffices to prove that A' is finite over R. This is Algebra, Lemma 7.144.16. $\square$

Next, we come to the normalization of a scheme X. We only define/construct it when X has locally finitely many irreducible components. Let X be a scheme such that every quasi-compact open has finitely many irreducible components. Let $X^{(0)} \subset X$ be the set of generic points of irreducible components of X. Let

$$f : Y = \coprod\nolimits_{\eta \in X^{(0)}} Spec(\kappa(\eta)) \longrightarrow X \tag{24.46.11.1}$$

be the inclusion of the generic points into X using the canonical maps of Schemes, Section 21.13. Note that this morphism is quasi-compact by assumption and quasi-separated as Y is separated (see Schemes, Section 21.21).

Definition 24.46.12. Let X be a scheme such that every quasi-compact open has finitely many irreducible components. We define the *normalization* of X as the morphism

$$\nu : X^\nu \longrightarrow X$$

which is the normalization of X in the morphism $f : Y \to X$ (24.46.11.1) constructed above.

Any locally Noetherian scheme has a locally finite set of irreducible components and the definition applies to it. Usually the normalization is defined only for reduced schemes. With the definition above the normalization of X is the same as the normalization of the reduction X_{red} of X.

[13]The proof shows that the lemma holds if f is quasi-compact and ``essentially of finite type''.

Lemma 24.46.13. *Let X be a scheme such that every quasi-compact open has finitely many irreducible components. The normalization morphism ν factors through the reduction X_{red} and $X^\nu \to X_{red}$ is the normalization of X_{red}.*

Proof. Let $f : Y \to X$ be the morphism (24.46.11.1). We get a factorization $Y \to X_{red} \to X$ of f from Schemes, Lemma 21.12.6. By Lemma 24.46.4 we obtain a canonical morphism $X^\nu \to X_{red}$ and that X^ν is the normalization of X_{red} in Y. The lemma follows as $Y \to X_{red}$ is identical to the morphism (24.46.11.1) constructed for X_{red}. □

If X is reduced, then the normalization of X is the same as the relative spectrum of the integral closure of $\mathcal{O}_X$ in the sheaf of meromorphic functions $\mathcal{K}_X$ (see Divisors, Section 26.15). Namely, $\mathcal{K}_X = f_*\mathcal{O}_Y$ in this case, see Divisors, Lemma 26.15.7 and its proof. We describe this here explicitly.

Lemma 24.46.14. *Let X be a reduced scheme such that every quasi-compact open has finitely many irreducible components. Let $Spec(A) = U \subset X$ be an affine open. Then*

1. *A has finitely many minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$,*
2. *the total ring of fractions $Q(A)$ of A is $Q(A/\mathfrak{q}_1) \times \ldots \times Q(A/\mathfrak{q}_t)$,*
3. *the integral closure A' of A in $Q(A)$ is the product of the integral closures of the domains $A/\mathfrak{q}_i$ in the fields $Q(A/\mathfrak{q}_i)$, and*
4. *$\nu^{-1}(U)$ is identified with the spectrum of A'.*

Proof. Minimal primes correspond to irreducible components (Algebra, Lemma 7.23.1), hence we have (1) by assumption. Then $(0) = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_t$ because A is reduced (Algebra, Lemma 7.16.2). Then we have $Q(A) = \prod A_{\mathfrak{q}_i} = \prod \kappa(\mathfrak{q}_i)$ by Algebra, Lemmas 7.22.2 and 7.23.3. This proves (2). Part (3) follows from Algebra, Lemma 7.33.14, or Lemma 24.46.7. Part (4) holds because it is clear that $f^{-1}(U) \to U$ is the morphism

$$Spec\left(\prod \kappa(\mathfrak{q}_i)\right) \longrightarrow Spec(A)$$

where $f : Y \to X$ is the morphism (24.46.11.1). □

Lemma 24.46.15. *Let X be a scheme such that every quasi-compact open has finitely many irreducible components.*

1. *The normalization X^ν is a normal scheme.*
2. *The morphism $\nu : X^\nu \to X$ is integral, surjective, and induces a bijection on irreducible components.*
3. *For any integral, birational[14] morphism $X' \to X$ there exists a factorization $X^\nu \to X' \to X$ and $X^\nu \to X'$ is the normalization of X'.*
4. *For any morphism $Z \to X$ with Z a normal scheme such that each irreducible component of Z dominates an irreducible component of X there exists a unique factorization $Z \to X^\nu \to X$.*

Proof. Let $f : Y \to X$ be as in (24.46.11.1). Part (1) follows from Lemma 24.46.10 and the fact that Y is normal. It also follows from the description of the affine opens in Lemma 24.46.14.

The morphism ν is integral by Lemma 24.46.4. By Lemma 24.46.10 the morphism $Y \to X^\nu$ induces a bijection on irreducible components, and by construction of Y this implies that $X^\nu \to X$ induces a bijection on irreducible components. By construction $f : Y \to X$ is

[14]It suffices if $X'_{red} \to X_{red}$ is birational.

dominant, hence also ν is dominant. Since an integral morphism is closed (Lemma 24.42.7) this implies that ν is surjective. This proves (2).

Suppose that $\alpha : X' \to X$ is integral and birational. Any quasi-compact open U' of X' maps to a quasi-compact open of X, hence we see that U' has only finitely many irreducible components. Let $f' : Y' \to X'$ be the morphism (24.46.11.1) constructed starting with X'. As α is birational it is clear that $Y' = Y$ and $f = \alpha \circ f'$. Hence the factorization $X^\nu \to X' \to X$ exists and $X^\nu \to X'$ is the normalization of X' by Lemma 24.46.4. This proves (3).

Let $g : Z \to X$ be a morphism whose domain is a normal scheme and such that every irreducible component dominates an irreducible component of X. By Lemma 24.46.13 we have $X^\nu = X^\nu_{red}$ and by Schemes, Lemma 21.12.6 $Z \to X$ factors through X_{red}. Hence we may replace X by X_{red} and assume X is reduced. Moreover, as the factorization is unique it suffices to construct it locally on Z. Let $W \subset Z$ and $U \subset X$ be affine opens such that $g(W) \subset U$. Write $U = Spec(A)$ and $W = Spec(B)$, with $g|_W$ given by $\varphi : A \to B$. We will use the results of Lemma 24.46.14 freely. Let $\mathfrak p_1, \ldots, \mathfrak p_t$ be the minimal primes of A. As Z is normal, we see that B is a normal ring, in particular reduced. Moreover, by assumption any minimal prime $\mathfrak q \subset B$ we have that $\varphi^{-1}(\mathfrak q)$ is a minimal prime of A. Hence if $x \in A$ is a nonzero divisor, i.e., $x \not\in \bigcup \mathfrak p_i$, then $\varphi(x)$ is a nonzero divisor in B. Thus we obtain a canonical ring map $Q(A) \to Q(B)$. As B is normal it is equal to its integral closure in $Q(B)$ (see Algebra, Lemma 7.33.11). Hence we see that the integral closure $A' \subset Q(A)$ of A maps into B via the canonical map $Q(A) \to Q(B)$. Since $\nu^{-1}(U) = Spec(A')$ this gives the canonical factorization $W \to \nu^{-1}(U) \to U$ of $\nu|_W$. We omit the verification that it is unique. □

Lemma 24.46.16. *Let X be an integral, Japanese scheme. The normalization $\nu : X^\nu \to X$ is a finite morphism.*

Proof. Follows from the definitions and Lemma 24.46.14. Namely, in this case the lemma says that $\nu^{-1}(Spec(A))$ is the spectrum of the integral closure of A in its field of fractions. □

Lemma 24.46.17. *Let X be a nagata scheme. The normalization $\nu : X^\nu \to X$ is a finite morphism.*

Proof. Note that a Nagata scheme is locally Noetherian, thus Definition 24.46.12 does apply. Write $X^\nu \to X$ as the composition $X^\nu \to X_{red} \to X$. As $X_{red} \to X$ is a closed immersion it is finite. Hence it suffices to prove the lemma for a reduced Nagata scheme (by Lemma 24.42.5). Let $Spec(A) = U \subset X$ be an affine open. By Lemma 24.46.14 we have $\nu^{-1}(U) = Spec(\prod A'_i)$ where A'_i is the integral closure of $A/\mathfrak q_i$ in its fraction field. As A is a Nagata ring (see Properties, Lemma 23.13.6) each of the ring extensions $A/\mathfrak q_i \subset A'_i$ are finite. Hence $A \to \prod A'_i$ is a finite ring map and we win. □

24.47. Zariski's Main Theorem (algebraic version)

This is the version you can prove using purely algebraic methods. Before we can prove more powerful versions (for non-affine morphisms) we need to develop more tools. See Coherent, Section 25.20 and More on Morphisms, Section 33.29.

Theorem 24.47.1. *(Algebraic version of Zariski's Main Theorem) Let $f : Y \to X$ be an affine morphism of schemes. Assume f is of finite type. Let X' be the normalization of X*

in Y. Picture:

$$\begin{tikzcd} Y \arrow[rr, "f'"] \arrow[dr, "f"'] & & X' \arrow[dl, "\nu"] \\ & X & \end{tikzcd}$$

Then there exists an open subscheme $U' \subset X'$ such that

(1) $(f')^{-1}(U') \to U'$ *is an isomorphism, and*
(2) $(f')^{-1}(U') \subset Y$ *is the set of points at which f is quasi-finite.*

Proof. There is an immediate reduction to the case where X and hence Y are affine. Say $X = Spec(R)$ and $Y = Spec(A)$. Then $X' = Spec(A')$, where A' is the integral closure of R in A, see Definitions 24.46.2 and 24.46.3. By Algebra, Theorem 7.114.13 for every $y \in Y$ at which f is quasi-finite, there exists an open $U'_y \subset X'$ such that $(f')^{-1}(U'_y) \to U'_y$ is an isomorphism. Set $U' = \bigcup U'_y$ where $y \in Y$ ranges over all points where f is quasi-finite. It remains to show that f is quasi-finite at all points of $(f')^{-1}(U')$. If $y \in (f')^{-1}(U')$ with image $x \in X$, then we see that $Y_x \to X'_x$ is an isomorphism in a neighbourhood of y. Hence there is no point of Y_x which specializes to y, since this is true for $f'(y)$ in X'_x, see Lemma 24.42.8. By Lemma 24.19.6 part (3) this implies f is quasi-finite at y. □

We can use the algebraic version of Zariski's Main Theorem to show that the set of points where a morphism is quasi-finite is open.

Lemma 24.47.2. *Let $f : X \to S$ be a morphism of schemes. The set of points of X where f is quasi-finite is an open $U \subset X$. The induced morphism $U \to S$ is locally quasi-finite.*

Proof. Suppose f is quasi-finite at x. Let $x \in U = Spec(R) \subset X$, $V = Spec(A) \subset S$ be affine opens as in Definition 24.19.1. By either Theorem 24.47.1 above or Algebra, Lemma 7.114.14, the set of primes $\mathfrak{q}$ at which $R \to A$ is quasi-finite is open in $Spec(A)$. Since these all correspond to points of X where f is quasi-finite we get the first statement. The second statement is obvious. □

We will improve the following lemma to general quasi-finite separated morphisms later, see More on Morphisms, Lemma 33.29.4.

Lemma 24.47.3. *Let $f : Y \to X$ be a morphism of schemes. Assume*

(1) *X and Y are affine, and*
(2) *f is quasi-finite.*

Then there exists a diagram

$$\begin{tikzcd} Y \arrow[rr, "j"] \arrow[dr, "f"'] & & Z \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

with Z affine, π finite and j an open immersion.

Proof. This is Algebra, Lemma 7.114.15 reformulated in the laguage of schemes. □

Lemma 24.47.4. *Let $f : Y \to X$ be a quasi-finite morphism of schemes. Let $T \subset Y$ be a closed nowhere dense subset of Y. Then $f(T) \subset X$ is a nowhere dense subset of X.*

Proof. As in the proof of Lemma 24.44.7 this reduces immediately to the case where the base X is affine. In this case $Y = \bigcup_{i=1,\ldots,n} Y_i$ is a finite union of affine opens (as f is quasi-compact). Since each $T \cap Y_i$ is nowhere dense, and since a finite union of nowhere dense sets is nowhere dense (see Topology, Lemma 5.17.2), it suffices to prove that the image $f(T \cap Y_i)$ is nowhere dense in X. This reduces us to the case where both X and Y are affine. At this point we apply Lemma 24.47.3 above to get a diagram

$$\begin{tikzcd} Y \arrow[rr, "j"] \arrow[dr, "f"'] & & Z \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

with Z affine, π finite and j an open immersion. Set $\overline{T} = \overline{j(T)} \subset Z$. By Topology, Lemma 5.17.3 we see $\overline{T}$ is nowhere dense in Z. Since $f(T) \subset \pi(\overline{T})$ the lemma follows from the corresponding result in the finite case, see Lemma 24.44.7. □

24.48. Universally bounded fibres

Let X be a scheme over a field k. If X is finite over k, then $X = Spec(A)$ where A is a finite k-algebra. Another way to say this is that X is finite locally free over $Spec(k)$, see Definition 24.44.1. Hence $X \to Spec(k)$ has a *degree* which is an integer $d \geq 0$, namely $d = \dim_k(A)$. We sometime call this the *degree* of the (finite) scheme X over k.

Definition 24.48.1. Let $f : X \to Y$ be a morphism of schemes.

(1) We say the integer *n bounds the degrees of the fibres of f* if for all $y \in Y$ the fibre X_y is a finite scheme over $\kappa(y)$ whose degree over $\kappa(y)$ is $\leq n$.
(2) We say the *fibres of f are universally bounded*[15] if there exists an integer n which bounds the degrees of the fibres of f.

Note that in particular the number of points in a fibre is bounded by n as well. (The converse does not hold, even if all fibres are finite reduced schemes.)

Lemma 24.48.2. *Let $f : X \to Y$ be a morphism of schemes. Let $n \geq 0$. The following are equivalent:*

(1) *the integer n bounds the degrees of the fibres of f, and*
(2) *for every morphism $Spec(k) \to Y$, where k is a field, the fibre product $X_k = Spec(k) \times_Y X$ is finite over k of degree $\leq n$.*

In this case f is universally bounded and the schemes X_k have at most n points.

Proof. The implication (2) ⇒ (1) is trivial. The other implication holds because if the image of $Spec(k) \to Y$ is y, then $X_k = Spec(k) \times_{Spec(\kappa(y))} X_y$. □

Lemma 24.48.3. *A composition of morphisms with universally bounded fibres is a morphism with universally bounded fibres. More precisely, assume that n bounds the degrees of the fibres of $f : X \to Y$ and m bounds the degrees of $g : Y \to Z$. Then nm bounds the degrees of the fibres of $g \circ f : X \to Z$.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ have universally bounded fibres. Say that $\deg(X_y/\kappa(y)) \leq n$ for all $y \in Y$, and that $\deg(Y_z/\kappa(z)) \leq m$ for all $z \in Z$. Let $z \in Z$ be a point. By assumption the scheme Y_z is finite over $Spec(\kappa(z))$. In particular, the underlying topological space of Y_z is a finite discrete set. The fibres of the morphism $f_z : X_z \to Y_z$ are the fibres of f at the corresponding points of Y, which are finite discrete sets by the reasoning above. Hence we conclude that the underlying topological space of X_z is a finite discrete set as well. Thus X_z is an affine scheme (this is a nice exercise; it also follows

[15]This is probably nonstandard notation.

for example from Properties, Lemma 23.26.1 applied to the set of all points of X_z). Write $X_z = Spec(A)$, $Y_z = Spec(B)$, and $k = \kappa(z)$. Then $k \to B \to A$ and we know that (a) $\dim_k(B) \leq m$, and (b) for every maximal ideal $\mathfrak{m} \subset B$ we have $\dim_{\kappa(\mathfrak{m})}(A/\mathfrak{m}A) \leq n$. We claim this implies that $\dim_k(A) \leq nm$. Note that B is the product of its localizations $B_{\mathfrak{m}}$, for example because Y_z is a disjoint union of 1-point schemes, or by Algebra, Lemmas 7.49.2 and 7.49.8. So we see that $\dim_k(B) = \sum_{\mathfrak{m}}(B_{\mathfrak{m}})$ and $\dim_k(A) = \sum_{\mathfrak{m}}(A_{\mathfrak{m}})$ where in both cases $\mathfrak{m}$ runs over the maximal ideals of B (not of A). By the above, and Nakayama's Lemma (Algebra, Lemma 7.14.5) we see that each $A_{\mathfrak{m}}$ is a quotient of $B_{\mathfrak{m}}^{\oplus n}$ as a $B_{\mathfrak{m}}$-module. Hence $\dim_k(A_{\mathfrak{m}}) \leq n \dim_k(B_{\mathfrak{m}})$. Putting everything together we see that

$$\dim_k(A) = \sum\nolimits_{\mathfrak{m}}(A_{\mathfrak{m}}) \leq \sum\nolimits_{\mathfrak{m}} n \dim_k(B_{\mathfrak{m}}) = n \dim_k(B) \leq nm$$

as desired. □

Lemma 24.48.4. *A base change of a morphism with universally bounded fibres is a morphism with universally bounded fibres. More precisely, if n bounds the degrees of the fibres of $f : X \to Y$ and $Y' \to Y$ is any morphism, then the degrees of the fibres of the base change $f' : Y' \times_Y X \to Y' \to Y'$ is also bounded by n.*

Proof. This is clear from the result of Lemma 24.48.2. □

Lemma 24.48.5. *Let $f : X \to Y$ be a morphism of schemes. Let $Y' \to Y$ be a morphism of schemes, and let $f' : X' = X_{Y'} \to Y'$ be the base change of f. If $Y' \to Y$ is surjective and f' has universally bounded fibres, then f has universally bounded fibres. More precisely, if n bounds the degree of the fibres of f', then also n bounds the degrees of the fibres of f.*

Proof. Let $n \geq 0$ be an integer bounding the degrees of the fibres of f'. We claim that n works for f also. Namely, if $y \in Y$ is a point, then choose a point $y' \in Y'$ lying over y and observe that

$$X'_{y'} = Spec(\kappa(y')) \times_{Spec(\kappa(y))} X_y.$$

Since $X'_{y'}$ is assumed finite of degree $\leq n$ over $\kappa(y')$ it follows that also X_y is finite of degree $\leq n$ over $\kappa(y)$. (Some details omitted.) □

Lemma 24.48.6. *An immersion has universally bounded fibres.*

Proof. The integer $n = 1$ works in the definition. □

Lemma 24.48.7. *Let $f : X \to Y$ be an étale morphism of schemes. Let $n \geq 0$. The following are equivalent*

(1) *the integer n bounds the degrees of the fibres,*
(2) *for every field k and morphism $Spec(k) \to Y$ the base change $X_k = Spec(k) \times_Y X$ has at most n points, and*
(3) *for every $y \in Y$ and every separable algebraic closure $\kappa(y) \subset \kappa(y)^{sep}$ the scheme $X_{\kappa(y)^{sep}}$ has at most n points.*

Proof. This follows from Lemma 24.48.2 and the fact that the fibres X_y are disjoint unions of spectra of finite separable field extensions of $\kappa(y)$, see Lemma 24.35.7. □

Having universally bounded fibres is an absolute notion and not a relative notion. This is why the condition in the following lemma is that X is quasi-compact, and not that f is quasi-compact.

Lemma 24.48.8. *Let $f : X \to Y$ be a morphism of schemes. Assume that*

(1) *f is locally quasi-finite, and*

(2) *X is quasi-compact.*

Then f has universally bounded fibres.

Proof. Since X is quasi-compact, there exists a finite affine open covering $X = \bigcup_{i=1,\ldots,n} U_i$ and affine opens $V_i \subset Y$, $i = 1, \ldots, n$ such that $f(U_i) \subset V_i$. Because of the local nature of ``local quasi-finiteness'' (see Lemma 24.19.6 part (4)) we see that the morphisms $f|_{U_i} : U_i \to V_i$ are locally quasi-finite morphisms of affines, hence quasi-finite, see Lemma 24.19.9. For $y \in Y$ it is clear that $X_y = \bigcup_{y \in V_i} (U_i)_y$ is an open covering. Hence it suffices to prove the lemma for a quasi-finite morphism of affines (namely, if n_i works for the morphism $f|_{U_i} : U_i \to V_i$, then $\sum n_i$ works for f).

Assume $f : X \to Y$ is a quasi-finite morphism of affines. By Lemma 24.47.3 we can find a diagram

$$\begin{tikzcd} X \arrow[rr, "j"] \arrow[dr, "f"'] & & Z \arrow[dl, "\pi"] \\ & Y & \end{tikzcd}$$

with Z affine, π finite and j an open immersion. Since j has universally bounded fibres (Lemma 24.48.6) this reduces us to showing that π has universally bounded fibres (Lemma 24.48.3).

This reduces us to a morphism of the form $Spec(B) \to Spec(A)$ where $A \to B$ is finite. Say B is generated by $x_1, \ldots, x_n$ over A and say $P_i(T) \in A[T]$ is a monic polynomial of degree d_i such that $P_i(x_i) = 0$ in B (a finite ring extension is integral, see Algebra, Lemma 7.32.3). With these notations it is clear that

$$\bigoplus_{0 \leq e_i < d_i, i=1,\ldots n} A \longrightarrow B, \quad (a_{(e_1,\ldots,e_n)}) \longmapsto \sum a_{(e_1,\ldots,e_n)} x_1^{e_1} \ldots x_n^{e_n}$$

is a surjective A-module map. Thus for any prime $\mathfrak{p} \subset A$ this induces a surjective map $\kappa(\mathfrak{p})$-vector spaces

$$\kappa(\mathfrak{p})^{\oplus d_1 \ldots d_n} \longrightarrow B \otimes_A \kappa(\mathfrak{p})$$

In other words, the integer $d_1 \ldots d_n$ works in the definition of a morphism with universally bounded fibres. $\square$

Lemma 24.48.9. *Consider a commutative diagram of morphisms of schemes*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \arrow[dl, "h"] \\ & Z & \end{tikzcd}$$

If g has universally bounded fibres, and f is surjective and flat, then also h has universally bounded fibres. More precisely, if n bounds the degree of the fibres of g, then also n bounds the degree of the fibres of h.

Proof. Assume g has universally bounded fibres, and f is surjective and flat. Say the degree of the fibres of g is bounded by $n \in \mathbf{N}$. We claim n also works for h. Let $z \in Z$. Consider the morphism of schemes $X_z \to Y_z$. It is flat and surjective. By assumption X_z is a finite scheme over $\kappa(z)$, in particular it is the spectrum of an Artinian ring (by Algebra, Lemma 7.49.2). By Lemma 24.11.13 the morphism $X_z \to Y_z$ is affine in particular quasi-compact. It follows from Lemma 24.24.10 that Y_z is a finite discrete as this holds for X_z. Hence Y_z is an affine scheme (this is a nice exercise; it also follows for example from Properties, Lemma

23.26.1 applied to the set of all points of Y_z). Write $Y_z = Spec(B)$ and $X_z = Spec(A)$. Then A is faithfully flat over B, so $B \subset A$. Hence $\dim_k(B) \leq \dim_k(A) \leq n$ as desired. □

24.49. Other chapters

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CHAPTER 25

Coherent Cohomology

25.1. Introduction

The title of this chapter is a bit of a lie because we will first prove a number of results on the cohomology of quasi-coherent sheaves. A fundamental reference is [**DG67**]. Having done this we wil elaborate on cohomology of coherent sheaves in the Noetherian setting. See also [**Ser55b**].

25.2. Cech cohomology of quasi-coherent sheaves

Let X be a scheme. Let $U \subset X$ be an affine open. Recall that a *standard open covering* of U is a covering of the form $\mathcal{U} : U = \bigcup_{i=1}^n D(f_i)$ where $f_1, \ldots, f_n \in \Gamma(U, \mathcal{O}_X)$ generate the unit ideal, see Schemes, Definition 21.5.2.

Lemma 25.2.1. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{U} : U = \bigcup_{i=1}^n D(f_i)$ be a standard open covering of an affine open of X. Then $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for all $p > 0$.*

Proof. Write $U = Spec(A)$ for some ring A. In other words, $f_1, \ldots, f_n$ are elements of A which generate the unit ideal of A. Write $\mathcal{F}|_U = \widetilde{M}$ for some A-module M. Clearly the Cech complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is identified with the complex

$$\prod_{i_0} M_{f_{i_0}} \to \prod_{i_0 i_1} M_{f_{i_0} f_{i_1}} \to \prod_{i_0 i_1 i_2} M_{f_{i_0} f_{i_1} f_{i_2}} \to \ldots$$

We are asked to show that the extended complex

$$0 \to M \to \prod_{i_0} M_{f_{i_0}} \to \prod_{i_0 i_1} M_{f_{i_0} f_{i_1}} \to \prod_{i_0 i_1 i_2} M_{f_{i_0} f_{i_1} f_{i_2}} \to \ldots \tag{25.2.1.1}$$

(whose truncation we have studied in Algebra, Lemma 7.20.2) is exact. It suffices to show that (25.2.1.1) is exact after localizing at a prime $\mathfrak{p}$, see Algebra, Lemma 7.21.1. In fact we will show that the extended complex localized at $\mathfrak{p}$ is homotopic to zero.

There exists an index i such that $f_i \not\in \mathfrak{p}$. Choose and fix such an element i_{fix}. Note that $M_{f_{i_{\text{fix}}}, \mathfrak{p}} = M_{\mathfrak{p}}$. Similarly for a localization at a product $f_{i_0} \ldots f_{i_p}$ and $\mathfrak{p}$ we can drop any f_{i_j} for which $i_j = i_{\text{fix}}$. Let us define a homotopy

$$h : \prod\nolimits_{i_0 \ldots i_{p+1}} M_{f_{i_0} \ldots f_{i_{p+1}}, \mathfrak{p}} \longrightarrow \prod\nolimits_{i_0 \ldots i_p} M_{f_{i_0} \ldots f_{i_p}, \mathfrak{p}}$$

by the rule

$$h(s)_{i_0 \ldots i_p} = s_{i_{\text{fix}} i_0 \ldots i_p}$$

(This is ``dual" to the homotopy in the proof of Cohomology, Lemma 18.10.4.) In other words, $h : \prod_{i_0} M_{f_{i_0}, \mathfrak{p}} \to M$ is projection onto the factor $M_{f_{i_{\text{fix}}}, \mathfrak{p}} = M_{\mathfrak{p}}$ and in general the map h equal projection onto the factors $M_{f_{i_{\text{fix}}} f_{i_1} \ldots f_{i_{p+1}}, \mathfrak{p}} = M_{f_{i_1} \ldots f_{i_{p+1}}, \mathfrak{p}}$. We compute

$$\begin{aligned}(dh + hd)(s)_{i_0 \ldots i_p} &= \sum_{j=0}^{p} (-1)^j h(s)_{i_0 \ldots \hat{i}_j \ldots i_p} + d(s)_{i_{\text{fix}} i_0 \ldots i_p} \\ &= \sum_{j=0}^{p} (-1)^j s_{i_{\text{fix}} i_0 \ldots \hat{i}_j \ldots i_p} + s_{i_0 \ldots i_p} + \sum_{j=0}^{p} (-1)^{j+1} s_{i_{\text{fix}} i_0 \ldots \hat{i}_j \ldots i_p} \\ &= s_{i_0 \ldots i_p}\end{aligned}$$

This proves the identity map is homotopic to zero as desired. □

The following lemma says in particular that for any affine scheme X and any quasi-coherent sheaf $\mathcal{F}$ on X we have

$$H^p(X, \mathcal{F}) = 0$$

for all $p > 0$.

Lemma 25.2.2. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. For any affine open $U \subset X$ we have $H^p(U, \mathcal{F}) = 0$ for all $p > 0$.*

Proof. We are going to apply Cohomology, Lemma 18.11.8. As our basis $\mathcal{B}$ for the topology of X we are going to use the affine opens of X. As our set Cov of open coverings we are going to use the standard open coverings of affine opens of X. Next we check that conditions (1), (2) and (3) of Cohomology, Lemma 18.11.8 hold. Note that the intersection of standard opens in an affine is another standard open. Hence property (1) holds. The coverings form a cofinal system of open coverings of any element of $\mathcal{B}$, see Schemes, Lemma 21.5.1. Hence (2) holds. Finally, condition (3) of the lemma follows from Lemma 25.2.1. □

Here is a relative version of the vanishing of cohomology of quasi-coherent sheaves on affines.

Lemma 25.2.3. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If f is affine then $R^i f_* \mathcal{F} = 0$ for all $i > 0$.*

Proof. According to Cohomology, Lemma 18.6.3 the sheaf $R^i f_* \mathcal{F}$ is the sheaf associated to the presheaf $V \mapsto H^i(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)})$. By assumption, whenever V is affine we have that $f^{-1}(V)$ is affine, see Morphisms, Definition 24.11.1. By Lemma 25.2.2 we conclude that $H^i(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)}) = 0$ whenever V is affine. Since S has a basis consisting of affine opens we win. □

Lemma 25.2.4. *Let X be a scheme. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be an open covering such that $U_{i_0 \ldots i_p}$ is affine open for all $p \geq 0$ and all $i_0, \ldots, i_p \in I$ In this case for any quasi-coherent sheaf $\mathcal{F}$ we have*

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(X, \mathcal{F})$$

as $\Gamma(X, \mathcal{O}_X)$-modules for all p.

Proof. In view of Lemma 25.2.2 this is a special case of Cohomology, Lemma 18.11.5. □

25.3. Vanishing of cohomology

We have seen that on an affine scheme the higher cohomology groups of any quasi-coherent sheaf vanish (Lemma 25.2.2). It turns out that this also characterizes affine schemes. We give two versions allthough the first covers all conceivable cases.

Lemma 25.3.1. *Let X be a scheme. Assume that*

(1) *X is quasi-compact,*
(2) *for every quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ we have $H^1(X, \mathcal{I}) = 0$.*

Then X is affine.

Proof. Let $x \in X$ be a closed point. Let $U \subset X$ be an affine open neighbourhood of x. Write $U = Spec(A)$ and let $\mathfrak{m} \subset A$ be the maximal ideal corresponding to x. Set $Z = X \setminus U$ and $Z' = Z \cup \{x\}$. By Schemes, Lemma 21.12.4 there are quasi-coherent sheaves of ideals $\mathcal{I}$, resp. $\mathcal{I}'$ cutting out the reduced closed subschemes Z, resp. Z'. Consider the short exact sequence

$$0 \to \mathcal{I}' \to \mathcal{I} \to \mathcal{I}/\mathcal{I}' \to 0.$$

Since x is a closed point of X and $x \notin Z$ we see that $\mathcal{I}/\mathcal{I}'$ is supported at x. In fact, the restriction of $\mathcal{I}/\mathcal{I}'$ to U corresponds to the A-module $A/\mathfrak{m}$. Hence we see that $\Gamma(X, \mathcal{I}/\mathcal{I}') = A/\mathfrak{m}$. Since by assumption $H^1(X, \mathcal{I}') = 0$ we see there exists a global section $f \in \Gamma(X, \mathcal{I})$ which maps to the element $1 \in A/\mathfrak{m}$ as a section of $\mathcal{I}/\mathcal{I}'$. Clearly we have $x \in X_f \subset U$. This implies that $X_f = D(f_A)$ where f_A is the image of f in $A = \Gamma(U, \mathcal{O}_X)$. In particular X_f is affine.

Consider the union $W = \bigcup X_f$ over all $f \in \Gamma(X, \mathcal{O}_X)$ such that X_f is affine. Obviously W is open in X. By the arguments above every closed point of X is contained in W. The closed subset $X \setminus W$ of X is also quasi-compact (see Topology, Lemma 5.9.3). Hence it has a closed point if it is nonempty (see Topology, Lemma 5.9.6). This is a would contradict the fact that all closed points are in W. Hence we conclude $X = W$.

Choose finitely many $f_1, \ldots, f_n \in \Gamma(X, \mathcal{O}_X)$ such that $X = X_{f_1} \cup \ldots \cup X_{f_n}$ and such that each X_{f_i} is affine. This is possible as we've seen above. By Properties, Lemma 23.24.2 to finish the proof it suffices to show that $f_1, \ldots, f_n$ generate the unit ideal in $\Gamma(X, \mathcal{O}_X)$. Conisder the short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_X^{\oplus n} \xrightarrow{f_1, \ldots, f_n} \mathcal{O}_X \longrightarrow 0$$

The arrow defined by $f_1, \ldots, f_n$ is surjective since the opens X_{f_i} cover X. We let $\mathcal{F}$ be the kernel of this surjective map. Observe that $\mathcal{F}$ has a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_n = \mathcal{F}$$

so that each subquotient $\mathcal{F}_i/\mathcal{F}_{i-1}$ is isomorphic to a quasi-coherent sheaf of ideals. Namely we can take $\mathcal{F}_i$ to be the intersection of the first i direct summands of $\mathcal{O}_X^{\oplus n}$. The assumption of the lemma implies that $H^1(X, \mathcal{F}_i/\mathcal{F}_{i-1}) = 0$ for all i. This implies that $H^1(X, \mathcal{F}_2) = 0$ because it is sandwiched between $H^1(X, \mathcal{F}_1)$ and $H^1(X, \mathcal{F}_2/\mathcal{F}_1)$. Continuing like this we deduce that $H^1(X, \mathcal{F}) = 0$. Therefore we conclude that the map

$$\bigoplus\nolimits_{i=1,\ldots,n} \Gamma(X, \mathcal{O}_X) \xrightarrow{f_1, \ldots, f_n} \Gamma(X, \mathcal{O}_X)$$

is surjective as desired. $\square$

Note that if X is a Noetherian scheme then every quasi-coherent sheaf of ideals is automatically a coherent sheaf of ideals and a finite type quasi-coherent sheaf of ideals. Hence the preceding lemma and the next lemma both apply in this case.

Lemma 25.3.2. *Let X be a scheme. Assume that*

(1) *X is quasi-compact,*
(2) *X is quasi-separated, and*
(3) *$H^1(X, \mathcal{I}) = 0$ for every quasi-coherent sheaf of ideals $\mathcal{I}$ of finite type.*

Then X is affine.

Proof. By Properties, Lemma 23.20.3 every quasi-coherent sheaf of ideals is a directed colimit of quasi-coherent sheaves of ideals of finite type. By Cohomology, Lemma 18.15.1 taking cohomology on X commutes with directed colimits. Hence we see that $H^1(X, \mathcal{I}) = 0$ for every quasi-coherent sheaf of ideals on X. In other words we see that Lemma 25.3.1 applies. $\square$

25.4. Derived category of quasi-coherent modules

In this section we briefly discuss the relationship between quasi-coherent modules and all modules on a scheme S. (This should be elaborated on and generalized.) A reference is [**TT90**, Appendix B]. By the discussion in Schemes, Section 21.24 the embedding $QCoh(\mathcal{O}_S) \subset Mod(\mathcal{O}_S)$ exhibits $QCoh(\mathcal{O}_S)$ as a weak Serre subcategory of the category of $\mathcal{O}_S$-modules. Denote $D_{QCoh}(\mathcal{O}_S) \subset D(\mathcal{O}_S)$ the subcategory of complexes whose cohomology sheaves are quasi-coherent, see Derived Categories, Section 11.12. Thus we obtain a canonical functor

$$D(QCoh(\mathcal{O}_S)) \longrightarrow D_{QCoh}(\mathcal{O}_S) \tag{25.4.0.1}$$

see Derived Categories, Equation (11.12.1.1).

Lemma 25.4.1. *If $S = Spec(A)$ is an affine scheme, then (25.4.0.1) is an equivalence.*

Proof. The key to this lemma is to prove that the functor $R\Gamma(S, -)$ gives a quasi-inverse. For complexes bounded below this is straightforward using the vanishing of cohomology of Lemma 25.2.2. To prove it also for unbounded complexes we have to be a little bit careful: namely, even if you accept that the unbounded derived functor $R\Gamma(S, -)$ exists, then it isn't obvious how to compute it!

Let $\mathcal{F}^\bullet$ be an object of $D_{QCoh}(\mathcal{O}_S))$ and denote $\mathcal{H}^i = H^i(\mathcal{F}^\bullet)$ its ith cohomology sheaf. Let $\mathcal{B}$ be the set of affine open subsets of S. Then $H^p(U, \mathcal{H}^i) = 0$ for all $p > 0$, all $i \in \mathbf{Z}$, and all $U \in \mathcal{B}$, see Lemma 25.2.2. According to Cohomology, Section 18.23 this implies there exists a quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ where $\mathcal{I}^\bullet$ is a K-injective complex, $\mathcal{I}^\bullet = lim\, \mathcal{I}_n^\bullet$, each $\mathcal{I}_n^\bullet$ is a bounded below complex of injectives, the maps in the system $\ldots \to \mathcal{I}_2^\bullet \to \mathcal{I}_1^\bullet$ are termwise split surjections, and each $\mathcal{I}_n^\bullet$ is quasi-isomorphic to $\tau_{\geq -n}\mathcal{F}^\bullet$. In particular, we conclude that $R\Gamma(S, -)$ is defined at each object of $D_{QCoh}(\mathcal{O}_S))$, see Derived Categories, Lemma 11.28.4, with values $R\Gamma(S, \mathcal{F}^\bullet) = \Gamma(S, \mathcal{I}^\bullet)$. This defines an exact functor of triangulated categories

$$R\Gamma(S, -) : D_{QCoh}(\mathcal{O}_S) \longrightarrow D(A), \tag{25.4.1.1}$$

see Derived Categories, Proposition 11.14.8. In the proof of Cohomology, Lemma 18.23.1 we have seen that $H^m(\Gamma(S, \mathcal{I}^\bullet))$ is the limit of the cohomology groups $H^m(\Gamma(S, \mathcal{I}_n^\bullet))$. For $n > -m$ these groups are equal to $\Gamma(S, \mathcal{H}^m)$ by the vanishing of higer cohomology and the spectral sequence of Derived Categories, Lemma 11.20.3. Combined with the (assumed) equality

$$\mathcal{H}^m = \Gamma(\widetilde{S, \mathcal{H}^m})$$

we conclude the canonical map of complexes

$$\Gamma(\widetilde{S, \mathcal{I}^\bullet}) \longrightarrow \mathcal{I}^\bullet$$

(see Schemes, Lemma 21.7.1) is a quasi-isomorphism. We claim the composition

$$D(A) \cong D(QCoh(\mathcal{O}_S)) \longrightarrow D_{QCoh}(\mathcal{O}_S) \longrightarrow D(A)$$

is isomorphic to the identity functor. Namely, given a complex of A-modules $M^\bullet$, let $\mathcal{F}^\bullet = \widetilde{M^\bullet}$, choose $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ as above, and finally take $\Gamma(S, \mathcal{I}^\bullet)$. The arguments above show that $M^\bullet = \Gamma(S, \mathcal{F}^\bullet) \to \Gamma(S, \mathcal{I}^\bullet)$ is a quasi-isomorphism. This is functorial in $M^\bullet$, hence we conclude that the composition of functors is isomorphic to the identity functor on $D(A)$. On the other hand, we have seen above that the composition

$$D_{QCoh}(\mathcal{O}_S) \longrightarrow D(A) \cong D(QCoh(\mathcal{O}_S)) \longrightarrow D_{QCoh}(\mathcal{O}_S)$$

is isomorphic to the identity functor, via the quasi-isomorphisms $\Gamma(\widetilde{S, \mathcal{I}^\bullet}) \to \mathcal{I}^\bullet$ above. This finishes the proof. $\square$

Actually it is true that the comparison map $D(QCoh(\mathcal{O}_S)) \to D_{QCoh}(\mathcal{O}_S)$ is an equivalence for any quasi-compact and (semi-)separated scheme (insert future reference here).

25.5. Quasi-coherence of higher direct images

We have seen that the higher cohomology groups of a quasi-coherent module on an affine is zero. For (quasi-)separated quasi-compact schemes X this implies vanishing of cohomology groups of quasi-coherent sheaves beyond a certain degree. However, it may not be the case that X has finite cohomological dimension, because that is defined in terms of vanishing of cohomology of *all* $\mathcal{O}_X$-modules.

Lemma 25.5.1. *Let X be a quasi-compact separated scheme. Let $t = t(X)$ be the minimal number of affine opens needed to cover X. Then $H^n(X, \mathcal{F}) = 0$ for all $n \geq t$ and all quasi-coherent sheaves $\mathcal{F}$.*

Proof. First proof. By induction on t. If $t = 1$ the result follows from Lemma 25.2.2. If $t > 1$ write $X = U \cup V$ with V affine open and $U = U_1 \cup \ldots \cup U_{t-1}$ a union of $t-1$ open affines. Note that in this case $U \cap V = (U_1 \cap V) \cup \ldots (U_{t-1} \cap V)$ is also a union of $t-1$ affine open subschemes, see Schemes, Lemma 21.21.8. We apply the Mayer-Vietoris long exact sequence

$$0 \to H^0(X, \mathcal{F}) \to H^0(U, \mathcal{F}) \oplus H^0(V, \mathcal{F}) \to H^0(U \cap V, \mathcal{F}) \to H^1(X, \mathcal{F}) \to \ldots$$

see Cohomology, Lemma 18.8.2. By induction we see that the groups $H^i(U, \mathcal{F})$, $H^i(U, \mathcal{F})$, $H^i(U, \mathcal{F})$ are zero for $i \geq t-1$. It follows immediately that $H^i(X, \mathcal{F})$ is zero for $i \geq t$.

Second proof. Let $\mathcal{U} : X = \bigcup_{i=1}^t U_i$ be a finite affine open covering. Since X is separated the multiple intersections $U_{i_0 \ldots i_p}$ are all affine, see Schemes, Lemma 21.21.8. By Lemma 25.2.4 the Cech cohomology groups $\check{H}^p(\mathcal{U}, \mathcal{F})$ agree with the cohomology groups. By Cohomology, Lemma 18.17.6 the Cech cohomology groups may be computed using the alternating Cech complex $\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F})$. As the covering consists of t elements we see immediately that $\check{\mathcal{C}}^p_{alt}(\mathcal{U}, \mathcal{F}) = 0$ for all $p \geq t$. Hence the result follows. $\square$

Lemma 25.5.2. *Let X be a quasi-compact quasi-separated scheme. Let $X = U_1 \cup \ldots \cup U_t$ be an affine open covering. Set*

$$d = \max_{I \subset \{1,\ldots,t\}} |I| + t(\bigcap\nolimits_{i \in I} U_i)$$

where $t(U)$ is the minimal number of affines needed to cover the scheme U. Then $H^n(X, \mathcal{F}) = 0$ for all $n \geq d$ and all quasi-coherent sheaves $\mathcal{F}$.

Proof. Note that since X is quasi-separated the numbers $t(\bigcap_{i \in I} U_i)$ are finite. Let $\mathcal{U} : X = \bigcup_{i=1}^t U_i$. By Cohomology, Lemma 18.11.4 there is a spectral sequence

$$E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F}))$$

converging to $H^{p+q}(U, \mathcal{F})$. By Cohomology, Lemma 18.17.6 we have

$$E_2^{p,q} = H^p(\check{\mathcal{C}}_{alt}^\bullet(\mathcal{U}, \underline{H}^q(\mathcal{F}))$$

The alternating Čech complex with values in the presheaf $\underline{H}^q(\mathcal{F})$ vanishes in high degrees by Lemma 25.5.1, more precisely $E_2^{p,q} = 0$ for $p + q \geq d$. Hence the result follows. □

Lemma 25.5.3. *Let $f : X \to S$ be a morphism of schemes. Assume that f is quasi-separated and quasi-compact.*

1. *For any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the higher direct images $R^p f_*\mathcal{F}$ are quasi-coherent on S.*
2. *If S is quasi-compact, there exists an integer $n = n(X, S, f)$ such that $R^p f_*\mathcal{F} = 0$ for all $p \geq n$ and any quasi-coherent sheaf $\mathcal{F}$ on X.*
3. *In fact, if S is quasi-compact we can find $n = n(X, S, f)$ such that for every morphism of schemes $S' \to S$ we have $R^p(f')_*\mathcal{F}' = 0$ for $p \geq n$ and any quasi-coherent sheaf $\mathcal{F}'$ on X'. Here $f' : X' = S' \times_S X \to S'$ is the base change of f.*

Proof. We first prove (1). Note that under the hypotheses of the lemma the sheaf $R^0 f_*\mathcal{F} = f_*\mathcal{F}$ is quasi-coherent by Schemes, Lemma 21.24.1. Using Cohomology, Lemma 18.6.4 we see that forming higher direct images commutes with restriction to open subschemes. Since being quasi-coherent is local on S we may assume S is affine.

Assume S is affine and f quasi-compact and separated. Let $t \geq 1$ be the minimal number of affine opens needed to cover X. We will prove this case of (1) by induction on t. If $t = 1$ then the morphism f is affine by Morphisms, Lemma 24.11.12 and (1) follows from Lemma 25.2.3. If $t > 1$ write $X = U \cup V$ with V affine open and $U = U_1 \cup \ldots \cup U_{t-1}$ a union of $t - 1$ open affines. Note that in this case $U \cap V = (U_1 \cap V) \cup \ldots (U_{t-1} \cap V)$ is also a union of $t - 1$ affine open subschemes, see Schemes, Lemma 21.21.8. We will apply the relative Mayer-Vietoris sequence

$$0 \to f_*\mathcal{F} \to a_*(\mathcal{F}|_U) \oplus b_*(\mathcal{F}|_V) \to c_*(\mathcal{F}|_{U \cap V}) \to R^1 f_*\mathcal{F} \to \ldots$$

see Cohomology, Lemma 18.8.3. By induction we see that $R^p a_*\mathcal{F}$, $R^p b_*\mathcal{F}$ and $R^p c_*\mathcal{F}$ are all quasi-coherent. This implies that each of the sheaves $R^p f_*\mathcal{F}$ is quasi-coherent since it sits in the middle of a short exact sequence with a cokernel of a map between quasi-coherent sheaves on the left and a kernel of a map between quasi-coherent sheaves on the right. Using the results on quasi-coherent sheaves in Schemes, Section 21.24 we see conclude $R^p f_*\mathcal{F}$ is quasi-coherent.

Assume S is affine and f quasi-compact and quasi-separated. Let $t \geq 1$ be the minimal number of affine opens needed to cover X. We will prove (1) by induction on t. In case

$t = 1$ the morphism f is separated and we are back in the previous case (see previous paragraph). If $t > 1$ write $X = U \cup V$ with V affine open and U a union of $t-1$ open affines. Note that in this case $U \cap V$ is an open subscheme of an affine scheme and hence separated (see Schemes, Lemma 21.21.6). We will apply the relative Mayer-Vietoris sequence

$$0 \to f_*\mathcal{F} \to a_*(\mathcal{F}|_U) \oplus b_*(\mathcal{F}|_V) \to c_*(\mathcal{F}|_{U\cap V}) \to R^1 f_*\mathcal{F} \to \ldots$$

see Cohomology, Lemma 18.8.3. By induction and the result of the previous paragraph we see that $R^p a_*\mathcal{F}$, $R^p b_*\mathcal{F}$ and $R^p c_*\mathcal{F}$ are quasi-coherent. As in the previous paragraph this implies each of sheaves $R^p f_*\mathcal{F}$ is quasi-coherent.

Next, we prove (3) and a fortiori (2). Choose a finite affine open covering $S = \bigcup_{j=1,\ldots m} S_j$. For each i choose a finite affine open covering $f^{-1}(S_j) = \bigcup_{i=1,\ldots t_j} U_{ji}$. Let

$$d_j = \max_{I \subset \{1,\ldots,t_j\}} |I| + t(\bigcap\nolimits_{i\in I} U_{ji})$$

be the integer found in Lemma 25.5.2. We claim that $n(X, S, f) = \max d_j$ works.

Namely, let $S' \to S$ be a morphism of schemes and let $\mathcal{F}'$ be a quasi-coherent sheaf on $X' = S' \times_S X$. We want to show that $R^p f'_*\mathcal{F}' = 0$ for $p \geq n(X, S, f)$. Since this question is local on S' we may assume that S' is affine and maps into S_j for some j. Then $X' = S' \times_{S_j} f^{-1}(S_j)$ is covered by the open affines $S' \times_{S_j} U_{ji}$, $i = 1, \ldots t_j$ and the intersections

$$\bigcap\nolimits_{i\in I} S' \times_{S_j} U_{ji} = S' \times_{S_j} \bigcap\nolimits_{i\in I} U_{ji}$$

are covered by the same number of affines as before the base change. Applying Lemma 25.5.2 we get $H^p(X', \mathcal{F}') = 0$. By the first part of the proof we already know that each $R^q f'_*\mathcal{F}'$ is quasi-coherent hence has vanishing higher cohomology groups on our affine scheme S', thus we see that $H^0(S', R^p f'_*\mathcal{F}') = H^p(X', \mathcal{F}') = 0$ by Cohomology, Lemma 18.12.6. Since $R^p f'_*\mathcal{F}'$ is quasi-coherent we conclude that $R^p f'_*\mathcal{F}' = 0$. □

Lemma 25.5.4. *Let $f : X \to S$ be a morphism of schemes. Assume that f is quasi-separated and quasi-compact. Assume S is affine. For any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have*

$$H^q(X, \mathcal{F}) = H^0(S, R^q f_*\mathcal{F})$$

for all $q \in \mathbf{Z}$.

Proof. Consider the Leray spectral sequence $E_2^{p,q} = H^p(S, R^q f_*\mathcal{F})$ converging to $H^{p+q}(X, \mathcal{F})$, see Cohomology, Lemma 18.12.4. By Lemma 25.5.3 we see that the sheaves $R^q f_*\mathcal{F}$ are quasi-coherent. By Lemma 25.2.2 we see that $E_2^{p,q} = 0$ when $p > 0$. Hence the spectral sequence degenerates at E_2 and we win. See also Cohomology, Lemma 18.12.6 (2) for the general principle. □

25.6. Cohomology and base change, I

Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Suppose further that $g : S' \to S$ is any morphism of schemes. Denote $X' = X_{S'} = S' \times_S X$ the base change of X and denote $f' : X' \to S'$ the base change of f. Also write $g' : X' \to X$ the projection, and set $\mathcal{F}' = (g')^*\mathcal{F}$. Here is a diagram representing the situation:

(25.6.0.1)
$$\begin{tikzcd} \mathcal{F}' = (g')^*\mathcal{F} & X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] & \mathcal{F} \\ Rf'_*\mathcal{F}' & S' \arrow[r, "g"] & S & Rf_*\mathcal{F} \end{tikzcd}$$

Here is the simplest case of the base change property we have in mind.

Lemma 25.6.1. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume f is affine. In this case $f_*\mathcal{F} \cong Rf_*\mathcal{F}$ is a quasi-coherent sheaf, and for every base change diagram (25.6.0.1) we have*

$$g^* f_*\mathcal{F} = f'_*(g')^*\mathcal{F}.$$

Proof. The vanishing of higher direct images is Lemma 25.2.3. The statement is local on S and S'. Hence we may assume $X = Spec(A)$, $S = Spec(R)$, $S' = Spec(R')$ and $\mathcal{F} = \widetilde{M}$ for some A-module M. We use Schemes, Lemma 21.7.3 to describe pullbacks and pushforwards of $\mathcal{F}$. Namely, $X' = Spec(R' \otimes_R A)$ and $\mathcal{F}'$ is the quasi-coherent sheaf associated to $(R' \otimes_R A) \otimes_A M$. Thus we see that the lemma boils down to the equality

$$(R' \otimes_R A) \otimes_A M = R' \otimes_R M$$

as R'-modules. $\square$

In many situations it is sufficient to know about the following special case of cohomology and base change. It follows immediately from the stronger results in the next section, but since it is so important it deserves its own proof.

Lemma 25.6.2. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $g : S' \to S$ be a morphism of schemes. Assume that g is flat and that f is quasi-compact and quasi-separated. Then for any $i \geq 0$ we have*

$$R^i f'_*\mathcal{F}' = g^* R^i f_*\mathcal{F}$$

with notation as in (25.6.0.1). Moreover, the induced isomorphism is the map given by the base change map of Cohomology, Lemma 18.14.1.

Proof. The statement is local on S' and hence we may assume S and S' are affine. Say $S = \text{Spec}(A)$ and $S' = \text{Spec}(B)$. In this case we are really trying to show that the map

$$H^i(X, \mathcal{F}) \otimes_A B \longrightarrow H^i(X_B, \mathcal{F}_B)$$

(given by the reference in the statement of the lemma) is an isomorphism where $X_B = \text{Spec}(B) \times_{\text{Spec}(A)} X$ and $\mathcal{F}_B$ is the pullback of $\mathcal{F}$ to X_B.

In case X is separated, choose an affine open covering $\mathcal{U} : X = U_1 \cup \ldots \cup U_t$ and recall that

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(X, \mathcal{F}),$$

see Lemma 25.2.4. If $\mathcal{U}_B : X_B = (U_1)_B \cup \ldots \cup (U_t)_B$ we obtain by base change, then it is still the case that each $(U_i)_B$ is affine and that X_B is separated. Thus we obtain

$$\check{H}^p(\mathcal{U}_B, \mathcal{F}_B) = H^p(X_B, \mathcal{F}_B).$$

We have the following relation between the Čech complexes

$$\check{\mathcal{C}}^\bullet(\mathcal{U}_B, \mathcal{F}_B) = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \otimes_A B$$

as follows from Lemma 25.6.1. Since $A \to B$ is flat, the same thing remains true on taking cohomology.

In case X is quasi-separated, choose an affine open covering $\mathcal{U} : X = U_1 \cup \ldots \cup U_t$. We will use the Čech-to-cohomology spectral sequence Cohomology, Lemma 18.11.4. The reader who wishes to avoid this spectral sequence can use Majer-Vietoris and induction on t as in the proof of Lemma 25.5.3. The spectral sequence has E_2-page $E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F}))$ and converges to $H^{p+q}(X, \mathcal{F})$. Similarly, we have a spectral sequence with

E_2-page $E_2^{p,q} = \check{H}^p(\mathcal{U}_B, \underline{H}^q(\mathcal{F}_B))$ and converges to $H^{p+q}(X_B, \mathcal{F}_B)$. Since the intersections $U_{i_0\ldots i_p}$ are quasi-compact and separated, the result of the second paragraph of the proof gives $\check{H}^p(\mathcal{U}_B, \underline{H}^q(\mathcal{F}_B)) = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F})) \otimes_A B$. Using that $A \to B$ is flat we conclude that $H^i(X, \mathcal{F}) \otimes_A B \to H^i(X_B, \mathcal{F}_B)$ is an isomorphism for all i and we win. □

25.7. Cohomology and base change, II

We would like to prove a little more in situation (25.6.0.1). Namely, if f is quasi-compact and quasi-separated we would like to represent $Rf_*\mathcal{F}$ by a complex of quasi-coherent sheaves on S. This can be done in some cases, for example if S is quasi-compact and (semi-)separated, by relating it to the question of whether $D^+_{QCoh}(S)$ is equivalent to $D^+(QCoh(\mathcal{O}_S))$, see Section 25.4.

In this section we will use a different approach which produces a complex having a good base change property. First of all the result is very easy if f and S are separated. Since this is the case which by far the most often used we treat it separately.

Lemma 25.7.1. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume X and S are separated and quasi-compact. In this case we can compute $Rf_*\mathcal{F}$ as follows:*

1. *Choose a finite affine open covering $\mathcal{U} : X = \bigcup_{i=1,\ldots,n} U_i$.*
2. *For $i_0, \ldots, i_p \in \{1, \ldots, n\}$ denote $f_{i_0\ldots i_p} : U_{i_0\ldots i_p} \to S$ the restriction of f to the intersection $U_{i_0\ldots i_p} = U_{i_0} \cap \ldots \cap U_{i_p}$.*
3. *Set $\mathcal{F}_{i_0\ldots i_p}$ equal to the restriction of $\mathcal{F}$ to $U_{i_0\ldots i_p}$.*
4. *Set*
$$\check{\mathcal{C}}^p(\mathcal{U}, f, \mathcal{F}) = \bigoplus\nolimits_{i_0\ldots i_p} f_{i_0\ldots i_p *}\mathcal{F}_{i_0\ldots i_p}$$
and define differentials $d : \check{\mathcal{C}}^p(\mathcal{U}, f, \mathcal{F}) \to \check{\mathcal{C}}^{p+1}(\mathcal{U}, f, \mathcal{F})$ as in Cohomology, Equation (18.9.0.1).

Then the complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F})$ is a complex of quasi-coherent sheaves on S which comes equipped with an isomorphism
$$\check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F}) \longrightarrow Rf_*\mathcal{F}$$
in $D^+(S)$. This isomorphism is functorial in the quasi-coherent sheaf $\mathcal{F}$.

Proof. Omitted. Hint: Use the resolution $\mathcal{F} \to \mathfrak{C}^\bullet(\mathcal{U}, \mathcal{F})$ of Cohomology, Lemma 18.18.3. Observe that $\check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F}) = f_*\mathfrak{C}^\bullet(\mathcal{U}, \mathcal{F})$. Also observe that both the inclusion morphisms $j_{i_0\ldots i_p} : U_{i_0\ldots i_p} \to X$ and the morphisms $f_{i_0\ldots i_p} : U_{i_0\ldots i_p} \to S$ are affine because S and X and $f : X \to S$ are separated, see Morphisms, Lemma 24.11.11. Hence $R^q(j_{i_0\ldots i_p})_*\mathcal{F}_{i_0\ldots i_p}$ as well as $R^q(f_{i_0\ldots i_p})_*\mathcal{F}_{i_0\ldots i_p}$ are zero for $q > 0$. Finally, put all of this information together (e.g. use a spectral sequence, for example by choosing a Cartan-Eilenberg resolution of the complex $\mathfrak{C}^\bullet(\mathcal{U}, \mathcal{F})$). □

Next, we are going to consider what happens if we do a base change.

Lemma 25.7.2. *With notation as in diagram (25.6.0.1). Assume $f : X \to S$ and $\mathcal{F}$ satisfy the hypotheses of Lemma 25.7.1. Choose a finite affine open covering $\mathcal{U} : X = \bigcup U_i$ of X. There is a canonical isomorphism*
$$g^*\check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F}) \longrightarrow Rf'_*\mathcal{F}'$$

in $D^+(S')$. Moreover, if $S' \to S$ is affine, then in fact

$$g^*\check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F}) = \check{\mathcal{C}}^\bullet(\mathcal{U}', f', \mathcal{F}')$$

with $\mathcal{U}' : X' = \bigcup U'_i$ where $U'_i = (g')^{-1}(U_i) = U_{i,S'}$ is also affine.

Proof. In fact we may define $U'_i = (g')^{-1}(U_i) = U_{i,S'}$ no matter whether S' is affine over S or not. Let $\mathcal{U}' : X' = \bigcup U'_i$ be the induced covering of X'. In this case we claim that

$$g^*\check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F}) = \check{\mathcal{C}}^\bullet(\mathcal{U}', f', \mathcal{F}')$$

with $\check{\mathcal{C}}^\bullet(\mathcal{U}', f', \mathcal{F}')$ defined in exactly the same manner as in Lemma 25.7.1. This is clear from the case of affine morphisms (Lemma 25.6.1) by working locally on S'. Moreover, exactly as in the proof of Lemma 25.7.1 one sees that there is an isomorphism

$$\check{\mathcal{C}}^\bullet(\mathcal{U}', f', \mathcal{F}') \longrightarrow Rf'_*\mathcal{F}'$$

in $D^+(S')$ since the morphisms $U'_i \to X'$ and $U'_i \to S'$ are still affine (being base changes of affine morphisms). Details omitted. □

The lemma above says that the complex

$$\mathcal{K}^\bullet = \check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F})$$

is a bounded below complex of quasi-coherent sheaves on S which *universally* computes the higher direct images of $f : X \to S$. This is something about this particular complex and it is not preserved by replacing $\check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F})$ by a quasi-isomorphic complex in general! In other words, this is not a statement that makes sense in the derived category. The reason is that the pullback $g^*\mathcal{K}^\bullet$ is *not* equal to the derived pullback $Lg^*\mathcal{K}^\bullet$ of $\mathcal{K}^\bullet$ in general!

Here is a more general case where we can prove this statement. We remark that the condition of S being separated is harmless in most applications, since this is usually used to prove some local property of the total derived image. The proof is significantly more involved and uses hypercoverings; it is a nice example of how you can use them sometimes.

Lemma 25.7.3. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Assume that f is quasi-compact and quasi-separated and that S is quasi-compact and separated. There exists a bounded below complex $\mathcal{K}^\bullet$ of quasi-coherent $\mathcal{O}_S$-modules with the following property: For every morphism $g : S' \to S$ the complex $g^*\mathcal{K}^\bullet$ is a representative for $Rf'_*\mathcal{F}'$ with notation as in diagram (25.6.0.1).*

Proof. (If f is separated as well, please see Lemma 25.7.2.) The assumptions imply in particular that X is quasi-compact and quasi-separated as a scheme. Let $\mathcal{B}$ be the set of affine opens of X. By Hypercoverings, Lemma 20.9.4 we can find a hypercovering $K = (I, \{U_i\})$ such that each I_n is finite and each U_i is an affine open of X. By Hypercoverings, Lemma 20.7.3 there is a spectral sequence with E_2-page

$$E_2^{p,q} = \check{H}^p(K, \underline{H}^q(\mathcal{F}))$$

converging to $H^{p+q}(X, \mathcal{F})$. Note that $\check{H}^p(K, \underline{H}^q(\mathcal{F}))$ is the pth cohomology group of the complex

$$\prod\nolimits_{i \in I_0} H^q(U_i, \mathcal{F}) \to \prod\nolimits_{i \in I_1} H^q(U_i, \mathcal{F}) \to \prod\nolimits_{i \in I_2} H^q(U_i, \mathcal{F}) \to \ldots$$

Since each U_i is affine we see that this is zero unless $q = 0$ in which case we obtain

$$\prod\nolimits_{i \in I_0} \mathcal{F}(U_i) \to \prod\nolimits_{i \in I_1} \mathcal{F}(U_i) \to \prod\nolimits_{i \in I_2} \mathcal{F}(U_i) \to \ldots$$

Thus we conclude that $R\Gamma(X, \mathcal{F})$ is computed by this complex.

For any n and $i \in I_n$ denote $f_i : U_i \to S$ the restriction of f to U_i. As S is separated and U_i is affine this morphism is affine. Consider the complex of quasi-coherent sheaves

$$\mathcal{K}^\bullet = (\prod\nolimits_{i\in I_0} f_{i,*}\mathcal{F}|_{U_i} \to \prod\nolimits_{i\in I_1} f_{i,*}\mathcal{F}|_{U_i} \to \prod\nolimits_{i\in I_2} f_{i,*}\mathcal{F}|_{U_i} \to \ldots)$$

on S. As in Hypercoverings, Lemma 20.7.3 we obtain a map $\mathcal{K}^\bullet \to Rf_*\mathcal{F}$ in $D(\mathcal{O}_S)$ by choosing an injective resolution of $\mathcal{F}$ (details omitted). Consider any affine scheme V and a morphism $g : V \to S$. Then the base change X_V has a hypercovering $K_V = (I, \{U_{i,V}\})$ obtained by base change. Moreover, $g^*f_{i,*}\mathcal{F} = f_{i,V,*}(g')^*\mathcal{F}|_{U_{i,V}}$. Thus the arguments above prove that $\Gamma(V, g^*\mathcal{K}^\bullet)$ computes $R\Gamma(X_V, (g')^*\mathcal{F})$. This finishes the proof of the lemma as it suffices to prove the equality of complexes Zariski locally on S'. □

25.8. Ample invertible sheaves and cohomology

Given a ringed space X, an invertible $\mathcal{O}_X$-module $\mathcal{L}$, a section $s \in \Gamma(X, \mathcal{L})$ and an $\mathcal{O}_X$-module $\mathcal{F}$ we get a map $\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}$, $t \mapsto t \otimes s$ which we call multiplication by s. We usually denote it $t \mapsto st$.

Lemma 25.8.1. *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$ be a section. Let $\mathcal{F}' \subset \mathcal{F}$ be quasi-coherent $\mathcal{O}_X$-modules. Assume that*

1. *X is quasi-compact,*
2. *$\mathcal{F}$ is of finite type, and*
3. *$\mathcal{F}'|_{X_s} = \mathcal{F}|_{X_s}$.*

Then there exists an $n \geq 0$ such that multiplication by s^n on $\mathcal{F}$ factors through $\mathcal{F}'$.

Proof. In other words we claim that $s^n\mathcal{F} \subset \mathcal{F}' \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ for some $n \geq 0$. If this is true for n_0 then it is true for all $n \geq n_0$. Hence it suffices to show there is a finite open covering such that the result holds for each of the members of this open covering. Since X is quasi-compact we may therefore assume that X is affine and that $\mathcal{L} \cong \mathcal{O}_X$. Thus the lemma translates into the following algebra problem (use Properties, Lemma 23.16.1): Let A be a ring. Let $f \in A$. Let $M' \subset M$ be A-modules. Assume M is a finite A-module, and assume that $(M')_f = M_f$. Then there exists an $n \geq 0$ such that $f^nM \subset M'$. The proof of this is omitted. □

Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$ be a section. Assume X quasi-compact and quasi-separated. The following lemma says roughly that the category of finitely presented $\mathcal{O}_{X_s}$-modules is the category of finitely presented $\mathcal{O}_X$-modules where the map multiplication by s has been inverted.

Lemma 25.8.2. *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$ be a section. Let $\mathcal{F}$, $\mathcal{F}'$ be quasi-coherent $\mathcal{O}_X$-modules. Let $\psi : \mathcal{F}|_{X_s} \to \mathcal{F}'|_{X_s}$ be a map of $\mathcal{O}_{X_s}$-modules. Assume that*

1. *X is quasi-compact and quasi-separated, and*
2. *$\mathcal{F}$ is of finitely presented.*

Then there exists an $n \geq 0$ and a morphism $\alpha : \mathcal{F} \to \mathcal{F}' \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ whose restriction to X_s equals ψ via the identification $\mathcal{L}^{\otimes n}|_{X_s} = \mathcal{O}_{X_s}$ coming from s. Moreover, given a pair of solutions (n, α) and (n', α') there exists an $m \geq \max(n, n')$ such that $s^{m-n}\alpha = s^{m-n'}\alpha'$.

Proof. If the lemma holds for n_0 with map α_0 then it holds for all $n \geq n_0$ simply by taking $\alpha = s^{n-n_0}\alpha_0$. Choose a finite affine open covering $X = \bigcup U_i$ such that $\mathcal{L}|_{U_i}$ is trivial. Choose finite affine open coverings $U_i \cap U_{i'} = \bigcup U_{ii'j}$. Suppose we can prove the lemma when X is affine and $\mathcal{L}$ is trivial. Then we can find $n_i \geq 0$ $\alpha_i : \mathcal{F}|_{U_i} \to \mathcal{F}'|_{U_i} \otimes_{\mathcal{O}_{U_i}} \mathcal{L}^{\otimes n_i}|_{U_i}$ satisfying the relation over U_i. By the uniqueness assertion of the lemma, and the finiteness of the number of affines $U_{ii'j}$ we can find a single large integer m such that the maps $s^{m-n_i}\alpha_i$ and $s^{m-n_{i'}}\alpha_{i'}$ agree over $U_{ii'j}$ and hence over $U_i \cap U_{i'}$. Thus the morphisms $s^{m-n_i}\alpha_i$ glue to give our global map α. Proof of the uniqueness statement is omitted.

Assume X affine and that $\mathcal{L} \cong \mathcal{O}_X$. Then the lemma translates into the following algebra problem (use Properties, Lemma 23.16.2): Let A be a ring. Let $f \in A$. Let $\psi : M_f \to (M')_f$ be a map of A_f-modules. Assume M is a finitely presented A-module. Then there exists an $n \geq 0$ and an A-module map $\alpha : M \to M'$ such that $\alpha \otimes 1_{A_f} = f^n\psi$. Moreover, given any second solution (n', α') there exists an $m \geq \max(n, n')$ such that $f^{m-n}\alpha = f^{m-n'}\alpha'$. The proof of this algebraic fact is omitted. □

Cohomology is functorial. In particular, given a ringed space X, an invertible $\mathcal{O}_X$-module $\mathcal{L}$, a section $s \in \Gamma(X, \mathcal{L})$ we get maps

$$H^p(X, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}), \quad \xi \longmapsto s\xi$$

induced by the map $\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}$ which is multiplication by s.

Lemma 25.8.3. *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$ be a section. Assume that*

(1) *X is quasi-compact and quasi-separated, and*
(2) *X_s is affine.*

Then for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ and every $p > 0$ and all $\xi \in H^p(X, \mathcal{F})$ there exists an $n \geq 0$ such that $s^n\xi = 0$ in $H^p(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$.

Proof. You can prove this lemma using a Mayer-Vietoris type argument and induction on the number of affines needed to cover X similar to the proof of Lemma 25.5.3. This may be preferable to the proof that follows.

Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Cohomology on X commutes with directed colimits of sheaves of $\mathcal{O}_X$-modules, see Cohomology, Lemma 18.15.1. By Properties, Lemma 23.20.7 we can write $\mathcal{F}$ as a directed colimit of $\mathcal{O}_X$-submodules of finite presentation. Hence every $\xi \in H^p(X, \mathcal{F})$ is the image of $\xi' \in H^p(X, \mathcal{F}')$ for some $\mathcal{O}_X$-submodule of finite presentation. Thus we may replace $\mathcal{F}$ by $\mathcal{F}'$ and assume $\mathcal{F}$ is of finite presentation.

Let $j : X_s \to X$ be the inclusion morphism. Morphisms, Lemma 24.11.10 says that j is an affine morphism. Hence $R^q j_*(j^*\mathcal{F}) = 0$ for all $q > 0$, see Lemma 25.2.3. Since also $H^p(X_s, j^*\mathcal{F}) = 0$ by Lemma 25.2.2, we conclude that $H^p(X, j_*j^*\mathcal{F}) = 0$ for all $p > 0$ for example by the Leray spectral sequence (Cohomology, Lemma 18.12.4). Write

$$j_*j^*\mathcal{F} = colim_{\lambda \in \Lambda} \mathcal{F}_\lambda$$

as a directed colimit of $\mathcal{O}_X$-modules $\mathcal{F}_\lambda$ of finite presentation (Properties, Lemma 23.20.7 again). By Modules, Lemma 15.11.6 there exists a $\lambda \in \Lambda$ such that $\mathcal{F} \to j_*j^*\mathcal{F}$ factors through $\mathcal{F}_\lambda$. After shrinking Λ we may assume that we have a compatible collection of morphisms $\chi_\lambda : \mathcal{F} \to \mathcal{F}_\lambda$ for all $\lambda \in \Lambda$ which when taking the colimit gives the canonical map $\mathcal{F} \to j_*j^*\mathcal{F}$.

With these preparations the proof goes as follows. Take $\xi \in H^p(X, \mathcal{F})$ for some $p > 0$. It maps to zero in $H^p(X, j_*j^*\mathcal{F})$ because we saw above this group is zero. By Cohomology, Lemma 18.15.1 again it follows that ξ maps to zero in $H^p(X, \mathcal{F}_\lambda)$ via the map χ_λ for some λ. Note that since $\mathcal{F} \to j_*j^*\mathcal{F}$ is an isomorphism over X_s we see that there is an $\mathcal{O}_{X_s}$-module map $\psi : \mathcal{F}_\lambda|_{X_s} \to \mathcal{F}|_{X_s}$ which is a left inverse to $\chi_\lambda : \mathcal{F} \to \mathcal{F}_\lambda$. By Lemma 25.8.2 there exists an n and a map $\alpha : \mathcal{F}_\lambda \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ such that α restricts to ψ on X_s (via $\mathcal{L}^{\otimes n}|_{X_s} \cong \mathcal{O}_{X_s}$). By the uniqueness part of Lemma 25.8.2 applied to $\alpha \circ \chi_\lambda$ which restricts to multiplication by s^n on X_s we may assume (after increasing n) that the composition

$$\mathcal{F} \xrightarrow[\chi_\lambda]{} \mathcal{F}_\lambda \xrightarrow[\alpha]{} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$$

is equal to multiplication by s^n on $\mathcal{F}$. Hence we see that $s^n\xi = 0$. □

25.9. Cohomology of projective space

In this section we compute the cohomology of the twists of the structure sheaf on $\mathbf{P}^n_S$ over a scheme S. Recall that $\mathbf{P}^n_S$ was defined as the fibre product $\mathbf{P}^n_S = S \times_{Spec(\mathbf{Z})} \mathbf{P}^n_{\mathbf{Z}}$ in Constructions, Definition 22.13.2. It was shown to be equal to

$$\mathbf{P}^n_S = \underline{\mathrm{Proj}}_S(\mathcal{O}_S[T_0, \ldots, T_n])$$

in Constructions, Lemma 22.20.4. In particular, projective space is a particular case of a projective bundle. If $S = Spec(R)$ is affine then we have

$$\mathbf{P}^n_S = \mathbf{P}^n_R = \mathrm{Proj}(R[T_0, \ldots, T_n]).$$

All these identifications are compatible and compatible with the constructions of the twisted structure sheaves $\mathcal{O}_{\mathbf{P}^n_S}(d)$.

Before we state the result we need some notation. Let R be a ring. Recall that $R[T_0, \ldots, T_n]$ is a graded R-algebra where each T_i is homogenous of degree 1. Denote $(R[T_0, \ldots, T_n])_d$ the degree d summand. It is a finite free R-module of rank $\binom{n+d}{d}$ when $d \geq 0$ and zero else. It has a basis consisting of monomials $T_0^{e_0} \ldots T_n^{e_n}$ with $\sum e_i = d$. We will also use the following notation: $R[\frac{1}{T_0}, \ldots, \frac{1}{T_n}]$ denotes the $\mathbf{Z}$-graded ring with $\frac{1}{T_i}$ in degree -1. In particular the $\mathbf{Z}$-graded $R[\frac{1}{T_0}, \ldots, \frac{1}{T_n}]$ module

$$\frac{1}{T_0 \ldots T_n} R[\frac{1}{T_0}, \ldots, \frac{1}{T_n}]$$

which shows up in the statement below is zero in degrees $\geq -n$, is free on the generator $\frac{1}{T_0 \ldots T_n}$ in degree $-n-1$ and is free of rank $(-1)^n\binom{n+d}{d}$ for $d \leq -n-1$.

Lemma 25.9.1. *Let R be a ring. Let $n \geq 0$ be an integer. We have*

$$H^q(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n_R}(d)) = \begin{cases} (R[T_0, \ldots, T_n])_d & \text{if} \quad q = 0 \\ 0 & \text{if} \quad q \neq 0, n \\ \left(\frac{1}{T_0 \ldots T_n} R[\frac{1}{T_0}, \ldots, \frac{1}{T_n}]\right)_d & \text{if} \quad q = n \end{cases}$$

as R-modules.

Proof. We will use the standard affine open convering

$$\mathcal{U} : \mathbf{P}^n_R = \bigcup\nolimits_{i=0}^n D_+(T_i)$$

to compute the cohomology using the Cech complex. This is permissible by Lemma 25.2.4 since any intersection of finitely many affine $D_+(T_i)$ is also a standard affine open (see Constructions, Section 22.8). In fact, we can use the alternating or ordered Cech complex according to Cohomology, Lemmas 18.17.3 and 18.17.6.

The ordering we will use on $\{0, \ldots, n\}$ is the usual one. Hence the complex we are looking at has terms

$$\check{\mathcal{C}}^p_{ord}(\mathcal{U}, \mathcal{O}_{\mathbf{P}_R}(d)) = \bigoplus\nolimits_{i_0 < \ldots < i_p} (R[T_0, \ldots, T_n, \frac{1}{T_{i_0} \ldots T_{i_p}}])_d$$

Moreover, the maps are given by the usual formula

$$d(s)_{i_0 \ldots i_{p+1}} = \sum\nolimits_{j=0}^{p+1} (-1)^j s_{i_0 \ldots \hat{i}_j \ldots i_{p+1}}$$

see Cohomology, Section 18.17. Note that each term of this complex has a natural $\mathbf{Z}^{n+1}$-grading. Namely, we get this by declaring a monomial $T_0^{e_0} \ldots T_n^{e_n}$ to be homogeneous with weight $(e_0, \ldots, e_n) \in \mathbf{Z}^{n+1}$. It is clear that the differential given above respects the grading. In a formula we have

$$\check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{O}_{\mathbf{P}_R}(d)) = \bigoplus\nolimits_{\vec{e} \in \mathbf{Z}^{n+1}} \check{\mathcal{C}}^\bullet(\vec{e})$$

where not all summand on the right hand side occur (see below). Hence in order to compute the cohomology modules of the complex it suffices to compute the cohomology of the graded pieces and take the direct sum at the end.

Fix $\vec{e} = (e_0, \ldots, e_n) \in \mathbf{Z}^{n+1}$. In order for this weight to occur in the complex above we need to assume $e_0 + \ldots + e_n = d$ (if not then it occurs for a different twist of the structure sheaf of course). Assuming this set

$$NEG(\vec{e}) = \{i \in \{0, \ldots, n\} \mid e_i < 0\}.$$

With this notation the weight $\vec{e}$ summand $\check{\mathcal{C}}^\bullet(\vec{e})$ of the Cech complex above has the following terms

$$\check{\mathcal{C}}^p(\vec{e}) = \bigoplus\nolimits_{i_0 < \ldots < i_p,\ NEG(\vec{e}) \subset \{i_0, \ldots, i_p\}} R \cdot T_0^{e_0} \ldots T_n^{e_n}$$

In other words, the terms corresponding to $i_0 < \ldots < i_p$ such that $NEG(\vec{e})$ is not contained in $\{i_0 \ldots i_p\}$ are zero. The differential of the complex $\check{\mathcal{C}}^\bullet(\vec{e})$ is still given by the exact same formula as above.

Suppose that $NEG(\vec{e}) = \{0, \ldots, n\}$, i.e., that all exponents e_i are negative. In this case the complex $\check{\mathcal{C}}^\bullet(\vec{e})$ has only one term, namely $\check{\mathcal{C}}^n(\vec{e}) = R \cdot \frac{1}{T^{-e_0} \ldots T^{-e_n}}$. Hence in this case

$$H^q(\check{\mathcal{C}}^\bullet(\vec{e})) = \begin{cases} R \cdot \frac{1}{T^{-e_0} \ldots T^{-e_n}} & \text{if} \quad q = n \\ 0 & \text{if} \quad \text{else} \end{cases}$$

The direct sum of all of these terms clearly gives the value

$$\left(\frac{1}{T_0 \ldots T_n} R[\frac{1}{T_0}, \ldots, \frac{1}{T_n}] \right)_d$$

in degree n as stated in the lemma. Moreover these terms do not contribute to cohomology in other degrees (also in accordance with the statement of the lemma).

Assume $NEG(\vec{e}) = \emptyset$. In this case the complex $\check{\mathcal{C}}^\bullet(\vec{e})$ has a summand R corresponding to all $i_0 < \ldots < i_p$. Let us compare the complex $\check{\mathcal{C}}^\bullet(\vec{e})$ to another complex. Namely, consider the affine open open covering

$$\mathcal{V} : Spec(R) = \bigcup\nolimits_{i \in \{0, \ldots, n\}} V_i$$

where $V_i = Spec(R)$ for all i. Consider the alternating Cech complex

$$\check{\mathcal{C}}^\bullet_{ord}(\mathcal{V}, \mathcal{O}_{Spec(R)})$$

By the same reasoning as above this computes the cohomology of the structure sheaf on $Spec(R)$. Hence we see that $H^p(\check{\mathcal{C}}^\bullet_{ord}(\mathcal{V}, \mathcal{O}_{Spec(R)})) = R$ if $p = 0$ and is 0 whenever $p > 0$. For these facts, see Lemma 25.2.1 and its proof. Note that also $\check{\mathcal{C}}^\bullet_{ord}(\mathcal{V}, \mathcal{O}_{Spec(R)})$ has a summand R for every $i_0 < \ldots < i_p$ and has exactly the same differential as $\check{\mathcal{C}}^\bullet(\vec{e})$. In other words these complexes are isomorphic complexes and hence have the same cohomology. We conclude that

$$H^q(\check{\mathcal{C}}^\bullet(\vec{e})) = \begin{cases} R \cdot T^{e_0} \ldots T^{e_n} & \text{if} \quad q = 0 \\ 0 & \text{if} \quad \text{else} \end{cases}$$

in the case that $NEG(\vec{e}) = \emptyset$. The direct sum of all of these terms clearly gives the value

$$(R[T_0, \ldots, T_n])_d$$

in degree 0 as stated in the lemma. Moreover these terms do not contribute to cohomology in other degrees (also in accordance with the statement of the lemma).

To finish the proof of the lemma we have to show that the complexes $\check{\mathcal{C}}^\bullet(\vec{e})$ are acyclic when $NEG(\vec{e})$ is neither empty nor equal to $\{0, \ldots, n\}$. Pick an index $i_{\text{fix}} \not\in NEG(\vec{e})$ (such an index exists). Consider the map

$$h : \check{\mathcal{C}}^{p+1}(\vec{e}) \to \check{\mathcal{C}}^p(\vec{e})$$

given by the rule

$$h(s)_{i_0 \ldots i_p} = s_{i_{\text{fix}} i_0 \ldots i_p}$$

(compare with the proof of Lemma 25.2.1). It is clear that this is well defined since

$$NEG(\vec{e}) \subset \{i_0, \ldots, i_p\} \Leftrightarrow NEG(\vec{e}) \subset \{i_{\text{fix}}, i_0, \ldots, i_p\}$$

Also $\check{\mathcal{C}}^0(\vec{e}) = 0$ so that this formula does work for all p including $p = -1$. The exact same (combinatorial) computation as in the proof of Lemma 25.2.1 shows that

$$(hd + dh)(s)_{i_0 \ldots i_p} = s_{i_0 \ldots i_p}$$

Hence we see that the identity map of the complex $\check{\mathcal{C}}^\bullet(\vec{e})$ is homotopic to zero which implies that it is acyclic. $\square$

In the following lemma we are going to use the pairing of free R-modules

$$R[T_0, \ldots, T_n] \times \frac{1}{T_0 \ldots T_n} R[\frac{1}{T_0}, \ldots, \frac{1}{T_n}] \longrightarrow R$$

which is defined by the rule

$$(f, g) \longmapsto \text{coefficient of } \frac{1}{T_0 \ldots T_n} \text{ in } fg.$$

In other words, the basis element $T_0^{e_0} \ldots T_n^{e_n}$ pairs with the basis element $T_0^{d_0} \ldots T_n^{d_n}$ to give 1 if and only if $e_i + d_i = -1$ for all i, and pairs to zero in all other cases. Using this pairing we get an identification

$$\left(\frac{1}{T_0 \ldots T_n} R[\frac{1}{T_0}, \ldots, \frac{1}{T_n}]\right)_d = Hom_R((R[T_0, \ldots, T_n])_{-n-1-d}, R)$$

Thus we can reformulate the result of Lemma 25.9.1 as saying that

(25.9.1.1) $$H^q(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n_R}(d)) = \begin{cases} (R[T_0, \ldots, T_n])_d & \text{if} \quad q = 0 \\ 0 & \text{if} \quad q \neq 0, n \\ Hom_R((R[T_0, \ldots, T_n])_{-n-1-d}, R) & \text{if} \quad q = n \end{cases}$$

Lemma 25.9.2. *The identifications of Equation (25.9.1.1) are compatible with base change w.r.t. ring maps $R \to R'$. Moreover, for any $f \in R[T_0, \ldots, T_n]$ homogeneous of degree m the map multiplication by f*

$$\mathcal{O}_{\mathbf{P}^n_R}(d) \longrightarrow \mathcal{O}_{\mathbf{P}^n_R}(d+m)$$

induces the map on the cohomology group via the identifications of Equation (25.9.1.1) which is multiplication by f for H^0 and the contragredient of multiplication by f

$$(R[T_0, \ldots, T_n])_{-n-1-(d+m)} \longrightarrow (R[T_0, \ldots, T_n])_{-n-1-d}$$

on H^n.

Proof. Suppose that $R \to R'$ is a ring map. Let $\mathcal{U}$ be the standard affine open convering of $\mathbf{P}^n_R$, and let $\mathcal{U}'$ be the standard affine open convering of $\mathbf{P}^n_{R'}$. Note that $\mathcal{U}'$ is the pullback of the covering $\mathcal{U}$ under the canonical morphism $\mathbf{P}^n_{R'} \to \mathbf{P}^n_R$. Hence there is a map of Cech complexes

$$\gamma : \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{O}_{\mathbf{P}_R}(d)) \longrightarrow \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}', \mathcal{O}_{\mathbf{P}_{R'}}(d))$$

which is compatible with the map on cohomology by Cohomology, Lemma 18.13.3. It is clear from the computations in the proof of Lemma 25.9.1 that this map of Cech complexes is compatible with the identifications of the cohomology groups in question. (Namely the basis elements for the Cech complex over R simply map to the corresponding basis elements for the Cech complex over R'.) Whence the first statement of the lemma.

Now fix the ring R and consider two homogeneous polynomials $f, g \in R[T_0, \ldots, T_n]$ both of the same degree m. Since cohomology is an additive functor, it is clear that the map induced by multiplication by $f + g$ is the same as the sum of the maps induced by multiplication by f and the map induced by multiplication by g. Moreover, since cohomology is a functor a similar result holds for multiplication by a product fg where f, g are both homogeneous (but not necessarily of the same degree). Hence to verify the second statement of the lemma it suffices to prove this when $f = x \in R$ or when $f = T_i$. In the case of multiplication by an element $x \in R$ the result follows since every cohomology groups or complex in sight has the structuce or an R-module or complex of R-modules. Finally, we consider the case of multiplication by T_i as a $\mathcal{O}_{\mathbf{P}^n_R}$-linear map

$$\mathcal{O}_{\mathbf{P}^n_R}(d) \longrightarrow \mathcal{O}_{\mathbf{P}^n_R}(d+1)$$

The statement on H^0 is clear. For the statement on H^n consider multiplication by T_i as a map on Cech complexes

$$\check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{O}_{\mathbf{P}_R}(d)) \longrightarrow \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{O}_{\mathbf{P}_R}(d+1))$$

We are going to use the notation introduced in the proof of Lemma 25.9.1. We consider the effect of multiplication by T_i in terms of the decompositions

$$\check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{O}_{\mathbf{P}_R}(d)) = \bigoplus\nolimits_{\vec{e} \in \mathbf{Z}^{n+1},\ \sum e_i = d} \check{\mathcal{C}}^\bullet(\vec{e})$$

and

$$\check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{O}_{\mathbf{P}_R}(d+1)) = \bigoplus\nolimits_{\vec{e} \in \mathbf{Z}^{n+1},\ \sum e_i = d+1} \check{\mathcal{C}}^\bullet(\vec{e})$$

It is clear that it maps the subcomplex $\check{\mathcal{C}}^\bullet(\vec{e})$ to the subcomplex $\check{\mathcal{C}}^\bullet(\vec{e} + \vec{b}_i)$ where $\vec{b}_i = (0, \ldots, 0, 1, 0, \ldots, 0))$ the ith basis vector. In other words, it maps the summand of H^n corresponding to $\vec{e}$ with $e_i < 0$ and $\sum e_i = d$ to the summand of H^n corresponding to $\vec{e} + \vec{b}_i$

(which is zero if $e_i + b_i \geq 0$). It is easy to see that this corresponds exactly to the action of the contragredient of multiplication by T_i as a map

$$(R[T_0, \ldots, T_n])_{-n-1-(d+1)} \longrightarrow (R[T_0, \ldots, T_n])_{-n-1-d}$$

This proves the lemma. □

Before we state the relative version we need some notation. Namely, recall that $\mathcal{O}_S[T_0, \ldots, T_n]$ is a graded $\mathcal{O}_S$-module where each T_i is homogenous of degree 1. Denote $(\mathcal{O}_S[T_0, \ldots, T_n])_d$ the degree d summand. It is a finite locally free sheaf of rank $\binom{n+d}{d}$ on S.

Lemma 25.9.3. *Let S be a scheme. Let $n \geq 0$ be an integer. Consider the structure morphism*

$$f : \mathbf{P}^n_S \longrightarrow S.$$

We have

$$R^q f_*(\mathcal{O}_{\mathbf{P}^n_S}(d)) = \begin{cases} (\mathcal{O}_S[T_0, \ldots, T_n])_d & \text{if } \quad q = 0 \\ 0 & \text{if } \quad q \neq 0, n \\ \mathcal{H}om_{\mathcal{O}_S}((\mathcal{O}_S[T_0, \ldots, T_n])_{-n-1-d}, \mathcal{O}_S) & \text{if } \quad q = n \end{cases}$$

Proof. Omitted. Hint: This follows since the identifications in (25.9.1.1) are compatible with affine base change by Lemma 25.9.2. □

Next we state the version for projective bundles associated to finite locally free sheaves. Let S be a scheme. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_S$-module of constant rank $n + 1$, see Modules, Section 15.14. In this case we think of $\text{Sym}(\mathcal{E})$ as a graded $\mathcal{O}_S$-module where $\mathcal{E}$ is the graded part of degree 1. And $\text{Sym}^d(\mathcal{E})$ is the degree d summand. It is a finite locally free sheaf of rank $\binom{n+d}{d}$ on S. Recall that our normalization is that

$$\pi : \mathbf{P}(\mathcal{E}) = \underline{\text{Proj}}_S(\text{Sym}(\mathcal{E})) \longrightarrow S$$

and that there are natural maps $\text{Sym}^d(\mathcal{E}) \to \pi_*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(d)$.

Lemma 25.9.4. *Let S be a scheme. Let $n \geq 1$. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_S$-module of constant rank $n + 1$. Consider the structure morphism*

$$\pi : \mathbf{P}(\mathcal{E}) \longrightarrow S.$$

We have

$$R^q \pi_*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(d)) = \begin{cases} Sym^d(\mathcal{E}) & \text{if } \quad q = 0 \\ 0 & \text{if } \quad q \neq 0, n \\ \mathcal{H}om_{\mathcal{O}_S}(Sym^{-n-1-d}(\mathcal{E}) \otimes_{\mathcal{O}_S} \wedge^{n+1}\mathcal{E}, \mathcal{O}_S) & \text{if } \quad q = n \end{cases}$$

These identifications are compatible with base change and isomorphism between locally free sheaves.

Proof. Consider the canonical map

$$\pi^*\mathcal{E} \longrightarrow \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$$

and twist down by 1 to get

$$\pi^*(\mathcal{E})(-1) \longrightarrow \mathcal{O}_{\mathbf{P}(\mathcal{E})}$$

This is a surjective map from a locally free rank $n+1$ sheaf onto the structure sheaf. Hence the corresponding Koszul complex is exact (insert future reference here). In other words there is an exact complex

$$0 \to \pi^*(\wedge^{n+1}\mathcal{E})(-n-1) \to \ldots \to \pi^*(\wedge^i\mathcal{E})(-i) \to \ldots \to \mathcal{E}(-1) \to \mathcal{O}_{\mathbf{P}(\mathcal{E})} \to 0$$

We will think of the term $\pi^*(\wedge^i \mathcal{E})(-i)$ as being in degree $-i$. We are going to compute the higher direct images of this acyclic complex using the first spectral sequence of Derived Categories, Lemma 11.20.3. Namely, we see that there is a spectral sequence with terms

$$E_2^{p,q} = H^p(L^{\bullet,q}) \quad \text{with} \quad L^{-i,q} = R^q\pi_*\left(\pi^*(\wedge^i\mathcal{E})(-i)\right)$$

converging to zero! By the projection formula (Cohomology, Lemma 18.7.2) we have

$$L^{-i,q} = \wedge^i\mathcal{E} \otimes_{\mathcal{O}_S} R^q\pi_*\left(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(-i)\right).$$

Note that locally on S the sheaf $\mathcal{E}$ is trivial, i.e., isomorphic to $\mathcal{O}_S^{\oplus n+1}$, hence locally on S the morphism $\mathbf{P}(\mathcal{E}) \to S$ can be identified with $\mathbf{P}^n_S \to S$. Hence locally on S we can use the result of Lemmas 25.9.1, 25.9.2, or 25.9.3. It follows that $L^{-i,q} = 0$ unless $i = q = 0$ or $i = n+1$ and $q = n$. This in turn implies that $E_2^{p,q} = 0$ unless $(p,q) = (0,0)$ or $(p,q) = (-n-1, n)$, and

$$E_2^{0,0} = \pi_*\mathcal{O}_{\mathbf{P}(\mathcal{E})} = \mathcal{O}_S$$
$$E_2^{-n-1,n} = \wedge^{n+1}\mathcal{E} \otimes_{\mathcal{O}_S} R^n\pi_*\left(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(-n-1)\right).$$

Hence there can only be one nonzero differential in the spectral sequence namely the map d_{n+1} inducing a map

$$d_{n+1}^{0,0} : \mathcal{O}_S \longrightarrow \wedge^{n+1}\mathcal{E} \otimes_{\mathcal{O}_S} R^n\pi_*\left(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(-n-1)\right)$$

which has to be an isomorphism (because the spectral sequence converges to the 0 sheaf). Since $\wedge^{n+1}\mathcal{E}$ is an invertible sheaf, this implies that $R^n\pi_*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(-n-1)$ is invertible as well and canonically isomorphic to the inverse of $\wedge^{n+1}\mathcal{E}$. In other words we have proved the case $d = -n-1$ of the lemma.

Working locally on S we see immediately from the computation of cohomology in Lemmas 25.9.1, 25.9.2, or 25.9.3 the statements on vanishing of the lemma. Moreover the result on $R^0\pi_*$ is clear as well, since there are canonical maps $\mathrm{Sym}^d(\mathcal{E}) \to \pi_*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(d)$ for all d. It remains to show that the description of $R^n\pi_*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(d)$ is correct for $d < -n-1$. In other to do this we consider the map

$$\pi^*(\mathrm{Sym}^{-d+n+1}(\mathcal{E})) \otimes_{\mathcal{O}_{\mathbf{P}(\mathcal{E})}} \mathcal{O}_{\mathbf{P}(\mathcal{E})}(d) \longrightarrow \mathcal{O}_{\mathbf{P}(\mathcal{E})}(-n-1)$$

Applying $R^n\pi_*$ and the projection formula (see above) we get a map

$$\mathrm{Sym}^{-d+n+1}(\mathcal{E}) \otimes_{\mathcal{O}_S} R^n\pi_*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(d)) \longrightarrow R^n\pi_*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(-n-1) = (\wedge^{n+1}\mathcal{E})^{\otimes -1}$$

(the last equality we have shown above). Again by the local calculations of Lemmas 25.9.1, 25.9.2, or 25.9.3 it follows that this map induces a perfect pairing between $R^n\pi_*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(d))$ and $\mathrm{Sym}^{-d+n+1}(\mathcal{E}) \otimes \wedge^{n+1}(\mathcal{E})$ as desired. $\square$

25.10. Supports of modules

In this section we collect some elementary results on supports of quasi-coherent modules on schemes. Recall that the support of a sheaf of modules has been defined in Modules, Section 15.5. On the other hand, the support of a module was defined in Algebra, Section 7.59. These match.

Lemma 25.10.1. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $Spec(A) = U \subset X$ be an affine open, and set $M = \Gamma(U, \mathcal{F})$. Let $x \in U$, and let $\mathfrak{p} \subset A$ be the corresponding prime. The following are equivalent*

(1) *$\mathfrak{p}$ is in the support of M, and*

(2) *x is in the support of $\mathcal{F}$.*

Proof. This follows from the equality $\mathcal{F}_x = M_{\mathfrak{p}}$, see Schemes, Lemma 21.5.4 and the definitions. □

Lemma 25.10.2. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. The support of $\mathcal{F}$ is closed under specialization.*

Proof. If $x' \rightsquigarrow x$ is a specialization and $\mathcal{F}_x = 0$ then $\mathcal{F}_{x'}$ is zero, as $\mathcal{F}_{x'}$ is a localization of the module $\mathcal{F}_x$. Hence the complement of $\text{Supp}(\mathcal{F})$ is closed under generalization. □

For finite type quasi-coherent modules the support is closed, can be checked on fibres, and commutes with base change.

Lemma 25.10.3. *Let $\mathcal{F}$ be a finite type quasi-coherent module on a scheme X. Then*

(1) *The support of $\mathcal{F}$ is closed.*
(2) *For $x \in X$ we have*

$$x \in Supp(\mathcal{F}) \Leftrightarrow \mathcal{F}_x \neq 0 \Leftrightarrow \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x) \neq 0.$$

(3) *For any morphism of schemes $f : Y \to X$ the pullback $f^*\mathcal{F}$ is of finite type as well and we have $Supp(f^*\mathcal{F}) = f^{-1}(Supp(\mathcal{F}))$.*

Proof. Part (1) is a reformulation of Modules, Lemma 15.9.6. You can also combine Lemma 25.10.1, Properties, Lemma 23.16.1, and Algebra, Lemma 7.59.4 to see this. The first equivalence in (2) is the definition of support, and the second equivalence follows from Nakayama's lemma, see Algebra, Lemma 7.14.5. Let $f : Y \to X$ be a morphism of schemes. Note that $f^*\mathcal{F}$ is of finite type by Modules, Lemma 15.9.2. For the final assertion, let $y \in Y$ with image $x \in X$. Recall that

$$(f^*\mathcal{F})_y = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{Y,y},$$

see Sheaves, Lemma 6.26.4. Hence $(f^*\mathcal{F})_y \otimes \kappa(y)$ is nonzero if and only if $\mathcal{F}_x \otimes \kappa(x)$ is nonzero. By (2) this implies $x \in \text{Supp}(\mathcal{F})$ if and only if $y \in \text{Supp}(f^*\mathcal{F})$, which is the content of assertion (3). □

Lemma 25.10.4. *Let $\mathcal{F}$ be a finite type quasi-coherent module on a scheme X. There exists a smallest closed subscheme $i : Z \to X$ such that there exists a quasi-coherent $\mathcal{O}_Z$-module $\mathcal{G}$ with $i_*\mathcal{G} \cong \mathcal{F}$. Moreover:*

(1) *If $Spec(A) \subset X$ is any affine open, and $\mathcal{F}|_{Spec(A)} = \widetilde{M}$ then $Z \cap Spec(A) = Spec(A/I)$ where $I = Ann_A(M)$.*
(2) *The quasi-coherent sheaf $\mathcal{G}$ is unique up to unique isomorphism.*
(3) *The quasi-coherent sheaf $\mathcal{G}$ is of finite type.*
(4) *The support of $\mathcal{G}$ and of $\mathcal{F}$ is Z.*

Proof. Suppose that $i' : Z' \to X$ is a closed subscheme which satisfies the description on open affines from the lemma. Then by Morphisms, Lemma 24.3.1 we see that $\mathcal{F} \cong i'_*\mathcal{G}'$ for some unique quasi-coherent sheaf $\mathcal{G}'$ on Z'. Furthermore, it is clear that Z' is the smallest closed subscheme with this property (by the same lemma). Finally, using Properties, Lemma 23.16.1 and Algebra, Lemma 7.5.6 it follows that $\mathcal{G}'$ is of finite type. We have $\text{Supp}(\mathcal{G}') = Z$ by Algebra, Lemma 7.59.4. Hence, in order to prove the lemma it suffices to show that the characterization in (1) actually does define a closed subscheme. And, in order to do this it suffices to prove that the given rule produces a quasi-coherent sheaf of ideals, see Morphisms, Lemma 24.2.3. This comes down to the following algebra fact: If A is a ring, $f \in A$, and M is a finite A-module, then $\text{Ann}_A(M)_f = \text{Ann}_{A_f}(M_f)$. We omit the proof. □

Definition 25.10.5. Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. The *scheme theoretic support of* $\mathcal{F}$ is the closed subscheme $Z \subset X$ constructed in Lemma 25.10.4.

In this situation we often think of $\mathcal{F}$ as a quasi-coherent sheaf of finite type on Z (via the equivalence of categories of Morphisms, Lemma 24.3.1).

25.11. Coherent sheaves on locally Noetherian schemes

We have defined the notion of a coherent module on any ringed space in Modules, Section 15.12. Allthough it is possible to consider coherent sheaves on non-Noetherian schemes we will always assume the base scheme is locally Noetherian when we consider coherent sheaves. Here is a characterization of coherent sheaves on locally Noetherian schemes.

Lemma 25.11.1. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. The following are equivalent*

(1) *$\mathcal{F}$ is coherent,*
(2) *$\mathcal{F}$ is a quasi-coherent, finite type $\mathcal{O}_X$-module,*
(3) *$\mathcal{F}$ is a finitely presented $\mathcal{O}_X$-module,*
(4) *for any affine open $Spec(A) = U \subset X$ we have $\mathcal{F}|_U = \widetilde{M}$ with M a finite A-module, and*
(5) *there exists an affine open covering $X = \bigcup U_i$, $U_i = Spec(A_i)$ such that each $\mathcal{F}|_{U_i} = \widetilde{M}_i$ with M_i a finite A_i-module.*

In particular $\mathcal{O}_X$ is coherent, any invertible $\mathcal{O}_X$-module is coherent, and more generally any finite locally free $\mathcal{O}_X$-module is invertible.

Proof. The implications (1) ⇒ (2) and (1) ⇒ (3) hold in general, see Modules, Lemma 15.12.2. If $\mathcal{F}$ is finitely presented then $\mathcal{F}$ is quasi-coherent, see Modules, Lemma 15.11.2. Hence also (3) ⇒ (2).

Assume $\mathcal{F}$ is a quasi-coherent, finite type $\mathcal{O}_X$-module. By Properties, Lemma 23.16.1 we see that on any affine open $Spec(A) = U \subset X$ we have $\mathcal{F}|_U = \widetilde{M}$ with M a finite A-module. Since A is Noetherian we see that M has a finite resolution

$$A^{\oplus m} \to A^{\oplus n} \to M \to 0.$$

Hence $\mathcal{F}$ is of finite presentation by Properties, Lemma 23.16.2. In other words (2) ⇒ (3).

By Modules, Lemma 15.12.5 it suffices to show that $\mathcal{O}_X$ is coherent in order to show that (3) implies (1). Thus we have to show: given any open $U \subset X$ and any finite collection of sections $f_i \in \mathcal{O}_X(U)$, $i = 1, \ldots, n$ the kernel of the map $\bigoplus_{i=1,\ldots,n} \mathcal{O}_U \to \mathcal{O}_U$ is of finite type. Since being of finite type is a local property it suffices to check this in a neighbourhood of any $x \in U$. Thus we may assume $U = Spec(A)$ is affine. In this case $f_1, \ldots, f_n \in A$ are elements of A. Since A is Noetherian, see Properties, Lemma 23.5.2 the kernel K of the map $\bigoplus_{i=1,\ldots,n} A \to A$ is a finite A-module. See for example Algebra, Lemma 7.47.1. As the functor~ is exact, see Schemes, Lemma 21.5.4 we get an exact sequence

$$\widetilde{K} \to \bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_U \to \mathcal{O}_U$$

and by Properties, Lemma 23.16.1 again we see that $\widetilde{K}$ is of finite type. We conclude that (1), (2) and (3) are all equivalent.

It follows from Properties, Lemma 23.16.1 that (2) implies (4). It is trivial that (4) implies (5). The discussion in Schemes, Section 21.24 show that (5) implies that $\mathcal{F}$ is quasi-coherent and it is clear that (5) implies that $\mathcal{F}$ is of finite type. Hence (5) implies (2) and we win. □

Lemma 25.11.2. *Let X be a locally Noetherian scheme. The category of coherent $\mathcal{O}_X$-modules is abelian. More precisely, the kernel and cokernel of a map of coherent $\mathcal{O}_X$-modules are coherent. Any extension of coherent sheaves is coherent.*

Proof. This is a restatement of Modules, Lemma 15.12.4 in a particular case. □

The following lemma does not always hold for the category of coherent $\mathcal{O}_X$-modules on a general ringed space X.

Lemma 25.11.3. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Any quasi-coherent submodule of $\mathcal{F}$ is coherent. Any quasi-coherent quotient module of $\mathcal{F}$ is coherent.*

Proof. We may assume that X is affine, say $X = Spec(A)$. Properties, Lemma 23.5.2 implies that A is Noetherian. Lemma 25.11.1 turns this into algebra. The algebraic counter part of the lemma is that a quotient, or a submodule of a finite A-module is a finite A-module, see for example Algebra, Lemma 7.47.1. □

Lemma 25.11.4. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. The $\mathcal{O}_X$-modules $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ are coherent.*

Proof. It is shown in Modules, Lemma 15.19.4 that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is coherent. The result for tensor products is Modules, Lemma 15.15.5 □

Lemma 25.11.5. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. Let $\varphi : \mathcal{G} \to \mathcal{F}$ be a homomorphism of $\mathcal{O}_X$-modules. Let $x \in X$.*

(1) *If $\mathcal{F}_x = 0$ then there exists an open neighbourhood $U \subset X$ of x such that $\mathcal{F}|_U = 0$.*
(2) *If $\varphi_x : \mathcal{G}_x \to \mathcal{F}_x$ is injective, then there exists an open neighbourhood $U \subset X$ of x such that $\varphi|_U$ is injective.*
(3) *If $\varphi_x : \mathcal{G}_x \to \mathcal{F}_x$ is surjective, then there exists an open neighbourhood $U \subset X$ of x such that $\varphi|_U$ is surjective.*
(4) *If $\varphi_x : \mathcal{G}_x \to \mathcal{F}_x$ is bijective, then there exists an open neighbourhood $U \subset X$ of x such that $\varphi|_U$ is an isomorphism.*

Proof. See Modules, Lemmas 15.9.4, 15.9.5, and 15.12.6. □

Lemma 25.11.6. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. Let $x \in X$. Suppose $\psi : \mathcal{G}_x \to \mathcal{F}_x$ is a map of $\mathcal{O}_{X,x}$-modules. Then there exists an open neighbourhood $U \subset X$ of x and a map $\varphi : \mathcal{G}|_U \to \mathcal{F}|_U$ such that $\varphi_x = \psi$.*

Proof. In view of Lemma 25.11.1 this is a reformulation of Modules, Lemma 15.19.3. □

Lemma 25.11.7. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $Supp(\mathcal{F})$ is closed, and $\mathcal{F}$ comes from a coherent sheaf on the scheme theoretic support of $\mathcal{F}$, see Definition 25.10.5.*

Proof. Let $i : Z \to X$ be the scheme theoretic support of $\mathcal{F}$ and let $\mathcal{G}$ be the finite type quasi-coherent sheaf on Z such that $i_*\mathcal{G} \cong \mathcal{F}$. Since $Z = \text{Supp}(\mathcal{F})$ we see that the support is closed. The scheme Z is locally Noetherian by Morphisms, Lemmas 24.14.5 and 24.14.6. Finally, $\mathcal{G}$ is a coherent $\mathcal{O}_Z$-module by Lemma 25.11.1 □

Lemma 25.11.8. *Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume f is finite and Y locally Noetherian. Then $R^p f_*\mathcal{F} = 0$ for $p > 0$ and $f_*\mathcal{F}$ is coherent if $\mathcal{F}$ is coherent.*

Proof. The higher direct images vanish by Lemma 25.2.3 and because a finite morphism is affine (by definition). Note that the assumptions imply that also X is locally Noetherian (see Morphisms, Lemma 24.14.6) and hence the statement makes sense. Let $Spec(A) = V \subset Y$ be an affine open subset. By Morphisms, Definition 24.42.1 we see that $f^{-1}(V) = Spec(B)$ with $A \to B$ finite. Lemma 25.11.1 turns the statement of the lemma into the following algebra fact: If M is a finite B-module, then M is also finite viewed as a A-module, see Algebra, Lemma 7.7.2. □

In the situation of the lemma also the higher direct images are coherent since they vanish. We will show that this is always the case for a proper morphism between locally Noetherian schemes (insert future reference here).

25.12. Coherent sheaves on Noetherian schemes

In this section we mention some properties of coherent sheaves on Noetherian schemes.

Lemma 25.12.1. *Let X be a Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The ascending chain condition holds for quasi-coherent submodules of $\mathcal{F}$. In other words, give any sequence*

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}$$

of quasi-coherent submodules, then $\mathcal{F}_n = \mathcal{F}_{n+1} = \ldots$ for some $n \geq 0$.

Proof. Choose a finite affine open covering. On each member of the covering we get stabilization by Algebra, Lemma 7.47.1. Hence the lemma follows. □

Lemma 25.12.2. *Let X be a Noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf on X. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals corresponding to a closed subscheme $Z \subset X$. Then there is some $n \geq 0$ such that $\mathcal{I}^n\mathcal{F} = 0$ if and only if $Supp(\mathcal{F}) \subset Z$ (set theoretically).*

Proof. This follows immediately from Algebra, Lemma 7.59.9 because X has a finite covering by spectra of Noetherian rings. □

Lemma 25.12.3. *(Artin-Rees.) Let X be a Noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf on X. Let $\mathcal{G} \subset \mathcal{F}$ be a quasi-coherent subsheaf. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Then there exists a $c \geq 0$ such that for all $n \geq c$ we have*

$$\mathcal{I}^{n-c}(\mathcal{I}^c\mathcal{F} \cap \mathcal{G}) = \mathcal{I}^n\mathcal{F}$$

Proof. This follows immediately from Algebra, Lemma 7.47.4 because X has a finite covering by spectra of Noetherian rings. □

Lemma 25.12.4. *Let X be a Noetherian scheme. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Denote $Z \subset X$ the corresponding closed subscheme and set $U = X \setminus Z$. There is a canonical isomorphism*

$$colim_n \, Hom_{\mathcal{O}_X}(\mathcal{I}^n\mathcal{G}, \mathcal{F}) \longrightarrow Hom_{\mathcal{O}_U}(\mathcal{G}|_U, \mathcal{F}|_U).$$

In particular we have an isomorphism

$$colim_n \, Hom_{\mathcal{O}_X}(\mathcal{I}^n, \mathcal{F}) \longrightarrow \Gamma(U, \mathcal{F}).$$

Proof. We first prove the second equality. Let $\mathcal{F}_n$ denote the quasi-coherent subsheaf of $\mathcal{F}$ consisting of sections annihilated by $\mathcal{I}^n$, see Properties, Lemma 23.22.5. Since $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots$ we see that $\mathcal{F}_n = \mathcal{F}_{n+1} = \ldots$ for some $n \geq 0$ by Lemma 25.12.1. Set $\mathcal{H} = \mathcal{F}_n$ for this n. By Artin-Rees (Lemma 25.12.3) there exists an $c \geq 0$ such that $\mathcal{I}^m\mathcal{F} \cap \mathcal{G} \subset \mathcal{I}^{m-c}\mathcal{G}$. Picking $m = n + c$ we get $\mathcal{I}^m\mathcal{F} \cap \mathcal{G} \subset \mathcal{I}^n\mathcal{G} = 0$. Thus if we set $\mathcal{F}' = \mathcal{I}^m\mathcal{F}$ then we see that $\mathcal{F}' \cap \mathcal{F}_n = 0$ and $\mathcal{F}'|_U = \mathcal{F}|_U$. Note in particular that the subsheaf $(\mathcal{F}')_N$ of sections annihilated by $\mathcal{I}^N$ is zero for all $N \geq 0$. Hence by Properties, Lemma 23.22.5 we deduce that the top horizontal arrow in the following commutative diagram is a bijection:

$$\begin{tikzcd} colim_n\, Hom_{\mathcal{O}_X}(\mathcal{I}^n, \mathcal{F}') \arrow[r] \arrow[d] & \Gamma(U, \mathcal{F}') \arrow[d] \\ colim_n\, Hom_{\mathcal{O}_X}(\mathcal{I}^n, \mathcal{F}) \arrow[r] & \Gamma(U, \mathcal{F}) \end{tikzcd}$$

Since also the right vertical arrow is a bijection we conclude that the bottom horizontal arrow is surjective. The bottom horizontal arrow is injective by Properties, Lemma 23.22.5. This proves the bottom arrow is a bijection as desired.

Next, we come to the general case. By Lemma 25.11.4 the sheaf $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F})$ is coherent. By definition we have

$$\mathcal{H}(U) = Hom_{\mathcal{O}_U}(\mathcal{G}|_U, \mathcal{F}|_U)$$

Pick a ψ in the right hand side of the first arrow of the lemma, i.e., $\psi \in \mathcal{H}(U)$. The result just proved applies to $\mathcal{H}$ and hence there exists an $n \geq 0$ and an $\varphi : \mathcal{I}^n \to \mathcal{H}$ which recovers ψ on restricition to U. By Modules, Lemma 15.19.1 φ corresponds to a map

$$\varphi : \mathcal{I}^{\otimes n} \otimes_{\mathcal{O}_X} \mathcal{G} \longrightarrow \mathcal{F}.$$

This is almost what we want except that the source of the arrow is the tensor product of $\mathcal{I}^n$ and $\mathcal{G}$ and not the product. We will show that, at the cost of increasing n, the difference is irrelevant. Consider the short exact sequence

$$0 \to \mathcal{K} \to \mathcal{I}^n \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{I}^n\mathcal{G} \to 0$$

where $\mathcal{K}$ is defined as the kernel. Note that $\mathcal{I}^n\mathcal{K} = 0$ (proof omitted). By Artin-Rees again we see that

$$\mathcal{K} \cap \mathcal{I}^m(\mathcal{I}^n \otimes_{\mathcal{O}_X} \mathcal{G}) = 0$$

for some m large enough. In other words we see that

$$\mathcal{I}^m(\mathcal{I}^n \otimes_{\mathcal{O}_X} \mathcal{G}) \longrightarrow \mathcal{I}^{n+m}\mathcal{G}$$

is an isomorphism. Let φ' be the restriction of φ to this submodule thought of as a map $\mathcal{I}^{m+n}\mathcal{G} \to \mathcal{F}$. Then φ' gives an element of the left hand side of the first arrow of the lemma which maps to ψ via the arrow. In other words we have prove surjectivity of the arrow. We omit the proof of injectivity. □

25.13. Depth

In this section we talk a little bit about depth and property (S_k) for coherent modules on locally Noetherian schemes. Note that we have already discussed this notion for locally Noetherian schemes in Properties, Section 23.12.

Definition 25.13.1. Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $k \geq 0$ be an integer.

(1) We say $\mathcal{F}$ has *depth k at a point x* of X if $\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) = k$.
(2) We say X has *depth k at a point x* of X if $\text{depth}(\mathcal{O}_{X,x}) = k$.
(3) We say $\mathcal{F}$ has property (S_k) if
$$\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) \geq \min(k, \dim(\mathcal{F}_x))$$
for all $x \in X$.
(4) We say X has property (S_k) if $\mathcal{O}_X$ has property (S_k).

Any coherent sheaf satisfies condition (S_0). Condition (S_1) is equivalent to having no embedded associated points, see Divisors, Lemma 26.4.3.

We have seen in Properties, Lemma 23.12.2 that a locally Noetherian scheme is Cohen-Macaulay if and only if (S_k) holds for all k. Thus it makes sense to introduce the following definition, which is equivalent to the condition that all stalks are Cohen-Macaulay modules.

Definition 25.13.2. Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. We say $\mathcal{F}$ is *Cohen-Macaulay* if and only if (S_k) holds for all $k \geq 0$.

25.14. Devissage of coherent sheaves

Let X be a Noetherian scheme. Consider an integral closed subscheme $i : Z \to X$. It is often convenient to consider coherent sheaves of the form $i_*\mathcal{G}$ where $\mathcal{G}$ is a coherent sheaf on Z. In particular we are interested in these sheaves when $\mathcal{G}$ is a torsion free rank 1 sheaf. For example $\mathcal{G}$ could be a nonzero sheaf of ideals on Z, or even more specifically $\mathcal{G} = \mathcal{O}_Z$.

Throughout this section we will use that a coherent sheaf is the same thing as a finite type quasi-coherent sheaf and that a quasi-coherent subquotient of a coherent sheaf is coherent, see Section 25.11. The support of a coherent sheaf is closed, see Modules, Lemma 15.9.6.

Lemma 25.14.1. *Let X be a Noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf on X. Suppose that $Supp(\mathcal{F}) = Z \cup Z'$ with Z, Z' closed. Then there exists a short exact sequence of coherent sheaves*
$$0 \to \mathcal{G}' \to \mathcal{F} \to \mathcal{G} \to 0$$
with $Supp(\mathcal{G}') \subset Z'$ and $Supp(\mathcal{G}) \subset Z$.

Proof. Throughout the proof we will use that a coherent sheaf is the same thing as a finite type quasi-coherent sheaf and that a quasi-coherent subquotient of a coherent sheaf is coherent, see Section 25.11. The support of a coherent sheaf is closed, see Modules, Lemma 15.9.6. Let $\mathcal{I} \subset \mathcal{O}_X$ be the sheaf of ideals defining the reduced induced closed subscheme structure on Z, see Schemes, Lemma 21.12.4. Consider the subsheaves $\mathcal{G}'_n = \mathcal{I}^n\mathcal{F}$ and the quotients $\mathcal{G}_n = \mathcal{F}/\mathcal{I}^n\mathcal{F}$. For each n we have a short exact sequence
$$0 \to \mathcal{G}'_n \to \mathcal{F} \to \mathcal{G}_n \to 0$$
For every point x of $Z' \setminus Z$ we have $\mathcal{I}_x = \mathcal{O}_{X,x}$ and hence $\mathcal{G}_{n,x} = 0$. Thus we see that $\text{Supp}(\mathcal{G}_n) \subset Z$. Note that $X \setminus Z'$ is a Noetherian scheme. Hence by Lemma 25.12.2 there exists an n such that $\mathcal{G}'_n|_{X\setminus Z'} = \mathcal{I}^n\mathcal{F}|_{X\setminus Z'} = 0$. For such an n we see that $\text{Supp}(\mathcal{G}'_n) \subset Z'$. Thus setting $\mathcal{G}' = \mathcal{G}'_n$ and $\mathcal{G} = \mathcal{G}_n$ works. □

Lemma 25.14.2. *Let X be a Noetherian scheme. Let $i : Z \to X$ be an integral closed subscheme. Let $\xi \in Z$ be the generic point. Let $\mathcal{F}$ be a coherent sheaf on X. Assume that $\mathcal{F}_\xi$ is annihilated by $\mathfrak{m}_\xi$. Then there exists an integer $r \geq 0$ and a sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$ and an injective map of coherent sheaves*
$$i_*\left(\mathcal{I}^{\oplus r}\right) \to \mathcal{F}$$

which is an isomorphism in a neighbourhood of ξ.

Proof. Let $\mathcal{J} \subset \mathcal{O}_X$ be the ideal sheaf of Z. Let $\mathcal{F}' \subset \mathcal{F}$ be the subsheaf of local sections of $\mathcal{F}$ which are annihilated by $\mathcal{J}$. It is a quasi-coherent sheaf by Properties, Lemma 23.22.3. Moreover, $\mathcal{F}'_\xi = \mathcal{F}_\xi$ because $\mathcal{J}_\xi = \mathfrak{m}_\xi$ and part (3) of Properties, Lemma 23.22.3. By Lemma 25.11.5 we see that $\mathcal{F}' \to \mathcal{F}$ induces an isomorphism in a neighbourhood of ξ. Hence we may replace $\mathcal{F}$ by $\mathcal{F}'$ and assume that $\mathcal{F}$ is annihilated by $\mathcal{J}$.

Assume $\mathcal{J}\mathcal{F} = 0$. By Morphisms, Lemma 24.3.1 we can write $\mathcal{F} = i_*\mathcal{G}$ for some quasi-coherent sheaf $\mathcal{G}$ on Z. By Modules, Lemma 15.13.3 combined with the results of Section 25.11 we also see $\mathcal{G}$ is coherent on Z. Suppose we can find a morphism $\mathcal{I}^{\oplus r} \to \mathcal{G}$ which is an isomorphism in a neighbourhood of the generic point ξ of Z. Then applying i_* (which is left exact) we get the result of the lemma. Hence we have reduced to the case $X = Z$.

Suppose $Z = X$ is an integral Noetherian scheme with generic point ξ. Note that $\mathcal{O}_{X,\xi} = \kappa(\xi)$ is the function field of X in this case. Since $\mathcal{F}_\xi$ is a finite $\mathcal{O}_\xi$-module we see that $r = \dim_{\kappa(\xi)} \mathcal{F}_\xi$ is finite. Hence the sheaves $\mathcal{O}_X^{\oplus r}$ and $\mathcal{F}$ have isomorphic stalks at ξ. By Lemma 25.11.6 there exists a nonempty open $U \subset X$ and a morphism $\psi : \mathcal{O}_X^{\oplus r}|_U \to \mathcal{F}|_U$ which is an isomorphism at ξ, and hence an isomorphism in a neighbourhood of ξ by Lemma 25.11.5. By Schemes, Lemma 21.12.4 there exists a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ whose associated closed subscheme $Z \subset X$ is the complement of U. By Lemma 25.12.4 there exists an $n \geq 0$ and a morphism $\mathcal{I}^n(\mathcal{O}_X)^{\oplus r}) \to \mathcal{F}$ which recovers our ψ over U. Since $\mathcal{I}^n(\mathcal{O}_X)^{\oplus r}) = (\mathcal{I}^n)^{\oplus r}$ we get a map as in the lemma. It is injective because X is integral and it is injective at the generic point of X (easy proof omitted). $\square$

Lemma 25.14.3. *Let X be a Noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf on X. There exists a filtration*

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_m = \mathcal{F}$$

by coherent subsheaves such that for each $j = 1, \ldots, m$ there exists an integral closed subscheme $Z_j \subset X$ and a sheaf of ideals $\mathcal{I}_j \subset \mathcal{O}_{Z_j}$ such that

$$\mathcal{F}_j/\mathcal{F}_{j-1} \cong (Z_j \to X)_*\mathcal{I}_j$$

Proof. Consider the collection

$$\mathcal{T} = \left\{ \begin{matrix} Z \subset X \text{ closed such that there exists a coherent sheaf } \mathcal{F} \\ \text{with } \operatorname{Supp}(\mathcal{F}) = Z \text{ for which the lemma is wrong} \end{matrix} \right\}$$

We are trying to show that $\mathcal{T}$ is empty. If not, then because X is Noetherian we can choose a minimal element $Z \in \mathcal{T}$. This means that there exists a coherent sheaf $\mathcal{F}$ on X whose support is Z and for which the lemma does not hold. Clearly $Z \neq \emptyset$ since the only sheaf whose support is empty is the zero sheaf for which the lemma does hold (with $m = 0$).

If Z is not irreducible, then we can write $Z = Z_1 \cup Z_2$ with Z_1, Z_2 closed and strictly smaller than Z. Then we can apply Lemma 25.14.1 to get a short exact sequence of coherent sheaves

$$0 \to \mathcal{G}_1 \to \mathcal{F} \to \mathcal{G}_2 \to 0$$

with $\operatorname{Supp}(\mathcal{G}_i) \subset Z_i$. By minimality of Z each of $\mathcal{G}_i$ has a filtration as in the statement of the lemma. By considering the induced filtration on $\mathcal{F}$ we arrive at a contradiction. Hence we conclude that Z is irreducible.

Suppose Z is irreducible. Let $\mathcal{J}$ be the sheaf of ideals cutting out the reduced induced closed subscheme structure of Z, see Schemes, Lemma 21.12.4. By Lemma 25.12.2 we see there exists an $n \geq 0$ such that $\mathcal{J}^n\mathcal{F} = 0$. Hence we obtain a filtration

$$0 = \mathcal{I}^n\mathcal{F} \subset \mathcal{I}^{n-1}\mathcal{F} \subset \ldots \subset \mathcal{I}\mathcal{F} \subset \mathcal{F}$$

each of whose succesive subquotients is annihilated by $\mathcal{J}$. Hence if each of these subquotients has a filtration as in the statement of the lemma then also $\mathcal{F}$ does. In other words we may assume that $\mathcal{J}$ does annihilate $\mathcal{F}$.

In the case where Z is irreducible and $\mathcal{J}\mathcal{F} = 0$ we can apply Lemma 25.14.2. This gives a short exact sequence

$$0 \to i_*(\mathcal{I}^{\oplus r}) \to \mathcal{F} \to \mathcal{Q} \to 0$$

where $\mathcal{Q}$ is defined as the quotient. Since $\mathcal{Q}$ is zero in a neighbourhood of ξ by the lemma just cited we see that the support of $\mathcal{Q}$ is strictly smaller than Z. Hence we see that $\mathcal{Q}$ has a filtration of the desired type by minimality of Z. But then clearly $\mathcal{F}$ does too, which is our final contradiction. $\square$

Lemma 25.14.4. *Let X be a Noetherian scheme. Let $\mathcal{P}$ be a property of coherent sheaves on X such that*

(1) *For any short exact sequence of coherent sheaves*

$$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$$

if $\mathcal{F}_i$, $i = 1, 2$ have property $\mathcal{P}$ then so does $\mathcal{F}$.

(2) *For every integral closed subscheme $Z \subset X$ and every quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$ we have $\mathcal{P}$ for $i_*\mathcal{I}$.*

Then property $\mathcal{P}$ holds for every coherent sheaf on X.

Proof. First note that if $\mathcal{F}$ is a coherent sheaf with a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_m = \mathcal{F}$$

by coherent subsheaves such that each of $\mathcal{F}_i/\mathcal{F}_{i-1}$ has property $\mathcal{P}$, then so does $\mathcal{F}$. This follows from the property (1) for $\mathcal{P}$. On the other hand, by Lemma 25.14.3 we can filter any $\mathcal{F}$ with succesive subquotients as in (2). Hence the lemma follows. $\square$

Lemma 25.14.5. *Let X be a Noetherian scheme. Let $Z_0 \subset X$ be an irreducible closed subset with generic point ξ. Let $\mathcal{P}$ be a property of coherent sheaves on X such that*

(1) *For any short exact sequence of coherent sheaves if two out of three of them have property $\mathcal{P}$ then so does the third.*

(2) *For every integral closed subscheme $Z \subset Z_0 \subset X$, $Z \neq Z_0$ and every quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$ we have $\mathcal{P}$ for $(Z \to X)_*\mathcal{I}$.*

(3) *There exists some coherent sheaf $\mathcal{G}$ on X such that*
   (a) *$Supp(\mathcal{G}) = Z_0$,*
   (b) *$\mathcal{G}_\xi$ is annihilated by $\mathfrak{m}_\xi$,*
   (c) $\dim_{\kappa(\xi)} \mathcal{G}_\xi = 1$, *and*
   (d) *property $\mathcal{P}$ holds for $\mathcal{G}$.*

Then property $\mathcal{P}$ holds for every coherent sheaf $\mathcal{F}$ on X whose support is contained in Z_0.

Proof. First note that if $\mathcal{F}$ is a coherent sheaf with a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_m = \mathcal{F}$$

by coherent subsheaves such that each of $\mathcal{F}_i/\mathcal{F}_{i-1}$ has property $\mathcal{P}$, then so does $\mathcal{F}$. Or, if $\mathcal{F}$ has property $\mathcal{P}$ and all but one of the $\mathcal{F}_i/\mathcal{F}_{i-1}$ has property $\mathcal{P}$ then so does the last one. This follows from the two-out-of-three property (1) for $\mathcal{P}$.

As a first application of these remarks we conclude that any coherent sheaf whose support is strictly contained in Z_0 has property $\mathcal{P}$. Namely, such a sheaf has a filtration (see Lemma 25.14.3) whose subquotients have property $\mathcal{P}$ according to (2).

As a second application we apply this remark to the sheaf $\mathcal{G}$ from assumption (3) and a filtration

$$0 = \mathcal{G}_0 \subset \mathcal{G}_1 \subset \ldots \subset \mathcal{G}_m = \mathcal{G}$$

by coherent subsheaves as in Lemma 25.14.3. Let $Z_j \to X$ be the integral closed subschemes, and $\mathcal{I}_i \subset \mathcal{O}_{Z_i}$ the quasi-coherent sheaves of ideals such that $\mathcal{G}_i/\mathcal{G}_{i-1} \cong (Z_i \to X)_*\mathcal{I}_i$. We may obviously assume all the $\mathcal{I}_i$ are nonzero. Since $\dim_{\kappa(\xi)} \mathcal{G}_\xi = 1$ we see that there is exactly one $i = i_0$ such that $Z_{i_0} = Z_0$ and all other $Z_i \subset Z_0$ are proper irreducible closed subsets of Z_0. By the remark above and (2) we see that $(Z_{i_0} \to X)_*\mathcal{I}_{i_0}$ has property $\mathcal{P}$. We conclude that there exists at least one quasi-coherent sheaf of ideals $\mathcal{I}$ (namely $\mathcal{I}_{i_0}$) on Z_0 such that $(Z_0 \to X)_*\mathcal{I}$ has property $\mathcal{P}$.

Next, suppose that $\mathcal{I}'$ is another quasi-coherent sheaf of ideals Z_0. Then we can consider the intersection $\mathcal{I}'' = \mathcal{I}' \cap \mathcal{I}$ and we get two short exact sequences

$$0 \to (Z_0 \to X)_*\mathcal{I}'' \to (Z_0 \to X)_*\mathcal{I} \to \mathcal{Q} \to 0$$

and

$$0 \to (Z_0 \to X)_*\mathcal{I}' \to (Z_0 \to X)_*\mathcal{I} \to \mathcal{Q}' \to 0.$$

Note that the support of the coherent sheaves $\mathcal{Q}$ and $\mathcal{Q}'$ are strictly contained in Z_0. Hence $\mathcal{Q}$ and $\mathcal{Q}$ have property $\mathcal{P}$ (see above). Hence we conclude using (1) that $(Z_0 \to X)_*\mathcal{I}''$ and $(Z_0 \to X)_*\mathcal{I}'$ both have $\mathcal{P}$ as well.

The final step of the proof is to note that any coherent sheaf $\mathcal{F}$ on X whose support is contained in Z_0 has a filtration (see Lemma 25.14.3 again) whose subquotients all have property $\mathcal{P}$ by what we just said. □

Lemma 25.14.6. *Let X be a Noetherian scheme. Let $\mathcal{P}$ be a property of coherent sheaves on X such that*

(1) *For any short exact sequence of coherent sheaves if two out of three of them have property $\mathcal{P}$ then so does the third.*
(2) *For every integral closed subscheme $Z \subset X$ with generic point ξ there exists some coherent sheaf $\mathcal{G}$ such that*
   (a) *$Supp(\mathcal{G}) = Z$,*
   (b) *$\mathcal{G}_\xi$ is annihilated by $\mathfrak{m}_\xi$,*
   (c) *$\dim_{\kappa(\xi)} \mathcal{G}_\xi = 1$, and*
   (d) *property $\mathcal{P}$ holds for $\mathcal{G}$.*

Then property $\mathcal{P}$ holds for every coherent sheaf on X.

Proof. According to Lemma 25.14.4 it suffices to show that given any integral closed subscheme $Z \subset X$ and every quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$ we have $\mathcal{P}$ for $(Z \to X)_*\mathcal{I}$. If this fails, then since X is Noetherian there is a minimal integral closed subscheme $Z_0 \subset X$ such that $\mathcal{P}$ fails for $(Z_0 \to X)_*\mathcal{I}$ for some quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_{Z_0}$. In other words, the result does hold for any integral closed subscheme of Z. According to Lemma 25.14.5 this cannot happen. □

Lemma 25.14.7. *(Variant of Lemma 25.14.5.) Let X be a Noetherian scheme. Let $Z_0 \subset X$ be an irreducible closed subset with generic point ξ. Let $\mathcal{P}$ be a property of coherent sheaves on X such that*

1. *For any short exact sequence of coherent sheaves if two out of three of them have property $\mathcal{P}$ then so does the third.*
2. *If $\mathcal{P}$ holds for a direct sum of coherent sheaves then it holds for both.*
3. *For every integral closed subscheme $Z \subset Z_0 \subset X$, $Z \neq Z_0$ and every quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$ we have $\mathcal{P}$ for $(Z \to X)_*\mathcal{I}$.*
4. *There exists some coherent sheaf $\mathcal{G}$ on X such that*
   (a) *$Supp(\mathcal{G}) = Z_0$,*
   (b) *$\mathcal{G}_\xi$ is annihilated by $\mathfrak{m}_\xi$, and*
   (c) *property $\mathcal{P}$ holds for $\mathcal{G}$.*

Then property $\mathcal{P}$ holds for every coherent sheaf $\mathcal{F}$ on X whose support is contained in Z_0.

Proof. The proof is a variant on the proof of Lemma 25.14.5. In exactly the same manner as in that proof we see that any coherent sheaf whose support is strictly contained in Z_0 has property $\mathcal{P}$.

Consider a coherent sheaf $\mathcal{G}$ as in (3). By Lemma 25.14.2 there exists a sheaf of ideals $\mathcal{I}$ on Z_0 and a short exact sequence

$$0 \to \left((Z_0 \to X)_*\mathcal{I}\right)^{\oplus r} \to \mathcal{G} \to \mathcal{Q} \to 0$$

where the support of $\mathcal{Q}$ is stricly contained in Z_0. In particular $r > 0$ and $\mathcal{I}$ is nonzero because the support of $\mathcal{G}$ is equal to Z. Since $\mathcal{Q}$ has property $\mathcal{P}$ we conclude that also $\left((Z_0 \to X)_*\mathcal{I}\right)^{\oplus r}$ has property $\mathcal{P}$. By (2) we deduce property $\mathcal{P}$ for $(Z_0 \to X)_*\mathcal{I}$. Slotting this into the proof of Lemma 25.14.5 at the appropriate point gives the lemma. Some details omitted. $\square$

Lemma 25.14.8. *(Variant of Lemma 25.14.6.) Let X be a Noetherian scheme. Let $\mathcal{P}$ be a property of coherent sheaves on X such that*

1. *For any short exact sequence of coherent sheaves if two out of three of them have property $\mathcal{P}$ then so does the third.*
2. *If $\mathcal{P}$ holds for a direct sum of coherent sheaves then it holds for both.*
3. *For every integral closed subscheme $Z \subset X$ with generic point ξ there exists some coherent sheaf $\mathcal{G}$ such that*
   (a) *$Supp(\mathcal{G}) = Z$,*
   (b) *$\mathcal{G}_\xi$ is annihilated by $\mathfrak{m}_\xi$, and*
   (c) *property $\mathcal{P}$ holds for $\mathcal{G}$.*

Then property $\mathcal{P}$ holds for every coherent sheaf on X.

Proof. This follows from Lemma 25.14.7 by exactly the same argument as used in the proof of Lemma 25.14.6. $\square$

Lemma 25.14.9. *(Cohomological variant of Lemma 25.14.7.) Let X be a Noetherian scheme. Let $Z_0 \subset X$ be an irreducible closed subset with generic point ξ. Let $\mathcal{P}$ be a property of coherent sheaves on X such that*

1. *For any short exact sequence of coherent sheaves*
   $$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$$
   if $\mathcal{F}_i$, $i = 1, 2$ have property $\mathcal{P}$ then so does $\mathcal{F}$.
2. *If $\mathcal{P}$ holds for a direct sum of coherent sheaves then it holds for both.*

(3) *For every integral closed subscheme $Z \subset Z_0 \subset X$, $Z \neq Z_0$ and every quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$ we have $\mathcal{P}$ for $(Z \to X)_*\mathcal{I}$.*
(4) *There exists some coherent sheaf $\mathcal{G}$ such that*
  (a) *$Supp(\mathcal{G}) = Z_0$,*
  (b) *$\mathcal{G}_\xi$ is annihilated by $\mathfrak{m}_\xi$, and*
  (c) *for every quasi-coherent sheaf of ideals $\mathcal{J} \subset \mathcal{O}_X$ such that $\mathcal{J}_\xi = \mathcal{O}_{X,\xi}$ there exists a quasi-coherent subsheaf $\mathcal{G}' \subset \mathcal{J}\mathcal{G}$ with $\mathcal{G}'_\xi = \mathcal{G}_\xi$ and such that $\mathcal{P}$ holds for $\mathcal{G}'$.*

Then property $\mathcal{P}$ holds for every coherent sheaf $\mathcal{F}$ on X whose support is contained in Z_0.

Proof. The proof is a variant on the proof of Lemma 25.14.5. In exactly the same manner as in that proof we see that any coherent sheaf whose support is strictly contained in Z_0 has property $\mathcal{P}$. Note that this does not use the full strength of the two-out-of-three property of that lemma, only the weaker variant (1) above which is in force in the current situation.

Let us denote $i : Z_0 \to X$ the closed immersion. Consider a coherent sheaf $\mathcal{G}$ as in (4). By Lemma 25.14.2 there exists a sheaf of ideals $\mathcal{I}$ on Z_0 and a short exact sequence

$$0 \to i_*\mathcal{I}^{\oplus r} \to \mathcal{G} \to \mathcal{Q} \to 0$$

where the support of $\mathcal{Q}$ is stricly contained in Z_0. In particular $r > 0$ and $\mathcal{I}$ is nonzero because the support of $\mathcal{G}$ is equal to Z_0. Let $\mathcal{I}' \subset \mathcal{I}$ be any nonzero quasi-coherent sheaf of ideals on Z contained in $\mathcal{I}$. Then we also get a short exact sequence

$$0 \to i_*(\mathcal{I}')^{\oplus r} \to \mathcal{G} \to \mathcal{Q}' \to 0$$

where $\mathcal{Q}'$ has support properly contained in Z_0. Let $\mathcal{J} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals cuttting out the support of $\mathcal{Q}'$ (for example the ideal corresponding to the reduced induced closed subscheme structure on the the support of $\mathcal{Q}'$). Then $\mathcal{J}_\xi = \mathcal{O}_{X,\xi}$. By Lemma 25.12.2 we see that $\mathcal{J}^n\mathcal{Q}' = 0$ for some n. Hence $\mathcal{J}^n\mathcal{G} \subset i_*(\mathcal{I}')^{\oplus r}$. By assumption (4)(c) of the lemma we see there exists a quasi-coherent subsheaf $\mathcal{G}' \subset \mathcal{J}^n\mathcal{G}$ with $\mathcal{G}'_\xi = \mathcal{G}_\xi$ for which property $\mathcal{P}$ holds. Hence we get a short exact sequence

$$0 \to \mathcal{G}' \to i_*(\mathcal{I}')^{\oplus r} \to \mathcal{Q}'' \to 0$$

where $\mathcal{Q}''$ has support properly contained in Z_0. Thus by our initial remarks and property (1) of the lemma we conclude that $i_*(\mathcal{I}')^{\oplus r}$ satisfies $\mathcal{P}$. Hence we see that $i_*\mathcal{I}'$ satisfies $\mathcal{P}$ by (2). Finally, for an arbitrary quasi-coherent sheaf of ideals $\mathcal{I}'' \subset \mathcal{O}_{Z_0}$ we can set $\mathcal{I} = \mathcal{I}'' \cap \mathcal{I}$ and we get a short exact sequence

$$0 \to i_*(\mathcal{I}') \to i_*(\mathcal{I}'') \to \mathcal{Q}''' \to 0$$

where $\mathcal{Q}'''$ has support properly contained in Z_0. Hence we conclude that property $\mathcal{P}$ holds for $i_*\mathcal{I}''$ as well. Slotting this into the proof of Lemma 25.14.5 at the appropriate point gives the lemma. Some details omitted. □

Lemma 25.14.10. *(Cohomological variant of Lemma 25.14.8.) Let X be a Noetherian scheme. Let $\mathcal{P}$ be a property of coherent sheaves on X such that*

(1) *For any short exact sequence of coherent sheaves*

$$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$$

if $\mathcal{F}_i$, $i = 1, 2$ have property $\mathcal{P}$ then so does $\mathcal{F}$.
(2) *If $\mathcal{P}$ holds for a direct sum of coherent sheaves then it holds for both.*
(3) *For every integral closed subscheme $Z \subset X$ with generic point ξ there exists some coherent sheaf $\mathcal{G}$ such that*

(a) *Supp*$(\mathcal{G}) = Z$,
(b) $\mathcal{G}_\xi$ *is annihilated by* $\mathfrak{m}_\xi$, *and*
(c) *for every quasi-coherent sheaf of ideals* $\mathcal{J} \subset \mathcal{O}_X$ *such that* $\mathcal{J}_\xi = \mathcal{O}_{X,\xi}$ *there exists a quasi-coherent subsheaf* $\mathcal{G}' \subset \mathcal{J}\mathcal{G}$ *with* $\mathcal{G}'_\xi = \mathcal{G}_\xi$ *and such that* $\mathcal{P}$ *holds for* $\mathcal{G}'$.

Then property $\mathcal{P}$ *holds for every coherent sheaf on* X.

Proof. Identical to the proofs of Lemmas 25.14.8 and 25.14.6. □

25.15. Finite morphisms and affines

In this section we use the results of the preceding sections to show that the image of a Noetherian affine scheme under a finite morphism is affine. We will see later that this result holds more generally (see Limits, Lemma 27.7.1).

Lemma 25.15.1. *Let* $f : Y \to X$ *be a morphism of schemes. Assume* f *is finite, surjective and* X *locally Noetherian. Let* $Z \subset X$ *be an integral closed subscheme with generic point* ξ. *Then there exists a coherent sheaf* $\mathcal{F}$ *on* Y *such that the support of* $f_*\mathcal{F}$ *is equal to* Z *and* $(f_*\mathcal{F})_\xi$ *is annihilated by* $\mathfrak{m}_\xi$.

Proof. Note that Y is locally Noetherian by Morphisms, Lemma 24.14.6. Because f is surjective the fibre Y_ξ is not empty. Pick $\xi' \in Y$ mapping to ξ. Let $Z' = \overline{\{\xi'\}}$. We may think of $Z' \subset Y$ as a reduced closed subscheme, see Schemes, Lemma 21.12.4. Hence the sheaf $\mathcal{F} = (Z' \to Y)_*\mathcal{O}_{Z'}$ is a coherent sheaf on Y (see Lemma 25.11.8). Look at the commutative diagram

$$\begin{tikzcd} Z' \arrow[r, "i'"] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ Z \arrow[r, "i"] & X \end{tikzcd}$$

We see that $f_*\mathcal{F} = i_*f'_*\mathcal{O}_{Z'}$. Hence the stalk of $f_*\mathcal{F}$ at ξ is the stalk of $f'_*\mathcal{O}_{Z'}$ at ξ. Note that since Z' is integral with generic point ξ' we have that ξ' is the only point of Z' lying over ξ, see Algebra, Lemmas 7.32.3 and 7.32.18. Hence the stalk of $f'_*\mathcal{O}_{Z'}$ at ξ equal $\mathcal{O}_{Z',\xi'} = \kappa(\xi')$. In particular the stalk of $f_*\mathcal{F}$ at ξ is not zero. This combined with the fact that $f_*\mathcal{F}$ is of the form $i_*f'_*$(something) implies the lemma. □

Lemma 25.15.2. *Let* $f : Y \to X$ *be a morphism of schemes. Let* $\mathcal{F}$ *be a quasi-coherent sheaf on* Y. *Let* $\mathcal{I}$ *be a quasi-coherent sheaf of ideals on* X. *If the morphism* f *is affine then* $\mathcal{I}f_*\mathcal{F} = f_*(f^{-1}\mathcal{I}\mathcal{F})$.

Proof. The notation means the following. Since f^{-1} is an exact functor we see that $f^{-1}\mathcal{I}$ is a sheaf of ideals of $f^{-1}\mathcal{O}_X$. Via the map $f^\sharp : f^{-1}\mathcal{O}_X \to \mathcal{O}_Y$ this acts on $\mathcal{F}$. Then $f^{-1}\mathcal{I}\mathcal{F}$ is the subsheaf generated by sums of local sections of the form as where a is a local section of $f^{-1}\mathcal{I}$ and s is a local section of $\mathcal{F}$. It is a quasi-coherent $\mathcal{O}_Y$-submodule of $\mathcal{F}$ because it is also the image of a natural map $f^*\mathcal{I} \otimes_{\mathcal{O}_Y} \mathcal{F} \to \mathcal{F}$.

Having said this the proof is straightforward. Namely, the question is local and hence we may assume X is affine. Since f is affine we see that Y is affine too. Thus we may write $Y = Spec(B)$, $X = Spec(A)$, $\mathcal{F} = \widetilde{M}$, and $\mathcal{I} = \tilde{I}$. The assertion of the lemma in this case boils down to the statement that

$$I(M_A) = ((IB)M)_A$$

where M_A indicates the A-module associated to the B-module M. □

Lemma 25.15.3. *Let $f : Y \to X$ be a morphism of schemes. Assume*

(1) *f finite,*
(2) *f surjective,*
(3) *Y affine, and*
(4) *X Noetherian.*

Then X is affine.

Proof. We will prove that under the assumptions of the lemma for any coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have $H^1(X, \mathcal{F}) = 0$. Since this will in particular imply that $H^1(X, \mathcal{I}) = 0$ for every quasi-coherent sheaf of ideals of $\mathcal{O}_X$. Then it will follows that X is affine from either Lemma 25.3.1 or Lemma 25.3.2.

Let $\mathcal{P}$ be the property of coherent sheaves $\mathcal{F}$ on X defined by the rule

$$\mathcal{P}(\mathcal{F}) \Leftrightarrow H^1(X, \mathcal{F}) = 0.$$

We are going to apply Lemma 25.14.10. Thus we have to verify (1), (2) and (3) of that lemma for $\mathcal{P}$. Property (1) follows from the long exact cohomology sequence associated to a short exact sequence of sheaves. Property (2) follows since $H^1(X, -)$ is an additive functor. To see (3) let $Z \subset X$ be an integral closed subscheme with generic point ξ. Let $\mathcal{F}$ be a coherent sheaf on Y such that the support of $f_*\mathcal{F}$ is equal to Z and $(f_*\mathcal{F})_\xi$ is annihilated by $\mathfrak{m}_\xi$, see Lemma 25.15.1. We claim that taking $\mathcal{G} = f_*\mathcal{F}$ works. We only have to verify part (3)(c) of Lemma 25.14.10. Hence assume that $\mathcal{J} \subset \mathcal{O}_X$ is a quasi-coherent sheaf of ideals such that $\mathcal{J}_\xi = \mathcal{O}_{X,\xi}$. A finite morphism is affine hence by Lemma 25.15.2 we see that $\mathcal{J}\mathcal{G} = f_*(f^{-1}\mathcal{J}\mathcal{F})$. Also, as pointed out in the proof of Lemma 25.15.2 the sheaf $f^{-1}\mathcal{J}\mathcal{F}$ is a quasi-coherent $\mathcal{O}_Y$-module. Since Y is affine we see that $H^1(Y, f^{-1}\mathcal{J}\mathcal{F}) = 0$, see Lemma 25.2.2. Since f is finite, hence affine, we see that $R^q f_*(f^{-1}\mathcal{J}\mathcal{F}) = 0$ for all $q \geq 1$ by Lemma 25.2.3. By Cohomology, Lemma 18.12.6 we see that

$$H^1(X, \mathcal{J}\mathcal{G}) = H^1(X, f_*(f^{-1}\mathcal{J}\mathcal{F})) = H^1(Y, f^{-1}\mathcal{J}\mathcal{F}) = 0.$$

Hence the quasi-coherent subsheaf $\mathcal{G}' = \mathcal{J}\mathcal{G}$ satisfies $\mathcal{P}$. This verifies property (3)(c) of Lemma 25.14.10 as desired. □

25.16. Coherent sheaves and projective morphisms

It seems illuminating to formulate an all-in-one result for projective space over a Noetherian ring.

Lemma 25.16.1. *Let R be a Noetherian ring. Let $n \geq 0$ be an integer. For every coherent sheaf $\mathcal{F}$ on $\mathbf{P}^n_R$ we have the following:*

(1) *There exists an $r \geq 0$ and $d_1, \ldots, d_r \in \mathbf{Z}$ and a surjection*
$$\bigoplus\nolimits_{j=1,\ldots,r} \mathcal{O}_{\mathbf{P}^n_R}(d_j) \longrightarrow \mathcal{F}.$$
(2) *We have $H^i(\mathbf{P}^n_R, \mathcal{F}) = 0$ unless $0 \leq i \leq n$.*
(3) *For any i the cohomology group $H^i(\mathbf{P}^n_R, \mathcal{F})$ is a finite R-module.*
(4) *If $i > 0$, then $H^i(\mathbf{P}^n_R, \mathcal{F}(d)) = 0$ for all d large enough.*
(5) *For any $k \in \mathbf{Z}$ the graded $R[T_0, \ldots, T_n]$-module*
$$\bigoplus\nolimits_{d \geq k} H^0(\mathbf{P}^n_R, \mathcal{F}(d))$$
is a finite $R[T_0, \ldots, T_n]$-module.

Proof. We will use that $\mathcal{O}_{\mathbf{P}^n_R}(1)$ is an ample invertible sheaf on the scheme $\mathbf{P}^n_R$. This follows directly from the definition since $\mathbf{P}^n_R$ covered by the standard affine opens $D_+(T_i)$. Hence by Properties, Proposition 23.23.12 every finite type quasi-coherent $\mathcal{O}_{\mathbf{P}^n_R}$-module is a quotient of a finite direct sum of tensor powers of $\mathcal{O}_{\mathbf{P}^n_R}(1)$. On the other hand a coherent sheaves and finite type quasi-coherent sheaves are the same thing on projective space over R by Lemma 25.11.1. Thus we see (1).

Projective n-space $\mathbf{P}^n_R$ is covered by $n+1$ affines, namely the standard opens $D_+(T_i)$, $i = 0, \ldots, n$, see Constructions, Lemma 22.13.3. Hence we see that for any quasi-coherent sheaf $\mathcal{F}$ on $\mathbf{P}^n_R$ we have $H^i(\mathbf{P}^n_R, \mathcal{F}) = 0$ for $i \geq n+1$, see Lemma 25.5.1. Hence (2) holds.

Let us prove (3) and (4) simultaneously for all coherent sheaves on $\mathbf{P}^n_R$ by descending induction on i. Clearly the result holds for $i \geq n+1$ by (2). Suppose we know the result for $i+1$ and we want to show the result for i. (If $i = 0$, then part (4) is vacuous.) Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_R$. Choose a surjection as in (1) and denote $\mathcal{G}$ the kernel so that we have a short exact sequence

$$0 \to \mathcal{G} \to \bigoplus\nolimits_{j=1,\ldots,r} \mathcal{O}_{\mathbf{P}^n_R}(d_j) \to \mathcal{F} \to 0$$

By Lemma 25.11.2 we see that $\mathcal{G}$ is coherent. The long exact cohomology sequence gives an exact sequence

$$H^i(\mathbf{P}^n_R, \bigoplus\nolimits_{j=1,\ldots,r} \mathcal{O}_{\mathbf{P}^n_R}(d_j)) \to H^i(\mathbf{P}^n_R, \mathcal{F}) \to H^{i+1}(\mathbf{P}^n_R, \mathcal{G}).$$

By induction assumption the right R-module is finite and by Lemma 25.9.1 the left R-module is finite. Since R is Noetherian it follows immediately that $H^i(\mathbf{P}^n_R, \mathcal{F})$ is a finite R-module. This proves the induction step for assertion (3). Since $\mathcal{O}_{\mathbf{P}^n_R}(d)$ is invertible we see that twisting on $\mathbf{P}^n_R$ is an exact functor (since you get it by tensoring with an invertible sheaf, see Constructions, Definition 22.10.1). This means that for all $d \in \mathbf{Z}$ the sequence

$$0 \to \mathcal{G}(d) \to \bigoplus\nolimits_{j=1,\ldots,r} \mathcal{O}_{\mathbf{P}^n_R}(d_j + d) \to \mathcal{F}(d) \to 0$$

is short exact. The resulting cohomology sequence is

$$H^i(\mathbf{P}^n_R, \bigoplus\nolimits_{j=1,\ldots,r} \mathcal{O}_{\mathbf{P}^n_R}(d_j + d)) \to H^i(\mathbf{P}^n_R, \mathcal{F}(d)) \to H^{i+1}(\mathbf{P}^n_R, \mathcal{G}(d)).$$

By induction assumption we see the module on the right is zero for $d \gg 0$ and by the computation in Lemma 25.9.1 the module on the left is zero as soon as $d \geq -\min\{d_j\}$ and $i \geq 1$. Hence the induction step for assertion (4). This concludes the proof of (3) and (4).

In order to prove (5) note that for all sufficiently large d the map

$$H^0(\mathbf{P}^n_R, \bigoplus\nolimits_{j=1,\ldots,r} \mathcal{O}_{\mathbf{P}^n_R}(d_j + d)) \to H^0(\mathbf{P}^n_R, \mathcal{F}(d))$$

is surjective by the vanishing of $H^1(\mathbf{P}^n_R, \mathcal{G}(d))$ we just proved. In other words, the module

$$M_k = \bigoplus\nolimits_{d \geq k} H^0(\mathbf{P}^n_R, \mathcal{F}(d))$$

is for k large enough a quotient of the corresponding module

$$N_k = \bigoplus\nolimits_{d \geq k} H^0(\mathbf{P}^n_R, \bigoplus\nolimits_{j=1,\ldots,r} \mathcal{O}_{\mathbf{P}^n_R}(d_j + d))$$

When k is sufficiently small (e.g. $k < -d_j$ for all j) then

$$N_k = \bigoplus\nolimits_{j=1,\ldots,r} R[T_0, \ldots, T_n](d_j)$$

by our computations in Section 25.9. In particular it is finitely generated. Suppose $k \in \mathbf{Z}$ is arbitrary. Choose $k_- \ll k \ll k_+$. Consider the diagram

$$\begin{tikzcd} N_{k_-} & N_{k_+} \arrow[l] \arrow[d] \\ M_k & M_{k_+} \arrow[l] \end{tikzcd}$$

where the vertical arrow is the surjective map above and the horizontal arrows are the obvious inclusion maps. By what was said above we see that N_{k_-} is a finitely generated $R[T_0, \ldots, T_n]$-module. Hence N_{k_+} is a finitely generated $R[T_0, \ldots, T_n]$-module because it is a submodule of a finitely generated module and the ring $R[T_0, \ldots, T_n]$ is Noetherian. Since the vertical arrow is surjective we conclude that M_{k_+} is a finitely generated $R[T_0, \ldots, T_n]$-module. The quotient M_k/M_{k_+} is finite as an R-module since it is a finite direct sum of the finite R-modules $H^0(\mathbf{P}^n_R, \mathcal{F}(d))$ for $k \leq d < k_+$. Note that we use part (3) for $i = 0$ here. Hence M_k/M_{k_+} is a fortiori a finite $R[T_0, \ldots, T_n]$-module. In other words, we have sandwiched M_k between two finite $R[T_0, \ldots, T_n]$-modules and we win. $\square$

Lemma 25.16.2. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{L}$ be an invertible sheaf on X. Assume that*

(1) *S is Noetherian,*
(2) *f is proper,*
(3) *$\mathcal{F}$ is coherent, and*
(4) *$\mathcal{L}$ is relatively ample on X/S.*

Then there exists an n_0 such that for all $n \geq n_0$ we have

$$R^p f_*\left(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}\right) = 0$$

for all $p > 0$.

Proof. A proper morphism is of finite type by definition. By Morphisms, Lemma 24.38.7 there exists an open covering $S = \bigcup V_j$ and immersions $i_j : X_j \to \mathbf{P}^{n_j}_{V_j}$, where $X_j = f^{-1}(V_j)$ such that $i_j^*\mathcal{O}(1)$ is a power of $\mathcal{L}$. Since S is quasi-compact we may assume the covering is finite. Clearly, if we solve the question for each of the finitely many systems $(X_j \to V_j, \mathcal{L}|_{X_j}, \mathcal{F}|_{V_j})$ then the result follows. Hence we may assume there exists an immersion $i : X \to \mathbf{P}^n_S$ such that $\mathcal{L}^{\otimes d} = i^*\mathcal{O}(1)$ for some $d \geq 1$.

Repeating the argument above with a finite affine open covering of S we see that we may also assume that S is affine. In this case the vanishing of $R^p f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes n})$ is equivalent to the vanishing of $H^p(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$, see Lemma 25.5.4.

Since f is proper we see that i is a closed immersion (Morphisms, Lemma 24.40.7). Hence we see that $R^p i_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) = 0$ for all $p \geq 1$ (see Lemma 25.11.8 for example). This implies that

$$H^p(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = H^p(\mathbf{P}^n_S, i_*(\mathcal{F} \otimes \mathcal{L}^{\otimes n}))$$

by the Leray spectral sequence (Cohomology, Lemma 18.12.4). Moreover, by the projection formula (Cohomology, Lemma 18.7.2) we have

$$i_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) = i_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes \langle n \rangle_d}) \otimes_{\mathcal{O}_{\mathbf{P}^n_S}} \mathcal{O}(\lfloor n/d \rfloor)$$

for all $n \in \mathbf{Z}$ where $\langle n \rangle_d \in \{0, 1, \ldots, d-1\}$ is the unique element congruent to n module d. The sheaves $\mathcal{F}_j = i_*(\mathcal{F} \otimes \mathcal{L}^{\otimes j})$, $j \in \{0, 1, \ldots, d-1\}$ are coherent by Lemma 25.11.8. Thus we see that for all n large enough the cohomology groups $H^p(\mathbf{P}^n_S, \mathcal{F}_j(n))$ vanish by Lemma 25.16.1. Putting everything together this implies the lemma. $\square$

25.17. Chow's Lemma

In this section we prove Chow's lemma in the Noetherian case (Lemma 25.17.1). In Limits, Section 27.8 we prove some variants for the non-Noetherian case.

Lemma 25.17.1. *Let S be a Noetherian scheme. Let $f : X \to S$ be a separated morphism of finite type. Then there exists an $n \geq 0$ and a diagram*

$$\begin{tikzcd} X \arrow[dr] & X' \arrow[l, "\pi"'] \arrow[r] \arrow[d] & \mathbf{P}^n_S \arrow[dl] \\ & S & \end{tikzcd}$$

where $X' \to \mathbf{P}^n_S$ is an immersion, and $\pi : X' \to X$ is proper and surjective. Moreover, we may arrange it such that there exists a dense open subscheme $U \subset X$ such that $\pi^{-1}(U) \to U$ is an isomorphism.

Proof. All of the schemes we will encounter during the rest of the proof are going to be of finite type over the Noetherian scheme S and hence Noetherian (see Morphisms, Lemma 24.14.6). All morphisms between them will automatically be quasi-compact, locally of finite type and quasi-separated, see Morphisms, Lemma 24.14.8 and Properties, Lemmas 23.5.4 and 23.5.6.

The underlying topological space of X is Noetherian (see Properties, Lemma 23.5.5) and we conclude that X has only finitely many irreducible components (see Topology, Lemma 5.6.2). Say $X = X_1 \cup \ldots \cup X_r$ is the decomposition of X into irreducible components. Let $\eta_i \in X_i$ be the generic point. For every point $x \in X$ there exists an affine open $U_x \subset X$ which contains x and each of the generic points η_i. See Properties, Lemma 23.26.4. Since X is quasi-compact, we can find a finite affine open covering $X = U_1 \cup \ldots \cup U_m$ such that each U_i contains $\eta_1, \ldots, \eta_r$. In particular we conclude that the open $U = U_1 \cap \ldots \cap U_m \subset X$ is a dense open. This and the fact that the U_i are affine opens covering X is all that we will use below.

Let $X^* \subset X$ be the scheme theoretic closure of $U \to X$, see Morphisms, Definition 24.4.2. Let $U_i^* = X^* \cap U_i$. Note that U_i^* is a closed subscheme of U_i. Hence U_i^* is affine. Since U is dense in X the morphism $X^* \to X$ is a surjective closed immersion. It is an isomorphism over U. Hence we may replace X by X^* and U_i by U_i^* and assume that U is scheme theoretically dense in X, see Morphisms, Definition 24.5.1.

By Morphisms, Lemma 24.38.3 we can find an immersion $j_i : U_i \to \mathbf{P}^{n_i}_S$ for each i. By Morphisms, Lemma 24.5.7 we can find closed subschemes $Z_i \subset \mathbf{P}^{n_i}_S$ such that $j_i : U_i \to Z_i$ is a scheme theoretically dense open immersion. Note that $Z_i \to S$ is proper, see Morphisms, Lemma 24.41.5. Consider the morphism

$$j = (j_1|_U, \ldots, j_n|_U) : U \longrightarrow \mathbf{P}^{n_1}_S \times_S \ldots \times_S \mathbf{P}^{n_n}_S.$$

By the lemma cited above we can find a closed subscheme Z of $\mathbf{P}^{n_1}_S \times_S \ldots \times_S \mathbf{P}^{n_n}_S$ such that $j : U \to Z$ is an open immersion and such that U is scheme theoretically dense in Z. The morphism $Z \to S$ is proper. Consider the ith projection

$$\mathrm{pr}_i|_Z : Z \longrightarrow \mathbf{P}^{n_i}_S.$$

This morphism factors through Z_i (see Morphisms, Lemma 24.4.6). Denote $p_i : Z \to Z_i$ the induced morphism. This is a proper morphism, see Morphisms, Lemma 24.40.7 for example. At this point we have that $U \subset U_i \subset Z_i$ are scheme theoretically dense open immersions. Moreover, we can think of Z as the scheme theoretic image of the ``diagonal'' morphism $U \to Z_1 \times_S \ldots \times_S Z_n$.

Set $V_i = p_i^{-1}(U_i)$. Note that $p_i|_{V_i} : V_i \to U_i$ is proper. Set $X' = V_1 \cup \ldots \cup V_n$. By construction X' has an immersion into the scheme $\mathbf{P}^{n_1}_S \times_S \ldots \times_S \mathbf{P}^{n_n}_S$. Thus by the Segre embedding (see Morphisms, Lemma 24.41.6) we see that X' has an immersion into a projective space over S.

We claim that the morphisms $p_i|_{V_i} : V_i \to U_i$ glue to a morphism $X' \to X$. Namely, it is clear that $p_i|_U$ is the identity map from U to U. Since $U \subset X'$ is scheme theoretically dense by construction, it is also scheme theoretically dense in the open subscheme $V_i \cap V_j$. Thus we see that $p_i|_{V_i \cap V_j} = p_j|_{V_i \cap V_j}$ as morphisms into the separated S-scheme X, see Morphisms, Lemma 24.5.10. We denote the resulting morphism $\pi : X' \to X$.

We claim that $\pi^{-1}(U_i) = V_i$. Since $\pi|_{V_i} = p_i|_{V_i}$ it follows that $V_i \subset \pi^{-1}(U_i)$. Consider the diagram

$$\begin{tikzcd} V_i \arrow[r] \arrow[dr, "p_i|_{V_i}"'] & \pi^{-1}(U_i) \arrow[d] \\ & U_i \end{tikzcd}$$

Since $V_i \to U_i$ is proper we see that the image of the horizontal arrow is closed, see Morphisms, Lemma 24.40.7. Since $V_i \subset \pi^{-1}(U_i)$ is scheme theoretically dense (as it contains U) we conclude that $V_i = \pi^{-1}(U_i)$ as claimed.

This shows that $\pi^{-1}(U_i) \to U_i$ is identified with the proper morphism $p_i|_{V_i} : V_i \to U_i$. Hence we see that X has a finite affine covering $X = \bigcup U_i$ such that the restriction of π is proper on each member of the covering. Thus by Morphisms, Lemma 24.40.3 we see that π is proper.

Finally we have to show that $\pi^{-1}(U) = U$. To see this we argue in the same way as above using the diagram

$$\begin{tikzcd} U \arrow[r] \arrow[dr] & \pi^{-1}(U) \arrow[d] \\ & U \end{tikzcd}$$

and using that $\mathrm{id}_U : U \to U$ is proper and that U is scheme theoretically dense in $\pi^{-1}(U)$. □

Remark 25.17.2. In the situation of Chow's Lemma 25.17.1:

(1) The morphism π is actually H-projective (hence projective, see Morphisms, Lemma 24.41.2) since the morphism $X' \to \mathbf{P}^n_S \times_S X = \mathbf{P}^n_X$ is a closed immersion (use the fact that π is proper, see Morphisms, Lemma 24.40.7).

(2) We may assume that $\pi^{-1}(U)$ is scheme theoretically dense in X'. Namely, we can simply replace X' by the scheme theoretic closure of $\pi^{-1}(U)$. In this case we can think of U as a scheme theoretically dense open subscheme of X'. See Morphisms, Section 24.4.
(3) If X is reduced then we may choose X' reduced. This is clear from (2).

25.18. Higher direct images of coherent sheaves

In this section we prove the fundamental fact that the higher direct images of a coherent sheaf under a proper morphism are coherent.

Lemma 25.18.1. *Let S be a locally Noetherian scheme. Let $f : X \to S$ be a locally projective morphism. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $R^i f_* \mathcal{F}$ is a coherent $\mathcal{O}_S$-module for all $i \geq 0$.*

Proof. We first remark that a locally projective morphism is proper (Morphisms, Lemma 24.41.5) and hence of finite type. In particular X is locally Noetherian (Morphisms, Lemma 24.14.6) and hence the statement makes sense. Moreover, by Lemma 25.5.3 the sheaves $R^p f_* \mathcal{F}$ are quasi-coherent.

Having said this the statement is local on S (for example by Cohomology, Lemma 18.6.4). Hence we may assume $S = Spec(R)$ is the spectrum of a Noetherian ring, and X is a closed subscheme of $\mathbf{P}^n_R$ for some n, see Morphisms, Lemma 24.41.4. In this case, the sheaves $R^p f_* \mathcal{F}$ are the quasi-coherent sheaves associated to the R-modules $H^p(X, \mathcal{F})$, see Lemma 25.5.4. Hence it suffices to show that R-modules $H^p(X, \mathcal{F})$ are finite R-modules (Lemma 25.11.1). Denote $i : X \to \mathbf{P}^n_R$ the closed immersion. Note that $R^p i_* \mathcal{F} = 0$ by Lemma 25.11.8. Hence the Leray spectral sequence (Cohomology, Lemma 18.12.4) for $i : X \to \mathbf{P}^n_R$ degenerates, and we see that $H^p(X, \mathcal{F}) = H^p(\mathbf{P}^n_R, i_* \mathcal{F})$. Since the sheaf $i_* \mathcal{F}$ is coherent by Lemma 25.11.8 we see that the lemma follows from Lemma 25.16.1. $\square$

Here is the general statement.

Lemma 25.18.2. *Let S be a locally Noetherian scheme. Let $f : X \to S$ be a proper morphism. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $R^i f_* \mathcal{F}$ is a coherent $\mathcal{O}_S$-module for all $i \geq 0$.*

Proof. Since the problem is local on S we may assume that S is a Noetherian scheme. Since a proper morphism is of finite type we see that in this case X is a Noetherian scheme also. Consider the property $\mathcal{P}$ of coherent sheaves on X defined by the rule

$$\mathcal{P}(\mathcal{F}) \Leftrightarrow R^p f_* \mathcal{F} \text{ is coherent for all } p \geq 0$$

We are going to use the result of Lemma 25.14.6 to prove that $\mathcal{P}$ holds for every coherent sheaf on X.

Let

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

be a short exact sequence of coherent sheaves on X. Consider the long exact sequence of higher direct images

$$R^{p-1} f_* \mathcal{F}_3 \to R^p f_* \mathcal{F}_1 \to R^p f_* \mathcal{F}_2 \to R^p f_* \mathcal{F}_3 \to R^{p+1} f_* \mathcal{F}_1$$

Then it is clear that if 2-out-of-3 of the sheaves $\mathcal{F}_i$ have property $\mathcal{P}$, then the higher direct images of the third are sandwiched in this exact complex between two coherent sheaves. Hence these higher direct images are also coherent by Lemma 25.11.2 and 25.11.3. Hence property $\mathcal{P}$ holds for the third as well.

Let $Z \subset X$ be an integral closed subscheme. We have to find a coherent sheaf $\mathcal{F}$ on X whose support is contained in Z, whose stalk at the generic point ξ of Z is a 1-dimensional vector space over $\kappa(\xi)$ such that $\mathcal{P}$holds for $\mathcal{F}$. Denote $g = f|_Z : Z \to S$ the restriction of f. Suppose we can find a coherent sheaf $\mathcal{G}$ on Z such that (a) $\mathcal{G}_\xi$ is a 1-dimensional vector space over $\kappa(\xi)$, (b) $R^p g_*\mathcal{G} = 0$ for $p > 0$, and (c) $g_*\mathcal{G}$ is coherent. Then we can consider $\mathcal{F} = (Z \to X)_*\mathcal{G}$. As $Z \to X$ is a closed immersion we see that $(Z \to X)_*\mathcal{G}$ is coherent on X and $R^p(Z \to X)_*\mathcal{G} = 0$ for $p > 0$ (Lemma 25.11.8). Hence by the relative Leray spectral sequence (Cohomology, Lemma 18.12.8) we will have $R^p f_*\mathcal{F} = R^p g_*\mathcal{G} = 0$ for $p > 0$ and $f_*\mathcal{F} = g_*\mathcal{G}$ is coherent. Finally $\mathcal{F}_\xi = ((Z \to X)_*\mathcal{G})_\xi = \mathcal{G}_\xi$ which verifies the condition on the stalk at ξ. Hence everything depends on finding a coherent sheaf $\mathcal{G}$ on Z which has properties (a), (b), and (c).

We can apply Chow's Lemma 25.17.1 to the morphism $Z \to S$. Thus we get a diagram

$$\begin{tikzcd} Z \arrow[dr, "g"'] & Z' \arrow[l, "\pi"'] \arrow[r, "i"] \arrow[d, "g'"] & \mathbf{P}^n_S \arrow[dl] \\ & S & \end{tikzcd}$$

as in the statement of Chow's lemma. Also, let $U \subset Z$ be the dense open subscheme such that $\pi^{-1}(U) \to U$ is an isomorphism. By the discussion in Remark 25.17.2 we see that $i' = (i, \pi) : \mathbf{P}^n_S \times_S Z' = \mathbf{P}^n_Z$ is a closed immersion. Hence

$$\mathcal{L} = i^*\mathcal{O}_{\mathbf{P}^n_X}(1) \cong (i')^*\mathcal{O}_{\mathbf{P}^n_Z}(1)$$

is g'-relatively ample and π-relatively ample (for example by Morphisms, Lemma 24.38.7). Hence by Lemma 25.16.2 there exists an $n \geq 0$ such that both $R^p\pi_*\mathcal{L}^{\otimes n} = 0$ for all $p > 0$ and $R^p(g')_*\mathcal{L}^{\otimes n} = 0$ for all $p > 0$. Set $\mathcal{G} = \pi_*\mathcal{L}^{\otimes n}$. Property (a) holds because $\pi_*\mathcal{L}^{\otimes}|_U$ is an invertible sheaf (as $\pi^{-1}(U) \to U$ is an isomorphism). Properties (b) and (c) hold because by the relative Leray spectral sequence (Cohomology, Lemma 18.12.8) we have

$$E_2^{p,q} = R^p g_* R^q \pi_* \mathcal{L}^{\otimes n} \Rightarrow R^{p+q}(g')_*\mathcal{L}^{\otimes n}$$

and by choice of n the only nonzero terms in $E_2^{p,q}$ are those with $q = 0$ and the only nonzero terms of $R^{p+q}(g')_*\mathcal{L}^{\otimes n}$ are those with $p = q = 0$. This implies that $R^p g_*\mathcal{G} = 0$ for $p > 0$ and that $g_*\mathcal{G} = (g')_*\mathcal{L}^{\otimes n}$. Finally, applying the previous Lemma 25.18.1 we see that $g_*\mathcal{G} = (g')_*\mathcal{L}^{\otimes n}$ is coherent as desired. □

Lemma 25.18.3. *Let $S = Spec(A)$ with A a Noetherian ring. Let $f : X \to S$ be a proper morphism. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $H^i(X, \mathcal{F})$ is finite A-module for all $i \geq 0$.*

Proof. This is just the affine case of Lemma 25.18.2. Namely, by Lemmas 25.5.3 and 25.5.4 we know that $R^i f_*\mathcal{F}$ is the quasi-coherent sheaf associated to the A-module $H^i(X, \mathcal{F})$ and by Lemma 25.11.1 this is a coherent sheaf if and only if $H^i(X, \mathcal{F})$ is an A-module of finite type. □

25.19. The theorem on formal functions

In this section we study the behaviour of cohomology of sequences of sheaves either of the form $\{I^n\mathcal{F}\}_{n\geq 0}$ or of the form $\{\mathcal{F}/I^n\mathcal{F}\}_{n\geq 0}$ as n-varies.

Here and below we use the following notation. Given a morphism of schemes $f : X \to Y$, a quasi-coherent sheaf $\mathcal{F}$ on X, and a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Y$ we denote

$\mathcal{I}^n\mathcal{F}$ the quasi-coherent subsheaf generated by products of local sections of $f^{-1}(\mathcal{I}^n)$ and $\mathcal{F}$. In a formula

$$\mathcal{I}^n\mathcal{F} = \mathrm{Im}\left(f^*(\mathcal{I}^n) \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow \mathcal{F}\right).$$

Note that there are natural maps

$$f^{-1}(\mathcal{I}^n) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{I}^m\mathcal{F} \longrightarrow f^*(\mathcal{I}^n) \otimes_{\mathcal{O}_X} \mathcal{I}^m\mathcal{F} \longrightarrow \mathcal{I}^{n+m}\mathcal{F}$$

Hence a section of $\mathcal{I}^n$ will give rise to a map $R^pf_*(\mathcal{I}^m\mathcal{F}) \to R^pf_*(\mathcal{I}^{n+m}\mathcal{F})$ by functoriality of higher direct images. Localizing and then sheafifying we see that there are $\mathcal{O}_Y$-module maps

$$\mathcal{I}^n \otimes_{\mathcal{O}_Y} R^pf_*(\mathcal{I}^m\mathcal{F}) \longrightarrow R^pf_*(\mathcal{I}^{n+m}\mathcal{F}).$$

In other words we see that $\bigoplus_{n\geq 0} R^pf_*(\mathcal{I}^n\mathcal{F})$ is a graded $\bigoplus_{n\geq 0}\mathcal{I}^n$-module.

If $Y = Spec(A)$ and $\mathcal{I} = \tilde{I}$ we denote $\mathcal{I}^n\mathcal{F}$ simply $I^n\mathcal{F}$. The maps introduced above give $M = \bigoplus H^p(X, I^n\mathcal{F})$ the structure of a graded $S = \bigoplus I^n$-module. If f is proper, A is Noetherian and $\mathcal{F}$ is coherent, then this turns out to be a module of finite type.

Lemma 25.19.1. *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Set $S = \bigoplus_{n\geq 0} I^n$. Let $f : X \to Spec(A)$ be a proper morphism. Let $\mathcal{F}$ be a coherent sheaf on X. Then for every $p \geq 0$ the graded S-module $\bigoplus_{n\geq 0} H^p(X, I^n\mathcal{F})$ is a finite S-module.*

Proof. To prove this we consider the fibre product diagram

$$\begin{array}{ccc} X' = Spec(S) \times_{Spec(A)} X & \longrightarrow & X \\ \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f} \\ Spec(S) & \longrightarrow & Spec(A) \end{array}$$

Note that f' is a proper morphism, see Morphisms, Lemma 24.40.5. Also, S is a finitely generated A-algebra, and hence Noetherian (Algebra, Lemma 7.28.1). Thus the result will follow from Lemma 25.18.3 if we can show there exists a coherent sheaf $\mathcal{F}'$ on X' whose cohomology groups $H^p(X', \mathcal{F}')$ are identified with $\bigoplus_{n\geq 0} H^p(X, I^n\mathcal{F})$.

To do this note that the morphism $\pi : X' \to X$ is affine, see Morphisms, Lemma 24.11.8. Hence $H^p(X', \mathcal{F}') = H^p(X, \pi_*\mathcal{F}')$. In other words, it suffices to construct a coherent $\mathcal{O}_{X'}$-module $\mathcal{F}'$ such that $\pi_*\mathcal{F}' = \bigoplus_{n\geq 0} I^n\mathcal{F}$. Note that $\pi_*\mathcal{O}_{X'} = \bigoplus_{n\geq 0} I^n \otimes_A \mathcal{O}_X$ hence the sheaf $\bigoplus_{n\geq 0} I^n\mathcal{F}$ has a natural structure of $\pi_*\mathcal{O}_{X'}$-module. By Morphisms, Lemma 24.11.6 we see that there is a unique quasi-coherent $\mathcal{O}_{X'}$-module $\mathcal{F}'$ such that $\pi_*\mathcal{F}' \cong \bigoplus_{n\geq 0} I^n\mathcal{F}$ as $\pi_*\mathcal{O}_{X'}$-modules. Finally, we have to show that $\mathcal{F}'$ is a coherent $\mathcal{O}_{X'}$-module.

Let $Spec(B) = U \subset X$ be any affine open. Say $\mathcal{F}|_U$ is the coherent $\mathcal{O}_U$-module associated to the finite B-module M. By definition $\pi^{-1}(U) = Spec(S \otimes_A B)$. Since $B' = S \otimes_A B = \bigoplus_{n\geq 0} I^n \otimes_A B$ it is clear that $\mathcal{F}'$ corresponds to the B'-module $\bigoplus I^n M$ which is clearly finitely generated. □

Lemma 25.19.2. *Given a morphism of schemes $f : X \to Y$, a quasi-coherent sheaf $\mathcal{F}$ on X, and a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Y$. Assume Y locally Noetherian, f proper, and $\mathcal{F}$ coherent. Then*

$$\mathcal{M} = \bigoplus_{n\geq 0} R^pf_*(\mathcal{I}^n\mathcal{F})$$

is a graded $\mathcal{A} = \bigoplus_{n\geq 0}\mathcal{I}^n$-module which is quasi-coherent and of finite type.

Proof. The statement is local on Y, hence this reduces to the case where Y is affine. In the affine case the result follows from Lemma 25.19.1. Details omitted. □

Lemma 25.19.3. *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let $f : X \to Spec(A)$ be a proper morphism. Let $\mathcal{F}$ be a coherent sheaf on X. Then for every $p \geq 0$ there exists an integer $c \geq 0$ such that*

1. *the multiplication map $I^{n-c} \otimes H^p(X, I^c\mathcal{F}) \to H^p(X, I^n\mathcal{F})$ is surjective for all $n \geq c$, and*
2. *the image of $H^p(X, I^{n+m}\mathcal{F}) \to H^p(X, I^n\mathcal{F})$ is contained in the submodule $I^{m-c}H^p(X, I^n\mathcal{F})$ for all $n \geq 0$, $m \geq c$.*

Proof. By Lemma 25.19.1 we can find $d_1, \ldots, d_t \geq 0$, and $x_i \in H^p(X, I^{d_i}\mathcal{F})$ such that $\bigoplus_{n\geq 0} H^p(X, I^n\mathcal{F})$ is generated by $x_1, \ldots, x_t$ over $S = \bigoplus_{n\geq 0} I^n$. Take $c = \max\{d_i\}$. It is clear that (1) holds. For (2) let $b = \max(0, n - c)$. Consider the commutative diagram of A-modules

$$\begin{tikzcd}
I^{n+m-c-b} \otimes I^b \otimes H^p(X, I^c\mathcal{F}) \arrow[r] \arrow[d] & I^{n+m-c} \otimes H^p(X, I^c\mathcal{F}) \arrow[r] & H^p(X, I^{n+m}\mathcal{F}) \arrow[d] \\
I^{n+m-c-b} \otimes H^p(X, I^n\mathcal{F}) \arrow[rr] & & H^p(X, I^n\mathcal{F})
\end{tikzcd}$$

By part (1) of the lemma the composition of the horizontal arrows is surjective if $n+m \geq c$. On the other hand, it is clear that $n + m - c - b \geq m - c$. Hence part (2). □

In the situation of Lemmas 25.19.1 and 25.19.3 consider the inverse system

$$\mathcal{F}/I\mathcal{F} \leftarrow \mathcal{F}/I^2\mathcal{F} \leftarrow \mathcal{F}/I^3\mathcal{F} \leftarrow \ldots$$

We would like to know what happens to the cohomology groups. Here is a first result.

Lemma 25.19.4. *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let $f : X \to Spec(A)$ be a proper morphism. Let $\mathcal{F}$ be a coherent sheaf on X. Fix $p \geq 0$.*

1. *There exists a $c_1 \geq 0$ such that for all $n \geq c_1$ we have*
$$Ker(H^p(X, \mathcal{F}) \to H^p(X, \mathcal{F}/I^n\mathcal{F})) \subset I^{n-c_1}H^p(X, \mathcal{F}).$$
2. *The inverse system*
$$\left(H^p(X, \mathcal{F}/I^n\mathcal{F})\right)_{n\in\mathbf{N}}$$
satisfies the Mittag-Leffler condition (see Homology, Definition 10.23.2).
3. *In fact for any p and n there exists a $c_2(n) \geq n$ such that*
$$Im(H^p(X, \mathcal{F}/I^k\mathcal{F}) \to H^p(X, \mathcal{F}/I^n\mathcal{F})) = Im(H^p(X, \mathcal{F}) \to H^p(X, \mathcal{F}/I^n\mathcal{F}))$$
for all $k \geq c_2(n)$.

Proof. Let $c_1 = \max\{c_p, c_{p+1}\}$, where c_p, c_{p+1} are the integers found in Lemma 25.19.3 for H^p and H^{p+1}. We will use this constant in the proofs of (1), (2) and (3).

Let us prove part (1). Consider the short exact sequence

$$0 \to I^n\mathcal{F} \to \mathcal{F} \to \mathcal{F}/I^n\mathcal{F} \to 0$$

From the long exact cohomology sequence we see that

$$\mathrm{Ker}(H^p(X, \mathcal{F}) \to H^p(X, \mathcal{F}/I^n\mathcal{F})) = \mathrm{Im}(H^p(X, I^n\mathcal{F}) \to H^p(X, \mathcal{F}))$$

Hence by our choice of c_1 we see that this is contained in $I^{n-c_1}H^p(X, \mathcal{F})$ for $n \geq c_1$.

Note that part (3) implies part (2) by definition of the Mittag-Leffler condition.

Let us prove part (3). Fix an n throughout the rest of the proof. Consider the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & I^n\mathcal{F} \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{F}/I^n\mathcal{F} \arrow[r] & 0 \\ 0 \arrow[r] & I^{n+m}\mathcal{F} \arrow[r] \arrow[u] & \mathcal{F} \arrow[r] \arrow[u] & \mathcal{F}/I^{n+m}\mathcal{F} \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

This gives rise to the following commutative diagram

$$\begin{tikzcd} H^p(X, I^n\mathcal{F}) \arrow[r] & H^p(X,\mathcal{F}) \arrow[r] & H^p(X,\mathcal{F}/I^n\mathcal{F}) \arrow[r, "\delta"'] & H^{p+1}(X, I^n\mathcal{F}) \\ H^p(X, I^{n+m}\mathcal{F}) \arrow[r] \arrow[u] & H^p(X,\mathcal{F}) \arrow[r] \arrow[u, "1"] & H^p(X,\mathcal{F}/I^{n+m}\mathcal{F}) \arrow[r] \arrow[u] & H^{p+1}(X, I^{n+m}\mathcal{F}) \arrow[u, "a"] \end{tikzcd}$$

If $m \geq c_1$ we see that the image of a is contained in $I^{m-c_1}H^{p+1}(X, I^n\mathcal{F})$. By the Artin-Rees lemma (see Algebra, Lemma 7.47.5) there exists an integer $c_3(n)$ such that

$$I^N H^{p+1}(X, I^n\mathcal{F}) \cap \mathrm{Im}(\delta) \subset \delta\left(I^{N-c_3(n)}H^p(X,\mathcal{F}/I^n\mathcal{F})\right)$$

for all $N \geq c_3(n)$. As $H^p(X,\mathcal{F}/I^n\mathcal{F})$ is annihilated by I^n, we see that if $m \geq c_3(n) + c_1 + n$, then

$$\mathrm{Im}(H^p(X,\mathcal{F}/I^{n+m}\mathcal{F}) \to H^p(X,\mathcal{F}/I^n\mathcal{F})) = \mathrm{Im}(H^p(X,\mathcal{F}) \to H^p(X,\mathcal{F}/I^n\mathcal{F}))$$

In other words, part (3) holds with $c_2(n) = c_3(n) + c_1 + n$. □

Theorem 25.19.5. *(Theorem on formal functions) Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let $f : X \to Spec(A)$ be a proper morphism. Let $\mathcal{F}$ be a coherent sheaf on X. Fix $p \geq 0$. The system of maps*

$$H^p(X,\mathcal{F})/I^nH^p(X,\mathcal{F}) \longrightarrow H^p(X,\mathcal{F}/I^n\mathcal{F})$$

define an isomorphism of limits

$$H^p(X,\mathcal{F})^\wedge \longrightarrow lim_n\, H^p(X,\mathcal{F}/I^n\mathcal{F})$$

where the left hand side is the completion of the A-module $H^p(X,\mathcal{F})$ with respect to the ideal I, see Algebra, Section 7.90. Moreover, this is in fact a homeomorphism for the limit topologies.

Proof. In fact, this follows immediately from Lemma 25.19.4. We spell out the details. Set $M = H^p(X,\mathcal{F})$ and $M_n = H^p(X,\mathcal{F}/I^n\mathcal{F})$. Denote $N_n = \mathrm{Im}(M \to M_n)$. By the description of the limit in Homology, Section 10.23 we have

$$lim_n\, M_n = \{(x_n) \in \prod M_n \mid \varphi_i(x_n) = x_{n-1},\ n = 2, 3, \ldots\}$$

Pick an element $x = (x_n) \in lim_n\, M_n$. By Lemma 25.19.4 part (3) we have $x_n \in N_n$ for all n since by definition x_n is the image of some $x_{n+m} \in M_{n+m}$ for all m. By Lemma 25.19.4 part (1) we see that there exists a factorization

$$M \to N_n \to M/I^{n-c_1}M$$

of the reduction map. Denote $y_n \in M/I^{n-c_1}M$ the image of x_n for $n \geq c_1$. Since for $n' \geq n$ the composition $M \to M_{n'} \to M_n$ is the given map $M \to M_n$ we see that $y_{n'}$ maps to y_n under the canonical map $M/I^{n'-c_1}M \to M/I^{n-c_1}M$. Hence $y = (y_{n+c_1})$ defines an element of $lim_n\, M/I^nM$. We omit the verification that y maps to x under the map

$$M^\wedge = lim_n\, M/I^nM \longrightarrow lim_n\, M_n$$

of the lemma. We also omit the verification on topologies. □

Lemma 25.19.6. *Given a morphism of schemes $f : X \to Y$, a quasi-coherent sheaf $\mathcal{F}$ on X, and a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Y$. Assume*

(1) *Y locally Noetherian,*
(2) *f proper, and*
(3) *$\mathcal{F}$ coherent.*

Let $y \in Y$ be a point. Consider the infinitesimal neighbourhoods

$$\begin{tikzcd} X_n = Spec(\mathcal{O}_{Y,y}/\mathfrak{m}_y^n) \times_Y X \arrow[r, "i_n"'] \arrow[d, "f_n"'] & X \arrow[d, "f"] \\ Spec(\mathcal{O}_{Y,y}/\mathfrak{m}_y^n) \arrow[r, "c_n"] & Y \end{tikzcd}$$

of the fibre $X_1 = X_y$ and set $\mathcal{F}_n = i_n^*\mathcal{F}$. Then we have

$$\left(R^p f_*\mathcal{F}\right)_y^\wedge \cong lim_n\, H^p(X_n, \mathcal{F}_n)$$

as $\mathcal{O}_{Y,y}^\wedge$-modules.

Proof. This is just a reformulation of a special case of the theorem on formal functions, Theorem 25.19.5. Let us spell it out. Note that $\mathcal{O}_{Y,y}$ is a Noetherian local ring. Consider the canonical morphism $c : Spec(\mathcal{O}_{Y,y}) \to Y$, see Schemes, Equation (21.13.1.1). This is a flat morphism as it identifies local rings. Denote momentarily $f' : X' \to Spec(\mathcal{O}_{Y,y})$ the base change of f to this local ring. We see that $c^*R^p f_*\mathcal{F} = R^p f'_*\mathcal{F}'$ by Lemma 25.6.2. Moreover, the infinitesimal neighbourhoods of the fibre X_y and X'_y are identified (verification omitted; hint: the morphisms c_n factor through c).

Hence we may assume that $Y = Spec(A)$ is the spectrum of a Noetherian local ring A with maximal ideal $\mathfrak{m}$ and that $y \in Y$ corresponds to the closed point (i.e., to $\mathfrak{m}$). In particular it follows that

$$\left(R^p f_*\mathcal{F}\right)_y = \Gamma(Y, R^p f_*\mathcal{F}) = H^p(X, \mathcal{F}).$$

In this case also, the morphisms c_n are each closed immersions. Hence their base changes i_n are closed immersions as well. Note that $i_{n,*}\mathcal{F}_n = i_{n,*}i_n^*\mathcal{F} = \mathcal{F}/\mathfrak{m}^n\mathcal{F}$. By the Leray spectral sequence for i_n, and Lemma 25.11.8 we see that

$$H^p(X_n, \mathcal{F}_n) = H^p(X, i_{n,*}\mathcal{F}) = H^p(X, \mathcal{F}/\mathfrak{m}^n\mathcal{F})$$

Hence we may indeed apply the theorem on formal functions to compute the limit in the statement of the lemma and we win. □

Here is a lemma which we will generalize later to fibres of dimension > 0, namely the next lemma.

Lemma 25.19.7. *Let $f : X \to Y$ be a morphism of schemes. Let $y \in Y$. Assume*

(1) *Y locally Noetherian,*
(2) *f is proper, and*
(3) *$f^{-1}(\{y\})$ is finite.*

Then for any coherent sheaf $\mathcal{F}$ on X we have $(R^p f_*\mathcal{F})_y = 0$ for all $p > 0$.

Proof. The fibre X_y is finite, and by Morphisms, Lemma 24.19.7 it is a finite discrete space. Moreover, the underlying topological space of each infinitesimal neighourhood X_n is the same. Hence each of the schemes X_n is affine according to Schemes, Lemma 21.11.7. Hence it follows that $H^p(X_n, \mathcal{F}_n) = 0$ for all $p > 0$. Hence we see that $(R^p f_* \mathcal{F})^\wedge_y = 0$ by Lemma 25.19.6. Note that $R^p f_* \mathcal{F}$ is coherent by Lemma 25.18.2 and hence $R^p f_* \mathcal{F}_y$ is a finite $\mathcal{O}_{Y,y}$-module. By Algebra, Lemma 7.90.2 this implies that $(R^p f_* \mathcal{F})_y = 0$. □

Lemma 25.19.8. *Let $f : X \to Y$ be a morphism of schemes. Let $y \in Y$. Assume*

(1) *Y locally Noetherian,*
(2) *f is proper, and*
(3) $\dim(X_y) = d$.

Then for any coherent sheaf $\mathcal{F}$ on X we have $(R^p f_* \mathcal{F})_y = 0$ for all $p > d$.

Proof. The fibre X_y is of finite type over $Spec(\kappa(y))$. Hence X_y is a Noetherian scheme by Morphisms, Lemma 24.14.6. Hence the underlying topological space of X_y is Noetherian, see Properties, Lemma 23.5.5. Moreover, the underlying topological space of each infinitesimal neighourhood X_n is the same as that of X_y. Hence $H^p(X_n, \mathcal{F}_n) = 0$ for all $p > d$ by Cohomology, Lemma 18.16.5. Hence we see that $(R^p f_* \mathcal{F})^\wedge_y = 0$ by Lemma 25.19.6 for $p > d$. Note that $R^p f_* \mathcal{F}$ is coherent by Lemma 25.18.2 and hence $R^p f_* \mathcal{F}_y$ is a finite $\mathcal{O}_{Y,y}$-module. By Algebra, Lemma 7.90.2 this implies that $(R^p f_* \mathcal{F})_y = 0$. □

25.20. Applications of the theorem on formal functions

We will add more here as needed. For the moment we need the following characterization of finite morphisms (in the Noetherian case -- for a more general version see the chapter More on Morphisms, Section 33.29).

Lemma 25.20.1. *(For a more general version see More on Morphisms, Lemma 33.29.5). Let $f : X \to S$ be a morphism of schemes. Assume S is locally Noetherian. The following are equivalent*

(1) *f is finite, and*
(2) *f is proper with finite fibres.*

Proof. A finite morphism is proper according to Morphisms, Lemma 24.42.10. A finite morphism is quasi-finite according to Morphisms, Lemma 24.42.9. A quasi-finite morphism has finite fibres, see Morphisms, Lemma 24.19.10. Hence a finite morphism is proper and has finite fibres.

Assume f is proper with finite fibres. We want to show f is finite. In fact it suffices to prove f is affine. Namely, if f is affine, then it follows that f is integral by Morphisms, Lemma 24.42.7 whereupon it follows from Morphisms, Lemma 24.42.4 that f is finite.

To show that f is affine we may assume that S is affine, and our goal is to show that X is affine too. Since f is proper we see that X is separated and quasi-compact. Hence we may use the criterion of Lemma 25.3.2 to prove that X is affine. To see this let $\mathcal{I} \subset \mathcal{O}_X$ be a finite type ideal sheaf. In particular $\mathcal{I}$ is a coherent sheaf on X. By Lemma 25.19.7 we conclude that $R^1 f_* \mathcal{I}_s = 0$ for all $s \in S$. In other words, $R^1 f_* \mathcal{I} = 0$. Hence we see from the Leray Spectral Sequence for f that $H^1(X, \mathcal{I}) = H^1(S, f_* \mathcal{I})$. Since S is affine, and $f_* \mathcal{I}$ is quasi-coherent (Schemes, Lemma 21.24.1) we conclude $H^1(S, f_* \mathcal{I}) = 0$ from Lemma 25.2.2 as desired. Hence $H^1(X, \mathcal{I}) = 0$ as desired. □

As a consequence we have the following useful result.

Lemma 25.20.2. *(For a more general version see More on Morphisms, Lemma 33.29.6). Let $f : X \to S$ be a morphism of schemes. Let $s \in S$. Assume*

1. *S is locally Noetherian,*
2. *f is proper, and*
3. *$f^{-1}(\{s\})$ is a finite set.*

Then there exists an open neighbourhood $V \subset S$ of s such that $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is finite.

Proof. The morphism f is quasi-finite at all the points of $f^{-1}(\{s\})$ by Morphisms, Lemma 24.19.7. By Morphisms, Lemma 24.47.2 the set of points at which f is quasi-finite is an open $U \subset X$. Let $Z = X \setminus U$. Then $s \notin f(Z)$. Since f is proper the set $f(Z) \subset S$ is closed. Choose any open neighbourhood $V \subset S$ of s with $Z \cap V = \emptyset$. Then $f^{-1}(V) \to V$ is locally quasi-finite and proper. Hence it is quasi-finite (Morphisms, Lemma 24.19.9), hence has finite fibres (Morphisms, Lemma 24.19.10), hence is finite by Lemma 25.20.1. □

25.21. Other chapters

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CHAPTER 26

Divisors

26.1. Introduction

In this chapter we study some very basic questions related to defining divisors, etc. A basic reference is [**DG67**].

26.2. Associated points

Let R be a ring and let M be an R-module. Recall that a prime $\mathfrak{p} \subset R$ is *associated* to M if there exists an element of M whose annihilator is $\mathfrak{p}$. See Algebra, Definition 7.60.1. Here is the definition of associated points for quasi-coherent sheaves on schemes as given in [**DG67**, IV Definition 3.1.1].

Definition 26.2.1. Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on X.

(1) We say $x \in X$ is *associated* to $\mathcal{F}$ if the maximal ideal $\mathfrak{m}_x$ is associated to the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$.
(2) We denote $\text{Ass}(\mathcal{F})$ or $\text{Ass}_X(\mathcal{F})$ the set of associated points of $\mathcal{F}$.
(3) The *associated points of* X are the associated points of $\mathcal{O}_X$.

These definitions are most useful when X is locally Noetherian and $\mathcal{F}$ of finite type. For example it may happen that a generic point of an irreducible component of X is not associated to X, see Example 26.2.7. In the non-Noetherian case it may be more convenient to use weakly associated points, see Section 26.5. Let us link the scheme theoretic notion with the algebraic notion on affine opens; note that this correspondence works perfectly only for locally Noetherian schemes.

Lemma 26.2.2. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $Spec(A) = U \subset X$ be an affine open, and set $M = \Gamma(U, \mathcal{F})$. Let $x \in U$, and let $\mathfrak{p} \subset A$ be the corresponding prime.*

(1) *If $\mathfrak{p}$ is associated to M, then x is associated to $\mathcal{F}$.*
(2) *If $\mathfrak{p}$ is finitely generated, then the coverse holds as well.*

In particular, if X is locally Noetherian, then the equivalence

$$\mathfrak{p} \in Ass(M) \Leftrightarrow x \in Ass(\mathcal{F})$$

holds for all pairs $(\mathfrak{p}, x)$ as above.

Proof. This follows from Algebra, Lemma 7.60.14. But we can also argue directly as follows. Suppose $\mathfrak{p}$ is associated to M. Then there exists an $m \in M$ whose annihilator is $\mathfrak{p}$. Since localization is exact we see that $\mathfrak{p}A_{\mathfrak{p}}$ is the annihilator of $m/1 \in M_{\mathfrak{p}}$. Since $M_{\mathfrak{p}} = \mathcal{F}_x$ (Schemes, Lemma 21.5.4) we conclude that x is associated to $\mathcal{F}$.

Conversely, assume that x is associated to $\mathcal{F}$, and $\mathfrak{p}$ is finitely generated. As x is associated to $\mathcal{F}$ there exists an element $m' \in M_{\mathfrak{p}}$ whose annihilator is $\mathfrak{p}A_{\mathfrak{p}}$. Write $m' = m/f$ for some $f \in A$, $f \notin \mathfrak{p}$. The annihilator I of m is an ideal of A such that $IA_{\mathfrak{p}} = \mathfrak{p}A_{\mathfrak{p}}$. Hence $I \subset \mathfrak{p}$, and $(\mathfrak{p}/I)_{\mathfrak{p}} = 0$. Since $\mathfrak{p}$ is finitely generated, there exists a $g \in A$, $g \notin \mathfrak{p}$ such that $g(\mathfrak{p}/I) = 0$. Hence the annihilator of gm is $\mathfrak{p}$ and we win.

If X is locally Noetherian, then A is Noetherian (Properties, Lemma 23.5.2) and $\mathfrak{p}$ is always finitely generated. □

Lemma 26.2.3. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $Ass(\mathcal{F}) \subset Supp(\mathcal{F})$.*

Proof. This is immediate from the definitions. □

Lemma 26.2.4. *Let X be a scheme. Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of quasi-coherent sheaves on X. Then $Ass(\mathcal{F}_2) \subset Ass(\mathcal{F}_1) \cup Ass(\mathcal{F}_3)$ and $Ass(\mathcal{F}_1) \subset Ass(\mathcal{F}_2)$.*

Proof. For every point $x \in X$ the sequence of stalks $0 \to \mathcal{F}_{1,x} \to \mathcal{F}_{2,x} \to \mathcal{F}_{3,x} \to 0$ is a short exact sequence of $\mathcal{O}_{X,x}$-modules. Hence the lemma follows from Algebra, Lemma 7.60.3. □

Lemma 26.2.5. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $Ass(\mathcal{F}) \cap U$ is finite for every quasi-compact open $U \subset X$.*

Proof. This is true because the set of associated primes of a finite module over a Noetherian ring is finite, see Algebra, Lemma 7.60.5. To translate from schemes to algebra use that U is a finite union of affine opens, each of these opens is the spectrum of a Noetherian ring (Properties, Lemma 23.5.2), $\mathcal{F}$ corresponds to a finite module over this ring (Coherent, Lemma 25.11.1), and finally use Lemma 26.2.2. □

Lemma 26.2.6. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then*

$$\mathcal{F} = 0 \Leftrightarrow Ass(\mathcal{F}) = \emptyset.$$

Proof. If $\mathcal{F} = 0$, then $\text{Ass}(\mathcal{F}) = \emptyset$ by definition. Conversely, if $\text{Ass}(\mathcal{F}) = \emptyset$, then $\mathcal{F} = 0$ by Algebra, Lemma 7.60.7. To translate from schemes to algebra, restrict to any affine and use Lemma 26.2.2. □

Example 26.2.7. Let k be a field. The ring $R = R[x_1, x_2, x_3, \ldots]/(x_i^2)$ is local with locally nilpotent maximal ideal $\mathfrak{m}$. There exists no element of R which has annihilator $\mathfrak{m}$. Hence $\text{Ass}(R) = \emptyset$, and $X = Spec(R)$ is an example of a scheme which has no associated points.

Lemma 26.2.8. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x \in Supp(\mathcal{F})$ be a point in the support of $\mathcal{F}$ which is not a specialization of another point of $Supp(\mathcal{F})$. Then $x \in Ass(\mathcal{F})$. In particular, any generic point of an irreducible component of X is an associated point of X.*

Proof. Since $x \in \text{Supp}(\mathcal{F})$ the module $\mathcal{F}_x$ is not zero. Hence $\text{Ass}(\mathcal{F}_x) \subset Spec(\mathcal{O}_{X,x})$ is nonempty by Algebra, Lemma 7.60.7. On the other hand, by assumption $\text{Supp}(\mathcal{F}_x) = \{\mathfrak{m}_x\}$. Since $\text{Ass}(\mathcal{F}_x) \subset \text{Supp}(\mathcal{F}_x)$ (Algebra, Lemma 7.60.2) we see that $\mathfrak{m}_x$ is associated to $\mathcal{F}_x$ and we win. □

26.3. Morphisms and associated points

Lemma 26.3.1. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X which is flat over S. Let $\mathcal{G}$ be a quasi-coherent sheaf on S. Then we have*

$$Ass_X(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}) \supset \bigcup\nolimits_{s \in Ass_S(\mathcal{G})} Ass_{X_s}(\mathcal{F}_s)$$

and equality holds if S is locally Noetherian.

Proof. Let $x \in X$ and let $s = f(x) \in S$. Set $B = \mathcal{O}_{X,x}$, $A = \mathcal{O}_{S,s}$, $N = \mathcal{F}_x$, and $M = \mathcal{G}_s$. Note that the stalk of $\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}$ at x is equal to the B-module $M \otimes_A N$. Hence $x \in \text{Ass}_X(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G})$ if and only if $\mathfrak{m}_B$ is in $\text{Ass}_B(M \otimes_A N)$. Similarly $s \in \text{Ass}_S(\mathcal{G})$ and $x \in \text{Ass}_{X_s}(\mathcal{F}_s)$ if and only if $\mathfrak{m}_A \in \text{Ass}_A(M)$ and $\mathfrak{m}_B/\mathfrak{m}_A B \in \text{Ass}_{B \otimes \kappa(\mathfrak{m}_A)}(N \otimes \kappa(\mathfrak{m}_A))$. Thus the lemma follows from Algebra, Lemma 7.62.5. □

26.4. Embedded points

Let R be a ring and let M be an R-module. Recall that a prime $\mathfrak{p} \subset R$ is an *embedded associated* to M if it is an associated prime of M which is not minimal among the associated primes of M. See Algebra, Definition 7.64.1. Here is the definition of embedded associated points for quasi-coherent sheaves on schemes as given in [**DG67**, IV Definition 3.1.1].

Definition 26.4.1. Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on X.

(1) An *embedded associated point* of $\mathcal{F}$ is an associated point which is not maximal among the associated points of $\mathcal{F}$, i.e., it is the specialization of another associated point of $\mathcal{F}$.
(2) A point x of X is called an *embedded point* if x is an embedded associated point of $\mathcal{O}_X$.
(3) An *embedded component* of X is an irreducible closed subset $Z = \overline{\{x\}}$ where x is an embedded point of X.

In the Noetherian case when $\mathcal{F}$ is coherent we have the following.

Lemma 26.4.2. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then*

(1) *the generic points of irreducible components of Supp($\mathcal{F}$) are associated points of $\mathcal{F}$, and*
(2) *an associated point of $\mathcal{F}$ is embedded if and only if it is not a generic point of an irreducible component of Supp($\mathcal{F}$).*

In particular an embedded point of X is an associated point of X which is not a generic point of an irreducible component of X.

Proof. Recall that in this case $Z = \text{Supp}(\mathcal{F})$ is closed, see Coherent, Lemma 25.10.3 and that the generic points of irreducible components of Z are associated points of $\mathcal{F}$, see Lemma 26.2.8. Finally, we have $\text{Ass}(\mathcal{F}) \subset Z$, by Lemma 26.2.3. These results, combined with the fact that Z is a sober topological space and hence every point of Z is a specialization of a generic point of Z, imply (1) and (2). □

Lemma 26.4.3. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf on X. Then the following are equivalent:*

(1) *$\mathcal{F}$ has no embedded associated points, and*
(2) *$\mathcal{F}$ has property (S_1).*

Proof. This is Algebra, Lemma 7.140.2, combined with Lemma 26.2.2 above. □

Lemma 26.4.4. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf on X. The set of coherent subsheaves*

$$\{\mathcal{K} \subset \mathcal{F} \mid Supp(\mathcal{K}) \text{ is nowhere dense in } Supp(\mathcal{F})\}$$

has a maximal element $\mathcal{K}$. Setting $\mathcal{F}' = \mathcal{F}/\mathcal{K}$ we have the following

(1) *$Supp(\mathcal{F}') = Supp(\mathcal{F})$,*
(2) *$\mathcal{F}'$ has no embedded associated points, and*
(3) *there exists a dense open $U \subset X$ such that $U \cap Supp(\mathcal{F})$ is dense in $Supp(\mathcal{F})$ and $\mathcal{F}'|_U \cong \mathcal{F}|_U$.*

Proof. This follows from Algebra, Lemmas 7.64.2 and 7.64.3. Note that U can be taken as the complement of the closure of the set of embedded associated points of $\mathcal{F}$. □

Lemma 26.4.5. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module without embedded associated points. Set*

$$\mathcal{I} = Ker(\mathcal{O}_X \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})).$$

This is a coherent sheaf of ideals which defines a closed subscheme $Z \subset X$ without embedded points. Moreover there exists a coherent sheaf $\mathcal{G}$ on Z such that (a) $\mathcal{F} = (Z \to X)_*\mathcal{G}$, (b) $\mathcal{G}$ has no associated embedded points, and (c) $Supp(\mathcal{G}) = Z$ (as sets).

Proof. Some of the statements we have seen in the proof of Coherent, Lemma 25.11.7. The others follow from Algebra, Lemma 7.64.4. □

26.5. Weakly associated points

Let R be a ring and let M be an R-module. Recall that a prime $\mathfrak{p} \subset R$ is *weakly associated* to M if there exists an element m of M such that $\mathfrak{p}$ is minimal among the primes containing the annihilator of m. See Algebra, Definition 7.63.1. If R is a local ring with maximal ideal $\mathfrak{m}$, then $\mathfrak{m}$ is associated to M if and only if there exists an element $m \in M$ whose annihilator has radical $\mathfrak{m}$, see Algebra, Lemma 7.63.2.

Definition 26.5.1. Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on X.

(1) We say $x \in X$ is *weakly associated* to $\mathcal{F}$ if the maximal ideal $\mathfrak{m}_x$ is weakly associated to the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$.
(2) We denote WeakAss($\mathcal{F}$) the set of weakly associated points of $\mathcal{F}$.
(3) The *weakly associated points of X* are the weakly associated points of $\mathcal{O}_X$.

In this case, on any affine open, this corresponds exactly to the weakly associated primes as defined above. Here is the precise statement.

Lemma 26.5.2. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $Spec(A) = U \subset X$ be an affine open, and set $M = \Gamma(U, \mathcal{F})$. Let $x \in U$, and let $\mathfrak{p} \subset A$ be the corresponding prime. The following are equivalent*

(1) *$\mathfrak{p}$ is weakly associated to M, and*
(2) *x is weakly associated to $\mathcal{F}$.*

Proof. This follows from Algebra, Lemma 7.63.2. □

Lemma 26.5.3. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then*

$$Ass(\mathcal{F}) \subset WeakAss(\mathcal{F}) \subset Supp(\mathcal{F}).$$

Proof. This is immediate from the definitions. □

Lemma 26.5.4. *Let X be a scheme. Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of quasi-coherent sheaves on X. Then $WeakAss(\mathcal{F}_2) \subset WeakAss(\mathcal{F}_1) \cup WeakAss(\mathcal{F}_3)$ and $WeakAss(\mathcal{F}_1) \subset WeakAss(\mathcal{F}_2)$.*

Proof. For every point $x \in X$ the sequence of stalks $0 \to \mathcal{F}_{1,x} \to \mathcal{F}_{2,x} \to \mathcal{F}_{3,x} \to 0$ is a short exact sequence of $\mathcal{O}_{X,x}$-modules. Hence the lemma follows from Algebra, Lemma 7.63.3. □

Lemma 26.5.5. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then*

$$\mathcal{F} = (0) \Leftrightarrow WeakAss(\mathcal{F}) = \emptyset$$

Proof. Follows from Lemma 26.5.2 and Algebra, Lemma 7.63.4 □

Lemma 26.5.6. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x \in Supp(\mathcal{F})$ be a point in the support of $\mathcal{F}$ which is not a specialization of another point of $Supp(\mathcal{F})$. Then $x \in WeakAss(\mathcal{F})$. In particular, any generic point of an irreducible component of X is weakly associated to $\mathcal{O}_X$.*

Proof. Since $x \in \text{Supp}(\mathcal{F})$ the module $\mathcal{F}_x$ is not zero. Hence $\text{WeakAss}(\mathcal{F}_x) \subset Spec(\mathcal{O}_{X,x})$ is nonempty by Algebra, Lemma 7.63.4. On the other hand, by assumption $\text{Supp}(\mathcal{F}_x) = \{\mathfrak{m}_x\}$. Since $\text{WeakAss}(\mathcal{F}_x) \subset \text{Supp}(\mathcal{F}_x)$ (Algebra, Lemma 7.63.5) we see that $\mathfrak{m}_x$ is weakly associated to $\mathcal{F}_x$ and we win. □

Lemma 26.5.7. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If $\mathfrak{m}_x$ is a finitely generated ideal of $\mathcal{O}_{X,x}$, then*

$$x \in Ass(\mathcal{F}) \Leftrightarrow x \in WeakAss(\mathcal{F}).$$

In particular, if X is locally Noetherian, then $Ass(\mathcal{F}) = WeakAss(\mathcal{F})$.

Proof. See Algebra, Lemma 7.63.8. □

26.6. Morphisms and weakly associated points

Lemma 26.6.1. *Let $f : X \to S$ be an affine morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then we have*

$$WeakAss_S(f_*\mathcal{F}) \subset f(WeakAss_X(\mathcal{F}))$$

Proof. We may assume X and S affine, so $X \to S$ comes from a ring map $A \to B$. Then $\mathcal{F} = \widetilde{M}$ for some B-module M. By Lemma 26.5.2 the weakly associated points of $\mathcal{F}$ correspond exactly to the weakly associated primes of M. Similarly, the weakly associated points of $f_*\mathcal{F}$ correspond exactly to the weakly associated primes of M as an A-module. Hence the lemma follows from Algebra, Lemma 7.63.10. □

Lemma 26.6.2. *Let $f : X \to S$ be an affine morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If X is locally Noetherian, then we have*

$$f(Ass_X(\mathcal{F})) = Ass_S(f_*\mathcal{F}) = WeakAss_S(f_*\mathcal{F}) = f(WeakAss_X(\mathcal{F}))$$

Proof. We may assume X and S affine, so $X \to S$ comes from a ring map $A \to B$. As X is locally Noetherian the ring B is Noetherian, see Properties, Lemma 23.5.2. Write $\mathcal{F} = \widetilde{M}$ for some B-module M. By Lemma 26.2.2 the associated points of $\mathcal{F}$ correspond exactly to the associated primes of M, and any associated prime of M as an A-module is an associated points of $f_*\mathcal{F}$. Hence the inclusion

$$f(\mathrm{Ass}_X(\mathcal{F})) \subset \mathrm{Ass}_S(f_*\mathcal{F})$$

follows from Algebra, Lemma 7.60.12. We have the inclusion

$$\mathrm{Ass}_S(f_*\mathcal{F}) \subset \mathrm{WeakAss}_S(f_*\mathcal{F})$$

by Lemma 26.5.3. We have the inclusion

$$\mathrm{WeakAss}_S(f_*\mathcal{F}) \subset f(\mathrm{WeakAss}_X(\mathcal{F}))$$

by Lemma 26.6.1. The outer sets are equal by Lemma 26.5.7 hence we have equality everywhere. □

Lemma 26.6.3. *Let $f : X \to S$ be a finite morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\mathit{WeakAss}(f_*\mathcal{F}) = f(\mathit{WeakAss}(\mathcal{F}))$.*

Proof. We may assume X and S affine, so $X \to S$ comes from a finite ring map $A \to B$. Write $\mathcal{F} = \widetilde{M}$ for some B-module M. By Lemma 26.5.2 the weakly associated points of $\mathcal{F}$ correspond exactly to the weakly associated primes of M. Similarly, the weakly associated points of $f_*\mathcal{F}$ correspond exactly to the weakly associated primes of M as an A-module. Hence the lemma follows from Algebra, Lemma 7.63.12. □

Lemma 26.6.4. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_S$-module. Let $x \in X$ with $s = f(x)$. If f is flat at x, the point x is a generic point of the fibre X_s, and $s \in \mathit{WeakAss}_S(\mathcal{G})$, then $x \in \mathit{WeakAss}(f^*\mathcal{G})$.*

Proof. Let $A = \mathcal{O}_{S,s}$, $B = \mathcal{O}_{X,x}$, and $M = \mathcal{G}_s$. Let $m \in M$ be an element whose annihilator $I = \{a \in A \mid am = 0\}$ has radical $\mathfrak{m}_A$. Then $m \otimes 1$ has annihilator IB as $A \to B$ is faithfully flat. Thus it suffices to see that $\sqrt{IB} = \mathfrak{m}_B$. This follows from the fact that the maximal ideal of $B/\mathfrak{m}_A B$ is locally nilpotent (see Algebra, Lemma 7.23.3) and the assumption that $\sqrt{I} = \mathfrak{m}_A$. Some details omitted. □

26.7. Relative assassin

Definition 26.7.1. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The *relative assassin of $\mathcal{F}$ in X over S* is the set

$$\mathrm{Ass}_{X/S}(\mathcal{F}) = \bigcup\nolimits_{s \in S} \mathrm{Ass}_{X_s}(\mathcal{F}_s)$$

where $\mathcal{F}_s = (X_s \to X)^*\mathcal{F}$ is the restriction of $\mathcal{F}$ to the fibre of f at s.

Again there is a caveat that this is best used when the fibres of f are locally Noetherian and $\mathcal{F}$ is of finite type. In the general case we should probably use the relative weak assassin (defined in the next section).

Lemma 26.7.2. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $g : S' \to S$ be a morphism of schemes. Consider the base change diagram*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow & & \downarrow \\ S' & \xrightarrow{g} & S \end{array}$$

and set $\mathcal{F}' = (g')^*\mathcal{F}$. Let $x' \in X'$ be a point with images $x \in X$, $s' \in S'$ and $s \in S$. Assume f locally of finite type. Then $x' \in Ass_{X'/S'}(\mathcal{F}')$ if and only if $x \in Ass_{X/S}(\mathcal{F})$ and x' corresponds to a generic point of an irreducible component of $Spec(\kappa(s') \otimes_{\kappa(s)} \kappa(x))$.

Proof. Consider the morphism $X'_{s'} \to X_s$ of fibres. As $X_{s'} = X_s \times_{Spec(\kappa(s))} Spec(\kappa(s'))$ this is a flat morphism. Moreover $\mathcal{F}'_{s'}$ is the pullback of $\mathcal{F}_s$ via this morphism. As X_s is locally of finite type over the Noetherian scheme $Spec(\kappa(s))$ we have that X_s is locally Noetherian, see Morphisms, Lemma 24.14.6. Thus we may apply Lemma 26.3.1 and we see that

$$\mathrm{Ass}_{X'_{s'}}(\mathcal{F}'_{s'}) = \bigcup\nolimits_{x \in \mathrm{Ass}(\mathcal{F}_s)} \mathrm{Ass}((X'_{s'})_x).$$

Thus to prove the lemma it suffices to show that the associated points of the fibre $(X'_{s'})_x$ of the morphism $X'_{s'} \to X_s$ over x are its generic points. Note that $(X'_{s'})_x = Spec(\kappa(s') \otimes_{\kappa(s)} \kappa(x))$ as schemes. By Algebra, Lemma 7.149.1 the ring $\kappa(s') \otimes_{\kappa(s)} \kappa(x)$ is a Noetherian Cohen-Macaulay ring. Hence its associated primes are its minimal primes, see Algebra, Proposition 7.60.6 (minimal primes are associated) and Algebra, Lemma 7.140.2 (no embedded primes). $\square$

Remark 26.7.3. With notation and assumptions as in Lemma 26.7.2 we see that it is always the case that $(g')^{-1}(\mathrm{Ass}_{X/S}(\mathcal{F})) \supset \mathrm{Ass}_{X'/S'}(\mathcal{F}')$. If the morphism $S' \to S$ is locally quasi-finite, then we actually have

$$(g')^{-1}(\mathrm{Ass}_{X/S}(\mathcal{F})) = \mathrm{Ass}_{X'/S'}(\mathcal{F}')$$

because in this case the field extensions $\kappa(s) \subset \kappa(s')$ are always finite. In fact, this holds more generally for any morphism $g : S' \to S$ such that all the field extensions $\kappa(s) \subset \kappa(s')$ are algebraic, because in this case all prime ideals of $\kappa(s') \otimes_{\kappa(s)} \kappa(x)$ are maximal (and minimal) primes, see Algebra, Lemma 7.32.17.

26.8. Relative weak assassin

Definition 26.8.1. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The *relative weak assassin of $\mathcal{F}$ in X over S* is the set

$$\mathrm{WeakAss}_{X/S}(\mathcal{F}) = \bigcup\nolimits_{s \in S} \mathrm{WeakAss}(\mathcal{F}_s)$$

where $\mathcal{F}_s = (X_s \to X)^*\mathcal{F}$ is the restriction of $\mathcal{F}$ to the fibre of f at s.

Lemma 26.8.2. *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $WeakAss_{X/S}(\mathcal{F}) = Ass_{X/S}(\mathcal{F})$.*

Proof. This is true becase the fibres of f are locally Noetherian schemes, and associated and weakly associated points agree on locally Noetherian schemes, see Lemma 26.5.7. $\square$

26.9. Effective Cartier divisors

For some reason it seem convenient to define the notion of an effective Cartier divisor before anything else.

Definition 26.9.1. Let S be a scheme.

(1) A *locally principal closed subscheme* of S is a closed subscheme whose sheaf of ideals is locally generated by a single element.
(2) An *effective Cartier divisor* on S is a closed subscheme $D \subset S$ such that the ideal sheaf $\mathcal{I}_D \subset \mathcal{O}_X$ is an invertible $\mathcal{O}_X$-module.

Thus an effective Cartier divisor is a locally principal closed subscheme, but the converse is not always true. Effective Cartier divisors are closed subschemes of pure codimension 1 in the strongest possible sense. Namely they are locally cut out by a single element which is not a zero divisor. In particular they are nowhere dense.

Lemma 26.9.2. *Let S be a scheme. Let $D \subset S$ be a closed subscheme. The following are equivalent:*

(1) *The subscheme D is an effective Cartier divisor on S.*
(2) *For every $x \in D$ there exists an affine open neighbourhood $Spec(A) = U \subset X$ of x such that $U \cap D = Spec(A/(f))$ with $f \in A$ not a zero divisor.*

Proof. Assume (1). For every $x \in D$ there exists an affine open neighbourhood $Spec(A) = U \subset X$ of x such that $\mathcal{I}_D|_U \cong \mathcal{O}_U$. In other words, there exists a section $f \in \Gamma(U, \mathcal{I}_D)$ which freely generates the restriction $\mathcal{I}_D|_U$. Hence $f \in A$, and the multiplication map $f : A \to A$ is injective. Also, since $\mathcal{I}_D$ is quasi-coherent we see that $D \cap U = Spec(A/(f))$.

Assume (2). Let $x \in D$. By assumption there exists an affine open neighbourhood $Spec(A) = U \subset X$ of x such that $U \cap D = Spec(A/(f))$ with $f \in A$ not a zero divisor. Then $\mathcal{I}_D|_U \cong \mathcal{O}_U$ since it is equal to $\widetilde{(f)} \cong \widetilde{A} \cong \mathcal{O}_U$. Of course $\mathcal{I}_D$ restricted to the open subscheme $S \setminus D$ is isomorphic to $\mathcal{O}_{X \setminus D}$. Hence $\mathcal{I}_D$ is an invertible $\mathcal{O}_S$-module. □

Lemma 26.9.3. *Let S be a scheme. Let $D \subset S$ be an effective Cartier divisor. Let $s \in D$. If $\dim_s(S) < \infty$, then $\dim_s(D) < \dim_s(S)$.*

Proof. Assume $\dim_s(S) < \infty$. Let $U = Spec(A) \subset S$ be an affine open neighbourhood of X such that $\dim(U) = \dim_s(S)$ and such that $D = V(f)$ for some nonzero divisor $f \in A$ (see Lemma 26.9.2). Recall that $\dim(U)$ is the Krull dimension of the ring A and that $\dim(U \cap D)$ is the Krull dimension of the ring $A/(f)$. Then f is not contained in any minimal prime of A. Hence any maximal chain of primes in $A/(f)$, viewed as a chain of primes in A, can be extended by adding a minimal prime. □

Definition 26.9.4. Let S be a scheme. Given effective Cartier divisors D_1, D_2 on S we set $D = D_1 + D_2$ equal to the closed subscheme of S corresponding to the quasi-coherent sheaf of ideals $\mathcal{I}_{D_1}\mathcal{I}_{D_2} \subset \mathcal{O}_S$. We call this the *sum of the effective Cartier divisors D_1 and D_2*.

It is clear that we may define the sum $\sum n_i D_i$ given finitely many effective Cartier divisors D_i on X and nonnegative integers n_i.

Lemma 26.9.5. *The sum of two effective Cartier divisors is an effective Cartier divisor.*

Proof. Omitted. Locally $f_1, f_2 \in A$ are nonzero divisors, then also $f_1 f_2 \in A$ is a nonzero divisor. □

Lemma 26.9.6. *Let X be a scheme. Let D, D' be two effective Cartier divisors on X. If $D \subset D'$ (as closed subschemes of X), then there exists an effective Cartier divisor D'' such that $D' = D + D''$.*

Proof. Omitted. □

Recall that we have defined the inverse image of a closed subscheme under any morphism of schemes in Schemes, Definition 21.17.7.

Lemma 26.9.7. *Let $f : S' \to S$ be a morphism of schemes. Let $Z \subset S$ be a locally principal closed subscheme. Then the inverse image $f^{-1}(Z)$ is a locally principal closed subscheme of S'.*

Proof. Omitted. □

Definition 26.9.8. Let $f : S' \to S$ be a morphism of schemes. Let $D \subset S$ be an effective Cartier divisor. We say the *pullback of D by f is defined* if the closed subscheme $f^{-1}(D) \subset S'$ is an effective Cartier divisor. In this case we denote it either f^*D or $f^{-1}(D)$ and we call it the *pullback of the effective Cartier divisor*.

The condition that $f^{-1}(D)$ is an effective Cartier divisor is often satisfied in practice. Here is an example lemma.

Lemma 26.9.9. *Let $f : X \to Y$ be a morphism of schemes. Let $D \subset Y$ be an effective Cartier divisor. The pullback of D by f is defined in each of the following cases:*

(1) *X, Y integral and f dominant,*
(2) *X reduced, and for any generic point ξ of any irreducible component of X we have $f(\xi) \notin D$,*
(3) *X is locally Noetherian and for any associated point x of X we have $f(x) \notin D$,*
(4) *X is locally Noetherian, has no embedded points, and for any generic point ξ of any irreducible component of X we have $f(\xi) \notin D$,*
(5) *f is flat, and*
(6) *add more here as needed.*

Proof. The question is local on X, and hence we reduce to the case where $X = Spec(A)$, $Y = Spec(R)$, f is given by $\varphi : R \to A$ and $D = Spec(R/(t))$ where $t \in R$ is not a zero divisor. The goal in each case is to show that $\varphi(t) \in A$ is not a zero divisor.

In case (2) this follows as the intersection of all minimal primes of a ring is the nilradical of the ring, see Algebra, Lemma 7.16.2.

Let us prove (3). By Lemma 26.2.2 the associated points of X correspond to the primes $\mathfrak{p} \in \text{Ass}(A)$. By Algebra, Lemma 7.60.9 we have $\bigcup_{\mathfrak{p} \in \text{Ass}(A)} \mathfrak{p}$ is the set of zero divisors of A. The hypothesis of (3) is that $\varphi(t) \notin \mathfrak{p}$ for all $\mathfrak{p} \in \text{Ass}(A)$. Hence $\varphi(t)$ is a nonzero divisor as desired.

Part (4) follows from (3) and the definitions. □

Lemma 26.9.10. *Let $f : S' \to S$ be a morphism of schemes. Let D_1, D_2 be effective Cartier divisors on S. If the pullbacks of D_1 and D_2 are defined then the pullback of $D = D_1 + D_2$ is defined and $f^*D = f^*D_1 + f^*D_2$.*

Proof. Omitted. □

Definition 26.9.11. Let S be a scheme and let D be an effective Cartier divisor. The *invertible sheaf $\mathcal{O}_S(D)$ associated to D* is given by

$$\mathcal{O}_S(D) := \mathcal{H}om_{\mathcal{O}_S}(\mathcal{I}_D, \mathcal{O}_S) = \mathcal{I}_D^{\otimes -1}.$$

The canonical section, usually denoted 1 or 1_D, is the global section of $\mathcal{O}_S(D)$ corresponding to the inclusion mapping $\mathcal{I}_D \to \mathcal{O}_S$.

Lemma 26.9.12. *Let S be a scheme. Let D_1, D_2 be effective Cartier divisors on S. Let $D = D_1 + d_2$. Then there is a unique isomorphism*

$$\mathcal{O}_S(D_1) \otimes_{\mathcal{O}_S} \mathcal{O}_S(D_2) \longrightarrow \mathcal{O}_S(D)$$

which maps $1_{D_1} \otimes 1_{D_2}$ to 1_D.

Proof. Omitted. □

Definition 26.9.13. Let $(X, \mathcal{O}_X)$ be a locally ringed space. Let $\mathcal{L}$ be an invertible sheaf on X. A global section $s \in \Gamma(X, \mathcal{L})$ is called a *regular section* if the map $\mathcal{O}_X \to \mathcal{L}$, $f \mapsto fs$ is injective.

Lemma 26.9.14. *Let X be a locally ringed space. Let $f \in \Gamma(X, \mathcal{O}_X)$. The following are equivalent:*

(1) *f is a regular section, and*
(2) *for any $x \in X$ the image $f \in \mathcal{O}_{X,x}$ is not a zero divisor.*

If X is a scheme these are also equivalent to

(3) *for any affine open $Spec(A) = U \subset X$ the image $f \in A$ is not a zero divisor, and*
(4) *there exists an affine open covering $X = \bigcup Spec(A_i)$ such that the image of f in A_i is not a zero divisor for all i.*

Proof. Omitted. □

Note that a global section s of an invertible $\mathcal{O}_X$-module $\mathcal{L}$ may be seen as an $\mathcal{O}_X$-module map $s : \mathcal{O}_X \to \mathcal{L}$. Its dual is therefore a map $s : \mathcal{L}^{\otimes -1} \to \mathcal{O}_X$. (See Modules, Definition 15.21.3 for the definition of the dual invertible sheaf.)

Definition 26.9.15. Let X be a scheme. Let $\mathcal{L}$ be an invertible sheaf. Let $s \in \Gamma(X, \mathcal{L})$. The *zero scheme* of s is the closed subscheme $Z(s) \subset X$ defined by the quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ which is the image of the map $s : \mathcal{L}^{\otimes -1} \to \mathcal{O}_X$.

Lemma 26.9.16. *Let X be a scheme. Let $\mathcal{L}$ be an invertible sheaf. Let $s \in \Gamma(X, \mathcal{L})$.*

(1) *Consider closed immersions $i : Z \to X$ such that $i^*s \in \Gamma(Z, i^*\mathcal{L}))$ is zero ordered by inclusion. The zero scheme $Z(s)$ is the minimal element of this set.*
(2) *For any morphism of schemes $f : Y \to X$ we have $f^*s = 0$ in $\Gamma(Y, f^*\mathcal{L})$ if and only if f factors through $Z(s)$.*
(3) *The zero scheme $Z(s)$ is a locally principal closed subscheme.*
(4) *The zero scheme $Z(s)$ is an effective Cartier divisor if and only if s is a regular section of $\mathcal{L}$.*

Proof. Omitted. □

Lemma 26.9.17. *Let S be a scheme.*

(1) *If $D \subset S$ is an effective Cartier divisor, then the canonical section 1_D of $\mathcal{O}_S(D)$ is regular.*
(2) *Conversely, if s is a regular section of the invertible sheaf $\mathcal{L}$, then there exists a unique effective Cartier divisor $D = Z(s) \subset S$ and a unique isomorphism $\mathcal{O}_S(D) \to \mathcal{L}$ which maps 1_D to s.*

The constructions $D \mapsto (\mathcal{O}_X(D), 1_D)$ and $(\mathcal{L}, s) \mapsto Z(s)$ give mutually inverse maps

$$\left\{\textit{effective Cartier divisors on } X\right\} \leftrightarrow \left\{\begin{array}{c}\textit{pairs } (\mathcal{L}, s) \textit{ consisting of an invertible} \\ \mathcal{O}_X\textit{-module and a regular global section}\end{array}\right\}$$

Proof. Omitted. □

Here is a way to produce effective Cartier divisors.

Lemma 26.9.18. *Let X be a scheme. Let $Z \subset X$ be a closed subscheme. The blow up $b : X' \to X$ of Z has the following properties:*

(1) *$b|_{b^{-1}(X \setminus Z)} : b^{-1}(X \setminus Z) \to X \setminus Z$ is an isomorphism, and*
(2) *$E = b^{-1}(Z)$ is an effective Cartier divisor on X'.*

Proof. Proof omitted. Here are some hints: If $X = Spec(A)$ and $Z = Spec(A/I)$, then $X' = \text{Proj}(\bigoplus_{n\geq 0} I^n)$. Write $S = \bigoplus_{n\geq 0} I^n$ as a graded ring. Pick an element $f \in I$ and denote $F \in S_1$ the corresponding element in degree one of S. It is clear that the standard opens $D_+(F)$ cover X' in this case. Each $D_+(F)$ is the spectrum of the ring $S_{(F)}$. Note that f is a nonzero divisor on $S_{(F)}$ since $fa/F^d = 0$ (some $a \in S_d$) implies also that Fa/F^{d+1} is zero. Moreover, $IS_{(F)}$ is generated by the elements $g = fG/F$ where $G \in S_1$ is the degree 1 element of S corresponding to g. Hence it is indeed the case that $IS_{(F)}$ is generated by a single nonzero divisor as desired. $\square$

26.10. Relative effective Cartier divisors

The following lemma shows that an effective Cartier divisor which is flat over the base is reall a ``family of effective Cartier divisors" over the base. For example the restriction to any fibre is an effective Cartier divisor.

Lemma 26.10.1. *Let $f : X \to S$ be a morphism of schemes. Let $D \subset X$ be a closed subscheme. Assume*

(1) *D is an effective Cartier divisor, and*
(2) *$D \to S$ is a flat morphism.*

Then for every morphism of schemes $g : S' \to S$ the pullback $(g')^{-1}D$ is an effective Cartier divisor on $X' = S' \times_S X$.

Proof. Using Lemma 26.9.2 we translate this as follows into algebra. Let $A \to B$ be a ring map and $h \in B$. Assume h is a nonzero divisor and that B/hB is flat over A. Then

$$0 \to B \xrightarrow{h} B \to B/hB \to 0$$

is a short exact sequence of A-modules with B/hB flat over A. By Algebra, Lemma 7.35.11 this sequence remains exact on tensoring over A with any module, in particular with any A-algebra A'. $\square$

This lemma is the motivation for the following definition.

Definition 26.10.2. Let $f : X \to S$ be a morphism of schemes. A *relative effective Cartier divisor* on X/S is an effective Cartier divisor $D \subset X$ such that $D \to S$ is a flat morphism of schemes.

We warn the reader that this may be nonstandard notation. In particular, in [**DG67**, IV, Section 21.15] the notion of a relative divisor is discussed only when $X \to S$ is flat and locally of finite presentation. Our definition is a bit more general. However, it turns out that if $x \in D$ then $X \to S$ is flat at x in many cases (but not always).

Lemma 26.10.3. *Let $f : X \to S$ be a morphism of schemes. Let $D \subset X$ be a relative effective Cartier divisor on X/S. If $x \in D$ and $\mathcal{O}_{X,x}$ is Noetherian, then f is flat at x.*

Proof. Set $A = \mathcal{O}_{S,f(x)}$ and $B = \mathcal{O}_{X,x}$. Let $h \in B$ be an element which generates the ideal of D. Then h is a nonzero divisor in B such that B/hB is a flat local A-algebra. Let $I \subset A$ be a finitely generated ideal. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r, "h"'] & B \arrow[r] & B/hB \arrow[r] & 0 \\
0 \arrow[r] & B \otimes_A I \arrow[r, "h"] \arrow[u] & B \otimes_A I \arrow[r] \arrow[u] & B/hB \otimes_A I \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

The lower sequence is short exact as B/hB is flat over A, see Algebra, Lemma 7.35.11. The right vertical arrow is injective as B/hB is flat over A, see Algebra, Lemma 7.35.4. Hence multiplication by h is surjective on the kernel K of the middle vertical arrow. By Nakayama's lemma, see Algebra, Lemma 7.14.5 we conclude that $K = 0$. Hence B is flat over A, see Algebra, Lemma 7.35.4. □

The following lemma relies on the algebraic version of openness of the flat locus. The scheme theoretic version can be found in More on Morphisms, Section 33.11.

Lemma 26.10.4. *Let $f : X \to S$ be a morphism of schemes. Let $D \subset X$ be a relative effective Cartier divisor. If f is locally of finite presentation, then there exists an open subscheme $U \subset X$ such that $D \subset U$ and such that $f|_U : U \to S$ is flat.*

Proof. Pick $x \in D$. It suffices to find an open neighbourhood $U \subset X$ of x such that $f|_U$ is flat. Hence the lemma reduces to the case that $X = Spec(B)$ and $S = Spec(A)$ are affine and that D is given by a nonzero divisor $h \in B$. By assumption B is a finitely presented A-algebra and B/hB is a flat A-algebra. We are going to use absolute Noetherian approximation.

Write $B = A[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$. Assume h is the image of $h' \in A[x_1, \ldots, x_n]$. Choose a finite type $\mathbf{Z}$-subalgebra $A_0 \subset A$ such that all the coefficients of the polynomials $h', g_1, \ldots, g_m$ are in A_0. Then we can set $B_0 = A_0[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$ and h_0 the image of h' in B_0. Then $B = B_0 \otimes_{A_0} A$. Set

$$J_0 = \{b \in B_0 \mid \exists n > 0,\ h_0^n b = 0\}$$

and $C_0 = B_0/J_0$. The image $\overline{h}_0$ of h_0 is a nonzero divisor in C_0 (see More on Algebra, Lemma 12.8.6). As h is a nonzero divisor in B we see that $B_0 \to B$ annihilates J_0. Hence the isomorphism $B_0 \otimes_{A_0} A \to B$ factors through the surjective map $B_0 \otimes_{A_0} A \to C_0 \otimes_{A_0} A$ whence also $C_0 \otimes_{A_0} A \cong B$. Thus the ring map $A \to B$ is approximated by the ring maps $A_0 \to C_0$. By Algebra, Lemma 7.120.5 we may, after enlarging A_0, assume that $C_0/\overline{h}_0 C_0$ is flat over A_0.

Set $f_0 : X_0 \to S_0$ equal to $Spec$ of the ring map $A_0 \to C_0$. Set $D_0 = Spec(C_0/\overline{h}_0 C_0)$. Since $B = C_0 \otimes_{A_0} A$, i.e., $X = X_0 \times_{S_0} S$, it now suffices to prove the lemma for $X_0 \to S_0$ and the relative effective Cartier divisor D_0, see Morphisms, Lemma 24.24.6. Hence we have reduced to the case where A is a Noetherian ring. In this case we know that the ring map $A \to B$ is flat at every prime $\mathfrak{q}$ of $V(h)$ by Lemma 26.10.3. Combined with the fact that the flat locus is open in this case, see Algebra, Theorem 7.120.4 we win. □

There is also the following lemma (whose idea is apparantly due to Michael Artin, see [**Nob77**]) which needs no finiteness assumptions at all.

Lemma 26.10.5. *Let $f : X \to S$ be a morphism of schemes. Let $D \subset X$ be a relative effective Cartier divisor on X/S. If f is flat at all points of $X \setminus D$, then f is flat.*

Proof. This translates into the following algebra fact: Let $A \to B$ be a ring map and $h \in B$. Assume h is a nonzero divisor, that B/hB is flat over A, and that the localization B_h is flat over A. Then B is flat over A. The reason is that we have a short exact sequence

$$0 \to B \to B_h \to colim_n (1/h^n)B/B \to 0$$

and that the second and third terms are flat over A, which implies that B is flat over A (see Algebra, Lemma 7.35.12). Note that a filtered colimit of flat modules is flat (see Algebra, Lemma 7.35.2) and that by induction on n each $(1/h^n)B/B \cong B/h^nB$ is flat over A since it fits into the short exact sequence

$$0 \to B/h^{n-1}B \xrightarrow{h} B/h^nB \to B/hB \to 0$$

Some details omitted. □

Example 26.10.6. Here is an example of a relative effective Cartier divisor D where the ambient scheme is not flat in a neighbourhood of D. Namely, let $A = k[t]$ and

$$B = k[t, x, y, x^{-1}y, x^{-2}y, \ldots]/(ty, tx^{-1}y, tx^{-2}y, \ldots)$$

Then B is not flat over A but $B/xB \cong A$ is flat over A. Moreover x is a nonzero divisor and hence defines a relative effective Cartier divisor in $Spec(B)$ over $Spec(A)$.

If the ambient scheme is flat and locally of finite presentation over the base, then we can characterize a relative effective Cartier divisor in terms of its fibres. See also More on Morphisms, Lemma 33.16.1 for a slightly different take on this lemma.

Lemma 26.10.7. *Let $\varphi : X \to S$ be a flat morphism which is locally of finite presentation. Let $Z \subset X$ be a closed subscheme. Let $x \in Z$ with image $s \in S$.*

1. *If $Z_s \subset X_s$ is a Cartier divisor in a neighbourhood of x, then there exists an open $U \subset X$ and a relative effective Cartier divisor $D \subset U$ such that $Z \cap U \subset D$.*
2. *If $Z_s \subset X_s$ is a Cartier divisor in a neighbourhood of x, the morphism $Z \to X$ is of finite presentation, and $Z \to S$ is flat at x, then we can choose U and D such that $Z \cap U = D$.*
3. *If $Z_s \subset X_s$ is a Cartier divisor in a neighbourhood of x and Z is a locally principal closed subscheme of X in a neighbourhood of x, then we can choose U and D such that $Z \cap U = D$.*

In particular, if $Z \to S$ is locally of finite presentation and flat and all fibres $Z_s \subset X_s$ are effective Cartier divisors, then Z is a relative effective Cartier divisor. Similarly, if Z is a locally principal closed subscheme of X such that all fibres $Z_s \subset X_s$ are effective Cartier divisors, then Z is a relative effective Cartier divisor.

Proof. Choose affine open neighbourhoods $Spec(A)$ of s and $Spec(B)$ of x such that $\varphi(Spec(B)) \subset Spec(A)$. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to s. Let $\mathfrak{q} \subset B$ be the prime ideal corresponding to x. Let $I \subset B$ be the ideal corresponding to Z. By the initial assumption of the lemma we know that $A \to B$ is flat and of finite presentation. The assumption in (1) means that, after shrinking $Spec(B)$, we may assume $I(B \otimes_A \kappa(\mathfrak{p}))$ is generated by a single element which is a nonzero divisor in $B \otimes_A \kappa(\mathfrak{p})$. Say $f \in I$ maps to this generator. We claim that after inverting an element $g \in B$, $g \notin \mathfrak{q}$ the closed subscheme $D = V(f) \subset Spec(B_g)$ is a relative effective Cartier divisor.

By Algebra, Lemma 7.120.5 we can find a flat finite type ring map $A_0 \to B_0$ of Noetherian rings, an element $f_0 \in B_0$, a ring map $A_0 \to A$ and an isomorphism $A \otimes_{A_0} B_0 \cong B$. If $\mathfrak{p}_0 = A_0 \cap \mathfrak{p}$ then we see that

$$B \otimes_A \kappa(\mathfrak{p}) = \left(B_0 \otimes_{A_0} \kappa(\mathfrak{p}_0)\right) \otimes_{\kappa(\mathfrak{p}_0))} \kappa(\mathfrak{p})$$

hence f_0 is a nonzero divisor in $B_0 \otimes_{A_0} \kappa(\mathfrak{p}_0)$. By Algebra, Lemma 7.91.2 we see that f_0 is a nonzero divisor in $(B_0)_{\mathfrak{q}_0}$ where $\mathfrak{q}_0 = B_0 \cap \mathfrak{q}$ and that $(B_0/f_0B_0)_{\mathfrak{q}_0}$ is flat over A_0. Hence by Algebra, Lemma 7.65.8 and Algebra, Theorem 7.120.4 there exists a $g_0 \in B_0$, $g_0 \notin \mathfrak{q}_0$ such that f_0 is a nonzero divisor in $(B_0)_{g_0}$ and such that $(B_0/f_0B_0)_{g_0}$ is flat over A_0. Hence we see that $D_0 = V(f_0) \subset Spec((B_0)_{g_0})$ is a relative effective Cartier divisor. Since we know that this property is preserved under base change, see Lemma 26.10.1, we obtain the claim mentioned above with g equal to the image of g_0 in B.

At this point we have proved (1). To see (2) consider the closed immersion $Z \to D$. The surjective ring map $u : \mathcal{O}_{D,x} \to \mathcal{O}_{Z,x}$ is a map of flat local $\mathcal{O}_{S,s}$-algebras which are essentially of finite presentation, and which becomes an isomorphisms after dividing by $\mathfrak{m}_s$. Hence it is an isomorphism, see Algebra, Lemma 7.119.4. It follows that $Z \to D$ is an isomorphism in a neighbourhood of x, see Algebra, Lemma 7.117.6. To see (3), after possibly shrinking U we may assume that the ideal of D is generated by a single nonzero divisor f and the ideal of Z is generated by an element g. Then $f = gh$. But $g|_{U_s}$ and $f|_{U_s}$ cut out the same effective Cartier divisor in a neighbourhood of x. Hence $h|_{X_s}$ is a unit in $\mathcal{O}_{X_s,x}$, hence h is a unit in $\mathcal{O}_{X,x}$ hence h is a unit in an open neighbourhood of x. I.e., $Z \cap U = D$ after shrinking U.

The final statements of the lemma follow immediately from parts (2) and (3), combined with the fact that $Z \to S$ is locally of finite presentation if and only if $Z \to X$ is of finite presentation, see Morphisms, Lemmas 24.20.3 and 24.20.11. □

26.11. The normal cone of an immersion

Let $i : Z \to X$ be a closed immersion. Let $\mathcal{I} \subset \mathcal{O}_X$ be the corresponding quasi-coherent sheaf of ideals. Consider the quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras $\bigoplus_{n\geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$. Since the sheaves $\mathcal{I}^n/\mathcal{I}^{n+1}$ are each annihilated by $\mathcal{I}$ this graded algebra corresponds to a quasi-coherent sheaf of graded $\mathcal{O}_Z$-algebras by Morphisms, Lemma 24.3.1. This quasi-coherent graded $\mathcal{O}_Z$-algebra is called the *conormal algebra of Z in X* and is often simply denoted $\bigoplus_{n\geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$ by the abuse of notation mentioned in Morphisms, Section 24.3.

Let $f : Z \to X$ be an immersion. We define the conormal algebra of f as the conormal sheaf of the closed immersion $i : Z \to X \setminus \partial Z$, where $\partial Z = \overline{Z} \setminus Z$. It is often denoted $\bigoplus_{n\geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$ where $\mathcal{I}$ is the ideal sheaf of the closed immersion $i : Z \to X \setminus \partial Z$.

Definition 26.11.1. Let $f : Z \to X$ be an immersion. The *conormal algebra $\mathcal{C}_{Z/X,*}$ of Z in X* or the *conormal algebra of f* is the quasi-coherent sheaf of graded $\mathcal{O}_Z$-algebras $\bigoplus_{n\geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$ described above.

Thus $\mathcal{C}_{Z/X,1} = \mathcal{C}_{Z/X}$ is the conormal sheaf of the immersion. Also $\mathcal{C}_{Z/X,0} = \mathcal{O}_Z$ and $\mathcal{C}_{Z/X,n}$ is a quasi-coherent $\mathcal{O}_Z$-module characterized by the property

$$i_*\mathcal{C}_{Z/X,n} = \mathcal{I}^n/\mathcal{I}^{n+1} \tag{26.11.1.1}$$

where $i : Z \to X \setminus \partial Z$ and $\mathcal{I}$ is the ideal sheaf of i as above. Finally, note that there is a canonical surjective map

$$\mathrm{Sym}^*(\mathcal{C}_{Z/X}) \longrightarrow \mathcal{C}_{Z/X,*} \tag{26.11.1.2}$$

of quasi-coherent graded $\mathcal{O}_Z$-algebras which is an isomorphism in degrees 0 and 1.

Lemma 26.11.2. *Let $i : Z \to X$ be an immersion. The conormal algebra of i has the following properties:*

(1) *Let $U \subset X$ be any open such that $i(Z)$ is a closed subset of U. Let $\mathcal{I} \subset \mathcal{O}_U$ be the sheaf of ideals corresponding to the closed subscheme $i(Z) \subset U$. Then*
$$\mathcal{C}_{Z/X,*} = i^*\left(\bigoplus\nolimits_{n\geq 0} \mathcal{I}^n\right) = i^{-1}\left(\bigoplus\nolimits_{n\geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}\right)$$
(2) *For any affine open $Spec(R) = U \subset X$ such that $Z \cap U = Spec(R/I)$ there is a canonical isomorphism $\Gamma(Z \cap U, \mathcal{C}_{Z/X,*}) = \bigoplus_{n\geq 0} I^n/I^{n+1}$.*

Proof. Mostly clear from the definitions. Note that given a ring R and an ideal I of R we have $I^n/I^{n+1} = I^n \otimes_R R/I$. Details omitted. $\square$

Lemma 26.11.3. *Let*

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ Z' \arrow[r, "i'"] & X' \end{tikzcd}$$

be a commutative diagram in the category of schemes. Assume i, i' immersions. There is a canonical map of graded $\mathcal{O}_Z$-algebras

$$f^*\mathcal{C}_{Z'/X',*} \longrightarrow \mathcal{C}_{Z/X,*}$$

characterized by the following property: For every pair of affine opens $(Spec(R) = U \subset X, Spec(R') = U' \subset X')$ with $f(U) \subset U'$ such that $Z \cap U = Spec(R/I)$ and $Z' \cap U' = Spec(R'/I')$ the induced map

$$\Gamma(Z' \cap U', \mathcal{C}_{Z'/X',*}) = \bigoplus (I')^n/(I')^{n+1} \longrightarrow \bigoplus\nolimits_{n\geq 0} I^n/I^{n+1} = \Gamma(Z \cap U, \mathcal{C}_{Z/X,*})$$

is the one induced by the ring map $f^\sharp : R' \to R$ which has the property $f^\sharp(I') \subset I$.

Proof. Let $\partial Z' = \overline{Z'} \setminus Z'$ and $\partial Z = \overline{Z} \setminus Z$. These are closed subsets of X' and of X. Replacing X' by $X' \setminus \partial Z'$ and X by $X \setminus \left(g^{-1}(\partial Z') \cup \partial Z\right)$ we see that we may assume that i and i' are closed immersions.

The fact that $g \circ i$ factors through i' implies that $g^*\mathcal{I}'$ maps into $\mathcal{I}$ under the canonical map $g^*\mathcal{I}' \to \mathcal{O}_X$, see Schemes, Lemmas 21.4.6 and 21.4.7. Hence we get an induced map of quasi-coherent sheaves $g^*((\mathcal{I}')^n/(\mathcal{I}')^{n+1}) \to \mathcal{I}^n/\mathcal{I}^{n+1}$. Pulling back by i gives $i^*g^*((\mathcal{I}')^n/(\mathcal{I}')^{n+1}) \to i^*(\mathcal{I}^n/\mathcal{I}^{n+1})$. Note that $i^*(\mathcal{I}^n/\mathcal{I}^{n+1}) = \mathcal{C}_{Z/X,n}$. On the other hand, $i^*g^*((\mathcal{I}')^n/(\mathcal{I}')^{n+1}) = f^*(i')^*((\mathcal{I}')^n/(\mathcal{I}')^{n+1}) = f^*\mathcal{C}_{Z'/X',n}$. This gives the desired map.

Checking that the map is locally described as the given map $(I')^n/(I')^{n+1} \to I^n/I^{n+1}$ is a matter of unwinding the definitions and is omitted. Another observation is that given any $x \in i(Z)$ there do exist affine open neighbourhoods U, U' with $f(U) \subset U'$ and $Z \cap U$ as well as $U' \cap Z'$ closed such that $x \in U$. Proof omitted. Hence the requirement of the lemma indeed characterizes the map (and could have been used to define it). $\square$

Lemma 26.11.4. *Let*

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ Z' \arrow[r, "i'"] & X' \end{tikzcd}$$

be a fibre product diagram in the category of schemes with i, i' immersions. Then the canonical map $f^*\mathcal{C}_{Z'/X',*} \to \mathcal{C}_{Z/X,*}$ of Lemma 26.11.3 is surjective. If g is flat, then it is an isomorphism.

Proof. Let $R' \to R$ be a ring map, and $I' \subset R'$ an ideal. Set $I = I'R$. Then $(I')^n/(I')^{n+1} \otimes_{R'} R \to I^n/I^{n+1}$ is surjective. If $R' \to R$ is flat, then $I^n = (I')^n \otimes_{R'} R$ and we see the map is an isomorphism. $\square$

Definition 26.11.5. Let $i : Z \to X$ be an immersion of schemes. The *normal cone* C_ZX of Z in X is

$$C_ZX = \underline{Spec}_Z(\mathcal{C}_{Z/X,*})$$

see Constructions, Definitions 22.7.1 and 22.7.2. The *normal bundle* of Z in X is the vector bundle

$$N_Z X = \underline{Spec}_Z(\mathrm{Sym}(\mathcal{C}_{Z/X}))$$

see Constructions, Definitions 22.6.1 and 22.6.2.

Thus $C_Z X \to Z$ is a cone over Z and $N_Z X \to Z$ is a vector bundle over Z (recall that in our terminology this does not imply that the conormal sheaf is a finite locally free sheaf). Moreover, the canonical surjection (26.11.1.2) of graded algebras defines a canonical closed immersion

$$C_Z X \longrightarrow N_Z X \tag{26.11.5.1}$$

of cones over Z.

26.12. Regular ideal sheaves

In this section we generalize the notion of an effective Cartier divisor to higher codimension. Recall that a sequence of elements $f_1, \ldots, f_r$ of a ring R is a *regular sequence* if for each $i = 1, \ldots, r$ the element f_i is a nonzero divisor on $R/(f_1, \ldots, f_{i-1})$ and $R/(f_1, \ldots, f_r) \neq 0$, see Algebra, Definition 7.65.1. There are three closely related weaker conditions that we can impose. The first is to assume that $f_1, \ldots, f_r$ is a *Koszul-regular sequence*, i.e., that $H_i(K_\bullet(f_1, \ldots, f_r)) = 0$ for $i > 0$, see More on Algebra, Definition 12.22.1. The sequence is called an *H_1-regular sequence* if $H_1(K_\bullet(f_1, \ldots, f_r)) = 0$. Another condition we can impose is that with $J = (f_1, \ldots, f_r)$, the map

$$R/J[T_1, \ldots, T_r] \longrightarrow \bigoplus\nolimits_{n \geq 0} J^n/J^{n+1}$$

which maps T_i to $f_i \bmod J^2$ is an isomorphism. In this case we say that $f_1, \ldots, f_r$ is a *quasi-regular sequence*, see Algebra, Definition 7.66.1. Given an R-module M there is also a notion of M-regular and M-quasi-regular sequence.

We can generalize this to the case of ringed spaces as follows. Let X be a ringed space and let $f_1, \ldots, f_r \in \Gamma(X, \mathcal{O}_X)$. We say that $f_1, \ldots, f_r$ is a *regular sequence* if for each $i = 1, \ldots, r$ the map

$$f_i : \mathcal{O}_X/(f_1, \ldots, f_{i-1}) \longrightarrow \mathcal{O}_X/(f_1, \ldots, f_{i-1}) \tag{26.12.0.2}$$

is an injective map of sheaves. We say that $f_1, \ldots, f_r$ is a *Koszul-regular sequence* if the Koszul complex

$$K_\bullet(\mathcal{O}_X, f_\bullet), \tag{26.12.0.3}$$

see Modules, Definition 15.20.2, is acyclic in degrees > 0. We say that $f_1, \ldots, f_r$ is a *H_1-regular sequence* if the Koszul complex $K_\bullet(\mathcal{O}_X, f_\bullet)$ is exact in degree 1. Finally, we say that $f_1, \ldots, f_r$ is a *quasi-regular* sequence if the map

$$\mathcal{O}_X/\mathcal{J}[T_1, \ldots, T_r] \longrightarrow \bigoplus\nolimits_{d \geq 0} \mathcal{J}^d/\mathcal{J}^{d+1} \tag{26.12.0.4}$$

is an isomorphism of sheaves where $\mathcal{J} \subset \mathcal{O}_X$ is the sheaf of ideals generated by $f_1, \ldots, f_r$. (There is also a notion of $\mathcal{F}$-regular and $\mathcal{F}$-quasi-regular sequence for a given $\mathcal{O}_X$-module $\mathcal{F}$ which we will introduce here if we ever need it.)

Lemma 26.12.1. *Let X be a ringed space. Let $f_1, \ldots, f_r \in \Gamma(X, \mathcal{O}_X)$. We have the following implications $f_1, \ldots, f_r$ is a regular sequence $\Rightarrow$ $f_1, \ldots, f_r$ is a Koszul-regular sequence $\Rightarrow$ $f_1, \ldots, f_r$ is an H_1-regular sequence $\Rightarrow$ $f_1, \ldots, f_r$ is a quasi-regular sequence.*

Proof. Since we may check exactness at stalks, a sequence $f_1, \ldots, f_r$ is a regular sequence if and only if the maps

$$f_i : \mathcal{O}_{X,x}/(f_1, \ldots, f_{i-1}) \longrightarrow \mathcal{O}_{X,x}/(f_1, \ldots, f_{i-1})$$

are injective for all $x \in X$. In other words, the image of the sequence $f_1, \ldots, f_r$ in the ring $\mathcal{O}_{X,x}$ is a regular sequence for all $x \in X$. The other types of regularity can be checked stalkwise as well (details omitted). Hence the implications follow from More on Algebra, Lemmas 12.22.2 and 12.22.5. $\square$

Definition 26.12.2. Let X be a ringed space. Let $\mathcal{J} \subset \mathcal{O}_X$ be a sheaf of ideals.

(1) We say $\mathcal{J}$ is *regular* if for every $x \in \text{Supp}(\mathcal{O}_X/\mathcal{J})$ there exists an open neighbourhood $x \in U \subset X$ and a regular sequence $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ such that $\mathcal{J}|_U$ is generated by $f_1, \ldots, f_r$.
(2) We say $\mathcal{J}$ is *Koszul-regular* if for every $x \in \text{Supp}(\mathcal{O}_X/\mathcal{J})$ there exists an open neighbourhood $x \in U \subset X$ and a Koszul-regular sequence $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ such that $\mathcal{J}|_U$ is generated by $f_1, \ldots, f_r$.
(3) We say $\mathcal{J}$ is *H_1-regular* if for every $x \in \text{Supp}(\mathcal{O}_X/\mathcal{J})$ there exists an open neighbourhood $x \in U \subset X$ and a H_1-regular sequence $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ such that $\mathcal{J}|_U$ is generated by $f_1, \ldots, f_r$.
(4) We say $\mathcal{J}$ is *quasi-regular* if for every $x \in \text{Supp}(\mathcal{O}_X/\mathcal{J})$ there exists an open neighbourhood $x \in U \subset X$ and a quasi-regular sequence $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ such that $\mathcal{J}|_U$ is generated by $f_1, \ldots, f_r$.

Many properties of this notion immediately follow from the corresponding notions for regular and quasi-regular sequences in rings.

Lemma 26.12.3. *Let X be a ringed space. Let $\mathcal{J}$ be a sheaf of ideals. We have the following implications: $\mathcal{J}$ is regular $\Rightarrow$ $\mathcal{J}$ is Koszul-regular $\Rightarrow$ $\mathcal{J}$ is H_1-regular $\Rightarrow$ $\mathcal{J}$ is quasi-regular.*

Proof. The lemma immediately reduces to Lemma 26.12.1. $\square$

Lemma 26.12.4. *Let X be a locally ringed space. Let $\mathcal{J} \subset \mathcal{O}_X$ be a sheaf of ideals. Then $\mathcal{J}$ is quasi-regular if and only if the following conditions are satisfied:*

(1) *$\mathcal{J}$ is an $\mathcal{O}_X$-module of finite type,*
(2) *$\mathcal{J}/\mathcal{J}^2$ is a finite locally free $\mathcal{O}_X/\mathcal{J}$-module, and*
(3) *the canonical maps*

$$Sym^n_{\mathcal{O}_X/\mathcal{J}}(\mathcal{J}/\mathcal{J}^2) \longrightarrow \mathcal{J}^n/\mathcal{J}^{n+1}$$

are isomorphisms for all $n \geq 0$.

Proof. It is clear that if $U \subset X$ is an open such that $\mathcal{J}|_U$ is generated by a quasi-regular sequence $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ then $\mathcal{J}|_U$ is of finite type, $\mathcal{J}|_U/\mathcal{J}^2|_U$ is free with basis $f_1, \ldots, f_r$, and the maps in (3) are isomorphisms because they are coordinate free formulation of the degree n part of (26.12.0.4). Hence it is clear that being quasi-regular implies conditions (1), (2), and (3).

Conversely, suppose that (1), (2), and (3) hold. Pick a point $x \in \text{Supp}(\mathcal{O}_X/\mathcal{J})$. Then there exists a neighbourhood $U \subset X$ of x such that $\mathcal{J}|_U/\mathcal{J}^2|_U$ is free of rank r over $\mathcal{O}_U/\mathcal{J}|_U$. After possibly shrinking U we may assume there exist $f_1, \ldots, f_r \in \mathcal{J}(U)$ which map to a basis of $\mathcal{J}|_U/\mathcal{J}^2|_U$ as an $\mathcal{O}_U/\mathcal{J}|_U$-module. In particular we see that the images of $f_1, \ldots, f_r$ in $\mathcal{J}_x/\mathcal{J}_x^2$ generate. Hence by Nakayama's lemma (Algebra, Lemma 7.14.5) we see that $f_1, \ldots, f_r$ generate the stalk $\mathcal{J}_x$. Hence, since $\mathcal{J}$ is of finite type, by Modules, Lemma

15.9.4 after shrinking U we may assume that $f_1, \ldots, f_r$ generate $\mathcal{J}$. Finally, from (3) and the isomorphism $\mathcal{J}|_U/\mathcal{J}^2|_U = \bigoplus \mathcal{O}_U/\mathcal{J}|_U f_i$ it is clear that $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ is a quasi-regular sequence. $\square$

Lemma 26.12.5. *Let $(X, \mathcal{O}_X)$ be a locally ringed space. Let $\mathcal{J} \subset \mathcal{O}_X$ be a sheaf of ideals. Let $x \in X$ and $f_1, \ldots, f_r \in \mathcal{J}_x$ whose images give a basis for the $\kappa(x)$-vector space $\mathcal{J}_x/\mathfrak{m}_x\mathcal{J}_x$.*

1. *If $\mathcal{J}$ is quasi-regular, then there exists an open neighbourhood such that $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ form a quasi-regular sequence generating $\mathcal{J}|_U$.*
2. *If $\mathcal{J}$ is H_1-regular, then there exists an open neighbourhood such that $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ form an H_1-regular sequence generating $\mathcal{J}|_U$.*
3. *If $\mathcal{J}$ is Koszul-regular, then there exists an open neighbourhood such that $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ form an Koszul-regular sequence generating $\mathcal{J}|_U$.*

Proof. First assume that $\mathcal{J}$ is quasi-regular. We may choose an open neighbourhood $U \subset X$ of x and a quasi-regular sequence $g_1, \ldots, g_s \in \mathcal{O}_X(U)$ which generates $\mathcal{J}|_U$. Note that this implies that $\mathcal{J}/\mathcal{J}^2$ is free of rank s over $\mathcal{O}_U/\mathcal{J}|_U$ (see Lemma 26.12.4 and its proof) and hence $r = s$. We may shrink U and assume $f_1, \ldots, f_r \in \mathcal{J}(U)$. Thus we may write

$$f_i = \sum a_{ij} g_j$$

for some $a_{ij} \in \mathcal{O}_X(U)$. By assumption the matrix $A = (a_{ij})$ maps to an invertible matrix over $\kappa(x)$. Hence, after shrinking U once more, we may assume that (a_{ij}) is invertible. Thus we see that $f_1, \ldots, f_r$ give a basis for $(\mathcal{J}/\mathcal{J}^2)|_U$ which proves that $f_1, \ldots, f_r$ is a quasi-regular sequence over U.

Note that in order to prove (2) and (3) we may, because the assumptions of (2) and (3) are stronger than the assumption in (1), already assume that $f_1, \ldots, f_r \in \mathcal{J}(U)$ and $f_i = \sum a_{ij} g_j$ with (a_{ij}) invertible as above, where now $g_1, \ldots, g_r$ is a H_1-regular or Koszul-regular sequence. Since the Koszul complex on $f_1, \ldots, f_r$ is isomorphic to the Koszul complex on $g_1, \ldots, g_r$ via the matrix (a_{ij}) (see More on Algebra, Lemma 12.21.4) we conclude that $f_1, \ldots, f_r$ is H_1-regular or Koszul-regular as desired. $\square$

Lemma 26.12.6. *Any regular, Koszul-regular, H_1-regular, or quasi-regular sheaf of ideals on a scheme is a finite type quasi-coherent sheaf of ideals.*

Proof. This follows as such a sheaf of ideals is locally generated by finitely many sections. And any sheaf of ideals locally generated by sections on a scheme is quasi-coherent, see Schemes, Lemma 21.10.1. $\square$

Lemma 26.12.7. *Let X be a scheme. Let $\mathcal{J}$ be a sheaf of ideals. Then $\mathcal{J}$ is regular (resp. Koszul-regular, H_1-regular, quasi-regular) if and only if for every $x \in Supp(\mathcal{O}_X/\mathcal{J})$ there exists an affine open neighbourhood $x \in U \subset X$, $U = Spec(A)$ such that $\mathcal{J}|_U = \tilde{I}$ and such that I is generated by a regular (resp. Koszul-regular, H_1-regular, quasi-regular) sequence $f_1, \ldots, f_r \in A$.*

Proof. By assumption we can find an open neighbourhood U of x over which $\mathcal{J}$ is generated by a regular (resp. Koszul-regular, H_1-regular, quasi-regular) sequence $f_1, \ldots, f_r \in \mathcal{O}_X(U)$. After shrinking U we may assume that U is affine, say $U = Spec(A)$. Since $\mathcal{J}$ is quasi-coherent by Lemma 26.12.6 we see that $\mathcal{J}|_U = \tilde{I}$ for some ideal $I \subset A$. Now we can use the fact that

$$\tilde{\ } : \text{Mod}_A \longrightarrow QCoh(U)$$

is an equivalence of categories which preserves exactness. For example the fact that the functions f_i generate $\mathcal{J}$ means that the f_i, seen as elements of A generate I. The fact that (26.12.0.2) is injective (resp. (26.12.0.3) is exact, (26.12.0.3) is exact in degree 1, (26.12.0.4) is an isomorphism) implies the correponding property of the map $A/(f_1, \ldots, f_{i-1}) \to A/(f_1, \ldots, f_{i-1})$ (resp. the complex $K_\bullet(A, f_1, \ldots, f_r)$, the map $A/I[T_1, \ldots, T_r] \to \bigoplus I^n/I^{n+1}$). Thus $f_1, \ldots, f_r \in A$ is a regular (resp. Koszul-regular, H_1-regular, quasi-regular) sequence of the ring A. □

Lemma 26.12.8. *Let X be a locally Noetherian scheme. Let $\mathcal{J} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let x be a point of the support of $\mathcal{O}_X/\mathcal{J}$. The following are equivalent*

(1) *$\mathcal{J}_x$ is generated by a regular sequence in $\mathcal{O}_{X,x}$,*
(2) *$\mathcal{J}_x$ is generated by a Koszul-regular sequence in $\mathcal{O}_{X,x}$,*
(3) *$\mathcal{J}_x$ is generated by an H_1-regular sequence in $\mathcal{O}_{X,x}$,*
(4) *$\mathcal{J}_x$ is generated by a quasi-regular sequence in $\mathcal{O}_{X,x}$,*
(5) *there exists an affine neighbourhood $U = Spec(A)$ of x such that $\mathcal{J}|_U = \tilde{I}$ and I is generated by a regular sequence in A, and*
(6) *there exists an affine neighbourhood $U = Spec(A)$ of x such that $\mathcal{J}|_U = \tilde{I}$ and I is generated by a Koszul-regular sequence in A, and*
(7) *there exists an affine neighbourhood $U = Spec(A)$ of x such that $\mathcal{J}|_U = \tilde{I}$ and I is generated by an H_1-regular sequence in A, and*
(8) *there exists an affine neighbourhood $U = Spec(A)$ of x such that $\mathcal{J}|_U = \tilde{I}$ and I is generated by a quasi-regular sequence in A,*
(9) *there exists a neighbourhood U of x such that $\mathcal{J}|_U$ is regular, and*
(10) *there exists a neighbourhood U of x such that $\mathcal{J}|_U$ is Koszul-regular, and*
(11) *there exists a neighbourhood U of x such that $\mathcal{J}|_U$ is H_1-regular, and*
(12) *there exists a neighbourhood U of x such that $\mathcal{J}|_U$ is quasi-regular.*

In particular, on a locally Noetherian scheme the notions of regular, Koszul-regular, H_1-regular, or quasi-regular ideal sheaf all agree.

Proof. It follows from Lemma 26.12.7 that (5) ⇔ (9), (6) ⇔ (10), (7) ⇔ (11), and (8) ⇔ (12). It is clear that (5) ⇒ (1), (6) ⇒ (2), (7) ⇒ (3), and (8) ⇒ (4). We have (1) ⇒ (5) by Algebra, Lemma 7.65.8. We have (9) ⇒ (10) ⇒ (11) ⇒ (12) by Lemma 26.12.3. Finally, (4) ⇒ (1) by Algebra, Lemma 7.66.6. Now all 12 statements are equivalent. □

26.13. Regular immersions

Let $i : Z \to X$ be an immersion of schemes. By definition this means there exists an open subscheme $U \subset X$ such that Z is identified with a closed subscheme of U. Let $\mathcal{I} \subset \mathcal{O}_U$ be the corresponding quasi-coherent sheaf of ideals. Suppose $U' \subset X$ is a second such open subscheme, and denote $\mathcal{I}' \subset \mathcal{O}_{U'}$ the corresponding quasi-coherent sheaf of ideals. Then $\mathcal{I}|_{U \cap U'} = \mathcal{I}'|_{U \cap U'}$. Moreover, the support of $\mathcal{O}_U/\mathcal{I}$ is Z which is contained in $U \cap U'$ and is also the support of $\mathcal{O}_{U'}/\mathcal{I}'$. Hence it follows from Definition 26.12.2 that $\mathcal{I}$ is a regular ideal if and only if $\mathcal{I}'$ is a regular ideal. Similarly for being Koszul-regular, H_1-regular, or quasi-regular.

Definition 26.13.1. Let $i : Z \to X$ be an immersion of schemes. Choose an open subscheme $U \subset X$ such that i identifies Z with a closed subscheme of U and denote $\mathcal{I} \subset \mathcal{O}_U$ the corresponding quasi-coherent sheaf of ideals.

(1) We say i is a *regular immersion* if $\mathcal{I}$ is regular.
(2) We say i is a *Koszul-regular immersion* if $\mathcal{I}$ is Koszul-regular.

(3) We say i is a *H_1-regular immersion* if $\mathcal{I}$ is H_1-regular.
(4) We say i is a *quasi-regular immersion* if $\mathcal{I}$ is quasi-regular.

The discussion above shows that this is independent of the choice of U. The conditions are listed in decreasing order of strength, see Lemma 26.13.2. A Koszul-regular closed immersion is smooth locally a regular immersion, see Lemma 26.13.11. In the locally Noetherian case all four notions agree, see Lemma 26.12.8.

Lemma 26.13.2. *Let $i : Z \to X$ be an immersion of schemes. We have the following implications: i is regular $\Rightarrow$ i is Koszul-regular $\Rightarrow$ i is H_1-regular $\Rightarrow$ i is quasi-regular.*

Proof. The lemma immediately reduces to Lemma 26.12.3. ☐

Lemma 26.13.3. *Let $i : Z \to X$ be an immersion of schemes. Assume X is locally Noetherian. Then i is regular $\Leftrightarrow$ i is Koszul-regular $\Leftrightarrow$ i is H_1-regular $\Leftrightarrow$ i is quasi-regular.*

Proof. Follows immediately from Lemma 26.13.2 and Lemma 26.12.8. ☐

Lemma 26.13.4. *Let $i : Z \to X$ be a regular (resp. Koszul-regular, H_1-regular, quasi-regular) immersion. Let $X' \to X$ be a flat morphism. Then the base change $i' : Z \times_X X' \to X'$ is a regular (resp. Koszul-regular, H_1-regular, quasi-regular) immersion.*

Proof. Via Lemma 26.12.7 this translates into the algebraic statements in Algebra, Lemmas 7.65.7 and 7.66.3 and More on Algebra, Lemma 12.22.4. ☐

Lemma 26.13.5. *Let $i : Z \to X$ be an immersion of schemes. Then i is a quasi-regular immersion if and only if the following conditions are satisfied*

(1) *i is locally of finite presentation,*
(2) *the conormal sheaf $\mathcal{C}_{Z/X}$ is finite locally free, and*
(3) *the map (26.11.1.2) is an isomorphism.*

Proof. An open immersion is locally of finite presentation. Hence we may replace X by an open subscheme $U \subset X$ such that i identifies Z with a closed subscheme of U, i.e., we may assume that i is a closed immersion. Let $\mathcal{I} \subset \mathcal{O}_X$ be the corresponding quasi-coherent sheaf of ideals. Recall, see Morphisms, Lemma 24.20.7 that $\mathcal{I}$ is of finite type if and only if i is locally of finite presentation. Hence the equivalence follows from Lemma 26.12.4 and unwinding the definitions. ☐

Lemma 26.13.6. *Let $Z \to Y \to X$ be immersions of schemes. Assume that $Z \to Y$ is H_1-regular. Then the canonical sequence of Morphisms, Lemma 24.31.5*

$$0 \to i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

is exact and locally split.

Proof. Since $\mathcal{C}_{Z/Y}$ is finite locally free (see Lemma 26.13.5 and Lemma 26.12.3) it suffices to prove that the sequence is exact. By what was proven in Morphisms, Lemma 24.31.5 it suffices to show that the first map is injective. Working affine locally this reduces to the following question: Suppose that we have a ring A and ideals $I \subset J \subset A$. Assume that $J/I \subset A/I$ is generated by an H_1-regular sequence. Does this imply that $I/I^2 \otimes_A A/J \to J/J^2$ is injective? Note that $I/I^2 \otimes_A A/J = I/IJ$. Hence we are trying to prove that $I \cap J^2 = IJ$. This is the result of More on Algebra, Lemma 12.22.7. ☐

A composition of quasi-regular immersions may not be quasi-regular, see Algebra, Remark 7.66.8. The other types of regular immersions are preserved under composition.

Lemma 26.13.7. *Let $i : Z \to Y$ and $j : Y \to X$ be immersions of schemes.*

(1) *If i and j are regular immersions, so is $j \circ i$.*
(2) *If i and j are Koszul-regular immersions, so is $j \circ i$.*
(3) *If i and j are H_1-regular immersions, so is $j \circ i$.*
(4) *If i is an H_1-regular immersion and j is a quasi-regular immersion, then $j \circ i$ is a quasi-regular immersion.*

Proof. The algebraic version of (1) is Algebra, Lemma 7.65.9. The algebraic version of (2) is More on Algebra, Lemma 12.22.11. The algebraic version of (3) is More on Algebra, Lemma 12.22.9. The algebraic version of (4) is More on Algebra, Lemma 12.22.8. □

Lemma 26.13.8. *Let $i : Z \to Y$ and $j : Y \to X$ be immersions of schemes. Assume that the sequence*

$$0 \to i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

of Morphisms, Lemma 24.31.5 is exact and locally split.

(1) *If $j \circ i$ is a quasi-regular immersion, so is i.*
(2) *If $j \circ i$ is a H_1-regular immersion, so is i.*
(3) *If both j and $j \circ i$ are Koszul-regular immersions, so is i.*

Proof. After shrinking Y and X we may assume that i and j are closed immersions. Denote $\mathcal{I} \subset \mathcal{O}_X$ the ideal sheaf of Y and $\mathcal{J} \subset \mathcal{O}_X$ the ideal sheaf of Z. The conormal sequence is $0 \to \mathcal{I}/\mathcal{I}\mathcal{J} \to \mathcal{J}/\mathcal{J}^2 \to \mathcal{J}/(\mathcal{I} + \mathcal{J}^2) \to 0$. Let $z \in Z$ and set $y = i(z)$, $x = j(y) = j(i(z))$. Choose $f_1, \ldots, f_n \in \mathcal{I}_x$ which map to a basis of $\mathcal{I}_x/\mathfrak{m}_z\mathcal{I}_x$. Extend this to $f_1, \ldots, f_n, g_1, \ldots, g_m \in \mathcal{J}_x$ which map to a basis of $\mathcal{J}_x/\mathfrak{m}_z\mathcal{J}_x$. This is possible as we have assumed that the sequence of conormal sheaves is split in a neighbourhood of z, hence $\mathcal{I}_x/\mathfrak{m}_x\mathcal{I}_x \to \mathcal{J}_x/\mathfrak{m}_x\mathcal{J}_x$ is injective.

Proof of (1). By Lemma 26.12.5 we can find an affine open neighbourhood U of x such that $f_1, \ldots, f_n, g_1, \ldots, g_m$ forms a quasi-regular sequence generating $\mathcal{J}$. Hence by Algebra, Lemma 7.66.5 we see that $g_1, \ldots, g_m$ induces a quasi-regular sequence on $Y \cap U$ cutting out Z.

Proof of (2). Exactly the same as the proof of (1) except using More on Algebra, Lemma 12.22.10.

Proof of (3). By Lemma 26.12.5 (applied twice) we can find an affine open neighbourhood U of x such that $f_1, \ldots, f_n$ forms a Koszul-regular sequence generating $\mathcal{I}$ and $f_1, \ldots, f_n, g_1, \ldots, g_m$ forms a Koszul-regular sequence generating $\mathcal{J}$. Hence by More on Algebra, Lemma 12.22.12 we see that $g_1, \ldots, g_m$ induces a Koszul-regular sequence on $Y \cap U$ cutting out Z. □

Lemma 26.13.9. *Let $i : Z \to Y$ and $j : Y \to X$ be immersions of schemes. Pick $z \in Z$ and denote $y \in Y$, $x \in X$ the corresponding points. Assume X is locally Noetherian. The following are equivalent*

(1) *i is a regular immersion in a neighbourhood of z and j is a regular immersion in a neighbourhood of y,*
(2) *i and $j \circ i$ are regular immersions in a neighbourhood of z,*
(3) *$j \circ i$ is a regular immersion in a neighbourhood of z and the conormal sequence*

$$0 \to i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

is split exact in a neighbourhood of z.

Proof. Since X (and hence Y) is locally Noetherian all 4 types of regular immersions agree, and moreover we may check whether a morphism is a regular immersion on the level of local rings, see Lemma 26.12.8. The implication (1) ⇒ (2) is Lemma 26.13.7. The implication (2) ⇒ (3) is Lemma 26.13.6. Thus it suffices to prove that (3) implies (1).

Assume (3). Set $A = \mathcal{O}_{X,x}$. Denote $I \subset A$ the kernel of the surjective map $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ and denote $J \subset A$ the kernel of the surjective map $\mathcal{O}_{X,x} \to \mathcal{O}_{Z,z}$. Note that any mimimal sequence of elements generating J in A is a quasi-regular hence regular sequence, see Lemma 26.12.5. By assumption the conormal sequence

$$0 \to I/IJ \to J/J^2 \to J/(I + J^2 \to 0$$

is split exact as a sequence of A/J-modules. Hence we can pick a minimal system of generators $f_1, \ldots, f_n, g_1, \ldots, g_m$ of J with $f_1, \ldots, f_n \in I$ a minimal system of generators of I. As pointed out above $f_1, \ldots, f_n, g_1, \ldots, g_m$ is a regular sequence in A. It follows directly from the definition of a regular sequence that $f_1, \ldots, f_n$ is a regular sequence in A and $\overline{g}_1, \ldots, \overline{g}_m$ is a regular sequence in A/I. Thus j is a regular immersion at y and i is a regular immersion at z. □

Remark 26.13.10. In the situation of Lemma 26.13.9 parts (1), (2), (3) are **not** equivalent to ``$j \circ i$ and j are regular immersions at z and y''. An example is $X = \mathbf{A}^1_k = Spec(k[x])$, $Y = Spec(k[x]/(x^2))$ and $Z = Spec(k[x]/(x))$.

Lemma 26.13.11. *Let $i : Z \to X$ be a Koszul regular closed immersion. Then there exists a surjective smooth morphism $X' \to X$ such that the base change $i' : Z \times_X X' \to X'$ of i is a regular immersion.*

Proof. We may assume that X is affine and the ideal of Z generated by a Koszul-regular sequence by replacing X by the members of a suitable affine open covering (affine opens as in Lemma 26.12.7). The affine case is More on Algebra, Lemma 12.22.16. □

26.14. Relative regular immersions

In this section we consider the base change property for regular immersions. The following lemma does not hold for regular immersions or for Koszul immersions, see Examples, Lemma 64.6.2.

Lemma 26.14.1. *Let $f : X \to S$ be a morphism of schemes. Let $i : Z \subset X$ be an immersion. Assume*

(1) *i is an H_1-regular (resp. quasi-regular) immersion, and*
(2) *$Z \to S$ is a flat morphism.*

Then for every morphism of schemes $g : S' \to S$ the base change $Z' = S' \times_S Z \to X' = S' \times_S X$ is an H_1-regular (resp. quasi-regular) immersion.

Proof. Unwinding the definitions and using Lemma 26.12.7 we translate this into algebra as follows. Let $A \to B$ be a ring map and $f_1, \ldots, f_r \in B$. Assume $B/(f_1, \ldots, f_r)B$ is flat over A. Consider a ring map $A \to A'$. Set $B' = B \otimes_A A'$ and $J' = JB'$.

Case I: $f_1, \ldots, f_r$ is quasi-regular. Set $J = (f_1, \ldots, f_r)$. By assumption J^n/J^{n+1} is isomorphic to a direct sum of copies of B/J hence flat over A. By induction and Algebra, Lemma 7.35.12 we conclude that B/J^n is flat over A. The ideal $(J')^n$ is equal to $J^n \otimes_A A'$, see Algebra, Lemma 7.35.11. Hence $(J')^n/(J')^{n+1} = J^n/J^{n+1} \otimes_A A'$ which clearly implies that $f_1, \ldots, f_r$ is a quasi-regular sequence in B'.

Case II: $f_1, \ldots, f_r$ is H_1-regular. By More on Algebra, Lemma 12.22.14 the vanishing of the Koszul homology group $H_1(K_\bullet(B, f_1, \ldots, f_r))$ implies the vanshing of $H_1(K_\bullet(B', f'_1, \ldots, f'_r))$ and we win. □

This lemma is the motivation for the following definition.

Definition 26.14.2. Let $f : X \to S$ be a morphism of schemes. Let $i : Z \to X$ be an immersion.

(1) We say i is a *relative quasi-regular immersion* if $Z \to S$ is flat and i is a quasi-regular immersion.
(2) We say i is a *relative H_1-regular immersion* if $Z \to S$ is flat and i is an H_1-regular immersion.

We warn the reader that this may be nonstandard notation. Lemma 26.14.1 guarantees that relative quasi-regular (resp. H_1-regular) immersions are preserved under any base change. A relative H_1-regular immersion is a relative quasi-regular immersion, see Lemma 26.13.2. Please take a look at Lemma 26.14.5 (or Lemma 26.14.4) which shows that if $Z \to X$ is a relative H_1-regular (or quasi-regular) immersion and the ambient scheme is (flat and) locally of finite presentation over S, then $Z \to X$ is actually a regular immersion and the same remains true after any base change.

Lemma 26.14.3. *Let $f : X \to S$ be a morphism of schemes. Let $Z \to X$ be a relative quasi-regular immersion. If $x \in Z$ and $\mathcal{O}_{X,x}$ is Noetherian, then f is flat at x.*

Proof. Let $f_1, \ldots, f_r \in \mathcal{O}_{X,x}$ be a quasi-regular sequence cutting out the ideal of Z at x. By Algebra, Lemma 7.66.6 we know that $f_1, \ldots, f_r$ is a regular sequence. Hence f_r is a nonzero divisor on $\mathcal{O}_{X,x}/(f_1, \ldots, f_{r-1})$ such that the quotient is a flat $\mathcal{O}_{S,f(x)}$-module. By Lemma 26.10.3 we conclude that $\mathcal{O}_{X,x}/(f_1, \ldots, f_{r-1})$ is a flat $\mathcal{O}_{S,f(x)}$-module. Continuing by induction we find that $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{S,s}$-module. □

Lemma 26.14.4. *Let $X \to S$ be a morphism of schemes. Let $Z \to X$ be an immersion. Assume*

(1) *$X \to S$ is flat and locally of finite presentation,*
(2) *$Z \to X$ is a relative quasi-regular immersion.*

Then $Z \to X$ is a regular immersion and the same remains true after any base change.

Proof. Pick $x \in Z$ with image $s \in S$. To prove this it suffices to find an affine neighbourhood of x contained in U such that the result holds on that affine open. Hence we may assume that X is affine and there exist a quasi-regular sequence $f_1, \ldots, f_r \in \Gamma(X, \mathcal{O}_X)$ such that $Z = V(f_1, \ldots, f_r)$. By Lemma 26.14.1 and its proof the sequence $f_1|_{X_s}, \ldots, f_r|_{X_s}$ is a quasi-regular sequence in $\Gamma(X_s, \mathcal{O}_{X_s})$. Since X_s is Noetherian, this implies, possibly after shrinking X a bit, that $f_1|_{X_s}, \ldots, f_r|_{X_s}$ is a regular sequence, see Algebra, Lemmas 7.66.6 and 7.65.8. By Lemma 26.10.7 it follows that $Z_1 = V(f_1) \subset X$ is a relative effective Cartier divisor, again after possibly shrinking X a bit. Applying the same lemma again, but now to $Z_2 = V(f_1, f_2) \subset Z_1$ we see that $Z_2 \subset Z_1$ is a relative effective Cartier divisor. And so on until on reaches $Z = Z_n = V(f_1, \ldots, f_n)$. Since being a relative effective Cartier divisor is preserved under arbitrary base change, see Lemma 26.10.1, we also see that the final statement of the lemma holds. □

Lemma 26.14.5. *Let $X \to S$ be a morphism of schemes. Let $Z \to X$ be a relative H_1-regular immersion. Assume $X \to S$ is locally of finite presentation. Then*

(1) *there exists an open subscheme $U \subset X$ such that $Z \subset U$ and such that $U \to S$ is flat, and*
(2) *$Z \to X$ is a regular immersion and the same remains true after any base change.*

Proof. Pick $x \in Z$. To prove (1) suffices to find an open neighbourhood $U \subset X$ of x such that $U \to S$ is flat. Hence the lemma reduces to the case that $X = Spec(B)$ and $S = Spec(A)$ are affine and that Z is given by an H_1-regular sequence $f_1, \ldots, f_r \in B$. By assumption B is a finitely presented A-algebra and $B/(f_1, \ldots, f_r)B$ is a flat A-algebra. We are going to use absolute Noetherian approximation.

Write $B = A[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$. Assume f_i is the image of $f_i' \in A[x_1, \ldots, x_n]$. Choose a finite type $\mathbf{Z}$-subalgebra $A_0 \subset A$ such that all the coefficients of the polynomials $f_1', \ldots, f_r', g_1, \ldots, g_m$ are in A_0. We set $B_0 = A_0[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$ and we denote $f_{i,0}$ the image of f_i' in B_0. Then $B = B_0 \otimes_{A_0} A$ and

$$B/(f_1, \ldots, f_r) = B_0/(f_{0,1}, \ldots, f_{0,r}) \otimes_{A_0} A.$$

By Algebra, Lemma 7.120.5 we may, after enlarging A_0, assume that $B_0/(f_{0,1}, \ldots, f_{0,r})$ is flat over A_0. It may not be the case at this point that the Koszul cohomology group $H_1(K_\bullet(B_0, f_{0,1}, \ldots, f_{0,r}))$ is zero. On the other hand, as B_0 is Noetherian, it is a finitely generated B_0-module. Let $\xi_1, \ldots, \xi_n \in H_1(K_\bullet(B_0, f_{0,1}, \ldots, f_{0,r}))$ be generators. Let $A_0 \subset A_1 \subset A$ be a larger finite type $\mathbf{Z}$-subalgebra of A. Denote $f_{1,i}$ the image of $f_{0,i}$ in $B_1 = B_0 \otimes_{A_0} A_1$. By More on Algebra, Lemma 12.22.14 the map

$$H_1(K_\bullet(B_0, f_{0,1}, \ldots, f_{0,r})) \otimes_{A_0} A_1 \longrightarrow H_1(K_\bullet(B_1, f_{1,1}, \ldots, f_{1,r}))$$

is surjective. Furthermore, it is clear that the colimit (over all choices of A_1 as above) of the complexes $K_\bullet(B_1, f_{1,1}, \ldots, f_{1,r})$ is the complex $K_\bullet(B, f_1, \ldots, f_r)$ which is acyclic in degree 1. Hence

$$colim_{A_0 \subset A_1 \subset A} H_1(K_\bullet(B_1, f_{1,1}, \ldots, f_{1,r})) = 0$$

by Algebra, Lemma 7.8.9. Thus we can find a choice of A_1 such that $\xi_1, \ldots, \xi_n$ all map to zero in $H_1(K_\bullet(B_1, f_{1,1}, \ldots, f_{1,r}))$. In other words, the Koszul cohomology group $H_1(K_\bullet(B_1, f_{1,1}, \ldots, f_{1,r}))$ is zero.

Consider the morphism of affine schemes $X_1 \to S_1$ equal to $Spec$ of the ring map $A_1 \to B_1$ and $Z_1 = Spec(B_1/(f_{1,1}, \ldots, f_{1,r}))$. Since $B = B_1 \otimes_{A_1} A$, i.e., $X = X_1 \times_{S_1} S$, and similarly $Z = Z_1 \times_S S_1$, it now suffices to prove (1) for $X_1 \to S_1$ and the relative H_1-regular immersion $Z_1 \to X_1$, see Morphisms, Lemma 24.24.6. Hence we have reduced to the case where $X \to S$ is a finite type morphism of Noetherian schemes. In this case we know that $X \to S$ is flat at every point of Z by Lemma 26.14.3. Combined with the fact that the flat locus is open in this case, see Algebra, Theorem 7.120.4 we see that (1) holds. Part (2) then follows from an application of Lemma 26.14.4. □

If the ambient scheme is flat and locally of finite presentation over the base, then we can characterize a relative quasi-regular immersion in terms of its fibres.

Lemma 26.14.6. *Let $\varphi : X \to S$ be a flat morphism which is locally of finite presentation. Let $T \subset X$ be a closed subscheme. Let $x \in T$ with image $s \in S$.*

(1) *If $T_s \subset X_s$ is a quasi-regular immersion in a neighbourhood of x, then there exists an open $U \subset X$ and a relative quasi-regular immersion $Z \subset U$ such that $Z_s = T_s \cap U_s$ and $T \cap U \subset Z$.*
(2) *If $T_s \subset X_s$ is a quasi-regular immersion in a neighbourhood of x, the morphism $T \to X$ is of finite presentation, and $T \to S$ is flat at x, then we can choose U and Z as in (1) such that $T \cap U = Z$.*

(3) *If $T_s \subset X_s$ is a quasi-regular immersion in a neighbourhood of x, and T is cut out by c equations in a neighbourhood of x, where $c = \dim_x(X_s) - \dim_x(T_s)$, then we can choose U and Z as in (1) such that $T \cap U = Z$.*

In each case $Z \to U$ is a regular immersion by Lemma 26.14.4. In particular, if $T \to S$ is locally of finite presentation and flat and all fibres $T_s \subset X_s$ are quasi-regular immersions, then $T \to X$ is a relative quasi-regular immersion.

Proof. Choose affine open neighbourhoods $Spec(A)$ of s and $Spec(B)$ of x such that $\varphi(Spec(B)) \subset Spec(A)$. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to s. Let $\mathfrak{q} \subset B$ be the prime ideal corresponding to x. Let $I \subset B$ be the ideal corresponding to T. By the initial assumption of the lemma we know that $A \to B$ is flat and of finite presentation. The assumption in (1) means that, after shrinking $Spec(B)$, we may assume $I(B \otimes_A \kappa(\mathfrak{p}))$ is generated by a quasi-regular sequence of elements. After possibly localizing B at some $g \in B$, $g \not\in \mathfrak{q}$ we may assume there exist $f_1, \ldots, f_r \in I$ which map to a quasi-regular sequence in $B \otimes_A \kappa(\mathfrak{p})$ which generates $I(B \otimes_A \kappa(\mathfrak{p}))$. By Algebra, Lemmas 7.66.6 and 7.65.8 we may assume after another localization that $f_1, \ldots, f_r \in I$ form a regular sequence in $B \otimes_A \kappa(\mathfrak{p})$. By Lemma 26.10.7 it follows that $Z_1 = V(f_1) \subset Spec(B)$ is a relative effective Cartier divisor, again after possibly localizing B. Applying the same lemma again, but now to $Z_2 = V(f_1, f_2) \subset Z_1$ we see that $Z_2 \subset Z_1$ is a relative effective Cartier divisor. And so on until one reaches $Z = Z_n = V(f_1, \ldots, f_n)$. Then $Z \to Spec(B)$ is a regular immersion and Z is flat over S, in particular $Z \to Spec(B)$ is a relative quasi-regular immersion over $Spec(A)$. This proves (1).

To see (2) consider the closed immersion $Z \to D$. The surjective ring map $u : \mathcal{O}_{D,x} \to \mathcal{O}_{Z,x}$ is a map of flat local $\mathcal{O}_{S,s}$-algebras which are essentially of finite presentation, and which becomes an isomorphisms after dividing by $\mathfrak{m}_s$. Hence it is an isomorphism, see Algebra, Lemma 7.119.4. It follows that $Z \to D$ is an isomorphism in a neighbourhood of x, see Algebra, Lemma 7.117.6.

To see (3), after possibly shrinking U we may assume that the ideal of Z is generated by a regular sequence $f_1, \ldots, f_r$ (see our construction of Z above) and the ideal of T is generated by $g_1, \ldots, g_c$. We claim that $c = r$. Namely,

$$\begin{aligned} \dim_x(X_s) &= \dim(\mathcal{O}_{X_s,x}) + \operatorname{trdeg}_{\kappa(s)}(\kappa(x)), \\ \dim_x(T_s) &= \dim(\mathcal{O}_{T_s,x}) + \operatorname{trdeg}_{\kappa(s)}(\kappa(x)), \\ \dim(\mathcal{O}_{X_s,x}) &= \dim(\mathcal{O}_{T_s,x}) + r \end{aligned}$$

the first two equalities by Algebra, Lemma 7.107.3 and the second by r times applying Algebra, Lemma 7.57.11. As $T \subset Z$ we see that $f_i = \sum b_{ij} g_j$. But the ideals of Z and T cut out the same quasi-regular closed subscheme of X_s in a neighbourhood of x. Hence the matrix (b_{ij}) mod $\mathfrak{m}_x$ is invertible (some details omitted). Hence (b_{ij}) is invertible in an open neighbourhood of x. In other words, $T \cap U = Z$ after shrinking U.

The final statements of the lemma follow immediately from part (2), combined with the fact that $Z \to S$ is locally of finite presentation if and only if $Z \to X$ is of finite presentation, see Morphisms, Lemmas 24.20.3 and 24.20.11. ☐

The following lemma is an enhancement of Morphisms, Lemma 24.33.20.

Lemma 26.14.7. *Let $f : X \to S$ be a smooth morphism of schemes. Let $\sigma : S \to X$ be a section of f. Then σ is a regular immersion.*

Proof. By Schemes, Lemma 21.21.11 the morphism σ is an immersion. After replacing X by an open neighbourhood of $\sigma(S)$ we may assume that σ is a closed immersion. Let $T = \sigma(S)$ be the corresponding closed subscheme of X. Since $T \to S$ is an isomorphism it is flat and of finite presentation. Also a smooth morphism is flat and locally of finite presentation, see Morphisms, Lemmas 24.33.9 and 24.33.8. Thus, according to Lemma 26.14.6, it suffices to show that $T_s \subset X_s$ is a quasi-regular closed subscheme. This follows immediately from Morphisms, Lemma 24.33.20 but we can also see it directly as follows. Let k be a field and let A be a smooth k-algebra. Let $\mathfrak m \subset A$ be a maximal ideal whose residue field is k. Then $\mathfrak m$ is generated by a quasi-regular sequence, possibly after replacing A by A_g for some $g \in A$, $g \notin \mathfrak m$. In Algebra, Lemma 7.129.3 we proved that $A_{\mathfrak m}$ is a regular local ring, hence $\mathfrak m A_{\mathfrak m}$ is generated by a regular sequence. This does indeed imply that $\mathfrak m$ is generated by a regular sequence (after replacing A by A_g for some $g \in A$, $g \notin \mathfrak m$), see Algebra, Lemma 7.65.8. □

The following lemma has a kind of converse, see Lemma 26.14.11.

Lemma 26.14.8. *Let*

$$\begin{tikzcd} Y \arrow[rr, "i"] \arrow[dr, "j"'] & & X \arrow[dl] \\ & S & \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume $X \to S$ smooth, and i, j immersions. If j is a regular (resp. Koszul-regular, H_1-regular, quasi-regular) immersion, then so is i.

Proof. We can write i as the composition

$$Y \to Y \times_S X \to X$$

By Lemma 26.14.7 the first arrow is a regular immersion. The second arrow is a flat base change of $Y \to S$, hence is a regular (resp. Koszul-regular, H_1-regular, quasi-regular) immersion, see Lemma 26.13.4. We conclude by an application of Lemma 26.13.7. □

Lemma 26.14.9. *Let*

$$\begin{tikzcd} Y \arrow[rr, "i"] \arrow[dr] & & X \arrow[dl] \\ & S & \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume that $Y \to S$ is syntomic, $X \to S$ smooth, and i an immersion. Then i is a regular immersion.

Proof. After replacing X by an open neighbourhood of $i(Y)$ we may assume that i is a closed immersion. Let $T = i(Y)$ be the corresponding closed subscheme of X. Since $T \cong Y$ the morphism $T \to S$ is flat and of finite presentation (Morphisms, Lemmas 24.30.6 and 24.30.7). Also a smooth morphism is flat and locally of finite presentation (Morphisms, Lemmas 24.33.9 and 24.33.8). Thus, according to Lemma 26.14.6, it suffices to show that $T_s \subset X_s$ is a quasi-regular closed subscheme. As X_s is locally of finite type over a field, it is Noetherian (Morphisms, Lemma 24.14.6). Thus we can check that $T_s \subset X_s$ is a quasi-regular immersion at points, see Lemma 26.12.8. Take $t \in T_s$. By Morphisms, Lemma 24.30.9 the local ring $\mathcal{O}_{T_s,t}$ is a local complete intersection over $\kappa(s)$. The local ring $\mathcal{O}_{X_s,t}$ is regular, see Algebra, Lemma 7.129.3. By Algebra, Lemma 7.124.7 we see that the kernel of the surjection $\mathcal{O}_{X_s,t} \to \mathcal{O}_{T_s,t}$ is generated by a regular sequence, which is what we had to show. $\square$

Lemma 26.14.10. *Let*

$$\begin{tikzcd} Y \arrow[rr, "i"] \arrow[dr] & & X \arrow[dl] \\ & S & \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume that $Y \to S$ is smooth, $X \to S$ smooth, and i an immersion. Then i is a regular immersion.

Proof. This is a special case of Lemma 26.14.9 because a smooth morphism is syntomic, see Morphisms, Lemma 24.33.7. $\square$

Lemma 26.14.11. *Let*

$$\begin{tikzcd} Y \arrow[rr, "i"] \arrow[dr, "j"'] & & X \arrow[dl] \\ & S & \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume $X \to S$ smooth, and i, j immersions. If i is a Koszul-regular (resp. H_1-regular, quasi-regular) immersion, then so is j.

Proof. Let $y \in Y$ be any point. Set $x = i(y)$ and set $s = j(y)$. It suffices to prove the result after replacing X, S by open neighbourhoods U, V of x, s and Y by an open neighbourhood of y in $i^{-1}(U) \cap j^{-1}(V)$. Hence we may assume that Y, X and S are affine. In this case we can choose a closed immersion $h : X \to \mathbf{A}^n_S$ over S for some n. Note that h is a regular immersion by Lemma 26.14.10. Hence $h \circ i$ is a Koszul-regular (resp. H_1-regular, quasi-regular) immersion, see Lemmas 26.13.7 and 26.13.2. In this way we reduce to the case $X = \mathbf{A}^n_S$ and S affine.

After replacing S by an affine open V and replacing Y by $j^{-1}(V)$ we may assume that i is a closed immersion and S affine. Write $S = Spec(A)$. Then $j : Y \to S$ defines an isomorphism of Y to the closed subscheme $Spec(A/I)$ for some ideal $I \subset A$. The map $i : Y = Spec(A/I) \to \mathbf{A}^n_S = Spec(A[x_1, \ldots, x_n])$ corresponds to an A-algebra homomorphism $i^\sharp : A[x_1, \ldots, x_n] \to A/I$. Choose $a_i \in A$ which map to $i^\sharp(x_i)$ in A/I. Observe that the ideal of the closed immersion i is

$$J = (x_1 - a_1, \ldots, x_n - a_n) + IA[x_1, \ldots, x_n].$$

Set $K = (x_1 - a_1, \ldots, x_n - a_n)$. We claim the sequence

$$0 \to K/KJ \to J/J^2 \to J/(K + J^2) \to 0$$

is split exact. To see this note that K/K^2 is free with basis $x_i - a_i$ over the ring $A[x_1, \ldots, x_n]/K \cong A$. Hence K/KJ is free with the same basis over the ring $A[x_1, \ldots, x_n]/J \cong A/I$. On the other hand, taking derivatives gives a map

$$\mathrm{d}_{A[x_1,\ldots,x_n]/A} : J/J^2 \longrightarrow \Omega_{A[x_1,\ldots,x_n]/A} \otimes_{A[x_1,\ldots,x_n]} A[x_1, \ldots, x_n]/J$$

which maps the generators $x_i - a_i$ to the basis elements $\mathrm{d}x_i$ of the free module on the right. The claim follows. Moreover, note that $x_1 - a_1, \ldots, x_n - a_n$ is a regular sequence in

$A[x_1, \ldots, x_n]$ with quotient ring $A[x_1, \ldots, x_n]/(x_1 - a_1, \ldots, x_n - a_n) \cong A$. Thus we have a factorization

$$Y \to V(x_1 - a_1, \ldots, x_n - a_n) \to \mathbf{A}^n_S$$

of our closed immersion i where the composition is Koszul-regular (resp. H_1-regular, quasi-regular), the second arrow is a regular immersion, and the associated conormal sequence is split. Now the result follows from Lemma 26.13.8. $\square$

26.15. Meromorphic functions and sections

See [**Kle79**] for some possible pitfalls[1].

Let $(X, \mathcal{O}_X)$ be a locally ringed space. For any open $U \subset X$ we have defined the set $\mathcal{S}(U) \subset \mathcal{O}_X(U)$ of regular sections of $\mathcal{O}_X$ over U, see Definition 26.9.13. The restriction of a regular section to a smaller open is regular. Hence $\mathcal{S} : U \mapsto \mathcal{S}(U)$ is a subsheaf (of sets) of $\mathcal{O}_X$. We sometimes denote $\mathcal{S} = \mathcal{S}_X$ if we want to indicate the dependence on X. Moreover, $\mathcal{S}(U)$ is a multiplicative subset of the ring $\mathcal{O}_X(U)$ for each U. Hence we may consider the presheaf of rings

$$U \longmapsto \mathcal{S}(U)^{-1}\mathcal{O}_X(U),$$

see Modules, Lemma 15.22.1.

Definition 26.15.1. Let $(X, \mathcal{O}_X)$ be a locally ringed space. The *sheaf of meromorphic functions on X* is the sheaf $\mathcal{K}_X$ associated to the presheaf displayed above. A *meromorphic function* on X is a global section of $\mathcal{K}_X$.

Since each element of each $\mathcal{S}(U)$ is a nonzero divisor on $\mathcal{O}_X(U)$ we see that the natural map of sheaves of rings $\mathcal{O}_X \to \mathcal{K}_X$ is injective.

Example 26.15.2. Let $A = \mathbf{C}[x, \{y_\alpha\}_{\alpha \in \mathbf{C}}]/((x - \alpha)y_\alpha, y_\alpha y_\beta)$. Any element of A can be written uniquely as $f(x) + \sum \lambda_\alpha y_\alpha$ with $f(x) \in \mathbf{C}[x]$ and $\lambda_\alpha \in \mathbf{C}$. Let $X = Spec(A)$. In this case $\mathcal{O}_X = \mathcal{K}_X$, since on any affine open $D(f)$ the ring A_f any nonzero divisor is a unit (proof omitted).

Definition 26.15.3. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of locally ringed spaces. We say that *pulbacks of meromorphic functions are defined for f* if for every pair of open $U \subset X$, $V \subset Y$ such that $f(U) \subset V$, and any section $s \in \Gamma(V, \mathcal{S}_Y)$ the pullback $f^\sharp(s) \in \Gamma(U, \mathcal{O}_X)$ is an element of $\Gamma(U, \mathcal{S}_X)$.

In this case there is an induced map $f^\sharp : f^{-1}\mathcal{K}_Y \to \mathcal{K}_X$, in other words we obtain a commutative diagram of morphisms of ringed spaces

$$\begin{tikzcd} (X, \mathcal{K}_X) \arrow[r] \arrow[d, "f"] & (X, \mathcal{O}_X) \arrow[d, "f"] \\ (Y, \mathcal{K}_Y) \arrow[r] & (Y, \mathcal{O}_X) \end{tikzcd}$$

We sometimes denote $f^*(s) = f^\sharp(s)$ for a section $s \in \Gamma(Y, \mathcal{K}_Y)$.

Lemma 26.15.4. *Let $f : X \to Y$ be a morphism of schemes. In each of the following cases pullbacks of meromorphic sections are defined.*

(1) *X, Y are integral and f is dominant,*

[1] Danger, Will Robinson!

(2) *X is integral and the generic point of X maps to a generic point of an irreducible component of Y,*
(3) *X is reduced and every generic point of every irreducible component of X maps to the generic point of an irreducible component of Y,*
(4) *X is locally Noetherian, and any associated point of X maps to a generic point of an irreducible component of Y, and*
(5) *X is locally Noetherian, has no embedded points and any generic point of an irreducible component of X maps to the generic point of an irreducible component of Y.*

Proof. Omitted. Hint: Similar to the proof of Lemma 26.9.9, using the following fact (on Y): if an element $x \in R$ maps to a nonzero divisor in $R_{\mathfrak p}$ for a minimal prime $\mathfrak p$ of R, then $x \not\in \mathfrak p$. See Algebra, Lemma 7.23.3. □

Let $(X, \mathcal{O}_X)$ be a locally ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Consider the presheaf $U \mapsto \mathcal{S}(U)^{-1}\mathcal{F}(U)$. Its sheafification is the sheaf $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K}_X$, see Modules, Lemma 15.22.2.

Definition 26.15.5. Let X be a locally ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules.

(1) We denote $\mathcal{K}_X(\mathcal{F})$ the sheaf of $\mathcal{K}_X$-modules which is the sheafification of the presheaf $U \mapsto \mathcal{S}(U)^{-1}\mathcal{F}(U)$. Equivalently $\mathcal{K}_X(\mathcal{F}) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K}_X$ (see above).
(2) A *meromorphic section of $\mathcal{F}$* is a global section of $\mathcal{K}_X(\mathcal{F})$.

In particular we have
$$\mathcal{K}_X(\mathcal{F})_x = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{K}_{X,x} = \mathcal{S}_x^{-1}\mathcal{F}_x$$
for any point $x \in X$. However, one has to be careful since it may not be the case that $\mathcal{S}_x$ is the set of nonzero divisors in the local ring $\mathcal{O}_{X,x}$. Namely, there is always an injective map
$$\mathcal{K}_{X,x} \longrightarrow Q(\mathcal{O}_{X,x})$$
to the total quotient ring. It is also surjective if and only if $\mathcal{S}_x$ is the set of nonzero divisors in $\mathcal{O}_{X,x}$.

Lemma 26.15.6. *Let X be a locally Noetherian scheme.*

(1) *For any $x \in X$ we have $\mathcal{S}_x \subset \mathcal{O}_{X,x}$ is the set of nonzero divisors, and $\mathcal{K}_{X,x}$ is the total quotient ring of $\mathcal{O}_{X,x}$.*
(2) *For any affine open $Spec(A) = U \subset X$ we have that $\mathcal{K}_X(U)$ equals the total quotient ring of A.*

Proof. Let A be a Noetherian ring. Let $\mathfrak p \subset A$ be a prime ideal. Let $f, g \in A$, $g \not\in \mathfrak p$. Let $I = \{x \in A \mid fx = 0\}$. Suppose f/g is a nonzero divisor in $A_{\mathfrak p}$. Then we see that $I_{\mathfrak p} = 0$ by exactness of localization. Since A is Noetherian we see that I is finitely generated and hence that $g'I = 0$ for some $g' \in A$, $g' \not\in \mathfrak p$. Hence f is a nonzero divisor in $A_{g'}$, i.e., in a Zariski open neighbourhood of $\mathfrak p$. This proves (1).

Let $f \in \Gamma(X, \mathcal{K}_{X,x})$ be a meromorphic function on $X = Spec(A)$. Set $I = \{x \in A \mid xf \in A\}$. For every prime $\mathfrak p \subset A$ we can write the image of f in the stalk at $\mathfrak p$ as a/b, $a, b \in A_{\mathfrak p}$ with $b \in A_{\mathfrak p}$ not a zero divisor. Hence, clearing denominators, we find there exists an element $x \in I$ such that x maps to a nonzero divisor on $A_{\mathfrak p}$. Let $\text{Ass}(A) = \{\mathfrak q_1, \ldots, \mathfrak q_t\}$ be the associated primes of A. By looking at $IA_{\mathfrak q_i}$ and using Algebra, Lemma 7.60.14 the above says that $I \not\subset \mathfrak q_i$ for each i. By Algebra, Lemma 7.14.3 there exists an element $x \in I$, $x \not\in \bigcup \mathfrak q_i$. By Algebra, Lemma 7.60.9 we see that x is not a zero divisor on A. Hence $f = (xf)/x$ is an element of the total ring of fractions of A. This proves (2). □

Lemma 26.15.7. *Let X be a scheme. Assume X is reduced and any quasi-compact open $U \subset X$ has a finite number of irreducible components.*

(1) *The sheaf $\mathcal{K}_X$ is a quasi-coherent sheaf of $\mathcal{O}_X$-algebras.*
(2) *For any $x \in X$ we have $\mathcal{S}_x \subset \mathcal{O}_{X,x}$ is the set of nonzero divisors. In particular $\mathcal{K}_{X,x}$ is the total quotient ring of $\mathcal{O}_{X,x}$.*
(3) *For any affine open $Spec(A) = U \subset X$ we have that $\mathcal{K}_X(U)$ equals the total quotient ring of A.*

Proof. Let X be as in the lemma. Let $X^{(0)} \subset X$ be the set of generic points of irreducible components of X. Let

$$f : Y = \coprod\nolimits_{\eta \in X^{(0)}} Spec(\kappa(\eta)) \longrightarrow X$$

be the inclusion of the generic points into X using the canonical maps of Schemes, Section 21.13. (This morphism was used in Morphisms, Definition 24.46.12 to define the normalization of X.) We claim that $\mathcal{K}_X = f_*\mathcal{O}_Y$. First note that $\mathcal{K}_Y = \mathcal{O}_Y$ as Y is a disjoint union of spectra of field. Next, note that pullbacks of meromorphic functions are defined for f, by Lemma 26.15.4. This gives a map

$$\mathcal{K}_X \longrightarrow f_*\mathcal{O}_Y.$$

Let $Spec(A) = U \subset X$ be an affine open. Then A is a reduced ring with finitely many minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$. Then we have $Q(A) = \prod A_{\mathfrak{q}_i} = \prod \kappa(\mathfrak{q}_i)$ by Algebra, Lemmas 7.22.2 and 7.23.3. In other words, already the value of the presheaf $U \mapsto \mathcal{S}(U)^{-1}\mathcal{O}_X(U)$ agrees with $f_*\mathcal{O}_Y(U)$ on our affine open U. Hence the displayed map is an isomorphism.

Now we are ready to prove (1), (2) and (3). The morphism f is quasi-compact by our assumption that the set of irreducible components of X is locally finite. Hence f is quasi-compact and quasi-separated (as Y is separated). By Schemes, Lemma 21.24.1 $f_*\mathcal{O}_Y$ is quasi-coherent. This proves (1). Let $x \in X$. Then

$$(f_*\mathcal{O}_Y)_x = \prod\nolimits_{\eta \in X^{(0)},\ x \in \overline{\{\eta\}}} \kappa(\eta)$$

On the other hand, $\mathcal{O}_{X,x}$ is reduced and has finitely minimal primes $\mathfrak{q}_i$ corresponding exactly to those $\eta \in X^{(0)}$ such that $x \in \overline{\{\eta\}}\kappa(\eta)$. Hence by Algebra, Lemmas 7.22.2 and 7.23.3 again we see that $Q(\mathcal{O}_{X,x}) = \prod \kappa(\mathfrak{q}_i)$ is the same as $(f_*\mathcal{O}_Y)_x$. This proves (2). Part (3) we saw during the course of the proof that $\mathcal{K}_X = f_*\mathcal{O}_Y$. $\square$

Lemma 26.15.8. *Let X be a scheme. Assume X is reduced and any quasi-compact open $U \subset X$ has a finite number of irreducible components. Then the normalization morphism $\nu : X^\nu \to X$ is the morphism*

$$\underline{Spec}_X(\mathcal{O}') \longrightarrow X$$

where $\mathcal{O}' \subset \mathcal{K}_X$ is the integral closure of $\mathcal{O}_X$ in the sheaf of meromorphic functions.

Proof. Compare the definition of the normalization morphism $\nu : X^\nu \to X$ (see Morphisms, Definition 24.46.12) with the result $\mathcal{K}_X = f_*\mathcal{O}_Y$ obtained in the proof of Lemma 26.15.7 above. $\square$

Lemma 26.15.9. *Let X be an integral scheme with generic point η. We have*

(1) *the sheaf of meromorphic functions is isomorphic to the constant sheaf with value the function field (see Morphisms, Definition 24.8.5) of X.*
(2) *for any quasi-coherent sheaf $\mathcal{F}$ on X the sheaf $\mathcal{K}_X(\mathcal{F})$ is isomorphic to the constant sheaf with value $\mathcal{F}_\eta$.*

Proof. Omitted. □

Definition 26.15.10. Let X be a locally ringed space. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. A meromorphic section s of $\mathcal{L}$ is said to be *regular* if the induced map $\mathcal{K}_X \to \mathcal{K}_X(\mathcal{L})$ is injective. (In other words, this means that s is a regular section of the invertible $\mathcal{K}_X$-module $\mathcal{K}_X(\mathcal{L})$. See Definition 26.9.13.)

First we spell out when (regular) meromorphic sections can be pulled back. After that we discuss the existence of regular meromorphic sections and consequences.

Lemma 26.15.11. *Let $f : X \to Y$ be a morphism of locally ringed spaces. Assume that pullbacks of meromorphic functions are defined for f (see Definition 26.15.3).*

(1) *Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_Y$-modules. There is a canonical pullback map $f^* : \Gamma(Y, \mathcal{K}_Y(\mathcal{F})) \to \Gamma(X, \mathcal{K}_X(f^*\mathcal{F}))$ for meromorphic sections of $\mathcal{F}$.*
(2) *Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. A regular meromorphic section s of $\mathcal{L}$ pulls back to a regular meromorphic section f^*s of $f^*\mathcal{L}$.*

Proof. Omitted. □

In some cases we can show regular meromorphic sections exist.

Lemma 26.15.12. *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. In each of the following cases $\mathcal{L}$ has a regular meromorphic section:*

(1) *X is integral,*
(2) *X is reduced and any quasi-compact open has a finite number of irreducible components, and*
(3) *X is locally Noetherian and has no embedded points.*

Proof. In case (1) we have seen in Lemma 26.15.9 that $\mathcal{K}_X(\mathcal{L})$ is a constant sheaf with value $\mathcal{L}_\eta$, and hence the result is clear.

Suppose X is a scheme. Let $G \subset X$ be the set of generic points of irreducible components of X. For each $\eta \in G$ denote $j_\eta : \eta \to X$ the canonical morphism of $\eta = Spec(\kappa(\eta))$ into X (see Schemes, Lemma 21.13.3). Consider the sheaf

$$\mathcal{G}_X(\mathcal{L}) = \prod\nolimits_{\eta \in G} j_{\eta,*}(\mathcal{L}_\eta).$$

There is a canonical map

$$\varphi : \mathcal{K}_X(\mathcal{L}) \longrightarrow \mathcal{G}_X(\mathcal{L})$$

coming from the maps $\mathcal{K}_X(\mathcal{L})_\eta \to \mathcal{L}_\eta$ and adjunction (see Sheaves, Lemma 6.27.3).

We claim that in cases (2) and (3) the map φ is an isomorphism for any invertible sheaf $\mathcal{L}$. Before proving this let us show that cases (2) and (3) follow from this. Namely, we can choose $s_\eta \in \mathcal{L}_\eta$ which generate $\mathcal{L}_\eta$, i.e., such that $\mathcal{L}_\eta = \mathcal{O}_{X,\eta}s_\eta$. Since the claim applied to $\mathcal{O}_X$ gives $\mathcal{K}_X = \mathcal{G}_X(\mathcal{O}_X)$ it is clear that the global section $s = \prod_{\eta \in G} s_\eta$ is regular as desired.

To prove that φ is an isomorphism we may work locally on X. For example it suffices to show that sections of $\mathcal{K}_X(\mathcal{L})$ and $\mathcal{G}_X(\mathcal{L})$ agree over small affine opens U. Say $U = Spec(A)$ and $\mathcal{L}|_U \cong \mathcal{O}_U$. By Lemmas 26.15.6 and 26.15.7 we see that $\Gamma(U, \mathcal{K}_X) = Q(A)$ is the total ring of fractions of A. On the other hand, $\Gamma(U, \mathcal{G}_X(\mathcal{O}_X)) = \prod_{\mathfrak{q} \subset A \text{ minimal}} A_\mathfrak{q}$. In both cases we see that the set of minimal primes of A is finite, say $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$, and that the set of zero divisors of A is equal to $\mathfrak{q}_1 \cup \ldots \cup \mathfrak{q}_t$ (see Algebra, Lemma 7.60.9). Hence the result follows from Algebra, Lemma 7.22.2. □

Lemma 26.15.13. *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let s be a regular meromorphic section of $\mathcal{L}$. Let us denote $\mathcal{I} \subset \mathcal{O}_X$ the sheaf of ideals defined by the rule*

$$\mathcal{I}(V) = \{f \in \mathcal{O}_Z(V) \mid fs \in \mathcal{L}(V)\}.$$

The formula makes sense since $\mathcal{L}(V) \subset \mathcal{K}_X(\mathcal{L})(V)$. Then $\mathcal{I}$ is a quasi-coherent sheaf of ideals and we have injective maps

$$1 : \mathcal{I} \longrightarrow \mathcal{O}_X, \quad s : \mathcal{I} \longrightarrow \mathcal{L}$$

whose cokernels are supported on closed nowhere dense subsets of X.

Proof. The question is local on X. Hence we may assume that $X = Spec(A)$, and $\mathcal{L} = \mathcal{O}_X$. After shrinking furhter we may assume that $s = x/y$ with $a, b \in A$ *both* nonzero divisors in A. Set $I = \{x \in A \mid x(a/b) \in A\}$.

To show that $\mathcal{I}$ is quasi-coherent we have to show that $I_f = \{x \in A_f \mid x(a/b) \in A_f\}$ for every $f \in A$. If $c/f^n \in A_f$, $(c/f^n)(a/b) \in A_f$, then we see that $f^m c(a/b) \in A$ for some m, hence $c/f^n \in I_f$. Conversely it is easy to see that I_f is contained in $\{x \in A_f \mid x(a/b) \in A_f\}$. This proves quasi-coherence.

Let us prove the final statement. It is clear that $(b) \subset I$. Hence $V(I) \subset V(b)$ is a nowhere dense subset as b is a nonzero divisor. Thus the cokernel of 1 is supported in a nowhere dense closed set. The same argument works for the cokerenel of s since $s(b) = (a) \subset sI \subset A$. □

Definition 26.15.14. Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let s be a regular meromorphic section of $\mathcal{L}$. The sheaf of ideals $\mathcal{I}$ constructed in Lemma 26.15.13 is called the *ideal sheaf of denominators of s*.

Here is a lemma which will be used later.

Lemma 26.15.15. *Suppose given*

(1) *X a locally Noetherian scheme,*
(2) *$\mathcal{L}$ an invertible $\mathcal{O}_X$-module,*
(3) *s a regular meromorphic section of $\mathcal{L}$, and*
(4) *$\mathcal{F}$ coherent on X without embedded associated points and $Supp(\mathcal{F}) = X$.*

Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal of denominators of s. Let $T \subset X$ be the union of the supports of $\mathcal{O}_X/\mathcal{I}$ and $\mathcal{L}/s(\mathcal{I})$ which is a nowhere dense closed subset $T \subset X$ according to Lemma 26.15.13. Then there are canonical injective maps

$$1 : \mathcal{I}\mathcal{F} \to \mathcal{F}, \quad s : \mathcal{I}\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}$$

whose cokernels are supported on T.

Proof. Reduce to the affine case with $\mathcal{L} \cong \mathcal{O}_X$, and $s = a/b$ with $a, b \in A$ both nonzero divisors. Proof of reduction step omitted. Write $\mathcal{F} = \widetilde{M}$. Let $I = \{x \in A \mid x(a/b) \in A\}$ so that $\mathcal{I} = \tilde{I}$ (see proof of Lemma 26.15.13). Note that $T = V(I) \cup V((a/b)I)$. For any A-module M consider the map $1 : IM \to M$; this is the map that gives rise to the map 1 of the lemma. Consider on the other hand the map $\sigma : IM \to M_b, x \mapsto ax/b$. Since b is not a zero divisor in A, and since M has support $Spec(A)$ and no embedded primes we see that b is a nonzero divisor on M also. Hence $M \subset M_b$. By definition of I we have $\sigma(IM) \subset M$ as submodules of M_b. Hence we get an A-module map $s : IM \to M$ (namely the unique map such that $s(z)/1 = \sigma(z)$ in M_b for all $z \in IM$). It is injective because a is a nonzero divisor also (on both A and M). It is clear that M/IM is annihilated by I and that $M/s(IM)$ is annihilated by $(a/b)I$. Thus the lemma follows. □

26.16. Other chapters

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CHAPTER 27

Limits of Schemes

27.1. Introduction

In this chapter we start proving some basic theorems of algebraic geometry. A basic reference is [**DG67**].

27.2. Directed limits of schemes with affine transition maps

In this section we construct the limit.

Lemma 27.2.1. *Let I be a directed partially ordered set. Let $(S_i, f_{ii'})$ be an inverse system of schemes over I. If all the schemes S_i are affine, then the limit $S = lim_i S_i$ exists in the category of schemes. In fact S is affine and $S = Spec(colim_i R_i)$ with $R_i = \Gamma(S_i, \mathcal{O})$.*

Proof. Just define $S = Spec(colim_i R_i)$. It follows from Schemes, Lemma 21.6.4 that S is the limit even in the category of locally ringed spaces. □

Lemma 27.2.2. *Let I be a directed partially ordered set. Let $(S_i, f_{ii'})$ be an inverse system of schemes over I. If all the morphisms $f_{ii'} : S_i \to S_{i'}$ are affine, then the limit $S = lim_i S_i$ exists in the category of schemes. Moreover,*

1. *each of the morphisms $f_i : S \to S_i$ is affine,*
2. *for any $i \in I$ and any open subscheme $U_i \subset S_i$ we have*

$$f_i^{-1}(U_i) = lim_{i' \geq i} f_{i'i}^{-1}(U_i).$$

Proof. Choose $i_0 \in I$. Note that I is nonempty as the limit is directed. For convenience write $S_0 = S_{i_0}$ and $i_0 = 0$. For every $i \geq 0$ consider the quasi-coherent sheaf of $\mathcal{O}_{S_0}$-algebras $\mathcal{A}_i = f_{i0,*}\mathcal{O}_{S_i}$. Recall that $S_i = \underline{Spec}_{S_0}(\mathcal{A})$, see Morphisms, Lemma 24.11.3. Set $\mathcal{A} = colim_{i \geq 0} \mathcal{A}_i$. This is a quasi-coherent sheaf of $\mathcal{O}_{S_0}$-algebras, see Schemes, Section 21.24. Set $S = \underline{Spec}_{S_0}(\mathcal{A})$. By Morphisms, Lemma 24.11.5 we get for $i \geq 0$ morphisms $f_i : S \to S_i$ compatible with the transition morphisms. Note that the morphisms f_i are affine by Morphisms, Lemma 24.11.11 for example. By Lemma 27.2.1 above we see that for any affine open $U_0 \subset S_0$ the inverse image $U = f_0^{-1}(U_0) \subset S$ is the limit of the system of opens $U_i = f_i^{-1}(U_0)$, $i \geq 0$ in the category of schemes.

Let T be a scheme. Let $g_i : T \to S_i$ be a compatible system of morphisms. To show that $S = lim_i S_i$ we have to prove there is a unique morphism $g : T \to S$ with $g_i = f_i \circ g$ for all $i \in I$. For every $t \in T$ there exists an affine open $U_0 \subset S_0$ containing $g_0(t)$. Let $V \subset g_0^{-1}(U_0)$ be an affine open neighbourhood containing t. By the remarks above we obtain a unique morphism $g_V : V \to U = f_0^{-1}(U_0)$ such that $f_i \circ g_V = g_i|_{U_i}$ for all i. The open sets $V \subset T$ so constructed form a basis for the topology of T. The morphisms g_V glue to a morphism $g : T \to S$ because of the uniqueness property. This gives the desired morphism $g : T \to S$. □

Lemma 27.2.3. *Let I be a directed partially ordered set. Let $(S_i, f_{ii'})$ be an inverse system of schemes over I. Assume all the morphisms $f_{ii'} : S_i \to S_{i'}$ are affine, Let $S = lim_i\, S_i$.*

(1) *We have $S_{set} = lim_i\, S_{i,set}$ where S_{set} indicates the underlying set of the scheme S.*
(2) *If $s, s' \in S$ and s' is not a specialization of s then for some $i \in I$ the image $s'_i \in S_i$ of s' is not a specialization of the image $s_i \in S_i$ of s.*
(3) *Add more easy facts on topology of S here. (Requirement: whatever is added should be easy in the affine case.)*

Proof. Proof of (1). Pick $i \in I$. Take $U_i \subset S_i$ an affine open. Denote $U_{i'} = f_{i'i}^{-1}(U_i)$ and $U = f_i^{-1}(U_i)$. Suppose we can show that $U_{set} = lim_{i' \geq i}\, U_{i',set}$. Then assertion (1) follows by a simple argument using an affine covering of S_i. Hence we may assume all S_i and S affine. This reduces us to the following algebra question: Suppose given a system of rings $(A_i, \varphi_{ii'})$ over I. Set $A = colim_i\, A_i$ with canonical maps $\varphi_i : A_i \to A$. Then

$$Spec(A) = lim_i\, Spec(A_i)$$

Namely, suppose that we are given primes $\mathfrak{p}_i \subset A_i$ such that $\mathfrak{p}_i = \varphi_{ii'}^{-1}(\mathfrak{p}_{i'})$ for all $i' \geq i$. Then we simply set

$$\mathfrak{p} = \{x \in A \mid \exists i, x_i \in \mathfrak{p}_i \text{ with } \varphi(x_i) = x\}$$

It is clear that this is an ideal and has the property that $\varphi_i^{-1}(\mathfrak{p}) = \mathfrak{p}_i$. Then it follows easily that it is a prime ideal as well. This proves (1).

Proof of (2). Pick $i \in I$. Pick an affine open $U_i \subset S_i$ containing $f_i(s')$. If $f_i(s) \notin S_i$ then we are done. Hence reduce to the affine case by considering the inverse images of U_i as above. This reduces us to the following algebra question: Suppose given a system of rings $(A_i, \varphi_{ii'})$ over I. Set $A = colim_i\, A_i$ with canonical maps $\varphi_i : A_i \to A$. Suppose given primes $\mathfrak{p}$, $\mathfrak{p}'$ of A. Suppose that $\mathfrak{p} \not\subset \mathfrak{p}'$. Then for some i we have $\varphi_i^{-1}(\mathfrak{p}) \not\subset \varphi_i^{-1}(\mathfrak{p}')$. This is clear. □

Lemma 27.2.4. *Let I be a directed partially ordered set. Let $(S_i, f_{ii'})$ be an inverse system of schemes over I. Assume all the morphisms $f_{ii'} : S_i \to S_{i'}$ are affine, Let $S = lim_i\, S_i$. Let $i \in I$. Suppose that X_i is a scheme over S_i. Set $X_j = S_j \times_{S_i} X_i$ for $j \geq i$ and set $X = S \times_{S_i} X_i$. Then*

$$X = colim_{j \geq i}\, X_j$$

Proof. The transition morphisms of the system $\{X_j\}_{j \geq i}$ are affine as they are base changes of the affine morphisms between the S_j, see Morphisms, Lemma 24.11.8. Hence we know the limit of the system $\{X_j\}_{j \geq i}$ exists. There is a canonical morphism $X \to lim\, X_j$. To see that it is an isomorphism we may work locally. Hence we may assume that $X_i = Spec(B_i)$ is an affine such that the morphism $X_i \to S_i$ has image contained in an affine open subscheme U of S_i. In this case we may also replace each S_j by the inverse image of U in S_j, in other words we may assume all the $S_j = Spec(A_j)$ are affine. Then we have $X_j = Spec(A_j \otimes_{A_i} B_i)$. In this case the statement becomes the equality

$$colim_{j \geq i}(A_j \otimes_{A_i} B_i) = (colim_{j \geq i}\, A_j) \otimes_{A_i} B_i$$

which follows from Algebra, Lemma 7.11.8. □

Lemma 27.2.5. *Let I be a directed partially ordered set. Let $(S_i, f_{ii'})$ be an inverse system of schemes over I. Assume*

(1) *all the morphisms $f_{ii'} : S_i \to S_{i'}$ are affine,*

(2) *all the schemes S_i are quasi-compact and quasi-separated.*

Let $S = lim_i S_i$. Let $i \in I$. Suppose that $\mathcal{F}_i$ is a quasi-coherent sheaf on S_i. Set $\mathcal{F}_j = f_{ji}^*\mathcal{F}_i$ for $j \geq i$ and set $\mathcal{F} = f_i^*\mathcal{F}_i$. Then

$$\Gamma(S, \mathcal{F}) = colim_{j \geq i} \Gamma(S_j, \mathcal{F}_j)$$

Proof. Write $\mathcal{A}_j = f_{ji,*}\mathcal{O}_{S_j}$. This is a quasi-coherent sheaf of $\mathcal{O}_{S_i}$-algebras (see Morphisms, Lemma 24.11.5) and S_j is the relative spectrum of $\mathcal{A}_j$ over S_i. In the proof of Lemma 27.2.2 we constructed S as the relative spectrum of $\mathcal{A} = colim_{j \geq i} \mathcal{A}_j$ over S_i. Set

$$\mathcal{M}_j = \mathcal{F}_i \otimes_{\mathcal{O}_{S_i}} \mathcal{A}_j$$

and

$$\mathcal{M} = \mathcal{F}_i \otimes_{\mathcal{O}_{S_i}} \mathcal{A}.$$

Then we have $f_{ji,*}\mathcal{F}_j = \mathcal{M}_j$ and $f_{i,*}\mathcal{F} = \mathcal{M}$. Since $\mathcal{A}$ is the colimit of the sheaves $\mathcal{A}_j$ and since tensor product commutes with directed colimits, we conclude that $\mathcal{M} = colim_{j \geq i} \mathcal{M}_j$. Since S_i is quasi-compact and quasi-separated we see that

$$\begin{aligned} \Gamma(S, \mathcal{F}) &= \Gamma(S_i, \mathcal{M}) \\ &= \Gamma(S_i, colim_{j \geq i} \mathcal{M}_j) \\ &= colim_{j \geq i} \Gamma(S_i, \mathcal{M}_j) \\ &= colim_{j \geq i} \Gamma(S_j, \mathcal{F}_j) \end{aligned}$$

see Sheaves, Lemma 6.29.1 and Topology, Lemma 5.18.2 for the middle equality. □

27.3. Absolute Noetherian Approximation

A nice reference for this section is Appendix C of the article by Thomason and Trobaugh [**TT90**]. See Categories, Section 4.19 for our conventions regarding directed systems. We will use the existence result and properties of the limit from Section 27.2 without further mention.

Lemma 27.3.1. *Let I be a directed partially ordered set. Let $(S_i, f_{ii'})$ be an inverse system of schemes over I. Assume*

(1) *all the morphisms $f_{ii'} : S_i \to S_{i'}$ are affine,*
(2) *all the schemes S_i are quasi-compact, and*
(3) *all the schemes S_i are nonempty.*

Then the limit $S = lim_i S_i$ is nonempty.

Proof. Choose $i_0 \in I$. Note that I is nonempty as the limit is directed. For convenience write $S_0 = S_{i_0}$ and $i_0 = 0$. Choose an affine open covering $S_0 = \bigcup_{j=1,\ldots,m} U_j$. Since I is directed there exists a $j \in \{1, \ldots, m\}$ such that $f_{i0}^{-1}(U_j) \neq \emptyset$ for all $i \geq 0$. Hence $lim_{i \geq 0} f_{i0}^{-1}(U_j)$ is not empty since a directed colimit of nonzero rings is nonzero (because $1 \neq 0$). As $lim_{i \geq 0} f_{i0}^{-1}(U_j)$ is an open subscheme of the limit we win. □

Lemma 27.3.2. *Let I be a directed partially ordered set. Let $(S_i, f_{ii'})$ be an inverse system of schemes over I. Assume*

(1) *all the morphisms $f_{ii'} : S_i \to S_{i'}$ are affine, and*
(2) *all the schemes S_i are quasi-compact.*

Let $S = lim_i S_i$. Suppose for each i we are given a nonempty closed subset $Z_i \subset S_i$ with $f_{ii'}(Z_i) \subset Z_{i'}$. Then there exists a point $s \in S$ with $f_i(s) \in Z_i$ for all i.

Proof. Let $Z_i \subset S_i$ also denote the reduced closed subscheme associated to Z_i, see Schemes, Definition 21.12.5. A closed immersion is affine, and a composition of affine morphisms is affine (see Morphisms, Lemmas 24.11.9 and 24.11.7), and hence $Z_i \to S_{i'}$ is affine when $i \geq i'$. We conclude that the morphism $f_{ii'} : Z_i \to Z_{i'}$ is affine by Morphisms, Lemma 24.11.11. Each of the schemes Z_i is quasi-compact as a closed subscheme of a quasi-compact scheme. Hence we may apply Lemma 27.3.1 to see that $Z = lim_i Z_i$ is nonempty. Since there is a canonical morphism $Z \to S$ we win. □

Lemma 27.3.3. *Let I be a directed partially ordered set. Let $(S_i, f_{ii'})$ be an inverse system of schemes over I. Assume all the morphisms $f_{ii'} : S_i \to S_{i'}$ are affine. Let $S = lim_i S_i$. Suppose we are given an i and a morphism $T \to S_i$ such that*

(1) *$T \times_{S_i} S = \emptyset$, and*
(2) *T is quasi-compact.*

Then $T \times_{S_i} S_{i'} = \emptyset$ for all sufficiently large i'.

Proof. By Lemma 27.2.4 we see that $T \times_{S_i} S = lim_{i' \geq i} T \times_{S_i} S_{i'}$. Hence the result follows from Lemma 27.3.1. □

Lemma 27.3.4. *Let I be a directed partially ordered set. Let $(S_i, f_{ii'})$ be an inverse system of schemes over I. Assume all the morphisms $f_{ii'} : S_i \to S_{i'}$ are affine, and all the schemes S_i are quasi-compact. Let $S = lim_i S_i$ with projection morphisms $f_i : S \to S_i$. Suppose we are given an i and a locally constructible subset $E \subset S_i$ such that $f_i(S) \subset E$. Then $f_{ii'}(S_{i'}) \subset E$ for all sufficiently large i'.*

Proof. Writing S_i as a finite union of open affine subschemes reduces the question to the case that S_i is affine and E is constructible, see Lemma 27.2.2 and Properties, Lemma 23.2.1. In this case the complement $S_i \setminus E$ is contstructible too. Hence there exists an affine scheme T and a morphism $T \to S_i$ whose image is $S_i \setminus E$, see Algebra, Lemma 7.26.3. By Lemma 27.3.3 we see that $T \times_{S_i} S_{i'}$ is empty for all sufficiently large i', and hence $f_{ii'}(S_{i'}) \subset E$ for all sufficiently large i'. □

Lemma 27.3.5. *Let I be a directed partially ordered set. Let $(S_i, f_{ii'})$ be an inverse system of schemes over I. Assume*

(1) *all the morphisms $f_{ii'} : S_i \to S_{i'}$ are affine,*
(2) *all the schemes S_i are quasi-compact and quasi-separated.*

Then we have the following:

(1) *Given any quasi-compact open $V \subset S = lim_i S_i$ there exists an $i \in I$ and a quasi-compact open $V_i \subset S_i$ such that $f_i^{-1}(V_i) = V$.*
(2) *Given $V_i \subset S_i$ and $V_{i'} \subset S_{i'}$ quasi-compact opens such that $f_i^{-1}(V_i) = f_{i'}^{-1}(V_{i'})$ there exists an index $i'' \geq i, i'$ such that $f_{i''i}^{-1}(V_i) = f_{i''i'}^{-1}(V_{i'})$.*
(3) *If $V_{1,i}, \ldots, V_{n,i} \subset S_i$ are quasi-compact opens and $S = f_i^{-1}(V_{1,i}) \cup \ldots \cup f_i^{-1}(V_{n,i})$ then $S_{i'} = f_{i'i}^{-1}(V_{1,i}) \cup \ldots \cup f_{i'i}^{-1}(V_{n,i})$ for some $i' \geq i$.*

Proof. Choose $i_0 \in I$. Note that I is nonempty as the limit is directed. For convenience we write $S_0 = S_{i_0}$ and $i_0 = 0$. Choose an affine open covering $S_0 = U_{1,0} \cup \ldots \cup U_{m,0}$. Denote $U_{j,i} \subset S_i$ the inverse image of $U_{j,0}$ under the transition morphism for $i \geq 0$. Denote U_j the inverse image of $U_{j,0}$ in S. Note that $U_j = lim_i U_{j,i}$ is a limit of affine schemes.

We first prove the uniqueness statement: Let $V_i \subset S_i$ and $V_{i'} \subset S_{i'}$ quasi-compact opens such that $f_i^{-1}(V_i) = f_{i'}^{-1}(V_{i'})$. It suffices to show that $f_{i''i}^{-1}(V_i \cap U_{j,i''})$ and $f_{i''i'}^{-1}(V_{i'} \cap U_{j,i''})$

become equal for i'' large enough. Hence we reduce to the case of a limit of affine schemes. In this case write $S = Spec(R)$ and $S_i = Spec(R_i)$ for all $i \in I$. We may write $V_i = S_i \setminus V(h_1, \ldots, h_m)$ and $V_{i'} = S_{i'} \setminus V(g_1, \ldots, g_n)$. The assumption means that the ideals $\sum g_j R$ and $\sum h_j R$ have the same radical in R. This means that $g_j^N = \sum a_{jj'} h_{j'}$ and $h_j^N = \sum b_{jj'} g_{j'}$ for some $N \gg 0$ and $a_{jj'}$ and $b_{jj'}$ in R. Since $R = colim_i R_i$ we can chose an index $i'' \geq i$ such that the equations $g_j^N = \sum a_{jj'} h_{j'}$ and $h_j^N = \sum b_{jj'} g_{j'}$ hold in $R_{i''}$ for some $a_{jj'}$ and $b_{jj'}$ in $R_{i''}$. This implies that the ideals $\sum g_j R_{i''}$ and $\sum h_j R_{i''}$ have the same radical in $R_{i''}$ as desired.

We prove existence. We may apply the uniqueness statement to the limit of schemes $U_{j_1} \cap U_{j_2} = lim_i U_{j_1,i} \cap U_{j_2,i}$ since these are still quasi-compact due to the fact that the S_i were assumed quasi-separated. Hence it is enough to prove existence in the affine case. In this case write $S = Spec(R)$ and $S_i = Spec(R_i)$ for all $i \in I$. Then $V = S \setminus V(g_1, \ldots, g_n)$ for some $g_1, \ldots, g_n \in R$. Choose any i large enough so that each of the g_j comes from an element $g_{j,i} \in R_i$ and take $V_i = S_i \setminus V(g_{1,i}, \ldots, g_{n,i})$.

The statement on coverings follows from the uniqueness statement for the opens $V_{1,i} \cup \ldots \cup V_{n,i}$ and S_i of S_i. □

Lemma 27.3.6. *Let I be a directed partially ordered set. Let $(S_i, f_{ii'})$ be an inverse system of schemes over I. Assume*

(1) *all the morphisms $f_{ii'} : S_i \to S_{i'}$ are affine,*
(2) *all the schemes S_i are quasi-compact and quasi-separated, and*
(3) *the limit $S = lim_i S_i$ is quasi-affine.*

Then for some $i_0 \in I$ the schemes S_i for $i \geq i_0$ are quasi-affine.

Proof. Choose $i_0 \in I$. Note that I is nonempty as the limit is directed. For convenience we write $S_0 = S_{i_0}$ and $i_0 = 0$. Let $s \in S$. We may choose an affine open $U_0 \subset S_0$ containing $f_0(s)$. Since S is quasi-affine we may choose an element $a \in \Gamma(S, \mathcal{O}_S)$ such that $s \in D(a) \subset f_0^{-1}(U_0)$, and such that $D(a)$ is affine. By Lemma 27.2.5 there exists an $i \geq 0$ such that a comes from an element $a_i \in \Gamma(S_i, \mathcal{O}_{S_i})$. For any index $j \geq i$ we denote a_j the image of a_i in the global sections of the structure sheaf of S_j. Consider the opens $D(a_j) \subset S_j$ and $U_j = f_{j0}^{-1}(U_0)$. Note that U_j is affine and $D(a_j)$ is a quasi-compact open of S_j, see Properties, Lemma 23.23.4 for example. Hence we may apply Lemma 27.3.5 to the opens U_j and $U_j \cup D(a_j)$ to conclude that $D(a_j) \subset U_j$ for some $j \geq i$. For such an index j we see that $D(a_j) \subset S_j$ is an affine open (because $D(a_j)$ is a standard affine open of the affine open U_j) containing the image $f_j(s)$.

We conclude that for every $s \in S$ there exist an index $i \in I$, and a global section $a \in \Gamma(S_i, \mathcal{O}_{S_i})$ such that $D(a) \subset S_i$ is an affine open containing $f_i(s)$. Because S is quasi-compact we may choose a single index $i \in I$ and global sections $a_1, \ldots, a_m \in \Gamma(S_i, \mathcal{O}_{S_i})$ such that each $D(a_j) \subset S_i$ is affine open and such that $f_i : S \to S_i$ has image contained in the union $W_i = \bigcup_{j=1,\ldots,m} D(a_j)$. For $i' \geq i$ set $W_{i'} = f_{i'i}^{-1}(W_i)$. Since $f_i^{-1}(W_i)$ is all of S we see (by Lemma 27.3.5 again) that for a suitable $i' \geq i$ we have $S_{i'} = W_{i'}$. Thus we may replace i by i' and assume that $S_i = \bigcup_{j=1,\ldots,m} D(a_j)$. This implies that $\mathcal{O}_{S_i}$ is an ample invertible sheaf on S_i (see Properties, Definition 23.23.1) and hence that S_i is quasi-affine, see Properties, Lemma 23.24.1. Hence we win. □

Lemma 27.3.7. *Let I be a directed partially ordered set. Let $(S_i, f_{ii'})$ be an inverse system of schemes over I. Assume*

(1) *all the morphisms $f_{ii'} : S_i \to S_{i'}$ are affine,*
(2) *all the schemes S_i are quasi-compact and quasi-separated, and*
(3) *the limit $S = lim_i S_i$ is affine.*

Then for some $i_0 \in I$ the schemes S_i for $i \geq i_0$ are affine.

Proof. By Lemma 27.3.6 we may assume that S_i is quasi-affine for all i. Set $R_i = \Gamma(S_i, \mathcal{O}_{S_i})$. Then S_i is a quasi-compact open of $\overline{S_i} := Spec(R_i)$. Write $S = Spec(R)$. We have $R = colim_i R_i$ by Lemma 27.2.5. Hence also $S = lim_i \overline{S_i}$. Let $Z_i \subset \overline{S_i}$ be the closed subset such that $\overline{S_i} = Z_i \coprod S_i$. We have to show that Z_i is empty for some i. Assume Z_i is nonempty for all i to get a contradiction. By Lemma 27.3.2 there exists a point s of S which maps to a point of Z_i for every i. But $S = lim_i S_i$, and hence we get a contradiction. □

Lemma 27.3.8. *Let W be a quasi-affine scheme of finite type over $\mathbf{Z}$. Suppose $W \to Spec(R)$ is an open immersion into an affine scheme. There exists a finite type $\mathbf{Z}$-algebra $A \subset R$ which induces an open immersion $W \to Spec(A)$. Moreover, R is the directed colimit of such subalgebras.*

Proof. Choose an affine open covering $W = \bigcup_{i=1,\ldots,n} W_i$ such that each W_i is a standard affine open in $Spec(R)$. In other words, if we write $W_i = Spec(R_i)$ then $R_i = R_{f_i}$ for some $f_i \in R$. Choose finitely many $x_{ij} \in R_i$ which generate R_i over $\mathbf{Z}$. Pick an $N \gg 0$ such that each $f_i^N x_{ij}$ comes from an element of R, say $y_{ij} \in R$. Set A equal to the $\mathbf{Z}$-algebra generated by the f_i and the y_{ij} and (optionally) finitely many additional elements of R. Then A works. Details omitted. □

Lemma 27.3.9. *Suppose given a cartesian diagram of rings*

$$\begin{tikzcd} B \arrow[r, "s"] & R \\ B' \arrow[u] \arrow[r] & R' \arrow[u, "t"'] \end{tikzcd}$$

Suppose $h \in B'$ corresponds to $g \in B$ and $f \in R'$ such that $s(g) = t(f)$. Then the diagram

$$\begin{tikzcd} B_g \arrow[r, "s"] & R_{s(g)} = R_{t(f)} \\ (B')_h \arrow[u] \arrow[r] & (R')_f \arrow[u, "t"'] \end{tikzcd}$$

is cartesian too.

Proof. Note that $B' = \{(b, r') \in B \times R' \mid s(b) = t(r')\}$. So $h = (g, f) \in B'$. First we show that $(B')_h$ maps injectively into $B_g \times (R')_f$. Namely, suppose that $(x, y)/h^n$ maps to zero. This means that $(g^N x, f^N y)$ is zero for some N. Which clearly implies that x/g^n and y/f^n are both zero. Next, suppose that x/g^n and y/f^m are elements which map to the same element of $R_{s(g)}$. This means that $s(g)^N(t(f)^m s(x) - s(g)^n t(y)) = 0$ in R' for some $N \gg 0$. We can rewrite this as $s(g^{m+N}x) = t(f^{n+N}y)$. Hence we see that the pair $(x/g^n, y/f^m)$ is the image of the element $(g^{m+N}x, t(f^{n+N}y))/(g, f)^{n+m+N}$ of $(B')_h$. □

Lemma 27.3.10. *Suppose given a cartesian diagram of rings*

$$\begin{tikzcd} B \arrow[r, "s"'] & R \\ B' \arrow[u] \arrow[r] & R' \arrow[u, "t"'] \end{tikzcd}$$

Let $W' \subset Spec(R')$ be an open of the form $W' = D(f_1) \cup \ldots \cup D(f_n)$ such that $t(f_i) = s(g_i)$ for some $g_i \in B$ and $B_{g_i} \cong R_{s(g_i)}$. Then $B' \to R'$ induces an open immersion of W' into $Spec(B')$.

Proof. Set $h_i = (g_i, f_i) \in B'$. Lemma 27.3.9 above shows that $(B')_{h_i} \cong (R')_{f_i}$ as desired. □

Proposition 27.3.11. *Let S be a quasi-compact and quasi-separated scheme. There exist a directed partially ordered set I and an inverse system of schemes $(S_i, f_{ii'})$ over I such that*

(1) *the transition morphisms $f_{ii'}$ are affine*
(2) *each S_i is of finite type over $\mathbf{Z}$, and*
(3) *$S = lim_i S_i$.*

Proof. Choose an affine open covering $S = \bigcup_{j=1,\ldots,m} U_j$ with m minimal. We will prove the lemma by induction on m. The lemma is obvious when $m = 1$ since any ring is the directed colimit of its finitely generated $\mathbf{Z}$-subalgebras.

Thus we may assume

(1) $S = U \cup V$,
(2) U affine open in S,
(3) V quasi-compact open in S, and
(4) $V = lim_i V_i$ with $(V_i, f_{ii'})$ an inverse system over a directed set I, each $f_{ii'}$ affine and each V_i of finite type over $\mathbf{Z}$.

Set $W = U \cap V$. This is a quasi-compact open of V. By Lemma 27.3.5 (and after shrinking I) we may assume that there exist opens $W_i \subset V_i$ such that $f_{ij}^{-1}(W_j) = W_i$ and such that $f_i^{-1}(W_i) = W$. Since W is a quasi-compact open of U it is quasi-affine. Hence we may assume (after shrinking I again) that W_i is quasi-affine for all i, see Lemma 27.3.6.

Write $U = Spec(B)$. Set $R = \Gamma(W, \mathcal{O}_W)$, and $R_i = \Gamma(W_i, \mathcal{O}_{W_i})$. By Lemma 27.2.5 we have $R = colim_i R_i$. Now we have the maps of rings

$$\begin{tikzcd} B \arrow[r, "s"'] & R \\ & R_i \arrow[u, "t_i"'] \end{tikzcd}$$

We set $B_i = \{(b, r) \in B \times R_i \mid s(b) = t_i(t)\}$ so that we have a cartesian diagram

$$\begin{tikzcd} B \arrow[r, "s"'] & R \\ B_i \arrow[u] \arrow[r] & R_i \arrow[u, "t_i"'] \end{tikzcd}$$

for each i. The transition maps $R_i \to R_{i'}$ induce maps $B_i \to B_{i'}$. It is clear that $B = colim_i B_i$.

As W is a quasi-compact open of $U = Spec(B)$ we can find a finitely many elements $g_j \in B$, $j = 1, \ldots, m$ such that $D(g_j) \subset W$ and such that $W = \bigcup_{j=1,\ldots,m} D(g_j)$. Note that this implies $D(g_j) = W_{s(g_j)}$ as open subsets of U, where $W_{s(g_j)}$ denotes the largest open subset of W on which $s(g_j)$ is invertible. Hence

$$B_{g_j} = \Gamma(D(g_j), \mathcal{O}_U) = \Gamma(W_{s(g_j)}, \mathcal{O}_W) = R_{s(g_j)},$$

where the last equality is Properties, Lemma 23.15.2. Since $W_{s(g_j)}$ is affine this also implies that $D(s(g_j)) = W_{s(g_j)}$ as open subsets of $Spec(R)$. Since $R = colim_i R_i$ we can (after shrinking I) assume there exist $g_{j,i} \in R_i$ for all $i \in I$ such that $s(g_j) = t_i(g_{j,i})$. Of course we choose the $g_{j,i}$ such that $g_{j,i}$ maps to $g_{j,i'}$ under the transition maps $R_i \to R_{i'}$. Then, by Lemma 27.3.5 we can (after shrinking I again) assume the corresponding opens $D(g_{j,i}) \subset Spec(R_i)$ are contained in W_i, $j = 1, \ldots, m$ and cover W_i. At this point we may apply Lemma 27.3.10 to conclude that the morphism $W_i \to Spec(R_i) \to Spec(B_i)$ is an open immersion.

By Lemma 27.3.8 we can write each B_i as a directed colimit of subalgebras $A_{i,l} \subset B_i$, $l \in L_i$ each of finite type over $\mathbf{Z}$ and such that W_i is identified with an open subscheme of $Spec(A_{i,l})$. Let $S_{i,l}$ be the scheme obtained by glueing V_i and $Spec(A_{i,l})$ along the open W_i, see Schemes, Section 21.14. Here is the resulting commutative diagram of schemes:

$$\begin{tikzcd}
 & & V \arrow[dll] \arrow[d] & W \arrow[l] \arrow[dll] \arrow[d] \\
V_i \arrow[d] & W_i \arrow[l] \arrow[d] & S \arrow[dll] & U \arrow[l] \arrow[dll] \\
S_{i,l} & Spec(A_{i,l}) \arrow[l] & &
\end{tikzcd}$$

The morphism $S \to S_{i,l}$ arises because the upper right square is a push out in the category of schemes. Note that $S_{i,l}$ is of finite type over $\mathbf{Z}$ since it has a finite affine open covering whose members are spectra of finite type $\mathbf{Z}$-algebras. We define a partial ordering on $J = \coprod_{i \in I} L_i$ by the rule $(i', l') \geq (i, l)$ if and only if $i' \geq i$ and the map $B_i \to B_{i'}$ maps $A_{i,l}$ into $A_{i',l'}$. This is exactly the condition needed to define a morphism $S_{i',l'} \to S_{i,l}$: namely make a commutative diagram as above using the transition morphisms $V_{i'} \to V_i$ and $W_{i'} \to W_i$ and the morphism $Spec(A_{i',l'}) \to Spec(A_{i,l})$ induced by the ring map $A_{i,l} \to A_{i',l'}$. The relevant commutativities have been built into the constructions. We claim that S is the directed limit of the schemes $S_{i,l}$. Since by construction the schemes V_i have limit V this boils down to the fact that B is the limit of the rings $A_{i,l}$ which is true by construction. $\square$

27.4. Limits and morphisms of finite presentation

The following is a generalization of Algebra, Lemma 7.118.2.

Proposition 27.4.1. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

1. *The morphism f is locally of finite presentation.*
2. *For any directed partially ordered set I, and any inverse system $(T_i, f_{ii'})$ of S-schemes over I with each T_i affine, we have*

$$Mor_S(lim_i T_i, X) = colim_i Mor_S(T_i, X)$$

(3) *For any directed partially ordered set I, and any inverse system $(T_i, f_{ii'})$ of S-schemes over I with each $f_{ii'}$ affine and every T_i quasi-compact and quasi-separated as a scheme, we have*
$$Mor_S(lim_i T_i, X) = colim_i Mor_S(T_i, X)$$

Proof. It is clear that (3) implies (2).

Let us prove that (2) implies (1). Assume (2). Choose any affine opens $U \subset X$ and $V \subset S$ such that $f(U) \subset V$. We have to show that $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is of finite presentation. Let $(A_i, \varphi_{ii'})$ be a directed system of $\mathcal{O}_S(V)$-algebras. Set $A = colim_i A_i$. According to Algebra, Lemma 7.118.2 we have to show that
$$Hom_{\mathcal{O}_S(V)}(\mathcal{O}_X(U), A) = colim_i Hom_{\mathcal{O}_S(V)}(\mathcal{O}_X(U), A_i)$$
Consider the schemes $T_i = Spec(A_i)$. They form an inverse system of V-schemes over I with transition morphisms $f_{ii'} : T_i \to T_{i'}$ induced by the $\mathcal{O}_S(V)$-algebra maps $\varphi_{i'i}$. Set $T := Spec(A) = lim_i T_i$. The formula above becomes in terms of morphism sets of schemes
$$Mor_V(lim_i T_i, U) = colim_i Mor_V(T_i, U).$$
We first observe that $Mor_V(T_i, U) = Mor_S(T_i, U)$ and $Mor_V(T, U) = Mor_S(T, U)$. Hence we have to show that
$$Mor_S(lim_i T_i, U) = colim_i Mor_S(T_i, U)$$
and we are given that
$$Mor_S(lim_i T_i, X) = colim_i Mor_S(T_i, X).$$
Hence it suffices to prove that given a morphism $g_i : T_i \to X$ over S such that the composition $T \to T_i \to X$ ends up in U there exists some $i' \geq i$ such that the composition $g_{i'} : T_{i'} \to T_i \to X$ ends up in U. Denote $Z_{i'} = g_{i'}^{-1}(X \setminus U)$. Assume each $Z_{i'}$ is nonempty to get a contradiction. By Lemma 27.3.2 there exists a point t of T which is mapped into $Z_{i'}$ for all $i' \geq i$. Such a point is not mapped into U. A contradiction.

Finally, let us prove that (1) implies (3). Assume (1). Let an inverse directed system $(T_i, f_{ii'})$ of S-schemes be given. Assume the morphisms $f_{ii'}$ are affine and each T_i is quasi-compact and quasi-separated as a scheme. Let $T = lim_i T_i$. Denote $f_i : T \to T_i$ the projection morphisms. We have to show:

(a) Given morphisms $g_i, g_i' : T_i \to X$ over S such that $g_i \circ f_i = g_i' \circ f_i$, then there exists an $i' \geq i$ such that $g_i \circ f_{i'i} = g_i' \circ f_{i'i}$.
(b) Given any morphism $g : T \to X$ over S there exists an $i \in I$ and a morphism $g_i : T_i \to X$ such that $g = f_i \circ g_i$.

First let us prove the uniqueness part (a). Let $g_i, g_i' : T_i \to X$ be morphisms such that $g_i \circ f_i = g_i' \circ f_i$. For any $i' \geq i$ we set $g_{i'} = g_i \circ f_{i'i}$ and $g_{i'}' = g_i' \circ f_{i'i}$. We also set $g = g_i \circ f_i = g_i' \circ f_i$. Consider the morphism $(g_i, g_i') : T_i \to X \times_S X$. Set
$$W = \bigcup\nolimits_{U \subset X \text{ affine open}, V \subset S \text{ affine open}, f(U) \subset V} U \times_V U.$$
This is an open in $X \times_S X$, with the property that the morphism $\Delta_{X/S}$ factors through a closed immersion into W, see the proof of Schemes, Lemma 21.21.2. Note that the composition $(g_i, g_i') \circ f_i : T \to X \times_S X$ is a morphism into W because it factors through the diagonal by assumption. Set $Z_{i'} = (g_{i'}, g_{i'}')^{-1}(X \times_S X \setminus W)$. If each $Z_{i'}$ is nonempty, then by Lemma 27.3.2 there exists a point $t \in T$ which maps to $Z_{i'}$ for all $i' \geq i$. This is a contradiction with the fact that T maps into W. Hence we may increase i and assume that $(g_i, g_i') : T_i \to X \times_S X$

is a morphism into W. By construction of W, and since T_i is quasi-compact we can find a finite affine open covering $T_i = T_{1,i} \cup \ldots \cup T_{n,i}$ such that $(g_i, g_i')|_{T_{j,i}}$ is a morphism into $U \times_V U$ for some pair (U, V) as in the definition of W above. Since it suffices to prove that $g_{i'}$ and $g'_{i'}$ agree on each of the $f_{i'i}^{-1}(T_{j,i})$ this reduces us to the affine case. The affine case follows from Algebra, Lemma 7.118.2 and the fact that the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is of finite presentation (see Morphisms, Lemma 24.20.2).

Finally, we prove the existence part (b). Let $g : T \to X$ be a morphism of schemes over S. We can find a finite affine open covering $T = W_1 \cup \ldots \cup W_n$ such that for each $j \in \{1, \ldots, n\}$ there exist affine opens $U_j \subset X$ and $V_j \subset S$ with $f(U_j) \subset V_j$ and $g(W_j) \subset U_j$. By Lemmas 27.3.5 and 27.3.7 (after possibly shrinking I) we may assume that there exist affine open coverings $T_i = W_{1,i} \cup \ldots \cup W_{n,i}$ compatible with transition maps such that $W_j = lim_i\, W_{j,i}$. We apply Algebra, Lemma 7.118.2 to the rings corresponding to the affine schemes U_j, V_j, $W_{j,i}$ and W_j using that $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_j)$ is of finite presentation (see Morphisms, Lemma 24.20.2). Thus we can find for each j an index $i_j \in I$ and a morphism $g_{j,i_j} : W_{j,i_j} \to X$ such that $g_{j,i_j} \circ f_i|_{W_j} : W_j \to W_{j,i} \to X$ equals $g|_{W_j}$. By part (a) proved above, using the quasi-compactness of $W_{j_1,i} \cap W_{j_2,i}$ which follows as T_i is quasi-separated, we can find an index $i' \in I$ larger than all i_j such that

$$g_{j_1,i_{j_1}} \circ f_{i'i_{j_1}}|_{W_{j_1,i'} \cap W_{j_2,i'}} = g_{j_2,i_{j_2}} \circ f_{i'i_{j_2}}|_{W_{j_1,i'} \cap W_{j_2,i'}}$$

for all $j_1, j_2 \in \{1, \ldots, n\}$. Hence the morphisms $g_{j,i_j} \circ f_{i'i_j}|_{W_{j,i'}}$ glue to given the desired morphism $T_{i'} \to X$. □

Remark 27.4.2. Let S be a scheme. Let us say that a functor $F : (Sch/S)^{opp} \to Sets$ is *limit preserving* if for every directed inverse system $\{T_i\}_{i \in I}$ of affine schemes with limit T we have $F(T) = colim_i\, F(T_i)$. Let X be a scheme over S, and let $h_X : (Sch/S)^{opp} \to Sets$ be its functor of points, see Schemes, Section 21.15. In this terminology Proposition 27.4.1 says that a scheme X is locally of finite presentation over S if and only if h_X is limit preserving.

27.5. Finite type closed in finite presentation

A reference is [**Con07**].

Lemma 27.5.1. *Let $f : X \to S$ be a morphism of schemes. Assume:*

(1) *The morphism f is locally of finite type.*
(2) *The scheme X is quasi-compact and quasi-separated.*

Then there exists a morphism of finite presentation $f' : X' \to S$ and an immersion $X \to X'$ of schemes over S.

Proof. By Proposition 27.3.11 we can write $X = lim_i\, X_i$ with each X_i of finite type over $\mathbf{Z}$ and with transition morphisms $f_{ii'} : X_i \to X_{i'}$ affine. Consider the commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & X_{i,S} \arrow[r] \arrow[d] & X_i \arrow[d] \\ & S \arrow[r] & Spec(\mathbf{Z}) \end{tikzcd}$$

Note that X_i is of finite presentation over $Spec(\mathbf{Z})$, see Morphisms, Lemma 24.20.9. Hence the base change $X_{i,S} \to S$ is of finite presentation by Morphisms, Lemma 24.20.4. Thus it suffices to show that the arrow $X \to X_{i,S}$ is an immersion for some i sufficiently large.

To do this we choose a finite affine open covering $X = V_1 \cup \ldots \cup V_n$ such that f maps each V_j into an affine open $U_j \subset S$. Let $h_{j,a} \in \mathcal{O}_X(V_j)$ be a finite set of elements which generate $\mathcal{O}_X(V_j)$ as an $\mathcal{O}_S(U_j)$-algebra, see Morphisms, Lemma 24.14.2. By Lemmas 27.3.5 and 27.3.7 (after possibly shrinking I) we may assume that there exist affine open coverings $X_i = V_{1,i} \cup \ldots \cup V_{n,i}$ compatible with transition maps such that $V_j = lim_i V_{j,i}$. By Lemma 27.2.5 we can choose i so large that each $h_{j,a}$ comes from an element $h_{j,a,i} \in \mathcal{O}_{X_i}(V_{j,i})$. At this point it is clear that

$$V_j \longrightarrow U_j \times_{Spec(\mathbf{Z})} V_{j,i} = (V_{j,i})_{U_j} \subset (V_{j,i})_S \subset X_{i,S}$$

is a closed immersion. Since the union of the schemes which appear as the targets of these morphisms form an open of $X_{i,S}$ we win. □

Remark 27.5.2. We cannot do better than this if we do not assume more on S and the morphism $f : X \to S$. For example, in general it will not be possible to find a *closed* immersion $X \to X'$ as in the lemma. The reason is that this would imply that f is quasi-compact which may not be the case. An example is to take S to be infinite dimensional affine space with 0 doubled and X to be one of the two infinite dimensional affine spaces.

Lemma 27.5.3. *Let $f : X \to S$ be a morphism of schemes. Assume:*

(1) *The morphism f is of locally of finite type.*
(2) *The scheme X is quasi-compact and quasi-separated, and*
(3) *The scheme S is quasi-separated.*

Then there exists a morphism of finite presentation $f' : X' \to S$ and a closed immersion $X \to X'$ of schemes over S.

Proof. By Lemma 27.5.1 above there exists a morphism $Y \to S$ of finite presentation and an immersion $i : X \to Y$ of schemes over S. For every point $x \in X$, there exists an affine open $V_x \subset Y$ such that $i^{-1}(V_x) \to V_x$ is a closed immersion. Since X is quasi-compact we can find finitely may affine opens $V_1, \ldots, V_n \subset Y$ such that $i(X) \subset V_1 \cup \ldots \cup V_n$ and $i^{-1}(V_j) \to V_j$ is a closed immersion. In other words such that $i : X \to X' = V_1 \cup \ldots \cup V_n$ is a closed immersion of schemes over S. Since S is quasi-separated and Y is quasi-separated over S we deduce that Y is quasi-separated, see Schemes, Lemma 21.21.13. Hence the open immersion $X' = V_1 \cup \ldots \cup V_n \to Y$ is quasi-compact. This implies that $X' \to Y$ is of finite presentation, see Morphisms, Lemma 24.20.6. We conclude since then $X' \to Y \to S$ is a composition of morphisms of finite presentation, and hence of finite presentation (see Morphisms, Lemma 24.20.3). □

Lemma 27.5.4. *Let S be a scheme. Let I be a directed partially ordered set. Let $(X_i, f_{ii'})$ be an inverse system of schemes over S indexed by I. Assume*

(1) *the scheme S is quasi-separated,*
(2) *each X_i is locally of finite type over S,*
(3) *all the morphisms $f_{ii'} : X_i \to X_{i'}$ are affine,*
(4) *all the schemes X_i are quasi-compact and quasi-separated,*
(5) *the morphism $X = lim_i X_i \to S$ is separated.*

Then $X_i \to S$ is separated for all i large enough.

Proof. Let $i_0 \in I$. Note that I is nonempty as the limit is directed. For convenience write $X_0 = X_{i_0}$ and $i_0 = 0$. As X_0 is quasi-compact we can find finitely many affine opens $U_1, \ldots, U_n \subset S$ such that $X_0 \to S$ maps into $U_1 \cup \ldots \cup U_n$. Denote $h_i : X_i \to S$ the structure morphism. It suffices to check that for some $i \geq 0$ the morphisms $h_i^{-1}(U_j) \to U_j$

are separated for all $j = 1, \ldots, n$. Since S is quasi-separated the morphisms $U_j \to S$ are quasi-compact. Hence $h_i^{-1}(U_j)$ is quasi-compact and quasi-separated. In this way we reduce to the case S affine.

Assume S affine. Choose a finite affine open covering $X_0 = V_{1,0} \cup \ldots \cup V_{m,0}$. As usual we denote $V_{j,i}$ the inverse image of $V_{j,0}$ in X_i for $i \geq 0$. We also denote V_j the inverse image of $V_{j,0}$ in X. By assumption the intersections $V_{j_1,i} \cap V_{j_2,i}$ are quasi-compact opens. Since X is separated we see that $V_{j_1} \cap V_{j_2}$ is affine. Hence we see that $V_{j_1,i} \cap V_{j_2,i}$ are all affine for i big enough by Lemma 27.3.7. After increasing $i_0 = 0$ we may assume this holds for all $i \geq 0$. By Schemes, Lemma 21.21.8 we have to show that for some i big enough the ring map

$$\mathcal{O}_{X_i}(V_{j_1,i}) \otimes_{\mathcal{O}_S(S)} \mathcal{O}_{X_i}(V_{j_2,i}) \longrightarrow \mathcal{O}_{X_i}(V_{j_1,i} \cap V_{j_2,i})$$

is surjective. Since $V_{j,i}$ is the inverse image of $V_{j,0}$ under the affine transition maps f_{i0} we see that

$$V_{j_1,i} \cap V_{j_2,i} = V_{j_1,i} \times_{V_{j_1,0}} (V_{j_1,0} \cap V_{j_2,0})$$

Choose generators $x_{j_1,j_2,\alpha} \in \mathcal{O}_{X_0}(V_{j_1,0} \cap V_{j_2,0})$ as an algebra over $\mathcal{O}_{X_0}(V_{j_1,0})$. We can choose finitely many of these since $\mathcal{O}_{X_0}(V_{j_1,0} \cap V_{j_2,0})$ is a finite type $\mathcal{O}_S(S)$-algebra, see Morphisms, Lemma 24.14.2. By the displayed equality of fibre products, the images of $x_{j_1,j_2,\alpha}$ generate $\mathcal{O}_{X_i}(V_{j_1,i} \cap V_{j_2,i})$ as an algebra over $\mathcal{O}_{X_i}(V_{j_1,i})$ also. Since X is separated the ring maps

$$\mathcal{O}_X(V_{j_1}) \otimes_{\mathcal{O}_S(S)} \mathcal{O}_X(V_{j_2,i}) \longrightarrow \mathcal{O}_X(V_{j_1} \cap V_{j_2})$$

are surjective. Hence we can find finite sums

$$\sum y_{j_1,j_2,\alpha,\beta} \otimes z_{j_1,j_2,\alpha,\beta}$$

in the left hand side which map to the elements $x_{j_1,j_2,\alpha}$ of the right hand side. Using Lemma 27.2.5 we may choose i large enough so that each of the (finitely many) elements $y_{j_1,j_2,\alpha,\beta}$ (resp. $z_{j_1,j_2,\alpha,\beta}$) comes from a corresponding element $y_{j_1,j_2,\alpha,\beta,i}$ (resp. $z_{j_1,j_2,\alpha,\beta,i}$) of $\mathcal{O}_{X_i}(V_{j_1,i})$ (resp. $\mathcal{O}_{X_i}(V_{j_2,i})$ and moreover such that the image of

$$\sum y_{j_1,j_2,\alpha,\beta,i} \otimes z_{j_1,j_2,\alpha,\beta,i}$$

is the image of the element $x_{j_1,j_2,\alpha}$ in $\mathcal{O}_{X_i}(V_{j_1,i} \cap V_{j_2,i})$. This clearly implies the desired surjectivity and we win. □

Remark 27.5.5. Is there an easy example to show that the finite type condition for the morphisms $X_i \to S$ is necessary? Email if you have one.

A less technical version of the results above is the following.

Proposition 27.5.6. *Let $f : X \to S$ be a morphism of schemes. Assume:*

1. *The morphism f is of finite type and separated.*
2. *The scheme S is quasi-compact and quasi-separated.*

Then there exists a separated morphism of finite presentation $f' : X' \to S$ and a closed immersion $X \to X'$ of schemes over S.

Proof. We have seen that there is a closed immersion $X \to Y$ with Y/S of finite presentation. Let $\mathcal{I} \subset \mathcal{O}_Y$ be the quasi-coherent sheaf of ideals defining X as a closed subscheme of Y. By Properties, Lemma 23.20.3 we can write $\mathcal{I}$ as a directed colimit $\mathcal{I} = colim_{a \in A} \mathcal{I}_a$ of its quasi-coherent sheaves of ideals of finite type. Let $X_a \subset Y$ be the closed subscheme defined by $\mathcal{I}_a$. These form an inverse system of schemes indexed by A. The transition morphisms $X_a \to X_{a'}$ are affine because they are closed immersions. Each X_a is quasi-compact and quasi-separated since it is a closed subscheme of Y and Y is quasi-compact and quasi-separated by our assumptions. We have $X = lim_a X_a$ as follows directly from the fact that $\mathcal{I} = colim_{a\in A} \mathcal{I}_a$. Each of the morphisms $X_a \to Y$ is of finite presentation, see Morphisms, Lemma 24.20.7. Hence the morphisms $X_a \to S$ are of finite presentation. Thus it suffices to show that $X_a \to S$ is separated for some $a \in A$. This follows from Lemma 27.5.4 as we have assumed that $X \to S$ is separated. □

We end this section with a variant concerning finite morphisms.

Lemma 27.5.7. *Let $f : X \to S$ be a morphism of schemes. Assume:*

(1) *The morphism f is finite.*
(2) *The scheme S is quasi-compact and quasi-separated.*

Then there exists a morphism which is finite and of finite presentation $f' : X' \to S$ and a closed immersion $X \to X'$ of schemes over S.

Proof. By Proposition 27.5.6 there is a closed immersion $X \to Y$ with $g : Y \to S$ separated and of finite presentation. Let $\mathcal{I} \subset \mathcal{O}_Y$ be the quasi-coherent sheaf of ideals defining X as a closed subscheme of Y. By Properties, Lemma 23.20.3 we can write $\mathcal{I}$ as a directed colimit $\mathcal{I} = colim_{a\in A} \mathcal{I}_a$ of its quasi-coherent sheaves of ideals of finite type. Let $X_a \subset Y$ be the closed subscheme defined by $\mathcal{I}_a$ and denote $f_a : X_a \to S$ the structure morphism. These form an inverse system of schemes indexed by A. The transition morphisms $X_a \to X_{a'}$ are affine because they are closed immersions. Each X_a is quasi-compact and separated over S since it is a closed subscheme of Y and Y is quasi-compact and separated over S. We have $X = lim_a X_a$ as follows directly from the fact that $\mathcal{I} = colim_{a\in A} \mathcal{I}_a$. Each of the morphisms $X_a \to Y$ is of finite presentation, see Morphisms, Lemma 24.20.7. Hence the morphisms $X_a \to S$ are of finite presentation. Thus it suffices to show that $f_a : X_a \to S$ is finite for some $a \in A$.

Choose a finite affine open covering $S = \bigcup_{j=1,\ldots,n} V_j$. For each j the scheme $f^{-1}(V_j) = lim_a f_a^{-1}(V_j)$ is affine (as a finite morphism is affine by definition). Hence by Lemma 27.3.7 there exists an $a \in A$ such that each $f_a^{-1}(V_j)$ is affine. In other words, $f_a : X_a \to S$ is affine, see Morphisms, Lemma 24.11.3. By replacing Y with X_a we may assume $g : Y \to S$ is affine.

For each $j = 1, \ldots, m$ the ring $\mathcal{O}_Y(g^{-1}(V_j))$ is a finitely presented $\mathcal{O}_S(V_j)$-algebra. Say it is generated by x_{ji}, $i = 1, \ldots, n_j$. Note that the images of x_{ji} in $\mathcal{O}_X(f_a^{-1}(V_j))$, resp. $\mathcal{O}_X(f^{-1}(V_j))$ generate over $\mathcal{O}_S(V_j)$ as well. Since $f : X \to S$ is finite, the image of x_{ji} in $\mathcal{O}_X(f^{-1}(V_j))$ satisfies a monic polynomial P_{ij} whose coefficients are elements of $\mathcal{O}_S(V_j)$. Since $\mathcal{O}_X(f^{-1}(V_j)) = colim_{a\in A} \mathcal{O}_{X_a}(f_a^{-1}(V_j))$ we see there exists an $a \in A$ such that $P_{ji}(x_{ij})$ maps to zero in $\mathcal{O}_{X_a}(f_a^{-1}(V_j))$ for all j, i. It follows from Morphisms, Lemma 24.42.3 that the morphism $f_a : X_a \to S$ is finite for this a. □

27.6. Descending relative objects

The following lemma is typical of the type of results in this section. We write out the ``standard'' proof completely. It may be faster to convince yourself that the result is true than to read this proof.

Lemma 27.6.1. *Let I be a directed partially ordered set. Let $(S_i, f_{ii'})$ be an inverse system of schemes over I. Assume*

(1) *all the morphisms $f_{ii'} : S_i \to S_{i'}$ are affine,*
(2) *all the schemes S_i are quasi-compact and quasi-separated.*

Let $S = lim_i S_i$. Then we have the following:

(1) *For any morphism of finite presentation $X \to S$ there exists an index $i \in I$ and a morphism of finite presentation $X_i \to S_i$ such that $X \cong X_{i,S}$ as schemes over S.*
(2) *Suppose given an index $i \in I$, schemes X_i, Y_i of finite presentation over S_i and a morphism $\varphi : X_{i,S} \to Y_{i,S}$ over S. Then there exists an index $i' \geq i$ and a morphism $\varphi_{i'} : X_{i,S_{i'}} \to Y_{i,S_{i'}}$ whose base change to S is φ.*
(3) *Suppose given an index $i \in I$, schemes X_i, Y_i of finite presentation over S_i and a pair of morphisms $\varphi_i, \psi_i : X_i \to Y_i$. Assume that the base changes are equal: $\varphi_{i,S} = \psi_{i,S}$. Then there exists an index $i' \geq i$ such that $\varphi_{i,S_{i'}} = \psi_{i,S_{i'}}$.*

In other words, the category of schemes of finite presentation over S is the colimit over I of the categories of schemes of finite presentation over S_i.

Proof. In case each of the schemes S_i is affine, and we consider only affine schemes of finite presentation over S_i, resp. S this lemma is equivalent to Algebra, Lemma 7.118.6. We claim that the affine case implies the lemma in general.

Let us prove (3). Suppose given an index $i \in I$, schemes X_i, Y_i of finite presentation over S_i and a pair of morphisms $\varphi_i, \psi_i : X_i \to Y_i$. Assume that the base changes are equal: $\varphi_{i,S} = \psi_{i,S}$. We will use the notation $X_{i'} = X_{i,S_{i'}}$ and $Y_{i'} = Y_{i,S_{i'}}$ for $i' \geq i$. We also set $X = X_{i,S}$ and $Y = Y_{i,S}$. Note that according to Lemma 27.2.4 we have $X = lim_{i' \geq i} X_{i'}$ and similarly for Y. Additionally we denote $\varphi_{i'}$ and $\psi_{i'}$ (resp. φ and ψ) the base change of φ_i and ψ_i to $S_{i'}$ (resp. S). So our assumption means that $\varphi = \psi$. Since Y_i and X_i are of finite presentation over S_i, and since S_i is quasi-compact and quasi-separated, also X_i and Y_i are quasi-compact and quasi-separated (see Morphisms, Lemma 24.20.10). Hence we may choose a finite affine open covering $Y_i = \bigcup V_{j,i}$ such that each $V_{j,i}$ maps into an affine open of S. As above, denote $V_{j,i'}$ the inverse image of $V_{j,i}$ in $Y_{i'}$ and V_j the inverse image in Y. The immersions $V_{j,i'} \to Y_{i'}$ are quasi-compact, and the inverse images $U_{j,i'} = \varphi_i^{-1}(V_{j,i'})$ and $U'_{j,i'} = \psi_i^{-1}(V_{j,i'})$ are quasi-compact opens of $X_{i'}$. By assumption the inverse images of V_j under φ and ψ in X are equal. Hence by Lemma 27.3.5 there exists an index $i' \geq i$ such that of $U_{j,i'} = U'_{j,i'}$ in $X_{i'}$. Choose an finite affine open covering $U_{j,i'} = U'_{j,i'} = \bigcup W_{j,k,i'}$ which induce coverings $U_{j,i''} = U'_{j,i''} = \bigcup W_{j,k,i''}$ for all $i'' \geq i'$. By the affine case there exists an index i'' such that $\varphi_{i''}|_{W_{j,k,i''}} = \psi_{i''}|_{W_{j,k,i''}}$ for all j, k. Then i'' is an index such that $\varphi_{i''} = \psi_{i''}$ and (3) is proved.

Let us prove (2). Suppose given an index $i \in I$, schemes X_i, Y_i of finite presentation over S_i and a morphism $\varphi : X_{i,S} \to Y_{i,S}$. We will use the notation $X_{i'} = X_{i,S_{i'}}$ and $Y_{i'} = Y_{i,S_{i'}}$ for $i' \geq i$. We also set $X = X_{i,S}$ and $Y = Y_{i,S}$. Note that according to Lemma 27.2.4 we have $X = lim_{i' \geq i} X_{i'}$ and similarly for Y. Since Y_i and X_i are of finite presentation over S_i, and since S_i is quasi-compact and quasi-separated, also X_i and Y_i are quasi-compact and quasi-separated (see Morphisms, Lemma 24.20.10). Hence we may choose a finite affine open covering $Y_i = \bigcup V_{j,i}$ such that each $V_{j,i}$ maps into an affine open of S. As above, denote $V_{j,i'}$ the inverse image of $V_{j,i}$ in $Y_{i'}$ and V_j the inverse image in Y. The immersions $V_j \to Y$ are quasi-compact, and the inverse images $U_j = \varphi^{-1}(V_j)$ are quasi-compact opens of X. Hence by Lemma 27.3.5 there exists an index $i' \geq i$ and quasi-compact opens $U_{j,i'}$ of $X_{i'}$ whose inverse image in X is U_j. Choose an finite affine open covering $U_{j,i'} = \bigcup W_{j,k,i'}$ which induce affine open coverings $U_{j,i''} = \bigcup W_{j,k,i''}$ for all $i'' \geq i'$ and an affine open covering $U_j = \bigcup W_{j,k}$. By the affine case there exists an index i'' and morphisms $\varphi_{j,k,i''} : W_{j,k,i''} \to V_{j,i''}$ such that $\varphi|_{W_{j,k}} = \varphi_{j,k,i'',S}$ for all j, k. By part (3) proved above, there is a further index $i''' \geq i''$ such that

$$\varphi_{j_1,k_1,i'',S_{i'''}}|_{W_{j_1,k_1,i'''} \cap W_{j_2,k_2,i'''}} = \varphi_{j_2,k_2,i'',S_{i'''}}|_{W_{j_1,k_1,i'''} \cap W_{j_2,k_2,i'''}}$$

for all j_1, j_2, k_1, k_2. Then i''' is an index such that there exists a morphism $\varphi_{i'''} : X_{i'''} \to Y_{i'''}$ whose base change to S gives φ. Hence (2) holds.

Let us prove (1). Suppose given a scheme X of finite presentation over S. Since X is of finite presentation over S, and since S is quasi-compact and quasi-separated, also X is quasi-compact and quasi-separated (see Morphisms, Lemma 24.20.10). Choose a finite affine open covering $X = \bigcup U_j$ such that each U_j maps into an affine open $V_j \subset S$. Denote $U_{j_1 j_2} = U_{j_1} \cap U_{j_2}$ and $U_{j_1 j_2 j_3} = U_{j_1} \cap U_{j_2} \cap U_{j_3}$. By Lemmas 27.3.5 and 27.3.7 we can find an index i_1 and affine opens $V_{j,i_1} \subset S_{i_1}$ such that each V_j is the inverse of this in S. Let $V_{j,i}$ be the inverse image of V_{j,i_1} in S_i for $i \geq i_1$. By the affine case we may find an index $i_2 \geq i_1$ and affine schemes $U_{j,i_2} \to V_{j,i_2}$ such that $U_j = S \times_{S_{i_2}} U_{j,i_2}$ is the base change. Denote $U_{j,i} = S_i \times_{S_{i_2}} U_{j,i_2}$ for $i \geq i_2$. By Lemma 27.3.5 there exists an index $i_3 \geq i_2$ and open subschemes $W_{j_1,j_2,i_3} \subset U_{j_1,i_3}$ whose base change to S is equal to $U_{j_1 j_2}$. Denote $W_{j_1,j_2,i} = S_i \times_{S_{i_3}} W_{j_1,j_2,i_3}$ for $i \geq i_3$. By part (2) shown above there exists an index $i_4 \geq i_3$ and morphisms $\varphi_{j_1,j_2,i_4} : W_{j_1,j_2,i_4} \to W_{j_2,j_1,i_4}$ whose base change to S gives the identity morphism $U_{j_1 j_2} = U_{j_2 j_1}$ for all j_1, j_2. For all $i \geq i_4$ denote $\varphi_{j_1,j_2,i} = \mathrm{id}_S \times \varphi_{j_1,j_2,i_4}$ the base change. We claim that for some $i_5 \geq i_4$ the system $((U_{j,i_5})_j, (W_{j_1,j_2,i_5})_{j_1,j_2}, (\varphi_{j_1,j_2,i_5})_{j_1,j_2})$ forms a glueing datum as in Schemes, Section 21.14. In order to see this we have to verify that for i large enough we have

$$\varphi_{j_1,j_2,i}^{-1}(W_{j_1,j_2,i} \cap W_{j_1,j_3,i}) = W_{j_1,j_2,i} \cap W_{j_1,j_3,i}$$

and that for large enough i the cocycle condition holds. The first condition follows from Lemma 27.3.5 and the fact that $U_{j_2 j_1 j_3} = U_{j_1 j_2 j_3}$. The second from part (1) of the lemma proved above and the fact that the cocycle condition holds for the maps $\mathrm{id} : U_{j_1 j_2} \to U_{j_2 j_1}$. Ok, so now we can use Schemes, Lemma 21.14.2 to glue the system $((U_{j,i_5})_j, (W_{j_1,j_2,i_5})_{j_1,j_2}, (\varphi_{j_1,j_2,i_5})_{j_1,j_2})$ to get a scheme $X_{i_5} \to S_{i_5}$. By construction the base change of X_{i_5} to S is formed by glueing the open affines U_j along the opens $U_{j_1} \leftarrow U_{j_1 j_2} \to U_{j_2}$. Hence $S \times_{S_{i_5}} X_{i_5} \cong X$ as desired. □

Lemma 27.6.2. *With notation and assumptions as in Lemma 27.6.1. Let $i \in I$. Suppose that $\varphi_i : X_i \to Y_i$ is a morphism of schemes of finite presentation over S_i. If the base change of φ_i to S is affine, then there exists an index $i' \geq i$ such that $id_{S_{i'}} \times \varphi_i : X_{i,S_{i'}} \to Y_{i,S_{i'}}$ is affine.*

Proof. For $i' \geq i$ denote $X_{i'} = S_{i'} \times_{S_i} X_i$ and similarly for $Y_{i'}$. Denote $\varphi_{i'}$ the base change of φ_i to $S_{i'}$. Also set $X = S \times_{S_i} X_i$, $Y = S \times_{S_i} X_i$, and φ the base change of φ_i to S. Let $Y_i = \bigcup V_{j,i}$ be a finite affine open covering. Set $U_{j,i} = \varphi_i^{-1}(V_{j,i})$. For $i' \geq i$ we denote $V_{j,i'}$ the inverse image of $V_{j,i}$ in $Y_{i'}$ and $U_{j,i'} = \varphi_{i'}^{-1}(V_{j,i'})$. Similarly we have $U_j = \varphi^{-1}(V_j)$. Then $U_j = lim_{i' \geq i} U_{j,i'}$ (see Lemma 27.2.2). Since U_j is affine by assumption we see that each $U_{j,i'}$ is affine for i' large enough, see Lemma 27.3.7. Thus $\varphi_{i'}$ is affine for i' large enough, see Morphisms, Lemma 24.11.3. □

Lemma 27.6.3. *With notation and assumptions as in Lemma 27.6.1. Let $i \in I$. Suppose that $\varphi_i : X_i \to Y_i$ is a morphism of schemes of finite presentation over S_i. If the base change* *of φ_i to S is flat, then there exists an index $i' \geq i$ such that $id_{S_{i'}} \times \varphi_i : X_{i,S_{i'}} \to Y_{i,S_{i'}}$ is flat.*

Proof. For $i' \geq i$ denote $X_{i'} = S_{i'} \times_{S_i} X_i$ and similarly for $Y_{i'}$. Denote $\varphi_{i'}$ the base change of φ_i to $S_{i'}$. Also set $X = S \times_{S_i} X_i$, $Y = S \times_{S_i} X_i$, and φ the base change of φ_i to S. Let $Y_i = \bigcup_{j=1,\ldots,m} V_{j,i}$ be a finite affine open covering such that each $V_{j,i}$ maps into some affine open of S_i. For each $j = 1, \ldots m$ let $\varphi_i^{-1}(V_{j,i}) = \bigcup_{k=1,\ldots,m(j)} U_{k,j,i}$ be a finite affine open covering. For $i' \geq i$ we denote $V_{j,i'}$ the inverse image of $V_{j,i}$ in $Y_{i'}$ and $U_{k,j,i'}$ the inverse image of $U_{k,j,i}$ in $X_{i'}$. Similarly we have $U_{k,j} \subset X$ and $V_j \subset Y$. Then $U_{k,j} = lim_{i' \geq i} U_{k,j,i'}$ and $V_j = lim_{i' \geq i} V_j$ (see Lemma 27.2.2). Hence we see that the lemma reduces to the case that X_i and Y_i are affine and map into an affine open of S_i, i.e., we may also assume that S is affine.

In the affine case we reduce to the following algebra result. Suppose that $R = colim_{i \in I} R_i$. For some $i \in I$ suppose given a map $A_i \to B_i$ of finitely presented R_i-algebras. Then, if $R \otimes_{R_i} A_i \to R \otimes_{R_i} B_i$ is flat, then for some $i' \geq i$ the map $R_{i'} \otimes_{R_i} A_i \to R_{i'} \otimes_{R_i} B_i$ is flat. This follows from Algebra, Lemma 7.120.5 part (3). □

Lemma 27.6.4. *With notation and assumptions as in Lemma 27.6.1. Let $i \in I$. Suppose that $\varphi_i : X_i \to Y_i$ is a morphism of schemes of finite presentation over S_i. If the base change of φ_i to S is a finite morphism, then there exists an index $i' \geq i$ such that $id_{S_{i'}} \times \varphi_i : X_{i,S_{i'}} \to Y_{i,S_{i'}}$ is a finite morphism.*

Proof. A finite morphism is affine, see Morphisms, Definition 24.42.1. Hence by Lemma 27.6.2 above we may assume that φ_i is affine. By writing Y_i as a finite union of affines we reduce to proving the result when X_i and Y_i are affine and map into a common affine $W_i \subset S_i$. The corresponding algebraic statement is the following: Suppose that $A = colim_i A_i$ is a directed colimit of rings, i an index, $A_i \to B_i$ and $A_i \to C_i$ ring maps of finite presentation, and $\varphi_i : B_i \to C_i$ a map of A_i-algebras such that

$$colim_{i' \geq i}(A_{i'} \otimes_{A_i} B_i) \longrightarrow colim_{i' \geq i}(A_{i'} \otimes_{A_i} C_i)$$

is finite. Then for some i' the map

$$A_{i'} \otimes_{A_i} B_i \longrightarrow A_{i'} \otimes_{A_i} C_i$$

is finite. The proof of this statement is omitted. (Hint: It suffices for C_i to be of finite type over A_i). □

Lemma 27.6.5. *With notation and assumptions as in Lemma 27.6.1. Let $i \in I$. Suppose that $\varphi_i : X_i \to Y_i$ is a morphism of schemes of finite presentation over S_i. If the base change of φ_i to S is a closed immersion, then there exists an index $i' \geq i$ such that $id_{S_{i'}} \times \varphi_i : X_{i,S_{i'}} \to Y_{i,S_{i'}}$ is a closed immersion.*

Proof. A closed immersion is affine, see Morphisms, Lemma 24.11.9. Hence by Lemma 27.6.2 above we may assume that φ_i is affine. By writing Y_i as a finite union of affines we reduce to proving the result when X_i and Y_i are affine and map into a common affine $W_i \subset S_i$. The corresponding algebraic statement is the following: Suppose that $A = colim_i A_i$ is a directed colimit of rings, i an index, $A_i \to B_i$ and $A_i \to C_i$ ring maps of finite presentation, and $\varphi_i : B_i \to C_i$ a map of A_i-algebras such that

$$colim_{i' \geq i}(A_{i'} \otimes_{A_i} B_i) \longrightarrow colim_{i' \geq i}(A_{i'} \otimes_{A_i} C_i)$$

is surjective. Then for some i' the map

$$A_{i'} \otimes_{A_i} B_i \longrightarrow A_{i'} \otimes_{A_i} C_i$$

is surjective. The proof of this statement is omitted. (Hint: It suffices for C_i to be of finite type over A_i). □

Lemma 27.6.6. *With notation and assumptions as in Lemma 27.6.1. Let $i \in I$. Suppose that X_i is a scheme of finite presentation over S_i. If the base change of X_i to S is separated over S then there exists an index $i' \geq i$ such that $X_{i,S_{i'}}$ is separated over $S_{i'}$.*

Proof. Apply Lemma 27.6.5 above to the diagonal morphism $\Delta_{X_i/S_i} : X_i \to X_i \times_{S_i} X_i$. □

Lemma 27.6.7. *With notation and assumptions as in Lemma 27.6.1. Let $i \in I$. Suppose that $\varphi_i : X_i \to Y_i$ is a morphism of schemes of finite presentation over S. If the base change of φ_i to S is finite locally free (of degree d) then there exists an index $i' \geq i$ such that the base change of φ_i to $S_{i'}$ is finite locally free (of degree d).*

Proof. By Lemmas 27.6.3 and 27.6.4 we see that we may reduce to the case that φ_i is flat and finite. On the other hand, φ_i is locally of finite presentation by Morphisms, Lemma 24.20.11. Hence φ_i is finite locally free by Morphisms, Lemma 24.44.2. If moreover $\varphi_i \times S$ is finite locally free of degree d, then the image of $Y_i \times_{S_i} S \to Y_i$ is contained in the open and closed locus $W_d \subset Y_i$ over which φ_i has degree d. By Lemma 27.3.4 we see that for some $i' \gg i$ the image of $Y_{i'} \to Y_i$ is contained in W_d. Then the base change of φ_i to $S_{i'}$ will be finite locally free of degree d. □

Lemma 27.6.8. *Let I be a directed partially ordered set. Let $(S_i, f_{ii'})$ be an inverse system of schemes over I. Assume*

1. *all the morphisms $f_{ii'} : S_i \to S_{i'}$ are affine,*
2. *all the schemes S_i are quasi-compact and quasi-separated.*

Let $S = lim_i S_i$. Then we have the following:

1. *For any sheaf of $\mathcal{O}_S$-modules $\mathcal{F}$ of finite presentation there exists an index $i \in I$ and a sheaf of $\mathcal{O}_{S_i}$-modules of finite presentation $\mathcal{F}_i$ such that $\mathcal{F} \cong f_i^*\mathcal{F}_i$.*
2. *Suppose given an index $i \in I$, sheaves of $\mathcal{O}_{S_i}$-modules $\mathcal{F}_i$, $\mathcal{G}_i$ of finite presentation and a morphism $\varphi : f_i^*\mathcal{F}_i \to f_i^*\mathcal{G}_i$ over S. Then there exists an index $i' \geq i$ and a morphism $\varphi_{i'} : f_{i'i}^*\mathcal{F}_i \to f_{i'i}^*\mathcal{G}_i$ whose base change to S is φ.*
3. *Suppose given an index $i \in I$, sheaves of $\mathcal{O}_{S_i}$-modules $\mathcal{F}_i$, $\mathcal{G}_i$ of finite presentation and a pair of morphisms $\varphi_i, \psi_i : \mathcal{F}_i \to \mathcal{G}_i$. Assume that the base changes are equal: $f_i^*\varphi_i = f_i^*\psi_i$. Then there exists an index $i' \geq i$ such that $f_{i'i}^*\varphi_i = f_{i'i}^*\psi_i$.*

In other words, the category of modules of finite presentation over S is the colimit over I of the categories modules of finite presentation over S_i.

Proof. Omitted. Since we have written out completely the proof of Lemma 27.6.1 above it seems wise to use this here and not completely write this proof out also. For example we can use:

1. there is an equivalence of categories between quasi-coherent $\mathcal{O}_S$-modules and vector bundles over S, see Constructions, Section 22.6.
2. a vector bundle $\mathbf{V}(\mathcal{F}) \to S$ is of finite presentation over S if and only if $\mathcal{F}$ is an $\mathcal{O}_S$-module of finite presentation.

Then you can descend morphisms in terms of morphisms of the associated vectorbundles. Similarly for objects. □

Lemma 27.6.9. *With notation and assumptions as in Lemma 27.6.1. Let $i \in I$. Suppose that $\varphi_i : X_i \to Y_i$ is a morphism of schemes of finite presentation over S_i and that $\mathcal{F}_i$ is a quasi-coherent $\mathcal{O}_{X_i}$-module of finite presentation. If the pullback of $\mathcal{F}_i$ to $X_i \times_{S_i} S$ is flat* *over $Y_i \times_{S_i} S$, then there exists an index $i' \geq i$ such that the pullback of $\mathcal{F}_i$ to $X_i \times_{S_i} S_{i'}$ is flat over $Y_i \times_{S_i} S_{i'}$.*

Proof. (This lemma is the analogue of Lemma 27.6.3 for modules.) For $i' \geq i$ denote $X_{i'} = S_{i'} \times_{S_i} X_i$, $\mathcal{F}_{i'} = (X_{i'} \to X_i)^*\mathcal{F}_i$ and similarly for $Y_{i'}$. Denote $\varphi_{i'}$ the base change of φ_i to $S_{i'}$. Also set $X = S \times_{S_i} X_i$, $Y = S \times_{S_i} X_i$, $\mathcal{F} = (X \to X_i)^*\mathcal{F}_i$ and φ the base change of φ_i to S. Let $Y_i = \bigcup_{j=1,\ldots,m} V_{j,i}$ be a finite affine open covering such that each $V_{j,i}$ maps into some affine open of S_i. For each $j = 1, \ldots m$ let $\varphi_i^{-1}(V_{j,i}) = \bigcup_{k=1,\ldots,m(j)} U_{k,j,i}$ be a finite affine open covering. For $i' \geq i$ we denote $V_{j,i'}$ the inverse image of $V_{j,i}$ in $Y_{i'}$ and $U_{k,j,i'}$ the inverse image of $U_{k,j,i}$ in $X_{i'}$. Similarly we have $U_{k,j} \subset X$ and $V_j \subset Y$. Then $U_{k,j} = lim_{i' \geq i} U_{k,j,i'}$ and $V_j = lim_{i' \geq i} V_j$ (see Lemma 27.2.2). Since $X_{i'} = \bigcup_{k,j} U_{k,j,i'}$ is a finite open covering it suffices to prove the lemma for each of the morphisms $U_{k,j,i} \to V_{j,i}$ and the sheaf $\mathcal{F}_i|_{U_{k,j,i}}$. Hence we see that the lemma reduces to the case that X_i and Y_i are affine and map into an affine open of S_i, i.e., we may also assume that S is affine.

In the affine case we reduce to the following algebra result. Suppose that $R = colim_{i \in I} R_i$. For some $i \in I$ suppose given a map $A_i \to B_i$ of finitely presented R_i-algebras. Let N_i be a finitely presented B_i-module. Then, if $R \otimes_{R_i} N_i$ is flat over $R \otimes_{R_i} A_i$, then for some $i' \geq i$ the module $R_{i'} \otimes_{R_i} N_i$ is flat over $R_{i'} \otimes_{R_i} A$. This is exactly the result proved in Algebra, Lemma 7.120.5 part (3). $\square$

27.7. Characterizing affine schemes

If $f : X \to S$ is a surjective integral morphism of schemes such that X is an affine scheme then S is affine too. See [**Con07**, A.2]. Our proof relies on the Noetherian case which we stated and proved in Coherent, Lemma 25.15.3. See also [**DG67**, II 6.7.1].

Lemma 27.7.1. *Let $f : X \to S$ be a morphism of schemes. Assume that f is surjective and finite, and assume that X is affine. Then S is affine.*

Proof. Since f is surjective and X is quasi-compact we see that S is quasi-compact. Consider the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\Delta"] \arrow[d] & X \times X \arrow[d] \\ S \arrow[r, "\Delta"] & S \times S \end{tikzcd}$$

(products over $Spec(\mathbf{Z})$). Since X is separated the image of the top horizontal arrow is closed. The right vertical arrow is the composition of $X \times X \to X \times S \to S \times S$ and hence is finite (see Morphisms, Lemmas 24.42.5 and 24.42.6). Hence it is proper (see Morphisms, Lemma 24.42.10). Thus the image of $\Delta(X)$ in $S \times S$ is closed. But as $X \to S$ is surjective we conclude that $\Delta(S)$ is closed as well. Hence S is separated.

By Lemma 27.5.7 there exists a factorization $X \to Y \to S$, with $X \to Y$ a closed immersion and $Y \to S$ finite and of finite presentation. Let $\mathcal{I} \subset \mathcal{O}_Y$ be the quasi-coherent sheaf of ideals cutting out the closed subscheme X in Y. By Properties, Lemma 23.20.3 we can write $\mathcal{I}$ as a directed colimit $\mathcal{I} = colim_{a \in A} \mathcal{I}_a$ of its quasi-coherent sheaves of ideals of finite type. Let $X_a \subset Y$ be the closed subscheme defined by $\mathcal{I}_a$. These form an inverse system of schemes indexed by A. The transition morphisms $X_a \to X_{a'}$ are affine because they are closed immersions. Each X_a is quasi-compact and quasi-separated since it is a closed subscheme of Y and Y is quasi-compact and quasi-separated. Each of the morphisms $X_a \to Y$ is of finite presentation, see Morphisms, Lemma 24.20.7. Hence the morphisms $X_a \to S$ are of finite presentation, and also finite as the composition of a closed immersion and a finite morphism. We have $X = lim_a X_a$ as follows directly from the fact that $\mathcal{I} = colim_{a \in A} \mathcal{I}_a$. Hence by Lemma 27.3.7 we see that X_a is affine for some $a \in A$. Replacing X by X_a we may assume that $X \to S$ is surjective, finite, of finite presentation and that X is affine.

By Proposition 27.3.11 we may write $S = lim_{i \in I} S_i$ as a directed limits as schemes of finite type over $\mathbf{Z}$. By Lemma 27.6.1 we can after shrinking I assume there exist schemes $X_i \to S_i$ of finite presentation such that $X_{i'} = X_i \times_S S_{i'}$ for $i' \geq i$ and such that $X = lim_i X_i$. By Lemma 27.6.4 we may assume that $X_i \to S_i$ is finite for all $i \in I$ as well. By Lemma 27.3.7 once again we may assume that X_i is affine for all $i \in I$. Hence the result follows from the Noetherian case, see Coherent, Lemma 25.15.3. □

Proposition 27.7.2. *Let $f : X \to S$ be a morphism of schemes. Assume that f is surjective and integral, and assume that X is affine. Then S is affine.*

Proof. Since f is surjective and X is quasi-compact we see that S is quasi-compact. Consider the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\Delta"'] \arrow[d] & X \times X \arrow[d] \\ S \arrow[r, "\Delta"] & S \times S \end{tikzcd}$$

(products over $Spec(\mathbf{Z})$). Since X is separated the image of the top horizontal arrow is closed. The right vertical arrow is the composition of $X \times X \to X \times S \to S \times S$ and hence is integral (see Morphisms, Lemmas 24.42.5 and 24.42.6). Hence it is universally closed (see Morphisms, Lemma 24.42.7). Thus the image of $\Delta(X)$ in $S \times S$ is closed. But as $X \to S$ is surjective we conclude that $\Delta(S)$ is closed as well. Hence S is separated. This in particular implies that f is an affine morphism, see Morphisms, Lemma 24.11.11.

Consider the sheaf $\mathcal{A} = f_* \mathcal{O}_X$. This is a quasi-coherent sheaf of $\mathcal{O}_S$-algebras, see Schemes, Lemma 21.24.1. By Properties, Lemma 23.20.3 we can write $\mathcal{A} = colim_i \mathcal{F}_i$ as a filtered colimit of finite type $\mathcal{O}_X$-modules. Let $\mathcal{A}_i \subset \mathcal{A}$ be the $\mathcal{O}_X$-subalgebra generated by $\mathcal{F}_i$. Since the map of algebras $\mathcal{O}_X \to \mathcal{A}$ is integral, we see that each $\mathcal{A}_i$ is a finite quasi-coherent $\mathcal{O}_S$-algebra. Hence

$$X_i = \underline{Spec}_S(\mathcal{A}_i) \longrightarrow S$$

is a finite morphism of schemes. It is clear that $X = lim_i X_i$. Hence by Lemma 27.3.7 we see that for i sufficiently large the scheme X_i is affine. Moreover, since $X \to S$ factors through each X_i we see that $X_i \to S$ is surjective. Hence we conclude that S is affine by Lemma 27.7.1. □

27.8. Variants of Chow's Lemma

In this section we prove a number of variants of Chow's lemma. The most interesting version is probably just the Noetherian case, which we stated and proved in Coherent, Section 25.17.

Lemma 27.8.1. *Let S be a quasi-compact and quasi-separated scheme. Let $f : X \to S$ be a separated morphism of finite type. Then there exists an $n \geq 0$ and a diagram*

$$\begin{tikzcd} X \arrow[dr] & X' \arrow[l, "\pi"'] \arrow[r] \arrow[d] & \mathbf{P}^n_S \arrow[dl] \\ & S & \end{tikzcd}$$

where $X' \to \mathbf{P}^n_S$ is an immersion, and $\pi : X' \to X$ is proper and surjective.

Proof. By Proposition 27.5.6 we can find a closed immersion $X \to Y$ where Y is separated and of finite presentation over S. Clearly, if we prove the assertion for Y, then the result follows for X. Hence we may assume that X is of finite presentation over S.

Write $S = lim_i S_i$ as a directed limit of Noetherian schemes, see Proposition 27.3.11. By Lemma 27.6.1 we can find an index $i \in I$ and a scheme $X_i \to S_i$ of finite presentation so that $X = S \times_{S_i} X_i$. By Lemma 27.6.6 we may assume that $X_i \to S_i$ is separated. Clearly, if we prove the assertion for X_i over S_i, then the assertion holds for X. The case $X_i \to S_i$ is treated by Coherent, Lemma 25.17.1. $\square$

Here is a variant of Chow's lemma where we assume the scheme on top has finitely many irreducible components.

Lemma 27.8.2. *Let S be a quasi-compact and quasi-separated scheme. Let $f : X \to S$ be a separated morphism of finite type. Assume that X has finitely many irreducible components. Then there exists an $n \geq 0$ and a diagram*

$$\begin{tikzcd} X \arrow[dr] & X' \arrow[l, "\pi"'] \arrow[r] \arrow[d] & \mathbf{P}^n_S \arrow[dl] \\ & S & \end{tikzcd}$$

where $X' \to \mathbf{P}^n_S$ is an immersion, and $\pi : X' \to X$ is proper and surjective. Moreover, there exists an open dense subscheme $U \subset X$ such that $\pi^{-1}(U) \to U$ is an isomorphism of schemes.

Proof. Let $X = Z_1 \cup \ldots \cup Z_n$ be the decomposition of X into irreducible components. Let $\eta_j \in Z_j$ be the generic point.

There are (at least) two ways to proceed with the proof. The first is to redo the proof of Coherent, Lemma 25.17.1 using the general Properties, Lemma 23.26.4 to find suitable affine opens in X. (This is the ``standard'' proof.) The second is to use absolute Noetherian approximation as in the proof of Lemma 27.8.1 above. This is what we will do here.

By Proposition 27.5.6 we can find a closed immersion $X \to Y$ where Y is separated and of finite presentation over S. Write $S = lim_i S_i$ as a directed limit of Noetherian schemes, see Proposition 27.3.11. By Lemma 27.6.1 we can find an index $i \in I$ and a scheme $Y_i \to S_i$ of finite presentation so that $Y = S \times_{S_i} Y_i$. By Lemma 27.6.6 we may assume that $Y_i \to S_i$ is separated. We have the following diagram

$$\begin{tikzcd} \eta_j \in Z_j \arrow[r] & X \arrow[r] \arrow[dr] & Y \arrow[r] \arrow[d] & Y_i \arrow[d] \\ & & S \arrow[r] & S_i \end{tikzcd}$$

Denote $h : X \to Y_i$ the composition.

For $i' \geq i$ write $Y_{i'} = S_{i'} \times_{S_i} Y_i$. Then $Y = lim_{i' \geq i} Y_{i'}$, see Lemma 27.2.4. Choose $j, j' \in \{1, \ldots, n\}$, $j \neq j'$. Note that η_j is not a specialization of $\eta_{j'}$. By Lemma 27.2.3 we can replace i by a bigger index and assume that $h(\eta_j)$ is not a specialization of $h(\eta_{j'})$ for all pairs (j, j') as above. For such an index, let $Y' \subset Y_i$ be the scheme theoretic image of $h : X \to Y_i$, see Morphisms, Definition 24.4.2. The morphism h is quasi-compact as the composition of the quasi-compact morphisms $X \to Y$ and $Y \to Y_i$ (which is affine). Hence by Morphisms, Lemma 24.4.3 the morphism $X \to Y'$ is dominant. Thus the generic points of Y' are all contained in the set $\{h(\eta_1), \ldots, h(\eta_n)\}$, see Morphisms, Lemma 24.6.3. Since none of the $h(\eta_j)$ is the specialization of another we see that the points $h(\eta_1), \ldots, h(\eta_n)$ are pairwise distinct and are each a generic point of Y'.

We apply Coherent, Lemma 25.17.1 above to the morphism $Y' \to S_i$. This gives a diagram

$$\begin{tikzcd} Y' \arrow[dr] & Y^* \arrow[l, "\pi"'] \arrow[r] \arrow[d] & \mathbf{P}^n_{S_i} \arrow[dl] \\ & S_i & \end{tikzcd}$$

such that π is proper and surjective and an isomorphism over a dense open subscheme $V \subset Y'$. By our choice of i above we know that $h(\eta_1), \ldots, h(\eta_n) \in V$. Consider the commutative diagram

$$\begin{tikzcd} X' \arrow[r, equal] & X \times_{Y'} Y^* \arrow[r] \arrow[d] & Y^* \arrow[r] \arrow[d] & \mathbf{P}^n_{S_i} \arrow[ddl] \\ & X \arrow[r] \arrow[d] & Y' \arrow[d] & \\ & S \arrow[r] & S_i & \end{tikzcd}$$

Note that $X' \to X$ is an isomorphism over the open subscheme $U = h^{-1}(V)$ which contains each of the η_j and hence is dense in X. We conclude $X \leftarrow X' \to \mathbf{P}^n_S$ is a solution to the problem posed in the lemma. □

27.9. Applications of Chow's lemma

We can use Chow's lemma to investigate the notions of proper and separated morphisms. As a first application we have the following.

Lemma 27.9.1. *Let S be a scheme. Let $f : X \to S$ be a separated morphism of finite type. The following are equivalent:*

(1) *The morphism f is proper.*
(2) *For any morphism $S' \to S$ which is locally of finite type the base change $X_{S'} \to S'$ is closed.*
(3) *For every $n \geq 0$ the morphism $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ is closed.*

Proof. Clearly (1) implies (2), and (2) implies (3), so we just need to show (3) implies (1). First we reduce to the case when S is affine. Assume that (3) implies (1) when the base is affine. Now let $f : X \to S$ be a separated morphism of finite type. Being proper is local on the base (see Morphisms, Lemma 24.40.3), so if $S = \bigcup_\alpha S_\alpha$ is an open affine cover, and if we denote $X_\alpha := f^{-1}(S_\alpha)$, then it is enough to show that $f|_{X_\alpha} : X_\alpha \to S_\alpha$ is proper for all α. Since S_α is affine, if the map $f|_{X_\alpha}$ satisfies (3), then it will satisfy (1) by assumption, and will be proper. To finish the reduction to the case S is affine, we must show that if $f : X \to S$ is separated of finite type satisfying (3), then $f|_{X_\alpha} : X_\alpha \to S_\alpha$ is separated of finite type satisfying (3). Separatedness and finite type are clear. To see (3), notice that $\mathbf{A}^n \times X_\alpha$ is the open preimage of $\mathbf{A}^n \times S_\alpha$ under the map $1 \times f$. Fix a closed set $Z \subset \mathbf{A}^n \times X_\alpha$. Let $\bar{Z}$ denote the closure of Z in $\mathbf{A}^n \times X$. Then for topological reasons,

$$1 \times f(\bar{Z}) \cap \mathbf{A}^n \times S_\alpha = 1 \times f(Z).$$

Hence $1 \times f(Z)$ is closed, and we have reduced the proof of (3) ⇒ (1) to the affine case.

Assume S affine, and $f : X \to S$ separated of finite type. We can apply Chow's Lemma 27.8.1 to get $\pi : X' \to X$ proper surjective and $X' \to \mathbf{P}^n_S$ an immersion. If X is proper over S, then $X' \to S$ is proper (Morphisms, Lemma 24.40.4). Since $\mathbf{P}^n_S \to S$ is separated, we conclude that $X' \to \mathbf{P}^n_S$ is proper (Morphisms, Lemma 24.40.7) and hence a closed immersion (Schemes, Lemma 21.10.4). Conversely, assume $X' \to \mathbf{P}^n_S$ is a closed immersion. Consider the diagram:

(27.9.1.1)
$$\begin{tikzcd} X' \arrow[r] \arrow[d, "\pi"'] & \mathbf{P}^n_S \arrow[d] \\ X \arrow[r, "f"] & S \end{tikzcd}$$

All maps are a priori proper except for $X \to S$. Hence we conclude that $X \to S$ is proper by Morphisms, Lemma 24.40.8. Therefore, we have shown that $X \to S$ is proper if and only if $X' \to \mathbf{P}^n_S$ is a closed immersion.

Assume S is affine and (3) holds, and let n, X', π be as above. Since being a closed morphism is local on the base, the map $X \times \mathbf{P}^n \to S \times \mathbf{P}^n$ is closed since by (3) $X \times \mathbf{A}^n \to S \times \mathbf{A}^n$ is closed and since projective space is covered by copies of affine n-space, see Constructions, Lemma 22.13.3. By Morphisms, Lemma 24.40.5 the morphism

$$X' \times_S \mathbf{P}^n_S \to X \times_S \mathbf{P}^n_S = X \times \mathbf{P}^n$$

is proper. Since $\mathbf{P}^n$ is separated, the projection

$$X' \times_S \mathbf{P}^n_S = \mathbf{P}^n_{X'} \to X'$$

will be separated as it is just a base change of a separated morphism. Therefore, the map $X' \to X' \times_S \mathbf{P}^n_S$ is proper, since it is a section to a separated map (see Schemes, Lemma 21.21.12). Composing all these proper morphisms

$$X' \to X' \times_S \mathbf{P}^n_S \to X \times_S \mathbf{P}^n_S = X \times \mathbf{P}^n \to S \times \mathbf{P}^n = \mathbf{P}^n_S$$

we see that the map $X' \to \mathbf{P}^n_S$ is proper, and hence a closed immersion. □

If the base is Noetherian we can show that the valuative criterion holds using only discrete valuation rings. First we state the result concerning separation. We will often use solid commutative diagrams of morphisms of schemes having the following shape

(27.9.1.2)
$$\begin{tikzcd} Spec(K) \arrow[r] \arrow[d] & X \arrow[d] \\ Spec(A) \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

with A a valuation ring and K its field of fractions.

Lemma 27.9.2. *Let S be a locally Noetherian scheme. Let $f : X \to S$ be a morphism of schemes. Assume f is locally of finite type. The following are equivalent:*

(1) *The morphism f is separated.*
(2) *For any diagram (27.9.1.2) there is at most one dotted arrow.*
(3) *For all diagrams (27.9.1.2) with A a discrete valuation ring there is at most one dotted arrow.*

(4) *For any irreducible component X_0 of X with generic point $\eta \in X_0$, for any discrete valuation ring $A \subset K = \kappa(\eta)$ with fraction field K and any diagram (27.9.1.2) such that the morphism $Spec(K) \to X$ is the canonical one (see Schemes, Section 21.13) there is at most one dotted arrow.*

Proof. Clearly (1) implies (2), (2) implies (3), and (3) implies (4). It remains to show (4) implies (1). Assume (4). We begin by reducing to S affine. Being separated is a local on the base (see Schemes, Lemma 21.21.8). Hence, as in the proof of Lemma 27.9.1, if we can show that whenever $X \to S$ has (4) that the restriction $X_\alpha \to S_\alpha$ has (4) where $S_\alpha \subset S$ is an (affine) open subset and $X_\alpha := f^{-1}(S_\alpha)$, then we will be done. The generic points of the irreducible components of X_α will be the generic points of irreducible components of X, since X_α is open in X. Therefore, any two distinct dotted arrows in the diagram

(27.9.2.1)
$$\begin{tikzcd} Spec(K) \arrow[r] \arrow[d] & X_\alpha \arrow[d] \\ Spec(A) \arrow[r] \arrow[ur, dashed] & S_\alpha \end{tikzcd}$$

would then give two distinct arrows in diagram (27.9.1.2) via the maps $X_\alpha \to X$ and $S_\alpha \to S$, which is a contradiction. Thus we have reduced to the case S is affine. We remark that in the course of this reduction, we prove that if $X \to S$ has (4) then the restriction $U \to V$ has (4) for opens $U \subset X$ and $V \subset S$ with $f(U) \subset V$.

We next wish to reduce to the case $X \to S$ is finite type. Assume that we know (4) implies (1) when X is finite type. Since S is Noetherian and X is locally of finite type over S we see X is locally Noetherian as well (see Morphisms, Lemma 24.14.6). Thus, $X \to S$ is quasi-separated (see Properties, Lemma 23.5.4), and therefore we may apply the valuative criterion to check whether X is separated (see Schemes, Lemma 21.22.2). Let $X = \bigcup_\alpha X_\alpha$ be an affine open cover of X. Given any two dotted arrows, in a diagram (27.9.1.2), the image of the closed points of Spec A will fall in two sets X_α and X_β. Since $X_\alpha \cup X_\beta$ is open, for topological reasons it must contain the image of $Spec(A)$ under both maps. Therefore, the two dotted arrows factor through $X_\alpha \cup X_\beta \to X$, which is a scheme of finite type over S. Since $X_\alpha \cup X_\beta$ is an open subset of X, by our previous remark, $X_\alpha \cup X_\beta$ satisfies (4), so by assumption, is separated. This implies the two given dotted arrows are the same. Therefore, we have reduced to $X \to S$ is finite type.

Assume $X \to S$ of finite type and assume (4). Since $X \to S$ is finite type, and S is an affine Noetherian scheme, X is also Noetherian (see Morphisms, Lemma 24.14.6). Therefore, $X \to X \times_S X$ will be a quasi-compact immersion of Noetherian schemes. We proceed by contradiction. Assume that $X \to X \times_S X$ is not closed. Then, there is some $y \in X \times_S X$ in the closure of the image that is not in the image. As X is Noetherian it has finitely many irreducible components. Therefore, y is in the closure of the image of one of the irreducible components $X_0 \subset X$. Give X_0 the reduced induced structure. The composition $X_0 \to X \to X \times_S X$ factors through the closed subscheme $X_0 \times_S X_0 \subset X \times_S X$. Denote the closure of $\Delta(X_0)$ in $X_0 \times_S X_0$ by $\bar{X}_0$ (again as a reduced closed subscheme). Thus $y \in \bar{X}_0$. Since $X_0 \to X_0 \times_S X_0$ is an immersion, the image of X_0 will be open in $\bar{X}_0$. Hence X_0 and $\bar{X}_0$ are birational. Since $\bar{X}_0$ is a closed subscheme of a Noetherian scheme, it is Noetherian. Thus, the local ring $\mathcal{O}_{\bar{X}_0,y}$ is a local Noetherian domain with fraction field K equal to the function field of X_0. By the Krull-Akizuki theorem (see Algebra, Lemma 7.110.11), there exists a discrete valuation ring A dominating $\mathcal{O}_{\bar{X}_0,y}$ with fraction field K. This allows to to construct a diagram:

(27.9.2.2)
$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X_0 \arrow[d, "\Delta"] \\ A \arrow[r] \arrow[ur, dashed] & X_0 \times_S X_0 \end{tikzcd}$$

which sends Spec K to the generic point of $\Delta(X_0)$ and the closed point of A to $y \in X_0 \times_S X_0$ (use the material in Schemes, Section 21.13 to construct the arrows). There cannot even exist a set theoretic dotted arrow, since y is not in the image of Δ by our choice of y. By categorical means, the existence of the dotted arrow in the above diagram is equivalent to the uniqueness of the dotted arrow in the following diagram:

(27.9.2.3)
$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X_0 \arrow[d] \\ A \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

Therefore, we have non-uniqueness in this latter diagram by the nonexistence in the first. Therefore, X_0 does not satisfy uniqueness for discrete valuation rings, and since X_0 is an irreducible component of X, we have that $X \to S$ does not satisfy (4). Therefore, we have shown (4) implies (1). □

Lemma 27.9.3. *Let S be a locally Noetherian scheme. Let $f : X \to S$ be a morphism of finite type. The following are equivalent:*

(1) *The morphism f is proper.*
(2) *For any diagram (27.9.1.2) there exists exactly one dotted arrow.*
(3) *For all diagrams (27.9.1.2) with A a discrete valuation ring there exists exactly one dotted arrow.*
(4) *For any irreducible component X_0 of X with generic point $\eta \in X_0$, for any discrete valuation ring $A \subset K = \kappa(\eta)$ with fraction field K and any diagram (27.9.1.2) such that the morphism $Spec(K) \to X$ is the canonical one (see Schemes, Section 21.13) there exists exactly one dotted arrow.*

Proof. (1) implies (2) implies (3) implies (4). We will now show (4) implies (1). As in the proof of Lemma 27.9.2, we can reduce to the case S is affine, since properness is local on the base, and if $X \to S$ satisfies (4), then $X_\alpha \to S_\alpha$ does as well for open $S_\alpha \subset S$ and $X_\alpha = f^{-1}(S_\alpha)$.

Now S is a Noetherian scheme, and so X is as well, since $X \to S$ is of finite type. Now we may use Chow's lemma (Coherent, Lemma 25.17.1) to get a surjective, proper, birational $X' \to X$ and an immersion $X' \to \mathbf{P}^n_S$. We wish to show $X \to S$ is universally closed. As in the proof of Lemma 27.9.1, it is enough to check that $X' \to \mathbf{P}^n_S$ is a closed immersion. For the sake of contradiction, assume that $X' \to \mathbf{P}^n_S$ is not a closed immersion. Then there is some $y \in \mathbf{P}^n_S$ that is in the closure of the image of X', but is not in the image. So y is in the closure of the image of an irreducible component X'_0 of X', but not in the image. Let $\bar{X}'_0 \subset \mathbf{P}^n_S$ be the closure of the image of X'_0. As $X' \to \mathbf{P}^n_S$ is an immersion of Noetherian schemes, the morphism $X'_0 \to \bar{X}'_0$ is open and dense. By Algebra, Lemma 7.110.11 or Properties, Lemma 23.5.9 we can find a discrete valuation ring A dominating $\mathcal{O}_{\bar{X}'_0,y}$ and with identical field of fractions K. It is clear that K is the residue field at the generic point of X'_0. Thus the solid commutative diagram

$$\begin{tikzcd} \mathrm{Spec}\, K \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d] & \mathbf{P}^n_S \arrow[d] \\ \mathrm{Spec}\, A \arrow[r, dashed] \arrow[ur, dashed] \arrow[urr] & X \arrow[r] & S \end{tikzcd} \tag{27.9.3.1}$$

Note that the closed point of A maps to $y \in \mathbf{P}^n_S$. By construction, there does not exist a set theoretic lift to X'. As $X' \to X$ is birational, the image of X'_0 in X is an irreducible component X_0 of X and K is also identified with the function field of X_0. Hence, as $X \to S$ is assumed to satisfy (4), the dotted arrow $Spec(A) \to X$ exists. Since $X' \to X$ is proper, the dotted arrow lifts to the dotted arrow $Spec(A) \to X'$ (use Schemes, Proposition 21.20.6). We can compose this with the immersion $X' \to \mathbf{P}^n_S$ to obtain another morphism (not depicted in the diagram) from $Spec(A) \to \mathbf{P}^n_S$. Since $\mathbf{P}^n_S$ is proper over S, it satisfies (2), and so these two morphisms agree. This is a contradiction, for we have constructed the forbidden lift of our original map $Spec(A) \to \mathbf{P}^n_S$ to X'. □

27.10. Universally closed morphisms

In this section we discuss when a quasi-compact but not necessarily separated morphism is universally closed. We first prove a lemma which will allow us to check universal closedness after a base change which is locally of finite presentation.

Lemma 27.10.1. *Let $f : X \to S$ be a quasi-compact morphism of schemes. Let $g : T \to S$ be a morphism of schemes. Let $t \in T$ be a point and $Z \subset X_T$ be a closed subscheme such that $Z \cap X_t = \emptyset$. Then there exists an open neighbourhood $V \subset T$ of t, a commutative diagram*

$$\begin{tikzcd} V \arrow[r, "a"] \arrow[d] & T' \arrow[d, "b"] \\ T \arrow[r, "g"] & S, \end{tikzcd}$$

and a closed subscheme $Z' \subset X_{T'}$ such that

1. *the morphism $b : T' \to S$ is locally of finite presentation,*
2. *with $t' = a(t)$ we have $Z' \cap X_{t'} = \emptyset$, and*
3. *$Z \cap X_V$ maps into Z' via the morphism $X_V \to X_{T'}$.*

Proof. Let $s = g(t)$. During the proof we may always replace T by an open neighbourhood of t. Hence we may also replace S by an open neighbourhood of s. Thus we may and do assume that T and S are affine. Say $S = Spec(A)$, $T = Spec(B)$, g is given by the ring map $A \to B$, and t correspond to the prime ideal $\mathfrak{q} \subset B$.

As $X \to S$ is quasi-compact and S is affine we may write $X = \bigcup_{i=1,\ldots,n} U_i$ as a finite union of affine opens. Write $U_i = Spec(C_i)$. In particular we have $X_T = \bigcup_{i=1,\ldots,n} U_{i,T} = \bigcup_{i=1,\ldots n} Spec(C_i \otimes_A B)$. Let $I_i \subset C_i \otimes_A B$ be the ideal corresponding to the closed subscheme $Z \cap U_{i,T}$. The condition that $Z \cap X_t = \emptyset$ signifies that I_i generates the unit ideal in the ring

$$C_i \otimes_A \kappa(\mathfrak{q}) = (B \setminus \mathfrak{q})^{-1} \left(C_i \otimes_A B/\mathfrak{q}C_i \otimes_A B\right)$$

Since $I_i(B \setminus \mathfrak{q})^{-1}(C_i \otimes_A B) = (B \setminus \mathfrak{q})^{-1} I_i$ this means that $1 = x_i/g_i$ for some $x_i \in I_i$ and $g_i \in B$, $g_i \not\in \mathfrak{q}$. Thus, clearing denominators we can find a relation of the form

$$x_i + \sum\nolimits_j f_{i,j} c_{i,j} = g_i$$

with $x_i \in I_i$, $f_{i,j} \in \mathfrak{q}$, $c_{i,j} \in C_i \otimes_A B$, and $g_i \in B$, $g_i \not\in \mathfrak{q}$. After replacing B by $B_{g_1 \ldots g_n}$, i.e., after replacing T by a smaller affine neighbourhood of t, we may assume the equations read

$$x_i + \sum\nolimits_j f_{i,j} c_{i,j} = 1$$

with $x_i \in I_i$, $f_{i,j} \in \mathfrak{q}$, $c_{i,j} \in C_i \otimes_A B$.

To finish the argument write B as a colimit of finitely presented A-algebras B_λ over a directed partially ordered set Λ. For each λ set $\mathfrak{q}_\lambda = (B_\lambda \to B)^{-1}(\mathfrak{q})$. For sufficiently large $\lambda \in \Lambda$ we can find

(1) an element $x_{i,\lambda} \in C_i \otimes_A B_\lambda$ which maps to x_i,
(2) elements $f_{i,j,\lambda} \in \mathfrak{q}_{i,\lambda}$ mapping to $f_{i,j}$, and
(3) elements $c_{i,j,\lambda} \in C_i \otimes_A B_\lambda$ mapping to $c_{i,j}$.

After increasing λ a bit more the equation

$$x_{i,\lambda} + \sum\nolimits_j f_{i,j,\lambda} c_{i,j,\lambda} = 1$$

will hold. Fix such a λ and set $T' = Spec(B_\lambda)$. Then $t' \in T'$ is the point corresponding to the prime $\mathfrak{q}_\lambda$. Finally, let $Z' \subset X_{T'}$ be the scheme theoretic closure of $Z \to X_T \to X_{T'}$. As $X_T \to X_{T'}$ is affine, we can compute Z' on the affine open pieces $U_{i,T'}$ as the closed subscheme associated to $\mathrm{Ker}(C_i \otimes_A B_\lambda \to C_i \otimes_A B/I_i)$, see Morphisms, Example 24.4.4. Hence $x_{i,\lambda}$ is in the ideal defining Z'. Thus the last displayed equation shows that $Z' \cap X_{t'}$ is empty. □

Lemma 27.10.2. *Let $f : X \to S$ be a quasi-compact morphism of schemes. The following are equivalent*

(1) *f is universally closed,*
(2) *for every morphism $S' \to S$ which is locally of finite presentation the base change $X_{S'} \to S'$ is closed, and*
(3) *for every n the morphism $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ is closed.*

Proof. It is clear that (1) implies (2). Let us prove that (2) implies (1). Suppose that the base change $X_T \to T$ is not closed for some scheme T over S. By Schemes, Lemma 21.19.8 this means that there exists some specialization $t_1 \rightsquigarrow t$ in T and a point $\xi \in X_T$ mapping to t_1 such that ξ does not specialize to a point in the fibre over t. Set $Z = \overline{\{\xi\}} \subset X_T$. Then $Z \cap X_t = \emptyset$. Apply Lemma 27.10.1. We find an open neighbourhood $V \subset T$ of t, a commutative diagram

$$\begin{tikzcd} V \arrow[r, "a"] \arrow[d] & T' \arrow[d, "b"] \\ T \arrow[r, "g"] & S, \end{tikzcd}$$

and a closed subscheme $Z' \subset X_{T'}$ such that

(1) the morphism $b : T' \to S$ is locally of finite presentation,
(2) with $t' = a(t)$ we have $Z' \cap X_{t'} = \emptyset$, and
(3) $Z \cap X_V$ maps into Z' via the morphism $X_V \to X_{T'}$.

Clearly this means that $X_{T'} \to T'$ maps the closed subset Z' to a subset of T' which contains $a(t_1)$ but not $t' = a(t)$. Since $a(t_1) \rightsquigarrow a(t) = t'$ we conclude that $X_{T'} \to T'$ is not closed. Hence we have shown that $X \to S$ not universally closed implies that $X_{T'} \to T'$ is not closed for some $T' \to S$ which is locally of finite presentation. In order words (2) implies (1).

Assume that $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ is closed for every integer n. We want to prove that $X_T \to T$ is closed for every scheme T which is locally of finite presentation over S. We may of course assume that T is affine and maps into an affine open V of S (since $X_T \to T$ being a closed is local on T). In this case there exists a closed immersion $T \to \mathbf{A}^n \times V$ because $\mathcal{O}_T(T)$ is a finitely presented $\mathcal{O}_S(V)$-algebra, see Morphisms, Lemma 24.20.2. Then $T \to \mathbf{A}^n \times S$ is a locally closed immersion. Hence we get a cartesian diagram

$$\begin{tikzcd} X_T \arrow[r] \arrow[d, "f_T"'] & \mathbf{A}^n \times X \arrow[d, "f_n"] \\ T \arrow[r] & \mathbf{A}^n \times S \end{tikzcd}$$

of schemes where the horizontal arrows are locally closed immersions. Hence any closed subset $Z \subset X_T$ can be written as $X_T \cap Z'$ for some closed subset $Z' \subset \mathbf{A}^n \times X$. Then $f_T(Z) = T \cap f_n(Z')$ and we see that if f_n is closed, then also f_T is closed. □

Lemma 27.10.3. *Let $f : X \to S$ be a finite type morphism of schemes. Assume S is locally Noetherian. Then the following are equivalent*

(1) *f is universally closed,*
(2) *for every n the morphism $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ is closed,*
(3) *for any diagram (27.9.1.2) there exists some dotted arrow,*
(4) *for all diagrams (27.9.1.2) with A a discrete valuation ring there exists some dotted arrow.*

Proof. The equivalence of (1) and (2) is a special case of Lemma 27.10.2. The equivalence of (1) and (3) is a special case of Schemes, Proposition 21.20.6. Trivially (3) implies (4). Thus all we have to do is prove that (4) implies (2). We will prove that $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ is closed by the criterion of Schemes, Lemma 21.19.8. Pick n and a specialization $z \rightsquigarrow z'$ of points in $\mathbf{A}^n \times S$ and a point $y \in \mathbf{A}^n \times X$ lying over z. Note that $\kappa(y)$ is a finitely generated field extension of $\kappa(z)$ as $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ is of finite type. Hence by Properties, Lemma 23.5.9 or Algebra, Lemma 7.110.11 implies that there exists a discrete valuation ring $A \subset \kappa(y)$ with fraction field $\kappa(z)$ dominating the image of $\mathcal{O}_{\mathbf{A}^n \times S, z'}$ in $\kappa(z)$. This gives a commutative diagram

$$\begin{tikzcd} Spec(\kappa(y)) \arrow[r] \arrow[d] & \mathbf{A}^n \times X \arrow[r] \arrow[d] & X \arrow[d] \\ Spec(A) \arrow[r] & \mathbf{A}^n \times S \arrow[r] & S \end{tikzcd}$$

Now property (4) implies that there exists a morphism $Spec(A) \to X$ which fits into this diagram. Since we already have the morphism $Spec(A) \to \mathbf{A}^n$ from the left lower horizontal arrow we also get a morphism $Spec(A) \to \mathbf{A}^n \times X$ fitting into the left square. Thus the image $y' \in \mathbf{A}^n \times X$ of the closed point is a specialization of y lying over z'. This proves that specializations lift along $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ and we win. □

27.11. Limits and dimensions of fibres

The following lemma is most often used in the situation of Lemma 27.6.1 to assure that if the fibres of the limit have dimension $\leq d$, then the fibres at some finite stage have dimension $\leq d$.

Lemma 27.11.1. *Let I be a directed partially ordered set. Let $(f_i : X_i \to S_i)$ be an inverse system of morphisms of schemes over I. Assume*

(1) *all the morphisms $S_{i'} \to S_i$ are affine,*
(2) *all the schemes S_i are quasi-compact and quasi-separated,*
(3) *the morphisms f_i are of finite type, and*
(4) *the morphisms $X_{i'} \to X_i \times_{S_i} S_{i'}$ are closed immersions.*

Let $f : X = lim_i X_i \to S = lim_i S_i$ be the limit. Let $d \geq 0$. If every fibre of f has dimension $\leq d$, then for some i every fibre of f_i has dimension $\leq d$.

Proof. For each i let $U_i = \{x \in X_i \mid \dim_x((X_i)_{f_i(x)}) \leq d\}$. This is an open subset of X_i, see Morphisms, Lemma 24.27.4. Set $Z_i = X_i \setminus U_i$ (with reduced induced scheme structure). We have to show that $Z_i = \emptyset$ for some i. If not, then $Z = lim\, Z_i \neq \emptyset$, see Lemma 27.3.1. Say $z \in Z$ is a point. Note that $Z \subset X$ is a closed subscheme. Set $s = f(z)$. For each i let $s_i \in S_i$ be the image of s. We remark that Z_s is the limit of the schemes $(Z_i)_{s_i}$ and Z_s is also the limit of the schemes $(Z_i)_{s_i}$ base changed to $\kappa(s)$. Moreover, all the morphisms

$$Z_s \longrightarrow (Z_{i'})_{s_{i'}} \times_{Spec(\kappa(s_{i'}))} Spec(\kappa(s)) \longrightarrow (Z_i)_{s_i} \times_{Spec(\kappa(s_i))} Spec(\kappa(s)) \longrightarrow X_s$$

are closed immersions by assumption (4). Hence Z_s is the scheme theoretic intersection of the closed subschemes $(Z_i)_{s_i} \times_{Spec(\kappa(s_i))} Spec(\kappa(s))$ in X_s. Since all the irreducible components of the schemes $(Z_i)_{s_i} \times_{Spec(\kappa(s_i))} Spec(\kappa(s))$ have dimension $> d$ and contain z we conclude that Z_s contains an irreducible component of dimension $> d$ passing through z which contradicts the fact that $Z_s \subset X_s$ and $\dim(X_s) \leq d$. □

Lemma 27.11.2. *Let S be a quasi-compact and quasi-separated scheme. Let $f : X \to S$ be a morphism of finite presentation. Let $d \geq 0$ be an integer. If $Z \subset X$ be a closed subscheme such that $\dim(Z_s) \leq d$ for all $s \in S$, then there exists a closed subscheme $Z' \subset X$ such that*

(1) *$Z \subset Z'$,*
(2) *$Z' \to X$ is of finite presentation, and*
(3) *$\dim(Z'_s) \leq d$ for all $s \in S$.*

Proof. By Proposition 27.3.11 we can write $S = lim\, S_i$ as the limit of a directed inverse system of Noetherian schemes with affine transition maps. By Lemma 27.6.1 we may assume that there exist a system of morphisms $f_i : X_i \to S_i$ of finite presentation such that $X_{i'} = X_i \times_{S_i} S_{i'}$ for all $i' \geq i$ and such that $X = X_i \times_{S_i} S$. Let $Z_i \subset X_i$ be the scheme theoretic image of $Z \to X \to X_i$. Then for $i' \geq i$ the morphism $X_{i'} \to X_i$ maps $Z_{i'}$ into Z_i and the induced morphism $Z_{i'} \to Z_i \times_{S_i} S_{i'}$ is a closed immersion. By Lemma 27.11.1 we see that the dimension of the fibres of $Z_i \to S_i$ all have dimension $\leq d$ for a suitable $i \in I$. Fix such an i and set $Z' = Z_i \times_{S_i} S \subset X$. Since S_i is Noetherian, we see that X_i is Noetherian, and hence the morphism $Z_i \to X_i$ is of finite presentation. Therefore also the base change $Z' \to X$ is of finite presentation. Moreover, the fibres of $Z' \to S$ are base changes of the fibres of $Z_i \to S_i$ and hence have dimension $\leq d$. □

27.12. Other chapters

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CHAPTER 28

Varieties

28.1. Introduction

In this chapter we start studying varieties and more generally schemes over a field. A fundamental reference is [**DG67**].

28.2. Notation

Throughout this chapter we use the letter k to denote the ground field.

28.3. Varieties

In the stacks project we will use the following as our definition of a variety.

Definition 28.3.1. Let k be a field. A *variety* is a scheme X over k such that X is integral and the structure morphism $X \to Spec(k)$ is separated and of finite type.

This definition has the following drawback. Suppose that $k \subset k'$ is an extension of fields. Suppose that X is a variety over k. Then the base change $X_{k'} = X \times_{Spec(k)} Spec(k')$ is not necessarily a variety over k'. This phenomenon (in greater generality) will be discussed in detail in the following sections. The product of two varieties need not be a variety (this is really the same phenomenon). Here is an example.

Example 28.3.2. Let $k = \mathbf{Q}$. Let $X = Spec(\mathbf{Q}(i))$ and $Y = Spec(\mathbf{Q}(i))$. Then the product $X \times_{Spec(k)} Y$ of the varieties X and Y is not a variety, since it is reducible. (It is isomorphic to the disjoint union of two copies of X.)

If the ground field is algebraically closed however, then the product of varieties is a variety. This follows from the results in the algebra chapter, but there we treat much more general situations. There is also a simple direct proof of it which we present here.

Lemma 28.3.3. *Let k be an algebraically closed field. Let X, Y be varieties over k. Then $X \times_{Spec(k)} Y$ is a variety over k.*

Proof. The morphism $X \times_{Spec(k)} Y \to Spec(k)$ is of finite type and separated because it is the composition of the morphisms $X \times_{Spec(k)} Y \to Y \to Spec(k)$ which are separated and of finite type, see Morphisms, Lemmas 24.14.4 and 24.14.3 and Schemes, Lemma 21.21.13. To finish the proof it suffices to show that $X \times_{Spec(k)} Y$ is integral. Let $X = \bigcup_{i=1,\ldots,n} U_i$, $Y = \bigcup_{j=1,\ldots,m} V_j$ be finite affine open coverings. If we can show that each $U_i \times_{Spec(k)} V_j$ is integral, then we are done by Properties, Lemmas 23.3.2, 23.3.3, and 23.3.4. This reduces us to the affine case.

The affine case translates into the following algebra statement: Suppose that A, B are integral domains and finitely generated k-algebras. Then $A \otimes_k B$ is an integral domain. To get a contradiction suppose that

$$(\sum\nolimits_{i=1,\ldots,n} a_i \otimes b_i)(\sum\nolimits_{j=1,\ldots,m} c_j \otimes d_j) = 0$$

in $A \otimes_k B$ with both factors nonzero in $A \otimes_k B$. We may assume that $b_1, \ldots, b_n$ are k-linearly independent in B, and that $d_1, \ldots, d_m$ are k-linearly independent in B. Of course we may also assume that a_1 and c_1 are nonzero in A. Hence $D(a_1c_1) \subset Spec(A)$ is nonempty. By the Hilbert Nullstellensatz (Algebra, Theorem 7.30.1) we can find a maximal ideal $\mathfrak{m} \subset A$ contained in $D(a_1c_1)$ and $A/\mathfrak{m} = k$ as k is algebraically closed. Denote $\overline{a}_i, \overline{c}_j$ the residue classes of a_i, c_j in $A/\mathfrak{m} = k$. Then equation above becomes

$$(\sum\nolimits_{i=1,\ldots,n} \overline{a}_i b_i)(\sum\nolimits_{j=1,\ldots,m} \overline{c}_j d_j) = 0$$

which is a contradiction with $\mathfrak{m} \in D(a_1c_1)$, the linear independence of $b_1, \ldots, b_n$ and $d_1, \ldots, d_m$, and the fact that B is a domain. □

28.4. Geometrically reduced schemes

If X is a reduced scheme over a field, then it can happen that X becomes nonreduced after extending the ground field. This does not happen for geometrically reduced schemes.

Definition 28.4.1. Let k be a field. Let X be a scheme over k. Let $x \in X$ be a point.

1. Let $x \in X$ be a point. We say X is *geometrically reduced at x* if for any field extension $k \subset k'$ and any point $x' \in X_{k'}$ lying over x the local ring $\mathcal{O}_{X_{k'},x'}$ is reduced.
2. We say X is *geometrically reduced* over k if X is geometrically reduced at every point of X.

This may seem a little mysterious at first, but it is really the same thing as the notion discussed in the algebra chapter. Here are some basic results explaining the connection.

Lemma 28.4.2. *Let k be a field. Let X be a scheme over k. Let $x \in X$. The following are equivalent*

1. *X is geometrically reduced at x, and*
2. *the ring $\mathcal{O}_{X,x}$ is geometrically reduced over k (see Algebra, Definition 7.40.1).*

Proof. Assume (1). This in particular implies that $\mathcal{O}_{X,x}$ is reduced. Let $k \subset k'$ be a finite purely inseparable field extension. Consider the ring $\mathcal{O}_{X,x} \otimes_k k'$. By Algebra, Lemma 7.43.2 its spectrum is the same as the spectrum of $\mathcal{O}_{X,x}$. Hence it is a local ring also (Algebra, Lemma 7.17.2). Therefore there is a unique point $x' \in X_{k'}$ lying over x and $\mathcal{O}_{X_{k'},x'} \cong \mathcal{O}_{X,x} \otimes_k k'$. By assumption this is a reduced ring. Hence we deduce (2) by Algebra, Lemma 7.41.3.

Assume (2). Let $k \subset k'$ be a field extension. Since $Spec(k') \to Spec(k)$ is surjective, also $X_{k'} \to X$ is surjective (Morphisms, Lemma 24.9.4). Let $x' \in X_{k'}$ be any point lying over x. The local ring $\mathcal{O}_{X_{k'},x'}$ is a localization of the ring $\mathcal{O}_{X,x} \otimes_k k'$. Hence it is reduced by assumption and (1) is proved. □

The notion isn't interesting in characteristic zero.

Lemma 28.4.3. *Let X be a scheme over a perfect field k (e.g. k has characteristic zero). Let $x \in X$. If $\mathcal{O}_{X,x}$ is reduced, then X is geometrically reduced at x. If X is reduced, then X is geometrically reduced over k.*

Proof. The first statement follows from Lemma 28.4.2 and Algebra, Lemma 7.40.6 and the definition of a perfect field (Algebra, Definition 7.42.1). The second statement follows from the first. □

Lemma 28.4.4. *Let k be a field of characteristic $p > 0$. Let X be a scheme over k. The following are equivalent*

(1) *X is geometrically reduced,*
(2) *$X_{k'}$ is reduced for every field extension $k \subset k'$,*
(3) *$X_{k'}$ is reduced for every finite purely inseparable field extension $k \subset k'$,*
(4) *$X_{k^{1/p}}$ is reduced,*
(5) *$X_{k^{perf}}$ is reduced,*
(6) *$X_{\bar{k}}$ is reduced,*
(7) *for every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is geometrically reduced (see Algebra, Definition 7.40.1).*

Proof. Assume (1). Then for every field extension $k \subset k'$ and every point $x' \in X_{k'}$ the local ring of $X_{k'}$ at x' is reduced. In other words $X_{k'}$ is reduced. Hence (2).

Assume (2). Let $U \subset X$ be an affine open. Then for every field extension $k \subset k'$ the scheme $X_{k'}$ is reduced, hence $U_{k'} = Spec(\mathcal{O}(U) \otimes_k k')$ is reduced, hence $\mathcal{O}(U) \otimes_k k'$ is reduced (see Properties, Section 23.3). In other words $\mathcal{O}(U)$ is geometrically reduced, so (7) holds.

Assume (7). For any field extension $k \subset k'$ the base change $X_{k'}$ is gotten by gluing the spectra of the rings $\mathcal{O}_X(U) \otimes_k k'$ where U is affine open in X (see Schemes, Section 21.17). Hence $X_{k'}$ is reduced. So (1) holds.

This proves that (1), (2), and (7) are equivalent. These are equivalent to (3), (4), (5), and (6) because we can apply Algebra, Lemma 7.41.3 to $\mathcal{O}_X(U)$ for $U \subset X$ affine open. □

Lemma 28.4.5. *Let k be a field of characteristic $p > 0$. Let X be a scheme over k. Let $x \in X$. The following are equivalent*

(1) *X is geometrically reduced at x,*
(2) *$\mathcal{O}_{X_{k'},x'}$ is reduced for every finite purely inseparable field extension k' of k and $x' \in X_{k'}$ the unique point lying over x,*
(3) *$\mathcal{O}_{X_{k^{1/p}},x'}$ is reduced for $x' \in X_{k'}$ the unique point lying over x, and*
(4) *$\mathcal{O}_{X_{k^{perf}},x'}$ is reduced for $x' \in X_{k^{perf}}$ the unique point lying over x.*

Proof. Note that if $k \subset k'$ is purely inseparable, then $X_{k'} \to X$ induces a homeomorphism on underlying topological spaces, see Algebra, Lemma 7.43.2. Whence the uniqueness of x' lying over x mentioned in the statement. Moreover, in this case $\mathcal{O}_{X_{k'},x'} = \mathcal{O}_{X,x} \otimes_k k'$. Hence the lemma follows from Lemma 28.4.2 above and Algebra, Lemma 7.41.3. □

Lemma 28.4.6. *Let k be a field. Let X be a scheme over k. Let k'/k be a field extension. Let $x \in X$ be a point, and let $x' \in X_{k'}$ be a point lying over x. The following are equivalent*

(1) *X is geometrically reduced at x,*
(2) *$X_{k'}$ is geometrically reduced at x'.*

In particular, X is geometrically reduced over k if and only if $X_{k'}$ is geometrically reduced over k'.

Proof. It is clear that (1) implies (2). Assume (2). Let $k \subset k''$ be a finite purely inseparable field extension and let $x'' \in X_{k''}$ be a point lying over x (actually it is unique). We can find a common field extension $k \subset k'''$ (i.e. with both $k' \subset k'''$ and $k'' \subset k'''$) and a point $x''' \in X_{k'''}$ lying over both x' and x''. Consider the map of local rings

$$\mathcal{O}_{X_{k''},x''} \longrightarrow \mathcal{O}_{X_{k'''},x'''}.$$

This is a flat local ring homomorphism and hence faithfully flat. By (2) we see that the local ring on the right is reduced. Thus by Algebra, Lemma 7.146.2 we conlude that $\mathcal{O}_{X_{k''},x''}$ is reduced. Thus by Lemma 28.4.5 we conclude that X is geometrically reduced at x. □

Lemma 28.4.7. *Let k be a field. Let X, Y be schemes over k.*

(1) *If X is geometrically reduced at x, and Y reduced, then $X \times_k Y$ is reduced at every point lying over x.*
(2) *If X geometrically reduced over k and Y reduced. Then $X \times_k Y$ is reduced.*

Proof. Combine, Lemmas 28.4.2 and 28.4.4 and Algebra, Lemma 7.40.5. □

Lemma 28.4.8. *Let k be a field. Let X be a scheme over k.*

(1) *If $x' \rightsquigarrow x$ is a specialization and X is geometrically reduced at x, then X is geometrically reduced at x'.*
(2) *If $x \in X$ such that (a) $\mathcal{O}_{X,x}$ is reduced, and (b) for each specialization $x' \rightsquigarrow x$ where x' is a generic point of an irreducible component of X the scheme X is geometrically reduced at x', then X is geometrically reduced at x.*
(3) *If X is reduced and geometrically reduced at all generic points of irreducible components of X, then X is geometrically reduced.*

Proof. Part (1) follows from Lemma 28.4.2 and the fact that if A is a geometrically reduced k-algebra, then $S^{-1}A$ is a geometrically reduced k-algebra for any multiplicative subset S of A, see Algebra, Lemma 7.40.3.

Let $A = \mathcal{O}_{X,x}$. The assumptions (a) and (b) of (2) imply that A is reduced, and that $A_{\mathfrak{q}}$ is geometrically reduced over k for every minimal prime $\mathfrak{q}$ of A. Hence A is geometrically reduced over k, see Algebra, Lemma 7.40.7. Thus X is geometrically reduced at x, see Lemma 28.4.2.

Part (3) follows trivially from part (2). □

Lemma 28.4.9. *Let k be a field. Let X be a scheme over k. Let $x \in X$. Assume X locally Noetherian and geometrically reduced at x. Then there exists an open neighbourhood $U \subset X$ of x which is geometrically reduced over k.*

Proof. Let R be a Noetherian k-algebra. Let $\mathfrak{p} \subset R$ be a prime. Let $I = \text{Ker}(R \to R_{\mathfrak{p}}$. Since $IR_{\mathfrak{p}} = R_{\mathfrak{p}}$ and I is finitely generated there exists an $f \in R$, $f \notin \mathfrak{p}$ such that $fI = 0$. Hence $R_f \subset R_{\mathfrak{p}}$.

Assume X locally Noetherian and geometrically reduced at x. If we apply the above to $R = \mathcal{O}_X(U)$ for some affine open neighbourhood of x, and $\mathfrak{p} \subset R$ the prime corresponding to x, then we see that after shrinking U we may assume $R \subset R_{\mathfrak{p}}$. By Lemma 28.4.2 the assumption means that $R_{\mathfrak{p}}$ is geometrically reduced over k. By Algebra, Lemma 7.40.2 this implies that R is geometrically reduced over k, which in turn implies that U is geometrically reduced. □

Example 28.4.10. Let $k = \mathbf{F}_p(s,t)$, i.e., a purely transcendental extension of the prime field. Consider the variety $X = Spec(k[x,y]/(1 + sx^p + ty^p))$. Let $k \subset k'$ be any extension such that both s and t have a pth root in k'. Then the base change $X_{k'}$ is not reduced. Namely, the ring $k'[x,y]/(1 + sx^p + ty^p)$ contains the element $1 + s^{1/p}x + t^{1/p}y$ whose pth power is zero but which is not zero (since the ideal $(1 + sx^p + ty^p)$ certainly does not contain any nonzero element of degree $< p$).

Lemma 28.4.11. *Let k be a field. Let $X \to Spec(k)$ be locally of finite type. Assume X has finitely many irreducible components. Then there exists a finite purely inseparable extension $k \subset k'$ such that $(X_{k'})_{red}$ is geometrically reduced over k'.*

Proof. To prove this lemma we may replace X by its reduction X_{red}. Hence we may assume that X is reduced and locally of finite type over k. Let $x_1, \ldots, x_n \in X$ be the generic points of the irreducible components of X. Note that for every purely inseparable algebraic extension $k \subset k'$ the morphism $(X_{k'})_{red} \to X$ is a homeomorphism, see Algebra, Lemma 7.43.2. Hence the points $x'_1, \ldots, x'_n$ lying over $x_1, \ldots, x_n$ are the generic points of the irreducible components of $(X_{k'})_{red}$. As X is reduced the local rings $K_i = \mathcal{O}_{X,x_i}$ are fields, see Algebra, Lemma 7.23.3. As X is locally of finite type over k the field extensions $k \subset K_i$ are finitely generated field extensions. Finally, the local rings $\mathcal{O}_{X_{k'},x'_i}$ are the fields $(K_i \otimes_k k')_{red}$. By Algebra, Lemma 7.42.3 we can find a finite purely inseparable extension $k \subset k'$ such that $(K_i \otimes_k k')_{red}$ are separable field extensions of k'. In particular each $(K_i \otimes_k k')_{red}$ is geometrically reduced over k' by Algebra, Lemma 7.41.1. At this point Lemma 28.4.8 part (3) implies that $(X_{k'})_{red}$ is geometrically reduced. $\square$

28.5. Geometrically connected schemes

If X is a connected scheme over a field, then it can happen that X becomes disconnected after extending the ground field. This does not happen for geometrically connected schemes.

Definition 28.5.1. Let X be a scheme over the field k. We say X is *geometrically connected* over k if the scheme $X_{k'}$ is connected[1] for every field extension k' of k.

Here is an example of a variety which is not geometrically connected.

Example 28.5.2. Let $k = \mathbf{Q}$. The scheme $X = Spec(\mathbf{Q}(i))$ is a variety over $Spec(\mathbf{Q})$. But the base change $X_{\mathbf{C}}$ is the spectrum of $\mathbf{C} \otimes_{\mathbf{Q}} \mathbf{Q}(i) \cong \mathbf{C} \times \mathbf{C}$ which is the disjoint union of two copies of $Spec(\mathbf{C})$. So in fact, this is an example of a non-geometrically connected variety.

Lemma 28.5.3. *Let X be a scheme over the field k. Let $k \subset k'$ be a field extension. Then X is geometrically connected over k if and only if $X_{k'}$ is geometrically connected over k'.*

Proof. If X is geometrically connected over k, then it is clear that $X_{k'}$ is geometrically connected over k'. For the converse, note that for any field extension $k \subset k''$ there exists a common field extension $k' \subset k'''$ and $k'' \subset k'''$. As the morphism $X_{k'''} \to X_{k''}$ is surjective (as a base change of a surjective morphism between spectra of fields) we see that the connectedness of $X_{k'''}$ implies the connectedness of $X_{k''}$. Thus if $X_{k'}$ is geometrically connected over k' then X is geometrically connected over k. $\square$

Lemma 28.5.4. *Let k be a field. Let X, Y be schemes over k. Assume X is geometrically connected over k. Then the projection morphism*

$$p : X \times_k Y \longrightarrow Y$$

induces a bijection between connected components.

Proof. The scheme theoretic fibres of p are connected and nonempty, since they are base changes of the geometrically connected scheme X by field extensions. Moreover the scheme theoretic fibres are homeomorphic to the set theoretic fibres, see Schemes, Lemma 21.18.5. By Morphisms, Lemma 24.22.4 the map p is open. Thus we may apply Topology, Lemma 5.4.5 to conclude. $\square$

[1] An empty topological space is connected.

Lemma 28.5.5. *Let k be a field. Let A be a k-algebra. Then $X = Spec(A)$ is geometrically connected over k if and only if A is geometrically connected over k (see Algebra, Definition 7.44.3).*

Proof. Immediate from the definitions. $\square$

Lemma 28.5.6. *Let $k \subset k'$ be an extension of fields. Let X be a scheme over k. Assume k separably algebraically closed. Then the morphism $X_{k'} \to X$ induces a bijection of connected components. In particular, X is geometrically connected over k if and only if X is connected.*

Proof. Since k is separably algebraically closed we see that k' is geometrically connected over k, see Algebra, Lemma 7.44.4. Hence $Z = Spec(k')$ is geometrically connected over k by Lemma 28.5.5 above. Since $X_{k'} = Z \times_k X$ the result is a special case of Lemma 28.5.4. $\square$

Lemma 28.5.7. *Let k be a field. Let X be a scheme over k. Let $\overline{k}$ be a separable algebraic closure of k. Then X is geometrically connected if and only if the base change $X_{\overline{k}}$ is connected.*

Proof. Assume $X_{\overline{k}}$ is connected. Let $k \subset k'$ be a field extension. There exists a field extension $\overline{k} \subset \overline{k}'$ such that k' embeds into $\overline{k}'$ as an extension of k. By Lemma 28.5.6 we see that $X_{\overline{k}'}$ is connected. Since $X_{\overline{k}'} \to X_{k'}$ is surjective we conclude that $X_{k'}$ is connected as desired. $\square$

Lemma 28.5.8. *Let k be a field. Let X be a scheme over k. Let A be a k-algebra. Let $V \subset X_A$ be a quasi-compact open. Then there exists a finitely generated k-subalgebra $A' \subset A$ and a quasi-compact open $V' \subset X_{A'}$ such that $V = V'_A$.*

Proof. We remark that if X is also quasi-separated this follows from Limits, Lemma 27.3.5. Let $U_1, \ldots, U_n$ be finitely many affine opens of X such that $V \subset \bigcup U_{i,A}$. Say $U_i = Spec(R_i)$. Since V is quasi-compact we can find finitely many $f_{ij} \in R_i \otimes_k A$, $j = 1, \ldots, n_i$ such that $V = \bigcup_i \bigcup_{j=1,\ldots,n_i} D(f_{ij})$ where $D(f_{ij}) \subset U_{i,A}$ is the corresponding standard open. (We do not claim that $V \cap U_{i,A}$ is the union of the $D(f_{ij})$, $j = 1, \ldots, n_i$.) It is clear that we can find a finitely generated k-subalgebra $A' \subset A$ such that f_{ij} is the image of some $f'_{ij} \in R_i \otimes_k A'$. Set $V' = \bigcup D(f'_{ij})$ which is a quasi-compact open of $X_{A'}$. Denote $\pi : X_A \to X_{A'}$ the canonical morphism. We have $\pi(V) \subset V'$ as $\pi(D(f_{ij})) \subset D(f'_{ij})$. If $x \in X_A$ with $\pi(x) \in V'$, then $\pi(x) \in D(f'_{ij})$ for some i, j and we see that $x \in D(f_{ij})$ as f'_{ij} maps to f_{ij}. Thus we see that $V = \pi^{-1}(V')$ as desired. $\square$

Let k be a field. Let $k \subset \overline{k}$ be a (possibly infinite) Galois extension. For example $\overline{k}$ could be the separable algebraic closure of k. For any $\sigma \in \mathrm{Gal}(\overline{k}/k)$ we get a corresponding automorphism $Spec(\sigma) : Spec(\overline{k}) \longrightarrow Spec(\overline{k})$. Note that $Spec(\sigma) \circ Spec(\tau) = Spec(\tau \circ \sigma)$. Hence we get an action

$$\mathrm{Gal}(\overline{k}/k)^{opp} \times Spec(\overline{k}) \longrightarrow Spec(\overline{k})$$

of the opposite group on the scheme $Spec(\overline{k})$. Let X be a scheme over k. Since $X_{\overline{k}} = Spec(\overline{k}) \times_{Spec(k)} X$ by definition we see that the action above induces a canonical action

$$\mathrm{Gal}(\overline{k}/k)^{opp} \times X_{\overline{k}} \longrightarrow X_{\overline{k}}. \tag{28.5.8.1}$$

Lemma 28.5.9. *Let k be a field. Let X be a scheme over k. Let $\overline{k}$ be a (possibly infinite) Galois extension of k. Let $V \subset X_{\overline{k}}$ be a quasi-compact open. Then*

(1) *there exists a finite subextension $k \subset k' \subset \overline{k}$ and a quasi-compact open $V' \subset X_{k'}$ such that $V = (V')_{\overline{k}}$,*
(2) *there exists an open subgroup $H \subset Gal(\overline{k}/k)$ such that $\sigma(V) = V$ for all $\sigma \in H$.*

Proof. By Lemma 28.5.8 there exists a finite subextension $k \subset k' \subset \overline{k}$ and an open $V' \subset X_{k'}$ which pulls back to V. This proves (1). Since $\text{Gal}(\overline{k}/k')$ is open in $\text{Gal}(\overline{k}/k)$ part (2) is clear as well. $\square$

Lemma 28.5.10. *Let k be a field. Let $k \subset \overline{k}$ be a (possibly infinite) Galois extension. Let X be a scheme over k. Let $\overline{T} \subset X_{\overline{k}}$ have the following properties*

(1) *$\overline{T}$ is a closed subset of $X_{\overline{k}}$,*
(2) *for every $\sigma \in Gal(\overline{k}/k)$ we have $\sigma(\overline{T}) = \overline{T}$.*

Then there exists a closed subset $T \subset X$ whose inverse image in $X_{k'}$ is $\overline{T}$.

Proof. This lemma immediately reduces to the case where $X = Spec(A)$ is affine. In this case, let $\overline{I} \subset A \otimes_k \overline{k}$ be the radical ideal corresponding to $\overline{T}$. Assumption (2) implies that $\sigma(\overline{I}) = \overline{I}$ for all $\sigma \in \text{Gal}(\overline{k}/k)$. Pick $x \in \overline{I}$. There exists a finite Galois extension $k \subset k'$ contained in $\overline{k}$ such that $x \in A \otimes_k k'$. Set $G = \text{Gal}(k'/k)$. Set

$$P(T) = \prod\nolimits_{\sigma \in G}(T - \sigma(x)) \in (A \otimes_k k')[T]$$

It is clear that $P(T)$ is monic and is actually an element of $(A \otimes_k k')^G[T] = A[T]$ (by basic Galois theory). Moreover, if we write $P(T) = T^d + a_1T^{d-1} + \ldots + a_0$ the we see that $a_i \in I := A \cap \overline{I}$. By Algebra, Lemma 7.34.5 we see that x is contained in the radical of $I(A \otimes_k \overline{k})$. Hence $\overline{I}$ is the radical of $I(A \otimes_k \overline{k})$ and setting $T = V(I)$ is a solution. $\square$

Lemma 28.5.11. *Let k be a field. Let X be a scheme over k. The following are equivalent*

(1) *X is geometrically connected,*
(2) *for every finite separable field extension $k \subset k'$ the scheme $X_{k'}$ is connected.*

Proof. It follows immediately from the definition that (1) implies (2). Assume that X is not geometrically connected. Let $k \subset \overline{k}$ be a separable algebraic closure of k. By Lemma 28.5.7 it follows that $X_{\overline{k}}$ is disconnected. Say $X_{\overline{k}} = \overline{U} \amalg \overline{V}$ with $\overline{U}$ and $\overline{V}$ open, closed, and nonempty.

Suppose that $W \subset X$ is any quasi-compact open. Then $W_{\overline{k}} \cap \overline{U}$ and $W_{\overline{k}} \cap \overline{V}$ are open and closed in $W_{\overline{k}}$. In particular $W_{\overline{k}} \cap \overline{U}$ and $W_{\overline{k}} \cap \overline{V}$ are quasi-compact, and by Lemma 28.5.9 both $W_{\overline{k}} \cap \overline{U}$ and $W_{\overline{k}} \cap \overline{V}$ are defined over a finite subextension and invariant under an open subgroup of $\text{Gal}(\overline{k}/k)$. We will use this without further mention in the following.

Pick $W_0 \subset X$ quasi-compact open such that both $W_{0,\overline{k}} \cap \overline{U}$ and $W_{0,\overline{k}} \cap \overline{V}$ are nonempty. Choose a finite subextension $k \subset k' \subset \overline{k}$ and a decompostion $W_{0,k'} = U'_0 \amalg V'_0$ into open and closed subsets such that $W_{0,\overline{k}} \cap \overline{U} = (U'_0)_{\overline{k}}$ and $W_{0,\overline{k}} \cap \overline{V} = (V'_0)_{\overline{k}}$. Let $H = \text{Gal}(\overline{k}/k') \subset \text{Gal}(\overline{k}/k)$. In particular $\sigma(W_{0,\overline{k}} \cap \overline{U}) = W_{0,\overline{k}} \cap \overline{U}$ and similarly for $\overline{V}$.

Having chosen W_0, k' as above, for every quasi-compact open $W \subset X$ we set

$$U_W = \bigcap\nolimits_{\sigma \in H} \sigma(W_{\overline{k}} \cap \overline{U}), \quad V_W = \bigcup\nolimits_{\sigma \in H} \sigma(W_{\overline{k}} \cap \overline{V}).$$

Now, since $W_{\overline{k}} \cap \overline{U}$ and $W_{\overline{k}} \cap \overline{V}$ are fixed by an open subgroup of $\text{Gal}(\overline{k}/k)$ we see that the union and intersection above are finite. Hence U_W and V_W are both open and closed. Also, by construction $W_{\bar{k}} = U_W \amalg V_W$.

We claim that if $W \subset W' \subset X$ are quasi-compact open, then $W_{\overline{k}} \cap U_{W'} = U_W$ and $W_{\overline{k}} \cap V_{W'} = V_W$. Verification omitted. Hence we see that upon defining $U = \bigcup_{W \subset X} U_W$ and $V = \bigcup_{W \subset X} V_W$ we obtain $X_{\overline{k}} = U \amalg V$ is a disjoint union of open and closed subsets. It is clear that V is nonempty as it is constructed by taking unions (locally). On the other hand, U is nonempty since it contains $W_0 \cap \overline{U}$ by construction. Finally, $U, V \subset X_{\bar{k}}$ are closed and H-invariant by construction. Hence by Lemma 28.5.10 we have $U = (U')_{\bar{k}}$, and $V = (V')_{\bar{k}}$ for some closed $U', V' \subset X_{k'}$. Clearly $X_{k'} = U' \amalg V'$ and we see that $X_{k'}$ is disconnected as desired. $\square$

Lemma 28.5.12. *Let k be a field. Let $k \subset \overline{k}$ be a (possibly infinite) Galois extension. Let $f : T \to X$ be a morphism of schemes over k. Assume $T_{\overline{k}}$ nonempty connected and $X_{\overline{k}}$ disconnected. Then X is disconnected.*

Proof. Write $X_{\overline{k}} = \overline{U} \coprod \overline{V}$ with $\overline{U}$ and $\overline{V}$ open and closed. Denote $\overline{f} : T_{\overline{k}} \to X_{\overline{k}}$ the base change of f. Since $T_{\overline{k}}$ is connected we see that $T_{\overline{k}}$ is contained in either $\overline{f}^{-1}(\overline{U})$ or $\overline{f}^{-1}(\overline{V})$. Say $T_{\overline{k}} \subset \overline{f}^{-1}(\overline{U})$.

Fix a quasi-compact open $W \subset X$. There exists a finite Galois subextension $k \subset k' \subset \overline{k}$ such that $\overline{U} \cap W_{\overline{k}}$ and $\overline{V} \cap W_{\overline{k}}$ come from quasi-compact opens $U', V' \subset W_{k'}$. Then also $W_{k'} = U' \coprod V'$. Consider

$$U'' = \bigcap\nolimits_{\sigma \in \text{Gal}(k'/k)} \sigma(U'), \quad V'' = \bigcup\nolimits_{\sigma \in \text{Gal}(k'/k)} \sigma(V').$$

These are Galois invariant, open and closed, and $W_{k'} = U'' \coprod V''$. By Lemma 28.5.10 we get open and closed subsets $U_W, V_W \subset W$ such that $U'' = (U_W)_{k'}$, $V'' = (V_W)_{k'}$ and $W = U_W \coprod V_W$.

We claim that if $W \subset W' \subset X$ are quasi-compact open, then $W \cap U_{W'} = U_W$ and $W \cap V_{W'} = V_W$. Verification omitted. Hence we see that upon defining $U = \bigcup_{W \subset X} U_W$ and $V = \bigcup_{W \subset X} V_W$ we obtain $X = U \coprod V$. It is clear that V is nonempty as it is constructed by taking unions (locally). On the other hand, U is nonempty since it contains $f(T)$ by construction. $\square$

Lemma 28.5.13. *Let k be a field. Let $T \to X$ be a morphism of schemes over k. Assume T is nonempty and geometrically connected and X connected. Then X is geometrically connected.*

Proof. This is a reformulation of Lemma 28.5.12. $\square$

Lemma 28.5.14. *Let k be a field. Let X be a scheme over k. Assume X is connected and has a point x such that k is algebraically closed in $\kappa(x)$. Then X is geometrically connected. In particular, if X has a k-rational point and X is connected, then X is geometrically connected.*

Proof. Set $T = Spec(\kappa(x))$. Let $k \subset \overline{k}$ be a separable algebraic closure of k. The assumption on $k \subset \kappa(x)$ implies that $T_{\overline{k}}$ is irreducible, see Algebra, Lemma 7.43.10. Hence by Lemma 28.5.13 we see that $X_{\overline{k}}$ is connected. By Lemma 28.5.7 we conclude that X is geometrically connected. $\square$

Lemma 28.5.15. *Let $k \subset K$ be an extension of fields. Let X be a scheme over k. For every connected component T of X the inverse image $T_K \subset X_K$ is a union of connected components of X_K.*

Proof. This is a purely topological statement. Denote $p : X_K \to X$ the projection morphism. Let $T \subset X$ be a connected component of X. Let $t \in T_K = p^{-1}(T)$. Let $C \subset X_K$ be a connected component containing t. Then $p(C)$ is a connected subset of X which meets T, hence $p(C) \subset T$. Hence $C \subset T_K$. $\square$

Remark 28.5.16. Let $k \subset K$ be an extension of fields. Let X be a scheme over k. Denote $p : X_K \to X$ the projection morphism. Let $\overline{T} \subset X_K$ be a connected component. Is it true that $p(\overline{T})$ is a connected component of X? We do not know the answer, even when $k \subset K$ is finite! If you do, or if you have a reference, please email stacks.project@gmail.com.

Let X be a scheme. We denote $\pi_0(X)$ the set of connected components of X.

Lemma 28.5.17. *Let k be a field, with separable algebraic closure $\overline{k}$. Let X be a scheme over k. There is an action*

$$Gal(\overline{k}/k)^{opp} \times \pi_0(X_{\overline{k}}) \longrightarrow \pi_0(X_{\overline{k}})$$

with the following properties:

(1) *An element $\overline{T} \in \pi_0(X_{\overline{k}})$ is fixed by the action if and only if there exists a connected component $T \subset X$, which is geometrically connected over k, such that $T_{\overline{k}} = \overline{T}$.*
(2) *For any field extension $k \subset k'$ with separable algebraic closure $\overline{k}'$ the diagram*

$$\begin{tikzcd} Gal(\overline{k}'/k') \times \pi_0(X_{\overline{k}'}) \arrow[r] \arrow[d] & \pi_0(X_{\overline{k}'}) \arrow[d] \\ Gal(\overline{k}/k) \times \pi_0(X_{\overline{k}}) \arrow[r] & \pi_0(X_{\overline{k}}) \end{tikzcd}$$

is commutative (where the right vertical arrow is a bijection according to Lemma 28.5.6).

Proof. The action (28.5.8.1) of $\mathrm{Gal}(\overline{k}/k)$ on $X_{\overline{k}}$ induces an action on its connected components. Connected components are always closed (Topology, Lemma 5.4.3). Hence if $\overline{T}$ is as in (1), then by Lemma 28.5.10 there exists a closed subset $T \subset X$ such that $\overline{T} = T_{\overline{k}}$. Note that T is geometrically connected over k, see Lemma 28.5.7. To see that T is a connected component of X, suppose that $T \subset T'$, $T \neq T'$ where T' is a connected component of X. In this case $T'_{\overline{k}}$, strictly contains $\overline{T}$ and hence is disconnnected. By Lemma 28.5.12 this means that T' is disconnected! Contradiction.

We omit the proof of the functoriality in (2). $\square$

Lemma 28.5.18. *Let k be a field, with separable algebraic closure $\overline{k}$. Let X be a scheme over k. Assume*

(1) *X is quasi-compact, and*
(2) *the connected components of $X_{\overline{k}}$ are open.*

Then

(a) *$\pi_0(X_{\overline{k}})$ is finite, and*
(b) *the action of $Gal(\overline{k}/k)$ on $\pi_0(X_{\overline{k}})$ is continuous.*

Moreover, assumptions (1) and (2) are satisfied when X is of finite type over k.

Proof. Since the connected components are open, cover $X_{\overline{k}}$ (Topology, Lemma 5.4.3) and $X_{\overline{k}}$ is quasi-compact, we conclude that there are only finitely many of them. Thus (a) holds. By Lemma 28.5.8 these connected components are each defined over a finite subextension of $k \subset \overline{k}$ and we get (b). If X is of finite type over k, then $X_{\overline{k}}$ is of finite type over $\overline{k}$ (Morphisms, Lemma 24.14.4). Hence $X_{\overline{k}}$ is a Noetherian scheme (Morphisms, Lemma 24.14.6) and has an underlying Noetherian topological space (Properties, Lemma 23.5.5). Thus $X_{\overline{k}}$ has finitely many irreducible components (Topology, Lemma 5.6.2) and a fortiori finitely many connected components (which are therefore open). □

28.6. Geometrically irreducible schemes

If X is an irreducible scheme over a field, then it can happen that X becomes reducible after extending the ground field. This does not happen for geometrically irreducible schemes.

Definition 28.6.1. Let X be a scheme over the field k. We say X is *geometrically irreducible* over k if the scheme $X_{k'}$ is irreducible[2] for any field extension k' of k.

Lemma 28.6.2. *Let X be a scheme over the field k. Let $k \subset k'$ be a field extension. Then X is geometrically irreducible over k if and only if $X_{k'}$ is geometrically irreducible over k'.*

Proof. If X is geometrically irreducible over k, then it is clear that $X_{k'}$ is geometrically irreducible over k'. For the converse, note that for any field extension $k \subset k''$ there exists a common field extension $k' \subset k'''$ and $k'' \subset k'''$. As the morphism $X_{k'''} \to X_{k''}$ is surjective (as a base change of a surjective morphism between spectra of fields) we see that the irreducibility of $X_{k'''}$ implies the irreducibility of $X_{k''}$. Thus if $X_{k'}$ is geometrically irreducible over k' then X is geometrically irreducible over k. □

Lemma 28.6.3. *Let X be a scheme over a separably closed field k. If X is irreducible, then X_K is irreducible for any field extension $k \subset K$. I.e., X is geometrically irreducible over k.*

Proof. Use Properties, Lemma 23.3.3 and Algebra, Lemma 7.43.4. □

Lemma 28.6.4. *Let k be a field. Let X, Y be schemes over k. Assume X is geometrically irreducible over k. Then the projection morphism*

$$p : X \times_k Y \longrightarrow Y$$

induces a bijection between irreducible components.

Proof. First, note that the scheme theoretic fibres of p are irreducible, since they are base changes of the geometrically irreducible scheme X by field extensions. Moreover the scheme theoretic fibres are homeomorphic to the set theoretic fibres, see Schemes, Lemma 21.18.5. By Morphisms, Lemma 24.22.4 the map p is open. Thus we may apply Topology, Lemma 5.5.8 to conclude. □

Lemma 28.6.5. *Let k be a field. Let X be a scheme over k. The following are equivalent*

(1) *X is geometrically irreducible over k,*
(2) *for every affine open U the k-algebra $\mathcal{O}_X(U)$ is geometrically irreducible over k (see Algebra, Definition 7.43.6),*
(3) *X is irreducible and there exists an affine open covering $X = \bigcup U_i$ such that each k-algebra $\mathcal{O}_X(U_i)$ is geometrically irreducible, and*

[2]An irreducible space is nonempty.

(4) *there exists an open covering $X = \bigcup_{i \in I} X_i$ such that X_i is geometrically irreducible for each i and such that $X_i \cap X_j \neq \emptyset$ for all $i, j \in I$.*

Moreover, if X is geometrically irreducible so is every open subscheme of X.

Proof. An affine scheme $Spec(A)$ over k is geometrically irreducible if and only if A is geometrically irreducible over k; this is immediate from the definitions. Recall that if a scheme is irreducible so is every nonempty open subscheme of X, any two nonempty open subsets have a nonempty intersection. Also, if every affine open is irreducible then the scheme is irreducible, see Properties, Lemma 23.3.3. Hence the final statement of the lemma is clear, as well as the implications (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3), and (3) $\Rightarrow$ (4). If (4) holds, then for any field extension k'/k the scheme $X_{k'}$ has a covering by irreducible opens which pairwise intersect. Hence $X_{k'}$ is irreducible. Hence (4) implies (1). $\square$

Lemma 28.6.6. *Let X be a geometrically irreducible scheme over the field k. Let $\xi \in X$ be its generic point. Then $\kappa(\xi)$ is a geometrically irreducible over k.*

Proof. Combining Lemma 28.6.5 and Algebra, Lemma 7.43.8 we see that $\mathcal{O}_{X,\xi}$ is geometrically irreducible over k. Since $\mathcal{O}_{X,\xi} \to \kappa(\xi)$ is a surjection with locally nilpotent kernel (see Algebra, Lemma 7.23.3) it follows that $\kappa(\xi)$ is geometrically irreducible, see Algebra, Lemma 7.43.2. $\square$

Lemma 28.6.7. *Let $k \subset k'$ be an extension of fields. Let X be a scheme over k. Set $X' = X_{k'}$. Assume k separably algebraically closed. Then the morphism $X' \to X$ induces a bijection of irreducible components.*

Proof. Since k is separably algebraically closed we see that k' is geometrically irreducible over k, see Algebra, Lemma 7.43.7. Hence $Z = Spec(k')$ is geometrically irreducible over k. by Lemma 28.6.5 above. Since $X' = Z \times_k X$ the result is a special case of Lemma 28.6.4. $\square$

Lemma 28.6.8. *Let k be a field. Let X be a scheme over k. Assume X is quasi-compact. The following are equivalent:*

(1) *X is geometrically irreducible over k,*
(2) *for every finite separable field extension $k \subset k'$ the scheme $X_{k'}$ is irreducible, and*
(3) *$X_{\overline{k}}$ is irreducible, where $k \subset \overline{k}$ is a separable algebraic closure of k.*

Proof. Assume $X_{\overline{k}}$ is irreducible, i.e., assume (3). Let $k \subset k'$ be a field extension. There exists a field extension $\overline{k} \subset \overline{k}'$ such that k' embeds into $\overline{k}'$ as an extension of k. By Lemma 28.6.7 we see that $X_{\overline{k}'}$ is irreducible. Since $X_{\overline{k}'} \to X_{k'}$ is surjective we conclude that $X_{k'}$ is irreducible. Hence (1) holds.

Let $k \subset \overline{k}$ be a separable algebraic closure of k. Assume not (3), i.e., assume $X_{\overline{k}}$ is reducible. Our goal is to show that also $X_{k'}$ is reducible for some finite subextension $k \subset k' \subset \overline{k}$. Let $X = \bigcup_{i \in I} U_i$ be an affine open covering with U_i not empty. If for some i the scheme U_i is reducible, or if for some pair $i \neq j$ the intersection $U_i \cap U_j$ is empty, then X is reducible (Properties, Lemma 23.3.3) and we are done. In particular we may assume that $U_{i,\overline{k}} \cap U_{j,\overline{k}}$ for all $i, j \in I$ is nonempty and we conclude that $U_{i,\overline{k}}$ has to be reducible for some i. According to Algebra, Lemma 7.43.5 this means that $U_{i,k'}$ is reducible for some finite separable field extension $k \subset k'$. Hence also $X_{k'}$ is reducible. Thus we see that (2) implies (3).

The implication (1) $\Rightarrow$ (2) is immediate. This proves the lemma. $\square$

Lemma 28.6.9. *Let $k \subset K$ be an extension of fields. Let X be a scheme over k. For every irreducible component T of X the inverse image $T_K \subset X_K$ is a union of irreducible components of X_K.*

Proof. Let $T \subset X$ be an irreducible component of X. The morphism $T_K \to T$ is flat, so generalizations lift along $T_K \to T$. Hence every $\xi \in T_K$ which is a generic point of an irreducible component of T_K maps to the generic point η of T. If $\xi' \rightsquigarrow \xi$ is a specialization in X_K then ξ' maps to η since there are no points specializing to η in X. Hence $\xi' \in T_K$ and we conclude that $\xi = \xi'$. In other words ξ is the generic point of an irreducible component of X_K. This means that the irreducible components of T_K are all irreducible components of X_K. □

For a scheme X we denote $\text{IrredComp}(X)$ the set of irreducible components of X.

Lemma 28.6.10. *Let $k \subset K$ be an extension of fields. Let X be a scheme over k. For every irreducible component $\overline{T} \subset X_K$ the image of $\overline{T}$ in X is an irreducible component in X. This defines a canonical map*

$$IrredComp(X_K) \longrightarrow IrredComp(X)$$

which is surjective.

Proof. Consider the diagram

$$\begin{tikzcd} X_K \arrow[d] & X_{\overline{K}} \arrow[l] \arrow[d] \\ X & X_{\overline{k}} \arrow[l] \end{tikzcd}$$

where $\overline{K}$ is the separable algebraic closure of K, and where $\overline{k}$ is the separable algebraic closure of k. By Lemma 28.6.7 the morphism $X_{\overline{K}} \to X_{\overline{k}}$ induces a bijection between irreducible components. Hence it suffices to show the lemma for the morphisms $X_{\overline{k}} \to X$ and $X_{\overline{K}} \to X_K$. In other words we may assume that $K = \overline{k}$.

The morphism $p : X_{\overline{k}} \to X$ is integral, flat and surjective. Flatness implies that generalizations lift along p, see Morphisms, Lemma 24.24.8. Hence generic points of irreducible components of $X_{\overline{k}}$ map to generic points of irreducible components of X. Integrality implies that p is universally closed, see Morphisms, Lemma 24.42.7. Hence we conclude that the image $p(\overline{T})$ of an irreducible component is a closed irreducible subset which contains a generic point of an irreducible component of X, hence $p(\overline{T})$ is an irreducible component of X. This proves the first assertion. If $T \subset X$ is an irreducible component, then $p^{-1}(T) = T_K$ is a nonempty union of irreducible components, see Lemma 28.6.9. Each of these necessarily maps onto T by the first part. Hence the map is surjective. □

Lemma 28.6.11. *Let k be a field, with separable algebraic closure $\overline{k}$. Let X be a scheme over k. There is an action*

$$Gal(\overline{k}/k)^{opp} \times IrredComp(X_{\overline{k}}) \longrightarrow IrredComp(X_{\overline{k}})$$

with the following properties:

(1) *An element $\overline{T} \in IrredComp(X_{\overline{k}})$ is fixed by the action if and only if there exists an irreducible component $T \subset X$, which is geometrically irreducible over k, such that $T_{\overline{k}} = \overline{T}$.*

(2) *For any field extension $k \subset k'$ with separable algebraic closure $\overline{k}'$ the diagram*

$$\begin{array}{ccc} Gal(\overline{k}'/k') \times IrredComp(X_{\overline{k}'}) & \longrightarrow & IrredComp(X_{\overline{k}'}) \\ \downarrow & & \downarrow \\ Gal(\overline{k}/k) \times IrredComp(X_{\overline{k}}) & \longrightarrow & IrredComp(X_{\overline{k}}) \end{array}$$

is commutative (where the right vertical arrow is a bijection according to Lemma 28.6.7).

Proof. The action (28.5.8.1) of $\text{Gal}(\overline{k}/k)$ on $X_{\overline{k}}$ induces an action on its irreducible components. Irreducible components are always closed (Topology, Lemma 5.4.3). Hence if $\overline{T}$ is as in (1), then by Lemma 28.5.10 there exists a closed subset $T \subset X$ such that $\overline{T} = T_{\overline{k}}$. Note that T is geometrically irreducible over k, see Lemma 28.6.8. To see that T is an irreducible component of X, suppose that $T \subset T'$, $T \neq T'$ where T' is an irreducible component of X. Let $\overline{\eta}$ be the generic point of $\overline{T}$. It maps to the generic point η of T. Then the generic point $\xi \in T'$ specializes to η. As $X_{\overline{k}} \to X$ is flat there exists a point $\overline{\xi} \in X_{\overline{k}}$ which maps to ξ and specializes to $\overline{\eta}$. It follows that the closure of the singleton $\{\overline{\xi}\}$ is an irreducible closed subset of $X_{\overline{\xi}}$ which strictly contains $\overline{T}$. This is the desired contradiction.

We omit the proof of the functoriality in (2). $\square$

Lemma 28.6.12. *Let k be a field, with separable algebraic closure $\overline{k}$. Let X be a scheme over k. The fibres of the map*

$$IrredComp(X_{\overline{k}}) \longrightarrow IrredComp(X)$$

of Lemma 28.6.10 are exactly the orbits of $Gal(\overline{k}/k)$ under the action of Lemma 28.6.11.

Proof. Let $T \subset X$ be an irreducible component of X. Let $\eta \in T$ be its generic point. By Lemmas 28.6.9 and 28.6.10 the generic points of irreducible components of $\overline{T}$ which map into T map to η. By Algebra, Lemma 7.43.12 the Galois group acts transitively on all of the points of $X_{\overline{k}}$ mapping to η. Hence the lemma follows. $\square$

Lemma 28.6.13. *Let k be a field. Assume $X \to Spec(k)$ locally of finite type. In this case*

(1) *the action*

$$Gal(\overline{k}/k)^{opp} \times IrredComp(X_{\overline{k}}) \longrightarrow IrredComp(X_{\overline{k}})$$

is continuous if we give $IrredComp(X_{\overline{k}})$ the discrete topology,

(2) *every irreducible component of $X_{\overline{k}}$ can be defined over a finite extension of k, and*

(3) *given any irreducible component $T \subset X$ the scheme $T_{\overline{k}}$ is a finite union of irreducible components of $X_{\overline{k}}$ which are all in the same $Gal(\overline{k}/k)$-orbit.*

Proof. Let $\overline{T}$ be an irreducible component of $X_{\overline{k}}$. We may choose an affine open $U \subset X$ such that $\overline{T} \cap U_{\overline{k}}$ is not empty. Write $U = Spec(A)$, so A is a finite type k-algebra, see Morphisms, Lemma 24.14.2. Hence $A_{\overline{k}}$ is a finite type $\overline{k}$-algebra, and in particular Noetherian. Let $\mathfrak{p} = (f_1, \ldots, f_n)$ be the prime ideal corresponding to $\overline{T} \cap U_{\overline{k}}$. Since $A_{\overline{k}} = A \otimes_k \overline{k}$ we see that there exists a finite subextension $k \subset k' \subset \overline{k}$ such that each $f_i \in A_{k'}$. It is clear that $\text{Gal}(\overline{k}/k')$ fixes $\overline{T}$, which proves (1).

Part (2) follows by applying Lemma 28.6.11 (1) to the situation over k' which implies the irreducible component $\overline{T}$ is of the form $T'_{\overline{k}}$ for some irreducible $T' \subset X_{k'}$.

To prove (3), let $T \subset X$ be an irreducible component. Choose an irreducible component $\overline{T} \subset X_{\overline{k}}$ which maps to T, see Lemma 28.6.10. By the above the orbit of $\overline{T}$ is finite, say it is $\overline{T}_1, \ldots, \overline{T}_n$. Then $\overline{T}_1 \cup \ldots \cup \overline{T}_n$ is a $\text{Gal}(\overline{k}/k)$-invariant closed subset of $X_{\overline{k}}$ hence of the form $W_{\overline{k}}$ for some $W \subset X$ closed by Lemma 28.5.10. Clearly $W = T$ and we win. □

Lemma 28.6.14. *Let k be a field. Let $X \to Spec(k)$ be locally of finite type. Assume X has finitely many irreducible components. Then there exists a finite separable extension $k \subset k'$ such that every irreducible component of $X_{k'}$ is geometrically irreducible over k'.*

Proof. Let $\overline{k}$ be a separable algebraic closure of k. The assumption that X has finitely many irreducible components combined with Lemma 28.6.13 (3) shows that $X_{\overline{k}}$ has finitely many irreducible components $\overline{T}_1, \ldots, \overline{T}_n$. By Lemma 28.6.13 (2) there exists a finite extension $k \subset k' \subset \overline{k}$ and irreducible components $T_i \subset X_{k'}$ such that $\overline{T}_i = T_{i,\overline{k}}$ and we win. □

Lemma 28.6.15. *Let X be a scheme over the field k. Assume X has finitely many irreducible components which are all geometrically irreducible. Then X has finitely many connected components each of which is geometrically connected.*

Proof. This is clear because a connected component is a union of irreducible components. Details omitted. □

28.7. Geometrically integral schemes

If X is an irreducible scheme over a field, then it can happen that X becomes reducible after extending the ground field. This does not happen for geometrically irreducible schemes.

Definition 28.7.1. Let X be a scheme over the field k.

(1) Let $x \in X$. We say X is *geometrically pointwise integral at x* if for every field extension $k \subset k'$ and every $x' \in X_{k'}$ lying over x the local ring $\mathcal{O}_{X_{k'},x'}$ is integral.
(2) We say X is *geometrically pointwise integral* if X is geometrically pointwise integral at every point.
(3) We say X is *geometrically integral* over k if the scheme $X_{k'}$ is integral for every field extension k' of k.

The distinction between notions (2) and (3) is necessary. For example if $k = \mathbf{R}$ and $X = Spec(\mathbf{C}[x])$, then X is geometrically pointwise integral over $\mathbf{R}$ but of course not geometrically integral.

Lemma 28.7.2. *Let k be a field. Let X be a scheme over k. Then X is geometrically integral over k if and only if X is both geometrically reduced and geometrically irreducible over k.*

Proof. See Properties, Lemma 23.3.4. □

28.8. Geometrically normal schemes

In Properties, Definition 23.7.1 we have defined the notion of a normal scheme. This notion is defined even for non-Noetherian schemes. Hence, contrary to our discussion of ``geometrically regular'' schemes we consider all field extensions of the ground field.

Definition 28.8.1. Let X be a scheme over the field k.

(1) Let $x \in X$. We say X is *geometrically normal at x* if for every field extension $k \subset k'$ and every $x' \in X_{k'}$ lying over x the local ring $\mathcal{O}_{X_{k'},x'}$ is normal.
(2) We say X is *geometrically normal* over k if X is geometrically normal at every $x \in X$.

Lemma 28.8.2. *Let k be a field. Let X be a scheme over k. Let $x \in X$. The following are equivalent*

(1) *X is geometrically normal at x,*
(2) *for every finite purely inseparable field extension k' of k and $x' \in X_{k'}$ lying over over x the local ring $\mathcal{O}_{X_{k'},x'}$ is normal, and*
(3) *the ring $\mathcal{O}_{X,x}$ is geometrically normal over k (see Algebra, Definition 7.147.2).*

Proof. It is clear that (1) implies (2). Assume (2). Let $k \subset k'$ be a finite purely inseparable field extension (for example $k = k'$). Consider the ring $\mathcal{O}_{X,x} \otimes_k k'$. By Algebra, Lemma 7.43.2 its spectrum is the same as the spectrum of $\mathcal{O}_{X,x}$. Hence it is a local ring also (Algebra, Lemma 7.17.2). Therefore there is a unique point $x' \in X_{k'}$ lying over x and $\mathcal{O}_{X_{k'},x'} \cong \mathcal{O}_{X,x} \otimes_k k'$. By assumption this is a normal ring. Hence we deduce (3) by Algebra, Lemma 7.147.1.

Assume (3). Let $k \subset k'$ be a field extension. Since $Spec(k') \to Spec(k)$ is surjective, also $X_{k'} \to X$ is surjective (Morphisms, Lemma 24.9.4). Let $x' \in X_{k'}$ be any point lying over x. The local ring $\mathcal{O}_{X_{k'},x'}$ is a localization of the ring $\mathcal{O}_{X,x} \otimes_k k'$. Hence it is normal by assumption and (1) is proved. $\square$

Lemma 28.8.3. *Let k be a field. Let X be a scheme over k. The following are equivalent*

(1) *X is geometrically normal,*
(2) *$X_{k'}$ is a normal scheme for every field extension $k \subset k'$,*
(3) *$X_{k'}$ is a normal scheme for every finitely generated field extension $k \subset k'$,*
(4) *$X_{k'}$ is a normal scheme for every finite purely inseparable field extension $k \subset k'$, and*
(5) *for every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is geometrically normal (see Algebra, Definition 7.147.2).*

Proof. Assume (1). Then for every field extension $k \subset k'$ and every point $x' \in X_{k'}$ the local ring of $X_{k'}$ at x' is normal. By definition this means that $X_{k'}$ is normal. Hence (2).

It is clear that (2) implies (3) implies (4).

Assume (4) and let $U \subset X$ be an affine open subscheme. Then $U_{k'}$ is a normal scheme for any finite purely inseparable extension $k \subset k'$ (including $k = k'$). This means that $k' \otimes_k \mathcal{O}(U)$ is a normal ring for all finite purely inseparable extensions $k \subset k'$. Hence $\mathcal{O}(U)$ is a geometrically normal k-algebra by definition.

Assume (5). For any field extension $k \subset k'$ the base change $X_{k'}$ is gotten by gluing the spectra of the rings $\mathcal{O}_X(U) \otimes_k k'$ where U is affine open in X (see Schemes, Section 21.17). Hence $X_{k'}$ is normal. So (1) holds. $\square$

Lemma 28.8.4. *Let k be a field. Let X be a scheme over k. Let k'/k be a field extension. Let $x \in X$ be a point, and let $x' \in X_{k'}$ be a point lying over x. The following are equivalent*

(1) *X is geometrically normal at x,*
(2) *$X_{k'}$ is geometrically normal at x'.*

In particular, X is geometrically normal over k if and only if $X_{k'}$ is geometrically normal over k'.

Proof. It is clear that (1) implies (2). Assume (2). Let $k \subset k''$ be a finite purely inseparable field extension and let $x'' \in X_{k''}$ be a point lying over x (actually it is unique). We can find a common field extension $k \subset k'''$ (i.e. with both $k' \subset k'''$ and $k'' \subset k'''$) and a point $x''' \in X_{k'''}$ lying over both x' and x''. Consider the map of local rings

$$\mathcal{O}_{X_{k''},x''} \longrightarrow \mathcal{O}_{X_{k'''},x''''}.$$

This is a flat local ring homomorphism and hence faithfully flat. By (2) we see that the local ring on the right is normal. Thus by Algebra, Lemma 7.146.3 we conlude that $\mathcal{O}_{X_{k''},x''}$ is normal. By Lemma 28.8.2 we see that X is geometrically normal at x. □

Lemma 28.8.5. *Let k be a field. Let X be a geometrically normal scheme over k and let Y be a normal scheme over k. Then $X \times_k Y$ is a normal scheme.*

Proof. This reduces to Algebra, Lemma 7.147.4 by Lemma 28.8.3. □

28.9. Change of fields and locally Noetherian schemes

Let X a locally Noetherian scheme over a field k. It is not always that case that $X_{k'}$ is locally Noetherian too. For example if $X = Spec(\overline{\mathbf{Q}})$ and $k = \mathbf{Q}$, then $X_{\overline{\mathbf{Q}}}$ is the spectrum of $\overline{\mathbf{Q}} \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}$ which is not Noetherian. (Hint: It has too many idempotents). But if we only base change using finitely generated field extensions then the Noetherian property is preserved. (Or if X is locally of finite type over k, since this proprety is preserved under base change.)

Lemma 28.9.1. *Let k be a field. Let X be a scheme over k. Let $k \subset k'$ be a finitely generated field extension. Then X is locally Noetherian if and only if $X_{k'}$ is locally Noetherian.*

Proof. Using Properties, Lemma 23.5.2 we reduce to the case where X is affine, say $X = Spec(A)$. In this case we have to prove that A is Noetherian if and only if $A_{k'}$ is Noetherian. Since $A \to A_{k'} = k' \otimes_k A$ is faithfully flat, we see that if $A_{k'}$ is Noetherian, then so is A, by Algebra, Lemma 7.146.1. Conversely, if A is Noetherian then $A_{k'}$ is Noetherian by Algebra, Lemma 7.28.7. □

28.10. Geometrically regular schemes

A geometrically regular scheme over a field k is a locally Noetherian scheme over k which remains regular upon suitable changes of base field. A finite type scheme over k is geometrically regular if and only if it is smooth over k (see Lemma 28.10.6). The notion of geometric regularity is most interesting in situations where smoothness cannot be used such as formal fibres (insert future reference here).

In the following definition we restrict ourselves to locally Noetherian schemes, since the property of being a regular local ring is only defined for Noetherian local rings. By Lemma 28.8.3 above, if we restrict ourselves to finitely generated field extensions then this property is preserved under change of base field. This comment will be used without further reference in this section. In particular the following definition makes sense.

Definition 28.10.1. Let k be a field. Let X be a locally Noetherian scheme over k.

(1) Let $x \in X$. We say X is *geometrically regular at x* over k if for every finitely generated field extension $k \subset k'$ and any $x' \in X_{k'}$ lying over x the local ring $\mathcal{O}_{X_{k'},x'}$ is regular.
(2) We say X is *geometrically regular over k* if X is geometrically regular at all of its points.

A similar definition works to define geometrically Cohen-Macaulay, (R_k), and (S_k) schemes over a field. We will add a section for these separately as needed.

Lemma 28.10.2. *Let k be a field. Let X be a locally Noetherian scheme over k. Let $x \in X$. The following are equivalent*

(1) *X is geometrically regular at x,*
(2) *for every finite purely inseparable field extension k' of k and $x' \in X_{k'}$ lying over over x the local ring $\mathcal{O}_{X_{k'},x'}$ is regular, and*
(3) *the ring $\mathcal{O}_{X,x}$ is geometrically regular over k (see Algebra, Definition 7.148.2).*

Proof. It is clear that (1) implies (2). Assume (2). This in particular implies that $\mathcal{O}_{X,x}$ is a regular local ring. Let $k \subset k'$ be a finite purely inseparable field extension. Consider the ring $\mathcal{O}_{X,x} \otimes_k k'$. By Algebra, Lemma 7.43.2 its spectrum is the same as the spectrum of $\mathcal{O}_{X,x}$. Hence it is a local ring also (Algebra, Lemma 7.17.2). Therefore there is a unique point $x' \in X_{k'}$ lying over x and $\mathcal{O}_{X_{k'},x'} \cong \mathcal{O}_{X,x} \otimes_k k'$. By assumption this is a regular ring. Hence we deduce (3) from the definition of a geometrically regular ring.

Assume (3). Let $k \subset k'$ be a field extension. Since $Spec(k') \to Spec(k)$ is surjective, also $X_{k'} \to X$ is surjective (Morphisms, Lemma 24.9.4). Let $x' \in X_{k'}$ be any point lying over x. The local ring $\mathcal{O}_{X_{k'},x'}$ is a localization of the ring $\mathcal{O}_{X,x} \otimes_k k'$. Hence it is regular by assumption and (1) is proved. □

Lemma 28.10.3. *Let k be a field. Let X be a locally Noetherian scheme over k. The following are equivalent*

(1) *X is geometrically regular,*
(2) *$X_{k'}$ is a regular scheme for every finitely generated field extension $k \subset k'$,*
(3) *$X_{k'}$ is a regular scheme for every finite purely inseparable field extension $k \subset k'$,*
(4) *for every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is geometrically regular (see Algebra, Definition 7.148.2), and*
(5) *there exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is geometrically regular over k.*

Proof. Assume (1). Then for every finitely generated field extension $k \subset k'$ and every point $x' \in X_{k'}$ the local ring of $X_{k'}$ at x' is regular. By Properties, Lemma 23.9.2 this means that $X_{k'}$ is regular. Hence (2).

It is clear that (2) implies (3).

Assume (3) and let $U \subset X$ be an affine open subscheme. Then $U_{k'}$ is a regular scheme for any finite purely inseparable extension $k \subset k'$ (including $k = k'$). This means that $k' \otimes_k \mathcal{O}(U)$ is a regular ring for all finite purely inseparable extensions $k \subset k'$. Hence $\mathcal{O}(U)$ is a geometrically regular k-algebra and we see that (4) holds.

It is clear that (4) implies (5). Let $X = \bigcup U_i$ be an affine open covering as in (5). For any field extension $k \subset k'$ the base change $X_{k'}$ is gotten by gluing the spectra of the rings $\mathcal{O}_X(U_i) \otimes_k k'$ (see Schemes, Section 21.17). Hence $X_{k'}$ is regular. So (1) holds. □

Lemma 28.10.4. *Let k be a field. Let X be a scheme over k. Let k'/k be a finitely generated field extension. Let $x \in X$ be a point, and let $x' \in X_{k'}$ be a point lying over x. The following are equivalent*

(1) *X is geometrically regular at x,*
(2) *$X_{k'}$ is geometrically regular at x'.*

In particular, X is geometrically regular over k if and only if $X_{k'}$ is geometrically regular over k'.

Proof. It is clear that (1) implies (2). Assume (2). Let $k \subset k''$ be a finite purely inseparable field extension and let $x'' \in X_{k''}$ be a point lying over x (actually it is unique). We can find a common, finitely generated, field extension $k \subset k'''$ (i.e. with both $k' \subset k'''$ and $k'' \subset k'''$) and a point $x''' \in X_{k'''}$ lying over both x' and x''. Consider the map of local rings

$$\mathcal{O}_{X_{k''},x''} \longrightarrow \mathcal{O}_{X_{k'''},x''''}.$$

This is a flat local ring homomorphism of Noetherian local rings and hence faithfully flat. By (2) we see that the local ring on the right is regular. Thus by Algebra, Lemma 7.102.8 we conlude that $\mathcal{O}_{X_{k''},x''}$ is regular. By Lemma 28.10.2 we see that X is geometrically regular at x. $\square$

The following lemma is a geometric variant of Algebra, Lemma 7.148.3.

Lemma 28.10.5. *Let k be a field. Let $f : X \to Y$ be a morphism of locally Noetherian schemes over k. Let $x \in X$ be a point and set $y = f(x)$. If X is geometrically regular at x and f is flat at x then Y is geometrically regular at y. In particular, if X is geometrically regular over k and f is flat and surjective, then Y is geometrically regular over k.*

Proof. Let k' be finite purely inseparable extension of k. Let $f' : X_{k'} \to Y_{k'}$ be the base change of f. Let $x' \in X_{k'}$ be the unique point lying over x. If we show that $Y_{k'}$ is regular at $y' = f'(x')$, then Y is geometrically regular over k at y', see Lemma 28.10.3. By Morphisms, Lemma 24.24.6 the morphism $X_{k'} \to Y_{k'}$ is flat at x'. Hence the ring map

$$\mathcal{O}_{Y_{k'},y'} \longrightarrow \mathcal{O}_{X_{k'},x'}$$

is a flat local homommorphism of local Noetherian rings with right hand side regular by assumption. Hence the left hand side is a regular local ring by Algebra, Lemma 7.102.8. $\square$

Lemma 28.10.6. *Let k be a field. Let X be a scheme of finite type over k. Let $x \in X$. Then X is geometrically regular at x if and only if $X \to Spec(k)$ is smooth at x (Morphisms, Definition 24.33.1).*

Proof. The question is local around x, hence we may assume that $X = Spec(A)$ for some finite type k-algebra. Let x correspond to the prime $\mathfrak{p}$.

If A is smooth over k at $\mathfrak{p}$, then we may localize A and assume that A is smooth over k. In this case $k' \otimes_k A$ is smooth over k' for all extension fields k'/k, and each of these Noetherian rings is regular by Algebra, Lemma 7.129.3.

Assume X is geometrically regular at x. Consider the residue field $K := \kappa(x) = \kappa(\mathfrak{p})$ of x. It is a finitely generated extension of k. By Algebra, Lemma 7.42.3 there exists a finite purely inseparable extension $k \subset k'$ such that the compositum $k'K$ is a separable field extension of k'. Let $\mathfrak{p}' \subset A' = k' \otimes_k A$ be a prime ideal lying over $\mathfrak{p}$. It is the unique prime lying over $\mathfrak{p}$, see Algebra, Lemma 7.43.2. Hence the residue field $K' := \kappa(\mathfrak{p}')$ is the compositum $k'K$. By assumption the local ring $(A')_{\mathfrak{p}'}$ is regular. Hence by Algebra, Lemma 7.129.5 we see that $k' \to A'$ is smooth at $\mathfrak{p}'$. This in turn implies that $k \to A$ is smooth at $\mathfrak{p}$ by Algebra, Lemma 7.126.18. The lemma is proved. $\square$

Example 28.10.7. Let $k = \mathbf{F}_p(t)$. It is quite easy to give an example of a regular variety V over k which is not geometrically reduced. For example we can take $Spec(k[x]/(x^p - t))$. In fact, there exists an example of a regular variety V which is geometrically reduced, but not even geometrically normal. Namely, take for $p > 2$ the scheme $V = Spec(k[x, y]/(y^2 - x^p + t))$. This is a variety as the polynomial $y^2 - x^p + t \in k[x, y]$ is irreducible. The morphism $V \to Spec(k)$ is smooth at all points except at the point $v_0 \in V$ corresponding to the maximal ideal $(y, x^p - t)$ (because $2y$ is invertible). In particular we see that V is (geometrically) regular at all points, except possibly v_0. The local ring

$$\mathcal{O}_{V,v_0} = \big(k[x, y]/(y^2 - x^p + t)\big)_{(y,x^p-t)}$$

is a domain of dimension 1. Its maximal ideal is generated by 1 element, namely $x^p - t$. Hence it is a discrete valuation ring and regular. Let $k' = k[t^{1/p}]$. Denote $t' = t^{1/p} \in k'$, $V' = V_{k'}$, $v'_0 \in V'$ the unique point lying over v_0. Over k' we can write $x^p - t = (x - t')^p$, but the polynomial $y^2 - (x - t')^p$ is still irreducible and V' is still a variety. But the element

$$\frac{y}{x - t'} \in f.f.(\mathcal{O}_{V',v'_0})$$

is integral over $\mathcal{O}_{V',v'_0}$ (just compute its square) and not contained in it, so V' is not normal at v'_0. This concludes the example.

28.11. Change of fields and the Cohen-Macaulay property

The following lemma says that it does not make sense to define geometrically Cohen-Macaulay schemes, since these would be the same as Cohen-Macaulay schemes.

Lemma 28.11.1. *Let X be a locally Noetherian scheme over the field k. Let $k \subset k'$ be a finitely generated field extension. Let $x \in X$ be a point, and let $x' \in X_{k'}$ be a point lying over x. Then we have*

$$\mathcal{O}_{X,x} \text{ is Cohen-Macaulay} \Leftrightarrow \mathcal{O}_{X_{k'},x'} \text{ is Cohen-Macaulay}$$

If X is locally of finite type over k, the same holds for any field extension $k \subset k'$.

Proof. The first case of the lemma follows from Algebra, Lemma 7.149.2. The second case of the lemma is equivalent to Algebra, Lemma 7.121.6. ☐

28.12. Change of fields and the Jacobson property

A scheme locally of finite type over a field has plenty of closed points, namely it is Jacobson. Moreover, the residue fields are finite extensions of the ground field.

Lemma 28.12.1. *Let X be a scheme which is locally of finite type over k. Then*

1. *for any closed point $x \in X$ the extension $k \subset \kappa(x)$ is algebraic, and*
2. *X is a Jacobson scheme (Properties, Definition 23.6.1).*

Proof. A scheme is Jacobson if and only if it has an affine open covering by Jacobson schemes, see Properties, Lemma 23.6.3. The property on residue fields at closed points is also local on X. Hence we may assume that X is affine. In this case the result is a consequence of the Hilbert Nullstellenstaz, see Algebra, Theorem 7.30.1. It also follows from a combination of Morphisms, Lemmas 24.15.8, 24.15.9, and 24.15.10. ☐

It turns out that if X is not locally of finite type, then we can achieve the same result after making a suitably large base field extension.

Lemma 28.12.2. *Let X be a scheme over a field k. For any field extension $k \subset K$ whose cardinality is large enough we have*

1. *for any closed point $x \in X_K$ the extension $K \subset \kappa(x)$ is algebraic, and*

(2) *X_K is a Jacobson scheme (Properties, Definition 23.6.1).*

Proof. Choose an affine open covering $X = \bigcup U_i$. By Algebra, Lemma 7.31.12 and Properties, Lemma 23.6.2 there exist cardinals κ_i such that $U_{i,K}$ has the desired properties over K if $\#(K) \geq \kappa_i$. Set $\kappa = \max\{\kappa_i\}$. Then if the cardinality of K is larger than κ we see that each $U_{i,K}$ satisfies the conclusions of the lemma. Hence X_K is Jacobson by Properties, Lemma 23.6.3. The statement on residue fields at closed points of X_K follows from the corresponding statements for residue fields of closed points of the $U_{i,K}$. □

28.13. Algebraic schemes

The following definition is taken from [**DG67**, I Definition 6.4.1].

Definition 28.13.1. Let k be a field. An *algebraic k-scheme* is a scheme X over k such that the structure morphism $X \to Spec(k)$ is of finite type. A *locally algebraic k-scheme* is a scheme X over k such that the structure morphism $X \to Spec(k)$ is locally of finite type.

Note that every (locally) algebraic k-scheme is (locally) Noetherian, see Morphisms, Lemma 24.14.6. The category of algebraic k-schemes has all products and fibre products (unlike the category of varieties over k). Similarly for the category of locally algebraic k-schemes.

Lemma 28.13.2. *Let k be a field. Let X be a locally algebraic k-scheme of dimension 0. Then X is a disjoint union of spectra of local Artinian k-algebras A with $\dim_k(A) < \infty$. If X is an algebraic k-scheme of dimension 0, then in addition X is affine and the morphism $X \to Spec(k)$ is finite.*

Proof. Let X be a locally algebraic k-scheme of dimension 0. Let $U = Spec(A) \subset X$ be an affine open subscheme. Since $\dim(X) = 0$ we see that $\dim(A) = 0$. By Noether normalization, see Algebra, Lemma 7.106.4 we see that there exists a finite injection $k \to A$, i.e., $\dim_k(A) < \infty$. Hence A is Artinian, see Algebra, Lemma 7.49.2. This implies that $A = A_1 \times \ldots \times A_r$ is a product of finitely many Artinian local rings, see Algebra, Lemma 7.49.8. Of course $\dim_k(A_i) < \infty$ for each i as the sum of these dimensions equals $\dim_k(A)$.

The arguments above show that X has an open covering whose members are finite discrete topological spaces. Hence X is a discrete topological space. It follows that X is isomorphic to the disjoint union of its connected components each of which is a singleton. Since a singleton scheme is affine we conclude (by the results of the paragraph above) that each of these singletons is the spectrum of a local Artinian k-algebra A with $\dim_k(A) < \infty$.

Finally, if X is an algebraic k-scheme of dimension 0, then X is quasi-compact hence is a finite disjoint union $X = Spec(A_1) \amalg \ldots \amalg Spec(A_r)$ hence affine (see Schemes, Lemma 21.6.8) and we have seen the finiteness of $X \to Spec(k)$ in the first paragraph of the proof. □

28.14. Closures of products

Some results on the relation between closure and products.

Lemma 28.14.1. *Let k be a field. Let X, Y be schemes over k, and let $A \subset X$, $B \subset Y$ be subsets. Set*

$$AB = \{z \in X \times_k Y \mid pr_X(\gamma) \in A,\ pr_Y(\gamma) \in B\} \subset X \times_k Y$$

Then set theoretically we have

$$\overline{A} \times_k \overline{B} = \overline{AB}$$

Proof. The inclusion $\overline{AB} \subset \overline{A} \times_k \overline{B}$ is immediate. We may replace X and Y by the reduced closed subschemes $\overline{A}$ and $\overline{B}$. Let $W \subset X \times_k Y$ be a nonempty open subset. By Morphisms, Lemma 24.22.4 the subset $U = \text{pr}_X(W)$ is nonempty open in X. Hence $A \cap U$ is nonempty. Pick $a \in A \cap U$. Denote $Y_{\kappa(a)} = \{a\} \times_k Y$ the fibre of $\text{pr}_X : X \times_k Y \to X$ over a. By Morphisms, Lemma 24.22.4 again the morphism $Y_a \to Y$ is open as $Spec(\kappa(a)) \to Spec(k)$ is universally open. Hence the nonempty open subset $W_a = W \times_{X \times_k Y} Y_a$ maps to a nonempty open subset of Y. We conclude there exists a $b \in B$ in the image. Hence $AB \cap W \neq \emptyset$ as desired. □

Lemma 28.14.2. *Let k be a field. Let $f : A \to X$, $g : B \to Y$ be morphisms of schemes over k. Then set theoretically we have*

$$\overline{f(A)} \times_k \overline{g(B)} = \overline{(f \times g)(A \times_k B)}$$

Proof. This follows from Lemma 28.14.1 as the image of $f \times g$ is $f(A)g(B)$ in the notation of that lemma. □

Lemma 28.14.3. *Let k be a field. Let $f : A \to X$, $g : B \to Y$ be quasi-compact morphisms of schemes over k. Let $Z \subset X$ be the scheme theoretic image of f, see Morphisms, Definition 24.4.2. Similarly, let $Z' \subset Y$ be the scheme theoretic image of g. Then $Z \times_k Z'$ is the scheme theoretic image of $f \times g$.*

Proof. Recall that Z is the smallest closed subscheme of X through which f factors. Similarly for Z'. Let $W \subset X \times_k Y$ be the scheme theoretic image of $f \times g$. As $f \times g$ factors through $Z \times_k Z'$ we see that $W \subset Z \times_k Z'$.

To prove the other inclusion let $U \subset X$ and $V \subset Y$ be affine opens. By Morphisms, Lemma 24.4.3 the scheme $Z \cap U$ is the scheme theoretic image of $f|_{f^{-1}(U)} : f^{-1}(U) \to U$, and similarly for $Z' \cap V$ and $W \cap U \times_k V$. Hence we may assume X and Y affine. As f and g are quasi-compact this implies that $A = \bigcup U_i$ is a finite union of affines and $B = \bigcup V_j$ is a finite union of affines. Then we may replace A by $\coprod U_i$ and B by $\coprod V_j$, i.e., we may assume that A and B are affine as well. In this case Z is cut out by $\text{Ker}(\Gamma(X, \mathcal{O}_X) \to \Gamma(A, \mathcal{O}_A))$ and similarly for Z' and W. Hence the result follows from the equality

$$\Gamma(A \times_k B, \mathcal{O}_{A \times_k B}) = \Gamma(A, \mathcal{O}_A) \otimes_k \Gamma(B, \mathcal{O}_B)$$

which holds as A and B are affine. Details omitted. □

28.15. Schemes smooth over fields

Here are two lemmas characterizing smooth schemes over fields.

Lemma 28.15.1. *Let k be a field. Let X be a scheme over k. Assume*

(1) *X is locally of finite type over k,*
(2) *$\Omega_{X/k}$ is locally free, and*
(3) *k has characteristic zero.*

Then the structure morphism $X \to Spec(k)$ is smooth.

Proof. This follows from Algebra, Lemma 7.129.7. □

In positive characteristic there exist nonreduced schemes of finite type whose sheaf of differentials is free, for example $Spec(\mathbf{F}_p[t]/(t^p))$ over $Spec(\mathbf{F}_p)$. If the ground field k is nonperfect of characteristic p, there exist reduced schemes X/k with free $\Omega_{X/k}$ which are nonsmooth, for example $Spec(k[t]/(t^p - a)$ where $a \in k$ is not a pth power.

Lemma 28.15.2. *Let k be a field. Let X be a scheme over k. Assume*

(1) *X is locally of finite type over k,*
(2) *$\Omega_{X/k}$ is locally free,*
(3) *X is reduced, and*
(4) *k is perfect.*

Then the structure morphism $X \to Spec(k)$ is smooth.

Proof. Let $x \in X$ be a point. As X is locally Noetherian (see Morphisms, Lemma 24.14.6) there are finitely many irreducible components $X_1, \ldots, X_n$ passing through x (see Properties, Lemma 23.5.5 and Topology, Lemma 5.6.2). Let $\eta_i \in X_i$ be the generic point. As X is reduced we have $\mathcal{O}_{X,\eta_i} = \kappa(\eta_i)$, see Algebra, Lemma 7.23.3. Moreover, $\kappa(\eta_i)$ is a finitely generated field extension of the perfect field k hence separably generated over k (see Algebra, Section 7.39). It follows that $\Omega_{X/k,\eta_i} = \Omega_{\kappa(\eta_i)/k}$ is free of rank the transcendence degree of $\kappa(\eta_i)$ over k. By Morphisms, Lemma 24.27.1 we conclude that $\dim_{\eta_i}(X_i) = \text{rank}_{\eta_i}(\Omega_{X/k})$. Since $x \in X_1 \cap \ldots \cap X_n$ we see that

$$\text{rank}_x(\Omega_{X/k}) = \text{rank}_{\eta_i}(\Omega_{X/k}) = \dim(X_i).$$

Therefore $\dim_x(X) = \text{rank}_x(\Omega_{X/k})$, see Algebra, Lemma 7.105.5. It follows that $X \to Spec(k)$ is smooth at x for example by Algebra, Lemma 7.129.3. □

Lemma 28.15.3. *Let $X \to Spec(k)$ be a smooth morphism where k is a field. Then X is a regular scheme.*

Proof. (See also Lemma 28.10.6.) By Algebra, Lemma 7.129.3 every local ring $\mathcal{O}_{X,x}$ is regular. And because X is locally of finite type over k it is locally Noetherian. Hence X is regular by Properties, Lemma 23.9.2. □

Lemma 28.15.4. *Let $X \to Spec(k)$ be a smooth morphism where k is a field. Then X is geometrically regular, geometrically normal, and geometrically reduced over k.*

Proof. (See also Lemma 28.10.6.) Let k' be a finite purely inseparable extension of k. It suffices to prove that $X_{k'}$ is regular, normal, reduced, see Lemmas 28.10.3, 28.8.3, and 28.4.5. By Morphisms, Lemma 24.33.5 the morphism $X_{k'} \to Spec(k')$ is smooth too. Hence it suffices to show that a scheme X smooth over a field is regular, normal, and reduced. We see that X is regular by Lemma 28.15.3. Hence Properties, Lemma 23.9.4 guarantees that X is normal. □

Lemma 28.15.5. *Let k be a field. Let $d \geq 0$. Let $W \subset \mathbf{A}^d_k$ be nonempty open. Then there exists a closed point $w \in W$ such that $k \subset \kappa(w)$ is finite separable.*

Proof. After possible shrinking W we may assume that $W = \mathbf{A}^d_k \setminus V(f)$ for some $f \in k[x_1, \ldots, x_n]$. If the lemma is wrong then $f(a_1, \ldots, a_n) = 0$ for all $(a_1, \ldots, a_n) \in (k^{sep})^n$. This is absurd as k^{sep} is an infinite field. □

Lemma 28.15.6. *Let k be a field. If X is smooth over $Spec(k)$ then the set*

$$\{x \in X \textit{ closed such that } k \subset \kappa(x) \textit{ is finite separable}\}$$

is dense in X.

Proof. It suffices to show that given a nonempty smooth X over k there exists at least one closed point whose residue field is finite separable over k. To see this, choose a diagram

$$X \longleftarrow U \xrightarrow{\pi} \mathbf{A}^d_k$$

with π étale, see Morphisms, Lemma 24.35.20. The morphism $\pi : U \to \mathbf{A}^d_k$ is open, see Morphisms, Lemma 24.35.13. By Lemma 28.15.5 we may choose a closed point $w \in \pi(V)$ whose residue field is finite separable over k. Pick any $x \in V$ with $\pi(x) = w$. By Morphisms, Lemma 24.35.7 the field extension $\kappa(w) \subset \kappa(x)$ is finite separable. Hence $k \subset \kappa(x)$ is finite separable. The point x is a closed point of X by Morphisms, Lemma 24.19.2. □

Lemma 28.15.7. *Let X be a scheme over a field k. If X is locally of finite type and geometrically reduced over k then X contains a dense open which is smooth over k.*

Proof. The problem is local on X, hence we may assume X is quasi-compact. Let $X = X_1 \cup \ldots \cup X_n$ be the irreducible components of X. Then $Z = \bigcup_{i \neq j} X_i \cap X_j$ is nowhere dense in X. Hence we may replace X by $X \setminus Z$. As $X \setminus Z$ is a disjoint union of irreducible schemes, this reduces us to the case where X is irreducible. As X is irreducible and reduced, it is integral, see Properties, Lemma 23.3.4. Let $\eta \in X$ be its generic point. Then the function field $K = k(X) = \kappa(\eta)$ is geometrically reduced over k, hence separable over k, see Algebra, Lemma 7.41.1. Let $U = Spec(A) \subset X$ be any nonempty affine open so that $K = f.f.(A) = A_{(0)}$. Apply Algebra, Lemma 7.129.5 to conclude that A is smooth at (0) over k. By definition this means that some principal localization of A is smooth over k and we win. □

Lemma 28.15.8. *Let k be a field. Let $f : X \to Y$ be a morphism of schemes locally of finite type over k. Let $x \in X$ be a point and set $y = f(x)$. If $X \to Spec(k)$ is smooth at x and f is flat at x then $Y \to Spec(k)$ is smooth at y. In particular, if X is smooth over k and f is flat and surjective, then Y is smooth over k.*

Proof. It suffices to show that Y is geometrically regular at y, see Lemma 28.10.6. This follows from Lemma 28.10.5 (and Lemma 28.10.6 applied to (X, x)). □

28.16. Types of varieties

Short section discussion some elementary global properties of varieties.

Definition 28.16.1. Let k be a field. Let X be a variety over k.

(1) We say X is an *affine variety* if X is an affine scheme. This is equivalent to requiring X it be isomorphic to a closed subscheme of $\mathbf{A}^n_k$ for some n.
(2) We say X is a *projective variety* if the structure morphism $X \to Spec(k)$ is projective. By Morphisms, Lemma 24.41.4 this is true if and only if X is isomorphic to a closed subscheme of $\mathbf{P}^n_k$ for some n.
(3) We say X is a *quasi-projective variety* if the structure morphism $X \to Spec(k)$ is quasi-projective. By Morphisms, Lemma 24.39.4 this is true if and only if X is isomorphic to a locally closed subscheme of $\mathbf{P}^n_k$ for some n.
(4) A *proper variety* is a variety such that the morphism $X \to Spec(k)$ is proper.

Note that a projective variety is a proper variety, see Morphisms, Lemma 24.41.5. Also, an affine variety is quasi-projective as $\mathbf{A}^n_k$ is isomorphic to an open subscheme of $\mathbf{P}^n_k$, see Constructions, Lemma 22.13.3.

Lemma 28.16.2. *Let X be a proper variety over k. Then $\Gamma(X, \mathcal{O}_X)$ is a field which is a finite extension of the field k.*

Proof. By Coherent, Lemma 25.18.2 we see that $\Gamma(X, \mathcal{O}_X)$ is a finite dimensional k-vector space. It is also a k-algebra without zero-divisors. Hence it is a field, see Algebra, Lemma 7.32.17. □

28.17. Groups of invertible functions

It is often (but not always) the case that $\mathcal{O}^*(X)/k^*$ is a finitely generated abelian group if X is a variety over k. We show this by a series of lemmas. Everything rests on the following special case.

Lemma 28.17.1. *Let k be an algebraically closed field. Let $\overline{X}$ be a proper variety over k. Let $X \subset \overline{X}$ be an open subscheme. Assume X is normal. Then $\mathcal{O}^*(X)/k^*$ is a finitely generated abelian group.*

Proof. We will use without further mention that for any affine open U of $\overline{X}$ the ring $\mathcal{O}(U)$ is a finitely generated k-algebra, which is Noetherian, a domain and normal, see Algebra, Lemma 7.28.1, Properties, Definition 23.3.1, Properties, Lemmas 23.5.2 and 23.7.2, Morphisms, Lemma 24.14.2.

Let $\xi_1, \ldots, \xi_r$ be the generic points of the complement of X in $\overline{X}$. There are finitely many since $\overline{X}$ has a Noetherian underlying topological space (see Morphisms, Lemma 24.14.6, Properties, Lemma 23.5.5, and Topology, Lemma 5.6.2). For each i the local ring $\mathcal{O}_i = \mathcal{O}_{X,\xi_i}$ is a normal Noetherian local domain (as a localization of a Noetherian normal domain). Let $J \subset \{1, \ldots, r\}$ be the set of indices i such that $\dim(\mathcal{O}_i) = 1$. For $j \in J$ the local ring $\mathcal{O}_j$ is a discrete valuation ring, see Algebra, Lemma 7.110.6. Hence we obtain a valuation

$$v_j : k(\overline{X})^* \longrightarrow \mathbf{Z}$$

with the property that $v_j(f) \geq 0 \Leftrightarrow f \in \mathcal{O}_j$.

Think of $\mathcal{O}(X)$ as a sub k-algebra of $k(X) = k(\overline{X})$. We claim that the kernel of the map

$$\mathcal{O}(X)^* \longrightarrow \prod\nolimits_{j \in J} \mathbf{Z}, \quad f \longmapsto \prod v_j(f)$$

is k^*. It is clear that this claim proves the lemma. Namely, suppose that $f \in \mathcal{O}(X)$ is an element of the kernel. Let $U = Spec(B) \subset \overline{X}$ be any affine open. Then B is a Noetherian normal domain. For every height one prime $\mathfrak{q} \subset B$ with corresponding point $\xi \in X$ we see that either $\xi = \xi_j$ for some $j \in J$ or that $\xi \in X$. The reason is that $\text{codim}(\overline{\{\xi\}}, \overline{X}) = 1$ by Properties, Lemma 23.11.4 and hence if $\xi \in \overline{X} \setminus X$ it must be a generic point of $\overline{X} \setminus X$, hence equal to some ξ_j, $j \in J$. We conclude that $f \in \mathcal{O}_{X,\xi} = B_{\mathfrak{q}}$ in either case as f is in the kernel of the map. Thus $f \in \bigcap_{\text{ht}(\mathfrak{q})=1} B_{\mathfrak{q}} = B$, see Algebra, Lemma 7.140.6. In other words, we see that $f \in \Gamma(\overline{X}, \mathcal{O}_{\overline{X}})$. But since k is algebraically closed we conclude that $f \in k$ by Lemma 28.16.2. $\square$

Next, we generalize the case above by some elementary arguments, still keeping the field algebraically closed.

Lemma 28.17.2. *Let k be an algebraically closed field. Let X be an integral scheme locally of finite type over k. Then $\mathcal{O}^*(X)/k^*$ is a finitely generated abelian group.*

Proof. As X is integral the restriction mapping $\mathcal{O}(X) \to \mathcal{O}(U)$ is injective for any nonempty open subscheme $U \subset X$. Hence we may assume that X is affine. Choose a closed immersion $X \to \mathbf{A}^n_k$ and denote $\overline{X}$ the closure of X in $\mathbf{P}^n_k$ via the usual immersion $\mathbf{A}^n_k \to \mathbf{P}^n_k$. Thus we may assume that X is an affine open of a projective variety $\overline{X}$.

Let $\nu : \overline{X}^\nu \to \overline{X}$ be the normalization morphism, see Morphisms, Definition 24.46.12. We know that ν is finite, dominant, and that $\overline{X}^\nu$ is a normal irreducible scheme, see Morphisms, Lemmas 24.46.15, 24.46.16, and 24.17.2. It follows that $\overline{X}^\nu$ is a proper variety, because $\overline{X} \to Spec(k)$ is proper as a composition of a finite and a proper morphism (see results in Morphisms, Sections 24.40 and 24.42). It also follows that ν is a surjective morphism, because the image of ν is closed and contains the generic point of $\overline{X}$. Hence setting $X^\nu = \nu^{-1}(X)$ we see that it suffices to prove the result for X^ν. In other words, we may assume that X is a nonempty open of a normal proper variety $\overline{X}$. This case is handled by Lemma 28.17.1. □

The preceding lemma implies the following slight generalization.

Lemma 28.17.3. *Let k be an algebraically closed field. Let X be a connected reduced scheme which is locally of finite type over k with finitely many irreducible components. Then $\mathcal{O}^*(X)/k^*$ is a finitely generated abelian group.*

Proof. Let $X = \bigcup X_i$ be the irreducible components. By Lemma 28.17.2 we see that $\mathcal{O}(X_i)^*/k^*$ is a finitely generated abelian group. Let $f \in \mathcal{O}(X)^*$ be in the kernel of the map

$$\mathcal{O}(X)^* \longrightarrow \prod \mathcal{O}(X_i)^*/k^*.$$

Then for each i there exists an element $\lambda_i \in k$ such that $f|_{X_i} = \lambda_i$. By restricting to $X_i \cap X_j$ we conclude that $\lambda_i = \lambda_j$ if $X_i \cap X_j \neq \emptyset$. Since X is connected we conclude that all λ_i agree and hence that $f \in k^*$. This proves that

$$\mathcal{O}(X)^*/k^* \subset \prod \mathcal{O}(X_i)^*/k^*$$

and the lemma follows as on the right we have a product of finitely many finitely generated abelian groups. □

Lemma 28.17.4. *Let k be a field. Let X be a scheme over k which is connected and reduced. Then the integral closure of k in $\Gamma(X, \mathcal{O}_X)$ is a field.*

Proof. Let $k' \subset \Gamma(X, \mathcal{O}_X)$ be the integral closure of k. Then $X \to Spec(k)$ factors through $Spec(k')$, see Schemes, Lemma 21.6.4. As X is reduced we see that k' has no nonzero nilpotent elements. As $k \to k'$ is integral we see that every prime ideal of k' is both a maximal ideal and a minimal prime, and $Spec(k')$ is totally disconnected, see Algebra, Lemmas 7.32.18 and 7.23.5. As X is connected the morphism $X \to Spec(k')$ is constant, say with image the point corresponding to $\mathfrak{p} \subset k'$. Then any $f \in k'$, $f \notin \mathfrak{p}$ maps to an invertible element of $\mathcal{O}_X$. By definition of k' this then forces f to be a unit of k'. Hence we see that k' is local with maximal ideal $\mathfrak{p}$, see Algebra, Lemma 7.17.2. Since we've already seen that k' is reduced this implies that k' is a field, see Algebra, Lemma 7.23.3. □

Proposition 28.17.5. *Let k be a field. Let X be a scheme over k. Assume that X is locally of finite type over k, connected, reduced, and has finitely many irreducible components. Then $\mathcal{O}(X)^*/k^*$ is a finitely generated abelian group if in addition to the conditions above at least one of the following conditions is satisfied:*

1. *the integral closure of k in $\Gamma(X, \mathcal{O}_X)$ is k,*
2. *X has a k-rational point, or*
3. *X is geometrically integral.*

Proof. Let $\overline{k}$ be an algebraic closure of k. Let Y be a connected component of $(X_{\overline{k}})_{red}$. Note that the canonical morphism $p : Y \to X$ is open (by Morphisms, Lemma 24.22.4) and closed (by Morphisms, Lemma 24.42.7). Hence $p(Y) = X$ as X was assumed connected. In particular, as X is reduced this implies $\mathcal{O}(X) \subset \mathcal{O}(Y)$. By Lemma 28.6.13 we see that Y has finitely many irreducible components. Thus Lemma 28.17.3 applies to Y. This implies that if $\mathcal{O}(X)^*/k^*$ is not a finitely generated abelian group, then there exist elements $f \in \mathcal{O}(X)$, $f \not\in k$ which map to an element of $\overline{k}$ via the map $\mathcal{O}(X) \to \mathcal{O}(Y)$. In this case f is algebraic over k, hence integral over k. Thus, if condition (1) holds, then this cannot happen. To finish the proof we show that conditions (2) and (3) imply (1).

Let $k \subset k' \subset \Gamma(X, \mathcal{O}_X)$ be the integral closure of k in $\Gamma(X, \mathcal{O}_X)$. By Lemma 28.17.4 we see that k' is a field. If $e : Spec(k) \to X$ is a k-rational point, then $e^\sharp : \Gamma(X, \mathcal{O}_X) \to k$ is a section to the inclusion map $k \to \Gamma(X, \mathcal{O}_X)$. In particular the restriction of $e^\sharp$ to k' is a field map $k' \to k$ over k, which clearly shows that (2) implies (1).

If the integral closure k' of k in $\Gamma(X, \mathcal{O}_X)$ is not trivial, then we see that X is either not geometrically connected (if $k \subset k'$ is not purely inseparable) or that X is not geometrically reduced (if $k \subset k'$ is nontrivial purely inseparable). Details omitted. Hence (3) implies (1). □

Lemma 28.17.6. *Let k be a field. Let X be a variety over k. The group $\mathcal{O}(X)^*/k^*$ is a finitely generated abelian group provided at least one of the following conditions holds:*

1. *k is integrally closed in $\Gamma(X, \mathcal{O}_X)$,*
2. *k is algebraically closed in $k(X)$,*
3. *X is geometrically integral over k, or*
4. *k is the ``intersection'' of the field extensions $k \subset \kappa(x)$ where x runs over the closed points of x.*

Proof. We see that (1) is enough by Proposition 28.17.5. We omit the verification that each of (2), (3), (4) implies (1). □

28.18. Uniqueness of base field

The phrase ``let X be a scheme over k'' means that X is a scheme which comes equipped with a morphism $X \to Spec(k)$. Now we can ask whether the field k is uniquely determined by the scheme X. Of course this is not the case, since for example $\mathbf{A}^1_{\mathbf{C}}$ which we ordinarily consider as a scheme over the field $\mathbf{C}$ of complex numbers, could also be considered as a scheme over $\mathbf{Q}$. But what if we ask that the morphism $X \to Spec(k)$ does not factor as $X \to Spec(k') \to Spec(k)$ for any nontrivial field extension $k \subset k'$? In other words we ask that k is somehow maximal such that X lives over k.

An example to show that this still does not garantee uniqueness of k is the scheme

$$X = Spec\left(\mathbf{Q}(x)[y]\left[\frac{1}{P(y)}, P \in \mathbf{Q}[y], P \neq 0\right]\right)$$

At first sight this seems to be a scheme over $\mathbf{Q}(x)$, but on a second look it is clear that it is also a scheme over $\mathbf{Q}(y)$. Moreover, the fields $\mathbf{Q}(x)$ and $\mathbf{Q}(y)$ are subfields of $R = \Gamma(X, \mathcal{O}_X)$ which are maximal among the subfields of R (details omitted). In particular, both $\mathbf{Q}(x)$ and $\mathbf{Q}(y)$ are maximal in the sense above. Note that both morphisms $X \to Spec(\mathbf{Q}(x))$ and $X \to Spec(\mathbf{Q}(y))$ are ``essentially of finite type'' (i.e., the corresponding ring map is essentially of finite type). Hence X is a Noetherian scheme of finite dimension, i.e., it is not completely pathological.

Another issue that can prevent uniqueness is that the scheme X may be nonreduced. In that case there can be many different morphisms from X to the spectrum of a given field.

As an explicit example consider the dual numbers $D = \mathbf{C}[y]/(y^2) = \mathbf{C} \oplus \epsilon\mathbf{C}$. Given any derivation $\theta : \mathbf{C} \to \mathbf{C}$ over $\mathbf{Q}$ we get a ring map

$$\mathbf{C} \longrightarrow D, \quad c \longmapsto c + \epsilon\theta(c).$$

The subfield of $\mathbf{C}$ on which all of these maps are the same is the algebraic closure of $\mathbf{Q}$. This means that taking the intersection of all the fields that X can live over may end up being a very small field if X is nonreduced.

One observation in this regard is the following: given a field k and two subfields k_1, k_2 of k such that k is finite over k_1 and over k_2, then in general it is *not* the case that k is finite over $k_1 \cap k_2$. An example is the field $k = \mathbf{Q}(t)$ and its subfields $k_1 = \mathbf{Q}(t^2)$ and $\mathbf{Q}((t+1)^2)$. Namely we have $k_1 \cap k_2 = \mathbf{Q}$ in this case. So in the following we have to be careful when taking intersections of fields.

Having said all of this we now show that if X is locally of finite type over a field, then some uniqueness holds. Here is the precise result.

Proposition 28.18.1. *Let X be a scheme. Let $a : X \to Spec(k_1)$ and $b : X \to Spec(k_2)$ be morphisms from X to spectra of fields. Assume a, b are locally of finite type, and X is reduced, and connected. Then we have $k_1' = k_2'$, where $k_i' \subset \Gamma(X, \mathcal{O}_X)$ is the integral closure of k_i in $\Gamma(X, \mathcal{O}_X)$.*

Proof. First, assume the lemma holds in case X is quasi-compact (we will do the quasi-compact case below). As X is locally of finite type over a field, it is locally Noetherian, see Morphisms, Lemma 24.14.6. In particular this means that it is locally connected, connected components of open subsets are open, and intersections of quasi-compact opens are quasi-compact, see Properties, Lemma 23.5.5, Topology, Lemma 5.4.8, Topology, Section 5.6, and Topology, Lemma 5.11.1. Pick an open covering $X = \bigcup_{i \in I} U_i$ such that each U_i is quasi-compact and connected. For each i let $K_i \subset \mathcal{O}_X(U_i)$ be the integral closure of k_1 and of k_2. For each pair $i, j \in I$ we decompose

$$U_i \cap U_j = \coprod U_{i,j,l}$$

into its finitely many connected components. Write $K_{i,j,l} \subset \mathcal{O}(U_{i,j,l})$ for the integral closure of k_1 and of k_2. By Lemma 28.17.4 the rings K_i and $K_{i,j,l}$ are fields. Now we claim that k_1' and k_2' both equal the kernel of the map

$$\prod K_i \longrightarrow \prod K_{i,j,l}, \quad (x_i)_i \longmapsto x_i|_{U_{i,j,l}} - x_j|_{U_{i,j,l}}$$

which proves what we want. Namely, it is clear that k_1' is contained in this kernel. On the other hand, suppose that $(x_i)_i$ is in the kernel. By the sheaf condition $(x_i)_i$ corresponds to $f \in \mathcal{O}(X)$. Pick some $i_0 \in I$ and let $P(T) \in k_1[T]$ be a monic polynomial with $P(x_{i_0}) = 0$. Then we claim that $P(f) = 0$ which proves that $f \in k_1$. To prove this we have to show that $P(x_i) = 0$ for all i. Pick $i \in I$. As X is connected there exists a sequence $i_0, i_1, \ldots, i_n = i \in I$ such that $U_{i_t} \cap U_{i_{t+1}} \neq \emptyset$. Now this means that for each t there exists an l_t such that x_{i_t} and $x_{i_{t+1}}$ map to the same element of the field $K_{i,j,l}$. Hence if $P(x_{i_t}) = 0$, then $P(x_{i_{t+1}}) = 0$. By induction, starting with $P(x_{i_0}) = 0$ we deduce that $P(x_i) = 0$ as desired.

To finish the proof of the lemma we prove the lemma under the additional hypothesis that X is quasi-compact. By Lemma 28.17.4 after replacing k_i by k_i' we may assume that k_i is integrally closed in $\Gamma(X, \mathcal{O}_X)$. This implies that $\mathcal{O}(X)^*/k_i^*$ is a finitely generated abelian group, see Proposition 28.17.5. Let $k_{12} = k_1 \cap k_2$ as a subring of $\mathcal{O}(X)$. Note that k_{12} is a field. Since

$$k_1^*/k_{12}^* \longrightarrow \mathcal{O}(X)^*/k_2^*$$

we see that k_1^*/k_{12}^* is a finitely generated abelian group as well. Hence there exist $\alpha_1, \ldots, \alpha_n \in k_1^*$ such that every element $\lambda \in k_1$ has the form

$$\lambda = c\alpha_1^{e_1} \ldots \alpha_n^{e_n}$$

for some $e_i \in \mathbf{Z}$ and $c \in k_{12}$. In particular, the ring map

$$k_{12}[x_1, \ldots, x_n, \frac{1}{x_1 \ldots x_n}] \longrightarrow k_1, \quad x_i \longmapsto \alpha_i$$

is surjective. By the Hilbert Nullstellensatz, Algebra, Theorem 7.30.1 we conclude that k_1 is a finite extension of k_{12}. In the same way we conclude that k_2 is a finite extension of k_{12}. In particular both k_1 and k_2 are contained in the integral closure k'_{12} of k_{12} in $\Gamma(X, \mathcal{O}_X)$. But since k'_{12} is a field by Lemma 28.17.4 and since we chose k_i to be integrally closed in $\Gamma(X, \mathcal{O}_X)$ we conclude that $k_1 = k_{12} = k_2$ as desired. □

28.19. Other chapters

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CHAPTER 29

Chow Homology and Chern Classes

29.1. Introduction

In this chapter we discuss Chow homology groups and the construction of chern classes of vector bundles as elements of operational Chow cohomology groups (everything with $\mathbf{Z}$-coefficients). We follow the first few chapters of [**Ful98**], except that we have been less precise about the supports of the cycles involved. More classical discussions of Chow groups can be found in [**Sam56**], [**Che58a**], and [**Che58b**]. Of course there are many others.

To make the material a little bit more challenging we decided to treat a somewhat more general case than is usually done. Namely we assume our schemes X are locally of finite type over a fixed locally Noetherian base scheme which is universally catenary and has a given dimension function. This seems to be all that is needed to be able to define the Chow homology groups $A_*(X)$ and the action of capping with chern classes on them. This is an indication that we should be able to define these also for algebraic stacks locally of finite type over such a base.

In another chapter we will define the intersection products on $A_*(X)$ using Serre's Tor-formula in case X is nonsingular (see [**Ser00**], or [**Ser65**]) and we have a good moving lemma. See (insert future reference here).

29.2. Determinants of finite length modules

The material in this section is related to the material in the paper [**KM76**] and to the material in the thesis [**Ros09**]. If you have a good reference then please email stacks.project@gmail.com.

Given any field κ and any finite dimensional κ-vector space V we set $\det_\kappa(V)$ equal to $\det_\kappa(V) = \wedge^n(V)$ where $n = \dim_\kappa(V)$. We want to generalize this slightly.

Definition 29.2.1. Let R be a local ring with maximal ideal $\mathfrak m$ and residue field κ. Let M be a finite length R-module. Say $l = \text{length}_R(M)$.

(1) Given elements $x_1, \ldots, x_r \in M$ we denote $\langle x_1, \ldots, x_r\rangle = Rx_1 + \ldots + Rx_r$ the R-submodule of R generated by $x_1, \ldots, x_r$.
(2) We will say an l-tuples of elements $(e_1, \ldots, e_l)$ of M is *admissible* if $\mathfrak m e_i \in \langle e_1, \ldots, e_{i-1}\rangle$ for $i = 1, \ldots, l$.
(3) A *symbol* $[e_1, \ldots, e_l]$ will mean $(e_1, \ldots, e_l)$ is an admissible l-tuple.
(4) An *admissible relation* between symbols is one of the following:
    (a) if $(e_1, \ldots, e_l)$ is an admissible sequence and for some $1 \leq a \leq l$ we have $e_a \in \langle e_1, \ldots, e_{a-1}\rangle$, then $[e_1, \ldots, e_l] = 0$,
    (b) if $(e_1, \ldots, e_l)$ is an admissible sequence and for some $1 \leq a \leq l$ we have $e_a = \lambda e'_a + x$ with $\lambda \in R^*$, and $x \in \langle e_1, \ldots, e_{a-1}\rangle$, then

$$[e_1, \ldots, e_l] = \overline{\lambda}[e_1, \ldots, e_{a-1}, e'_a, e_{a+1}, \ldots, e_l]$$

where $\overline{\lambda} \in \kappa^*$ is the image of λ in the residue field, and

(c) if $(e_1, \ldots, e_l)$ is an admissible sequence and $\mathfrak{m}e_a \subset \langle e_1, \ldots, e_{a-2}\rangle$ then

$$[e_1, \ldots, e_l] = -[e_1, \ldots, e_{a-2}, e_a, e_{a-1}, e_{a+1}, \ldots, e_l].$$

(5) We define the *determinant of the finite length R-module* to be

$$\det_\kappa(M) = \left\{ \frac{\kappa\text{-vector space generated by symbols}}{\kappa\text{-linear combinations of admissible relations}} \right\}$$

We stress that always $l = \text{length}_R(M)$. We also stress that it does not follow that the symbol $[e_1, \ldots, e_l]$ is additive in the entries (this will typically not be the case). Before we can show that the determinant $\det_\kappa(M)$ actually has dimension 1 we have to show that it has dimension at most 1.

Lemma 29.2.2. *With notations as above we have* $\dim_\kappa(\det_\kappa(M)) \leq 1$.

Proof. Fix an admissible sequence $(f_1, \ldots, f_l)$ of M such that

$$\text{length}_R(\langle f_1, \ldots, f_i\rangle) = i$$

for $i = 1, \ldots, l$. Such an admissible sequence exists exactly because M has length l. We will show that any element of $\det_\kappa(M)$ is a κ-multiple of the symbol $[f_1, \ldots, f_l]$. This will prove the lemma.

Let $(e_1, \ldots, e_l)$ be an admissible sequence of M. It suffices to show that $[e_1, \ldots, e_l$ is a multiple of $[f_1, \ldots, f_l]$. First assume that $\langle e_1, \ldots, e_l\rangle \neq M$. Then there exists an $i \in [1, \ldots, l]$ such that $e_i \in \langle e_1, \ldots, e_{i-1}\rangle$. It immediately follows from the first admissible relation that $[e_1, \ldots, e_n] = 0$ in $\det_\kappa(M)$. Hence we may assume that $\langle e_1, \ldots, e_l\rangle = M$. In particular there exists a smallest index $i \in \{1, \ldots, l\}$ such that $f_1 \in \langle e_1, \ldots, e_i\rangle$. This means that $e_i = \lambda f_1 + x$ with $x \in \langle e_1, \ldots, e_{i-1}\rangle$ and $\lambda \in R^*$. By the second admissible relation this means that $[e_1, \ldots, e_l] = \overline{\lambda}[e_1, \ldots, e_{i-1}, f_1, e_{i+1}, \ldots, e_l]$. Note that $\mathfrak{m}f_1 = 0$. Hence by applying the third admissible relation $i-1$ times we see that

$$[e_1, \ldots, e_l] = (-1)^{i-1}\overline{\lambda}[f_1, e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_l].$$

Note that it is also the case that $\langle f_1, e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_l\rangle = M$. By induction suppose we have proven that our original symbol is equal to a scalar times

$$[f_1, \ldots, f_j, e_{j+1}, \ldots, e_l]$$

for some admissible sequence $(f_1, \ldots, f_j, e_{j+1}, \ldots, e_l)$ whose elements generate M, i.e., with $\langle f_1, \ldots, f_j, e_{j+1}, \ldots, e_l\rangle = M$. Then we find the smallest i such that $f_{j+1} \in \langle f_1, \ldots, f_j, e_{j+1}, \ldots, e_i\rangle$ and we go through the same process as above to see that

$$[f_1, \ldots, f_j, e_{j+1}, \ldots, e_l] = (\text{scalar})[f_1, \ldots, f_j, f_{j+1}, e_{j+1}, \ldots, \hat{e_i}, \ldots, e_l]$$

Continuing in this vein we obtain the desired result. □

Before we show that $\det_\kappa(M)$ always has dimension 1, let us show that it agree with the usual top exterior power in the case the module is a vector space over κ.

Lemma 29.2.3. *Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field κ. Let M be a finite length R-module which is annihilated by $\mathfrak{m}$. Let $l = n = \dim_\kappa(M)$. Then the map*

$$\det_\kappa(M) \longrightarrow \wedge^l_\kappa(M), \quad [e_1, \ldots, e_l] \longmapsto e_1 \wedge \ldots \wedge e_l$$

is an isomorphism.

Proof. It is clear that the rule described in the lemma gives a κ-linear map since all of the admissible relations are satisfied by the usual symbols $e_1 \wedge \ldots \wedge e_l$. It is also clearly a surjective map. Since by Lemma 29.2.2 the left hand side has dimension at most one we see that the map is an isomorphism. $\square$

Lemma 29.2.4. *Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field κ. Let M be a finite length R-module. The determinant $\det_\kappa(M)$ defined above is a κ-vector space of dimension 1. It is generated by the symbol $[f_1, \ldots, f_l]$ for any admissible sequence such that $\langle f_1, \ldots f_l\rangle = M$.*

Proof. We know $\det_\kappa(M)$ has dimension at most 1, and in fact that it is generated by $[f_1, \ldots, f_l]$, by Lemma 29.2.2 and its proof. We will show by induction on $l = \text{length}(M)$ that it is nonzero. For $l = 1$ it follows from Lemma 29.2.3. Choose a nonzero element $f \in M$ with $\mathfrak{m}f = 0$. Set $\overline{M} = M/\langle f\rangle$, and denote the quotient map $x \mapsto \overline{x}$. We will define a surjective map

$$\psi : \det_k(M) \to \det_\kappa(\overline{M})$$

which will prove the lemma since by induction the determinant of $\overline{M}$ is nonzero.

We define ψ on symbols as follows. Let $(e_1, \ldots, e_l)$ be an admissible sequence. If $f \not\in \langle e_1, \ldots, e_l\rangle$ then we simply set $\psi([e_1, \ldots, e_l]) = 0$. If $f \in \langle e_1, \ldots, e_l\rangle$ then we choose an i minimal such that $f \in \langle e_1, \ldots, e_i\rangle$ and write $e_i = \lambda f + x$ for some $\lambda \in R$ and $x \in \langle e_1, \ldots, e_{i-1}\rangle$. In this case we set

$$\psi([e_1, \ldots, e_l]) = \overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l].$$

Note that it is indeed the case that $(\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l)$ is an admissible sequence in $\overline{M}$, so this makes sense. Let us show that extending this rule κ-linearly to linear combinations of symbolds does indeed lead to a map on determinants. To do this we have to show that the admissible relations are mapped to zero.

Type (a) relations. Suppose we have $(e_1, \ldots, e_l)$ an admissible sequence and for some $1 \leq a \leq l$ we have $e_a \in \langle e_1, \ldots, e_{a-1}\rangle$. Suppose that $f \in \langle e_1, \ldots, e_i\rangle$ with i minimal. Then it is immediate that $i \neq a$. Since it is also the case that $\overline{e}_a \in \langle \overline{e}_1, \ldots, \hat{\overline{e}_i}, \ldots, \overline{e}_{a-1}\rangle$ we see immediately that the same admissible relation for $\det_\kappa(\overline{M})$ forces the symbol $[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l]$ to be zero as desired.

Type (b) relations. Suppose we have $(e_1, \ldots, e_l)$ an admissible sequence and for some $1 \leq a \leq l$ we have $e_a = \lambda e'_a + x$ with $\lambda \in R^*$, and $x \in \langle e_1, \ldots, e_{a-1}\rangle$. Suppose that $f \in \langle e_1, \ldots, e_i\rangle$ with i minimal. Say $e_i = \mu f + y$ with $y \in \langle e_1, \ldots, e_{i-1}\rangle$. If $i < a$ then the desired equality is

$$\overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l] = \overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_{a-1}, \overline{e}'_a, \overline{e}_{a+1}, \ldots, \overline{e}_l]$$

which follows from $\overline{e}_a = \lambda\overline{e}'_a + \overline{x}$ and the corresponding admissible relation for $\det_\kappa(\overline{M})$. If $i > a$ then the desired equality is

$$\overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l] = \overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{a-1}, \overline{e}'_a, \overline{e}_{a+1}, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l]$$

which follows from $\overline{e}_a = \lambda\overline{e}'_a + \overline{x}$ and the corresponding admissible relation for $\det_\kappa(\overline{M})$. The interesting case is when $i = a$. In this case we have $e_a = \lambda e'_a + x = \mu f + y$. Hence also $e'_a = \lambda^{-1}(\mu f + y - x)$. Thus we see that

$$\psi([e_1, \ldots, e_l]) = \overline{\mu}[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l] = \psi(\overline{\lambda}[e_1, \ldots, e_{a-1}, e'_a, e_{a+1}, \ldots, e_l])$$

as desired.

Type (c) relations. Suppose that $(e_1, \ldots, e_l)$ is an admissible sequence and $\mathfrak{m}e_a \subset \langle e_1, \ldots, e_{a-2}\rangle$. Suppose that $f \in \langle e_1, \ldots, e_i\rangle$ with i minimal. Say $e_i = \lambda f + x$ with $x \in \langle e_1, \ldots, e_{i-1}\rangle$. If $i < a - 1$, then the desired equality is

$$\overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l] = \overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_{a-2}, \overline{e}_a, \overline{e}_{a-1}, \overline{e}_{a+1}, \ldots, \overline{e}_l]$$

which follows from the type (c) admissible relation for $\det_\kappa(\overline{M})$. Similarly, if $i > a$, then the desired equality is

$$\overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l] = \overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{a-2}, \overline{e}_a, \overline{e}_{a-1}, \overline{e}_{a+1}, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l]$$

which follows from the type (c) admissible relation for $\det_\kappa(\overline{M})$. If $i = a$, then the desired equality is

$$\overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{a-1}, \overline{e}_{a+1}, \ldots, \overline{e}_l] = \overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{a-2}, \overline{e}_{a-1}, \overline{e}_{a+1}, \ldots, \overline{e}_l]$$

which is tautological. Finally, the interesting case is $i = a - 1$. This case itself splits into two cases as to whether $f \in \langle e_1, \ldots, e_{a-2}, e_a\rangle$ or not. If not, then we see that the desired equality is

$$\overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{a-2}, \overline{e}_a, \ldots, \overline{e}_l] = \overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{a-2}, \overline{e}_a, \overline{e}_{a+1}, \ldots, \overline{e}_l]$$

which is tautological since after switching e_{a-1} and e_a the smallest index such that f is in the becomes equal to $i' = a$ and it is again e_a which is removed. On the other hand, suppose that $f \in \langle e_1, \ldots, e_{a-2}, e_a\rangle$. In this case we see that we can, besides the equality $e_{a-1} = \lambda f + x$ of above, also write $e_a = \mu f + y$ with $y \in \langle e_1, \ldots, e_{a-2}\rangle$. Clearly this means that both $e_a \in \langle e_1, \ldots, e_{a-1}\rangle$ and $e_{a-1} \in \langle e_1, \ldots, e_{a-2}, e_a\rangle$. Thus we can use relations of type (a) and the compatibility of ψ with these shown above to see that both

$$\psi([e_1, \ldots, e_l]) \quad \text{and} \quad \psi([e_1, \ldots, e_{a-2}, e_a, e_{a-1}, e_{a+1}, \ldots, e_l])$$

are zero, as desired.

At this point we have shown that ψ is well defined, and all that remains is to show that it is surjective. To see this let $(\overline{f}_2, \ldots, \overline{f}_l)$ be an admissible sequence in $\overline{M}$. We can choose lifts $f_2, \ldots, f_l \in M$, and then $(f, f_2, \ldots, f_l)$ is an admissible sequence in M. Since $\psi([f, f_2, \ldots, f_l]) = [f_2, \ldots, f_l]$ we win. □

Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field κ. Note that if $\varphi : M \to N$ is an isomorphism of finite length R-modules, then we get an isomorphism

$$\det_\kappa(\varphi) : \det_\kappa(M) \to \det_\kappa(N)$$

simply by the rule

$$\det_\kappa(\varphi)([e_1, \ldots, e_l]) = [\varphi(e_1), \ldots, \varphi(e_l)]$$

for any symbol $[e_1, \ldots, e_l]$ for M. Hence we see that $\det_\kappa$ is a functor

$$\left\{ \begin{matrix} \text{fi} \\ \text{with isomorphisms} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{1-dimensional } \kappa\text{-vector spaces} \\ \text{with isomorphisms} \end{matrix} \right\} \tag{29.2.4.1}$$

This is typical for a ``determinant functor'' (see [**Knu02**]), as is the following additivity property.

Lemma 29.2.5. *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. For every short exact sequence*

$$0 \to K \to L \to M \to 0$$

of finite length R-modules there exists a canonical isomorphism

$$\gamma_{K \to L \to M} : \det_\kappa(K) \otimes_\kappa \det_\kappa(M) \longrightarrow \det_\kappa(L)$$

defined by the rule on nonzero symbols

$$[e_1, \ldots, e_k] \otimes [\overline{f}_1, \ldots, \overline{f}_m] \longrightarrow [e_1, \ldots, e_k, f_1, \ldots, f_m]$$

with the following properties:

(1) *For every isomorphism of short exact sequences, i.e., for every commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d, "u"] & L \arrow[r] \arrow[d, "v"] & M \arrow[r] \arrow[d, "w"] & 0 \\ 0 \arrow[r] & K' \arrow[r] & L' \arrow[r] & M' \arrow[r] & 0 \end{tikzcd}$$

with short exact rows and isomorphisms u, v, w we have

$$\gamma_{K' \to L' \to M'} \circ (\det_\kappa(u) \otimes \det_\kappa(w)) = \det_\kappa(v) \circ \gamma_{K \to L \to M},$$

(2) *for every commutative square of finite length R-modules with exact rows and columns*

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & D \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & G \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & I \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

the following diagram is commutative

$$\begin{tikzcd} \det_\kappa(A) \otimes \det_\kappa(C) \otimes \det_\kappa(G) \otimes \det_\kappa(I) \arrow[r, "\gamma_{A \to B \to C} \otimes \gamma_{G \to H \to I}"'] \arrow[dd, "\epsilon"'] & \det_\kappa(B) \otimes \det_\kappa(H) \arrow[d, "\gamma_{B \to E \to H}"] \\ & \det_\kappa(E) \\ \det_\kappa(A) \otimes \det_\kappa(G) \otimes \det_\kappa(C) \otimes \det_\kappa(I) \arrow[r, "\gamma_{A \to D \to G} \otimes \gamma_{C \to F \to I}"] & \det_\kappa(D) \otimes \det_\kappa(F) \arrow[u, "\gamma_{D \to E \to F}"'] \end{tikzcd}$$

where ϵ is the switch of the factors in the tensor product times $(-1)^{cg}$ with $c = length_R(C)$ and $g = length_R(G)$, and

(3) *the map $\gamma_{K \to L \to M}$ agrees with the usual isomorphism if $0 \to K \to L \to M \to 0$ is actually a short exact sequence of κ-vector spaces.*

Proof. The significance of taking nonzero symbols in the explicit description of the map $\gamma_{K \to L \to M}$ is simply that if $(e_1, \ldots, e_l)$ is an admissible sequence in K, and $(\overline{f}_1, \ldots, \overline{f}_m)$ is an admissible sequence in M, then it is not garanteed that $(e_1, \ldots, e_l, f_1, \ldots, f_m)$ is an admissible sequence in L (where of course $f_i \in L$ signifies a lift of $\overline{f}_i$). However, if the symbol $[e_1, \ldots, e_l]$ is nonzero in $\det_\kappa(K)$, then necessarily $K = \langle e_1, \ldots, e_k\rangle$ (see proof of

Lemma 29.2.2), and in this case it is true that $(e_1, \ldots, e_k, f_1, \ldots, f_m)$ is an admissible sequence. Moreover, by the admissible relations of type (b) for $\det_\kappa(L)$ we see that the value of $[e_1, \ldots, e_k, f_1, \ldots, f_m]$ in $\det_\kappa(L)$ is independent of the choice of the lifts f_i in this case also. Given this remark, it is clear that an admissible relation for $e_1, \ldots, e_k$ in K translates into an admissible relation among $e_1, \ldots, e_k, f_1, \ldots, f_m$ in L, and similarly for an admissible relation among the $\overline{f}_1, \ldots, \overline{f}_m$. Thus γ defines a linear map of vector spaces as claimed in the lemma.

By Lemma 29.2.4 we know $\det_\kappa(L)$ is generated by any single symbol $[x_1, \ldots, x_{k+m}]$ such that $(x_1, \ldots, x_{k+m})$ is an admissible sequence with $L = \langle x_1, \ldots, x_{k+m}\rangle$. Hence it is clear that the map $\gamma_{K\to L\to M}$ is surjective and hence an isomorphism.

Property (1) holds because

$$\begin{aligned} & \det_\kappa(v)([e_1, \ldots, e_k, f_1, \ldots, f_m]) \\ = \; & [v(e_1), \ldots, v(e_k), v(f_1), \ldots, v(f_m)] \\ = \; & \gamma_{K'\to L'\to M'}([u(e_1), \ldots, u(e_k)] \otimes [w(f_1), \ldots, w(f_m)]). \end{aligned}$$

Property (2) means that given a symbol $[\alpha_1, \ldots, \alpha_a]$ generating $\det_\kappa(A)$, a symbol $[\gamma_1, \ldots, \gamma_c]$ generating $\det_\kappa(C)$, a symbol $[\zeta_1, \ldots, \zeta_g]$ generating $\det_\kappa(G)$, and a symbol $[\iota_1, \ldots, \iota_i]$ generating $\det_\kappa(I)$ we have

$$\begin{aligned} & [\alpha_1, \ldots, \alpha_a, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_c, \tilde{\zeta}_1, \ldots, \tilde{\zeta}_g, \tilde{\iota}_1, \ldots, \tilde{\iota}_i] \\ = \; & (-1)^{cg}[\alpha_1, \ldots, \alpha_a, \tilde{\zeta}_1, \ldots, \tilde{\zeta}_g, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_c, \tilde{\iota}_1, \ldots, \tilde{\iota}_i] \end{aligned}$$

(for suitable lifts $\tilde{x}$ in E) in $\det_\kappa(E)$. This holds because we may use the admissible relations of type (c) cg times in the following order: move the $\tilde{\zeta}_1$ past the elements $\tilde{\gamma}_c, \ldots, \tilde{\gamma}_1$ (allowed since $\mathfrak{m}\tilde{\zeta}_1 \subset A$), then move $\tilde{\zeta}_2$ past the elements $\tilde{\gamma}_c, \ldots, \tilde{\gamma}_1$ (allowed since $\mathfrak{m}\tilde{\zeta}_2 \subset A + R\tilde{\zeta}_1$), and so on.

Part (3) of the lemma is obvious. This finishes the proof. $\square$

We can use the maps γ of the lemma to define more general maps γ as follows. Suppose that $(R, \mathfrak{m}, \kappa)$ is a local ring. Let M be a finite length R-module and suppose we are given a finite filtration (see Homology, Definition 10.13.1)

$$M = F^n \supset F^{n+1} \supset \ldots \supset F^{m-1} \supset F^m = 0.$$

Then there is a canonical isomorphism

$$\gamma_{(M,F)} : \bigotimes\nolimits_i \det_\kappa(F^i/F^{i+1}) \longrightarrow \det_\kappa(M)$$

well defined up to sign(!). One can make the sign explicit either by giving a well defined order of the terms in the tensor product (starting with higher indices unfortunately), and by thinking of the target category for the functor $\det_\kappa$ as the category of 1-dimensional super vector spaces. See [**KM76**, Section 1].

Here is another typical result for determinant functors. It is not hard to show. The tricky part is usually to show the existence of a determinant functor.

Lemma 29.2.6. *Let $(R, \mathfrak{m}, \kappa)$ be any local ring. The functor*

$$\det_\kappa : \left\{ \begin{matrix} fi R\text{-modules} \\ \text{with isomorphisms} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} 1\text{-dimensional } \kappa\text{-vector spaces} \\ \text{with isomorphisms} \end{matrix} \right\}$$

endowed with the maps $\gamma_{K\to L\to M}$ is characterized by the following properties

(1) *its restriction to the subcategory of modules annihilated by $\mathfrak{m}$ is isomorphic to the usual determinant functor (see Lemma 29.2.3), and*
(2) *(1), (2) and (3) of Lemma 29.2.5 hold.*

Proof. Omitted. □

Lemma 29.2.7. *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Let $I \subset \mathfrak{m}$ be an ideal, and set $R' = R/I$. Let $\det_{R,\kappa}$ denote the determinant functor on the category Mod^f_R finite length R-modules and denote $\det_{R',\kappa}$ the determinant on the category $Mod^f_{R'}$ of finite length R'-modules. Then $Mod^f_{R'} \subset Mod^f_R$ is a full subcategory and there exists an isomorphism of functors*

$$\det_{R,\kappa}|_{Mod^f_{R'}} = \det_{R',\kappa}$$

compatible with the isomorphisms $\gamma_{K\to L\to M}$ for either of these functors.

Proof. This can be shown by using the characterization of the pair $(\det_{R',\kappa}, \gamma)$ in Lemma 29.2.6. But really the isomorphism is obtained by mapping a symbol $[x_1, \ldots, x_l] \in \det_{R,\kappa}(M)$ to the corresponding symbol $[x_1, \ldots, x_l] \in \det_{R',\kappa}(M)$ which ``obviously'' works. □

Here is a case where we can compute the determinant of a linear map. In fact there is nothing mysterious about this in any case, see Example 29.2.9 for a random example.

Lemma 29.2.8. *Let R be a local ring with residue field κ. Let $u \in R^*$ be a unit. Let M be a module of finite length over R. Denote $u_M : M \to M$ the map multiplication by u. Then*

$$\det_\kappa(u_M) : \det_\kappa(M) \longrightarrow \det_\kappa(M)$$

is multiplication by $\overline{u}^l$ where $l = length_R(M)$ and $\overline{u} \in \kappa^*$ is the image of u.

Proof. Denote $f_M \in \kappa^*$ the element such that $\det_\kappa(u_M) = f_M \mathrm{id}_{\det_\kappa(M)}$. Suppose that $0 \to K \to L \to M \to 0$ is a short exact sequence of finite R-modules. Then we see that u_k, u_L, u_M give an isomorphism of short exact sequences. Hence by Lemma 29.2.5 (1) we conclude that $f_K f_M = f_L$. This means that by induction on length it suffices to prove the lemma in the case of length 1 where it is trivial. □

Example 29.2.9. Consider the local ring $R = \mathbf{Z}_p$. Set $M = \mathbf{Z}_p/(p^2) \oplus \mathbf{Z}_p/(p^3)$. Let $u : M \to M$ be the map given by the matrix

$$u = \begin{pmatrix} a & b \\ pc & d \end{pmatrix}$$

where $a, b, c, d \in \mathbf{Z}_p$, and $a, d \in \mathbf{Z}_p^*$. In this case $\det_\kappa(u)$ equals multiplication by $a^2 d^3 \bmod p \in \mathbf{F}_p^*$. This can easily be seen by consider the effect of u on the symbol $[p^2 e, pe, pf, e, f]$ where $e = (0, 1) \in M$ and $f = (1, 0) \in M$.

29.3. Periodic complexes

Of course there is a very general notion of periodic complexes. We can require periodicity of the maps, or periodicity of the objects. We will add these here as needed. For the moment we only need the following cases.

Definition 29.3.1. Let R be a ring.

(1) A 2-*periodic complex* over R is given by a quadruple (M, N, φ, ψ) consisting of R-modules M, N and R-module maps $\varphi : M \to N$, $\psi : N \to M$ such that

$$\ldots \longrightarrow M \xrightarrow{\varphi} N \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow \ldots$$

is a complex. In this setting we define the *cohomology modules* of the complex to be the R-modules

$$H^0(M, N, \varphi, \psi) = \mathrm{Ker}(\varphi)/\mathrm{Im}(\psi), \quad \text{and} \quad H^1(M, N, \varphi, \psi) = \mathrm{Ker}(\psi)/\mathrm{Im}(\varphi).$$

We say the 2-periodic complex is *exact* if the cohomology groups are zero.

(2) A $(2,1)$-*periodic complex* over R is given by a triple (M, φ, ψ) consisting of an R-module M and R-module maps $\varphi : M \to M$, $\psi : M \to M$ such that

$$\ldots \longrightarrow M \xrightarrow{\varphi} M \xrightarrow{\psi} M \xrightarrow{\varphi} M \longrightarrow \ldots$$

is a complex. Since this is a special case of a 2-periodic complex we have its *cohomology modules* $H^0(M, \varphi, \psi)$, $H^1(M, \varphi, \psi)$ and a notion of exactness.

In the following we will use any result proved for 2-periodic complexes without further mention for $(2,1)$-periodic complexes. It is clear that the collection of 2-periodic complexes (resp. $(2,1)$-periodic complexes) forms a category with morphisms $(f, g) : (M, N, \varphi, \psi) \to (M', N', \varphi', \psi')$ pairs of morphisms $f : M \to M'$ and $g : N \to N'$ such that $\varphi' \circ f = f \circ \varphi$ and $\psi' \circ g = g \circ \psi$. In fact it is an abelian category, with kernels and cokernels as in Homology, Lemma 10.10.3. Also, note that a special case are the $(2,1)$-periodic complexes of the form $(M, 0, \psi)$. In this special case we have

$$H^0(M, 0, \psi) = \mathrm{Coker}(\psi), \quad \text{and} \quad H^1(M, 0, \psi) = \mathrm{Ker}(\psi).$$

Definition 29.3.2. Let R be a local ring. Let (M, N, φ, ψ) be a 2-periodic complex over R whose cohomology groups have finite length over R. In this case we define the *multiplicity* of (M, N, φ, ψ) to be the integer

$$e_R(M, N, \varphi, \psi) = \mathrm{length}_R(H^0(M, N, \varphi, \psi)) - \mathrm{length}_R(H^1(M, N, \varphi, \psi))$$

We will sometimes (especially in the case of a $(2,1)$-periodic complex with $\varphi = 0$) call this the *Herbrand quotient*[1].

Lemma 29.3.3. *Let R be a local ring.*

(1) *If (M, N, φ, ψ) is a 2-periodic complex such that M, N have finite length. Then $e_R(M, N, \varphi, \psi) = \mathrm{length}_R(M) - \mathrm{length}_R(N)$.*
(2) *If (M, φ, ψ) is a $(2,1)$-periodic complex such that M has finite length. Then $e_R(M, \varphi, \psi) = 0$.*
(3) *Suppose that we have a short exact sequence of $(2,1)$-periodic complexes*

$$0 \to (M_1, N_1, \varphi_1, \psi_1) \to (M_2, N_2, \varphi_2, \psi_2) \to (M_3, N_3, \varphi_3, \psi_3) \to 0$$

If two out of three have cohomology modules of finite length so does the third and we have

$$e_R(M_2, N_2, \varphi_2, \psi_2) = e_R(M_1, N_1, \varphi_1, \psi_1) + e_R(M_3, N_3, \varphi_3, \psi_3).$$

[1] If the residue field of R is finite with q elements it is customary to call the Herbrand quotient $h(M, N, \varphi, \psi) = q^{e_R(M,N,\varphi,\psi)}$ which is equal to the number of elements of H^0 divided by the number of elements of H^1.

Proof. Proof of (3). Abbreviate $A = (M_1, N_1, \varphi_1, \psi_1)$, $B = (M_2, N_2, \varphi_2, \psi_2)$ and $C = (M_3, N_3, \varphi_3, \psi_3)$. We have a long exact cohomology sequence

$$\ldots \to H^1(C) \to H^0(A) \to H^0(B) \to H^0(C) \to H^1(A) \to H^1(B) \to H^1(C) \to \ldots$$

This gives a finite exact sequence

$$0 \to I \to H^0(A) \to H^0(B) \to H^0(C) \to H^1(A) \to H^1(B) \to K \to 0$$

with $0 \to K \to H^1(C) \to I \to 0$ a filtration. By additivity of the length function (Algebra, Lemma 7.48.3) we see the result. The proofs of (1) and (2) are omitted. □

Let R be a local ring with residue field κ. Let (M, φ, ψ) be a $(2,1)$-periodic complex over R. Assume that M has finite length and that (M, φ, ψ) is exact. We are going to use the determinant construction to define an invariant of this situation. See Section 29.2. Let us abbreviate $K_\varphi = \mathrm{Ker}(\varphi)$, $I_\varphi = \mathrm{Im}(\varphi)$, $K_\psi = \mathrm{Ker}(\psi)$, and $I_\psi = \mathrm{Im}(\psi)$. The short exact sequences

$$0 \to K_\varphi \to M \to I_\varphi \to 0, \quad 0 \to K_\psi \to M \to I_\psi \to 0$$

give isomorphisms

$$\gamma_\varphi : \det_\kappa(K_\varphi) \otimes \det_\kappa(I_\varphi) \longrightarrow \det_\kappa(M), \quad \gamma_\psi : \det_\kappa(K_\psi) \otimes \det_\kappa(I_\psi) \longrightarrow \det_\kappa(M),$$

see Lemma 29.2.5. On the other hand the exactness of the complex gives equalities $K_\varphi = I_\psi$, and $K_\psi = I_\varphi$ and hence an isomorphism

$$\sigma : \det_\kappa(K_\varphi) \otimes \det_\kappa(I_\varphi) \longrightarrow \det_\kappa(K_\psi) \otimes \det_\kappa(I_\psi)$$

by switching the factors. Using this notation we can define our invariant.

Definition 29.3.4. Let R be a local ring with residue field κ. Let (M, φ, ψ) be a $(2,1)$-periodic complex over R. Assume that M has finite length and that (M, φ, ψ) is exact. The *determinant of* (M, φ, ψ) is the element

$$\det_\kappa(M, \varphi, \psi) \in \kappa^*$$

such that the composition

$$\det_\kappa(M) \xrightarrow{\gamma_\psi \circ \sigma \circ \gamma_\varphi^{-1}} \det_\kappa(M)$$

is multiplication by $(-1)^{\mathrm{length}_R(I_\varphi)\mathrm{length}_R(I_\psi)} \det_\kappa(M, \varphi, \psi)$.

Remark 29.3.5. Here is a more down to earth description of the determinant introduced above. Let R be a local ring with residue field κ. Let (M, φ, ψ) be a $(2,1)$-periodic complex over R. Assume that M has finite length and that (M, φ, ψ) is exact. Let us abbreviate $I_\varphi = \mathrm{Im}(\varphi)$, $I_\psi = \mathrm{Im}(\psi)$ as above. Assume that $\mathrm{length}_R(I_\varphi) = a$ and $\mathrm{length}_R(I_\psi) = b$, so that $a + b = \mathrm{length}_R(M)$ by exactness. Choose admissible sequences $x_1, \ldots, x_a \in I_\varphi$ and $y_1, \ldots, y_b \in I_\psi$ such that the symbol $[x_1, \ldots, x_a]$ generates $\det_\kappa(I_\varphi)$ and the symbol $[x_1, \ldots, x_b]$ generates $\det_\kappa(I_\psi)$. Choose $\tilde{x}_i \in M$ such that $\varphi(\tilde{x}_i) = x_i$. Choose $\tilde{y}_j \in M$ such that $\psi(\tilde{y}_j) = y_j$. Then $\det_\kappa(M, \varphi, \psi)$ is characterized by the equality

$$[x_1, \ldots, x_a, \tilde{y}_1, \ldots, \tilde{y}_b] = (-1)^{ab} \det_\kappa(M, \varphi, \psi)[y_1, \ldots, y_b, \tilde{x}_1, \ldots, \tilde{x}_a]$$

in $\det_\kappa(M)$. This also explains the sign.

Lemma 29.3.6. *Let R be a local ring with residue field κ. Let (M, φ, ψ) be a $(2,1)$-periodic complex over R. Assume that M has finite length and that (M, φ, ψ) is exact. Then*

$$\det_\kappa(M, \varphi, \psi) \det_\kappa(M, \psi, \varphi) = 1.$$

Proof. Omitted. □

Lemma 29.3.7. *Let R be a local ring with residue field κ. Let (M, φ, φ) be a $(2,1)$-periodic complex over R. Assume that M has finite length and that (M, φ, ψ) is exact. Then $length_R(M) = 2length_R(Im(\varphi))$ and*

$$\det_\kappa(M, \varphi, \psi) = (-1)^{length_R(Im(\varphi))} = (-1)^{\frac{1}{2}length_R(M)}$$

Proof. Follows directly from the sign rule in the definitions. $\square$

Lemma 29.3.8. *Let R be a local ring with residue field κ. Let M be a finite length R-module.*

(1) *if $\varphi : M \to M$ is an isomorphism then* $\det_\kappa(M, \varphi, 0) = \det_\kappa(\varphi)$.
(2) *if $\psi : M \to M$ is an isomorphism then* $\det_\kappa(M, 0, \psi) = \det_\kappa(\psi)^{-1}$.

Proof. Let us prove (1). Set $\psi = 0$. Then we may, with notation as above Definition 29.3.4, identify $K_\varphi = I_\psi = 0$, $I_\varphi = K_\psi = M$. With these identifications, the map

$$\gamma_\varphi : \kappa \otimes \det_\kappa(M) = \det_\kappa(K_\varphi) \otimes \det_\kappa(I_\varphi) \longrightarrow \det_\kappa(M)$$

is identified with $\det_\kappa(\varphi^{-1})$. On the other hand the map γ_ψ is identified with the identity map. Hence $\gamma_\psi \circ \sigma \circ \gamma_\varphi^{-1}$ is equal to $\det_\kappa(\varphi)$ in this case. Whence the result. We omit the proof of (2). $\square$

Lemma 29.3.9. *Let R be a local ring with residue field κ. Suppose that we have a short exact sequence of $(2,1)$-periodic complexes*

$$0 \to (M_1, \varphi_1, \psi_1) \to (M_2, \varphi_2, \psi_2) \to (M_3, \varphi_3, \psi_3) \to 0$$

with all M_i of finite length, and each (M_1, φ_1, ψ_1) exact. Then

$$\det_\kappa(M_2, \varphi_2, \psi_2) = \det_\kappa(M_1, \varphi_1, \psi_1) \det_\kappa(M_3, \varphi_3, \psi_3).$$

in κ^*.

Proof. Let us abbreviate $I_{\varphi,i} = \mathrm{Im}(\varphi_i)$, $K_{\varphi,i} = \mathrm{Ker}(\varphi_i)$, $I_{\psi,i} = \mathrm{Im}(\psi_i)$, and $K_{\psi,i} = \mathrm{Ker}(\psi_i)$. Observe that we have a commutative square

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K_{\varphi,1} \arrow[r] \arrow[d] & K_{\varphi,2} \arrow[r] \arrow[d] & K_{\varphi,3} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_1 \arrow[r] \arrow[d] & M_2 \arrow[r] \arrow[d] & M_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I_{\varphi,1} \arrow[r] \arrow[d] & I_{\varphi,2} \arrow[r] \arrow[d] & I_{\varphi,3} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

of finite length R-modules with exact rows and columns. The top row is exact since it can be identified with the sequence $I_{\psi,1} \to I_{\psi,2} \to I_{\psi,3} \to 0$ of images, and similarly for the bottom row. There is a similar diagram involving the modules $I_{\psi,i}$ and $K_{\psi,i}$. By definition

$\det_\kappa(M_2, \varphi_2, \psi_2)$ corresponds, up to a sign, to the composition of the left vertical maps in the following diagram

$$\begin{tikzcd}
\det_\kappa(M_1) \otimes \det_\kappa(M_3) \arrow[r, "\gamma"] \arrow[d, "\gamma^{-1}\otimes\gamma^{-1}"] & \det_\kappa(M_2) \arrow[d, "\gamma^{-1}"] \\
\det_\kappa(K_{\varphi,1}) \otimes \det_\kappa(I_{\varphi,1}) \otimes \det_\kappa(K_{\varphi,3}) \otimes \det_\kappa(I_{\varphi,3}) \arrow[r, "\gamma\otimes\gamma"] \arrow[d, "\sigma\otimes\sigma"] & \det_\kappa(K_{\varphi,2}) \otimes \det_\kappa(I_{\varphi,2}) \arrow[d, "\sigma"] \\
\det_\kappa(K_{\psi,1}) \otimes \det_\kappa(I_{\psi,1}) \otimes \det_\kappa(K_{\psi,3}) \otimes \det_\kappa(I_{\psi,3}) \arrow[r, "\gamma\otimes\gamma"] \arrow[d, "\gamma\otimes\gamma"] & \det_\kappa(K_{\psi,2}) \otimes \det_\kappa(I_{\psi,2}) \arrow[d, "\gamma"] \\
\det_\kappa(M_1) \otimes \det_\kappa(M_3) \arrow[r, "\gamma"] & \det_\kappa(M_2)
\end{tikzcd}$$

The top and bottom squares are commutative up to sign by applying Lemma 29.2.5 (2). The middle square is trivially commutative (we are just switching factors). Hence we see that $\det_\kappa(M_2, \varphi_2, \psi_2) = \epsilon \det_\kappa(M_1, \varphi_1, \psi_1) \det_\kappa(M_3, \varphi_3, \psi_3)$ for some sign ϵ. And the sign can be worked out, namely the outer rectangle in the diagram above commutes up to

$$\begin{aligned}
\epsilon &= (-1)^{\text{length}(I_{\varphi,1})\text{length}(K_{\varphi,3}) + \text{length}(I_{\psi,1})\text{length}(K_{\psi,3})} \\
&= (-1)^{\text{length}(I_{\varphi,1})\text{length}(I_{\psi,3}) + \text{length}(I_{\psi,1})\text{length}(I_{\varphi,3})}
\end{aligned}$$

(proof omitted). It follows easily from this that the signs work out as well. ☐

Example 29.3.10. Let k be a field. Consider the ring $R = k[T]/(T^2)$ of dual numbers over k. Denote t the class of T in R. Let $M = R$ and $\varphi = ut$, $\psi = vt$ with $u, v \in k^*$. In this case $\det_k(M)$ has generator $e = [t, 1]$. We identify $I_\varphi = K_\varphi = I_\psi = K_\psi = (t)$. Then $\gamma_\varphi(t \otimes t) = u^{-1}[t, 1]$ (since $u^{-1} \in M$ is a lift of $t \in I_\varphi$) and $\gamma_\psi(t \otimes t) = v^{-1}[t, 1]$ (same reason). Hence we see that $\det_k(M, \varphi, \psi) = -u/v \in k^*$.

Example 29.3.11. Let $R = \mathbf{Z}_p$ and let $M = \mathbf{Z}_p/(p^l)$. Let $\varphi = p^b u$ and $\varphi = p^a v$ with $a, b \geq 0$, $a + b = l$ and $u, v \in \mathbf{Z}_p^*$. Then a computation as in Example 29.3.10 shows that

$$\begin{aligned}
\det_{\mathbf{F}_p}(\mathbf{Z}_p/(p^l), p^b u, p^a v) &= (-1)^{ab} u^a / v^b \bmod p \\
&= (-1)^{\text{ord}_p(\alpha)\text{ord}_p(\beta)} \frac{\alpha^{\text{ord}_p(\beta)}}{\beta^{\text{ord}_p(\alpha)}} \bmod p
\end{aligned}$$

with $\alpha = p^b u, \beta = p^a v \in \mathbf{Z}_p$. See Lemma 29.4.10 for a more general case (and a proof).

Example 29.3.12. Let $R = k$ be a field. Let $M = k^{\oplus a} \oplus k^{\oplus b}$ be $l = a + b$ dimensional. Let φ and ψ be the following diagonal matrices

$$\varphi = \text{diag}(u_1, \ldots, u_a, 0, \ldots, 0), \quad \psi = \text{diag}(0, \ldots, 0, v_1, \ldots, v_b)$$

with $u_i, v_j \in k^*$. In this case we have

$$\det_k(M, \varphi, \psi) = \frac{u_1 \ldots u_a}{v_1 \ldots v_b}.$$

This can be seen by a direct computation or by computing in case $l = 1$ and using the additivity of Lemma 29.3.9.

Example 29.3.13. Let $R = k$ be a field. Let $M = k^{\oplus a} \oplus k^{\oplus a}$ be $l = 2a$ dimensional. Let φ and ψ be the following block matrices

$$\varphi = \begin{pmatrix} 0 & U \\ 0 & 0 \end{pmatrix}, \quad \psi = \begin{pmatrix} 0 & V \\ 0 & 0 \end{pmatrix},$$

with $U, V \in \mathrm{Mat}(a \times a, k)$ invertible. In this case we have

$$\det_k(M, \varphi, \psi) = (-1)^a \frac{\det(U)}{\det(V)}.$$

This can be seen by a direct computation. The case $a = 1$ is similar to the computation in Example 29.3.10.

Example 29.3.14. Let $R = k$ be a field. Let $M = k^{\oplus 4}$. Let

$$\varphi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ u_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & u_2 & 0 \end{pmatrix} \quad \varphi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & v_2 & 0 \\ 0 & 0 & 0 & 0 \\ v_1 & 0 & 0 & 0 \end{pmatrix}$$

with $u_1, u_2, v_1, v_2 \in k^*$. Then we have

$$\det_k(M, \varphi, \psi) = -\frac{u_1 u_2}{v_1 v_2}.$$

Next we come to the analogue of the fact that the determinant of a composition of linear endomorphisms is the product of the determinants. To avoid very long formulae we write $I_\varphi = \mathrm{Im}(\varphi)$, and $K_\varphi = \mathrm{Ker}(\varphi)$ for any R-module map $\varphi : M \to M$. We also denote $\varphi\psi = \varphi \circ \psi$ for a pair of morphisms $\varphi, \psi : M \to M$.

Lemma 29.3.15. *Let R be a local ring with residue field κ. Let M be a finite length R-module. Let α, β, γ be endomorphisms of M. Assume that*

1. *$I_\alpha = K_{\beta\gamma}$, and similarly for any permutation of α, β, γ,*
2. *$K_\alpha = I_{\beta\gamma}$, and similarly for any permutation of α, β, γ.*

Then

1. *The triple $(M, \alpha, \beta\gamma)$ is an exact $(2, 1)$-periodic complex.*
2. *The triple $(I_\gamma, \alpha, \beta)$ is an exact $(2, 1)$-periodic complex.*
3. *The triple $(M/K_\beta, \alpha, \gamma)$ is an exact $(2, 1)$-periodic complex.*
4. *We have*

$$\det_\kappa(M, \alpha, \beta\gamma) = \det_\kappa(I_\gamma, \alpha, \beta) \det_\kappa(M/K_\beta, \alpha, \gamma).$$

Proof. It is clear that the assumptions imply part (1) of the lemma.

To see part (1) note that the assumptions imply that $I_{\gamma\alpha} = I_{\alpha\gamma}$, and similarly for kernels and any other pair of morphisms. Moreover, we see that $I_{\gamma\beta} = I_{\beta\gamma} = K_\alpha \subset I_\gamma$ and similarly for any other pair. In particular we get a short exact sequence

$$0 \to I_{\beta\gamma} \to I_\gamma \xrightarrow{\alpha} I_{\alpha\gamma} \to 0$$

and similarly we get a short exact sequence

$$0 \to I_{\alpha\gamma} \to I_\gamma \xrightarrow{\beta} I_{\beta\gamma} \to 0.$$

This proves $(I_\gamma, \alpha, \beta)$ is an exact $(2, 1)$-periodic complex. Hence part (2) of the lemma holds.

To see that α, γ give well defined endomorphisms of M/K_β we have to check that $\alpha(K_\beta) \subset K_\beta$ and $\gamma(K_\beta) \subset K_\beta$. This is true because $\alpha(K_\beta) = \alpha(I_{\gamma\alpha}) = I_{\alpha\gamma\alpha} \subset I_{\alpha\gamma} = K_\beta$, and similarly in the other case. The kernel of the map $\alpha : M/K_\beta \to M/K_\beta$ is $K_{\beta\alpha}/K_\beta = I_\gamma/K_\beta$. Similarly, the kernel of $\gamma : M/K_\beta \to M/K_\beta$ is equal to I_α/K_β. Hence we conclude that (3) holds.

We introduce $r = \text{length}_R(K_\alpha)$, $s = \text{length}_R(K_\beta)$ and $t = \text{length}_R(K_\gamma)$. By the exact sequences above and our hypotheses we have $\text{length}_R(I_\alpha) = s + t$, $\text{length}_R(I_\beta) = r + t$, $\text{length}_R(I_\gamma) = r + s$, and $\text{length}(M) = r + s + t$. Choose

(1) an admissible sequence $x_1, \ldots, x_r \in K_\alpha$ generating K_α
(2) an admissible sequence $y_1, \ldots, y_s \in K_\beta$ generating K_β,
(3) an admissible sequence $z_1, \ldots, z_t \in K_\gamma$ generating K_γ,
(4) elements $\tilde{x}_i \in M$ such that $\beta\gamma\tilde{x}_i = x_i$,
(5) elements $\tilde{y}_i \in M$ such that $\alpha\gamma\tilde{y}_i = y_i$,
(6) elements $\tilde{z}_i \in M$ such that $\beta\alpha\tilde{z}_i = z_i$.

With these choices the sequence $y_1, \ldots, y_s, \alpha\tilde{z}_1, \ldots, \alpha\tilde{z}_t$ is an admissible sequence in I_α generating it. Hence, by Remark 29.3.5 the determinant $D = \det_\kappa(M, \alpha, \beta\gamma)$ is the unique element of κ^* such that

$$\begin{aligned}&[y_1, \ldots, y_s, \alpha\tilde{z}_1, \ldots, \alpha\tilde{z}_s, \tilde{x}_1, \ldots, \tilde{x}_r]\\ &= (-1)^{r(s+t)} D[x_1, \ldots, x_r, \gamma\tilde{y}_1, \ldots, \gamma\tilde{y}_s, \tilde{z}_1, \ldots, \tilde{z}_t]\end{aligned}$$

By the same remark, we see that $D_1 = \det_\kappa(M/K_\beta, \alpha, \gamma)$ is characterized by

$$[y_1, \ldots, y_s, \alpha\tilde{z}_1, \ldots, \alpha\tilde{z}_t, \tilde{x}_1, \ldots, \tilde{x}_r] = (-1)^{rt} D_1[y_1, \ldots, y_s, \gamma\tilde{x}_1, \ldots, \gamma\tilde{x}_r, \tilde{z}_1, \ldots, \tilde{z}_t]$$

By the same remark, we see that $D_2 = \det_\kappa(I_\gamma, \alpha, \beta)$ is characterized by

$$[y_1, \ldots, y_s, \gamma\tilde{x}_1, \ldots, \gamma\tilde{x}_r, \tilde{z}_1, \ldots, \tilde{z}_t] = (-1)^{rs} D_2[x_1, \ldots, x_r, \gamma\tilde{y}_1, \ldots, \gamma\tilde{y}_s, \tilde{z}_1, \ldots, \tilde{z}_t]$$

Combining the formulas above we see that $D = D_1 D_2$ as desired. □

Lemma 29.3.16. *Let R be a local ring with residue field κ. Let $\alpha : (M, \varphi, \psi) \to (M', \varphi', \psi')$ be a morphism of $(2, 1)$-periodic complexes over R. Assume*

(1) *M, M' have finite length,*
(2) *(M, φ, ψ), (M', φ', ψ') are exact,*
(3) *the maps φ, ψ induce the zero map on $K = Ker(\alpha)$, and*
(4) *the maps φ, ψ induce the zero map on $Q = Coker(\alpha)$.*

Denote $N = \alpha(M) \subset M'$. We obtain two short exact sequences of $(2, 1)$-periodic complexes

$$\begin{gathered}0 \to (N, \varphi', \psi') \to (M', \varphi', \psi') \to (Q, 0, 0) \to 0\\ 0 \to (K, 0, 0) \to (M, \varphi, \psi) \to (N, \varphi', \psi') \to 0\end{gathered}$$

which induce two isomorphisms $\alpha_i : Q \to K$, $i = 0, 1$. Then

$$\det_\kappa(M, \varphi, \psi) = \det_\kappa(\alpha_0^{-1} \circ \alpha_1) \det_\kappa(M', \varphi', \psi')$$

In particular, if $\alpha_0 = \alpha_1$, then $\det_\kappa(M, \varphi, \psi) = \det_\kappa(M', \varphi', \psi')$.

Proof. There are (at least) two ways to prove this lemma. One is to produce an enormous commutative diagram using the properties of the determinants. The other is to use the characterization of the determinants in terms of admissible sequences of elements. It is the second approach that we will use.

First let us explain precisely what the maps α_i are. Namely, α_0 is the composition

$$\alpha_0 : Q = H^0(Q, 0, 0) \to H^1(N, \varphi', \psi') \to H^2(K, 0, 0) = K$$

and α_1 is the composition

$$\alpha_1 : Q = H^1(Q, 0, 0) \to H^2(N, \varphi', \psi') \to H^3(K, 0, 0) = K$$

coming from the boundary maps of the short exact sequences of complexes displayed in the lemma. The fact that the complexes (M, φ, ψ), (M', φ', ψ') are exact implies these maps are isomorphisms.

We will use the notation $I_\varphi = \mathrm{Im}(\varphi)$, $K_\varphi = \mathrm{Ker}(\varphi)$ and similarly for the other maps. Exactness for M and M' means that $K_\varphi = I_\psi$ and three similar equalities. We introduce $k = \text{length}_R(K)$, $a = \text{length}_R(I_\varphi)$, $b = \text{length}_R(I_\psi)$. Then we see that $\text{length}_R(M) = a + b$, and $\text{length}_R(N) = a+b-k$, $\text{length}_R(Q) = k$ and $\text{length}_R(M') = a+b$. The exact sequences below will show that also $\text{length}_R(I_{\varphi'}) = a$ and $\text{length}_R(I_{\psi'}) = b$.

The assumption that $K \subset K_\varphi = I_\psi$ means that φ factors through N to give an exact sequence

$$0 \to \alpha(I_\psi) \to N \xrightarrow{\varphi\alpha^{-1}} I_\psi \to 0.$$

Here $\varphi\alpha^{-1}(x') = y$ means $x' = \alpha(x)$ and $y = \varphi(x)$. Similarly, we have

$$0 \to \alpha(I_\varphi) \to N \xrightarrow{\psi\alpha^{-1}} I_\varphi \to 0.$$

The assumption that ψ' induces the zero map on Q means that $I_{\psi'} = K_{\varphi'} \subset N$. This means the quotient $\varphi'(N) \subset I_{\varphi'}$ is identified with Q. Note that $\varphi'(N) = \alpha(I_\varphi)$. Hence we conclude there is an isomorphism

$$\varphi' : Q \to I_{\varphi'}/\alpha(I_\varphi)$$

simply described by $\varphi'(x' \bmod N) = \varphi'(x') \bmod \alpha(I_\varphi)$. In exactly the same way we get

$$\psi' : Q \to I_{\psi'}/\alpha(I_\psi)$$

Finally, note that α_0 is the composition

$$Q \xrightarrow{\varphi'} I_{\varphi'}/\alpha(I_\varphi) \xrightarrow{\psi\alpha^{-1}|_{I_{\varphi'}/\alpha(I_\varphi)}} K$$

and similarly $\alpha_1 = \varphi\alpha^{-1}|_{I_{\psi'}/\alpha(I_\psi)} \circ \psi'$.

To shorten the formulas below we are going to write αx instead of $\alpha(x)$ in the following. No confusion should result since all maps are indicated by greek letters and elements by roman letters. We are going to choose

(1) an admissible sequence $z_1, \ldots, z_k \in K$ generating K,
(2) elements $z_i' \in M$ such that $\varphi z_i' = z_i$,
(3) elements $z_i'' \in M$ such that $\psi z_i'' = z_i$,
(4) elements $x_{k+1}, \ldots, x_a \in I_\varphi$ such that $z_1, \ldots, z_k, x_{k+1}, \ldots, x_a$ is an admissible sequence generating I_φ,
(5) elements $\tilde{x}_i \in M$ such that $\varphi\tilde{x}_i = x_i$,
(6) elements $y_{k+1}, \ldots, y_b \in I_\psi$ such that $z_1, \ldots, z_k, y_{k+1}, \ldots, y_b$ is an admissible sequence generating I_ψ,
(7) elements $\tilde{y}_i \in M$ such that $\psi\tilde{y}_i = y_i$, and
(8) elements $w_1, \ldots, w_k \in M'$ such that $w_1 \bmod N, \ldots, w_k \bmod N$ are an admissible sequence in Q generating Q.

By Remark 29.3.5 the element $D = \det_\kappa(M, \varphi, \psi) \in \kappa^*$ is characterized by

$$\begin{aligned} & [z_1, \ldots, z_k, x_{k+1}, \ldots, x_a, z_1'', \ldots, z_k'', \tilde{y}_{k+1}, \ldots, \tilde{y}_b] \\ = \quad & (-1)^{ab} D[z_1, \ldots, z_k, y_{k+1}, \ldots, y_b, z_1', \ldots, z_k', \tilde{x}_{k+1}, \ldots, \tilde{x}_a] \end{aligned}$$

Note that by the discussion above $\alpha x_{k+1}, \ldots, \alpha x_a, \varphi w_1, \ldots, \varphi w_k$ is an admissible sequence generating $I_{\varphi'}$ and $\alpha y_{k+1}, \ldots, \alpha y_b, \psi w_1, \ldots, \psi w_k$ is an admissible sequence generating $I_{\psi'}$. Hence by Remark 29.3.5 the element $D' = \det_\kappa(M', \varphi', \psi') \in \kappa^*$ is characterized by

$$\begin{aligned} & [\alpha x_{k+1}, \ldots, \alpha x_a, \varphi' w_1, \ldots, \varphi' w_k, \alpha \tilde{y}_{k+1}, \ldots, \alpha \tilde{y}_b, w_1, \ldots, w_k] \\ = \quad & (-1)^{ab} D' [\alpha y_{k+1}, \ldots, \alpha y_b, \psi' w_1, \ldots, \psi' w_k, \alpha \tilde{x}_{k+1}, \ldots, \alpha \tilde{x}_a, w_1, \ldots, w_k] \end{aligned}$$

Note how in the first, resp. second displayed formula the the first, resp. last k entries of the symbols on both sides are the same. Hence these formulas are really equivalent to the equalities

$$\begin{aligned} & [\alpha x_{k+1}, \ldots, \alpha x_a, \alpha z''_1, \ldots, \alpha z''_k, \alpha \tilde{y}_{k+1}, \ldots, \alpha \tilde{y}_b] \\ = \quad & (-1)^{ab} D [\alpha y_{k+1}, \ldots, \alpha y_b, \alpha z'_1, \ldots, \alpha z'_k, \alpha \tilde{x}_{k+1}, \ldots, \alpha \tilde{x}_a] \end{aligned}$$

and

$$\begin{aligned} & [\alpha x_{k+1}, \ldots, \alpha x_a, \varphi' w_1, \ldots, \varphi' w_k, \alpha \tilde{y}_{k+1}, \ldots, \alpha \tilde{y}_b] \\ = \quad & (-1)^{ab} D' [\alpha y_{k+1}, \ldots, \alpha y_b, \psi' w_1, \ldots, \psi' w_k, \alpha \tilde{x}_{k+1}, \ldots, \alpha \tilde{x}_a] \end{aligned}$$

in $\det_\kappa(N)$. Note that $\varphi' w_1, \ldots, \varphi' w_k$ and $\alpha z''_1, \ldots, z''_k$ are admissible sequences generating the module $I_{\varphi'}/\alpha(I_\varphi)$. Write

$$[\varphi' w_1, \ldots, \varphi' w_k] = \lambda_0 [\alpha z''_1, \ldots, \alpha z''_k]$$

in $\det_\kappa(I_{\varphi'}/\alpha(I_\varphi))$ for some $\lambda_0 \in \kappa^*$. Similarly, write

$$[\psi' w_1, \ldots, \psi' w_k] = \lambda_1 [\alpha z'_1, \ldots, \alpha z'_k]$$

in $\det_\kappa(I_{\psi'}/\alpha(I_\psi))$ for some $\lambda_1 \in \kappa^*$. On the one hand it is clear that

$$\alpha_i([w_1, \ldots, w_k]) = \lambda_i [z_1, \ldots, z_k]$$

for $i = 0, 1$ by our description of α_i above, which means that

$$\det_\kappa(\alpha_0^{-1} \circ \alpha_1) = \lambda_1/\lambda_0$$

and on the other hand it is clear that

$$\begin{aligned} & \lambda_0 [\alpha x_{k+1}, \ldots, \alpha x_a, \alpha z''_1, \ldots, \alpha z''_k, \alpha \tilde{y}_{k+1}, \ldots, \alpha \tilde{y}_b] \\ = \quad & [\alpha x_{k+1}, \ldots, \alpha x_a, \varphi' w_1, \ldots, \varphi' w_k, \alpha \tilde{y}_{k+1}, \ldots, \alpha \tilde{y}_b] \end{aligned}$$

and

$$\begin{aligned} & \lambda_1 [\alpha y_{k+1}, \ldots, \alpha y_b, \alpha z'_1, \ldots, \alpha z'_k, \alpha \tilde{x}_{k+1}, \ldots, \alpha \tilde{x}_a] \\ = \quad & [\alpha y_{k+1}, \ldots, \alpha y_b, \psi' w_1, \ldots, \psi' w_k, \alpha \tilde{x}_{k+1}, \ldots, \alpha \tilde{x}_a] \end{aligned}$$

which imply $\lambda_0 D = \lambda_1 D'$. The lemma follows. □

29.4. Symbols

The correct generality for this construction is perhaps the situation of the following lemma.

Lemma 29.4.1. *Let A be a Noetherian local ring. Let M be a finite A-module of dimension 1. Assume $\varphi, \psi : M \to M$ are two injective A-module maps, and assume $\varphi(\psi(M)) = \psi(\varphi(M))$, for example if φ and ψ commute. Then $\text{length}_R(M/\varphi\psi M) < \infty$ and $(M/\varphi\psi M, \varphi, \psi)$ is an exact $(2,1)$-periodic complex.*

Proof. Let $\mathfrak{q}$ be a minimal prime of the support of M. Then $M_\mathfrak{q}$ is a finite length $A_\mathfrak{q}$-module, see Algebra, Lemma 7.59.8. Hence both φ and ψ induce isomorphisms $M_\mathfrak{q} \to M_\mathfrak{q}$. Thus the support of $M/\varphi\psi M$ is $\{\mathfrak{m}_A\}$ and hence it has finite length (see lemma cited above). Finally, the kernel of φ on $M/\varphi\psi M$ is clearly $\psi M/\varphi\psi M$, and hence the kernel of φ is the image of ψ on $M/\varphi\psi M$. Similarly the other way since $M/\varphi\psi M = M/\psi\varphi M$ by assumption. □

Lemma 29.4.2. *Let A be a Noetherian local ring. Let $a, b \in A$.*

(1) *if M is a finite A-module of dimension 1 such that a, b are nonzero divisors on M, then $length_A(M/abM) < \infty$ and $(M/abM, a, b)$ is a $(2,1)$-periodic exact complex.*
(2) *if a, b are nonzero divisors and $\dim(A) = 1$ then $length_A(A/(ab)) < \infty$ and $(A/(ab), a, b)$ is a $(2,1)$-periodic exact complex.*

In particular, in these case $\det_\kappa(M/abM, a, b) \in \kappa^*$, resp. $\det_\kappa(A/(ab), a, b) \in \kappa^*$ are defined.

Proof. Follows from Lemma 29.4.1. □

Definition 29.4.3. Let A be a Noetherian local ring with residue field κ. Let $a, b \in A$. Let M be a finite A-module of dimension 1 such that a, b are nonzero-divisors on M. We define the *symbol associated to* M, a, b to be the element

$$d_M(a, b) = \det_\kappa(M/abM, a, b) \in \kappa^*$$

Lemma 29.4.4. *Let A be a Noetherian local ring. Let $a, b, c \in A$. Let M be a finite A-module with $\dim(M) = 1$. Assume a, b, c are nonzero divisors on M. Then*

$$d_M(a, bc) = d_M(a, b)d_M(a, c)$$

and $d_M(a, b)d_M(b, a) = 1$.

Proof. The first statement is immediate from Lemma 29.3.15 above. The second comes from Lemma 29.3.6. □

Definition 29.4.5. Let A be a Noetherian local domain of dimension 1 with residue field κ. Let K be the fraction field of A. We define the *tame symbol* of A to be the map

$$K^* \times K^* \longrightarrow \kappa^*, \quad (x, y) \longmapsto d_A(x, y)$$

where $d_A(x, y)$ is extended to $K^* \times K^*$ by the multiplicativity of Lemma 29.4.4.

It is clear that we may extend more generally $d_M(-, -)$ to certain rings of fractions of A (even if A is not a domain).

Lemma 29.4.6. *Let A be a Noetherian local ring. Let M be a finite A-module of dimension 1. Let $b \in A$ be a nonzero divisor on M, and let $u \in A^*$. Then*

$$d_M(u, b) = u^{length_M(M/bM)} \bmod \mathfrak{m}_A.$$

In particular, if $M = A$, then $d_A(u, b) = u^{ord_A(b)} \bmod \mathfrak{m}_A$.

Proof. Note that in this case $M/ubM = M/bM$ on which multiplication by b is zero. Hence $d_M(u, b) = \det_\kappa(u|_{M/bM})$ by Lemma 29.3.8. The lemma then follows from Lemma 29.2.8. □

Lemma 29.4.7. *Let A be a Noetherian local ring. Let $a, b \in A$. Let*

$$0 \to M \to M' \to M'' \to 0$$

be a short exact sequence of A-modules of dimension 1 such that a, b are nonzero divisors on all three A-modules. Then

$$d_{M'}(a, b) = d_M(a, b) d_{M''}(a, b)$$

in κ^*.

Proof. It is easy to see that this leads to a short exact sequence of exact $(2, 1)$-periodic complexes

$$0 \to (M/abM, a, b) \to (M'/abM', a, b) \to (M''/abM'', a, b) \to 0$$

Hence the lemma follows from Lemma 29.3.9. □

Lemma 29.4.8. *Let A be a Noetherian local ring. Let $\alpha : M \to M'$ be a homomorphism of finite A-modules of dimension 1. Let $a, b \in A$. Assume*

(1) *a, b are nonzero divisors on both M and M', and*
(2) $\dim(Ker(\alpha)), \dim(Coker(\alpha)) \leq 0$.

Then $d_M(a, b) = d_{M'}(a, b)$.

Proof. If $a \in A^*$, then the equality follows from the equality $\text{length}(M/bM) = \text{length}(M'/bM')$ and Lemma 29.4.6. Similarly if b is a unit the lemma holds as well (by the symmetry of Lemma 29.4.4). Hence we may assume that $a, b \in \mathfrak{m}_A$. This in particular implies that $\mathfrak{m}$ is not an associated prime of M, and hence $\alpha : M \to M'$ is injective. This permits us to think of M as a submodule of M'. By assumption M'/M is a finite A-module with support $\{\mathfrak{m}_A\}$ and hence has finite length. Note that for any third module M'' with $M \subset M'' \subset M'$ the maps $M \to M''$ and $M'' \to M'$ satisfy the assumptions of the lemma as well. This reduces us, by induction on the length of M'/M, to the case where $\text{length}_A(M'/M) = 1$. Finally, in this case consider the map

$$\overline{\alpha} : M/abM \longrightarrow M'/abM'.$$

By construction the cokernel Q of $\overline{\alpha}$ has length 1. Since $a, b \in \mathfrak{m}_A$, they act trvially on Q. It also follows that the kernel K of $\overline{\alpha}$ has length 1 and hence also a, b act trivially on K. Hence we may apply Lemma 29.3.16. Thus it suffices to see that the two maps $\alpha_i : Q \to K$ are the same. In fact, both maps are equal to the map $q = x' \bmod \text{Im}(\overline{\alpha}) \mapsto abx' \in K$. We omit the verification. □

Lemma 29.4.9. *Let A be a Noetherian local ring. Let M be a finite A-module with $\dim(M) = 1$. Let $a, b \in A$ nonzero divisors on M. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ be the minimal primes in the support of M. Then*

$$d_M(a, b) = \prod\nolimits_{i=1,\ldots,t} d_{A/\mathfrak{q}_i}(a, b)^{length_{A_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i})}$$

as elements of κ^*.

Proof. Choose a filtration by A-submodules

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$$

such that each quotient M_j/M_{j-1} is isomorphic to $A/\mathfrak{p}_j$ for some prime ideal $\mathfrak{p}_j$ of A. See Algebra, Lemma 7.59.1. For each j we have either $\mathfrak{p}_j = \mathfrak{q}_i$ for some i, or $\mathfrak{p}_j = \mathfrak{m}_A$.

Moreover, for a fixed i, the number of j such that $\mathfrak{p}_j = \mathfrak{q}_i$ is equal to $\text{length}_{A_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i})$ by Algebra, Lemma 7.59.10. Hence $d_{M_j}(a, b)$ is defined for each j and

$$d_{M_j}(a, b) = \begin{cases} d_{M_{j-1}}(a, b) d_{A/\mathfrak{q}_i}(a, b) & \text{if} \quad \mathfrak{p}_j = \mathfrak{q}_i \\ d_{M_{j-1}}(a, b) & \text{if} \quad \mathfrak{p}_j = \mathfrak{m}_A \end{cases}$$

by Lemma 29.4.7 in the first instance and Lemma 29.4.8 in the second. Hence the lemma. $\square$

Lemma 29.4.10. *Let A be a discrete valuation ring with fraction field K. For nonzero $x, y \in K$ we have*

$$d_A(x, y) = (-1)^{ord_A(x) ord_A(y)} \frac{x^{ord_A(y)}}{y^{ord_A(x)}} \bmod \mathfrak{m}_A,$$

in other words the symbol is equal to the usual tame symbol.

Proof. By multiplicativity it suffices to prove this when $x, y \in A$. Let $t \in A$ be a uniformizer. Write $x = t^b u$ and $y = t^b v$ for some $a, b \geq 0$ and $u, v \in A^*$. Set $l = a + b$. Then $t^{l-1}, \ldots, t^b$ is an admissible sequence in $(x)/(xy)$ and $t^{l-1}, \ldots, t^a$ is an admissible sequence in $(y)/(xy)$. Hence by Remark 29.3.5 we see that $d_A(x, y)$ is characterized by the equation

$$[t^{l-1}, \ldots, t^b, v^{-1} t^{b-1}, \ldots, v^{-1}] = (-1)^{ab} d_A(x, y) [t^{l-1}, \ldots, t^a, u^{-1} t^{a-1}, \ldots, u^{-1}].$$

Hence by the admissible relations for the symbols $[x_1, \ldots, x_l]$ we see that

$$d_A(x, y) = (-1)^{ab} u^a / v^b \bmod \mathfrak{m}_A$$

as desired. $\square$

We add the following lemma here. It is very similar to Algebra, Lemma 7.110.2.

Lemma 29.4.11. *Let R be a local Noetherian domain of dimension 1 with maximal ideal $\mathfrak{m}$. Let $a, b \in \mathfrak{m}$ be nonzero. There exists a finite ring extension $R \subset R'$ with same field of fractions, and $t, a', b' \in R'$ such that $a = ta'$ and $b = tb'$ and $R' = a'R' + b'R'$.*

Proof. Set $I = (a, b)$. The idea is to blow up R in I as in the proof of Algebra, Lemma 7.110.2. Instead of doing the algebraic argument we work geometrically. Let $X = \text{Proj}(\bigoplus I^d / I^{d+1})$. By Constructions, Lemma 22.21.2 this is an integral scheme. The morphism $X \to Spec(R)$ is projective by Morphisms, Lemma 24.41.10. By Algebra, Lemma 7.104.2 and the fact that X is quasi-compact we see that the fibre of $X \to Spec(R)$ over $\mathfrak{m}$ is finite. By Properties, Lemma 23.26.5 there exists an affine open $U \subset X$ containing this fibre. Hence $X = U$ because $X \to Spec(R)$ is closed. In other words X is affine, say $X = Spec(R')$. By Morphisms, Lemma 24.14.2 we see that $R \to R'$ is of finite type. Since $X \to Spec(R)$ is proper and affine it is integral (see Morphisms, Lemma 24.42.7). Hence $R \to R'$ is of finite type and integral, hence finite (Algebra, Lemma 7.32.5). By Divisors, Lemma 26.9.18 we see that IR' is a locally principal ideal. Since R' is semi-local we see that IR' is principal, see Algebra, Lemma 7.72.6, say $IR' = (t)$. Then we have $a = a't$ and $b = b't$ and everything is clear. $\square$

Lemma 29.4.12. *Let A be a Noetherian local ring. Let $a, b \in A$. Let M be a finite A-module of dimension 1 on which each of a, b, $b - a$ are nonzero divisors. Then*

$$d_M(a, b - a) d_M(b, b) = d_M(b, b - a) d_M(a, b)$$

in κ^*.

Proof. By Lemma 29.4.9 it suffices to show the relation when $M = A/\mathfrak{q}$ for some prime $\mathfrak{q} \subset A$ with $\dim(A/\mathfrak{q}) = 1$.

In case $M = A/\mathfrak{q}$ we may replace A by $A/\mathfrak{q}$ and a, b by their images in $A/\mathfrak{q}$. Hence we may assume $A = M$ and A a local Noetherian domain of dimension 1. The reason is that the residue field κ of A and $A/\mathfrak{q}$ are the same and that for any $A/\mathfrak{q}$-module M the determinant taken over A or over $A/\mathfrak{q}$ are canonically identified. See Lemma 29.2.7.

It suffices to show the relation when both a, b are in the maximal ideal. Namely, the case where one or both are units follows from Lemma 29.4.6.

Choose an extension $A \subset A'$ and factorizations $a = ta'$, $b = tb'$ as in Lemma 29.4.11. Note that also $b - a = t(b' - a')$ and that $A' = (a', b') = (a', b' - a') = (b' - a', b')$. Here and in the following we think of A' as an A-module and a, b, a', b', t as A-module endomorphisms of A'. We will use the notation $d^A_{A'}(a', b')$ and so on to indicate

$$d^A_{A'}(a', b') = \det_\kappa(A'/a'b'A', a', b')$$

which is defined by Lemma 29.4.1. The upper index ^{A} is used to distinguish this from the already defined symbol $d_{A'}(a', b')$ which is different (for example because it has values in the residue field of A' which may be different from κ). By Lemma 29.4.8 we see that $d_A(a, b) = d^A_{A'}(a, b)$, and similarly for the other combinations. Using this and multiplicativity we see that it suffices to prove

$$d^A_{A'}(a', b' - a')d^A_{A'}(b', b') = d^A_{A'}(b', b' - a')d^A_{A'}(a', b')$$

Now, since $(a', b') = A'$ and so on we have

$$\begin{array}{rcl} A'/(a'(b' - a')) & \cong & A'/(a') \oplus A'/(b' - a') \\ A'/(b'(b' - a')) & \cong & A'/(b') \oplus A'/(b' - a') \\ A'/(a'b') & \cong & A'/(a') \oplus A'/(b') \end{array}$$

Moreover, note that multiplication by $b' - a'$ on $A/(a')$ is equal to multiplication by b', and that multiplication by $b' - a'$ on $A/(b')$ is equal to multiplication by $-a'$. Using Lemmas 29.3.8 and 29.3.9 we conclude

$$\begin{array}{rcl} d^A_{A'}(a', b' - a') & = & \det_\kappa(b'|_{A'/(a')})^{-1} \det_\kappa(a'|_{A'/(b'-a')}) \\ d^A_{A'}(b', b' - a') & = & \det_\kappa(-a'|_{A'/(b')})^{-1} \det_\kappa(b'|_{A'/(b'-a')}) \\ d^A_{A'}(a', b') & = & \det_\kappa(b'|_{A'/(a')})^{-1} \det_\kappa(a'|_{A'/(b')}) \end{array}$$

Hence we conclude that

$$(-1)^{\text{length}_A(A'/(b'))}d^A_{A'}(a', b' - a') = d^A_{A'}(b', b' - a')d^A_{A'}(a', b')$$

the sign coming from the $-a'$ in the second equality above. On the other hand, by Lemma 29.3.7 we have $d^A_{A'}(b', b') = (-1)^{\text{length}_A(A'/(b'))}$, and the lemma is proved. □

The tame symbol is a Steinberg symbol.

Lemma 29.4.13. *Let A be a Noetherian local domain of dimension 1. Let $K = f.f.(A)$. For $x \in K \setminus \{0, 1\}$ we have*

$$d_A(x, 1 - x) = 1$$

Proof. Write $x = a/b$ with $a, b \in A$. The hypothesis implies, since $1 - x = (b - a)/b$, that also $b - a \neq 0$. Hence we compute

$$d_A(x, 1 - x) = d_A(a, b - a)d_A(a, b)^{-1}d_A(b, b - a)^{-1}d_A(b, b)$$

Thus we have to show that $d_A(a, b-a)d_A(b, b) = d_A(b, b-a)d_A(a, b)$. This is Lemma 29.4.12. □

29.5. Lengths and determinants

In this section we use the determinant to compare lattices. The key lemma is the following.

Lemma 29.5.1. *Let R be a noetherian local ring. Let $\mathfrak q \subset R$ be a prime with $\dim(R/\mathfrak q) = 1$. Let $\varphi : M \to N$ be a homomorphism of finite R-modules. Assume there exist $x_1, \ldots, x_l \in M$ and $y_1, \ldots, y_l \in M$ with the following properties*

(1) $M = \langle x_1, \ldots, x_l\rangle$,
(2) $\langle x_1, \ldots, x_i\rangle/\langle x_1, \ldots, x_{i-1}\rangle \cong R/\mathfrak q$ *for* $i = 1, \ldots, l$,
(3) $N = \langle y_1, \ldots, y_l\rangle$, *and*
(4) $\langle y_1, \ldots, y_i\rangle/\langle y_1, \ldots, y_{i-1}\rangle \cong R/\mathfrak q$ *for* $i = 1, \ldots, l$.

Then φ is injective if and only if $\varphi_{\mathfrak q}$ is an isomorphism, and in this case we have

$$length_R(Coker(\varphi)) = ord_{R/\mathfrak q}(f)$$

where $f \in \kappa(\mathfrak q)$ is the element such that

$$[\varphi(x_1), \ldots, \varphi(x_l)] = f[y_1, \ldots, y_l]$$

in $\det_{\kappa(\mathfrak q)}(N_{\mathfrak q})$.

Proof. First, note that the lemma holds in case $l = 1$. Namely, in this case x_1 is a basis of M over $R/\mathfrak q$ and y_1 is a basis of N over $R/\mathfrak q$ and we have $\varphi(x_1) = fy_1$ for some $f \in R$. Thus φ is injective if and only if $f \notin \mathfrak q$. Moreover, $\mathrm{Coker}(\varphi) = R/(f, \mathfrak q)$ and hence the lemma holds by definition of $\mathrm{ord}_{R/q}(f)$ (see Algebra, Definition 7.112.2).

In fact, suppose more generally that $\varphi(x_i) = f_i y_i$ for some $f_i \in R$, $f_i \notin \mathfrak q$. Then the induced maps

$$\langle x_1, \ldots, x_i\rangle/\langle x_1, \ldots, x_{i-1}\rangle \longrightarrow \langle y_1, \ldots, y_i\rangle/\langle y_1, \ldots, y_{i-1}\rangle$$

are all injective and have cokernels isomorphic to $R/(f_i, \mathfrak q)$. Hence we see that

$$\mathrm{length}_R(\mathrm{Coker}(\varphi)) = \sum \mathrm{ord}_{R/\mathfrak q}(f_i).$$

On the other hand it is clear that

$$[\varphi(x_1), \ldots, \varphi(x_l)] = f_1 \ldots f_l[y_1, \ldots, y_l]$$

in this case from the admissible relation (b) for symbols. Hence we see the result holds in this case also.

We prove the general case by induction on l. Assume $l > 1$. Let $i \in \{1, \ldots, l\}$ be minimal such that $\varphi(x_1) \in \langle y_1, \ldots, y_i\rangle$. We will argue by induction on i. If $i = 1$, then we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \langle x_1\rangle \arrow[r] \arrow[d] & \langle x_1, \ldots, x_l\rangle \arrow[r] \arrow[d] & \langle x_1, \ldots, x_l\rangle/\langle x_1\rangle \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \langle y_1\rangle \arrow[r] & \langle y_1, \ldots, y_l\rangle \arrow[r] & \langle y_1, \ldots, y_l\rangle/\langle y_1\rangle \arrow[r] & 0
\end{tikzcd}$$

and the lemma follows from the snake lemma and induction on l. Assume now that $i > 1$. Write $\varphi(x_1) = a_1y_1 + \ldots + a_{i-1}y_{i-1} + ay_i$ with $a_j, a \in R$ and $a \notin \mathfrak q$ (since otherwise i was not minimal). Set

$$x'_j = \begin{cases} x_j & \text{if} \quad j = 1 \\ ax_j & \text{if} \quad j \geq 2 \end{cases} \quad \text{and} \quad y'_j = \begin{cases} y_j & \text{if} \quad j < i \\ ay_j & \text{if} \quad j \geq i \end{cases}$$

Let $M' = \langle x'_1, \ldots, x'_l\rangle$ and $N' = \langle y'_1, \ldots, y'_l\rangle$. Since $\varphi(x'_1) = a_1 y'_1 + \ldots + a_{i-1}y'_{i-1} + y'_i$ by construction and since for $j > 1$ we have $\varphi(x'_j) = a\varphi(x_i) \in \langle y'_1, \ldots, y'_l\rangle$ we get a commutative diagram of R-modules and maps

$$\begin{tikzcd} M' \arrow[r, "\varphi'"'] \arrow[d] & N' \arrow[d] \\ M \arrow[r, "\varphi"] & N \end{tikzcd}$$

By the result of the second paragraph of the proof we know that $\text{length}_R(M/M') = (l - 1)\text{ord}_{R/\mathfrak q}(a)$ and similarly $\text{length}_R(M/M') = (l - i + 1)\text{ord}_{R/\mathfrak q}(a)$. By a diagram chase this implies that

$$\text{length}_R(\text{Coker}(\varphi')) = \text{length}_R(\text{Coker}(\varphi)) + i\ \text{ord}_{R/\mathfrak q}(a).$$

On the other hand, it is clear that writing

$$[\varphi(x_1), \ldots, \varphi(x_l)] = f[y_1, \ldots, y_l], \quad [\varphi'(x'_1), \ldots, \varphi(x'_l)] = f'[y'_1, \ldots, y'_l]$$

we have $f' = a^i f$. Hence it suffices to prove the lemma for the case that $\varphi(x_1) = a_1y_1 + \ldots a_{i-1}y_{i-1} + y_i$, i.e., in the case that $a = 1$. Next, recall that

$$[y_1, \ldots, y_l] = [y_1, \ldots, y_{i-1}, a_1y_1 + \ldots a_{i-1}y_{i-1} + y_i, y_{i+1}, \ldots, y_l]$$

by the admissible relations for symbols. The sequence $y_1, \ldots, y_{i-1}, a_1y_1 + \ldots + a_{i-1}y_{i-1} + y_i, y_{i+1}, \ldots, y_l$ satisfies the conditions (3), (4) of the lemma also. Hence, we may actually assume that $\varphi(x_1) = y_i$. In this case, note that we have $\mathfrak q x_1 = 0$ which implies also $\mathfrak q y_i = 0$. We have

$$[y_1, \ldots, y_l] = -[y_1, \ldots, y_{i-2}, y_i, y_{i-1}, y_{i+1}, \ldots, y_l]$$

by the third of the admissible relations defining $\det_{\kappa(\mathfrak q)}(N_{\mathfrak q})$. Hence we may replace $y_1, \ldots, y_l$ by the sequence $y'_1, \ldots, y'_l = y_1, \ldots, y_{i-2}, y_i, y_{i-1}, y_{i+1}, \ldots, y_l$ (which also satisfies conditions (3) and (4) of the lemma). Clearly this decreases the invariant i by 1 and we win by induction on i. □

To use the previous lemma we show that often sequences of elements with the required properties exist.

Lemma 29.5.2. *Let R be a local Noetherian ring. Let $\mathfrak q \subset R$ be a prime ideal. Let M be a finite R-module such that $\mathfrak q$ is one of the minimal primes of the support of M. Then there exist $x_1, \ldots, x_l \in M$ such that*

(1) *the support of $M/\langle x_1, \ldots, x_l\rangle$ does not contain $\mathfrak q$, and*
(2) *$\langle x_1, \ldots, x_i\rangle/\langle x_1, \ldots, x_{i-1}\rangle \cong R/\mathfrak q$ for $i = 1, \ldots, l$.*

Moreover, in this case $l = \text{length}_{R_{\mathfrak q}}(M_{\mathfrak q})$.

Proof. The condition that $\mathfrak q$ is a minimal prime in the support of M implies that $l = \text{length}_{R_{\mathfrak q}}(M_{\mathfrak q})$ is finite (see Algebra, Lemma 7.59.8). Hence we can find $y_1, \ldots, y_l \in M_{\mathfrak q}$ such that $\langle y_1, \ldots, y_i\rangle/\langle y_1, \ldots, y_{i-1}\rangle \cong \kappa(\mathfrak q)$ for $i = 1, \ldots, l$. We can find $f_i \in R$, $f_i \notin \mathfrak q$ such that f_iy_i is the image of some element $z_i \in M$. Moreover, as R is Noetherian we can write $\mathfrak q = (g_1, \ldots, g_t)$ for some $g_j \in R$. By assumption $g_jy_i \in \langle y_1, \ldots, y_{i-1}\rangle$ inside the module $M_{\mathfrak q}$. By our choice of z_i we can find some further elements $f_{ji} \in R$, $f_{ij} \notin \mathfrak q$ such that $f_{ij}g_jz_i \in \langle z_1, \ldots, z_{i-1}\rangle$ (equality in the module M). The lemma follows by taking

$$x_1 = f_{11}f_{12}\ldots f_{1t}z_1, \quad x_2 = f_{11}f_{12}\ldots f_{1t}f_{21}f_{22}\ldots f_{2t}z_2,$$

and so on. Namely, since all the elements f_i, f_{ij} are invertible in $R_\mathfrak{q}$ we still have that $R_\mathfrak{q}x_1 + \ldots + R_\mathfrak{q}x_i/R_\mathfrak{q}x_1 + \ldots + R_\mathfrak{q}x_{i-1} \cong \kappa(\mathfrak{q})$ for $i = 1, \ldots, l$. By construction, $\mathfrak{q}x_i \in \langle x_1, \ldots, x_{i-1}\rangle$. Thus $\langle x_1, \ldots, x_i\rangle/\langle x_1, \ldots, x_{i-1}\rangle$ is an R-module generated by one element, annihilated $\mathfrak{q}$ such that localizing at $\mathfrak{q}$ gives a q-dimensional vector space over $\kappa(\mathfrak{q})$. Hence it is isomorphic to $R/\mathfrak{q}$. □

Here is the main result of this section. We will see below the various different consequences of this proposition. The reader is encouraged to first prove the easier Lemma 29.5.4 his/herself.

Proposition 29.5.3. *Let R be a local Noetherian ring with residue field κ. Suppose that (M, φ, ψ) is a $(2,1)$-periodic complex over R. Assume*

(1) *M is a finite R-module,*
(2) *the cohomology modules of (M, φ, ψ) are of finite length, and*
(3) $\dim(Supp(M)) = 1$.

Let $\mathfrak{q}_i$, $i = 1, \ldots, t$ be the minimal primes of the support of M. Then we have[2]

$$-e_R(M, \varphi, \psi) = \sum\nolimits_{i=1,\ldots,t} ord_{R/\mathfrak{q}_i}\left(\det\nolimits_{\kappa(\mathfrak{q}_i)}(M_{\mathfrak{q}_i}, \varphi_{\mathfrak{q}_i}, \psi_{\mathfrak{q}_i})\right)$$

Proof. We first reduce to the case $t = 1$ in the following way. Note that $\text{Supp}(M) = \{\mathfrak{m}, \mathfrak{q}_1, \ldots, \mathfrak{q}_t\}$, where $\mathfrak{m} \subset R$ is the maximal ideal. Let M_i denote the image of $M \to M_{\mathfrak{q}_i}$, so $\text{Supp}(M_i) = \{\mathfrak{m}, \mathfrak{q}_i\}$. The map φ (resp. ψ) induces an R-module map $\varphi_i : M_i \to M_i$ (resp. $\psi_i : M_i \to M_i$). Thus we get a morphism of $(2,1)$-periodic complexes

$$(M, \varphi, \psi) \longrightarrow \bigoplus\nolimits_{i=1,\ldots,t}(M_i, \varphi_i, \psi_i).$$

The kernel and cokernel of this map have support equal to $\{\mathfrak{m}\}$ (or are zero). Hence by Lemma 29.3.3 these $(2,1)$-periodic complexes have multiplicity 0. In other words we have

$$e_R(M, \varphi, \psi) = \sum\nolimits_{i=1,\ldots,t} e_R(M_i, \varphi_i, \psi_i)$$

On the other hand we clearly have $M_{\mathfrak{q}_i} = M_{i,\mathfrak{q}_i}$, and hence the terms of the right hand side of the formula of the lemma are equal to the expressions

$$\text{ord}_{R/\mathfrak{q}_i}\left(\det\nolimits_{\kappa(\mathfrak{q}_i)}(M_{i,\mathfrak{q}_i}, \varphi_{i,\mathfrak{q}_i}, \psi_{i,\mathfrak{q}_i})\right)$$

In other words, if we can prove the lemma for each of the modules M_i, then the lemma holds. This reduces us to the case $t = 1$.

Assume we have a $(2,1)$-periodic complex (M, φ, ψ) over a Noetherian local ring with M a finite R-module, $\text{Supp}(M) = \{\mathfrak{m}, \mathfrak{q}\}$, and finite length cohomology modules. The proof in this case follows from Lemma 29.5.1 and careful bookkeeping. Denote $K_\varphi = \text{Ker}(\varphi)$, $I_\varphi = \text{Im}(\varphi)$, $K_\psi = \text{Ker}(\psi)$, and $I_\psi = \text{Im}(\psi)$. Since R is Noetherian these are all finite R-modules. Set

$$a = \text{length}_{R_\mathfrak{q}}(I_{\varphi,\mathfrak{q}}) = \text{length}_{R_\mathfrak{q}}(K_{\psi,\mathfrak{q}}), \quad b = \text{length}_{R_\mathfrak{q}}(I_{\psi,\mathfrak{q}}) = \text{length}_{R_\mathfrak{q}}(K_{\varphi,\mathfrak{q}}).$$

Equalities because the complex becomes exact after localizing at $\mathfrak{q}$. Note that $l = \text{length}_{R_\mathfrak{q}}(M_\mathfrak{q})$ is equal to $l = a + b$.

We are going to use Lemma 29.5.2 to choose sequences of elements in finite R-modules N with support contained in $\{\mathfrak{m}, \mathfrak{q}\}$. In this case $N_\mathfrak{q}$ has finite length, say $n \in \mathbf{N}$. Let us call a

[2] Obviously we could get rid of the minus sign by redefining $\det_\kappa(M, \varphi, \psi)$ as the inverse of its current value, see Definition 29.3.4.

sequence $w_1, \ldots, w_n \in N$ with properties (1) and (2) of Lemma 29.5.2 a ``good sequence". Note that the quotient $N/\langle w_1, \ldots, w_n\rangle$ of N by the submodule generated by a good sequence has support (contained in) $\{\mathfrak m\}$ and hence has finite length (Algebra, Lemma 7.59.8). Moreover, the symbol $[w_1, \ldots, w_n] \in \det_{\kappa(\mathfrak q)}(N_{\mathfrak q})$ is a generator, see Lemma 29.2.4.

Having said this we choose good sequences

$$\begin{matrix} x_1, \ldots, x_b & \text{in} & K_\varphi, & t_1, \ldots, t_a & \text{in} & K_\psi, \\ y_1, \ldots, y_a & \text{in} & I_\varphi \cap \langle t_1, \ldots t_a\rangle, & s_1, \ldots, s_b & \text{in} & I_\psi \cap \langle x_1, \ldots, x_b\rangle. \end{matrix}$$

We will adjust our choices a little bit as follows. Choose lifts $\tilde y_i \in M$ of $y_i \in I_\varphi$ and $\tilde s_i \in M$ of $s_i \in I_\psi$. It may not be the case that $\mathfrak q \tilde y_1 \subset \langle x_1, \ldots, x_b\rangle$ and it may not be the case that $\mathfrak q \tilde s_1 \subset \langle t_1, \ldots, t_a\rangle$. However, using that $\mathfrak q$ is finitely generated (as in the proof of Lemma 29.5.2) we can find a $d \in R$, $d \not\in \mathfrak q$ such that $\mathfrak q d\tilde y_1 \subset \langle x_1, \ldots, x_b\rangle$ and $\mathfrak q d\tilde s_1 \subset \langle t_1, \ldots, t_a\rangle$. Thus after replacing y_i by dy_i, $\tilde y_i$ by $d\tilde y_i$, s_i by ds_i and $\tilde s_i$ by $d\tilde s_i$ we see that we may assume also that $x_1, \ldots, x_b, \tilde y_1, \ldots, \tilde y_b$ and $t_1, \ldots, t_a, \tilde s_1, \ldots, \tilde s_b$ are good sequences in M.

Finally, we choose a good sequence $z_1, \ldots, z_l$ in the finite R-module

$$\langle x_1, \ldots, x_b, \tilde y_1, \ldots, \tilde y_a\rangle \cap \langle t_1, \ldots, t_a, \tilde s_1, \ldots, \tilde s_b\rangle.$$

Note that this is also a good sequence in M.

Since $I_{\varphi,\mathfrak q} = K_{\psi,\mathfrak q}$ there is a unique element $h \in \kappa(\mathfrak q)$ such that $[y_1, \ldots, y_a] = h[t_1, \ldots, t_a]$ inside $\det_{\kappa(\mathfrak q)}(K_{\psi,\mathfrak q})$. Similarly, as $I_{\psi,\mathfrak q} = K_{\varphi,\mathfrak q}$ there is a unique element $h \in \kappa(\mathfrak q)$ such that $[s_1, \ldots, s_b] = g[x_1, \ldots, x_b]$ inside $\det_{\kappa(\mathfrak q)}(K_{\varphi,\mathfrak q})$. We can also do this with the three good sequences we have in M. All in all we get the following identities

$$\begin{aligned} [y_1, \ldots, y_a] &= h[t_1, \ldots, t_a] \\ [s_1, \ldots, s_b] &= g[x_1, \ldots, x_b] \\ [z_1, \ldots, z_l] &= f_\varphi[x_1, \ldots, x_b, \tilde y_1, \ldots, \tilde y_a] \\ [z_1, \ldots, z_l] &= f_\psi[t_1, \ldots, t_a, \tilde s_1, \ldots, \tilde s_b] \end{aligned}$$

for some $g, h, f_\varphi, f_\psi \in \kappa(\mathfrak q)$.

Having set up all this notation let us compute $\det_{\kappa(\mathfrak q)}(M, \varphi, \psi)$. Namely, consider the element $[z_1, \ldots, z_l]$. Under the map $\gamma_\psi \circ \sigma \circ \gamma_\varphi^{-1}$ of Definition 29.3.4 we have

$$\begin{aligned} [z_1, \ldots, z_l] &= f_\varphi[x_1, \ldots, x_b, \tilde y_1, \ldots, \tilde y_a] \\ &\mapsto f_\varphi[x_1, \ldots, x_b] \otimes [y_1, \ldots, y_a] \\ &\mapsto f_\varphi h/g[t_1, \ldots, t_a] \otimes [s_1, \ldots, s_b] \\ &\mapsto f_\varphi h/g[t_1, \ldots, t_a, \tilde s_1, \ldots, \tilde s_b] \\ &= f_\varphi h/f_\psi g[z_1, \ldots, z_l] \end{aligned}$$

This means that $\det_{\kappa(\mathfrak q)}(M_{\mathfrak q}, \varphi_{\mathfrak q}, \psi_{\mathfrak q})$ is equal to $f_\varphi h/f_\psi g$ up to a sign.

We abbreviate the following quantities

$$\begin{aligned}
k_\varphi &= \text{length}_R(K_\varphi/\langle x_1, \ldots, x_b\rangle) \\
k_\psi &= \text{length}_R(K_\psi/\langle t_1, \ldots, t_a\rangle) \\
i_\varphi &= \text{length}_R(I_\varphi/\langle y_1, \ldots, y_a\rangle) \\
i_\psi &= \text{length}_R(I_\psi/\langle s_1, \ldots, s_a\rangle) \\
m_\varphi &= \text{length}_R(M/\langle x_1, \ldots, x_b, \tilde{y}_1, \ldots, \tilde{y}_a\rangle) \\
m_\psi &= \text{length}_R(M/\langle t_1, \ldots, t_a, \tilde{s}_1, \ldots, \tilde{s}_b\rangle) \\
\delta_\varphi &= \text{length}_R(\langle x_1, \ldots, x_b, \tilde{y}_1, \ldots, \tilde{y}_a\rangle\langle z_1, \ldots, z_l\rangle) \\
\delta_\psi &= \text{length}_R(\langle t_1, \ldots, t_a, \tilde{s}_1, \ldots, \tilde{s}_b\rangle\langle z_1, \ldots, z_l\rangle)
\end{aligned}$$

Using the exact sequences $0 \to K_\varphi \to M \to I_\varphi \to 0$ we get $m_\varphi = k_\varphi + i_\varphi$. Similarly we have $m_\psi = k_\psi + i_\psi$. We have $\delta_\varphi + m_\varphi = \delta_\psi + m_\psi$ since this is equal to the colength of $\langle z_1, \ldots, z_l\rangle$ in M. Finally, we have

$$\delta_\varphi = \text{ord}_{R/\mathfrak{q}}(f_\varphi), \quad \delta_\psi = \text{ord}_{R/\mathfrak{q}}(f_\psi)$$

by our first application of the key Lemma 29.5.1.

Next, let us compute the multiplicity of the periodic complex

$$\begin{aligned}
e_R(M, \varphi, \psi) &= \text{length}_R(K_\varphi/I_\psi) - \text{length}_R(K_\psi/I_\varphi) \\
&= \text{length}_R(\langle x_1, \ldots, x_b\rangle/\langle s_1, \ldots, s_b\rangle) + k_\varphi - i_\psi \\
&\quad -\text{length}_R(\langle t_1, \ldots, t_a\rangle/\langle y_1, \ldots, y_a\rangle) - k_\psi + i_\varphi \\
&= \text{ord}_{R/\mathfrak{q}}(g/h) + k_\varphi - i_\psi - k_\psi + i_\varphi \\
&= \text{ord}_{R/\mathfrak{q}}(g/h) + m_\varphi - m_\psi \\
&= \text{ord}_{R/\mathfrak{q}}(g/h) + \delta_\psi - \delta_\varphi \\
&= \text{ord}_{R/\mathfrak{q}}(f_\psi g/f_\varphi h)
\end{aligned}$$

where we used the key Lemma 29.5.1 twice in the third equality. By our computation of $\det_{\kappa(\mathfrak{q})}(M_\mathfrak{q}, \varphi_\mathfrak{q}, \psi_\mathfrak{q})$ this proves the proposition. □

In most applications the following lemma suffices.

Lemma 29.5.4. *Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Let M be a finite R-module, and let $\psi : M \to M$ be an R-module map. Assume that*

(1) *Ker(ψ) and Coker(ψ) have finite length, and*
(2) $\dim(Supp(M)) \leq 1$.

Write $Supp(M) = \{\mathfrak{m}, \mathfrak{q}_1, \ldots, \mathfrak{q}_t\}$ and denote $f_i \in \kappa(\mathfrak{q}_i)^*$ the element such that $\det_{\kappa(\mathfrak{q}_i)}(\psi_{\mathfrak{q}_i}) : \det_{\kappa(\mathfrak{q}_i)}(M_{\mathfrak{q}_i}) \to \det_{\kappa(\mathfrak{q}_i)}(M_{\mathfrak{q}_i})$ is multiplication by f_i. Then we have

$$length_R(Coker(\psi)) - length_R(Ker(\psi)) = \sum\nolimits_{i=1,\ldots,t} ord_{R/\mathfrak{q}_i}(f_i).$$

Proof. Recall that $H^0(M, 0, \psi) = \text{Coker}(\psi)$ and $H^1(M, 0, \psi) = \text{Ker}(\psi)$, see remarks above Definition 29.3.2. The lemma follows by combining Proposition 29.5.3 with Lemma 29.3.8.

Alternative proof. Reduce to the case $\text{Supp}(M) = \{\mathfrak{m}, \mathfrak{q}\}$ as in the proof of Proposition 29.5.3. Then directly combine Lemmas 29.5.1 and 29.5.2 to prove this specific case of Proposition 29.5.3. There is much less bookkeeping in this case, and the reader is encouraged to work this out. Details omitted. □

Lemma 29.5.5. *Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Let M be a finite R-module. Let $x \in R$. Assume that*

(1) $\dim(Supp(M)) \leq 1$, *and*
(2) $\dim(M/xM) \leq 0$.

Write $Supp(M) = \{\mathfrak{m}, \mathfrak{q}_1, \ldots, \mathfrak{q}_t\}$. Then

$$length_R(M_x) - length_R({}_xM) = \sum\nolimits_{i=1,\ldots,t} ord_{R/\mathfrak{q}_i}(x) length_{R_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i}).$$

where $M_x = M/xM$ and ${}_xM = Ker(x : M \to M)$.

Proof. This is a special case of Lemma 29.5.4. To see that $f_i = x^{\text{length}_{R_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i})}$ see Lemma 29.2.8. □

Lemma 29.5.6. *Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Let $I \subset R$ be an ideal and let $x \in R$. Assume x is a nonzero divisor on R/I and that* $\dim(R/I) = 1$. *Then*

$$length_R(R/(x, I)) = \sum\nolimits_i length_R(R/(x, \mathfrak{q}_i)) length_{R_{\mathfrak{q}_i}}((R/I)_{\mathfrak{q}_i})$$

where $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ are the minimal primes over I. More generally if M is any finite Cohen-Macaulay module of dimension 1 *over R and* $\dim(M/xM) = 0$, *then*

$$length_R(M/xM) = \sum\nolimits_i length_R(R/(x, \mathfrak{q}_i)) length_{R_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i}).$$

where $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ are the minimal primes of the support of M.

Proof. These are special cases of Lemma 29.5.5. □

Lemma 29.5.7. *Let A be a Noetherian local ring. Let M be a finite A-module. Let $a, b \in A$. Assume*

(1) $\dim(A) = 1$,
(2) *both a and b are nonzero divisors in A,*
(3) *A has no embedded primes,*
(4) *M has no embedded associated primes,*
(5) *$Supp(M) = Spec(A)$.*

Let $I = \{x \in A \mid x(a/b) \in A\}$. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ be the minimal primes of A. Then $(a/b)IM \subset M$ and

$$length_A(M/(a/b)IM) - length_A(M/IM) = \sum\nolimits_i length_{A_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i}) ord_{A/\mathfrak{q}_i}(a/b)$$

Proof. Since M has no embedded associated primes, and since the support of M is $Spec(A)$ we see that $\text{Ass}(M) = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_t\}$. Hence a, b are nonzero divisors on M. Note that

$$\begin{aligned}
&\text{length}_A(M/(a/b)IM) \\
&= \text{length}_A(bM/aIM) \\
&= \text{length}_A(M/aIM) - \text{length}_A(M/bM) \\
&= \text{length}_A(M/aM) + \text{length}_A(aM/aIM) - \text{length}_A(M/bM) \\
&= \text{length}_A(M/aM) + \text{length}_A(M/IM) - \text{length}_A(M/bM)
\end{aligned}$$

as the injective map $b : M \to bM$ maps $(a/b)IM$ to aIM and the injective map $a : M \to aM$ maps IM to aIM. Hence the left hand side of the equation of the lemma is equal to

$$\text{length}_A(M/aM) - \text{length}_A(M/bM).$$

Applying the second formula of Lemma 29.5.6 with $x = a, b$ respectively and using Algebra, Definition 7.112.2 of the ord-functions we get the result. □

29.6. Application to tame symbol

In this section we apply the results above to show the following lemma.

Lemma 29.6.1. *Let A be a 2-dimensional Noetherian local domain. Let $K = f.f.(A)$. Let $f, g \in K^*$. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ be the height 1 primes $\mathfrak{q}$ of A such that either f or g is not an element of $A_{\mathfrak{q}}^*$. Then we have*

$$\sum\nolimits_{i=1,\ldots,t} ord_{A/\mathfrak{q}_i}(d_{A_{\mathfrak{q}_i}}(f, g)) = 0$$

We can also write this as

$$\sum\nolimits_{height(\mathfrak{q})=1} ord_{A/\mathfrak{q}}(d_{A_{\mathfrak{q}}}(f, g)) = 0$$

since at any height one prime $\mathfrak{q}$ of A where $f, g \in A_{\mathfrak{q}}^*$ we have $d_{A_{\mathfrak{q}}}(f, g) = 1$ by Lemma 29.4.6.

Proof. Since the tame symbols $d_{A_{\mathfrak{q}}}(f, g)$ are additive (Lemma 29.4.4) and the order functions $\text{ord}_{A/\mathfrak{q}}$ are additive (Algebra, Lemma 7.112.1) it suffices to prove the formula when $f = a \in A$ and $g = b \in A$. In this case we see that we have to show

$$\sum\nolimits_{\text{height}(\mathfrak{q})=1} \text{ord}_{A/\mathfrak{q}}(\det\nolimits_\kappa(A_{\mathfrak{q}}/(ab), a, b)) = 0$$

By Proposition 29.5.3 this is equivalent to showing that

$$e_A(A/(ab), a, b) = 0.$$

Since the complex $A/(ab) \xrightarrow{a} A/(ab) \xrightarrow{b} A/(ab) \xrightarrow{a} A/(ab)$ is exact we win. $\square$

29.7. Setup

We will throughout work over a locally Noetherian universally catenary base S endowed with a dimension function δ. Allthough it is likely possible to generalize (parts of) the discussion in the chapter, it seems that this is a good first approximation. We usually do not assume our schemes are separated or quasi-compact. Many interesting algebraic stacks are non-separated and/or non-quasi-compact and this is a good case study to see how to develop a reasonable theory for those as well. In order to reference these hypotheses we give it a number.

Situation 29.7.1. Here S is a locally Noetherian, and universally catenary scheme. Moreover, we assume S is endowed with a dimension function $\delta : S \longrightarrow \mathbf{Z}$.

See Morphisms, Definition 24.16.1 for the notion of a universally catenary scheme, and see Topology, Definition 5.16.1 for the notion of a dimension function. Recall that any locally Noetherian catenary scheme locally has a dimension function, see Properties, Lemma 23.11.3. Moreover, there are lots of schemes which are universally catenary, see Morphisms, Lemma 24.16.4.

Let (S, δ) be as in Situation 29.7.1. Any scheme X locally of finite type over S is locally Noetherian and catenary. In fact, X has a canonical dimension function

$$\delta = \delta_{X/S} : X \longrightarrow \mathbf{Z}$$

associated to $(f : X \to S, \delta)$ given by the rule $\delta_{X/S}(x) = \delta(f(x)) + \text{trdeg}_{\kappa(f(x))}\kappa(x)$. See Morphisms, Lemma 24.29.2. Moreover, if $h : X \to Y$ is a morphism of schemes locally of finite type over S, and $x \in X$, $y = h(x)$, then obviously $\delta_{X/S}(x) = \delta_{Y/S}(y) + \text{trdeg}_{\kappa(y)}\kappa(x)$. We will freely use this function and its properties in the following.

Here are the basic examples of setups as above. In fact, the main interest lies in the case where the base is the spectrum of a field, or the case where the base is the spectrum of a Dedekind ring (e.g. $\mathbf{Z}$, or a discrete valuation ring).

Example 29.7.2. Here $S = Spec(k)$ and k is a field. We set $\delta(pt) = 0$ where pt indicates the unique point of S. The pair (S, δ) is an example of a situation as in Situation 29.7.1 by Morphisms, Lemma 24.16.4.

Example 29.7.3. Here $S = Spec(A)$, where A is a Noetherian domain of dimension 1. For example we could consider $A = \mathbf{Z}$. We set $\delta(\mathfrak{p}) = 0$ if $\mathfrak{p}$ is a maximal ideal and $\delta(\mathfrak{p}) = 1$ if $\mathfrak{p} = (0)$ corresponds to the generic point. This is an example of Situation 29.7.1 by Morphisms, Lemma 24.16.4.

In good cases δ corresponds to the dimension function.

Lemma 29.7.4. *Let (S, δ) be as in Situation 29.7.1. Assume in addition S is a Jacobson scheme, and $\delta(s) = 0$ for every closed point s of S. Let X be locally of finite type over S. Let $Z \subset X$ be an integral closed subscheme and let $\xi \in Z$ be its generic point. The following integers are the same:*

(1) $\delta_{X/S}(\xi)$,
(2) $\dim(Z)$, *and*
(3) $\dim(\mathcal{O}_{Z,z})$ *where z is a closed point of Z.*

Proof. Let $X \to S$, $\xi \in Z \subset X$ be as in the lemma. Since X is locally of finite type over S we see that X is Jacobson, see Morphisms, Lemma 24.15.9. Hence closed points of X are dense in every closed subset of Z and map to closed points of S. Hence given any chain of irreducible closed subsets of Z we can end it with a closed point of Z. It follows that $\dim(Z) = \sup_z(\dim(\mathcal{O}_{Z,z})$ (see Properties, Lemma 23.11.4) where $z \in Z$ runs over the closed points of Z. Note that $\dim(\mathcal{O}_{Z,z}) = \delta(\xi) - \delta(z))$ by the properties of a dimension function. For each closed $z \in Z$ the field extension $\kappa(z) \supset \kappa(f(z))$ is finite, see Morphisms, Lemma 24.15.8. Hence $\delta_{X/S}(z) = \delta(f(z)) = 0$ for $z \in Z$ closed. It follows that all three integers are equal. □

In the situation of the lemma above the value of δ at the generic point of a closed irreducible subset is the dimension of the irreducible closed subset. However, in general we cannot expect the equality to hold. For example if $S = Spec(\mathbf{C}[[t]])$ and $X = Spec(\mathbf{C}((t)))$ then we would get $\delta(x) = 1$ for the unique point of X, but $\dim(X) = 0$. Still we want to think of $\delta_{X/S}$ as giving the dimension of the irreducible closed subschemes. Thus we introduce the following terminology.

Definition 29.7.5. Let (S, δ) as in Situation 29.7.1. For any scheme X locally of finite type over S and any irreducible closed subset $Z \subset X$ we define

$$\dim_\delta(Z) = \delta(\xi)$$

where $\xi \in Z$ is the generic point of Z. We will call this the *δ-dimension of Z*. If Z is a closed subscheme of X, then we define $\dim_\delta(Z)$ as the supremum of the δ-dimensions of its irreducible components.

29.8. Cycles

Since we are not assuming our schemes are quasi-compact we have to be a little careful when defining cycles. We have to allow infinite sums because a rational function may have infinitely many poles for example. In any case, if X is quasi-compact then a cycle is a finite sum as usual.

Definition 29.8.1. Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Let $k \in \mathbf{Z}$.

(1) A collection of closed subschemes $\{Z_i\}_{i \in I}$ of X is said to be *locally finite* if for every quasi-compact open $U \subset X$ the set
$$\#\{i \in I \mid Z_i \cap U \neq \emptyset\}$$
is finite.

(2) A *cycle on* X is a formal sum
$$\alpha = \sum n_Z [Z]$$
where the sum is over integral closed subschemes $Z \subset X$, each $n_Z \in \mathbf{Z}$, and the collection $\{Z; n_Z \neq 0\}$ is locally finite.

(3) A *k-cycle*, on X is a cycle
$$\alpha = \sum n_Z [Z]$$
where $n_Z \neq 0 \Rightarrow \dim_\delta(Z) = k$.

(4) The abelian group of all k-cycles on X is denoted $Z_k(X)$.

In other words, a k-cycle on X is a locally finite formal $\mathbf{Z}$-linear combination of integral closed subschemes of δ-dimension k. Addition of k-cycles $\alpha = \sum n_Z[Z]$ and $\beta = \sum m_Z[Z]$ is given by
$$\alpha + \beta = \sum (n_Z + m_Z)[Z],$$
i.e., by adding the coefficients.

29.9. Cycle associated to a closed subscheme

Lemma 29.9.1. *Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Let $Z \subset X$ be a closed subscheme.*

(1) *The collection of irreducible components of Z is locally finite.*

(2) *Let $Z' \subset Z$ be an irreducible component and let $\xi \in Z'$ be its generic point. Then*
$$length_{\mathcal{O}_{X,\xi}} \mathcal{O}_{Z,\xi} < \infty$$

(3) *If $\dim_\delta(Z) \leq k$ and $\xi \in Z$ with $\delta(\xi) = k$, then ξ is a generic point of an irreducible component of Z.*

Proof. Let $U \subset X$ be a quasi-compact open subscheme. Then U is a Noetherian scheme, and hence has a Noetherian underlying topological space (Properties, Lemma 23.5.5). Hence every subspace is Noetherian and has finitely many irreducible components (see Topology, Lemma 5.6.2). This proves (1).

Let $Z' \subset Z$, $\xi \in Z'$ be as in (2). Then $\dim(\mathcal{O}_{Z,\xi}) = 0$ (for example by Properties, Lemma 23.11.4). Hence $\mathcal{O}_{Z,\xi}$ is Noetherian local ring of dimension zero, and hence has finite length over itself (see Algebra, Proposition 7.57.6). Hence, it also has finite length over $\mathcal{O}_{X,\xi}$, see Algebra, Lemma 7.48.12.

Assume $\xi \in Z$ and $\delta(\xi) = k$. Consider the closure $Z' = \overline{\{\xi\}}$. It is an irreducible closed subscheme with $\dim_\delta(Z') = k$ by definition. Since $\dim_\delta(Z) = k$ it must be an irreducible component of Z. Hence we see (3) holds. $\square$

Definition 29.9.2. Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Let $Z \subset X$ be a closed subscheme.

(1) For any irreducible component $Z' \subset Z$ with generic point ξ the integer $m_{Z',Z} = \text{length}_{\mathcal{O}_{X,\xi}} \mathcal{O}_{Z,\xi}$ (Lemma 29.9.1) is called the *multiplicity of Z' in Z*.
(2) Assume $\dim_\delta(Z) \leq k$. The *k-cycle associated to Z* is

$$[Z]_k = \sum m_{Z',Z}[Z']$$

where the sum is over the irreducible components of Z of δ-dimension k. (This is a k-cycle by Lemma 29.9.1.)

It is important to note that we only define $[Z]_k$ if the δ-dimension of Z does not exceed k. In other words, by convention, if we write $[Z]_k$ then this implies that $\dim_\delta(Z) \leq k$.

29.10. Cycle associated to a coherent sheaf

Lemma 29.10.1. *Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module.*

(1) *The collection of irreducible components of the support of $\mathcal{F}$ is locally finite.*
(2) *Let $Z' \subset Supp(\mathcal{F})$ be an irreducible component and let $\xi \in Z'$ be its generic point. Then*

$$length_{\mathcal{O}_{X,\xi}} \mathcal{F}_\xi < \infty$$

(3) *If $\dim_\delta(Supp(\mathcal{F})) \leq k$ and $\xi \in Z$ with $\delta(\xi) = k$, then ξ is a generic point of an irreducible component of $Supp(\mathcal{F})$.*

Proof. By Coherent, Lemma 25.11.7 the support Z of $\mathcal{F}$ is a closed subset of X. We may think of Z as a reduced closed subscheme of X (Schemes, Lemma 21.12.4). Hence (1) and (3) follow immediately by applying Lemma 29.9.1 to $Z \subset X$.

Let $\xi \in Z'$ be as in (2). In this case for any specialization $\xi' \rightsquigarrow \xi$ in X we have $\mathcal{F}_{\xi'} = 0$. Recall that the non-maximal primes of $\mathcal{O}_{X,\xi}$ correspond to the points of X specializing to ξ (Schemes, Lemma 21.13.2). Hence $\mathcal{F}_\xi$ is a finite $\mathcal{O}_{X,\xi}$-module whose support is $\{\mathfrak{m}_\xi\}$. Hence it has finite length by Algebra, Lemma 7.59.8. □

Definition 29.10.2. Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module.

(1) For any irreducible component $Z' \subset \text{Supp}(\mathcal{F})$ with generic point ξ the integer $m_{Z',\mathcal{F}} = \text{length}_{\mathcal{O}_{X,\xi}} \mathcal{F}_\xi$ (Lemma 29.10.1) is called the *multiplicity of Z' in $\mathcal{F}$*.
(2) Assume $\dim_\delta(\text{Supp}(\mathcal{F})) \leq k$. The *$k$-cycle associated to $\mathcal{F}$* is

$$[\mathcal{F}]_k = \sum m_{Z',\mathcal{F}}[Z']$$

where the sum is over the irreducible components of $\text{Supp}(\mathcal{F})$ of δ-dimension k. (This is a k-cycle by Lemma 29.10.1.)

It is important to note that we only define $[\mathcal{F}]_k$ if $\mathcal{F}$ is coherent and the δ-dimension of $\text{Supp}(\mathcal{F})$ does not exceed k. In other words, by convention, if we write $[\mathcal{F}]_k$ then this implies that $\mathcal{F}$ is coherent on X and $\dim_\delta(\text{Supp}(\mathcal{F})) \leq k$.

Lemma 29.10.3. *Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Let $Z \subset X$ be a closed subscheme. If $\dim_\delta(Z) \leq k$, then $[Z]_k = [\mathcal{O}_Z]_k$.*

Proof. This is because in this case the multiplicities $m_{Z',Z}$ and $m_{Z',\mathcal{O}_Z}$ agree by definition. □

Lemma 29.10.4. *Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be a short exact sequence of coherent sheaves on X. Assume that the δ-dimension of the supports of $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ is $\leq k$. Then $[\mathcal{G}]_k = [\mathcal{F}]_k + [\mathcal{H}]_k$.*

Proof. Follows immediately from additivity of lengths, see Algebra, Lemma 7.48.3. □

29.11. Preparation for proper pushforward

Lemma 29.11.1. *Let (S, δ) be as in Situation 29.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a morphism. Assume X, Y integral and $\dim_\delta(X) = \dim_\delta(Y)$. Then either $f(X)$ is contained in a proper closed subscheme of Y, or f is dominant and the extension of function fields $R(Y) \subset R(X)$ is finite.*

Proof. The closure $\overline{f(X)} \subset Y$ is irreducible as X is irreducible. If $\overline{f(X)} \neq Y$, then we are done. If $\overline{f(X)} = Y$, then f is dominant and by Morphisms, Lemma 24.6.5 we see that the generic point η_Y of Y is in the image of f. Of course this implies that $f(\eta_X) = \eta_Y$, where $\eta_X \in X$ is the generic point of X. Since $\delta(\eta_X) = \delta(\eta_Y)$ we see that $R(Y) = \kappa(\eta_Y) \subset \kappa(\eta_X) = R(X)$ is an extension of transcendence degree 0. Hence Morphisms, Lemma 24.45.4 applies. □

Lemma 29.11.2. *Let (S, δ) be as in Situation 29.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a morphism. Assume f is quasi-compact, and $\{Z_i\}_{i \in I}$ is a locally finite collection of closed subsets of X. Then $\{\overline{f(Z_i)}\}_{i \in I}$ is a locally finite collection of closed subsets of Y.*

Proof. Let $V \subset Y$ be a quasi-compact open subset. Since f is quasi-compact the open $f^{-1}(V)$ is quasi-compact. Hence the set $\{i \in I \mid Z_i \cap f^{-1}(V) \neq \emptyset\}$ is finite by assumption. Since this is the same as the set $\{i \in I \mid f(Z_i) \cap V \neq \emptyset\}$ we win. □

29.12. Proper pushforward

Definition 29.12.1. Let (S, δ) be as in Situation 29.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a morphism. Assume f is proper.

(1) Let $Z \subset X$ be an integral closed subscheme with $\dim_\delta(Z) = k$. We define
$$f_*[Z] = \begin{cases} 0 & \text{if } \dim_\delta(f(Z)) < k, \\ \deg(Z/f(Z))[f(Z)] & \text{if } \dim_\delta(f(Z)) = k. \end{cases}$$
Here we think of $f(Z) \subset Y$ as an integral closed subscheme. The degree of Z over $f(Z)$ is finite if $\dim_\delta(f(Z)) = \dim_\delta(Z)$ by Lemma 29.11.1.

(2) Let $\alpha = \sum n_Z[Z]$ be a k-cycle on X. The *pushforward* of α as the sum
$$f_*\alpha = \sum n_Z f_*[Z]$$
where each $f_*[Z]$ is defined as above. The sum is locally finite by Lemma 29.11.2 above.

By definition the proper pushforward of cycles
$$f_* : Z_k(X) \longrightarrow Z_k(Y)$$
is a homomorphism of abelian groups. It turns $X \mapsto Z_k(X)$ into a covariant functor on the category of schemes locally of finite type over S with morphisms equal to proper morphisms.

Lemma 29.12.2. *Let (S, δ) be as in Situation 29.7.1. Let X, Y, and Z be locally of finite type over S. Let $f : X \to Y$ and $g : Y \to Z$ be proper morphisms. Then $g_* \circ f_* = (g \circ f)_*$ as maps $Z_k(X) \to Z_k(Z)$.*

Proof. Let $W \subset X$ be an integral closed subscheme of dimension k. Consider $W' = f(Z) \subset Y$ and $W'' = g(f(Z)) \subset Z$. Since f, g are proper we see that W' (resp. W'') is an integral closed subscheme of Y (resp. Z). We have to show that $g_*(f_*[W]) = (f \circ g)_*[W]$. If $\dim_\delta(W'') < k$, then both sides are zero. If $\dim_\delta(W'') = k$, then we see the induced morphisms

$$W \longrightarrow W' \longrightarrow W''$$

both satisfy the hypotheses of Lemma 29.11.1. Hence

$$g_*(f_*[W]) = \deg(W/W')\deg(W'/W'')[W''], \quad (f \circ g)_*[W] = \deg(W/W'')[W''].$$

Then we can apply Morphisms, Lemma 24.45.6 to conclude. □

Lemma 29.12.3. *Let (S, δ) be as in Situation 29.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a morphism. Assume f is proper.*

1. *Let $Z \subset X$ be a closed subscheme with $\dim_\delta(Z) \leq k$. Then*
$$f_*[Z]_k = [f_*\mathcal{O}_Z]_k.$$
2. *Let $\mathcal{F}$ be a coherent sheaf on X such that $\dim_\delta(Supp(\mathcal{F})) \leq k$. Then*
$$f_*[\mathcal{F}]_k = [f_*\mathcal{F}]_k.$$

Note that the statement makes sense since $f_*\mathcal{F}$ and $f_*\mathcal{O}_Z$ are coherent $\mathcal{O}_Y$-modules by Coherent, Lemma 25.18.2.

Proof. Part (1) follows from (2) and Lemma 29.10.3. Let $\mathcal{F}$ be a coherent sheaf on X. Assume that $\dim_\delta(\mathrm{Supp}(\mathcal{F})) \leq k$. By Coherent, Lemma 25.11.7 there exists a closed subscheme $i : Z \to X$ and a coherent $\mathcal{O}_Z$-module $\mathcal{G}$ such that $i_*\mathcal{G} \cong \mathcal{F}$ and such that the support of $\mathcal{F}$ is Z. Let $Z' \subset Y$ be the scheme theoretic image of $f|_Z : Z \to Y$.Consider the commutative diagram of schemes

$$\begin{tikzcd} Z \arrow[r, "i"'] \arrow[d, "f|_Z"'] & X \arrow[d, "f"] \\ Z' \arrow[r, "i'"] & Y \end{tikzcd}$$

We have $f_*\mathcal{F} = f_*i_*\mathcal{G} = i'_*(f|_Z)_*\mathcal{G}$ by going around the diagram in two ways. Suppose we know the result holds for closed immersions and for $f|_Z$. Then we see that

$$f_*[\mathcal{F}]_k = f_*i_*[\mathcal{G}]_k = (i')_*(f|_Z)_*[\mathcal{G}]_k = (i')_*[(f|_Z)_*\mathcal{G}]_k = [(i')_*(f|_Z)_*\mathcal{G}]_k = [f_*\mathcal{F}]_k$$

as desired. The case of a closed immersion is straightforward (omitted). Note that $f|_Z : Z \to Z'$ is a dominant morphism (see Morphisms, Lemma 24.4.3). Thus we have reduced to the case where $\dim_\delta(X) \leq k$ and $f : X \to Y$ is proper and dominant.

Assume $\dim_\delta(X) \leq k$ and $f : X \to Y$ is proper and dominant. Since f is dominant, for every irreducible component $Z \subset Y$ with generic point η there exists a point $\xi \in X$ such that $f(\xi) = \eta$. Hence $\delta(\eta) \leq \delta(\xi) \leq k$. Thus we see that in the expressions

$$f_*[\mathcal{F}]_k = \sum n_Z[Z], \quad \text{and} \quad [f_*\mathcal{F}]_k = \sum m_Z[Z].$$

whenever $n_Z \neq 0$, or $m_Z \neq 0$ the integral closed subscheme Z is actually an irreducible component of Y of δ-dimension k. Pick such an integral closed subscheme $Z \subset Y$ and denote η its generic point. Note that for any $\xi \in X$ with $f(\xi) = \eta$ we have $\delta(\xi) \geq k$ and hence ξ is a generic point of an irreducible component of X of δ-dimension k as well (see Lemma 29.9.1). Since f is quasi-compact and X is locally Noetherian, there can be only finitely many of these and hence $f^{-1}(\{\eta\})$ is finite. By Morphisms, Lemma 24.45.1 there exists an open neighbourhood $\eta \in V \subset Y$ such that $f^{-1}(V) \to V$ is finite. Replacing Y by V and X by $f^{-1}(V)$ we reduce to the case where Y is affine, and f is finite.

Write $Y = Spec(R)$ and $X = Spec(A)$ (possible as a finite morphism is affine). Then R and A are Noetherian rings and A is finite over R. Moreover $\mathcal{F} = \widetilde{M}$ for some finite A-module M. Note that $f_*\mathcal{F}$ corresponds to M viewed as an R-module. Let $\mathfrak{p} \subset R$ be the minimal prime corresponding to $\eta \in Y$. The coefficient of Z in $[f_*\mathcal{F}]_k$ is clearly $\text{length}_{R_\mathfrak{p}}(M_\mathfrak{p})$. Let $\mathfrak{q}_i$, $i = 1, \ldots, t$ be the primes of A lying over $\mathfrak{p}$. Then $A_\mathfrak{p} = \prod A_{\mathfrak{q}_i}$ since $A_\mathfrak{p}$ is an Artinian ring being finite over the dimension zero local Noetherian ring $R_\mathfrak{p}$. Clearly the coefficient of Z in $f_*[\mathcal{F}]_k$ is

$$\sum\nolimits_{i=1,\ldots,t}[\kappa(\mathfrak{q}_i) : \kappa(\mathfrak{p})]\text{length}_{A_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i})$$

Hence the desired equality follows from Algebra, Lemma 7.48.12. □

29.13. Preparation for flat pullback

Recall that a morphism $f : X \to Y$ which is locally of finite type is said to have relative dimension r if every nonempty fibre is equidimensional of dimension r. See Morphisms, Definition 24.28.1.

Lemma 29.13.1. *Let (S, δ) be as in Situation 29.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a morphism. Assume f is flat of relative dimension r. For any closed subset $Z \subset Y$ we have*

$$\dim_\delta(f^{-1}(Z)) = \dim_\delta(Z) + r.$$

If Z is irreducible and $Z' \subset f^{-1}(Z)$ is an irreducible component, then Z' dominates Z and $\dim_\delta(Z') = \dim_\delta(Z) + r$.

Proof. It suffices to prove the final statement. We may replace Y by the integral closed subscheme Z and X by the scheme theoretic inverse image $f^{-1}(Z) = Z \times_Y X$. Hence we may assume $Z = Y$ is integral and f is a flat morphism of relative dimension r. Since Y is locally Noetherian the morphism f which is locally of finite type, is actually locally of finite presentation. Hence Morphisms, Lemma 24.24.9 applies and we see that f is open. Let $\xi \in X$ be a generic point of an irreducible component of X. By the openness of f we see that $f(\xi)$ is the generic point η of $Z = Y$. Note that $\dim_\xi(X_\eta) = r$ by assumption that f has relative dimension r. On the other hand, since ξ is a generic point of X we see that $\mathcal{O}_{X,\xi} = \mathcal{O}_{X_\eta,\xi}$ has only one prime ideal and hence has dimension 0. Thus by Morphisms, Lemma 24.27.1 we conclude that the transcendence degree of $\kappa(\xi)$ over $\kappa(\eta)$ is r. In other words, $\delta(\xi) = \delta(\eta) + r$ as desired. □

Here is the lemma that we will use to prove that the flat pullback of a locally finite collection of closed subschemes is locally finite.

Lemma 29.13.2. *Let (S, δ) be as in Situation 29.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a morphism. Assume $\{Z_i\}_{i \in I}$ is a locally finite collection of closed subsets of Y. Then $\{f^{-1}(Z_i)\}_{i \in I}$ is a locally finite collection of closed subsets of Y.*

Proof. Let $U \subset X$ be a quasi-compact open subset. Since the image $f(U) \subset Y$ is a quasi-compact subset there exists a quasi-compact open $V \subset Y$ such that $f(U) \subset V$. Note that

$$\{i \in I \mid f^{-1}(Z_i) \cap U \neq \emptyset\} \subset \{i \in I \mid Z_i \cap V \neq \emptyset\}.$$

Since the right hand side is finite by assumption we win. □

29.14. Flat pullback

In the following we use $f^{-1}(Z)$ to denote the *scheme theoretic inverse image* of a closed subscheme $Z \subset Y$ for a morphism of schemes $f : X \to Y$. We recall that the scheme theoretic inverse image is the fibre product

$$\begin{array}{ccc} f^{-1}(Z) & \longrightarrow & X \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Y \end{array}$$

and it is also the closed subscheme of X cut out by the quasi-coherent sheaf of ideals $f^{-1}(\mathcal{I})\mathcal{O}_X$, if $\mathcal{I} \subset \mathcal{O}_Y$ is the quasi-coherent sheaf of ideals corresponding to Z in Y. (This is discussed in Schemes, Section 21.4 and Lemma 21.17.6 and Definition 21.17.7.)

Definition 29.14.1. Let (S, δ) be as in Situation 29.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a morphism. Assume f is flat of relative dimension r.

(1) Let $Z \subset Y$ be an integral closed subscheme of δ-dimension k. We define $f^*[Z]$ to be the $(k + r)$-cycle on X to the scheme theoretic inverse image

$$f^*[Z] = [f^{-1}(Z)]_{k+r}.$$

This makes sense since $\dim_\delta(f^{-1}(Z)) = k + r$ by Lemma 29.13.1.

(2) Let $\alpha = \sum n_i[Z_i]$ be a k-cycle on Y. The *flat pullback of α by f* is the sum

$$f^*\alpha = \sum n_i f^*[Z_i]$$

where each $f^*[Z_i]$ is defined as above. The sum is locally finite by Lemma 29.13.2.

(3) We denote $f^* : Z_k(Y) \to Z_{k+r}(Y)$ the map of abelian groups so obtained.

An open immersion is flat. This is an important though trivial special case of a flat morphism. If $U \subset X$ is open then sometimes the pullback by $j : U \to X$ of a cycle is called the *restriction* of the cycle to U. Note that in this case the maps

$$j^* : Z_k(X) \longrightarrow Z_k(U)$$

are all *surjective*. The reason is that given any integral closed subscheme $Z' \subset U$, we can take the closure of Z of Z' in X and think of it as a reduced closed subscheme of X (see Schemes, Lemma 21.12.4). And clearly $Z \cap U = Z'$, in other words $j^*[Z] = [Z']$ whence the surjectivity. In fact a little bit more is true.

Lemma 29.14.2. *Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Let $U \subset X$ be an open subscheme, and denote $i : Y = X \setminus U \to X$ as a reduced closed subscheme of X. For every $k \in \mathbf{Z}$ the sequence*

$$Z_k(Y) \xrightarrow{i_*} Z_k(X) \xrightarrow{j_*} Z_k(U) \longrightarrow 0$$

is an exact complex of abelian groups.

Proof. By the description above the basis elements $[Z]$ of the free abelian group $Z_k(X)$ map either to (distinct) basis elements $[Z \cap U]$ or to zero if $Z \subset Y$. Hence the lemma is clear. $\square$

Lemma 29.14.3. *Let (S, δ) be as in Situation 29.7.1. Let X, Y, Z be locally of finite type over S. Let $f : X \to Y$ and $g : Y \to Z$ be flat morphisms of relative dimensions r and s. Then $g \circ f$ is flat of relative dimension $r + s$ and*

$$f^* \circ g^* = (g \circ f)^*$$

as maps $Z_k(Z) \to Z_{k+r+s}(X)$.

Proof. The composition is flat of relative dimension $r + s$ by Morphisms, Lemma 24.28.3. Suppose that

(1) $W \subset Z$ is a closed integral subscheme of δ-dimension k,
(2) $W' \subset Y$ is a closed integral subscheme of δ-dimension $k + s$ with $W' \subset g^{-1}(W)$, and
(3) $W'' \subset Y$ is a closed integral subscheme of δ-dimension $k + s + r$ with $W'' \subset f^{-1}(W')$.

We have to show that the coefficient n of $[W'']$ in $(g \circ f)^*[W]$ agrees with the coefficient m of $[W'']$ in $f^*(g^*[W])$. That it suffices to check the lemma in these cases follows from Lemma 29.13.1. Let $\xi'' \in W''$, $\xi' \in W'$ and $\xi \in W$ be the generic points. Consider the local rings $A = \mathcal{O}_{Z,\xi}$, $B = \mathcal{O}_{Y,\xi'}$ and $C = \mathcal{O}_{X,\xi''}$. Then we have local flat ring maps $A \to B$, $B \to C$ and moreover

$$n = \text{length}_C(C/\mathfrak{m}_A C), \quad \text{and} \quad m = \text{length}_C(C/\mathfrak{m}_B C)\text{length}_B(B/\mathfrak{m}_A B)$$

Hence the equality follows from Algebra, Lemma 7.48.14. □

Lemma 29.14.4. *Let (S, δ) be as in Situation 29.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a flat morphism of relative dimension r.*

(1) *Let $Z \subset Y$ be a closed subscheme with $\dim_\delta(Z) \leq k$. Then we have $\dim_\delta(f^{-1}(Z)) \leq k + r$ and $[f^{-1}(Z)]_{k+r} = f^*[Z]_k$ in $Z_{k+r}(X)$.*
(2) *Let $\mathcal{F}$ be a coherent sheaf on Y with $\dim_\delta(Supp(\mathcal{F})) \leq k$. Then we have $\dim_\delta(Supp(f^*\mathcal{F})) \leq k + r$ and*

$$f^*[\mathcal{F}]_k = [f^*\mathcal{F}]_{k+r}$$

in $Z_{k+r}(X)$.

Proof. Part (1) follows from part (2) by Lemma 29.10.3 and the fact that $f^*\mathcal{O}_Z = \mathcal{O}_{f^{-1}(Z)}$.

Proof of (2). As X, Y are locally Noetherian we may apply Coherent, Lemma 25.11.1 to see that $\mathcal{F}$ is of finite type, hence $f^*\mathcal{F}$ is of finite type (Modules, Lemma 15.9.2), hence $f^*\mathcal{F}$ is coherent (Coherent, Lemma 25.11.1 again). Thus the lemma makes sense. Let $W \subset Y$ be an integral closed subscheme of δ-dimension k, and let $W' \subset X$ be an integral closed subscheme of dimension $k + r$ mapping into W under f. We have to show that the coefficient n of $[W]$ in $f^*[\mathcal{F}]_k$ agrees with the coefficient m of $[W]$ in $[f^*\mathcal{F}]_{k+r}$. Let $\xi \in W$ and $\xi' \in W'$ be the generic points. Let $A = \mathcal{O}_{Y,\xi}$, $B = \mathcal{O}_{X,\xi'}$ and set $M = \mathcal{F}_\xi$ as an A-module. (Note that M has finite length by our dimension assumptions, but we actually do not need to verify this. See Lemma 29.10.1.) We have $f^*\mathcal{F}_{\xi'} = B \otimes_A M$. Thus we see that

$$n = \text{length}_B(B \otimes_A M) \quad \text{and} \quad m = \text{length}_A(M)\text{length}_B(B/\mathfrak{m}_A B)$$

Thus the equality follows from Algebra, Lemma 7.48.13. □

29.15. Push and pull

In this section we verify that proper pushforward and flat pullback are compatible when this makes sense. By the work we did above this is a consequence of cohomology and base change.

Lemma 29.15.1. *Let (S, δ) be as in Situation 29.7.1. Let*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

be a fibre product diagram of schemes locally of finite type over S. Assume $f : X \to Y$ proper and $g : Y' \to Y$ flat of relative dimension r. Then also f' is proper and g' is flat of relative dimension r. For any k-cycle α on X we have

$$g^* f_* \alpha = f'_*(g')^* \alpha$$

in $Z_{k+r}(Y')$.

Proof. The assertion that f' is proper follows from Morphisms, Lemma 24.40.5. The assertion that g' is flat of relative dimension r follows from Morphisms, Lemmas 24.28.2 and 24.24.7. It suffices to prove the equality of cycles when $\alpha = [W]$ for some integral closed subscheme $W \subset X$ of δ-dimension k. Note that in this case we have $\alpha = [\mathcal{O}_W]_k$, see Lemma 29.10.3. By Lemmas 29.12.3 and 29.14.4 it therefore suffices to show that $f'_*(g')^*\mathcal{O}_W$ is isomorphic to $g^* f_* \mathcal{O}_W$. This follows from cohomology and base change, see Coherent, Lemma 25.6.2. □

Lemma 29.15.2. *Let (S, δ) be as in Situation 29.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a finite locally free morphism of degree d (see Morphisms, Definition 24.44.1). Then f is both proper and flat of relative dimension 0, and*

$$f_* f^* \alpha = d\alpha$$

for every $\alpha \in Z_k(Y)$.

Proof. A finite locally free morphism is flat and finite by Morphisms, Lemma 24.44.2, and a finite morphism is proper by Morphisms, Lemma 24.42.10. We omit showing that a finite morphism has relative dimension 0. Thus the formula makes sense. To prove it, let $Z \subset Y$ be an integral closed subscheme of δ-dimension k. It suffices to prove the formula for $\alpha = [Z]$. Since the base change of a finite locally free morphism is finite locally free (Morphisms, Lemma 24.44.4) we see that $f_* f^* \mathcal{O}_Z$ is a finite locally free sheaf of rank d on Z. Hence

$$f_* f^*[Z] = f_* f^*[\mathcal{O}_Z]_k = [f_* f^* \mathcal{O}_Z]_k = d[Z]$$

where we have used Lemmas 29.14.4 and 29.12.3. □

29.16. Preparation for principal divisors

Recall that if Z is an irreducible closed subset of a scheme X, then the codimension of Z in X is equal to the dimension of the local ring $\mathcal{O}_{X,\xi}$, where $\xi \in Z$ is the generic point. See Properties, Lemma 23.11.4.

Definition 29.16.1. Let X be a locally Noetherian scheme. Assume X is integral. Let $f \in R(X)^*$. For every integral closed subscheme $Z \subset X$ of codimension 1 we define the *order of vanishing of f along Z* as the integer

$$\text{ord}_Z(f) = \text{ord}_{\mathcal{O}_{X,\xi}}(f)$$

where the right hand side is the notion of Algebra, Definition 7.112.2 and ξ is the generic point of Z.

Of course it can happen that $\text{ord}_Z(f) < 0$. In this case we say that f has a *pole* along Z and that $-\text{ord}_Z(f) > 0$ is the *order of pole of f along Z*. Note that for $f, g \in R(X)^*$ we have

$$\text{ord}_Z(fg) = \text{ord}_Z(f) + \text{ord}_Z(g).$$

Lemma 29.16.2. *Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Assume X is integral. If $Z \subset X$ is an integral closed subscheme of codimension 1, then $\dim_\delta(Z) = \dim_\delta(X) - 1$.*

Proof. This is more or less the defining property of a dimension function. □

Lemma 29.16.3. *Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Assume X is integral. Let $f \in R(X)^*$. Then the set*

$$\{Z \subset X \mid Z \textit{ is integral, closed of codimension } 1 \textit{ and } \textit{ord}_Z(f) \neq 0\}$$

is locally finite in X.

Proof. This is true simply because there exists a nonempty open subscheme $U \subset X$ such that f corresponds to a section of $\Gamma(U, \mathcal{O}_X^*)$, and hence the codimension 1 irreducibles which can occur in the set of the lemma are all irreducible components of $X \setminus U$. Hence Lemma 29.9.1 gives the desired result. □

Lemma 29.16.4. *Let $f : X \to Y$ be a morphism of schemes. Let $\xi \in Y$ be a point. Assume that*

1. *X, Y are integral,*
2. *X is locally Noetherian*
3. *f is proper, dominant and $R(X) \subset R(Y)$ is finite, and*
4. *$\dim(\mathcal{O}_{Y,\xi}) = 1$.*

Then there exists an open neighbourhood $V \subset Y$ of ξ such that $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is finite.

Proof. By Coherent, Lemma 25.20.2 it suffices to prove that $f^{-1}(\{\xi\})$ is finite. We replace Y by an affine open, say $Y = Spec(R)$. Note that R is Noetherian, as X is assumed locally Noetherian. Since f is proper it is quasi-compact. Hence we can find a finite affine open covering $X = U_1 \cup \ldots \cup U_n$ with each $U_i = Spec(A_i)$. Note that $R \to A_i$ is a finite type injective homomorphism of domains with $f.f.(R) \subset f.f.(A_i)$ finite. Thus the lemma follows from Algebra, Lemma 7.104.2. □

29.17. Principal divisors

The following definition is the key to everything that follows.

Definition 29.17.1. Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Assume X is integral with $\dim_\delta(X) = n$. Let $f \in R(X)^*$. The *principal divisor associated to f* is the $(n-1)$-cycle

$$\text{div}(f) = \text{div}_X(f) = \sum \text{ord}_Z(f)[Z]$$

where the sum is over integral closed subschemes of codimension 1 and $\text{ord}_Z(f)$ is as in Definition 29.16.1. This makes sense by Lemmas 29.16.2 and 29.16.3.

Lemma 29.17.2. *Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Assume X is integral with $\dim_\delta(X) = n$. Let $f, g \in R(X)^*$. Then*

$$div(fg) = div(f) + div(g)$$

in $Z_{n-1}(X)$.

Proof. This is clear from the additivity of the ord functions. □

An important role in the discussion of principal divisors is played by the ``universal" principal divisor $[0] - [\infty]$ on $\mathbf{P}^1_S$. To make this more precise, let us denote

$$D_0, D_\infty \subset \mathbf{P}^1_S = \underline{\text{Proj}}_S(\mathcal{O}_S[X_0, X_1])$$

the closed subscheme cut out by the section X_1, resp. X_0 of $\mathcal{O}(1)$. These are effective Cartier divisors, see Divisors, Definition 26.9.1 and Lemma 26.9.17. The following lemma says that loosely speaking we have ``$\text{div}(X_1/X_0) = [D_0] - [D_1]$" and that this is the universal principal divisor.

Lemma 29.17.3. *Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Assume X is integral and $n = \dim_\delta(X)$. Let $f \in R(X)^*$. Let $U \subset X$ be a nonempty open such that f corresponds to a section $f \in \Gamma(U, \mathcal{O}_X^*)$. Let $Y \subset X \times_S \mathbf{P}^1_S$ be the closure of the graph of $f : U \to \mathbf{P}^1_S$. Then*

1. *the projection morphism $p : Y \to X$ is proper,*
2. *$p|_{p^{-1}(U)} : p^{-1}(U) \to U$ is an isomorphism,*
3. *the pullbacks $q^{-1}D_0$ and $q^{-1}D_\infty$ via the morphism $q : Y \to \mathbf{P}^1_S$ are effective Cartier divisors on Y,*
4. *we have*
$$div_Y(f) = [q^{-1}D_0]_{n-1} - [q^{-1}D_\infty]_{n-1}$$
5. *we have*
$$div_X(f) = p_* div_Y(f)$$
6. *if we view $Y_0 = q^{-1}D_0$, and $Y_\infty = q^{-1}D_\infty$ as closed subschemes of X via the morphism p then we have*
$$div_X(f) = [Y_0]_{n-1} - [Y_\infty]_{n-1}$$

Proof. Since X is integral, we see that U is integral. Hence Y is integral, and $(1, f)(U) \subset Y$ is an open dense subscheme. Also, note that the closed subscheme $Y \subset X \times_S \mathbf{P}^1_S$ does not depend on the choice of the open U, since after all it is the closure of the one point set $\{\eta'\} = \{(1, f)(\eta)\}$ where $\eta \in X$ is the generic point. Having said this let us prove the assertions of the lemma.

For (1) note that p is the composition of the closed immersion $Y \to X \times_S \mathbf{P}^1_S = \mathbf{P}^1_X$ with the proper morphism $\mathbf{P}^1_X \to X$. As a composition of proper morphisms is proper (Morphisms, Lemma 24.40.4) we conclude.

It is clear that $Y \cap U \times_S \mathbf{P}^1_S = (1, f)(U)$. Thus (2) follows. It also follows that $\dim_\delta(Y) = n$.

Note that $q(\eta') = f(\eta)$ is not contained in D_0 or D_∞ since $f \in R(X)^*$. Hence $q^{-1}D_0$ and $q^{-1}D_\infty$ are effective Cartier divisors on Y by Divisors, Lemma 26.9.9. Thus we see (3). It also follows that $\dim_\delta(q^{-1}D_0) = n - 1$ and $\dim_\delta(q^{-1}D_\infty) = n - 1$.

Consider the effective Cartier divisor $q^{-1}D_0$. At every point $\xi \in q^{-1}D_0$ we have $f \in \mathcal{O}_{Y,\xi}$ and the local equation for $q^{-1}D_0$ is given by f. In particular, if $\delta(\xi) = n-1$ so ξ is the generic point of a integral closed subscheme Z of δ-dimension $n-1$, then we see that the coefficient of $[Z]$ in $\text{div}_Y(f)$ is

$$\text{ord}_Z(f) = \text{length}_{\mathcal{O}_{Y,\xi}}(\mathcal{O}_{Y,\xi}/f\mathcal{O}_{Y,\xi}) = \text{length}_{\mathcal{O}_{Y,\xi}}(\mathcal{O}_{q^{-1}D_0,\xi})$$

which is the coefficient of $[Z]$ in $[q^{-1}D_0]_{n-1}$. A similar argument using the rational function $1/f$ shows that $-[q^{-1}D_\infty]$ agrees with the terms with negative coefficients in the expression for $\text{div}_Y(f)$. Hence (4) follows.

Note that $D_0 \to S$ is an isomorphism. Hence we see that $X \times_S D_0 \to X$ is an isomorphism as well. Clearly we have $q^{-1}D_0 = Y \cap X \times_S D_0$ (scheme theoretic intersection) inside $X \times_S \mathbf{P}^1_S$. Hence it is really the case that $Y_0 \to X$ is a closed immersion. By the same token we see that

$$p_*\mathcal{O}_{q^{-1}D_0} = \mathcal{O}_{Y_0}$$

and hence by Lemma 29.12.3 we have $p_*[q^{-1}D_0]_{n-1} = [Y_0]_{n-1}$. Of course the same is true for D_∞ and Y_∞. Hence to finish the proof of the lemma it suffices to prove the last assertion.

Let $Z \subset X$ be an integral closed subscheme of δ-dimension $n-1$. We want to show that the coefficient of $[Z]$ in $\text{div}(f)$ is the same as the coefficient of $[Z]$ in $[Y_0]_{n-1} - [Y_\infty]_{n-1}$. We may apply Lemma 29.16.4 to the morphism $p : Y \to X$ and the generic point $\xi \in Z$. Hence we may replace X by an affine open neighbourhood of ξ and assume that $p : Y \to X$ is finite. Write $X = Spec(R)$ and $Y = Spec(A)$ with p induced by a finite homomorphism $R \to A$ of Noetherian domains which induces an isomorphism $f.f.(R) \cong f.f.(A)$ of fraction fields. Now we have $f \in f.f.(R)$ and a prime $\mathfrak{p} \subset R$ with $\dim(R_\mathfrak{p}) = 1$. The coefficient of $[Z]$ in $\text{div}_X(f)$ is $\text{ord}_{R_\mathfrak{p}}(f)$. The coefficient of $[Z]$ in $p_*\text{div}_Y(f)$ is

$$\sum\nolimits_{\mathfrak{q} \text{ lying over } \mathfrak{p}} [\kappa(\mathfrak{q}) : \kappa(\mathfrak{p})]\text{ord}_{A_\mathfrak{q}}(f)$$

The desired equality therefore follows from Algebra, Lemma 7.112.8. □

This lemma will be superseded by the more general Lemma 29.20.1.

Lemma 29.17.4. *Let (S, δ) be as in Situation 29.7.1. Let X, Y be locally of finite type over S. Assume X, Y are integral and $n = \dim_\delta(Y)$. Let $f : X \to Y$ be a flat morphism of relative dimension r. Let $g \in R(Y)^*$. Then*

$$f^*(div_Y(g)) = div_X(g)$$

in $Z_{n+r-1}(X)$.

Proof. Note that since f is flat it is dominant so that f induces an embedding $R(Y) \subset R(X)$, and hence we may think of g as an element of $R(X)^*$. Let $Z \subset X$ be an integral closed subscheme of δ-dimension $n+r-1$. Let $\xi \in Z$ be its generic point. If $\dim_\delta(f(Z)) > n-1$, then we see that the coefficient of $[Z]$ in the left and right hand side of the equation is zero. Hence we may assume that $Z' = \overline{f(Z)}$ is an intral closed subscheme of Y of δ-dimension $n-1$. Let $\xi' = f(\xi)$. It is the generic point of Z'. Set $A = \mathcal{O}_{Y,\xi'}$, $B = \mathcal{O}_{X,\xi}$. The ring map $A \to B$ is a flat local homomorphism of Noetherian local domains of dimension 1. We have $g \in f.f.(A)$. What we have to show is that

$$\text{ord}_A(g)\text{length}_B(B/\mathfrak{m}_A B) = \text{ord}_B(g).$$

This follows from Algebra, Lemma 7.48.13 (details omitted). □

29.18. Two fun results on principal divisors

The first lemma implies that the pushforward of a principal divisor along a generically finite morphism is a principal divisor.

Lemma 29.18.1. *Let (S, δ) be as in Situation 29.7.1. Let X, Y be locally of finite type over S. Assume X, Y are integral and $n = \dim_\delta(X) = \dim_\delta(Y)$. Let $p : X \to Y$ be a dominant proper morphism. Let $f \in R(X)^*$. Set*

$$g = Nm_{R(X)/R(Y)}(f).$$

Then we have $p_*div(f) = div(g)$.

Proof. Let $Z \subset Y$ be an integral closed subscheme of δ-dimension $n-1$. We want to show that the coefficient of $[Z]$ in $p_*\text{div}(f)$ and $\text{div}(g)$ are equal. We may apply Lemma 29.16.4 to the morphism $p : X \to X$ and the generic point $\xi \in Z$. Hence we may replace X by an affine open neighbourhood of ξ and assume that $p : Y \to X$ is finite. Write $X = Spec(R)$ and $Y = Spec(A)$ with p induced by a finite homomorphism $R \to A$ of Noetherian domains which induces an finite field extension $f.f.(R) \subset f.f.(A)$ of fraction fields. Now we have $f \in f.f.(A)$, $g = \text{Nm}(f) \in f.f.(R)$, and a prime $\mathfrak{p} \subset R$ with $\dim(R_\mathfrak{p}) = 1$. The coefficient of $[Z]$ in $\text{div}_Y(g)$ is $\text{ord}_{R_\mathfrak{p}}(g)$. The coefficient of $[Z]$ in $p_*\text{div}_X(f)$ is

$$\sum\nolimits_{\mathfrak{q} \text{ lying over } \mathfrak{p}} [\kappa(\mathfrak{q}) : \kappa(\mathfrak{p})]\text{ord}_{A_\mathfrak{q}}(f)$$

The desired equality therefore follows from Algebra, Lemma 7.112.8. □

The following lemma says that the degree of a principal divisor on a proper curve is zero.

Lemma 29.18.2. *Let K be any field. Let X be a 1-dimensional integral scheme endowed with a proper morphism $c : X \to Spec(K)$. Let $f \in K(X)^*$ be an invertible rational function. Then*

$$\sum\nolimits_{x \in X \ closed} [\kappa(x) : K] ord_{\mathcal{O}_{X,x}}(f) = 0$$

where ord is as in Algebra, Definition 7.112.2. In other words, $c_*div(f) = 0$.

Proof. Consider the diagram

$$\begin{tikzcd} Y \arrow[r, "p"] \arrow[d, "q"] & X \arrow[d, "c"] \\ \mathbf{P}^1_K \arrow[r, "c'"] & Spec(K) \end{tikzcd}$$

that we constructed in Lemma 29.17.3 starting with X and the rational function f over $S = Spec(K)$. We will use all the results of this lemma without further mention. We have to show that $c_*\text{div}_X(f) = p_*c_*\text{div}_Y(f) = 0$. This is the same as proving that $c'_*q_*\text{div}_Y(f) = 0$. If $q(Y)$ is a closed point of $\mathbf{P}^1_K$ then we see that $\text{div}_X(f) = 0$ and the lemma holds. Thus we may assume that q is dominant. Since $\text{div}_Y(f) = [q^{-1}D_0]_0 - [q^{-1}D_\infty]_0$ we see (by definition of flat pullback) that $\text{div}_Y(f) = q^*([D_0]_0 - [D_\infty]_0)$. Suppose we can show that $q : Y \to \mathbf{P}^1_K$ is finite locally free of degree d (see Morphisms, Definition 24.44.1). Then byy Lemma 29.15.2 we get $q_*\text{div}_Y(f) = d([D_0]_0 - [D_\infty]_0)$. Since clearly $c'_*[D_0]_0 = c'_*[D_\infty]_0$ we win.

It remains to show that q is finite locally free. (It will automatically have some given degree as $\mathbf{P}^1_K$ is connected.) Since $\dim(\mathbf{P}^1_K) = 1$ we see that q is finite for example by Lemma 29.16.4. All local rings of $\mathbf{P}^1_K$ at closed points are regular local rings of dimension 1 (in other words discrete valuation rings), since they are localizations of $K[T]$ (see Algebra, Lemma 7.105.1). Hence for $y \in Y$ closed the local ring $\mathcal{O}_{Y,y}$ will be flat over $\mathcal{O}_{\mathbf{P}^1_K, q(y)}$ as soon as it is torsion free. This is obviously the case as $\mathcal{O}_{Y,y}$ is a domain and q is dominant. Thus q is flat. Hence q is finite locally free by Morphisms, Lemma 24.44.2. □

29.19. Rational equivalence

In this section we define *rational equivalence* on k-cycles. We will allow locally finite sums of images of principal divisors (under closed immersions). This leads to some pretty strange phenomena, see Example 29.19.3. However, if we do not allow these then we do not know how to prove that capping with chern classes of line bundles factors through rational equivalence.

Definition 29.19.1. Let (S, δ) be as in Situation 29.7.1. Let X be a scheme locally of finite type over S. Let $k \in \mathbf{Z}$.

(1) Given any locally finite collection $\{W_j \subset X\}$ of integral closed subschemes with $\dim_\delta(W_j) = k + 1$, and any $f_j \in R(W_j)^*$ we may consider
$$\sum (i_j)_*\text{div}(f_j) \in Z_k(X)$$
where $i_j : W_j \to X$ is the inclusion morphism. This makes sense as the morphism $\coprod i_j : \coprod W_j \to X$ is proper.
(2) We say that $\alpha \in Z_k(X)$ is *rationally equivalent to zero* if α is a cycle of the form displayed above.
(3) We say $\alpha, \beta \in Z_k(X)$ are *rationally equivalent* and we write $\alpha \sim_{rat} \beta$ if $\alpha - \beta$ is rationally equivalent to zero.
(4) We define
$$A_k(X) = Z_k(X)/ \sim_{rat}$$
to be the *Chow group of k-cycles on X*. This is sometimes called the *Chow group of k-cycles module rational equivalence on X*.

There are many other interesting (adequate) equivalence relations. Rational equivalence is the coarsest one of them all. A very simple but important lemma is the following.

Lemma 29.19.2. *Let (S, δ) be as in Situation 29.7.1. Let X be a scheme locally of finite type over S. Let $U \subset X$ be an open subscheme, and denote $i : Y = X \setminus U \to X$ as a reduced closed subscheme of X. Let $k \in \mathbf{Z}$. Suppose $\alpha, \beta \in Z_k(X)$. If $\alpha|_U \sim_{rat} \beta|_U$ then there exist a cycle $\gamma \in Z_k(Y)$ such that*
$$\alpha \sim_{rat} \beta + i_*\gamma.$$
In other words, the sequence
$$A_k(Y) \xrightarrow{i_*} A_k(X) \xrightarrow{j_*} A_k(U) \longrightarrow 0$$
is an exact complex of abelian groups.

Proof. Let $\{W_j\}_{j\in J}$ be a locally finite collection of integral closed subschemes of δ-dimension $k + 1$, and let $f_j \in R(W_j)^*$ be elements such that $(\alpha - \beta)|_U = \sum (i_j)_*\text{div}(f_j)$ as in the definition. Set $W_j' \subset X$ equal to the closure of W_j. Suppose that $V \subset X$ is a quasi-compact open. Then also $V \cap U$ is quasi-compact open in U as V is Noetherian. Hence the set $\{j \in J \mid W_j \cap V \neq \emptyset\} = \{j \in J \mid W_j' \cap V \neq \emptyset\}$ is finite since $\{W_j\}$ is locally finite. In other words we see that $\{W_j'\}$ is also locally finite. Since $R(W_j) = R(W_j')$ we see that
$$\alpha - \beta - \sum (i_j')_*\text{div}(f_j)$$

is a cycle supported on Y and the lemma follows (see Lemma 29.14.2). □

Example 29.19.3. Here is a ``strange" example. Suppose that S is the spectrum of a field k with δ as in Example 29.7.2. Suppose that $X = C_1 \cup C_2 \cup \ldots$ is an infinite union of curves $C_j \cong \mathbf{P}^1_k$ glued together in the following way: The point $\infty \in C_j$ is glued transversally to the point $0 \in C_{j+1}$ for $j = 1, 2, 3, \ldots$. Take the point $0 \in C_1$. This gives a zero cycle $[0] \in Z_0(X)$. The ``strangeness" in this situation is that actually $[0] \sim_{rat} 0$! Namely we can choose the rational function $f_j \in R(C_j)$ to be the function which has a simple zero at 0 and a simple pole at ∞ and no other zeros or poles. Then we see that the sum $\sum (i_j)_*\text{div}(f_j)$ is exactly the 0-cycle [0]. In fact it turns out that $A_0(X) = 0$ in this example. If you find this too bizarre, then you can just make sure your spaces are always quasi-compact (so X does not even exist for you).

Remark 29.19.4. Let (S, δ) be as in Situation 29.7.1. Let X be a scheme locally of finite type over S. Suppose we have infinite collections $\alpha_i, \beta_i \in Z_k(X)$, $i \in I$ of k-cycles on X. Suppose that the supports of α_i and β_i form locally finite collections of closed subsets of X so that $\sum \alpha_i$ and $\sum \beta_i$ are defined as cycles. Moreover, assume that $\alpha_i \sim_{rat} \beta_i$ for each i. Then it is not clear that $\sum \alpha_i \sim_{rat} \sum \beta_i$. Namely, the problem is that the rational equivalences may be given by locally finite families $\{W_{i,j}, f_{i,j} \in R(W_{i,j})^*\}_{j \in J_i}$ but the union $\{W_{i,j}\}_{i \in I, j \in J_i}$ may not be locally finite.

In many cases in practice, one has a locally finite family of closed subsets $\{T_i\}_{i \in I}$ such that α_i, β_i are supported on T_i and such that $\alpha_i = \beta_i$ in $A_k(T_i)$, in other words, the families $\{W_{i,j}, f_{i,j} \in R(W_{i,j})^*\}_{j \in J_i}$ consist of subschemes $W_{i,j} \subset T_i$. In this case it is true that $\sum \alpha_i \sim_{rat} \sum \beta_i$ on X, simply because the family $\{W_{i,j}\}_{i \in I, j \in J_i}$ is automatically locally finite in this case.

29.20. Properties of rational equivalence

Lemma 29.20.1. *Let (S, δ) be as in Situation 29.7.1. Let X, Y be schemes locally of finite type over S. Let $f : X \to Y$ be a flat morphism of relative dimension r. Let $\alpha \sim_{rat} \beta$ be rationally equivalent k-cycles on Y. Then $f^*\alpha \sim_{rat} f^*\beta$ as $(k + r)$-cycles on X.*

Proof. What do we have to show? Well, suppose we are given a collection

$$i_j : W_j \longrightarrow Y$$

of closed immersions, with each W_j integral of δ-dimension $k + 1$ and rational functions $f_j \in R(W_j)^*$. Moreover, assume that the collection $\{i_j(W_j)\}_{j \in J}$ is locally finite on Y. Then we have to show that

$$f^*(\sum i_{j,*}\text{div}(f_j))$$

is rationally equivalent to zero on X.

Consider the fibre products

$$i'_j : W'_j = W_j \times_Y X \longrightarrow X.$$

For each j, consider the collection $\{W'_{j,l}\}_{l \in L_j}$ of irreducible components $W'_{j,l} \subset W'_j$ having δ-dimension $k + 1$. We may write

$$[W'_j]_{k+1} = \sum\nolimits_{l \in L_j} n_{j,l}[W'_{j,l}]_{k+1}$$

for some $n_{j,l} > 0$. By Lemma 29.13.1 we see that $W'_{j,l} \to W_j$ is dominant and hence we can let $f_{j,l} \in R(W'_{j,l})^*$ denote the image of f_j under the map of fields $R(W_j) \to R(W'_{j,l})$. We claim that

(1) the collection $\{W'_{j,l}\}_{j\in J, l\in L_j}$ is locally finite on X, and
(2) with obvious notation $f^*(\sum i_{j,*}\mathrm{div}(f_j)) = \sum i'_{j,l,*}\mathrm{div}(f_{j,l}^{n_{j,l}})$.

Clearly this claim implies the lemma.

To show (1), note that $\{W'_j\}$ is a locally finite collection of closed subschemes of X by Lemma 29.13.2. Hence if $U \subset X$ is quasi-compact, then U meets only finitely many W'_j. By Lemma 29.9.1 the collection of irreducible components of each W_j is locally finite as well. Hence we see only finitely many $W'_{j,l}$ meet U as desired.

Let $Z \subset X$ be an integral closed subscheme of δ-dimension $k + r$. We have to show that the coefficient n of $[Z]$ in $f^*(\sum i_{j,*}\mathrm{div}(f_j))$ is equal to the coefficient m of $[Z]$ in $\sum i'_{j,l,*}\mathrm{div}(f_{j,l}^{n_{j,l}})$. Let Z' be the closure of $f(Z)$ which is an integral closed subscheme of Y. By Lemma 29.13.1 we have $\dim_\delta(Z') \geq k$. If $\dim_\delta(Z') > k$, then the coefficients n and m are both zero, since the generic point of Z will not be contained in any W'_j or $W'_{j,l}$. Hence we may assume that $\dim_\delta(Z') = k$.

We are going to translate the equality of n and m into algebra. Namely, let $\xi' \in Z'$ and $\xi \in Z$ be the generic points. Set $A = \mathcal{O}_{Y,\xi'}$ and $B = \mathcal{O}_{X,\xi}$. Note that A, B are Noetherian, $A \to B$ is flat, local, and that $\mathfrak{m}_A B$ is an ideal of definition of the local ring B. There are finitely many j such that W_j passes through ξ', and these correspond to prime ideals

$$\mathfrak{p}_1, \ldots, \mathfrak{p}_T \subset A$$

with the property that $\dim(A/\mathfrak{p}_t) = 1$ for each $t = 1, \ldots, T$. The rational functions f_j correspond to elements $f_t \in \kappa(\mathfrak{p}_t)^*$. Say $\mathfrak{p}_t$ corresponds to W_j. By construction, the closed subschemes $W'_{j,l}$ which meet ξ correspond $1-1$ with minimal primes

$$\mathfrak{p}_t B \subset \mathfrak{q}_{t,1}, \ldots, \mathfrak{q}_{t,S_t} \subset B$$

over $\mathfrak{p}_t B$. The integers $n_{j,l}$ correspond to the integers

$$n_{t,s} = \mathrm{length}_{B_{\mathfrak{q}_{t,s}}}((B/\mathfrak{p}_t B)_{B_{\mathfrak{q}_{t,s}}})$$

The rational functions $f_{j,l}$ correspond to the images $f_{t,s} \in \kappa(\mathfrak{q}_{t,s})^*$ of the elements $f_t \in \kappa(\mathfrak{p}_t)^*$. Putting everything together we see that

$$n = \sum \mathrm{ord}_{A/\mathfrak{p}_t}(f_t)\mathrm{length}_B(B/\mathfrak{m}_A B)$$

and that

$$m = \sum \mathrm{ord}_{B/\mathfrak{q}_{t,s}}(f_{t,s})\mathrm{length}_{B_{\mathfrak{q}_{t,s}}}((B/\mathfrak{p}_t B)_{B_{\mathfrak{q}_{t,s}}})$$

Note that it suffices to prove the equality for each $t \in \{1, \ldots, T\}$ separately. Writing $f_t = x/y$ for some nonzero $\overline{x}, \overline{y} \in A/\mathfrak{p}_t$ coming from $x, y \in A$ we see that it suffices to prove

$$\mathrm{length}_{A/\mathfrak{p}_t}(A/(\mathfrak{p}_t, x))\mathrm{length}_B(B/\mathfrak{m}_A B) = \mathrm{length}_B(B/(x, \mathfrak{p}_t)B)$$

(equality uses Algebra, Lemma 7.48.13) equals

$$\sum\nolimits_{s=1,\ldots,S_t} \mathrm{ord}_{B/\mathfrak{q}_{t,s}}(B/(x, \mathfrak{q}_{t,s}))\mathrm{length}_{B_{\mathfrak{q}_{t,s}}}((B/\mathfrak{p}_t B)_{B_{\mathfrak{q}_{t,s}}})$$

and similarly for y. Note that as $x \notin \mathfrak{p}_t$ we see that x is a nonzero divisor on $A/\mathfrak{p}_t$. As $A \to B$ is flat it follows that x is a nonzero divisor on the module $M = B/\mathfrak{p}_t B$. Hence the equality above follows from Lemma 29.5.6. ☐

Lemma 29.20.2. *Let (S, δ) be as in Situation 29.7.1. Let X, Y be schemes locally of finite type over S. Let $p : X \to Y$ be a proper morphism. Suppose $\alpha, \beta \in Z_k(X)$ are rationally equivalent. Then $p_*\alpha$ is rationally equivalent to $p_*\beta$.*

Proof. What do we have to show? Well, suppose we are given a collection

$$i_j : W_j \longrightarrow X$$

of closed immersions, with each W_j integral of δ-dimension $k+1$ and rational functions $f_j \in R(W_j)^*$. Moreover, assume that the collection $\{i_j(W_j)\}_{j \in J}$ is locally finite on X. Then we have to show that

$$p_*\left(\sum i_{j,*}\mathrm{div}(f_j)\right)$$

is rationally equivalent to zero on X.

Note that the sum is equal to

$$\sum p_* i_{j,*}\mathrm{div}(f_j).$$

Let $W'_j \subset Y$ be the integral closed subscheme which is the image of $p \circ i_j$. The collection $\{W'_j\}$ is locally finite in Y by Lemma 29.11.2. Hence it suffices to show, for a given j, that either $p_* i_{j,*}\mathrm{div}(f_j) = 0$ or that it is equal to $i'_{j,*}\mathrm{div}(g_j)$ for some $g_j \in R(W'_j)^*$.

The arguments above therefore reduce us to the case of a since integral closed subscheme $W \subset X$ of δ-dimension $k+1$. Let $f \in R(W)^*$. Let $W' = p(W)$ as above. We get a commutative diagram of morphisms

$$\begin{tikzcd} W \arrow[r, "i"'] \arrow[d, "p'"'] & X \arrow[d, "p"] \\ W' \arrow[r, "i'"] & Y \end{tikzcd}$$

Note that $p_* i_* \mathrm{div}(f) = i'_*(p')_*\mathrm{div}(f)$ by Lemma 29.12.2. As explained above we have to show that $(p')_*\mathrm{div}(f)$ is the divisor of a rational function on W' or zero. There are three cases to distinguish.

The case $\dim_\delta(W') < k$. In this case automatically $(p')_*\mathrm{div}(f) = 0$ and there is nothing to prove.

The case $\dim_\delta(W') = k$. Let us show that $(p')_*\mathrm{div}(f) = 0$ in this case. Let $\eta \in W'$ be the generic point. Note that $c : W_\eta \to Spec(K)$ is a proper integral curve over $K = \kappa(\eta)$ whose function field $K(W_\eta)$ is identified with $R(W)$. Here is a diagram

$$\begin{tikzcd} W_\eta \arrow[r] \arrow[d, "c"'] & W \arrow[d, "p"] \\ Spec(K) \arrow[r] & W' \end{tikzcd}$$

Let us denote $f_\eta \in K(W_\eta)^*$ the rational function corresponding to $f \in R(W)^*$. Moreover, the closed points ξ of W_η correspond $1-1$ to the closed integral subschemes $Z = Z_\xi \subset W$ of δ-dimension k with $f(Z) = W'$. Note that the multiplicity of Z_ξ in $\mathrm{div}(f)$ is equal to $\mathrm{ord}_{\mathcal{O}_{W_\eta,\xi}}(f_\eta)$ simply because the local rings $\mathcal{O}_{W_\eta,\xi}$ and $\mathcal{O}_{W,\xi}$ are identified (as subrings of their fraction fields). Hence we see that the multiplicity of $[W']$ in $(p')_*\mathrm{div}(f)$ is equal to the multiplicity of $[Spec(K)]$ in $c_*\mathrm{div}(f_\eta)$. By Lemma 29.18.2 this is zero.

The case $\dim_\delta(W') = k+1$. In this case Lemma 29.18.1 applies, and we see that indeed $p'_*\mathrm{div}(f) = \mathrm{div}(g)$ for some $g \in R(W')^*$ as desired. $\square$

29.21. Different characterizations of rational equivalence

Let (S, δ) be as in Situation 29.7.1. Let X be a scheme locally of finite type over S. Given any closed subscheme $Z \subset X \times_S \mathbf{P}^1_S = X \times \mathbf{P}^1$ we let Z_0, resp. Z_∞ be the scheme theoretic closed subscheme $Z_0 = \mathrm{pr}_2^{-1}(D_0)$, resp. $Z_\infty = \mathrm{pr}_2^{-1}(D_\infty)$. Here D_0, D_∞ are as defined just above Lemma 29.17.3.

Lemma 29.21.1. *Let (S, δ) be as in Situation 29.7.1. Let X be a scheme locally of finite type over S. Let $W \subset X \times_S \mathbf{P}^1_S$ be an integral closed subscheme of δ-dimension $k + 1$. Assume $W \neq W_0$, and $W \neq W_\infty$. Then*

(1) *W_0, W_∞ are effective Cartier divisors of W,*
(2) *W_0, W_∞ can be viewed as closed subschemes of X and*
$$[W_0]_k \sim_{rat} [W_\infty]_k,$$
(3) *for any locally finite family of integral closed subschemes $W_i \subset X \times_S \mathbf{P}^1_S$ of δ-dimension $k+1$ with $W_i \neq (W_i)_0$ and $W_i \neq (W_i)_\infty$ we have $\sum([(W_i)_0]_k - [(W_i)_\infty]_k) \sim_{rat} 0$ on X, and*
(4) *for any $\alpha \in Z_k(X)$ with $\alpha \sim_{rat} 0$ there exists a locally finite family of integral closed subschemes $W_i \subset X \times_S \mathbf{P}^1_S$ as above such that $\alpha = \sum([(W_i)_0]_k - [(W_i)_\infty]_k)$.*

Proof. Part (1) follows from Divisors, Lemma 26.9.9 since the generic point of W is not mapped into D_0 or D_∞ under the projection $X \times_S \mathbf{P}^1_S \to \mathbf{P}^1_S$ by assumtion.

Since $X \times_S D_0 \to X$ is an isomorphism we see that W_0 is isomorphic to a closed subscheme of X. Similarly for W_∞. Consider the morphism $p : W \to X$. It is proper and on W we have $[W_0]_k \sim_{rat} [W_\infty]_k$. Hence part (2) follows from Lemma 29.20.2 as clearly $p_*[W_0]_k = [W_0]_k$ and similarly for W_∞.

The only content of statement (3) is, given parts (1) and (2), that the collection $\{(W_i)_0, (W_i)_\infty\}$ is a locally finite collection of closed subschemes of X. This is clear.

Suppose that $\alpha \sim_{rat} 0$. By definition this means there exist integral closed subschemes $V_i \subset X$ of δ-dimension $k + 1$ and rational functions $f_i \in R(V_i)^*$ such that the family $\{V_i\}_{i \in I}$ is locally finite in X and such that $\alpha = \sum (V_i \to X)_* \mathrm{div}(f_i)$. Let
$$W_i \subset V_i \times_S \mathbf{P}^1_S \subset X \times_S \mathbf{P}^1_S$$
be the closure of the graph of the rational map f_i as in Lemma 29.17.3. Then we have that $(V_i \to X)_* \mathrm{div}(f_i)$ is equal to $[(W_i)_0]_k - [(W_i)_\infty]_k$ by that same lemma. Hence the result is clear. □

Lemma 29.21.2. *Let (S, δ) be as in Situation 29.7.1. Let X be a scheme locally of finite type over S. Let Z be a closed subscheme of $X \times \mathbf{P}^1$. Assume $\dim_\delta(Z) \leq k + 1$, $\dim_\delta(Z_0) \leq k$, $\dim_\delta(Z_\infty) \leq k$ and assume any embedded point ξ (Divisors, Definition 26.4.1) of Z has $\delta(\xi) < k$. Then*
$$[Z_0]_k \sim_{rat} [Z_\infty]_k$$
as k-cycles on X.

Proof. Let $\{W_i\}_{i \in I}$ be the collection of irreducible components of Z which have δ-dimension $k + 1$. Write
$$[Z]_{k+1} = \sum n_i [W_i]$$

with $n_i > 0$ as per definition. Note that $\{W_i\}$ is a locally finite collection of closed subsets of $X \times_S \mathbf{P}^1_S$ by Lemma 29.9.1. We claim that

$$[Z_0]_k = \sum n_i[(W_i)_0]_k$$

and similarly for $[Z_\infty]_k$. If we prove this then the lemma follows from Lemma 29.21.1.

Let $Z' \subset X$ be an integral closed subscheme of δ-dimension k. To prove the equality above it suffices to show that the coefficient n of $[Z']$ in $[Z_0]_k$ is the same as the coefficient m of $[Z']$ in $\sum n_i[(W_i)_0]_k$. Let $\xi' \in Z'$ be the generic point. Set $\xi = (\xi', 0) \in X \times_S \mathbf{P}^1_S$. Consider the local ring $A = \mathcal{O}_{X\times_S\mathbf{P}^1_S,\xi}$. Let $I \subset A$ be the ideal cutting out Z, in other words so that $A/I = \mathcal{O}_{Z,\xi}$. Let $t \in A$ be the element cutting out $X \times_S D_0$ (i.e., the coordinate of $\mathbf{P}^1$ at zero pulled back). By our choice of $\xi' \in Z'$ we have $\delta(\xi) = k$ and hence $\dim(A/I) = 1$. Since ξ is not an embedded point by definition we see that A/I is Cohen-Macaulay. Since $\dim_\delta(Z_0) = k$ we see that $\dim(A/(t, I)) = 0$ which implies that t is a nonzero divisor on A/I. Finally, the irreducible closed subschemes W_i passing through ξ correspond to the minimal primes $I \subset \mathfrak{q}_i$ over I. The multiplicities n_i correspond to the lengths $\text{length}_{A_{\mathfrak{q}_i}}(A/I)_{\mathfrak{q}_i}$. Hence we see that

$$n = \text{length}_A(A/(t, I))$$

and

$$m = \sum \text{length}_A(A/(t, \mathfrak{q}_i))\text{length}_{A_{\mathfrak{q}_i}}(A/I)_{\mathfrak{q}_i}$$

Thus the result follows from Lemma 29.5.6. □

Lemma 29.21.3. *Let (S, δ) be as in Situation 29.7.1. Let X be a scheme locally of finite type over S. Let $\mathcal{F}$ be a coherent sheaf on $X \times \mathbf{P}^1$. Let $i_0, i_\infty : X \to X \times \mathbf{P}^1$ be the closed immersion such that $i_t(x) = (x, t)$. Denote $\mathcal{F}_0 = i_0^*\mathcal{F}$ and $\mathcal{F}_\infty = i_\infty^*\mathcal{F}$. Assume*

1. *$\dim_\delta(Supp(\mathcal{F})) \leq k + 1$,*
2. *$\dim_\delta(Supp(\mathcal{F}_0)) \leq k$, $\dim_\delta(Supp(\mathcal{F}_\infty)) \leq k$, and*
3. *any nonmaximal associated point (insert future reference here) $\xi \in Supp(\mathcal{F})$ of $\mathcal{F}$ has $\delta(\xi) < k$.*

Then

$$[\mathcal{F}_0]_k \sim_{rat} [\mathcal{F}_\infty]_k$$

as k-cycles on X.

Proof. Let $\{W_i\}_{i\in I}$ be the collection of irreducible components of $\text{Supp}(\mathcal{F})$ which have δ-dimension $k + 1$. Write

$$[\mathcal{F}]_{k+1} = \sum n_i[W_i]$$

with $n_i > 0$ as per definition. Note that $\{W_i\}$ is a locally finite collection of closed subsets of $X \times_S \mathbf{P}^1_S$ by Lemma 29.10.1. We claim that

$$[\mathcal{F}_0]_k = \sum n_i[(W_i)_0]_k$$

and similarly for $[\mathcal{F}_\infty]_k$. If we prove this then the lemma follows from Lemma 29.21.1.

Let $Z' \subset X$ be an integral closed subscheme of δ-dimension k. To prove the equality above it suffices to show that the coefficient n of $[Z']$ in $[\mathcal{F}_0]_k$ is the same as the coefficient m of $[Z']$ in $\sum n_i[(W_i)_0]_k$. Let $\xi' \in Z'$ be the generic point. Set $\xi = (\xi', 0) \in X \times_S \mathbf{P}^1_S$. Consider the local ring $A = \mathcal{O}_{X\times_S\mathbf{P}^1_S,\xi}$. Let $M = \mathcal{F}_\xi$ as an A-module. Let $t \in A$ be the element cutting out $X \times_S D_0$ (i.e., the coordinate of $\mathbf{P}^1$ at zero pulled back). By our choice of $\xi' \in Z'$ we have $\delta(\xi) = k$ and hence $\dim(M) = 1$. Since ξ is not an associated point of $\mathcal{F}$ by definition we see that M is Cohen-Macaulay module. Since $\dim_\delta(\text{Supp}(\mathcal{F}_0)) = k$ we see that $\dim(M/tM) = 0$ which implies that t is a nonzero divisor on M. Finally, the irreducible closed subschemes W_i passing through ξ correspond to the minimal primes $\mathfrak{q}_i$ of $\text{Ass}(M)$. The multiplicities n_i correspond to the lengths $\text{length}_{A_{\mathfrak{q}_i}} M_{\mathfrak{q}_i}$. Hence we see that

$$n = \text{length}_A(M/tM)$$

and

$$m = \sum \text{length}_A(A/(t, \mathfrak{q}_i)A)\text{length}_{A_{\mathfrak{q}_i}} M_{\mathfrak{q}_i}$$

Thus the result follows from Lemma 29.5.6. □

29.22. Rational equivalence and K-groups

In this section we compare the cycle groups $Z_k(X)$ and the Chow groups $A_k(X)$ with certain K_0-groups of abelian categories of coherent sheaves on X. We avoid having to talk about $K_1(\mathcal{A})$ for an abelian category $\mathcal{A}$ by dint of Homology, Lemma 10.8.3. In particular, the motivation for the precise form of Lemma 29.22.4 is that lemma.

Let us introduce the following notation. Let (S, δ) be as in Situation 29.7.1. Let X be a scheme locally of finite type over S. We denote $\text{Coh}(X) = \text{Coh}(\mathcal{O}_X)$ the category of coherent sheaves on X. It is an abelian category, see Coherent, Lemma 25.11.2. For any $k \in \mathbf{Z}$ we let $\text{Coh}_{\leq k}(X)$ be the full subcategory of $\text{Coh}(X)$ consisting of those coherent sheaves $\mathcal{F}$ having $\dim_\delta(\text{Supp}(\mathcal{F})) \leq k$.

Lemma 29.22.1. *Let us introduce the following notation. Let (S, δ) be as in Situation 29.7.1. Let X be a scheme locally of finite type over S. The categories $Coh_{\leq k}(X)$ are Serre subcategories of the abelian category $Coh(X)$.*

Proof. Omitted. The definition of a Serre subcateory is Homology, Definition 10.7.1. □

Lemma 29.22.2. *Let (S, δ) be as in Situation 29.7.1. Let X be a scheme locally of finite type over S. There are maps*

$$Z_k(X) \longrightarrow K_0(Coh_{\leq k}(X)/Coh_{\leq k-1}(X)) \longrightarrow Z_k(X)$$

whose composition is the identity. The first is the map

$$\sum n_Z[Z] \mapsto \left[\bigoplus\nolimits_{n_Z>0} \mathcal{O}_Z^{\oplus n_Z}\right] - \left[\bigoplus\nolimits_{n_Z<0} \mathcal{O}_Z^{\oplus -n_Z}\right]$$

and the second comes from the map $\mathcal{F} \mapsto [\mathcal{F}]_k$. If X is quasi-compact, then both maps are isomorphisms.

Proof. Note that the direct sum $\bigoplus_{n_Z>0} \mathcal{O}_Z^{\oplus n_Z}$ is indeed a coherent sheaf on X since the family $\{Z \mid n_Z > 0\}$ is locally finite on X. The map $\mathcal{F} \to [\mathcal{F}]_k$ is additive on $\text{Coh}_{\leq k}(X)$, see Lemma 29.10.4. And $[\mathcal{F}]_k = 0$ if $\mathcal{F} \in \text{Coh}_{\leq k-1}(X)$. This implies we have the left map as shown in the lemma. It is clear that their composition is the identity.

In case X is quasi-compact we will show that the right arrow is injective. Suppose that $q \in K_0(\text{Coh}_{\leq k}(X)/\text{Coh}_{\leq k+1}(X))$ maps to zero in $Z_k(X)$. By Homology, Lemma 10.8.3 we can find a $\tilde{q} \in K_0(\text{Coh}_{\leq k}(X))$ mapping to q. Write $\tilde{q} = [\mathcal{F}] - [\mathcal{G}]$ for some $\mathcal{F}, \mathcal{G} \in K_0(\text{Coh}_{\leq k}(X))$. Since X is quasi-compact we may apply Coherent, Lemma 25.14.3. This shows that there exist integral closed subschemes $Z_j, T_i \subset X$ and (nonzero) ideal sheaves $\mathcal{I}_j \subset \mathcal{O}_{Z_j}$, $\mathcal{J}_i \subset \mathcal{O}_{T_i}$ such that $\mathcal{F}$, resp. $\mathcal{G}$ have filtrations whose succesive quotients are the sheaves $\mathcal{I}_j$, resp. $\mathcal{J}_i$. In particular we see that $\dim_\delta(Z_j), \dim_\delta(T_i) \leq k$. In other words we have

$$[\mathcal{F}] = \sum\nolimits_j [\mathcal{I}_j], \quad [\mathcal{G}] = \sum\nolimits_i [\mathcal{J}_i],$$

in $K_0(\textit{Coh}_{\leq k}(X))$. Our assumption is that $\sum_j [\mathcal{I}_j]_k - \sum_i [\mathcal{J}_i]_k = 0$. It is clear that we may throw out the indices j, resp. i such that $\dim_\delta(Z_j) < k$, resp. $\dim_\delta(T_i) < k$, since the corresponding sheaves are in $\textit{Coh}_{k-1}(X)$ and also do not contribute to the cycle. Moreover, the exact sequences $0 \to \mathcal{I}_j \to \mathcal{O}_{Z_j} \to \mathcal{O}_{Z_j}/\mathcal{I}_j \to 0$ and $0 \to \mathcal{J}_i \to \mathcal{O}_{T_i} \to \mathcal{O}_{Z_i}/\mathcal{J}_i \to 0$ show similarly that we may replace $\mathcal{I}_j$, resp. $\mathcal{J}_i$ by $\mathcal{O}_{Z_j}$, resp. $\mathcal{O}_{T_i}$. OK, and finally, at this point it is clear that our assumption

$$\sum\nolimits_j [\mathcal{O}_{Z_j}]_k - \sum\nolimits_i [\mathcal{O}_{T_i}]_k = 0$$

implies that in $K_0(\textit{Coh}_k(X))$ we have also $\sum_j [\mathcal{O}_{Z_j}] - \sum_i [\mathcal{O}_{T_i}] = 0$ as desired. □

Remark 29.22.3. It seems likely that the arrows of Lemma 29.22.2 are not isomorphisms if X is not quasi-compact. For example, suppose X is an infinite disjoint union $X = \coprod_{n \in \mathbf{N}} \mathbf{P}^1_k$ over a field k. Let $\mathcal{F}$, resp. $\mathcal{G}$ be the coherent sheaf on X whose restriction to the nth summand is equal to the skyscraper sheaf at 0 associated to $\mathcal{O}_{\mathbf{P}^1_k, 0}/\mathfrak{m}_0^n$, resp. $\kappa(0)^{\oplus n}$. The cycle associated to $\mathcal{F}$ is equal to the cycle associated to $\mathcal{G}$, namely both are equal to $\sum n[0_n]$ where $0_n \in X$ denotes 0 on the nth component of X. But there seems to be no way to show that $[\mathcal{F}] = [\mathcal{G}]$ in $K_0(\textit{Coh}(X))$ since any proof we can envision uses infinitely many relations.

Lemma 29.22.4. *Let (S, δ) be as in Situation 29.7.1. Let X be a scheme locally of finite type over S. Let $\mathcal{F}$ be a coherent sheaf on X. Let*

$$\ldots \longrightarrow \mathcal{F} \xrightarrow{\varphi} \mathcal{F} \xrightarrow{\psi} \mathcal{F} \xrightarrow{\varphi} \mathcal{F} \longrightarrow \ldots$$

be a complex as in Homology, Equation (10.8.2.1). Assume that

(1) $\dim_\delta(\textit{Supp}(\mathcal{F})) \leq k + 1$.
(2) $\dim_\delta(\textit{Supp}(H^i(\mathcal{F}, \varphi, \psi))) \leq k$ *for* $i = 0, 1$.

Then we have

$$[H^0(\mathcal{F}, \varphi, \psi)]_k \sim_{rat} [H^1(\mathcal{F}, \varphi, \psi)]_k$$

as k-cycles on X.

Proof. Let $\{W_j\}_{j \in J}$ be the collection of irreducible components of $\text{Supp}(\mathcal{F})$ which have δ-dimension $k + 1$. Note that $\{W_j\}$ is a locally finite collection of closed subsets of X by Lemma 29.10.1. For every j, let $\xi_j \in W_j$ be the generic point. Set

$$f_j = \det\nolimits_{\kappa(\xi_j)}(\mathcal{F}_{\xi_j}, \varphi_{\xi_j}, \psi_{\xi_j}) \in R(W_j)^*.$$

See Definition 29.3.4 for notation. We claim that

$$-[H^0(\mathcal{F}, \varphi, \psi)]_k + [H^1(\mathcal{F}, \varphi, \psi)]_k = \sum (W_j \to X)_* \text{div}(f_j)$$

If we prove this then the lemma follows.

Let $Z \subset X$ be an integral closed subscheme of δ-dimension k. To prove the equality above it suffices to show that the coefficient n of $[Z]$ in $[H^0(\mathcal{F}, \varphi, \psi)]_k - [H^1(\mathcal{F}, \varphi, \psi)]_k$ is the same as the coefficient m of $[Z]$ in $\sum (W_j \to X)_* \text{div}(f_j)$. Let $\xi \in Z$ be the generic point. Consider the local ring $A = \mathcal{O}_{X,\xi}$. Let $M = \mathcal{F}_\xi$ as an A-module. Denote $\varphi, \psi : M \to M$ the action of φ, ψ on the stalk. By our choice of $\xi \in Z$ we have $\delta(\xi) = k$ and hence $\dim(M) = 1$. Finally, the integral closed subschemes W_j passing through ξ correspond to the minimal primes $\mathfrak{q}_i$ of $\text{Supp}(M)$. In each case the element $f_j \in R(W_j)^*$ corresponds to the element $\det_{\kappa(\mathfrak{q}_i)}(M_{\mathfrak{q}_i}, \varphi, \psi)$ in $\kappa(\mathfrak{q}_i)^*$. Hence we see that

$$n = -e_A(M, \varphi, \psi)$$

and

$$m = \sum \text{ord}_{A/\mathfrak{q}_i}(\det_{\kappa(\mathfrak{q}_i)}(M_{\mathfrak{q}_i}, \varphi, \psi))$$

Thus the result follows from Proposition 29.5.3. $\square$

Lemma 29.22.5. *Let (S, δ) be as in Situation 29.7.1. Let X be a scheme locally of finite type over S. Denote $B_k(X)$ the image of the map*

$$K_0(Coh_{\leq k}(X)/Coh_{\leq k-1}(X)) \longrightarrow K_0(Coh_{\leq k+1}(X)/Coh_{\leq k-1}(X)).$$

There is a commutative diagram

$$\begin{tikzcd} K_0\left(\frac{Coh_{\leq k}(X)}{Coh_{\leq k-1}(X)}\right) \arrow[r] \arrow[d] & B_k(X) \arrow[r, hook] \arrow[d] & K_0\left(\frac{Coh_{\leq k+1}(X)}{Coh_{\leq k-1}(X)}\right) \\ Z_k(X) \arrow[r] & A_k(X) & \end{tikzcd}$$

where the left vertical arrow is the one from Lemma 29.22.2. If X is quasi-compact then both vertical arrows are isomorphisms.

Proof. Suppose we have an element $[A] - [B]$ of $K_0(\text{Coh}_{\leq k}(X)/\text{Coh}_{\leq k-1}(X))$ which maps to zero in $B_k(X)$, i.e., in $K_0(\text{Coh}_{\leq k+1}(X)/\text{Coh}_{\leq k-1}(X))$. Suppose $[A] = [\mathcal{A}]$ and $[B] = [\mathcal{B}]$ for some coherent sheaves $\mathcal{A}, \mathcal{B}$ on X supported in δ-dimension $\leq k$. The assumption that $[A] - [B]$ maps to zero in the group $K_0(\text{Coh}_{\leq k+1}(X)/\text{Coh}_{\leq k-1}(X))$ means that there exists coherent sheaves $\mathcal{A}', \mathcal{B}'$ on X supported in δ-dimension $\leq k-1$ such that $[\mathcal{A} \oplus \mathcal{A}'] - [\mathcal{B} \oplus \mathcal{B}']$ is zero in $K_0(\text{Coh}_{k+1}(X))$ (use part (1) of Homology, Lemma 10.8.3). By part (2) of Homology, Lemma 10.8.3 this means there exists a $(2, 1)$-periodic complex $(\mathcal{F}, \varphi, \psi)$ in the category $\text{Coh}_{\leq k+1}(X)$ such that $\mathcal{A} \oplus \mathcal{A}' = H^0(\mathcal{F}, \varphi, \psi)$ and $\mathcal{B} \oplus \mathcal{B}' = H^1(\mathcal{F}, \varphi, \psi)$. By Lemma 29.22.4 this implies that

$$[\mathcal{A} \oplus \mathcal{A}']_k \sim_{rat} [\mathcal{B} \oplus \mathcal{B}']_k$$

This proves that $[A] - [B]$ maps to zero via the composition

$$K_0(\text{Coh}_{\leq k}(X)/\text{Coh}_{\leq k-1}(X)) \longrightarrow Z_k(X) \longrightarrow A_k(X).$$

In other words this proves the commutative diagram exists.

Next, assume that X is quasi-compact. By Lemma 29.22.2 the left vertical arrow is bijective. Hence it suffices to show any $\alpha \in Z_k(X)$ which is rationally equivalent to zero maps to zero in $B_k(X)$ via the inverse of the left vertical arrow composed with the horizontal arrow. By Lemma 29.21.1 we see that $\alpha = \sum([(W_i)_0]_k - [(W_i)_\infty]_k)$ for some closed integral subschemes $W_i \subset X \times_S \mathbf{P}^1_S$ of δ-dimension $k+1$. Moreover the family $\{W_i\}$ is finite because X is quasi-compact. Note that the ideal sheaves $\mathcal{I}_i, \mathcal{J}_i \subset \mathcal{O}_{W_i}$ of the effective Cartier divisors $(W_i)_0, (W_i)_\infty$ are isomorphic (as $\mathcal{O}_{W_i}$-modules). This is true because the ideal sheaves of D_0 and D_∞ on $\mathbf{P}^1$ are isomorphic and $\mathcal{I}_i, \mathcal{J}_i$ are the pullbacks of these. (Some details omitted.) Hence we have short exact sequences

$$0 \to \mathcal{I}_i \to \mathcal{O}_{W_i} \to \mathcal{O}_{(W_i)_0} \to 0, \quad 0 \to \mathcal{J}_i \to \mathcal{O}_{W_i} \to \mathcal{O}_{(W_i)_\infty} \to 0$$

of coherent $\mathcal{O}_{W_i}$-modules. Also, since $[(W_i)_0]_k = [p_*\mathcal{O}_{(W_i)_0}]_k$ in $Z_k(X)$ we see that the inverse of the left vertical arrow maps $[(W_i)_0]_k$ to the element $[p_*\mathcal{O}_{(W_i)_0}]$ in $K_0(\textit{Coh}_{\leq k}(X)/\textit{Coh}_{\leq k-1}(X))$. Thus we have

$$\begin{aligned} \alpha &= \sum \left([(W_i)_0]_k - [(W_i)_\infty]_k\right) \\ &\mapsto \sum \left([p_*\mathcal{O}_{(W_i)_0}] - [p_*\mathcal{O}_{(W_i)_\infty}]\right) \\ &= \sum \left([p_*\mathcal{O}_{W_i}] - [p_*\mathcal{I}_i] - [p_*\mathcal{O}_{W_i}] + [p_*\mathcal{J}_i]\right) \end{aligned}$$

in $K_0(\textit{Coh}_{\leq k+1}(X)/\textit{Coh}_{\leq k-1}(X))$. By what was said above this is zero, and we win. □

Remark 29.22.6. Let (S, δ) be as in Situation 29.7.1. Let X be a scheme locally of finite type over S. Assume X is quasi-compact. The result of Lemma 29.22.5 in particular gives a map

$$A_k(X) \longrightarrow K_0(\textit{Coh}(X)/\textit{Coh}_{\leq k-1}(X)).$$

We have not been able to find a statement or conjecture in the literature as to whether this map is should be injective or not. If X is connected nonsingular, then, using the isomorphism $K_0(X) = K^0(X)$ (see insert future reference here) and chern classes (see below), one can show that the map is an isomorphism up to $(p-1)!$-torsion where $p = \dim_\delta(X) - k$.

29.23. Preparation for the divisor associated to an invertible sheaf

For the following remarks, see Divisors, Section 26.15. Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $\xi \in X$ be a point. If $s_\xi, s'_\xi \in \mathcal{L}_\xi$ generate $\mathcal{L}_\xi$ as $\mathcal{O}_{X,\xi}$-module, then there exists a unit $u \in \mathcal{O}_{X,\xi}^*$ such that $s_\xi = u s'_\xi$. The stalk of the sheaf of meromorphic sections $\mathcal{K}_X(\mathcal{L})$ of $\mathcal{L}$ at x is equal to $\mathcal{K}_{X,x} \otimes_{\mathcal{O}_{X,x}} \mathcal{L}_x$. Thus the image of any meromorphic section s of $\mathcal{L}$ in the stalk at x can be written as $s = f s_\xi$ with $f \in \mathcal{K}_{X,x}$. Below we will abbreviate this by saying $f = s/s_\xi$. Also, if X is integral we have $\mathcal{K}_{X,x} = R(X)$ is equal to the function field of X, so $s/s_\xi \in R(X)$. If s is a *regular* meromorphic section (see Divisors, Definition 26.15.10), then actually $f \in R(X)^*$. (On an integral scheme a regular meromorphic section is the same thing as a nonzero meromorphic section.) Hence the following definition makes sense.

Definition 29.23.1. Let X be a locally Noetherian scheme. Assume X is integral. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{K}_X(\mathcal{L}))$ be a regular meromorphic section of $\mathcal{L}$. For every integral closed subscheme $Z \subset X$ of codimension 1 we define the *order of vanishing of s along Z* as the integer

$$\text{ord}_{Z,\mathcal{L}}(s) = \text{ord}_{\mathcal{O}_{X,\xi}}(s/s_\xi)$$

where the right hand side is the notion of Algebra, Definition 7.112.2, $\xi \in Z$ is the generic point, and $s_\xi \in \mathcal{L}_\xi$ is a generator.

Lemma 29.23.2. *Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Assume X is integral. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \mathcal{K}_X(\mathcal{L})$ be a regular (i.e., nonzero) meromorphic section of $\mathcal{L}$. Then the set*

$$\{Z \subset X \mid Z \text{ is irreducible, closed of codimension } 1 \text{ and } \text{ord}_{Z,\mathcal{L}}(s) \neq 0\}$$

is locally finite in X.

Proof. This is true simply because there exists a nonempty open subscheme $U \subset X$ such that s corresponds to a section of $\Gamma(U, \mathcal{L})$ which generates $\mathcal{L}$ over U. Hence the codimension 1 irreducibles which can occur in the set of the lemma are all irreducible components of $X \setminus U$. Hence Lemma 29.9.1 gives the desired result. $\square$

Lemma 29.23.3. *Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Assume X is integral and $n = \dim_\delta(X)$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s, s' \in \mathcal{K}_X(\mathcal{L})$ be nonzero meromorphic sections of $\mathcal{L}$. Then $f = s/s'$ is an element of $R(X)^*$ and we have*

$$\sum ord_{Z,\mathcal{L}}(s)[Z] = \sum ord_{Z,\mathcal{L}}(s')[Z] + div(f)$$

(where the sums are over integral closed subschemes $Z \subset X$ of δ-dimension $n-1$) as elements of $Z_{n-1}(X)$.

Proof. This is clear from the definitions. Note that Lemma 29.23.2 garantees that the sums are indeed elements of $Z_{n-1}(X)$. $\square$

29.24. The divisor associated to an invertible sheaf

The material above allows us to define the divisor associated to an invertible sheaf.

Definition 29.24.1. Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Assume X is integral and $n = \dim_\delta(X)$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module.

(1) For any nonzero meromorphic section s of $\mathcal{L}$ we define the *Weil divisor associated to s* as
$$\operatorname{div}_\mathcal{L}(s) := \sum \operatorname{ord}_{Z,\mathcal{L}}(s)[Z] \in Z_{n-1}(X)$$
where the sum is over integral closed subschemes $Z \subset X$ of δ-dimension $n-1$.

(2) We define *Weil divisor associated to $\mathcal{L}$*
$$c_1(\mathcal{L}) \cap [X] = \text{class of } \operatorname{div}_\mathcal{L}(s) \in A_{n-1}(X)$$
where s is any nonzero meromorphic section of $\mathcal{L}$ over X. This is well defined by Lemma 29.23.3.

There are some cases where it is easy to compute the Weil divisor associated to an invertible sheaf.

Lemma 29.24.2. *Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Assume X is integral and $n = \dim_\delta(X)$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$ be a nonzero global section. Then*

$$div_\mathcal{L}(s) = [Z(s)]_{n-1}$$

in $Z_{n-1}(X)$ and

$$c_1(\mathcal{L}) \cap [X] = [Z(s)]_{n-1}$$

in $A_{n-1}(X)$.

Proof. Let $Z \subset X$ be an integral closed subscheme of δ-dimension $n-1$. Let $\xi \in Z$ be its generic point. Choose a generator $s_\xi \in \mathcal{L}_\xi$. Write $s = f s_\xi$ for some $f \in \mathcal{O}_{X,\xi}$. By definition of $Z(s)$, see Divisors, Definition 26.9.15 we see that $Z(s)$ is cut out by a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ such that $\mathcal{I}_\xi = (f)$. Hence $\text{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{Z(s),\xi}) = \text{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,\xi}/(f)) = \operatorname{ord}_{\mathcal{O}_{X,x}}(f)$ as desired. $\square$

Lemma 29.24.3. *Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Assume X is integral and $n = \dim_\delta(X)$. Let $\mathcal{L}$, $\mathcal{N}$ be invertible $\mathcal{O}_X$-modules. Then*

(1) *Let s, resp. t be a nonzero meromorphic section of $\mathcal{L}$, resp. $\mathcal{N}$. Then st is a nonzero meromorphic section of $\mathcal{L} \otimes \mathcal{N}$, and*
$$div_{\mathcal{L}\otimes\mathcal{N}}(st) = div_{\mathcal{L}}(s) + div_{\mathcal{N}}(t)$$
in $Z_{n-1}(X)$.
(2) *We have*
$$c_1(\mathcal{L}) \cap [X] + c_1(\mathcal{N}) \cap [X] = c_1(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N}) \cap [X]$$
in $A_{n-1}(X)$.

Proof. Let s, resp. t be a nonzero meromorphic section of $\mathcal{L}$, resp. $\mathcal{N}$. Then st is a nonzero meromorphic section of $\mathcal{L}\otimes\mathcal{N}$. Let $Z \subset X$ be an integral closed subscheme of δ-dimension $n-1$. Let $\xi \in Z$ be its generic point. Choose generators $s_\xi \in \mathcal{L}_\xi$, and $t_\xi \in \mathcal{N}_\xi$. Then $s_\xi t_\xi$ is a generator for $(\mathcal{L} \otimes \mathcal{N})_\xi$. So $st/(s_\xi t_\xi) = (s/s_\xi)(t/t_\xi)$. Hence we see that
$$\text{div}_{\mathcal{L}\otimes\mathcal{N},Z}(st) = \text{div}_{\mathcal{L},Z}(s) + \text{div}_{\mathcal{N},Z}(t)$$
by the additivity of the ord_Z function. □

The following lemma will be superseded by the more general Lemma 29.25.4.

Lemma 29.24.4. *Let (S, δ) be as in Situation 29.7.1. Let X, Y be locally of finite type over S. Assume X, Y are integral and $n = \dim_\delta(Y)$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_Y$-module. Let $f : X \to Y$ be a flat morphism of relative dimension r. Let $\mathcal{L}$ be an invertible sheaf on Y. Then*
$$f^*(c_1(\mathcal{L}) \cap [Y]) = c_1(f^*\mathcal{L}) \cap [X]$$
in $A_{n+r-1}(X)$.

Proof. Let s be a nonzero meromorphic section of $\mathcal{L}$. We will show that actually $f^*\text{div}_{\mathcal{L}}(s) = \text{div}_{f^*\mathcal{L}}(f^*s)$ and hence the lemma holds. To see this let $\xi \in Y$ be a point and let $s_\xi \in \mathcal{L}_\xi$ be a generator. Write $s = gs_\xi$ with $g \in R(X)^*$. Then there is an open neighbourhood $V \subset Y$ of ξ such that $s_\xi \in \mathcal{L}(V)$ and such that s_ξ generates $\mathcal{L}|_V$. Hence we see that
$$\text{div}_{\mathcal{L}}(s)|_V = \text{div}(g)|_V.$$
In exactly the same way, since f^*s_ξ generates $\mathcal{L}$ over $f^{-1}(V)$ and since $f^*s = gf^*s_\xi$ we also have
$$\text{div}_{\mathcal{L}}(f^*s)|_{f^{-1}(V)} = \text{div}(g)|_{f^{-1}(V)}.$$
Thus the desired equality of cycles over $f^{-1}(V)$ follows from the corresponding result for pull backs of principal divisors, see Lemma 29.17.4. □

29.25. Intersecting with Cartier divisors

Definition 29.25.1. Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. We define, for every integer k, an operation
$$c_1(\mathcal{L}) \cap - : Z_{k+1}(X) \to A_k(X)$$
called *intersection with the first chern class of $\mathcal{L}$*.

(1) Given an integral closed subscheme $i : W \to X$ with $\dim_\delta(W) = k+1$ we define
$$c_1(\mathcal{L}) \cap [W] = i_*(c_1(i^*\mathcal{L}) \cap [W])$$
where the right hand side is defined in Definition 29.24.1.
(2) For a general $(k+1)$-cycle $\alpha = \sum n_i[W_i]$ we set
$$c_1(\mathcal{L}) \cap \alpha = \sum n_i c_1(\mathcal{L}) \cap [W_i]$$

Write each $c_1(\mathcal{L}) \cap W_i = \sum_j n_{i,j}[Z_{i,j}]$ with $\{Z_{i,j}\}_j$ a locally finite sum of integral closed subschemes of W_i. Since $\{W_i\}$ is a locally finite collection of integral closed subschemes on X, it follows easily that $\{Z_{i,j}\}_{i,j}$ is a locally finite collection of closed subschemes of X. Hence $c_1(\mathcal{L}) \cap \alpha = \sum n_i n_{i,j}[Z_{i,j}]$ is a cycle. Another, more convenient, way to think about this is to observe that the morphism $\coprod W_i \to X$ is proper. Hence $c_1(\mathcal{L}) \cap \alpha$ can be viewed as the pushforward of a class in $A_k(\coprod W_i) = \prod A_k(W_i)$. This also explains why the result is well defined up to rational equivalence on X.

The main goal for the next few sections is to show that intersecting with $c_1(\mathcal{L})$ factors through rational equivalence, and is commutative. This is not a trviality.

Lemma 29.25.2. *Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Let $\mathcal{L}$, $\mathcal{N}$ be an invertible sheaves on X. Then*

$$c_1(\mathcal{L}) \cap \alpha + c_1(\mathcal{N}) \cap \alpha = c_1(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N}) \cap \alpha$$

in $A_k(X)$ for every $\alpha \in Z_{k-1}(X)$. Moreover, $c_1(\mathcal{O}_X) \cap \alpha = 0$ for all α.

Proof. The additivity follows directly from Lemma 29.24.3 and the definitions. To see that $c_1(\mathcal{O}_X) \cap \alpha = 0$ consider the section $1 \in \Gamma(X, \mathcal{O}_X)$. This restricts to an everywhere nonzero section on any integral closed subscheme $W \subset X$. Hence $c_1(\mathcal{O}_X) \cap [W] = 0$ as desired. $\square$

The following lemma is a useful result in order to compute the intersection product of the c_1 of an invertible sheaf and the cycle associated to a closed subscheme. Recall that $Z(s) \subset X$ denotes the zero scheme of a global section s of an invertible sheaf on a scheme X, see Divisors, Definition 26.9.15.

Lemma 29.25.3. *Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $Z \subset X$ be a closed subscheme. Assume* $\dim_\delta(Z) \leq k + 1$. *Let $s \in \Gamma(Z, \mathcal{L}|_Z)$. Assume*

(1) $\dim_\delta(Z(s)) \leq k$, *and*
(2) *for every generic point ξ of an irreducible component of $Z(s)$ of dimension k the multiplication by s induces an injection $\mathcal{O}_{Z,\xi} \to (\mathcal{L}|_Z)_\xi$.*

This holds for example if s is a regular section of $\mathcal{L}|_Z$. Then

$$[Z(s)]_k = c_1(\mathcal{L}) \cap [Z]_{k+1}$$

in $A_k(X)$.

Proof. Write

$$[Z]_{k+1} = \sum n_i[W_i]$$

where $W_i \subset Z$ are the irreducible components of Z of δ-dimension $k + 1$ and $n_i > 0$. By assumption the restriction $s_i = s|_{W_i} \in \Gamma(W_i, \mathcal{L}|_{W_i})$ is not zero, and hence is a regular section. By Lemma 29.24.2 we see that $[Z(s_i)]_k$ represents $c_1(\mathcal{L}|_{W_i})$. Hence by definition

$$c_1(\mathcal{L}) \cap [Z]_{k+1} = \sum n_i[Z(s_i)]_k$$

In fact, the proof below will show that we have

$$[Z(s)]_k = \sum n_i[Z(s_i)]_k \quad (29.25.3.1)$$

as k-cycles on X.

Let $Z' \subset X$ be an integral closed subscheme of δ-dimension k. Let $\xi' \in Z'$ be its generic point. We want to compare the coefficient n of $[Z']$ in the expression $\sum n_i[Z(s_i)]_k$ with the coefficient m of $[Z']$ in the expression $[Z(s)]_k$. Choose a generator $s_{\xi'} \in \mathcal{L}_\xi$. Let $\mathcal{I} \subset \mathcal{O}_X$

be the ideal sheaf of Z. Write $A = \mathcal{O}_{X,\xi'}$, $L = \mathcal{L}_{\xi'}$ and $I = \mathcal{I}_{\xi'}$. Then $L = As_{\xi'}$ and $L/IL = (A/I)s_{\xi'} = (\mathcal{L}|_Z)_{\xi'}$. Write $s = fs_{\xi'}$ for some (unique) $f \in A/I$. Hypothesis (2) means that $f : A/I \to A/I$ is injective. Since $\dim_\delta(Z) \leq k+1$ and $\dim_\delta(Z') = k$ we have $\dim(A/I) = 0$ or 1. We have

$$m = \text{length}_A(A/(f, I))$$

which is finite in either case.

If $\dim(A/I) = 0$, then $f : A/I \to A/I$ being injective implies that $f \in (A/I)^*$. Hence in this case m is zero. Moreover, the condition $\dim(A/I) = 0$ means that ξ' does not lie on any irreducible component of δ-dimension $k+1$, i.e., $n = 0$ as well.

Now, let $\dim(A/I) = 1$. Since A is a Noetherian local ring there are finitely many minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_t \supset I$ over I. These correspond 1-1 with W_i passing through ξ'. Moreover $n_i = \text{length}_{A_{\mathfrak{q}_i}}((A/I)_{\mathfrak{q}_i})$. Also, the multiplicity of $[Z']$ in $[Z(s_i)]_k$ is $\text{length}_A(A/(f, \mathfrak{q}_i))$. Hence the equation to prove in this case is

$$\text{length}_A(A/(f, I)) = \sum \text{length}_{A_{\mathfrak{q}_i}}((A/I)_{\mathfrak{q}_i})\text{length}_A(A/(f, \mathfrak{q}_i))$$

which follows from Lemma 29.5.6. □

Lemma 29.25.4. *Let (S, δ) be as in Situation 29.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a flat morphism of relative dimension r. Let $\mathcal{L}$ be an invertible sheaf on Y. Let α be a k-cycle on Y. Then*

$$f^*(c_1(\mathcal{L}) \cap \alpha) = c_1(f^*\mathcal{L}) \cap f^*\alpha$$

in $A_{k+r-1}(X)$.

Proof. Write $\alpha = \sum n_i[W_i]$. We claim it suffices to show that $f^*(c_1(\mathcal{L}) \cap [W_i]) = c_1(f^*\mathcal{L}) \cap f^*[W_i]$ for each i. Proof of this claim is omitted. (Remarks: it is clear in the quasi-compact case. Something similar happened in the proof of Lemma 29.20.1, and one can copy the method used there here. Another possibility is to check the cycles and rational equivalences used for all W_i combined at each step form a locally finite collection).

Let $W \subset Y$ be an integral closed subscheme of δ-dimension k. We have to show that $f^*(c_1(\mathcal{L}) \cap [W]) = c_1(f^*\mathcal{L}) \cap f^*[W]$. Consider the following fibre product diagram

$$\begin{tikzcd} W' = W \times_Y X \arrow[r] \arrow[d] & X \arrow[d] \\ W \arrow[r] & Y \end{tikzcd}$$

and let $W'_i \subset W'$ be the irreducible components of δ-dimension $k+r$. Write $[W']_{k+r} = \sum n_i[W'_i]$ with $n_i > 0$ as per definition. So $f^*[W] = \sum n_i[W'_i]$. Choose a nonzero meromorphic section s of $\mathcal{L}|_W$. Since each $W'_i \to W$ is dominant we see that $s_i = s|_{W'_i}$ is a nonzero meromorphic section for each i. We claim that we have the following equality of cycles

$$\sum n_i \text{div}_{\mathcal{L}|_{W_i}}(s_i) = f^*\text{div}_{\mathcal{L}|_W}(s)$$

in $Z_{k+r-1}(X)$.

Having formulated the problem as an equality of cycles we may work locally on Y. Hence we may assume Y and also W affine, and $s = p/q$ for some nonzero sections $p \in \Gamma(W, \mathcal{L})$ and $q \in \Gamma(W, \mathcal{O})$. If we can show both

$$\sum n_i \text{div}_{\mathcal{L}|_{W_i}}(p_i) = f^*\text{div}_{\mathcal{L}|_W}(p), \quad \text{and} \quad \sum n_i \text{div}_{\mathcal{O}|_{W_i}}(q_i) = f^*\text{div}_{\mathcal{O}|_W}(q)$$

(with obvious notations) then we win by the additivity, see Lemma 29.24.3. Thus we may assume that $s \in \Gamma(W, \mathcal{L}|_W)$. In this case we may apply the equality (29.25.3.1) obtained in the proof of Lemma 29.25.3 to see that

$$\sum n_i \text{div}_{\mathcal{L}|_{W_i}}(s_i) = [Z(s')]_{k+r-1}$$

where $s' \in f^*\mathcal{L}|_{W'}$ denotes the pull back of s to W'. On the other hand we have

$$f^*\text{div}_{\mathcal{L}|_W}(s) = f^*[Z(s)]_{k-1} = [f^{-1}(Z(s))]_{k+r-1},$$

by Lemmas 29.24.2 and 29.14.4. Since $Z(s') = f^{-1}(Z(s))$ we win. $\square$

Lemma 29.25.5. *Let (S, δ) be as in Situation 29.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a proper morphism. Let $\mathcal{L}$ be an invertible sheaf on Y. Let s be a nonzero meromorphic section s of $\mathcal{L}$ on Y. Assume X, Y integral, f dominant, and* $\dim_\delta(X) = \dim_\delta(Y)$. *Then*

$$f_*\left(div_{f^*\mathcal{L}}(f^*s)\right) = [R(X) : R(Y)]div_{\mathcal{L}}(s).$$

In particular

$$f_*(c_1(f^*\mathcal{L}) \cap [X]) = c_1(\mathcal{L}) \cap f_*[Y].$$

Proof. The last equation follows from the first since $f_*[X] = [R(X) : R(Y)][Y]$ by definition. It turns out that we can re-use Lemma 29.18.1 to prove this. Namely, since we are trying to prove an equality of cycles, we may work locally on Y. Hence we may assume that $\mathcal{L} = \mathcal{O}_Y$. In this case s corresponds to a rational function $g \in R(Y)$, and we are simply trying to prove

$$f_*\left(\text{div}_X(g)\right) = [R(X) : R(Y)]\text{div}_Y(g).$$

Comparing with the result of the aforementioned Lemma 29.18.1 we see this true since $\text{Nm}_{R(X)/R(Y)}(g) = g^{[R(X):R(Y)]}$ as $g \in R(Y)^*$. $\square$

Lemma 29.25.6. *Let (S, δ) be as in Situation 29.7.1. Let X, Y be locally of finite type over S. Let $p : X \to Y$ be a proper morphism. Let $\alpha \in Z_{k+1}(X)$. Let $\mathcal{L}$ be an invertible sheaf on Y. Then*

$$p_*(c_1(p^*\mathcal{L}) \cap \alpha) = c_1(\mathcal{L}) \cap p_*\alpha$$

in $A_k(Y)$.

Proof. Suppose that p has the property that for every integral closed subscheme $W \subset X$ the map $p|_W : W \to Y$ is a closed immersion. Then, by definition of capping wiht $c_1(\mathcal{L})$ the lemma holds.

We will use this remark to reduce to a special case. Namely, write $\alpha = \sum n_i[W_i]$ with $n_i \neq 0$ and W_i pairwise distinct. Let $W_i' \subset Y$ be the image of W_i (as an integral closed subscheme). Consider the diagram

$$\begin{tikzcd}
X' = \coprod W_i \arrow[r, "q"'] \arrow[d, "p'"'] & X \arrow[d, "p"] \\
Y' = \coprod W_i' \arrow[r, "q'"] & Y.
\end{tikzcd}$$

Since $\{W_i\}$ is locally finite on X, and p is proper we see that $\{W_i'\}$ is locally finite on Y and that q, q', p' are also proper morphisms. We may think of $\sum n_i[W_i]$ also as a k-cycle $\alpha' \in Z_k(X')$. Clearly $q_*\alpha' = \alpha$. We have $q_*(c_1(q^*p^*\mathcal{L}) \cap \alpha') = c_1(p^*\mathcal{L}) \cap q_*\alpha'$ and $(q')_*(c_1((q')^*\mathcal{L}) \cap p'_*\alpha') = c_1(\mathcal{L}) \cap q'_*p'_*\alpha'$ by the initial remark of the proof. Hence it suffices to prove the lemma for the morphism p' and the cycle $\sum n_i[W_i]$. Clearly, this means we may assume X, Y integral, $f : X \to Y$ dominant and $\alpha = [X]$. In this case the result follows from Lemma 29.25.5. □

29.26. Cartier divisors and K-groups

In this section we describe how the intersection with the first chern class of an invertible sheaf $\mathcal{L}$ corresponds to tensoring with $\mathcal{L} - \mathcal{O}$ in K-groups.

Lemma 29.26.1. *Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{K}_X(\mathcal{L}))$ be a meromorphic section of $\mathcal{L}$. Assume*

(1) $\dim_\delta(X) \leq k + 1$,
(2) *X has no embedded points,*
(3) *$\mathcal{F}$ has no embedded associated points,*
(4) *the support of $\mathcal{F}$ is X, and*
(5) *the section s is regular meromorphic.*

In this situation let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal of denominators of s, see Divisors, Definition 26.15.14. Then we have the following:

(1) *there are short exact sequences*
$$\begin{matrix} 0 & \to & \mathcal{I}\mathcal{F} & \xrightarrow{1} & \mathcal{F} & \to & \mathcal{Q}_1 & \to & 0 \\ 0 & \to & \mathcal{I}\mathcal{F} & \xrightarrow{s} & \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L} & \to & \mathcal{Q}_2 & \to & 0 \end{matrix}$$
(2) *the coherent sheaves $\mathcal{Q}_1$, $\mathcal{Q}_2$ are supported in δ-dimension $\leq k$,*
(3) *the section s restricts to a regular meromorphic section s_i on every irreducible component X_i of X of δ-dimension $k + 1$, and*
(4) *writing $[\mathcal{F}]_{k+1} = \sum m_i[X_i]$ we have*
$$[\mathcal{Q}_2]_k - [\mathcal{Q}_1]_k = \sum m_i(X_i \to X)_* div_{\mathcal{L}|_{X_i}}(s_i)$$
in $Z_k(X)$, in particular
$$[\mathcal{Q}_2]_k - [\mathcal{Q}_1]_k = c_1(\mathcal{L}) \cap [\mathcal{F}]_{k+1}$$
in $A_k(X)$.

Proof. Recall from Divisors, Lemma 26.15.15 the existence of injective maps $1 : \mathcal{I}\mathcal{F} \to \mathcal{F}$ and $s : \mathcal{I}\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}$ whose cokernels are supported on a closed nowhere dense subsets T. Denote $\mathcal{Q}_i$ there cokernels as in the lemma. We conclude that $\dim_\delta(\text{Supp}(\mathcal{Q}_i)) \leq k$. By Divisors, Lemmas 26.15.4 and 26.15.11 the pullbacks s_i are defined and are regular meromorphic sections for $\mathcal{L}|_{X_i}$. The equality of cycles in (4) implies the equality of cycle classes in (4). Hence the only remaining thing to show is that
$$[\mathcal{Q}_2]_k - [\mathcal{Q}_1]_k = \sum m_i(X_i \to X)_* \text{div}_{\mathcal{L}|_{X_i}}(s_i)$$
holds in $Z_k(X)$. To see this, let $Z \subset X$ be an integral closed subscheme of δ-dimension k. Let $\xi \in Z$ be the generic point. Let $A = \mathcal{O}_{X,\xi}$ and $M = \mathcal{F}_\xi$. Moreover, choose a generator $s_\xi \in \mathcal{L}_\xi$. Then we can write $s = (a/b)s_\xi$ where $a, b \in A$ are nonzero divisors. In this case $I = \mathcal{I}_\xi = \{x \in A \mid x(a/b) \in A\}$. In this case the coefficient of $[Z]$ in the left hand side is
$$\text{length}_A(M/(a/b)IM) - \text{length}_A(M/IM)$$
and the coefficient of $[Z]$ in the right hand side is
$$\sum \text{length}_{A_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i}) \text{ord}_{A/\mathfrak{q}_i}(a/b)$$

where $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ are the minimal primes of the 1-dimensional local ring A. Hence the result follows from Lemma 29.5.7. □

Lemma 29.26.2. *Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Assume* $\dim_\delta(\mathit{Support}(\mathcal{F})) \leq k+1$. *Then the element*

$$[\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{F}] \in K_0(\mathit{Coh}_{\leq k+1}(X)/\mathit{Coh}_{\leq k-1}(X))$$

lies in the subgroup $B_k(X)$ of Lemma 29.22.5 and maps to the element $c_1(\mathcal{L}) \cap [\mathcal{F}]_{k+1}$ via the map $B_k(X) \to A_k(X)$.

Proof. Let

$$0 \to \mathcal{K} \to \mathcal{F} \to \mathcal{F}' \to 0$$

be the short exact sequence constructed in Divisors, Lemma 26.4.4. This in particular means that $\mathcal{F}'$ has no embedded associated points. Since the support of $\mathcal{K}$ is nowhere dense in the support of $\mathcal{F}$ we see that $\dim_\delta(\text{Supp}(\mathcal{K})) \leq k$. We may re-apply Divisors, Lemma 26.4.4 starting with $\mathcal{K}$ to get a short exact sequence

$$0 \to \mathcal{K}'' \to \mathcal{K} \to \mathcal{K}' \to 0$$

where now $\dim_\delta(\text{Supp}(\mathcal{K}'')) < k$ and $\mathcal{K}'$ has no embedded associated points. Suppose we can prove the lemma for the coherent sheaves $\mathcal{F}'$ and $\mathcal{K}'$. Then we see from the equations

$$[\mathcal{F}]_{k+1} = [\mathcal{F}']_{k+1} + [\mathcal{K}']_{k+1} + [\mathcal{K}'']_{k+1}$$

(use Lemma 29.10.4),

$$[\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{F}] = [\mathcal{F}' \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{F}'] + [\mathcal{K}' \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{K}'] + [\mathcal{K}'' \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{K}'']$$

(use the $\otimes \mathcal{L}$ is exact) and the trivial vanishing of $[\mathcal{K}'']_{k+1}$ and $[\mathcal{K}'' \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{K}'']$ in $K_0(\text{Coh}_{\leq k+1}(X)/\text{Coh}_{\leq k-1}(X))$ that the result holds for $\mathcal{F}$. What this means is that we may assume that the sheaf $\mathcal{F}$ has no embedded associated points.

Assume X, $\mathcal{F}$ as in the lemma, and assume in addition that $\mathcal{F}$ has no embedded associated points. Consider the sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$, the corresponding closed subscheme $i : Z \to X$ and the coherent $\mathcal{O}_Z$-module $\mathcal{G}$ constructed in Divisors, Lemma 26.4.5. Recall that Z is a locally Noetherian scheme without embedded points, $\mathcal{G}$ is a coherent sheaf without embedded associated points, with $\text{Supp}(\mathcal{G}) = Z$ and such that $i_*\mathcal{G} = \mathcal{F}$. Moreover, set $\mathcal{N} = \mathcal{L}|_Z$.

By Divisors, Lemma 26.15.12 the invertible sheaf $\mathcal{N}$ has a regular meromorphic section s over Z. Let us denote $\mathcal{J} \subset \mathcal{O}_Z$ the sheaf of denominators of s. By Lemma 29.26.1 there exist short exact sequences

$$\begin{matrix} 0 & \to & \mathcal{J}\mathcal{G} & \xrightarrow{1} & \mathcal{G} & \to & \mathcal{Q}_1 & \to & 0 \\ 0 & \to & \mathcal{J}\mathcal{G} & \xrightarrow{s} & \mathcal{G} \otimes_{\mathcal{O}_Z} \mathcal{N} & \to & \mathcal{Q}_2 & \to & 0 \end{matrix}$$

such that $\dim_\delta(\text{Supp}(\mathcal{Q}_i)) \leq k$ and such that the cycle $[\mathcal{Q}_2]_k - [\mathcal{Q}_1]_k$ is a representative of $c_1(\mathcal{N}) \cap [\mathcal{G}]_{k+1}$. We see (using the fact that $i_*(\mathcal{G} \otimes \mathcal{N}) = \mathcal{F} \otimes \mathcal{L}$ by the projection formula, see Cohomology, Lemma 18.7.2) that

$$[\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{F}] = [i_*\mathcal{Q}_2] - [i_*\mathcal{Q}_1]$$

in $K_0(\mathrm{Coh}_{\leq k+1}(X)/\mathrm{Coh}_{\leq k-1}(X))$. This already shows that $[\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{L}]-[\mathcal{F}]$ is an element of $B_k(X)$. Moreover we have

$$\begin{aligned} [i_*\mathcal{Q}_2]_k - [i_*\mathcal{Q}_1]_k &= i_*\left([\mathcal{Q}_2]_k - [\mathcal{Q}_1]_k\right) \\ &= i_*\left(c_1(\mathcal{N})\cap[\mathcal{G}]_{k+1}\right) \\ &= c_1(\mathcal{L})\cap i_*[\mathcal{G}]_{k+1} \\ &= c_1(\mathcal{L})\cap[\mathcal{F}]_{k+1} \end{aligned}$$

by the above and Lemmas 29.25.6 and 29.12.3. And this agree with the image of the element under $B_k(X)\to A_k(X)$ by definition. Hence the lemma is proved. □

29.27. Blowing up lemmas

In this section we prove some lemmas on representing Cartier divisors by suitable effective Cartier divisors on blow-ups. These lemmas can be found in [**Ful98**, Section 2.4]. We have adapted the formulation so they also work in the non-finite type setting. It may happen that the morphism b of Lemma 29.27.7 is a composition of infinitely many blow ups, but over any given quasi-compact open $W\subset X$ one needs only finitely many blow-ups (and this is the result of loc. cit.).

Lemma 29.27.1. *Let (S,δ) be as in Situation 29.7.1. Let X, Y be locally of finite type over S. Let $f : X\to Y$ be a proper morphism. Let $D\subset Y$ be an effective Cartier divisor. Assume X, Y integral, $n=\dim_\delta(X)=\dim_\delta(Y)$ and f dominant. Then*

$$f_*[f^{-1}(D)]_{n-1} = [R(X) : R(Y)][D]_{n-1}.$$

In particular if f is birational then $f_*[f^{-1}(D)]_{n-1}=[D]_{n-1}$.

Proof. Immediate from Lemma 29.25.5 and the fact that D is the zero scheme of the canonical section 1_D of $\mathcal{O}_X(D)$. □

Lemma 29.27.2. *Let (S,δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Assume X integral with $\dim_\delta(X)=n$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let s be a nonzero meromorphic section of $\mathcal{L}$. Let $U\subset X$ be the maximal open subscheme such that s corresponds to a section of $\mathcal{L}$ over U. There exists a projective morphism*

$$\pi : X'\longrightarrow X$$

such that

1. *X' is integral,*
2. *$\pi|_{\pi^{-1}(U)} : \pi^{-1}(U)\to U$ is an isomorphism,*
3. *there exist effective Cartier divisors $D, E\subset X'$ such that*
$$\pi^*\mathcal{L}=\mathcal{O}_{X'}(D-E),$$
4. *the meromorphic section s corresponds, via the isomorphism above, to the meromorphic section $1_D\otimes(1_E)^{-1}$ (see Divisors, Definition 26.9.11),*
5. *we have*
$$\pi_*([D]_{n-1}-[E]_{n-1}) = div_{\mathcal{L}}(s)$$
in $Z_{n-1}(X)$.

Proof. Let $\mathcal{I}\subset\mathcal{O}_X$ be the quasi-coherent ideal sheaf of denominators of s. Namely, we declare a local section f of $\mathcal{O}_X$ to be a local section of $\mathcal{I}$ if and only if fs is a local section of $\mathcal{L}$. On any affine open $U=Spec(A)$ of X write $\mathcal{L}|_U=\widetilde{L}$ for some invertible A-module L. Then A is a Noetherian domain with fraction field $K=R(X)$ and we may think of $s|_U$

as an element of $L \otimes_A K$ (see Divisors, Lemma 26.15.6). Let $I = \{x \in A \mid xs \in L\}$. Then we see that $\mathcal{I}|_U = \tilde{I}$ (details omitted) and hence $\mathcal{I}$ is quasi-coherent.

Consider the closed subscheme $Z \subset X$ defined by $\mathcal{I}$. It is clear that $U = X \setminus Z$. This suggests we should blow up Z. Let

$$\pi : X' = \underline{\text{Proj}}_X \left(\bigoplus\nolimits_{n \geq 0} \mathcal{I}^n \right) \longrightarrow X$$

be the blowing up of X along Z. The quasi-coherent sheaf of $\mathcal{O}_X$-algebras $\bigoplus_{n \geq 0} \mathcal{I}^n$ is generated in degree 1 over $\mathcal{O}_X$. Moreover, the degree 1 part is a coherent $\mathcal{O}_X$-module, in particular of finite type. Hence we see that π is projective and $\mathcal{O}_{X'}(1)$ is relatively very ample.

By Constructions, Lemma 22.21.2 we have X' is integral. By Divisors, Lemma 26.9.18 there exists an effective Cartier divisor $E \subset X'$ such that $\pi^{-1}\mathcal{I} \cdot \mathcal{O}_{X'} = \mathcal{I}_E$. Also, by the same lemma we see that $\pi^{-1}(U) \cong U$.

Denote s' the pullback of the meromorphic section s to a meromorphic section of $\mathcal{L}' = \pi^*\mathcal{L}$ over X'. It follows from the fact that $\mathcal{I}s \subset \mathcal{L}$ that $\mathcal{I}_E s' \subset \mathcal{L}'$. In other words, s' gives rise to an $\mathcal{O}_{X'}$-linear map $\mathcal{I}_E \to \mathcal{L}'$, or in other words a section $t \in \mathcal{L}' \otimes \mathcal{O}_{X'}(E)$. By Divisors, Lemma 26.9.17 we obtain a unique effective Cartier divisor $D \subset X'$ such that $\mathcal{L}' \otimes \mathcal{O}_{X'}(E) \cong \mathcal{O}_{X'}(D)$ with t corresponding to 1_D. Reversing this procedure we conclude that $\mathcal{L}' = \mathcal{O}_{X'}(-E) \cong \mathcal{O}_{X'}(D)$ with s' corresponding to $1_D \otimes 1_E^{-1}$ as in (4).

We still have to prove (5). By Lemma 29.25.5 we have

$$\pi_*(\text{div}_{\mathcal{L}'}(s')) = \text{div}_{\mathcal{L}}(s).$$

Hence it suffices to show that $\text{div}_{\mathcal{L}'}(s') = [D]_{n-1} - [E]_{n-1}$. This follows from the equality $s' = 1_D \otimes 1_E^{-1}$ and additivity, see Lemma 29.24.3. $\square$

Definition 29.27.3. Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Assume X integral and $\dim_\delta(X) = n$. Let D_1, D_2 be two effective Cartier divisors in X. Let $Z \subset X$ be an integral closed subscheme with $\dim_\delta(Z) = n - 1$. The *ϵ-invariant* of this situation is

$$\epsilon_Z(D_1, D_2) = n_Z \cdot m_Z$$

where n_Z, resp. m_Z is the coefficient of Z in the $(n-1)$-cycle $[D_1]_{n-1}$, resp. $[D_2]_{n-1}$.

Lemma 29.27.4. *Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Assume X integral and $\dim_\delta(X) = n$. Let D_1, D_2 be two effective Cartier divisors in X. Let Z be an open and closed subscheme of the scheme $D_1 \cap D_2$. Assume $\dim_\delta(D_1 \cap D_2 \setminus Z) \leq n - 2$. Then there exists a morphism $b : X' \to X$, and Cartier divisors D_1', D_2', E on X' with the following properties*

1. *X' is integral,*
2. *b is projective,*
3. *b is the blow up of X in the closed subscheme Z,*
4. *$E = b^{-1}(Z)$,*
5. *$b^{-1}(D_1) = D_1' + E$, and $b^{-1}D_2 = D_2' + E$,*
6. *$\dim_\delta(D_1' \cap D_2') \leq n - 2$, and if $Z = D_1 \cap D_2$ then $D_1' \cap D_2' = \emptyset$,*
7. *for every integral closed subscheme W' with $\dim_\delta(W') = n - 1$ we have*
   (a) *if $\epsilon_{W'}(D_1', E) > 0$, then setting $W = b(W')$ we have $\dim_\delta(W) = n - 1$ and*

$$\epsilon_{W'}(D_1', E) < \epsilon_W(D_1, D_2),$$

(b) *if* $\epsilon_{W'}(D'_2, E) > 0$*, then setting* $W = b(W')$ *we have* $\dim_\delta(W) = n - 1$ *and*
$$\epsilon_{W'}(D'_2, E) < \epsilon_W(D_1, D_2),$$

Proof. Note that the quasi-coherent ideal sheaf $\mathcal{I} = \mathcal{I}_{D_1} + \mathcal{I}_{D_2}$ defines the scheme theoretic intersection $D_1 \cap D_2 \subset X$. Since Z is a union of connected components of $D_1 \cap D_2$ we see that for every $z \in Z$ the kernel of $\mathcal{O}_{X,z} \to \mathcal{O}_{Z,z}$ is equal to $\mathcal{I}_z$. Let $b : X' \to X$ be the blow up of X in Z. (So Zariski locally around Z it is the blow up of X in $\mathcal{I}$.) Denote $E = b^{-1}(Z)$ the corresponding effective Cartier divisor, see Divisors, Lemma 26.9.18. Since $Z \subset D_1$ we have $E \subset f^{-1}(D_1)$ and hence $D_1 = D'_1 + E$ for some effective Cartier divisor $D'_1 \subset X'$, see Divisors, Lemma 26.9.6. Similarly $D_2 = D'_2 + E$. This takes care of assertions (1) -- (5).

Note that if W' is as in (7) (a) or (7) (b), then the image W of W' is contained in $D_1 \cap D_2$. If W is not contained in Z, then b is an isomorphism at the generic point of W and we see that $\dim_\delta(W) = \dim_\delta(W') = n - 1$ which contradicts the assumption that $\dim_\delta(D_1 \cap D_2 \setminus Z) \leq n - 2$. Hence $W \subset Z$. This means that to prove (6) and (7) we may work locally around Z on X.

Thus we may assume that $X = Spec(A)$ with A a Noetherian domain, and $D_1 = Spec(A/a)$, $D_2 = Spec(A/b)$ and $Z = D_1 \cap D_2$. Set $I = (a, b)$. Since A is a domain and $a, b \neq 0$ we can cover the blow up by two patches, namely $U = Spec(A[s]/(as - b))$ and $V = Spec(A[t]/(bt - a))$. These patches are glued using the isomorphism $A[s, s^{-1}]/(as - b) \cong A[t, t^{-1}]/(bt - a)$ which maps s to t^{-1}. The effective Cartier divisor E is described by $Spec(A[s]/(as - b, a)) \subset U$ and $Spec(A[t]/(bt - a, b)) \subset V$. The closed subscheme D'_1 corresponds to $Spec(A[t]/(bt - a, t)) \subset U$. The closed subscheme D'_2 corresponds to $Spec(A[s]/(as - b, s)) \subset V$. Since ``$ts = 1$'' we see that $D'_1 \cap D'_2 = \emptyset$.

Suppose we have a prime $\mathfrak{q} \subset A[s]/(as - b)$ of height one with $s, a \in \mathfrak{q}$. Let $\mathfrak{p} \subset A$ be the corresponding prime of A. Observe that $a, b \in \mathfrak{p}$. By the dimension formula we see that $\dim(A_\mathfrak{p}) = 1$ as well. The final assertion to be shown is that
$$\text{ord}_{A_\mathfrak{p}}(a) \text{ ord}_{A_\mathfrak{p}}(b) > \text{ord}_{B_\mathfrak{q}}(a) \text{ ord}_{B_\mathfrak{q}}(s)$$
where $B = A[s]/(as - b)$. By Algebra, Lemma 7.115.1 we have $\text{ord}_{A_\mathfrak{p}}(x) \geq \text{ord}_{B_\mathfrak{q}}(x)$ for $x = a, b$. Since $\text{ord}_{B_\mathfrak{q}}(s) > 0$ we win by additivity of the ord function and the fact that $as = b$. □

Definition 29.27.5. Let X be a scheme. Let $\{D_i\}_{i \in I}$ be a locally finite collection of effective Cartier divisors on X. Suppose given a function $I \to \mathbf{Z}_{\geq 0}$, $i \mapsto n_i$. The *sum of the effective Cartier divisors* $D = \sum n_i D_i$, is the unique effective Cartier divisor $D \subset X$ such that on any quasi-compact open $U \subset X$ we have $D|_U = \sum_{D_i \cap U \neq \emptyset} n_i D_i|_U$ is the sum as in Divisors, Definition 26.9.4.

Lemma 29.27.6. *Let* (S, δ) *be as in Situation 29.7.1. Let* X *be locally of finite type over* S*. Assume* X *integral and* $\dim_\delta(X) = n$*. Let* $\{D_i\}_{i \in I}$ *be a locally finite collection of effective Cartier divisors on* X*. Suppose given* $n_i \geq 0$ *for* $i \in I$*. Then*
$$[D]_{n-1} = \sum\nolimits_i n_i [D_i]_{n-1}$$
in $Z_{n-1}(X)$*.*

Proof. Since we are proving an equality of cycles we may work locally on X. Hence this reduces to a finite sum, and by induction to a sum of two effective Cartier divisors $D = D_1 + D_2$. By Lemma 29.24.2 we see that $D_1 = \text{div}_{\mathcal{O}_X(D_1)}(1_{D_1})$ where 1_{D_1} denotes the canonical section of $\mathcal{O}_X(D_1)$. Of course we have the same statement for D_2 and D. Since $1_D = 1_{D_1} \otimes 1_{D_2}$ via the identification $\mathcal{O}_X(D) = \mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)$ we win by Lemma 29.24.3. □

Lemma 29.27.7. *Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Assume X integral and $\dim_\delta(X) = d$. Let $\{D_i\}_{i\in I}$ be a locally finite collection of effective Cartier divisors on X. Assume that for all $\{i, j, k\} \subset I$, $\#\{i, j, k\} = 3$ we have $D_i \cap D_j \cap D_k = \emptyset$. Then there exist*

1. *an open subscheme $U \subset X$ with $\dim_\delta(X \setminus U) \leq d - 3$,*
2. *a morphism $b : U' \to U$, and*
3. *effective Cartier divisors $\{D'_j\}_{j\in J}$ on U'*

with the following properties:

1. *b is proper morphism $b : U' \to U$,*
2. *U' is integral,*
3. *b is an isomorphism over the complement of the union of the pairwise intersections of the $D_i|_U$,*
4. *$\{D'_j\}_{j\in J}$ is a locally finite collection of effective Cartier divisors on U',*
5. *$\dim_\delta(D'_j \cap D'_{j'}) \leq d - 2$ if $j \neq j'$, and*
6. *$b^{-1}(D_i|_U) = \sum n_{ij} D'_j$ for certain $n_{ij} \geq 0$.*

Moreover, if X is quasi-compact, then we may assume $U = X$ in the above.

Proof. Let us first prove this in the quasi-compact case, since it is perhaps the most interesting case. In this case we produce inductively a sequence of blowups

$$X = X_0 \xleftarrow{b_0} X_1 \xleftarrow{b_1} X_2 \leftarrow \ldots$$

and finite sets of effective Cartier divisors $\{D_{n,i}\}_{i\in I_n}$. At each stage these will have the property that any triple intersection $D_{n,i} \cap D_{n,j} \cap D_{n,k}$ is empty. Moreover, for each $n \geq 0$ we will have $I_{n+1} = I_n \coprod P(I_n)$ where $P(I_n)$ denotes the set of pairs of elements of I_n. Finally, we will have

$$b_n^{-1}(D_{n,i}) = D_{n+1,i} + \sum\nolimits_{i'\in I_n, i'\neq i} D_{n+1,\{i,i'\}}$$

We conclude that for each $n \geq 0$ we have $(b_0 \circ \ldots \circ b_n)^{-1}(D_i)$ is a nonnegative integer combination of the divisors $D_{n+1,j}$, $j \in I_{n+1}$.

To start the induction we set $X_0 = X$ and $I_0 = I$ and $D_{0,i} = D_i$.

Given $(X_n, \{D_{n,i}\}_{i\in I_n})$ let X_{n+1} be the blow up of X_n in the closed subscheme $Z_n = \bigcup_{\{i,i'\}\in P(I_n)} D_{n,i} \cap D_{n,i'}$. Note that the closed subschemes $D_{n,i} \cap D_{n,i'}$ are pairwise disjoint by our assumption on triple intersections. In other words we may write $Z_n = \coprod_{\{i,i'\}\in P(I_n)} D_{n,i} \cap D_{n,i'}$. Moreover, in a Zariski neighbourhood of $D_{n,i} \cap D_{n,i'}$ the morphism b_n is equal to the blow up of the scheme X_n in the closed subscheme $D_{n,i} \cap D_{n,i'}$, and the results of Lemma 29.27.4 apply. Hence setting $D_{n+1,\{i,i'\}} = b_n^{-1}(D_i \cap D_{i'})$ we get an effective Cartier divisor. The Cartier divisors $D_{n+1,\{i,i'\}}$ are pairwise disjoint. Clearly we have $b_n^{-1}(D_{n,i}) \supset D_{n+1,\{i,i'\}}$ for every $i' \in I_n$, $i' \neq i$. Hence, applying Divisors, Lemma 26.9.6 we see that indeed $b^{-1}(D_{n,i}) = D_{n+1,i} + \sum_{i'\in I_n, i'\neq i} D_{n+1,\{i,i'\}}$ for some effective Cartier divisor $D_{n+1,i}$ on X_{n+1}. In a neighbourhood of $D_{n+1,\{i,i'\}}$ these divisors $D_{n+1,i}$ play the role of the primed divisors of Lemma 29.27.4. In particular we conclude that $D_{n+1,i} \cap D_{n+1,i'} = \emptyset$ if $i \neq i'$,

$i, i' \in I_n$ by part (6) of Lemma 29.27.4. This already implies that triple intersections of the divisors $D_{n+1,i}$ are zero.

OK, and at this point we can use the quasi-compactness of X to conclude that the invariant

$$\epsilon(X, \{D_i\}_{i \in I}) = \max\{\epsilon_Z(D_i, D_{i'}) \mid Z \subset X, \dim_\delta(Z) = d - 1, \{i, i'\} \in \mathcal{P}(I)\} \quad (29.27.7.1)$$

is finite, since after all each D_i has at most finitely many irreducible components. We claim that for some n the invariant $\epsilon(X_n, \{D_{n,i}\}_{i \in I_n})$ is zero. Namely, if not then by Lemma 29.27.4 we have a strictly decreasing sequence

$$\epsilon(X, \{D_i\}_{i \in I}) = \epsilon(X_0, \{D_{0,i}\}_{i \in I_0}) > \epsilon(X_1, \{D_{1,i}\}_{i \in I_1}) > \ldots$$

of positive integers which is a contradiction. Take n with invariant $\epsilon(X_n, \{D_{n,i}\}_{i \in I_n})$ equal to zero. This means that there is no integral closed subscheme $Z \subset X_n$ and no pair of indices $i, i' \in I_n$ such that $\epsilon_Z(D_{n,i}, D_{n,i'}) > 0$. In other words, $\dim_\delta(D_{n,i}, D_{n,i'}) \leq d - 2$ for all pairs $\{i, i'\} \in \mathcal{P}(I_n)$ as desired.

Next, we come to the general case where we no longer assume that the scheme X is quasi-compact. The problem with the idea from the first part of the proof is that we may get and infinite sequence of blow ups with centers dominating a fixed point of X. In order to avoid this we cut out suitable closed subsets of codimension ≥ 3 at each stage. Namely, we will construct by induction a sequence of morphisms having the following shape

$$\begin{tikzcd}
X = X_0 & & & \\
U_0 \arrow[u, "j_0"] & X_1 \arrow[l, "b_0"'] & & \\
& U_1 \arrow[u, "j_1"] & X_2 \arrow[l, "b_1"'] & \\
& & U_2 \arrow[u, "j_2"] & X_3 \arrow[l, "b_2"']
\end{tikzcd}$$

Each of the morphisms $j_n : U_n \to X_n$ will be an open immersion. Each of the morphisms $b_n : X_{n+1} \to U_n$ will be a proper birational morphism of integral schemes. As in the quasi-compact case we will have effective Cartier divisors $\{D_{n,i}\}_{i \in I_n}$ on X_n. At each stage these will have the property that any triple intersection $D_{n,i} \cap D_{n,j} \cap D_{n,k}$ is empty. Moreover, for each $n \geq 0$ we will have $I_{n+1} = I_n \coprod \mathcal{P}(I_n)$ where $\mathcal{P}(I_n)$ denotes the set of pairs of elements of I_n. Finally, we will arrange it so that

$$b_n^{-1}(D_{n,i}|_{U_n}) = D_{n+1,i} + \sum\nolimits_{i' \in I_n, i' \neq i} D_{n+1,\{i,i'\}}$$

We start the induction by setting $X_0 = X$, $I_0 = I$ and $D_{0,i} = D_i$.

Given $(X_n, \{D_{n,i}\})$ we construct the open subscheme U_n as follows. For each pair $\{i, i'\} \in \mathcal{P}(I_n)$ consider the closed subscheme $D_{n,i} \cap D_{n,i'}$. This has ``good'' irreducible components which have $\delta$-dimension $d - 2$ and ``bad'' irreducible components which have δ-dimension $d - 1$. Let us set

$$\text{Bad}(i, i') = \bigcup\nolimits_{W \subset D_{n,i} \cap D_{n,i'} \text{ irred. comp. with } \dim_\delta(W) = d-1} W$$

and similarly

$$\mathrm{Good}(i,i') = \bigcup\nolimits_{W \subset D_{n,i} \cap D_{n,i'} \text{ irred. comp. with } \dim_\delta(W)=d-2} W.$$

Then $D_{n,i} \cap D_{n,i'} = \mathrm{Bad}(i,i') \cup \mathrm{Good}(i,i')$ and moreover we have $\dim_\delta(\mathrm{Bad}(i,i') \cap \mathrm{Good}(i,i')) \leq d-3$. Here is our choice of U_n:

$$U_n = X_n \setminus \bigcup\nolimits_{\{i,i'\} \in P(I_n)} \mathrm{Bad}(i,i') \cap \mathrm{Good}(i,i').$$

By our condition on triple intersections of the divisors $D_{n,i}$ we see that the union is actually a disjoint union. Moreover, we see that (as a scheme)

$$D_{n,i}|_{U_n} \cap D_{n,i'}|_{U_n} = Z_{n,i,i'} \coprod G_{n,i,i'}$$

where $Z_{n,i,i'}$ is δ-equidimension of dimension $d-1$ and $G_{n,i,i'}$ is δ-equidimensional of dimension $d-2$. (So toplogically $Z_{n,i,i'}$ is the union of the bad components but throw out intersections with good components.) Finally we set

$$Z_n = \bigcup\nolimits_{\{i,i'\} \in P(I_n)} Z_{n,i,i'} = \coprod\nolimits_{\{i,i'\} \in P(I_n)} Z_{n,i,i'},$$

and we let $b_n : X_{n+1} \to X_n$ be the blow up in Z_n. Note that Lemma 29.27.4 applies to the morphism $b_n : X_{n+1} \to X_n$ locally around each of the loci $D_{n,i}|_{U_n} \cap D_{n,i'}|_{U_n}$. Hence, exactly as in the first part of the proof we obtain effective Cartier divisors $D_{n+1,\{i,i'\}}$ for $\{i,i'\} \in P(I_n)$ and effective Cartier divisors $D_{n+1,i}$ for $i \in I_n$ such that $b_n^{-1}(D_{n,i}|_{U_n}) = D_{n+1,i} + \sum_{i' \in I_n, i' \neq i} D_{n+1,\{i,i'\}}$. For each n denote $\pi_n : X_n \to X$ the morphism obtained as the composition $j_0 \circ \ldots \circ j_{n-1} \circ b_{n-1}$.

Claim: given any quasi-compact open $V \subset X$ for all sufficiently large n the maps

$$\pi_n^{-1}(V) \leftarrow \pi_{n+1}^{-1}(V) \leftarrow \ldots$$

are all isomorphisms. Namely, if the map $\pi_n^{-1}(V) \leftarrow \pi_{n+1}^{-1}(V)$ is not an isomorphism, then $Z_{n,i,i'} \cap \pi_n^{-1}(V) \neq \emptyset$ for some $\{i,i'\} \in P(I_n)$. Hence there exists an irreducible component $W \subset D_{n,i} \cap D_{n,i'}$ with $\dim_\delta(W) = d-1$. In particular we see that $\epsilon_W(D_{n,i}, D_{n,i'}) > 0$. Applying Lemma 29.27.4 repeatedly we see that

$$\epsilon_W(D_{n,i}, D_{n,i'}) < \epsilon(V, \{D_i|_V\}) - n$$

with $\epsilon(V, \{D_i|_V\})$ as in (29.27.7.1). Since V is quasi-compact, we have $\epsilon(V, \{D_i|_V\}) < \infty$ and taking $n > \epsilon(V, \{D_i|_V\})$ we see the result.

Note that by construction the difference $X_n \setminus U_n$ has $\dim_\delta(X_n \setminus U_n) \leq d-3$. Let $T_n = \pi_n(X_n \setminus U_n)$ be its image in X. Traversing in the diagram of maps above using each b_n is closed it follows that $T_0 \cup \ldots \cup T_n$ is a closed subset of X for each n. Any $t \in T_n$ satisfies $\delta(t) \leq d-3$ by construction. Hence $\overline{T_n} \subset X$ is a closed subset with $\dim_\delta(T_n) \leq d-3$. By the claim above we see that for any quasi-compact open $V \subset X$ we have $T_n \cap V \neq \emptyset$ for at most finitely many n. Hence $\{\overline{T_n}\}_{n \geq 0}$ is a locally finite collection of closed subsets, and we may set $U = X \setminus \bigcup \overline{T_n}$. This will be U as in the lemma.

Note that $U_n \cap \pi_n^{-1}(U) = \pi_n^{-1}(U)$ by construction of U. Hence all the morphisms

$$b_n : \pi_{n+1}^{-1}(U) \longrightarrow \pi_n^{-1}(U)$$

are proper. Moreover, by the claim they eventually become isomorphisms over each quasi-compact open of X. Hence we can define

$$U' = lim_n \, \pi_n^{-1}(U).$$

The induced morphism $b : U' \to U$ is proper since this is local on U, and over each compact open the limit stabilizes. Similarly we set $J = \bigcup_{n\geq 0} I_n$ using the inclusions $I_n \to I_{n+1}$ from the construction. For $j \in J$ choose an n_0 such that j corresponds to $i \in I_{n_0}$ and define $D'_j = lim_{n\geq n_0} D_{n,i}$. Again this makes sense as locally over X the morphisms stabilize. The other claims of the lemma are verified as in the case of a quasi-compact X. □

29.28. Intersecting with effective Cartier divisors

To be able to prove the commutativity of intersection products we need a little more precision in terms of supports of the cycles. Here is the relevant notion.

Definition 29.28.1. Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Let D be an effective Cartier divisor on X, and denote $i : D \to X$ the closed immersion. We define, for every integer k, a *Gysin homomorphism*

$$i^* : Z_{k+1}(X) \to A_k(D).$$

(1) Given a integral closed subscheme $W \subset X$ with $\dim_\delta(W) = k + 1$ we define
 (a) if $W \not\subset D$, then $i^*[W] = [D \cap W]_k$ as a k-cycle on D, and
 (b) if $W \subset D$, then $i^*[W] = i'_*(c_1(\mathcal{O}_X(D)|_W) \cap [W])$, where $i' : W \to D$ is the induced closed immersion.
(2) For a general $(k + 1)$-cycle $\alpha = \sum n_j[W_j]$ we set
$$i^*\alpha = \sum n_j i^*[W_j]$$
(3) We denote $D \cdot \alpha = i_* i^*\alpha$ the pushforward of the class to a class on X.

In fact, as we will see later, this Gysin homomorphism i^* can be viewed as an example of a non-flat pull back. Thus we will sometimes informally call the class $i^*\alpha$ the *pullback* of the class α.

Lemma 29.28.2. *Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Let D be an effective Cartier divisor on X. Let α be a $(k + 1)$-cycle on X. Then $D \cdot \alpha = c_1(\mathcal{O}_X(D)) \cap \alpha$ in $A_k(X)$.*

Proof. Write $\alpha = \sum n_j[W_j]$ where $i_j : W_j \to X$ are integral closed subschemes with $\dim_\delta(W_j) = k$. Since D is the zero scheme of the canonical section 1_D of $\mathcal{O}_X(D)$ we see that $D \cap W_j$ is the zero scheme of the restriction $1_D|_{W_j}$. Hence for each j such that $W_j \not\subset D$ we have $c_1(\mathcal{O}_X(D)) \cap [W_j] = [D \cap W_j]_k$ by Lemma 29.25.3. So we have

$$c_1(\mathcal{O}_X(D)) \cap \alpha = \sum\nolimits_{W_j \not\subset D} n_j[D \cap W_j]_k + \sum\nolimits_{W_j \subset D} n_j i_{j,*}(c_1(\mathcal{O}_X(D)|_{W_j}) \cap [W_j])$$

in $A_k(X)$ by Definition 29.25.1. The right hand side matches (termwise) the push forward of the class $i^*\alpha$ on D from Definition 29.28.1. Hence we win. □

The following lemma will be superseded later.

Lemma 29.28.3. *Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Let D be an effective Cartier divisor on X. Let $W \subset X$ be a closed subscheme such that $D' = W \cap D$ is an effective Cartier divisor on W.*

$$\begin{tikzcd} D' \arrow[r, "i'"] \arrow[d, "i''"'] & W \arrow[d] \\ D \arrow[r, "i"] & X \end{tikzcd}$$

For any $(k + 1)$-cycle on W we have $i^*\alpha = (i'')_*(i')^*\alpha$ in $A_k(D)$.

Proof. Suppose $\alpha = [Z]$ for some integral closed subscheme $Z \subset W$. In case $Z \not\subset D$ we have $Z \cap D' = Z \cap D$ scheme theoretically. Hence the equality holds as cycles. In case $Z \subset D$ we also have $Z \subset D'$ and the equality holds since $\mathcal{O}_X(D)|_Z \cong \mathcal{O}_W(D')|_Z$ and the definition of i^* and $(i')^*$ in these cases. □

Lemma 29.28.4. *Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Let $i : D \to X$ be an effective Cartier divisor on X.*

(1) *Let $Z \subset X$ be a closed subscheme such that $\dim_\delta(Z) \leq k+1$ and such that $D \cap Z$ is an effective Cartier divisor on Z. Then $i^*[Z]_{k+1} = [D \cap Z]_k$.*
(2) *Let $\mathcal{F}$ be a coherent sheaf on X such that $\dim_\delta(\mathit{Support}(\mathcal{F})) \leq k+1$ and $1_D : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$ is injective. Then*
$$i^*[\mathcal{F}]_{k+1} = [i^*\mathcal{F}]_k$$
in $A_k(D)$.

Proof. Assume $Z \subset X$ as in (1). Then set $\mathcal{F} = \mathcal{O}_Z$. The assumption that $D \cap Z$ is an effective Cartier divisor is equivalent to the assumption that $1_D : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$ is injective. Moreover $[Z]_{k+1} = [\mathcal{F}]_{k+1}]$ and $[D \cap Z]_k = [\mathcal{O}_{D\cap Z}]_k = [i^*\mathcal{F}]_k$. See Lemma 29.10.3. Hence part (1) follows from part (2).

Write $[\mathcal{F}]_{k+1} = \sum m_j[W_j]$ with $m_j > 0$ and pairwise distinct integral closed subschemes $W_j \subset X$ of δ-dimension $k+1$. The assumption that $1_D : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$ is injective implies that $W_j \not\subset D$ for all j. By definition we see that
$$i^*[\mathcal{F}]_{k+1} = \sum [D \cap W_j]_k.$$
We claim that
$$\sum [D \cap W_j]_k = [i^*\mathcal{F}]_k$$
as cycles. Let $Z \subset D$ be an integral closed subscheme of δ-dimension k. Let $\xi \in Z$ be its generic point. Let $A = \mathcal{O}_{X,\xi}$. Let $M = \mathcal{F}_\xi$. Let $f \in A$ be an element generating the ideal of D, i.e., such that $\mathcal{O}_{D,\xi} = A/fA$. By assumption $\dim(M) = 1$, $f : M \to M$ is injective, and $\text{length}_A(M/fM) < \infty$. Moreover, $\text{length}_A(M/fM)$ is the coefficient of $[Z]$ in $[i^*\mathcal{F}]_k$. On the other hand, let $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ be the minimal primes in the support of M. Then
$$\sum \text{length}_{A_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i}) \text{ord}_{A/\mathfrak{q}_i}(f)$$
is the coefficient of $[Z]$ in $\sum [D \cap W_j]_k$. Hence we see the equality by Lemma 29.5.6. □

Lemma 29.28.5. *Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Let $\{i_j : D_j \to X\}_{j \in J}$ be a locally finite collection of effective Cartier divisors on X. Let $n_j > 0$, $j \in J$. Set $D = \sum_{j\in J} n_j D_j$, and denote $i : D \to X$ the inclusion morphism. Let $\alpha \in Z_{k+1}(X)$. Then*
$$p : \coprod\nolimits_{j \in J} D_j \longrightarrow D$$
is proper and
$$i^*\alpha = p_*\left(\sum n_j i_j^*\alpha\right)$$
in $A_k(D)$.

Proof. The proof of this lemma is made a bit longer than expected by a subtlety concerning infinite sums of rational equivalences. In the quasi-compact case the family D_j is finite and the result is altogether easy and a straightforward consequence of Lemmas 29.24.2 and 29.24.3 and the definitions.

The morphism p is proper since the family $\{D_j\}_{j\in J}$ is locally finite. Write $\alpha = \sum_{a\in A} m_a[W_a]$ with $W_a \subset X$ an integral closed subscheme of δ-dimension $k+1$. Denote $i_a : W_a \to X$ the closed immersion. We assume that $m_a \neq 0$ for all $a \in A$ such that $\{W_a\}_{a\in A}$ is locally finite on X.

Observe that by Definition 29.28.1 the class $i^*\alpha$ is the class of a cycle $\sum m_a\beta_a$ for certain $\beta_a \in Z_k(W_a \cap D)$. Namely, if $W_a \not\subset D$ then $\beta_a = [D \cap W_a]_k$ and if $W_a \subset D$, then β_a is a cycle representing $c_1(\mathcal{O}_X(D)) \cap [W_a]$.

For each $a \in A$ write $J = J_{a,1} \amalg J_{a,2} \amalg J_{a,3}$ where

(1) $j \in J_{a,1}$ if and only if $W_a \cap D_j = \emptyset$,
(2) $j \in J_{a,2}$ if and only if $W_a \neq W_a \cap D_1 \neq \emptyset$, and
(3) $j \in J_{a,3}$ if and only if $W_a \subset D_j$.

Since the family $\{D_j\}$ is locally finite we see that $J_{a,3}$ is a finite set. For every $a \in A$ and $j \in J$ we choose a cycle $\beta_{a,j} \in Z_k(W_a \cap D_j)$ as follows

(1) if $j \in J_{a,1}$ we set $\beta_{a,j} = 0$,
(2) if $j \in J_{a,2}$ we set $\beta_{a,j} = [D_j \cap W_a]_k$, and
(3) if $j \in J_{a,3}$ we choose $\beta_{a,j} \in Z_k(W_a)$ representing $c_1(i_a^*\mathcal{O}_X(D_j)) \cap [W_j]$.

We claim that

$$\beta_a \sim_{rat} \sum\nolimits_{j\in J} n_j\beta_{a,j}$$

in $A_k(W_a \cap D)$.

Case I: $W_a \not\subset D$. In this case $J_{a,3} = \emptyset$. Thus it suffices to show that $[D\cap W_a]_k = \sum n_j[D_j \cap W_a]_k$ as cycles. This is Lemma 29.27.6.

Case II: $W_a \subset D$. In this case β_a is a cycle representing $c_1(i_a^*\mathcal{O}_X(D)) \cap [W_a]$. Write $D = D_{a,1} + D_{a,2} + D_{a,3}$ with $D_{a,s} = \sum_{j\in J_{a,s}} n_jD_j$. By Lemma 29.24.3 we have

$$\begin{aligned} c_1(i_a^*\mathcal{O}_X(D)) \cap [W_a] &= c_1(i_a^*\mathcal{O}_X(D_{a,1})) \cap [W_a] + c_1(i_a^*\mathcal{O}_X(D_{a,2})) \cap [W_a] \\ &\quad + c_1(i_a^*\mathcal{O}_X(D_{a,3})) \cap [W_a]. \end{aligned}$$

It is clear that the first term of the sum is zero. Since $J_{a,3}$ is finite we see that the last term agrees with $\sum_{j\in J_{a,3}} n_jc_1(i_a^*\mathcal{L}_j) \cap [W_a]$, see Lemma 29.24.3. This is represented by $\sum_{j\in J_{a,3}} n_j\beta_{a,j}$. Finally, by Case I we see that the middle term is represented by the cycle $\sum_{j\in J_{a,2}} n_j[D_j \cap W_a]_k = \sum_{j\in J_{a,2}} n_j\beta_{a,j}$. Whence the claim in this case.

At this point we are ready to finish the proof of the lemma. Namely, we have $i^*D \sim_{rat} \sum m_a\beta_a$ by our choice of β_a. For each a we have $\beta_a \sim_{rat} \sum_j \beta_{a,j}$ with the rational equivalence taking place on $D \cap W_a$. Since the collection of closed subschemes $D \cap W_a$ is locally finite on D, we see that also $\sum m_a\beta_a \sim_{rat} \sum_{a,j} m_a\beta_{a,j}$ on D! (See Remark 29.19.4.) Ok, and now it is clear that $\sum_a m_a\beta_{a,j}$ (viewed as a cycle on D_j) represents $i_j^*\alpha$ and hence $\sum_{a,j} m_a\beta_{a,j}$ represents $p_*\sum_j i_j^*\alpha$ and we win. □

Lemma 29.28.6. *Let (S,δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Assume X integral and* $\dim_\delta(X) = n$. *Let D, D' be effective Cartier divisors on X. Assume* $\dim_\delta(D \cap D') = n-2$. *Let $i : D \to X$, resp. $i' : D' \to X$ be the corresponding closed immersions. Then*

(1) *there exists a cycle $\alpha \in Z_{n-2}(D \cap D')$ whose pushforward to D represents $i^*[D']_{n-1} \in A_{n-2}(D)$ and whose pushforward to D' represents $(i')^*[D]_{n-1} \in A_{n-2}(D')$, and*

(2) *we have*
$$D \cdot [D']_{n-1} = D' \cdot [D]_{n-1}$$
in $A_{n-2}(X)$.

Proof. Part (3) is a trivial consequence of parts (1) and (2). Because of symmetry we only need to prove (1). Let us write $[D]_{n-1} = \sum n_a[Z_a]$ and $[D']_{n-1} = \sum m_b[Z_b]$ with Z_a the irreducible components of D and $[Z_b]$ the irreducible components of D'. According to Definition 29.28.1, we have $i^*D' = \sum m_b i^*[Z_b]$ and $(i')^*D = \sum n_a(i')^*[Z_a]$. By assumption, none of the irreducible components Z_b is contained in D, and hence $i^*[Z_b] = [Z_b \cap D]_{n-2}$ by definition. Similarly $(i')^*[Z_a] = [Z_a \cap D']_{n-2}$. Hence we are trying to prove the equality of cycles
$$\sum n_a[Z_a \cap D']_{n-2} = \sum m_b[Z_b \cap D]_{n-2}$$
which are indeed supported on $D \cap D'$. Let $W \subset X$ be an integral closed subscheme with $\dim_\delta(W) = n-2$. Let $\xi \in W$ be its generic point. Set $R = \mathcal{O}_{X,\xi}$. It is a Noetherian local domain. Note that $\dim(R) = 2$. Let $f \in R$, resp. $f' \in R$ be an element defining the ideal of D, resp. D'. By assumption $\dim(R/(f, f')) = 0$. Let $\mathfrak{q}'_1, \ldots, \mathfrak{q}'_t \subset R$ be the minimal primes over (f'), let $\mathfrak{q}_1, \ldots, \mathfrak{q}_s \subset R$ be the minimal primes over (f). The equality above comes down to the equality
$$\sum_{i=1,\ldots,s} \text{length}_{R_{\mathfrak{q}_i}}(R_{\mathfrak{q}_i}/(f))\text{ord}_{R/\mathfrak{q}_i}(f') = \sum_{j=1,\ldots,t} \text{length}_{R_{\mathfrak{q}_j}}(R_{\mathfrak{q}'_j}/(f'))\text{ord}_{R/\mathfrak{q}_j}(f).$$
By Lemma 29.5.5 applied with $M = R/(f)$ the left hand side of this equation is equal to
$$\text{length}_R(R/(f, f')) - \text{length}_R(\text{Ker}(f' : R/(f) \to R/(f)))$$
OK, and now we note that $\text{Ker}(f' : R/(f) \to R/(f))$ is canonically isomorphic to $((f) \cap (f'))/(ff')$ via the map $x \bmod (f) \mapsto f'x \bmod (ff')$. Hence the left hand side is
$$\text{length}_R(R/(f, f')) - \text{length}_R((f) \cap (f')/(ff'))$$
Since this is symmetric in f and f' we win. □

Lemma 29.28.7. *Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Assume X integral and $\dim_\delta(X) = n$. Let $\{D_j\}_{j \in J}$ be a locally finite collection of effective Cartier divisors on X. Let $n_j, m_j \geq 0$ be collections of nonnegative integers. Set $D = \sum n_j D_j$ and $D' = \sum m_j D_j$. Assume that $\dim_\delta(D_j \cap D_{j'}) = n-2$ for every $j \neq j'$. Then $D \cdot [D']_{n-1} = D' \cdot [D]_{n-1}$ in $A_{n-2}(X)$.*

Proof. This lemma is a trivial consequence of Lemmas 29.27.6 and 29.28.6 in case the sums are finite, e.g., if X is quasi-compact. Hence we suggest the reader skip the proof.

Here is the proof in the general case. Let $i_j : D_j \to X$ be the closed immersions Let $p : \coprod D_j \to X$ denote coproduct of the morphisms i_j. Let $\{Z_a\}_{a \in A}$ be the collection of irreducible components of $\bigcup D_j$. For each j we write
$$[D_j]_{n-1} = \sum d_{j,a}[Z_a].$$
By Lemma 29.27.6 we have
$$[D]_{n-1} = \sum n_j d_{j,a}[Z_a], \quad [D']_{n-1} = \sum m_j d_{j,a}[Z_a].$$
By Lemma 29.28.5 we have
$$D \cdot [D']_{n-1} = p_*\left(\sum n_j i_j^*[D']_{n-1}\right), \quad D' \cdot [D]_{n-1} = p_*\left(\sum m_{j'} i_{j'}^*[D]_{n-1}\right).$$

As in the definition of the Gysin homomorphisms (see Definition 29.28.1) we choose cycles $\beta_{a,j}$ on $D_j \cap Z_a$ representing $i_j^*[Z_a]$. (Note that in fact $\beta_{a,j} = [D_j \cap Z_a]_{n-2}$ if Z_a is not contained in D_j, i.e., there is no choice in that case.) Now since p is a closed immersion when restricted to each of the D_j we can (and we will) view $\beta_{a,j}$ as a cycle on X. Plugging in the formulas for $[D]_{n-1}$ and $[D']_{n-1}$ obtained above we see that

$$D \cdot [D']_{n-1} = \sum\nolimits_{j,j',a} n_j m_{j'} d_{j',a} \beta_{a,j}, \quad D' \cdot [D]_{n-1} = \sum\nolimits_{j,j',a} m_{j'} n_j d_{j,a} \beta_{a,j'}.$$

Moreover, with the same conventions we also have

$$D_j \cdot [D_{j'}]_{n-1} = \sum d_{j',a} \beta_{a,j}.$$

In these terms Lemma 29.28.6 (see also its proof) says that for $j \neq j'$ the cycles $\sum d_{j',a}\beta_{a,j}$ and $\sum d_{j,a}\beta_{a,j'}$ are equal as cycles! Hence we see that

$$\begin{aligned} D \cdot [D']_{n-1} &= \sum\nolimits_{j,j',a} n_j m_{j'} d_{j',a} \beta_{a,j} \\ &= \sum\nolimits_{j \neq j'} n_j m_{j'} \left(\sum\nolimits_a d_{j',a} \beta_{a,j} \right) + \sum\nolimits_{j,a} n_j m_j d_{j,a} \beta_{a,j} \\ &= \sum\nolimits_{j \neq j'} n_j m_{j'} \left(\sum\nolimits_a d_{j,a} \beta_{a,j'} \right) + \sum\nolimits_{j,a} n_j m_j d_{j,a} \beta_{a,j} \\ &= \sum\nolimits_{j,j',a} m_{j'} n_j d_{j,a} \beta_{a,j'} \\ &= D' \cdot [D]_{n-1} \end{aligned}$$

and we win. □

Here is the key lemma of this chapter. A stronger version of this lemma asserts that $D \cdot [D']_{n-1} = D' \cdot [D]_{n-1}$ holds in $A_{n-2}(D \cap D')$ for suitable representatives of the dot products involved. The first proof of the lemma together with Lemmas 29.28.5, 29.28.6, and 29.28.7 can be modified to show this (see [**Ful98**]). It is not so clear how to modify the second proof to prove the refined version. An application of the refined version is a proof that the Gysin homomorphism factors through rational equivalence. We will show this by another method later.

Lemma 29.28.8. *Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Assume X integral and $\dim_\delta(X) = n$. Let D, D' be effective Cartier divisors on X. Then*

$$D \cdot [D']_{n-1} = D' \cdot [D]_{n-1}$$

in $A_{n-2}(X)$.

First proof of Lemma 29.28.8. First, let us prove this in case X is quasi-compact. In this case, apply Lemma 29.27.7 to X and the two element set $\{D, D'\}$ of effective Cartier divisors. Thus we get a proper morphism $b : X' \to X$, a finite collection of effective Cartier divisors $D'_j \subset X'$ intersecting pairwise in codimension ≥ 2, with $b^{-1}(D) = \sum n_j D'_j$, and $b^{-1}(D') = \sum m_j D'_j$. Note that $b_*[b^{-1}(D)]_{n-1} = [D]_{n-1}$ in $Z_{n-1}(X)$ and similarly for D', see Lemma 29.27.1. Hence, by Lemma 29.25.6 we have

$$D \cdot [D']_{n-1} = b_* \left(b^{-1}(D) \cdot [b^{-1}(D')]_{n-1} \right)$$

in $A_{n-2}(X)$ and similarly for the other term. Hence the lemma follows from the equality $b^{-1}(D) \cdot [b^{-1}(D')]_{n-1} = b^{-1}(D') \cdot [b^{-1}(D)]_{n-1}$ in $A_{n-2}(X')$ of Lemma 29.28.7.

Note that in the proof above, each referenced lemma works also in the general case (when X is not assumed quasi-compact). The only minor change in the general case is that the morphism $b : U' \to U$ we get from applying Lemma 29.27.7 has as its target an open $U \subset X$

whose complement has codimension ≥ 3. Hence by Lemma 29.19.2 we see that $A_{n-2}(U) = A_{n-2}(X)$ and after replacing X by U the rest of the proof goes through unchanged. $\square$

Second proof of Lemma 29.28.8. Let $\mathcal{I} = \mathcal{O}_X(-D)$ and $\mathcal{I}' = \mathcal{O}_X(-D')$ be the invertible ideal sheaves of D and D'. We denote $\mathcal{I}_{D'} = \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{O}_{D'}$ and $\mathcal{I}'_D = \mathcal{I}' \otimes_{\mathcal{O}_X} \mathcal{O}_D$. We can restrict the inclusion map $\mathcal{I} \to \mathcal{O}_X$ to D' to get a map

$$\varphi : \mathcal{I}_{D'} \longrightarrow \mathcal{O}_{D'}$$

and similarly

$$\psi : \mathcal{I}'_D \longrightarrow \mathcal{O}_D$$

It is clear that

$$\mathrm{Coker}(\varphi) \cong \mathcal{O}_{D \cap D'} \cong \mathrm{Coker}(\psi)$$

and

$$\mathrm{Ker}(\varphi) \cong \frac{\mathcal{I} \cap \mathcal{I}'}{\mathcal{I}\mathcal{I}'} \cong \mathrm{Ker}(\psi).$$

Hence we see that

$$\gamma = [\mathcal{I}_{D'}] - [\mathcal{O}_{D'}] = [\mathcal{I}'_D] - [\mathcal{O}_D]$$

in $K_0(\mathrm{Coh}_{\leq n-1}(X))$. On the other hand it is clear that

$$[\mathcal{I}'_D]_{n-1} = [D]_{n-1}, \quad [\mathcal{I}_{D'}]_{n-1} = [D']_{n-1}.$$

and that

$$\mathcal{O}_X(D') \otimes \mathcal{I}'_D = \mathcal{O}_D, \quad \mathcal{O}_X(D) \otimes \mathcal{I}_{D'} = \mathcal{O}_{D'}.$$

By Lemma 29.26.2 (applied two times) this means that the element γ is an element of $B_{n-2}(X)$, and maps to both $c_1(\mathcal{O}_X(D')) \cap [D]_{n-1}$ and to $c_1(\mathcal{O}_X(D)) \cap [D']_{n-1}$ and we win (since the map $B_{n-2}(X) \to A_{n-2}(X)$ is well defined -- which is the key to this proof). $\square$

29.29. Commutativity

At this point we can start using the material above and start proving more interesting results.

Lemma 29.29.1. *Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Assume X integral and $\dim_\delta(X) = n$. Let $\mathcal{L}$, $\mathcal{N}$ be invertible on X. Choose a nonzero meromorphic section s of $\mathcal{L}$ and a nonzero meromorphic section t of $\mathcal{N}$. Set $\alpha = div_{\mathcal{L}}(s)$ and $\beta = div_{\mathcal{N}}(t)$. Then*

$$c_1(\mathcal{N}) \cap \alpha = c_1(\mathcal{L}) \cap \beta$$

in $A_{n-2}(X)$.

Proof. By Lemma 29.27.2 (applied twice) there exists a proper morphism $\pi : X' \to X$ and effective Cartier divisors D_1, E_1, D_2, E_2 on X' such that

$$b^*\mathcal{L} = \mathcal{O}_{X'}(D_1 - E_1), \quad b^*\mathcal{N} = \mathcal{O}_{X'}(D_2 - E_2),$$

and such that

$$\alpha = \pi_*([D_1]_{n-1} - [E_1]_{n-1}), \quad \beta = \pi_*([D_2]_{n-1} - [E_2]_{n-1}).$$

By the projection formula of Lemma 29.25.6 and the additivity of Lemma 29.25.2 it is enough to show the equality

$$c_1(\mathcal{O}_{X'}(D_1)) \cap [D_2]_{n-1} = c_1(\mathcal{O}_{X'}(D_2)) \cap [D_1]_{n-1}$$

and three other similar equalities involving D_i and E_j. By Lemma 29.28.2 this is the same as showing that $D_1 \cdot [D_2]_{n-1} = D_2 \cdot [D_1]_{n-1}$ and so on. Thus the result follows from Lemma 29.28.8. $\square$

Lemma 29.29.2. *Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Let $\mathcal{L}$ be invertible on X. The operation $\alpha \mapsto c_1(\mathcal{L}) \cap \alpha$ factors through rational equivalence to give an operation*

$$c_1(\mathcal{L}) \cap - : A_{k+1}(X) \to A_k(X)$$

Proof. Let $\alpha \in Z_{k+1}(X)$, and $\alpha \sim_{rat} 0$. We have to show that $c_1(\mathcal{L}) \cap \alpha$ as defined in Definition 29.25.1 is zero. By Definition 29.19.1 there exists a locally finite family $\{W_j\}$ of integral closed subschemes with $\dim_\delta(W_j) = k+2$ and rational functions $f_j \in R(W_j)^*$ such that

$$\alpha = \sum (i_j)_* \mathrm{div}_{W_j}(f_j)$$

Note that $p : \coprod W_j \to X$ is a proper morphism, and hence $\alpha = p_*\alpha'$ where $\alpha' \in Z_{k+1}(\coprod W_j)$ is the sum of the principal divisors $\mathrm{div}_{W_j}(f_j)$. By the projection formula (Lemma 29.25.6) we have $c_1(\mathcal{L}) \cap \alpha = p_*(c_1(p^*\mathcal{L}) \cap \alpha')$. Hence it suffices to show that each $c_1(\mathcal{L}|_{W_j}) \cap \mathrm{div}_{W_j}(f_j)$ is zero. In other words we may assume that X is integral and $\alpha = \mathrm{div}_X(f)$ for some $f \in R(X)^*$.

Assume X is integral and $\alpha = \mathrm{div}_X(f)$ for some $f \in R(X)^*$. We can think of f as a regular meromorphic section of the invertible sheaf $\mathcal{N} = \mathcal{O}_X$. Choose a meromorphic section s of $\mathcal{L}$ and denote $\beta = \mathrm{div}_{\mathcal{L}}(s)$. By Lemma 29.29.1 we conclude that

$$c_1(\mathcal{L}) \cap \alpha = c_1(\mathcal{O}_X) \cap \beta.$$

However, by Lemma 29.25.2 we see that the right hand side is zero in $A_k(X)$ as desired. ☐

For any integer $s \geq 0$ we will denote

$$c_1(\mathcal{L})^s \cap - : A_{k+s}(X) \to A_k(X)$$

the s-fold iterate of the operation $c_1(\mathcal{L}) \cap -$. This makes sense by the lemma above.

Lemma 29.29.3. *Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Let $\mathcal{L}$, $\mathcal{N}$ be invertible on X. For any $\alpha \in A_{k+2}(X)$ we have*

$$c_1(\mathcal{L}) \cap c_1(\mathcal{N}) \cap \alpha = c_1(\mathcal{N}) \cap c_1(\mathcal{L}) \cap \alpha$$

as elements of $A_k(X)$.

Proof. Write $\alpha = \sum m_j[Z_j]$ for some lolcally finite collection of integral closed subschemes $Z_j \subset X$ with $\dim_\delta(Z_j) = k+2$. Consider the proper morphism $p : \coprod Z_j \to X$. Set $\alpha' = \sum m_j[Z_j]$ as a $(k+2)$-cycle on $\coprod Z_j$. By several applications of Lemma 29.25.6 we see that $c_1(\mathcal{L}) \cap c_1(\mathcal{N}) \cap \alpha = p_*(c_1(p^*\mathcal{L}) \cap c_1(p^*\mathcal{N}) \cap \alpha')$ and $c_1(\mathcal{N}) \cap c_1(\mathcal{L}) \cap \alpha = p_*(c_1(p^*\mathcal{N}) \cap c_1(p^*\mathcal{L}) \cap \alpha')$. Hence it suffices to prove the formula in case X is integral and $\alpha = [X]$. In this case the result follows from Lemma 29.29.1 and the definitions. ☐

29.30. Gysin homomorphisms

We want to show the Gysin homomorphisms factor through rational equivalence. One method (see [**Ful98**]) is to prove a more precise version of the key Lemma 29.28.8 keeping track of supports. Having obtained this one can find anlogues of the lemmas of Section 29.29 for the Gysin homomorphism and get the result. We will use another method.

Lemma 29.30.1. *Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Let X be integral and $n = \dim_\delta(X)$. Let $a \in \Gamma(X, \mathcal{O}_X)$ be a nonzero function. Let $i : D = Z(a) \to X$ be the closed immersion of the zero scheme of a. Let $f \in R(X)^*$. In this case $i^*\mathrm{div}_X(f) = 0$ in $A_{n-2}(D)$.*

Proof. Write $\text{div}_X(f) = \sum n_j[Z_j]$ for some integral closed subschemes $Z_j \subset X$ of δ-dimension $n-1$. We may assume that the family $\{Z_j\}_{j\in J}$ is locally finite and that $f \in \Gamma(U, \mathcal{O}_U^*)$ where $U = X \setminus \bigcup Z_j$ (see Lemma 29.16.3 and its proof).

Write $J = J_1 \coprod J_2$ where $J_1 = \{j \in J \mid Z_j \subset D\}$. Note that $\mathcal{O}_X(D) \cong \mathcal{O}_X$ because a^{-1} is a trivializing global section. Hence by Definition 29.28.1 of i^* we see that $i^*\text{div}_X(f)$ is represented by

$$\sum\nolimits_{j\in J_2} n_j[D \cap Z_j]_{n-2}.$$

Namely, the terms involving $c_1(\mathcal{O}_X(D)|_{Z_j}) \cap Z_j$ may be dropped since $c_1(\mathcal{O}) \cap -$ is the zero operation anyway (see Lemma 29.25.2).

For each j let $\xi_j \in Z_j$ be its generic point. Let $B_j = \mathcal{O}_{X,\xi_j}$, which has residue field $\kappa_j = \kappa(\xi_j) = R(Z_j)$. For $j \in J_1$, let

$$f_j = d_{B_j}(f, a)$$

be the tame symbol, see Definition 29.4.5. We claim that we have the following equality of cycles

$$\sum\nolimits_{j\in J_2} n_j[D \cap Z_j]_{n-2} = \sum\nolimits_{j\in J_1} (Z_j \to D)_*\text{div}_{Z_j}(f_j)$$

on D. Indeed, note that $[D \cap Z_j]_{n-2} = \text{div}_{Z_j}(a)$. Hence $n_j[D \cap Z_j]_{n-2} = \text{div}_{Z_j}(a^{n_j})$. Since $n_j = \text{ord}_{B_j}(f)$ we see that in fact also $n_j[D \cap Z_j]_{n-2} = \text{div}_{Z_j}(d_{B_j}(a, f))$, as a is a unit in B_j see Lemma 29.4.6. Note that $d_{B_j}(f, a) = d_{B_j}(a, f)^{-1}$, see Lemma 29.4.4. Hence altogether we are trying to show that

$$\sum\nolimits_{j\in J} (Z_j \to D)_*\text{div}_{Z_j}(d_{B_j}(a, f)) = 0$$

as an $(n-2)$-cycle. Consider any codimension 2 integral closed subscheme $W \subset X$ with generic point $\zeta \in X$. Set $A = \mathcal{O}_{X,\zeta}$. Applying Lemma 29.6.1 to (A, a, f) we see that the coefficient of $[W]$ in the expression above is zero as desired. □

Lemma 29.30.2. *Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Let X be integral and $n = \dim_\delta(X)$. Let $i : D \to X$ be an effective Cartier divisor. Let $f \in R(X)^*$. In this case $i^*div_X(f) = 0$ in $A_{n-2}(D)$.*

Proof. This proof is a repeat of the proof of Lemma 29.30.1. So make sure you've read that one first.

Write $\text{div}_X(f) = \sum n_j[Z_j]$ for some integral closed subschemes $Z_j \subset X$ of δ-dimension $n-1$. We may assume that the family $\{Z_j\}_{j\in J}$ is locally finite and that $f \in \Gamma(U, \mathcal{O}_U^*)$ where $U = X \setminus \bigcup Z_j$ (see Lemma 29.16.3 and its proof).

Write $J = J_1 \coprod J_2$ where $J_1 = \{j \in J \mid Z_j \subset D\}$. For each j let $\xi_j \in Z_j$ be its generic point. Let us write $\mathcal{L} = \mathcal{O}_X(D)$. Choose $\tilde{s}_j \in \mathcal{L}_{\xi_j}$ a generator. Denote $s_j \in \mathcal{L}_{\xi_j} \otimes \kappa(\xi_j)$ the corresponding nonzero meromorphic section of $\mathcal{L}|_{Z_j}$. Then by Definition 29.28.1 of i^* we see that $i^*\text{div}_X(f)$ is represented by the cycle

$$\sum\nolimits_{j\in J_2} n_j[D \cap Z_j]_{n-2} + \sum\nolimits_{j\in J_2} n_j\text{div}_{\mathcal{L}|_{Z_j}}(s_j)$$

on D. Our goal is to show that this is rationally equivalent to zero on D.

Let $B_j = \mathcal{O}_{X,\xi_j}$, which has residue field $\kappa_j = \kappa(\xi_j) = R(Z_j)$. Write $s = a_j\tilde{s}_j$ for some $a_j \in B_j$. For $j \in J_1$ let

$$f_j = d_{B_j}(f, a_j) \in \kappa_j^* = R(Z_j)^*$$

be the tame symbol, see Definition 29.4.5. We claim that we have the following equality of cycles

$$\sum\nolimits_{j\in J_2} n_j[D\cap Z_j]_{n-2} + \sum\nolimits_{j\in J_2} n_j\mathrm{div}_{\mathcal{L}|_{Z_j}}(s_j) = \sum\nolimits_{j\in J_1}(Z_j\to D)_*\mathrm{div}_{Z_j}(f_j)$$

on D. This will clearly prove the lemma.

Note that for $j\in J_2$ we have $[D\cap Z_j]_{n-2} = \mathrm{div}_{\mathcal{L}|_{Z_j}}(s|_{Z_j})$. Since $s|_{Z_j} = a_j|_{Z_j}s_j$ we see that $[D\cap Z_j]_{n-2} = \mathrm{div}_{\mathcal{L}|_{Z_j}}(s_j) + \mathrm{div}_{Z_j}(a_j|_{Z_j})$. Hence, still for $j\in J_2$, we have

$$n_j[D\cap Z_j]_{n-2} = n_j\mathrm{div}_{\mathcal{L}|_{Z_j}}(s_j) + \mathrm{div}_{Z_j}((a_j|_{Z_j})^{n_j})$$

Since $n_j = \mathrm{ord}_{B_j}(f)$ we see that $\mathrm{div}_{Z_j}((a_j|_{Z_j})^{n_j}) = \mathrm{div}_{Z_j}(d_{B_j}(a_j, f))$, as a_j is a unit in B_j (since $j\in J_2$), see Lemma 29.4.6. Note that $d_{B_j}(f, a_j) = d_{B_j}(a_j, f)^{-1}$, see Lemma 29.4.4. Hence altogether we are trying to show that

$$\sum\nolimits_{j\in J} n_j\mathrm{div}_{\mathcal{L}|_{Z_j}}(s_j) = \sum\nolimits_{j\in J}(Z_j\to D)_*\mathrm{div}_{Z_j}(d_{B_j}(a_j, f)) \tag{29.30.2.1}$$

as an $(n-2)$-cycle.

Consider any codimension 2 integral closed subscheme $W\subset X$ with generic point $\zeta\in X$. Set $A = \mathcal{O}_{X,\zeta}$. Choose a generator $s_\zeta\in\mathcal{L}_\zeta$. For those j such that $\zeta\in Z_j$ we may write $\tilde{s}_j = b_js_\zeta$ with $b_j\in B_j^*$. We may also write $s = a_\zeta s_\zeta$ for some $a_\zeta\in A$. Then we see that $a_j = b_ja_\zeta$. The coefficient of $[W]$ on the right hand side of Equation (29.30.2.1) is

$$\sum\nolimits_{\zeta\in Z_j} n_j\mathrm{ord}_{A/\mathfrak{q}_j}(\overline{b_j}).$$

where $\mathfrak{q}_j\subset A$ is the height one prime corresponding to Z_j. Note that $B_j = A_{\mathfrak{q}_j}$ in this case. The coefficient of $[W]$ on the left hand side of Equation (29.30.2.1) is

$$\sum\nolimits_{\zeta\in Z_j}\mathrm{ord}_{A/\mathfrak{q}_j}(d_{A_{\mathfrak{q}_j}}(b_ja_\zeta, f)).$$

Since b_j is a unit, and $n_j = \mathrm{ord}_{A_{\mathfrak{q}_j}}(f)$ we see that $d_{A_{\mathfrak{q}_j}}(b_ja_\zeta, f) = \overline{b_j}^{n_j}d_{A_{\mathfrak{q}_j}}(a_\zeta, f)$ by Lemmas 29.4.4 and 29.4.6. By additivity of ord we see that it suffices to prove

$$0 = \sum\nolimits_{\zeta\in Z_j}\mathrm{ord}_{A/\mathfrak{q}_j}(d_{A_{\mathfrak{q}_j}}(a_\zeta, f))$$

which is Lemma 29.6.1. □

Lemma 29.30.3. *Let (S,δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Let $i : D\to X$ be an effective Cartier divisor on X. The Gysin homomorphism factors through rational equivalence to give a map $i^* : A_{k+1}(X)\to A_k(D)$.*

Proof. Let $\alpha\in Z_{k+1}(X)$ and assume that $\alpha\sim_{rat} 0$. This means there exists a locally finite collection of integral closed subschemes $W_j\subset X$ of δ-dimension $k+2$ and $f_j\in R(W_j)^*$ such that $\alpha = \sum i_{j,*}\mathrm{div}_{W_j}(f_j)$. By construction of the map i^* we see that $i^*\alpha = \sum i^*i_{j,*}\mathrm{div}_{W_j}(f_j)$ where each cycle $i^*i_{j,*}\mathrm{div}_{W_j}(f_j)$ is supported on $D\cap W_j$. If we can show that each $i^*i_{j,*}\mathrm{div}_{W_j}(f_j)$ is rationally equivalent on $W_j\cap D$, then we see that $i^*\alpha\sim_{rat} 0$ (this is clear if the sum is finite, in general see Remark 29.19.4).

Pick an index j. If $W_j\subset D$, then we see that $i^*i_{j,*}\mathrm{div}_{W_j}(f_j)$ is simply equal to

$$i'_{j,*}c_1(\mathcal{O}_X(D)|_{W_j})\cap\mathrm{div}_{W_j}(f_j)$$

where $i'_j : W_j \to D$ is the inclusion map. This is rationally equivalent to zero by Lemma 29.29.2. If $W_j \not\subset D$, then we see that $i^* i_{j,*}\text{div}_{W_j}(f_j)$ is simply equal to

$$(i')^* \text{div}_{W_j}(f_j)$$

where $i' : D \cap W_j \to W_j$ is the corresponding closed immersion (see Lemma 29.28.3). Hence in this case Lemma 29.30.2 applies, and we win. □

29.31. Relative effective Cartier divisors

Lemma 29.31.1. *Let $A \to B$ be a ring map. Let $f \in B$. Assume that*

(1) *$A \to B$ is flat,*
(2) *f is a nonzero divisor, and*
(3) *$A \to B/fB$ is flat.*

Then for every ideal $I \subset A$ the map $f : B/IB \to B/IB$ is injective.

Proof. Note that $IB = I \otimes_A B$ and $I(B/fB) = I \otimes_A B/fB$ by the flatness of B and B/fB over A. In particular $IB/fIB \cong I \otimes_A B/fB$ maps injectively into B/fB. Hence the result follows from the snake lemma applied to the diagram

$$\begin{tikzcd} 0 \arrow[r] & I \otimes_A B \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "f"] & B/IB \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & I \otimes_A B \arrow[r] & B \arrow[r] & B/IB \arrow[r] & 0 \end{tikzcd}$$

with exact rows. □

Lemma 29.31.2. *Let (S, δ) be as in Situation 29.7.1. Let X, Y be locally of finite type over S. Let $p : X \to Y$ be a flat morphism of relative dimension r. Let $i : D \to X$ be an effective Cartier divisor with the property that $p|_D : D \to Y$ is flat of relative dimension $r - 1$. Let $\mathcal{L} = \mathcal{O}_X(D)$. For any $\alpha \in A_{k+1}(Y)$ we have*

$$i^* p^* \alpha = (p|_D)^* \alpha$$

in $A_{k+r}(D)$ and

$$c_1(\mathcal{L}) \cap p^* \alpha = i_*((p|_D)^* \alpha)$$

in $A_{k+r}(X)$.

Proof. Let $W \subset Y$ be an integral closed subvariety of δ-dimension $k + 1$. By Lemma 29.31.1 we see that $D \cap p^{-1}W$ is an effective Cartier divisor on $p^{-1}W$. By Lemma 29.28.4 we see that $i^*[p^{-1}W]_{k+r+1} = [D \cap W]_{k+r} = [(p|_D)^{-1}(W)]_{k+r}$. Since by definition $p^*[W] = [p^{-1}W]_{k+r+1}$ and $(p|_D)^*[W] = [(p|_D)^{-1}(W)]_{k+r}$ we see we have equality of cycles. Hence if $\alpha = \sum m_j[W_j]$, then we get $i^*\alpha = \sum m_j i^*[W_j] = \sum m_j (p|_D)^*[W_j]$ as cycles. This proves then first equality. To deduce the second from the first apply Lemma 29.28.2. □

29.32. Affine bundles

Lemma 29.32.1. *Let (S, δ) be as in Situation 29.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a flat morphism of relative dimension r. Assume that for every $y \in Y$, there exists an open neighbourhood $U \subset Y$ such that $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is identified with the morphism $U \times \mathbf{A}^r \to U$. Then $f^* : A_k(Y) \to A_{k+r}(X)$ is surjective for all $k \in \mathbf{Z}$.*

Proof. Let $\alpha \in A_{k+r}(X)$. Write $\alpha = \sum m_j[W_j]$ with $m_j \neq 0$ and W_j pairwise distinct integral closed subschemes of δ-dimension $k+r$. Then the family $\{W_j\}$ is locally finite in X. For any quasi-compact open $V \subset Y$ we see that $f^{-1}(V) \cap W_j$ is nonempty only for finitely many j. Hence the collection $Z_j = \overline{f(W_j)}$ of closures of images is a locally finite collection of integral closed subschemes of Y.

Consider the fibre product diagrams

$$\begin{tikzcd} f^{-1}(Z_j) \arrow[r] \arrow[d, "f_j"'] & X \arrow[d, "f"] \\ Z_j \arrow[r] & Y \end{tikzcd}$$

Suppose that $[W_j] \in Z_{k+r}(f^{-1}(Z_j))$ is rationally equivalent to $f_j^*\beta_j$ for some k-cycle $\beta_j \in A_k(Z_j)$. Then $\beta = \sum m_j\beta_j$ will be a k-cycle on Y and $f^*\beta = \sum m_j f_j^*\beta_j$ will be rationally equivalent to α (see Remark 29.19.4). This reduces us to the case Y integral, and $\alpha = [W]$ for some integral closed subscheme of X dominating Y. In particular we may assume that $d = \dim_\delta(Y) < \infty$.

Hence we can use induction on $d = \dim_\delta(Y)$. If $d < k$, then $A_{k+r}(X) = 0$ and the lemma holds. By assumption there exists a dense open $V \subset Y$ such that $f^{-1}(V) \cong V \times \mathbf{A}^r$ as schemes over V. Suppose that we can show that $\alpha|_{f^{-1}(V)} = f^*\beta$ for some $\beta \in Z_k(V)$. By Lemma 29.14.2 we see that $\beta = \beta'|_V$ for some $\beta' \in Z_k(Y)$. By the exact sequence $A_k(f^{-1}(Y \setminus V)) \to A_k(X) \to A_k(f^{-1}(V))$ of Lemma 29.19.2 we see that $\alpha - f^*\beta'$ comes from a cycle $\alpha' \in A_{k+r}(f^{-1}(Y \setminus V))$. Since $\dim_\delta(Y \setminus V) < d$ we win by induction on d.

Thus we may assume that $X = Y \times \mathbf{A}^r$. In this case we can factor f as

$$X = Y \times \mathbf{A}^r \to Y \times \mathbf{A}^{r-1} \to \ldots \to Y \times \mathbf{A}^1 \to Y.$$

Hence it suffices to do the case $r = 1$. By the argument in the second paragraph of the proof we are reduced to the case $\alpha = [W]$, Y integral, and $W \to Y$ dominant. Again we can do induction on $d = \dim_\delta(Y)$. If $W = Y \times \mathbf{A}^1$, then $[W] = f^*[Y]$. Lastly, $W \subset Y \times \mathbf{A}^1$ is a proper inclusion, then $W \to Y$ induces a finite field extension $R(Y) \subset R(W)$. Let $P(T) \in R(Y)[T]$ be the monic irreducible polynomial such that the generic fibre of $W \to Y$ is cut out by P in $\mathbf{A}^1_{R(Y)}$. Let $V \subset Y$ be a nonempty open such that $P \in \Gamma(V, \mathcal{O}_Y)[T]$, and such that $W \cap f^{-1}(V)$ is still cut out by P. Then we see that $\alpha|_{f^{-1}(V)} \sim_{rat} 0$ and hence $\alpha \sim_{rat} \alpha'$ for some cycle α' on $(Y \setminus V) \times \mathbf{A}^1$. By induction on the dimension we win. $\square$

Remark 29.32.2. We will see later (Lemma 29.33.3) that if X is a vectorbundle over Y then the pullback map $A_k(Y) \to A_{k+r}(X)$ is an isomorphism. Is this true in general?

29.33. Projective space bundle formula

Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Consider a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ of rank r. Our convention is that the *projective bundle associated to* $\mathcal{E}$ is the morphism

$$\mathbf{P}(\mathcal{E}) = \underline{\mathrm{Proj}}_X(\mathrm{Sym}^*(\mathcal{E})) \xrightarrow{\pi} X$$

over X with $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ normalized so that $\pi_*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)) = \mathcal{E}$. In particular there is a surjection $\pi^*\mathcal{E} \to \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$. We will say informally ``let $(\pi : P \to X, \mathcal{O}_P(1))$ be the projective bundle associated to $\mathcal{E}$'' to denote the situation where $P = \mathbf{P}(\mathcal{E})$ and $\mathcal{O}_P(1) = \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$.

Lemma 29.33.1. *Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ of rank r. Let $(\pi : P \to X, \mathcal{O}_P(1))$ be the projective bundle associated to $\mathcal{E}$. For any $\alpha \in A_k(X)$ we the element*

$$\pi_* \left(c_1(\mathcal{O}_P(1))^s \cap \pi^*\alpha \right) \in A_{k+r-1-s}(X)$$

is 0 if $s < r - 1$ and is equal to α when $s = r - 1$.

Proof. Let $Z \subset X$ be an integral closed subscheme of δ-dimension k. Note that $\pi^*[Z] = [\pi^{-1}(Z)]$ as $\pi^{-1}(Z)$ is integral of δ-dimension $r - 1$. If $s < r - 1$, then by construction $c_1(\mathcal{O}_P(1))^s \cap \pi^*[Z]$ is represented by a $(k + r - 1 - s)$-cycle supported on $\pi^{-1}(Z)$. Hence the pushforward of this cycle is zero for dimension reasons.

Let $s = r - 1$. By the argument given above we see that $\pi_*(c_1(\mathcal{O}_P(1))^s \cap \pi^*\alpha) = n[Z]$ for some $n \in \mathbf{Z}$. We want to show that $n = 1$. For the same dimension reasons as above it suffices to prove this result after replacing X by $X \setminus T$ where $T \subset Z$ is a proper closed subset. Let ξ be the generic point of Z. We can choose elements $e_1, \ldots, e_{r-1} \in \mathcal{E}_\xi$ which form part of a basis of $\mathcal{E}_\xi$. These give rational sections $s_1, \ldots, s_{r-1}$ of $\mathcal{O}_P(1)|_{\pi^{-1}(Z)}$ whose common zero set is the closure of the image a rational section of $\mathbf{P}(\mathcal{E}|_Z) \to Z$ union a closed subset whose support maps to a proper closed subset T of Z. After removing T from X (and correspondingly $\pi^{-1}(T)$ from P), we see that $s_1, \ldots, s_n$ form a sequence of global sections $s_i \in \Gamma(\pi^{-1}(Z), \mathcal{O}_{\pi^{-1}(Z)}(1))$ whose common zero set is the image of a section $Z \to \pi^{-1}(Z)$. Hence we see succesively that

$$\begin{array}{rcl} \pi^*[Z] & = & [\pi^{-1}(Z)] \\ c_1(\mathcal{O}_P(1)) \cap \pi^*[Z] & = & [Z(s_1)] \\ c_1(\mathcal{O}_P(1))^2 \cap \pi^*[Z] & = & [Z(s_1) \cap Z(s_2)] \\ \ldots & = & \ldots \\ c_1(\mathcal{O}_P(1))^{r-1} \cap \pi^*[Z] & = & [Z(s_1) \cap \ldots \cap Z(s_{r-1})] \end{array}$$

by repeated applications of Lemma 29.25.3. Since the pushforward by π of the image of a section of π over Z is clearly $[Z]$ we see the result when $\alpha = [Z]$. We omit the verification that these arguments imply the result for a general cycle $\alpha = \sum n_j[Z_j]$. $\square$

Lemma 29.33.2. *(Projective space bundle formula.) Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ of rank r. Let $(\pi : P \to X, \mathcal{O}_P(1))$ be the projective bundle associated to $\mathcal{E}$. The map*

$$\bigoplus\nolimits_{i=0}^{r-1} A_{k+i}(X) \longrightarrow A_{k+r-1}(P),$$

$$(\alpha_0, \ldots, \alpha_{r-1}) \longmapsto \pi^*\alpha_0 + c_1(\mathcal{O}_P(1)) \cap \pi^*\alpha_1 + \ldots + c_1(\mathcal{O}_P(1))^{r-1} \cap \pi^*\alpha_{r-1}$$

is an isomorphism.

Proof. Fix $k \in \mathbf{Z}$. We first show the map is injective. Suppose that $(\alpha_0, \ldots, \alpha_{r-1})$ is an element of the left hand side that maps to zero. By Lemma 29.33.1 we see that

$$0 = \pi_*(\pi^*\alpha_0 + c_1(\mathcal{O}_P(1)) \cap \pi^*\alpha_1 + \ldots + c_1(\mathcal{O}_P(1))^{r-1} \cap \pi^*\alpha_{r-1}) = \alpha_{r-1}$$

Next, we see that

$$0 = \pi_*(c_1(\mathcal{O}_P(1)) \cap (\pi^*\alpha_0 + c_1(\mathcal{O}_P(1)) \cap \pi^*\alpha_1 + \ldots + c_1(\mathcal{O}_P(1))^{r-2} \cap \pi^*\alpha_{r-2})) = \alpha_{r-2}$$

and so on. Hence the map is injective.

It remains to show the map is surjective. Let X_i, $i \in I$ be the irreducible components of X. Then $P_i = \mathbf{P}(\mathcal{E}|_{X_i})$, $i \in I$ are the irreducible components of P. If the map is surjective for each of the morphisms $P_i \to X_i$, then the map is surjective for $\pi : P \to X$. Details omitted. Hence we may assume X is irreducible. Thus $\dim_\delta(X) < \infty$ and in particular we may use induction on $\dim_\delta(X)$.

The result is clear if $\dim_\delta(X) < k$. Let $\alpha \in A_{k+r-1}(P)$. For any locally closed subscheme $T \subset X$ denote $\gamma_T : \bigoplus A_{k+i}(T) \to A_{k+r-1}(\pi^{-1}(T))$ the map

$$\gamma_T(\alpha_0, \ldots, \alpha_{r-1}) = \pi^*\alpha_0 + \ldots + c_1(\mathcal{O}_{\pi^{-1}(T)}(1))^{r-1} \cap \pi^*\alpha_{r-1}.$$

Suppose for some nonempty open $U \subset X$ we have $\alpha|_{\pi^{-1}(U)} = \gamma_U(\alpha_0, \ldots, \alpha_{r-1})$. Then we may choose lifts $\alpha'_i \in A_{k+i}(X)$ and we see that $\alpha - \gamma_X(\alpha'_0, \ldots, \alpha'_{r-1})$ is by Lemma 29.19.2 rationally equivalent to a k-cycle on $P_Y = \mathbf{P}(\mathcal{E}|_Y)$ where $Y = X \setminus U$ as a reduced closed subscheme. Note that $\dim_\delta(Y) < \dim_\delta(X)$. By induction the result holds for $P_Y \to Y$ and hence the result holds for α. Hence we may replace X by any nonempty open of X.

In particular we may assume that $\mathcal{E} \cong \mathcal{O}_X^{\oplus r}$. In this case $\mathbf{P}(\mathcal{E}) = X \times \mathbf{P}^{r-1}$. Let us use the stratification

$$\mathbf{P}^{r-1} = \mathbf{A}^{r-1} \coprod \mathbf{A}^{r-2} \coprod \ldots \coprod \mathbf{A}^0$$

The closure of each stratum is a $\mathbf{P}^{r-1-i}$ which is a representative of $c_1(\mathcal{O}(1))^i \cap [\mathbf{P}^{r-1}]$. Hence P has a similar stratification

$$P = U^{r-1} \coprod U^{r-2} \coprod \ldots \coprod U^0$$

Let P^i be the closure of U^i. Let $\pi^i : P^i \to X$ be the restriction of π to P^i. Let $\alpha \in A_{k+r-1}(P)$. By Lemma 29.32.1 we can write $\alpha|_{U^{r-1}} = \pi^*\alpha_0|_{U^{r-1}}$ for some $\alpha_0 \in A_k(X)$. Hence the difference $\alpha - \pi^*\alpha_0$ is the image of some $\alpha' \in A_{k+r-1}(P^{r-2})$. By Lemma 29.32.1 again we can write $\alpha'|_{U^{r-2}} = (\pi^{r-2})^*\alpha_1|_{U^{r-2}}$ for some $\alpha_1 \in A_{k+1}(X)$. By Lemma 29.31.2 we see that the image of $(\pi^{r-2})^*\alpha_1$ represents $c_1(\mathcal{O}_P(1)) \cap \pi^*\alpha_1$. We also see that $\alpha - \pi^*\alpha_0 - c_1(\mathcal{O}_P(1)) \cap \pi^*\alpha_1$ is the image of some $\alpha'' \in A_{k+r-1}(P^{r-3})$. And so on. □

Lemma 29.33.3. *Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. Let*

$$p : E = \underline{Spec}(Sym^*(\mathcal{E})) \longrightarrow X$$

be the associated vector bundle over X. Then $p^* : A_k(X) \to A_{k+r}(E)$ is an isomorphism for all k.

Proof. For surjectivity see Lemma 29.32.1. Let $(\pi : P \to X, \mathcal{O}_P(1))$ be the projective space bundle associated to the finite locally free sheaf $\mathcal{E} \oplus \mathcal{O}_X$. Let $s \in \Gamma(P, \mathcal{O}_P(1))$ correspond to the global section $(0, 1) \in \Gamma(X, \mathcal{E} \oplus \mathcal{O}_X)$. Let $D = Z(s) \subset P$. Note that $(\pi|_D : D \to X, \mathcal{O}_P(1)|_D)$ is the projective space bundle associated to $\mathcal{E}$. We denote $\pi_D = \pi|_D$ and $\mathcal{O}_D(1) = \mathcal{O}_P(1)|_D$. Moreover, D is an effective Cartier divisor on P. Hence $\mathcal{O}_P(D) = \mathcal{O}_P(1)$ (see Divisors, Lemma 26.9.17). Also there is an isomorphism $E \cong P \setminus D$. Denote $j : E \to P$ the corresponding open immersion. For injectivity we use that the kernel of

$$j^* : A_{k+r}(P) \longrightarrow A_{k+r}(E)$$

are the cycles supported in the effective Cartier divisor D, see Lemma 29.19.2. So if $p^*\alpha = 0$, then $\pi^*\alpha = i_*\beta$ for some $\beta \in A_{k+r}(D)$. By Lemma 29.33.2 we may write

$$\beta = \pi_D^*\beta_0 + \ldots + c_1(\mathcal{O}_D(1))^{r-1} \cap \pi_D^*\beta_{r-1}.$$

for some $\beta_i \in A_{k+i}(X)$. By Lemmas 29.31.2 and 29.25.6 this implies

$$\pi^*\alpha = i_*\beta = c_1(\mathcal{O}_P(1)) \cap \pi^*\beta_0 + \ldots + c_1(\mathcal{O}_D(1))^r \cap \pi^*\beta_{r-1}.$$

Since the rank of $\mathcal{E} \oplus \mathcal{O}_X$ is $r+1$ this contradicts Lemma 29.25.6 unless all α and all β_i are zero. □

29.34. The Chern classes of a vector bundle

We can use the projective space bundle formula to define the chern classes of a rank r vector bundle in terms of the expansion of $c_1(\mathcal{O}(1))^r$ in terms of the lower powers, see formula (29.34.1.1). The reason for the signs will be explained later.

Definition 29.34.1. Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Assume X is integral and $n = \dim_\delta(X)$. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. Let $(\pi : P \to X, \mathcal{O}_P(1))$ be the projective space bundle associated to $\mathcal{E}$.

(1) By Lemma 29.33.2 there are elements $c_i \in A_{n-i}(X)$, $i = 0, \ldots, r$ such that $c_0 = [X]$, and

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi^* c_{r-i} = 0. \tag{29.34.1.1}$$

(2) With notation as above we set $c_i(\mathcal{E}) \cap [X] = c_i$ as an element of $A_{n-i}(X)$. We call these the *chern classes of $\mathcal{E}$ on X*.

(3) The *total chern class of $\mathcal{E}$ on X* is the combination

$$c(\mathcal{E}) \cap [X] = c_0(\mathcal{E}) \cap [X] + c_1(\mathcal{E}) \cap [X] + \ldots + c_r(\mathcal{E}) \cap [X]$$

which is an element of $A_*(X) = \bigoplus_{k \in \mathbf{Z}} A_k(X)$.

Let us check that this does not give a new notion in case the vector bundle has rank 1.

Lemma 29.34.2. *Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Assume X is integral and $n = \dim_\delta(X)$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. The first chern class of $\mathcal{L}$ on X of Definition 29.34.1 is equal to the Weil divisor associated to $\mathcal{L}$ by Definition 29.24.1.*

Proof. In this proof we use $c_1(\mathcal{L}) \cap [X]$ to denote the construction of Definition 29.24.1. Since $\mathcal{L}$ has rank 1 we have $\mathbf{P}(\mathcal{L}) = X$ and $\mathcal{O}_{\mathbf{P}(\mathcal{L})}(1) = \mathcal{L}$ by our normalizations. Hence (29.34.1.1) reads

$$(-1)^1 c_1(\mathcal{L}) \cap c_0 + (-1)^0 c_1 = 0$$

Since $c_0 = [X]$, we conclude $c_1 = c_1(\mathcal{L}) \cap [X]$ as desired. □

Remark 29.34.3. We could also rewrite equation 29.34.1.1 as

$$\sum\nolimits_{i=0}^{r} c_1(\mathcal{O}_P(-1))^i \cap \pi^* c_{r-i} = 0. \tag{29.34.3.1}$$

but we find it easier to work with the tautological quotient sheaf $\mathcal{O}_P(1)$ instead of its dual.

29.35. Intersecting with chern classes

Definition 29.35.1. Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. We define, for every integer k and any $0 \leq j \leq r$, an operation

$$c_j(\mathcal{E}) \cap - : Z_k(X) \to A_{k-j}(X)$$

called *intersection with the jth chern class of $\mathcal{E}$*.

(1) Given an integral closed subscheme $i : W \to X$ of δ-dimension k we define

$$c_j(\mathcal{E}) \cap [W] = i_*(c_j(i^*\mathcal{E}) \cap [W]) \in A_{k-j}(X)$$

where $c_j(i^*\mathcal{E}) \cap [W]$ is as defined in Definition 29.34.1.

(2) For a general k-cycle $\alpha = \sum n_i[W_i]$ we set
$$c_j(\mathcal{E}) \cap \alpha = \sum n_i c_j(\mathcal{E}) \cap [W_i]$$

Again, if $\mathcal{E}$ has rank 1 then this agrees with our previous definition.

Lemma 29.35.2. *Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. Let $(\pi : P \to X, \mathcal{O}_P(1))$ be the projective bundle associated to $\mathcal{E}$. For $\alpha \in Z_k(X)$ the elements $c_j(\mathcal{E}) \cap \alpha$ are the unique elements α_j of $A_{k-j}(X)$ such that $\alpha_0 = \alpha$ and*
$$\sum\nolimits_{i=0}^r (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi^*(\alpha_{r-i}) = 0$$
holds in the Chow group of P.

Proof. The uniqueness of $\alpha_0, \ldots, \alpha_r$ such that $\alpha_0 = \alpha$ and such that the displayed equation holds follows from the projective space bundle formula Lemma 29.33.2. The identity holds by definition for $\alpha = [X]$. For a general k-cycle α on X write $\alpha = \sum n_a[W_a]$ with $n_a \neq 0$, and $i_a : W_a \to X$ pairwise distinct integral closed subschemes. Then the family $\{W_a\}$ is locally finite on X. Set $P_a = \pi^{-1}(W_a) = \mathbf{P}(\mathcal{E}|_{W_a})$. Denote $i'_a : P_a \to P$ the corresponding closed immersions. Consider the fibre product diagram
$$\begin{tikzcd} P' \arrow[r, equal] \arrow[d, "\pi'"'] & \coprod P_a \arrow[r, "i'_a"] \arrow[d, "\pi_a"'] & P \arrow[d, "\pi"] \\ X' \arrow[r, equal] & \coprod W_a \arrow[r, "i_a"] & X \end{tikzcd}$$
The morphism $p : X' \to X$ is proper. Moreover $\pi' : P' \to X'$ together with the invertible sheaf $\mathcal{O}_{P'}(1) = \coprod \mathcal{O}_{P_a}(1)$ which is also the pullback of $\mathcal{O}_P(1)$ is the projective bundle associated to $\mathcal{E}' = p^*\mathcal{E}$. By definition
$$c_j(\mathcal{E}) \cap [\alpha] = \sum i_{a,*}(c_j(\mathcal{E}|_{W_a}) \cap [W_a]).$$
Write $\beta_{a,j} = c_j(\mathcal{E}|_{W_a}) \cap [W_a]$ which is an element of $A_{k-j}(W_a)$. We have
$$\sum\nolimits_{i=0}^r (-1)^i c_1(\mathcal{O}_{P_a}(1))^i \cap \pi_a^*(\beta_{a,r-i}) = 0$$
for each a by definition. Thus clearly we have
$$\sum\nolimits_{i=0}^r (-1)^i c_1(\mathcal{O}_{P'}(1))^i \cap (\pi')^*(\beta_{r-i}) = 0$$
with $\beta_j = \sum n_a \beta_{a,j} \in A_{k-j}(X')$. Denote $p' : P' \to P$ the morphism $\coprod i'_a$. We have $\pi^* p_* \beta_j = p'_*(\pi')^*\beta_j$ by Lemma 29.15.1. By the projection formula of Lemma 29.25.6 we conclude that
$$\sum\nolimits_{i=0}^r (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi^*(p_*\beta_j) = 0$$
Since $p_*\beta_j$ is a representative of $c_j(\mathcal{E}) \cap \alpha$ we win. $\square$

This characterization of chern classes allows us to prove many more properties.

Lemma 29.35.3. *Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. If $\alpha \sim_{rat} \beta$ are rationally equivalent k-cycles on X then $c_j(\mathcal{E}) \cap \alpha = c_j(\mathcal{E}) \cap \beta$ in $A_{k-j}(X)$.*

Proof. By Lemma 29.35.2 the elements $\alpha_j = c_j(\mathcal{E}) \cap \alpha$, $j \geq 1$ and $\beta_j = c_j(\mathcal{E}) \cap \beta$, $j \geq 1$ are uniquely determined by the *same* equation in the chow group of the projective bundle associated to $\mathcal{E}$. (This of course relies on the fact that flat pullback is compatible with rational equivalence, see Lemma 29.20.1.) Hence they are equal. $\square$

In other words capping with chern classes of finite locally free sheaves factors through rational equivalence to give maps

$$c_j(\mathcal{E}) \cap - : A_k(X) \to A_{k-j}(X).$$

Lemma 29.35.4. *Let (S, δ) be as in Situation 29.7.1. Let X, Y be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on Y. Let $f : X \to Y$ be a flat morphism of relative dimension r. Let α be a k-cycle on Y. Then*

$$f^*(c_j(\mathcal{E}) \cap \alpha) = c_j(f^*\mathcal{E}) \cap f^*\alpha$$

Proof. Write $\alpha_j = c_j(\mathcal{E}) \cap \alpha$, so $\alpha_0 = \alpha$. By Lemma 29.35.2 we have

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi^*(\alpha_{r-i}) = 0$$

in the chow group of the projective bundle $(\pi : P \to Y, \mathcal{O}_P(1))$ associated to $\mathcal{E}$. Consider the fibre product diagram

$$\begin{tikzcd} P_X = \mathbf{P}(f^*\mathcal{E}) \arrow[r, "f_P"'] \arrow[d, "\pi_X"'] & P \arrow[d, "\pi"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Note that $\mathcal{O}_{P_X}(1) = f_P^*\mathcal{O}_P(1)$. By Lemmas 29.25.4 and 29.14.3 we see that

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_{P_X}(1))^i \cap \pi_X^*(f^*\alpha_{r-i}) = 0$$

holds in the chow group of P_X. Since $f^*\alpha_0 = f^*\alpha$ the lemma follows from the uniqueness in Lemma 29.35.2. $\square$

Lemma 29.35.5. *Let (S, δ) be as in Situation 29.7.1. Let X, Y be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. Let $p : X \to Y$ be a proper morphism. Let α be a k-cycle on X. Let $\mathcal{E}$ be a finite locally free sheaf on Y. Then*

$$p_*(c_j(p^*\mathcal{E}) \cap \alpha) = c_j(\mathcal{E}) \cap p_*\alpha$$

Proof. Write $\alpha_j = c_j(p^*\mathcal{E}) \cap \alpha$, so $\alpha_0 = \alpha$. By Lemma 29.35.2 we have

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi_X^*(\alpha_{r-i}) = 0$$

in the chow group of the projective bundle $(\pi_X : P_X \to X, \mathcal{O}_{P_X}(1))$ associated to $p^*\mathcal{E}$. Let $(\pi : P \to Y, \mathcal{O}_P(1))$ be the projective bundle associated to $\mathcal{E}$. Consider the fibre product diagram

$$\begin{tikzcd} P_X = \mathbf{P}(p^*\mathcal{E}) \arrow[r, "p_P"'] \arrow[d, "\pi_X"'] & P \arrow[d, "\pi"] \\ X \arrow[r, "p"] & Y \end{tikzcd}$$

Note that $\mathcal{O}_{P_X}(1) = p_P^*\mathcal{O}_P(1)$. Pushing the displayed equality above to P and using Lemmas 29.15.1, 29.25.6 and 29.14.3 we see that

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi^*(p_*\alpha_{r-i}) = 0$$

holds in the chow group of P. Since $p_*\alpha_0 = p_*\alpha$ the lemma follows from the uniqueness in Lemma 29.35.2. $\square$

Lemma 29.35.6. *Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$, $\mathcal{F}$ be finite locally free sheaves on X of ranks r and s. For any $\alpha \in A_k(X)$ we have*

$$c_i(\mathcal{E}) \cap c_j(\mathcal{F}) \cap \alpha = c_j(\mathcal{F}) \cap c_i(\mathcal{E}) \cap \alpha$$

as elements of $A_{k-i-j}(X)$.

Proof. Consider

$$\pi : \mathbf{P}(\mathcal{E}) \times_X \mathbf{P}(\mathcal{F}) \longrightarrow X$$

with invertible sheaves $\mathcal{L} = \mathrm{pr}_1^*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ and $\mathcal{N} = \mathrm{pr}_2^*\mathcal{O}_{\mathbf{P}(\mathcal{F})}(1)$. Write $\alpha_{i,j}$ for the left hand side and $\beta_{i,j}$ for the right hand side. Also write $\alpha_j = c_j(\mathcal{F}) \cap \alpha$ and $\beta_i = c_i(\mathcal{E}) \cap \alpha$. In particular this means that $\alpha_0 = \alpha = \beta_0$, and $\alpha_{0,j} = \alpha_j = \beta_{0,j}$, $\alpha_{i,0} = \beta_i = \beta_{i,0}$. From Lemma 29.35.2 (pulled back to the space above using Lemma 29.25.4 for the first two) we see that

$$\begin{aligned}
0 &= \sum\nolimits_{j=0,\ldots,s} (-1)^j c_1(\mathcal{N})^j \cap \pi^*\alpha_{s-j} \\
0 &= \sum\nolimits_{i=0,\ldots,r} (-1)^i c_1(\mathcal{L})^i \cap \pi^*\beta_{r-i} \\
0 &= \sum\nolimits_{i=0,\ldots,r} (-1)^i c_1(\mathcal{L})^i \cap \pi^*\alpha_{r-i,s-j} \\
0 &= \sum\nolimits_{j=0,\ldots,s} (-1)^j c_1(\mathcal{N})^j \cap \pi^*\beta_{r-i,s-j}
\end{aligned}$$

We can combine the first and the third of these to get

$$\begin{aligned}
& (-1)^{r+s} c_1(\mathcal{L})^r \cap c_1(\mathcal{N})^s \cap \pi^*\alpha \\
& = \sum\nolimits_{j=1,\ldots,s} (-1)^{r+j-1} c_1(\mathcal{L})^r \cap c_1(\mathcal{N})^j \cap \pi^*\alpha_{s-j} \\
& = \sum\nolimits_{j=1,\ldots,s} (-1)^{j-1+r} c_1(\mathcal{N})^j \cap c_1(\mathcal{L})^r \cap \pi^*\alpha_{0,s-j} \\
& = \sum\nolimits_{j=1}^s \sum\nolimits_{i=1}^r (-1)^{i+j} c_1(\mathcal{N})^j \cap c_1(\mathcal{L})^i \cap \pi^*\alpha_{r-i,s-j}
\end{aligned}$$

using that capping with $c_1(\mathcal{L})$ commutes with capping with $c_1(\mathcal{N})$. In exactly the same way one shows that

$$(-1)^{r+s} c_1(\mathcal{L})^r \cap c_1(\mathcal{N})^s \cap \pi^*\alpha = \sum\nolimits_{j=1}^s \sum\nolimits_{i=1}^r (-1)^{i+j} c_1(\mathcal{N})^j \cap c_1(\mathcal{L})^i \cap \pi^*\beta_{r-i,s-j}$$

By the projective space bundle formula Lemma 29.33.2 applied twice these representations are unique. Whence the result. ☐

29.36. Polynomial relations among chern classes

Definition 29.36.1. Let $P(x_{i,j}) \in \mathbf{Z}[x_{i,j}]$ be a polynomial. We write P as a finite sum

$$\sum\nolimits_s \sum\nolimits_{I=((i_1,j_1),(i_2,j_2),\ldots,(i_s,j_s))} a_I x_{i_1,j_1} \ldots x_{i_s,j_s}.$$

Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Let $\mathcal{E}_i$ be a finite collection of finite locally free sheaves on X. We say that P is a *polynomial relation among the chern classes* and we write $P(c_j(\mathcal{E}_i)) = 0$ if for any morphism $f : Y \to X$ of an integral scheme locally of finite type over S the cycle

$$\sum\nolimits_s \sum\nolimits_{I=((i_1,j_1),(i_2,j_2),\ldots,(i_s,j_s))} a_I \, c_{j_1}(f^*\mathcal{E}_{i_1}) \cap \ldots \cap c_{j_s}(f^*\mathcal{E}_{i_s}) \cap [Y]$$

is zero in $A_*(Y)$.

This is not an elegant definition but it will do for now. It makes sense because we showed in Lemma 29.35.6 that capping with chern classes of vector bundles is commutative. By our definitions and results above this is equivalent with requiring all the operations

$$\sum_s \sum_I a_I\, c_{j_1}(f^*\mathcal{E}_{i_1}) \cap \ldots \cap c_{j_s}(f^*\mathcal{E}_{i_s}) \cap - : A_*(Y) \to A_*(Y)$$

to be zero for all morphisms $f : Y \to X$ which are locally of finite type.

An example of such a relation is the relation

$$c_1(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N}) = c_1(\mathcal{L}) + c_1(\mathcal{N})$$

proved in Lemma 29.25.2. More generally, here is what happens when we tensor an arbitrary locally free sheaf by an invertible sheaf.

Lemma 29.36.2. *Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. Let $\mathcal{L}$ be an invertible sheaf on X. Then*

$$c_i(\mathcal{E} \otimes \mathcal{L}) = \sum\nolimits_{j=0}^{i} \binom{r-i+j}{j} c_{i-j}(\mathcal{E}) c_1(\mathcal{L})^j \quad (29.36.2.1)$$

is a valid polynomial relation in the sense described above.

Proof. This should hold for any triple $(X, \mathcal{E}, \mathcal{L})$. In particular it should hold when X is integral, and in fact by definition of a polynomial relation it is enough to prove it holds when capping with $[X]$ for such X. Thus assume that X is integral. Let $(\pi : P \to X, \mathcal{O}_P(1))$, resp. $(\pi' : P' \to X, \mathcal{O}_{P'}(1))$ be the projective space bundle associated to $\mathcal{E}$, resp. $\mathcal{E} \otimes \mathcal{L}$. Consider the canonical morphsm

$$\begin{tikzcd} P \arrow[rr, "g"] \arrow[dr, "\pi"'] & & P' \arrow[dl, "\pi'"] \\ & X & \end{tikzcd}$$

see Constructions, Lemma 22.19.1. It has the property that $g^*\mathcal{O}_{P'}(1) = \mathcal{O}_P(1) \otimes \pi^*\mathcal{L}$. This means that we have

$$\sum\nolimits_{i=0}^{r} (-1)^i (\xi + x)^i \cap \pi^*(c_{r-i}(\mathcal{E} \otimes \mathcal{L}) \cap [X]) = 0$$

in $A_*(P)$, where ξ represents $c_1(\mathcal{O}_P(1))$ and x represents $c_1(\pi^*\mathcal{L})$. By simple algebra this is equivalent to

$$\sum\nolimits_{i=0}^{r} (-1)^i \xi^i \left(\sum\nolimits_{j=i}^{r} (-1)^{j-i} \binom{j}{i} x^{j-i} \cap \pi^*(c_{r-j}(\mathcal{E} \otimes \mathcal{L}) \cap [X]) \right) = 0$$

Comparing with Equation (29.34.1.1) it follows from this that

$$c_{r-i}(\mathcal{E}) \cap [X] = \sum\nolimits_{j=i}^{r} \binom{j}{i} (-c_1(\mathcal{L}))^{j-i} \cap c_{r-j}(\mathcal{E} \otimes \mathcal{L}) \cap [X]$$

Reworking this (getting rid of minus signs, and renumbering) we get the desired relation. $\square$

Some example cases of (29.36.2.1) are

$$c_1(\mathcal{E}\otimes\mathcal{L}) = c_1(\mathcal{E}) + rc_1(\mathcal{L})$$

$$c_2(\mathcal{E}\otimes\mathcal{L}) = c_2(\mathcal{E}) + (r-1)c_1(\mathcal{E})c_1(\mathcal{L}) + \binom{r}{2}c_1(\mathcal{L})^2$$

$$c_3(\mathcal{E}\otimes\mathcal{L}) = c_3(\mathcal{E}) + (r-2)c_2(\mathcal{E})c_1(\mathcal{L}) + \binom{r-1}{2}c_1(\mathcal{E})c_1(\mathcal{L})^2 + \binom{r}{3}c_1(\mathcal{L})^3$$

29.37. Additivity of chern classes

All of the preliminary lemmas follow trivially from the final result.

Lemma 29.37.1. *Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$, $\mathcal{F}$ be finite locally free sheaves on X of ranks r, $r-1$ which fit into a short exact sequence*

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{F} \to 0$$

Then

$$c_r(\mathcal{E}) = 0, \quad c_j(\mathcal{E}) = c_j(\mathcal{F}), \quad j = 0, \ldots, r-1$$

are valid polynomial relations among chern classes.

Proof. By Definition 29.36.1 it suffices to show that if X is integral then $c_j(\mathcal{E}) \cap [X] = c_j(\mathcal{F}) \cap [X]$. Let $(\pi : P \to X, \mathcal{O}_P(1))$, resp. $(\pi' : P' \to X, \mathcal{O}_{P'}(1))$ denote the projective space bundle associated to $\mathcal{E}$, resp. $\mathcal{F}$. The surjection $\mathcal{E} \to \mathcal{F}$ gives rise to a closed immersion

$$i : P' \longrightarrow P$$

over X. Moreover, the element $1 \in \Gamma(X, \mathcal{O}_X) \subset \Gamma(X, \mathcal{E})$ gives rise to a global section $s \in \Gamma(P, \mathcal{O}_P(1))$ whose zero set is exactly P'. Hence P' is an effective Cartier divisor on P such that $\mathcal{O}_P(P') \cong \mathcal{O}_P(1)$. Hence we see that

$$c_1(\mathcal{O}_P(1)) \cap \pi^*\alpha = i_*((\pi')^*\alpha)$$

for any cycle class α on X by Lemma 29.31.2. By Lemma 29.35.2 we see that $\alpha_j = c_j(\mathcal{F}) \cap [X]$, $j = 0, \ldots, r-1$ satisfy

$$\sum\nolimits_{j=0}^{r-1}(-1)^j c_1(\mathcal{O}_{P'}(1))^j \cap (\pi')^*\alpha_j = 0$$

Pushing this to P and using the remark above as well as Lemma 29.25.6 we get

$$\sum\nolimits_{j=0}^{r-1}(-1)^j c_1(\mathcal{O}_P(1))^{j+1} \cap \pi^*\alpha_j = 0$$

By the uniqueness of Lemma 29.35.2 we conclude that $c_r(\mathcal{E}) \cap [X] = 0$ and $c_j(\mathcal{E}) \cap [X] = \alpha_j = c_j(\mathcal{F}) \cap [X]$ for $j = 0, \ldots, r-1$. Hence the lemma holds. $\square$

Lemma 29.37.2. *Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$, $\mathcal{F}$ be finite locally free sheaves on X of ranks r, $r-1$ which fit into a short exact sequence*

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0$$

where $\mathcal{L}$ is an invertible sheaf Then

$$c(\mathcal{E}) = c(\mathcal{L})c(\mathcal{F})$$

is a valid polynomial relation among chern classes.

Proof. This relation really just says that $c_i(\mathcal{E}) = c_i(\mathcal{F}) + c_1(\mathcal{L})c_{i-1}(\mathcal{F})$. By Lemma 29.37.1 we have $c_j(\mathcal{E} \otimes \mathcal{L}^{\otimes -1}) = c_j(\mathcal{E} \otimes \mathcal{L}^{\otimes -1})$ for $j = 0, \ldots, r$ (were we set $c_r(\mathcal{F}) = 0$ by convention). Applying Lemma 29.36.2 we deduce

$$\sum_{j=0}^{i} \binom{r-i+j}{j} (-1)^j c_{i-j}(\mathcal{E}) c_1(\mathcal{L})^j = \sum_{j=0}^{i} \binom{r-1-i+j}{j} (-1)^j c_{i-j}(\mathcal{F}) c_1(\mathcal{L})^j$$

Setting $c_i(\mathcal{E}) = c_i(\mathcal{F}) + c_1(\mathcal{L})c_{i-1}(\mathcal{F})$ gives a ``solution'' of this equation. The lemma follows if we show that this is the only possible solution. We omit the verification. □

Lemma 29.37.3. *Let (S, δ) be as in Situation 29.7.1. Let X be a scheme locally of finite type over S. Suppose that $\mathcal{E}$ sits in an exact sequence*

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$$

of finite locally free sheaves $\mathcal{E}_i$ of rank r_i. Then

$$c(\mathcal{E}) = c(\mathcal{E}_1)c(\mathcal{E}_2)$$

is a polynomial relation among chern classes.

Proof. We may assume that X is integral and we have to show the identity when capping against $[X]$. By induction on r_1. The case $r_1 = 1$ is Lemma 29.37.2. Assuem $r_1 > 1$. Let $(\pi : P \to X, \mathcal{O}_P(1))$ denote the projective space bundle associated to $\mathcal{E}_1$. Note that

(1) $\pi^* : A_*(X) \to A_*(P)$ is injective, and
(2) $\pi^*\mathcal{E}_1$ sits in a short exact sequence $0 \to \mathcal{F} \to \pi^*\mathcal{E}_1 \to \mathcal{L} \to 0$ where $\mathcal{L}$ is invertible.

The first assertion follows from the projective space bundle formula and the second follows from the definition of a projective space bundle. (In fact $\mathcal{L} = \mathcal{O}_P(1)$.) Let $Q = \pi^*\mathcal{E}/\mathcal{F}$, which sits in an exact sequence $0 \to \mathcal{L} \to Q \to \pi^*\mathcal{E}_2 \to 0$. By induction we have

$$\begin{aligned} c(\pi^*\mathcal{E}) \cap [P] &= c(\mathcal{F}) \cap c(\pi^*\mathcal{E}/\mathcal{F}) \cap [P] \\ &= c(\mathcal{F}) \cap c(\mathcal{L}) \cap c(\pi^*\mathcal{E}_2) \cap [P] \\ &= c(\pi^*\mathcal{E}_1) \cap c(\pi^*\mathcal{E}_2) \cap [P] \end{aligned}$$

Since $[P] = \pi^*[X]$ we win by Lemma 29.35.4. □

Lemma 29.37.4. *Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Let $\mathcal{L}_i$, $i = 1, \ldots, r$ be invertible $\mathcal{O}_X$-modules on X. Let $\mathcal{E}$ be a finite locally free rank r $\mathcal{O}_X$-module endowed with a filtration*

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \ldots \subset \mathcal{E}_r = \mathcal{E}$$

such that $\mathcal{E}_i/\mathcal{E}_{i-1} \cong \mathcal{L}_i$. Set $c_1(\mathcal{L}_i) = x_i$. Then

$$c(\mathcal{E}) = \prod\nolimits_{i=1}^{r}(1 + x_i)$$

is a valid polynomial relation among chern classes in the sense of Definition 29.36.1.

Proof. Apply Lemma 29.37.2 and induction. □

29.38. The splitting principle

In our setting it is not so easy to say what the splitting principle exactly says/is. Here is a possible formulation.

Lemma 29.38.1. *Let (S, δ) be as in Situation 29.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf $\mathcal{E}$ on X of rank r. There exists a projective flat morphism of relative dimension d $\pi : P \to X$ such that*

(1) *for any morphism $f : Y \to X$ the map $\pi_Y^* : A_*(Y) \to A_{*+r}(Y \times_X P)$ is injective, and*
(2) *$\pi^*\mathcal{E}$ has a filtration with succesive quotients $\mathcal{L}_1, \ldots, \mathcal{L}_r$ for some invertible $\mathcal{O}_P$-modules $\mathcal{L}_i$.*

Proof. Omitted. Hint: Use a composition of projective space bundles, i.e., a flag variety over X. □

The splitting principle refers to the practice of symbolically writing

$$c(\mathcal{E}) = \prod (1 + x_i)$$

with $x_i = c_1(\mathcal{L}_i)$. The expressions x_i are then called the *Chern roots* of $\mathcal{E}$. In order to prove polynomial relations amoing chern classes of vector bundles it is permissible to do calculations using the chern roots.

For example, let us calculate the chern classes of the dual vector bundle $\mathcal{E}^\wedge$. Note that if $\mathcal{E}$ has a filtration with subquotients invertible sheaves $\mathcal{L}_i$ then $\mathcal{E}^\wedge$ has a filtration with subquotients the invertible sheaves $\mathcal{L}_i^{-1}$. Hence if x_i are the chern roots of $\mathcal{E}$, then the $-x_i$ are the chern roots of $\mathcal{E}^\wedge$. It follows that

$$c_j(\mathcal{E}^\wedge) = (-1)^j c_j(\mathcal{E})$$

is a valid polynomial relation among chern classes.

In the same vain, let us compute the chern classes of a tensor product of vector bundles. Namely, suppose that $\mathcal{E}$, $\mathcal{F}$ are finite locally free of ranks r, s. Write

$$c(\mathcal{E}) = \prod\nolimits_{i=1}^r (1 + x_i), \quad c(\mathcal{E}) = \prod\nolimits_{j=1}^s (1 + y_j)$$

where x_i, y_j are the chern roots of $\mathcal{E}$, $\mathcal{F}$. Then we see that

$$c(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}) = \prod\nolimits_{i,j} (1 + x_i + y_j)$$

Here are some examples of what this means in terms of chern classes

$$c_1(\mathcal{E} \otimes \mathcal{F}) = rc_1(\mathcal{F}) + sc_1(\mathcal{E})$$
$$c_2(\mathcal{E} \otimes \mathcal{F}) = r^2 c_2(\mathcal{F}) + rsc_1(\mathcal{F})c_1(\mathcal{E}) + s^2 c_2(\mathcal{E})$$

29.39. Chern classes and tensor product

We define the *Chern character* of a finite locally free sheaf of rank r to be the formal expression

$$ch(\mathcal{E}) := \sum\nolimits_{i=1}^r e^{x_i}$$

if the x_i are the chern roots of $\mathcal{E}$. Writing this in terms of chern classes $c_i = c_i(\mathcal{E})$ we see that

$$ch(\mathcal{E}) = r + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) + \frac{1}{24}(c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4) + \ldots$$

What does it mean that the coefficients are rational numbers? Well this simply means that we think of these as operations

$$ch_j(\mathcal{E}) \cap - : A_k(X) \longrightarrow A_{k-j}(X) \otimes_{\mathbf{Z}} \mathbf{Q}$$

and we think of polynomial relations among them as relations between these operations with values in the groups $A_{k-j}(Y) \otimes_{\mathbf{Z}} \mathbf{Q}$ for varying Y. By the above we have in case of an exact sequence

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$$

that

$$ch(\mathcal{E}) = ch(\mathcal{E}_1) + ch(\mathcal{E}_2)$$

Using the Chern character we can express the compatibility of the chern classes and tensor product as follows:

$$ch(\mathcal{E}_1 \otimes_{\mathcal{O}_X} \mathcal{E}_2) = ch(\mathcal{E}_1)ch(\mathcal{E}_2)$$

This follows directly from the discussion of the chern roots of the tensor product in the previous section.

29.40. Todd classes

A final class associated to a vector bundle $\mathcal{E}$ of rank r is its *Todd class* $Todd(\mathcal{E})$. In terms of the chern roots $x_1, \ldots, x_r$ it is defined as

$$Todd(\mathcal{E}) = \prod\nolimits_{i=1}^{r} \frac{x_i}{1 - e^{-x_i}}$$

In terms of the chern classes $c_i = c_i(\mathcal{E})$ we have

$$Todd(\mathcal{E}) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 + \frac{1}{720}(-c_1^4 + 4c_1^2c_2 + 3c_2^2 + c_1c_3 - c_4) + \ldots$$

We have made the appropriate remaks about denominators in the previous section. It is the case that given an exact sequence

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$$

we have

$$Todd(\mathcal{E}) = Todd(\mathcal{E}_1)Todd(\mathcal{E}_2).$$

29.41. Grothendieck-Riemann-Roch

Let (S, δ) be as in Situation 29.7.1. Let X, Y be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf $\mathcal{E}$ on X of rank r. Let $f : X \to Y$ be a proper smooth morphism. Assume that $R^if_*\mathcal{E}$ are locally free sheaves on Y of finite rank (for example if Y is a point). The Grothendieck-Riemann-Roch theorem implies that in this case we have

$$f_*(Todd(T_{X/Y})ch(\mathcal{E})) = \sum (-1)^i ch(R^if_*\mathcal{E})$$

Here

$$T_{X/Y} = \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/Y}, \mathcal{O}_X)$$

is the relative tangent bundle of X over Y. The theorem is more general and becomes easier to prove when formulated in correct generality. We will return to this elsewhere (insert future reference here).

29.42. Other chapters

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CHAPTER 30

Topologies on Schemes

30.1. Introduction

In this document we explain what the different topologies on the category of schemes are. Some references are [**Gro71**] and [**BLR90**]. Before doing so we would like to point out that there are many different choices of sites (as defined in Sites, Definition 9.6.2) which give rise to the same notion of sheaf on the underlying category. Hence our choices may be slightly different from those in the references but ultimately lead to the same cohomology groups, etc.

30.2. The general procedure

In this section we explain a general procedure for producing the sites we will be working with. Suppose we want to study sheaves over schemes with respect to some topology τ. In order to get a site, as in Sites, Definition 9.6.2, of schemes with that topology we have to do some work. Namely, we cannot simply say ``consider all schemes with the Zariski topology'' since that would give a ``big'' category. Instead, in each section of this chapter we will proceed as follows:

(1) We define a class Cov_τ of coverings of schemes satisfying the axioms of Sites, Definition 9.6.2. It will always be the case that a Zariski open covering of a scheme is a covering for τ.
(2) We single out a notion of standard τ-covering within the category of affine schemes.
(3) We define what is an ``absolute'' big τ-site Sch_τ. These are the sites one gets by appropriately choosing a set of schemes and a set of coverings.
(4) For any object S of Sch_τ we define the big τ-site $(Sch/S)_\tau$ and for suitable τ the small[1] τ-site S_τ.
(5) In addition there is a site $(Aff/S)_\tau$ using the notion of standard τ-covering of affines whose category of sheaves is equivalent to the category of sheaves on $(Sch/S)_\tau$.

The above is a little clumsy in that we do not end up with a canonical choice for the big τ-site of a scheme, or even the small τ-site of a scheme. If you are willing to ignore set theoretic difficulties, then you can work with classes and end up with canonical big and small sites...

30.3. The Zariski topology

Definition 30.3.1. Let T be a scheme. A *Zariski covering of T* is a family of morphisms $\{f_i : T_i \to T\}_{i\in I}$ of schemes such that each f_i is an open immersion and such that $T = \bigcup f_i(T_i)$.

This defines a (proper) class of coverings. Next, we show that this notion satisfies the conditions of Sites, Definition 9.6.2.

[1]The words big and small here do not relate to bigness/smallness of the corresponding categories.

Lemma 30.3.2. *Let T be a scheme.*

(1) *If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is a Zariski covering of T.*
(2) *If $\{T_i \to T\}_{i\in I}$ is a Zariski covering and for each i we have a Zariski covering $\{T_{ij} \to T_i\}_{j\in J_i}$, then $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is a Zariski covering.*
(3) *If $\{T_i \to T\}_{i\in I}$ is a Zariski covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i\in I}$ is a Zariski covering.*

Proof. Omitted. □

Lemma 30.3.3. *Let T be an affine scheme. Let $\{T_i \to T\}_{i\in I}$ be a Zariski covering of T. Then there exists a Zariski covering $\{U_j \to T\}_{j=1,\ldots,m}$ which is a refinement of $\{T_i \to T\}_{i\in I}$ such that each U_j is a standard open of T, see Schemes, Definition 21.5.2. Moreover, we may choose each U_j to be an open of one of the T_i.*

Proof. Follows as T is quasi-compact and standard opens form a basis for its topology. This is also proved in Schemes, Lemma 21.5.1. □

Thus we define the corresponding standard coverings of affines as follows.

Definition 30.3.4. Compare Schemes, Definition 21.5.2. Let T be an affine scheme. A *standard Zariski covering* of T is a a Zariski covering $\{U_j \to T\}_{j=1,\ldots,m}$ with each $U_j \to T$ inducing an isomorphism with a standard affine open of T.

Definition 30.3.5. A *big Zariski site* is any site Sch_{Zar} as in Sites, Definition 9.6.2 constructed as follows:

(1) Choose any set of schemes S_0, and any set of Zariski coverings Cov_0 among these schemes.
(2) As underlying category of Sch_{Zar} take any category Sch_α constructed as in Sets, Lemma 3.9.2 starting with the set S_0.
(3) As coverings of Sch_{Zar} choose any set of coverings as in Sets, Lemma 3.11.1 starting with the category Sch_α and the class of Zariski coverings, and the set Cov_0 chosen above.

It is shown in Sites, Lemma 9.8.6 that, after having chosen the category Sch_α, the category of sheaves on Sch_α does not depend on the choice of coverings chosen in (3) above. In other words, the topos $Sh(Sch_{Zar})$ only depends on the choice of the category Sch_α. It is shown in Sets, Lemma 3.9.9 that these categories are closed under many constructions of algebraic geometry, e.g., fibre products and taking open and closed subschemes. We can also show that the exact choice of Sch_α does not matter too much, see Section 30.10.

Another approach would be to assume the existence of a strongly inaccessible cardinal and to define Sch_{Zar} to be the category of schemes contained in a chosen universe with set of coverings the Zariski coverings contained in that same universe.

Before we continue with the introduction of the big Zariski site of a scheme S, let us point out that the topology on a big Zariski site Sch_{Zar} is in some sense induced from the Zariski topology on the category of all schemes.

Lemma 30.3.6. *Let Sch_{Zar} be a big Zariski site as in Definition 30.3.5. Let $T \in Ob(Sch_{Zar})$. Let $\{T_i \to T\}_{i\in I}$ be an arbitrary Zariski covering of T. There exists a covering $\{U_j \to T\}_{j\in J}$ of T in the site Sch_{Zar} which is tautologically equivalent (see Sites, Definition 9.8.2) to $\{T_i \to T\}_{i\in I}$*

Proof. Since each $T_i \to T$ is an open immersion, we see by Sets, Lemma 3.9.9 that each T_i is isomorphic to an object V_i of Sch_{Zar}. The covering $\{V_i \to T\}_{i \in I}$ is tautologically equivalent to $\{T_i \to T\}_{i \in I}$ (using the identity map on I both ways). Moreover, $\{V_i \to T\}_{i \in I}$ is combinatorially equivalent to a covering $\{U_j \to T\}_{j \in J}$ of T in the site Sch_{Zar} by Sets, Lemma 3.11.1. □

Definition 30.3.7. Let S be a scheme. Let Sch_{Zar} be a big Zariski site containing S.

(1) The *big Zariski site of S*, denoted $(Sch/S)_{Zar}$, is the site Sch_{Zar}/S introduced in Sites, Section 9.21.
(2) The *small Zariski site of S*, which we denote S_{Zar}, is the full subcategory of $(Sch/S)_{Zar}$ whose objects are those U/S such that $U \to S$ is an open immersion. A covering of S_{Zar} is any covering $\{U_i \to U\}$ of $(Sch/S)_{Zar}$ with $U \in Ob(S_{Zar})$.
(3) The *big affine Zariski site of S*, denoted $(Aff/S)_{Zar}$, is the full subcategory of $(Sch/S)_{Zar}$ whose objects are affine U/S. A covering of $(Aff/S)_{Zar}$ is any covering $\{U_i \to U\}$ of $(Sch/S)_{Zar}$ which is a standard Zariski covering.

It is not completely clear that the small Zariski site and the big affine Zariski site are sites. We check this now.

Lemma 30.3.8. *Let S be a scheme. Let Sch_{Zar} be a big Zariski site containing S. Both S_{Zar} and $(Aff/S)_{Zar}$ are sites.*

Proof. Let us show that S_{Zar} is a site. It is a category with a given set of families of morphisms with fixed target. Thus we have to show properties (1), (2) and (3) of Sites, Definition 9.6.2. Since $(Sch/S)_{Zar}$ is a site, it suffices to prove that given any covering $\{U_i \to U\}$ of $(Sch/S)_{Zar}$ with $U \in Ob(S_{Zar})$ we also have $U_i \in Ob(S_{Zar})$. This follows from the definitions as the composition of open immersions is an open immersion.

Let us show that $(Aff/S)_{Zar}$ is a site. Reasoning as above, it suffices to show that the collection of standard Zariski coverings of affines satisfies properties (1), (2) and (3) of Sites, Definition 9.6.2. Let R be a ring. Let $f_1, \ldots, f_n \in R$ generate the unit ideal. For each $i \in \{1, \ldots, n\}$ let $g_{i1}, \ldots, g_{in_i} \in R_{f_i}$ be elements generating the unit ideal of R_{f_i}. Write $g_{ij} = f_{ij}/f_i^{e_{ij}}$ which is possible. After replacing f_{ij} by $f_i f_{ij}$ if necessary, we have that $D(f_{ij}) \subset D(f_i) \cong Spec(R_{f_i})$ is equal to $D(g_{ij}) \subset Spec(R_{f_i})$. Hence we see that the family of morphisms $\{D(g_{ij}) \to Spec(R)\}$ is a standard Zariski covering. From these considerations it follows that (2) holds for standard Zariski coverings. We omit the verification of (1) and (3). □

Lemma 30.3.9. *Let S be a scheme. Let Sch_{Zar} be a big Zariski site containing S. The underlying categories of the sites Sch_{Zar}, $(Sch/S)_{Zar}$, S_{Zar}, and $(Aff/S)_{Zar}$ have fibre products. In each case the obvious functor into the category Sch of all schemes commutes with taking fibre products. The categories $(Sch/S)_{Zar}$, and S_{Zar} both have a final object, namely S/S.*

Proof. For Sch_{Zar} it is true by construction, see Sets, Lemma 3.9.9. Suppose we have $U \to S$, $V \to U$, $W \to U$ morphisms of schemes with $U, V, W \in Ob(Sch_{Zar})$. The fibre product $V \times_U W$ in Sch_{Zar} is a fibre product in Sch and is the fibre product of V/S with W/S over U/S in the category of all schemes over S, and hence also a fibre product in $(Sch/S)_{Zar}$. This proves the result for $(Sch/S)_{Zar}$. If $U \to S$, $V \to U$ and $W \to U$ are open immersions then so is $V \times_U W \to S$ and hence we get the result for S_{Zar}. If U, V, W are affine, so is $V \times_U W$ and hence the result for $(Aff/S)_{Zar}$. □

Next, we check that the big affine site defines the same topos as the big site.

Lemma 30.3.10. *Let S be a scheme. Let Sch_{Zar} be a big Zariski site containing S. The functor $(Aff/S)_{Zar} \to (Sch/S)_{Zar}$ is a special cocontinuous functor. Hence it induces an equivalence of topoi from $Sh((Aff/S)_{Zar})$ to $Sh((Sch/S)_{Zar})$.*

Proof. The notion of a special cocontinuous functor is introduced in Sites, Definition 9.25.2. Thus we have to verify assumptions (1) -- (5) of Sites, Lemma 9.25.1. Denote the inclusion functor $u : (Aff/S)_{Zar} \to (Sch/S)_{Zar}$. Being cocontinuous just means that any Zariski covering of T/S, T affine, can be refined by a standard Zariski covering of T. This is the content of Lemma 30.3.3. Hence (1) holds. We see u is continuous simply because a standard Zariski covering is a Zariski covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that u is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering. □

Let us check that the notion of a sheaf on the small Zariski site corresponds to notion of a sheaf on S.

Lemma 30.3.11. *The category of sheaves on S_{Zar} is equivalent to the category of sheaves on the underlying topological space of S.*

Proof. We will use repeatedly that for any object U/S of S_{Zar} the morphism $U \to S$ is an isomorphism onto an open subscheme. Let $\mathcal{F}$ be a sheaf on S. Then we define a sheaf on S_{Zar} by the rule $\mathcal{F}'(U/S) = \mathcal{F}(\mathrm{Im}(U \to S))$. For the converse, we choose for every open subscheme $U \subset S$ an object $U'/S \in Ob(S_{Zar})$ with $\mathrm{Im}(U' \to S) = U$ (here you have to use Sets, Lemma 3.9.9). Given a sheaf $\mathcal{G}$ on S_{Zar} we define a sheaf on S by setting $\mathcal{G}(U) = \mathcal{G}(U'/S)$. To see that $\mathcal{G}'$ is a sheaf we use that for any open covering $U = \bigcup_{i\in I} U_i$ the covering $\{U_i \to U\}_{i\in I}$ is combinatorially equivalent to a covering $\{U'_j \to U'\}_{j\in J}$ in S_{Zar} by Sets, Lemma 3.11.1, and we use Sites, Lemma 9.8.4. Details omitted. □

From now on we will not make any distinction between a sheaf on S_{Zar} or a sheaf on S. We will always use the procedures of the proof of the lemma to go between the two notions. Next, we esthablish some relationships between the topoi associated to these sites.

Lemma 30.3.12. *Let Sch_{Zar} be a big Zariski site. Let $f : T \to S$ be a morphism in Sch_{Zar}. The functor $T_{Zar} \to (Sch/S)_{Zar}$ is cocontinuous and induces a morphism of topoi*

$$i_f : Sh(T_{Zar}) \longrightarrow Sh((Sch/S)_{Zar})$$

For a sheaf $\mathcal{G}$ on $(Sch/S)_{Zar}$ we have the formula $(i_f^{-1}\mathcal{G})(U/T) = \mathcal{G}(U/S)$. The functor i_f^{-1} also has a left adjoint $i_{f,!}$ which commutes with fibre products and equalizers.

Proof. Denote the functor $u : T_{Zar} \to (Sch/S)_{Zar}$. In other words, given and open immersion $j : U \to T$ corresponding to an object of T_{Zar} we set $u(U \to T) = (f \circ j : U \to S)$. This functor commutes with fibre products, see Lemma 30.3.9. Moreover, T_{Zar} has equalizers (as any two morphisms with the same source and target are the same) and u commutes with them. It is clearly cocontinuous. It is also continuous as u transforms coverings to coverings and commutes with fibre products. Hence the lemma follows from Sites, Lemmas 9.19.5 and 9.19.6. □

Lemma 30.3.13. *Let S be a scheme. Let Sch_{Zar} be a big Zariski site containing S. The inclusion functor $S_{Zar} \to (Sch/S)_{Zar}$ satisfies the hypotheses of Sites, Lemma 9.19.8 and hence induces a morphism of sites*

$$\pi_S : (Sch/S)_{Zar} \longrightarrow S_{Zar}$$

and a morphism of topoi

$$i_S : Sh(S_{Zar}) \longrightarrow Sh((Sch/S)_{Zar})$$

such that $\pi_S \circ i_S = id$. Moreover, $i_S = i_{id_S}$ with i_{id_S} as in Lemma 30.3.12. In particular the functor $i_S^{-1} = \pi_{S,*}$ is described by the rule $i_S^{-1}(\mathcal{G})(U/S) = \mathcal{G}(U/S)$.

Proof. In this case the functor $u : S_{Zar} \to (Sch/S)_{Zar}$, in addition to the properties seen in the proof of Lemma 30.3.12 above, also is fully faithful and transforms the final object into the final object. The lemma follows. □

Definition 30.3.14. In the situation of Lemma 30.3.13 the functor $i_S^{-1} = \pi_{S,*}$ is often called the *restriction to the small Zariski site*, and for a sheaf $\mathcal{F}$ on the big Zariski site we denote $\mathcal{F}|_{S_{Zar}}$ this restriction.

With this notation in place we have for a sheaf $\mathcal{F}$ on the big site and a sheaf $\mathcal{G}$ on the big site that

$$\begin{aligned} Mor_{Sh(S_{Zar})}(\mathcal{F}|_{S_{Zar}}, \mathcal{G}) &= Mor_{Sh((Sch/S)_{Zar})}(\mathcal{F}, i_{S,*}\mathcal{G}) \\ Mor_{Sh(S_{Zar})}(\mathcal{G}, \mathcal{F}|_{S_{Zar}}) &= Mor_{Sh((Sch/S)_{Zar})}(\pi_S^{-1}\mathcal{G}, \mathcal{F}) \end{aligned}$$

Moreover, we have $(i_{S,*}\mathcal{G})|_{S_{Zar}} = \mathcal{G}$ and we have $(\pi_S^{-1}\mathcal{G})|_{S_{Zar}} = \mathcal{G}$.

Lemma 30.3.15. *Let Sch_{Zar} be a big Zariski site. Let $f : T \to S$ be a morphism in Sch_{Zar}. The functor*

$$u : (Sch/T)_{Zar} \longrightarrow (Sch/S)_{Zar}, \quad V/T \longmapsto V/S$$

is cocontinuous, and has a continuous right adjoint

$$v : (Sch/S)_{Zar} \longrightarrow (Sch/T)_{Zar}, \quad (U \to S) \longmapsto (U \times_S T \to T).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Sch/T)_{Zar}) \longrightarrow Sh((Sch/S)_{Zar})$$

We have $f_{big}^{-1}(\mathcal{G})(U/T) = \mathcal{G}(U/S)$. We have $f_{big,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$. Also, f_{big}^{-1} has a left adjoint $f_{big!}$ which commutes with fibre products and equalizers.

Proof. The functor u is cocontinuous, continuous, and commutes with fibre products and equalizers (details omitted; compare with proof of Lemma 30.3.12). Hence Sites, Lemmas 9.19.5 and 9.19.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given U/T and V/S we have $Mor_S(u(U), V) = Mor_T(U, V \times_S T)$ as desired. Thus we may apply Sites, Lemmas 9.20.1 and 9.20.2 to get the formula for $f_{big,*}$. □

Lemma 30.3.16. *Let Sch_{Zar} be a big Zariski site. Let $f : T \to S$ be a morphism in Sch_{Zar}.*

1. *We have $i_f = f_{big} \circ i_T$ with i_f as in Lemma 30.3.12 and i_T as in Lemma 30.3.13.*
2. *The functor $S_{Zar} \to T_{Zar}$, $(U \to S) \mapsto (U \times_S T \to T)$ is continuous and induces a morphism of topoi*

$$f_{small} : Sh(T_{Zar}) \longrightarrow Sh(S_{Zar}).$$

The functors f_{small}^{-1} and $f_{small,*}$ agree with the usual notions f^{-1} and f_* is we identify sheaves on T_{Zar}, resp. S_{Zar} with sheaves on T, resp. S via Lemma 30.3.11.

(3) *We have a commutative diagram of morphisms of sites*

$$\begin{tikzcd} T_{Zar} \arrow[d, "f_{small}"'] & (Sch/T)_{Zar} \arrow[l, "\pi_T"'] \arrow[d, "f_{big}"] \\ S_{Zar} & (Sch/S)_{Zar} \arrow[l, "\pi_S"] \end{tikzcd}$$

so that $f_{small} \circ \pi_T = \pi_S \circ f_{big}$ as morphisms of topoi.

(4) *We have $f_{small} = \pi_S \circ f_{big} \circ i_T = \pi_S \circ i_f$.*

Proof. The equality $i_f = f_{big} \circ i_T$ follows from the equality $i_f^{-1} = i_T^{-1} \circ f_{big}^{-1}$ which is clear from the descriptions of these functors above. Thus we see (1).

Statement (2): See Sites, Example 9.14.2.

Part (3) follows because π_S and π_T are given by the inclusion functors and f_{small} and f_{big} by the base change functor $U \mapsto U \times_S T$.

Statement (4) follows from (3) by precomposing with i_T. □

In the situation of the lemma, using the terminology of Definition 30.3.14 we have: for $\mathcal{F}$ a sheaf on the big Zariski site of T

$$(f_{big,*}\mathcal{F})|_{S_{Zar}} = f_{small,*}(\mathcal{F}|_{T_{Zar}}),$$

This equality is clear from the commutativity of the diagram of sites of the lemma, since restriction to the small Zariski site of T, resp. S is given by $\pi_{T,*}$, resp. $\pi_{S,*}$. A similar formula involving pullbacks and restrictions is false.

Lemma 30.3.17. *Given schemes X, Y, Y in $(Sch/S)_{Zar}$ and morphisms $f : X \to Y$, $g : Y \to Z$ we have $g_{big} \circ f_{big} = (g \circ f)_{big}$ and $g_{small} \circ f_{small} = (g \circ f)_{small}$.*

Proof. This follows from the simple description of push forward and pull back for the functors on the big sites from Lemma 30.3.15. For the functors on the small sites this is Sheaves, Lemma 6.21.2 via the identification of Lemma 30.3.11. □

We can think about a sheaf on the big Zariski site of S as a collection of ``usual'' sheaves on all schemes over S.

Lemma 30.3.18. *Let S be a scheme contained in a big Zariski site Sch_{Zar}. A sheaf $\mathcal{F}$ on the big Zariski site $(Sch/S)_{Zar}$ is given by the following data:*

(1) *for every $T/S \in Ob((Sch/S)_{Zar})$ a sheaf $\mathcal{F}_T$ on T,*
(2) *for every $f : T' \to T$ in $(Sch/S)_{Zar}$ a map $c_f : f^{-1}\mathcal{F}_T \to \mathcal{F}_{T'}$.*

These data are subject to the following conditions:

(i) *given any $f : T' \to T$ and $g : T'' \to T'$ in $(Sch/S)_{Zar}$ the composition $g^{-1}c_f \circ c_g$ is equal to $c_{f\circ g}$, and*
(ii) *if $f : T' \to T$ in $(Sch/S)_{Zar}$ is an open immersion then c_f is an isomorphism.*

Proof. Given a sheaf $\mathcal{F}$ on $Sh((Sch/S)_{Zar})$ we set $\mathcal{F}_T = i_p^{-1}\mathcal{F}$ where $p : T \to S$ is the structure morphism. Note that $\mathcal{F}_T(U) = \mathcal{F}(U'/S)$ for any open $U \subset T$, and $U' \to T$ an open immersion in $(Sch/T)_{Zar}$ with image U, see Lemmas 30.3.11 and 30.3.12. Hence given $f : T' \to T$ over S and $U, U' \to T$ we get a canonical map $\mathcal{F}_T(U) = \mathcal{F}(U'/S) \to \mathcal{F}(U' \times_T T'/S) = \mathcal{F}_{T'}(f^{-1}(U))$ where the middle is the restriction map of $\mathcal{F}$ with respect to the morphism $U' \times_T T' \to U'$ over S. The collection of these maps are compatible with restrictions, and hence define an f-map c_f from $\mathcal{F}_T$ to $\mathcal{F}_{T'}$, see Sheaves, Definition 6.21.7

and the discussion surrounding it. It is clear that $c_{f\circ g}$ is the composition of c_f and c_g, since composition of restriction maps of $\mathcal{F}$ gives restriction maps.

Conversely, given a system $(\mathcal{F}_T, c_f)$ as in the lemma we may define a presheaf $\mathcal{F}$ on $Sh((Sch/S)_{Zar})$ by simply setting $\mathcal{F}(T/S) = \mathcal{F}_T(T)$. As restriction mapping, given $f : T' \to T$ we set for $s \in \mathcal{F}(T)$ the pull back $f^*(s)$ equal to $c_f(s)$ (where we think of c_f as an f-map again). The condition on the c_f garantees that pullbacks satisfy the required functoriality property. We omit the verification that this is a sheaf. It is clear that the constructions so defined are mutually inverse. □

30.4. The étale topology

Let S be a scheme. We would like to define the étale-topology on the category of schemes over S. According to our general principle we first introduce the notion of an étale covering.

Definition 30.4.1. Let T be a scheme. An *étale covering of T* is a family of morphisms $\{f_i : T_i \to T\}_{i\in I}$ of schemes such that each f_i is étale and such that $T = \bigcup f_i(T_i)$.

Lemma 30.4.2. *Any Zariski covering is an étale covering.*

Proof. This is clear from the definitions and the fact that an open immersion is an étale morphism, see Morphisms, Lemma 24.35.9. □

Next, we show that this notion satisfies the conditions of Sites, Definition 9.6.2.

Lemma 30.4.3. *Let T be a scheme.*

(1) *If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is an étale covering of T.*
(2) *If $\{T_i \to T\}_{i\in I}$ is an étale covering and for each i we have an étale covering $\{T_{ij} \to T_i\}_{j\in J_i}$, then $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is an étale covering.*
(3) *If $\{T_i \to T\}_{i\in I}$ is an étale covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i\in I}$ is an étale covering.*

Proof. Omitted. □

Lemma 30.4.4. *Let T be an affine scheme. Let $\{T_i \to T\}_{i\in I}$ be an étale covering of T. Then there exists an étale covering $\{U_j \to T\}_{j=1,\ldots,m}$ which is a refinement of $\{T_i \to T\}_{i\in I}$ such that each U_j is an affine scheme. Moreover, we may choose each U_j to be open affine in one of the T_i.*

Proof. Omitted. □

Thus we define the corresponding standard coverings of affines as follows.

Definition 30.4.5. Let T be an affine scheme. A *standard étale covering* of T is a family $\{f_j : U_j \to T\}_{j=1,\ldots,m}$ with each U_j is affine and étale over T and $T = \bigcup f_j(U_j)$.

In the definition above we do **not** assume the morphisms f_j are standard étale. The reason is that if we did then the standard étale coverings would not define a site on Aff/S, for example because of Algebra, Lemma 7.132.14 part (4). On the other hand, an étale morphism of affines is automatically standard smooth, see Algebra, Lemma 7.132.2. Hence a standard étale covering is a standard smooth covering and a standard syntomic covering.

Definition 30.4.6. A *big étale site* is any site $Sch_{étale}$ as in Sites, Definition 9.6.2 constructed as follows:

(1) Choose any set of schemes S_0, and any set of étale coverings Cov_0 among these schemes.

(2) As underlying category take any category Sch_α constructed as in Sets, Lemma 3.9.2 starting with the set S_0.
(3) Choose any set of coverings as in Sets, Lemma 3.11.1 starting with the category Sch_α and the class of étale coverings, and the set Cov_0 chosen above.

See the remarks following Definition 30.3.5 for motivation and explanation regarding the definition of big sites.

Before we continue with the introduction of the big étale site of a scheme S, let us point out that the topology on a big étale site $Sch_{étale}$ is in some sense induced from the étale topology on the category of all schemes.

Lemma 30.4.7. *Let $Sch_{étale}$ be a big étale site as in Definition 30.4.6. Let $T \in Ob(Sch_{étale})$. Let $\{T_i \to T\}_{i\in I}$ be an arbitrary étale covering of T.*

(1) *There exists a covering $\{U_j \to T\}_{j\in J}$ of T in the site $Sch_{étale}$ which refines $\{T_i \to T\}_{i\in I}$.*
(2) *If $\{T_i \to T\}_{i\in I}$ is a standard étale covering, then it is tautologically equivalent to a covering in $Sch_{étale}$.*
(3) *If $\{T_i \to T\}_{i\in I}$ is a Zariski covering, then it is tautologically equivalent to a covering in $Sch_{étale}$.*

Proof. For each i choose an affine open covering $T_i = \bigcup_{j\in J_i} T_{ij}$ such that each T_{ij} maps into an affine open subscheme of T. By Lemma 30.4.3 the refinement $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is an étale covering of T as well. Hence we may assume each T_i is affine, and maps into an affine open W_i of T. Applying Sets, Lemma 3.9.9 we see that W_i is isomorphic to an object of Sch_{Zar}. But then T_i as a finite type scheme over W_i is isomorphic to an object V_i of Sch_{Zar} by a second application of Sets, Lemma 3.9.9. The covering $\{V_i \to T\}_{i\in I}$ refines $\{T_i \to T\}_{i\in I}$ (because they are isomorphic). Moreover, $\{V_i \to T\}_{i\in I}$ is combinatorially equivalent to a covering $\{U_j \to T\}_{j\in J}$ of T in the site Sch_{Zar} by Sets, Lemma 3.9.9. The covering $\{U_j \to T\}_{j\in J}$ is a refinement as in (1). In the situation of (2), (3) each of the schemes T_i is isomorphic to an object of $Sch_{étale}$ by Sets, Lemma 3.9.9, and another application of Sets, Lemma 3.11.1 gives what we want. □

Definition 30.4.8. Let S be a scheme. Let $Sch_{étale}$ be a big étale site containing S.

(1) The *big étale site of S*, denoted $(Sch/S)_{étale}$, is the site $Sch_{étale}/S$ introduced in Sites, Section 9.21.
(2) The *small étale site of S*, which we denote $S_{étale}$, is the full subcategory of $(Sch/S)_{étale}$ whose objects are those U/S such that $U \to S$ is étale. A covering of $S_{étale}$ is any covering $\{U_i \to U\}$ of $(Sch/S)_{étale}$ with $U \in Ob(S_{étale})$.
(3) The *big affine étale site of S*, denoted $(Aff/S)_{étale}$, is the full subcategory of $(Sch/S)_{étale}$ whose objects are affine U/S. A covering of $(Aff/S)_{étale}$ is any covering $\{U_i \to U\}$ of $(Sch/S)_{étale}$ which is a standard étale covering.

It is not completely clear that the big affine étale site or the small étale site are sites. We check this now.

Lemma 30.4.9. *Let S be a scheme. Let $Sch_{étale}$ be a big étale site containing S. Both $S_{étale}$ and $(Aff/S)_{étale}$ are sites.*

Proof. Let us show that $S_{étale}$ is a site. It is a category with a given set of families of morphisms with fixed target. Thus we have to show properties (1), (2) and (3) of Sites, Definition 9.6.2. Since $(Sch/S)_{étale}$ is a site, it suffices to prove that given any covering

$\{U_i \to U\}$ of $(Sch/S)_{Zar}$ with $U \in Ob(S_{étale})$ we also have $U_i \in Ob(S_{étale})$. This follows from the definitions as the composition of étale morphisms is an étale morphism.

Let us show that $(Aff/S)_{étale}$ is a site. Reasoning as above, it suffices to show that the collection of standard étale coverings of affines satisfies properties (1), (2) and (3) of Sites, Definition 9.6.2. This is clear since for example, given a standard étale covering $\{T_i \to T\}_{i \in I}$ and for each i we have a standard étale covering $\{T_{ij} \to T_i\}_{j \in J_i}$, then $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is a standard étale covering because $\bigcup_{i \in I} J_i$ is finite and each T_{ij} is affine. □

Lemma 30.4.10. *Let S be a scheme. Let $Sch_{étale}$ be a big étale site containing S. The underlying categories of the sites $Sch_{étale}$, $(Sch/S)_{étale}$, $S_{étale}$, and $(Aff/S)_{étale}$ have fibre products. In each case the obvious functor into the category Sch of all schemes commutes with taking fibre products. The categories $(Sch/S)_{étale}$, and $S_{étale}$ both have a final object, namely S/S.*

Proof. For $Sch_{étale}$ it is true by construction, see Sets, Lemma 3.9.9. Suppose we have $U \to S$, $V \to U$, $W \to U$ morphisms of schemes with $U, V, W \in Ob(Sch_{étale})$. The fibre product $V \times_U W$ in $Sch_{étale}$ is a fibre product in Sch and is the fibre product of V/S with W/S over U/S in the category of all schemes over S, and hence also a fibre product in $(Sch/S)_{étale}$. This proves the result for $(Sch/S)_{étale}$. If $U \to S$, $V \to U$ and $W \to U$ are étale then so is $V \times_U W \to S$ and hence we get the result for $S_{étale}$. If U, V, W are affine, so is $V \times_U W$ and hence the result for $(Aff/S)_{étale}$. □

Next, we check that the big affine site defines the same topos as the big site.

Lemma 30.4.11. *Let S be a scheme. Let $Sch_{étale}$ be a big étale site containing S. The functor $(Aff/S)_{étale} \to (Sch/S)_{étale}$ is special cocontinuous and induces an equivalence of topoi from $Sh((Aff/S)_{étale})$ to $Sh((Sch/S)_{étale})$.*

Proof. The notion of a special cocontinuous functor is introduced in Sites, Definition 9.25.2. Thus we have to verify assumptions (1) -- (5) of Sites, Lemma 9.25.1. Denote the inclusion functor $u : (Aff/S)_{étale} \to (Sch/S)_{étale}$. Being cocontinuous just means that any étale covering of T/S, T affine, can be refined by a standard étale covering of T. This is the content of Lemma 30.4.4. Hence (1) holds. We see u is continuous simply because a standard étale covering is a étale covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that u is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering. □

Next, we esthablish some relationships between the topoi associated to these sites.

Lemma 30.4.12. *Let $Sch_{étale}$ be a big étale site. Let $f : T \to S$ be a morphism in $Sch_{étale}$. The functor $T_{étale} \to (Sch/S)_{étale}$ is cocontinuous and induces a morphism of topoi*

$$i_f : Sh(T_{étale}) \longrightarrow Sh((Sch/S)_{étale})$$

For a sheaf $\mathcal{G}$ on $(Sch/S)_{étale}$ we have the formula $(i_f^{-1}\mathcal{G})(U/T) = \mathcal{G}(U/S)$. The functor i_f^{-1} also has a left adjoint $i_{f,!}$ which commutes with fibre products and equalizers.

Proof. Denote the functor $u : T_{étale} \to (Sch/S)_{étale}$. In other words, given an étale morphism $j : U \to T$ corresponding to an object of $T_{étale}$ we set $u(U \to T) = (f \circ j : U \to S)$. This functor commutes with fibre products, see Lemma 30.4.10. Let $a, b : U \to V$ be two morphisms in $T_{étale}$. In this case the equalizer of a and b (in the category of schemes) is

$$V \times_{\Delta_{V/T}, V \times_T V, (a,b)} U \times_T U$$

which is a fibre product of schemes étale over T, hence étale over T. Thus $T_{étale}$ has equalizers and u commutes with them. It is clearly cocontinuous. It is also continuous as u transforms coverings to coverings and commutes with fibre products. Hence the Lemma follows from Sites, Lemmas 9.19.5 and 9.19.6. □

Lemma 30.4.13. *Let S be a scheme. Let $Sch_{étale}$ be a big étale site containing S. The inclusion functor $S_{étale} \to (Sch/S)_{étale}$ satisfies the hypotheses of Sites, Lemma 9.19.8 and hence induces a morphism of sites*

$$\pi_S : (Sch/S)_{étale} \longrightarrow S_{étale}$$

and a morphism of topoi

$$i_S : Sh(S_{étale}) \longrightarrow Sh((Sch/S)_{étale})$$

such that $\pi_S \circ i_S = id$. Moreover, $i_S = i_{id_S}$ with i_{id_S} as in Lemma 30.4.12. In particular the functor $i_S^{-1} = \pi_{S,*}$ is described by the rule $i_S^{-1}(\mathcal{G})(U/S) = \mathcal{G}(U/S)$.

Proof. In this case the functor $u : S_{étale} \to (Sch/S)_{étale}$, in addition to the properties seen in the proof of Lemma 30.4.12 above, also is fully faithful and transforms the final object into the final object. The lemma follows from Sites, Lemma 9.19.8. □

Definition 30.4.14. In the situation of Lemma 30.4.13 the functor $i_S^{-1} = \pi_{S,*}$ is often called the *restriction to the small étale site*, and for a sheaf $\mathcal{F}$ on the big étale site we denote $\mathcal{F}|_{S_{étale}}$ this restriction.

With this notation in place we have for a sheaf $\mathcal{F}$ on the big site and a sheaf $\mathcal{G}$ on the big site that

$$\begin{aligned} Mor_{Sh(S_{étale})}(\mathcal{F}|_{S_{étale}}, \mathcal{G}) &= Mor_{Sh((Sch/S)_{étale})}(\mathcal{F}, i_{S,*}\mathcal{G}) \\ Mor_{Sh(S_{étale})}(\mathcal{G}, \mathcal{F}|_{S_{étale}}) &= Mor_{Sh((Sch/S)_{étale})}(\pi_S^{-1}\mathcal{G}, \mathcal{F}) \end{aligned}$$

Moreover, we have $(i_{S,*}\mathcal{G})|_{S_{étale}} = \mathcal{G}$ and we have $(\pi_S^{-1}\mathcal{G})|_{S_{étale}} = \mathcal{G}$.

Lemma 30.4.15. *Let $Sch_{étale}$ be a big étale site. Let $f : T \to S$ be a morphism in $Sch_{étale}$. The functor*

$$u : (Sch/T)_{étale} \longrightarrow (Sch/S)_{étale}, \quad V/T \longmapsto V/S$$

is cocontinuous, and has a continuous right adjoint

$$v : (Sch/S)_{étale} \longrightarrow (Sch/T)_{étale}, \quad (U \to S) \longmapsto (U \times_S T \to T).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Sch/T)_{étale}) \longrightarrow Sh((Sch/S)_{étale})$$

We have $f_{big}^{-1}(\mathcal{G})(U/T) = \mathcal{G}(U/S)$. We have $f_{big,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$. Also, f_{big}^{-1} has a left adjoint $f_{big!}$ which commutes with fibre products and equalizers.

Proof. The functor u is cocontinuous, continuous and commutes with fibre products and equalizers (details omitted; compare with the proof of Lemma 30.4.12). Hence Sites, Lemmas 9.19.5 and 9.19.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given U/T and V/S we have $Mor_S(u(U), V) = Mor_T(U, V \times_S T)$ as desired. Thus we may apply Sites, Lemmas 9.20.1 and 9.20.2 to get the formula for $f_{big,*}$. □

Lemma 30.4.16. *Let $Sch_{étale}$ be a big étale site. Let $f : T \to S$ be a morphism in $Sch_{étale}$.*

(1) *We have $i_f = f_{big} \circ i_T$ with i_f as in Lemma 30.4.12 and i_T as in Lemma 30.4.13.*

(2) *The functor* $S_{étale} \to T_{étale}$, $(U \to S) \mapsto (U \times_S T \to T)$ *is continuous and induces a morphism of topoi*

$$f_{small} : Sh(T_{étale}) \longrightarrow Sh(S_{étale}).$$

We have $f_{small,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$.

(3) *We have a commutative diagram of morphisms of sites*

$$\begin{tikzcd} T_{étale} \arrow[d, "f_{small}"'] & (Sch/T)_{étale} \arrow[l, "\pi_T"] \arrow[d, "f_{big}"] \\ S_{étale} & (Sch/S)_{étale} \arrow[l, "\pi_S"'] \end{tikzcd}$$

so that $f_{small} \circ \pi_T = \pi_S \circ f_{big}$ *as morphisms of topoi.*

(4) *We have* $f_{small} = \pi_S \circ f_{big} \circ i_T = \pi_S \circ i_f$.

Proof. The equality $i_f = f_{big} \circ i_T$ follows from the equality $i_f^{-1} = i_T^{-1} \circ f_{big}^{-1}$ which is clear from the descriptions of these functors above. Thus we see (1).

The functor $u : S_{étale} \to T_{étale}$, $u(U \to S) = (U \times_S T \to T)$ transforms coverings into coverings and commutes with fibre products, see Lemma 30.4.3 (3) and 30.4.10. Moreover, both $S_{étale}, T_{étale}$ have final objects, namely S/S and T/T and $u(S/S) = T/T$. Hence by Sites, Proposition 9.14.6 the functor u corresponds to a morphism of sites $T_{étale} \to S_{étale}$. This in turn gives rise to the morphism of topoi, see Sites, Lemma 9.15.3. The description of the pushforward is clear from these references.

Part (3) follows because π_S and π_T are given by the inclusion functors and f_{small} and f_{big} by the base change functors $U \mapsto U \times_S T$.

Statement (4) follows from (3) by precomposing with i_T. □

In the situation of the lemma, using the terminology of Definition 30.4.14 we have: for $\mathcal{F}$ a sheaf on the big étale site of T

$$(f_{big,*}\mathcal{F})|_{S_{étale}} = f_{small,*}(\mathcal{F}|_{T_{étale}}),$$

This equality is clear from the commutativity of the diagram of sites of the lemma, since restriction to the small étale site of T, resp. S is given by $\pi_{T,*}$, resp. $\pi_{S,*}$. A similar formula involving pullbacks and restrictions is false.

Lemma 30.4.17. *Given schemes X, Y, Y in* $Sch_{étale}$ *and morphisms* $f : X \to Y$, $g : Y \to Z$ *we have* $g_{big} \circ f_{big} = (g \circ f)_{big}$ *and* $g_{small} \circ f_{small} = (g \circ f)_{small}$.

Proof. This follows from the simple description of push forward and pull back for the functors on the big sites from Lemma 30.4.15. For the functors on the small sites this follows from the description of the pushforward functors in Lemma 30.4.16. □

We can think about a sheaf on the big étale site of S as a collection of ``usual" sheaves on all schemes over S.

Lemma 30.4.18. *Let S be a scheme contained in a big étale site* $Sch_{étale}$. *A sheaf* $\mathcal{F}$ *on the big étale site* $(Sch/S)_{étale}$ *is given by the following data:*

(1) *for every* $T/S \in Ob((Sch/S)_{étale})$ *a sheaf* $\mathcal{F}_T$ *on* $T_{étale}$,
(2) *for every* $f : T' \to T$ *in* $(Sch/S)_{étale}$ *a map* $c_f : f_{small}^{-1}\mathcal{F}_T \to \mathcal{F}_{T'}$.

These data are subject to the following conditions:

(i) *given any* $f : T' \to T$ *and* $g : T'' \to T'$ *in* $(Sch/S)_{étale}$ *the composition* $g_{small}^{-1}c_f \circ c_g$ *is equal to* $c_{f\circ g}$, *and*
(ii) *if* $f : T' \to T$ *in* $(Sch/S)_{étale}$ *is étale then* c_f *is an isomorphism.*

Proof. Given a sheaf $\mathcal{F}$ on $Sh((Sch/S)_{étale})$ we set $\mathcal{F}_T = i_p^{-1}\mathcal{F}$ where $p : T \to S$ is the structure morphism. Note that $\mathcal{F}_T(U) = \mathcal{F}(U/S)$ for any $U \to T$ in $T_{étale}$ see Lemma 30.4.12. Hence given $f : T' \to T$ over S and $U \to T$ we get a canonical map $\mathcal{F}_T(U) = \mathcal{F}(U/S) \to \mathcal{F}(U \times_T T'/S) = \mathcal{F}_{T'}(U \times_T T')$ where the middle is the restriction map of $\mathcal{F}$ with respect to the morphism $U \times_T T' \to U$ over S. The collection of these maps are compatible with restrictions, and hence define a map $c'_f : \mathcal{F}_T \to f_{small,*}\mathcal{F}_{T'}$ where $u : T_{étale} \to T'_{étale}$ is the base change functor associated to f. By adjunction of $f_{small,*}$ (see Sites, Section 9.13) with f_{small}^{-1} this is the same as a map $c_f : f_{small}^{-1}\mathcal{F}_T \to \mathcal{F}_{T'}$. It is clear that $c'_{f\circ g}$ is the composition of c'_f and $f_{small,*}c'_g$, since composition of restriction maps of $\mathcal{F}$ gives restriction maps, and this gives the desired relationship among c_f, c_g and $c_{f\circ g}$.

Conversely, given a system $(\mathcal{F}_T, c_f)$ as in the lemma we may define a presheaf $\mathcal{F}$ on $Sh((Sch/S)_{étale})$ by simply setting $\mathcal{F}(T/S) = \mathcal{F}_T(T)$. As restriction mapping, given $f : T' \to T$ we set for $s \in \mathcal{F}(T)$ the pull back $f^*(s)$ equal to $c_f(s)$ where we think of c_f as a map $\mathcal{F}_T \to f_{small,*}\mathcal{F}_{T'}$ again. The condition on the c_f garantees that pullbacks satisfy the required functoriality property. We omit the verification that this is a sheaf. It is clear that the constructions so defined are mutually inverse. □

30.5. The smooth topology

In this section we define the smooth topology. This is a bit pointless as it will turn out later (see More on Morphisms, Section 33.26) that this topology defines the same topos as the étale topology. But still it makes sense and it is used occasionally.

Definition 30.5.1. Let T be a scheme. An *smooth covering of* T is a family of morphisms $\{f_i : T_i \to T\}_{i\in I}$ of schemes such that each f_i is smooth and such that $T = \bigcup f_i(T_i)$.

Lemma 30.5.2. *Any étale covering is a smooth covering, and a fortiori, any Zariski covering is a smooth covering.*

Proof. This is clear from the definitions, the fact that an étale morphism is smooth see Morphisms, Definition 24.35.1 and Lemma 30.4.2. □

Next, we show that this notion satisfies the conditions of Sites, Definition 9.6.2.

Lemma 30.5.3. *Let* T *be a scheme.*

(1) *If* $T' \to T$ *is an isomorphism then* $\{T' \to T\}$ *is an smooth covering of* T.
(2) *If* $\{T_i \to T\}_{i\in I}$ *is a smooth covering and for each i we have a smooth covering* $\{T_{ij} \to T_i\}_{j\in J_i}$, *then* $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ *is a smooth covering.*
(3) *If* $\{T_i \to T\}_{i\in I}$ *is a smooth covering and* $T' \to T$ *is a morphism of schemes then* $\{T' \times_T T_i \to T'\}_{i\in I}$ *is a smooth covering.*

Proof. Omitted. □

Lemma 30.5.4. *Let* T *be an affine scheme. Let* $\{T_i \to T\}_{i\in I}$ *be a smooth covering of* T. *Then there exists a smooth covering* $\{U_j \to T\}_{j=1,\ldots,m}$ *which is a refinement of* $\{T_i \to T\}_{i\in I}$ *such that each* U_j *is an affine scheme, and such that each morphism* $U_j \to T$ *is standard smooth, see Morphisms, Definition 24.33.1. Moreover, we may choose each* U_j *to be open affine in one of the* T_i.

Proof. Omitted, but see Algebra, Lemma 7.126.10. □

Thus we define the corresponding standard coverings of affines as follows.

Definition 30.5.5. Let T be an affine scheme. A *standard smooth covering* of T is a family $\{f_j : U_j \to T\}_{j=1,\ldots,m}$ with each U_j is affine, $U_j \to T$ standard smooth and $T = \bigcup f_j(U_j)$.

Definition 30.5.6. A *big smooth site* is any site Sch_{smooth} as in Sites, Definition 9.6.2 constructed as follows:

(1) Choose any set of schemes S_0, and any set of smooth coverings Cov_0 among these schemes.
(2) As underlying category take any category Sch_α constructed as in Sets, Lemma 3.9.2 starting with the set S_0.
(3) Choose any set of coverings as in Sets, Lemma 3.11.1 starting with the category Sch_α and the class of smooth coverings, and the set Cov_0 chosen above.

See the remarks following Definition 30.3.5 for motivation and explanation regarding the definition of big sites.

Before we continue with the introduction of the big smooth site of a scheme S, let us point out that the topology on a big smooth site Sch_{smooth} is in some sense induced from the smooth topology on the category of all schemes.

Lemma 30.5.7. *Let Sch_{smooth} be a big smooth site as in Definition 30.5.6. Let $T \in Ob(Sch_{smooth})$. Let $\{T_i \to T\}_{i\in I}$ be an arbitrary smooth covering of T.*

(1) *There exists a covering $\{U_j \to T\}_{j\in J}$ of T in the site Sch_{smooth} which refines $\{T_i \to T\}_{i\in I}$.*
(2) *If $\{T_i \to T\}_{i\in I}$ is a standard smooth covering, then it is tautologically equivalent to a covering of Sch_{smooth}.*
(3) *If $\{T_i \to T\}_{i\in I}$ is a Zariski covering, then it is tautologically equivalent to a covering of Sch_{smooth}.*

Proof. For each i choose an affine open covering $T_i = \bigcup_{j\in J_i} T_{ij}$ such that each T_{ij} maps into an affine open subscheme of T. By Lemma 30.5.3 the refinement $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is an smooth covering of T as well. Hence we may assume each T_i is affine, and maps into an affine open W_i of T. Applying Sets, Lemma 3.9.9 we see that W_i is isomorphic to an object of Sch_{Zar}. But then T_i as a finite type scheme over W_i is isomorphic to an object V_i of Sch_{Zar} by a second application of Sets, Lemma 3.9.9. The covering $\{V_i \to T\}_{i\in I}$ refines $\{T_i \to T\}_{i\in I}$ (because they are isomorphic). Moreover, $\{V_i \to T\}_{i\in I}$ is combinatorially equivalent to a covering $\{U_j \to T\}_{j\in J}$ of T in the site Sch_{Zar} by Sets, Lemma 3.9.9. The covering $\{U_j \to T\}_{j\in J}$ is a refinement as in (1). In the situation of (2), (3) each of the schemes T_i is isomorphic to an object of Sch_{smooth} by Sets, Lemma 3.9.9, and another application of Sets, Lemma 3.11.1 gives what we want. □

Definition 30.5.8. Let S be a scheme. Let Sch_{smooth} be a big smooth site containing S.

(1) The *big smooth site of S*, denoted $(Sch/S)_{smooth}$, is the site Sch_{smooth}/S introduced in Sites, Section 9.21.
(2) The *big affine smooth site of S*, denoted $(Aff/S)_{smooth}$, is the full subcategory of $(Sch/S)_{smooth}$ whose objects are affine U/S. A covering of $(Aff/S)_{smooth}$ is any covering $\{U_i \to U\}$ of $(Sch/S)_{smooth}$ which is a standard smooth covering.

Next, we check that the big affine site defines the same topos as the big site.

Lemma 30.5.9. *Let S be a scheme. Let $Sch_{étale}$ be a big smooth site containing S. The functor $(Aff/S)_{smooth} \to (Sch/S)_{smooth}$ is special cocontinuous and induces an equivalence of topoi from $Sh((Aff/S)_{smooth})$ to $Sh((Sch/S)_{smooth})$.*

Proof. The notion of a special cocontinuous functor is introduced in Sites, Definition 9.25.2. Thus we have to verify assumptions (1) -- (5) of Sites, Lemma 9.25.1. Denote the inclusion functor $u : (Aff/S)_{smooth} \to (Sch/S)_{smooth}$. Being cocontinuous just means that any smooth covering of T/S, T affine, can be refined by a standard smooth covering of T. This is the content of Lemma 30.5.4. Hence (1) holds. We see u is continuous simply because a standard smooth covering is a smooth covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that u is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering. □

To be continued...

Lemma 30.5.10. *Let Sch_{smooth} be a big smooth site. Let $f : T \to S$ be a morphism in Sch_{smooth}. The functor*

$$u : (Sch/T)_{smooth} \longrightarrow (Sch/S)_{smooth}, \quad V/T \longmapsto V/S$$

is cocontinuous, and has a continuous right adjoint

$$v : (Sch/S)_{smooth} \longrightarrow (Sch/T)_{smooth}, \quad (U \to S) \longmapsto (U \times_S T \to T).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Sch/T)_{smooth}) \longrightarrow Sh((Sch/S)_{smooth})$$

We have $f_{big}^{-1}(\mathcal{G})(U/T) = \mathcal{G}(U/S)$. We have $f_{big,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$. Also, f_{big}^{-1} has a left adjoint $f_{big!}$ which commutes with fibre products and equalizers.

Proof. The functor u is cocontinuous, continuous, and commutes with fibre products and equalizers. Hence Sites, Lemmas 9.19.5 and 9.19.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given U/T and V/S we have $Mor_S(u(U), V) = Mor_T(U, V \times_S T)$ as desired. Thus we may apply Sites, Lemmas 9.20.1 and 9.20.2 to get the formula for $f_{big,*}$. □

30.6. The syntomic topology

In this section we define the syntomic topology. This topology is quite interesting in that it often has the same cohomology groups as the fppf topology but is technically easier to deal with.

Definition 30.6.1. Let T be a scheme. An *syntomic covering of T* is a family of morphisms $\{f_i : T_i \to T\}_{i \in I}$ of schemes such that each f_i is syntomic and such that $T = \bigcup f_i(T_i)$.

Lemma 30.6.2. *Any smooth covering is a syntomic covering, and a fortiori, any étale or Zariski covering is a syntomic covering.*

Proof. This is clear from the definitions and the fact that a smooth morphism is syntomic, see Morphisms, Lemma 24.33.7 and Lemma 30.5.2. □

Next, we show that this notion satisfies the conditions of Sites, Definition 9.6.2.

Lemma 30.6.3. *Let T be a scheme.*

(1) *If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is an syntomic covering of T.*

(2) *If $\{T_i \to T\}_{i\in I}$ is a syntomic covering and for each i we have a syntomic covering $\{T_{ij} \to T_i\}_{j\in J_i}$, then $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is a syntomic covering.*
(3) *If $\{T_i \to T\}_{i\in I}$ is a syntomic covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i\in I}$ is a syntomic covering.*

Proof. Omitted. □

Lemma 30.6.4. *Let T be an affine scheme. Let $\{T_i \to T\}_{i\in I}$ be a syntomic covering of T. Then there exists a syntomic covering $\{U_j \to T\}_{j=1,\ldots,m}$ which is a refinement of $\{T_i \to T\}_{i\in I}$ such that each U_j is an affine scheme, and such that each morphism $U_j \to T$ is standard syntomic, see Morphisms, Definition 24.30.1. Moreover, we may choose each U_j to be open affine in one of the T_i.*

Proof. Omitted, but see Algebra, Lemma 7.125.16. □

Thus we define the corresponding standard coverings of affines as follows.

Definition 30.6.5. Let T be an affine scheme. A *standard syntomic covering* of T is a family $\{f_j : U_j \to T\}_{j=1,\ldots,m}$ with each U_j is affine, $U_j \to T$ standard syntomic and $T = \bigcup f_j(U_j)$.

Definition 30.6.6. A *big syntomic site* is any site $Sch_{syntomic}$ as in Sites, Definition 9.6.2 constructed as follows:

(1) Choose any set of schemes S_0, and any set of syntomic coverings Cov_0 among these schemes.
(2) As underlying category take any category Sch_α constructed as in Sets, Lemma 3.9.2 starting with the set S_0.
(3) Choose any set of coverings as in Sets, Lemma 3.11.1 starting with the category Sch_α and the class of syntomic coverings, and the set Cov_0 chosen above.

See the remarks following Definition 30.3.5 for motivation and explanation regarding the definition of big sites.

Before we continue with the introduction of the big syntomic site of a scheme S, let us point out that the topology on a big syntomic site $Sch_{syntomic}$ is in some sense induced from the syntomic topology on the category of all schemes.

Lemma 30.6.7. *Let $Sch_{syntomic}$ be a big syntomic site as in Definition 30.6.6. Let $T \in Ob(Sch_{syntomic})$. Let $\{T_i \to T\}_{i\in I}$ be an arbitrary syntomic covering of T.*

(1) *There exists a covering $\{U_j \to T\}_{j\in J}$ of T in the site $Sch_{syntomic}$ which refines $\{T_i \to T\}_{i\in I}$.*
(2) *If $\{T_i \to T\}_{i\in I}$ is a standard syntomic covering, then it is tautologically equivalent to a covering in $Sch_{syntomic}$.*
(3) *If $\{T_i \to T\}_{i\in I}$ is a Zariski covering, then it is tautologically equivalent to a covering in $Sch_{syntomic}$.*

Proof. For each i choose an affine open covering $T_i = \bigcup_{j\in J_i} T_{ij}$ such that each T_{ij} maps into an affine open subscheme of T. By Lemma 30.6.3 the refinement $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is an syntomic covering of T as well. Hence we may assume each T_i is affine, and maps into an affine open W_i of T. Applying Sets, Lemma 3.9.9 we see that W_i is isomorphic to an object of Sch_{Zar}. But then T_i as a finite type scheme over W_i is isomorphic to an object V_i of Sch_{Zar} by a second application of Sets, Lemma 3.9.9. The covering $\{V_i \to T\}_{i\in I}$ refines $\{T_i \to T\}_{i\in I}$ (because they are isomorphic). Moreover, $\{V_i \to T\}_{i\in I}$ is combinatorially equivalent to a covering $\{U_j \to T\}_{j\in J}$ of T in the site Sch_{Zar} by Sets, Lemma 3.9.9. The covering $\{U_j \to T\}_{j\in J}$ is a covering as in (1). In the situation of (2), (3) each of the schemes T_i is isomorphic to an object of Sch_{Zar} by Sets, Lemma 3.9.9, and another application of Sets, Lemma 3.11.1 gives what we want. □

Definition 30.6.8. Let S be a scheme. Let $Sch_{syntomic}$ be a big syntomic site containing S.

(1) The *big syntomic site of S*, denoted $(Sch/S)_{syntomic}$, is the site $Sch_{syntomic}/S$ introduced in Sites, Section 9.21.
(2) The *big affine syntomic site of S*, denoted $(Aff/S)_{syntomic}$, is the full subcategory of $(Sch/S)_{syntomic}$ whose objects are affine U/S. A covering of $(Aff/S)_{syntomic}$ is any covering $\{U_i \to U\}$ of $(Sch/S)_{syntomic}$ which is a standard syntomic covering.

Next, we check that the big affine site defines the same topos as the big site.

Lemma 30.6.9. *Let S be a scheme. Let $Sch_{syntomic}$ be a big syntomic site containing S. The functor $(Aff/S)_{syntomic} \to (Sch/S)_{syntomic}$ is special cocontinuous and induces an equivalence of topoi from $Sh((Aff/S)_{syntomic})$ to $Sh((Sch/S)_{syntomic})$.*

Proof. The notion of a special cocontinuous functor is introduced in Sites, Definition 9.25.2. Thus we have to verify assumptions (1) -- (5) of Sites, Lemma 9.25.1. Denote the inclusion functor $u : (Aff/S)_{syntomic} \to (Sch/S)_{syntomic}$. Being cocontinuous just means that any syntomic covering of T/S, T affine, can be refined by a standard syntomic covering of T. This is the content of Lemma 30.6.4. Hence (1) holds. We see u is continuous simply because a standard syntomic covering is a syntomic covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that u is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering. □

To be continued...

Lemma 30.6.10. *Let $Sch_{syntomic}$ be a big syntomic site. Let $f : T \to S$ be a morphism in $Sch_{syntomic}$. The functor*

$$u : (Sch/T)_{syntomic} \longrightarrow (Sch/S)_{syntomic}, \quad V/T \longmapsto V/S$$

is cocontinuous, and has a continuous right adjoint

$$v : (Sch/S)_{syntomic} \longrightarrow (Sch/T)_{syntomic}, \quad (U \to S) \longmapsto (U \times_S T \to T).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Sch/T)_{syntomic}) \longrightarrow Sh((Sch/S)_{syntomic})$$

We have $f_{big}^{-1}(\mathcal{G})(U/T) = \mathcal{G}(U/S)$. We have $f_{big,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$. Also, f_{big}^{-1} has a left adjoint $f_{big!}$ which commutes with fibre products and equalizers.

Proof. The functor u is cocontinuous, continuous, and commutes with fibre products and equalizers. Hence Sites, Lemmas 9.19.5 and 9.19.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given U/T and V/S we have $Mor_S(u(U), V) = Mor_T(U, V \times_S T)$ as desired. Thus we may apply Sites, Lemmas 9.20.1 and 9.20.2 to get the formula for $f_{big,*}$. □

30.7. The fppf topology

Let S be a scheme. We would like to define the fppf-topology[2] on the category of schemes over S. According to our general principle we first introduce the notion of an fppf-covering.

[2] The letters fppf stand for ``fidèlement plat de présentation finie''.

Definition 30.7.1. Let T be a scheme. An *fppf covering of T* is a family of morphisms $\{f_i : T_i \to T\}_{i\in I}$ of schemes such that each f_i is flat, locally of finite presentation and such that $T = \bigcup f_i(T_i)$.

Lemma 30.7.2. *Any syntomic covering is an fppf covering, and a fortiori, any smooth, étale, or Zariski covering is an fppf covering.*

Proof. This is clear from the definitions, the fact that a synomtic morphism is flat and locally of finite presentation, see Morphisms, Lemmas 24.30.6 and 24.30.7, and Lemma 30.6.2. ☐

Next, we show that this notion satisfies the conditions of Sites, Definition 9.6.2.

Lemma 30.7.3. *Let T be a scheme.*

1. *If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is an fppf covering of T.*
2. *If $\{T_i \to T\}_{i\in I}$ is an fppf covering and for each i we have an fppf covering $\{T_{ij} \to T_i\}_{j\in J_i}$, then $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is an fppf covering.*
3. *If $\{T_i \to T\}_{i\in I}$ is an fppf covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i\in I}$ is an fppf covering.*

Proof. The first assertion is clear. The second follows as the composition of flat morphisms is flat (see Morphisms, Lemma 24.24.5) and the composition of morphisms of finite presentation is of finite presentation (see Morphisms, Lemma 24.20.3). The third follows as the base change of a flat morphism is flat (see Morphisms, Lemma 24.24.7) and the base change of a morphism of finite presentation is of finite presentation (see Morphisms, Lemma 24.20.4). Moreover, the base change of a surjective family of morphisms is surjective (proof omitted). ☐

Lemma 30.7.4. *Let T be an affine scheme. Let $\{T_i \to T\}_{i\in I}$ be an fppf covering of T. Then there exists an fppf covering $\{U_j \to T\}_{j=1,\ldots,m}$ which is a refinement of $\{T_i \to T\}_{i\in I}$ such that each U_j is an affine scheme. Moreover, we may choose each U_j to be open affine in one of the T_i.*

Proof. This follows directly from the definitions using that a morphism which is flat and locally of finite presentation is open, see Morphisms, Lemma 24.24.9. ☐

Thus we define the corresponding standard coverings of affines as follows.

Definition 30.7.5. Let T be an affine scheme. A *standard fppf covering* of T is a family $\{f_j : U_j \to T\}_{j=1,\ldots,m}$ with each U_j is affine, flat and of finite presentation over T and $T = \bigcup f_j(U_j)$.

Definition 30.7.6. A *big fppf site* is any site Sch_{fppf} as in Sites, Definition 9.6.2 constructed as follows:

1. Choose any set of schemes S_0, and any set of fppf coverings Cov_0 among these schemes.
2. As underlying category take any category Sch_α constructed as in Sets, Lemma 3.9.2 starting with the set S_0.
3. Choose any set of coverings as in Sets, Lemma 3.11.1 starting with the category Sch_α and the class of fppf coverings, and the set Cov_0 chosen above.

See the remarks following Definition 30.3.5 for motivation and explanation regarding the definition of big sites.

Before we continue with the introduction of the big fppf site of a scheme S, let us point out that the topology on a big fppf site Sch_{fppf} is in some sense induced from the fppf topology on the category of all schemes.

Lemma 30.7.7. *Let Sch_{fppf} be a big fppf site as in Definition 30.7.6. Let $T \in Ob(Sch_{fppf})$. Let $\{T_i \to T\}_{i\in I}$ be an arbitrary fppf covering of T.*

(1) *There exists a covering $\{U_j \to T\}_{j\in J}$ of T in the site Sch_{fppf} which refines $\{T_i \to T\}_{i\in I}$.*
(2) *If $\{T_i \to T\}_{i\in I}$ is a standard fppf covering, then it is tautologically equivalent to a covering of Sch_{fppf}.*
(3) *If $\{T_i \to T\}_{i\in I}$ is a Zariski covering, then it is tautologically equivalent to a covering of Sch_{fppf}.*

Proof. For each i choose an affine open covering $T_i = \bigcup_{j\in J_i} T_{ij}$ such that each T_{ij} maps into an affine open subscheme of T. By Lemma 30.7.3 the refinement $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is an fppf covering of T as well. Hence we may assume each T_i is affine, and maps into an affine open W_i of T. Applying Sets, Lemma 3.9.9 we see that W_i is isomorphic to an object of Sch_{Zar}. But then T_i as a finite type scheme over W_i is isomorphic to an object V_i of Sch_{Zar} by a second application of Sets, Lemma 3.9.9. The covering $\{V_i \to T\}_{i\in I}$ refines $\{T_i \to T\}_{i\in I}$ (because they are isomorphic). Moreover, $\{V_i \to T\}_{i\in I}$ is combinatorially equivalent to a covering $\{U_j \to T\}_{j\in J}$ of T in the site Sch_{Zar} by Sets, Lemma 3.9.9. The covering $\{U_j \to T\}_{j\in J}$ is a refinement as in (1). In the situation of (2), (3) each of the schemes T_i is isomorphic to an object of Sch_{fppf} by Sets, Lemma 3.9.9, and another application of Sets, Lemma 3.11.1 gives what we want. □

Definition 30.7.8. Let S be a scheme. Let Sch_{fppf} be a big fppf site containing S.

(1) The *big fppf site of S*, denoted $(Sch/S)_{fppf}$, is the site Sch_{fppf}/S introduced in Sites, Section 9.21.
(2) The *big affine fppf site of S*, denoted $(Aff/S)_{fppf}$, is the full subcategory of $(Sch/S)_{fppf}$ whose objects are affine U/S. A covering of $(Aff/S)_{fppf}$ is any covering $\{U_i \to U\}$ of $(Sch/S)_{fppf}$ which is a standard fppf covering.

It is not completely clear that the big affine fppf site is a site. We check this now.

Lemma 30.7.9. *Let S be a scheme. Let Sch_{fppf} be a big fppf site containing S. Then $(Aff/S)_{fppf}$ is a site.*

Proof. Let us show that $(Aff/S)_{fppf}$ is a site. Reasoning as in the proof of Lemma 30.4.9 it suffices to show that the collection of standard fppf coverings of affines satisfies properties (1), (2) and (3) of Sites, Definition 9.6.2. This is clear since for example, given a standard fppf covering $\{T_i \to T\}_{i\in I}$ and for each i we have a standard fppf covering $\{T_{ij} \to T_i\}_{j\in J_i}$, then $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is a standard fppf covering because $\bigcup_{i\in I} J_i$ is finite and each T_{ij} is affine. □

Lemma 30.7.10. *Let S be a scheme. Let Sch_{fppf} be a big fppf site containing S. The underlying categories of the sites Sch_{fppf}, $(Sch/S)_{fppf}$, and $(Aff/S)_{fppf}$ have fibre products. In each case the obvious functor into the category Sch of all schemes commutes with taking fibre products. The category $(Sch/S)_{fppf}$ has a final object, namely S/S.*

Proof. For Sch_{fppf} it is true by construction, see Sets, Lemma 3.9.9. Suppose we have $U \to S$, $V \to U$, $W \to U$ morphisms of schemes with $U, V, W \in Ob(Sch_{fppf})$. The fibre product $V \times_U W$ in Sch_{fppf} is a fibre product in Sch and is the fibre product of V/S with

W/S over U/S in the category of all schemes over S, and hence also a fibre product in $(Sch/S)_{fppf}$. This proves the result for $(Sch/S)_{fppf}$. If U, V, W are affine, so is $V \times_U W$ and hence the result for $(Aff/S)_{fppf}$. □

Next, we check that the big affine site defines the same topos as the big site.

Lemma 30.7.11. *Let S be a scheme. Let Sch_{fppf} be a big fppf site containing S. The functor $(Aff/S)_{fppf} \to (Sch/S)_{fppf}$ is cocontinuous and induces an equivalence of topoi from $Sh((Aff/S)_{fppf})$ to $Sh((Sch/S)_{fppf})$.*

Proof. The notion of a special cocontinuous functor is introduced in Sites, Definition 9.25.2. Thus we have to verify assumptions (1) -- (5) of Sites, Lemma 9.25.1. Denote the inclusion functor $u : (Aff/S)_{fppf} \to (Sch/S)_{fppf}$. Being cocontinuous just means that any fppf covering of T/S, T affine, can be refined by a standard fppf covering of T. This is the content of Lemma 30.7.4. Hence (1) holds. We see u is continuous simply because a standard fppf covering is a fppf covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that u is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering. □

Next, we esthablish some relationships between the topoi associated to these sites.

Lemma 30.7.12. *Let Sch_{fppf} be a big fppf site. Let $f : T \to S$ be a morphism in Sch_{fppf}. The functor*

$$u : (Sch/T)_{fppf} \longrightarrow (Sch/S)_{fppf}, \quad V/T \longmapsto V/S$$

is cocontinuous, and has a continuous right adjoint

$$v : (Sch/S)_{fppf} \longrightarrow (Sch/T)_{fppf}, \quad (U \to S) \longmapsto (U \times_S T \to T).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Sch/T)_{fppf}) \longrightarrow Sh((Sch/S)_{fppf})$$

We have $f_{big}^{-1}(\mathcal{G})(U/T) = \mathcal{G}(U/S)$. We have $f_{big,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$. Also, f_{big}^{-1} has a left adjoint $f_{big!}$ which commutes with fibre products and equalizers.

Proof. The functor u is cocontinuous, continuous, and commutes with fibre products and equalizers. Hence Sites, Lemmas 9.19.5 and 9.19.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given U/T and V/S we have $Mor_S(u(U), V) = Mor_T(U, V \times_S T)$ as desired. Thus we may apply Sites, Lemmas 9.20.1 and 9.20.2 to get the formula for $f_{big,*}$. □

Lemma 30.7.13. *Given schemes X, Y, Y in $(Sch/S)_{fppf}$ and morphisms $f : X \to Y$, $g : Y \to Z$ we have $g_{big} \circ f_{big} = (g \circ f)_{big}$.*

Proof. This follows from the simple description of push forward and pull back for the functors on the big sites from Lemma 30.7.12. □

30.8. The fpqc topology

Definition 30.8.1. Let T be a scheme. An *fpqc covering of T* is a family of morphisms $\{f_i : T_i \to T\}_{i \in I}$ of schemes such that each f_i is flat and such that for every affine open $U \subset T$ there exists $n \geq 0$, a map $a : \{1, \ldots, n\} \to I$ and affine opens $V_j \subset T_{a(j)}$, $j = 1, \ldots, n$ with $\bigcup_{j=1}^n f_{a(j)}(V_j) = U$.

To be sure this condition implies that $T = \bigcup f_i(T_i)$. It is slightly harder to recognize an fpqc covering, hence we provide some lemmas to do so.

Lemma 30.8.2. *Let T be a scheme. Let $\{f_i : T_i \to T\}_{i \in I}$ be a family of morphisms of schemes with target T. The following are equivalent*

(1) *$\{f_i : T_i \to T\}_{i \in I}$ is an fpqc covering,*
(2) *each f_i is flat and for every affine open $U \subset T$ there exist $i_1, \ldots, i_n \in I$ and quasi-compact opens $U_j \subset T_{i_j}$ such that $U = \bigcup_{j=1,\ldots,n} f_{i_j}(U_j)$,*
(3) *each f_i is flat and there exists an affine open covering $T = \bigcup_{\alpha \in A} U_\alpha$ and for each $\alpha \in A$ there exist $i_{\alpha,1}, \ldots, i_{\alpha,n(\alpha)} \in I$ and quasi-compact opens $U_{\alpha,j} \subset T_{i_{\alpha,j}}$ such that $U = \bigcup_{j=1,\ldots,n(\alpha)} f_{i_{\alpha,j}}(U_{\alpha,j})$, and*
(4) *each f_i is flat, and for every $t \in T$ there exist $i_1, \ldots, i_n \in I$ and quasi-compact opens $U_j \subset T_{i_j}$ such that $\bigcup_{j=1,\ldots,n} f_{i_j}(U_j)$ is a (not necessarily open) neighbourhood of t in T.*

Proof. Omitted. □

Lemma 30.8.3. *Let T be a scheme. Let $\{f_i : T_i \to T\}_{i \in I}$ be a family of morphisms of schemes with target T. The following are equivalent*

(1) *$\{f_i : T_i \to T\}_{i \in I}$ is an fpqc covering, and*
(2) *setting $T' = \coprod_{i \in I} T_i$, and $f = \coprod_{i \in I} f_i$ the family $\{f : T' \to T\}$ is an fpqc covering.*

Proof. Omitted. □

Lemma 30.8.4. *Let T be a scheme. Let $\{f_i : T_i \to T\}_{i \in I}$ be a family of morphisms of schemes with target T. Assume that*

(1) *each f_i is flat, and*
(2) *the family $\{f_i : T_i \to T\}_{i \in I}$ can be refined by a fpqc covering of T.*

Then $\{f_i : T_i \to T\}_{i \in I}$ is a fpqc covering of T.

Proof. Omitted. □

Lemma 30.8.5. *Let T be a scheme. Let $\{f_i : T_i \to T\}_{i \in I}$ be a family of morphisms of schemes with target T. Assume that*

(1) *each f_i is flat, and*
(2) *there exists an fpqc covering $\{g_j : S_j \to T\}_{j \in J}$ such that each $\{S_j \times_T T_i \to S_j\}_{i \in I}$ is an fpqc covering.*

Then $\{f_i : T_i \to T\}_{i \in I}$ is a fpqc covering of T.

Proof. Omitted. Hint: Follows from Lemma 30.8.4. □

Lemma 30.8.6. *Any fppf covering is an fpqc covering, and a fortiori, any syntomic, smooth, étale or Zariski covering is an fpqc covering.*

Proof. We will show that an fppf covering is an fpqc covering, and then the rest follows from Lemma 30.7.2. Let $\{f_i : U_i \to U\}_{i \in I}$ be an fppf covering. By definition this means that the f_i are flat which checks the first condition of Definition 30.8.1. To check the second, let $V \subset U$ be an affine open subset. Write $f_i^{-1}(V) = \bigcup_{j \in J_i} V_{ij}$ for some affine opens $V_{ij} \subset U_i$. Since each f_i is open (Morphisms, Lemma 24.24.9), we see that $V = \bigcup_{i \in I} \bigcup_{j \in J_i} f_i(V_{ij})$ is an open covering of V. Since V is quasi-compact, this covering has a finite refinement. This finishes the proof. □

The fpqc[3] topology cannot be treated in the same way as the fppf topology[4]. Namely, suppose that R is a nonzero ring. For any faithfully flat ring map $R \to R'$ the morphism $Spec(R') \to Spec(R)$ is an fpqc-covering. We claim that there does not exist a set A of fpqc-coverings of $Spec(R)$ such that every fpqc-covering can be refined by an element of A. For example, if $R = k$ is a field, then for any set I we can consider the purely transcendental field extension $k \subset k(\{t_i\}_{i \in I})$. We leave it to the reader to show that there does not exist a set of morphisms of schemes $\{S_j \to Spec(k)\}_{j \in J}$ such that every morphism $Spec(k(\{t_i\}_{i \in I}))$ is dominated by one of the schemes S_j.

A mildly interesting option is to consider only those faithfully flat ring extensions $R \to R'$ where the cardinality of R' is suitably bounded. (And if you consider all schemes in a fixed universe as in SGA4 then you are bounding the cardinality by a strongly inaccessible cardinal.) However, it is not so clear what happens if you change the cardinal to a bigger one.

For these reasons we do not introduce fpqc sites and we will not consider cohomology with respect to the fpqc-topology.

On the other hand, given a contravariant functor $F : Sch^{opp} \to Sets$ it does make sense to ask whether F satisfies the sheaf property for the fpqc topology, see below. Moreover, we can wonder about descent of object in the fpqc topology, etc. Simply put, for certain results the correct generality is to work with fpqc coverings.

Lemma 30.8.7. *Let T be a scheme.*

(1) *If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is an fpqc covering of T.*
(2) *If $\{T_i \to T\}_{i \in I}$ is an fpqc covering and for each i we have an fpqc covering $\{T_{ij} \to T_i\}_{j \in J_i}$, then $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is an fpqc covering.*
(3) *If $\{T_i \to T\}_{i \in I}$ is an fpqc covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i \in I}$ is an fpqc covering.*

Proof. Omitted. □

Lemma 30.8.8. *Let T be an affine scheme. Let $\{T_i \to T\}_{i \in I}$ be an fpqc covering of T. Then there exists an fpqc covering $\{U_j \to T\}_{j=1,\ldots,n}$ which is a refinement of $\{T_i \to T\}_{i \in I}$ such that each U_j is an affine scheme. Moreover, we may choose each U_j to be open affine in one of the T_i.*

Proof. This follows directly from the definition. □

Definition 30.8.9. Let T be an affine scheme. A *standard fpqc covering* of T is a family $\{f_j : U_j \to T\}_{j=1,\ldots,n}$ with each U_j is affine, flat over T and $T = \bigcup f_j(U_j)$.

Since we do not introduce the affine site we have to show directly that the collection of all standard fpqc coverings satisfies the axioms.

Lemma 30.8.10. *Let T be an affine scheme.*

(1) *If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is a standard fpqc covering of T.*
(2) *If $\{T_i \to T\}_{i \in I}$ is a standard fpqc covering and for each i we have a standard fpqc covering $\{T_{ij} \to T_i\}_{j \in J_i}$, then $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is a standard fpqc covering.*
(3) *If $\{T_i \to T\}_{i \in I}$ is a standard fpqc covering and $T' \to T$ is a morphism of affine schemes then $\{T' \times_T T_i \to T'\}_{i \in I}$ is a standard fpqc covering.*

[3]The letters fpqc stand for ``fidèlement plat quasi-compacte''.

[4]A more precise statement would be that the analogue of Lemma 30.7.7 for the fpqc topology does not hold.

Proof. Omitted. □

Lemma 30.8.11. *Let T be a scheme. Let $\{f_i : T_i \to T\}_{i\in I}$ be a family of morphisms of schemes with target T. Assume that*

1. *each f_i is flat, and*
2. *every affine scheme Z and morphism $h : Z \to T$ there exists a standard fpqc covering $\{Z_j \to Z\}_{j=1,\ldots,n}$ which refines the family $\{T_i \times_T Z \to Z\}_{i\in I}$.*

Then $\{f_i : T_i \to T\}_{i\in I}$ is a fpqc covering of T.

Proof. Omitted. Hint: Follows from Lemmas 30.8.4 and 30.8.5. □

Definition 30.8.12. Let F be a contravariant functor on the category of schemes with values in sets.

1. Let $\{U_i \to T\}_{i\in I}$ be a family of morphisms of schemes with fixed target. We say that F *satisfies the sheaf property for the given family* if for any collection of elements $\xi_i \in F(U_i)$ such that $\xi_i|_{U_i\times_T U_j} = \xi_j|_{U_i\times_T U_j}$ there exists a unique element $\xi \in F(T)$ such that $\xi_i = \xi|_{U_i}$ in $F(U_i)$.
2. We say that F *satisfies the sheaf property for the fpqc topology* if it satisfies the sheaf property for any fpqc covering.

We try to avoid using the terminology ``F is a sheaf'' in this situation since we are not defining a category of fpqc sheaves as we explained above.

Lemma 30.8.13. *Let F be a contravariant functor on the category of schemes with values in sets. Then F satisfies the sheaf property for the fpqc topology if and only if it satisfies*

1. *the sheaf property for every Zariski covering, and*
2. *the sheaf property for any standard fpqc covering.*

Moreover, in the presence of (1) property (2) is equivalent to property

(2') *the sheaf property for $\{V \to U\}$ with V, U affine and $V \to U$ faithfully flat.*

Proof. Omitted. □

30.9. Change of topologies

Let $f : X \to Y$ be a morphism of schemes over a base change S. In this case we have the following morphisms of sites (with suitable choices of sites as in Remark 30.9.1 below):

1. $(Sch/X)_{fppf} \longrightarrow (Sch/Y)_{fppf}$,
2. $(Sch/X)_{fppf} \longrightarrow (Sch/Y)_{syntomic}$,
3. $(Sch/X)_{fppf} \longrightarrow (Sch/Y)_{smooth}$,
4. $(Sch/X)_{fppf} \longrightarrow (Sch/Y)_{étale}$,
5. $(Sch/X)_{fppf} \longrightarrow (Sch/Y)_{Zar}$,
6. $(Sch/X)_{syntomic} \longrightarrow (Sch/Y)_{syntomic}$,
7. $(Sch/X)_{syntomic} \longrightarrow (Sch/Y)_{smooth}$,
8. $(Sch/X)_{syntomic} \longrightarrow (Sch/Y)_{étale}$,
9. $(Sch/X)_{syntomic} \longrightarrow (Sch/Y)_{Zar}$,
10. $(Sch/X)_{smooth} \longrightarrow (Sch/Y)_{smooth}$,
11. $(Sch/X)_{smooth} \longrightarrow (Sch/Y)_{étale}$,
12. $(Sch/X)_{smooth} \longrightarrow (Sch/Y)_{Zar}$,
13. $(Sch/X)_{étale} \longrightarrow (Sch/Y)_{étale}$,
14. $(Sch/X)_{étale} \longrightarrow (Sch/Y)_{Zar}$,
15. $(Sch/X)_{Zar} \longrightarrow (Sch/Y)_{Zar}$,

(16) $(Sch/X)_{fppf} \longrightarrow Y_{\acute{e}tale}$,
(17) $(Sch/X)_{syntomic} \longrightarrow Y_{\acute{e}tale}$,
(18) $(Sch/X)_{smooth} \longrightarrow Y_{\acute{e}tale}$,
(19) $(Sch/X)_{\acute{e}tale} \longrightarrow Y_{\acute{e}tale}$,
(20) $(Sch/X)_{fppf} \longrightarrow Y_{Zar}$,
(21) $(Sch/X)_{syntomic} \longrightarrow Y_{Zar}$,
(22) $(Sch/X)_{smooth} \longrightarrow Y_{Zar}$,
(23) $(Sch/X)_{\acute{e}tale} \longrightarrow Y_{Zar}$,
(24) $(Sch/X)_{Zariski} \longrightarrow Y_{Zar}$,
(25) $X_{\acute{e}tale} \longrightarrow Y_{\acute{e}tale}$,
(26) $X_{\acute{e}tale} \longrightarrow Y_{Zar}$,
(27) $X_{Zar} \longrightarrow Y_{Zar}$,

In each case the underlying continuous functor $Sch/Y \to Sch/X$, or $Y_\tau \to Sch/X$ is the functor $Y'/Y \mapsto X \times_Y Y'/Y$. Namely, in the sections above we have seen the morphisms $f_{big} : (Sch/X)_\tau \to (Sch/Y)_\tau$ and $f_{small} : X_\tau \to Y_\tau$ for τ as above. We also have seen the morphisms of sites $\pi_Y : (Sch/Y)_\tau \to Y_\tau$ for $\tau \in \{\acute{e}tale, Zariski\}$. On the other hand, it is clear that the identity functor $(Sch/X)_\tau \to (Sch/X)_{\tau'}$ defines a morphism of sites when τ is a stronger topology than τ'. Hence composing these gives the list of possible morphisms above.

Because of the simple description of the underlying functor it is clear that given morphisms of schemes $X \to Y \to Z$ the composition of two of the morphisms of sites above, e.g.,

$$(Sch/X)_{\tau_0} \longrightarrow (Sch/Y)_{\tau_1} \longrightarrow (Sch/Z)_{\tau_2}$$

is the corresponding morphism of sites associated to the morphism of schemes $X \to Z$.

Remark 30.9.1. Take any category Sch_α constructed as in Sets, Lemma 3.9.2 starting with the set of schemes $\{X, Y, S\}$. Choose any set of coverings Cov_{fppf} on Sch_α as in Sets, Lemma 3.11.1 starting with the category Sch_α and the class of fppf coverings. Let Sch_{fppf} denote the big fppf site so obtained. Next, for $\tau \in \{Zariski, \acute{e}tale, smooth, syntomic\}$ let Sch_τ have the same underlying category as Sch_{fppf} with coverings $\text{Cov}_\tau \subset \text{Cov}_{fppf}$ simply the subset of τ-coverings. It is straightforward to check that this gives rise to a big site Sch_τ.

30.10. Change of big sites

In this section we explain what happens on changing the big Zariski/fppf/étale sites.

Let $\tau, \tau' \in \{Zariski, \acute{e}tale, smooth, syntomic, fppf\}$. Given two big sites Sch_τ and $Sch'_{\tau'}$ we say that *Sch_τ is contained in $Sch'_{\tau'}$* if $Ob(Sch_\tau) \subset Ob(Sch'_{\tau'})$ and $\text{Cov}(Sch_\tau) \subset \text{Cov}(Sch'_{\tau'})$. In this case τ is stronger than τ', for example, no fppf site can be contained in an étale site.

Lemma 30.10.1. *Any set of big Zariski sites is contained in a common big Zariski site. The same is true, mutatis mutandis, for big fppf and big étale sites.*

Proof. This is true because the union of a set of sets is a set, and the constructions in the chapter on sets. □

Lemma 30.10.2. *Let $\tau \in \{Zariski, \acute{e}tale, smooth, syntomic, fppf\}$. Suppose given big sites Sch_τ and Sch'_τ. Assume that Sch_τ is contained in Sch'_τ. The inclusion functor $Sch_\tau \to Sch'_\tau$ satisfies the assumptions of Sites, Lemma 9.19.8. There are morphisms of topoi*

$$\begin{aligned} g : Sh(Sch_\tau) &\longrightarrow Sh(Sch'_\tau) \\ f : Sh(Sch'_\tau) &\longrightarrow Sh(Sch_\tau) \end{aligned}$$

such that $f \circ g \cong id$. For any object S of Sch_τ the inclusion functor $(Sch/S)_\tau \to (Sch'/S)_\tau$ satisfies the assumptions of Sites, Lemma 9.19.8 also. Hence similarly we obtain morphisms

$$\begin{aligned} g : Sh((Sch/S)_\tau) &\longrightarrow Sh((Sch'/S)_\tau) \\ f : Sh((Sch'/S)_\tau) &\longrightarrow Sh((Sch/S)_\tau) \end{aligned}$$

with $f \circ g \cong id$.

Proof. Assumptions (b), (c), and (e) of Sites, Lemma 9.19.8 are immediate for the functors $Sch_\tau \to Sch'_\tau$ and $(Sch/S)_\tau \to (Sch'/S)_\tau$. Property (a) holds by Lemma 30.3.6, 30.4.7, 30.5.7, 30.6.7, or 30.7.7. Property (d) holds because fibre products in the categories Sch_τ, Sch'_τ exist and are compatible with fibre products in the category of schemes. $\square$

Discussion: The functor $g^{-1} = f_*$ is simply the restriction functor which associates to a sheaf $\mathcal{G}$ on Sch'_τ the restriction $\mathcal{G}|_{Sch_\tau}$. Hence this lemma simply says that given any sheaf of sets $\mathcal{F}$ on Sch_τ there exists a canonical sheaf $\mathcal{F}'$ on Sch'_τ such that $\mathcal{F}|_{Sch'_\tau} = \mathcal{F}'$. In fact the sheaf $\mathcal{F}'$ has the following description: it is the sheafification of the presheaf

$$Sch'_\tau \longrightarrow Sets, \quad V \longmapsto colim_{V \to U} \mathcal{F}(U)$$

where U is an object of Sch_τ. This is true because $\mathcal{F}' = f^{-1}\mathcal{F} = (u_p\mathcal{F})^\#$ according to Sites, Lemmas 9.19.5 and 9.19.8.

30.11. Other chapters

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CHAPTER 31

Descent

31.1. Introduction

In the chapter on topologies on schemes (see Topologies, Section 30.1) we introduced Zariski, étale, fppf, smooth, syntomic and fpqc coverings of schemes. In this chapter we discuss what kind of structures over schemes can be descended through such coverings. See for example [**Gro95a**], [**Gro95b**], [**Gro95e**], [**Gro95f**], [**Gro95c**], and [**Gro95d**]. This is also meant to introduce the notions of descent, descent data, effective descent data, in the less formal setting of descent questions for quasi-coherent sheaves, schemes, etc. The formal notion, that of a stack over a site, is discussed in the chapter on stacks (see Stacks, Section 50.1).

31.2. Descent data for quasi-coherent sheaves

In this chapter we will use the convention where the projection maps $\mathrm{pr}_i : X \times \ldots \times X \to X$ are labeled starting with $i = 0$. Hence we have $\mathrm{pr}_0, \mathrm{pr}_1 : X \times X \to X$, $\mathrm{pr}_0, \mathrm{pr}_1, \mathrm{pr}_2 : X \times X \times X \to X$, etc.

Definition 31.2.1. Let S be a scheme. Let $\{f_i : S_i \to S\}_{i \in I}$ be a family of morphisms with target S.

(1) A *descent datum* $(\mathcal{F}_i, \varphi_{ij})$ *for quasi-coherent sheaves* with respect to the given family is given by a quasi-coherent sheaf $\mathcal{F}_i$ on S_i for each $i \in I$, an isomorphism of quasi-coherent $\mathcal{O}_{S_i \times_S S_j}$-modules $\varphi_{ij} : \mathrm{pr}_0^*\mathcal{F}_i \to \mathrm{pr}_1^*\mathcal{F}_j$ for each pair $(i, j) \in I^2$ such that for every triple of indices $(i, j, k) \in I^3$ the diagram
$$\begin{tikzcd} \mathrm{pr}_0^*\mathcal{F}_i \arrow[rr, "\mathrm{pr}_{02}^*\varphi_{ik}"'] \arrow[dr, "\mathrm{pr}_{01}^*\varphi_{ij}"'] & & \mathrm{pr}_2^*\mathcal{F}_k \\ & \mathrm{pr}_1^*\mathcal{F}_j \arrow[ur, "\mathrm{pr}_{12}^*\varphi_{jk}"'] & \end{tikzcd}$$
of $\mathcal{O}_{S_i \times_S S_j \times_S S_k}$-modules commutes. This is called the *cocycle condition*.

(2) A *morphism* $\psi : (\mathcal{F}_i, \varphi_{ij}) \to (\mathcal{F}'_i, \varphi'_{ij})$ *of descent data* is given by a family $\psi = (\psi_i)_{i \in I}$ of morphisms of $\mathcal{O}_{S_i}$-modules $\psi_i : \mathcal{F}_i \to \mathcal{F}'_i$ such that all the diagrams
$$\begin{tikzcd} \mathrm{pr}_0^*\mathcal{F}_i \arrow[r, "\varphi_{ij}"'] \arrow[d, "\mathrm{pr}_0^*\psi_i"'] & \mathrm{pr}_1^*\mathcal{F}_j \arrow[d, "\mathrm{pr}_1^*\psi_j"] \\ \mathrm{pr}_0^*\mathcal{F}'_i \arrow[r, "\varphi'_{ij}"] & \mathrm{pr}_1^*\mathcal{F}'_j \end{tikzcd}$$
commute.

A good example to keep in mind is the following. Suppose that $S = \bigcup S_i$ is an open covering. In that case we have seen descent data for sheaves of sets in Sheaves, Section 6.33 where we called them ``glueing data for sheaves of sets with respect to the given covering''. Moreover, we proved that the category of glueing data is equivalent to the category of sheaves on S. We will show the analogue in the setting above when $\{S_i \to S\}_{i\in I}$ is an fpqc covering.

In the extreme case where the covering $\{S \to S\}$ is given by id_S a descent datum is necessarily of the form $(\mathcal{F}, \mathrm{id}_{\mathcal{F}})$. The cocycle condition garantees that the identity on $\mathcal{F}$ is the only permitted map in this case. The following lemma shows in particular that to every quasi-coherent sheaf of $\mathcal{O}_S$-modules there is associated a unique descent datum with respect to any given family.

Lemma 31.2.2. *Let $\mathcal{U} = \{U_i \to U\}_{i\in I}$ and $\mathcal{V} = \{V_j \to V\}_{j\in J}$ be families of morphisms of schemes with fixed target. Let $(g, \alpha : I \to J, (g_i)) : \mathcal{U} \to \mathcal{V}$ be a morphism of families of maps with fixed target, see Sites, Definition 9.8.1. Let $(\mathcal{F}_j, \varphi_{jj'})$ be a descent datum for quasi-coherent sheaves with respect to the family $\{V_j \to V\}_{j\in J}$. Then*

(1) *The system*
$$\left(g_i^*\mathcal{F}_{\alpha(i)}, (g_i \times g_{i'})^*\varphi_{\alpha(i)\alpha(i')}\right)$$
is a descent datum with respect to the family $\{U_i \to U\}_{i\in I}$.
(2) *This construction is functorial in the descent datum $(\mathcal{F}_j, \varphi_{jj'})$.*
(3) *Given a second morphism $(g', \alpha' : I \to J, (g_i'))$ of families of maps with fixed target with $g = g'$ there exists a functorial isomorphism of descent data*
$$(g_i^*\mathcal{F}_{\alpha(i)}, (g_i \times g_{i'})^*\varphi_{\alpha(i)\alpha(i')}) \cong ((g_i')^*\mathcal{F}_{\alpha'(i)}, (g_i' \times g_{i'}')^*\varphi_{\alpha'(i)\alpha'(i')}).$$

Proof. Omitted. Hint: The maps $g_i^*\mathcal{F}_{\alpha(i)} \to (g_i')^*\mathcal{F}_{\alpha'(i)}$ which give the ismorphism of descent data in part (3) are the pullbacks of the maps $\varphi_{\alpha(i)\alpha'(i)}$ by the morphisms $(g_i, g_i') : U_i \to V_{\alpha(i)} \times_V V_{\alpha'(i)}$. $\square$

Any family $\mathcal{U} = \{S_i \to S\}_{i\in I}$ is a refinement of the trivial covering $\{S \to S\}$ in a unique way. For a quasi-coherent sheaf $\mathcal{F}$ on S we denote simply $(\mathcal{F}|_{S_i}, can)$ the descent datum with respect to $\mathcal{U}$ obtained by the procedure above.

Definition 31.2.3. Let S be a scheme. Let $\{S_i \to S\}_{i\in I}$ be a family of morphisms with target S.

(1) Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_S$-module. We call the unique descent on $\mathcal{F}$ datum with respect to the covering $\{S \to S\}$ the *trivial descent datum*.
(2) The pullback of the trival descent datum to $\{S_i \to S\}$ is called the *canonical descent datum*. Notation: $(\mathcal{F}|_{S_i}, can)$.
(3) A descent datum $(\mathcal{F}_i, \varphi_{ij})$ for quasi-coherent sheaves with respect to the given covering is said to be *effective* if there exists a quasi-coherent sheaf $\mathcal{F}$ on S such that $(\mathcal{F}_i, \varphi_{ij})$ is isomorphic to $(\mathcal{F}|_{S_i}, can)$.

Lemma 31.2.4. *Let S be a scheme. Let $S = \bigcup U_i$ be an open covering. Any descent datum on quasi-coherent sheaves for the family $\mathcal{U} = \{U_i \to S\}$ is effective. Moreover, the functor from the category of quasi-coherent $\mathcal{O}_S$-modules to the category of descent data with respect to $\mathcal{U}$ is fully faithful.*

Proof. This follows immediately from Sheaves, Section 6.33 and the fact that being quasi-coherent is a local property, see Modules, Definition 15.10.1. $\square$

To prove more we first need to study the case of modules over rings.

31.3. Descent for modules

Let $R \to A$ be a ring map. By Simplicial, Example 14.5.5 this gives rise to a cosimplicial R-algebra

$$A \rightleftarrows A \otimes_R A \rightleftarrows A \otimes_R A \otimes_R A$$

Let us denote this $(A/R)_\bullet$ so that $(A/R)_n$ is the $(n+1)$-fold tensor product of A over R. Given a map $\varphi : [n] \to [m]$ the R-algebra map $(A/R)_\bullet(\varphi)$ is the map

$$a_0 \otimes \ldots \otimes a_n \longmapsto \prod\nolimits_{\varphi(i)=0} a_i \otimes \prod\nolimits_{\varphi(i)=1} a_i \otimes \ldots \otimes \prod\nolimits_{\varphi(i)=m} a_i$$

where we use the convention that the empty product is 1. Thus the first few maps, notation as in Simplicial, Section 14.5, are

$$\begin{array}{lcccc}
\delta^1_0 & : & a_0 & \mapsto & 1 \otimes a_0 \\
\delta^1_1 & : & a_0 & \mapsto & a_0 \otimes 1 \\
\sigma^0_0 & : & a_0 \otimes a_1 & \mapsto & a_0 a_1 \\
\delta^2_0 & : & a_0 \otimes a_1 & \mapsto & 1 \otimes a_0 \otimes a_1 \\
\delta^2_1 & : & a_0 \otimes a_1 & \mapsto & a_0 \otimes 1 \otimes a_1 \\
\delta^2_2 & : & a_0 \otimes a_1 & \mapsto & a_0 \otimes a_1 \otimes 1 \\
\sigma^1_0 & : & a_0 \otimes a_1 \otimes a_2 & \mapsto & a_0 a_1 \otimes a_2 \\
\sigma^1_1 & : & a_0 \otimes a_1 \otimes a_2 & \mapsto & a_0 \otimes a_1 a_2
\end{array}$$

and so on.

An R-module M gives rise to a cosimplicial $(A/R)_\bullet$-module $(A/R)_\bullet \otimes_R M$. In other words $M_n = (A/R)_n \otimes_R M$ and using the R-algebra maps $(A/R)_n \to (A/R)_m$ to define the corresponding maps on $M \otimes_R (A/R)_\bullet$.

The analogue to a descent datum for quasi-coherent sheaves in the setting of modules is the following.

Definition 31.3.1. Let $R \to A$ be a ring map.

(1) A *descent datum* (N, φ) *for modules with respect to* $R \to A$ is given by an A-module N and a isomorphism of $A \otimes_R A$-modules

$$\varphi : N \otimes_R A \to A \otimes_R N$$

such that the *cocycle condition* holds: the diagram of $A \otimes_R A \otimes_R A$-module maps

$$\begin{tikzcd}
N \otimes_R A \otimes_R A \arrow[rr, "\varphi_{02}"'] \arrow[dr, "\varphi_{01}"'] & & A \otimes_R A \otimes_R N \\
& A \otimes_R N \otimes_R A \arrow[ur, "\varphi_{12}"'] &
\end{tikzcd}$$

commutes (see below for notation).

(2) A *morphism* $(N, \varphi) \to (N', \varphi')$ *of descent data* is a morphism of A-modules $\psi : N \to N'$ such that the diagram

$$\begin{tikzcd}
N \otimes_R A \arrow[r, "\varphi"'] \arrow[d, "\psi \otimes \mathrm{id}_A"'] & A \otimes_R N \arrow[d, "\mathrm{id}_A \otimes \psi"] \\
N' \otimes_R A \arrow[r, "\varphi'"] & A \otimes_R N'
\end{tikzcd}$$

is commutative.

In the definition we use the notation that $\varphi_{01} = \varphi \otimes \mathrm{id}_A$, $\varphi_{12} = \mathrm{id}_A \otimes \varphi$, and $\varphi_{02}(n \otimes 1 \otimes 1) = \sum a_i \otimes 1 \otimes n_i$ if $\varphi(n) = \sum a_i \otimes n_i$. All three are $A \otimes_R A \otimes_R A$-module homomorphisms. Equivalently we have

$$\varphi_{ij} = \varphi \otimes_{(A/R)_1,\ (A/R)_\bullet(\tau^2_{ij})} (A/R)_2$$

where $\tau^2_{ij} : [1] \to [2]$ is the map $0 \mapsto i$, $1 \mapsto j$. Namely, $(A/R)_\bullet(\tau^2_{02})(a_0 \otimes a_1) = a_0 \otimes 1 \otimes a_1$, and similarly for the others[1].

We need some more notation to be able to state the next lemma. Let (N, φ) be a descent datum with respect to a ring map $R \to A$. For $n \geq 0$ and $i \in [n]$ we set

$$N_{n,i} = A \otimes_R \ldots \otimes_R A \otimes_R N \otimes_R A \otimes_R \ldots \otimes_R A$$

with the factor N in the ith spot. It is an $(A/R)_n$-module. If we introduce the maps $\tau^n_i : [0] \to [n]$, $0 \mapsto i$ then we see that

$$N_{n,i} = N \otimes_{(A/R)_0,\ (A/R)_\bullet(\tau^n_i)} (A/R)_n$$

For $0 \leq i \leq j \leq n$ we let $\tau^n_{ij} : [1] \to [n]$ be the map such that 0 maps to i and 1 to j. Similarly to the above the homomorphism φ induces isomorphisms

$$\varphi^n_{ij} = \varphi \otimes_{(A/R)_1,\ (A/R)_\bullet(\tau^n_{ij})} (A/R)_n : N_{n,i} \longrightarrow N_{n,j}$$

of $(A/R)_n$-modules when $i < j$. If $i = j$ we set $\varphi^n_{ij} = \mathrm{id}$. Since these are all isomorphisms they allow us to move the factor N to any spot we like. And the cocycle condition exactly means that it does not matter how we do this (e.g., as a composition of two of these or at once). Finally, for any $\beta : [n] \to [m]$ we define the morphism

$$N_{\beta,i} : N_{n,i} \to N_{m,\beta(i)}$$

as the unique $(A/R)_\bullet(\beta)$-semi linear map such that

$$N_{\beta,i}(1 \otimes \ldots \otimes n \otimes \ldots \otimes 1) = 1 \otimes \ldots \otimes n \otimes \ldots \otimes 1$$

for all $n \in N$. This hints at the following lemma.

Lemma 31.3.2. *Let $R \to A$ be a ring map. Given a descent datum (N, φ) we can associate to it a cosimplicial $(A/R)_\bullet$-module $N_\bullet$[2] by the rules $N_n = N_{n,n}$ and given $\beta : [n] \to [m]$ setting we define*

$$N_\bullet(\beta) = (\varphi^m_{\beta(n)m}) \circ N_{\beta,n} : N_{n,n} \longrightarrow N_{m,m}.$$

This procedure is functorial in the descent datum.

Proof. Here are the first few maps where $\varphi(n \otimes 1) = \sum \alpha_i \otimes x_i$

$$\begin{array}{ccccccc}
\delta^1_0 & : & N & \to & A \otimes N & n & \mapsto & 1 \otimes n \\
\delta^1_1 & : & N & \to & A \otimes N & n & \mapsto & \sum \alpha_i \otimes x_i \\
\sigma^0_0 & : & A \otimes N & \to & N & a_0 \otimes n & \mapsto & a_0 n \\
\delta^2_0 & : & A \otimes N & \to & A \otimes A \otimes N & a_0 \otimes n & \mapsto & 1 \otimes a_0 \otimes n \\
\delta^2_1 & : & A \otimes N & \to & A \otimes A \otimes N & a_0 \otimes n & \mapsto & a_0 \otimes 1 \otimes n \\
\delta^2_2 & : & A \otimes N & \to & A \otimes A \otimes N & a_0 \otimes n & \mapsto & \sum a_0 \otimes \alpha_i \otimes x_i \\
\sigma^1_0 & : & A \otimes A \otimes N & \to & A \otimes N & a_0 \otimes a_1 \otimes n & \mapsto & a_0 a_1 \otimes n \\
\sigma^1_1 & : & A \otimes A \otimes N & \to & A \otimes N & a_0 \otimes a_1 \otimes n & \mapsto & a_0 \otimes a_1 n
\end{array}$$

[1]Note that $\tau^2_{ij} = \delta^2_k$, if $\{i, j, k\} = [2] = \{0, 1, 2\}$, see Simplicial, Definition 14.2.1.

[2]We should really write $(N, \varphi)_\bullet$.

with notation as in Simplicial, Section 14.5. We first verify the two properties $\sigma_0^0 \circ \delta_0^1 = \text{id}$ and $\sigma_0^0 \circ \delta_1^1 = \text{id}$. The first one, $\sigma_0^0 \circ \delta_0^1 = \text{id}$, is clear from the explicit description of the morphisms above. To prove the second relation we have to use the cocycle condition (because it does not holds for an arbitrary isomorphism $\varphi : N \otimes_R A \to A \otimes_R N$). Write $p = \sigma_0^0 \circ \delta_1^1 : N \to N$. By the description of the maps above we deduce that p is also equal to

$$p = \varphi \otimes \text{id} : N = (N \otimes_R A) \otimes_{(A \otimes_R A)} A \longrightarrow (A \otimes_R N) \otimes_{(A \otimes_R A)} A = N$$

Since φ is an isomorphism we see that p is an isomorphism. Write $\varphi(n \otimes 1) = \sum \alpha_i \otimes x_i$ for certain $\alpha_i \in A$ and $x_i \in N$. Then $p(n) = \sum \alpha_i x_i$. Next, write $\varphi(x_i \otimes 1) = \sum \alpha_{ij} \otimes y_j$ for certain $\alpha_{ij} \in A$ and $y_j \in N$. Then the cocycle condition says that

$$\sum \alpha_i \otimes \alpha_{ij} \otimes y_j = \sum \alpha_i \otimes 1 \otimes x_i.$$

This means that $p(n) = \sum \alpha_i x_i = \sum \alpha_i \alpha_{ij} y_j = \sum \alpha_i p(x_i) = p(p(n))$. Thus p is a projector, and since it is an isomorphism it is the identity.

To prove fully that $N_\bullet$ is a cosimplicial module we have to check all 5 types of relations of Simplicial, Remark 14.5.3. The relations on composing σ's are obvious. The relations on composing δ's come down to the cocycle condition for φ. In exactly the same way as above one checks the relations $\sigma_j \circ \delta_j = \sigma_j \circ \delta_{j+1} = \text{id}$. Finally, the other relations on compositions of δ's and σ's hold for any φ whatsoever. □

Note that to an R-module M we can associate a canonical descent datum, namely $(M \otimes_R A, can)$ where $can : (M \otimes_R A) \otimes_R A \to A \otimes_R (M \otimes_R A)$ is the obvious map: $(m \otimes a) \otimes a' \mapsto a \otimes (m \otimes a')$.

Lemma 31.3.3. *Let $R \to A$ be a ring map. Let M be an R-module. The cosimplicial $(A/R)_\bullet$-module associated to the canonical descent datum is isomorphic to the cosimplicial module $(A/R)_\bullet \otimes_R M$.*

Proof. Omitted. □

Definition 31.3.4. Let $R \to A$ be a ring map. We say a descent datum (N, φ) is *effective* if there exists an R-module M and an isomorphism of descent data from $(M \otimes_R A, can)$ to (N, φ).

Let $R \to A$ be a ring map. Let (N, φ) be a descent datum. We may take the cochain complex $s(N_\bullet)$ associated with $N_\bullet$ (see Simplicial, Section 14.23). It has the following shape:

$$N \to A \otimes_R N \to A \otimes_R A \otimes_R N \to \ldots$$

We can describe the maps. The first map is the map

$$n \longmapsto 1 \otimes n - \varphi(n \otimes 1).$$

The second map on pure tensors has the values

$$a \otimes n \longmapsto 1 \otimes a \otimes n - a \otimes 1 \otimes n + a \otimes \varphi(n \otimes 1).$$

It is clear how the pattern continues.

In the special case where $N = A \otimes_R M$ we see that for any $m \in M$ the element $1 \otimes m$ is in the kernel of the first map of the cochain complex associated to the cosimplicial module $(A/R)_\bullet \otimes_R M$. Hence we get an extended cochain complex

$$0 \to M \to A \otimes_R M \to A \otimes_R A \otimes_R M \to \ldots \tag{31.3.4.1}$$

Here we think of the 0 as being in degree -2, the module M in degree -1, the module $A \otimes_R M$ in degree 0, etc. Note that this complex has the shape

$$0 \to R \to A \to A \otimes_R A \to A \otimes_R A \otimes_R A \to \ldots$$

when $M = R$.

Lemma 31.3.5. *Suppose that $R \to A$ has a section. Then for any R-module M the extended cochain complex (31.3.4.1) is exact.*

Proof. By Simplicial, Lemma 14.26.4 the map $R \to (A/R)_\bullet$ is a homotopy equivalence of cosimplicial R-algebras (here R denotes the constant cosimplicial R-algebra). Hence $M \to (A/R)_\bullet \otimes_R M$ is a homotopy equivalence in the category of cosimplicial R-modules, because $\otimes_R M$ is a functor from the category of R-algebras to the category of R-modules, see Simplicial, Lemma 14.26.3. This implies that the induced map of associated complexes is a homotopy equivalence, see Simplicial, Lemma 14.26.5. Since the complex associated to the constant cosimplicial R-module M is the complex

$$M \xrightarrow{0} M \xrightarrow{1} M \xrightarrow{0} M \xrightarrow{1} M \ldots$$

we win (since the extended version simply puts an extra M at the beginning). $\square$

Lemma 31.3.6. *Suppose that $R \to A$ is faithfully flat, see Algebra, Definition 7.35.1. Then for any R-module M the extended cochain complex (31.3.4.1) is exact.*

Proof. Suppose we can show there exists a faithfully flat ring map $R \to R'$ such that the result holds for the ring map $R' \to A' = R' \otimes_R A$. Then the result follows for $R \to A$. Namely, for any R-module M the cosimplicial module $(M \otimes_R R') \otimes_{R'} (A'/R')_\bullet$ is just the cosimplicial module $R' \otimes_R (M \otimes_R (A/R)_\bullet)$. Hence the vanishing of cohomology of the complex associated to $(M \otimes_R R') \otimes_{R'} (A'/R')_\bullet$ implies the vanishing of the cohomology of the complex associated to $M \otimes_R (A/R)_\bullet$ by faithful flatness of $R \to R'$. Similarly for the vanishing of cohomology groups in degrees -1 and 0 of the extended complex (proof omitted).

But we have such a faithfull flat extension. Namely $R' = A$ works because the ring map $R' = A \to A' = A \otimes_A A$ has a section $a \otimes a' \mapsto aa'$ and Lemma 31.3.5 applies. $\square$

Here is how the complex relates to the question of effectivity.

Lemma 31.3.7. *Let $R \to A$ be a faithfully flat ring map. Let (N, φ) be a descent datum. Then (N, φ) is effective if and only if the canonical map*

$$A \otimes_R H^0(s(N_\bullet)) \longrightarrow N$$

is an isomorphism.

Proof. If (N, φ) is effective, then we may write $N = A \otimes_R M$ with $\varphi = can$. It follows that $H^0(s(N_\bullet)) = M$ by Lemmas 31.3.3 and 31.3.6. Conversely, suppose the map of the lemma is an isomorphism. In this case set $M = H^0(s(N_\bullet))$. This is an R-submodule of N, namely $M = \{n \in N \mid 1 \otimes n = \varphi(n \otimes 1)\}$. The only thing to check is that via the isomorphism $A \otimes_R M \to N$ the canonical descent data agrees with φ. We omit the verification. $\square$

Lemma 31.3.8. *Let $R \to A$ be a ring map, and let $R \to R'$ be faithfully flat. Set $A' = R' \otimes_R A$. If all descent data for $R' \to A'$ are effective, then so are all descent data for $R \to A$.*

Proof. Let (N, φ) be a descent datum for $R \to A$. Set $N' = R' \otimes_R N = A' \otimes_A N$, and denote $\varphi' = \mathrm{id}_{R'} \otimes \varphi$ the base change of the descend datum φ. Then (N', φ') is a descent datum for $R' \to A'$ and $H^0(s(N'_\bullet)) = R' \otimes_R H^0(s(N_\bullet))$. Moreover, the map $A' \otimes_{R'} H^0(s(N'_\bullet)) \to N'$ is identified with the base change of the A-module map $A \otimes_R H^0(s(N)) \to N$ via the faithfully flat map $A \to A'$. Hence we conclude by Lemma 31.3.7. $\square$

Here is the main result of this section. Its proof may seem a little clumsy; for a more highbrow approach see Remark 31.3.11 below.

Proposition 31.3.9. *Let $R \to A$ be a faithfully flat ring map. Then*

(1) *any descent datum on modules with respect to $R \to A$ is effective,*
(2) *the functor $M \mapsto (A \otimes_R M, can)$ from R-modules to the category of descent data is an equivalence, and*
(3) *the inverse functor is given by $(N, \varphi) \mapsto H^0(s(N_\bullet))$.*

Proof. We only prove (1) and omit the proofs of (2) and (3). As $R \to A$ is faithfully flat, there exists a faithfully flat base change $R \to R'$ such that $R' \to A' = R' \otimes_R A$ has a section (namely take $R' = A$ as in the proof of Lemma 31.3.6). Hence, using Lemma 31.3.8 we may assume that $R \to A$ as a section, say $\sigma : A \to R$. Let (N, φ) be a descent datum relative to $R \to A$. Set

$$M = H^0(s(N_\bullet)) = \{n \in N \mid 1 \otimes n = \varphi(n \otimes 1)\} \subset N$$

By Lemma 31.3.7 it suffices to show that $A \otimes_R M \to N$ is an isomorphism.

Take an element $n \in N$. Write $\varphi(n \otimes 1) = \sum a_i \otimes x_i$ for certain $a_i \in A$ and $x_i \in N$. By Lemma 31.3.2 we have $n = \sum a_i x_i$ in N (because $\sigma^0_0 \circ \delta^1_0 = \mathrm{id}$ in any cosimplicial object). Next, write $\varphi(x_i \otimes 1) = \sum a_{ij} \otimes y_j$ for certain $a_{ij} \in A$ and $y_j \in N$. The cocycle condition means that

$$\sum a_i \otimes a_{ij} \otimes y_j = \sum a_i \otimes 1 \otimes x_i$$

in $A \otimes_R A \otimes_R N$. We conclude two things from this. First, by applying σ to the first A we conclude that $\sum \sigma(a_i)\varphi(x_i \otimes 1) = \sum \sigma(a_i) \otimes x_i$ which means that $\sum \sigma(a_i)x_i \in M$. Next, by applying σ to the middle A and multiplying out we conclude that $\sum_i a_i(\sum_j \sigma(a_{ij})y_j) = \sum a_i x_i = n$. Hence by the first conclusion we see that $A \otimes_R M \to N$ is surjective. Finally, suppose that $m_i \in M$ and $\sum a_i m_i = 0$. Then we see by applying φ to $\sum a_i m_i \otimes 1$ that $\sum a_i \otimes m_i = 0$. In other words $A \otimes_R M \to N$ is injective and we win. $\square$

Remark 31.3.10. Let R be a ring. Let $f_1, \ldots, f_n \in R$ generate the unit ideal. The ring $A = \prod_i R_{f_i}$ is a faithfully flat R-algebra. We remark that the cosimplicial ring $(A/R)_\bullet$ has the following ring in degree n:

$$\prod\nolimits_{i_0, \ldots, i_n} R_{f_{i_0} \ldots f_{i_n}}$$

Hence the results above recover Algebra, Lemmas 7.20.1, 7.20.2 and 7.21.4. But the results above actually say more because of exactness in higher degrees. Namely, it implies that Cech cohomology of quasi-coherent sheaves on affines is trivial, see (insert future reference here).

Remark 31.3.11. Let R be a ring. Let $A_\bullet$ be a cosimplicial R-algebra. In this setting a descent datum corresponds to an cosimplicial $A_\bullet$-module $M_\bullet$ with the property that for every $n, m \geq 0$ and every $\varphi : [n] \to [m]$ the map $M(\varphi) : M_n \to M_m$ induces an isomorphism

$$M_n \otimes_{A_n, A(\varphi)} A_m \longrightarrow M_m.$$

Let us call such a cosimplicial module a *cartesian module*. In this setting, the proof of Proposition 31.3.9 can be split in the following steps

(1) If $R \to R'$ is faithfully flat, $R \to A$ any ring map, then descent data for A/R are effective if descent data for $(R' \otimes_R A)/R'$ are effective.
(2) Let A be an R-algebra. Descent data for A/R correspond to cartesian $(A/R)_\bullet$-modules.
(3) If $R \to A$ has a section then $(A/R)_\bullet$ is homotopy equivalent to R, the constant cosimplicial R-algebra with value R.
(4) If $A_\bullet \to B_\bullet$ is a homotopy equivalence of cosimplicial R-algebras then the functor $M_\bullet \mapsto M_\bullet \otimes_{A_\bullet} B_\bullet$ induces an equivalence of categories between cartesian $A_\bullet$-modules and cartesian $B_\bullet$-modules.

For (1) see Lemma 31.3.8. Part (2) uses Lemma 31.3.2. Part (3) we have seen in the proof of Lemma 31.3.5 (it relies on Simplicial, Lemma 14.26.4). Moreover, part (4) is a triviality if you think about it right!

31.4. Fpqc descent of quasi-coherent sheaves

The main application of flat descent for modules is the corresponding descent statement for quasi-coherent sheaves with respect to fpqc-coverings.

Lemma 31.4.1. *Let S be an affine scheme. Let $\mathcal{U} = \{f_i : U_i \to S\}_{i=1,\ldots,n}$ be a standard fpqc covering of S, see Topologies, Definition 30.8.1. Any descent datum on quasi-coherent sheaves for $\mathcal{U} = \{U_i \to S\}$ is effective. Moreover, the functor from the category of quasi-coherent $\mathcal{O}_S$-modules to the category of descent data with respect to $\mathcal{U}$ is fully faithful.*

Proof. This is a restatement of Proposition 31.3.9 in terms of schemes. First, note that a descent datum ξ for quasi-coherent sheaves with respect to $\mathcal{U}$ is exactly the same as a descent datum ξ' for quasi-coherent sheaves with respect to the covering $\mathcal{U}' = \{\coprod_{i=1,\ldots,n} U_i \to S\}$. Moreover, effectivity for ξ is the same as effectivity for ξ'. Hence we may assume $n = 1$, i.e., $\mathcal{U} = \{U \to S\}$ where U and S are affine. In this case descent data correspond to descent data on modules with respect to the ring map

$$\Gamma(S, \mathcal{O}) \longrightarrow \Gamma(U, \mathcal{O}).$$

Since $U \to S$ is surjective and flat, we see that this ring map is faithfully flat. In other words, Proposition 31.3.9 applies and we win. $\square$

Proposition 31.4.2. *Let S be a scheme. Let $\mathcal{U} = \{\varphi_i : U_i \to S\}$ be an fpqc covering, see Topologies, Definition 30.8.1. Any descent datum on quasi-coherent sheaves for $\mathcal{U} = \{U_i \to S\}$ is effective. Moreover, the functor from the category of quasi-coherent $\mathcal{O}_S$-modules to the category of descent data with respect to $\mathcal{U}$ is fully faithful.*

Proof. Let $S = \bigcup_{j \in J} V_j$ be an affine open covering. For $j, j' \in J$ we denote $V_{jj'} = V_j \cap V_{j'}$ the intersection (which need not be affine). For $V \subset S$ open we denote $\mathcal{U}_V = \{V \times_S U_i \to V\}_{i \in I}$ which is a fpqc-covering (Topologies, Lemma 30.8.7). By definition of an fpqc covering, we can find for each $j \in J$ a finite set K_j, a map $\underline{i} : K_j \to I$, affine opens $U_{\underline{i}(k),k} \subset U_{\underline{i}(k)}$, $k \in K_j$ such that $\mathcal{V}_j = \{U_{\underline{i}(k),k} \to V_j\}_{k \in K_j}$ is a standard fpqc covering of V_j. And of course, $\mathcal{V}_j$ is a refinement of $\mathcal{U}_{V_j}$. Picture

$$\begin{tikzcd}
\mathcal{V}_j \arrow[r] \arrow[d, squiggly] & \mathcal{U}_{V_j} \arrow[r] \arrow[d, squiggly] & \mathcal{U} \arrow[d, squiggly] \\
V_j \arrow[r, equal] & V_j \arrow[r] & S
\end{tikzcd}$$

where the top horizontal arrows are morphisms of families of morphisms with fixed target (see Sites, Definition 9.8.1).

To prove the proposition you show successively the faithfulness, fullyness, and essential surjectivity of the functor from quasi-coherent sheaves to descent data.

Faithfulness. Let $\mathcal{F}$, $\mathcal{G}$ be quasi-coherent sheaves on S and let $a, b : \mathcal{F} \to \mathcal{G}$ be homomorphisms of $\mathcal{O}_S$-modules. Suppose $\varphi_i^*(a) = \varphi^*(b)$ for all i. Pick $s \in S$. Then $s = \varphi_i(u)$ for some $i \in I$ and $u \in U_i$. Since $\mathcal{O}_{S,s} \to \mathcal{O}_{U_i,u}$ is flat, hence faithfully flat (Algebra, Lemma 7.35.16) we see that $a_s = b_s : \mathcal{F}_s \to \mathcal{G}_s$. Hence $a = b$.

Fully faithfulness. Let $\mathcal{F}$, $\mathcal{G}$ be quasi-coherent sheaves on S and let $a_i : \varphi_i^*\mathcal{F} \to \varphi_i^*\mathcal{G}$ be homomorphisms of $\mathcal{O}_{U_i}$-modules such that $\mathrm{pr}_0^* a_i = \mathrm{pr}_1^* a_j$ on $U_i \times_U U_j$. We can pull back these morphisms to get morphisms

$$a_k : \mathcal{F}|_{U_{\underline{i}(k),k}} \longrightarrow \mathcal{G}|_{U_{\underline{i}(k),k}}$$

$k \in K_j$ with notation as above. Moreover, Lemma 31.2.2 assures us that these define a morphism between (canonical) descent data on $\mathcal{V}_j$. Hence, by Lemma 31.4.1, we get correspondingly unique morphisms $a_j : \mathcal{F}|_{V_j} \to \mathcal{G}|_{V_j}$. To see that $a_j|_{V_{jj'}} = a_{j'}|_{V_{jj'}}$ we use that both a_j and $a_{j'}$ agree with the pullback of the morphism $(a_i)_{i\in I}$ of (canonical) descent data to any covering refining both $\mathcal{V}_{j,V_{jj'}}$ and $\mathcal{V}_{j',V_{jj'}}$, and using the faithfullness already shown. For example the covering $\mathcal{V}_{jj'} = \{V_k \times_S V_{k'} \to V_{jj'}\}_{k\in K_j, k'\in K_{j'}}$ will do.

Essential surjectivity. Let $\xi = (\mathcal{F}_i, \varphi_{ii'})$ be a descent datum for quasi-coherent sheaves relative to the covering $\mathcal{U}$. Pull back this descent datum to get descent data ξ_j for quasi-coherent sheaves relative to the coverings $\mathcal{V}_j$ of V_j. By Lemma 31.4.1 once again there exist quasi-coherent sheaves $\mathcal{F}_j$ on V_j whose associated canonical descent datum is isomorphic to ξ_j. By fully faithfulness (proved above) we see there are isomorphisms

$$\phi_{jj'} : \mathcal{F}_j|_{V_{jj'}} \longrightarrow \mathcal{F}_{j'}|_{V_{jj'}}$$

corresponding to the isomorphism of descent data between the pull back of ξ_j and $\xi_{j'}$ to $\mathcal{V}_{jj'}$. To see that these maps $\phi_{jj'}$ satisfy the cocycle condition we use faithfullness (proved above) over the triple intersections $V_{jj'j''}$. Hence, by Lemma 31.2.4 we see that the sheaves $\mathcal{F}_j$ glue to a quasi-coherent sheaf $\mathcal{F}$ as desired. We still have to verify that the canonical descent datum relative to $\mathcal{U}$ associated to $\mathcal{F}$ is isomorphic to the descent datum we started out with. This verification is omitted. □

31.5. Descent of finiteness properties of modules

In this section we prove that one can check quasi-coherent module has a certain finiteness conditions by checking on the members of a covering.

Lemma 31.5.1. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i\in I}$ be an fpqc covering such that each $f_i^*\mathcal{F}$ is a finite type $\mathcal{O}_{X_i}$-module. Then $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module.*

Proof. Omitted. For the affine case, see Algebra, Lemma 7.77.2. □

Lemma 31.5.2. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i\in I}$ be an fpqc covering such that each $f_i^*\mathcal{F}$ is an $\mathcal{O}_{X_i}$-module of finite presentation. Then $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation.*

Proof. Omitted. For the affine case, see Algebra, Lemma 7.77.2. □

Lemma 31.5.3. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i\in I}$ be an fpqc covering such that each $f_i^*\mathcal{F}$ is a flat $\mathcal{O}_{X_i}$-module. Then $\mathcal{F}$ is a flat $\mathcal{O}_X$-module.*

Proof. Omitted. For the affine case, see Algebra, Lemma 7.77.2. $\square$

Lemma 31.5.4. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i\in I}$ be an fpqc covering such that each $f_i^*\mathcal{F}$ is a finite locally free $\mathcal{O}_{X_i}$-module. Then $\mathcal{F}$ is a finite locally free $\mathcal{O}_X$-module.*

Proof. This follows from the fact that a quasi-coherent sheaf is finite locally free if and only if it is of finite presentation and flat, see Algebra, Lemma 7.72.2. Namely, if each $f_i^*\mathcal{F}$ is flat and of finite presentation, then so is $\mathcal{F}$ by Lemmas 31.5.3 and 31.5.2. $\square$

The definition of a locally projective quasi-coherent sheaf can be found in Properties, Section 23.19.

Lemma 31.5.5. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i\in I}$ be an fpqc covering such that each $f_i^*\mathcal{F}$ is a locally projective $\mathcal{O}_{X_i}$-module. Then $\mathcal{F}$ is a locally projective $\mathcal{O}_X$-module.*

Proof. Omitted. For Zariski coverings this is Properties, Lemma 23.19.2. For the affine case this is Algebra, Theorem 7.89.5. $\square$

Remark 31.5.6. Being locally free is a property of quasi-coherent modules which does not descend in the fpqc topology. Namely, suppose that R is a ring and that M is a projective R-module which is a countable direct sum $M = \bigoplus L_n$ of rank 1 locally free modules, but not locally free, see Examples, Lemma 64.15.5. Then M becomes free on making the faithfully flat base change

$$R \longrightarrow \bigoplus\nolimits_{m\geq 1} \bigoplus\nolimits_{(i_1,\ldots,i_m)\in \mathbf{Z}^{\oplus m}} L_1^{\otimes i_1} \otimes_R \ldots \otimes_R L_m^{\otimes i_m}$$

But we don't know what happens for fppf coverings. In other words, we don't know the answer to the following question: Suppose $A \to B$ is a faithfully flat ring map of finite presentation. Let M be an A-module such that $M \otimes_A B$ is free. Is M a locally free A-module? It turns out that if A is Noetherian, then the answer is yes. This follows from the results of [**Bas63**]. But in general we don't know the answer. If you know the answer, or have a reference, please email stacks.project@gmail.com.

We also add here two results which are related to the results above, but are of a slightly different nature.

Lemma 31.5.7. *Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume f is a finite morphism. Then $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite type if and only if $f_*\mathcal{F}$ is an $\mathcal{O}_Y$-module of finite type.*

Proof. As f is finite it is affine. This reduces us to the case where f is the morphism $Spec(B) \to Spec(A)$ given by a finite ring map $A \to B$. Moreover, then $\mathcal{F} = \widetilde{M}$ is the sheaf of modules associated to the B-module M. Note that M is finite as a B-module if and only if M is finite as an A-module, see Algebra, Lemma 7.7.2. Combined with Properties, Lemma 23.16.1 this proves the lemma. $\square$

Lemma 31.5.8. *Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume f is finite and of finite presentation. Then $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation if and only if $f_*\mathcal{F}$ is an $\mathcal{O}_Y$-module of finite presentation.*

Proof. As f is finite it is affine. This reduces us to the case where f is the morphism $Spec(B) \to Spec(A)$ given by a finite and finitely presented ring map $A \to B$. Moreover, then $\mathcal{F} = \widetilde{M}$ is the sheaf of modules associated to the B-module M. Note that M is finitely presented as a B-module if and only if M is finitely presented as an A-module, see Algebra, Lemma 7.7.4. Combined with Properties, Lemma 23.16.2 this proves the lemma. $\square$

31.6. Quasi-coherent sheaves and topologies

Let S be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_S$-module. Consider the functor

$$(Sch/S)^{opp} \longrightarrow Ab, \quad (f : T \to S) \longmapsto \Gamma(T, f^*\mathcal{F}). \tag{31.6.0.1}$$

Lemma 31.6.1. *Let S be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_S$-module. Let $\tau \in \{Zariski, fpqc, fppf, \acute{e}tale, smooth, syntomic\}$. The functor defined in (31.6.0.1) satisfies the sheaf condition with respect to any τ-covering $\{T_i \to T\}_{i \in I}$ of any scheme T over S.*

Proof. For $\tau \in \{Zariski, fppf, \acute{e}tale, smooth, syntomic\}$ a τ-covering is also a fpqc-covering, see the results in Topologies, Lemmas 30.4.2, 30.5.2, 30.6.2, 30.7.2, and 30.8.6. Hence it suffices to prove the theorem for a fpqc covering. Assume that $\{f_i : T_i \to T\}_{i \in I}$ is an fpqc covering where $f : T \to S$ is given. Suppose that we have a family of sections $s_i \in \Gamma(T_i, f_i^* f^* \mathcal{F})$ such that $s_i|_{T_i \times_T T_j} = s_j|_{T_i \times_T T_j}$. We have to find the correspond section $s \in \Gamma(T, f^*\mathcal{F})$. We can reinterpret the s_i as a family of maps $\varphi_i : f_i^*\mathcal{O}_T = \mathcal{O}_{T_i} \to f_i^* f^* \mathcal{F}$ compatible with the canonical descent data associated to the quasi-coherent sheaves $\mathcal{O}_T$ and $f^*\mathcal{F}$ on T. Hence by Proposition 31.4.2 we see that we may (uniquely) descend these to a map $\mathcal{O}_T \to f^*\mathcal{F}$ which gives us our section s. $\square$

We may in particular make the following definition.

Definition 31.6.2. Let $\tau \in \{Zariski, fppf, \acute{e}tale, smooth, syntomic\}$. Let S be a scheme. Let Sch_τ be a big site containing S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_S$-module.

1. The *structure sheaf of the big site* $(Sch/S)_\tau$ is the sheaf of rings $T/S \mapsto \Gamma(T, \mathcal{O}_T)$ which is denoted $\mathcal{O}$ or $\mathcal{O}_S$.
2. If $\tau = \acute{e}tale$ the structure sheaf of the small site $S_{\acute{e}tale}$ is the sheaf of rings $T/S \mapsto \Gamma(T, \mathcal{O}_T)$ which is denoted $\mathcal{O}$ or $\mathcal{O}_S$.
3. The *sheaf of $\mathcal{O}$-modules associated to $\mathcal{F}$* on the big site $(Sch/S)_\tau$ is the sheaf of $\mathcal{O}$-modules $(f : T \to S) \mapsto \Gamma(T, f^*\mathcal{F})$ which is denoted $\mathcal{F}^a$ (and often simply $\mathcal{F}$).
4. Let $\tau = \acute{e}tale$ (resp. $\tau = Zariski$). The *sheaf of $\mathcal{O}$-modules associated to $\mathcal{F}$* on the small site $S_{\acute{e}tale}$ (resp. S_{Zar}) is the sheaf of $\mathcal{O}$-modules $(f : T \to S) \mapsto \Gamma(T, f^*\mathcal{F})$ which is denoted $\mathcal{F}^a$ (and often simply $\mathcal{F}$).

Note how we use the same notation $\mathcal{F}^a$ in each case. No confusion can really arise from this as by definition the rule that defines the sheaf $\mathcal{F}^a$ is independent of the site we choose to look at.

Remark 31.6.3. In Topologies, Lemma 30.3.11 we have seen that the small Zariski site of a scheme S is equivalent to S as a topological space in the sense that the category of sheaves are naturally equivalent. Now that S_{Zar} is also endowed with a structure sheaf $\mathcal{O}$ we see that sheaves of modules on the ringed site $(S_{Zar}, \mathcal{O})$ agree with sheaves of modules on the ringed space $(S, \mathcal{O}_S)$.

Remark 31.6.4. Let $f : T \to S$ be a morphism of schemes. Each of the morphisms of sites f_{sites} listed in Topologies, Section 30.9 becomes a morphism of ringed sites. Namely, each of these morphisms of sites $f_{sites} : (Sch/T)_\tau \to (Sch/S)_{\tau'}$, or $f_{sites} : (Sch/S)_\tau \to S_{\tau'}$ is given by the continuous functor $S'/S \mapsto T \times_S S'/S$. Hence, given S'/S we let

$$f_{sites}^\sharp : \mathcal{O}(S'/S) \longrightarrow f_{sites,*}\mathcal{O}(S'/S) = \mathcal{O}(S \times_S S'/T)$$

be the usual map $\text{pr}_{S'}^\sharp : \mathcal{O}(S') \to \mathcal{O}(T \times_S S')$. Similarly, the morphism $i_f : Sh(T_\tau) \to Sh((Sch/S)_\tau)$ for $\tau \in \{Zar, étale\}$, see Topologies, Lemmas 30.3.12 and 30.4.12, becomes a morphism of ringed topoi because $i_f^{-1}\mathcal{O} = \mathcal{O}$. Here are some special cases:

(1) The morphism of big sites $f_{big} : (Sch/X)_{fppf} \to (Sch/Y)_{fppf}$, becomes a morphism of ringed sites
$$(f_{big}, f_{big}^\sharp) : ((Sch/X)_{fppf}, \mathcal{O}_X) \longrightarrow ((Sch/Y)_{fppf}, \mathcal{O}_Y)$$
as in Modules on Sites, Definition 16.6.1. Similarly for the big syntomic, smooth, étale and Zariski sites.

(2) The morphism of small sites $f_{small} : X_{étale} \to Y_{étale}$, becomes a morphism of ringed sites
$$(f_{small}, f_{small}^\sharp) : (X_{étale}, \mathcal{O}_X) \longrightarrow (Y_{étale}, \mathcal{O}_Y)$$
as in Modules on Sites, Definition 16.6.1. Similarly for the small Zariski site.

Let S be a scheme. It is clear that given an $\mathcal{O}$-module on (say) $(Sch/S)_{Zar}$ the pullback to (say) $(Sch/S)_{fppf}$ is just the fppf-sheafification. To see what happens when comparing big and small sites we have the following.

Lemma 31.6.5. *Let S be a scheme. Denote*

$$\begin{array}{rcll} id_{\tau,Zar} & : & (Sch/S)_\tau \to S_{Zar}, & \tau \in \{Zar, étale, smooth, syntomic, fppf\} \\ id_{\tau,étale} & : & (Sch/S)_\tau \to S_{étale}, & \tau \in \{étale, smooth, syntomic, fppf\} \\ id_{small,étale,Zar} & : & S_{étale}, \to S_{Zar}, & \end{array}$$

the morphisms of ringed sites of Remark 31.6.4.

(1) *Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_S$-modules which we view a sheaf of $\mathcal{O}$-modules on S_{Zar}. Then $(id_{\tau,Zar})^*\mathcal{F}$ is the τ-sheafification of the Zariski sheaf*
$$(f : T \to S) \longmapsto \Gamma(T, f^*\mathcal{F})$$
on $(Sch/S)_\tau$.

(2) *Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_S$-modules which we view a sheaf of $\mathcal{O}$-modules on S_{Zar}. Then $(id_{small,étale,Zar})^*\mathcal{F}$ is the étale sheafification of the Zariski sheaf*
$$(f : T \to S) \longmapsto \Gamma(T, f^*\mathcal{F})$$
on $S_{étale}$.

(3) *Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules on $S_{étale}$. Then $(id_{\tau,étale})^*\mathcal{F}$ is the τ-sheafification of the étale sheaf*
$$(f : T \to S) \longmapsto \Gamma(T, f_{small}^*\mathcal{F})$$
where $f_{small} : T_{étale} \to S_{étale}$ is the morphism of ringed small étale sites of Remark 31.6.4.

Proof. Proof of (1). We first note that the result is true when $\tau = Zar$ because in that case we have the morphism of topoi $i_f : Sh(T_{Zar}) \to Sh(Sch/S)_{Zar})$ such that $\text{id}_{\tau,Zar} \circ i_f = f_{small}$ as morphisms $T_{Zar} \to S_{Zar}$, see Topologies, Lemmas 30.3.12 and 30.3.16. Since pullback is transitive (see Modules on Sites, Lemma 16.13.3) we see that $i_f^*(\text{id}_{\tau,Zar})^*\mathcal{F} =$

$f_{small}^*\mathcal{F}$ as desired. Hence, by the remark preceding this lemma we see that $(\mathrm{id}_{\tau,Zar})^*\mathcal{F}$ is the τ-sheafification of the presheaf $T \mapsto \Gamma(T, f^*\mathcal{F})$.

The proof of (3) is exactly the same as the proof of (1), except that it uses Topologies, Lemmas 30.4.12 and 30.4.16. We omit the proof of (2). $\square$

Remark 31.6.6. Remark 31.6.4 and Lemma 31.6.5 have the following applications:

(1) Let S be a scheme. The construction $\mathcal{F} \mapsto \mathcal{F}^a$ is the pullback under the morphism of ringed sites $\mathrm{id}_{\tau,Zar} : ((Sch/S)_\tau, \mathcal{O}) \to (S_{Zar}, \mathcal{O})$ or the morphism $(S_{\acute{e}tale}, \mathcal{O}) \to (S_{Zar}, \mathcal{O})$.

(2) Let $f : X \to Y$ be a morphism of schemes. For any of the morphisms f_{sites} of ringed sites of Remark 31.6.4 we have

$$(f^*\mathcal{F})^a = f_{sites}^*\mathcal{F}^a.$$

This follows from (1) and the fact that pullbacks are compatible with compositions of morphisms of ringed sites, see Modules on Sites, Lemma 16.13.3.

Lemma 31.6.7. *Let S be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_S$-module. Let $\tau \in \{Zariski, fppf, \acute{e}tale, smooth, syntomic\}$.*

(1) *The sheaf $\mathcal{F}^a$ is a quasi-coherent $\mathcal{O}$-module on $(Sch/S)_\tau$, as defined in Modules on Sites, Definition 16.23.1.*

(2) *If $\tau = \acute{e}tale$ (resp. $\tau = Zariski$), then the sheaf $\mathcal{F}^a$ is a quasi-coherent $\mathcal{O}$-module on $S_{\acute{e}tale}$ (resp. S_{Zar}) as defined in Modules on Sites, Definition 16.23.1.*

Proof. Let $\{S_i \to S\}$ be a Zariski covering such that we have exact sequences

$$\bigoplus\nolimits_{k\in K_i} \mathcal{O}_{S_i} \longrightarrow \bigoplus\nolimits_{j\in J_i} \mathcal{O}_{S_i} \longrightarrow \mathcal{F} \longrightarrow 0$$

for some index sets K_i and J_i. This is possible by the definition of a quasi-coherent sheaf on a ringed space (See Modules, Definition 15.10.1).

Proof of (1). Let $\tau \in \{Zariski, fppf, \acute{e}tale, smooth, syntomic\}$. It is clear that $\mathcal{F}^a|_{(Sch/S_i)_\tau}$ also sits in an exact sequence

$$\bigoplus\nolimits_{k\in K_i} \mathcal{O}|_{(Sch/S_i)_\tau} \longrightarrow \bigoplus\nolimits_{j\in J_i} \mathcal{O}|_{(Sch/S_i)_\tau} \longrightarrow \mathcal{F}^a|_{(Sch/S_i)_\tau} \longrightarrow 0$$

Hence $\mathcal{F}^a$ is quasi-coherent by Modules on Sites, Lemma 16.23.3.

Proof of (2). Let $\tau = \acute{e}tale$. It is clear that $\mathcal{F}^a|_{(S_i)_{\acute{e}tale}}$ also sits in an exact sequence

$$\bigoplus\nolimits_{k\in K_i} \mathcal{O}|_{(S_i)_{\acute{e}tale}} \longrightarrow \bigoplus\nolimits_{j\in J_i} \mathcal{O}|_{(S_i)_{\acute{e}tale}} \longrightarrow \mathcal{F}^a|_{(S_i)_{\acute{e}tale}} \longrightarrow 0$$

Hence $\mathcal{F}^a$ is quasi-coherent by Modules on Sites, Lemma 16.23.3. The case $\tau = Zariski$ is similar (actually, it is really tautological since the corresponding ringed topoi agree). $\square$

Lemma 31.6.8. *Let S be a scheme. Let*

(a) *$\tau \in \{Zariski, fppf, \acute{e}tale, smooth, syntomic\}$ and $\mathcal{C} = (Sch/S)_\tau$, or*

(b) *let $\tau = \acute{e}tale$ and $\mathcal{C} = S_{\acute{e}tale}$, or*

(c) *let $\tau = Zariski$ and $\mathcal{C} = S_{Zar}$.*

Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}$. Let $U \in Ob(\mathcal{C})$ be affine. Let $\{U_i \to U\}_{i=1,\ldots,n}$ be a standard affine τ-covering in $\mathcal{C}$. Then

(1) *$\mathcal{V} = \{\coprod_{i=1,\ldots,n} U_i \to U\}$ is a τ-covering of U,*

(2) *$\mathcal{U}$ is a refinement of $\mathcal{V}$, and*

(3) *the induced map on Cech complexes (Cohomology on Sites, Equation (19.9.2.1))*
$$\check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$
is an isomorphism of complexes.

Proof. This follows because
$$(\coprod\nolimits_{i_0=1,\ldots,n} U_{i_0}) \times_U \ldots \times_U (\coprod\nolimits_{i_p=1,\ldots,n} U_{i_p}) = \coprod\nolimits_{i_0,\ldots,i_p \in \{1,\ldots,n\}} U_{i_0} \times_U \ldots \times_U U_{i_p}$$
and the fact that $\mathcal{F}(\coprod_a V_a) = \prod_a \mathcal{F}(V_a)$ since disjoint unions are τ-coverings. $\square$

Lemma 31.6.9. *Let S be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on S. Let τ, $\mathcal{C}$, U, $\mathcal{U}$ be as in Lemma 31.6.8. Then there is an isomorphism of complexes*
$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^a) \cong s((A/R)_\bullet \otimes_R M)$$
(see Section 31.3) where $R = \Gamma(U, \mathcal{O}_U)$, $M = \Gamma(U, \mathcal{F}^a)$ and $R \to A$ is a faithfully flat ring map. In particular
$$\check{H}^p(\mathcal{U}, \mathcal{F}^a) = 0$$
for all $p \geq 1$.

Proof. By Lemma 31.6.8 we see that $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^a)$ is isomorphic to $\check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F}^a)$ where $\mathcal{V} = \{V \to U\}$ with $V = \coprod_{i=1,\ldots n} U_i$ affine also. Set $A = \Gamma(V, \mathcal{O}_V)$. Since $\{V \to U\}$ is a τ-covering we see that $R \to A$ is faithfully flat. On the other hand, by definition of $\mathcal{F}^a$ we have that the degree p term $\check{\mathcal{C}}^p(\mathcal{V}, \mathcal{F}^a)$ is
$$\Gamma(V \times_U \ldots \times_U V, \mathcal{F}^a) = \Gamma(Spec(A \otimes_R \ldots \otimes_R A), \mathcal{F}^a) = A \otimes_R \ldots \otimes_R A \otimes_R M$$
We omit the verification that the maps of the chech complex agree with the maps in the complex $s((A/R)_\bullet \otimes_R M)$. The vanishing of cohomology is Lemma 31.3.6. $\square$

Proposition 31.6.10. *Let S be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on S. Let $\tau \in \{Zariski, fppf, étale, smooth, syntomic\}$.*

(1) *There is a canonical isomorphism*
$$H^q(S, \mathcal{F}) = H^q((Sch/S)_\tau, \mathcal{F}^a).$$
(2) *There are canonical isomorphisms*
$$H^q(S, \mathcal{F}) = H^q(S_{Zar}, \mathcal{F}^a) = H^q(S_{étale}, \mathcal{F}^a).$$

Proof. The result for $q = 0$ is clear from the definition of $\mathcal{F}^a$. Let $\mathcal{C} = (Sch/S)_\tau$, or $\mathcal{C} = S_{étale}$, or $\mathcal{C} = S_{Zar}$.

We are going to apply Cohomology on Sites, Lemma 19.11.8 with $\mathcal{F} = \mathcal{F}^a$, $\mathcal{B} \subset Ob(\mathcal{C})$ the set of affine schemes in $\mathcal{C}$, and $\text{Cov} \subset \text{Cov}_\mathcal{C}$ the set of standard affine τ-coverings. Assumption (3) of the lemma is satisfied by Lemma 31.6.9. Hence we conclude that $H^p(U, \mathcal{F}^a) = 0$ for every affine object U of $\mathcal{C}$.

Next, let $U \in Ob(\mathcal{C})$ be any separated object. Denote $f : U \to S$ the structure morphism. Let $U = \bigcup U_i$ be an affine open covering. We may also think of this as a τ-covering $\mathcal{U} = \{U_i \to U\}$ of U in $\mathcal{C}$. Note that $U_{i_0} \times_U \ldots \times_U U_{i_p} = U_{i_0} \cap \ldots \cap U_{i_p}$ is affine as we assumed U separated. By Cohomology on Sites, Lemma 19.11.6 and the result above we see that
$$H^p(U, \mathcal{F}^a) = \check{H}^p(\mathcal{U}, \mathcal{F}^a) = H^p(U, f^*\mathcal{F})$$
the last equality by Coherent, Lemma 25.2.4. In particular, if S is separated we can take $U = S$ and $f = \text{id}_S$ and the proposition is proved. We suggest the reader skip the rest of the proof (or rewrite it to give a clearer exposition).

Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ on S. Choose an injective resolution $\mathcal{F}^a \to \mathcal{J}^\bullet$ on $\mathcal{C}$. Denote $\mathcal{J}^n|_S$ the restriction of $\mathcal{J}^n$ to opens of S; this is a sheaf on the topological space S as open coverings are τ-coverings. We get a complex

$$0 \to \mathcal{F} \to \mathcal{J}^0|_S \to \mathcal{J}^1|_S \to \ldots$$

which is exact since its sections over any affine open $U \subset S$ is exact (by the vanshing of $H^p(U, \mathcal{F}^a)$, $p > 0$ seen above). Hence by Derived Categories, Lemma 11.17.6 there exists map of complexes $\mathcal{J}^\bullet|_S \to \mathcal{I}^\bullet$ which in particular induces a map

$$R\Gamma(\mathcal{C}, \mathcal{F}^a) = \Gamma(S, \mathcal{J}^\bullet) \longrightarrow \Gamma(S, \mathcal{I}^\bullet) = R\Gamma(S, \mathcal{F}).$$

Taking cohomology gives the map $H^n(\mathcal{C}, \mathcal{F}^a) \to H^n(S, \mathcal{F})$ which we have to prove is an isomorphism. Let $\mathcal{U} : S = \bigcup U_i$ be an affine open covering which we may think of as a τ-covering also. By the above we get a map of double complexes

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{J}) = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{J}|_S) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}).$$

This map induces a map of spectral sequences

$${}^\tau E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F}^a)) \longrightarrow E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F}))$$

The first spectral sequence converges to $H^{p+q}(\mathcal{C}, \mathcal{F})$ and the second to $H^{p+q}(S, \mathcal{F})$. On the other hand, we have seen that the induced maps ${}^\tau E_2^{p,q} \to E_2^{p,q}$ are bijections (as all the intersections are separated being opens in affines). Whence also the maps $H^n(\mathcal{C}, \mathcal{F}^a) \to H^n(S, \mathcal{F})$ are isomorphisms, and we win. □

Proposition 31.6.11. *Let S be a scheme. Let $\tau \in \{Zariski, fppf, étale, smooth, syntomic\}$.*

(1) *The functor $\mathcal{F} \mapsto \mathcal{F}^a$ defines an equivalence of categories*

$$QCoh(\mathcal{O}_S) \longrightarrow QCoh((Sch/S)_\tau, \mathcal{O})$$

between the category of quasi-coherent sheaves on S and the category of quasi-coherent $\mathcal{O}$-modules on the big τ site of S.

(2) *Let $\tau = étale$, or $\tau = Zariski$. The functor $\mathcal{F} \mapsto \mathcal{F}^a$ defines an equivalence of categories*

$$QCoh(\mathcal{O}_S) \longrightarrow QCoh(S_\tau, \mathcal{O})$$

between the category of quasi-coherent sheaves on S and the category of quasi-coherent $\mathcal{O}$-modules on the small τ site of S.

Proof. We have seen in Lemma 31.6.7 that the functor is well defined. It is straightforward to show that the functor is fully faithful (we omit the verification). To finish the proof we will show that a quasi-cohernet $\mathcal{O}$-module on $(Sch/S)_\tau$ gives rise to a descent datum for quasi-coherent sheaves relative to a τ-covering of S. Having produced this descent datum we will appeal to Proposition 31.4.2 to get the corresponding quasi-coherent sheaf on S.

Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}$-modules on the big τ site of S. By Modules on Sites, Definition 16.23.1 there exists a τ-covering $\{S_i \to S\}_{i \in I}$ of S such that each of the restrictions $\mathcal{G}|_{(Sch/S_i)_\tau}$ has a global presentation

$$\bigoplus\nolimits_{k \in K_i} \mathcal{O}|_{(Sch/S_i)_\tau} \longrightarrow \bigoplus\nolimits_{j \in J_i} \mathcal{O}|_{(Sch/S_i)_\tau} \longrightarrow \mathcal{G}|_{(Sch/S_i)_\tau} \longrightarrow 0$$

for some index sets J_i and K_i. We claim that this implies that $\mathcal{G}|_{(Sch/S_i)_\tau}$ is $\mathcal{F}_i^a$ for some quasi-coherent sheaf $\mathcal{F}_i$ on S_i. Namely, this is clear for the direct sums $\bigoplus_{k \in K_i} \mathcal{O}|_{(Sch/S_i)_\tau}$ and $\bigoplus_{j \in J_i} \mathcal{O}|_{(Sch/S_i)_\tau}$. Hence we see that $\mathcal{G}|_{(Sch/S_i)_\tau}$ is a cokernel of a map $\varphi : \mathcal{K}_i^a \to \mathcal{L}_i^a$ for some quasi-coherent sheaves $\mathcal{K}_i$, $\mathcal{L}_i$ on S_i. By the fully faithfulness of $(\ )^a$ we see that

$\varphi = \phi^a$ for some map of quasi-coherent sheaves $\phi : \mathcal{K}_i \to \mathcal{L}_i$ on S_i. Then it is clear that $\mathcal{G}|_{(Sch/S_i)_\tau} \cong \text{Coker}(\phi)^a$ as claimed.

Since $\mathcal{G}$ lives on all of the category $(Sch/S_i)_\tau$ we see that

$$(\text{pr}_0^*\mathcal{F}_i)^a \cong \mathcal{G}|_{(Sch/(S_i \times_S S_j))_\tau} \cong (\text{pr}_1^*\mathcal{F})^a$$

as $\mathcal{O}$-modules on $(Sch/(S_i \times_S S_j))_\tau$. Hence, using fully faithfulness again we get canonical isomorphisms

$$\phi_{ij} : \text{pr}_0^*\mathcal{F}_i \longrightarrow \text{pr}_1^*\mathcal{F}_j$$

of quasi-coherent modules over $S_i \times_S S_j$. We omit the verification that these satisfy the cocycle condition. Since they do we see by effectivity of descent for quasi-coherent sheaves and the covering $\{S_i \to S\}$ (Proposition 31.4.2) that there exists a quasi-coherent sheaf $\mathcal{F}$ on S with $\mathcal{F}|_{S_i} \cong \mathcal{F}_i$ compatible with the given descent data. In other words we are given $\mathcal{O}$-module isomorphisms

$$\phi_i : \mathcal{F}^a|_{(Sch/S_i)_\tau} \longrightarrow \mathcal{G}|_{(Sch/S_i)_\tau}$$

which agree over $S_i \times_S S_j$. Hence, since $\mathcal{H}om_\mathcal{O}(\mathcal{F}^a, \mathcal{G})$ is a sheaf (Modules on Sites, Lemma 16.25.1), we conclude that there is a morphism of $\mathcal{O}$-modules $\mathcal{F}^a \to \mathcal{G}$ recovering the isomorphisms ϕ_i above. Hence this is an isomorphism and we win.

The case of the sites $S_{étale}$ and S_{Zar} is proved in the exact same manner. □

Lemma 31.6.12. *Let S be a scheme. Let $\tau \in \{Zariski, fppf, étale, smooth, syntomic\}$. Let $\mathcal{P}$ be one of the properties of modules[3] defined in Modules on Sites, Definitions 16.17.1, 16.23.1, and 16.26.1. The equivalences of categories*

$$QCoh(\mathcal{O}_S) \longrightarrow QCoh((Sch/S)_\tau, \mathcal{O}) \quad \textit{and} \quad QCoh(\mathcal{O}_S) \longrightarrow QCoh(S_\tau, \mathcal{O})$$

defined by the rule $\mathcal{F} \mapsto \mathcal{F}^a$ seen in Proposition 31.6.11 have the property

$$\mathcal{F} \textit{ has } \mathcal{P} \Leftrightarrow \mathcal{F}^a \textit{ has } \mathcal{P} \textit{ as an } \mathcal{O}\textit{-module}$$

except (possibly) when $\mathcal{P}$ is ``locally free'' or ``coherent''. If $\mathcal{P}$ =``coherent'' the equivalence holds for $QCoh(\mathcal{O}_S) \to QCoh(S_\tau, \mathcal{O})$ when S is locally Noetherian and τ is Zariski or étale.

Proof. This is immediate for the global properties, i.e., those defined in Modules on Sites, Definition 16.17.1. For the local properties we can use Modules on Sites, Lemma 16.23.3 to translate ``$\mathcal{F}^a$ has $\mathcal{P}$'' into a property on the members of a covering of X. Hence the result follows from Lemmas 31.5.1, 31.5.2, 31.5.3, and 31.5.4. Being coherent for a quasi-coherent module is the same as being of finite type over a locally Noetherian scheme (see Coherent, Lemma 25.11.1) hence this reduces to the case of finite type modules (details omitted). □

Lemma 31.6.13. *Let S be a scheme. Let $\tau \in \{Zariski, fppf, étale, smooth, syntomic\}$. The functors*

$$QCoh(\mathcal{O}_S) \longrightarrow Mod((Sch/S)_\tau, \mathcal{O}) \quad \textit{and} \quad QCoh(\mathcal{O}_S) \longrightarrow Mod(S_\tau, \mathcal{O})$$

defined by the rule $\mathcal{F} \mapsto \mathcal{F}^a$ seen in Proposition 31.6.11 are

(1) *fully faithful,*
(2) *compatible with direct sums,*

[3]The list is: free, finite free, generated by global sections, generated by finitely many global sections, having a global presentation, having a global finite presentation, locally free, finite locally free, locally generated by sections, finite type, of finite presentation, coherent, or flat.

(3) *compatible with colimits,*
(4) *right exact,*
(5) *exact as a functor* $QCoh(\mathcal{O}_S) \to Mod(S_\tau, \mathcal{O})$*,*
(6) **not** *exact as a functor* $QCoh(\mathcal{O}_S) \to Mod((Sch/S)_\tau, \mathcal{O})$ *in general,*
(7) *given two quasi-coherent* $\mathcal{O}_S$*-modules* $\mathcal{F}, \mathcal{G}$ *we have* $(\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{G})^a = \mathcal{F}^a \otimes_{\mathcal{O}} \mathcal{G}^a$*,*
(8) *given two quasi-coherent* $\mathcal{O}_S$*-modules* $\mathcal{F}, \mathcal{G}$ *such that* $\mathcal{F}$ *is of finite presentation we have* $(\mathcal{H}om_{\mathcal{O}_S}(\mathcal{F}, \mathcal{G}))^a = \mathcal{H}om_{\mathcal{O}}(\mathcal{F}^a, \mathcal{G}^a)$*, and*
(9) *given a short exact sequence* $0 \to \mathcal{F}_1^a \to \mathcal{E} \to \mathcal{F}_2^a \to 0$ *of* $\mathcal{O}$*-modules then* $\mathcal{E}$ *is quasi-coherent*[4]*, i.e.,* $\mathcal{E}$ *is in the essential image of the functor.*

Proof. Part (1) we saw in Proposition 31.6.11.

We have seen in Schemes, Section 21.24 that a colimit of quasi-coherent sheaves on a scheme is a quasi-coherent sheaf. Moreover, in Remark 31.6.6 we saw that $\mathcal{F} \mapsto \mathcal{F}^a$ is the pullback functor for a morphism of ringed sites, hence commutes with all colimits, see Modules on Sites, Lemma 16.14.3. Thus (3) and its special case (3) hold.

This also shows that the functor is right exact (i.e., commutes with finite colimits), hence (4).

The functor $QCoh(\mathcal{O}_S) \to QCoh(S_{étale}, \mathcal{O})$, $\mathcal{F} \mapsto \mathcal{F}^a$ is left exact because an étale morphism is flat, see Morphisms, Lemma 24.35.12. This proves (5).

To see (6), suppose that $S = Spec(\mathbf{Z})$. Then $2 : \mathcal{O}_S \to \mathcal{O}_S$ is injective but the associated map of $\mathcal{O}$-modules on $(Sch/S)_\tau$ isn't injective because $2 : \mathbf{F}_2 \to \mathbf{F}_2$ isn't injective and $Spec(\mathbf{F}_2)$ is an object of $(Sch/S)_\tau$.

We omit the proofs of (7) and (8).

Let $0 \to \mathcal{F}_1^a \to \mathcal{E} \to \mathcal{F}_2^a \to 0$ be a short exact sequence of $\mathcal{O}$-modules with $\mathcal{F}_1$ and $\mathcal{F}_2$ quasi-coherent on S. Consider the restriction

$$0 \to \mathcal{F}_1 \to \mathcal{E}|_{S_{Zar}} \to \mathcal{F}_2$$

to S_{Zar}. By Proposition 31.6.10 we see that on any affine $U \subset S$ we have $H^1(U, \mathcal{F}_1^a) = H^1(U, \mathcal{F}_1) = 0$. Hence the sequence above is also exact on the right. By Schemes, Section 21.24 we conclude that $\mathcal{F} = \mathcal{E}|_{S_{Zar}}$ is quasi-coherent. Thus we obtain a commutative diagram

$$\begin{tikzcd}
 & \mathcal{F}_1^a \arrow[r] \arrow[d] & \mathcal{F}^a \arrow[r] \arrow[d] & \mathcal{F}_2^a \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F}_1^a \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{F}_2^a \arrow[r] & 0
\end{tikzcd}$$

To finish the proof it suffices to show that the top row is also right exact. To do this, denote once more $U = Spec(A) \subset S$ an affine open of S. We have seen above that $0 \to \mathcal{F}_1(U) \to \mathcal{E}(U) \to \mathcal{F}_2(U) \to 0$ is exact. For any affine scheme V/U, $V = Spec(B)$ the map $\mathcal{F}_1^a(V) \to \mathcal{E}(V)$ is injective. We have $\mathcal{F}_1^a(V) = \mathcal{F}_1(U) \otimes_A B$ by definition. The injection $\mathcal{F}_1^a(V) \to \mathcal{E}(V)$ factors as

$$\mathcal{F}_1(U) \otimes_A B \to \mathcal{E}(U) \otimes_A B \to \mathcal{E}(U)$$

[4]Warning: This is misleading. See part (6).

Considering A-algebras B of the form $B = A \oplus M$ we see that $\mathcal{F}_1(U) \to \mathcal{E}(U)$ is universally injective (see Algebra, Definition 7.76.1). Since $\mathcal{E}(U) = \mathcal{F}(U)$ we conclude that $\mathcal{F}_1 \to \mathcal{F}$ remains injective after any base change, or equivalently that $\mathcal{F}_1^a \to \mathcal{F}^a$ is injective. $\square$

Proposition 31.6.14. *Let $f : T \to S$ be a morphism of schemes.*

(1) *The equivalences of categories of Proposition 31.6.11 are compatible with pullback. More precisely, we have $f^*(\mathcal{G}^a) = (f^*\mathcal{G})^a$ for any quasi-coherent sheaf $\mathcal{G}$ on S.*
(2) *The equivalences of categories of Proposition 31.6.11 part (1) are* **not** *compatible with pushforward in general.*
(3) *If f is quasi-compact and quasi-separated, and $\tau \in \{Zariski, étale\}$ then f_* and $f_{small,*}$ preserve quasi-coherent sheaves and the diagram*

$$\begin{tikzcd} QCoh(\mathcal{O}_T) \arrow[r, "f_*"'] \arrow[d, "\mathcal{F} \mapsto \mathcal{F}^a"'] & QCoh(\mathcal{O}_S) \arrow[d, "\mathcal{G} \mapsto \mathcal{G}^a"] \\ QCoh(T_\tau, \mathcal{O}) \arrow[r, "f_{small,*}"] & QCoh(S_\tau, \mathcal{O}) \end{tikzcd}$$

is commutative, i.e., $f_{small,*}(\mathcal{F}^a) = (f_*\mathcal{F})^a$.

Proof. Part (1) follows from the discussion in Remark 31.6.6. Part (2) is just a warning, and can be explained in the following way: First the statement cannot be made precise since f_* does not transform quasi-coherent sheaves into quasi-coherent sheaves in general. Even if this is the case for f (and any base change of f), then the compatibility over the big sites would mean that formation of $f_*\mathcal{F}$ commutes with any base change, which does not hold in general. An explicit example is the quasi-compact open immersion $j : X = \mathbf{A}^2_k \setminus \{0\} \to \mathbf{A}^2_k = Y$ where k is a field. We have $j_*\mathcal{O}_X = \mathcal{O}_Y$ but after base change to $Spec(k)$ by the 0 map we see that the pushfoward is zero.

Let us prove (3) in case $\tau = étale$. Note that f, and any base change of f, transforms quasi-coherent sheaves into quasi-coherent sheaves, see Schemes, Lemma 21.24.1. The equality $f_{small,*}(\mathcal{F}^a) = (f_*\mathcal{F})^a$ means that for any étale morphism $g : U \to S$ we have $\Gamma(U, g^*f_*\mathcal{F}) = \Gamma(U \times_S T, (g')^*\mathcal{F})$ where $g' : U \times_S T \to T$ is the projection. This is true by Coherent, Lemma 25.6.2. $\square$

Lemma 31.6.15. *Let $f : T \to S$ be a quasi-compact and quasi-separated morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on T. For either the étale or Zariski topology, there are canonical isomorphisms $R^if_{small,*}(\mathcal{F}^a) = (R^if_*\mathcal{F})^a$.*

Proof. We prove this for the étale topology; we omit the proof in the case of the Zariski topology. By Coherent, Lemma 25.5.3 the sheaves $R^if_*\mathcal{F}$ are quasi-coherent so that the assertion makes sense. The sheaf $R^if_{small,*}\mathcal{F}^a$ is the sheaf associated to the presheaf

$$U \longmapsto H^i(U \times_S T, \mathcal{F}^a)$$

where $g : U \to S$ is an object of $S_{étale}$, see Cohomology on Sites, Lemma 19.8.4. By our conventions the right hand side is the étale cohomology of the restriction of $\mathcal{F}^a$ to the localization $T_{étale}/U \times_S T$ which equals $(U \times_S T)_{étale}$. By Proposition 31.6.10 this is presheaf the same as the presheaf

$$U \longmapsto H^i(U \times_S T, (g')^*\mathcal{F}),$$

where $g' : U \times_S T \to T$ is the projection. If U is affine then this is the same as $H^0(U, R^if'_*(g')^*\mathcal{F})$, see Coherent, Lemma 25.5.4. By Coherent, Lemma 25.6.2 this is equal to $H^0(U, g^*R^if_*\mathcal{F})$

which is the value of $(R^i f_*\mathcal{F})^a$ on U. Thus the values of the sheaves of modules $R^i f_{small,*}(\mathcal{F}^a)$ and $(R^i f_*\mathcal{F})^a$ on every affine object of $S_{\acute{e}tale}$ are canonically isomorphic which implies they are canonically isomorphic. $\square$

The results in this section say there is virtually no difference between quasi-coherent sheaves on S and quasi-coherent sheaves on any of the sites associated to S in the chapter on topologies. Hence one often sees statements on quasi-coherent sheaves formulated in either language, without restatements in the other.

31.7. Parasitic modules

Parasitic modules are those which are zero when restricted to schemes flat over the base scheme. Here is the formal definition.

Definition 31.7.1. Let S be a scheme. Let $\tau \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\}$. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules on $(Sch/S)_\tau$.

1. $\mathcal{F}$ is called *parasitic*[5] if for every flat morphism $U \to S$ we have $\mathcal{F}(U) = 0$.
2. $\mathcal{F}$ is called *parasitic for the τ-topology* if for every τ-covering $\{U_i \to S\}_{i\in I}$ we have $\mathcal{F}(U_i) = 0$ for all i.

If $\tau = fppf$ this means that $\mathcal{F}|_{U_{Zar}} = 0$ whenever $U \to S$ is flat and locally of finite presentation; similar for the other cases.

Lemma 31.7.2. *Let S be a scheme. Let $\tau \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\}$. Let $\mathcal{G}$ be a presheaf of $\mathcal{O}$-modules on $(Sch/S)_\tau$.*

1. *If $\mathcal{G}$ is parasitic for the τ-topology, then $H^p_\tau(U, \mathcal{G}) = 0$ for every U open in S, resp. étale over S, resp. smooth over S, resp. syntomic over S, resp. flat and locally of finite presentation over S.*
2. *If $\mathcal{G}$ is parasitic then $H^p_\tau(U, \mathcal{G}) = 0$ for every U flat over S.*

Proof. Proof in case $\tau = fppf$; the other cases are proved in the exact same way. The assumption means that $\mathcal{G}(U) = 0$ for any $U \to S$ flat and locally of finite presentation. Apply Cohomology on Sites, Lemma 19.11.8 to the subset $\mathcal{B} \subset Ob((Sch/S)_{fppf})$ consisting of $U \to S$ flat and locally of finite presentation and the collection Cov of all fppf coverings of elements of $\mathcal{B}$. $\square$

Lemma 31.7.3. *Let $f : T \to S$ be a morphism of schemes. For any parasitic $\mathcal{O}$-module on $(Sch/T)_\tau$ the pushforward $f_*\mathcal{F}$ and the higher direct images $R^i f_*\mathcal{F}$ are parasitic $\mathcal{O}$-modules on $(Sch/S)_\tau$.*

Proof. Recall that $R^i f_*\mathcal{F}$ is the sheaf associated to the presheaf

$$U \mapsto H^i((Sch/U \times_S T)_\tau, \mathcal{F})$$

see Cohomology on Sites, Lemma 19.8.4. If $U \to S$ is flat, then $U \times_S T \to T$ is flat as a base change. Hence the displayed group is zero by Lemma 31.7.2. If $\{U_i \to U\}$ is a τ-covering then $U_i \times_S T \to T$ is also flat. Hence it is clear that the sheafification of the displayed presheaf is zero on schemes U flat over S. $\square$

Lemma 31.7.4. *Let S be a scheme. Let $\tau \in \{Zar, \acute{e}tale\}$. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}$-modules on $(Sch/S)_{fppf}$ such that*

1. *$\mathcal{G}|_{S_\tau}$ is quasi-coherent, and*

[5] This may be nonstandard notation.

(2) *for every flat, locally finitely presented morphism $g : U \to S$ the canonical map $g_{\tau,small}^*(\mathcal{G}|_{S_\tau}) \to \mathcal{G}|_{U_\tau}$ is an isomorphism.*

Then $H^p(U, \mathcal{G}) = H^p(U, \mathcal{G}|_{U_\tau})$ for every U flat and locally of finite presentation over S.

Proof. Let $\mathcal{F}$ be the pullback of $\mathcal{G}|_{S_\tau}$ to the big fppf site $(Sch/S)_{fppf}$. Note that $\mathcal{F}$ is quasi-coherent. There is a canonical comparison map $\varphi : \mathcal{F} \to \mathcal{G}$ which by assumptions (1) and (2) induces an isomorphism $\mathcal{F}|_{U_\tau} \to \mathcal{G}|_{U_\tau}$ for all $g : U \to S$ flat and locally of finite presentation. Hence in the short exact sequences

$$0 \to \mathrm{Ker}(\varphi) \to \mathcal{F} \to \mathrm{Im}(\varphi) \to 0$$

and

$$0 \to \mathrm{Im}(\varphi) \to \mathcal{G} \to \mathrm{Coker}(\varphi) \to 0$$

the sheaves $\mathrm{Ker}(\varphi)$ and $\mathrm{Coker}(\varphi)$ are parasitic for the fppf topology. By Lemma 31.7.2 we conclude that $H^p(U, \mathcal{F}) \to H^p(U, \mathcal{G})$ is an isomorphism for $g : U \to S$ flat and locally of finite presentation. Since the result holds for $\mathcal{F}$ by Proposition 31.6.10 we win. □

31.8. Derived category of quasi-coherent modules

Let S be a scheme. Often the phrase ``the derived category of quasi-coherent modules on S'' refers to the category $D_{QCoh}(\mathcal{O}_S)$ and not the derived category $D(QCoh(\mathcal{O}_S))$. It turns out that $D_{QCoh}(\mathcal{O}_S)$ is often easier to work with.

In this section we show that $D_{QCoh}(\mathcal{O}_S)$ can be defined in terms of the small étale site of S. Namely, denote $\mathcal{O}_{small}$ the structure sheaf on $S_{étale}$. Recall that $QCoh(\mathcal{O}_S)$ is also a Serre subcategory of $Mod(\mathcal{O}_{étale})$, see Lemma 31.6.13. Hence we can let $D_{QCoh}(\mathcal{O}_{étale})$ be the trangulated subcategory of $D(\mathcal{O}_{étale})$ whose objects are the complexes with quasi-coherent cohomology sheaves, see Derived Categories, Section 11.12.

Lemma 31.8.1. *Let S be a scheme. There is a canonical equivalence $D_{QCoh}(\mathcal{O}_S) = D_{QCoh}(\mathcal{O}_{étale})$.*

Proof. Consider the morphism of ringed sites $\epsilon : S_{étale} \to S_{Zar}$, see Remark 31.6.4. This is the morphism $\mathrm{id}_{small,étale,Zar}$ of Lemma 31.6.5. Since every etale morphism $T \to S$ is flat (Morphisms, Lemma 24.35.12) the description of $\epsilon^* = \mathrm{id}_{small,étale,Zar}^*$ in Lemma 31.6.5 shows that ϵ^* is an exact functor. Hence it induces $\epsilon^* : D(\mathcal{O}_S) \to D(\mathcal{O}_{small})$. By the material in Section 31.6 given a quasi-coherent sheaf $\mathcal{F}$ on S the sheaf $\mathcal{F}^a = \epsilon^*\mathcal{F}$ is the corresonding quasi-coherent module on $S_{étale}$. Thus ϵ^* induces

$$\epsilon^* : D_{QCoh}(\mathcal{O}_S) \to D_{QCoh}(\mathcal{O}_{small})$$

We are going to construct a quasi-inverse functor.

Let $\mathcal{F}^\bullet$ be an object of $D_{QCoh}(\mathcal{O}_{small})$ and denote $\mathcal{H}^i = H^i(\mathcal{F}^\bullet)$ its ith cohomology sheaf. Let $\mathcal{B}$ be the set of affine objects of $S_{étale}$. Then $H^p(U, \mathcal{H}^i) = 0$ for all $p > 0$, all $i \in \mathbf{Z}$, and all $U \in \mathcal{B}$, see Proposition 31.6.10 and Coherent, Lemma 25.2.2. According to Cohomology on Sites, Section 19.20 this implies there exists a quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ where $\mathcal{I}^\bullet$ is a K-injective complex, $\mathcal{I}^\bullet = lim\, \mathcal{I}_n^\bullet$, each $\mathcal{I}_n^\bullet$ is a bounded below complex of injectives, the maps in the system $\ldots \to \mathcal{I}_2^\bullet \to \mathcal{I}_1^\bullet$ are termwise split surjections, and each $\mathcal{I}_n^\bullet$ is quasi-isomorphic to $\tau_{\geq -n}\mathcal{F}^\bullet$. In particular, we conclude that $R\epsilon_*$ is defined at each object of $D_{QCoh}(\mathcal{O}_{small})$, see Derived Categories, Lemma 11.28.4, with values $R\epsilon_*\mathcal{F}^\bullet = \epsilon_*\mathcal{I}^\bullet$. This defines an exact functor of triangulated categories

$$R\epsilon_* : D_{QCoh}(\mathcal{O}_{small}) \longrightarrow D(\mathcal{O}_S)$$

see Derived Categories, Proposition 11.14.8. Let V be an affine object of $S_{étale}$. In the proof of Cohomology on Sites, Lemma 19.20.1 we have seen that $H^m(\mathcal{I}^\bullet(V))$ is the limit of the cohomology groups $H^m(\mathcal{I}_n^\bullet(V))$. For $n > -m$ these groups are equal to $\mathcal{H}^m(V)$ by the vanishing of higer cohomology and the spectral sequence of Derived Categories, Lemma 11.20.3. If we apply this to all $V = U \subset S$ affine open, then we conclude that the mth cohomology sheaf of $\epsilon_*\mathcal{I}^\bullet$ is $\epsilon_*\mathcal{H}^m$. This implies that $R\epsilon_*\mathcal{F}^\bullet = \epsilon_*\mathcal{I}^\bullet$ is an object of $D_{QCoh}(\mathcal{O}_S)$ and we get our functor

$$R\epsilon_* : D_{QCoh}(\mathcal{O}_{small}) \longrightarrow D_{QCoh}(\mathcal{O}_S)$$

in the other direction! Since also for arbitrary V we have $\mathcal{H}^m(V) = (\epsilon^*\epsilon_*\mathcal{H}^m)(V)$ as $\mathcal{H}^m$ is quasi-coherent, we conclude the canonical map of complexes

$$\epsilon^*\epsilon_*\mathcal{I}^\bullet \longrightarrow \mathcal{I}^\bullet$$

is a quasi-isomorphism. This implies that the composition

$$D_{QCoh}(\mathcal{O}_{small}) \longrightarrow D_{QCoh}(\mathcal{O}_S) \longrightarrow D_{QCoh}(\mathcal{O}_{small})$$

is isomorphic to the identity functor. Finally, we claim that

$$D_{QCoh}(\mathcal{O}_S) \longrightarrow D_{QCoh}(\mathcal{O}_{small}) \longrightarrow D_{QCoh}(\mathcal{O}_S)$$

is isomorphic to the identity as well. Namely, for $\mathcal{G}^\bullet$ an object of $D_{QCoh}(\mathcal{O}_S)$ we choose a map $\epsilon^*\mathcal{G}^\bullet \to \mathcal{I}^\bullet$ into a K-injective complex as above and consider the map

$$\mathcal{G}^\bullet \to \epsilon_*\epsilon^*\mathcal{G}^\bullet \to \epsilon_*\mathcal{I}^\bullet$$

This is a quasi-isomorphism as we've just seen above that the cohomology sheaves of $\epsilon_*\epsilon_*\mathcal{I}^\bullet$ are exactly the quasi-coherent cohomology sheaves of the complex $\mathcal{G}^\bullet$ we started out with. □

31.9. Fpqc coverings are universal effective epimorphisms

We apply the material above to prove an interesting result, namely Lemma 31.9.3. By Sites, Section 9.12 this lemma implies that the representable presheaves on any of the sites $(Sch/S)_\tau$ are sheaves for $\tau \in \{Zarski, fppf, étale, smooth, syntomic\}$. First we prove a helper lemma.

Lemma 31.9.1. *For a scheme X denote $|X|$ the underlying set. Let $f : X \to S$ be a morphism of schemes. Then*

$$|X \times_S X| \to |X| \times_{|S|} |X|$$

is surjective.

Proof. Follows immediately from the description of points on the fibre product in Schemes, Lemma 21.17.5. □

Lemma 31.9.2. *Let $\{f_i : T_i \to T\}_{i \in I}$ be a fpqc covering. Suppose that for each i we have an open subset $W_i \subset T_i$ such that for all $i, j \in I$ we have $pr_0^{-1}(W_i) = pr_1^{-1}(W_j)$ as open subsets of $T_i \times_T T_j$. Then there exists a unique open subset $W \subset T$ such that $W_i = f_i^{-1}(W)$ for each i.*

Proof. Apply Lemma 31.9.1 to the map $\coprod_{i \in I} T_i \to T$. It implies there exists a subset $W \subset T$ such that $W_i = f_i^{-1}(W)$ for each i, namely $W = \bigcup f_i(W_i)$. To see that W is open we may work Zariski locally on T. Hence we may assume that T is affine. Using the definition of a fpqc covering, this reduces us to the case where $\{f_i : T_i \to T\}$ is a standard fpqc covering. In this case we may apply Morphisms, Lemma 24.24.10 to the morphism $\coprod T_i \to T$ to conclude that W is open. □

Lemma 31.9.3. *Let $\{T_i \to T\}$ be an fpqc covering, see Topologies, Definition 30.8.1. Then $\{T_i \to T\}$ is a universal effective epimorphism in the category of schemes, see Sites, Definition 9.12.1. In other words, every representable functor on the category of schemes satisfies the sheaf condition for the fpqc topology, see Topologies, Definition 30.8.12.*

Proof. Let S be a scheme. We have to show the following: Given morphisms $\varphi_i : T_i \to S$ such that $\varphi_i|_{T_i \times_T T_j} = \varphi_j|_{T_i \times_T T_j}$ there exists a unique morphism $T \to S$ which restricts to φ_i on each T_i. In other words, we have to show that the functor $h_S = Mor_{Sch}(-, S)$ satisfies the sheaf property for the fpqc topology.

Thus Topologies, Lemma 30.8.13 reduces us to the case of a Zariski covering and a covering $\{Spec(A) \to Spec(R)\}$ with $R \to A$ faithfully flat. The case of a Zariski covering follows from Schemes, Lemma 21.14.1.

Suppose that $R \to A$ is a faithfully flat ring map. Denote $\pi : Spec(A) \to Spec(R)$ the corresponding morphism of schemes. It is surjective and flat. Let $f : Spec(A) \to S$ be a morphism such that $f \circ \mathrm{pr}_1 = f \circ \mathrm{pr}_2$ as maps $Spec(A \otimes_R A) \to S$. By Lemma 31.9.1 we see that as a map on the underlying sets f is of the form $f = g \circ \pi$ for some (set theoretic) map $g : Spec(R) \to S$. By Morphisms, Lemma 24.24.10 and the fact that f is continuous we see that g is continuous.

Pick $x \in Spec(R)$. Choose $U \subset S$ affine open containing $g(x)$. Say $U = Spec(B)$. By the above we may choose an $r \in R$ such that $x \in D(r) \subset g^{-1}(U)$. The restriction of f to $\pi^{-1}(D(r))$ into U corresponds to a ring map $B \to A_r$. The two induced ring maps $B \to A_r \otimes_{R_r} A_r = (A \otimes_R A)_r$ are equal by assumption on f. Note that $R_r \to A_r$ is faithfully flat. By Lemma 31.3.6 the equalizer of the two arrows $A_r \to A_r \otimes_{R_r} A_r$ is R_r. We conclude that $B \to A_r$ factors uniquely through a map $B \to R_r$. This map in turn gives a morphism of schemes $D(r) \to U \to S$, see Schemes, Lemma 21.6.4.

What have we proved sofar? We have shown that for any prime $\mathfrak{p} \subset R$, there exists a standard affine open $D(r) \subset Spec(R)$ such that the morphism $f|_{\pi^{-1}(D(r))} : \pi^{-1}(D(r)) \to S$ factors uniquely though some morphism of schemes $D(r) \to S$. We omit the verification that these morphisms glue to the desired morphism $Spec(R) \to S$. □

31.10. Descent of finiteness properties of morphisms

Another application of flat descent for modules is the following amusing and useful result. There is an algebraic version and a scheme theoretic version. (The ``Noetherian'' reader should consult Lemma 31.10.2 instead of the next lemma.)

Lemma 31.10.1. *Let $R \to A \to B$ be ring maps. Assume $R \to B$ is of finite presentation and $A \to B$ faithfully flat and of finite presentation. Then $R \to A$ is of finite presentation.*

Proof. Consider the algebra $C = B \otimes_A B$ together with the pair of maps $p, q : B \to C$ given by $p(b) = b \otimes 1$ and $q(b) = 1 \otimes b$. Of course the two compositions $A \to B \to C$ are the same. Note that as $p : B \to C$ is flat and of finite presentation (base change of $A \to B$), the ring map $R \to C$ is of finite presentation (as the composite of $R \to B \to C$).

We are going to use the criterion Algebra, Lemma 7.118.2 to show that $R \to A$ is of finite presentation. Let S be any R-algebra, and suppose that $S = colim_{\lambda \in \Lambda} S_\lambda$ is written as a directed colimit of R-algebras. Let $A \to S$ be an R-algebra homomorphism. We have to show that $A \to S$ factors through one of the S_λ. Consider the rings $B' = S \otimes_A B$ and $C' = S \otimes_A C = B' \otimes_S B'$. As B is faithfully flat of finite presentation over A, also B' is faithfully flat of finite presentation over S. By Algebra, Lemma 7.120.5 part (2) applied to the pair $(S \to B', B')$ and the system (S_λ) there exists a $\lambda_0 \in \Lambda$ and a flat, finitely presented S_{λ_0}-algebra B_{λ_0} such that $B' = S \otimes_{S_{\lambda_0}} B_{\lambda_0}$. For $\lambda \geq \lambda_0$ set $B_\lambda = S_\lambda \otimes_{S_{\lambda_0}} B_{\lambda_0}$ and $C_\lambda = B_\lambda \otimes_{S_\lambda} B_\lambda$.

We interupt the flow of the argument to show that $S_\lambda \to B_\lambda$ is faithfully flat for λ large enough. (This should really be a separate lemma somewhere else, maybe in the chapter on limits.) Since $Spec(B_{\lambda_0}) \to Spec(S_{\lambda_0})$ is flat and of finite presentation it is open (see Morphisms, Lemma 24.24.9). Let $I \subset S_{\lambda_0}$ be an ideal such that $V(I) \subset Spec(S_{\lambda_0})$ is the complement of the image. Note that formation of the image commutes with base change. Hence, since $Spec(B') \to Spec(S)$ is surjective, and $B' = B_{\lambda_0} \otimes_{S_{\lambda_0}} S$ we see that $IS = S$. Thus for some $\lambda \geq \lambda_0$ we have $IS_\lambda = S_\lambda$. For this and all greater λ the morphism $Spec(B_\lambda) \to Spec(S_\lambda)$ is surjective.

By analogy with the notation in the first paragraph of the proof denote $p_\lambda, q_\lambda : B_\lambda \to C_\lambda$ the two canonical maps. Then $B' = colim_{\lambda \geq \lambda_0} B_\lambda$ and $C' = colim_{\lambda \geq \lambda_0} C_\lambda$. Since B and C are finitely presented over R there exist (by Algebra, Lemma 7.118.2 applied several times) a $\lambda \geq \lambda_0$ and an R-algebra maps $B \to B_\lambda$, $C \to C_\lambda$ such that the diagram

$$\begin{tikzcd} C \arrow[r] & C_\lambda \\ B \arrow[u, shift left, "q"'] \arrow[u, shift right, "p"] \arrow[r] & B_\lambda \arrow[u, shift left, "q_\lambda"'] \arrow[u, shift right, "p_\lambda"] \end{tikzcd}$$

is commutative. OK, and this means that $A \to B \to B_\lambda$ maps into the equalizer of p_λ and q_λ. By By Lemma 31.3.6 we see that S_λ is the equalizer of p_λ and q_λ. Thus we get the desired ring map $A \to S_\lambda$ and we win. $\square$

Here is an easier version of this dealing with the property of being of finite type.

Lemma 31.10.2. *Let $R \to A \to B$ be ring maps. Assume $R \to B$ is of finite type and $A \to B$ faithfully flat and of finite presentation. Then $R \to A$ is of finite type.*

Proof. By Algebra, Lemma 7.120.6 there exists a commtutative diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d, equal] & A_0 \arrow[r] \arrow[d] & B_0 \arrow[d] \\ R \arrow[r] & A \arrow[r] & B \end{tikzcd}$$

with $R \to A_0$ of finite presentation, $A_0 \to B_0$ faithfully flat of finite presentation and $B = A \otimes_{A_0} B_0$. Since $R \to B$ is of finite type by assumption, we may add some elements to A_0 and assume that the map $B_0 \to B$ is surjective! In this case, since $A_0 \to B_0$ is faithfully flat, we see that as

$$(A_0 \to A) \otimes_{A_0} B_0 \cong (B_0 \to B)$$

is surjective, also $A_0 \to A$ is surjective. Hence we win. $\square$

Lemma 31.10.3. *Let*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & S & \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume that f is surjective, flat and locally of finite presentation and assume that p is locally of finite presentation (resp. locally of finite type). Then q is locally of finite presentation (resp. locally of finite type).

Proof. The problem is local on S and Y. Hence we may assume that S and Y are affine. Since f is flat and locally of finite presentation, we see that f is open (Morphisms, Lemma 24.24.9). Hence, since Y is quasi-compact, there exist finitely many affine opens $X_i \subset X$ such that $Y = \bigcup f(X_i)$. Clearly we may replace X by $\coprod X_i$, and hence we may assume X is affine as well. In this case the lemma is equivalent to Lemma 31.10.1 (resp. Lemma 31.10.2) above. $\square$

We use this to improve some of the results on morphisms obtained earlier.

Lemma 31.10.4. *Let*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & S & \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume that

(1) *f is surjective, and syntomic (resp. smooth, resp. étale),*
(2) *p is syntomic (resp. smooth, resp. étale).*

Then q is syntomic (resp. smooth, resp. étale).

Proof. Combine Morphisms, Lemmas 24.30.16, 24.33.19, and 24.35.19 with Lemma 31.10.3 above. $\square$

Actually we can strengthen this result as follows.

Lemma 31.10.5. *Let*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & S & \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume that

(1) *f is surjective, flat, and locally of finite presentation,*
(2) *p is smooth (resp. étale).*

Then q is smooth (resp. étale).

Proof. Assume (1) and that p is smooth. By Lemma 31.10.3 we see that q is locally of finite presentation. By Morphisms, Lemma 24.24.11 we see that q is flat. Hence now it suffices to show that the fibres of q are smooth, see Morphisms, Lemma 24.33.3. Apply Varieties, Lemma 28.15.8 to the flat surjective morphisms $X_s \to Y_s$ for $s \in S$ to conclude. We omit the proof of the étale case. $\square$

Remark 31.10.6. With the assumptions (1) and p smooth in Lemma 31.10.5 it is not automatically the case that $X \to Y$ is smooth. A counter example is $S = Spec(k)$, $X = Spec(k[s])$, $Y = Spec(k[t])$ and f given by $t \mapsto s^2$. But see also Remark 31.10.7 for some information on the structure of f.

Remark 31.10.7. Let

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & S & \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume that

(1) f is surjective, flat, and locally of finite presentation,
(2) p is syntomic.

Then *both q and f* are syntomic. This follows from the following result of Avramov: If $\varphi : A \to B$ is a local homomorphism of local Noetherian rings, and φ is flat and B is a complete intersection, then both A and $B/\mathfrak{m}_A B$ are complete intersections. See [**Avr75**]. If we need this result we will add its proof to the stacks project.

The following type of lemma is occasionally useful.

Lemma 31.10.8. *Let $X \to Y \to Z$ be morphism of schemes. Let P be one of the following properties of morphisms of schemes: flat, locally finite type, locally finite presentation. Assume that $X \to Z$ has P and that $\{X \to Y\}$ can be refined by an fppf covering of Y. Then $Y \to Z$ is P.*

Proof. Let $Spec(C) \subset Z$ be an affine open and let $Spec(B) \subset Y$ be an affine open which maps into $Spec(C)$. The assumption on $X \to Y$ implies we can find a standard affine fppf covering $\{Spec(B_j) \to Spec(B)\}$ and lifts $x_j : Spec(B_j) \to X$. Since $Spec(B_j)$ is quasi-compact we can find finitely many affine opens $Spec(A_i) \subset X$ lying over $Spec(B)$ such that the image of each x_j is contained in the union $\bigcup Spec(A_i)$. Hence after replacing each $Spec(B_j)$ by a standard affine Zariski coverings of itself we may assume we have a standard affine fppf covering $\{Spec(B_i) \to Spec(B)\}$ such that each $Spec(B_i) \to Y$ factors through an affine open $Spec(A_i) \subset X$ lying over $Spec(B)$. In other words, we have ring maps $C \to B \to A_i \to B_i$ for each i. Note that we can also consider

$$C \to B \to A = \prod A_i \to B' = \prod B_i$$

and that the ring map $B \to \prod B_i$ is faithfully flat and of finite presentation.

The case $P = flat$. In this case we know that $C \to A$ is flat and we have to prove that $C \to B$ is flat. Suppose that $N \to N' \to N''$ is an exact sequence of C-modules. We want to show that $N \otimes_C B \to N' \otimes_C B \to N'' \otimes_C B$ is exact. Let H be its cohomology and let H' be the cohomology of $N \otimes_C B' \to N' \otimes_C B' \to N'' \otimes_C B'$. As $B \to B'$ is flat we know that $H' = H \otimes_B B'$. On the other hand $N \otimes_C A \to N' \otimes_C A \to N'' \otimes_C A$ is exact hence has zero cohomology. Hence the map $H \to H'$ is zero (as it factors through the zero module). Thus $H' = 0$. As $B \to B'$ is faithfully flat we conclude that $H = 0$ as desired.

The case $P = locally\ finite\ type$. In this case we know that $C \to A$ is of finite type and we have to prove that $C \to B$ is of finite type. Because $B \to B'$ is of finite presentation (hence of finite type) we see that $A \to B'$ is of finite type, see Algebra, Lemma 7.6.2. Therefore $C \to B'$ is of finite type and we conclude by Lemma 31.10.2.

The case $P = locally\ finite\ presentation$. In this case we know that $C \to A$ is of finite presentation and we have to prove that $C \to B$ is of finite presentation. Because $B \to B'$ is of finite presentation and $B \to A$ of finite type we see that $A \to B'$ is of finite presentation, see Algebra, Lemma 7.6.2. Therefore $C \to B'$ is of finite presentation and we conclude by Lemma 31.10.1. □

31.11. Local properties of schemes

It often happens one can prove the members of a covering of a scheme have a certain property. In many cases this implies the scheme has the property too. For example, if S is a scheme, and $f : S' \to S$ is a surjective flat morphism such that S' is a reduced scheme, then S is reduced. You can prove this by looking at local rings and using Algebra, Lemma 7.146.2. We say that the property of being reduced *descends through flat surjective morphisms*. Some results of this type are collected in Algebra, Section 7.146.

On the other hand, there are examples of surjective flat morphisms $f : S' \to S$ with S reduced and S' not, for example the morphism $Spec(k[x]/(x^2)) \to Spec(k)$. Hence the property of being reduced does not *ascend along flat morphisms*. Having infinite residue fields is a property which does ascend along flat morphisms (but does not descend along surjective flat morphisms of course). Some results of this type are collected in Algebra, Section 7.145.

Finally, we say that a property is *local for the flat topology* if it ascends along flat morphisms and descends along flat surjective morphisms. A somewhat silly example is the property of having residue fields of a given characteristic. To be more precise, and to tie this in with the various topologies on schemes, we make the following formal definition.

Definition 31.11.1. Let $\mathcal{P}$ be a property of schemes. Let $\tau \in \{fpqc, fppf, syntomic, smooth, étale, Zariski\}$. We say $\mathcal{P}$ is *local in the τ-topology* if for any τ-covering $\{S_i \to S\}_{i \in I}$ (see Topologies, Section 30.2) we have

$$S \text{ has } \mathcal{P} \Leftrightarrow \text{each } S_i \text{ has } \mathcal{P}.$$

To be sure, since isomorphisms are always coverings we see (or require) that property $\mathcal{P}$ holds for S if and only if it holds for any scheme S' isomorphic to S. In fact, if $\tau = fpqc$, $fppf, syntomic, smooth, étale$, or $Zariski$, then if S has $\mathcal{P}$ and $S' \to S$ is flat, flat and locally of finite presentation, syntomic, smooth, étale, or an open immersion, then S' has $\mathcal{P}$. This is true because we can always extend $\{S' \to S\}$ to a τ-covering.

We have the following implications: $\mathcal{P}$ is local in the fpqc topology $\Rightarrow$ $\mathcal{P}$ is local in the fppf topology $\Rightarrow$ $\mathcal{P}$ is local in the syntomic topology $\Rightarrow$ $\mathcal{P}$ is local in the smooth topology $\Rightarrow$ $\mathcal{P}$ is local in the étale topology $\Rightarrow$ $\mathcal{P}$ is local in the Zariski topology. This follows from Topologies, Lemmas 30.4.2, 30.5.2, 30.6.2, 30.7.2, and 30.8.6.

Lemma 31.11.2. *Let $\mathcal{P}$ be a property of schemes. Let $\tau \in \{fpqc, fppf, étale, smooth, syntomic\}$. Assume that*

(1) *the property is local in the Zariski topology,*
(2) *for any morphism of affine schemes $S' \to S$ which is flat, flat of finite presentation, étale, smooth or syntomic depending on whether τ is fpqc, fppf, étale, smooth, or syntomic, property $\mathcal{P}$ holds for S' if property $\mathcal{P}$ holds for S, and*
(3) *for any surjective morphism of affine schemes $S' \to S$ which is flat, flat of finite presentation, étale, smooth or syntomic depending on whether τ is fpqc, fppf, étale, smooth, or syntomic, property $\mathcal{P}$ holds for S if property $\mathcal{P}$ holds for S'.*

Then $\mathcal{P}$ is τ local on the base.

Proof. This follows almost immediately from the definition of a τ-covering, see Topologies, Definition 30.8.1 30.7.1 30.4.1 30.5.1, or 30.6.1 and Topologies, Lemma 30.8.8, 30.7.4, 30.4.4, 30.5.4, or 30.6.4. Details omitted. $\square$

Remark 31.11.3. In Lemma 31.11.2 above if $\tau = smooth$ then in condition (3) we may assume that the morphism is a (surjective) standard smooth morphism. Similarly, when $\tau = syntomic$ or $\tau = \acute{e}tale$.

31.12. Properties of schemes local in the fppf topology

In this section we find some properties of schemes which are local on the base in the fppf topology.

Lemma 31.12.1. *The property $\mathcal{P}(S) =$``S is locally Noetherian" is local in the fppf topology.*

Proof. We will use Lemma 31.11.2. First we note that ``being locally Noetherian" is local in the Zariski topology. This is clear from the definition, see Properties, Definition 23.5.1. Next, we show that if $S' \to S$ is a flat, finitely presented morphism of affines and S is locally Noetherian, then S' is locally Noetherian. This is Morphisms, Lemma 24.14.6. Finally, we have to show that if $S' \to S$ is a surjective flat, finitely presented morphism of affines and S' is locally Noetherian, then S is locally Noetherian. This follows from Algebra, Lemma 7.146.1. Thus (1), (2) and (3) of Lemma 31.11.2 hold and we win. $\square$

Lemma 31.12.2. *The property $\mathcal{P}(S) =$``S is Jacobson" is local in the fppf topology.*

Proof. We will use Lemma 31.11.2. First we note that ``being Jacobson" is local in the Zariski topology. This is Properties, Lemma 23.6.3. Next, we show that if $S' \to S$ is a flat, finitely presented morphism of affines and S is Jacobson, then S' is Jacobson. This is Morphisms, Lemma 24.15.9. Finally, we have to show that if $f : S' \to S$ is a surjective flat, finitely presented morphism of affines and S' is Jacobson, then S is Jacobson. Say $S = Spec(A)$ and $S' = Spec(B)$ and $S' \to S$ given by $A \to B$. Then $A \to B$ is finitely presented and faithfully flat. Moreover, the ring B is Jacobson, see Properties, Lemma 23.6.3.

By Algebra, Lemma 7.121.9 there exists a diagram

$$\begin{tikzcd} B \arrow[rr] & & B' \\ & A \arrow[ul] \arrow[ur] & \end{tikzcd}$$

with $A \to B'$ finitely presented, faithfully flat and quasi-finite. In particular, $B \to B'$ is finite type, and we see from Algebra, Proposition 7.31.18 that B' is Jacobson. Hence we may assume that $A \to B$ is quasi-finite as well as faithfully flat and of finite presentation.

Assume A is not Jacobson to get a contradiction. According to Algebra, Lemma 7.31.5 there exists a nonmaximal prime $\mathfrak{p} \subset A$ and an element $f \in A$, $f \notin \mathfrak{p}$ such that $V(\mathfrak{p}) \cap D(f) = \{\mathfrak{p}\}$.

This leads to a contradiction as follows. First let $\mathfrak{p} \subset \mathfrak{m}$ be a maximal ideal of A. Pick a prime $\mathfrak{m}' \subset B$ lying over $\mathfrak{m}$ (exists because $A \to B$ is faithfully flat, see Algebra, Lemma 7.35.15). As $A \to B$ is flat, by going down see Algebra, Lemma 7.35.17, we can find a prime $\mathfrak{q} \subset \mathfrak{m}'$ lying over $\mathfrak{p}$. In particular we see that $\mathfrak{q}$ is not maximal. Hence according to

Algebra, Lemma 7.31.5 again the set $V(\mathfrak{q}) \cap D(f)$ is infinite (here we finally use that B is Jacobson). All points of $V(\mathfrak{q}) \cap D(f)$ map to $V(\mathfrak{p}) \cap D(f) = \{\mathfrak{p}\}$. Hence the fibre over $\mathfrak{p}$ is infinite. This contradicts the fact that $A \to B$ is quasi-finite (see Algebra, Lemma 7.113.4 or more explicitly Morphisms, Lemma 24.19.10). Thus the lemma is proved. □

31.13. Properties of schemes local in the syntomic topology

In this section we find some properties of schemes which are local on the base in the syntomic topology.

Lemma 31.13.1. *The property $\mathcal{P}(S) =$``S is locally Noetherian and (S_k)'' is local in the syntomic topology.*

Proof. We will check (1), (2) and (3) of Lemma 31.11.2. As a syntomic morphism is flat of finite presentation (Morphisms, Lemmas 24.30.7 and 24.30.6) we have already checked this for ``being locally Noetherian'' in the proof of Lemma 31.12.1. We will use this without further mention in the proof. First we note that $\mathcal{P}$ is local in the Zariski topology. This is clear from the definition, see Coherent, Definition 25.13.1. Next, we show that if $S' \to S$ is a syntomic morphism of affines and S has $\mathcal{P}$, then S' has $\mathcal{P}$. This is Algebra, Lemma 7.145.4 (use Morphisms, Lemma 24.30.2 and Algebra, Definition 7.125.1 and Lemma 7.124.3). Finally, we show that if $S' \to S$ is a surjective syntomic morphism of affines and S' has $\mathcal{P}$, then S has $\mathcal{P}$. This is Algebra, Lemma 7.146.5. Thus (1), (2) and (3) of Lemma 31.11.2 hold and we win. □

Lemma 31.13.2. *The property $\mathcal{P}(S) =$``S is Cohen-Macaulay'' is local in the syntomic topology.*

Proof. This is clear from Lemma 31.13.1 above since a scheme is Cohen-Macaulay if and only if it is locally Noetherian and (S_k) for all $k \geq 0$, see Properties, Lemma 23.12.2. □

31.14. Properties of schemes local in the smooth topology

In this section we find some properties of schemes which are local on the base in the smooth topology.

Lemma 31.14.1. *The property $\mathcal{P}(S) =$``S is reduced'' is local in the smooth topology.*

Proof. We will use Lemma 31.11.2. First we note that ``being reduced'' is local in the Zariski topology. This is clear from the definition, see Schemes, Definition 21.12.1. Next, we show that if $S' \to S$ is a smooth morphism of affines and S is reduced, then S' is reduced. This is Algebra, Lemma 7.145.6. Finally, we show that if $S' \to S$ is a surjective smooth morphism of affines and S' is reduced, then S is reduced. This is Algebra, Lemma 7.146.2. Thus (1), (2) and (3) of Lemma 31.11.2 hold and we win. □

Lemma 31.14.2. *The property $\mathcal{P}(S) =$``S is normal'' is local in the smooth topology.*

Proof. We will use Lemma 31.11.2. First we show ``being normal'' is local in the Zariski topology. This is clear from the definition, see Properties, Definition 23.7.1. Next, we show that if $S' \to S$ is a smooth morphism of affines and S is normal, then S' is normal. This is Algebra, Lemma 7.145.7. Finally, we show that if $S' \to S$ is a surjective smooth morphism of affines and S' is normal, then S is normal. This is Algebra, Lemma 7.146.3. Thus (1), (2) and (3) of Lemma 31.11.2 hold and we win. □

Lemma 31.14.3. *The property $\mathcal{P}(S) =$``S is locally Noetherian and (R_k)'' is local in the smooth topology.*

Proof. We will check (1), (2) and (3) of Lemma 31.11.2. As a smooth morphism is flat of finite presentation (Morphisms, Lemmas 24.33.9 and 24.33.8) we have already checked this for ``being locally Noetherian'' in the proof of Lemma 31.12.1. We will use this without further mention in the proof. First we note that $\mathcal{P}$ is local in the Zariski topology. This is clear from the definition, see Properties, Definition 23.12.1. Next, we show that if $S' \to S$ is a smooth morphism of affines and S has $\mathcal{P}$, then S' has $\mathcal{P}$. This is Algebra, Lemmas 7.145.5 (use Morphisms, Lemma 24.33.2, Algebra, Lemmas 7.126.4 and 7.129.3). Finally, we show that if $S' \to S$ is a surjective smooth morphism of affines and S' has $\mathcal{P}$, then S has $\mathcal{P}$. This is Algebra, Lemma 7.146.5. Thus (1), (2) and (3) of Lemma 31.11.2 hold and we win. $\square$

Lemma 31.14.4. *The property $\mathcal{P}(S) =$``S is regular'' is local in the smooth topology.*

Proof. This is clear from Lemma 31.14.3 above since a locally Noetherian scheme is regular if and only if it is locally Noetherian and (R_k) for all $k \geq 0$. $\square$

Lemma 31.14.5. *The property $\mathcal{P}(S) =$``S is Nagata'' is local in the smooth topology.*

Proof. We will check (1), (2) and (3) of Lemma 31.11.2. First we note that being Nagata is local in the Zariski topology. This is Properties, Lemma 23.13.6. Next, we show that if $S' \to S$ is a smooth morphism of affines and S is Nagata, then S' is Nagata. This is Morphisms, Lemma 24.17.1. Finally, we show that if $S' \to S$ is a surjective smooth morphism of affines and S' is Nagata, then S is Nagata. This is Algebra, Lemma 7.146.7. Thus (1), (2) and (3) of Lemma 31.11.2 hold and we win. $\square$

31.15. Variants on descending properties

Sometimes one can descend properties, which are not local. We put results of this kind in this section.

Lemma 31.15.1. *If $f : X \to Y$ is a flat and surjective morphism of schemes and X is reduced, then Y is reduced.*

Proof. The result follows by looking at local rings (Schemes, Definition 21.12.1) and Algebra, Lemma 7.146.2. $\square$

Lemma 31.15.2. *Let $f : X \to Y$ be a morphism of algebraic spaces. If f is locally of finite presentation, flat, and surjective and X is regular, then Y is regular.*

Proof. This lemma reduces to the following algebra statement: If $A \to B$ is a faithfully flat, finitely presentated ring homomorphism with B Noetherian and regular, then A is Noetherian and regular. We see that A is Noetherian by Algebra, Lemma 7.146.1 and regular by Algebra, Lemma 7.102.8. $\square$

31.16. Germs of schemes

Definition 31.16.1. Germs of schemes.

1. A pair (X, x) consisting of a scheme X and a point $x \in X$ is called the *germ of X at x*.
2. A *morphism of germs* $f : (X, x) \to (S, s)$ is an equivalence class of morphisms of schemes $f : U \to S$ with $f(x) = s$ where $U \subset X$ is an open neighbourhood of x. Two such f, f' are said to be equivalent if and only if f and f' agree in some open neighbourhood of x.

(3) We define the *composition of morphisms of germs* by composing representatives (this is well defined).

Before we continue we need one more definition.

Definition 31.16.2. Let $f : (X, x) \to (S, s)$ be a morphism of germs. We say f is *étale* (resp. *smooth*) if there exists a representative $f : U \to S$ of f which is an étale morphism (resp. a smooth morphism) of schemes.

31.17. Local properties of germs

Definition 31.17.1. Let $\mathcal{P}$ be a property of germs of schemes. We say that $\mathcal{P}$ is *étale local* (resp. *smooth local*) if for any étale (resp. smooth) morphism of germs $(U', u') \to (U, u)$ we have $\mathcal{P}(U, u) \Leftrightarrow \mathcal{P}(U', u')$.

Let (X, x) be a germ of a scheme. The dimension of X at x is the minimum of the dimensions of open neighbourhoods of x in X, and any small enough open neighbourhood has this dimension. Hence this is an invariant of the isomorphism class of the germ. We denote this simply $\dim_x(X)$. The following lemma tells us that the assertion $\dim_x(X) = d$ is an étale local property of germs.

Lemma 31.17.2. *Let $f : U \to V$ be an étale morphism of schemes. Let $u \in U$ and $v = f(u)$. Then* $\dim_u(U) = \dim_v(V)$.

Proof. In the statement $\dim_u(U)$ is the dimension of U at u as defined in Topology, Definition 5.7.1 as the minimum of the Krull dimensions of open neighbourhoods of u in U. Similarly for $\dim_v(V)$.

Let us show that $\dim_v(V) \geq \dim_u(U)$. Let V' be an open neighbourhood of v in V. Then there exists an open neighbourhood U' of u in U contained in $f^{-1}(V')$ such that $\dim_u(U) = \dim(U')$. Suppose that $Z_0 \subset Z_1 \subset \ldots \subset Z_n$ is a chain of irreducible closed subschemes of U'. If $\xi_i \in Z_i$ is the generic point then we have specializations $\xi_n \rightsquigarrow \xi_{n-1} \rightsquigarrow \ldots \rightsquigarrow \xi_0$. This gives specializations $f(\xi_n) \rightsquigarrow f(\xi_{n-1}) \rightsquigarrow \ldots \rightsquigarrow f(\xi_0)$ in V'. Note that $f(\xi_j) \neq f(\xi_i)$ if $i \neq j$ as the fibres of f are discrete (see Morphisms, Lemma 24.35.7). Hence we see that $\dim(V') \geq n$. The inequality $\dim_v(V) \geq \dim_u(U)$ follows formally.

Let us show that $\dim_u(U) \geq \dim_v(V)$. Let U' be an open neighbourhood of u in U. Note that $V' = f(U')$ is an open neighbourhood of v by Morphisms, Lemma 24.24.9. Hence $\dim(V') \geq \dim_v(V)$. Pick a chain $Z_0 \subset Z_1 \subset \ldots \subset Z_n$ of irreducible closed subschemes of V'. Let $\xi_i \in Z_i$ be the generic point, so we have specializations $\xi_n \rightsquigarrow \xi_{n-1} \rightsquigarrow \ldots \rightsquigarrow \xi_0$. Since $\xi_0 \in f(U')$ we can find a point $\eta_0 \in U'$ with $f(\eta_0) = \xi_0$. Consider the map of local rings

$$\mathcal{O}_{V', \xi_0} \longrightarrow \mathcal{O}_{U', \eta_0}$$

which is a flat local ring map by Morphisms, Lemma 24.35.12. Note that the points ξ_i correspond to primes of the ring on the left by Schemes, Lemma 21.13.2. Hence by going down (see Algebra, Section 7.36) for the displayed ring map we can find a sequence of specializations $\eta_n \rightsquigarrow \eta_{n-1} \rightsquigarrow \ldots \rightsquigarrow \eta_0$ in U' mapping to the sequence $\xi_n \rightsquigarrow \xi_{n-1} \rightsquigarrow \ldots \rightsquigarrow \xi_0$ under f. This implies that $\dim_u(U) \geq \dim_v(V)$. $\square$

Let (X, x) be a germ of a scheme. The isomorphism class of the local ring $\mathcal{O}_{X,x}$ is an invariant of the germ. The following lemma says that the property $\dim(\mathcal{O}_{X,x}) = d$ is an étale local property of germs.

Lemma 31.17.3. *Let $f : U \to V$ be an étale morphism of schemes. Let $u \in U$ and $v = f(u)$. Then* $\dim(\mathcal{O}_{U,u}) = \dim(\mathcal{O}_{V,v})$.

Proof. The algebraic statement we are asked to prove is the following: If $A \to B$ is an étale ring map and $\mathfrak{q}$ is a prime of B lying over $\mathfrak{p} \subset A$, then $\dim(A_{\mathfrak{p}}) = \dim(B_{\mathfrak{q}})$.

Namely, because $A_{\mathfrak{p}} \to B_{\mathfrak{q}}$ is flat we have going down, and hence the inequality $\dim(A_{\mathfrak{p}}) \leq \dim(B_{\mathfrak{q}})$. On the other hand, suppose that $\mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \ldots \subset \mathfrak{q}_n$ is a chain of primes in $B_{\mathfrak{q}}$. Then the corresponding sequence of primes $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_n$ (with $\mathfrak{p}_i = \mathfrak{q}_i \cap A_{\mathfrak{p}}$) is chain also (i.e., no equalities in the sequence) as there are no specializations among the points in a fibre of and étale morphism, see Morphisms, Lemma 24.35.7. This means that $\dim(A_{\mathfrak{p}}) \geq \dim(B_{\mathfrak{q}})$ as desired. $\square$

31.18. Properties of morphisms local on the target

Suppose that $f : X \to Y$ is a morphism of schemes. Let $g : Y' \to Y$ be a morphism of schemes. Let $f' : X' \to Y'$ be the base change of f by g:

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

Let $\mathcal{P}$ be a property of morphisms of schemes. Then we can wonder if (a) $\mathcal{P}(f) \Rightarrow \mathcal{P}(f')$, and also whether the converse (b) $\mathcal{P}(f') \Rightarrow \mathcal{P}(f)$ is true. If (a) holds whenever g is flat, then we say $\mathcal{P}$ is preserved under flat base change. If (b) holds whenever g is surjective and flat, then we say $\mathcal{P}$ descends through flat surjective base changes. If $\mathcal{P}$ is preserved under flat base changes and descends through flat surjective base changes, then we say $\mathcal{P}$ is flat local on the target. Compare with the discussion in Section 31.11. This turns out to be a very important notion which we formalize in the following definition.

Definition 31.18.1. Let $\mathcal{P}$ be a property of morphisms of schemes over a base. Let $\tau \in \{fpqc, fppf, syntomic, smooth, étale, Zariski\}$. We say $\mathcal{P}$ is *τ local on the base*, or *τ local on the target*, or *local on the base for the τ-topology* if for any τ-covering $\{Y_i \to Y\}_{i \in I}$ (see Topologies, Section 30.2) and any morphism of schemes $f : X \to Y$ over S we have

$$f \text{ has } \mathcal{P} \Leftrightarrow \text{each } Y_i \times_Y X \to Y_i \text{ has } \mathcal{P}.$$

To be sure, since isomorphisms are always coverings we see (or require) that property $\mathcal{P}$ holds for $X \to Y$ if and only if it holds for any arrow $X' \to Y'$ isomorphic to $X \to Y$. If a property is τ-local on the target then it is preserved by base changes by morphisms which occur in τ-coverings. Here is a formal statement.

Lemma 31.18.2. *Let $\tau \in \{fpqc, fppf, syntomic, smooth, étale, Zariski\}$. Let $\mathcal{P}$ be a property of morphisms which is τ local on the target. Let $f : X \to Y$ have property $\mathcal{P}$. For any morphism $Y' \to Y$ which is flat, resp. flat and locally of finite presentation, resp. syntomic, resp. étale, resp. an open immersion, the base change $f' : Y' \times_Y X \to Y'$ of f has property $\mathcal{P}$.*

Proof. This is true because we can fit $Y' \to Y$ into a family of morphisms which forms a τ-covering. $\square$

A simple often used consequence of the above is that if $f : X \to Y$ has property $\mathcal{P}$ which is τ-local on the target and $f(X) \subset V$ for some open subscheme $V \subset Y$, then also the induced morphism $X \to V$ has $\mathcal{P}$. Proof: The base change f by $V \to Y$ gives $X \to V$.

Lemma 31.18.3. *Let $\tau \in \{fppf, syntomic, smooth, étale\}$. Let $\mathcal{P}$ be a property of morphisms which is τ local on the target. For any morphism of schemes $f : X \to Y$ there exists a largest open $W(f) \subset Y$ such that the restriction $X_{W(f)} \to W(f)$ has $\mathcal{P}$. Moreover,*

(1) *if $g : Y' \to Y$ is flat and locally of finite presentation, syntomic, smooth, or étale and the base change $f' : X_{Y'} \to Y'$ has $\mathcal{P}$, then $g(Y') \subset W(f)$,*
(2) *if $g : Y' \to Y$ is flat and locally of finite presentation, syntomic, smooth, or étale, then $W(f') = g^{-1}(W(f))$, and*
(3) *if $\{g_i : Y_i \to Y\}$ is a τ-covering, then $g_i^{-1}(W(f)) = W(f_i)$, where f_i is the base change of f by $Y_i \to Y$.*

Proof. Consider the union W of the images $g(Y') \subset Y$ of morphisms $g : Y' \to Y$ with the properties:

(1) g is flat and locally of finite presentation, syntomic, smooth, or étale, and
(2) the base change $Y' \times_{g,Y} X \to Y'$ has property $\mathcal{P}$.

Since such a morphism g is open (see Morphisms, Lemma 24.24.9) we see that $W \subset Y$ is an open subset of Y. Since $\mathcal{P}$ is local in the τ topology the restriction $X_W \to W$ has property $\mathcal{P}$ because we are given a covering $\{Y' \to W\}$ of W such that the pullbacks have $\mathcal{P}$. This proves the existence and proves that $W(f)$ has property (1). To see property (2) note that $W(f') \supset g^{-1}(W(f))$ because $\mathcal{P}$ is stable under base change by flat and locally of finite presentation, syntomic, smooth, or étale morphisms, see Lemma 31.18.2. On the other hand, if $Y'' \subset Y'$ is an open such that $X_{Y''} \to Y''$ has property $\mathcal{P}$, then $Y'' \to Y$ factors through W by construction, i.e., $Y'' \subset g^{-1}(W(f))$. This proves (2). Assertion (3) follows from (2) because each morphism $Y_i \to Y$ is flat and locally of finite presentation, syntomic, smooth, or étale by our definition of a τ-covering. $\square$

Lemma 31.18.4. *Let $\mathcal{P}$ be a property of morphisms of schemes over a base. Let $\tau \in \{fpqc, fppf, étale, smooth, syntomic\}$. Assume that*

(1) *the property is preserved under flat, flat of finite presentation, étale, smooth or syntomic base change depending on whether τ is fpqc, fppf, étale, smooth, or syntomic (compare with Schemes, Definition 21.18.3),*
(2) *the property is Zariski local on the base.*
(3) *for any surjective morphism of affine schemes $S' \to S$ which is flat, flat of finite presentation, étale, smooth or syntomic depending on whether τ is fpqc, fppf, étale, smooth, or syntomic, and any morphism of schemes $f : X \to S$ property $\mathcal{P}$ holds for f if property $\mathcal{P}$ holds for the base change $f' : X' = S' \times_S X \to S'$.*

Then $\mathcal{P}$ is τ local on the base.

Proof. This follows almost immediately from the definition of a τ-covering, see Topologies, Definition 30.8.1 30.7.1 30.4.1 30.5.1, or 30.6.1 and Topologies, Lemma 30.8.8, 30.7.4, 30.4.4, 30.5.4, or 30.6.4. Details omitted. $\square$

Remark 31.18.5. (This is a repeat of Remark 31.11.3 above.) In Lemma 31.18.4 above if $\tau = smooth$ then in condition (3) we may assume that the morphism is a (surjective) standard smooth morphism. Similarly, when $\tau = syntomic$ or $\tau = étale$.

31.19. Properties of morphisms local in the fpqc topology on the target

In this section we find a large number of properties of morphisms of schemes which are local on the base in the fpqc topology.

Lemma 31.19.1. *The property $\mathcal{P}(f) =$``f is quasi-compact'' is fpqc local on the base.*

Proof. A base change of a quasi-compact morphism is quasi-compact, see Schemes, Lemma 21.19.3. Being quasi-compact is Zariski local on the base, see Schemes, Lemma 21.19.2. Finally, let $S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is quasi-compact. Then X' is quasi-compact, and $X' \to X$ is surjective. Hence X is quasi-compact. This implies that f is quasi-compact. Therefore Lemma 31.18.4 applies and we win. $\square$

Lemma 31.19.2. *The property $\mathcal{P}(f) =$``f is quasi-separated'' is fpqc local on the base.*

Proof. Any base change of a quasi-separated morphism is quasi-separated, see Schemes, Lemma 21.21.13. Being quasi-separated is Zariski local on the base (from the definition or by Schemes, Lemma 21.21.7). Finally, let $S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is quasi-separated. This means that $\Delta' : X' \to X' \times_{S'} X'$ is quasi-compact. Note that Δ' is the base change of $\Delta : X \to X \times_S X$ via $S' \to S$. By Lemma 31.19.1 this implies Δ is quasi-compact, and hence f is quasi-separated. Therefore Lemma 31.18.4 applies and we win. $\square$

Lemma 31.19.3. *The property $\mathcal{P}(f) =$``f is universally closed'' is fpqc local on the base.*

Proof. A base change of a universally closed morphism is universally closed by definition. Being universally closed is Zariski local on the base (from the definition or by Morphisms, Lemma 24.40.2). Finally, let $S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is universally closed. Let $T \to S$ be any morphism. Consider the diagram

$$\begin{tikzcd} X' \arrow[d] & S' \times_S T \times_S X \arrow[l] \arrow[r] \arrow[d] & T \times_S X \arrow[d] \\ S' & S' \times_S T \arrow[l] \arrow[r] & T \end{tikzcd}$$

in which both squares are cartesian. Thus the assumption implies that the middle vertical arrow is closed. The right horizontal arrows are flat, quasi-compact and surjective (as base changes of $S' \to S$). Hence a subset of T is closed if and only if its inverse image in $S' \times_S T$ is closed, see Morphisms, Lemma 24.24.10. An easy diagram chase shows that the right vertical arrow is closed too, and we conclude $X \to S$ is universally closed. Therefore Lemma 31.18.4 applies and we win. $\square$

Lemma 31.19.4. *The property $\mathcal{P}(f) =$``f is universally open'' is fpqc local on the base.*

Proof. The proof is the same as the proof of Lemma 31.19.3. $\square$

Lemma 31.19.5. *The property $\mathcal{P}(f) =$``f is separated'' is fpqc local on the base.*

Proof. A base change of a separated morphism is separated, see Schemes, Lemma 21.21.13. Being separated is Zariski local on the base (from the definition or by Schemes, Lemma 21.21.8). Finally, let $S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is separated. This means that $\Delta' : X' \to X' \times_{S'} X'$ is a closed immersion, hence universally closed. Note that Δ' is the base change of $\Delta : X \to X \times_S X$ via $S' \to S$. By Lemma 31.19.3 this implies Δ is universally closed. Since it is an immersion (Schemes, Lemma 21.21.2) we conclude Δ is a closed immersion. Hence f is separated. Therefore Lemma 31.18.4 applies and we win. $\square$

Lemma 31.19.6. *The property $\mathcal{P}(f)=$"f is surjective" is fpqc local on the base.*

Proof. This is clear. □

Lemma 31.19.7. *The property $\mathcal{P}(f)=$"f is universally injective" is fpqc local on the base.*

Proof. A base change of a universally injective morphism is universally injective (this is formal). Being universally injective is Zariski local on the base; this is clear from the definition. Finally, let $S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is universally injective. Let K be a field, and let $a, b : Spec(K) \to X$ be two morphisms such that $f \circ a = f \circ b$. As $S' \to S$ is surjective and by the discussion in Schemes, Section 21.13 there exists a field extension $K \subset K'$ and a morphism $Spec(K') \to S'$ such that the following solid diagram commutes

$$\begin{tikzcd}
Spec(K') \arrow[dr, dashed, "{a', b'}"'] \arrow[drr] \arrow[dd] & & \\
& X' \arrow[r] \arrow[d] & S' \arrow[d] \\
Spec(K) \arrow[r, "{a,b}"] & X \arrow[r] & S
\end{tikzcd}$$

As the square is cartesian we get the two dotted arrows a', b' making the diagram commute. Since $X' \to S'$ is universally injective we get $a' = b'$, by Morphisms, Lemma 24.10.2. Clearly this forces $a = b$ (by the discussion in Schemes, Section 21.13). Therefore Lemma 31.18.4 applies and we win.

An alternative proof would be to use the characterization of a universally injective morphism as one whose diagonal is surjective, see Morphisms, Lemma 24.10.2. The lemma then follows from the fact that the property of being surjective is fpqc local on the base, see Lemma 31.19.6. (Hint: use that the base change of the diagonal is the diagonal of the base change.) □

Lemma 31.19.8. *The property $\mathcal{P}(f)=$"f is locally of finite type" is fpqc local on the base.*

Proof. Being locally of finite type is preserved under base change, see Morphisms, Lemma 24.14.4. Being locally of finite type is Zariski local on the base, see Morphisms, Lemma 24.14.2. Finally, let $S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is locally of finite type. Let $U \subset X$ be an affine open. Then $U' = S' \times_S U$ is affine and of finite type over S'. Write $S = Spec(R)$, $S' = Spec(R')$, $U = Spec(A)$, and $U' = Spec(A')$. We know that $R \to R'$ is faithfully flat, $A' = R' \otimes_R A$ and $R' \to A'$ is of finite type. We have to show that $R \to A$ is of finite type. This is the result of Algebra, Lemma 7.117.1. It follows that f is locally of finite type. Therefore Lemma 31.18.4 applies and we win. □

Lemma 31.19.9. *The property $\mathcal{P}(f)=$"f is locally of finite presentation" is fpqc local on the base.*

Proof. Being locally of finite presentation is preserved under base change, see Morphisms, Lemma 24.20.4. Being locally of finite type is Zariski local on the base, see Morphisms, Lemma 24.20.2. Finally, let $S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is locally of finite presentation. Let $U \subset X$ be an affine open. Then $U' = S' \times_S U$ is affine and of finite type over S'. Write $S = Spec(R)$, $S' = Spec(R')$, $U = Spec(A)$, and $U' = Spec(A')$. We know that $R \to R'$ is faithfully flat, $A' = R' \otimes_R A$ and $R' \to A'$ is of finite presentation. We have to show that $R \to A$ is of finite presentation. This is the result of Algebra, Lemma 7.117.2. It follows that f is locally of finite presentation. Therefore Lemma 31.18.4 applies and we win. □

Lemma 31.19.10. *The property $\mathcal{P}(f) =$``f is of finite type'' is fpqc local on the base.*

Proof. Combine Lemmas 31.19.1 and 31.19.8. □

Lemma 31.19.11. *The property $\mathcal{P}(f) =$``f is of finite presentation'' is fpqc local on the base.*

Proof. Combine Lemmas 31.19.1, 31.19.2 and 31.19.9. □

Lemma 31.19.12. *The property $\mathcal{P}(f) =$``f is proper'' is fpqc local on the base.*

Proof. The lemma follows by combining Lemmas 31.19.3, 31.19.5 and 31.19.10. □

Lemma 31.19.13. *The property $\mathcal{P}(f) =$``f is flat'' is fpqc local on the base.*

Proof. Being flat is preserved under arbitrary base change, see Morphisms, Lemma 24.24.7. Being flat is Zariski local on the base by definition. Finally, let $S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is flat. Let $U \subset X$ be an affine open. Then $U' = S' \times_S U$ is affine and of finite type over S'. Write $S = Spec(R)$, $S' = Spec(R')$, $U = Spec(A)$, and $U' = Spec(A')$. We know that $R \to R'$ is faithfully flat, $A' = R' \otimes_R A$ and $R' \to A'$ is flat. Goal: Show that $R \to A$ is flat. This follows immediately from Algebra, Lemma 7.35.7. Hence f is flat. Therefore Lemma 31.18.4 applies and we win. □

Lemma 31.19.14. *The property $\mathcal{P}(f) =$``f is an open immersion'' is fpqc local on the base.*

Proof. The property of being an open immersion is stable under base change, see Schemes, Lemma 21.18.2. The property of being an open immersion is Zariski local on the base (this is obvious). Finally, let $S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is an open immersion. Then f' is universally open, and universally injective. Hence we conclude that f is universally open by Lemma 31.19.4, and universally injective by Lemma 31.19.7. In particular $f(X) \subset S$ is open, and we may replace S by $f(S)$ and assume that f is surjective. This implies that f' is an isomorphism and we have to show that f is an isomorphism also. Since f is universally injective we see that f is bijective. Hence f is a homeomorphism. Let $x \in X$ and choose $U \subset X$ an affine open neighbourhood of x. Since $f(U) \subset S$ is open, and S is affine we may choose a standard open $D(g) \subset f(U)$ containing $f(x)$ where $g \in \Gamma(S, \mathcal{O}_S)$. It is clear that $U \cap f^{-1}(D(g))$ is still affine and still an open neighbourhood of x. Replace U by $U \cap f^{-1}(D(g))$ and write $V = D(g) \subset S$ and V' the inverse image of V in S'. Note that V' is a standard open of S' as well and in particular that V' is affine. Since f' is an isomorphism we have $V' \times_V U \to V'$ is an isomorphism. In terms of rings this means that

$$\mathcal{O}(V') \longrightarrow \mathcal{O}(V') \otimes_{\mathcal{O}(V)} \mathcal{O}(U)$$

is an isomorphism. Since $\mathcal{O}(V) \to \mathcal{O}(V')$ is faithfully flat this implies that $\mathcal{O}(V) \to \mathcal{O}(U)$ is an isomorphism. Hence $U \cong V$ and we see that f is an isomorphism. Therefore Lemma 31.18.4 applies and we win. □

Lemma 31.19.15. *The property $\mathcal{P}(f) =$``f is an isomorphism" is fpqc local on the base.*

Proof. Combine Lemmas 31.19.6 and 31.19.14. □

Lemma 31.19.16. *The property $\mathcal{P}(f) =$``f is affine" is fpqc local on the base.*

Proof. A base change of an affine morphism is affine, see Morphisms, Lemma 24.11.8. Being affine is Zariski local on the base, see Morphisms, Lemma 24.11.3. Finally, let $g : S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is affine. In other words, X' is affine, say $X' = Spec(A')$. Also write $S = Spec(R)$ and $S' = Spec(R')$. We have to show that X is affine.

By Lemmas 31.19.1 and 31.19.5 we see that $X \to S$ is separated and quasi-compact. Thus $f_*\mathcal{O}_X$ is a quasi-coherent sheaf of $\mathcal{O}_S$-algebras, see Schemes, Lemma 21.24.1. Hence $f_*\mathcal{O}_X = \widetilde{A}$ for some R-algebra A. In fact $A = \Gamma(X, \mathcal{O}_X)$ of course. Also, by flat base change (see for example Coherent, Lemma 25.6.2) we have $g^*f_*\mathcal{O}_X = f'_*\mathcal{O}_{X'}$. In other words, we have $A' = R' \otimes_R A$. Consider the canonical morphism

$$X \longrightarrow Spec(A)$$

over S from Schemes, Lemma 21.6.4. By the above the base change of this morphism to S' is an isomorphism. Hence it is an isomorphism by Lemma 31.19.15. Therefore Lemma 31.18.4 applies and we win. □

Lemma 31.19.17. *The property $\mathcal{P}(f) =$``f is a closed immersion" is fpqc local on the base.*

Proof. Let $f : X \to Y$ be a morphism of schemes. Let $\{Y_i \to Y\}$ be an fpqc covering. Assume that each $f_i : Y_i \times_Y X \to Y_i$ is a closed immersion. This implies that each f_i is affine, see Morphisms, Lemma 24.11.9. By Lemma 31.19.16 we conclude that f is affine. It remains to show that $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective. For every $y \in Y$ there exists an i and a point $y_i \in Y_i$ mapping to y. By Coherent, Lemma 25.6.2 the sheaf $f_{i,*}(\mathcal{O}_{Y_i \times_Y X})$ is the pullback of $f_*\mathcal{O}_X$. By assumption it is a quotient of $\mathcal{O}_{Y_i}$. Hence we see that

$$\left(\mathcal{O}_{Y,y} \longrightarrow (f_*\mathcal{O}_X)_y\right) \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{Y_i,y_i}$$

is surjective. Since $\mathcal{O}_{Y_i,y_i}$ is faithfully flat over $\mathcal{O}_{Y,y}$ this implies the surjectivity of $\mathcal{O}_{Y,y} \longrightarrow (f_*\mathcal{O}_X)_y$ as desired. □

Lemma 31.19.18. *The property $\mathcal{P}(f) =$``f is quasi-affine" is fpqc local on the base.*

Proof. Let $f : X \to Y$ be a morphism of schemes. Let $\{g_i : Y_i \to Y\}$ be an fpqc covering. Assume that each $f_i : Y_i \times_Y X \to Y_i$ is quasi-affine. This implies that each f_i is quasi-compact and separated. By Lemmas 31.19.1 and 31.19.5 this implies that f is quasi-compact and separated. Consider the sheaf of $\mathcal{O}_Y$-algebras $\mathcal{A} = f_*\mathcal{O}_X$. By Schemes, Lemma 21.24.1 it is a quasi-coherent $\mathcal{O}_Y$-algebra. Consider the canonical morphism

$$j : X \longrightarrow \underline{Spec}_Y(\mathcal{A})$$

see Constructions, Lemma 22.4.7. By flat base change (see for example Coherent, Lemma 25.6.2) we have $g_i^*f_*\mathcal{O}_X = f_{i,*}\mathcal{O}_{X'}$ where $g_i : Y_i \to Y$ are the given flat maps. Hence the base change j_i of j by g_i is the canonical morphism of Constructions, Lemma 22.4.7 for the morphism f_i. By assumption and Morphisms, Lemma 24.12.3 all of these morphisms j_i are quasi-compact open immersions. Hence, by Lemmas 31.19.1 and 31.19.14 we see that j is a quasi-compact open immersion. Hence by Morphisms, Lemma 24.12.3 again we conclude that f is quasi-affine. □

Lemma 31.19.19. *The property $\mathcal{P}(f) =$``f is a quasi-compact immersion" is fpqc local on the base.*

Proof. Let $f : X \to Y$ be a morphism of schemes. Let $\{Y_i \to Y\}$ be an fpqc covering. Write $X_i = Y_i \times_Y X$ and $f_i : X_i \to Y_i$ the base change of f. Also denote $q_i : Y_i \to Y$ the given flat morphisms. Assume each f_i is a quasi-compact immersion. By Schemes, Lemma 21.23.7 each f_i is separated. By Lemmas 31.19.1 and 31.19.5 this implies that f is quasi-compact and separated. Let $X \to Z \to Y$ be the factorization of f through its scheme theoretic image. By Morphisms, Lemma 24.4.3 the closed subscheme $Z \subset Y$ is cut out by the quasi-coherent sheaf of ideals $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_Y \to f_*\mathcal{O}_X)$ as f is quasi-compact. By flat base change (see for example Coherent, Lemma 25.6.2; here we use f is separated) we see $f_{i,*}\mathcal{O}_{X_i}$ is the pullback $q_i^*f_*\mathcal{O}_X$. Hence $Y_i \times_Y Z$ is cut out by the quasi-coherent sheaf of ideals $q_i^*\mathcal{I} = \mathrm{Ker}(\mathcal{O}_{Y_i} \to f_{i,*}\mathcal{O}_{X_i})$. By Morphisms, Lemma 24.5.7 the morphisms $X_i \to Y_i \times_Y Z$ are open immersions. Hence by Lemma 31.19.14 we see that $X \to Z$ is an open immersion and hence f is a immersion as desired (we already saw it was quasi-compact). $\square$

Lemma 31.19.20. *The property $\mathcal{P}(f) =$``f is integral" is fpqc local on the base.*

Proof. An integral morphism is the same thing as an affine, universally closed morphism. See Morphisms, Lemma 24.42.7. Hence the lemma follows on combining Lemmas 31.19.3 and 31.19.16. $\square$

Lemma 31.19.21. *The property $\mathcal{P}(f) =$``f is finite" is fpqc local on the base.*

Proof. An finite morphism is the same thing as an integral, morphism which is locally of finite type. See Morphisms, Lemma 24.42.4. Hence the lemma follows on combining Lemmas 31.19.8 and 31.19.20. $\square$

Lemma 31.19.22. *The properties $\mathcal{P}(f) =$``$f$ is locally quasi-finite" and $\mathcal{P}(f) =$``f is quasi-finite" are fpqc local on the base.*

Proof. Let $f : X \to S$ be a morphism of schemes, and let $\{S_i \to S\}$ be an fpqc covering such that each base change $f_i : X_i \to S_i$ is locally quasi-finite. We have already seen (Lemma 31.19.8) that ``locally of finite type" is fpqc local on the base, and hence we see that $f$ is locally of finite type. Then it follows from Morphisms, Lemma 24.19.13 that $f$ is locally quasi-finite. The quasi-finite case follows as we have already seen that ``quasi-compact" is fpqc local on the base (Lemma 31.19.1). $\square$

Lemma 31.19.23. *The property $\mathcal{P}(f) =$``f is locally of finite type of relative dimension d" is fpqc local on the base.*

Proof. This follows immediately from the fact that being locally of finite type is fpqc local on the base and Morphisms, Lemma 24.27.3. $\square$

Lemma 31.19.24. *The property $\mathcal{P}(f) =$``f is syntomic" is fpqc local on the base.*

Proof. A morphism is syntomic if and only if it is locally of finite presentation, flat, and has locally complete intersections as fibres. We have seen already that being flat and locally of finite presentation are fpqc local on the base (Lemmas 31.19.13, and 31.19.9). Hence the result follows for syntomic from Morphisms, Lemma 24.30.12. $\square$

Lemma 31.19.25. *The property $\mathcal{P}(f) =$``f is smooth" is fpqc local on the base.*

Proof. A morphism is smooth if and only if it is locally of finite presentation, flat, and has smooth fibres. We have seen already that being flat and locally of finite presentation are fpqc local on the base (Lemmas 31.19.13, and 31.19.9). Hence the result follows for smooth from Morphisms, Lemma 24.33.15. □

Lemma 31.19.26. *The property $\mathcal{P}(f) =$``$f$ is unramified'' is fpqc local on the base. The property $\mathcal{P}(f) =$``f is G-unramified'' is fpqc local on the base.*

Proof. A morphism is unramified (resp. G-unramified) if and only if it is locally of finite type (resp. finite presentation) and its diagonal morphism is an open immersion (see Morphisms, Lemma 24.34.13). We have seen already that being locally of finite type (resp. locally of finite presentation) and an open immersion is fpqc local on the base (Lemmas 31.19.9, 31.19.8, and 31.19.14). Hence the result follows formally. □

Lemma 31.19.27. *The property $\mathcal{P}(f) =$``f is étale'' is fpqc local on the base.*

Proof. A morphism is étale if and only if it flat and G-unramified. See Morphisms, Lemma 24.35.16. We have seen already that being flat and G-unramified are fpqc local on the base (Lemmas 31.19.13, and 31.19.26). Hence the result follows. □

Lemma 31.19.28. *The property $\mathcal{P}(f) =$``$f$ is finite locally free'' is fpqc local on the base. Let $d \geq 0$. The property $\mathcal{P}(f) =$``f is finite locally free of degree d'' is fpqc local on the base.*

Proof. Being finite locally free is equivalent to being finite, flat and locally of finite presentation (Morphisms, Lemma 24.44.2). Hence this follows from Lemmas 31.19.21, 31.19.13, and 31.19.9. If $f : Z \to U$ is finite locally free, and $\{U_i \to U\}$ is a surjective family of morphisms such that each pullback $Z \times_U U_i \to U_i$ has degree d, then $Z \to U$ has degree d, for example because we can read off the degree in a point $u \in U$ from the fibre $(f_*\mathcal{O}_Z)_u \otimes_{\mathcal{O}_{U,u}} \kappa(u)$. □

Lemma 31.19.29. *The property $\mathcal{P}(f) =$``f is a monomorphism'' is fpqc local on the base.*

Proof. Let $f : X \to S$ be a morphism of schemes. Let $\{S_i \to S\}$ be an fpqc covering, and assume each of the base changes $f_i : X_i \to S_i$ of f is a monomorphism. Let $a, b : T \to X$ be two morphisms such that $f \circ a = f \circ b$. We have to show that $a = b$. Since f_i is a monomorphism we see that $a_i = b_i$, where $a_i, b_i : S_i \times_S T \to X_i$ are the base changes. In particular the compositions $S_i \times_S T \to T \to X$ are equal. Since $\coprod S_i \times_S T \to T$ is an epimorphism (see e.g. Lemma 31.9.3) we conclude $a = b$. □

Lemma 31.19.30. *The properties*

$\mathcal{P}(f) =$``f is a Koszul-regular immersion'',
$\mathcal{P}(f) =$``f is an H_1-regular immersion'', and
$\mathcal{P}(f) =$``f is a quasi-regular immersion''

are fpqc local on the base.

Proof. We will use the criterion of Lemma 31.18.4 to prove this. By Divisors, Definition 26.13.1 being a Koszul-regular (resp. H_1-regular, quasi-regular) immersion is Zariski local on the base. By Divisors, Lemma 26.13.4 being a Koszul-regular (resp. H_1-regular, quasi-regular) immersion is preserved under flat base change. The final hypothesis (3) of Lemma 31.18.4 translates into the following algebra statement: Let $A \to B$ be a faithfully flat ring map. Let $I \subset A$ be an ideal. If IB is locally on $Spec(B)$ generated by a Koszul-regular (resp. H_1-regular, quasi-regular) sequence in B, then $I \subset A$ is locally on $Spec(A)$ generated by a Koszul-regular (resp. H_1-regular, quasi-regular) sequence in A. This is More on Algebra, Lemma 12.23.3. □

31.20. Properties of morphisms local in the fppf topology on the target

In this section we find some properties of morphisms of schemes for which we could not (yet) show they are local on the base in the fpqc topology which, however, are local on the base in the fppf topology.

Lemma 31.20.1. *The property $\mathcal{P}(f) =$``f is an immersion'' is fppf local on the base.*

Proof. The property of being an immersion is stable under base change, see Schemes, Lemma 21.18.2. The property of being an immersion is Zariski local on the base. Finally, let $\pi : S' \to S$ be a surjective morphism of affine schemes, which is flat and locally of finite presentation. Note that $\pi : S' \to S$ is open by Morphisms, Lemma 24.24.9. Let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is an immersion. In particular we see that $f'(X') = \pi^{-1}(f(X))$ is locally closed. Hence by Topology, Lemma 5.15.2 we see that $f(X) \subset S$ is locally closed. Let $Z \subset S$ be the closed subset $Z = \overline{f(X)} \setminus f(X)$. By Topology, Lemma 5.15.2 again we see that $f'(X')$ is closed in $S' \setminus Z'$. Hence we may apply Lemma 31.19.17 to the fpqc covering $\{S' \setminus Z' \to S \setminus Z\}$ and conclude that $f : X \to S \setminus Z$ is a closed immersion. In other words, f is an immersion. Therefore Lemma 31.18.4 applies and we win. □

31.21. Application of fpqc descent of properties of morphisms

The following lemma may seem a bit frivolous but turns out is a useful tool in studying étale and unramified morphisms.

Lemma 31.21.1. *Let $f : X \to Y$ be a flat, quasi-compact, surjective monomorphism. Then f is an isomorphism.*

Proof. As f is a flat, quasi-compact, surjective morphism we see $\{X \to Y\}$ is an fpqc covering of Y. The diagonal $\Delta : X \to X \times_Y X$ is an isomorphism. This implies that the base change of f by f is an isomorphism. Hence we see f is an isomorphism by Lemma 31.19.15. □

We can use this lemma to show the following important result. We will discuss this and related results in more detail in Étale Morphisms, Section 37.14.

Lemma 31.21.2. *A universally injective étale morphism is an open immersion.*

Proof. Let $f : X \to Y$ be an étale morphism which is universally injective. Then f is open (Morphisms, Lemma 24.35.13) hence we can replace Y by $f(X)$ and we may assume that f is surjective. Then f is bijective and open hence a homeomorphism. Hence f is quasi-compact. Thus by Lemma 31.21.1 it suffices to show that f is a monomorphism. As $X \to Y$ is étale the morphism $\Delta_{X/Y} : X \to X \times_Y X$ is an open immersion by Morphisms, Lemma 24.34.13 (and Morphisms, Lemma 24.35.16). As f is universally injective $\Delta_{X/Y}$ is also surjective, see Morphisms, Lemma 24.10.2. Hence $\Delta_{X/Y}$ is an isomorphism, i.e., $X \to Y$ is a monomorphism. □

31.22. Properties of morphisms local on the source

It often happens one can prove a morphism has a certain property after precomposing with some other morphism. In many cases this implies the morphism has the property too. We formalize this in the following definition.

Definition 31.22.1. Let $\mathcal{P}$ be a property of morphisms of schemes. Let $\tau \in \{Zariski, fpqc, fppf, étale, smooth, syntomic\}$. We say $\mathcal{P}$ is *τ local on the source*, or *local on the source for the τ-topology* if for any morphism of schemes $f : X \to Y$ over S, and any τ-covering $\{X_i \to X\}_{i \in I}$ we have

$$f \text{ has } \mathcal{P} \Leftrightarrow \text{each } X_i \to Y \text{ has } \mathcal{P}.$$

To be sure, since isomorphisms are always coverings we see (or require) that property $\mathcal{P}$ holds for $X \to Y$ if and only if it holds for any arrow $X' \to Y'$ isomorphic to $X \to Y$. If a property is τ-local on the source then it is preserved by precomposing with morphisms which occur in τ-coverings. Here is a formal statement.

Lemma 31.22.2. *Let $\tau \in \{fpqc, fppf, syntomic, smooth, étale, Zariski\}$. Let $\mathcal{P}$ be a property of morphisms which is τ local on the source. Let $f : X \to Y$ have property $\mathcal{P}$. For any morphism $a : X' \to X$ which is flat, resp. flat and locally of finite presentation, resp. syntomic, resp. étale, resp. an open immersion, the composition $f \circ a : X' \to Y$ has property $\mathcal{P}$.*

Proof. This is true because we can fit $X' \to X$ into a family of morphisms which forms a τ-covering. □

Lemma 31.22.3. *Let $\mathcal{P}$ be a property of morphisms of schemes. Let $\tau \in \{fpqc, fppf, étale, smooth, syntomic\}$. Assume that*

(1) *the property is preserved under precomposing with flat, flat locally of finite presentation, étale, smooth or syntomic morphisms depending on whether τ is fpqc, fppf, étale, smooth, or syntomic,*
(2) *the property is Zariski local on the source,*
(3) *the property is Zariski local on the target,*
(4) *for any morphism of affine schemes $X \to Y$, and any surjective morphism of affine schemes $X' \to X$ which is flat, flat of finite presentation, étale, smooth or syntomic depending on whether τ is fpqc, fppf, étale, smooth, or syntomic, property $\mathcal{P}$ holds for f if property $\mathcal{P}$ holds for the composition $f' : X' \to Y$.*

Then $\mathcal{P}$ is τ local on the source.

Proof. This follows almost immediately from the definition of a τ-covering, see Topologies, Definition 30.8.1 30.7.1 30.4.1 30.5.1, or 30.6.1 and Topologies, Lemma 30.8.8, 30.7.4, 30.4.4, 30.5.4, or 30.6.4. Details omitted. (Hint: Use locality on the source and target to reduce the verification of property $\mathcal{P}$ to the case of a morphism between affines. Then apply (1) and (4).) □

Remark 31.22.4. (This is a repeat of Remarks 31.11.3 and 31.18.5 above.) In Lemma 31.22.3 above if $\tau = smooth$ then in condition (4) we may assume that the morphism is a (surjective) standard smooth morphism. Similarly, when $\tau = syntomic$ or $\tau = étale$.

31.23. Properties of morphisms local in the fpqc topology on the source

Here are some properties of morphisms that are fpqc local on the source.

Lemma 31.23.1. *The property $\mathcal{P}(f)$ =``f is flat'' is fpqc local on the source.*

Proof. Since flatness is defined in terms of the maps of local rings (Morphisms, Definition 24.24.1) what has to be shown is the following algebraic fact: Suppose $A \to B \to C$ are local homomorphisms of local rings, and assume $B \to C$ are flat. Then $A \to B$ is flat if and only if $A \to C$ is flat. If $A \to B$ is flat, then $A \to C$ is flat by Algebra, Lemma 7.35.3. Conversely, assume $A \to C$ is flat. Note that $B \to C$ is faithfully flat, see Algebra, Lemma 7.35.16. Hence $A \to B$ is flat by Algebra, Lemma 7.35.9. (Also see Morphisms, Lemma 24.24.11 for a direct proof.) □

Lemma 31.23.2. *Then property $\mathcal{P}(f : X \to Y) =$``for every $x \in X$ the map of local rings $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is injective'' is fpqc local on the source.*

Proof. Omitted. This is just a (probably misguided) attempt to be playful. □

31.24. Properties of morphisms local in the fppf topology on the source

Here are some properties of morphisms that are fppf local on the source.

Lemma 31.24.1. *The property $\mathcal{P}(f) =$``f is locally of finite presentation'' is fppf local on the source.*

Proof. Being locally of finite presentation is Zariski local on the source and the target, see Morphisms, Lemma 24.20.2. It is a property which is preserved under composition, see Morphisms, Lemma 24.20.3. This proves (1), (2) and (3) of Lemma 31.22.3. The final condition (4) is Lemma 31.10.1. Hence we win. □

Lemma 31.24.2. *The property $\mathcal{P}(f) =$``f is locally of finite type'' is fppf local on the source.*

Proof. Being locally of finite type is Zariski local on the source and the target, see Morphisms, Lemma 24.14.2. It is a property which is preserved under composition, see Morphisms, Lemma 24.14.3, and a flat morphism locally of finite presentation is locally of finite type, see Morphisms, Lemma 24.20.8. This proves (1), (2) and (3) of Lemma 31.22.3. The final condition (4) is Lemma 31.10.2. Hence we win. □

Lemma 31.24.3. *The property $\mathcal{P}(f) =$``f is open'' is fppf local on the source.*

Proof. Being an open morphism is clearly Zariski local on the source and the target. It is a property which is preserved under composition, see Morphisms, Lemma 24.22.3, and a flat morphism of finite presentation is open, see Morphisms, Lemma 24.24.9 This proves (1), (2) and (3) of Lemma 31.22.3. The final condition (4) follows from Morphisms, Lemma 24.24.10. Hence we win. □

Lemma 31.24.4. *The property $\mathcal{P}(f) =$``f is universally open'' is fppf local on the source.*

Proof. Let $f : X \to Y$ be a morphism of schemes. Let $\{X_i \to X\}_{i \in I}$ be an fppf covering. Denote $f_i : X_i \to X$ the compositions. We have to show that f is universally open if and only if each f_i is universally open. If f is universally open, then also each f_i is universally open since the maps $X_i \to X$ are universally open and compositions of universally open morphisms are universally open (Morphisms, Lemmas 24.24.9 and 24.22.3). Conversely, assume each f_i is universally open. Let $Y' \to Y$ be a morphism of schemes. Denote $X' = Y' \times_Y X$ and $X'_i = Y' \times_Y X_i$. Note that $\{X'_i \to X'\}_{i \in I}$ is an fppf covering also. The morphisms $f'_i : X'_i \to Y'$ are open by assumption. Hence by the Lemma 31.24.3 above we conclude that $f' : X' \to Y'$ is open as desired. □

31.25. Properties of morphisms local in the syntomic toplogy on the source

Here are some properties of morphisms that are syntomic local on the source.

Lemma 31.25.1. *The property $\mathcal{P}(f)$ =``f is syntomic'' is syntomic local on the source.*

Proof. Combine Lemma 31.22.3 with Morphisms, Lemma 24.30.2 (local for Zariski on source and target), Morphisms, Lemma 24.30.3 (pre-composing), and Lemma 31.10.4 (part (4)). □

31.26. Properties of morphisms local in the smooth topology on the source

Here are some properties of morphisms that are smooth local on the source.

Lemma 31.26.1. *The property $\mathcal{P}(f)$ =``f is smooth'' is smooth local on the source.*

Proof. Combine Lemma 31.22.3 with Morphisms, Lemma 24.33.2 (local for Zariski on source and target), Morphisms, Lemma 24.33.4 (pre-composing), and Lemma 31.10.4 (part (4)). □

31.27. Properties of morphisms local in the étale topology on the source

Here are some properties of morphisms that are étale local on the source.

Lemma 31.27.1. *The property $\mathcal{P}(f)$ =``f is étale'' is étale local on the source.*

Proof. Combine Lemma 31.22.3 with Morphisms, Lemma 24.35.2 (local for Zariski on source and target), Morphisms, Lemma 24.35.3 (pre-composing), and Lemma 31.10.4 (part (4)). □

Lemma 31.27.2. *The property $\mathcal{P}(f)$ =``f is locally quasi-finite'' is étale local on the source.*

Proof. We are going to use Lemma 31.22.3. By Morphisms, Lemma 24.19.11 the property of being locally quasi-finite is local for Zariski on source and target. By Morphisms, Lemmas 24.19.12 and 24.35.6 we see the precomposition of a locally quasi-finite morphism by an étale morphism is locally quasi-finite. Finally, suppose that $X \to Y$ is a morphism of affine schemes and that $X' \to X$ is a surjective étale morphism of affine schemes such that $X' \to Y$ is locally quasi-finite. Then $X' \to Y$ is of finite type, and by Lemma 31.10.2 we see that $X \to Y$ is of finite type also. Moreover, by assumption $X' \to Y$ has finite fibres, and hence $X \to Y$ has finite fibres also. We conclude that $X \to Y$ is quasi-finite by Morphisms, Lemma 24.19.10. This proves the last assumption of Lemma 31.22.3 and finishes the proof. □

Lemma 31.27.3. *The property $\mathcal{P}(f)$ =``$f$ is unramified'' is étale local on the source. The property $\mathcal{P}(f)$ =``f is G-unramified'' is étale local on the source.*

Proof. We are going to use Lemma 31.22.3. By Morphisms, Lemma 24.34.3 the property of being unramified (resp. G-unramified) is local for Zariski on source and target. By Morphisms, Lemmas 24.34.4 and 24.35.5 we see the precomposition of an unramified (resp. G-unramified) morphism by an étale morphism is unramified (resp. G-unramified). Finally, suppose that $X \to Y$ is a morphism of affine schemes and that $f : X' \to X$ is a surjective étale morphism of affine schemes such that $X' \to Y$ is unramified (resp. G-unramified). Then $X' \to Y$ is of finite type (resp. finite presentation), and by Lemma 31.10.2 (resp. Lemma 31.10.1) we see that $X \to Y$ is of finite type (resp. finite presentation) also. By Morphisms, Lemma 24.33.16 we have a short exact sequence

$$0 \to f^*\Omega_{X/Y} \to \Omega_{X'/Y} \to \Omega_{X'/X} \to 0.$$

As $X' \to Y$ is unramified we see that the middle term is zero. Hence, as f is faithfully flat we see that $\Omega_{X/Y} = 0$. Hence $X \to Y$ is unramified (resp. G-unramified), see Morphisms, Lemma 24.34.2. This proves the last assumption of Lemma 31.22.3 and finishes the proof. □

31.28. Properties of morphisms étale local on source-and-target

Let $\mathcal{P}$ be a property of morphisms of schemes. There is an intuitive meaning to the phrase ``$\mathcal{P}$ is étale local on the source and target". However, it turns out that this notion is not the same as asking $\mathcal{P}$ to be both étale local on the source and étale local on the target. Before we discuss this further we give two silly examples.

Example 31.28.1. Consider the property $\mathcal{P}$ of morphisms of schemes defined by the rule $\mathcal{P}(X \to Y) =$``Y is locally Noetherian". The reader can verify that this is étale local on the source and étale local on the target (omitted, see Lemma 31.12.1). But it is **not** true that if $f : X \to Y$ has $\mathcal{P}$ and $g : Y \to Z$ is étale, then $g \circ f$ has $\mathcal{P}$. Namely, f could be the identity on Y and g could be an open immersion of a locally Noetherian scheme Y into a non locally Noetherian scheme Z.

The following example is in some sense worse.

Example 31.28.2. Consider the property $\mathcal{P}$ of morphisms of schemes defined by the rule $\mathcal{P}(f : X \to Y) =$``for every $y \in Y$ which is a specialization of some $f(x)$, $x \in X$ the local ring $\mathcal{O}_{Y,y}$ is Noetherian". Let us verify that this is étale local on the source and étale local on the target. We will freely use Schemes, Lemma 21.13.2.

Local on the target: Let $\{g_i : Y_i \to Y\}$ be an étale covering. Let $f_i : X_i \to Y_i$ be the base change of f, and denote $h_i : X_i \to X$ the projection. Assume $\mathcal{P}(f)$. Let $f(x_i) \rightsquigarrow y_i$ be a specialization. Then $f(h_i(x_i)) \rightsquigarrow g_i(y_i)$ so $\mathcal{P}(f)$ implies $\mathcal{O}_{Y,g_i(y_i)}$ is Noetherian. Also $\mathcal{O}_{Y,g_i(y_i)} \to \mathcal{O}_{Y_i,y_i}$ is a localization of an étale ring map. Hence $\mathcal{O}_{Y_i,y_i}$ is Noetherian by Algebra, Lemma 7.28.1. Conversely, assume $\mathcal{P}(f_i)$ for all i. Let $f(x) \rightsquigarrow y$ be a specialization. Choose an i and $y_i \in Y_i$ mapping to y. Since x can be viewed as a point of $Spec(\mathcal{O}_{Y,y}) \times_Y X$ and $\mathcal{O}_{Y,y} \to \mathcal{O}_{Y_i,y_i}$ is faithfully flat, there exists a point $x_i \in Spec(\mathcal{O}_{Y_i,y_i}) \times_Y X$ mapping to x. Then $x_i \in X_i$, and $f_i(x_i)$ specializes to y_i. Thus we see that $\mathcal{O}_{Y_i,y_i}$ is Noetherian by $\mathcal{P}(f_i)$ which implies that $\mathcal{O}_{Y,y}$ is Noetherian by Algebra, Lemma 7.146.1.

Local on the source: Let $\{h_i : X_i \to X\}$ be an étale covering. Let $f_i : X_i \to Y$ be the composition $f \circ h_i$. Assume $\mathcal{P}(f)$. Let $f(x_i) \rightsquigarrow y$ be a specialization. Then $f(h_i(x_i)) \rightsquigarrow y$ so $\mathcal{P}(f)$ implies $\mathcal{O}_{Y,y}$ is Noetherian. Thus $\mathcal{P}(f_i)$ holds. Conversely, assume $\mathcal{P}(f_i)$ for all i. Let $f(x) \rightsquigarrow y$ be a specialization. Choose an i and $x_i \in X_i$ mapping to x. Then y is a specialization of $f_i(x_i) = f(x)$. Hence $\mathcal{P}(f_i)$ implies $\mathcal{O}_{Y,y}$ is Noetherian as desired.

We claim that there exists a commutative diagram

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "a"'] & V \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with surjective étale vertical arrows, such that h has $\mathcal{P}$ and f does not have $\mathcal{P}$. Namely, let

$$Y = Spec\left(\mathbf{C}[x_n; n \in \mathbf{Z}]/(x_n x_m; n \neq m)\right)$$

and let $X \subset Y$ be the open subscheme which is the complement of the point all of whose coordinates $x_n = 0$. Let $U = X$, let $V = X \coprod Y$, let a, b the obvious map, and let $h : U \to V$

be the inclusion of $U = X$ into the first summand of V. The claim above holds because U is locally Noetherian, but Y is not.

What should be the correct notion of a property which is étale local on the source-and-target? We think that, by analogy with Morphisms, Definition 24.13.1 it should be the following.

Definition 31.28.3. Let $\mathcal{P}$ be a property of morphisms of schemes. We say $\mathcal{P}$ is *étale local on source-and-target* if

(1) (stable under precomposing with étale maps) if $f : X \to Y$ is étale and $g : Y \to Z$ has $\mathcal{P}$, then $g \circ f$ has $\mathcal{P}$,
(2) (stable under étale base change) if $f : X \to Y$ has $\mathcal{P}$ and $Y' \to Y$ is étale, then the base change $f' : Y' \times_Y X \to Y'$ has $\mathcal{P}$, and
(3) (locality) given a morphism $f : X \to Y$ the following are equivalent
    (a) f has $\mathcal{P}$,
    (b) for every $x \in X$ there exists a commutative diagram

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "a"'] & V \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

    with étale vertical arrows and $u \in U$ with $a(u) = x$ such that h has $\mathcal{P}$.

It turns out this definition excludes the behavior seen in Examples 31.28.1 and 31.28.2. We will compare this to the definition in the paper [**DM69a**] by Deligne and Mumford in Remark 31.28.8. Moreover, a property which is étale local on the source-and-target is étale local on the source and étale local on the target. Finally, the converse is almost true as we will see in Lemma 31.28.5.

Lemma 31.28.4. *Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on source-and-target. Then*

(1) *$\mathcal{P}$ is étale local on the source,*
(2) *$\mathcal{P}$ is étale local on the target,*
(3) *$\mathcal{P}$ is stable under postcomposing with étale morphisms: if $f : X \to Y$ has $\mathcal{P}$ and $g : Y \to Z$ is étale, then $g \circ f$ has $\mathcal{P}$, and*
(4) *$\mathcal{P}$ has a permanence property: given $f : X \to Y$ and $g : Y \to Z$ étale such that $g \circ f$ has $\mathcal{P}$, then f has $\mathcal{P}$.*

Proof. We write everything out completely.

Proof of (1). Let $f : X \to Y$ be a morphism of schemes. Let $\{X_i \to X\}_{i \in I}$ be an étale covering of X. If each composition $h_i : X_i \to Y$ has $\mathcal{P}$, then for each $x \in X$ we can find an $i \in I$ and a point $x_i \in X_i$ mapping to x. Then $(X_i, x_i) \to (X, x)$ is an étale morphism of germs, and $\mathrm{id}_Y : Y \to Y$ is an étale morphism, and h_i is as in part (3) of Definition 31.28.3. Thus we see that f has $\mathcal{P}$. Conversely, if f has $\mathcal{P}$ then each $X_i \to Y$ has $\mathcal{P}$ by Definition 31.28.3 part (1).

Proof of (2). Let $f : X \to Y$ be a morphism of schemes. Let $\{Y_i \to Y\}_{i \in I}$ be an étale covering of Y. Write $X_i = Y_i \times_Y X$ and $h_i : X_i \to Y_i$ for the base change of f. If each $h_i : X_i \to Y_i$ has $\mathcal{P}$, then for each $x \in X$ we pick an $i \in I$ and a point $x_i \in X_i$ mapping to x. Then $(X_i, x_i) \to (X, x)$ is an étale morphism of germs, $Y_i \to Y$ is étale, and h_i is as in part (3) of Definition 31.28.3. Thus we see that f has $\mathcal{P}$. Conversely, if f has $\mathcal{P}$, then each $X_i \to Y_i$ has $\mathcal{P}$ by Definition 31.28.3 part (2).

Proof of (3). Assume $f : X \to Y$ has $\mathcal{P}$ and $g : Y \to Z$ is étale. For every $x \in X$ we can think of $(X, x) \to (X, x)$ as an étale morphism of germs, $Y \to Z$ is an étale morphism, and $h = f$ is as in part (3) of Definition 31.28.3. Thus we see that $g \circ f$ has $\mathcal{P}$.

Proof of (4). Let $f : X \to Y$ be a morphism and $g : Y \to Z$ étale such that $g \circ f$ has $\mathcal{P}$. Then by Definition 31.28.3 part (2) we see that $\mathrm{pr}_Y : Y \times_Z X \to Y$ has $\mathcal{P}$. But the morphism $(f, 1) : X \to Y \times_Z X$ is étale as a section to the étale projection $\mathrm{pr}_X : Y \times_Z X \to X$, see Morphisms, Lemma 24.35.18. Hence $f = \mathrm{pr}_Y \circ (f, 1)$ has $\mathcal{P}$ by Definition 31.28.3 part (1). □

The following lemma is the analogue of Morphisms, Lemma 24.13.4.

Lemma 31.28.5. *Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on source-and-target. Let $f : X \to Y$ be a morphism of schemes. The following are equivalent:*

(a) *f has property $\mathcal{P}$,*
(b) *for every $x \in X$ there exists an étale morphism of germs $a : (U, u) \to (X, x)$, an étale morphism $b : V \to Y$, and a morphism $h : U \to V$ such that $f \circ a = b \circ h$ and h has $\mathcal{P}$,*
(c) *for any commutative diagram*

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "a"'] & V \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with a, b étale the morphism h has $\mathcal{P}$,
(d) *for some diagram as in (c) with $a : U \to X$ surjective h has $\mathcal{P}$,*
(e) *there exists an étale covering $\{Y_i \to Y\}_{i \in I}$ such that each base change $Y_i \times_Y X \to Y_i$ has $\mathcal{P}$,*
(f) *there exists an étale covering $\{X_i \to X\}_{i \in I}$ such that each composition $X_i \to Y$ has $\mathcal{P}$,*
(g) *there exists an étale covering $\{Y_i \to Y\}_{i \in I}$ and for each $i \in I$ an étale covering $\{X_{ij} \to Y_i \times_Y X\}_{j \in J_i}$ such that each morphism $X_{ij} \to Y_i$ has $\mathcal{P}$.*

Proof. The equivalence of (a) and (b) is part of Definition 31.28.3. The equivalence of (a) and (e) is Lemma 31.28.4 part (2). The equivalence of (a) and (f) is Lemma 31.28.4 part (1). As (a) is now equivalent to (e) and (f) it follows that (a) equivalent to (g).

It is clear that (c) implies (a). If (a) holds, then for any diagram as in (c) the morphism $f \circ a$ has $\mathcal{P}$ by Definition 31.28.3 part (1), whereupon h has $\mathcal{P}$ by Lemma 31.28.4 part (4). Thus (a) and (c) are equivalent. It is clear that (c) implies (d). To see that (d) implies (a) assume we have a diagram as in (c) with $a : U \to X$ surjective and h having $\mathcal{P}$. Then $b \circ h$ has $\mathcal{P}$ by Lemma 31.28.4 part (3). Since $\{a : U \to X\}$ is an étale covering we conclude that f has $\mathcal{P}$ by Lemma 31.28.4 part (1). □

It seems that the result of the following lemma is not a formality, i.e., it actually uses something about the geometry of étale morphisms.

Lemma 31.28.6. *Let $\mathcal{P}$ be a property of morphisms of schemes. Assume*

(1) *$\mathcal{P}$ is étale local on the source,*
(2) *$\mathcal{P}$ is étale local on the target, and*

(3) *$\mathcal{P}$ is stable under postcomposing with open immersions: if $f : X \to Y$ has $\mathcal{P}$ and $Y \subset Z$ is an open subscheme then $X \to Z$ has $\mathcal{P}$.*

Then $\mathcal{P}$ is étale local on the source-and-target.

Proof. Let $\mathcal{P}$ be a property of morphisms of schemes which satisfies conditions (1), (2) and (3) of the lemma. By Lemma 31.22.2 we see that $\mathcal{P}$ is stable under precomposing with étale morphisms. By Lemma 31.18.2 we see that $\mathcal{P}$ is stable under étale base change. Hence it suffices to prove part (3) of Definition 31.28.3 holds.

More precisely, suppose that $f : X \to Y$ is a morphism of schemes which satisfies Definition 31.28.3 part (3)(b). In other words, for every $x \in X$ there exists an étale morphism $a_x : U_x \to X$, a point $u_x \in U_x$ mapping to x, an étale morphism $b_x : V_x \to Y$, and a morphism $h_x : U_x \to V_x$ such that $f \circ a_x = b_x \circ h_x$ and h_x has $\mathcal{P}$. The proof of the lemma is complete once we show that f has $\mathcal{P}$. Set $U = \coprod U_x$, $a = \coprod a_x$, $V = \coprod V_x$, $b = \coprod b_x$, and $h = \coprod h_x$. We obtain a commutative diagram

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "a"'] & V \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with a, b étale, a surjective. Note that h has $\mathcal{P}$ as each h_x does and $\mathcal{P}$ is étale local on the target. Because a is surjective and $\mathcal{P}$ is étale local on the source, it suffices to prove that $b \circ h$ has $\mathcal{P}$. This reduces the lemma to proving that $\mathcal{P}$ is stable under postcomposing with an étale morphism.

During the rest of the proof we let $f : X \to Y$ be a morphism with property $\mathcal{P}$ and $g : Y \to Z$ is an étale morphism. Consider the following statements:

(∅) With no additional assumptions $g \circ f$ has property $\mathcal{P}$.
(A) Whenever Z is affine $g \circ f$ has property $\mathcal{P}$.
(AA) Whenever X and Z are affine $g \circ f$ has property $\mathcal{P}$.
(AAA) Whenever X, Y, and Z are affine $g \circ f$ has property $\mathcal{P}$.

Once we have proved (∅) the proof of the lemma will be complete.

Claim 1: (AAA) ⇒ (AA). Namely, let $f : X \to Y$, $g : Y \to Z$ be as above with X, Z affine. As X is affine hence quasi-compact we can find finitely many affine open $Y_i \subset Y$, $i = 1, \ldots, n$ such that $X = \bigcup_{i=1,\ldots,n} f^{-1}(Y_i)$. Set $X_i = f^{-1}(Y_i)$. By Lemma 31.18.2 each of the morphisms $X_i \to Y_i$ has $\mathcal{P}$. Hence $\coprod_{i=1,\ldots,n} X_i \to \coprod_{i=1,\ldots,n} Y_i$ has $\mathcal{P}$ as $\mathcal{P}$ is étale local on the target. By (AAA) applied to $\coprod_{i=1,\ldots,n} X_i \to \coprod_{i=1,\ldots,n} Y_i$ and the étale morphism $\coprod_{i=1,\ldots,n} Y_i \to Z$ we see that $\coprod_{i=1,\ldots,n} X_i \to Z$ has $\mathcal{P}$. Now $\{\coprod_{i=1,\ldots,n} X_i \to X\}$ is an étale covering, hence as $\mathcal{P}$ is étale local on the source we conclude that $X \to Z$ has $\mathcal{P}$ as desired.

Claim 2: (AAA) ⇒ (A). Namely, let $f : X \to Y$, $g : Y \to Z$ be as above with Z affine. Choose an affine open covering $X = \bigcup X_i$. As $\mathcal{P}$ is étale local on the source we see that each $f|_{X_i} : X_i \to Y$ has $\mathcal{P}$. By (AA), which follows from (AAA) according to Claim 1, we see that $X_i \to Z$ has $\mathcal{P}$ for each i. Since $\{X_i \to X\}$ is an étale covering and $\mathcal{P}$ is étale local on the source we conclude that $X \to Z$ has $\mathcal{P}$.

Claim 3: (AAA) ⇒ (∅). Namely, let $f : X \to Y$, $g : Y \to Z$ be as above. Choose an affine open covering $Z = \bigcup Z_i$. Set $Y_i = g^{-1}(Z_i)$ and $X_i = f^{-1}(Y_i)$. By Lemma 31.18.2 each of the morphisms $X_i \to Y_i$ has $\mathcal{P}$. By (A), which follows from (AAA) according to Claim 2, we see that $X_i \to Z_i$ has $\mathcal{P}$ for each i. Since $\mathcal{P}$ is local on the target and $X_i = (g \circ f)^{-1}(Z_i)$ we conclude that $X \to Z$ has $\mathcal{P}$.

Thus to prove the lemma it suffices to prove (AAA). Let $f : X \to Y$ and $g : Y \to Z$ be as above X, Y, Z affine. Note that an étale morphism of affines has universally bounded fibres, see Morphisms, Lemma 24.35.6 and Lemma 24.48.8. Hence we can do induction on the integer n bounding the degree of the fibres of $Y \to Z$. See Morphisms, Lemma 24.48.7 for a description of this integer in the case of an étale morphism. If $n = 1$, then $Y \to Z$ is an open immersion, see Lemma 31.21.2, and the result follows from assumption (3) of the lemma. Assume $n > 1$.

Consider the following commutative diagram

$$\begin{tikzcd} X \times_Z Y \arrow[r, "f_Y"'] \arrow[d] & Y \times_Z Y \arrow[r, "\mathrm{pr}"'] \arrow[d] & Y \arrow[d] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

Note that we have a decomposition into open and closed subschemes $Y \times_Z Y = \Delta_{Y/Z}(Y) \amalg Y'$, see Morphisms, Lemma 24.34.13. As a base change the degrees of the fibres of the second projection $\mathrm{pr} : Y \times_Z Y \to Y$ are bounded by n, see Morphisms, Lemma 24.48.4. On the other hand, $\mathrm{pr}|_{\Delta(Y)} : \Delta(Y) \to Y$ is an isomorphism and every fibre has exactly one point. Thus, on applying Morphisms, Lemma 24.48.7 we conclude the degrees of the fibres of the restriction $\mathrm{pr}|_{Y'} : Y' \to Y$ are bounded by $n - 1$. Set $X' = f_Y^{-1}(Y')$. Picture

$$\begin{tikzcd} X \amalg X' \arrow[r, "f \amalg f'"'] \arrow[d, equal] & \Delta(Y) \amalg Y' \arrow[r] \arrow[d, equal] & Y \arrow[d, equal] \\ X \times_Z Y \arrow[r, "f_Y"] & Y \times_Z Y \arrow[r, "\mathrm{pr}"] & Y \end{tikzcd}$$

As $\mathcal{P}$ is étale local on the target and hence stable under étale base change (see Lemma 31.18.2) we see that f_Y has $\mathcal{P}$. Hence, as $\mathcal{P}$ is étale local on the source, $f' = f_Y|_{X'}$ has $\mathcal{P}$. By induction hypothesis we see that $X' \to Y$ has $\mathcal{P}$. As $\mathcal{P}$ is local on the source, and $\{X \to X \times_Z Y, X' \to X \times_Y Z\}$ is an étale covering, we conclude that $\mathrm{pr} \circ f_Y$ has $\mathcal{P}$. Note that $g \circ f$ can be viewed as a morphism $g \circ f : X \to g(Y)$. As $\mathrm{pr} \circ f_Y$ is the pullback of $g \circ f : X \to g(Y)$ via the étale covering $\{Y \to g(Y)\}$, and as $\mathcal{P}$ is étale local on the target, we conclude that $g \circ f : X \to g(Y)$ has property $\mathcal{P}$. Finally, applying assumption (3) of the lemma once more we conclude that $g \circ f : X \to Z$ has property $\mathcal{P}$. $\square$

Remark 31.28.7. Using Lemma 31.28.6 and the work done in the earlier sections of this chapter it is easy to make a list of types of morphisms which are étale local on the source-and-target. In each case we list the lemma which implies the property is étale local on the source and the lemma which implies the property is étale local on the target. In each case the third assumption of Lemma 31.28.6 is trivial to check, and we omit it. Here is the list:

1. flat, see Lemmas 31.23.1 and 31.19.13,
2. locally of finite presentation, see Lemmas 31.24.1 and 31.19.9,
3. locally finite type, see Lemmas 31.24.2 and 31.19.8,
4. universally open, see Lemmas 31.24.4 and 31.19.4,
5. syntomic, see Lemmas 31.25.1 and 31.19.24,
6. smooth, see Lemmas 31.26.1 and 31.19.25,
7. étale, see Lemmas 31.27.1 and 31.19.27,
8. locally quasi-finite, see Lemmas 31.27.2 and 31.19.22,

(9) unramified, see Lemmas 31.27.3 and 31.19.26,
(10) G-unramified, see Lemmas 31.27.3 and 31.19.26, and
(11) add more here as needed.

Remark 31.28.8. At this point we have three possible definitions of what it means for a property $\mathcal{P}$ of morphisms to be ``étale local on the source and target'':

(ST) $\mathcal{P}$ is étale local on the source and $\mathcal{P}$ is étale local on the target,
(DM) (the definition in the paper [**DM69a**, Page 100] by Deligne and Mumford) for every diagram

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "a"] & V \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with surjective étale vertical arrows we have $\mathcal{P}(h) \Leftrightarrow \mathcal{P}(f)$, and
(SP) $\mathcal{P}$ is étale local on the source-and-target.

In this section we have seen that (SP) $\Rightarrow$ (DM) $\Rightarrow$ (ST). The Examples 31.28.1 and 31.28.2 show that neither implication can be reversed. Finally, Lemma 31.28.6 shows that the difference disappears when looking at properties of morphisms which are stable under postcomposing with open immersions, which in practive will always be the case.

31.29. Properties of morphisms of germs local on source-and-target

In this section we discuss the analogue of the material in Section 31.28 for morphisms of germs of schemes.

Definition 31.29.1. Let $\mathcal{Q}$ be a property of morphisms of germs of schemes. We say $\mathcal{Q}$ is *étale local on the source-and-target* if for any commutative diagram

$$\begin{tikzcd} (U', u') \arrow[r, "h'"] \arrow[d, "a"] & (V', v') \arrow[d, "b"] \\ (U, u) \arrow[r, "h"] & (V, v) \end{tikzcd}$$

with étale vertical arrows we have $\mathcal{Q}(h) \Leftrightarrow \mathcal{Q}(h')$.

Lemma 31.29.2. *Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on the source-and-target. Consider the property $\mathcal{Q}$ of morphisms of germs defined by the rule*

$$\mathcal{Q}((X, x) \to (S, s)) \Leftrightarrow \textit{there exists a representative } U \to S \textit{ which has } \mathcal{P}$$

Then $\mathcal{Q}$ is étale local on the source-and-target as in Definition 31.29.1.

Proof. We first remark that as $\mathcal{P}$ is étale local on the source, see Lemma 31.28.4, if $(X, x) \to (S, s)$ has $\mathcal{P}$, then there are arbitrarily small neighbourhoods U of x in X such that a representative $U \to S$ of $(X, x) \to (S, s)$ has $\mathcal{P}$. We will use this without further mention. Let

$$\begin{tikzcd} (U', u') \arrow[r, "h'"] \arrow[d, "a"] & (V', v') \arrow[d, "b"] \\ (U, u) \arrow[r, "h"] & (V, v) \end{tikzcd}$$

be as in Definition 31.29.1. We will use a rather pedantic notation in order to distinguish between morphisms of germs and their representatives in this proof.

If $\mathcal{A}(h)$ holds, then $\mathcal{P}$ holds for a representative $h_1 : U_1 \to V$ of h. Let $a_1 : U_1' \to U$ be a representative of a which is étale with $a_1(U_1') \subset U_1$. As $\mathcal{P}$ is stable under precomposing with étale morphisms we see that $h_1 \circ a_1 : U_1' \to V$ has $\mathcal{P}$. Moreover, $h_1 \circ a_1 : U_1' \to V$ is a representative of $b \circ h'$ by the commutativity of the diagram. Choose a representative $b_1 : V_1' \to V$ of b. Choose a representative $h_1' : U_2' \to V'$ with $h_1'(U_1') \subset V_1'$, $U_2' \subset U_1'$, and $(h_1 \circ a_1)|_{U_2'} = b_1 \circ h_1'$. Then we see that $b_1 \circ h_1'$ has $\mathcal{P}$. Hence h' has $\mathcal{P}$ by Lemma 31.28.4 part (4).

Conversely, suppose $\mathcal{A}(h')$ holds. Choose a representative $b_1 : V_1' \to V$ of b. Choose a representative $h_1' : U_1' \to V'$ with $\mathcal{P}$ and with $h_1'(U_1') \subset V_1'$. Then $b_1 \circ h_1'$ has $\mathcal{P}$ by Lemma 31.28.4 part (3). Moreover, $b_1 \circ h_1' : U_1' \to V$ is a representative of $h \circ a$ by the commutativity of the diagram. Choose a representative $h_1 : U_1 \to V$ of h. Choose a representative $a_1 : U_2' \to U$ with $a_1(U_2') \subset U_1$, $U_2' \subset U_1'$, and $h_1 \circ a_1 = (b_1 \circ h_1')|_{U_2'}$. The we see that $h_1 \circ a_1$ has $\mathcal{P}$. As $\mathcal{P}$ is étale local on the source we conclude that $h_1|_{a_1(U_2')}$ has $\mathcal{P}$ and we win. □

Lemma 31.29.3. *Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on source-and-target. Let Q be the associated property of morphisms of germs, see Lemma 31.29.2. Let $f : X \to Y$ be a morphism of schemes. The following are equivalent:*

(1) *f has property $\mathcal{P}$, and*
(2) *for every $x \in X$ the morphism of germs $(X, x) \to (Y, f(x))$ has property $\mathcal{Q}$.*

Proof. The implication (1) ⇒ (2) is direct from the definitions. The implication (2) ⇒ (1) also follows from part (3) of Definition 31.28.3. □

A morphism of germs $(X, x) \to (S, s)$ determines a well defined map of local rings. Hence the following lemma makes sense.

Lemma 31.29.4. *The property of morphisms of germs*

$$\mathcal{A}((X, x) \to (S, s)) = \mathcal{O}_{S,s} \to \mathcal{O}_{X,x} \text{ is flat}$$

is étale local on the source-and-target.

Proof. Given a diagram as in Definition 31.29.1 we obtain the following diagram of local homomorphisms of local rings

$$\begin{tikzcd} \mathcal{O}_{U',u'} & \mathcal{O}_{V',v'} \arrow[l] \\ \mathcal{O}_{U,u} \arrow[u] & \mathcal{O}_{V,v} \arrow[l] \arrow[u] \end{tikzcd}$$

Note that the vertical arrows are localizations of étale ring maps, in particular they are essentially of finite presentation, flat, and unramified (see Algebra, Section 7.132). In particular the vertical maps are faithfully flat, see Algebra, Lemma 7.35.16. Now, if the upper horizontal arrow is flat, then the lower horizontal arrow is flat by an application of Algebra, Lemma 7.35.9 with $R = \mathcal{O}_{V,v}$, $S = \mathcal{O}_{U,u}$ and $M = \mathcal{O}_{U',u'}$. If the lower horizontal arrow is flat, then the ring map

$$\mathcal{O}_{V',v'} \otimes_{\mathcal{O}_{V,v}} \mathcal{O}_{U,u} \longleftarrow \mathcal{O}_{V',v'}$$

is flat by Algebra, Lemma 7.35.6. And the ring map

$$\mathcal{O}_{U',u'} \longleftarrow \mathcal{O}_{V',v'} \otimes_{\mathcal{O}_{V,v}} \mathcal{O}_{U,u}$$

is a localization of a map between étale ring extensions of $\mathcal{O}_{U,u}$, hence flat by Algebra, Lemma 7.132.8. □

Lemma 31.29.5. *Consider a commutative diagram of morphisms of schemes*

$$\begin{tikzcd} U' \arrow[r] \arrow[d] & V' \arrow[d] \\ U \arrow[r] & V \end{tikzcd}$$

with étale vertical arrows and a point $v' \in U'$ mapping to $v \in U$. Then the morphism of fibres $U'_{v'} \to U_v$ is étale.

Proof. Note that $U'_v \to U_v$ is étale as a base change of the étale morphism $U' \to U$. The scheme U'_v is a scheme over V'_v. By Morphisms, Lemma 24.35.7 the scheme V'_v is a disjoint union of spectra of finite separable field extensions of $\kappa(v)$. One of these is $v' = Spec(\kappa(v'))$. Hence $U'_{v'}$ is an open and closed subscheme of U'_v and it follows that $U'_{v'} \to U'_v \to U_v$ is étale (as a composition of an open immersion and an étale morphism, see Morphisms, Section 24.35). □

Given a morphism of germs of schemes $(X, x) \to (S, s)$ we can define the *fibre* as the isomorphism class of germs (U_s, x) where $U \to S$ is any representative. We will often abuse notation and just write (X_s, x).

Lemma 31.29.6. *Let $d \in \{0, 1, 2, \ldots, \infty\}$. The property of morphisms of germs*

$$\mathcal{P}_d((X, x) \to (S, s)) = \textit{the local ring } \mathcal{O}_{X_s,x} \textit{ of the fibre has dimension } d$$

is étale local on the source-and-target.

Proof. Given a diagram as in Definition 31.29.1 we obtain an étale morphism of fibres $U'_{v'} \to U_v$ mapping u' to u, see Lemma 31.29.5. Hence the result follows from Lemma 31.17.3. □

Lemma 31.29.7. *Let $r \in \{0, 1, 2, \ldots, \infty\}$. The property of morphisms of germs*

$$\mathcal{P}_r((X, x) \to (S, s)) \Leftrightarrow trdeg_{\kappa(s)}\kappa(x) = r$$

is étale local on the source-and-target.

Proof. Given a diagram as in Definition 31.29.1 we obtain the following diagram of local homomorphisms of local rings

$$\begin{tikzcd} \mathcal{O}_{U',u'} & \mathcal{O}_{V',v'} \arrow[l] \\ \mathcal{O}_{U,u} \arrow[u] & \mathcal{O}_{V,v} \arrow[l] \arrow[u] \end{tikzcd}$$

Note that the vertical arrows are localizations of étale ring maps, in particular they are unramified (see Algebra, Section 7.132). Hence $\kappa(u) \subset \kappa(u')$ and $\kappa(v) \subset \kappa(v')$ are finite separable field extensions. Thus we have $\text{trdeg}_{\kappa(v)}\kappa(u) = \text{trdeg}_{\kappa(v')}\kappa(u)$ which proves the lemma. □

Let (X, x) be a germ of a scheme. The dimension of X at x is the minimum of the dimensions of open neighbourhoods of x in X, and any small enough open neighbourhood has this dimension. Hence this is an invariant of the isomorphism class of the germ. We denote this simply $\dim_x(X)$.

Lemma 31.29.8. *Let $d \in \{0, 1, 2, \ldots, \infty\}$. The property of morphisms of germs*

$$\mathcal{P}_d((X, x) \to (S, s)) \Leftrightarrow \dim_x(X_s) = d$$

is étale local on the source-and-target.

Proof. Given a diagram as in Definition 31.29.1 we obtain an étale morphism of fibres $U'_{v'} \to U_v$ mapping u' to u, see Lemma 31.29.5. Hence now the equality $\dim_u(U_v) = \dim_{u'}(U'_{v'})$ follows from Lemma 31.17.2. □

31.30. Descent data for schemes over schemes

Most of the arguments in this section are formal relying only on the definition of a descent datum. In Section 31.36 we will examine the relationship with simplicial schemes which will somewhat clarify the situation. Hopefully the reader will be convinced by the end of Section 31.36 that the language of descent is awkward and the setting of simplicial schemes is natural for the questions being considered here.

Definition 31.30.1. Let $f : X \to S$ be a morphism of schemes.

(1) Let $V \to X$ be a scheme over X. A *descent datum for $V/X/S$* is an isomorphism $\varphi : V \times_S X \to X \times_S V$ of schemes over $X \times_S X$ satisfying the *cocycle condition* that the diagram

$$\begin{tikzcd} V \times_S X \times_S X \arrow[rr, "\varphi_{02}"] \arrow[dr, "\varphi_{01}"'] & & X \times_S X \times_S V \\ & X \times_S V \times_S X \arrow[ur, "\varphi_{12}"'] & \end{tikzcd}$$

commutes (with obvious notation).

(2) We also say that the pair $(V/X, \varphi)$ is a *descent datum relative to $X \to S$.*

(3) A *morphism $f : (V/X, \varphi) \to (V'/X, \varphi')$ of descent data relative to $X \to S$* is a morphism $f : V \to V'$ of schemes over X such that the diagram

$$\begin{tikzcd} V \times_S X \arrow[r, "\varphi"'] \arrow[d, "f \times \mathrm{id}_X"'] & X \times_S V \arrow[d, "\mathrm{id}_X \times f"] \\ V' \times_S X \arrow[r, "\varphi'"] & X \times_S V' \end{tikzcd}$$

commutes.

There are all kinds of ``miraculous'' identities which arise out of the definition above. For example the pullback of φ via the diagonal morphism $\Delta : X \to X \times_S X$ can be seen as a morphism $\Delta^*\varphi : V \to V$. This because $X \times_{\Delta, X \times_S X} (V \times_S X) = V$ and also $X \times_{\Delta, X \times_S X} (X \times_S V) = V$. In fact, $\Delta^*\varphi$ is equal to the identity. This is a good exercise if you are unfamiliar with this material.

Remark 31.30.2. Let $X \to S$ be a morphism of schemes. Let $(V/X, \varphi)$ be a descent datum relative to $X \to S$. We may think of the isomorphism φ as an isomorphism

$$(X \times_S X) \times_{\mathrm{pr}_0, X} V \longrightarrow (X \times_S X) \times_{\mathrm{pr}_1, X} V$$

of schemes over $X \times_S X$. So loosely speaking one may think of φ as a map $\varphi : \mathrm{pr}_0^*V \to \mathrm{pr}_1^*V$[6]. The cocycle condition then says that $\mathrm{pr}_{02}^*\varphi = \mathrm{pr}_{12}^*\varphi \circ \mathrm{pr}_{01}^*\varphi$. In this way it is very similar to the case of a descent datum on quasi-coherent sheaves.

Here is the definition in case you have a family of morphisms with fixed target.

Definition 31.30.3. Let S be a scheme. Let $\{X_i \to S\}_{i\in I}$ be a family of morphisms with target S.

(1) A *descent datum* (V_i, φ_{ij}) *relative to the family* $\{X_i \to S\}$ is given by a scheme V_i over X_i for each $i \in I$, an isomorphism $\varphi_{ij} : V_i \times_S X_j \to X_i \times_S V_j$ of schemes over $X_i \times_S X_j$ for each pair $(i, j) \in I^2$ such that for every triple of indices $(i, j, k) \in I^3$ the diagram

$$\begin{tikzcd} V_i \times_S X_j \times_S X_k \arrow[rr, "\mathrm{pr}_{02}^*\varphi_{ik}"] \arrow[dr, "\mathrm{pr}_{01}^*\varphi_{ij}"'] & & X_i \times_S X_j \times_S V_k \\ & X_i \times_S V_j \times_S X_k \arrow[ur, "\mathrm{pr}_{12}^*\varphi_{jk}"'] & \end{tikzcd}$$

of schemes over $X_i \times_S X_j \times_S X_k$ commutes (with obvious notation).

(2) A *morphism* $\psi : (V_i, \varphi_{ij}) \to (V_i', \varphi_{ij}')$ *of descent data* is given by a family $\psi = (\psi_i)_{i\in I}$ of morphisms of X_i-schemes $\psi_i : V_i \to V_i'$ such that all the diagrams

$$\begin{tikzcd} V_i \times_S X_j \arrow[r, "\varphi_{ij}"'] \arrow[d, "\psi_i\times\mathrm{id}"'] & X_i \times_S V_j \arrow[d, "\mathrm{id}\times\psi_j"] \\ V_i' \times_S X_j \arrow[r, "\varphi_{ij}'"] & X_i \times_S V_j' \end{tikzcd}$$

commute.

This is the notion that comes up naturally for example when the question arises whether the fibred category of relative curves is a stack in the fpqc topology (it isn't -- at least not if you stick to schemes).

Remark 31.30.4. Let S be a scheme. Let $\{X_i \to S\}_{i\in I}$ be a family of morphisms with target S. Let (V_i, φ_{ij}) be a descent datum relative to $\{X_i \to S\}$. We may think of the isomorphisms φ_{ij} as isomorphisms

$$(X_i \times_S X_j) \times_{\mathrm{pr}_0, X_i} V_i \longrightarrow (X_i \times_S X_j) \times_{\mathrm{pr}_1, X_j} V_j$$

of schemes over $X_i \times_S X_j$. So loosely speaking one may think of φ_{ij} as an isomorphism $\mathrm{pr}_0^*V_i \to \mathrm{pr}_1^*V_j$ over $X_i \times_S X_j$. The cocycle condition then says that $\mathrm{pr}_{02}^*\varphi_{ik} = \mathrm{pr}_{12}^*\varphi_{jk} \circ \mathrm{pr}_{01}^*\varphi_{ij}$. In this way it is very similar to the case of a descent datum on quasi-coherent sheaves.

The reason we will usually work with the version of a family consisting of a single morphism is the following lemma.

[6] Unfortunately, we have chosen the ``wrong'' direction for our arrow here. In Definitions 31.30.1 and 31.30.3 we should have the opposite direction to what was done in Definition 31.2.1 by the general principle that ``functions'' and ``spaces'' are dual.

Lemma 31.30.5. *Let S be a scheme. Let $\{X_i \to S\}_{i\in I}$ be a family of morphisms with target S. Set $X = \coprod_{i\in I} X_i$, and consider it as an S-scheme. There is a canonical equivalence of categories*

$$\begin{matrix}\text{category of descent data} \\ \text{relative to the family } \{X_i \to S\}_{i\in I}\end{matrix} \longrightarrow \begin{matrix}\text{category of descent data} \\ \text{relative to } X/S\end{matrix}$$

which maps (V_i, φ_{ij}) to (V, φ) with $V = \coprod_{i\in I} V_i$ and $\varphi = \coprod \varphi_{ij}$.

Proof. Observe that $X \times_S X = \coprod_{ij} X_i \times_S X_j$ and similarly for higher fibre products. Giving a morphism $V \to X$ is exactly the same as giving a family $V_i \to X_i$. And giving a descent datum φ is exactly the same as giving a family φ_{ij}. $\square$

Lemma 31.30.6. *(Pullback of descent data for schemes over schemes.)*

(1) *Let*

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d, "a'"'] & X \arrow[d, "a"] \\ S' \arrow[r, "h"] & S \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. The construction

$$(V \to X, \varphi) \longmapsto f^*(V \to X, \varphi) = (V' \to X', \varphi')$$

where $V' = X' \times_X V$ and where φ' is defined as the composition

$$\begin{tikzcd} V' \times_{S'} X' \arrow[r, equal] & (X' \times_X V) \times_{S'} X' \arrow[r, equal] & (X' \times_{S'} X') \times_{X \times_S X} (V \times_S X) \arrow[d, "id \times \varphi"] \\ X' \times_{S'} V' \arrow[r, equal] & X' \times_{S'} (X' \times_X V) \arrow[r, equal] & (X' \times_S X') \times_{X \times_S X} (X \times_S V) \end{tikzcd}$$

defines a functor from the category of descent data relative to $X \to S$ to the category of descent data relative to $X' \to S$.

(2) *Given two morphisms $f_i : X' \to X$, $i = 0, 1$ making the diagram commute the functors f_0^* and f_1^* are canonically isomorphic.*

Proof. We omit the proof of (1), but we remark that the morphism φ' is the morphism $(f' \times f')^*\varphi$ in the notation introduced in Remark 31.30.2. For (2) we indicate which morphism $f_0^*V \to f_1^*V$ gives the functorial isomorphism. Namely, since f_0 and f_1 both fit into the commutative diagram we see there is a unique morphism $r : X' \to X \times_S X$ with $f_i = \text{pr}_i \circ r$. Then we take

$$\begin{aligned} f_0^*V &= X' \times_{f_0, X} V \\ &= X' \times_{\text{pr}_0 \circ r, X} V \\ &= X' \times_{r, X \times_S X} (X \times_S X) \times_{\text{pr}_0 X} V \\ &\xrightarrow{\varphi} X' \times_{r, X \times_S X} (X \times_S X) \times_{\text{pr}_1 X} V \\ &= X' \times_{\text{pr}_1 \circ r, X} V \\ &= X' \times_{f_1, X} V \\ &= f_1^*V \end{aligned}$$

We omit the verification that this works. $\square$

Definition 31.30.7. With $S, S', X, X', f, a, a', h$ as in Lemma 31.30.6 the functor

$$(V, \varphi) \longmapsto f^*(V, \varphi)$$

constructed in that lemma is called the *pullback functor* on descent data.

Lemma 31.30.8. *(Pullback of descent data for schemes over families.) Let $\mathcal{U} = \{U_i \to S'\}_{i \in I}$ and $\mathcal{V} = \{V_j \to S\}_{j \in J}$ be families of morphisms with fixed target. Let $\alpha : I \to J$, $h : S' \to S$ and $g_i : U_i \to V_{\alpha(i)}$ be a morphism of families of maps with fixed target, see Sites, Definition 9.8.1.*

1. *Let $(Y_j, \varphi_{jj'})$ be a descent datum relative to the family $\{V_j \to S'\}$. The system*
$$\left(g_i^* Y_{\alpha(i)}, (g_i \times g_{i'})^* \varphi_{\alpha(i)\alpha(i')}\right)$$
(with notation as in Remark 31.30.4) is a descent datum relative to $\mathcal{V}$.
2. *This construction defines a functor between descent data relative to $\mathcal{U}$ and descent data relative to $\mathcal{V}$.*
3. *Given a second $\alpha' : I \to J$, $h' : S' \to S$ and $g'_i : U_i \to V_{\alpha'(i)}$ morphism of families of maps with fixed target, then if $h = h'$ the two resulting functors between descent data are canonically isomorphic.*
4. *These functors agree, via Lemma 31.30.5, with the pullback functors constructed in Lemma 31.30.6.*

Proof. This follows from Lemma 31.30.6 via the correspondence of Lemma 31.30.5. □

Definition 31.30.9. With $\mathcal{U} = \{U_i \to S'\}_{i \in I}$, $\mathcal{V} = \{V_j \to S\}_{j \in J}$, $\alpha : I \to J$, $h : S' \to S$, and $g_i : U_i \to V_{\alpha(i)}$ as in Lemma 31.30.8 the functor

$$(Y_j, \varphi_{jj'}) \longmapsto (g_i^* Y_{\alpha(i)}, (g_i \times g_{i'})^* \varphi_{\alpha(i)\alpha(i')})$$

constructed in that lemma is called the *pullback functor* on descent data.

If $\mathcal{U}$ and $\mathcal{V}$ have the same target S, and if $\mathcal{U}$ refines $\mathcal{V}$ (see Sites, Definition 9.8.1) but no explicit pair (α, g_i) is given, then we can still talk about the pullback functor since we have seen in Lemma 31.30.8 that the choice of the pair does not matter (up to a canonical isomorphism).

Definition 31.30.10. Let S be a scheme. Let $f : X \to S$ be a morphism of schemes.

1. Given a scheme U over S we have the *trivial descent datum* of U relative to $\mathrm{id} : S \to S$, namely the identity morphism on U.
2. By Lemma 31.30.6 we get a *canonical descent datum* on $X \times_S U$ relative to $X \to S$ by pulling back the trivial descent datum via f. We often denote $(X \times_S U, can)$ this descent datum.
3. Let $f : X \to S$ be a morphism of schemes. A descent datum (V, φ) relative to X/S is called *effective* if (V, φ) is isomorphic to the canonical descent datum $(X \times_S U, can)$ for some scheme U over S.

Thus being effective means there exists a scheme U over S and an isomorphism $\psi : V \to X \times_S U$ of X-schemes such that φ is equal to the composition

$$V \times_S X \xrightarrow{\psi \times \mathrm{id}_X} X \times_S U \times_S X = X \times_S X \times_S U \xrightarrow{\mathrm{id}_X \times \psi^{-1}} X \times_S V$$

Definition 31.30.11. Let S be a scheme. Let $\{X_i \to S\}$ be a family of morphisms with target S.

(1) Given a scheme U over S we have a *canonical descent datum* on the family of schemes $X_i \times_S U$ by pulling back the trivial descent datum for U relative to $\{\mathrm{id} : S \to S\}$. We denote this descent datum $(X_i \times_S U, can)$.
(2) A descent datum (V_i, φ_{ij}) relative to $\{X_i \to S\}$ is called *effective* if there exists a scheme U over S such that (V_i, φ_{ij}) is isomorphic to $(X_i \times_S U, can)$.

31.31. Fully faithfulness of the pullback functors

It turns out that the pullback functor between descent data for fpqc-coverings is fully faithful. In other words, morphisms of schemes satisfy fpqc descent. The goal of this section is to prove this. The reader is encouraged instead to prove this him/herself. The key is to use Lemma 31.9.3.

Lemma 31.31.1. *A surjective and flat morphism is an epimorphism in the category of schemes.*

Proof. Suppose we have $h : X' \to X$ surjective and flat and $a, b : X \to Y$ morphisms such that $a \circ h = b \circ h$. As h is surjective we see that a and b agree on underlying topological spaces. Pick $x' \in X'$ and set $x = h(x')$ and $y = a(x) = b(x)$. Consider the local ring maps

$$a^\sharp_x, b^\sharp_x : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$$

These become equal when composed with the flat local homomorphism $h^\sharp_{x'} : \mathcal{O}_{X,x} \to \mathcal{O}_{X',x'}$. Since a flat local homomorphism is faithfully flat (Algebra, Lemma 7.35.16) we conclude that $h^\sharp_{x'}$ is injective. Hence $a^\sharp_x = b^\sharp_x$ which implies $a = b$ as desired. □

Lemma 31.31.2. *Let $h : S' \to S$ be a surjective, flat morphism of schemes. The base change functor*

$$Sch/S \longrightarrow Sch/S', \quad X \longmapsto S' \times_S X$$

is faithful.

Proof. Let X_1, X_2 be schemes over S. Let $\alpha, \beta : X_2 \to X_1$ be morphisms over S. If α, β base change to the same morphism then we get a commutative diagram as follows

$$\begin{tikzcd}
X_2 \arrow[d, "\alpha"] & S' \times_S X_2 \arrow[l] \arrow[r] \arrow[d] & X_2 \arrow[d, "\beta"] \\
X_1 & S' \times_S X_1 \arrow[l] \arrow[r] & X_1
\end{tikzcd}$$

Hence it suffices to show that $S' \times_S X_2 \to X_2$ is an epimorphism. As the base change of a surjective and flat morphism it is surjective and flat (see Morphisms, Lemmas 24.9.4 and 24.24.7). Hence the lemma follows from Lemma 31.31.1. □

Lemma 31.31.3. *In the situation of Lemma 31.30.6 assume that $f : X' \to X$ is surjective and flat. Then the pullback functor is faithful.*

Proof. Let (V_i, φ_i), $i = 1, 2$ be descend data for $X \to S$. Let $\alpha, \beta : V_1 \to V_2$ be morphisms of descent data. Suppose that $f^*\alpha = f^*\beta$. Our task is to show that $\alpha = \beta$. Note that α, β are morphisms of schemes over X, and that $f^*\alpha$, $f^*\beta$ are simply the base changes of α, β to morphisms over X'. Hence the lemma follows from Lemma 31.31.2. □

Here is the key lemma of this section.

Lemma 31.31.4. *In the situation of Lemma 31.30.6 assume*

(1) *$\{f : X' \to X\}$ is an fpqc covering (for example if f is surjective, flat, and quasi-compact), and*
(2) *$f \times f : X' \times_{S'} X' \to X \times_S X$ is surjective and flat*[7].

Then the pullback functor is fully faithful.

Proof. Assumption (1) implies that f is surjective and flat. Hence the pullback functor is faithful by Lemma 31.31.3. Let (V, φ) and (W, ψ) be two descent data relative to $X \to S$. Set $(V', \varphi') = f^*(V, \varphi)$ and $(W', \psi') = f^*(W, \psi)$. Let $\alpha' : V' \to W'$ be a morphism of descent data for X' over S'. We have to show there exists a morphism $\alpha : V \to W$ of descent data for X over S whose pullback is α'.

Recall that V' is the base change of V by f and that φ' is the base change of φ by $f \times f$ (see Remark 31.30.2). By assumption the diagram

$$\begin{tikzcd} V' \times_{S'} X' \arrow[r, "\varphi'"'] \arrow[d, "\alpha' \times \mathrm{id}"'] & X' \times_{S'} V' \arrow[d, "\mathrm{id} \times \alpha'"] \\ W' \times_{S'} X' \arrow[r, "\psi'"] & X' \times_{S'} W' \end{tikzcd}$$

commutes. We claim the two compositions

$$V' \times_V V' \xrightarrow{\mathrm{pr}_i} V' \xrightarrow{\alpha'} W' \longrightarrow W, \quad i = 0, 1$$

are the same. The reader is advised to prove this themselves rather than read the rest of this paragraph. (Please email if you find a nice clean argument.) Let v_0, v_1 be points of V' which map to the same point $v \in V$. Let $x_i \in X'$ be the image of v_i, and let x be the point of X which is the image of v in X. In other words, $v_i = (x_i, v)$ in $V' = X' \times_X V$. Write $\varphi(v, x) = (x, v')$ for some point v' of V. This is possible because φ is a morphism over $X \times_S X$. Denote $v_i' = (x_i, v')$ which is a point of V'. Then a calculation (using the definition of φ') shows that $\varphi'(v_i, x_j) = (x_i, v_j')$. Denote $w_i = \alpha'(v_i)$ and $w_i' = \alpha'(v_i')$. Now we may write $w_i = (x_i, u_i)$ for some point u_i of W, and $w_i' = (x_i, u_i')$ for some point u_i' of W. The claim is equivalent to the assertion: $u_0 = u_1$. A formal calculation using the definition of ψ' (see Lemma 31.30.6) shows that the commutativity of the diagram displayed above says that

$$((x_i, x_j), \psi(u_i, x)) = ((x_i, x_j), (x, u_j'))$$

as points of $(X' \times_{S'} X') \times_{X \times_S X} (X \times_S W)$ for all $i, j \in \{0, 1\}$. This shows that $\psi(u_0, x) = \psi(u_1, x)$ and hence $u_0 = u_1$ by taking ψ^{-1}. This proves the claim because the argument above was formal and we can take scheme points (in other words, we may take $(v_0, v_1) = \mathrm{id}_{V' \times_V V'}$).

At this point we can use Lemma 31.9.3. Namely, $\{V' \to V\}$ is a fpqc covering as the base change of the morphism $f : X' \to X$. Hence, by Lemma 31.9.3 the morphism $\alpha' : V' \to W' \to W$ factors through a unique morphism $\alpha : V \to W$ whose base change is necessarily α'. Finally, we see the diagram

$$\begin{tikzcd} V \times_S X \arrow[r, "\varphi"'] \arrow[d, "\alpha \times \mathrm{id}"'] & X \times_S V \arrow[d, "\mathrm{id} \times \alpha"] \\ W \times_S X \arrow[r, "\psi"] & X \times_S W \end{tikzcd}$$

[7] This follows from (1) if $S = S'$.

commutes because its base change to $X' \times_{S'} X'$ commutes and the morphism $X' \times_{S'} X' \to X \times_S X$ is surjective and flat (use Lemma 31.31.2). Hence α is a morphism of descent data $(V, \varphi) \to (W, \psi)$ as desired. □

The following two lemmas have been obsoleted by the improved exposition of the previous material. But they are stilll true!

Lemma 31.31.5. *Let $X \to S$ be a morphism of schemes. Let $f : X \to X$ be a selfmap of X over S. In this case pullback by f is isomorphic to the identity functor on the category of descent data relative to $X \to S$.*

Proof. This is clear from Lemma 31.30.6 since it tells us that $f^* \cong \mathrm{id}^*$. □

Lemma 31.31.6. *Let $f : X' \to X$ be a morphism of schemes over a base scheme S. Assume there exists a morphism $g : X \to X'$ over S, for example if f has a section. Then the pullback functor of Lemma 31.30.6 defines an equivalence of categories between the category of descent data relative to X/S and X'/S.*

Proof. Let $g : X \to X'$ be a morphism over S. Lemma 31.31.5 above shows that the functors $f^* \circ g^* = (g \circ f)^*$ and $g^* \circ f^* = (f \circ g)^*$ are isomorphic to the respective identity functors as desired. □

Lemma 31.31.7. *Let $f : X \to X'$ be a morphism of schemes over a base scheme S. Assume $X \to S$ is surjective and flat. Then the pullback functor of of Lemma 31.30.6 is a faithful functor from the category of descent data relative to X'/S to the category of descent data relative to X/S.*

Proof. We may factor $X \to X'$ as $X \to X \times_S X' \to X'$. The first morphism has a section, hence induces an equivalence of categories of descent data by Lemma 31.31.6. The second morphism is surjective and flat, hence induces a faithful functor by Lemma 31.31.3. □

Lemma 31.31.8. *Let $f : X \to X'$ be a morphism of schemes over a base scheme S. Assume $\{X \to S\}$ is an fpqc covering (for example if f is surjective, flat and quasi-compact). Then the pullback functor of of Lemma 31.30.6 is a fully faithful functor from the category of descent data relative to X'/S to the category of descent data relative to X/S.*

Proof. We may factor $X \to X'$ as $X \to X \times_S X' \to X'$. The first morphism has a section, hence induces an equivalence of categories of descent data by Lemma 31.31.6. The second morphism is an fpqc covering hence induces a fully faithful functor by Lemma 31.31.4. □

Lemma 31.31.9. *Let S be a scheme. Let $\mathcal{U} = \{U_i \to S\}_{i \in I}$, and $\mathcal{V} = \{V_j \to S\}_{j \in J}$, be families of morphisms with target S. Let $\alpha : I \to J$, $id : S \to S$ and $g_i : U_i \to V_{\alpha(i)}$ be a morphism of families of maps with fixed target, see Sites, Definition 9.8.1. Assume that for each $j \in J$ the family $\{g_i : U_i \to V_j\}_{\alpha(i)=j}$ is an fpqc covering of V_j. Then the pullback functor*

$$\textit{descent data relative to } \mathcal{V} \longrightarrow \textit{descent data relative to } \mathcal{U}$$

of Lemma 31.30.8 is fully faithful.

Proof. Consider the morphism of schemes

$$g : X = \coprod\nolimits_{i \in I} U_i \longrightarrow Y = \coprod\nolimits_{j \in J} V_j$$

over S which on the ith component maps into the $\alpha(i)$th component via the morphism $g_{\alpha(i)}$. We claim that $\{g : X \to Y\}$ is an fpqc covering of schemes. Namely, by Topologies, Lemma 30.8.3 for each j the morphism $\{\coprod_{\alpha(i)=j} U_i \to V_j\}$ is an fpqc covering. Thus for every affine open $V \subset V_j$ (which we may think of as an affine open of Y) we can find finitely many affine opens $W_1, \ldots, W_n \subset \coprod_{\alpha(i)=j} U_i$ (which we may think of as affine opens of X) such that $V = \bigcup_{i=1,\ldots,n} g(W_i)$. This provides enough affine opens of Y which can be covered by finitely many affine opens of X so that Topologies, Lemma 30.8.2 part (3) applies, and the claim follows. Let us write $DD(X/S)$, resp. $DD(\mathcal{U})$ for the category of descent data with respect to X/S, resp. $\mathcal{U}$, and similarly for Y/S and $\mathcal{V}$. Consider the diagram

$$\begin{tikzcd} DD(Y/S) \arrow[r] & DD(X/S) \\ DD(\mathcal{V}) \arrow[r] \arrow[u, "\text{Lemma 31.30.5}"] & DD(\mathcal{U}) \arrow[u, "\text{Lemma 31.30.5}"'] \end{tikzcd}$$

This diagram is commutative, see the proof of Lemma 31.30.8. The vertical arrows are equivalences. Hence the lemma follows from Lemma 31.31.4 which shows the top horizontal arrow of the diagram is fully faithful. $\square$

The next lemma shows that, in order to check effectiveness, we may always Zariski refine the given family of morphisms with target S.

Lemma 31.31.10. *Let S be a scheme. Let $\mathcal{U} = \{U_i \to S\}_{i \in I}$, and $\mathcal{V} = \{V_j \to S\}_{j \in J}$, be families of morphisms with target S. Let $\alpha : I \to J$, $id : S \to S$ and $g_i : U_i \to V_{\alpha(i)}$ be a morphism of families of maps with fixed target, see Sites, Definition 9.8.1. Assume that for each $j \in J$ the family $\{g_i : U_i \to V_j\}_{\alpha(i)=j}$ is a Zariski covering (see Topologies, Definition 30.3.1) of V_j. Then the pullback functor*

$$\textit{descent data relative to } \mathcal{V} \longrightarrow \textit{descent data relative to } \mathcal{U}$$

of Lemma 31.30.8 is an equivalence of categories. In particular, the category of schemes over S is equivalent to the category of descent data relative to any Zariski covering of S.

Proof. The functor is faithful and fully faithful by Lemma 31.31.9. Let us indicate how to prove that it is essentially surjective. Let $(X_i, \varphi_{ii'})$ be a descend datum relative to $\mathcal{U}$. Fix $j \in J$ and set $I_j = \{i \in I \mid \alpha(i) = j\}$. For $i, i' \in I_j$ note that there is a canonical morphism

$$c_{ii'} : U_i \times_{g_i, V_j, g_{i'}} U_{i'} \to U_i \times_S U_{i'}.$$

Hence we can pullback $\varphi_{ii'}$ by this morphism and set $\psi_{ii'} = c_{ii'}^* \varphi_{ii'}$ for $i, i' \in I_j$. In this way we obtain a descent datum $(X_i, \psi_{ii'})$ relative to the Zariski covering $\{g_i : U_i \to V_i\}_{i \in I_j}$. Note that $\psi_{ii'}$ is an isomorphism from the open $X_{i, U_i \times_{V_j} U_{i'}}$ of X_i to the corresponding open of $X_{i'}$. It follows from Schemes, Section 21.14 that we may glue $(X_i, \psi_{ii'})$ into a scheme Y_j over V_j. Moreover, the morphisms $\varphi_{ii'}$ for $i \in I_j$ and $i' \in I_{j'}$ glue to a morphism $\varphi_{jj'} : Y_j \times_S V_{j'} \to V_j \times_S Y_{j'}$ satisfying the cocycle condition (details omitted). Hence we obtain the desired descent datum $(Y_j, \varphi_{jj'})$ relative to $\mathcal{V}$. $\square$

Lemma 31.31.11. *Let S be a scheme. Let $\mathcal{U} = \{U_i \to S\}_{i \in I}$, and $\mathcal{V} = \{V_j \to S\}_{j \in J}$, be fpqc-coverings of S. If $\mathcal{U}$ is a refinement of $\mathcal{V}$, then the pullback functor*

$$\textit{descent data relative to } \mathcal{V} \longrightarrow \textit{descent data relative to } \mathcal{U}$$

is fully faithfull. In particular, the category of schemes over S is identified with a full subcategory of the category of descent data relative to any fpqc-covering of S.

Proof. Consider the fpqc-covering $\mathcal{W} = \{U_i \times_S V_j \to S\}_{(i,j)\in I\times J}$ of S. It is a refinement of both $\mathcal{U}$ and $\mathcal{V}$. Hence we have a 2-commutative diagram of functors and categories

$$\begin{tikzcd} DD(\mathcal{V}) \arrow[rr] \arrow[dr] & & DD(\mathcal{U}) \arrow[dl] \\ & DD(\mathcal{W}) & \end{tikzcd}$$

Notation as in the proof of Lemma 31.31.9 and commutativity by Lemma 31.30.8 part (3). Hence clearly it suffices to prove the functors $DD(\mathcal{V}) \to DD(\mathcal{W})$ and $DD(\mathcal{U}) \to DD(\mathcal{W})$ are fully faithful. This follows from Lemma 31.31.9 as desired. □

Remark 31.31.12. Lemma 31.31.11 says that morphisms of schemes satisfy fpqc descent. In other words, given a scheme S and schemes X, Y over S the functor

$$(Sch/S)^{opp} \longrightarrow Sets, \quad T \longmapsto Mor_T(X_T, Y_T)$$

satisfies the sheaf condition for the fpqc topology. The simplest case of this is the following. Suppose that $T \to S$ is a surjective flat morphism of affines. Let $\psi_0 : X_T \to Y_T$ be a morphism of schemes over T which is compatible with the canonical descent data. Then there exists a unique morphism $\psi : X \to Y$ whose base change to T is ψ_0. In fact this special case follows in a straightforward manner from Lemma 31.31.4. And, in turn, that lemma is a formal consequence of the following two facts: (a) the base change functor by a faihfully flat morphism is faithful, see Lemma 31.31.2 and (b) a scheme satisfies the sheaf condition for the fpqc topology, see Lemma 31.9.3.

31.32. Descending types of morphisms

In the following we study the question as to whether descent data for schemes relative to a fpqc-covering are effective. The first remark to make is that this is not always the case. We will see this (insert future reference here).

On the other hand, if the schemes we are trying to descend are particularly simple, then it is sometime the case that for whole classes of schemes descent data are effective. We will introduce terminology here that describes this phenomenon abstractly, even though it may lead to confusion if not used correctly later on.

Definition 31.32.1. Let $\mathcal{P}$ be a property of morphisms of schemes over a base. Let $\tau \in \{Zariski, fpqc, fppf, étale, smooth, syntomic\}$. We say *morphisms of type $\mathcal{P}$ satisfy descent for τ-coverings* if for any τ-covering $\mathcal{U} : \{U_i \to S\}_{i\in I}$ (see Topologies, Section 30.2), any descent datum (X_i, φ_{ij}) relative to $\mathcal{U}$ such that each morphism $X_i \to U_i$ has property $\mathcal{P}$ is effective.

Note that in each of the cases we have already seen that the functor from schemes over S to descent data over $\mathcal{U}$ is fully faithful (Lemma 31.31.11 combined with the results in Topologies that any τ-covering is also a fpqc-covering). We have also seen that descent data are always effective with respect to Zariski coverings (Lemma 31.31.10). It may be prudent to only study the notion just introduced when $\mathcal{P}$ is either stable under any base change or at least local on the base in the τ-topology (see Definition 31.18.1) in order to avoid erroneous arguments (relying on $\mathcal{P}$ when descending halfway).

Here is the obligatory lemma reducing this question to the case of a covering given by a single morphism of affines.

Lemma 31.32.2. *Let $\mathcal{P}$ be a property of morphisms of schemes over a base. Let $\tau \in \{fpqc, fppf, étale, smooth, syntomic\}$. Suppose that*

(1) *$\mathcal{P}$ is stable under any base change (see Schemes, Definition 21.18.3), and*
(2) *for any surjective morphism of affines $X \to S$ which is flat, flat of finite presentation, étale, smooth or syntomic depending on whether τ is fpqc, fppf, étale, smooth, or syntomic, any descent datum (V, φ) relative to X over S such that $\mathcal{P}$ holds for $V \to X$ is effective.*

Then morphisms of type $\mathcal{P}$ satisfy descent for τ-coverings.

Proof. Let S be a scheme. Let $\mathcal{U} = \{\varphi_i : U_i \to S\}_{i\in I}$ be a τ-covering of S. Let $(X_i, \varphi_{ii'})$ be a descent datum relative to $\mathcal{U}$ and assume that each morphism $X_i \to U_i$ has property $\mathcal{P}$. We have to show there exists a scheme $X \to S$ such that $(X_i, \varphi_{ii'}) \cong (U_i \times_S X, can)$.

Before we start the proof proper we remark that for any family of morphisms $\mathcal{V} : \{V_j \to S\}$ and any morphism of families $\mathcal{V} \to \mathcal{U}$, if we pullback the descent datum $(X_i, \varphi_{ii'})$ to a descent datum $(Y_j, \varphi_{jj'})$ over $\mathcal{V}$, then each of the morphisms $Y_j \to V_j$ has property $\mathcal{P}$ also. This is true because we assumed that $\mathcal{P}$ is stable under any base change and the definition of pullback (see Definition 31.30.9). We will use this without further mention.

First, let us prove the lemma when S is affine. By Topologies, Lemma 30.8.8, 30.7.4, 30.4.4, 30.5.4, or 30.6.4 there exists a standard τ-covering $\mathcal{V} : \{V_j \to S\}_{j=1,\ldots,m}$ which refines $\mathcal{U}$. The pullback functor $DD(\mathcal{U}) \to DD(\mathcal{V})$ between categories of descent data is fully faithful by Lemma 31.31.11. Hence it suffices to prove that the descend datum over the standard τ-covering $\mathcal{V}$ is effective. By Lemma 31.30.5 this reduces to the covering $\{\coprod_{j=1,\ldots,m} V_j \to S\}$ for which we have assumed the result in property (2) of the lemma. Hence the lemma holds when S is affine.

Assume S is general. Let $V \subset S$ be an affine open. By the properties of site the family $\mathcal{U}_V = \{V \times_S U_i \to V\}_{i\in I}$ is a τ-covering of V. Denote $(X_i, \varphi_{ii'})_V$ the restriction (or pullback) of the given descent datum to $\mathcal{U}_V$. Hence by what we just saw we obtain a scheme X_V over V whose canonical descent datum with respect to $\mathcal{U}_V$ is isomorphic to $(X_i, \varphi_{ii'})_V$. Suppose that $V' \subset V$ is an affine open of V. Then both $X_{V'}$ and $V' \times_V X_V$ have canonical descent data isomorphic to $(X_i, \varphi_{ii'})_{V'}$. Hence, by Lemma 31.31.11 again we obtain a canonical morphism $\rho^V_{V'} : X_{V'} \to X_V$ over S which identifies $X_{V'}$ with the inverse image of V' in X_V. We omit the verification that given affine opens $V'' \subset V' \subset V$ of S we have $\rho^V_{V''} = \rho^V_{V'} \circ \rho^{V'}_{V''}$.

By Constructions, Lemma 22.2.1 the data $(X_V, \rho^V_{V'})$ glue to a scheme $X \to S$. Moreover, we are given isomorphisms $V \times_S X \to X_V$ which recover the maps $\rho^V_{V'}$. Unwinding the construction of the schemes X_V we obtain isomorphisms

$$V \times_S U_i \times_S X \longrightarrow V \times_S X_i$$

compatible with the maps $\varphi_{ii'}$ and compatible with restricting to smaller affine opens in X. This implies that the canonical descent datum on $U_i \times_S X$ is isomorphic to the given descent datum and we win. $\square$

31.33. Descending affine morphisms

In this section we show that ``affine morphisms satisfy descent for fpqc-coverings''. Here is the formal statement.

Lemma 31.33.1. *Let S be a scheme. Let $\{X_i \to S\}_{i\in I}$ be an fpqc covering, see Topologies, Definition 30.8.1. Let $(V_i/X_i, \varphi_{ij})$ be a descent datum relative to $\{X_i \to S\}$. If each morphism $V_i \to X_i$ is affine, then the descent datum is effective.*

Proof. Being affine is a property of morphisms of schemes which is preserved under any base change, see Morphisms, Lemma 24.11.8. Hence Lemma 31.32.2 applies and it suffices to prove the statement of the lemma in case the fpqc-covering is given by a single $\{X \to S\}$ flat surjective morphism of affines. Say $X = Spec(A)$ and $S = Spec(R)$ so that $R \to A$ is a faithfully flat ring map. Let (V, φ) be a descent datum relative to X over S and assume that $V \to X$ is affine. Then $V \to X$ being affine implies that $V = Spec(B)$ for some A-algebra B (see Morphisms, Definition 24.11.1). The isomorphism φ corresponds to an isomorphism of rings

$$\varphi^\sharp : B \otimes_R A \longleftarrow A \otimes_R B$$

as $A \otimes_R A$-algebras. The cocycle condition on φ says that

$$\begin{tikzcd} B \otimes_R A \otimes_R A & & A \otimes_R A \otimes_R B \arrow[ll] \arrow[dl] \\ & A \otimes_R B \otimes_R A \arrow[ul] & \end{tikzcd}$$

is commutative. Inverting these arrows we see that we have a descent datum for modules with respect to $R \to A$ as in Definition 31.3.1. Hence we may apply Proposition 31.3.9 to obtain an R-module $C = \mathrm{Ker}(B \to A \otimes_R B)$ and an isomorphism $A \otimes_R C \cong B$ respecting descent data. Given any pair $c, c' \in C$ the produc cc' in B lies in C since the map φ is an algebra homomorphism. Hence C is an R-algebra whose base change to A is isomorphic to B compatibly with descent data. Applying $Spec$ we obtain a scheme U over S such that $(V, \varphi) \cong (X \times_S U, can)$ as desired. $\square$

Lemma 31.33.2. *Let S be a scheme. Let $\{X_i \to S\}_{i\in I}$ be an fpqc covering, see Topologies, Definition 30.8.1. Let $(V_i/X_i, \varphi_{ij})$ be a descent datum relative to $\{X_i \to S\}$. If each morphism $V_i \to X_i$ is a closed immersion, then the descent datum is effective.*

Proof. This is true because a closed immersion is an affine morphism (Morphisms, Lemma 24.11.9), and hence Lemma 31.33.1 applies. $\square$

31.34. Descending quasi-affine morphisms

In this section we show that ``quasi-affine morphisms satisfy descent for fpqc-coverings". Here is the formal statement.

Lemma 31.34.1. *Let S be a scheme. Let $\{X_i \to S\}_{i\in I}$ be an fpqc covering, see Topologies, Definition 30.8.1. Let $(V_i/X_i, \varphi_{ij})$ be a descent datum relative to $\{X_i \to S\}$. If each morphism $V_i \to X_i$ is quasi-affine, then the descent datum is effective.*

Proof. Being quasi-affine is a property of morphisms of schemes which is preserved under any base change, see Morphisms, Lemma 24.12.5. Hence Lemma 31.32.2 applies and it suffices to prove the statement of the lemma in case the fpqc-covering is given by a single $\{X \to S\}$ flat surjective morphism of affines. Say $X = Spec(A)$ and $S = Spec(R)$ so that $R \to A$ is a faithfully flat ring map. Let (V, φ) be a descent datum relative to X over S and assume that $\pi : V \to X$ is quasi-affine.

According to Morphisms, Lemma 24.12.3 this means that

$$V \longrightarrow \underline{Spec}_X(\pi_* \mathcal{O}_V) = W$$

is a quasi-compact open immersion of schemes over X. The projections $\mathrm{pr}_i : X \times_S X \to X$ are flat and hence we have

$$\mathrm{pr}_0^*\pi_*\mathcal{O}_V = (\pi \times \mathrm{id}_X)_*\mathcal{O}_{V\times_S X}, \quad \mathrm{pr}_1^*\pi_*\mathcal{O}_V = (\mathrm{id}_X \times \pi)_*\mathcal{O}_{X\times_S V}$$

by flat base change (Coherent, Lemma 25.6.2). Thus the isomorphism $\varphi : V\times_S X \to X\times_S V$ (which is an isomorphism over $X \times_S X$) induces an isomorphism of quasi-coherent sheaves of algebras

$$\varphi^\sharp : \mathrm{pr}_0^*\pi_*\mathcal{O}_V \longrightarrow \mathrm{pr}_1^*\pi_*\mathcal{O}_V$$

on $X \times_S X$. The cocycle condition for φ implies the cocyle condition for $\varphi^\sharp$. Another way to say this is that it produces a descent datum φ' on the affine scheme W relative to X over S, which moreover has the property that the morphism $V \to W$ is a morphism of descent data. Hence by Lemma 31.33.1 (or by effectivity of descent for quasi-coherent algebras) we obtain a scheme $U' \to S$ with an isomorphism $(W, \varphi') \cong (X \times_S U', can)$ of descent data. We note in passing that U' is affine by Lemma 31.19.16.

And now we can think of V as a (quasi-compact) open $V \subset X \times_S U'$ with the property that it is stable under the descent datum

$$can : X \times_S U' \times_S X \to X \times_S X \times_S U', (x_0, u', x_1) \mapsto (x_0, x_1, u').$$

In other words $(x_0, u') \in V \Rightarrow (x_1, u') \in V$ for any x_0, x_1, u' mapping to the same point of S. Because $X \to S$ is surjective we immediately find that V is the inverse image of a subset $U \subset U'$ under the morphism $X \times_S U' \to U'$. Because $X \to S$ is quasi-compact, flat and surjective also $X \times_S U' \to U'$ is quasi-compact flat and surjective. Hence by Morphisms, Lemma 24.24.10 this subset $U \subset U'$ is open and we win. □

31.35. Descent data in terms of sheaves

Here is another way to think about descent data in case of a covering on a site.

Lemma 31.35.1. *Let $\tau \in \{Zariski, fppf, étale, smooth, syntomic$[8]$\}$. Let Sch_τ be a big τ-site. Let $S \in Ob(Sch_\tau)$. Let $\{S_i \to S\}_{i\in I}$ be a covering in the site $(Sch/S)_\tau$. There is an equivalence of categories*

$$\left\{\begin{matrix}\textit{descent data } (X_i, \varphi_{ii'}) \textit{ such that} \\ \textit{each } X_i \in Ob((Sch/S)_\tau)\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\textit{sheaves } F \textit{ on } (Sch/S)_\tau \textit{ such that} \\ \textit{each } h_{S_i} \times F \textit{ is representable}\end{matrix}\right\}.$$

Moreover,

(1) *the objects representing $h_{S_i} \times F$ on the right hand side correspond to the schemes X_i on the left hand side, and*
(2) *the sheaf F is representable if and only if the corresponding descent datum $(X_i, \varphi_{ii'})$ is effective.*

Proof. We have seen in Section 31.9 that representable presheaves are sheaves on the site $(Sch/S)_\tau$. Moreover, the Yonea lemma (Categories, Lemma 4.3.5) garantees that maps between representable sheaves correspond one to one with maps between the representing objects. We will use these remarks without further mention during the proof.

Let us construct the functor from right to left. Let F be a sheaf on $(Sch/S)_\tau$ such that each $h_{S_i} \times F$ is representable. In this case let X_i be a representing object in $(Sch/S)_\tau$. It comes

[8] The fact that fpqc is missing is not a typo. See discussion in Topologies, Section 30.8.

equipped with a morphism $X_i \to S_i$. Then both $X_i \times_S S_{i'}$ and $S_i \times_S X_{i'}$ represent the sheaf $h_{S_i} \times F \times h_{S_{i'}}$ and hence we obtain an isomorphism

$$\varphi_{ii'} : X_i \times_S S_{i'} \to S_i \times_S X_{i'}$$

It is straightforward to see that the maps $\varphi_{ii'}$ are morphisms over $S_i \times_S S_{i'}$ and satsify the cocycle condition. The functor from right to left is given by this construction $F \mapsto (X_i, \varphi_{ii'})$.

Let us construct a functor from left to right. For each i denote F_i the sheaf h_{X_i}. The isomorphisms $\varphi_{ii'}$ give isomorphisms

$$\varphi_{ii'} : F_i \times h_{S_{i'}} \longrightarrow h_{S_i} \times F_{i'}$$

over $h_{S_i} \times h_{S_{i'}}$. Set F equal to the coequalizer in the following diagram

$$\coprod\nolimits_{i,i'} F_i \times h_{S_{i'}} \underset{\mathrm{pr}_1 \circ \varphi_{ii'}}{\overset{\mathrm{pr}_0}{\rightrightarrows}} \coprod\nolimits_i F_i \longrightarrow F$$

The cocylce condition garantees that $h_{S_i} \times F$ is isomorphic to F_i and hence representable. The functor from left to right is given by this construction $(X_i, \varphi_{ii'}) \mapsto F$.

We omit the verification that these constructions are mutually quasi-inverse functors. The final statements (1) and (2) follow from the constructions. $\square$

Remark 31.35.2. In the statement of Lemma 31.35.1 the condition that $h_{S_i} \times F$ is representable is equivalent to the condition that the restriction of F to $(Sch/S_i)_\tau$ is representable.

31.36. Descent in terms of simplicial schemes

A *simplicial scheme* is a simplicial object in the category of schemes, see Simplicial, Definition 14.3.1. In this chapter we will use a subscript $\bullet$ to denote simplicial objects. Recall that a simplicial scheme looks like

$$\begin{tikzcd} X_2 \arrow[r, shift left=3] \arrow[r, leftarrow, shift left=1] \arrow[r, shift right=1] \arrow[r, leftarrow, shift right=3] \arrow[r, shift right=5] & X_1 \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] & X_0 \end{tikzcd}$$

Here there are two morphisms $d^1_0, d^1_1 : X_1 \to X_0$ and a single morphism $s^0_0 : X_0 \to X_1$, etc. It is important to remember that $d^n_i : X_n \to X_{n-1}$ should be thought of as a ``projection forgetting the ith coordinate''.

Definition 31.36.1. Let $a : V_\bullet \to X_\bullet$ be a morphism of simplicial schemes. We say a is *cartesian*, or that $V_\bullet$ *is cartesian over* $X_\bullet$, if for every morphism $\varphi : [n] \to [m]$ of Δ the corresponding diagram

$$\begin{tikzcd} V_m \arrow[r, "a"] \arrow[d, "V_\bullet(\varphi)"'] & X_m \arrow[d, "X_\bullet(\varphi)"] \\ V_n \arrow[r, "a"] & X_n \end{tikzcd}$$

is a fibre square in the category of schemes.

Definition 31.36.2. Let $f : X \to S$ be a morphism of schemes. The *simplicial scheme associated to* f, denoted $(X/S)_\bullet$, is the functor $\Delta^{opp} \to Sch$, $[n] \mapsto X \times_S \ldots \times_S X$ described in Simplicial, Example 14.3.5.

Thus $(X/S)_n$ is the $(n+1)$-fold fibre product of X over S. The morphism $d^1_0 : X \times_S X \to X$ is the map $(x_0, x_1) \mapsto x_1$ and the morphism d^1_1 is the other projection. The morphism s^0_0 is the diagonal morphism $X \to X \times_S X$.

Lemma 31.36.3. *Let $f : X \to S$ be a morphism of schemes. Let $\pi : V_\bullet \to (X/S)_\bullet$ be a cartesian morphism. Set $V = V_0$ considered as a scheme over X. The morphisms $d^1_0, d^1_1 : V_1 \to V_0$ and the morphism $\pi_1 : V_1 \to X \times_S X$ induce isomorphisms*

$$V \times_S X \xleftarrow{(d^1_1, pr_1 \circ \pi_1)} V_1 \xrightarrow{(pr_0 \circ \pi_1, d^1_0)} X \times_S V.$$

Denote $\varphi : V \times_S X \to X \times_S V$ the resulting isomorphism. Then the pair (V, φ) is a descent datum relative to $X \to S$.

Proof. The statement that the displayed morphisms are isomorphisms is exactly the cartesian property for the maps $\delta^1_0, \delta^1_1 : [0] \to [1]$. The fact that the diagram of Definition 31.30.1 (1) commutes follows from the fact that each of the induced morphisms $V_2 \to V \times_{X, \mathrm{pr}_i} (X \times_S X \times_S X)$ associated to $[0] \to [2]$, $0 \mapsto i$ is an isomorphism. Details omitted. □

Lemma 31.36.4. *Let $f : X \to S$ be a morphism of schemes. The construction*

$$\begin{matrix} \textit{category of cartesian} \\ \textit{schemes over } (X/S)_\bullet \end{matrix} \longrightarrow \begin{matrix} \textit{category of descent data} \\ \textit{relative to } X/S \end{matrix}$$

of Lemma 31.36.3 is an equivalence of categories.

Proof. Here you have to show that given a descent datum (V, φ) you can canonically construct a cartesian morphism of simplicial schemes $V_\bullet \to (X/S)_\bullet$ so that if you apply the construction of Lemma 31.36.3 then you get back (V, φ). This we did carefully in Section 31.3 for the case of descent data for modules over rings and their associated cosimplicial rings, see especially Lemma 31.3.2. We can easily translate this to the current context. Namely, set

$$V_n = X \times_S \ldots \times_S X \times_S V.$$

Given a point $(x_0, \ldots, x_{n-1}, v)$ of V_n we use the convention that $x_n = \pi(v)$. Using this convention, given a morphism $\beta : [m] \to [n]$ the associated morphism

$$V_\bullet(\beta) : V_n \longrightarrow V_m$$

maps $(x_0, \ldots, x_{n-1}, v)$ to $(x_{\beta(0)}, \ldots, x_{\beta(m-1)}, v')$ where $\varphi^{-1}(x_{\beta(m)}, v) = (v', x_n)$. (It is a fact that $v' = v$ if $n = \beta(m)$; see discussion following Definition 31.30.1.) We omit the verification that this defines a simplicial scheme which is cartesian over $(X/S)_\bullet$. □

We may reinterpret the pullback of Lemma 31.30.6 as follows. Suppose given a commutative diagram of morphisms of schemes

$$\begin{tikzcd} X' \arrow[r, "f"'] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S. \end{tikzcd}$$

This gives rise to a morphism of simplicial schemes

$$f_\bullet : (X'/S')_\bullet \longrightarrow (X/S)_\bullet.$$

It is a pleasant exercise to check that given any morphism of simplical schemes $f_\bullet : Y_\bullet \to X_\bullet$ and a cartesian simplicial scheme $V_\bullet \to X_\bullet$ the fibre product

$$f^*_\bullet V_\bullet = Y_\bullet \times_{X_\bullet} V_\bullet$$

is a cartesian simplicial scheme over $Y_\bullet$. We omit the verification that this applied to the morphism $(X'/S')_\bullet \to (X/S)_\bullet$ corresponds via Lemma 31.36.4 with the pullback defined in terms of descent data.

31.37. Other chapters

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CHAPTER 32

Adequate Modules

32.1. Introduction

For any scheme X the category $QCoh(\mathcal{O}_X)$ of quasi-coherent modules is abelian and a weak Serre subcategory of the abelian category of all $\mathcal{O}_X$-modules. The same thing works for the category of quasi-coherent modules on an algebraic space X viewed as a subcategory of the category of all $\mathcal{O}_X$-modules on the small étale site of X. Moreover, for a quasi-compact and quasi-separated morphism $f : X \to Y$ the pushforward f_* and higher direct images preserve quasi-coherency.

Next, let X be a scheme and let $\mathcal{O}$ be the structure sheaf on one of the big sites of X, say, the big fppf site. The category of quasi-coherent $\mathcal{O}$-modules is abelian (in fact it is equivalent to the category of usual quasi-coherent $\mathcal{O}_X$-modules on the scheme X we mentioned above) but its imbedding into $Mod(\mathcal{O})$ is not exact. An example is the map of quasi-coherent modules

$$\mathcal{O}_{\mathbf{A}^1_k} \longrightarrow \mathcal{O}_{\mathbf{A}^1_k}$$

on $\mathbf{A}^1_k = Spec(k[x])$ given by multiplication by x. In the abelian category of quasi-coherent sheaves this map is injective, whereas in the abelian category of all $\mathcal{O}$-modules on the big site of $\mathbf{A}^1_k$ this map has a nontrivial kernel as we see by evaluating on sections over $Spec(k[x]/(x)) = Spec(k)$. Moreover, for a quasi-compact and quasi-separated morphism $f : X \to Y$ the functor $f_{big,*}$ does not preserve quasi-coherency.

In this chapter we introduce a larger category of modules, closely related to quasi-coherent modules, which ``fixes'' the two problems mentioned above.

32.2. Conventions

In this chapter we fix $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$ and we fix a big τ-site Sch_τ as in Topologies, Section 30.2. All schemes will be objects of Sch_τ. In particular, given a scheme S we obtain sites $(Aff/S)_\tau \subset (Sch/S)_\tau$. The structure sheaf $\mathcal{O}$ on these sites is defined by the rule $\mathcal{O}(T) = \Gamma(T, \mathcal{O}_T)$.

All rings A will be such that $Spec(A)$ is (isomorphic to) an object of Sch_τ. Given a ring A we denote Alg_A the category of A-algebras whose objects are the A-algebras B of the form $B = \Gamma(U, \mathcal{O}_U)$ where S is an affine object of Sch_τ. Thus given an affine scheme $S = Spec(A)$ the functor

$$(Aff/S)_\tau \longrightarrow Alg_A, \quad U \longmapsto \mathcal{O}(U)$$

is an equivalence.

32.3. Adequate functors

In this section we discuss a topic closely related to direct images of quasi-coherent sheaves. Most of this material was taken from the paper [**Jaf97**].

Definition 32.3.1. Let A be a ring. A *module-valued functor* is a functor $F : Alg_A \to Ab$ such that

(1) for every object B of Alg_A the group $F(B)$ is endowed with the structure of a B-module, and
(2) for any morphism $B \to B'$ of Alg_A the map $F(B) \to F(B')$ is B-linear.

A *morphism of module-valued functors* is a transformation of functors $\varphi : F \to G$ such that $F(B) \to G(B)$ is B-linear for all $B \in Ob(Alg_A)$.

Let $S = Spec(A)$ be an affine scheme. The category of module-valued functors on Alg_A is equivalent to the category $PMod((Aff/S)_\tau, \mathcal{O})$ of presheaves of $\mathcal{O}$-modules. The equivalence is given by the rule which assigns to the module-valued functor F the presheaf $\mathcal{F}$ defined by the rule $\mathcal{F}(U) = F(\mathcal{O}(U))$. This is clear from the equivalence $(Aff/S)_\tau \to Alg_A$, $U \mapsto \mathcal{O}(U)$ given in Section 32.2. The quasi-inverse sets $F(B) = \mathcal{F}(Spec(B))$.

An important special case of a module-valued functor comes about as follows. Let M be an A-module. Then we will denote $\underline{M}$ the module-valued functor $B \mapsto M \otimes_A B$ (with obvious B-module structure). Note that if $M \to N$ is a map of A-modules then there is an associated morphism $\underline{M} \to \underline{N}$ of module-valued functors. Conversely, any morphism of module-valued functors $\underline{M} \to \underline{N}$ comes from an A-module map $M \to N$ as the reader can see by evaluating on $B = A$. In other words Mod_A is a full subcategory of the category of module-valued functors on Alg_A.

Given and A-module map $\varphi : M \to N$ then $\text{Coker}(\underline{M} \to \underline{N}) = \underline{Q}$ where $Q = \text{Coker}(M \to N)$ because $\otimes$ is right exact. But this isn't the case for the kernel in general: for example an injective map of A-modules need not be injective after base change. Thus the following definition makes sense.

Definition 32.3.2. Let A be a ring. A module-valued functor F on Alg_A is called

(1) *adequate* if there exists a map of A-modules $M \to N$ such that F is isomorphic to $\text{Ker}(\underline{M} \to \underline{N})$.
(2) *linearly adequate* if F is isomorphic to the kernel of a map $\underline{A^{\oplus n}} \to \underline{A^{\oplus m}}$.

Note that F is adequate if and only if there exists an exact sequence $0 \to F \to \underline{M} \to \underline{N}$ and F is linearly adequate if and only if there exists an exact sequence $0 \to F \to \underline{A^{\oplus n}} \to \underline{A^{\oplus m}}$.

Let A be a ring. In this section we will show the category of adequate functors on Alg_A is abelian (Lemmas 32.3.10 and 32.3.11) and has a set of generators (Lemma 32.3.6). We will also see that it is a weak Serre subcategory of the category of all module-valued functors on Alg_A (Lemma 32.3.16) and that it has arbitrary colimits (Lemma 32.3.12).

Lemma 32.3.3. *Let A be a ring. Let F be an adequate functor on Alg_A. If $B = colim\, B_i$ is a filtered colimit of A-algebras, then $F(B) = colim\, F(B_i)$.*

Proof. This holds because for any A-module M we have $M \otimes_A B = colim\, M \otimes_A B_i$ (see Algebra, Lemma 7.11.8) and because filtered colimits commute with exact sequences, see Algebra, Lemma 7.8.9. □

Remark 32.3.4. Consider the category $Alg_{fp,A}$ whose objects are A-algebras B of the form $B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ and whose morphisms are A-algebra maps. Every A-algebra B is a filtered colimit of finitely presented A-algebra, i.e., a filtered colimit of objects of $Alg_{fp,A}$. By Lemma 32.3.3 we conclude every adequate functor F is determined by its restriction to $Alg_{fp,A}$. For some questions we can therefore restrict to functors on

$Alg_{fp,A}$. For example, the category of adequate functors does not depend on the choice of the big τ-site chosen in Section 32.2.

Lemma 32.3.5. *Let A be a ring. Let F be an adequate functor on Alg_A. If $B \to B'$ is flat, then $F(B) \otimes_B B' \to F(B')$ is an isomorphism.*

Proof. Choose an exact sequence $0 \to F \to \underline{M} \to \underline{N}$. This gives the diagram

$$\begin{tikzcd} 0 \arrow[r] & F(B) \otimes_B B' \arrow[r] \arrow[d] & (M \otimes_A B) \otimes_B B' \arrow[r] \arrow[d] & (N \otimes_A B) \otimes_B B' \arrow[d] \\ 0 \arrow[r] & F(B') \arrow[r] & M \otimes_A B' \arrow[r] & N \otimes_A B' \end{tikzcd}$$

where the rows are exact (the top one because $B \to B'$ is flat). Since the right two vertical arrows are isomorphisms, so is the left one. □

Lemma 32.3.6. *Let A be a ring. Let F be an adequate functor on Alg_A. Then there exists a surjection $L \to F$ with L a direct sum of linearly adequate functors.*

Proof. Choose an exact sequence $0 \to F \to \underline{M} \to \underline{N}$ where $\underline{M} \to \underline{N}$ is given by $\varphi : M \to N$. By Lemma 32.3.3 it suffices to construct $L \to F$ such that $L(B) \to F(B)$ is surjective for every finitely presented A-algebra B. Hence it suffices to construct, given a finitely presented A-algebra B and an element $\xi \in F(B)$ a map $L \to F$ with L linearly adequate such that ξ is in the image of $L(B) \to F(B)$. (Because there is a set worth of such pairs (B, ξ) up to isomorphism.)

To do this write $\sum_{i=1,\ldots,n} m_i \otimes b_i$ the image of ξ in $\underline{M}(B) = M \otimes_A B$. We know that $\sum \varphi(m_i) \otimes b_i = 0$ in $N \otimes_A B$. As N is a filtered colimit of finitely presented A-modules, we can find a finitely presented A-module N', a commutative diagram of A-modules

$$\begin{tikzcd} A^{\oplus n} \arrow[r] \arrow[d, "m_1, \ldots, m_n"'] & N' \arrow[d] \\ M \arrow[r] & N \end{tikzcd}$$

such that $(b_1, \ldots, b_n)$ maps to zero in $N' \otimes_A B$. Choose a presentation $A^{\oplus l} \to A^{\oplus k} \to N' \to 0$. Choose a lift $A^{\oplus n} \to A^{\oplus k}$ of the map $A^{\oplus n} \to N'$ of the diagram. Then we see that there exist $(c_1, \ldots, c_l) \in B^{\oplus l}$ such that $(b_1, \ldots, b_n, c_1, \ldots, c_l)$ maps to zero in $B^{\oplus k}$ under the map $B^{\oplus n} \oplus B^{\oplus l} \to B^{\oplus k}$. Consider the commutative diagram

$$\begin{tikzcd} A^{\oplus n} \oplus A^{\oplus l} \arrow[r] \arrow[d] & A^{\oplus k} \arrow[d] \\ M \arrow[r] & N \end{tikzcd}$$

where the left vertical arrow is zero on the summand $A^{\oplus l}$. Then we see that L equal to the kernel of $\underline{A^{\oplus n+l}} \to \underline{A^{\oplus k}}$ works because the element $(b_1, \ldots, b_n, c_1, \ldots, c_l) \in L(B)$ maps to ξ. □

Consider a graded A-algebra $B = \bigoplus_{d \geq 0} B_d$. Then there are two A-algebra maps $p, a : B \to B[t, t^{-1}]$, namely $p : b \mapsto b$ and $a : b \mapsto t^{\deg(b)} b$ where b is homogeneous. If F is a module-valued functor on Alg_A, then we define

$$F(B)^{(k)} = \{\xi \in F(B) \mid t^k F(p)(\xi) = F(a)(\xi)\}. \quad (32.3.6.1)$$

For functors which behave well with respect to flat ring extensions this gives a direct sum decompostion. This amounts to the fact that representations of $\mathbf{G}_m$ are completely reducible.

Lemma 32.3.7. *Let A be a ring. Let F be a module-valued functor on Alg_A. Assume that for $B \to B'$ flat the map $F(B) \otimes_B B' \to F(B')$ is an isomorphism. Let B be a graded A-algebra. Then*

1. *$F(B) = \bigoplus_{k \in \mathbf{Z}} F(B)^{(k)}$, and*
2. *the map $B \to B_0 \to B$ induces map $F(B) \to F(B)$ whose image is contained in $F(B)^{(0)}$.*

Proof. Let $x \in F(B)$. The map $p : B \to B[t, t^{-1}]$ is free hence we know that

$$F(B[t, t^{-1}]) = \bigoplus\nolimits_{k \in \mathbf{Z}} F(p)(F(B)) \cdot t^k = \bigoplus\nolimits_{k \in \mathbf{Z}} F(B) \cdot t^k$$

as indicated we drop the $F(p)$ in the rest of the proof. Write $F(a)(x) = \sum t^k x_k$ for some $x_k \in F(B)$. Denote $\epsilon : B[t, t^{-1}] \to B$ the B-algebra map $t \mapsto 1$. Note that the compositions $\epsilon \circ p, \epsilon \circ a : B \to B[t, t^{-1}] \to B$ are the identity. Hence we see that

$$x = F(\epsilon)(F(a)(x)) = F(\epsilon)(\sum t^k x_k) = \sum x_k.$$

On the other hand, we claim that $x_k \in F(B)^{(k)}$. Namely, consider the commutative diagram

$$\begin{tikzcd} B \arrow[r, "a"'] \arrow[d, "a'"'] & B[t, t^{-1}] \arrow[d, "f"] \\ B[s, s^{-1}] \arrow[r, "g"] & B[t, s, t^{-1}, s^{-1}] \end{tikzcd}$$

where $a'(b) = s^{\deg(b)} b$, $f(b) = b$, $f(t) = st$ and $g(b) = t^{\deg(b)} b$ and $g(s) = s$. Then

$$F(g)(F(a'))(x) = F(g)(\sum s^k x_k) = \sum s^k F(a)(x_k)$$

and going the other way we see

$$F(f)(F(a))(x) = F(f)(\sum t^k x_k) = \sum (st)^k x_k.$$

Since $B \to B[s, t, s^{-1}, t^{-1}]$ is free we see that $F(B[t, s, t^{-1}, s^{-1}]) = \bigoplus_{k,l \in \mathbf{Z}} F(B) \cdot t^k s^l$ and comparing coefficients in the expressions above we find $F(a)(x_k) = t^k x_k$ as desired.

Finally, the image of $F(B_0) \to F(B)$ is contained in $F(B)^{(0)}$ because $B_0 \to B \xrightarrow{a} B[t, t^{-1}]$ is equal to $B_0 \to B \xrightarrow{p} B[t, t^{-1}]$. $\square$

As a particular case of Lemma 32.3.7 note that

$$\underline{M}(B)^{(k)} = M \otimes_A B_k$$

where B_k is the degree k part of the graded A-algebra B.

Lemma 32.3.8. *Let A be a ring. Given a solid diagram*

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] \arrow[d, "\varphi"'] & \underline{A^{\oplus n}} \arrow[r] \arrow[dl, dotted] & \underline{A^{\oplus m}} \\ & \underline{M} & & \end{tikzcd}$$

of module-valued functors on Alg_A with exact row there exists a dotted arrow making the diagram commute.

Proof. Suppose that the map $A^{\oplus n} \to A^{\oplus m}$ is given by the $m \times n$-matrix (a_{ij}). Consider the ring $B = A[x_1, \ldots, x_n]/(\sum a_{ij}x_j)$. The element $(x_1, \ldots, x_n) \in \underline{A}^{\oplus n}(B)$ maps to zero in $\underline{A}^{\oplus m}(B)$ hence is the image of a unique element $\xi \in L(B)$. Note that ξ has the following universal property: for any A-algebra C and any $\xi' \in L(C)$ there exists an A-algebra map $B \to C$ such that ξ maps to ξ' via the map $L(B) \to L(C)$.

Note that B is a graded A-algebra, hence we can use Lemmas 32.3.7 and 32.3.5 to decompose the values of our functors on B into graded pieces. Note that $\xi \in L(B)^{(1)}$ as $(x_1, \ldots, x_n)$ is an element of degree one in $\underline{A}^{\oplus n}(B)$. Hence we see that $\varphi(\xi) \in \underline{M}(B)^{(1)} = M \otimes_A B_1$. Since B_1 is generated by $x_1, \ldots, x_n$ as an A-module we can write $\varphi(\xi) = \sum m_i \otimes x_i$. Consider the map $A^{\oplus n} \to M$ which maps the ith basis vector to m_i. By construction the associated map $\underline{A}^{\oplus n} \to \underline{M}$ maps the element ξ to $\varphi(\xi)$. It follows from the universal property mentioned above that the diagram commutes. $\square$

Lemma 32.3.9. *Let A be a ring. Let $\varphi : F \to \underline{M}$ be a map of module-valued functors on Alg_A with F adequate. Then Coker(φ) is adequate.*

Proof. By Lemma 32.3.6 we may assume that $F = \bigoplus L_i$ is a direct sum of linearly adequate functors. Choose exact sequences $0 \to L_i \to \underline{A}^{\oplus n_i} \to \underline{A}^{\oplus m_i}$. For each i choose a map $A^{\oplus n_i} \to M$ as in Lemma 32.3.8. Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus L_i \arrow[r] \arrow[d] & \bigoplus \underline{A}^{\oplus n_i} \arrow[r] \arrow[dl] & \bigoplus \underline{A}^{\oplus m_i} \\
& \underline{M} & &
\end{tikzcd}$$

Consider the A-modules

$$Q = \text{Coker}(\bigoplus A^{\oplus n_i} \to M \oplus \bigoplus A^{\oplus m_i}) \quad \text{and} \quad P = \text{Coker}(\bigoplus A^{\oplus n_i} \to \bigoplus A^{\oplus m_i}).$$

Then we see that Coker(φ) is isomorphic to the kernel of $\underline{Q} \to \underline{P}$. $\square$

Lemma 32.3.10. *Let A be a ring. Let $\varphi : F \to G$ be a map of adequate functors on Alg_A. Then Coker(φ) is adequate.*

Proof. Choose an injection $G \to \underline{M}$. Then we have an injection $G/F \to \underline{M}/F$. By Lemma 32.3.9 we see that $\underline{M}/F$ is adequate, hence we can find an injection $\underline{M}/F \to \underline{N}$. Composing we obtain an injection $G/F \to \underline{N}$. By Lemma 32.3.9 the cokernel of the induced map $G \to \underline{N}$ is adequate hence we can find an injection $\underline{N}/G \to \underline{K}$. Then $0 \to G/F \to \underline{N} \to \underline{K}$ is exact and we win. $\square$

Lemma 32.3.11. *Let A be a ring. Let $\varphi : F \to G$ be a map of adequate functors on Alg_A. Then Ker(φ) is adequate.*

Proof. Choose an injection $F \to \underline{M}$ and an injection $G \to \underline{N}$. Denote $F \to \underline{M \oplus N}$ the diagonal map so that

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & G \arrow[d] \\
\underline{M \oplus N} \arrow[r] & \underline{N}
\end{tikzcd}$$

commutes. By Lemma 32.3.10 we can find a module map $M \oplus N \to K$ such that F is the kernel of $\underline{M \oplus N} \to \underline{K}$. Then Ker$(\varphi)$ is the kernel of $\underline{M \oplus N} \to \underline{K \oplus N}$. $\square$

Lemma 32.3.12. *Let A be a ring. An arbitrary direct sum of adequate functors on Alg_A is adequate. A colimit of adequate functors is adequate.*

Proof. The statement on direct sums is immediate. A general colimit can be written as a kernel of a map between direct sums, see Categories, Lemma 4.13.11. Hence this follows from Lemma 32.3.11. □

Lemma 32.3.13. *Let A be a ring. Let F, G be module-valued functors on Alg_A. Let $\varphi : F \to G$ be a transformation of functors. Assume*

(1) *φ is additive,*
(2) *for every A-algebra B and $\xi \in F(B)$ and unit $u \in B^*$ we have $\varphi(u\xi) = u\varphi(\xi)$ in $G(B)$, and*
(3) *for any flat ring map $B \to B'$ we have $G(B) \otimes_B B' = G(B')$.*

Then φ is a morphism of module-valued functors.

Proof. Let B be an A-algebra, $\xi \in F(B)$, and $b \in B$. We have to show that $\varphi(b\xi) = b\varphi(\xi)$. Consider the ring map

$$B \to B' = B[x, y, x^{-1}, y^{-1}]/(x + y - b).$$

This ring map is faithfully flat, hence $G(B) \subset G(B')$. On the other hand

$$\varphi(b\xi) = \varphi((x + y)\xi) = \varphi(x\xi) + \varphi(y\xi) = x\varphi(\xi) + y\varphi(\xi) = (x + y)\varphi(\xi) = b\varphi(\xi)$$

because x, y are units in B'. Hence we win. □

Lemma 32.3.14. *Let A be a ring. Let $0 \to \underline{M} \to G \to L \to 0$ be a short exact sequence of module-valued functors on Alg_A with L linearly adequate. Then G is adequate.*

Proof. We first point out that for any flat A-algebra map $B \to B'$ the map $G(B) \otimes_B B' \to G(B')$ is an isomorphism. Namely, this holds for $\underline{M}$ and L, see Lemma 32.3.5 and hence follows for G by the five lemma. In particular, by Lemma 32.3.7 we see that $G(B) = \bigoplus_{k \in \mathbf{Z}} G(B)^{(k)}$ for any graded A-algebra B.

Choose an exact sequence $0 \to L \to \underline{A^{\oplus n}} \to \underline{A^{\oplus m}}$. Suppose that the map $A^{\oplus n} \to A^{\oplus m}$ is given by the $m \times n$-matrix (a_{ij}). Consider the graded A-algebra $B = A[x_1, \ldots, x_n]/(\sum a_{ij}x_j)$. The element $(x_1, \ldots, x_n) \in \underline{A^{\oplus n}}(B)$ maps to zero in $\underline{A^{\oplus m}}(B)$ hence is the image of a unique element $\xi \in L(B)$. Observe that $\xi \in L(B)^{(1)}$. The map

$$Hom_A(B, C) \longrightarrow L(C), \quad f \longmapsto L(f)(\xi)$$

defines an isomorphism of functors. The reason is that f is determined by the images $c_i = f(x_i) \in C$ which have to satisfy the relations $\sum a_{ij}c_j = 0$. And $L(C)$ is the set of n-tuples $(c_1, \ldots, c_n)$ satisfying the relations $\sum a_{ij}c_j = 0$.

Since the value of each of the functors $\underline{M}$, G, L on B is a direct sum of its weight spaces (by the lemma mentioned above) exactness of $0 \to \underline{M} \to G \to L \to 0$ implies the sequence $0 \to \underline{M}(B)^{(1)} \to G(B)^{(1)} \to L(B)^{(1)} \to 0$ is exact. Thus we may choose an element $\theta \in G(B)^{(1)}$ mapping to ξ.

Consider the graded A-algebra

$$C = A[x_1, \ldots, x_n, y_1, \ldots, y_n]/(\sum a_{ij}x_j, \sum a_{ij}y_j)$$

There are three graded A-algebra homomorphisms $p_1, p_2, m : B \to C$ defined by the rules

$$p_1(x_i) = x_i, \quad p_1(x_i) = y_i, \quad m(x_i) = x_i + y_i.$$

We will show that the element

$$\tau = G(m)(\theta) - G(p_1)(\theta) - G(p_2)(\theta) \in G(C)$$

is zero. First, τ maps to zero in $L(C)$ by a direct calculation. Hence τ is an element of $\underline{M}(C)$. Moreover, since m, p_1, p_2 are graded algebra maps we see that $\tau \in G(C)^{(1)}$ and since $\underline{M} \subset G$ we conclude

$$\tau \in \underline{M}(C)^{(1)} = M \otimes_A C_1.$$

We may write uniquely $\tau = \underline{M}(p_1)(\tau_1) + \underline{M}(p_2)(\tau_2)$ with $\tau_i \in M \otimes_A B_1 = \underline{M}(B)^{(1)}$ because $C_1 = p_1(B_1) \oplus p_2(B_1)$. Consider the ring map $q_1 : C \to B$ defined by $x_i \mapsto x_i$ and $y_i \mapsto 0$. Then $\underline{M}(q_1)(\tau) = \underline{M}(q_1)(\underline{M}(p_1)(\tau_1) + \underline{M}(p_2)(\tau_2)) = \tau_1$. On the other hand, because $q_1 \circ m = q_1 \circ p_1$ we see that $G(q_1)(\tau) = -G(q_1 \circ p_2)(\tau)$. Since $q_1 \circ p_2$ factors as $B \to A \to B$ we see that $G(q_1 \circ p_2)(\tau)$ is in $G(B)^{(0)}$, see Lemma 32.3.7. Hence $\tau_1 = 0$ because it is in $G(B)^{(0)} \cap \underline{M}(B)^{(1)} \subset G(B)^{(0)} \cap G(B)^{(1)} = 0$. Similarly $\tau_2 = 0$, whence $\tau = 0$.

Since $\theta \in G(B)$ we obtain a transformation of functors

$$\psi : L(-) = Hom_A(B, -) \longrightarrow G(-)$$

by mapping $f : B \to C$ to $G(f)(\theta)$. Since θ is a lift of ξ the map ψ is a right inverse of $G \to L$. In terms of ψ the statements proved above have the following meaning: $\tau = 0$ means that ψ is additive and $\theta \in G(B)^{(1)}$ implies that for any A-algebra D we have $\psi(ul) = u\psi(l)$ in $G(D)$ for $l \in L(D)$ and $u \in D^*$ a unit. This implies that ψ is a morphism of module-valued functors, see Lemma 32.3.13. Clearly this implies that $G \cong \underline{M} \oplus L$ and we win. □

Remark 32.3.15. Let A be a ring. The proof of Lemma 32.3.14 shows that any extension $0 \to \underline{M} \to E \to L \to 0$ of module-valued functors on Alg_A with L linearly adequate splits. It uses only the following properties of the module-valued functor $F = \underline{M}$:

(1) $F(B) \otimes_B B' \to F(B')$ is an isomorphism for a flat ring map $B \to B'$, and
(2) $F(C)^{(1)} = F(p_1)(F(B)^{(1)}) \oplus F(p_2)(F(B)^{(1)})$ where $B = A[x_1, \ldots, x_n]/(\sum a_{ij}x_j)$ and $C = A[x_1, \ldots, x_n, y_1, \ldots, y_n]/(\sum a_{ij}x_j, \sum a_{ij}y_j)$.

These two properties hold for any adequate functor F; details omitted. Hence we see that L is a projective object of the abelian category of adequate functors.

Lemma 32.3.16. *Let A be a ring. Let $0 \to F \to G \to H \to 0$ be a short exact sequence of module-valued functors on Alg_A. If F and H are adequate, so is G.*

Proof. Choose an exact sequence $0 \to F \to \underline{M} \to \underline{N}$. If we can show that $(\underline{M} \oplus G)/F$ is adequate, then G is the kernel of the map of adequate functors $(\underline{M} \oplus G)/F \to \underline{N}$, hence adequate by Lemma 32.3.11. Thus we may assume $F = \underline{M}$.

We can choose a surjection $L \to H$ where L is a direct sum of linearly adequate functors, see Lemma 32.3.6. If we can show that the pullback $G \times_H L$ is adequate, then G is the cokernel of the map $\mathrm{Ker}(L \to H) \to G \times_H L$ hence adequate by Lemma 32.3.10. Thus we may assume that $H = \bigoplus L_i$ is a direct sum of linearly adequate functors. By Lemma 32.3.14 each of the pullbacks $G \times_H L_i$ is adequate. By Lemma 32.3.12 we see that $\bigoplus G \times_H L_i$ is adequate. Then G is the cokernel of

$$\bigoplus\nolimits_{i \neq i'} F \longrightarrow \bigoplus G \times_H L_i$$

where ξ in the summand (i, i') maps to $(0, \ldots, 0, \xi, 0, \ldots, 0, -\xi, 0, \ldots, 0)$ with nonzero entries in the summands i and i'. Thus G is adequate by Lemma 32.3.10. □

Lemma 32.3.17. *Let $A \to A'$ be a ring map. If F is an adequate functor on Alg_A, then its restriction F' to $Alg_{A'}$ is adequate too.*

Proof. Choose an exact sequence $0 \to F \to \underline{M} \to \underline{N}$. Then $F'(B') = F(B') = \text{Ker}(M \otimes_A B' \to N \otimes_A B')$. Since $M \otimes_A B' = M \otimes_A A' \otimes_{A'} B'$ and similarly for N we see that F' is the kernel of $\underline{M \otimes_A A'} \to \underline{N \otimes_A A'}$. $\square$

Lemma 32.3.18. *Let $A \to A'$ be a ring map. If F' is an adequate functor on $Alg_{A'}$, then the module-valued functor $F : B \mapsto F'(A' \otimes_A B)$ on Alg_A is adequate too.*

Proof. Choose an exact sequence $0 \to F' \to \underline{M'} \to \underline{N'}$. Then

$$\begin{aligned} F(B) &= F'(A' \otimes_A B) \\ &= \text{Ker}(M' \otimes_{A'} (A' \otimes_A B) \to N' \otimes_{A'} (A' \otimes_A B)) \\ &= \text{Ker}(M' \otimes_A B \to N' \otimes_A B) \end{aligned}$$

Thus F is the kernel of $\underline{M} \to \underline{N}$ where $M = M'$ and $N = N'$ viewed as A-modules. $\square$

Lemma 32.3.19. *Let $A = A_1 \times \ldots \times A_n$ be a product of rings. An adequate functor over A is the same thing as a sequence $F_1, \ldots, F_n$ of adequate functors F_i over A_i.*

Proof. This is true because an A-algebra B is canonically a product $B_1 \times \ldots \times B_n$ and the same thing holds for A-modules. Setting $F(B) = \coprod F_i(B_i)$ gives the correspondence. Details omitted. $\square$

Lemma 32.3.20. *Let $A \to A'$ be a ring map and let F be a module-valued functor on Alg_A such that*

(1) *the restriction F' of F to the category of A'-algebras is adequate, and*
(2) *for any A-algebra B the sequence*

$$0 \to F(B) \to F(B \otimes_A A') \to F(B \otimes_A A' \otimes_A A')$$

is exact.

Then F is adequate.

Proof. The functors $B \to F(B \otimes_A A')$ and $B \mapsto F(B \otimes_A A' \otimes_A A')$ are adequate, see Lemmas 32.3.18 and 32.3.17. Hence F as a kernel of a map of adequate functors is adequate, see Lemma 32.3.11. $\square$

32.4. Higher exts of adequate functors

Let A be a ring. In Lemma 32.3.16 we have seen that any extension of adequate functors in the category of module-valued functors on Alg_A is adequate. In this section we show that the same remains true for higher ext groups.

Lemma 32.4.1. *Let A be a ring. For every module-valued functor F on Alg_A there exists a morphism $Q(F) \to F$ of module-valued functors on Alg_A such that (1) $Q(F)$ is adequate and (2) for every adequate functor G the map $Hom(G, Q(F)) \to Hom(G, F)$ is a bijection.*

Proof. Choose a set $\{L_i\}_{i \in I}$ of linearly adequate functors such that every linearly adequate functor is isomorphic to one of the L_i. This is possible. Suppose that we can find $Q(F) \to F$ with (1) and (2)' or every $i \in I$ the map $Hom(L_i, Q(F)) \to Hom(L_i, F)$ is a bijection. Then (2) holds. Namely, combining Lemmas 32.3.6 and 32.3.11 we see that every adequate functor G sits in an exact sequence

$$K \to L \to G \to 0$$

with K and L direct sums of linearly adequate functors. Hence (2)' implies that $Hom(L, Q(F)) \to Hom(L, F)$ and $Hom(K, Q(F)) \to Hom(K, F)$ are bijections, whence the same thing for G.

Consider the category $\mathcal{J}$ whose objects are pairs (i, φ) where $i \in I$ and $\varphi : L_i \to F$ is a morphism. A morphism $(i, \varphi) \to (i', \varphi')$ is a map $\psi : L_i \to L_{i'}$ such that $\varphi' \circ \psi = \varphi$. Set

$$Q(F) = colim_{(i,\varphi)\in Ob(\mathcal{J})} L_i$$

There is a natural map $Q(F) \to F$, by Lemma 32.3.12 it is adequate, and by construction it has property (2)'. $\square$

Lemma 32.4.2. *Let A be a ring. Denote $\mathcal{P}$ the category of module-valued functors on Alg_A and $\mathcal{A}$ the category of adequate functors on Alg_A. Denote $i : \mathcal{A} \to \mathcal{P}$ the inclusion functor. Denote $Q : \mathcal{P} \to \mathcal{A}$ the construction of Lemma 32.4.1. Then*

(1) *i is fully faithful, exact, and its image is a weak Serre subcategory,*
(2) *$\mathcal{P}$ has enough injectives,*
(3) *the functor Q is a right adjoint to i hence left exact,*
(4) *Q transforms injectives into injectives,*
(5) *$\mathcal{A}$ has enough injectives.*

Proof. This lemma just collects some facts we have already seen sofar. Part (1) is clear from the definitions, the characterization of weak Serre subcategories (see Homology, Lemma 10.7.3), and Lemmas 32.3.10, 32.3.11, and 32.3.16. Recall that $\mathcal{P}$ is equivalent to the category $PMod((Aff/Spec(A))_\tau, \mathcal{O})$. Hence (2) by Injectives, Proposition 17.12.5. Part (3) follows from Lemma 32.4.1 and Categories, Lemma 4.22.2. Parts (4) and (5) follow from Homology, Lemmas 10.22.1 and 10.22.3. $\square$

Let A be a ring. As in Formal Deformation Theory, Section 51.10 given an A-algebra B and an B-module N we set $B[N]$ equal to the R-algebra with underlying B-module $B \oplus N$ with multiplication given by $(b, m)(b', m') = (bb', bm' + b'm)$. Note that this construction is functorial in the pair (B, N) where morphism $(B, N) \to (B', N')$ is given by an A-algebra map $B \to B'$ and an B-module map $N \to N'$. In some sense the functor TF of pairs defined in the following lemma is the tangent space of F. Below we will only consider pairs (B, N) such that $B[N]$ is an object of Alg_A.

Lemma 32.4.3. *Let A be a ring. Let F be a module valued functor. For every $B \in Ob(Alg_A)$ and B-module N there is a canonical decomposition*

$$F(B[N]) = F(B) \oplus TF(B, N)$$

characterized by the following properties

(1) *$TF(B, N) = Ker(F(B[N]) \to F(B))$,*
(2) *there is a B-module structure $TF(B, N)$ compatible with $B[N]$-module structure on $F(B[N])$,*
(3) *TF is a functor from the category of pairs (B, N),*
(4) *there are canonical maps $N \otimes_B F(B) \to TF(B, N)$ inducing a transformation between functors defined on the category of pairs (B, N),*
(5) *$TF(B, 0) = 0$ and the map $TF(B, N) \to TF(B, N')$ is zero when $N \to N'$ is the zero map.*

Proof. Since $B \to B[N] \to B$ is the identity we see that $F(B) \to F(B[N])$ is a direct summand whose complement is $TF(N, B)$ as defined in (1). This construction is functorial in the pair (B, N) simply because given a morphism of pairs $(B, N) \to (B', N')$ we obtain a commutative diagram

$$\begin{tikzcd} B' \arrow[r] & B'[N'] \arrow[r] & B' \\ B \arrow[r] \arrow[u] & B[N] \arrow[r] \arrow[u] & B \arrow[u] \end{tikzcd}$$

in Alg_A. The B-module structure comes from the $B[N]$-module structure and the ring map $B \to B[N]$. The map in (4) is the composition

$$N \otimes_B F(B) \longrightarrow B[N] \otimes_{B[N]} F(B[N]) \longrightarrow F(B[N])$$

whose image is contained in $TF(B, N)$. (The first arrow uses the inclusions $N \to B[N]$ and $F(B) \to F(B[N])$ and the second arrow is the multiplication map.) If $N = 0$, then $B = B[N]$ hence $TF(B, 0) = 0$. If $N \to N'$ is zero then it factors as $N \to 0 \to N'$ hence the induced map is zero since $TF(B, 0) = 0$. $\square$

Let A be a ring. Let M be an A-module. Then the module-valued functor $\underline{M}$ has tangent space $T\underline{M}$ given by the rule $T\underline{M}(B, N) = N \otimes_A M$. In particular, for B given, the functor $N \mapsto T\underline{M}(B, N)$ is additive and right exact. It turns out this also holds for injective module-valued functors.

Lemma 32.4.4. *Let A be a ring. Let I be an injective object of the category of module-valued functors. Then for any $B \in Ob(Alg_A)$ and short exact sequence $0 \to N_1 \to N \to N_2 \to 0$ of B-modules the sequence*

$$TI(B, N_1) \to TI(B, N) \to TI(B, N_2) \to 0$$

is exact.

Proof. We will use the results of Lemma 32.4.3 without further mention. Denote $h : Alg_A \to Sets$ the functor given by $h(C) = Mor_A(B[N], C)$. Similarly for h_1 and h_2. The map $B[N] \to B[N_2]$ corresponding to the surjection $N \to N_2$ is surjective. It corresponds to a map $h_2 \to h$ such that $h_2(C) \to h(C)$ is injective for all A-algebras C. On the other hand, there are two maps $p, q : h \to h_1$, corresponding to the zero map $N_1 \to N$ and the injection $N_1 \to N$. Note that

$$\begin{tikzcd} h_2 \arrow[r] & h \arrow[r, shift left] \arrow[r, shift right] & h_1 \end{tikzcd}$$

is an equalizer diagram. Denote $\mathcal{O}_h$ the module-valued functor $C \mapsto \bigoplus_{h(C)} C$. Similarly for $\mathcal{O}_{h_1}$ and $\mathcal{O}_{h_2}$. Note that

$$Hom_{\mathcal{P}}(\mathcal{O}_h, F) = F(B[N])$$

where $\mathcal{P}$ is the category of of module-valued functors on Alg_A. We claim there is an equalizer diagram

$$\begin{tikzcd} \mathcal{O}_{h_2} \arrow[r] & \mathcal{O}_h \arrow[r, shift left] \arrow[r, shift right] & \mathcal{O}_{h_1} \end{tikzcd}$$

in $\mathcal{P}$. Namely, suppose that $C \in Ob(Alg_A)$ and $\xi = \sum_{i=1,\ldots,n} c_i \cdot f_i$ where $c_i \in C$ and $f_i : B[N] \to C$ is an element of $\mathcal{O}_h(C)$. If $p(\xi) = q(\xi)$, then we see that

$$\sum c_i \cdot f_i \circ z = \sum c_i \cdot f_i \circ y$$

where $z, y : B[N_1] \to B[N]$ are the maps $z : (b, m_1) \mapsto (b, 0)$ and $y : (b, m_1) \mapsto (b, m_1)$. This means that for every i there exists a j such that $f_j \circ z = f_i \circ y$. Clearly, this implies that $f_i(N_1) = 0$, i.e., f_i factors through a unique map $\overline{f}_i : B[N_2] \to C$. Hence ξ is the image of $\bar{\xi} = \sum c_i \cdot \bar{f}_i$. Since I is injective, it transforms this equalizer diagram into a coequalizer diagram

$$I(B[N_1]) \rightrightarrows I(B[N]) \longrightarrow I(B[N_2])$$

This diagram is compatible with the direct sum decompositions $I(B[N]) = I(B) \oplus TI(B, N)$ and $I(B[N_i]) = I(B) \oplus TI(B, N_i)$. The zero map $N \to N_1$ induces the zero map $TI(B, N) \to TI(B, N_1)$. Thus we see that the coequalizer property above means we have an exact sequence $TI(B, N_1) \to TI(B, N) \to TI(B, N_2) \to 0$ as desired. $\square$

Lemma 32.4.5. *Let A be a ring. Let F be a module-valued functor such that for any $B \in Ob(Alg_A)$ the functor $TF(B, -)$ on B-modules transforms a short exact sequence of B-modules into a right exact sequence. Then*

1. *$TF(B, N_1 \oplus N_2) = TF(B, N_1) \oplus TF(B, N_2)$,*
2. *there is a second functorial B-module structure on $TF(B, N)$ defined by setting $x \cdot b = TF(B, b \cdot 1_N)(x)$ for $x \in TF(B, N)$ and $b \in B$,*
3. *the canonical map $N \otimes_B F(B) \to TF(B, N)$ of Lemma 32.4.3 is B-linear also with respect to the second B-module structure,*
4. *given a finitely presented B-module N there is a canonical isomorphism $TF(B, B) \otimes_B N \to TF(B, N)$ where the tensor product uses the second B-module structure on $TF(B, B)$.*

Proof. We will use the results of Lemma 32.4.3 without further mention. The maps $N_1 \to N_1 \oplus N_2$ and $N_2 \to N_1 \oplus N_2$ give a map $TF(B, N_1) \oplus TF(B, N_2) \to TF(B, N_1 \oplus N_2)$ which is injective since the maps $N_1 \oplus N_2 \to N_1$ and $N_1 \oplus N_2 \to N_2$ induce an inverse. Since TF is right exact we see that $TF(B, N_1) \to TF(B, N_1 \oplus N_2) \to TF(B, N_2) \to 0$ is exact. Hence $TF(B, N_1) \oplus TF(B, N_2) \to TF(B, N_1 \oplus N_2)$ is an isomorphism. This proves (1).

To see (2) the only thing we need to show is that $x \cdot (b_1 + b_2) = x \cdot b_1 + x \cdot b_2$. (Associativity and additivity are clear.) To see this consider

$$N \xrightarrow{(b_1, b_2)} N \oplus N \xrightarrow{+} N$$

and apply $TF(B, -)$.

Part (3) follows immediately from the fact that $N \otimes_B F(B) \to TF(B, N)$ is functorial in the pair (B, N).

Suppose N is a finitely presented B-module. Choose a presentation $B^{\oplus m} \to B^{\oplus n} \to N \to 0$. This gives an exact sequence

$$TF(B, B^{\oplus m}) \to TF(B, B^{\oplus n}) \to TF(B, N) \to 0$$

by right exactness of $TF(B, -)$. By part (1) we can write $TF(B, B^{\oplus m}) = TF(B, B)^{\oplus m}$ and $TF(B, B^{\oplus n}) = TF(B, B)^{\oplus n}$. Next, suppose that $B^{\oplus m} \to B^{\oplus n}$ is given by the matrix $T = (b_{ij})$. Then the induced map $TF(B, B)^{\oplus m} \to TF(B, B)^{\oplus n}$ is given by the matrix with entries $TF(B, b_{ij} \cdot 1_B)$. This combined with right exactness of $\otimes$ proves (4). $\square$

Example 32.4.6. Let F be a module-valued functor as in Lemma 32.4.5. It is not always the case that the two module structures on $TF(B, N)$ agree. Here is an example. Suppose $A = \mathbf{F}_p$ where p is a prime. Set $F(B) = B$ but with B-module structure given by $b \cdot x = b^p x$. Then $TF(B, N) = N$ with B-module structure given by $b \cdot x = b^p x$ for $x \in N$. However, the second B-module structure is given by $x \cdot b = bx$. Note that in this case the canonical map $N \otimes_B F(B) \to TF(B, N)$ is zero as raising an element $n \in B[N]$ to the pth power is zero.

In the following lemma we will frequently use the observation that if $0 \to F \to G \to H \to 0$ is an exact sequence of module-valued functors on Alg_A, then for any pair (B, N) the sequence $0 \to TF(B, N) \to TG(B, N) \to TH(B, N) \to 0$ is exact. This follows from the fact that $0 \to F(B[N]) \to G(B[N]) \to H(B[N]) \to 0$ is exact.

Lemma 32.4.7. *Let A be a ring. For F a module-valued functor on Alg_A say $(*)$ holds if for all $B \in Ob(Alg_A)$ the functor $TF(B, -)$ on B-modules tranforms a short exact sequence of B-modules into a right exact sequence. Let $0 \to F \to G \to H \to 0$ be a short exact sequence of module-valued functors on Alg_A.*

1. *If $(*)$ holds for F, G then $(*)$ holds for H.*
2. *If $(*)$ holds for F, H then $(*)$ holds for G.*
3. *If $H' \to H$ is morphism of module-valued functors on Alg_A and $(*)$ holds for F, G, H, and H', then $(*)$ holds for $G \times_H H'$.*

Proof. Let B be given. Let $0 \to N_1 \to N_2 \to N_3 \to 0$ be a short exact sequence of B-modules. Part (1) follows from a diagram chase in the diagram

$$\begin{tikzcd}
0 \arrow[r] & TF(B, N_1) \arrow[r] \arrow[d] & TG(B, N_1) \arrow[r] \arrow[d] & TH(B, N_1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & TF(B, N_2) \arrow[r] \arrow[d] & TG(B, N_2) \arrow[r] \arrow[d] & TH(B, N_2) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & TF(B, N_3) \arrow[r] \arrow[d] & TG(B, N_3) \arrow[r] \arrow[d] & TH(B, N_3) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

with exact horizontal rows and exact columns involving TF and TG. To prove part (2) we do a diagram chase in the diagram

$$\begin{tikzcd}
0 \arrow[r] & TF(B, N_1) \arrow[r] \arrow[d] & TG(B, N_1) \arrow[r] \arrow[d] & TH(B, N_1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & TF(B, N_2) \arrow[r] \arrow[d] & TG(B, N_2) \arrow[r] \arrow[d] & TH(B, N_2) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & TF(B, N_3) \arrow[r] \arrow[d] & TG(B, N_3) \arrow[r] & TH(B, N_3) \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 &
\end{tikzcd}$$

with exact horizontal rows and exact columns involving TF and TH. Part (3) follows from part (2) as $G \times_H H'$ sits in the exact sequence $0 \to F \to G \times_H H' \to H' \to 0$. □

Most of the work in this section was done in order to prove the following key vanishing result.

Lemma 32.4.8. *Let A be a ring. Let M, P be A-modules with P of finite presentation. Then $Ext^i_{\mathcal{P}}(\underline{P}, \underline{M}) = 0$ for $i > 0$ where $\mathcal{P}$ is the category of module-valued functors on Alg_A.*

Proof. Choose an injective resolution $\underline{M} \to I^\bullet$ in $\mathcal{P}$, see Lemma 32.4.2. By Derived Categories, Lemma 11.26.2 any element of $\mathrm{Ext}^i_{\mathcal{P}}(\underline{P}, \underline{M})$ comes from a morphsm $\varphi : \underline{P} \to I^i$ with $d^i \circ \varphi = 0$. We will prove that the Yoneda extension

$$E : 0 \to \underline{M} \to I^0 \to \ldots \to I^{i-1} \times_{\mathrm{Ker}(d^i)} \underline{P} \to \underline{P} \to 0$$

of $\underline{P}$ by $\underline{M}$ associated to φ is trivial, which will prove the lemma by Derived Categories, Lemma 11.26.5.

For F a module-valued functor on Alg_A say $(*)$ holds if for all $B \in Ob(Alg_A)$ the functor $TF(B, -)$ on B-modules tranforms a short exact sequence of B-modules into a right exact sequence. Recall that the module-valued functors $\underline{M}, I^n, \underline{P}$ each have property $(*)$, see Lemma 32.4.4 and the remarks preceding it. By splitting $0 \to \underline{M} \to I^\bullet$ into short exact sequences we find that each of the functors $\mathrm{Im}(d^{n-1}) = \mathrm{Ker}(d^n) \subset I^n$ has property $(*)$ by Lemma 32.4.7 and also that $I^{i-1} \times_{\mathrm{Ker}(d^i)} \underline{P}$ has property $(*)$.

Thus we may assume the Yoneda extension is given as

$$E : 0 \to \underline{M} \to F_{i-1} \to \ldots \to F_0 \to \underline{P} \to 0$$

where each of the module-valued functors F_j has property $(*)$. Set $G_j(B) = TF_j(B, B)$ viewed as a B-module via the *second* B-module structure defined in Lemma 32.4.5. Since TF_j is a functor on pairs we see that G_j is a module-valued functor on Alg_A. Moreover, since E is an exact sequence the sequence $G_{j+1} \to G_j \to G_{j-1}$ is exact (see remark preceding Lemma 32.4.7). Observe that $T\underline{M}(B, B) = M \otimes_A B = \underline{M}(B)$ and that the two B-module structures agree on this. Thus we obtain a Yoneda extension

$$E' : 0 \to \underline{M} \to G_{i-1} \to \ldots \to G_0 \to \underline{P} \to 0$$

Moreover, the canonical maps

$$F_j(B) = B \otimes_B F_j(B) \longrightarrow TF_j(B, B) = G_j(B)$$

of Lemma 32.4.3 (4) are B-linear by Lemma 32.4.5 (3) and functorial in B. Hence a map

$$\begin{tikzcd} 0 \arrow[r] & \underline{M} \arrow[r] \arrow[d, "1"] & F_{i-1} \arrow[r] \arrow[d] & \ldots \arrow[r] & F_0 \arrow[r] \arrow[d] & \underline{P} \arrow[r] \arrow[d, "1"] & 0 \\ 0 \arrow[r] & \underline{M} \arrow[r] & G_{i-1} \arrow[r] & \ldots \arrow[r] & G_0 \arrow[r] & \underline{P} \arrow[r] & 0 \end{tikzcd}$$

of Yoneda extensions. In particular we see that E and E' have the same class in $\mathrm{Ext}^i_{\mathcal{P}}(\underline{P}, \underline{M})$ by the lemma on Yoneda Exts mentioned above. Finally, let N be a A-module of finite presentation. Then we see that

$$0 \to T\underline{M}(A, N) \to TF_{i-1}(A, N) \to \ldots \to TF_0(A, N) \to T\underline{P}(A, N) \to 0$$

is exact. By Lemma 32.4.5 (4) with $B = A$ this translates into the exactness of the sequence of A-modules

$$0 \to M \otimes_A N \to G_{i-1}(A) \otimes_A N \to \ldots \to G_0(A) \otimes_A N \to P \otimes_A N \to 0$$

Hence the sequence of A-modules $0 \to M \to G_{i-1}(A) \to \ldots \to G_0(A) \to P \to 0$ is universally exact, in the sense that it remains exact on tensoring with any finitely presented A-module N. Let $K = \mathrm{Ker}(G_0(A) \to P)$ so that we have exact sequences

$$0 \to K \to G_0(A) \to P \to 0 \quad \text{and} \quad G_2(A) \to G_1(A) \to K \to 0$$

Tensoring the second sequence with N we obtain that $K \otimes_A N = \mathrm{Coker}(G_2(A) \otimes_A N \to G_1(A) \otimes_A N)$. Exactness of $G_2(A) \otimes_A N \to G_1(A) \otimes_A N \to G_0(A) \otimes_A N$ then implies that $K \otimes_A N \to G_0(A) \otimes_A N$ is injective. By Algebra, Theorem 7.76.3 this means that the A-module extension $0 \to K \to G_0(A) \to P \to 0$ is exact, and because P is assumed of finite presentation this means the sequence is split, see Algebra, Lemma 7.76.4. Any splitting $P \to G_0(A)$ defines a map $\underline{P} \to G_0$ which splits the surjection $G_0 \to \underline{P}$. Thus the Yoneda extension E' is equivalent to the trivial Yoneda extension and we win. $\square$

Lemma 32.4.9. *Let A be a ring. Let M be an A-module. Let L be a linearly adequate functor on Alg_A. Then $Ext^i_{\mathcal{P}}(L, \underline{M}) = 0$ for $i > 0$ where $\mathcal{P}$ is the category of module-valued functors on Alg_A.*

Proof. Since L is linearly adequate there exists an exact sequence

$$0 \to L \to \underline{A^{\oplus m}} \to \underline{A^{\oplus n}} \to \underline{P} \to 0$$

Here $P = \text{Coker}(A^{\oplus m} \to A^{\oplus n})$ is the cokernel of the map of finite free A-modules which is given by the definition of linearly adequate functors. By Lemma 32.4.8 we have the vanishing of $\text{Ext}^i_{\mathcal{P}}(\underline{P}, \underline{M})$ and $\text{Ext}^i_{\mathcal{P}}(\underline{A}, \underline{M})$ for $i > 0$. Let $K = \text{Ker}(\underline{A^{\oplus n}} \to \underline{P})$. By the long exact sequence of Ext groups associated to the exact sequence $0 \to K \to \underline{A^{\oplus n}} \to \underline{P} \to 0$ we conclude that $\text{Ext}^i_{\mathcal{P}}(K, \underline{M}) = 0$ for $i > 0$. Repeating with the sequence $0 \to L \to \underline{A^{\oplus m}} \to K \to 0$ we win. $\square$

Lemma 32.4.10. *With notation as in Lemma 32.4.2 we have $R^pQ(F) = 0$ for all $p > 0$ and any adequate functor F.*

Proof. Choose an exact sequence $0 \to F \to \underline{M^0} \to \underline{M^1}$. Set $M^2 = \text{Coker}(M^0 \to M^1)$ so that $0 \to F \to \underline{M^0} \to \underline{M^1} \to \underline{M^2} \to 0$ is a resolution. By Derived Categories, Lemma 11.20.3 we obtain a spectral sequence

$$R^pQ(\underline{M^q}) \Rightarrow R^{p+q}Q(F)$$

Since $Q(\underline{M^q}) = \underline{M^q}$ it suffices to prove $R^pQ(\underline{M}) = 0$, $p > 0$ for any A-module M.

Choose an injective resolution $\underline{M} \to I^\bullet$ in the category $\mathcal{P}$. Suppose that $R^iQ(\underline{M})$ is nonzero. Then $\text{Ker}(Q(I^i) \to Q(I^{i+1}))$ is strictly bigger than the image of $Q(I^{i-1}) \to Q(I^i)$. Hence by Lemma 32.3.6 there exists a linearly adequate functor L and a map $\varphi : L \to Q(I^i)$ mapping into the kernel of $Q(I^i) \to Q(I^{i+1})$ which does not factor through the image of $Q(I^{i-1}) \to Q(I^i)$. Because Q is a left adjoint to the inclusion functor the map φ corresponds to a map $\varphi' : L \to I^i$ with the same properties. Thus φ' gives a nonzero element of $\text{Ext}^i_{\mathcal{P}}(L, \underline{M})$ contradicting Lemma 32.4.9. $\square$

32.5. Adequate modules

In Descent, Section 31.6 we have seen that quasi-coherent modules on a scheme S are the same as quasi-coherent modules on any of the big sites $(Sch/S)_\tau$ associated to S. We have seen that there are two issues with this identification:

1. $QCoh(\mathcal{O}_S) \to Mod((Sch/S)_\tau, \mathcal{O})$, $\mathcal{F} \mapsto \mathcal{F}^a$ is not exact in general, and
2. given a quasi-compact and quasi-separated morphism $f : X \to S$ the functor f_* does not preserve quasi-coherent sheaves on the big sites in general.

Part (1) means that we cannot define a triangulated subcategory of $D(\mathcal{O})$ consisting of complexes whose cohomology sheaves are quasi-coherent. Part (2) means that $Rf_*\mathcal{F}$ isn't a complex with quasi-coherent cohomology sheaves even when $\mathcal{F}$ is quasi-coherent and f is quasi-compact and quasi-separated. Moreover, the examples given in the proofs of Descent, Lemma 31.6.13 and Descent, Proposition 31.6.14 are not of a pathological nature.

In this section we discuss a slightly larger category of $\mathcal{O}$-modules on $(Sch/S)_\tau$ with contains the quasi-coherent modules, is abelian, and is preserved under f_* when f is quasi-compact and quasi-separated. To do this, suppose that S is a scheme. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules on $(Sch/S)_\tau$. For any affine object $U = Spec(A)$ of $(Sch/S)_\tau$ we can restrict $\mathcal{F}$ to $(Aff/U)_\tau$ to get a presheaf of $\mathcal{O}$-modules on this site. The corresponding module-valued functor, see Section 32.3, will be denoted

$$F = F_{\mathcal{F},A} : Alg_A \longrightarrow Ab, \quad B \longmapsto \mathcal{F}(Spec(B))$$

The assignment $\mathcal{F} \mapsto F_{\mathcal{F},A}$ is an exact functor of abelian categories.

Definition 32.5.1. A sheaf of $\mathcal{O}$-modules $\mathcal{F}$ on $(Sch/S)_\tau$ is *adequate* if there exists a τ-covering $\{Spec(A_i) \to S\}_{i\in I}$ such that $F_{\mathcal{F},A_i}$ is adequate for all $i \in I$.

We will see below that the category of adequate $\mathcal{O}$-modules is independent of the chosen topology τ.

Lemma 32.5.2. *Let S be a scheme. Let $\mathcal{F}$ be an adequate $\mathcal{O}$-module on $(Sch/S)_\tau$. For any affine scheme $Spec(A)$ over S the functor $F_{\mathcal{F},A}$ is adequate.*

Proof. Let $\{Spec(A_i) \to S\}_{i\in I}$ be a τ-covering such that $F_{\mathcal{F},A_i}$ is adequate for all $i \in I$. We can find a standard affine τ-covering $\{Spec(A'_j) \to Spec(A)\}_{j=1,\ldots,m}$ such that $Spec(A'_j) \to Spec(A) \to S$ factors through $Spec(A_{i(j)})$ for some $i(j) \in I$. Then we see that $F_{\mathcal{F},A'_j}$ is the restriction of $F_{\mathcal{F},A_{i(j)}}$ to the category of A'_j-algebras. Hence $F_{\mathcal{F},A'_j}$ is adequate by Lemma 32.3.17. By Lemma 32.3.19 the sequence $F_{\mathcal{F},A'_j}$ corresponds to an adequate ``product'' functor F' over $A' = A'_1 \times \ldots \times A'_m$. As $\mathcal{F}$ is a sheaf (for the Zariski topology) this product functor F' is equal to $F_{\mathcal{F},A'}$, i.e., is the restriction of F to A'-algebras. Finally, $\{Spec(A') \to Spec(A)\}$ is a τ-covering. It follows from Lemma 32.3.20 that $F_{\mathcal{F},A}$ is adequate. □

Lemma 32.5.3. *Let $S = Spec(A)$ be an affine scheme. The category of adequate $\mathcal{O}$-modules on $(Sch/S)_\tau$ is equivalent to the category of adequate module-valued functors on Alg_A.*

Proof. Given an adequate module $\mathcal{F}$ the functor $F_{\mathcal{F},A}$ is adequate by Lemma 32.5.2. Given an adequate functor F we choose an exact sequence $0 \to F \to \underline{M} \to \underline{N}$ and we consider the $\mathcal{O}$-module $\mathcal{F} = \text{Ker}(M^a \to N^a)$ where M^a denotes the quasi-coherent $\mathcal{O}$-module on $(Sch/S)_\tau$ associated to the quasi-coherent sheaf $\widetilde{M}$ on S. Note that $F = F_{\mathcal{F},A}$, in particular the module $\mathcal{F}$ is adequate by definition. We omit the proof that the constructions define mutually inverse equivalences of categories. □

Lemma 32.5.4. *Let $f : T \to S$ be a morphism of schemes. The pullback $f^*\mathcal{F}$ of an adequate $\mathcal{O}$-module $\mathcal{F}$ on $(Sch/S)_\tau$ is an adequate $\mathcal{O}$-module on $(Sch/T)_\tau$.*

Proof. The pullback map $f^* : Mod((Sch/S)_\tau, \mathcal{O}) \to Mod((Sch/T)_\tau, \mathcal{O})$ is given by restriction, i.e., $f^*\mathcal{F}(V) = \mathcal{F}(V)$ for any scheme V over T. Hence this lemma follows immediately from Lemma 32.5.2 and the definition. □

Here is a characterization of the category of adequate $\mathcal{O}$-modules. To understand the significance, consider a map $\mathcal{G} \to \mathcal{H}$ of quasi-coherent $\mathcal{O}_S$-modules on a scheme S. The cokernel of the associated map $\mathcal{G}^a \to \mathcal{H}^a$ of $\mathcal{O}$-modules is quasi-coherent because it is equal to $(\mathcal{H}/\mathcal{G})^a$. But the kernel of $\mathcal{G}^a \to \mathcal{H}^a$ in general isn't quasi-coherent. However, it is adequate.

Lemma 32.5.5. *Let S be a scheme. Let $\mathcal{F}$ be an $\mathcal{O}$-module on $(Sch/S)_\tau$. The following are equivalent*

(1) *$\mathcal{F}$ is adequate,*
(2) *there exists an affine open covering $S = \bigcup S_i$ and maps of quasi-coherent $\mathcal{O}_{S_i}$-modules $\mathcal{G}_i \to \mathcal{H}_i$ such that $\mathcal{F}|_{(Sch/S_i)_\tau}$ is the kernel of $\mathcal{G}_i^a \to \mathcal{H}_i^a$*
(3) *there exists a τ-covering $\{S_i \to S\}_{i \in I}$ and maps of $\mathcal{O}_{S_i}$-quasi-coherent modules $\mathcal{G}_i \to \mathcal{H}_i$ such that $\mathcal{F}|_{(Sch/S_i)_\tau}$ is the kernel of $\mathcal{G}_i^a \to \mathcal{H}_i^a$,*
(4) *there exists a τ-covering $\{f_i : S_i \to S\}_{i \in I}$ such that each $f_i^*\mathcal{F}$ is adequate,*
(5) *for any affine scheme U over S the restriction $\mathcal{F}|_{(Sch/U)_\tau}$ is the kernel of a map $\mathcal{G}^a \to \mathcal{H}^a$ of quasi-coherent $\mathcal{O}_U$-modules.*

Proof. Let $U = Spec(A)$ be an affine scheme over S. Set $F = F_{\mathcal{F},A}$. By definition, the functor F is adequate if and only if there exists a map of A-modules $M \to N$ such that $F = \text{Ker}(\underline{M} \to \underline{N})$. Combining with Lemmas 32.5.2 and 32.5.3 we see that (1) and (5) are equivalent.

It is clear that (5) implies (2) and (2) implies (3). If (3) holds then we can refine the covering $\{S_i \to S\}$ such that each $S_i = Spec(A_i)$ is affine. Then we see, by the prelimiary remarks of the proof, that $F_{\mathcal{F},A_i}$ is adequate. Thus $\mathcal{F}$ is adequate by definition. Hence (3) implies (1).

Finally, (4) is equivalent to (1) using Lemma 32.5.4 for one direction and that a composition of τ-coverings is a τ-covering for the other. □

Just like is true for quasi-coherent sheaves the category of adequate modules is independent of the topology.

Lemma 32.5.6. *Let $\mathcal{F}$ be an adequate $\mathcal{O}$-module on $(Sch/S)_\tau$. For any surjective flat morphism $Spec(B) \to Spec(A)$ of affines over S the extended Čech complex*

$$0 \to \mathcal{F}(Spec(A)) \to \mathcal{F}(Spec(B)) \to \mathcal{F}(Spec(B \otimes_A B)) \to \ldots$$

is exact. In particular $\mathcal{F}$ satisfies the sheaf condition for fpqc coverings, and is a sheaf of $\mathcal{O}$-modules on $(Sch/S)_{fppf}$.

Proof. With $A \to B$ as in the lemma let $F = F_{\mathcal{F},A}$. This functor is adequate by Lemma 32.5.2. By Lemma 32.3.5 since $A \to B$, $A \to B \otimes_A B$, etc are flat we see that $F(B) = F(A) \otimes_A B$, $F(B \otimes_A B) = F(A) \otimes_A B \otimes_A B$, etc. Exactness follows from Descent, Lemma 31.3.6.

Thus $\mathcal{F}$ satisfies the sheaf condition for τ-coverings (in particular Zariski coverings) and any faithfully flat covering of an affine by an affine. Arguing as in the proofs of Descent, Lemma 31.4.1 and Descent, Proposition 31.4.2 we conclude that $\mathcal{F}$ satisfies the sheaf condition for all fpqc coverings (made out of objects of $(Sch/S)_\tau$). Details omitted. □

Lemma 32.5.6 shows in particular that for any pair of topologies τ, τ' the collection of adequate modules for the τ-topology and the τ'-topology are identical (as presheaves of modules on the underlying category Sch/S).

Definition 32.5.7. Let S be a scheme. The category of adequate $\mathcal{O}$-modules on $(Sch/S)_\tau$ is denoted $Adeq(\mathcal{O})$ or $Adeq((Sch/S)_\tau, \mathcal{O})$. If we want to think just about the abelian category of adequate modules without choosing a topology we simply write $Adeq(S)$.

Lemma 32.5.8. *Let S be a scheme. Let $\mathcal{F}$ be an adequate $\mathcal{O}$-module on $(Sch/S)_\tau$.*
(1) *The restriction $\mathcal{F}|_{S_{Zar}}$ is a quasi-coherent $\mathcal{O}_S$-module on the scheme S.*
(2) *The restriction $\mathcal{F}|_{S_{étale}}$ is the quasi-coherent module associated to $\mathcal{F}|_{S_{Zar}}$.*
(3) *For any affine scheme U over S we have $H^q(U, \mathcal{F}) = 0$ for all $q > 0$.*

(4) *There is a canonical isomorphism*
$$H^q(S, \mathcal{F}|_{S_{Zar}}) = H^q((Sch/S)_\tau, \mathcal{F}).$$

Proof. By Lemma 32.3.5 and Lemma 32.5.2 we see that for any flat morphism of affines $U \to V$ over S we have $\mathcal{F}(U) = \mathcal{F}(V) \otimes_{\mathcal{O}(V)} \mathcal{O}(U)$. This works in particular if $U \subset V \subset S$ are affine opens of S, hence $\mathcal{F}|_{S_{Zar}}$ is quasi-coherent. Thus (1) holds.

Let $S' \to S$ be an étale morphism of schemes. Then for $U \subset S'$ affine open mapping into an affine open $V \subset S$ we see that $\mathcal{F}(U) = \mathcal{F}(V) \otimes_{\mathcal{O}(V)} \mathcal{O}(U)$ because $U \to V$ is étale, hence flat. Therefore $\mathcal{F}|_{S'_{Zar}}$ is the pullback of $\mathcal{F}|_{S_{Zar}}$. This proves (2).

We are going to apply Cohomology on Sites, Lemma 19.11.8 to the site $(Sch/S)_\tau$ with $\mathcal{B}$ the set of affine schemes over S and Cov the set of standard affine τ-coverings. Assumption (3) of the lemma is satisfied by Descent, Lemma 31.6.8 and Lemma 32.5.6 for the case of a covering by a single affine. Hence we conclude that $H^p(U, \mathcal{F}) = 0$ for every affine scheme U over S. This proves (3). In exactly the same way as in the proof of Descent, Proposition 31.6.10 this implies the equality of cohomologies (4). □

Remark 32.5.9. Let S be a scheme. We have functors $u : QCoh(\mathcal{O}_S) \to Adeq(\mathcal{O})$ and $v : Adeq(\mathcal{O}) \to QCoh(\mathcal{O}_S)$. Namely, the functor $u : \mathcal{F} \mapsto \mathcal{F}^a$ comes from taking the associated $\mathcal{O}$-module which is adequate by Lemma 32.5.5. Conversely, the functor v comes from restriction $v : \mathcal{G} \mapsto \mathcal{G}|_{S_{Zar}}$, see Lemma 32.5.8. Since $\mathcal{F}^a$ can be described as the pullback of $\mathcal{F}$ under a morphism of ringed topoi $((Sch/S)_\tau, \mathcal{O}) \to (S_{Zar}, \mathcal{O}_S)$, see Descent, Remark 31.6.6 and since restriction is the pushforward we see that u and v are adjoint as follows
$$\mathcal{H}om_{\mathcal{O}_S}(\mathcal{F}, v\mathcal{G}) = \mathcal{H}om_{\mathcal{O}}(u\mathcal{F}, \mathcal{G})$$
where $\mathcal{O}$ denotes the structure sheaf on the big site. It is immediate from the description that the adjunction mapping $\mathcal{F} \to vu\mathcal{F}$ is an isomorphism for all quasi-coherent sheaves.

Lemma 32.5.10. *Let S be a scheme. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules on $(Sch/S)_\tau$. If for every affine scheme $Spec(A)$ over S the functor $F_{\mathcal{F},A}$ is adequate, then the sheafification of $\mathcal{F}$ is an adequate $\mathcal{O}$-module.*

Proof. Let $U = Spec(A)$ be an affine scheme over S. Set $F = F_{\mathcal{F},A}$. The sheafification $\mathcal{F}^\# = (\mathcal{F}^+)^+$, see Sites, Section 9.10. By construction
$$(\mathcal{F})^+(U) = colim_{\mathcal{U}}\, \check{H}^0(\mathcal{U}, \mathcal{F})$$
where the colimit is over coverings in the site $(Sch/S)_\tau$. Since U is affine it suffices to take the limit over standard affine τ-coverings $\mathcal{U} = \{U_i \to U\}_{i\in I} = \{Spec(A_i) \to Spec(A)\}_{i\in I}$ of U. Since each $A \to A_i$ and $A \to A_i \otimes_A A_j$ is flat we see that
$$\check{H}^0(\mathcal{U}, \mathcal{F}) = \mathrm{Ker}(\prod F(A) \otimes_A A_i \to \prod F(A) \otimes_A A_i \otimes_A A_j)$$
by Lemma 32.3.5. Since $A \to \prod A_i$ is faithfully flat we see that this always is canonically isomorphic to $F(A)$ by Descent, Lemma 31.3.6. Thus the presheaf $(\mathcal{F})^+$ has the same value as $\mathcal{F}$ on all affine schemes over S. Repeating the argument once more we deduce the same thing for $\mathcal{F}^\# = ((\mathcal{F})^+)^+$. Thus $F_{\mathcal{F},A} = F_{\mathcal{F}^\#,A}$ and we conclude that $\mathcal{F}^\#$ is adequate. □

Lemma 32.5.11. *Let S be a scheme.*

(1) *The category $Adeq(\mathcal{O})$ is abelian.*
(2) *The functor $Adeq(\mathcal{O}) \to Mod((Sch/S)_\tau, \mathcal{O})$ is exact.*

(3) *If* $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ *is a short exact sequence of* $\mathcal{O}$*-modules and* $\mathcal{F}_1$ *and* $\mathcal{F}_3$ *are adequate, then* $\mathcal{F}_2$ *is adequate.*
(4) *The category* $Adeq(\mathcal{O})$ *has colimits and* $Adeq(\mathcal{O}) \to Mod((Sch/S)_\tau, \mathcal{O})$ *commutes with them.*

Proof. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of adequate $\mathcal{O}$-modules. To prove (1) and (2) it suffices to show that $\mathcal{K} = \text{Ker}(\varphi)$ and $\mathcal{Q} = \text{Coker}(\varphi)$ computed in $Mod((Sch/S)_\tau, \mathcal{O})$ are adequate. Let $U = Spec(A)$ be an affine scheme over S. Let $F = F_{\mathcal{F},A}$ and $G = F_{\mathcal{G},A}$. By Lemmas 32.3.11 and 32.3.10 the kernel K and cokernel Q of the induced map $F \to G$ are adequate functors. Because the kernel is computed on the level of presheaves, we see that $K = F_{\mathcal{K},A}$ and we conclude $\mathcal{K}$ is adequate. To prove the result for the cokernel, denote $\mathcal{Q}'$ the presheaf cokernel of φ. Then $Q = F_{\mathcal{Q}',A}$ and $\mathcal{Q} = (\mathcal{Q}')^\#$. Hence $\mathcal{Q}$ is adequate by Lemma 32.5.10.

Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is a short exact sequence of $\mathcal{O}$-modules and $\mathcal{F}_1$ and $\mathcal{F}_3$ are adequate. Let $U = Spec(A)$ be an affine scheme over S. Let $F_i = F_{\mathcal{F}_i,A}$. The sequence of functors

$$0 \to F_1 \to F_2 \to F_3 \to 0$$

is exact, because for $V = Spec(B)$ affine over U we have $H^1(V, \mathcal{F}_1) = 0$ by Lemma 32.5.8. Since F_1 and F_3 are adequate functors by Lemma 32.5.2 we see that F_2 is adequate by Lemma 32.3.16. Thus $\mathcal{F}_2$ is adequate.

Let $\mathcal{I} \to Adeq(\mathcal{O})$, $i \mapsto \mathcal{F}_i$ be a diagram. Denote $\mathcal{F} = colim_i\, \mathcal{F}_i$ the colimit computed in $Mod((Sch/S)_\tau, \mathcal{O})$. To prove (4) it suffices to show that $\mathcal{F}$ is adequate. Let $\mathcal{F}' = colim_i\, \mathcal{F}_i$ be the colimit computed in presheaves of $\mathcal{O}$-modules. Then $\mathcal{F} = (\mathcal{F}')^\#$. Let $U = Spec(A)$ be an affine scheme over S. Let $F_i = F_{\mathcal{F}_i,A}$. By Lemma 32.3.12 the functor $colim_i\, F_i = F_{\mathcal{F}',A}$ is adequate. Lemma 32.5.10 shows that $\mathcal{F}$ is adequate. $\square$

The following lemma tells us that the total direct image $Rf_*\mathcal{F}$ of an adequate module under a quasi-compact and quasi-separated morphism is a complex whose cohomology sheaves are adequate.

Lemma 32.5.12. *Let* $f : T \to S$ *be a quasi-compact and quasi-separated morphism of schemes. For any adequate* $\mathcal{O}_T$*-module on* $(Sch/T)_\tau$ *the pushforward* $f_*\mathcal{F}$ *and the higher direct images* $R^if_*\mathcal{F}$ *are adequate* $\mathcal{O}_S$*-modules on* $(Sch/S)_\tau$*.*

Proof. First we explain how to compute the higher direct images. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. Then $R^if_*\mathcal{F}$ is the ith cohomology sheaf of the complex $f_*\mathcal{I}^\bullet$. Hence $R^if_*\mathcal{F}$ is the sheaf associated to the presheaf which associates to an object U/S of $(Sch/S)_\tau$ the module

$$\begin{aligned}
\frac{\text{Ker}(f_*\mathcal{I}^i(U) \to f_*\mathcal{I}^{i+1}(U))}{\text{Im}(f_*\mathcal{I}^{i-1}(U) \to f_*\mathcal{I}^i(U))} &= \frac{\text{Ker}(\mathcal{I}^i(U \times_S T) \to \mathcal{I}^{i+1}(U \times_S T))}{\text{Im}(\mathcal{I}^{i-1}(U \times_S T) \to \mathcal{I}^i(U \times_S T))} \\
&= H^i(U \times_S T, \mathcal{F}) \\
&= H^i((Sch/U \times_S T)_\tau, \mathcal{F}|_{(Sch/U \times_S T)_\tau}) \\
&= H^i(U \times_S T, \mathcal{F}|_{(U \times_S T)_{Zar}})
\end{aligned}$$

The first equality by Topologies, Lemma 30.7.12 (and its analogues for other topologies), the second equality by definition of cohomology of $\mathcal{F}$ over an object of $(Sch/T)_\tau$, the third equality by Cohomology on Sites, Lemma 19.8.1, and the last equality by Lemma 32.5.8. Thus by Lemma 32.5.10 it suffices to prove the claim stated in the following paragraph.

Let A be a ring. Let T be a scheme quasi-compact and quasi-separated over A. Let $\mathcal{F}$ be an adequate $\mathcal{O}_T$-module on $(Sch/T)_\tau$. For an A-algebra B set $T_B = T \times_{Spec(A)} Spec(B)$ and denote $\mathcal{F}_B = \mathcal{F}|_{(T_B)_{Zar}}$ the restriction of $\mathcal{F}$ to the small Zariski site of T_B. (Recall that this is a ``usual'' quasi-coherent sheaf on the scheme T_B, see Lemma 32.5.8.) Claim: The functor

$$B \longmapsto H^q(T_B, \mathcal{F}_B)$$

is adequate. We will prove the lemma by the usual procedure of cutting T into pieces.

Case I: T is affine. In this case the schemes T_B are all affine and $H^q(T_B, \mathcal{F}_B) = 0$ for all $q \geq 1$. The functor $B \mapsto H^0(T_B, \mathcal{F}_B)$ is adequate by Lemma 32.3.18.

Case II: T is separated. Let n be the minimal number of affines needed to cover T. We argue by induction on n. The base case is Case I. Choose an affine open covering $T = V_1 \cup \ldots \cup V_n$. Set $V = V_1 \cup \ldots \cup V_{n-1}$ and $U = V_n$. Observe that

$$U \cap V = (V_1 \cap V_n) \cup \ldots \cup (V_{n-1} \cap V_n)$$

is also a union of $n-1$ affine opens as T is separated, see Schemes, Lemma 21.21.8. Note that for each B the base changes U_B, V_B and $(U \cap V)_B = U_B \cap V_B$ behave in the same way. Hence we see that for each B we have a long exact sequence

$$0 \to H^0(T_B, \mathcal{F}_B) \to H^0(U_B, \mathcal{F}_B) \oplus H^0(V_B, \mathcal{F}_B) \to H^0((U \cap V)_B, \mathcal{F}_B) \to H^1(T_B, \mathcal{F}_B) \to \ldots$$

functorial in B, see Cohomology, Lemma 18.8.2. By induction hypothesis the functors $B \mapsto H^q(U_B, \mathcal{F}_B)$, $B \mapsto H^q(V_B, \mathcal{F}_B)$, and $B \mapsto H^q((U \cap V)_B, \mathcal{F}_B)$ are adequate. Using Lemmas 32.3.11 and 32.3.10 we see that our functor $B \mapsto H^q(T_B, \mathcal{F}_B)$ sits in the middle of a short exact sequence whose outer terms are adequate. Thus the claim follows from Lemma 32.3.16.

Case III: General quasi-compact and quasi-separated case. The proof is again by induction on the number n of affines needed to cover T. The base case $n = 1$ is Case I. Choose an affine open covering $T = V_1 \cup \ldots \cup V_n$. Set $V = V_1 \cup \ldots \cup V_{n-1}$ and $U = V_n$. Note that since T is quasi-separated $U \cap V$ is a quasi-compact open of an affine scheme, hence Case II applies to it. The rest of the argument proceeds in exactly the same manner as in the paragraph above and is omitted. □

32.6. Parasitic adequate modules

In this section we start comparing adequate modules and quasi-coherent modules on a scheme S. Recall that there are functors $u : QCoh(\mathcal{O}_S) \to Adeq(\mathcal{O})$ and $v : Adeq(\mathcal{O}) \to QCoh(\mathcal{O}_S)$ satisfying the adjunction

$$\mathcal{H}om_{QCoh(\mathcal{O}_S)}(\mathcal{F}, v\mathcal{G}) = \mathcal{H}om_{Adeq(\mathcal{O})}(u\mathcal{F}, \mathcal{G})$$

and such that $\mathcal{F} \to vu\mathcal{F}$ is an isomorphism for every quasi-coherent sheaf $\mathcal{F}$, see Remark 32.5.9. Hence u is a fully faithfull embedding and we can identify $QCoh(\mathcal{O}_S)$ with a full subcategory of $Adeq(\mathcal{O})$. The functor v is exact but u is not left exact in general. The kernel of v is the subcategory of parasitic adequate modules.

In Descent, Definition 31.7.1 we give the definition of a parasitic module. For adequate modules the notion does not depend on the chosen topology.

Lemma 32.6.1. *Let S be a scheme. Let $\mathcal{F}$ be an adequate $\mathcal{O}$-module on $(Sch/S)_\tau$. The following are equivalent:*

(1) $v\mathcal{F} = 0$,
(2) *$\mathcal{F}$ is parasitic,*

(3) *$\mathcal{F}$ is parasitic for the τ-topology,*
(4) *$\mathcal{F}(U) = 0$ for all $U \subset S$ open, and*
(5) *there exists an affine open covering $S = \bigcup U_i$ such that $\mathcal{F}(U_i) = 0$ for all i.*

Proof. The implications (2) ⇒ (3) ⇒ (4) ⇒ (5) are immediate from the definitions. Assume (5). Suppose that $S = \bigcup U_i$ is an affine open covering such that $\mathcal{F}(U_i) = 0$ for all i. Let $V \to S$ be a flat morphism. There exists an affine open covering $V = \bigcup V_j$ such that each V_j maps into some U_i. As the morphism $V_j \to S$ is flat, also $V_j \to U_i$ is flat. Hence the corresponding ring map $A_i = \mathcal{O}(U_i) \to \mathcal{O}(V_j) = B_j$ is flat. Thus by Lemma 32.5.2 and Lemma 32.3.5 we see that $\mathcal{F}(U_i) \otimes_{A_i} B_j \to \mathcal{F}(V_j)$ is an isomorphism. Hence $\mathcal{F}(V_j) = 0$. Since $\mathcal{F}$ is a sheaf for the Zariski topology we conclude that $\mathcal{F}(V) = 0$. In this way we see that (5) implies (2).

This proves the equivalence of (2), (3), (4), and (5). As (1) is equivalent to (3) (see Remark 32.5.9) we conclude that all five conditions are equivalent. □

Let S be a scheme. The subcategory of parasitic adequate modules is a Serre subcategory of *Adeq*$(\mathcal{O})$. The quotient is the category of quasi-coherent modules.

Lemma 32.6.2. *Let S be a scheme. The subcategory $\mathcal{C} \subset Adeq(\mathcal{O})$ of parasitic adequate modules is a Serre subcategory. Moreover, the functor v induces an equivalence of categories*

$$Adeq(\mathcal{O})/\mathcal{C} = QCoh(\mathcal{O}_S).$$

Proof. The category $\mathcal{C}$ is the kernel of the exact functor $v : Adeq(\mathcal{O}) \to QCoh(\mathcal{O}_S)$, see Lemma 32.6.1. Hence it is a Serre subcategory by Homology, Lemma 10.7.4. By Homology, Lemma 10.7.6 we obtain an induced exact functor $\overline{v} : Adeq(\mathcal{O})/\mathcal{C} \to QCoh(\mathcal{O}_S)$. Because u is a right inverse to v we see right away that $\overline{v}$ is essentially surjective. We see that $\overline{v}$ is faithful by Homology, Lemma 10.7.7. Because u is a right inverse to v we finally conclude that $\overline{v}$ is fully faithful. □

Lemma 32.6.3. *Let $f : T \to S$ be a quasi-compact and quasi-separated morphism of schemes. For any parasitic adequate $\mathcal{O}_T$-module on $(Sch/T)_\tau$ the pushforward $f_*\mathcal{F}$ and the higher direct images $R^i f_*\mathcal{F}$ are parasitic adequate $\mathcal{O}_S$-modules on $(Sch/S)_\tau$.*

Proof. We have already seen in Lemma 32.5.12 that these higher direct images are adequate. Hence it suffices to show that $(R^i f_*\mathcal{F})(U_i) = 0$ for any τ-covering $\{U_i \to S\}$ open. And $R^i f_*\mathcal{F}$ is parasitic by Descent, Lemma 31.7.3. □

32.7. Derived categories of adequate modules, I

Let S be a scheme. We continue the discussion started in Section 32.6. The exact functor v induces a functor

$$D(Adeq(\mathcal{O})) \longrightarrow D(QCoh(\mathcal{O}_S))$$

and similarly for bounded versions.

Lemma 32.7.1. *Let S be a scheme. Let $\mathcal{C} \subset Adeq(\mathcal{O})$ denote the full subcategory consisting of parasitic adequate modules. Then*

$$D(Adeq(\mathcal{O}))/D_{\mathcal{C}}(Adeq(\mathcal{O})) = D(QCoh(\mathcal{O}_S))$$

and similarly for the bounded versions.

Proof. Follows immediately from Derived Categories, Lemma 11.12.3. □

Next, we look for a description the other way around by looking at the functors

$$K^+(QCoh(\mathcal{O}_S)) \longrightarrow K^+(Adeq(\mathcal{O})) \longrightarrow D^+(Adeq(\mathcal{O})) \longrightarrow D^+(QCoh(\mathcal{O}_S)).$$

In some cases the derived category of adequate modules is a localization of the homotopy category of complexes of quasi-coherent modules at universal quasi-isomorphisms. Let S be a scheme. A map of complexes $\varphi : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ of quasi-coherent $\mathcal{O}_S$-modules is said to be a *universal quasi-isomorphism* if for every morphism of schemes $f : T \to S$ the pullback $f^*\varphi$ is a quasi-isomorphism.

Lemma 32.7.2. *Let $U = Spec(A)$ be an affine scheme. The bounded below derived category $D^+(Adeq(\mathcal{O}))$ is the localization of $K^+(QCoh(\mathcal{O}_U))$ at the multiplicative subset of universal quasi-isomorphisms.*

Proof. If $\varphi : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ is a morphism of complexes of quasi-coherent $\mathcal{O}_U$-modules, then $u\varphi : u\mathcal{F}^\bullet \to u\mathcal{G}^\bullet$ is a quasi-isomorphism if and only if φ is a universal quasi-isomorphism. Hence the collection S of universal quasi-isomorphisms is a saturated multiplicative system compatible with the triangulated structure by Derived Categories, Lemma 11.5.3. Hence $S^{-1}K^+(QCoh(\mathcal{O}_U))$ exists and is a triangulated category, see Derived Categories, Proposition 11.5.5. We obtain a canonical functor $can : S^{-1}K^+(QCoh(\mathcal{O}_U)) \to D^+(Adeq(\mathcal{O}))$ by Derived Categories, Lemma 11.5.6.

Note that, almost by definition, every adequate module on U has an embedding into a quasi-coherent sheaf, see Lemma 32.5.5. Hence by Derived Categories, Lemma 11.15.4 given $\mathcal{F}^\bullet \in Ob(K^+(Adeq(\mathcal{O})))$ there exists a quasi-isomorphism $\mathcal{F}^\bullet \to u\mathcal{G}^\bullet$ where $\mathcal{G}^\bullet \in Ob(K^+(QCoh(\mathcal{O}_U)))$. This proves that can is essentially surjective.

Similarly, suppose that $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ are bounded below complexes of quasi-coherent $\mathcal{O}_U$-modules. A morphism in $D^+(Adeq(\mathcal{O}))$ between these consists of a pair $f : u\mathcal{F}^\bullet \to \mathcal{H}^\bullet$ and $s : u\mathcal{G}^\bullet \to \mathcal{H}^\bullet$ where s is a quasi-isomorphism. Pick a quasi-isomorphism $s' : \mathcal{H}^\bullet \to u\mathcal{E}^\bullet$. Then we see that $s' \circ f : \mathcal{F} \to \mathcal{E}^\bullet$ and the universal quasi-isomorphism $s' \circ s : \mathcal{G}^\bullet \to \mathcal{E}^\bullet$ give a morphism in $S^{-1}K^+(QCoh(\mathcal{O}_U))$ mapping to the given morphism. This proves the "fully" part of full faithfulness. Faithfulness is proved similarly. ☐

Lemma 32.7.3. *Let $U = Spec(A)$ be an affine scheme. The inclusion functor*

$$Adeq(\mathcal{O}) \to Mod((Sch/U)_\tau, \mathcal{O})$$

has a right adjoint A[1]*. Moreover, the adjunction mapping $A(\mathcal{F}) \to \mathcal{F}$ is an isomorphism for every adequate module $\mathcal{F}$.*

Proof. By Topologies, Lemma 30.7.11 (and similarly for the other topologies) we may work with $\mathcal{O}$-modules on $(Aff/U)_\tau$. Denote $\mathcal{P}$ the category of module-valued functors on Alg_A and $\mathcal{A}$ the category of adequate functors on Alg_A. Denote $i : \mathcal{A} \to \mathcal{P}$ the inclusion functor. Denote $Q : \mathcal{P} \to \mathcal{A}$ the construction of Lemma 32.4.1. We have the commutative diagram

(32.7.3.1)
$$\begin{tikzcd} Adeq(\mathcal{O}) \arrow[r, "k"'] \arrow[d, equal] & Mod((Aff/U)_\tau, \mathcal{O}) \arrow[r, "j"'] & PMod((Aff/U)_\tau, \mathcal{O}) \arrow[d, equal] \\ \mathcal{A} \arrow[rr, "i"] & & \mathcal{P} \end{tikzcd}$$

The left vertical equality is Lemma 32.5.3 and the right vertical equality was explained in Section 32.3. Define $A(\mathcal{F}) = Q(j(\mathcal{F}))$. Since j is fully faithful it follows immediately that

[1] This is the ``adequator''.

A is a right adjoint of the inclusion functor k. Also, since k is fully faithful too, the final assertion follows formally. □

The functor A is a right adjoint hence left exact. Since the inclusion functor is exact, see Lemma 32.5.11 we conclude that A transforms injectives into injectives, and that the category $Adeq(\mathcal{O})$ has enough injectives, see Homology, Lemma 10.22.3 and Injectives, Theorem 17.12.4. This also follows from the equivalence in (32.7.3.1) and Lemma 32.4.2.

Lemma 32.7.4. *Let $U = Spec(A)$ be an affine scheme. For any object $\mathcal{F}$ of $Adeq(\mathcal{O})$ we have $R^pA(\mathcal{F}) = 0$ for all $p > 0$ where A is as in Lemma 32.7.3.*

Proof. With notation as in the proof of Lemma 32.7.3 choose an injective resolution $k(\mathcal{F}) \to \mathcal{I}^\bullet$ in the category of $\mathcal{O}$-modules on $(Aff/U)_\tau$. By Cohomology on Sites, Lemmas 19.12.2 and Lemma 32.5.8 the complex $j(\mathcal{I}^\bullet)$ is exact. On the other hand, each $j(\mathcal{I}^n)$ is an injective object of the category of presheaves of modules by Cohomology on Sites, Lemma 19.12.1. It follows that $R^pA(\mathcal{F}) = R^pQ(j(k(\mathcal{F})))$. Hence the result now follows from Lemma 32.4.10. □

Let S be a scheme. By the discussion in Section 32.5 the embedding $Adeq(\mathcal{O}) \subset Mod((Sch/S)_\tau, \mathcal{O})$ exhibits $Adeq(\mathcal{O})$ as a weak Serre subcategory of the category of all $\mathcal{O}$-modules. Denote

$$D_{Adeq}(\mathcal{O}) \subset D(\mathcal{O}) = D(Mod((Sch/S)_\tau, \mathcal{O}))$$

the triangulated subcategory of complexes whose cohomology sheaves are adequate, see Derived Categories, Section 11.12. We obtain a canonical functor

$$D(Adeq(\mathcal{O})) \longrightarrow D_{Adeq}(\mathcal{O})$$

see Derived Categories, Equation (11.12.1.1).

Lemma 32.7.5. *If $U = Spec(A)$ is an affine scheme, then the bounded below version*

$$D^+(Adeq(\mathcal{O})) \longrightarrow D^+_{Adeq}(\mathcal{O}) \tag{32.7.5.1}$$

of the functor above is an equivalence.

Proof. Let $A : Mod(\mathcal{O}) \to Adeq(\mathcal{O})$ be the right adjoint to the inclusion functor constructed in Lemma 32.7.3. Since A is left exact and since $Mod(\mathcal{O})$ has enough injectives, A has a right derived functor $RA : D^+_{Adeq}(\mathcal{O}) \to D^+(Adeq(\mathcal{O}))$. We claim that RA is a quasi-inverse to (32.7.5.1). To see this the key fact is that if $\mathcal{F}$ is an adequate module, then the adjunction map $\mathcal{F} \to RA(\mathcal{F})$ is a quasi-isomorphism by Lemma 32.7.4.

Namely, to prove the lemma in full it suffices to show:

(1) Given $\mathcal{F}^\bullet \in K^+(Adeq(\mathcal{O}))$ the canonical map $\mathcal{F}^\bullet \to RA(\mathcal{F}^\bullet)$ is a quasi-isomorphism, and
(2) given $\mathcal{G}^\bullet \in K^+(Mod(\mathcal{O}))$ the canonical map $RA(\mathcal{G}^\bullet) \to \mathcal{G}^\bullet$ is a quasi-isomorphism.

Both (1) and (2) follow from the key fact via a spectral sequence argument using one of the spectral sequences of Derived Categories, Lemma 11.20.3. Some details omitted. □

Lemma 32.7.6. *Let $U = Spec(A)$ be an affine scheme. Let $\mathcal{F}$ and $\mathcal{G}$ be adequate $\mathcal{O}$-modules. For any $i \geq 0$ the natural map*

$$Ext^i_{Adeq(\mathcal{O})}(\mathcal{F}, \mathcal{G}) \longrightarrow Ext^i_{Mod(\mathcal{O})}(\mathcal{F}, \mathcal{G})$$

is an isomorphism.

Proof. By definition these ext groups are computed as hom sets in the derived category. Hence this follows immediately from Lemma 32.7.5. □

32.8. Pure extensions

We want to characterize extensions of quasi-coherent sheaves on the big site of an affine schemes in terms of algebra. To do this we introduce the following notion.

Definition 32.8.1. Let A be a ring.

(1) An A-module P is said to be *pure projective* if for every universally exact sequence $0 \to K \to M \to N \to 0$ of A-module the sequence $0 \to Hom_A(P, K) \to Hom_A(P, M) \to Hom_A(P, N) \to 0$ is exact.
(2) An A-module I is said to be *pure injective* if for every universally exact sequence $0 \to K \to M \to N \to 0$ of A-module the sequence $0 \to Hom_A(N, I) \to Hom_A(M, I) \to Hom_A(K, I) \to 0$ is exact.

Let's characterize pure projectives.

Lemma 32.8.2. *Let A be a ring.*

(1) *A module is pure projective if and only if it is a direct summand of a direct sum of finitely presented A-modules.*
(2) *For any module M there exists a universally exact sequence $0 \to N \to P \to M \to 0$ with P pure projective.*

Proof. First note that a finitely presented A-module is pure projective by Algebra, Theorem 7.76.3. Hence a direct summand of a direct sum of finitely presented A-modules is indeed pure projective. Let M be any A-module. Write $M = colim_{i \in I} P_i$ as a filtered colimit of finitely presented A-modules. Consider the sequence

$$0 \to N \to \bigoplus P_i \to M \to 0.$$

For any finitely presented A-module P the map $Hom_A(P, \bigoplus P_i) \to Hom_A(P, M)$ is surjective, as any map $P \to M$ factors through some P_i. Hence by Algebra, Theorem 7.76.3 this sequence is universally exact. This proves (2). If now M is pure projective, then the sequence is split and we see that M is a direct summand of $\bigoplus P_i$. $\square$

Let's characterize pure injectives.

Lemma 32.8.3. *Let A be a ring. For any A-module M set $M^\wedge = Hom_{\mathbf{Z}}(M, \mathbf{Q}/\mathbf{Z})$.*

(1) *For any A-module M the A-module $M^\wedge$ is pure injective.*
(2) *An A-module I is pure injective if and only if the map $I \to (I^\wedge)^\wedge$ splits.*
(3) *For any module M there exists a universally exact sequence $0 \to M \to I \to N \to 0$ with I pure injective.*

Proof. We will use the properties of the functor $M \mapsto M^\wedge$ found in Injectives, Section 17.3 without further mention. Part (1) holds because $Hom_A(N, M^\wedge) = Hom_{\mathbf{Z}}(N \otimes_A M, \mathbf{Q}/\mathbf{Z})$ and because $\mathbf{Q}/\mathbf{Z}$ is injective in the category of abelian groups. Hence if $I \to (I^\wedge)^\wedge$ is split, then I is pure injective. We claim that for any A-module M the evaluation map $ev : M \to (M^\wedge)^\wedge$ is universally injective. To see this note that $ev^\wedge : ((M^\wedge)^\wedge)^\wedge \to M^\wedge$ has a right inverse, namely $ev' : M^\wedge \to ((M^\wedge)^\wedge)^\wedge$. Then for any A-module N applying the exact faithful functor $^\wedge$ to the map $N \otimes_A M \to N \otimes_A (M^\wedge)^\wedge$ gives

$$Hom_A(N, ((M^\wedge)^\wedge)^\wedge) = \left(N \otimes_A (M^\wedge)^\wedge\right)^\wedge \to \left(N \otimes_A M\right)^\wedge = Hom_A(N, M^\wedge)$$

which is surjective by the existence of the right inverse. The claim follows. The claim implies (3) and the necessity of the condition in (2). $\square$

Before we continue we make the following observation which we will use frequently in the rest of this section.

Lemma 32.8.4. *Let A be a ring.*

1. *Let $L \to M \to N$ be a universally exact sequence of A-modules. Let $K = Im(M \to N)$. Then $K \to N$ is universally injective.*
2. *Any universally exact complex can be split into universally exact short exact sequences.*

Proof. Proof of (1). For any A-module T the sequence $L \otimes_A T \to M \otimes_A T \to K \otimes_A T \to 0$ is exact by right exactness of $\otimes$. By assumption the sequence $L \otimes_A T \to M \otimes_A T \to N \otimes_A T$ is exact. Combined this shows that $K \otimes_A T \to N \otimes_A T$ is injective.

Part (2) means the following: Suppose that $M^\bullet$ is a universally exact complex of A-modules. Set $K^i = \text{Ker}(d^i) \subset M^i$. Then the short exact sequences $0 \to K^i \to M^i \to K^{i+1} \to 0$ are universally exact. This follows immediately from part (1). □

Definition 32.8.5. Let A be a ring. Let M be an A-module.

1. A *pure projective resolution* $P_\bullet \to M$ is a universally exact sequence
$$\ldots \to P_1 \to P_0 \to M \to 0$$
with each P_i pure projective.
2. A *pure injective resolution* $M \to I^\bullet$ is a universally exact sequence
$$0 \to M \to I^0 \to I^1 \to \ldots$$
with each I^i pure injective.

These resolutions satisfy the usual uniqueness properties among the class of all universally exact left or right resolutions.

Lemma 32.8.6. *Let A be a ring.*

1. *Any A-module has a pure projective resolution.*

Let $M \to N$ be a map of A-modules. Let $P_\bullet \to M$ be a pure projective resolution and let $N_\bullet \to N$ be a universally exact resolution.

2. *There exists a map of complexes $P_\bullet \to N_\bullet$ inducing the given map*
$$M = Coker(P_1 \to P_0) \to Coker(N_1 \to N_0) = N$$
3. *two maps $\alpha, \beta : P_\bullet \to N_\bullet$ inducing the same map $M \to N$ are homotopic.*

Proof. Part (1) follows immediately from Lemma 32.8.2. Before we prove (2) and (3) note that by Lemma 32.8.4 we can split the universally exact complex $N_\bullet \to N \to 0$ into universally exact short exact sequences $0 \to K_0 \to N_0 \to N \to 0$ and $0 \to K_i \to N_i \to K_{i-1} \to 0$.

Proof of (2). Because P_0 is pure projective we can find a map $P_0 \to N_0$ lifting the map $P_0 \to M \to N$. We obtain an induced map $P_1 \to F_0 \to N_0$ wich ends up in K_0. Since P_1 is pure projective we may lift this to a map $P_1 \to N_1$. This in turn induces a map $P_2 \to P_1 \to N_1$ which maps to zero into N_0, i.e., into K_1. Hence we may lift to get a map $P_2 \to N_2$. Repeat.

Proof of (3). To show that α, β are homotopic it suffices to show the difference $\gamma = \alpha - \beta$ is homotopic to zero. Note that the image of $\gamma_0 : P_0 \to N_0$ is contained in K_0. Hence we may lift γ_0 to a map $h_0 : P_0 \to N_1$. Consider the map $\gamma_1' = \gamma_1 - h_0 \circ d_{P,1} : P_1 \to N_1$. By our choice of h_0 we see that the image of γ_1' is contained in K_1. Since P_1 is pure projective may lift γ_1' to a map $h_1 : P_1 \to N_2$. At this point we have $\gamma_1 = h_0 \circ d_{F,1} + d_{N,2} \circ h_1$. Repeat. □

Lemma 32.8.7. *Let A be a ring.*

(1) *Any A-module has a pure injective resolution.*

Let $M \to N$ be a map of A-modules. Let $M \to M^\bullet$ be a universally exact resolution and let $N \to I^\bullet$ be a pure injective resolution.

(2) *There exists a map of complexes $M^\bullet \to I^\bullet$ inducing the given map*
$$M = Ker(M^0 \to M^1) \to Ker(I^0 \to I^1) = N$$

(3) *two maps $\alpha, \beta : M^\bullet \to I^\bullet$ inducing the same map $M \to N$ are homotopic.*

Proof. This lemma is dual to Lemma 32.8.6. The proof is identical, except one has to reverse all the arrows. □

Using the material above we can define pure extension groups as follows. Let A be a ring and let M, N be A-modules. Choose a pure injective resolution $N \to I^\bullet$. By Lemma 32.8.7 the complex
$$Hom_A(M, I^\bullet)$$
is well defined up to homotopy. Hence its ith cohomology module is a well defined invariant of M and N.

Definition 32.8.8. Let A be a ring and let M, N be A-modules. The ith *pure extension module* $\text{Pext}^i_A(M, N)$ is the ith cohomology module of the complex $Hom_A(M, I^\bullet)$ where $I^\bullet$ is a pure injective resolution of N.

Warning: It is not true that an exact sequence of A-modules gives rise to a long exact sequence of pure extensions groups. (You need a universally exact sequence for this.) We collect some facts which are obvious from the material above.

Lemma 32.8.9. *Let A be a ring.*

(1) *$Pext^i_A(M, N) = 0$ for $i > 0$ whenever N is pure injective,*
(2) *$Pext^i_A(M, N) = 0$ for $i > 0$ whenever M is pure projective, in particular if M is an A-module of finite presentation,*
(3) *$Pext^i_A(M, N)$ is also the ith cohomology module of the complex $Hom_A(P_\bullet, N)$ where $P_\bullet$ is a pure projective resolution of M.*

Proof. To see (3) consider the double complex
$$A^{\bullet,\bullet} = Hom_A(P_\bullet, I^\bullet)$$
Each of its rows is exact except in degree 0 where its cohomology is $Hom_A(M, I^q)$. Each of its columns is exact except in degree 0 where its cohomology is $Hom_A(P_p, N)$. Hence the two spectral sequences associated to this complex in Homology, Section 10.19 degenerate, giving the equality. □

32.9. Higher exts of quasi-coherent sheaves on the big site

It turns out that the module-valued functor $\underline{I}$ associated to a pure injective module I gives rise to an injective object in the category of adequate functors on Alg_A. Warning: It is not true that a pure projective module gives rise to a projective object in the category of adequate functors. We do have plenty of projective objects, namely, the linearly adequate functors.

Lemma 32.9.1. *Let A be a ring. Let $\mathcal{A}$ be the category of adequate functors on Alg_A. The injective objects of $\mathcal{A}$ are exactly the functors $\underline{I}$ where I is a pure injective A-module.*

Proof. Let I be an injective object of $\mathcal{A}$. Choose an embedding $I \to \underline{M}$ for some A-module M. As I is injective we see that $\underline{M} = I \oplus F$ for some module-valued functor F. Then $M = I(A) \oplus F(A)$ and it follows that $I = \underline{I(A)}$. Thus we see that any injective object is of the form $\underline{I}$ for some A-module I. It is clear that the module I has to be pure injective since any universally exact sequence $0 \to M \to N \to L \to 0$ gives rise to an exact sequence $0 \to \underline{M} \to \underline{N} \to \underline{L} \to 0$ of $\mathcal{A}$.

Finally, suppose that I is a pure injective A-module. Choose an embedding $\underline{I} \to J$ into an injective object of $\mathcal{A}$ (see Lemma 32.4.2). We have seen above that $J = \underline{I'}$ for some A-module I' which is pure injective. As $\underline{I} \to \underline{I'}$ is injective the map $I \to I'$ is universally injective. By assumption on I it splits. Hence $\underline{I}$ is a summand of $J = \underline{I'}$ whence an injective object of the category $\mathcal{A}$. ☐

Let $U = Spec(A)$ be an affine scheme. Let M be an A-module. We will use the notation M^a to denote the quasi-coherent sheaf of $\mathcal{O}$-modules on $(Sch/U)_\tau$ associated to the quasi-coherent sheaf $\widetilde{M}$ on U. Now we have all the notation in place to formulate the following lemma.

Lemma 32.9.2. *Let $U = Spec(A)$ be an affine scheme. Let M, N be A-modules. For all i we have a canonical isomorphism*

$$Ext^i_{Mod(\mathcal{O})}(M^a, N^a) = Pext^i_A(M, N)$$

functorial in M and N.

Proof. Let us construct a canonical arrow from right to left. Namely, if $N \to I^\bullet$ is a pure injective resolution, then $M^a \to (I^\bullet)^a$ is an exact complex of (adequate) $\mathcal{O}$-modules. Hence any element of $\mathrm{Pext}^i_A(M, N)$ gives rise to a map $N^a \to M^a[i]$ in $D(\mathcal{O})$, i.e., an element of the group on the left.

To prove this map is an isomorphism, note that we may replace $\mathrm{Ext}^i_{Mod(\mathcal{O})}(M^a, N^a)$ by $\mathrm{Ext}^i_{Adeq(\mathcal{O})}(M^a, N^a)$, see Lemma 32.7.6. Let $\mathcal{A}$ be the category of adequate functors on Alg_A. We have seen that $\mathcal{A}$ is equivalent to $Adeq(\mathcal{O})$, see Lemma 32.5.3; see also the proof of Lemma 32.7.3. Hence now it suffices to prove that

$$\mathrm{Ext}^i_{\mathcal{A}}(\underline{M}, \underline{N}) = \mathrm{Pext}^i_A(M, N)$$

However, this is clear from Lemma 32.9.1 as a pure injective resolution $N \to I^\bullet$ exactly corresponds to an injective resolution of $\underline{N}$ in $\mathcal{A}$. ☐

32.10. Derived categories of adequate modules, II

Let S be a scheme. Denote $\mathcal{O}_S$ the structure sheaf of S and $\mathcal{O}$ the structure sheaf of the big site $(Sch/S)_\tau$. In Descent, Remark 31.6.4 we constructed a morphism of ringed sites

$$f : ((Sch/S)_\tau, \mathcal{O}) \longrightarrow (S_{Zar}, \mathcal{O}_S). \tag{32.10.0.1}$$

In the previous sections have seen that the functor $f_* : Mod(\mathcal{O}) \to Mod(\mathcal{O}_S)$ transforms adequate sheaves into quasi-coherent sheaves, and induces an exact functor $v : Adeq(\mathcal{O}) \to QCoh(\mathcal{O}_S)$, and in fact that $f_* = v$ induces an equivalence $Adeq(\mathcal{O})/\mathcal{C} \to QCoh(\mathcal{O}_S)$ where $\mathcal{C}$ is the subcategory of parasitic adequate modules. Moreover, the functor f^* transforms quasi-coherent modules into adequate modules, and induces a functor $u : QCoh(\mathcal{O}_S) \to Adeq(\mathcal{O})$ which is a left adjoint to v.

There is a very similar relationship between $D_{Adeq}(\mathcal{O})$ and $D_{QCoh}(S)$. First we explain why the category $D_{Adeq}(\mathcal{O})$ is independent of the chosen topology.

Remark 32.10.1. Let S be a scheme. Let $\tau, \tau' \in \{Zar, étale, smooth, syntomic, fppf\}$. Denote $\mathcal{O}_\tau$, resp. $\mathcal{O}_{\tau'}$ the structure sheaf $\mathcal{O}$ viewed as a sheaf on $(Sch/S)_\tau$, resp. $(Sch/S)_{\tau'}$. Then $D_{Adeq}(\mathcal{O}_\tau)$ and $D_{Adeq}(\mathcal{O}_{\tau'})$ are canonically isomorphic. This follows from Cohomology on Sites, Lemma 19.20.2. Namely, assume τ is stronger than the topology τ', let $\mathcal{C} = (Sch/S)_{fppf}$, and let $\mathcal{B}$ the collection of affine schemes over S. Assumptions (1) and (2) we've seen above. Assumption (3) is clear and assumption (4) follows from Lemma 32.5.8.

Remark 32.10.2. Let S be a scheme. The morphism f see (32.10.0.1) induces adjoint functors $Rf_* : D_{Adeq}(\mathcal{O}) \to D_{QCoh}(S)$ and $Lf^* : D_{QCoh}(S) \to D_{Adeq}(\mathcal{O})$. Moreover $Rf_* Lf^* \cong \text{id}_{D_{QCoh}(S)}$.

We sketch the proof. By Remark 32.10.1 we may assume the topology τ is the Zariski topology. We will use the existence of the unbounded total derived functors Lf^* and Rf_* on $\mathcal{O}$-modules and their adjointness, see Cohomology on Sites, Lemma 19.19.1. In this case f_* is just the restriction to the subcategory S_{Zar} of $(Sch/S)_{Zar}$. Hence it is clear that $Rf_* = f_*$ induces $Rf_* : D_{Adeq}(\mathcal{O}) \to D_{QCoh}(S)$. Suppose that $\mathcal{G}^\bullet$ is an object of $D_{QCoh}(S)$. We may choose a system $\mathcal{K}_1^\bullet \to \mathcal{K}_2^\bullet \to \ldots$ of bounded above complexes of flat $\mathcal{O}_S$-modules whose transition maps are termwise split injectives and a diagram

$$\begin{tikzcd} \mathcal{K}_1^\bullet \arrow[r] \arrow[d] & \mathcal{K}_2^\bullet \arrow[r] \arrow[d] & \ldots \\ \tau_{\leq 1}\mathcal{G}^\bullet \arrow[r] & \tau_{\leq 2}\mathcal{G}^\bullet \arrow[r] & \ldots \end{tikzcd}$$

with the properties (1), (2), (3) listed in Derived Categories, Lemma 11.27.1 where $\mathcal{P}$is the collection of flat $\mathcal{O}_S$-modules. Then $Lf^*\mathcal{G}^\bullet$ is computed by $colim\, f^*\mathcal{K}_n^\bullet$, see Cohomology on Sites, Lemmas 19.18.1 and 19.18.3 (note that our sites have enough points by Étale Cohomology, Lemma 38.30.1). We have to see that $H^i(Lf^*\mathcal{G}^\bullet) = colim\, H^i(f^*\mathcal{K}_n^\bullet)$ is adequate for each i. By Lemma 32.5.11 we conclude that it suffices to show that each $H^i(f^*\mathcal{K}_n^\bullet)$ is adequate.

The adequacy of $H^i(f^*\mathcal{K}_n^\bullet)$ is local on S, hence we may assume that $S = Spec(A)$ is affine. Because S is affine $D_{QCoh}(S) = D(QCoh(\mathcal{O}_S))$, see the discussion in Coherent, Section 25.4. Hence there exists a quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{K}_n^\bullet$ where $\mathcal{F}^\bullet$ is a bounded above complex of flat quasi-coherent modules. Then $f^*\mathcal{F}^\bullet \to f^*\mathcal{K}_n^\bullet$ is a quasi-isomorphism, and the cohomology sheaves of $f^*\mathcal{F}^\bullet$ are adequate.

The final assertion $Rf_* Lf^* \cong \text{id}_{D_{QCoh}(S)}$ follows from the explicit description of the functors above. (In plain english: if $\mathcal{F}$ is quasi-coherent and $p > 0$, then $L_pf^*\mathcal{F}$ is a parasitic adequate module.)

Remark 32.10.3. Remark 32.10.2 above implies we have an equivalence of derived categories

$$D_{Adeq}(\mathcal{O})/D_{\mathcal{C}}(\mathcal{O}) \longrightarrow D_{QCoh}(S)$$

where $\mathcal{C}$ is the category of parasitic adequate modules. Namely, it is clear that $D_{\mathcal{C}}(\mathcal{O})$ is the kernel of Rf_*, hence a functor as indicated. For any object X of $D_{Adeq}(\mathcal{O})$ the map

$Lf^*Rf_*X \to X$ maps to a quasi-isomorphism in $D_{QCoh}(S)$, hence $Lf^*Rf_*X \to X$ is an isomorphism in $D_{Adeq}(\mathcal{O})/D_{\mathcal{C}}(\mathcal{O})$. Finally, for X, Y objects of $D_{Adeq}(\mathcal{O})$ the map

$$Rf_* : Hom_{D_{Adeq}(\mathcal{O})/D_{\mathcal{C}}(\mathcal{O})}(X, Y) \to Hom_{D_{QCoh}(S)}(Rf_*X, Rf_*Y)$$

is bijective as Lf^* gives an inverse (by the remarks above).

32.11. Other chapters

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CHAPTER 33

More on Morphisms

33.1. Introduction

In this chapter we continue our study of properties of morphisms of schemes. A fundamental reference is [**DG67**].

33.2. Thickenings

The following terminology may not be completely standard, but it is convenient.

Definition 33.2.1. Thickenings.

(1) We say a scheme X' is a *thickening* of a scheme X if X is a closed subscheme of X' and the underlying topological spaces are equal.
(2) We say a scheme X' is a *first order thickening* of a scheme X if X is a closed subscheme of X' and the quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_{X'}$ defining X has square zero.
(3) Given two thickenings $X \subset X'$ and $Y \subset Y'$ a *morphism of thickenings* is a morphism $f' : X' \to Y'$ such that $f(X) \subset Y$, i.e., such that $f'|_X$ factors through the closed subscheme Y. In this situation we set $f = f'|_X : X \to Y$ and we say that $(f, f') : (X \subset X') \to (Y \subset Y')$ is a morphism of thickenings.
(4) Let S be a scheme. We similarly define *thickenings over S*, and *morphisms of thickenings over S*. This means that the schemes X, X', Y, Y' above are schemes over S, and that the morphisms $X \to X'$, $Y \to Y'$ and $f' : X' \to Y'$ are morphisms over S.

Finite order thickenings. Let $i_X : X \to X'$ be a thickening. Any local section of the kernel $\mathcal{I} = \text{Ker}(i_X^\sharp)$ is locally nilpotent. Let us say that $X \subset X'$ is a *finite order thickening* if the ideal sheaf $\mathcal{I}$ is ``globally'' nilpotent, i.e., if there exists an $n \geq 0$ such that $\mathcal{I}^{n+1} = 0$. Technically the class of finite order thickenings $X \subset X'$ is much easier to handle than the general case. Namely, in this case we have a filtration

$$0 \subset \mathcal{I}^n \subset \mathcal{I}^{n-1} \subset \ldots \subset \mathcal{I} \subset \mathcal{O}_{X'}$$

and we see that X' is filtered by closed subspaces

$$X = X_0 \subset X_1 \subset \ldots \subset X_{n-1} \subset X_{n+1} = X'$$

such that each pair $X_i \subset X_{i+1}$ is a first order thickening over B. Using simple induction arguments many results proved for first order thickenings can be rephrased as results on finite order thickenings.
First order thickening are described as follows (see Morphisms, Lemma 24.32.3).

Lemma 33.2.2. *Let X be a scheme over a base S. Consider a short exact sequence*

$$0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{O}_X \to 0$$

of sheaves on X where $\mathcal{A}$ is a sheaf of $f^{-1}\mathcal{O}_S$-algebras, $\mathcal{A} \to \mathcal{O}_X$ is a surjection of sheaves of $f^{-1}\mathcal{O}_S$-algebras, and $\mathcal{I}$ is its kernel. If

(1) *$\mathcal{I}$ is an ideal of square zero in $\mathcal{A}$, and*
(2) *$\mathcal{I}$ is quasi-coherent as an $\mathcal{O}_X$-module*

then $X' = (X, \mathcal{A})$ is a scheme and $X \to X'$ is a first order thickening over S. Moreover, any first order thickening over S is of this form.

Proof. It is clear that X' is a locally ringed space. Let $U = Spec(B)$ be an affine open of X. Set $A = \Gamma(U, \mathcal{A})$. Note that since $H^1(U, \mathcal{I}) = 0$ (see Coherent, Lemma 25.2.2) the map $A \to B$ is surjective. By assumption the kernel $I = \mathcal{I}(U)$ is an ideal of square zero in the ring A. By Schemes, Lemma 21.6.4 there is a canonical morphism of locally ringed spaces

$$(U, \mathcal{A}|_U) \longrightarrow Spec(A)$$

coming from the map $B \to \Gamma(U, \mathcal{A})$. Since this morphism fits into the commutative diagram

$$\begin{tikzcd}
(U, \mathcal{O}_X|_U) \arrow[r] \arrow[d] & Spec(B) \arrow[d] \\
(U, \mathcal{A}|_U) \arrow[r] & Spec(A)
\end{tikzcd}$$

we see that it is a homeomorphism on underlying topological spaces. Thus to see that it is an isomorphism, it suffices to check it induces an isomorphism on the local rings. For $u \in U$ corresponding to the prime $\mathfrak{p} \subset A$ we obtain a commutative diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & I_{\mathfrak{p}} \arrow[r] \arrow[d] & A_{\mathfrak{p}} \arrow[r] \arrow[d] & B_{\mathfrak{p}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{I}_u \arrow[r] & \mathcal{A}_u \arrow[r] & \mathcal{O}_{X,u} \arrow[r] & 0.
\end{tikzcd}$$

The left and right vertical arrows are isomorphisms because $\mathcal{I}$ and $\mathcal{O}_X$ are quasi-coherent sheaves. Hence also the middle map is an isomorphism. Hence every point of $X' = (X, \mathcal{A})$ has an affine neighbourhood and X' is a scheme as desired. □

Lemma 33.2.3. *Any thickening of an affine scheme is affine.*

Proof. This is a special case of Limits, Proposition 27.7.2. □

33.3. First order infinitesimal neighbourhood

A natural construction of first order thickenings is the following. Suppose that $i : Z \to X$ be an immersion of schemes. Choose an open subscheme $U \subset X$ such that i identifies Z with a closed subscheme $Z \subset U$. Let $\mathcal{I} \subset \mathcal{O}_U$ be the quasi-coherent sheaf of ideals defining Z in U. Then we can consider the closed subscheme $Z' \subset U$ defined by the quasi-coherent sheaf of ideals $\mathcal{I}^2$.

Definition 33.3.1. Let $i : Z \to X$ be an immersion of schemes. The *first order infinitesimal neighbourhood* of Z in X is the first order thickening $Z \subset Z'$ over X described above.

This thickening has the following universal property (which will assuage any fears that the construction above depends on the choice of the open U).

Lemma 33.3.2. *Let $i : Z \to X$ be an immersion of schemes. The first order infinitesimal neighbourhood Z' of Z in X has the following universal property: Given any commutative diagram*

$$\begin{tikzcd} Z \arrow[d, "i"'] & T \arrow[l, "a"'] \arrow[d] \\ X & T' \arrow[l, "b"] \end{tikzcd}$$

where $T \subset T'$ is a first order thickening over X, there exists a unique morphism $(a', a) : (T \subset T') \to (Z \subset Z')$ of thickenings over X.

Proof. Let $U \subset X$ be the open used in the construction of Z', i.e., an open such that Z is identified with a closed subscheme of U cut out by the quasi-coherent sheaf of ideals $\mathcal{I}$. Since $|T| = |T'|$ we see that $b(T') \subset U$. Hence we can think of b as a morphism into U. Let $\mathcal{J} \subset \mathcal{O}_{T'}$ be the ideal cutting out T. Since $b(T) \subset Z$ by the diagram above we see that $b^\sharp(b^{-1}\mathcal{I}) \subset \mathcal{J}$. As T' is a first order thickening of T we see that $\mathcal{J}^2 = 0$ hence $b^\sharp(b^{-1}(\mathcal{I}^2)) = 0$. By Schemes, Lemma 21.4.6 this implies that b factors through Z'. Denote $a' : T' \to Z'$ this factorization and everything is clear. $\square$

Lemma 33.3.3. *Let $i : Z \to X$ be an immersion of schemes. Let $Z \subset Z'$ be the first order infinitesimal neighbourhood of Z in X. Then the diagram*

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & Z' \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

induces a map of conormal sheaves $\mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Z'}$ by Morphisms, Lemma 24.31.3. This map is an isomorphism.

Proof. This is clear from the construction of Z' above. $\square$

33.4. Formally unramified morphisms

Recall that a ring map $R \to A$ is called *formally unramified* (see Algebra, Definition 7.135.1) if for every commutative solid diagram

$$\begin{tikzcd} A \arrow[r] \arrow[rd, dashed] & B/I \\ R \arrow[u] \arrow[r] & B \arrow[u] \end{tikzcd}$$

where $I \subset B$ is an ideal of square zero, at most one dotted arrow exists which makes the diagram commute. This motivates the following analogue for morphisms of schemes.

Definition 33.4.1. Let $f : X \to S$ be a morphism of schemes. We say f is *formally unramified* if given any solid commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & T \arrow[l] \arrow[d, "i"] \\ S & T' \arrow[l] \arrow[lu, dashed] \end{tikzcd}$$

where $T \subset T'$ is a first order thickening of affine schemes over S there exists at most one dotted arrow making the diagram commute.

We first prove some formal lemmas, i.e., lemmas which can be proved by drawing the corresponding diagrams.

Lemma 33.4.2. *If $f : X \to S$ is a formally unramified morphism, then given any solid commutative diagram*

$$\begin{tikzcd} X \arrow[d, "f"'] & T \arrow[l] \arrow[d, "i"] \\ S & T' \arrow[l] \arrow[ul, dashed] \end{tikzcd}$$

where $T \subset T'$ is a first order thickening of schemes over S there exists at most one dotted arrow making the diagram commute. In other words, in Definition 33.4.1 the condition that T be affine may be dropped.

Proof. This is true because a morphism is determined by its restrictions to affine opens. □

Lemma 33.4.3. *A composition of formally unramified morphisms is formally unramified.*

Proof. This is formal. □

Lemma 33.4.4. *A base change of a formally unramified morphism is formally unramified.*

Proof. This is formal. □

Lemma 33.4.5. *Let $f : X \to S$ be a morphism of schemes. Let $U \subset X$ and $V \subset S$ be open such that $f(U) \subset V$. If f is formally unramified, so is $f|_U : U \to V$.*

Proof. Consider a solid diagram

$$\begin{tikzcd} U \arrow[d, "f|_U"'] & T \arrow[l, "a"'] \arrow[d, "i"] \\ V & T' \arrow[l] \arrow[ul, dashed] \end{tikzcd}$$

as in Definition 33.4.1. If f is formally ramified, then there exists at most one S-morphism $a' : T' \to X$ such that $a'|_T = a$. Hence clearly there exists at most one such morphism into U. □

Lemma 33.4.6. *Let $f : X \to S$ be a morphism of schemes. Assume X and S are affine. Then f is formally unramified if and only if $\mathcal{O}_S(S) \to \mathcal{O}_X(X)$ is a formally unramified ring map.*

Proof. This is immediate from the definitions (Definition 33.4.1 and Algebra, Definition 7.135.1) by the equivalence of categories of rings and affine schemes, see Schemes, Lemma 21.6.5. □

Here is a characterization in terms of the sheaf of differentials.

Lemma 33.4.7. *Let $f : X \to S$ be a morphism of schemes. Then f is formally unramified if and only if $\Omega_{X/S} = 0$.*

Proof. We give two proofs.

First proof. It suffices to show that $\Omega_{X/S}$ is zero on the members of an affine open covering of X. Choose an affine open $U \subset X$ with $f(U) \subset V$ where $V \subset S$ is an affine open of S. By Lemma 33.4.5 the restriction $f_U : U \to V$ is formally unramified. By Morphisms, Lemma 24.32.7 we see that $\Omega_{X/S}|_U$ is the quasi-coherent sheaf associated to the $\mathcal{O}_X(U)$-module

$\Omega_{\mathcal{O}_X(U)/\mathcal{O}_S(V)}$. By Lemma 33.4.6 we see that $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is a formally unramified ring map. Hence by Algebra, Lemma 7.135.2 we conclude that $\Omega_{X/S}|_U = 0$ as desired.

Second proof. We recall some of the arguments of Morphisms, Section 24.32. Let $U \subset X \times_S X$ be an open such that $\Delta : X \to X \times_S X$ induces a closed immersion into U. Let $\mathcal{J} \subset \mathcal{O}_U$ be the ideal sheaf of this closed immersion. Let $X' \subset U$ be the closed subscheme defined by the quasi-coherent sheaf of ideals $\mathcal{J}^2$. Consider the two morphisms $p_1, p_2 : X' \to X$ induced by the two projections $X \times_S X \to X$. Note that p_1 and p_2 agree when composed with $\Delta : X \to X'$ and that $X \to X'$ is a closed immersion defined by a an ideal whose square is zero. Moreover there is a short exact sequence

$$0 \to \mathcal{J}/\mathcal{J}^2 \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$$

and $\Omega_{X/S} = \mathcal{J}/\mathcal{J}^2$. Moreover, $\mathcal{J}/\mathcal{J}^2$ is generated by the local sections $p_1^\sharp(f) - p_2^\sharp(f)$ for f a local section of $\mathcal{O}_X$.

Suppose that $f : X \to S$ is formally unramified. By assumption this means that $p_1 = p_2$ when restricted to any affine open $T' \subset X'$. Hence $p_1 = p_2$. By what was said above we conclude that $\Omega_{X/S} = \mathcal{J}/\mathcal{J}^2 = 0$.

Conversely, suppose that $\Omega_{X/S} = 0$. Then $X' = X$. Take any pair of morphisms $f'_1, f'_2 : T' \to X$ fitting as dotted arrows in the diagram of Definition 33.4.1. This gives a morphism $(f'_1, f'_2) : T' \to X \times_S X$. Since $f'_1|_T = f'_2|_T$ and $|T| = |T'|$ we see that the image of T' under (f'_1, f'_2) is contained in the open U chosen above. Since $(f'_1, f'_2)(T) \subset \Delta(X)$ and since T is defined by an ideal of square zero in T' we see that (f'_1, f'_2) factors through X'. As $X' = X$ we conclude $f'_1 = f'_2$ as desired. $\square$

Lemma 33.4.8. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *The morphism f is unramified (resp. G-unramified), and*
(2) *the morphism f is locally of finite type (resp. locally of finite presentation) and formally unramified.*

Proof. Use Lemma 33.4.7 and Morphisms, Lemma 24.34.2. $\square$

33.5. Universal first order thickenings

Let $h : Z \to X$ be a morphism of schemes. A *universal first order thickening* of Z over X is a first order thickening $Z \subset Z'$ over X such that given any first order thickening $T \subset T'$ over X and a solid commutative diagram

$$\begin{tikzcd} & Z \arrow[dl] & & T \arrow[ll, "a"'] \arrow[dr] & \\ Z' \arrow[drr] & & & & T' \arrow[llll, dotted, "a'"'] \arrow[dll, "b"] \\ & & X & & \end{tikzcd}$$

there exists a unique dotted arrow making the diagram commute. Note that in this situation $(a, a') : (T \subset T') \to (Z \subset Z')$ is a morphism of thickenings over X. Thus if a universal first order thickening exists, then it is unique up to unique isomorphism. In general a universal first order thickening does not exist, but if h is formally unramified then it does.

Lemma 33.5.1. *Let $h : Z \to X$ be a formally unramified morphism of schemes. There exists a universal first order thickening $Z \subset Z'$ of Z over X.*

Proof. During this proof we will say $Z \subset Z'$ is a universal first order thickening of Z over X if it satisfies the condition of the lemma. We will construct the universal first order thickening $Z \subset Z'$ over X by glueing, starting with the affine case which is Algebra, Lemma 7.136.1. We begin with some general remarks.

If a universal first order thickening of Z over X exists, then it is unique up to unique isomorphism. Moreover, suppose that $V \subset Z$ and $U \subset X$ are open subschemes such that $h(V) \subset U$. Let $Z \subset Z'$ be a universal first order thickening of Z over X. Let $V' \subset Z'$ be the open subscheme such that $V = Z \cap V'$. Then we claim that $V \subset V'$ is the universal first order thickening of V over U. Namely, suppose given any diagram

$$\begin{tikzcd} V \arrow[d, "h"'] & T \arrow[l, "a"'] \arrow[d] \\ U & T' \arrow[l, "b"'] \end{tikzcd}$$

where $T \subset T'$ is a first order thickening over U. By the universal property of Z' we obtain $(a, a') : (T \subset T') \to (Z \subset Z')$. But since we have equality $|T| = |T'|$ of underlying topological spaces we see that $a'(T') \subset V'$. Hence we may think of (a, a') as a morphism of thickenings $(a, a') : (T \subset T') \to (V \subset V')$ over U. Uniqueness is clear also. In a completely similar manner one proves that if $h(Z) \subset U$ and $Z \subset Z'$ is a universal first order thickening over U, then $Z \subset Z'$ is a universal first order thickening over X.

Before we glue affine pieces let us show that the lemma holds if Z and X are affine. Say $X = Spec(R)$ and $Z = Spec(S)$. By Algebra, Lemma 7.136.1 there exists a first order thickening $Z \subset Z'$ over X which has the universal property of the lemma for diagrams

$$\begin{tikzcd} Z \arrow[d, "h"'] & T \arrow[l, "a"'] \arrow[d] \\ X & T' \arrow[l, "b"'] \end{tikzcd}$$

where T, T' are affine. Given a general diagram we can choose an affine open covering $T' = \bigcup T'_i$ and we obtain morphisms $a'_i : T'_i \to Z'$ over X such that $a'_i|_{T_i} = a|_{T_i}$. By uniqueness we see that a'_i and a'_j agree on any affine open of $T'_i \cap T'_j$. Hence the morphisms a'_i glue to a global morphism $a' : T' \to Z'$ over X as desired. Thus the lemma holds if X and Z are affine.

Choose an affine open covering $Z = \bigcup Z_i$ such that each Z_i maps into an affine open U_i of X. By Lemma 33.4.5 the morphisms $Z_i \to U_i$ are formally unramified. Hence by the affine case we obtain universal first order thickenings $Z_i \subset Z'_i$ over U_i. By the general remarks above $Z_i \subset Z'_i$ is also a universal first order thickening of Z_i over X. Let $Z'_{i,j} \subset Z'_i$ be the open subscheme such that $Z_i \cap Z_j = Z'_{i,j} \cap Z_i$. By the general remarks we see that both $Z'_{i,j}$ and $Z'_{j,i}$ are universal first order thickenings of $Z_i \cap Z_j$ over X. Thus, by the first of our general remarks, we see that there is a canonical isomorphism $\varphi_{ij} : Z'_{i,j} \to Z'_{j,i}$ inducing the identity on $Z_i \cap Z_j$. We claim that these morphisms satisfy the cocycle condition of Schemes, Section 21.14. (Verification omitted. Hint: Use that $Z'_{i,j} \cap Z'_{i,k}$ is the universal first order thickening of $Z_i \cap Z_j \cap Z_k$ which determines it up to unique isomorphism by what was said above.) Hence we can use the results of Schemes, Section 21.14 to get a first order thickengin $Z \subset Z'$ over X which the property that the open subscheme $Z'_i \subset Z'$ with $Z_i = Z'_i \cap Z$ is a universal first order thickening of Z_i over X.

It turns out that this implies formally that Z' is a universal first order thickening of Z over X. Namely, we have the universal property for any diagram

$$\begin{tikzcd} Z \arrow[d, "h"'] & T \arrow[l, "a"'] \arrow[d] \\ X & T' \arrow[l, "b"] \end{tikzcd}$$

where $a(T)$ is contained in some Z_i. Given a general diagram we can choose an open covering $T' = \bigcup T'_i$ such that $a(T_i) \subset Z_i$. We obtain morphisms $a'_i : T'_i \to Z'$ over X such that $a'_i|_{T_i} = a|_{T_i}$. We see that a'_i and a'_j necassarily agree on $T'_i \cap T'_j$ since both $a'_i|_{T'_i \cap T'_j}$ and $a'_j|_{T'_i \cap T'_j}$ are solutions of the problem of mapping into the universal first oder thickening $Z'_i \cap Z'_j$ of $Z_i \cap Z_j$ over X. Hence the morphisms a'_i glue to a global morphism $a' : T' \to Z'$ over X as desired. This finishes the proof. □

Definition 33.5.2. Let $h : Z \to X$ be a formally unramified morphism of schemes.

(1) The *universal first order thickening* of Z over X is the thickening $Z \subset Z'$ constructed in Lemma 33.5.1.
(2) The *conormal sheaf of Z over X* is the conormal sheaf of Z in its universal first order thickening Z' over X.

We often denote the conormal sheaf $\mathcal{C}_{Z/X}$ in this situation.

Thus we see that there is a short exact sequence of sheaves

$$0 \to \mathcal{C}_{Z/X} \to \mathcal{O}_{Z'} \to \mathcal{O}_Z \to 0$$

on Z. The following lemma proves that there is no conflict between this definition and the definition in case $Z \to X$ is an immersion.

Lemma 33.5.3. *Let $i : Z \to X$ be an immersion of schemes. Then*

(1) *i is formally unramified,*
(2) *the universal first order thickening of Z over X is the first order infinitesimal neighbourhood of Z in X of Definition 33.3.1, and*
(3) *the conormal sheaf of i in the sense of Morphisms, Definition 24.31.1 agrees with the conormal sheaf of i in the sense of Definition 33.5.2.*

Proof. By Morphisms, Lemmas 24.34.7 and 24.34.8 an immersion is unramified, hence formally unramified by Lemma 33.4.8. The other assertions follow by combining Lemmas 33.3.2 and 33.3.3 and the definitions. □

Lemma 33.5.4. *Let $Z \to X$ be a formally unramified morphism of schemes. Then the universal first order thickening Z' is formally unramified over X.*

Proof. There are two proofs. The first is to show that $\Omega_{Z'/X} = 0$ by working affine locally and applying Algebra, Lemma 7.136.5. Then Lemma 33.4.7 implies what we want. The second is a direct argument as follows.

Let $T \subset T'$ be a first order thickening. Let

$$\begin{tikzcd} Z' \arrow[d] & T \arrow[l, "c"'] \arrow[d] \\ X & T' \arrow[l] \arrow[ul, "{a,b}"] \end{tikzcd}$$

be a commutative diagram. Consider two morphisms $a, b : T' \to Z'$ fitting into the diagram. Set $T_0 = c^{-1}(Z) \subset T$ and $T'_a = a^{-1}(Z)$ (scheme theoretically). Since Z' is a first order thickening of Z, we see that T' is a first order thickening of T'_a. Moreover, since $c = a|_T$ we see that $T_0 = T \cap T'_a$ (scheme theoretically). As T' is a first order thickening of T it follows that T'_a is a first order thickening of T_0. Now $a|_{T'_a}$ and $b|_{T'_a}$ are morphisms of T'_a into Z' over X which agree on T_0 as morphisms into Z. Hence by the universal property of Z' we conclude that $a|_{T'_a} = b|_{T'_a}$. Thus a and b are morphism from the first order thickening T' of T'_a whose restrictions to T'_a agree as morphisms into Z. Thus using the universal property of Z' once more we conclude that $a = b$. In other words, the defining property of a formally unramified morphism holds for $Z' \to X$ as desired. □

Lemma 33.5.5. *Consider a commutative diagram of schemes*

$$\begin{tikzcd} Z \arrow[r, "h"'] \arrow[d, "f"'] & X \arrow[d, "g"] \\ W \arrow[r, "h'"] & Y \end{tikzcd}$$

with h and h' formally unramified. Let $Z \subset Z'$ be the universal first order thickening of Z over X. Let $W \subset W'$ be the universal first order thickening of W over Y. There exists a canonical morphism $(f, f') : (Z, Z') \to (W, W')$ of thickenings over Y which fits into the following commutative diagram

$$\begin{tikzcd} & & Z' \arrow[dl] \arrow[d, "f'"] \\ Z \arrow[urr] \arrow[r] \arrow[d, "f"'] & X \arrow[d] & W' \arrow[dl] \\ W \arrow[urr] \arrow[r] & Y & \end{tikzcd}$$

In particular the morphism (f, f') of thickenings induces a morphism of conormal sheaves $f^*\mathcal{C}_{W/Y} \to \mathcal{C}_{Z/X}$.

Proof. The first assertion is clear from the universal property of W'. The induced map on conormal sheaves is the map of Morphisms, Lemma 24.31.3 applied to $(Z \subset Z') \to (W \subset W')$. □

Lemma 33.5.6. *Let*

$$\begin{tikzcd} Z \arrow[r, "h"'] \arrow[d, "f"'] & X \arrow[d, "g"] \\ W \arrow[r, "h'"] & Y \end{tikzcd}$$

be a fibre product diagram in the category of schemes with h' formally unramified. Then h is formally unramified and if $W \subset W'$ is the universal first order thickening of W over Y, then $Z = X \times_Y W \subset X \times_Y W'$ is the universal first order thickening of Z over X. In particular the canonical map $f^*\mathcal{C}_{W/Y} \to \mathcal{C}_{Z/X}$ of Lemma 33.5.5 is surjective.

Proof. The morphism h is formally unramified by Lemma 33.4.4. It is clear that $X \times_Y W'$ is a first order thickening. It is straightforward to check that it has the universal property because W' has the universal property (by mapping properties of fibre products). See

Morphisms, Lemma 24.31.4 for why this implies that the map of conormal sheaves is surjective. ☐

Lemma 33.5.7. *Let*

$$\begin{tikzcd} Z \arrow[r, "h"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ W \arrow[r, "h'"] & Y \end{tikzcd}$$

be a fibre product diagram in the category of schemes with h' formally unramified and g flat. In this case the corresponding map $Z' \to W'$ of universal first order thickenings is flat, and $f^*\mathcal{C}_{W/Y} \to \mathcal{C}_{Z/X}$ is an isomorphism.

Proof. Flatness is preserved under base change, see Morphisms, Lemma 24.24.7. Hence the first statement follows from the description of W' in Lemma 33.5.6. It is clear that $X \times_Y W'$ is a first order thickening. It is straightforward to check that it has the universal property because W' has the universal property (by mapping properties of fibre products). See Morphisms, Lemma 24.31.4 for why this implies that the map of conormal sheaves is an isomorphism. ☐

Lemma 33.5.8. *Taking the universal first order thickenings commutes with taking opens. More precisely, let $h : Z \to X$ be a formally unramified morphism of schemes. Let $V \subset Z$, $U \subset X$ be opens such that $h(V) \subset U$. Let Z' be the universal first order thickening of Z over X. Then $h|_V : V \to U$ is formally unramified and the universal first order thickening of V over U is the open subscheme $V' \subset Z'$ such that $V = Z \cap V'$. In particular, $\mathcal{C}_{Z/X}|_V = \mathcal{C}_{V/U}$.*

Proof. The first statement is Lemma 33.4.5. The compatibility of universal thickenings can be deduced from the proof of Lemma 33.5.1, or from Algebra, Lemma 7.136.4 or deduced from Lemma 33.5.7. ☐

Lemma 33.5.9. *Let $h : Z \to X$ be a formally unramified morphism of schemes over S. Let $Z \subset Z'$ be the universal first order thickening of Z over X with structure morphism $h' : Z' \to X$. The canonical map*

$$c_{h'} : (h')^*\Omega_{X/S} \longrightarrow \Omega_{Z'/S}$$

induces an isomorphism $h^*\Omega_{X/S} \to \Omega_{Z'/S} \otimes \mathcal{O}_Z$.

Proof. The map $c_{h'}$ is the map defined in Morphisms, Lemma 24.32.9. If $i : Z \to Z'$ is the given closed immersion, then $i^*c_{h'}$ is a map $h^*\Omega_{X/S} \to \Omega_{Z'/S} \otimes \mathcal{O}_Z$. Checking that it is an isomorphism reduces to the affine case by localization, see Lemma 33.5.8 and Morphisms, Lemma 24.32.6. In this case the result is Algebra, Lemma 7.136.5. ☐

Lemma 33.5.10. *Let $h : Z \to X$ be a formally unramified morphism of schemes over S. There is a canonical exact sequence*

$$\mathcal{C}_{Z/X} \to h^*\Omega_{X/S} \to \Omega_{Z/S} \to 0.$$

The first arrow is induced by $d_{Z'/S}$ where Z' is the universal first order neighbourhood of Z over X.

Proof. We know that there is a canonical exact sequence

$$\mathcal{C}_{Z/Z'} \to \Omega_{Z'/S} \otimes \mathcal{O}_Z \to \Omega_{Z/S} \to 0.$$

see Morphisms, Lemma 24.32.17. Hence the result follows on applying Lemma 33.5.9. ☐

Lemma 33.5.11. *Let*

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[dr, "j"'] & X \arrow[d] \\ & Y \end{tikzcd}$$

be a commutative diagram of schemes where i and j are formally unramified. Then there is a canonical exact sequence

$$\mathcal{C}_{Z/Y} \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/Y} \to 0$$

where the first arrow comes from Lemma 33.5.5 and the second from Lemma 33.5.10.

Proof. Denote $Z \to Z'$ the universal first order thickening of Z over X. Denote $Z \to Z''$ the universal first order thickening of Z over Y. By Lemma 33.5.10 here is a canonical morpism $Z' \to Z''$ so that we have a commutative diagram

$$\begin{tikzcd} Z \arrow[r, "i'"] \arrow[dr, "j'"'] & Z' \arrow[r] \arrow[d] & X \arrow[d] \\ & Z'' \arrow[r] & Y \end{tikzcd}$$

Apply Morphisms, Lemma 24.32.20 to the left triangle to get an exact sequence

$$\mathcal{C}_{Z/Z''} \to \mathcal{C}_{Z/Z'} \to (i')^*\Omega_{Z'/Z''} \to 0$$

As Z'' is formally unramified over Y (see Lemma 33.5.4) we have $\Omega_{Z'/Z''} = \Omega_{Z/Y}$ (by combining Lemma 33.4.7 and Morphisms, Lemma 24.32.11). Then we have $(i')^*\Omega_{Z'/Y} = i^*\Omega_{X/Y}$ by Lemma 33.5.9. $\square$

Lemma 33.5.12. *Let $Z \to Y \to X$ be formally unramified morphisms of schemes.*

(1) *If $Z \subset Z'$ is the universal first order thickening of Z over X and $Y \subset Y'$ is the universal first order thickening of Y over X, then there is a morphism $Z' \to Y'$ and $Y \times_{Y'} Z'$ is the universal first order thickening of Z over Y.*
(2) *There is a canonical exact sequence*

$$i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

where the maps come from Lemma 33.5.5 and $i : Z \to Y$ is the first morphism.

Proof. The map $h : Z' \to Y'$ in (1) comes from Lemma 33.5.5. The assertion that $Y \times_{Y'} Z'$ is the universal first order thickening of Z over Y is clear from the universal properties of Z' and Y'. By Morphisms, Lemma 24.31.5 we have an exact sequence

$$(i')^*\mathcal{C}_{Y\times_{Y'} Z'/Z'} \to \mathcal{C}_{Z/Z'} \to \mathcal{C}_{Z/Y\times_{Y'} Z'} \to 0$$

where $i' : Z \to Y \times_{Y'} Z'$ is the given morphism. By Morphisms, Lemma 24.31.4 there exists a surjection $h^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{Y\times_{Y'} Z'/Z'}$. Combined with the equalities $\mathcal{C}_{Y/Y'} = \mathcal{C}_{Y/X}$, $\mathcal{C}_{Z/Z'} = \mathcal{C}_{Z/X}$, and $\mathcal{C}_{Z/Y\times_{Y'} Z'} = \mathcal{C}_{Z/Y}$ this proves the lemma. $\square$

33.6. Formally étale morphisms

Recall that a ring map $R \to A$ is called *formally étale* (see Algebra, Definition 7.137.1) if for every commutative solid diagram

$$\begin{tikzcd} A \arrow[r] \arrow[rd, dashed] & B/I \\ R \arrow[u] \arrow[r] & B \arrow[u] \end{tikzcd}$$

where $I \subset B$ is an ideal of square zero, there exists exactly one dotted arrow which makes the diagram commute. This motivates the following analogue for morphisms of schemes.

Definition 33.6.1. Let $f : X \to S$ be a morphism of schemes. We say f is *formally étale* if given any solid commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & T \arrow[l] \arrow[d, "i"] \\ S & T' \arrow[l] \arrow[lu, dashed] \end{tikzcd}$$

where $T \subset T'$ is a first order thickening of affine schemes over S there exists exactly one dotted arrow making the diagram commute.

It is clear that a formally étale morphism is formally unramified. Hence if $f : X \to S$ is formally étale, then $\Omega_{X/S}$ is zero, see Lemma 33.4.7.

Lemma 33.6.2. *If $f : X \to S$ is a formally étale morphism, then given any solid commutative diagram*

$$\begin{tikzcd} X \arrow[d, "f"'] & T \arrow[l] \arrow[d, "i"] \\ S & T' \arrow[l] \arrow[lu, dashed] \end{tikzcd}$$

where $T \subset T'$ is a first order thickening of schemes over S there exists exactly one dotted arrow making the diagram commute. In other words, in Definition 33.6.1 the condition that T be affine may be dropped.

Proof. Let $T' = \bigcup T'_i$ be an affine open covering, and let $T_i = T \cap T'_i$. Then we get morphisms $a'_i : T'_i \to X$ fitting into the diagram. By uniqueness we see that a'_i and a'_j agree on any affine open subscheme of $T'_i \cap T'_j$. Hence a'_i and a'_j agree on $T'_i \cap T'_j$. Thus we see that the morphisms a'_i glue to a global morphism $a' : T' \to X$. The uniqueness of a' we have seen in Lemma 33.4.2. □

Lemma 33.6.3. *A composition of formally étale morphisms is formally étale.*

Proof. This is formal. □

Lemma 33.6.4. *A base change of a formally étale morphism is formally étale.*

Proof. This is formal. □

Lemma 33.6.5. *Let $f : X \to S$ be a morphism of schemes. Let $U \subset X$ and $V \subset S$ be open subschemes such that $f(U) \subset V$. If f is formally étale, so is $f|_U : U \to V$.*

Proof. Consider a solid diagram

$$\begin{tikzcd} U \arrow[d, "f|_U"'] & T \arrow[l, "a"'] \arrow[d, "i"] \\ V & T' \arrow[l] \arrow[ul, dashed] \end{tikzcd}$$

as in Definition 33.6.1. If f is formally ramified, then there exists exactly one S-morphism $a' : T' \to X$ such that $a'|_T = a$. Since $|T'| = |T|$ we conclude that $a'(T') \subset U$ which gives our unique morphism from T' into U. $\square$

Lemma 33.6.6. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *f is formally étale,*
(2) *f is formally unramified and the universal first order thickening of X over S is equal to X,*
(3) *f is formally unramified and $\mathcal{C}_{X/S} = 0$, and*
(4) *$\Omega_{X/S} = 0$ and $\mathcal{C}_{X/S} = 0$.*

Proof. Actually, the last assertion only make sense because $\Omega_{X/S} = 0$ implies that $\mathcal{C}_{X/S}$ is defined via Lemma 33.4.7 and Definition 33.5.2. This also makes it clear that (3) and (4) are equivalent.

Either of the assumptions (1), (2), and (3) imply that f is formally unramified. Hence we may assume f is formally unramified. The equivalence of (1), (2), and (3) follow from the universal property of the universal first order thickening X' of X over S and the fact that $X = X' \Leftrightarrow \mathcal{C}_{X/S} = 0$ since after all by definition $\mathcal{C}_{X/S} = \mathcal{C}_{X/X'}$ is the ideal sheaf of X in X'. $\square$

Lemma 33.6.7. *An unramified flat morphism is formally étale.*

Proof. Say $X \to S$ is unramified and flat. Then $\Delta : X \to X \times_S X$ is an open immersion, see Morphisms, Lemma 24.34.13. We have to show that $\mathcal{C}_{X/S}$ is zero. Consider the two projections $p, q : X \times_S X \to X$. As f is formally unramified (see Lemma 33.4.8), q is formally unramified (see Lemma 33.4.4). As f is flat, p is flat, see Morphisms, Lemma 24.24.7. Hence $p^*\mathcal{C}_{X/S} = \mathcal{C}_q$ by Lemma 33.5.7 where $\mathcal{C}_q$ denotes the conormal sheaf of the formally unramified morphism $q : X \times_S X \to X$. But $\Delta(X) \subset X \times_S X$ is an open subscheme which maps isomorphically to X via q. Hence by Lemma 33.5.8 we see that $\mathcal{C}_q|_{\Delta(X)} = \mathcal{C}_{X/X} = 0$. In other words, the pullback of $\mathcal{C}_{X/S}$ to X via the identity morphism is zero, i.e., $\mathcal{C}_{X/S} = 0$. $\square$

Lemma 33.6.8. *Let $f : X \to S$ be a morphism of schemes. Assume X and S are affine. Then f is formally étale if and only if $\mathcal{O}_S(S) \to \mathcal{O}_X(X)$ is a formally étale ring map.*

Proof. This is immediate from the definitions (Definition 33.6.1 and Algebra, Definition 7.137.1) by the equivalence of categories of rings and affine schemes, see Schemes, Lemma 21.6.5. $\square$

Lemma 33.6.9. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *The morphism f is étale, and*
(2) *the morphism f is locally of finite presentation and formally étale.*

Proof. Assume f is étale. An étale morphism is locally of finite presentation, flat and unramified, see Morphisms, Section 24.35. Hence f is locally of finite presentation and formally étale, see Lemma 33.6.7.

Conversely, suppose that f is locally of finite presentation and formally étale. Being étale is local in the Zariski topology on X and S, see Morphisms, Lemma 24.35.2. By Lemma 33.6.5 we can cover X by affine opens U which map into affine opens V such that $U \to V$ is formally étale (and of finite presentation, see Morphisms, Lemma 24.20.2). By Lemma 33.6.8 we see that the ring maps $\mathcal{O}(V) \to \mathcal{O}(U)$ are formally étale (and of finite presentation). We win by Algebra, Lemma 7.137.2. (We will give another proof of this implication when we discuss formally smooth morphisms.) □

33.7. Infinitesimal deformations of maps

In this section we explain how a derivation can be used to infinitesimally move a map. Throughout this section we use that a sheaf on a thickening X' of X can be seen as a sheaf on X.

Lemma 33.7.1. *Let S be a scheme. Let $X \subset X'$ and $Y \subset Y'$ be two first order thickenings over S. Let $(a, a'), (b, b') : (X \subset X') \to (Y \subset Y')$ be two morphisms of thickenings over S. Assume that*

(1) *$a = b$, and*
(2) *the two maps $a^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{X/X'}$ (Morphisms, Lemma 24.31.3) are equal.*

Then the map $(a')^\sharp - (b')^\sharp$ factors as

$$\mathcal{O}_{Y'} \to \mathcal{O}_Y \xrightarrow{D} a_*\mathcal{C}_{X/X'} \to a_*\mathcal{O}_{X'}$$

where D is an $\mathcal{O}_S$-derivation.

Proof. Instead of working on Y we work on X. The advantage is that the pullback functor a^{-1} is exact. Using (1) and (2) we obtain a commutive diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{C}_{X/X'} \arrow[r] & \mathcal{O}_{X'} \arrow[r] & \mathcal{O}_X \arrow[r] & 0 \\ 0 \arrow[r] & a^{-1}\mathcal{C}_{Y/Y'} \arrow[r] \arrow[u] & a^{-1}\mathcal{O}_{Y'} \arrow[r] \arrow[u, shift left, "(a')^\sharp"'] \arrow[u, shift right, "(b')^\sharp"] & a^{-1}\mathcal{O}_Y \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

Now it is a general fact that in such a situation the difference of the $\mathcal{O}_S$-algebra maps $(a')^\sharp$ and $(b')^\sharp$ is an $\mathcal{O}_S$-derivation from $a^{-1}\mathcal{O}_Y$ to $\mathcal{C}_{X/X'}$. By adjointness of the functors a^{-1} and a_* this is the same thing as an $\mathcal{O}_S$-derivation from $\mathcal{O}_Y$ into $a_*\mathcal{C}_{X/X'}$. Some details omitted. □

Note that in the situation of the lemma above we may write D as

$$D = \mathrm{d}_{Y/S} \circ \theta \tag{33.7.1.1}$$

where θ is an $\mathcal{O}_Y$-linear map $\theta : \Omega_{Y/S} \to a_*\mathcal{C}_{X/X'}$. Of course, then by adjunction again we may view θ as an $\mathcal{O}_X$-linear map $\theta : a^*\Omega_{Y/S} \to \mathcal{C}_{X/X'}$.

Lemma 33.7.2. *Let S be a scheme. Let $(a, a') : (X \subset X') \to (Y \subset Y')$ be a morphism of first order thickenings over S. Let*

$$\theta : a^*\Omega_{Y/S} \to \mathcal{C}_{X/X'}$$

be an $\mathcal{O}_X$-linear map. Then there exists a unique morphism of pairs $(b, b') : (X \subset X') \to (Y \subset Y')$ such that (1) and (2) of Lemma 33.7.1 hold and the derivation D and θ are related by Equation (33.7.1.1).

Proof. We simply set $b = a$ and we define $(b')^\sharp$ to be the map

$$(a')^\sharp + D : a^{-1}\mathcal{O}_{Y'} \to \mathcal{O}_{X'}$$

where D is as in Equation (33.7.1.1). We omit the verification that $(b')^\sharp$ is a map of sheaves of $\mathcal{O}_S$-algebras and that (1) and (2) of Lemma 33.7.1 hold. Equation (33.7.1.1) holds by construction. $\square$

Lemma 33.7.3. *Let S be a scheme. Let $X \subset X'$ and $Y \subset Y'$ be first order thickenings over S. Assume given a morphism $a : X \to Y$ and a map $A : a^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{X/X'}$ of $\mathcal{O}_X$-modules. For an open subscheme $U' \subset X'$ consider morphisms $a' : U' \to Y'$ such that*

1. *a' is a morphism over S,*
2. *$a'|_U = a|_U$, and*
3. *the induced map $a^*\mathcal{C}_{Y/Y'}|_U \to \mathcal{C}_{X/X'}|_U$ is the restriction of A to U.*

Here $U = X \cap U'$. Then the rule

$$U' \mapsto \{a' : U' \to Y' \text{ such that (1), (2), (3) hold.}\} \quad (33.7.3.1)$$

defines a sheaf of sets on X'.

Proof. Denote $\mathcal{F}$ the rule of the lemma. The restriction mapping $\mathcal{F}(U') \to \mathcal{F}(V')$ for $V' \subset U' \subset X'$ of $\mathcal{F}$ is really the restriction map $a' \mapsto a'|_{V'}$. With this definition in place it is clear that $\mathcal{F}$ is a sheaf since morphisms are defined locally. $\square$

In the following lemma we identify sheaves on X and any thickening of X.

Lemma 33.7.4. *Same notation and assumptions as in Lemma 33.7.3. There is an action of the sheaf*

$$\mathcal{H}om_{\mathcal{O}_X}(a^*\Omega_{Y/S}, \mathcal{C}_{X/X'})$$

on the sheaf (33.7.3.1). Moreover, the action is simply transitive for any open $U' \subset X'$ over which the sheaf (33.7.3.1) has a section.

Proof. This is a combination of Lemmas 33.7.1, 33.7.2, and 33.7.3. $\square$

Remark 33.7.5. A special case of Lemmas 33.7.1, 33.7.2, 33.7.3, and 33.7.4 is where $Y = Y'$. In this case the map A is always zero. The sheaf of Lemma 33.7.3 is just given by the rule

$$U' \mapsto \{a' : U' \to Y \text{ over } S \text{ with } a'|_U = a|_U\}$$

and we act on this by the sheaf $\mathcal{H}om_{\mathcal{O}_X}(a^*\Omega_{Y/S}, \mathcal{C}_{X/X'})$. The action of a local section θ on a' is sometimes indicated by $\theta \cdot a'$. Note that this means nothing else than the fact that $(a')^\sharp$ and $(\theta \cdot a')^\sharp$ differ by a derivation D which is related to θ by Equation (33.7.1.1).

Lemma 33.7.6. *Let S be a scheme. Let $X \subset X'$ be a first order thickening over S. Let Y be a scheme over S. Let $a', b' : X' \to Y$ be two morphisms over S with $a = a'|_X = b'|_X$. This gives rise to a commutative diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "a"'] & X' \arrow[d, "{(b',a')}"] \\ Y \arrow[r, "\Delta_{Y/S}"] & Y \times_S Y \end{tikzcd}$$

Since the horizontal arrows are immersions with conormal sheaves $\mathcal{C}_{X/X'}$ and $\Omega_{Y/S}$, by Morphisms, Lemma 24.31.3, we obtain a map $\theta : a^*\Omega_{Y/S} \to \mathcal{C}_{X/X'}$. Then this θ and the derivation D of Lemma 33.7.1 are related by Equation (33.7.1.1).

Proof. Omitted. Hint: The equality may be checked on affine opens where it comes from the following computation. If f is a local section of $\mathcal{O}_Y$, then $1 \otimes f - f \otimes 1$ is a local section of $\mathcal{C}_{Y/(Y \times_S Y)}$ corresponding to $\mathrm{d}_{Y/S}(f)$. It is mapped to the local section $(a')^\sharp(f) - (b')^\sharp(f) = D(f)$ of $\mathcal{C}_{X/X'}$. In other words, $\theta(\mathrm{d}_{Y/S}(f)) = D(f)$. $\square$

For later purposes we need a result that roughly states that the construction of Lemma 33.7.2 is compatible with étale localization.

Lemma 33.7.7. *Let*

$$\begin{tikzcd} X_1 \arrow[d] & X_2 \arrow[l, "f"'] \arrow[d] \\ S_1 & S_2 \arrow[l] \end{tikzcd}$$

be a commutative diagram of schemes with $X_2 \to X_1$ and $S_2 \to S_1$ étale. Then the map $c_f : f^*\Omega_{X_1/S_1} \to \Omega_{X_2/S_2}$ of Morphisms, Lemma 24.32.9 is an isomorphism.

Proof. We recall that an étale morphism $U \to V$ is a smooth morphism with $\Omega_{U/V} = 0$. Using this we see that Morphisms, Lemma 24.32.11 implies $\Omega_{X_2/S_2} = \Omega_{X_2/S_1}$ and Morphisms, Lemma 24.33.16 implies that the map $f^*\Omega_{X_1/S_1} \to \Omega_{X_2/S_1}$ (for the morphism f seen as a morphism over S_1) is an isomorphism. Hence the lemma follows. $\square$

Lemma 33.7.8. *Consider a commutative diagram of schemes*

$$\begin{tikzcd} T_2 \arrow[r] \arrow[d, "h"'] & T_2' \arrow[rr, "a_2'"'] \arrow[d, "h'"] & & X_2 \arrow[dl, "f"] \arrow[dd] \\ T_1 \arrow[r] & T_1' \arrow[r, "a_1'"'] & X_1 \arrow[d] & \\ & & S_1 & S_2 \arrow[l] \end{tikzcd}$$

and assume that

(1) *$i_1 : T_1 \to T_1'$ is a first order thickening,*
(2) *$i_2 : T_2 \to T_2'$ is a first order thickening, and*
(3) *$X_2 \to X_1$ and $S_2 \to S_1$ are étale.*

Write $a_i = a_i' \circ i_k$ for $k = 1, 2$. For any $\mathcal{O}_{T_1}$-linear map $\theta_1 : a_1^*\Omega_{X_1/S_1} \to \mathcal{C}_{T_1/T_1'}$ let θ_2 be the composition

$$a_2^*\Omega_{X_2/S_2} = h^*a_1^*\Omega_{X_1/S_1} \xrightarrow{h^*\theta_1} h^*\mathcal{C}_{T_1/T_1'} \longrightarrow \mathcal{C}_{T_2/T_2'}$$

(equality sign is explained in the proof). Then the diagram

$$\begin{tikzcd} T_2' \arrow[r, "\theta_2 \cdot a_2'"'] \arrow[d] & X_2 \arrow[d] \\ T_1' \arrow[r, "\theta_1 \cdot a_1'"] & X_1 \end{tikzcd}$$

commutes where the actions $\theta_2 \cdot a_2'$ and $\theta_1 \cdot a_1'$ are as in Remark 33.7.5.

Proof. The equality sign comes from the identification $f^*\Omega_{X_1/S_1} = \Omega_{X_2/S_2}$ of Lemma 33.7.7. Namely, using this we have $a_2^*\Omega_{X_2/S_2} = a_2^* f^*\Omega_{X_1/S_1} = h^* a_1^*\Omega_{X_1/S_1}$ because $f \circ a_2 = a_1 \circ h$. Having said this, the commutativity of the diagram may be checked on affine opens. Hence we may assume the schemes in the initial big diagram are affine. Thus we obtain a commutative diagram of rings

$$\begin{tikzcd} B_2/I_2 & B_2 \arrow[l] & & A_2 \arrow[ll, "a_2'"'] \\ B_1/I_1 \arrow[u] & B_1 \arrow[l] \arrow[u, "h'"] & A_1 \arrow[l, "a_1'"'] \arrow[ur, "f"'] & \\ & & R_1 \arrow[u] \arrow[r] & R_2 \arrow[uu] \end{tikzcd}$$

with $I_1^2 = 0$ and $I_2^2 = 0$ and moreover with the property that $A_2 \otimes_{A_1} \Omega_{A_1/R_1} \to \Omega_{A_2/R_2}$ is an isomorphism. Then $\theta_1 : B_1/I_1 \otimes_{A_1} \Omega_{A_1/R_1} \to I_1$ is B_1-linear. This gives an R_1-derivation $D_1 = \theta_1 \circ \mathrm{d}_{A_1/R_1} : A_1 \to I_1$. In a similar way we see that $\theta_2 : B_2/I_2 \otimes_{A_2} \Omega_{A_2/R_2} \to I_2$ gives rise to a R_2-derivation $D_2 = \theta_2 \circ \mathrm{d}_{A_2/R_2} : A_2 \to I_2$. The construction of θ_2 implies the following compatibility between θ_1 and θ_2: for every $x \in A_1$ we have

$$h'(D_1(x)) = D_2(f(x))$$

as elements of I_2. Now by the construction of the action in Lemma 33.7.2 and Remark 33.7.5 we know that $\theta_1 \cdot a_1'$ corresponds to the ring map $a_1' + D_1 : A_1 \to B_1$ and $\theta_2 \cdot a_2'$ corresponds to the ring map $a_2' + D_2 : A_2 \to B_2$. By the displayed equality above we obtain that $h' \circ (a_1' + D_1) = (a_2' + D_2) \circ f$ as desired. □

Remark 33.7.9. Lemma 33.7.8 can be improved in the following way. Suppose that we have a commutative diagram of schemes as in Lemma 33.7.8 but we do not assume that $X_2 \to X_1$ and $S_2 \to S_1$ are étale. Next, suppose we have $\theta_1 : a_1^*\Omega_{X_1/S_1} \to \mathcal{I}_1$ and $\theta_2 : a_2^*\Omega_{X_2/S_2} \to \mathcal{I}_2$ such that for a local section t of $\mathcal{O}_{X_1}$ we have $(h')^*\theta_1(a_1^*(\mathrm{d}_{X_1/S_1}(t))) = \theta_2(a_2^*(\mathrm{d}_{X_2/S_2}(f^*t)))$, i.e., such that

$$\begin{tikzcd} f_*\mathcal{O}_{X_2} \arrow[r, "f_*D_2"'] & f_*a_{2,*}\mathcal{C}_{T_2/T_2'} \\ \mathcal{O}_{X_1} \arrow[u, "f^\sharp"] \arrow[r, "D_1"] & a_{1,*}\mathcal{C}_{T_1/T_1'} \arrow[u, "\text{induced by } (h')^\sharp"'] \end{tikzcd}$$

is commutative where D_i corresponds to θ_i as in Equation (33.7.1.1). Then we have the conclusion of Lemma 33.7.8. The importance of the condition that both $X_2 \to X_1$ and $S_2 \to S_1$ are étale is that it allows us to construct a θ_2 from θ_1.

33.8. Infinitesimal deformations of schemes

The following simple lemma is often a convenient tool to check whether an infinitesimal deformation of a map is flat.

Lemma 33.8.1. *Let $(f, f') : (X \subset X') \to (S \subset S')$ be a morphism of first order thickenings. Assume that f is flat. Then the following are equivalent*

(1) *f' is flat and $X = S \times_{S'} X'$, and*
(2) *the canonical map $f^*\mathcal{C}_{S/S'} \to \mathcal{C}_{X/X'}$ is an isomorphism.*

Proof. As the problem is local on X' we may assume that X, X', S, S' are affine schemes. Say $S' = Spec(A')$, $X' = Spec(B')$, $S = Spec(A)$, $X = Spec(B)$ with $A = A'/I$ and $B = B'/J$ for some square zero ideals. Then we obtain the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r] & B' \arrow[r] & B \arrow[r] & 0 \\ 0 \arrow[r] & I \arrow[r] \arrow[u] & A' \arrow[r] \arrow[u] & A \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

with exact rows. The canonical map of the lemma is the map

$$I \otimes_A B = I \otimes_{A'} B' \longrightarrow J.$$

The assumption that f is flat signifies that $A \to B$ is flat.

Assume (1). Then $A' \to B'$ is flat and $J = IB'$. Flatness implies $\mathrm{Tor}_1^{A'}(B', A) = 0$ (see Algebra, Lemma 7.69.7). This means $I \otimes_{A'} B' \to B'$ is injective (see Algebra, Remark 7.69.8). Hence we see that $I \otimes_A B \to J$ is an isomorphism.

Assume (2). Then it follows that $J = IB'$, so that $X = S \times_{S'} X'$. Moreover, we get $\mathrm{Tor}_1^{A'}(B', A'/I) = 0$ by reversing the implications in the previous paragraph. Hence B' is flat over A' by Algebra, Lemma 7.91.8. ☐

The following lemma is the ``nilpotent'' version of the ``critère de platitude par fibres'', see Section 33.12.

Lemma 33.8.2. *Consider a commutative diagram*

$$\begin{tikzcd} (X \subset X') \arrow[rr, "{(f,f')}"'] \arrow[dr] & & (Y \subset Y') \arrow[dl] \\ & (S \subset S') & \end{tikzcd}$$

of thickenings. Assume

(1) *X' is flat over S',*
(2) *f is flat,*
(3) *$S \subset S'$ is a finite order thickening, and*
(4) *$X = S \times_{S'} X'$ and $Y = S \times_{S'} Y'$.*

Then f' is flat and Y' is flat over S' at all points in the image of f'.

Proof. Immediate consequence of Algebra, Lemma 7.93.8. ☐

Many properties of morphisms of schemes are preserved under flat deformations.

Lemma 33.8.3. *Consider a commutative diagram*

$$\begin{tikzcd} (X \subset X') \arrow[rr, "{(f,f')}"'] \arrow[dr] & & (Y \subset Y') \arrow[dl] \\ & (S \subset S') & \end{tikzcd}$$

of thickenings. Assume $S \subset S'$ is a finite order thickening, X' and Y' flat over S' and $X = S \times_{S'} X'$ and $Y = S \times_{S'} Y'$. Then

(1) *f is flat if and only if f' is flat,*
(2) *f is an isomorphism if and only if f' is an isomorphism,*

(3) *f is an open immersion if and only if f' is an open immersion,*
(4) *f is quasi-compact if and only if f' is quasi-compact,*
(5) *f is universally closed if and only if f' is universally closed,*
(6) *f is (quasi-)separated if and only if f' is (quasi-)separated,*
(7) *f is a monomorphism if and only if f' is a monomorphism,*
(8) *f is surjective if and only if f' is surjective,*
(9) *f is universally injective if and only if f' is universally injective,*
(10) *f is affine if and only if f' is affine,*
(11) *f is locally of finite type if and only if f' is locally of finite type,*
(12) *f is quasi-finite if and only if f' is quasi-finite,*
(13) *f is locally of finite presentation if and only if f' is locally of finite presentation,*
(14) *f is locally of finite type of relative dimension d if and only if f' is locally of finite type of relative dimension d,*
(15) *f is universally open if and only if f' is universally open,*
(16) *f is syntomic if and only if f' is syntomic,*
(17) *f is smooth if and only if f' is smooth,*
(18) *f is unramified if and only if f' is unramified,*
(19) *f is étale if and only if f' is étale,*
(20) *f is proper if and only if f' is proper,*
(21) *f is integral if and only if f' is integral,*
(22) *f is finite if and only if f' is finite,*
(23) *f is finite locally free (of rank d) if and only if f' is finite locally free (of rank d), and*
(24) *add more here.*

Proof. The assumptions on X and Y mean that f is the base change of f' by $X \to X'$. The properties $\mathcal{P}$ listed in (1) -- (23) above are all stable under base change, hence if f' has property $\mathcal{P}$, then so does f. See Schemes, Lemmas 21.18.2, 21.19.3, 21.21.13, and 21.23.5 and Morphisms, Lemmas 24.9.4, 24.10.4, 24.11.8, 24.14.4, 24.19.13, 24.20.4, 24.28.2, 24.30.4, 24.33.5, 24.34.5, 24.35.4, 24.40.5, 24.42.6, and 24.44.4.

The interesting direction in each case is therefore to assume that f has the property and deduce that f' has it too. By induction on the order of the thickening we may assume that $S \subset S'$ is a first order thickening, see discussion immediately following Definition 33.2.1. We make a couple of general remarks which we will use without further mention in the arguments below. (I) Let $W' \subset S'$ be an affine open and let $U' \subset X'$ and $V' \subset S'$ be affine opens lying over W' with $f'(U') \subset V'$. Let $W' = Spec(R')$ and denote $I \subset R'$ be the ideal defining the closed subscheme $W' \cap S$. Say $U' = Spec(B')$ and $V' = Spec(A')$. Then we get a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & IB' \arrow[r] & B' \arrow[r] & B \arrow[r] & 0 \\ 0 \arrow[r] & IA' \arrow[r] \arrow[u] & A' \arrow[r] \arrow[u] & A \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

with exact rows. Moreover, $IA' \cong I \otimes_R A$ and $IB' \cong I \otimes_R B$, see proof of Lemma 33.8.1. (II) The morphisms $X \to X'$ and $Y \to Y'$ are universal homeomorphisms. Hence the topology of the maps f and f' (after any base change) is identical. (III) If f is flat, then f' is flat, see Lemma 33.8.2.

Ad (1). This is general remark (III).

Ad (2). Assume f is an isomorphism. Choose an affine open $V' \subset Y'$ and set $U' = (f')^{-1}(V')$. Then $V = Y \cap V'$ is affine which implies that $V \cong f^{-1}(V) = U = Y \times_{Y'} U'$ is affine. By Lemma 33.2.3 we see that U' is affine. Hence $IB' \cong I \otimes_R B \cong I \otimes_R A \cong IA'$ and $A \cong B$. By the exactness of the rows in the diagram above we see that $A' \cong B'$, i.e., $U' \cong V'$. Thus f' is an isomorphism.

Ad (3). Assume f is an open immersion. Then f is an isomorphism of X with an open subscheme $V \subset Y$. Let $V' \subset Y'$ be the open subscheme whose underlying topological space is V. Then f' is a map from X' to V' which is an isomorphism by (2). Hence f' is an open immersion.

Ad (4). Immediate from remark (II).

Ad (5). Immediate from remark (II).

Ad (6). Note that $X \times_Y X = Y \times_{Y'} (X' \times_{Y'} X')$ so that $X' \times_{Y'} X'$ is a thickening of $X \times_Y X$. Hence the topology of the maps $\Delta_{X/Y}$ and $\Delta_{X'/Y'}$ matches and we win.

Ad (7). Assume f is a monomorphism. Consider the diagonal morphism $\Delta_{X'/Y'} : X' \to X' \times_{Y'} X'$. Because f is a monomorphism and because $X' \times_{Y'} X'$ is a thickening of $X \times_Y X$ we see that $\Delta_{X'/Y'}$ is surjective. Hence Lemma 33.8.2 implies that $X' \times_{Y'} X'$ is flat over S'. Then (2) shows that $\Delta_{X'/Y'}$ is an isomorphism.

Ad (8). This is clear.

Ad (9). Immediate from remark (II).

Ad (10). Assume f is affine. Choose an affine open $V' \subset Y'$ and set $U' = (f')^{-1}(V')$. Then $V = Y \cap V'$ is affine which implies that $U = Y \times_{Y'} U'$ is affine. By Lemma 33.2.3 we see that U' is affine. Hence f' is affine.

Ad (11). Via remark (I) comes down to proving $A' \to B'$ is of finite type if $A \to B$ is of finite type. Suppose that $x_1, \ldots, x_n \in B'$ are elements whose images in B generate B as an A-algebra. Then $A'[x_1, \ldots, x_n] \to B$ is surjective as both $A'[x_1, \ldots, x_n] \to B$ is surjective and $I \otimes_R A[x_1, \ldots, x_n] \to I \otimes_R B$ is surjective.

Ad (12). Follows from (11) and that quasi-finiteness of a morphism of finite type can be checked on fibres, see Morphisms, Lemma 24.19.6.

Ad (13). Via remark (I) comes down to proving $A' \to B'$ is of finite presentation if $A \to B$ is of finite presentation. We may assume that $B' = A'[x_1, \ldots, x_n]/K'$ for some ideal K' by (11). We get a short exact sequence

$$0 \to K' \to A'[x_1, \ldots, x_n] \to B' \to 0$$

As B' is flat over R' we see that $K' \otimes_{R'} R$ is the kernel of the surjection $A[x_1, \ldots, x_n] \to B$. By assumption on $A \to B$ there exist finitely many $f'_1, \ldots, f'_m \in K'$ whose images in $A[x_1, \ldots, x_n]$ generate this kernel. Since I is nilpotent we see that $f'_1, \ldots, f'_m$ generate K' by Nakayama's lemma, see Algebra, Lemma 7.14.5.

Ad (14). Follows from (11) and general remark (II).

Ad (15). Immediate from general remark (II).

Ad (16). Assume f is syntomic. By (13) f' is locally of finite presentation, by general remark (III) f' is flat and the fibres of f' are the fibres of f. Hence f' is syntomic by Morphisms, Lemma 24.30.11.

Ad (17). Assume f is smooth. By (13) f' is locally of finite presentation, by general remark (III) f' is flat, and the fibres of f' are the fibres of f. Hence f' is smooth by Morphisms, Lemma 24.33.3.

Ad (18). Assume f unramified. By (11) f' is locally of finite type and the fibres of f' are the fibres of f. Hence f' is unramified by Morphisms, Lemma 24.34.12.

Ad (19). Assume f étale. By (13) f' is locally of finite presentation, by general remark (III) f' is flat, and the fibres of f' are the fibres of f. Hence f' is étale by Morphisms, Lemma 24.35.8.

Ad 20). This follows from a combination of (6), (11), (4), and (5).

Ad (21). Combine (5) and (10) with Morphisms, Lemma 24.42.7.

Ad (22). Combine (21), and (11) with Morphisms, Lemma 24.42.4.

Ad (23). Assume f finite locally free. By (22) we see that f' is finite, by general remark (III) f' is flat, and by (13) f' is locally of finite presentation. Hence f' is finite locally free by Morphisms, Lemma 24.44.2. □

33.9. Formally smooth morphisms

Michael Artin's position on differential criteria of smoothness (e.g., Morphisms, Lemma 24.33.14) is that they are basically useless (in practice). In this section we introduce the notion of a formally smooth morphism $X \to S$. Such a morphism is characterized by the property that T-valued points of X lift to inifinitesimal thickenings of T provided T is affine. The main result is that a morphism which is formally smooth and locally of finite presentation is smooth, see Lemma 33.9.7. It turns out that this criterion is often easier to use than the differential criteria mentioned above.

Recall that a ring map $R \to A$ is called *formally smooth* (see Algebra, Definition 7.127.1) if for every commutative solid diagram

$$\begin{tikzcd} A \arrow[r] \arrow[dr, dashed] & B/I \\ R \arrow[u] \arrow[r] & B \arrow[u] \end{tikzcd}$$

where $I \subset B$ is an ideal of square zero, a dotted arrow exists which makes the diagram commute. This motivates the following analogue for morphisms of schemes.

Definition 33.9.1. Let $f : X \to S$ be a morphism of schemes. We say f is *formally smooth* if given any solid commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & T \arrow[l] \arrow[d, "i"] \\ S & T' \arrow[l] \arrow[ul, dashed] \end{tikzcd}$$

where $T \subset T'$ is a first order thickening of affine schemes over S there exists a dotted arrow making the diagram commute.

In the cases of formally unramified and formally étale morphisms the condition that T' be affine could be dropped, see Lemmas 33.4.2 and 33.6.2. This is no longer true in the case of formally smooth morphisms. In fact, a slightly more natural condition would be that we should be able to fill in the dotted arrow Zariski locally on T'. In fact, analyzing the proof of Lemma 33.9.7 shows that this would be equivalent to the definition as it currently stands.

Lemma 33.9.2. *A composition of formally smooth morphisms is formally smooth.*

Proof. Omitted. □

Lemma 33.9.3. *A base change of a formally smooth morphism is formally smooth.*

Proof. Omitted, but see Algebra, Lemma 7.127.2 for the algebraic version. □

Lemma 33.9.4. *Let $f : X \to S$ be a morphism of schemes. Then f is formally étale if and only if f is formally smooth and formally unramified.*

Proof. Omitted. □

Lemma 33.9.5. *Let $f : X \to S$ be a morphism of schemes. Let $U \subset X$ and $V \subset S$ be open subschemes such that $f(U) \subset V$. If f is formally smooth, so is $f|_U : U \to V$.*

Proof. Consider a solid diagram

$$\begin{tikzcd} U \arrow[d, "f|_U"'] & T \arrow[l, "a"'] \arrow[d, "i"] \\ V & T' \arrow[l] \arrow[lu, dashed] \end{tikzcd}$$

as in Definition 33.9.1. If f is formally smooth, then there exists an S-morphism $a' : T' \to X$ such that $a'|_T = a$. Since the underlying sets of T and T' are the same we see that a' is a morphism into U (see Schemes, Section 21.3). And it clearly is a V-morphism as well. Hence the dotted arrow above as desired. □

Lemma 33.9.6. *Let $f : X \to S$ be a morphism of schemes. Assume X and S are affine. Then f is formally smooth if and only if $\mathcal{O}_S(S) \to \mathcal{O}_X(X)$ is a formally smooth ring map.*

Proof. This is immediate from the definitions (Definition 33.9.1 and Algebra, Definition 7.127.1) by the equivalence of categories of rings and affine schemes, see Schemes, Lemma 21.6.5. □

The following lemma is the main result of this section. It is a victory of the functorial point of view in that it implies (combined with Limits, Proposition 27.4.1) that we can recognize whether a morphism $f : X \to S$ is smooth in terms of ``simple" properties of the functor $h_X : Sch/S \to Sets$.

Lemma 33.9.7. *(Infinitesimal lifting criterion) Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *The morphism f is smooth, and*
(2) *the morphism f is locally of finite presentation and formally smooth.*

Proof. Assume $f : X \to S$ is locally of finite presentation and formally smooth. Consider a pair of affine opens $Spec(A) = U \subset X$ and $Spec(R) = V \subset S$ such that $f(U) \subset V$. By Lemma 33.9.5 we see that $U \to V$ is formally smooth. By Lemma 33.9.6 we see that $R \to A$ is formally smooth. By Morphisms, Lemma 24.20.2 we see that $R \to A$ is of finite presentation. By Algebra, Proposition 7.127.13 we see that $R \to A$ is smooth. Hence by the definition of a smooth morphism we see that $X \to S$ is smooth.

Conversely, assume that $f : X \to S$ is smooth. Consider a solid commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & T \arrow[l, "a"'] \arrow[d, "i"] \\ S & T' \arrow[l] \arrow[ul, dashed] \end{tikzcd}$$

as in Definition 33.9.1. We will show the dotted arrow exists thereby proving that f is formally smooth.

Let $\mathcal{F}$ be the sheaf of sets on T' of Lemma 33.7.3, see also Remark 33.7.5. Let

$$\mathcal{H} = \mathcal{H}om_{\mathcal{O}_T}(a^*\Omega_{X/S}, \mathcal{C}_{T/T'})$$

be the sheaf of $\mathcal{O}_T$-modules on T introduced in Lemma 33.7.4. Our goal is simply to show that $\mathcal{F}(T) \neq \emptyset$. In other words we are trying to show that $\mathcal{F}$ is a trivial $\mathcal{H}$-torsor on T (see Cohomology, Section 18.5). There are two steps: (I) To show that $\mathcal{F}$ is a torsor we have to show that $\mathcal{F}_t \neq \emptyset$ for all $t \in T$ (see Cohomology, Definition 18.5.1). (II) To show that $\mathcal{F}$ is the trivial torsor it suffices to show that $H^1(T, \mathcal{H}) = 0$ (see Cohomology, Lemma 18.5.3 -- we may use either cohomology of $\mathcal{H}$ as an abelian sheaf or as an $\mathcal{O}_T$-module, see Cohomology, Lemma 18.12.3).

First we prove (I). To see this, for every $t \in T$ we can choose an affine open $U \subset T$ neighbourhood of t such that $a(U)$ is contained in an affine open $Spec(A) = W \subset X$ which maps to an affine open $Spec(R) = V \subset S$. By Morphisms, Lemma 24.33.2 the ring map $R \to A$ is smooth. Hence by Algebra, Proposition 7.127.13 the ring map $R \to A$ is formally smooth. Lemma 33.9.6 in turn implies that $W \to V$ is formally smooth. Hence we can lift $a|_U : U \to W$ to a V-morphism $a' : U' \to W \subset X$ showing that $\mathcal{F}(U) \neq \emptyset$.

Finally we prove (II). By Morphisms, Lemma 24.32.15 we see that $\Omega_{X/S}$ is of finite presentation (it is even finite locally free by Morphisms, Lemma 24.33.12). Hence $a^*\Omega_{X/S}$ is of finite presentation (see Modules, Lemma 15.11.4). Hence the sheaf $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_T}(a^*\Omega_{X/S}, \mathcal{C}_{T/T'})$ is quasi-coherent by the discussion in Schemes, Section 21.24. Thus by Coherent, Lemma 25.2.2 we have $H^1(T, \mathcal{H}) = 0$ as desired. □

Locally projective quasi-coherent modules are defined in Properties, Section 23.19.

Lemma 33.9.8. *Let $f : X \to Y$ be a formally smooth morphism of schemes. Then $\Omega_{X/Y}$ is locally projective on X.*

Proof. Choose $U \subset X$ and $V \subset Y$ affine open such that $f(U) \subset V$. By Lemma 33.9.5 $f|_U : U \to V$ is formally smooth. Hence $\Gamma(V, \mathcal{O}_V) \to \Gamma(U, \mathcal{O}_U)$ is a formally smooth ring map, see Lemma 33.9.6. Hence by Algebra, Lemma 7.127.7 the $\Gamma(U, \mathcal{O}_U)$-module $\Omega_{\Gamma(U,\mathcal{O}_U)/\Gamma(V,\mathcal{O}_V)}$ is projective. Hence $\Omega_{U/V}$ is locally projective, see Properties, Section 23.19. □

Lemma 33.9.9. *Let $f : X \to Y$, $g : Y \to S$ be morphisms of schemes. Assume f is formally smooth. Then*

$$0 \to f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0$$

(see Morphisms, Lemma 24.32.11) is short exact.

Proof. The algebraic version of this lemma is the following: Given ring maps $A \to B \to C$ with $B \to C$ formally smooth, then the sequence

$$0 \to C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

of Algebra, Lemma 7.122.7 is exact. This is Algebra, Lemma 7.127.9. □

Lemma 33.9.10. *Let $h : Z \to X$ be a formally unramified morphism of schemes over S. Assume that Z is formally smooth over S. Then the canonical exact sequence*

$$0 \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/S} \to \Omega_{Z/S} \to 0$$

of Lemma 33.5.10 is short exact.

Proof. Let $Z \to Z'$ be the universal first order thickening of Z over X. From the proof of Lemma 33.5.10 we see that our sequence is identified with the sequence

$$\mathcal{C}_{Z/Z'} \to \Omega_{Z'/S} \otimes \mathcal{O}_Z \to \Omega_{Z/S} \to 0.$$

Since $Z \to S$ is formally smooth we can locally on Z' find a left inverse $Z' \to Z$ over S to the inclusion map $Z \to Z'$. Thus the sequence is locally split, see Morphisms, Lemma 24.32.18. □

Lemma 33.9.11. *Let*

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[dr, "j"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

be a commutative diagram of schemes where i and j are formally unramified and f is formally smooth. Then the canonical exact sequence

$$0 \to \mathcal{C}_{Z/Y} \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/Y} \to 0$$

of Lemma 33.5.11 is exact and locally split.

Proof. Denote $Z \to Z'$ the universal first order thickening of Z over X. Denote $Z \to Z''$ the universal first order thickening of Z over Y. By Lemma 33.5.10 here is a canonical morpism $Z' \to Z''$ so that we have a commutative diagram

$$\begin{tikzcd} Z \arrow[r, "i'"] \arrow[dr, "j'"'] & Z' \arrow[r, "a"] \arrow[d, "k"] & X \arrow[d, "f"] \\ & Z'' \arrow[r, "b"] & Y \end{tikzcd}$$

In the proof of Lemma 33.5.11 we identified the sequence above with the sequence

$$\mathcal{C}_{Z/Z''} \to \mathcal{C}_{Z/Z'} \to (i')^*\Omega_{Z'/Z''} \to 0$$

Let $U'' \subset Z''$ be an affine open. Denote $U \subset Z$ and $U' \subset Z'$ the corresponding affine open subschemes. As f is formally smooth there exists a morphism $h : U'' \to X$ which agrees with i on U and such that $f \circ h$ equals $b|_{U''}$. Since Z' is the universal first order thickening we obtain a unique morphism $g : U'' \to Z'$ such that $g = a \circ h$. The universal property of Z'' implies that $k \circ g$ is the inclusion map $U'' \to Z''$. Hence g is a left inverse to k. Picture

$$\begin{tikzcd} U \arrow[r] \arrow[d] & Z' \arrow[d, "k"] \\ U'' \arrow[r] \arrow[ur, "g"] & Z'' \end{tikzcd}$$

Thus g induces a map $\mathcal{C}_{Z/Z'}|_U \to \mathcal{C}_{Z/Z''}|_U$ which is a left inverse to the map $\mathcal{C}_{Z/Z''} \to \mathcal{C}_{Z/Z'}$ over U. □

33.10. Smoothness over a Noetherian base

It turns out that if the base is Noetherian then we can get away with less in the formulation of formal smoothness. In some sense the following lemmas are the beggining of deformation theory.

Lemma 33.10.1. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. Assume that S is locally Noetherian and f locally of finite type. The following are equivalent:*

1. *f is smooth at x,*
2. *for every solid commutative diagram*

$$\begin{tikzcd} X \arrow[d, "f"'] & Spec(B) \arrow[l, "\alpha"'] \arrow[d, "i"] \\ S & Spec(B') \arrow[l, "\beta"] \arrow[ul, dashed] \end{tikzcd}$$

   where $B' \to B$ is a surjection of local rings with $Ker(B' \to B)$ of square zero, and α mapping the closed point of $Spec(B)$ to x there exists a dotted arrow making the diagram commute,
3. *same as in (2) but with $B' \to B$ ranging over small extensions (see Algebra, Definition 7.130.1), and*
4. *same as in (2) but with $B' \to B$ ranging over small extensions such that α induces an isomorphism $\kappa(x) \to \kappa(\mathfrak{m})$ where $\mathfrak{m} \subset B$ is the maximal ideal.*

Proof. Choose an affine neighbourhood $V \subset S$ of $f(x)$ and choose an affine neighbourhood $U \subset X$ of x such that $f(U) \subset V$. For any ``test" diagram as in (2) the morphism α will map $Spec(B)$ into U and the morphism β will map $Spec(B')$ into V (see Schemes, Section 21.13). Hence the lemma reduces to the morphism $f|_U : U \to V$ of affines. (Indeed, V is Noetherian and $f|_U$ is of finite type, see Properties, Lemma 23.5.2 and Morphisms, Lemma 24.14.2.) In this affine case the lemma is identical to Algebra, Lemma 7.130.2. □

Sometimes it is useful to know that one only needs to check the lifting criterion for small extensions ``centered" at points of finite type (see Morphisms, Section 24.15).

Lemma 33.10.2. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. Assume that S is locally Noetherian and f locally of finite type. The following are equivalent:*

1. *f is smooth,*
2. *for every solid commutative diagram*

$$\begin{tikzcd} X \arrow[d, "f"'] & Spec(B) \arrow[l, "\alpha"'] \arrow[d, "i"] \\ S & Spec(B') \arrow[l, "\beta"] \arrow[ul, dashed] \end{tikzcd}$$

   where $B' \to B$ is a small extension of Artinian local rings and β of finite type (!) there exists a dotted arrow making the diagram commute.

Proof. If f is smooth, then the infinitesimal lifting criterion (Lemma 33.9.7) says f is formally smooth and (2) holds.

Assume (2). The set of points $x \in X$ where f is not smooth forms a closed subset T of X. By the discussion in Morphisms, Section 24.15, if $T \neq \emptyset$ there exists a point $x \in T \subset X$ such that the morphism

$$Spec(\kappa(x)) \to X \to S$$

is of finite type (namely, pick any point x of T which is closed in an affine open of X). By Morphisms, Lemma 24.15.2 given any local Artinian ring B' with residue field $\kappa(x)$ then any morphism $\beta : Spec(B') \to S$ is of finite type. Thus we see that all the diagrams used in Lemma 33.10.1 (4) correspond to diagrams as in the current lemma (2). Whence $X \to S$ is smooth a x a contradiction. ☐

33.11. Openness of the flat locus

This result takes some work to prove, and (perhaps) deserves its own section. Here it is.

Theorem 33.11.1. *Let S be a scheme. Let $f : X \to S$ be a morphism which is locally of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module which is locally of finite presentation. Then*

$$U = \{x \in X \mid \mathcal{F} \text{ is flat over } S \text{ at } x\}$$

is open in X.

Proof. We may test for openness locally on X hence we may assume that f is a morphism of affine schemes. In this case the theorem is exactly Algebra, Theorem 7.120.4. ☐

Lemma 33.11.2. *Let S be a scheme. Let*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

be a cartesian diagram of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x' \in X'$ with images $x = g'(x')$ and $s' = g'(x')$.

(1) *If $\mathcal{F}$ is flat over S at x, then $(g')^*\mathcal{F}$ is flat over S' at x'.*
(2) *If g is flat at s' and $(g')^*\mathcal{F}$ is flat over S' at x', then $\mathcal{F}$ is flat over S at x.*

In particular, if g is flat, f is locally of finite presentation, and $\mathcal{F}$ is locally of finite presentation, then formation of the open subset of Theorem 33.11.1 commutes with base change.

Proof. Consider the commutative diagram of local rings

$$\begin{tikzcd} \mathcal{O}_{X',x'} & \mathcal{O}_{X,x} \arrow[l] \\ \mathcal{O}_{S',s'} \arrow[u] & \mathcal{O}_{S,s} \arrow[l] \arrow[u] \end{tikzcd}$$

Note that $\mathcal{O}_{X',x'}$ is a localization of $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{S',s'}$, and that $((g')^*\mathcal{F})_{x'}$ is equal to $\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X',x'}$. Hence the lemma follows from Algebra, Lemma 7.92.1. ☐

33.12. Critère de platitude par fibres

Consider a commutative diagram of schemes (left hand diagram)

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd} \qquad \begin{tikzcd} X_s \arrow[rr, "f_s"] \arrow[dr] & & Y_s \arrow[dl] \\ & Spec(\kappa(s)) & \end{tikzcd}$$

and a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$. Given a point $x \in X$ lying over $s \in S$ with image $y = f(x)$ we consider the question: Is $\mathcal{F}$ flat over Y at x? If $\mathcal{F}$ is flat over S at x, then the theorem states this question is intimately related to the question of whether the restriction of $\mathcal{F}$ to the fibre

$$\mathcal{F}_s = (X_s \to X)^*\mathcal{F}$$

is flat over Y_s at x. Below you will find a ``Noetherian" version, a ``finitely presented" version, and earlier we treated a ``nilpotent" version, see Lemma 33.8.2.

Theorem 33.12.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x \in X$. Set $y = f(x)$ and $s \in S$ the image of x in S. Assume S, X, Y locally Noetherian, $\mathcal{F}$ coherent, and $\mathcal{F}_x \neq 0$. Then the following are equivalent:*

(1) *$\mathcal{F}$ is flat over S at x, and $\mathcal{F}_s$ is flat over Y_s at x, and*
(2) *Y is flat over S at y and $\mathcal{F}$ is flat over Y at x.*

Proof. Consider the ring maps

$$\mathcal{O}_{S,s} \longrightarrow \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,x}$$

and the module $\mathcal{F}_x$. The stalk of $\mathcal{F}_s$ at x is the module $\mathcal{F}_x/\mathfrak{m}_s\mathcal{F}_x$ and the local ring of Y_s at y is $\mathcal{O}_{Y,y}/\mathfrak{m}_s\mathcal{O}_{Y,y}$. Thus the implication (1) $\Rightarrow$ (2) is Algebra, Lemma 7.91.14. If (2) holds, then the first ring map is faithfully flat and $\mathcal{F}_x$ is flat over $\mathcal{O}_{Y,y}$ so by Algebra, Lemma 7.35.3 we see that $\mathcal{F}_x$ is flat over $\mathcal{O}_{S,s}$. Moreover, $\mathcal{F}_x/\mathfrak{m}_s\mathcal{F}_x$ is the base change of the flat module $\mathcal{F}_x$ by $\mathcal{O}_{Y,y} \to \mathcal{O}_{Y,y}/\mathfrak{m}_s\mathcal{O}_{Y,y}$, hence flat by Algebra, Lemma 7.35.6. $\square$

Here is the non-Noetherian version.

Theorem 33.12.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume*

(1) *X is locally of finite presentation over S,*
(2) *$\mathcal{F}$ an $\mathcal{O}_X$-module of finite presentation, and*
(3) *Y is locally of finite type over S.*

Let $x \in X$. Set $y = f(x)$ and let $s \in S$ be the image of x in S. If $\mathcal{F}_x \neq 0$, then the following are equivalent:

(1) *$\mathcal{F}$ is flat over S at x, and $\mathcal{F}_s$ is flat over Y_s at x, and*
(2) *Y is flat over S at y and $\mathcal{F}$ is flat over Y at x.*

Moreover, the set of points x where (1) and (2) hold is open in Supp($\mathcal{F}$).

Proof. Consider the ring maps

$$\mathcal{O}_{S,s} \longrightarrow \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,x}$$

and the module $\mathcal{F}_x$. The stalk of $\mathcal{F}_s$ at x is the module $\mathcal{F}_x/\mathfrak{m}_s\mathcal{F}_x$ and the local ring of Y_s at y is $\mathcal{O}_{Y,y}/\mathfrak{m}_s\mathcal{O}_{Y,y}$. Thus the implication (1) $\Rightarrow$ (2) is Algebra, Lemma 7.119.9. If (2) holds, then the first ring map is faithfully flat and $\mathcal{F}_x$ is flat over $\mathcal{O}_{Y,y}$ so by Algebra, Lemma 7.35.3 we see that $\mathcal{F}_x$ is flat over $\mathcal{O}_{S,s}$. Moreover, $\mathcal{F}_x/\mathfrak{m}_s\mathcal{F}_x$ is the base change of the flat module $\mathcal{F}_x$ by $\mathcal{O}_{Y,y} \to \mathcal{O}_{Y,y}/\mathfrak{m}_s\mathcal{O}_{Y,y}$, hence flat by Algebra, Lemma 7.35.6.

By Morphisms, Lemma 24.20.11 the morphism f is locally of finite presentation. Consider the set

$$U = \{x \in X \mid \mathcal{F} \text{ flat at } x \text{ over both } Y \text{ and } S\}. \tag{33.12.2.1}$$

This set is open in X by Theorem 33.11.1. Note that if $x \in U$, then $\mathcal{F}_s$ is flat at x over Y_s as a base change of a flat module under the morphism $Y_s \to Y$, see Morphisms, Lemma 24.24.6. Hence at every point of $U \cap \text{Supp}(\mathcal{F})$ condition (1) is satisfied. On the other hand, it is clear that if $x \in \text{Supp}(\mathcal{F})$ satisfies (1) and (2), then $x \in U$. Thus the open set we are looking for is $U \cap \text{Supp}(\mathcal{F})$. $\square$

These theorems are often used in the following simplified forms. We give only the global statements -- of course there are also pointwise versions.

Lemma 33.12.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes over S. Assume*

1. *S, X, Y are locally Noetherian,*
2. *X is flat over S,*
3. *for every $s \in S$ the morphism $f_s : X_s \to Y_s$ is flat.*

Then f is flat. If f is also surjective, then Y is flat over S.

Proof. This is a special case of Theorem 33.12.1. $\square$

Lemma 33.12.4. *Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes over S. Assume*

1. *X is locally of finite presentation over S,*
2. *X is flat over S,*
3. *for every $s \in S$ the morphism $f_s : X_s \to Y_s$ is flat, and*
4. *Y is locally of finite type over S.*

Then f is flat. If f is also surjective, then Y is flat over S.

Proof. This is a special case of Theorem 33.12.2. $\square$

Lemma 33.12.5. *Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume*

1. *X is locally of finite presentation over S,*
2. *$\mathcal{F}$ an $\mathcal{O}_X$-module of finite presentation,*
3. *$\mathcal{F}$ is flat over S, and*
4. *Y is locally of finite type over S.*

Then the set

$$U = \{x \in X \mid \mathcal{F} \textit{ flat at } x \textit{ over } Y\}.$$

is open in X and its formation commutes with arbitrary base change: If $S' \to S$ is a morphism of schemes, and U' is the set of points of $X' = X \times_S S'$ where $\mathcal{F}' = \mathcal{F} \times_S S'$ is flat over $Y' = Y \times_S S'$, then $U' = U \times_S S'$.

Proof. By Morphisms, Lemma 24.20.11 the morphism f is locally of finite presentation. Hence U is open by Theorem 33.11.1. Because we have assumed that $\mathcal{F}$ is flat over S we see that Theorem 33.12.2 implies

$$U = \{x \in X \mid \mathcal{F}_s \text{ flat at } x \text{ over } Y_s\}.$$

where s always denotes the image of x in S. (This description also works trivially when $\mathcal{F}_x = 0$.) Moreover, the assumptions of the lemma remain in force for the morphism $f' : X' \to Y'$ and the sheaf $\mathcal{F}'$. Hence U' has a similar description. In other words, it suffices to prove that given $s' \in S'$ mapping to $s \in S$ we have

$$\{x' \in X'_{s'} \mid \mathcal{F}'_{s'} \text{ flat at } x' \text{ over } Y'_{s'}\}$$

is the inverse image of the corresponding locus in X_s. This is true by Lemma 33.11.2 because in the cartesian diagram

$$\begin{tikzcd} X'_{s'} \arrow[r] \arrow[d] & X_s \arrow[d] \\ Y'_{s'} \arrow[r] & Y_s \end{tikzcd}$$

the horizontal morphisms are flat as they are base changes by the flat morphism $Spec(\kappa(s')) \to Spec(\kappa(s))$. □

Lemma 33.12.6. *Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes over S. Assume*

(1) *X is locally of finite presentation over S,*
(2) *X is flat over S, and*
(3) *Y is locally of finite type over S.*

Then the set

$$U = \{x \in X \mid X \text{ flat at } x \text{ over } Y\}.$$

is open in X and its formation commutes with arbitrary base change.

Proof. This is a special case of Lemma 33.12.5. □

33.13. Normal morphisms

In the article [**DM69a**] of Deligne and Mumford the notion of a normal morphism is mentioned. This is just one in a series of types[1] of morphisms that can all be defined similarly. Over time we will add these in their own sections as needed.

Definition 33.13.1. Let $f : X \to Y$ be a morphism of schemes. Assume that all the fibres X_y are locally Noetherian schemes.

(1) Let $x \in X$, and $y = f(x)$. We say that f is *normal at x* if f is flat at x, and the scheme X_y is geometrically normal at x over $\kappa(y)$ (see Varieties, Definition 28.8.1).
(2) We say f is a *normal morphism* if f is normal at every point of X.

So the condition that the morphism $X \to Y$ is normal is stronger than just requiring all the fibres to be normal locally Noetherian schemes.

Lemma 33.13.2. *Let $f : X \to Y$ be a morphism of schemes. Assume all fibres of f are locally Noetherian. The following are equivalent*

(1) *f is normal, and*
(2) *f is flat and its fibres are geometrically normal schemes.*

Proof. This follows directly from the definitions. □

Lemma 33.13.3. *A smooth morphism is normal.*

[1] The other types are coprof $\leq k$, Cohen-Macaulay, (S_k), regular, (R_k), and reduced. See [**DG67**, IV Definition 6.8.1.].

Proof. Let $f : X \to Y$ be a smooth morphism. As f is locally of finite presentation, see Morphisms, Lemma 24.33.8 the fibres X_y are locally of finite type over a field, hence locally Noetherian. Moreover, f is flat, see Morphisms, Lemma 24.33.9. Finally, the fibres X_y are smooth over a field (by Morphisms, Lemma 24.33.5) and hence geometrically normal by Varieties, Lemma 28.15.4. Thus f is normal by Lemma 33.13.2. $\square$

We want to show that this notion is local on the source and target for the smooth topology. First we deal with the property of having locally Noetherian fibres.

Lemma 33.13.4. *The property $\mathcal{P}(f)$ =``the fibres of f are locally Noetherian'' is local in the fppf topology on the source and the target.*

Proof. Let $f : X \to Y$ be a morphism of schemes. Let $\{\varphi_i : Y_i \to Y\}_{i \in I}$ be an fppf covering of Y. Denote $f_i : X_i \to Y_i$ the base change of f by φ_i. Let $i \in I$ and let $y_i \in Y_i$ be a point. Set $y = \varphi_i(y_i)$. Note that

$$X_{i,y_i} = Spec(\kappa(y_i)) \times_{Spec(\kappa(y))} X_y.$$

Moreover, as φ_i is of finite presentation the field extension $\kappa(y) \subset \kappa(y_i)$ is finitely generated. Hence in this situation we have that X_y is locally Noetherian if and only if X_{i,y_i} is locally Noetherian, see Varieties, Lemma 28.9.1. This fact implies locality on the target.

Let $\{X_i \to X\}$ be an fppf covering of X. Let $y \in Y$. In this case $\{X_{i,y} \to X_y\}$ is an fppf covering of the fibre. Hence the locality on the source follows from Descent, Lemma 31.12.1. $\square$

Lemma 33.13.5. *The property $\mathcal{P}(f)$ =``the fibres of f are locally Noetherian and f is normal'' is local in the fppf topology on the target and local in the smooth topology on the source.*

Proof. We have $\mathcal{P}(f) = \mathcal{P}_1(f) \wedge \mathcal{P}_2(f) \wedge \mathcal{P}_3(f)$ where $\mathcal{P}_1(f)$ =``the fibres of $f$ are locally Noetherian'', $\mathcal{P}_2(f)$ =``f is flat'', and $\mathcal{P}_3(f)$ =``the fibres of f are geometrically normal''. We have already seen that $\mathcal{P}_1$ and $\mathcal{P}_2$ are local in the fppf topology on the source and the target, see Lemma 33.13.4, and Descent, Lemmas 31.19.13 and 31.23.1. Thus we have to deal with $\mathcal{P}_3$.

Let $f : X \to Y$ be a morphism of schemes. Let $\{\varphi_i : Y_i \to Y\}_{i \in I}$ be an fpqc covering of Y. Denote $f_i : X_i \to Y_i$ the base change of f by φ_i. Let $i \in I$ and let $y_i \in Y_i$ be a point. Set $y = \varphi_i(y_i)$. Note that

$$X_{i,y_i} = Spec(\kappa(y_i)) \times_{Spec(\kappa(y))} X_y.$$

Hence in this situation we have that X_y is geometrically normal if and only if X_{i,y_i} is geometrically normal, see Varieties, Lemma 28.8.4. This fact implies $\mathcal{P}_3$ is fpqc local on the target.

Let $\{X_i \to X\}$ be a smooth covering of X. Let $y \in Y$. In this case $\{X_{i,y} \to X_y\}$ is a smooth covering of the fibre. Hence the locality of $\mathcal{P}_3$ for the smooth topology on the source follows from Descent, Lemma 31.14.2. Combining the above the lemma follows. $\square$

33.14. Regular morphisms

Compare with Section 33.13. The algebraic version of this notion is discussed in More on Algebra, Section 12.31.

Definition 33.14.1. Let $f : X \to Y$ be a morphism of schemes. Assume that all the fibres X_y are locally Noetherian schemes.

(1) Let $x \in X$, and $y = f(x)$. We say that f is *regular at* x if f is flat at x, and the scheme X_y is geometrically regular at x over $\kappa(y)$ (see Varieties, Definition 28.10.1).
(2) We say f is a *regular morphism* if f is regular at every point of X.

The condition that the morphism $X \to Y$ is regular is stronger than just requiring all the fibres to be regular locally Noetherian schemes.

Lemma 33.14.2. *Let $f : X \to Y$ be a morphism of schemes. Assume all fibres of f are locally Noetherian. The following are equivalent*

(1) *f is regular,*
(2) *f is flat and its fibres are geometrically regular schemes,*
(3) *for every pair of affine opens $U \subset X$, $V \subset Y$ with $f(U) \subset V$ the ring map $\mathcal{O}(V) \to \mathcal{O}(U)$ is regular,*
(4) *there exists an open covering $Y = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$ is regular, and*
(5) *there exists an affine open covering $Y = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring maps $\mathcal{O}(V_j) \to \mathcal{O}(U_i)$ are regular.*

Proof. The equivalence of (1) and (2) is immediate from the definitions. Let $x \in X$ with $y = f(x)$. By definition f is flat at x if and only if $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is a flat ring map, and X_y is geometrically regular at x over $\kappa(y)$ if and only if $\mathcal{O}_{X_y,x} = \mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$ is a geometrically regular algebra over $\kappa(y)$. Hence Whether or not f is regular at x depends only on the local homomorphism of local rings $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$. Thus the equivalence of (1) and (4) is clear.

Recall (More on Algebra, Definition 12.31.1) that a ring map $A \to B$ is regular if and only if it is flat and the fibre rings $B \otimes_A \kappa(\mathfrak{p})$ are Noetherian and geometrically regular for all primes $\mathfrak{p} \subset A$. By Varieties, Lemma 28.10.3 this is equivalent to $Spec(B \otimes_A \kappa(\mathfrak{p}))$ being a geometrically regular scheme over $\kappa(\mathfrak{p})$. Thus we see that (2) implies (3). It is clear that (3) implies (5). Finally, assume (5). This implies that f is flat (see Morphisms, Lemma 24.24.3). Moreover, if $y \in Y$, then $y \in V_j$ for some j and we see that $X_y = \bigcup_{i \in I_j} U_{i,y}$ with each $U_{i,y}$ geometrically regular over $\kappa(y)$ by Varieties, Lemma 28.10.3. Another application of Varieties, Lemma 28.10.3 shows that X_y is geometrically regular. Hence (2) holds and the proof of the lemma is finished. □

Lemma 33.14.3. *A smooth morphism is regular.*

Proof. Let $f : X \to Y$ be a smooth morphism. As f is locally of finite presentation, see Morphisms, Lemma 24.33.8 the fibres X_y are locally of finite type over a field, hence locally Noetherian. Moreover, f is flat, see Morphisms, Lemma 24.33.9. Finally, the fibres X_y are smooth over a field (by Morphisms, Lemma 24.33.5) and hence geometrically regular by Varieties, Lemma 28.15.4. Thus f is regular by Lemma 33.14.2. □

Lemma 33.14.4. *The property $\mathcal{P}(f)$ =``the fibres of f are locally Noetherian and f is regular'' is local in the fppf topology on the target and local in the smooth topology on the source.*

Proof. We have $\mathcal{P}(f) = \mathcal{P}_1(f) \wedge \mathcal{P}_2(f) \wedge \mathcal{P}_3(f)$ where $\mathcal{P}_1(f)$ =``the fibres of $f$ are locally Noetherian'', $\mathcal{P}_2(f)$ =``f is flat'', and $\mathcal{P}_3(f)$ =``the fibres of f are geometrically regular''. We have already seen that $\mathcal{P}_1$ and $\mathcal{P}_2$ are local in the fppf topology on the source and the target, see Lemma 33.13.4, and Descent, Lemmas 31.19.13 and 31.23.1. Thus we have to deal with $\mathcal{P}_3$.

Let $f : X \to Y$ be a morphism of schemes. Let $\{\varphi_i : Y_i \to Y\}_{i\in I}$ be an fpqc covering of Y. Denote $f_i : X_i \to Y_i$ the base change of f by φ_i. Let $i \in I$ and let $y_i \in Y_i$ be a point. Set $y = \varphi_i(y_i)$. Note that

$$X_{i,y_i} = Spec(\kappa(y_i)) \times_{Spec(\kappa(y))} X_y.$$

Hence in this situation we have that X_y is geometrically regular if and only if X_{i,y_i} is geometrically regular, see Varieties, Lemma 28.10.4. This fact implies $\mathcal{P}_3$ is fpqc local on the target.

Let $\{X_i \to X\}$ be a smooth covering of X. Let $y \in Y$. In this case $\{X_{i,y} \to X_y\}$ is a smooth covering of the fibre. Hence the locality of $\mathcal{P}_3$ for the smooth topology on the source follows from Descent, Lemma 31.14.4. Combining the above the lemma follows. □

33.15. Cohen-Macaulay morphisms

Compare with Section 33.13. Note that, as pointed out in Algebra, Section 7.149 and Varieties, Section 28.11 ``geometrically Cohen-Macaulay'' is the same as plain Cohen-Macaulay.

Definition 33.15.1. Let $f : X \to Y$ be a morphism of schemes. Assume that all the fibres X_y are locally Noetherian schemes.

(1) Let $x \in X$, and $y = f(x)$. We say that f is *Cohen-Macaulay at x* if f is flat at x, and the local ring of the scheme X_y at x is Cohen-Macaulay.
(2) We say f is a *Cohen-Macaulay morphism* if f is Cohen-Macaulay at every point of X.

Here is a translation.

Lemma 33.15.2. *Let $f : X \to Y$ be a morphism of schemes. Assume all fibres of f are locally Noetherian. The following are equivalent*

(1) *f is Cohen-Macaulay, and*
(2) *f is flat and its fibres are Cohen-Macaulay schemes.*

Proof. This follows directly from the definitions. □

Lemma 33.15.3. *Let $f : X \to Y$ be a morphism of schemes. Assume that all the fibres X_y are locally Noetherian schemes. Let $Y' \to Y$ be locally of finite type. Let $f' : X' = X_{Y'} \to Y$ be the base change of f. Let $x' \in X'$ be a point with image $x \in X$.*

(1) *If f is Cohen-Macaulay at x, then the base change $f' : X' \to Y'$ is Cohen-Macaulay at x'.*
(2) *If $Y' \to Y$ is flat at $f'(x')$ and f' is Cohen-Macaulay at x', then f is Cohen-Macaulay at x.*

Proof. Note that the assumption on $Y' \to Y$ means that for $y' \in Y'$ mapping to $y \in Y$ the field extension $\kappa(y) \subset \kappa(y')$ is finitely generated. Hence also all the fibres $X'_{y'} = (X_y)_{\kappa(y')}$ are locally Noetherian, see Varieties, Lemma 28.9.1. Thus the lemma makes sense. Set $y' = f'(x')$ and $y = f(x)$. Hence we get the following commutative diagram of local rings

$$\begin{tikzcd} \mathcal{O}_{X',x'} & \mathcal{O}_{X,x} \arrow[l] \\ \mathcal{O}_{Y',y'} \arrow[u] & \mathcal{O}_{Y,y} \arrow[l] \arrow[u] \end{tikzcd}$$

where the upper left corner is a localization of the tensor product of the upper right and lower left corners over the lower right corner.

Assume f is Cohen-Macaulay at x. The flatness of $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ implies the flatness of $\mathcal{O}_{Y',y'} \to \mathcal{O}_{X',x'}$, see Algebra, Lemma 7.92.1. The fact that $\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$ is Cohen-Macaulay implies that $\mathcal{O}_{X',x'}/\mathfrak{m}_{y'}\mathcal{O}_{X',x'}$, see Varieties, Lemma 28.11.1. Hence we see that f' is Cohen-Macaulay at x'.

Assume $Y' \to Y$ is flat at y' and f' is Cohen-Macaulay at x'. The flatness of $\mathcal{O}_{Y',y'} \to \mathcal{O}_{X',x'}$ and $\mathcal{O}_{Y,y} \to \mathcal{O}_{Y',y'}$ implies the flatness of $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$, see Algebra, Lemma 7.92.1. The fact that $\mathcal{O}_{X',x'}/\mathfrak{m}_{y'}\mathcal{O}_{X',x'}$ is Cohen-Macaulay implies that $\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$, see Varieties, Lemma 28.11.1. Hence we see that f is Cohen-Macaulay at x. □

Lemma 33.15.4. *Let $f : X \to S$ be a flat morphism of finite presentation. Let*

$$W = \{x \in X \mid f \text{ is Cohen-Macaulay at } x\}$$

Then

(1) *we have*

$$W = \{x \in X \mid \mathcal{O}_{X_{f(x)},x} \text{ is Cohen-Macaulay}\},$$

(2) *W is open in X,*
(3) *W dense in every fibre of $X \to S$,*
(4) *the formation of W commutes with arbitrary base change of f: For any morphism $g : S' \to S$, consider the base change $f' : X' \to S'$ of f and the projection $g' : X' \to X$. Then the corresponding set W' for the morphism f' is equal to $W' = (g')^{-1}(W)$.*

Proof. As f is flat with locally Noetherian fibres the equality in (1) holds by definition. Parts (2) and (3) follow from Algebra, Lemma 7.121.5. Part (4) follows either from Algebra, Lemma 7.121.7 or Varieties, Lemma 28.11.1. □

Lemma 33.15.5. *Let $f : X \to S$ be a flat morphism of finite presentation. For $d \geq 0$ there exist opens $U_d \subset X$ with the following properties*

(1) *$W = \bigcup_{d\geq 0} U_d$ is dense in every fibre of f, and*
(2) *$U_d \to S$ is of relative dimension d (see Morphisms, Definition 24.28.1).*

Proof. This follows by combining Lemma 33.15.4 with Morphisms, Lemma 24.28.4. □

Lemma 33.15.6. *Let $f : X \to S$ be a flat morphism of finite presentation. Suppose $x' \rightsquigarrow x$ is a specialization of points of X with image $s' \rightsquigarrow s$ in S. If x is a generic point of an irreducible component of X_s then $\dim_{x'}(X_{s'}) = \dim_x(X_s)$.*

Proof. The point x is contained in U_d for some d, where U_d as in Lemma 33.15.5. □

Lemma 33.15.7. *The property $\mathcal{P}(f) =$``the fibres of f are locally Noetherian and f is Cohen-Macaulay" is local in the fppf topology on the target and local in the syntomic topology on the source.*

Proof. We have $\mathcal{P}(f) = \mathcal{P}_1(f) \wedge \mathcal{P}_2(f)$ where $\mathcal{P}_1(f) =$``$f$ is flat", and $\mathcal{P}_2(f) =$``the fibres of f are locally Noetherian and Cohen-Macaulay". We know that $\mathcal{P}_1$ is local in the fppf topology on the source and the target, see Descent, Lemmas 31.19.13 and 31.23.1. Thus we have to deal with $\mathcal{P}_2$.

Let $f : X \to Y$ be a morphism of schemes. Let $\{\varphi_i : Y_i \to Y\}_{i\in I}$ be an fppf covering of Y. Denote $f_i : X_i \to Y_i$ the base change of f by φ_i. Let $i \in I$ and let $y_i \in Y_i$ be a point. Set $y = \varphi_i(y_i)$. Note that

$$X_{i,y_i} = Spec(\kappa(y_i)) \times_{Spec(\kappa(y))} X_y.$$

and that $\kappa(y) \subset \kappa(y_i)$ is a finitely generated field extension. Hence if X_y is locally Noetherian, then X_{i,y_i} is locally Noetherian, see Varieties, Lemma 28.9.1. And if in addition X_y is Cohen-Macaulay, then X_{i,y_i} is Cohen-Macaulay, see Varieties, Lemma 28.11.1. Thus $\mathcal{P}_2$ is fppf local on the target.

Let $\{X_i \to X\}$ be a syntomic covering of X. Let $y \in Y$. In this case $\{X_{i,y} \to X_y\}$ is a syntomic covering of the fibre. Hence the locality of $\mathcal{P}_2$ for the syntomic topology on the source follows from Descent, Lemma 31.13.2. Combining the above the lemma follows. $\square$

33.16. Slicing Cohen-Macaulay morphisms

The results in this section eventually lead to the assertion that the fppf topology is the same as the ``finitely presented, flat, quasi-finite" topology. The following lemma is very closely related to Divisors, Lemma 26.10.7.

Lemma 33.16.1. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Let $h \in \mathfrak{m}_x \subset \mathcal{O}_{X,x}$. Assume*

(1) *f is locally of finite presentation,*
(2) *f is flat at x, and*
(3) *the image $\overline{h}$ of h in $\mathcal{O}_{X_s,x} = \mathcal{O}_{X,x}/\mathfrak{m}_s\mathcal{O}_{X,x}$ is a nonzero divisor.*

Then there exists an affine open neighbourhood $U \subset X$ of x such that h comes from $h \in \Gamma(U, \mathcal{O}_U)$ and such that $D = V(h)$ is an effective Cartier divisor in U with $x \in D$ and $D \to S$ flat and locally of finite presentation.

Proof. We are going to prove this by reducing to the Noetherian case. By openness of flatness (see Theorem 33.11.1) we may assume, after replacing X by an open neighbourhood of x, that $X \to S$ is flat. We may also assume that X and S are affine. After possible shrinking X a bit we may assume that there exists an $h \in \Gamma(X, \mathcal{O}_X)$ which maps to our given h.

We may write $S = Spec(A)$ and we may write $A = colim_i A_i$ as a directed colimit of finite type $\mathbf{Z}$ algebras. Then by Algebra, Lemma 7.120.5 or Limits, Lemmas 27.6.1, 27.6.2, and 27.6.1 we can find a cartesian diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X_0 \arrow[d, "f_0"] \\ S \arrow[r] & S_0 \end{tikzcd}$$

with f_0 flat and of finite presentation, X_0 affine, and S_0 affine and Noetherian. Let $x_0 \in X_0$, resp. $s_0 \in S_0$ be the image of x, resp. s. We may also assume there exists an element $h_0 \in \Gamma(X_0, \mathcal{O}_{X_0})$ which restricts to h on X. (If you used the algebra reference above then this is clear; if you used the references to the chapter on limits then this follows from Limits, Lemma 27.6.1 by thinking of h as a morphism $X \to \mathbf{A}^1_S$.) Note that $\mathcal{O}_{X_s,x}$ is a localization of $\mathcal{O}_{(X_0)_{s_0},x_0} \otimes_{\kappa(s_0)} \kappa(s)$, so that $\mathcal{O}_{(X_0)_{s_0},x_0} \to \mathcal{O}_{X_s,x}$ is a flat local ring map, in particular faithfully flat. Hence the image $\overline{h}_0 \in \mathcal{O}_{(X_0)_{s_0},x_0}$ is contained in $\mathfrak{m}_{(X_0)_{s_0},x_0}$ and is a nonzero divisor. We claim that after replacing X_0 by a principal open neighbourhood of x_0 the element h_0 is a nonzero divisor in $B_0 = \Gamma(X_0, \mathcal{O}_{X_0})$ such that B_0/h_0B_0 is flat over $A_0 = \Gamma(S_0, \mathcal{O}_{S_0})$. If so then

$$0 \to B_0 \xrightarrow{h_0} B_0 \to B_0/h_0B_0 \to 0$$

is a short exact sequence of flat A_0-modules. Hence this remains exact on tensoring with A (by Algebra, Lemma 7.35.11) and the lemma follows.

It remains to prove the claim above. The corresponding algebra statement is the following (we drop the subscript $_0$ here): Let $A \to B$ be a flat, finite type ring map of Noetherian rings. Let $\mathfrak{q} \subset B$ be a prime lying over $\mathfrak{p} \subset A$. Assume $h \in \mathfrak{q}$ maps to a nonzero divisor in $B_\mathfrak{q}/\mathfrak{p}B_\mathfrak{q}$. Goal: show that after possible replacing B by B_g for some $g \in B$, $g \not\in \mathfrak{q}$ the element h becomes a nonzero divisor and B/hB becomes flat over A. By Algebra, Lemma 7.91.2 we see that h is a nonzero divisor in $B_\mathfrak{q}$ and that $B_\mathfrak{q}/hB_\mathfrak{q}$ is flat over A. By openess of flatness, see Algebra, Theorem 7.120.4 or Theorem 33.11.1 we see that B/hB is flat over A after replacing B by B_g for some $g \in B$, $g \not\in \mathfrak{q}$. Finally, let $I = \{b \in B \mid hb = 0\}$ be the annihilator of h. Then $IB_\mathfrak{q} = 0$ as h is a nonzero divisor in $B_\mathfrak{q}$. Also I is finitely generated as B is Noetherian. Hence there exists a $g \in B$, $g \not\in \mathfrak{q}$ such that $IB_g = 0$. After replacing B by B_g we see that h is a nonzero divisor. □

Lemma 33.16.2. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Let $h_1, \ldots, h_r \in \mathcal{O}_{X,x}$. Assume*

(1) *f is locally of finite presentation,*
(2) *f is flat at x, and*
(3) *the images of $h_1, \ldots, h_r$ in $\mathcal{O}_{X_s,x} = \mathcal{O}_{X,x}/\mathfrak{m}_s\mathcal{O}_{X,x}$ form a regular sequence.*

Then there exists an affine open neighbourhood $U \subset X$ of x such that $h_1, \ldots, h_r$ come from $h_1, \ldots, h_r \in \Gamma(U, \mathcal{O}_U)$ and such that $Z = V(h_1, \ldots, h_r) \to U$ is a regular immersion with $x \in Z$ and $Z \to S$ flat and locally of finite presentation. Moreover, the base change $Z_{S'} \to U_{S'}$ is a regular immersion for any scheme S' over S.

Proof. (Our conventions on regular sequences imply that $h_i \in \mathfrak{m}_x$ for each i.) The case $r = 1$ follows from Lemma 33.16.1 combined with Divisors, Lemma 26.10.1 to see that $V(h_1)$ remains an effective Cartier divisor after base change. The case $r > 1$ follows from a straightforward induction on r (applying the result for $r = 1$ exactly r times; details omitted).

Another way to prove the lemma is using the material from Divisors, Section 26.14. Namely, first by openness of flatness (see Theorem 33.11.1) we may assume, after replacing X by an open neighbourhood of x, that $X \to S$ is flat. We may also assume that X and S are affine. After possible shrinking X a bit we may assume that we have $h_1, \ldots, h_r \in \Gamma(X, \mathcal{O}_X)$. Set $Z = V(h_1, \ldots, h_r)$. Note that X_s is a Noetherian scheme (because it is an algebraic $\kappa(s)$-scheme, see Varieties, Section 28.13) and that the topology on X_s is induced from the topology on X (see Schemes, Lemma 21.18.5). Hence after shrinking X a bit more we may assume that $Z_s \subset X_s$ is a regular immersion cut out by the r elements $h_i|_{X_s}$, see Divisors, Lemma 26.12.8 and its proof. It is also clear that $r = \dim_x(X_s) - \dim_x(Z_s)$ because

$$\begin{aligned} \dim_x(X_s) &= \dim(\mathcal{O}_{X_s,x}) + \text{trdeg}_{\kappa(s)}(\kappa(x)), \\ \dim_x(Z_s) &= \dim(\mathcal{O}_{Z_s,x}) + \text{trdeg}_{\kappa(s)}(\kappa(x)), \\ \dim(\mathcal{O}_{X_s,x}) &= \dim(\mathcal{O}_{Z_s,x}) + r \end{aligned}$$

the first two equalities by Algebra, Lemma 7.107.3 and the second by r times applying Algebra, Lemma 7.57.11. Hence Divisors, Lemma 26.14.6 part (3) applies to show that (after Zariski shrinking X) the morphism $Z \to X$ is a regular immersion to which Divisors, Lemma 26.14.4 applies (which gives the flatness and the statement on base change). □

Lemma 33.16.3. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Assume*

(1) *f is locally of finite presentation,*
(2) *f is flat at x, and*
(3) *$\mathcal{O}_{X_s,x}$ has depth ≥ 1.*

Then there exists an affine open neighbourhood $U \subset X$ of x and an effective Cartier divisor $D \subset U$ containing x such that $D \to S$ is flat and of finite presentation.

Proof. Pick any $h \in \mathfrak{m}_x \subset \mathcal{O}_{X,x}$ which maps to a nonzero divisor in $\mathcal{O}_{X_s,x}$ and apply Lemma 33.16.1. □

Lemma 33.16.4. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Assume*

(1) *f is locally of finite presentation,*
(2) *f is Cohen-Macaulay at x, and*
(3) *x is a closed point of X_s.*

Then there exists a regular immersion $Z \to X$ containing x such that

(a) *$Z \to S$ is flat and locally of finite presentation,*
(b) *$Z \to S$ is locally quasi-finite, and*
(c) *$Z_s = \{x\}$ set theoretically.*

Proof. We may and do replace S by an affine open neighbourhoof of s. We will prove the lemma for affine S by induction on $d = \dim_x(X_s)$.

The case $d = 0$. In this case we show that we may take Z to be an open neighbourhood of x. (Note that an open immersion is a regular immersion.) Namely, if $d = 0$, then $X \to S$ is quasi-finite at x, see Morphisms, Lemma 24.28.5. Hence there exists an affine open neighbourhood $U \subset X$ such that $U \to S$ is quasi-finite, see Morphisms, Lemma 24.47.2. Thus after replacing X by U we see that the fibre X_s is a finite discrete set. Hence after replacing X by a further affine open neigbourhood of X we see that that $f^{-1}(\{s\}) = \{x\}$ (because the topology on X_s is induced from the topology on X, see Schemes, Lemma 21.18.5). This proves the lemma in this case.

Next, assume $d > 0$. Note that because x is a closed point of its fibre the extension $\kappa(s) \subset \kappa(x)$ is finite (by the Hilbert Nullstellensatz, see Morphisms, Lemma 24.19.3). Thus we see

$$\mathrm{depth}(\mathcal{O}_{X_s,x}) = \dim(\mathcal{O}_{X_s,x}) = d > 0$$

the first equality as $\mathcal{O}_{X_s,x}$ is Cohen-Macaulay and the second by Morphisms, Lemma 24.27.1. Thus we may apply Lemma 33.16.3 to find a diagram

$$\begin{tikzcd} D \arrow[r] \arrow[rrd] & U \arrow[r] \arrow[rd] & X \arrow[d] \\ & & S \end{tikzcd}$$

with $x \in D$. Note that $\mathcal{O}_{D_s,x} = \mathcal{O}_{X_s,x}/(\overline{h})$ for some nonzero divisor $\overline{h}$, see Divisors, Lemma 26.10.1. Hence $\mathcal{O}_{D_s,x}$ is Cohen-Macaulay of dimension one less than the dimension of

$\mathcal{O}_{X_s,x}$, see Algebra, Lemma 7.96.2 for example. Thus the morphism $D \to S$ is flat, locally of finite presentation, and Cohen-Macaulay at x with $\dim_x(D_s) = \dim_x(X_s) - 1 = d - 1$. By induction hypothesis we can find a regular immersion $Z \to D$ having properties (a), (b), (c). As $Z \to D \to U$ are both regular immersions, we see that also $Z \to U$ is a regular immersion by Divisors, Lemma 26.13.7. This finishes the proof. □

Lemma 33.16.5. *Let $f : X \to S$ be a flat morphism of schemes which is locally of finite presentation Let $s \in S$ be a point in the image of f. Then there exists a commutative diagram*

$$\begin{tikzcd} S' \arrow[rr] \arrow[dr, "g"'] & & X \arrow[dl, "f"] \\ & S & \end{tikzcd}$$

where $g : S' \to S$ is flat, locally of finite presentation, locally quasi-finite, and $s \in g(S')$.

Proof. The fibre X_s is not empty by assumption. Hence there exists a closed point $x \in X_s$ where f is Cohen-Macaulay, see Lemma 33.15.4. Apply Lemma 33.16.4 and set $S' = S$. □

The following lemma shows that sheaves for the fppf topology are the same thing as sheaves for the ``quasi-finite, flat, finite presentation" toplogy.

Lemma 33.16.6. *Let S be a scheme. Let $\mathcal{U} = \{S_i \to S\}_{i \in I}$ be an fppf covering of S, see Topologies, Definition 30.5.1. Then there exists an fppf covering $\mathcal{V} = \{T_j \to S\}_{j \in J}$ which refines (see Sites, Definition 9.8.1) $\mathcal{U}$ such that each $T_j \to S$ is locally quasi-finite.*

Proof. For every $s \in S$ there exists an $i \in I$ such that s is in the image of $S_i \to S$. By Lemma 33.16.5 we can find a morphism $g_s : T_s \to S$ such that $s \in g_s(T)_s$ which is flat, locally of finite presentation and locally quasi finite and such that g_s factors through $S_i \to S$. Hence $\{T_s \to S\}$ is the desired covering of S that refines $\mathcal{U}$. □

33.17. Generic fibres

Some results on the relationship between generic fibres and nearby fibres.

Lemma 33.17.1. *Let $f : X \to Y$ be a finite type morphism of schemes. Assume Y irreducible with generic point η. If $X_\eta = \emptyset$ then there exists a nonempty open $V \subset Y$ such that $X_V = V \times_Y X = \emptyset$.*

Proof. Follows immediately from the more general Morphisms, Lemma 24.6.4. □

Lemma 33.17.2. *Let $f : X \to Y$ be a finite type morphism of schemes. Assume Y irreducible with generic point η. If $X_\eta \neq \emptyset$ then there exists a nonempty open $V \subset Y$ such that $X_V = V \times_Y X \to V$ is surjective.*

Proof. This follows, upon taking affine opens, from Algebra, Lemma 7.27.2. (Of course it also follows from generic flatness.) □

Lemma 33.17.3. *Let $f : X \to Y$ be a finite type morphism of schemes. Assume Y irreducible with generic point η. If $Z \subset X$ is a closed subset with Z_η nowhere dense in X_η, then there exists a nonempty open $V \subset Y$ such that Z_y is nowhere dense in X_y for all $y \in V$.*

Proof. Let $Y' \subset Y$ be the reduction of Y. Set $X' = Y' \times_Y X$ and $Z' = Y' \times_Y Z$. As $Y' \to Y$ is a universal homeomorphism by Morphisms, Lemma 24.43.4 we see that it suffices to prove the lemma for $Z' \subset X' \to Y'$. Thus we may assume that Y is integral, see Properties, Lemma 23.3.4. By Morphisms, Proposition 24.26.1 there exists a nonempty affine open $V \subset Y$ such that $X_V \to V$ and $Z_V \to Z$ are flat and of finite presentation. We claim that V works. Pick $y \in V$. If Z_y has a nonempty interior, then Z_y contains a generic point ξ of an irreducible component of X_y. Note that $\eta \rightsquigarrow f(\xi)$. Since $Z_V \to V$ is flat we can choose a specialization $\xi' \rightsquigarrow \xi$, $\xi' \in Z$ with $f(\xi') = \eta$, see Morphisms, Lemma 24.24.8. By Lemma 33.15.6 we see that

$$\dim_{\xi'}(Z_\eta) = \dim_\xi(Z_y) = \dim_\xi(X_y) = \dim_{\xi'}(X_\eta).$$

Hence some irreducible component of Z_η passing through ξ' has dimension $\dim_{\xi'}(X_\eta)$ which contradicts the assumption that Z_η is nowhere dense in X_η and we win. $\square$

Lemma 33.17.4. *Let $f : X \to Y$ be a finite type morphism of schemes. Assume Y irreducible with generic point η. Let $U \subset X$ be an open subscheme such that U_η is scheme theoretically dense in X_η. Then there exists a nonempty open $V \subset Y$ such that U_y is scheme theoretically dense in X_y for all $y \in V$.*

Proof. Let $Y' \subset Y$ be the reduction of Y. Let $X' = Y' \times_Y X$ and $U' = Y' \times_Y U$. As $Y' \to Y$ induces a bijection on points, and as $U' \to U$ and $X' \to X$ induce isomorphisms of scheme theoretic fibres, we may replace Y by Y' and X by X'. Thus we may assume that Y is integral, see Properties, Lemma 23.3.4. We may also replace Y by a nonempty affine open. In other words we may assume that $Y = Spec(A)$ where A is a domain with fraction field K.

As f is of finite type we see that X is quasi-compact. Write $X = X_1 \cup \ldots \cup X_n$ for some affine opens X_i. By Morphisms, Definition 24.5.1 we see that $U_i = X_i \cap U$ is an open subscheme of X_i such that $U_{i,\eta}$ is scheme theoretically dense in $X_{i,\eta}$. Thus it suffices to prove the result for the pairs (X_i, U_i), in other words we may assume that X is affine.

Write $X = Spec(B)$. Note that B_K is Noetherian as it is a finite type K-algebra. Hence U_η is quasi-compact. Thus we can find finitely many $g_1, \ldots, g_m \in B$ such that $D(g_j) \subset U$ and such that $U_\eta = D(g_1)_\eta \cup \ldots \cup D(g_m)_\eta$. The fact that U_η is scheme theoretically dense in X_η means that $B_K \to \bigoplus_j (B_K)_{g_j}$ is injective, see Morphisms, Example 24.5.4. By Algebra, Lemma 7.20.4 this is equivalent to the injectivity of $B_K \to \bigoplus_{j=1,\ldots,m} B_K$, $b \mapsto (g_1 b, \ldots, g_m b)$. Let M be the cokernel of this map over A, i.e., such that we have an exact sequence

$$0 \to I \to B \xrightarrow{(g_1,\ldots,g_m)} \bigoplus\nolimits_{j=1,\ldots,m} B \to M \to 0$$

After replacing A by A_h for some nonzero h we may assume that B is a flat, finitely presented A-algebra, and that M is flat over A, see Algebra, Lemma 7.109.3. The flatness of B over A implies that B is torsion free as an A-module, see More on Algebra, Lemma 12.17.3. Hence $B \subset B_K$. By assumption $I_K = 0$ which implies that $I = 0$ (as $I \subset B \subset B_K$ is a subset of I_K). Hence now we have a short exact sequence

$$0 \to B \xrightarrow{(g_1,\ldots,g_m)} \bigoplus\nolimits_{j=1,\ldots,m} B \to M \to 0$$

with M flat over A. Hence for every homomorphism $A \to \kappa$ where κ is a field, we obtain a short exact sequence

$$0 \to B \otimes_A \kappa \xrightarrow{(g_1\otimes 1,\ldots,g_m\otimes 1)} \bigoplus\nolimits_{j=1,\ldots,m} B \otimes_A \kappa \to M \otimes_A \kappa \to 0$$

see Algebra, Lemma 7.35.11. Reversing the arguments above this means that $\bigcup D(g_j \otimes 1)$ is scheme theoretically dense in $Spec(B \otimes_A \kappa)$. As $\bigcup D(g_j \otimes 1) = \bigcup D(g_j)_\kappa \subset U_\kappa$ we obtain that U_κ is scheme theoretically dense in X_κ which is what we wanted to prove. $\square$

Suppose given a morphism of schemes $f : X \to Y$ and a point $y \in Y$. Recally that the fibre X_y is homeomorphic to the subset $f^{-1}(\{y\})$ of X with induced topology, see Schemes, Lemma 21.18.5. Suppose given a closed subset $T(y) \subset X_y$. Let T be the closure of $T(y)$ in X. Endow T with the induced reduced scheme structure. Then T is a closed subscheme of X with the property that $T_y = T(y)$ set-theoretically. In fact T is the smallest closed subscheme of X with this property. Thus it is ``harmless" to denote a closed subset of X_y by T_y if we so desire. In the following lemma we apply this to the generic fibre of f.

Lemma 33.17.5. *Let $f : X \to Y$ be a finite type morphism of schemes. Assume Y irreducible with generic point η. Let $X_\eta = Z_{1,\eta} \cup \ldots \cup Z_{n,\eta}$ be a covering of the generic fibre by closed subsets of X_η. Let Z_i be the closure of $Z_{i,\eta}$ in X (see discussion above). Then there exists a nonempty open $V \subset Y$ such that $X_y = Z_{1,y} \cup \ldots \cup Z_{n,y}$ for all $y \in V$.*

Proof. If Y is Noetherian then $U = X \setminus (Z_1 \cup \ldots \cup Z_n)$ is of finite type over Y and we can directly apply Lemma 33.17.1 to get that $U_V = \emptyset$ for a nonempty open $V \subset Y$. In general we argue as follows. As the question is topological we may replace Y by its reduction. Thus Y is integral, see Properties, Lemma 23.3.4. After shrinking Y we may assume that $X \to Y$ is flat, see Morphisms, Proposition 24.26.1. In this case every point x in X_y is a specialization of a point $x' \in X_\eta$ by Morphisms, Lemma 24.24.8. As the Z_i are closed in X and cover the generic fibre this implies that $X_y = \bigcup Z_{i,y}$ for $y \in Y$ as desired. $\square$

The following lemma says that generic fibres of morphisms whose source is reduced are reduced.

Lemma 33.17.6. *Let $f : X \to Y$ be a morphism of schemes. Let $\eta \in Y$ be a generic point of an irreducible component of Y. Then $(X_\eta)_{red} = (X_{red})_\eta$.*

Proof. Choose an affine neighbourhood $Spec(A) \subset Y$ of η. Choose an affine open $Spec(B) \subset X$ mapping into $Spec(A)$ via the morphism f. Let $\mathfrak p \subset A$ be the minimal prime corresponding to η. Let B_{red} be the quotient of B by $\sqrt{(0)}$. The algebraic content of the lemma is that $B_{red} \otimes_A \kappa(\mathfrak p)$ is reduced. To prove this, suppose that $x \in B_{red} \otimes_A \kappa(\mathfrak p)$ is nilpotent. Say $x^n = 0$ for some $n > 0$. Pick an $f \in A$, $f \not\in \mathfrak p$ such that fx is the image of $y \in B_{red}$. Then $gy^n \in \mathfrak p B_{red}$ for some $g \in A$, $g \not\in \mathfrak p$. By Algebra, Lemma 7.23.3 we see that $\mathfrak p A_{\mathfrak p}$ is locally nilpotent. By Algebra, Lemma 7.14.1 we see that $\mathfrak p (B_{red})_{\mathfrak p}$ is locally nilpotent. Hence we conclude that gy^n is nilpotent in $(B_{red})_{\mathfrak p}$. Thus there exists a $h \in A$, $h \not\in \mathfrak p$ and an $m > 0$ such that $h(gy^n)^m = 0$ in B_{red}. This implies that hgy is nilpotent in B_{red}, i.e., that $hgy = 0$. Of course this means that $x = 0$ as desired. $\square$

Lemma 33.17.7. *Let $f : X \to Y$ be a morphism of schemes. Assume that Y is irreducible and f is of finite type. There exists a diagram*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X_V \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & V \arrow[r] & Y \end{tikzcd}$$

where

(1) *V is a nonempty open of Y,*

(2) $X_V = V \times_Y X$,
(3) $g : Y' \to V$ *is a finite universal homeomorphism,*
(4) $X' = (Y' \times_Y X)_{red} = (Y' \times_V X_V)_{red}$,
(5) g' *is a finite universal homeomorphism,*
(6) Y' *is an integral affine scheme,*
(7) f' *is flat and of finite presentation, and*
(8) *the generic fibre of* f' *is geometrically reduced.*

Proof. Let $V = Spec(A)$ be a nonempty affine open of Y. By assumption the radical of A is a prime ideal $\mathfrak{p}$. Let $K = f.f(A/\mathfrak{p})$ be the fraction field. Let p be the characteristic of K if positive and 1 if the characteristic is zero. By Varieties, Lemma 28.4.11 there exists a finite purely inseparable field extension $K \subset K'$ such that $X_{K'}$ is geometrically reduced over K'. Choose elements $x_1, \ldots, x_n \in K'$ which generate K' over K and such that some p-power of x_i is in $A/\mathfrak{p}$. Let $A' \subset K'$ be the finite A-subalgebra of K' generated by $x_1, \ldots, x_n$. Note that A' is a domain with fraction field K'. By Algebra, Lemma 7.43.2 we see that $A \to A'$ is a universal homeomorphism. Set $Y' = Spec(A')$. Set $X' = (Y' \times_Y X)_{red}$. The generic fibre of $X' \to Y'$ is $(X_K)_{red}$ by Lemma 33.17.6 which is geometrically reduced by construction. Note that $X' \to X_V$ is a finite universal homeomorphism as the composition of the reduction morphism $X' \to Y' \times_Y X$ (see Morphisms, Lemma 24.43.4) and the base change of g. At this point all of the properties of the lemma hold except for possibly (7). This can be achieved by shrinking Y' and hence V, see Morphisms, Proposition 24.26.1. □

Lemma 33.17.8. *Let $f : X \to Y$ be a morphism of schemes. Assume that Y is irreducible and f is of finite type. There exists a diagram*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X_V \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & V \arrow[r] & Y \end{tikzcd}$$

where

(1) V *is a nonempty open of* Y,
(2) $X_V = V \times_Y X$,
(3) $g : Y' \to V$ *is surjective finite étale,*
(4) $X' = Y' \times_Y X = Y' \times_V X_V$,
(5) g' *is surjective finite étale,*
(6) Y' *is an irreducible affine scheme, and*
(7) *all irreducible components of the generic fibre of* f' *are geometrically irreducible.*

Proof. Let $V = Spec(A)$ be a nonempty affine open of Y. By assumption the radical of A is a prime ideal $\mathfrak{p}$. Let $K = f.f(A/\mathfrak{p})$ be the fraction field. By Varieties, Lemma 28.6.14 there exists a finite separable field extension $K \subset K'$ such that all irreducible components of $X_{K'}$ are geometrically irreducible over K'. Choose an element $\alpha \in K'$ which generates K' over K, see Algebra, Lemma 7.38.5. Let $P(T) \in K[T]$ be the minimal polynomial for α over K. After replacing α by $f\alpha$ for some $f \in A$, $f \not\in \mathfrak{p}$ we may assume that there exists a monic polynomial $T^d + a_1T^{d-1} + \ldots + a_d \in A[T]$ which maps to $P(T) \in K[T]$ under the map $A[T] \to K[T]$. Set $A' = A[T]/(P)$. Then $A \to A'$ is a finite free ring map such that there exists a unique prime $\mathfrak{q}$ lying over $\mathfrak{p}$, such that $K = \kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q}) = K'$ is finite separable, and such that $\mathfrak{p}A'_{\mathfrak{q}}$ is the maximal ideal of $A'_{\mathfrak{q}}$. Hence $g : Y' = Spec(A') \to V = Spec(A)$ is étale at $\mathfrak{q}$, see Algebra, Lemma 7.132.7. This means that there exists an open $W \subset Spec(A')$ such that $g|_W : W \to Spec(A)$ is étale. Since g is finite and since $\mathfrak{q}$ is the only point lying over $\mathfrak{p}$ we see that $Z = g(Y' \setminus W)$ is a closed subset of V not containing $\mathfrak{p}$. Hence after replacing V by a principal affine open of V which does not meet Z we obtain that g is finite étale. $\square$

Lemma 33.17.9. *Let S be an integral scheme with generic point η. Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes such that*

(1) *f, g are locally of finite type,*
(2) *X_η, Y_η are integral with generic points x, y, and*
(3) *$\kappa(x) \cong \kappa(y)$ as $\kappa(\eta)$-extensions.*

Then there exist open subschemes $x \in U \subset X$, $y \in V \subset Y$ and an S-isomorphism $U \to V$ which induces the given isomorphism of residue fields.

Proof. The question is local around the points η, x, y. Hence we may replace S, X, Y by affine neighbourhoods of η, x, y and hence reduce to the case that S, X, Y are affine. Say $S = Spec(R)$ and $X = Spec(A)$, $Y = Spec(B)$. By Algebra, Lemma 7.109.3 we may also assume that A and B are flat and of finite presentation over R. Denote $K = f.f.(R)$. The rings A, B are torsion free as R-modules because A, B are flat over R, see More on Algebra, Lemma 12.17.3. Since $A \otimes_R K$ and $B \otimes_R K$ are domains by assumption it follows that A and B are domains. Set $L = f.f.(A)$ and $M = f.f.(B)$. Let $\varphi : L \to M$ be the given isomorphism of K-extensions.

Choose elements $x_1, \ldots, x_n \in A$ which generate A as an R-algebra, and choose elements $y_1, \ldots, y_m \in B$ which generate B as an R-algebra. Write $\varphi(x_i) = b_i/b$ for some $b, b_i \in B$. In other words, b is a common denominatior for the elements $\varphi(x_i) \in M = f.f.(B)$. Similarly, write $\varphi^{-1}(y_j) = a_j/a$ for some $a, a_j \in A$. Note that $\varphi(a) \in B_b$ because a can be written as a polynomial in the x_i. Similarly we have $\varphi^{-1}(b) \in A_a$. Thus φ gives an isomorphism

$$A_a \longrightarrow B_b$$

of R-algebras and the lemma is proven. $\square$

33.18. Relative assassins

Lemma 33.18.1. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\xi \in Ass_{X/S}(\mathcal{F})$ and set $Z = \overline{\{\xi\}} \subset X$. If f is locally of finite type and $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module, then there exists a nonempty open $V \subset Z$ such that for every $s \in f(V)$ the generic points of V_s are elements of $Ass_{X/S}(\mathcal{F})$.*

Proof. We may replace S by an affine open neighbourhood of $f(\xi)$ and X by an affine open neighbourhood of ξ. Hence we may assume $S = Spec(A)$, $X = Spec(B)$ and that f is given by the finite type ring map $A \to B$, see Morphisms, Lemma 24.14.2. Moreover, we may write $\mathcal{F} = \widetilde{M}$ for some finite B-module M, see Properties, Lemma 23.16.1. Let $\mathfrak{q} \subset B$ be the prime corresponding to ξ and let $\mathfrak{p} \subset A$ be the corresponding prime of A. By assumption $\mathfrak{q} \in \text{Ass}_B(M \otimes_A \kappa(\mathfrak{p}))$, see Algebra, Remark 7.62.6 and Divisors, Lemma 26.2.2. With this notation $Z = V(\mathfrak{q}) \subset Spec(B)$. In particular $f(Z) \subset V(\mathfrak{p})$. Hence clearly it suffices to prove the lemma after replacing A, B, and M by $A/\mathfrak{p}A$, $B/\mathfrak{p}B$, and $M/\mathfrak{p}M$. In other words we may assume that A is a domain with fraction field K and $\mathfrak{q} \subset B$ is an associated prime of $M \otimes_A K$.

At this point we can use generic flatness. Namely, by Algebra, Lemma 7.109.3 there exists a nonzero $g \in A$ such that M_g is flat as an A_g-module. After replacing A by A_g we may assume that M is flat as an A-module.

In this case, by Algebra, Lemma 7.62.4 we see that $\mathfrak{q}$ is also an associated prime of M. Hence we obtain an injective B-module map $B/\mathfrak{q} \to M$. Let Q be the cokernel so that we obtain a short exact sequence

$$0 \to B/\mathfrak{q} \to M \to Q \to 0$$

of finite B-modules. After applying generic flatness Algebra, Lemma 7.109.3 once more, this time to the B-module Q, we may assume that Q is a flat A-module. In particular we may assume the short exact sequence above is universally injective, see Algebra, Lemma 7.35.11. In this situation $(B/\mathfrak{q}) \otimes_A \kappa(\mathfrak{p}') \subset M \otimes_A \kappa(\mathfrak{p}')$ for any prime $\mathfrak{p}'$ of A. The lemma follows as a minimal prime $\mathfrak{q}'$ of the support of $(B/\mathfrak{q}) \otimes_A \kappa(\mathfrak{p}')$ is an associated prime of $(B/\mathfrak{q}) \otimes_A \kappa(\mathfrak{p}')$ by Divisors, Lemma 26.2.8. □

Lemma 33.18.2. *Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $U \subset X$ be an open subscheme. Assume*

1. *f is of finite type,*
2. *$\mathcal{F}$ is of finite type,*
3. *Y is irreducible with generic point η, and*
4. *$Ass_{X_\eta}(\mathcal{F}_\eta)$ is not contained in U_η.*

Then there exists a nonempty open subscheme $V \subset Y$ such that for all $y \in V$ the set $Ass_{X_y}(\mathcal{F}_y)$ is not contained in U_y.

Proof. Let $\xi \in \text{Ass}_{X_\eta}(\mathcal{F}_\eta)$ be a point which is not contained in U_η. Set $Z = \overline{\{\xi\}}$. By assumption $U \cap Z$ is not dense in the irreducible scheme Z_η. Hence by Lemma 33.17.3 after replacing Y by a nonempty open we may assume that $U_y \cap Z_y$ is nowhere dense in Z_y. On the other hand, by Lemma 33.18.1 there exists a nonempty open $V \subset Z$ such that every generic point of V_y is an associated point of $\mathcal{F}_y$. By Lemma 33.17.2 the set $f(V)$ contains a nonempty open subset of Y and we win. □

Lemma 33.18.3. *Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $U \subset X$ be an open subscheme. Assume*

1. *f is of finite type,*
2. *$\mathcal{F}$ is of finite type,*
3. *Y is irreducible with generic point η, and*
4. *$Ass_{X_\eta}(\mathcal{F}_\eta) \subset U_\eta$.*

Then there exists a nonempty open subscheme $V \subset Y$ such that for all $y \in V$ we have $Ass_{X_y}(\mathcal{F}_y) \subset U_y$.

Proof. (This proof is the same as the proof of Lemma 33.17.4. We urge the reader to read that proof first.) Since the statement is about fibres it is clear that we may replace Y by its reduction. Hence we may assume that Y is integral, see Properties, Lemma 23.3.4. We may also assume that $Y = Spec(A)$ is affine. Then A is a domain with fraction field K.

As f is of finite type we see that X is quasi-compact. Write $X = X_1 \cup \ldots \cup X_n$ for some affine opens X_i and set $\mathcal{F}_i = \mathcal{F}|_{X_i}$. By assumption the generic fibre of $U_i = X_i \cap U$ contains $\text{Ass}_{X_{i,\eta}}(\mathcal{F}_{i,\eta})$. Thus it suffices to prove the result for the triples $(X_i, \mathcal{F}_i, U_i)$, in other words we may assume that X is affine.

Write $X = Spec(B)$. Let N be a finite B-module such that $\mathcal{F} = \widetilde{N}$. Note that B_K is Noetherian as it is a finite type K-algebra. Hence U_η is quasi-compact. Thus we can find finitely many $g_1, \ldots, g_m \in B$ such that $D(g_j) \subset U$ and such that $U_\eta = D(g_1)_\eta \cup \ldots \cup D(g_m)_\eta$.

Since $\text{Ass}_{X_\eta}(\mathcal{F}_\eta) \subset U_\eta$ we see that $N_K \to \bigoplus_j (N_K)_{g_j}$ is injective. By Algebra, Lemma 7.20.4 this is equivalent to the injectivity of $N_K \to \bigoplus_{j=1,\ldots,m} N_K$, $n \mapsto (g_1 n, \ldots, g_m n)$. Let I and M be the kernel and cokernel of this map over A, i.e., such that we have an exact sequence

$$0 \to I \to N \xrightarrow{(g_1,\ldots,g_m)} \bigoplus\nolimits_{j=1,\ldots,m} N \to M \to 0$$

After replacing A by A_h for some nonzero h we may assume that B is a flat, finitely presented A-algebra and that both M and N are flat over A, see Algebra, Lemma 7.109.3. The flatness of N over A implies that N is torsion free as an A-module, see More on Algebra, Lemma 12.17.3. Hence $N \subset N_K$. By construction $I_K = 0$ which implies that $I = 0$ (as $I \subset N \subset N_K$ is a subset of I_K). Hence now we have a short exact sequence

$$0 \to N \xrightarrow{(g_1,\ldots,g_m)} \bigoplus\nolimits_{j=1,\ldots,m} N \to M \to 0$$

with M flat over A. Hence for every homomorphism $A \to \kappa$ where κ is a field, we obtain a short exact sequence

$$0 \to N \otimes_A \kappa \xrightarrow{(g_1\otimes 1,\ldots,g_m\otimes 1)} \bigoplus\nolimits_{j=1,\ldots,m} N \otimes_A \kappa \to M \otimes_A \kappa \to 0$$

see Algebra, Lemma 7.35.11. Reversing the arguments above this means that $\bigcup D(g_j \otimes 1)$ contains $\text{Ass}_{B\otimes_A\kappa}(N \otimes_A \kappa)$. As $\bigcup D(g_j \otimes 1) = \bigcup D(g_j)_\kappa \subset U_\kappa$ we obtain that U_κ contains $\text{Ass}_{X\otimes\kappa}(\mathcal{F} \otimes \kappa)$ which is what we wanted to prove. $\square$

Lemma 33.18.4. *Let $f : X \to S$ be a morphism which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $U \subset X$ be an open subscheme. Let $g : S' \to S$ be a morphism of schemes, let $f' : X' = X_{S'} \to S'$ be the base change of f, let $g' : X' \to X$ be the projection, set $\mathcal{F}' = (g')^*\mathcal{F}$, and set $U' = (g')^{-1}(U)$. Finally, let $s' \in S'$ with image $s = g(s')$. In this case*

$$Ass_{X_s}(\mathcal{F}_s) \subset U_s \Leftrightarrow Ass_{X'_{s'}}(\mathcal{F}'_{s'}) \subset U'_{s'}.$$

Proof. This follows immediately from Divisors, Lemma 26.7.2. See also Divisors, Remark 26.7.3. $\square$

Lemma 33.18.5. *Let $f : X \to Y$ be a morphism of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite presentation. Let $U \subset X$ be an open subscheme such that $U \to Y$ is quasi-compact. Then the set*

$$E = \{y \in Y \mid Ass_{X_y}(\mathcal{F}_y) \subset U_y\}$$

is locally constructible in Y.

Proof. Let $y \in Y$. We have to show that there exists an open neighbourhood V of y in Y such that $E \cap V$ is constructible in V. Thus we may assume that Y is affine. Write $Y = Spec(A)$ and $A = colim\, A_i$ as a directed limit of finite type $\mathbf{Z}$-algebras. By Limits, Lemma 27.6.1 we can find an i and a morphism $f_i : X_i \to Spec(A_i)$ of finite presentation whose base change to Y recovers f. After possibly increasing i we may assume there exists a quasi-coherent $\mathcal{O}_{X_i}$-module $\mathcal{F}_i$ of finite presentation whose pullback to X is isomorphic to $\mathcal{F}$, see Limits, Lemma 27.6.8. After possibly increasing i one more time we may assume there exists an open subscheme $U_i \subset X_i$ whose inverse image in X is U, see Limits, Lemma 27.3.5. By Lemma 33.18.4 it suffices to prove the lemma for f_i. Thus we reduce to the case where Y is the spectrum of a Noetherian ring.

We will use the criterion of Topology, Lemma 5.11.3 to prove that E is constructible in case Y is a Noetherian scheme. To see this let $Z \subset Y$ be an irreducible closed subscheme. We have to show that $E \cap Z$ either contains a nonempty open subset or is not dense in Z. This follows from Lemmas 33.18.2 and 33.18.3 applied to the base change $(X, \mathcal{F}, U) \times_Y Z$ over Z. ☐

33.19. Reduced fibres

Lemma 33.19.1. *Let $f : X \to Y$ be a morphism of schemes. Assume Y irreducible with generic point η and f of finite type. If X_η is nonreduced, then there exists a nonempty open $V \subset Y$ such that for all $y \in V$ the fibre X_y is nonreduced.*

Proof. Let $Y' \subset Y$ be the reduction of Y. Let $X' \to Y'$ be the base change of f. Note that $Y' \to Y$ induces a bijection on points and that $X' \to X$ identifies fibres. Hence we may assume that Y' is reduced, i.e., integral, see Properties, Lemma 23.3.4. We may also replace Y by an affine open. Hence we may assume that $Y = Spec(A)$ with A a domain. Denote $K = f.f.(A)$ the fraction field of A. Pick an affine open $Spec(B) = U \subset X$ and a section $h_\eta \in \Gamma(U_\eta, \mathcal{O}_{U_\eta}) = B_K$ which is nonzero and nilpotent. After shrinking Y we may assume that h comes from $h \in \Gamma(U, \mathcal{O}_U) = B$. After shrinking Y a bit more we may assume that h is nilpotent. Let $I = \{b \in B \mid hb = 0\}$ be the annihilator of h. Then $C = B/I$ is a finite type A-algebra whose generic fiber $(B/I)_K$ is nonzero (as $h_\eta \neq 0$). We apply generic flatness to $A \to C$ and $A \to B/hB$, see Algebra, Lemma 7.109.3, and we obtain a $g \in A$, $g \neq 0$ such that C_g is free as an A_g-module and $(B/hB)_g$ is flat as an A_g-module. Replace Y by $D(g) \subset Y$. Now we have the short exact sequence

$$0 \to C \to B \to B/hB \to 0.$$

with B/hB flat over A and with C nonzero free as an A-module. It follows that for any homomorphism $A \to \kappa$ to a field the ring $C \otimes_A \kappa$ is nonzero and the sequence

$$0 \to C \otimes_A \kappa \to B \otimes_A \kappa \to B/hB \otimes_A \kappa \to 0$$

is exact, see Algebra, Lemma 7.35.11. Note that $B/hB \otimes_A \kappa = (B \otimes_A \kappa)/h(B \otimes_A \kappa)$ by right exactness of tensor product. Thus we conclude that multiplication by h is not zero on $B \otimes_A \kappa$. This clearly means that for any point $y \in Y$ the element h restricts to a nonzero element of U_y, whence X_y is nonreduced. ☐

Lemma 33.19.2. *Let $f : X \to Y$ be a morphism of schemes. Let $g : Y' \to Y$ be any morphism, and denote $f' : X' \to Y'$ the base change of f. Then*

$$\{y' \in Y' \mid X'_{y'} \textit{ is geometrically reduced}\}$$
$$= g^{-1}(\{y \in Y \mid X_y \textit{ is geometrically reduced}\}).$$

Proof. This comes down to the statement that for $y' \in Y'$ with image $y \in Y$ the fibre $X'_{y'} = X_y \times_y y'$ is geometrically reduced over $\kappa(y')$ if and only if X_y is geometrically reduced over $\kappa(y)$. This follows from Varieties, Lemma 28.4.6. ☐

Lemma 33.19.3. *Let $f : X \to Y$ be a morphism of schemes. Assume Y irreducible with generic point η and f of finite type. If X_η is not geometrically reduced, then there exists a nonempty open $V \subset Y$ such that for all $y \in V$ the fibre X_y is not geometrically reduced.*

Proof. Apply Lemma 33.17.7 to get

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X_V \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & V \arrow[r] & Y \end{tikzcd}$$

with all the properties mentioned in that lemma. Let η' be the generic point of Y'. Consider the morphism $X' \to X_{Y'}$ (which is the reduction morphism) and the resulting morphism of generic fibres $X'_{\eta'} \to X_{\eta'}$. Since $X'_{\eta'}$ is geometrically reduced, and X_η is not this cannot be an isomorphism, see Varieties, Lemma 28.4.6. Hence $X_{\eta'}$ is nonreduced. Hence by Lemma 33.19.1 the fibres of $X_{Y'} \to Y'$ are nonreduced at all points $y' \in V'$ of a nonempty open $V' \subset Y'$. Since $g : Y' \to V$ is a homeomorphism Lemma 33.19.2 proves that $g(V')$ is the open we are looking for. $\square$

Lemma 33.19.4. *Let $f : X \to Y$ be a morphism of schemes. Assume*

(1) *Y is irreducible with generic point η,*
(2) *X_η is geometrically reduced, and*
(3) *f is of finite type.*

Then there exists a nonempty open subscheme $V \subset Y$ such that $X_V \to V$ has geometrically reduced fibres.

Proof. Let $Y' \subset Y$ be the reduction of Y. Let $X' \to Y'$ be the base change of f. Note that $Y' \to Y$ induces a bijection on points and that $X' \to X$ identifies fibres. Hence we may assume that Y' is reduced, i.e., integral, see Properties, Lemma 23.3.4. We may also replace Y by an affine open. Hence we may assume that $Y = Spec(A)$ with A a domain. Denote $K = f.f.(A)$ the fraction field of A. After shrinking Y a bit we may also assume that $X \to Y$ is flat and of finite presentation, see Morphisms, Proposition 24.26.1.

As X_η is geometrically reduced there exists an open dense subset $V \subset X_\eta$ such that $V \to Spec(K)$ is smooth, see Varieties, Lemma 28.15.7. Let $U \subset X$ be the set of points where f is smooth. By Morphisms, Lemma 24.33.15 we see that $V \subset U_\eta$. Thus the generic fibre of U is dense in the generic fibre of X. Since X_η is reduced, it follows that U_η is scheme theoretically dense in X_η, see Morphisms, Lemma 24.5.8. We note that as $U \to Y$ is smooth all the fibres of $U \to Y$ are geometrically reduced. Thus it suffices to show that, after shrinking Y, for all $y \in Y$ the scheme U_y is scheme theoretically dense in X_y, see Morphisms, Lemma 24.5.9. This follows from Lemma 33.17.4. $\square$

Lemma 33.19.5. *Let $f : X \to Y$ be a morphism of finite presentation. Then the set*

$$E = \{y \in Y \mid X_y \text{ is geometrically reduced}\}$$

is locally constructible in Y.

Proof. Let $y \in Y$. We have to show that there exists an open neighbourhood V of y in Y such that $E \cap V$ is constructible in V. Thus we may assume that Y is affine. Write $Y = Spec(A)$ and $A = colim\, A_i$ as a directed limit of finite type $\mathbf{Z}$-algebras. By Limits, Lemma 27.6.1 we can find an i and a morphism $f_i : X_i \to Spec(A_i)$ of finite presentation whose base change to Y recovers f. By Lemma 33.19.2 it suffices to prove the lemma for f_i. Thus we reduce to the case where Y is the spectrum of a Noetherian ring.

We will use the criterion of Topology, Lemma 5.11.3 to prove that E is constructible in case Y is a Noetherian scheme. To see this let $Z \subset Y$ be an irreducible closed subscheme.

We have to show that $E \cap Z$ either contains a nonempty open subset or is not dense in Z. If X_ξ is geometrically reduced, then Lemma 33.19.4 (applied to the morphism $X_Z \to Z$) implies that all fibres X_y are geometrically reduced for a nonempty open $V \subset Z$. If X_ξ is not geometrically reduced, then Lemma 33.19.3 (applied to the morphism $X_Z \to Z$) implies that all fibres X_y are geometrically reduced for a nonempty open $V \subset Z$. Thus we win. □

33.20. Irreducible components of fibres

Lemma 33.20.1. *Let $f : X \to Y$ be a morphism of schemes. Assume Y irreducible with generic point η and f of finite type. If X_η has n irreducible components, then there exists a nonempty open $V \subset Y$ such that for all $y \in V$ the fibre X_y has at least n irreducible components.*

Proof. As the question is purely topological we may replace X and Y by their reductions. In particular this implies that Y is integral, see Properties, Lemma 23.3.4. Let $X_\eta = X_{1,\eta} \cup \ldots \cup X_{n,\eta}$ be the decomposition of X_η into irreducible components. Let $X_i \subset X$ be the reduced closed subscheme whose generic fibre is $X_{i,\eta}$. Note that $Z_{i,j} = X_i \cap X_j$ is a closed subset of X_i whose generic fibre $Z_{i,j,\eta}$ is nowhere dense in $X_{i,\eta}$. Hence after shrinking Y we may assume that $Z_{i,j,y}$ is nowhere dense in $X_{i,y}$ for every $y \in Y$, see Lemma 33.17.3. After shrinking Y some more we may assume that $X_y = \bigcup X_{i,y}$ for $y \in Y$, see Lemma 33.17.5. Moreover, after shrinking Y we may assume that each $X_i \to Y$ is flat and of finite presentation, see Morphisms, Proposition 24.26.1. The morphisms $X_i \to Y$ are open, see Morphisms, Lemma 24.24.9. Thus there exists an open neighbourhood V of η which is contained in $f(X_i)$ for each i. For each $y \in V$ the schemes $X_{i,y}$ are nonempty closed subsets of X_y, we have $X_y = \bigcup X_{i,y}$ and the intersections $Z_{i,j,y} = X_{i,y} \cap X_{j,y}$ are not dense in $X_{i,y}$. Clearly this implies that X_y has at least n irreducible components. □

Lemma 33.20.2. *Let $f : X \to Y$ be a morphism of schemes. Let $g : Y' \to Y$ be any morphism, and denote $f' : X' \to Y'$ the base change of f. Then*

$$\begin{aligned} & \{y' \in Y' \mid X'_{y'} \text{ is geometrically irreducible}\} \\ = {} & g^{-1}(\{y \in Y \mid X_y \text{ is geometrically irreducible}\}). \end{aligned}$$

Proof. This comes down to the statement that for $y' \in Y'$ with image $y \in Y$ the fibre $X'_{y'} = X_y \times_y y'$ is geometrically irreducible over $\kappa(y')$ if and only if X_y is geometrically irreducible over $\kappa(y)$. This follows from Varieties, Lemma 28.6.2. □

Lemma 33.20.3. *Let $f : X \to Y$ be a morphism of schemes. Let*

$$n_{X/Y} : Y \to \{0, 1, 2, 3, \ldots, \infty\}$$

be the function which associates to $y \in Y$ the number of irreducible components of $(X_y)_K$ where K is a separably closed extension of $\kappa(y)$. This is well defined and if $g : Y' \to Y$ is a morphism then

$$n_{X'/Y'} = n_{X/Y} \circ g$$

where $X' \to Y'$ is the base change of f.

Proof. Suppose that $y' \in Y'$ has image $y \in Y$. Suppose $K \supset \kappa(y)$ and $K' \supset \kappa(y')$ are separably closed extensions. Then we may choose a commutative diagram

$$\begin{tikzcd} K \arrow[r] & K'' & K' \arrow[l] \\ \kappa(y) \arrow[u] \arrow[rr] & & \kappa(y') \arrow[u] \end{tikzcd}$$

of fields. The result follows as the morphisms of schemes

$$(X'_{y'})_{K'} \longleftarrow (X'_{y'})_{K''} = (X_y)_{K''} \longrightarrow (X_y)_K$$

induce bijections between irreducible components, see Varieties, Lemma 28.6.7. □

Lemma 33.20.4. *Let A be a domain with fraction field K. Let $P \in A[x_1, \ldots, x_n]$. Denote $\overline{K}$ the algebraic closure of K. Assume P is irreducible in $\overline{K}[x_1, \ldots, x_n]$. Then there exists a $f \in A$ such that $P^\varphi \in \kappa[x_1, \ldots, x_n]$ is irreducible for all homomorphisms $\varphi : A_f \to \kappa$ into fields.*

Proof. There exists an automorphism Ψ of $A[x_1, \ldots, x_n]$ over A such that $\Psi(P) = ax_n^d +$ lower order terms in x_n with $a \neq 0$, see Algebra, Lemma 7.106.2. We may replace P by $\Psi(P)$ and we may replace A by A_a. Thus we may assume that P is monic in x_n of degree $d > 0$. For $i = 1, \ldots, n-1$ let d_i be the degree of P in x_i. Note that this implies that P^φ is monic of degree d in x_n and has degree $\leq d_i$ in x_i for every homomorphism $\varphi : A \to \kappa$ where κ is a field. Thus if P^φ is reducible, then we can write

$$P^\varphi = Q_1 Q_2$$

with Q_1, Q_2 monic of degree $e_1, e_2 \geq 0$ in x_n with $e_1 + e_2 = d$ and having degree $\leq d_i$ in x_i for $i = 1, \ldots, n-1$. In other words we can write

$$Q_j = x_n^{e_j} + \sum\nolimits_{0 \leq l < e_j} \left(\sum\nolimits_{L \in \mathcal{L}} a_{j,l,L} x^L \right) x_n^l \quad (33.20.4.1)$$

where the sum is over the set $\mathcal{L}$ of multi-indices L of the form $L = (l_1, \ldots, l_{n-1})$ with $0 \leq l_i \leq d_i$. For any $e_1, e_2 \geq 0$ with $e_1 + e_2 = d$ we consider the A-algebra

$$B_{e_1,e_2} = A[\{a_{1,l,L}\}_{0 \leq l < e_1, L \in \mathcal{L}}, \{a_{2,l,L}\}_{0 \leq l < e_2, L \in \mathcal{L}}]/(\text{relations})$$

where the (relations) is the ideal generated by the coefficients of the polynomial

$$P - Q_1 Q_2 \in A[\{a_{1,l,L}\}_{0 \leq l < e_1, L \in \mathcal{L}}, \{a_{2,l,L}\}_{0 \leq l < e_2, L \in \mathcal{L}}][x_1, \ldots, x_n]$$

with Q_1 and Q_2 defined as in (33.20.4.1). OK, and the assumption that P is irreducible over $\overline{K}$ implies that there does not exist any A-algebra homomorphism $B_{e_1,e_2} \to \overline{K}$. By the Hilbert Nullstellensatz, see Algebra, Theorem 7.30.1 this means that $B_{e_1,e_2} \otimes_A K = 0$. As B_{e_1,e_2} is a finitely generated A-algebra this signifies that we can find an $f_{e_1,e_2} \in A$ such that $(B_{e_1,e_2})_{f_{e_1,e_2}} = 0$. By construction this means that if $\varphi : A_{f_{e_1,e_2}} \to \kappa$ is a homomorphism to a field, then P^φ does not have a factorization $P^\varphi = Q_1 Q_2$ with Q_1 of degree e_1 in x_n and Q_2 of degree e_2 in x_n. Thus taking $f = \prod_{e1,e_2 \geq 0, e_1+e_2=d} f_{e_1,e_2}$ we win. □

Lemma 33.20.5. *Let $f : X \to Y$ be a morphism of schemes. Assume*

1. *Y is irreducible with generic point η,*
2. *X_η is geometrically irreducible, and*
3. *f is of finite type.*

Then there exists a nonempty open subscheme $V \subset Y$ such that $X_V \to V$ has geometrically irreducible fibres.

First proof of Lemma 33.20.5. We give two proofs of the lemma. These are essentially equivalent; the second is more self contained but a bit longer. Choose a diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X_V \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & V \arrow[r] & Y \end{tikzcd}$$

as in Lemma 33.17.7. Note that the generic fibre of f' is the reduction of the geometric fibre of f (see Lemma 33.17.6) and hence is geometrically irreducible. Suppose that the lemma holds for the morphism f'. Then after shrinking V all the fibres of f' are geometrically irreducible. As $X' = (Y' \times_V X_V)_{red}$ this implies that all the fibres of $Y' \times_V X_V$ are geometrically irreducible. Hence by Lemma 33.20.2 all the fibres of $X_V \to V$ are geometrically irreducible and we win. In this way we see that we may assume that the generic fibre is geometrically reduced as well as geometrically irreducible and we may assume $Y = Spec(A)$ with A a domain.

Let $x \in X_\eta$ be the generic point. As X_η is geometrically irreducible and reduced we see that $L = \kappa(x)$ is a finitely generated extension of $K = \kappa(\eta) = f.f.(A)$ which is geometrically reduced and geometrically irreducible, see Varieties, Lemmas 28.4.2 and 28.6.6. In particular the field extension $K \subset L$ is separable, see Algebra, Lemma 7.41.1. Hence we can find $x_1, \ldots, x_{r+1} \in L$ which generate L over K and such that $x_1, \ldots, x_r$ is a transcendence basis for L over K, see Algebra, Lemma 7.39.3. Let $P \in K(x_1, \ldots, x_r)[T]$ be the minimal polynomial for x_{r+1}. Clearing denominators we may assume that P has coefficients in $A[x_1, \ldots, x_r]$. Note that as L is geometrically reduced and geometrically irreducible over K, the polynomial P is irreducible in $\overline{K}[x_1, \ldots, x_r, T]$ where $\overline{K}$ is the algebraic closure of K. Denote

$$B' = A[x_1, \ldots, x_{r+1}]/(P(x_{r+1}))$$

and set $X' = Spec(B')$. By construction the fraction field of B' is isomorphic to $L = \kappa(x)$ as K-extensions. Hence there exists an open $U \subset X$, and open $U' \subset X'$ and a Y-isomorphism $U \to U'$, see Lemma 33.17.9. Here is a diagram:

$$\begin{tikzcd} X \arrow[dr] & U \arrow[l] \arrow[r, equal] \arrow[d] & U' \arrow[r] \arrow[d] & X' \arrow[r, equal] \arrow[dl] & Spec(B') \\ & Y \arrow[r, equal] & Y & & \end{tikzcd}$$

Note that $U_\eta \subset X_\eta$ and $U'_\eta \subset X'_\eta$ are dense opens. Thus after shrinking Y by applying Lemma 33.17.3 we obtain that U_y is dense in X_y and U'_y is dense in X'_y for all $y \in Y$. Thus it suffices to prove the lemma for $X' \to Y$ which is the content of Lemma 33.20.4. $\square$

Second proof of Lemma 33.20.5. Let $Y' \subset Y$ be the reduction of Y. Let $X' \to X$ be the reduction of X. Note that $X' \to X \to Y$ factors through Y', see Schemes, Lemma 21.12.6. As $Y' \to Y$ and $X' \to X$ are universal homeomorphisms by Morphisms, Lemma 24.43.4 we see that it suffices to prove the lemma for $X' \to Y'$. Thus we may assume that X and Y are reduced. In particular Y is integral, see Properties, Lemma 23.3.4. Thus by Morphisms, Proposition 24.26.1 there exists a nonempty affine open $V \subset Y$ such that $X_V \to V$ is flat and of finite presentation. After replacing Y by V we may assume, in addition to (1), (2), (3) that Y is integral affine, X is reduced, and f is flat and of finite presentation. In particular f is universally open, see Morphisms, Lemma 24.24.9.

Pick a nonempty affine open $U \subset X$. Then $U \to Y$ is flat and of finite presentation with geometrically irreducible generic fibre. The complement $X_\eta \setminus U_\eta$ is nowhere dense. Thus after shrinking Y we may assume $U_y \subset X_y$ is open dense for all $y \in Y$, see Lemma 33.17.3. Thus we may replace X by U and we reduce to the case where Y is integral affine and X is reduced affine, flat and of finite presentation over Y with geometrically irreducible generic fibre X_η.

Write $X = Spec(B)$ and $Y = Spec(A)$. Then A is a domain, B is reduced, $A \to B$ is flat of finite presentation, and B_K is geometrically irreducible over $K = f.f.(A)$. In particular we see that B_K is a domain. Let $L = f.f.(B_K)$ be its fraction field. Note that L is a finitely generated field extension of K as B is an A-algebra of finite presentation. Let $K \subset K'$ be a finite purely inseparable extension such that $(L \otimes_K K')_{red}$ is a separably generated field extension, see Algebra, Lemma 7.42.3. Choose $x_1, \ldots, x_n \in K'$ which generate the field extension K' over K, and such that $x_i^{q_i} \in A$ for some prime power q_i (proof existence x_i omitted). Let A' be the A-subalgebra of K' generated by $x_1, \ldots, x_n$. Then A' is a finite A-subalgebra $A' \subset K'$ whose fraction field is K'. Note that $Spec(A') \to Spec(A)$ is a universal homeomorphism, see Algebra, Lemma 7.43.2. Hence it suffices to prove the result after base changing to $Spec(A')$. We are going to replace A by A' and B by $(B \otimes_A A')_{red}$ to arrive at the situation where L is a separably generated field extension of K. Of course it may happen that $(B \otimes_A A')_{red}$ is no longer flat, or of finite presentation over A', but this can be remedied by replacing A' by A'_f for a suitable $f \in A'$, see Algebra, Lemma 7.109.3.

At this point we know that A is a domain, B is reduced, $A \to B$ is flat and of finite presentation, B_K is a domain, and $L = f.f.(B_K)$ is a separably generated field extension of $K = f.f.(A)$. By Algebra, Lemma 7.39.3 we may write $L = K(x_1, \ldots, x_{r+1})$ where $x_1, \ldots, x_r$ are algebraically independent over K, and x_{r+1} is separable over $K(x_1, \ldots, x_r)$. After clearing denominators we may assume that the minimal polynomial $P \in K(x_1, \ldots, x_r)[T]$ of x_{r+1} over $K(x_1, \ldots, x_r)$ has coefficients in $A[x_1, \ldots, x_r]$. Note that since L/K is separable and since L is geometrically irreducible over K, the polynomial P is irreducible over the algebraic closure $\overline{K}$ of K. Denote

$$B' = A[x_1, \ldots, x_{r+1}]/(P(x_{r+1})).$$

By construction the fraction fields of B and B' are isomorphic as K-extensions. Hence there exists an isomorphism of A-algebras $B_h \cong B'_{h'}$ for suitable $h \in B$ and $h' \in B'$, see Lemma 33.17.9. In other words X and $X' = Spec(B')$ have a common affine open U. Here is a diagram:

$$\begin{tikzcd}
X = Spec(B) \arrow[dr] & U \arrow[l] \arrow[r] \arrow[d] & Spec(B') = X' \arrow[dl] \\
 & Y = Spec(A) &
\end{tikzcd}$$

After shrinking Y once more (by applying Lemma 33.17.3 to $Z = X \setminus U$ in X and $Z' = X' \setminus U$ in X') we see that U_y is dense in X_y and U_y is dense in X'_y for all $y \in Y$. Thus it suffices to prove the lemma for $X' \to Y$ which is the content of Lemma 33.20.4. □

Lemma 33.20.6. *Let $f : X \to Y$ be a morphism of schemes. Let $n_{X/Y}$ be the function on Y counting the numbers of geometrically irreducible components of fibres of f introduced in*

Lemma 33.20.3. Assume f of finite type. Let $y \in Y$ be a point. Then there exists a nonempty open $V \subset \overline{\{y\}}$ such that $n_{X/Y}|_V$ is constant.

Proof. Let Z be the reduced induced scheme structure on $\overline{\{y\}}$. Let $f_Z : X_Z \to Z$ be the base change of f. Clearly it suffices to prove the lemma for f_Z and the generic point of Z. Hence we may assume that Y is an integral scheme, see Properties, Lemma 23.3.4. Our goal in this case is to produce a nonempty open $V \subset Y$ such that $n_{X/Y}|_V$ is constant.

We apply Lemma 33.17.8 to $f : X \to Y$ and we get $g : Y' \to V \subset Y$. As $g : Y' \to V$ is surjective finite étale, in particular open (see Morphisms, Lemma 24.35.13), it suffices to prove that there exists an open $V' \subset Y'$ such that $n_{X'/Y'}|_{V'}$ is constant, see Lemma 33.20.3. Thus we see that we may assume that all irreducible components of the generic fibre X_η are geometrically irreducible over $\kappa(\eta)$.

At this point suppose that $X_\eta = X_{1,\eta} \bigcup \ldots \bigcup X_{n,\eta}$ is the decomposition of the generic fibre into (geometrically) irreducible components. In particular $n_{X/Y}(\eta) = n$. Let X_i be the closure of $X_{i,\eta}$ in X. After shrinking Y we may assume that $X = \bigcup X_i$, see Lemma 33.17.5. After shrinking Y some more we see that each fibre of f has at least n irreducible components, see Lemma 33.20.1. Hence $n_{X/Y}(y) \geq n$ for all $y \in Y$. After shrinking Y some more we obtain that $X_{i,y}$ is geometrically irreducible for each i and all $y \in Y$, see Lemma 33.20.5. Since $X_y = \bigcup X_{i,y}$ this shows that $n_{X/Y}(y) \leq n$ and finishes the proof. □

Lemma 33.20.7. *Let $f : X \to Y$ be a morphism of schemes. Let $n_{X/Y}$ be the function on Y counting the numbers of geometrically irreducible components of fibres of f introduced in Lemma 33.20.3. Assume f of finite presentation. Then the level sets*

$$E_n = \{y \in Y \mid n_{X/Y}(y) = n\}$$

of $n_{X/Y}$ are locally constructible in Y.

Proof. Fix n. Let $y \in Y$. We have to show that there exists an open neighbourhood V of y in Y such that $E_n \cap V$ is constructible in V. Thus we may assume that Y is affine. Write $Y = Spec(A)$ and $A = colim\, A_i$ as a directed limit of finite type $\mathbf{Z}$-algebras. By Limits, Lemma 27.6.1 we can find an i and a morphism $f_i : X_i \to Spec(A_i)$ of finite presentation whose base change to Y recovers f. By Lemma 33.20.3 it suffices to prove the lemma for f_i. Thus we reduce to the case where Y is the spectrum of a Noetherian ring.

We will use the criterion of Topology, Lemma 5.11.3 to prove that E_n is constructible in case Y is a Noetherian scheme. To see this let $Z \subset Y$ be an irreducible closed subscheme. We have to show that $E_n \cap Z$ either contains a nonempty open subset or is not dense in Z. Let $\xi \in Z$ be the generic point. Then Lemma 33.20.6 shows that $n_{X/Y}$ is constant in a neighbourhood of ξ in Z. This clearly implies what we want. □

33.21. Connected components of fibres

Lemma 33.21.1. *Let $f : X \to Y$ be a morphism of schemes. Assume Y irreducible with generic point η and f of finite type. If X_η has n connected components, then there exists a nonempty open $V \subset Y$ such that for all $y \in V$ the fibre X_y has at least n connected components.*

Proof. As the question is purely topological we may replace X and Y by their reductions. In particular this implies that Y is integral, see Properties, Lemma 23.3.4. Let $X_\eta = X_{1,\eta} \cup \ldots \cup X_{n,\eta}$ be the decomposition of X_η into connected components. Let $X_i \subset X$ be the reduced closed subscheme whose generic fibre is $X_{i,\eta}$. Note that $Z_{i,j} = X_i \cap X_j$ is a closed subset of

X whose generic fibre $Z_{i,j,\eta}$ is empty. Hence after shrinking Y we may assume that $Z_{i,j} = \emptyset$, see Lemma 33.17.1. After shrinking Y some more we may assume that $X_y = \bigcup X_{i,y}$ for $y \in Y$, see Lemma 33.17.5. Moreover, after shrinking Y we may assume that each $X_i \to Y$ is flat and of finite presentation, see Morphisms, Proposition 24.26.1. The morphisms $X_i \to Y$ are open, see Morphisms, Lemma 24.24.9. Thus there exists an open neighbourhood V of η which is contained in $f(X_i)$ for each i. For each $y \in V$ the schemes $X_{i,y}$ are nonempty closed subsets of X_y, we have $X_y = \bigcup X_{i,y}$ and the intersections $Z_{i,j,y} = X_{i,y} \cap X_{j,y}$ are empty! Clearly this implies that X_y has at least n connected components. □

Lemma 33.21.2. *Let $f : X \to Y$ be a morphism of schemes. Let $g : Y' \to Y$ be any morphism, and denote $f' : X' \to Y'$ the base change of f. Then*

$$\begin{aligned} &\{y' \in Y' \mid X'_{y'} \text{ is geometrically connected}\} \\ &= g^{-1}(\{y \in Y \mid X_y \text{ is geometrically connected}\}). \end{aligned}$$

Proof. This comes down to the statement that for $y' \in Y'$ with image $y \in Y$ the fibre $X'_{y'} = X_y \times_y y'$ is geometrically connected over $\kappa(y')$ if and only if X_y is geometrically connected over $\kappa(y)$. This follows from Varieties, Lemma 28.5.3. □

Lemma 33.21.3. *Let $f : X \to Y$ be a morphism of schemes. Let*

$$n_{X/Y} : Y \to \{0, 1, 2, 3, \ldots, \infty\}$$

be the function which associates to $y \in Y$ the number of connected components of $(X_y)_K$ where K is a separably closed extension of $\kappa(y)$. This is well defined and if $g : Y' \to Y$ is a morphism then

$$n_{X'/Y'} = n_{X/Y} \circ g$$

where $X' \to Y'$ is the base change of f.

Proof. Suppose that $y' \in Y'$ has image $y \in Y$. Suppose $K \supset \kappa(y)$ and $K' \supset \kappa(y')$ are separably closed extensions. Then we may choose a commutative diagram

$$\begin{tikzcd} K \arrow[r] & K'' & K' \arrow[l] \\ \kappa(y) \arrow[u] \arrow[rr] & & \kappa(y') \arrow[u] \end{tikzcd}$$

of fields. The result follows as the morphisms of schemes

$$(X'_{y'})_{K'} \longleftarrow (X'_{y'})_{K''} = (X_y)_{K''} \longrightarrow (X_y)_K$$

induce bijections between connected components, see Varieties, Lemma 28.5.6. □

Lemma 33.21.4. *Let $f : X \to Y$ be a morphism of schemes. Assume*

(1) *Y is irreducible with generic point η,*
(2) *X_η is nonempty and geometrically connected, and*
(3) *f is of finite type.*

Then there exists a nonempty open subscheme $V \subset Y$ such that $X_V \to V$ has nonempty geometrically connected fibres.

Proof. Choose a diagram

$$\begin{tikzcd} X' \arrow[r, "g'"'] \arrow[d, "f'"'] & X_V \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & V \arrow[r] & Y \end{tikzcd}$$

as in Lemma 33.17.8. Note that the generic fibre of f' is nonempty and geometrically connected (for example by Lemma 33.21.3). Suppose that the lemma holds for the morphism f'. This means that there exists a nonempty open $W \subset Y'$ such that every fibre of $X' \to Y'$ over W is nonempty and geometrically connected. Then, as g is an open morphism by Morphisms, Lemma 24.35.13 all the fibres of f at point of the nonempty open $V = g(W)$ are nonempty and geometrically connected, see Lemma 33.21.3. In this way we see that we may assume that the irreducible components of the generic fibre X_η are geometrically irreducible.

Let Y' be the reduction of Y, and set $X' = Y' \times_Y X$. Then it suffices to prove the lemma for the morphism $X' \to Y'$ (for example by Lemma 33.21.3 once again). Since the generic fibre of $X' \to Y'$ is the same as the generic fibre of $X \to Y$ we see that we may assume that Y is irreducible and reduced (i.e., integral, see Properties, Lemma 23.3.4) and that the irreducible components of the generic fibre X_η are geometrically irreducible.

At this point suppose that $X_\eta = X_{1,\eta} \bigcup \ldots \bigcup X_{n,\eta}$ is the decomposition of the generic fibre into (geometrically) irreducible components. Let X_i be the closure of $X_{i,\eta}$ in X. After shrinking Y we may assume that $X = \bigcup X_i$, see Lemma 33.17.5. Let $Z_{i,j} = X_i \cap X_j$. Let

$$\{1, \ldots, n\} \times \{1, \ldots, n\} = I \coprod J$$

where $(i, j) \in I$ if $Z_{i,j,\eta} = \emptyset$ and $(i, j) \in J$ if $Z_{i,j,\eta} \neq \emptyset$. After shrinking Y we may assume that $Z_{i,j} = \emptyset$ for all $(i, j) \in I$, see Lemma 33.17.1. After shrinking Y we obtain that $X_{i,y}$ is geometrically irreducible for each i and all $y \in Y$, see Lemma 33.20.5. After shrinking Y some more we achieve the situation where each $Z_{i,j} \to Y$ is flat and of finite presentation for all $(i, j) \in J$, see Morphisms, Proposition 24.26.1. This means that $f(Z_{i,j}) \subset Y$ is open, see Morphisms, Lemma 24.24.9. We claim that

$$V = \bigcap\nolimits_{(i,j)\in J} f(Z_{i,j})$$

works, i.e., that X_y is geometrically connected for each $y \in V$. Namely, the fact that X_η is connected implies that the equivalence relation generated by the pairs in J has only one equivalence class. Now if $y \in V$ and $K \supset \kappa(y)$ is a separably closed extension, then the irreducible components of $(X_y)_K$ are the fibres $(X_{i,y})_K$. Moreover, we see by construction and $y \in V$ that $(X_{i,y})_K$ meets $(X_{j,y})_K$ if and only $(i, j) \in J$. Hence the remark on equivalence classes shows that $(X_y)_K$ is connected and we win. $\square$

Lemma 33.21.5. *Let $f : X \to Y$ be a morphism of schemes. Let $n_{X/Y}$ be the function on Y counting the numbers of geometrically connected components of fibres of f introduced in Lemma 33.21.3. Assume f of finite type. Let $y \in Y$ be a point. Then there exists a nonempty open $V \subset \overline{\{y\}}$ such that $n_{X/Y}|_V$ is constant.*

Proof. Let Z be the reduced induced scheme structure on $\overline{\{y\}}$. Let $f_Z : X_Z \to Z$ be the base change of f. Clearly it suffices to prove the lemma for f_Z and the generic point of Z. Hence we may assume that Y is an integral scheme, see Properties, Lemma 23.3.4. Our goal in this case is to produce a nonempty open $V \subset Y$ such that $n_{X/Y}|_V$ is constant.

We apply Lemma 33.17.8 to $f : X \to Y$ and we get $g : Y' \to V \subset Y$. As $g : Y' \to V$ is surjective finite étale, in particular open (see Morphisms, Lemma 24.35.13), it suffices to prove that there exists an open $V' \subset Y'$ such that $n_{X'/Y'}|_{V'}$ is constant, see Lemma 33.20.3. Thus we see that we may assume that all irreducible components of the generic fibre X_η are geometrically irreducible over $\kappa(\eta)$. By Varieties, Lemma 28.6.15 this implies that also the connected components of X_η are geometrically connected.

At this point suppose that $X_\eta = X_{1,\eta} \bigcup \ldots \bigcup X_{n,\eta}$ is the decomposition of the generic fibre into (geometrically) connected components. In particular $n_{X/Y}(\eta) = n$. Let X_i be the closure of $X_{i,\eta}$ in X. After shrinking Y we may assume that $X = \bigcup X_i$, see Lemma 33.17.5. After shrinking Y some more we see that each fibre of f has at least n connected components, see Lemma 33.21.1. Hence $n_{X/Y}(y) \geq n$ for all $y \in Y$. After shrinking Y some more we obtain that $X_{i,y}$ is geometrically connected for each i and all $y \in Y$, see Lemma 33.21.4. Since $X_y = \bigcup X_{i,y}$ this shows that $n_{X/Y}(y) \leq n$ and finishes the proof. □

Lemma 33.21.6. *Let $f : X \to Y$ be a morphism of schemes. Let $n_{X/Y}$ be the function on Y counting the numbers of geometrically connected components of fibres of f introduced in Lemma 33.21.3. Assume f of finite presentation. Then the level sets*

$$E_n = \{y \in Y \mid n_{X/Y}(y) = n\}$$

of $n_{X/Y}$ are locally constructible in Y.

Proof. Fix n. Let $y \in Y$. We have to show that there exists an open neighbourhood V of y in Y such that $E_n \cap V$ is constructible in V. Thus we may assume that Y is affine. Write $Y = Spec(A)$ and $A = colim\, A_i$ as a directed limit of finite type $\mathbf{Z}$-algebras. By Limits, Lemma 27.6.1 we can find an i and a morphism $f_i : X_i \to Spec(A_i)$ of finite presentation whose base change to Y recovers f. By Lemma 33.21.3 it suffices to prove the lemma for f_i. Thus we reduce to the case where Y is the spectrum of a Noetherian ring.

We will use the criterion of Topology, Lemma 5.11.3 to prove that E_n is constructible in case Y is a Noetherian scheme. To see this let $Z \subset Y$ be an irreducible closed subscheme. We have to show that $E_n \cap Z$ either contains a nonempty open subset or is not dense in Z. Let $\xi \in Z$ be the generic point. Then Lemma 33.21.5 shows that $n_{X/Y}$ is constant in a neighbourhood of ξ in Z. This clearly implies what we want. □

Lemma 33.21.7. *Let $f : X \to S$ be a morphism of schemes. Assume that*

1. *S is the spectrum of a discrete valuation ring,*
2. *f is flat,*
3. *X is connected,*
4. *the closed fibre X_s is reduced.*

Then the generic fibre X_η is connected.

Proof. Write $Y = Spec(R)$ and let $\pi \in R$ be a uniformizer. To get a contradiction assume that X_η is disconnected. This means there exists a nontrivial idempotent $e \in \Gamma(X_\eta, \mathcal{O}_{X_\eta})$. Let $U = Spec(A)$ be any affine open in X. Note that π is a nonzero divisor on A as A is flat over R, see More on Algebra, Lemma 12.17.3 for example. Then $e|_{U_\eta}$ corresponds to an element $e \in A[1/\pi]$. Let $z \in A$ be an element such that $e = z/\pi^n$ with $n \geq 0$ minimal. Note that $z^2 = \pi^n z$. This means that $z \bmod \pi A$ is nilpotent if $n > 0$. By assumption $A/\pi A$ is reduced, and hence minimality of n implies $n = 0$. Thus we conclude that $e \in A$! In other words $e \in \Gamma(X, \mathcal{O}_X)$. As X is connected it follows that e is a trivial idempotent which is a contradiction. □

33.22. Connected components meeting a section

The results in this section are in particular applicable to a group scheme $G \to S$ and its neutral section $e : S \to G$.

Situation 33.22.1. Here $f : X \to Y$ be a morphism of schemes, and $s : Y \to X$ is a section of f. For every $y \in Y$ we denote X^0_y the connected component of X_y containing $s(y)$. Finally, we set $X^0 = \bigcup_{y \in Y} X^0_y$.

Lemma 33.22.2. *Let $f : X \to Y$, $s : Y \to X$ be as in Situation 33.22.1. If $g : Y' \to Y$ is any morphism, consider the base change diagram*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"'] \\ Y' \arrow[r, "g"] \arrow[u, bend left, "s'"] & Y \arrow[u, bend right, "s"'] \end{tikzcd}$$

so that we obtain $(X')^0 \subset X'$. Then $(X')^0 = (g')^{-1}(X^0)$.

Proof. Let $y' \in Y'$ with image $y \in Y$. We may think of X^0_y as a closed subscheme of X_y, see for example Morphisms, Definition 24.25.2. As $s(y) \in X^0_y$ we conclude from Varieties, Lemma 28.5.14 that X^0_y is a geometrically connected scheme over $\kappa(y)$. Hence $X^0_y \times_y y' \to X'_{y'}$ is a connected closed subscheme which contains $s'(y')$. Thus $X^0_y \times_y y' \subset (X'_{y'})^0$. The other inclusion $X^0_y \times_y y' \supset (X'_{y'})^0$ is clear as the image of $(X'_{y'})^0$ in X_y is a connected subset of X_y which contains $s(y)$. □

Lemma 33.22.3. *Let $f : X \to Y$, $s : Y \to X$ be as in Situation 33.22.1. Assume f of finite type. Let $y \in Y$ be a point. Then there exists a nonempty open $V \subset \overline{\{y\}}$ such that the inverse image of X^0 in the base change X_V is open and closed in X_V.*

Proof. Let $Z \subset Y$ be the induced reduced closed subscheme structure on $\overline{\{y\}}$. Let $f_Z : X_Z \to Z$ and $s_Z : Z \to X_Z$ be the base changes of f and s. By Lemma 33.22.2 we have $(X_Z)^0 = (X^0)_Z$. Hence it suffices to prove the lemma for the morphism $X_Z \to Z$ and the point $x \in X_Z$ which maps to the generic point of Z. In other words we have reduced the problem to the case where Y is an integral scheme (see Properties, Lemma 23.3.4) with generic point η. Our goal is to show that after shrinking Y the subset X^0 becomes an open and closed subset of X.

Note that the scheme X_η is of finite type over a field, hence Noetherian. Thus its connected components are open as well as closed. Hence we may write $X_\eta = X^0_\eta \coprod T_\eta$ for some open and closed subset T_η of X_η. Next, let $T \subset X$ be the closure of T_η and let $X^{00} \subset X$ be the closure of X^0_η. Note that T_η, resp. X^0_η is the generic fibre of T, resp. X^{00}, see discussion preceding Lemma 33.17.5. Moreover, that lemma implies that after shrinking Y we may assume that $X = X^{00} \cup T$ (set theoretically). Note that $(T \cap X^{00})_\eta = T_\eta \cap X^0_\eta = \emptyset$. Hence after shrinking Y we may assume that $T \cap X^{00} = \emptyset$, see Lemma 33.17.1. In particular X^{00} is open in X. Note that X^0_η is connected and has a rational point, namely $s(\eta)$, hence it is geometrically connected, see Varieties, Lemma 28.5.14. Thus after shrinking Y we may assume that all fibres of $X^{00} \to Y$ are geometrically connected, see Lemma 33.21.4. At this point it follows that the fibres X^{00}_y are open, closed, and connected subsets of X_y containing $\sigma(y)$. It follows that $X^0 = X^{00}$ and we win. □

Lemma 33.22.4. *Let $f : X \to Y$, $s : Y \to X$ be as in Situation 33.22.1. If f is of finite presentation then X^0 is locally constructible in X.*

Proof. Let $x \in X$. We have to show that there exists an open neighbourhood U of x such that $X^0 \cap U$ is constructible in U. This reduces us to the case where Y is affine. Write $Y = Spec(A)$ and $A = colim\, A_i$ as a directed limit of finite type $\mathbf{Z}$-algebras. By Limits, Lemma 27.6.1 we can find an i and a morphism $f_i : X_i \to Spec(A_i)$ of finite presentation, endowed with a section $s_i : Spec(A_i) \to X_i$ whose base change to Y recovers f and the section s. By Lemma 33.22.2 it suffices to prove the lemma for f_i, s_i. Thus we reduce to the case where Y is the spectrum of a Noetherian ring.

Assume Y is a Noetherian affine scheme. Since f is of finite presentation, i.e., of finite type, we see that X is a Noetherian scheme too, see Morphisms, Lemma 24.14.6. In order to prove the lemma in this case it suffices to show that for every irreducible closed subset $Z \subset X$ the intersection $Z \cap X^0$ either contains a nonempty open of Z or is not dense in Z, see Topology, Lemma 5.11.3. Let $x \in Z$ be the generic point, and let $y = f(x)$. By Lemma 33.22.3 there exists a nonempty open subset $V \subset \overline{\{y\}}$ such that $X^0 \cap X_V$ is open and closed in X_V. Since $f(Z) \subset \overline{\{y\}}$ and $f(x) = y \in V$ we see that $W = f^{-1}(V) \cap Z$ is a nonempty open subset of Z. It follows that $X^0 \cap W$ is open and closed in W. Since W is irreducible we see that $X^0 \cap W$ is either empty or equal to W. This proves the lemma. ☐

Lemma 33.22.5. *Let $f : X \to Y$, $s : Y \to X$ be as in Situation 33.22.1. Let $y \in Y$ be a point. Assume*

(1) *f is of finite presentation and flat, and*
(2) *the fibre X_y is geometrically reduced.*

Then X^0 is a neighbourhood of X^0_y in X.

Proof. We may replace Y with an affine open neighbourhood of y. Write $Y = Spec(A)$ and $A = colim\, A_i$ as a directed limit of finite type $\mathbf{Z}$-algebras. By Limits, Lemma 27.6.1 we can find an i and a morphism $f_i : X_i \to Spec(A_i)$ of finite presentation, endowed with a section $s_i : Spec(A_i) \to X_i$ whose base change to Y recovers f and the section s. After possibly increasing i we may also assume that f_i is flat, see Limits, Lemma 27.6.3. Let y_i be the image of y in Y_i. Note that $X_y = (X_{i,y_i}) \times_{y_i} y$. Hence X_{i,y_i} is geometrically reduced, see Varieties, Lemma 28.4.6. By Lemma 33.22.2 it suffices to prove the lemma for the system $f_i, s_i, y_i \in Y_i$. Thus we reduce to the case where Y is the spectrum of a Noetherian ring.

Assume Y is the spectrum of a Noetherian ring. Since f is of finite presentation, i.e., of finite type, we see that X is a Noetherian scheme too, see Morphisms, Lemma 24.14.6. Let $x \in X^0$ be a point lying over y. By Topology, Lemma 5.11.4 it suffices to prove that for any irreducible closed $Z \subset X$ passing through x the intersection $X^0 \cap Z$ is dense in Z. In particular it suffices to prove that the generic point $x' \in Z$ is in X^0. By Properties, Lemma 23.5.9 we can find a discrete valuation ring R and a morphism $Spec(R) \to X$ which maps the special point to x and the generic point to x'. We are going to think of $Spec(R)$ as a scheme over Y via the composition $Spec(R) \to X \to Y$. By Lemma 33.22.2 we have that $(X_R)^0$ is the inverse image of X^0. By construction we have a second section $t : Spec(R) \to X_R$ (besides the base change s_R of s) of the structure morphism $X_R \to Spec(R)$ such that $t(\eta_R)$ is a point of X_R which maps to x' and $t(0_R)$ is a point of X_R which maps to x. Note that $t(0_R)$ is in $(X_R)^0$ and that $t(\eta_R) \rightsquigarrow t(0_R)$. Thus it suffices to prove that this implies that

$t(\eta_R) \in (X_R)^0$. Hence it suffices to prove the lemma in the case where Y is the spectrum of a discrete valuation ring and y its closed point.

Assume Y is the spectrum of a discrete valuation ring and y is its closed point. Our goal is to prove that X^0 is a neighbourhood of X^0_y. Note that X^0_y is open and closed in X_y as X_y has finitely many irreducible components. Hence the complement $C = X_y \setminus X^0_y$ is closed in X. Thus $U = X \setminus C$ is an open neighbourhood of X^0_y and $U^0 = X^0$. Hence it suffices to prove the result for the morphism $U \to Y$. In other words, we may assume that X_y is connected. Suppose that X is disconnected, say $X = X_1 \amalg \ldots \amalg X_n$ is a decomposition into connected components. Then $s(Y)$ is completely contained in one of the X_i. Say $s(Y) \subset X_1$. Then $X^0 \subset X_1$. Hence we may replace X by X_1 and assume that X is connected. At this point Lemma 33.21.7 implies that X_η is connected, i.e., $X^0 = X$ and we win. □

Lemma 33.22.6. *Let $f : X \to Y$, $s : Y \to X$ be as in Situation 33.22.1. Assume*

(1) *f is of finite presentation and flat, and*
(2) *all fibres of f are geometrically reduced.*

Then X^0 is open in X.

Proof. This is an immediate consequence of Lemma 33.22.5. □

33.23. Dimension of fibres

Lemma 33.23.1. *Let $f : X \to Y$ be a morphism of schemes. Assume Y irreducible with generic point η and f of finite type. If X_η has dimension n, then there exists a nonempty open $V \subset Y$ such that for all $y \in V$ the fibre X_y has dimension n.*

Proof. Let $Z = \{x \in X \mid \dim_x(X_{f(x)}) > n\}$. By Morphisms, Lemma 24.27.4 this is a closed subset of X. By assumption $Z_\eta = \emptyset$. Hence by Lemma 33.17.1 we may shrink Y and assume that $Z = \emptyset$. Let $Z' = \{x \in X \mid \dim_x(X_{f(x)}) > n - 1\} = \{x \in X \mid \dim_x(X_{f(x)}) = n\}$. As before this is a closed subset of X. By assumption we have $Z'_\eta \neq \emptyset$. Hence after shrinking Y we may assume that $Z' \to Y$ is surjective, see Lemma 33.17.2. Hence we win. □

Lemma 33.23.2. *Let $f : X \to Y$ be a morphism of finite type. Let*

$$n_{X/Y} : Y \to \{0, 1, 2, 3, \ldots, \infty\}$$

be the function which associates to $y \in Y$ the dimension of X_y. If $g : Y' \to Y$ is a morphism then

$$n_{X'/Y'} = n_{X/Y} \circ g$$

where $X' \to Y'$ is the base change of f.

Proof. This follows from Morphisms, Lemma 24.27.3. □

Lemma 33.23.3. *Let $f : X \to Y$ be a morphism of schemes. Let $n_{X/Y}$ be the function on Y giving the dimension of fibres of f introduced in Lemma 33.23.2. Assume f of finite presentation. Then the level sets*

$$E_n = \{y \in Y \mid n_{X/Y}(y) = n\}$$

of $n_{X/Y}$ are locally constructible in Y.

Proof. Fix n. Let $y \in Y$. We have to show that there exists an open neighbourhood V of y in Y such that $E_n \cap V$ is constructible in V. Thus we may assume that Y is affine. Write $Y = Spec(A)$ and $A = colim\, A_i$ as a directed limit of finite type $\mathbf{Z}$-algebras. By Limits, Lemma 27.6.1 we can find an i and a morphism $f_i : X_i \to Spec(A_i)$ of finite presentation whose base change to Y recovers f. By Lemma 33.23.2 it suffices to prove the lemma for f_i. Thus we reduce to the case where Y is the spectrum of a Noetherian ring.

We will use the criterion of Topology, Lemma 5.11.3 to prove that E_n is constructible in case Y is a Noetherian scheme. To see this let $Z \subset Y$ be an irreducible closed subscheme. We have to show that $E_n \cap Z$ either contains a nonempty open subset or is not dense in Z. Let $\xi \in Z$ be the generic point. Then Lemma 33.23.1 shows that $n_{X/Y}$ is constant in a neighbourhood of ξ in Z. This implies what we want. □

33.24. Limit arguments

Some lemmas involving limits of schemes, and Noetherian approximation. We stick mostly to the affine case. Some of these lemmas are special cases of lemmas in the chapter on limits.

Lemma 33.24.1. *Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation. Then there exists a cartesian diagram*

$$\begin{tikzcd} X_0 \arrow[d, "f_0"'] & X \arrow[l, "g"'] \arrow[d, "f"] \\ S_0 & S \arrow[l] \end{tikzcd}$$

such that

(1) *X_0, S_0 are affine schemes,*
(2) *S_0 of finite type over $\mathbf{Z}$,*
(3) *f_0 is finite of finite type.*

Proof. Write $S = Spec(A)$ and $X = Spec(B)$. As f is of finite presentation we see that B is of finite presentation as an A-algebra, see Morphisms, Lemma 24.20.2. Thus the lemma follows from Algebra, Lemma 7.118.15. □

Lemma 33.24.2. *Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite presentation. Then there exists a diagram as in Lemma 33.24.1 such that there exists a coherent $\mathcal{O}_{X_0}$-module $\mathcal{F}_0$ with $g^*\mathcal{F}_0 = \mathcal{F}$.*

Proof. Write $S = Spec(A)$, $X = Spec(B)$, and $\mathcal{F} = \widetilde{M}$. As f is of finite presentation we see that B is of finite presentation as an A-algebra, see Morphisms, Lemma 24.20.2. As $\mathcal{F}$ is of finite presentation over $\mathcal{O}_X$ we see that M is of finite presentation as a B-module, see Properties, Lemma 23.16.2. Thus the lemma follows from Algebra, Lemma 7.118.15. □

Lemma 33.24.3. *Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite presentation and flat over S. Then we may choose a diagram as in Lemma 33.24.2 and sheaf $\mathcal{F}_0$ such that in addition $\mathcal{F}_0$ is flat over S_0.*

Proof. Write $S = Spec(A)$, $X = Spec(B)$, and $\mathcal{F} = \widetilde{M}$. As f is of finite presentation we see that B is of finite presentation as an A-algebra, see Morphisms, Lemma 24.20.2. As $\mathcal{F}$ is of finite presentation over $\mathcal{O}_X$ we see that M is of finite presentation as a B-module, see

Properties, Lemma 23.16.2. As $\mathcal{F}$ is flat over S we see that M is flat over A, see Morphisms, Lemma 24.24.2. Thus the lemma follows from Algebra, Lemma 7.120.5. □

Lemma 33.24.4. *Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation and flat. Then there exists a diagram as in Lemma 33.24.1 such that in addition f_0 is flat.*

Proof. This is a special case of Lemma 33.24.3. □

Lemma 33.24.5. *Let $f : X \to S$ be a morphism of affine schemes, which is smooth. Then there exists a diagram as in Lemma 33.24.1 such that in addition f_0 is smooth.*

Proof. Write $S = Spec(A)$, $X = Spec(B)$, and as f is smooth we see that B is smooth as an A-algebra, see Morphisms, Lemma 24.33.2. Hence the lemma follows from Algebra, Lemma 7.127.14. □

Lemma 33.24.6. *Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation with geometrically reduced fibres. Then there exists a diagram as in Lemma 33.24.1 such that in addition f_0 has geometrically reduced fibres.*

Proof. Apply Lemma 33.24.1 to get a cartesian diagram

$$\begin{tikzcd} X_0 \arrow[d, "f_0"'] & X \arrow[l, "g"'] \arrow[d, "f"] \\ S_0 & S \arrow[l, "h"'] \end{tikzcd}$$

of affine schemes with $X_0 \to S_0$ a finite type morphism of schemes of finite type over $\mathbf{Z}$. By Lemma 33.19.5 the set $E \subset S_0$ of points where the fibre of f_0 is geometrically reduced is a constructible subset. By Lemma 33.19.2 we have $h(S) \subset E$. Write $S_0 = Spec(A_0)$ and $S = Spec(A)$. Write $A = colim_i A_i$ as a direct colimit of finite type A_0-algebras. By Limits, Lemma 27.3.4 we see that $Spec(A_i) \to S_0$ has image contained in E for some i. After replacing S_0 by $Spec(A_i)$ and X_0 by $X_0 \times_{S_0} Spec(A_i)$ we see that all fibres of f_0 are geometrically reduced. □

Lemma 33.24.7. *Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation with geometrically irreducible fibres. Then there exists a diagram as in Lemma 33.24.1 such that in addition f_0 has geometrically irreducible fibres.*

Proof. Apply Lemma 33.24.1 to get a cartesian diagram

$$\begin{tikzcd} X_0 \arrow[d, "f_0"'] & X \arrow[l, "g"'] \arrow[d, "f"] \\ S_0 & S \arrow[l, "h"'] \end{tikzcd}$$

of affine schemes with $X_0 \to S_0$ a finite type morphism of schemes of finite type over $\mathbf{Z}$. By Lemma 33.20.7 the set $E \subset S_0$ of points where the fibre of f_0 is geometrically irreducible is a constructible subset. By Lemma 33.20.2 we have $h(S) \subset E$. Write $S_0 = Spec(A_0)$ and $S = Spec(A)$. Write $A = colim_i A_i$ as a direct colimit of finite type A_0-algebras. By Limits, Lemma 27.3.4 we see that $Spec(A_i) \to S_0$ has image contained in E for some i. After replacing S_0 by $Spec(A_i)$ and X_0 by $X_0 \times_{S_0} Spec(A_i)$ we see that all fibres of f_0 are geometrically irreducible. □

Lemma 33.24.8. *Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation with geometrically connected fibres. Then there exists a diagram as in Lemma 33.24.1 such that in addition f_0 has geometrically connected fibres.*

Proof. Apply Lemma 33.24.1 to get a cartesian diagram

$$\begin{tikzcd} X_0 \arrow[d, "f_0"'] & X \arrow[l, "g"'] \arrow[d, "f"] \\ S_0 & S \arrow[l, "h"'] \end{tikzcd}$$

of affine schemes with $X_0 \to S_0$ a finite type morphism of schemes of finite type over $\mathbf{Z}$. By Lemma 33.21.6 the set $E \subset S_0$ of points where the fibre of f_0 is geometrically connected is a constructible subset. By Lemma 33.21.2 we have $h(S) \subset E$. Write $S_0 = Spec(A_0)$ and $S = Spec(A)$. Write $A = colim_i A_i$ as a direct colimit of finite type A_0-algebras. By Limits, Lemma 27.3.4 we see that $Spec(A_i) \to S_0$ has image contained in E for some i. After replacing S_0 by $Spec(A_i)$ and X_0 by $X_0 \times_{S_0} Spec(A_i)$ we see that all fibres of f_0 are geometrically connected. $\square$

Lemma 33.24.9. *Let $d \geq 0$ be an integer. Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation all of whose fibres have dimension d. Then there exists a diagram as in Lemma 33.24.1 such that in addition all fibres of f_0 have dimension d.*

Proof. Apply Lemma 33.24.1 to get a cartesian diagram

$$\begin{tikzcd} X_0 \arrow[d, "f_0"'] & X \arrow[l, "g"'] \arrow[d, "f"] \\ S_0 & S \arrow[l, "h"'] \end{tikzcd}$$

of affine schemes with $X_0 \to S_0$ a finite type morphism of schemes of finite type over $\mathbf{Z}$. By Lemma 33.23.3 the set $E \subset S_0$ of points where the fibre of f_0 has dimension d is a constructible subset. By Lemma 33.23.2 we have $h(S) \subset E$. Write $S_0 = Spec(A_0)$ and $S = Spec(A)$. Write $A = colim_i A_i$ as a direct colimit of finite type A_0-algebras. By Limits, Lemma 27.3.4 we see that $Spec(A_i) \to S_0$ has image contained in E for some i. After replacing S_0 by $Spec(A_i)$ and X_0 by $X_0 \times_{S_0} Spec(A_i)$ we see that all fibres of f_0 have dimension d. $\square$

Lemma 33.24.10. *Let $f : X \to S$ be a morphism of affine schemes, which is standard syntomic (see Morphisms, Definition 24.30.1). Then there exists a diagram as in Lemma 33.24.1 such that in addition f_0 is standard syntomic.*

Proof. This lemma is an improvement of the awkward Algebra, Lemma 7.125.12. Write $X = Spec(B)$ and $S = Spec(A)$. By assumption we may write $B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ such that every nonzero fibre ring of $A \to B$ has dimension $n - c$. Let $A_0 \subset A$ be a finite type $\mathbf{Z}$-subalgebra such that the coefficients of the polynomials f_j are contained in A_0. Set $B_0 = A_0[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$. Let

$$E = \{\mathfrak{p}_0 \subset A_0 \mid \dim(B_0 \otimes_{A_0} \kappa(\mathfrak{p}_0) = n - c\}$$

By Lemma 33.23.3 the set E is constructible. By assumption and Lemma 33.23.2 the morphism $Spec(A) \to Spec(A_0)$ has image contained in E. Write $A = colim_i A_i$ as a direct colimit of finite type A_0-algebras. By Limits, Lemma 27.3.4 we see that $Spec(A_i) \to$

$Spec(A_0)$ has image contained in E for some i. After replacing A_0 by A_i and B_0 by $B_0 \otimes_{A_0} A_i$ we see that all nonempty fibre rings of $A_0 \to B_0$ have dimension $n - c$. Hence $A_0 \to B_0$ is a relative global complete intersection. □

Lemma 33.24.11. *(Noetherian approximation and combining properties.) Let P, Q be properties of morphisms of schemes which are stable under base change. Let $f : X \to S$ be a morphism of finite presentation of affine schemes. Assume we can find cartesian diagrams*

$$\begin{tikzcd} X_1 \arrow[d, "f_1"'] & X \arrow[l] \arrow[d, "f"] \\ S_1 & S \arrow[l] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X_2 \arrow[d, "f_2"'] & X \arrow[l] \arrow[d, "f"] \\ S_2 & S \arrow[l] \end{tikzcd}$$

of affine schemes, with S_1, S_2 of finite type over $\mathbf{Z}$ and f_1, f_2 of finite type such that f_1 has property P and f_2 has property Q. Then we can find a cartesian diagram

$$\begin{tikzcd} X_0 \arrow[d, "f_0"'] & X \arrow[l] \arrow[d, "f"] \\ S_0 & S \arrow[l] \end{tikzcd}$$

of affine schemes with S_0 of finite type over $\mathbf{Z}$ and f_0 of finite type such that f_0 has both property P and property Q.

Proof. The given pair of diagrams correspond to cocartesian diagrams of rings

$$\begin{tikzcd} B_1 \arrow[r] & B \\ A_1 \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} B_2 \arrow[r] & B \\ A_2 \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

Let $A_0 \subset A$ be a finite type $\mathbf{Z}$-subalgebra of A containing the image of both $A_1 \to A$ and $A_2 \to A$. Such a subalgebra exists because by assumption both A_1 and A_2 are of finite type over $\mathbf{Z}$. Note that the rings $B_{0,1} = B_1 \otimes_{A_1} A_0$ and $B_{0,2} = B_2 \otimes_{A_2} A_0$ are finite type A_0-algebras with the property that $B_{0,1} \otimes_{A_0} A \cong B \cong B_{0,2} \otimes_{A_0} A$ as A-algebras. As A is the directed colimit of its finite type A_0-subalgebras, by Limits, Lemma 27.6.1 we may assume after enlarging A_0 that there exists an isomorphism $B_{0,1} \cong B_{0,2}$ as A_0-algebras. Since properties P and Q are assumed stable under base change we conclude that setting $S_0 = Spec(A_0)$ and

$$X_0 = X_1 \times_{S_1} S_0 = Spec(B_{0,1}) \cong Spec(B_{0,2}) = X_2 \times_{S_2} S_0$$

works. □

33.25. Étale neighbourhoods

It turns out that some properties of morphisms are easier to study after doing an étale base change. It is convenient to introduce the following terminology.

Definition 33.25.1. Let S be a scheme. Let $s \in S$ be a point.

(1) An *étale neighbourhood of* (S, s) is a pair (U, u) together with an étale morphism of schemes $\varphi : U \to S$ such that $\varphi(u) = s$.
(2) A *morphism of étale neighbourhoods* $f : (V, v) \to (U, u)$ of (S, s) is simply a morphism of S-schemes $f : V \to U$ such that $f(v) = u$.

(3) An *elementary étale neighbourhood* is an étale neighbourhood $\varphi : (U, u) \to (S, s)$ such that $\kappa(s) = \kappa(u)$.

If $f : (V, v) \to (U, u)$ is a morphism of étale neighbourhoods, then f is automatically étale, see Morphisms, Lemma 24.35.18. Hence it turns (V, v) into an étale neighbourhood of (U, u). Of course, since the composition of étale morphisms is étale (Morphisms, Lemma 24.35.3) we see that conversely any étale neighbourhood (V, v) of (U, u) is an étale neighbourhood of (S, s) as well. We also remark that if $U \subset S$ is an open neighbourood of s, then $(U, s) \to (S, s)$ is an étale neighbourhood. This follows from the fact that an open immersion is étale (Morphisms, Lemma 24.35.9). We will use these remarks without further mention throughout this section.

Note that $\kappa(s) \subset \kappa(u)$ is a finite separable extension if $(U, u) \to (S, s)$ is an étale neighbourhood, see Morphisms, Lemma 24.35.15.

Lemma 33.25.2. *Let S be a scheme. Let $s \in S$. Let $\kappa(s) \subset k$ be a finite separable field extension. Then there exists an étale neighbourhood $(U, u) \to (S, s)$ such that the field extension $\kappa(s) \subset \kappa(u)$ is isomorphic to $\kappa(s) \subset k$.*

Proof. We may assume S is affine. In this case the lemma follows from Algebra, Lemma 7.132.15. □

Lemma 33.25.3. *Let S be a scheme, and let s be a geometric point of S. The category of étale neighborhoods has the following properties:*

(1) *Let $(U_i, u_i)_{i=1,2}$ be two étale neighborhoods of s in S. Then there exists a third étale neighborhood (U, u) and morphisms $(U, u) \to (U_i, u_i)$, $i = 1, 2$.*
(2) *Let $h_1, h_2 : (U, u) \to (U', u')$ be two morphisms between étale neighborhoods of s. Assume h_1, h_2 induce the same map $\kappa(u') \to \kappa(u)$ of residue fields. Then there exist an étale neighborhood (U'', u'') and a morphism $h : (U'', u'') \to (U, u)$ which equalizes h_1 and h_2, i.e., such that $h_1 \circ h = h_2 \circ h$.*

Proof. For part (1), consider the fibre product $U = U_1 \times_S U_2$. It is étale over both U_1 and U_2 because étale morphisms are preserved under base change, see Morphisms, Lemma 24.35.4. There is a point of U mapping to both u_1 and u_2 for example by the description of points of a fibre product in Schemes, Lemma 21.17.5. For part (2), define U'' as the fibre product

$$\begin{tikzcd} U'' \arrow[r] \arrow[d] & U \arrow[d, "{(h_1,h_2)}"] \\ U' \arrow[r, "\Delta"] & U' \times_S U'. \end{tikzcd}$$

Since h_1 and h_2 induce the same map of residue fields $\kappa(u') \to \kappa(u)$ there exists a point $u'' \in U''$ lying over u' with $\kappa(u'') = \kappa(u')$. In particular $U'' \neq \emptyset$. Moreover, since U' is étale over S, so is the fibre product $U' \times_S U'$ (see Morphisms, Lemmas 24.35.4 and 24.35.3). Hence the vertical arrow (h_1, h_2) is étale by Morphisms, Lemma 24.35.18. Therefore U'' is étale over U' by base change, and hence also étale over S (because compositions of étale morphisms are étale). Thus (U'', u'') is a solution to the problem. □

Lemma 33.25.4. *Let S be a scheme, and let s be a geometric point of S. The category of elementary étale neighborhoods of (S, s) is cofiltered (see Categories, Definition 4.18.1).*

Proof. This is immediate from the definitions and Lemma 33.25.3. □

Lemma 33.25.5. *Let S be a scheme. Let $s \in S$. Then we have*

$$\mathcal{O}^h_{S,s} = colim_{(U,u)} \mathcal{O}(U)$$

where the colimit is over the filtered category which is opposite to the category of elementary étale neighbourhoods (U, u) of (S, s).

Proof. Let $Spec(A) \subset S$ be an affine neighbourhood of s. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to s. With these choices we have canonical isomorphisms $\mathcal{O}_{S,s} = A_{\mathfrak{p}}$ and $\kappa(s) = \kappa(\mathfrak{p})$. A cofinal system of elementary étale neighbourhoods is given by those elementary étale neighbourhoods (U, u) such that U is affine and $U \to S$ factors through $Spec(A)$. In other words, we see that the right hand side is equal to $colim_{(B,\mathfrak{q})} B$ where the colimit is over étale A-algebras B endowed with a prime $\mathfrak{q}$ lying over $\mathfrak{p}$ with $\kappa(\mathfrak{p}) = \kappa(\mathfrak{q})$. Thus the lemma follows from Algebra, Lemma 7.139.23. $\square$

33.26. Slicing smooth morphisms

In this section we explain a result that roughly states that smooth coverings of a scheme S can be refined by étale coverings. The technique to prove this relies on a slicing argument.

Lemma 33.26.1. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Let $h \in \mathfrak{m}_x \subset \mathcal{O}_{X,x}$. Assume*

(1) *f is smooth at x, and*
(2) *the image $d\overline{h}$ of dh in*

$$\Omega_{X_s/s,x} \otimes_{\mathcal{O}_{X_s,x}} \kappa(x) = \Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$$

is nonzero.

Then there exists an affine open neighbourhood $U \subset X$ of x such that h comes from $h \in \Gamma(U, \mathcal{O}_U)$ and such that $D = V(h)$ is an effective Cartier divisor in U with $x \in D$ and $D \to S$ smooth.

Proof. AS f is smooth at x we may assume, after replacing X by an open neighbourhood of x that f is smooth. In particular we see that f is flat and locally of finite presentation. By Lemma 33.16.1 we already know there exists an open neighbourhood $U \subset X$ of x such that h comes from $h \in \Gamma(U, \mathcal{O}_U)$ and such that $D = V(h)$ is an effective Cartier divisor in U with $x \in D$ and $D \to S$ flat and of finite presentation. By Morphisms, Lemma 24.32.17 we have a short exact sequence

$$\mathcal{C}_{D/U} \to i^*\Omega_{U/S} \to \Omega_{D/S} \to 0$$

where $i : D \to U$ is the closed immersion and $\mathcal{C}_{D/U}$ is the conormal sheaf of D in U. As D is an effective Cartier divisor cut out by $h \in \Gamma(U, \mathcal{O}_U)$ we see that $\mathcal{C}_{D/U} = h \cdot \mathcal{O}_S$. Since $U \to S$ is smooth the sheaf $\Omega_{U/S}$ is finite locally free, hence its pullback $i^*\Omega_{U/S}$ is finite locally free also. The first arrow of the sequence maps the free generator h to the section $dh|_D$ of $i^*\Omega_{U/S}$ which has nonzero value in the fibre $\Omega_{U/S,x} \otimes \kappa(x)$ by assumption. By right exactness of $\otimes \kappa(x)$ we conclude that

$$\dim_{\kappa(x)} \left(\Omega_{D/S,x} \otimes \kappa(x)\right) = \dim_{\kappa(x)} \left(\Omega_{U/S,x} \otimes \kappa(x)\right) - 1.$$

By Morphisms, Lemma 24.33.14 we see that $\Omega_{U/S,x} \otimes \kappa(x)$ can be generated by at most $\dim_x(U_s)$ elements. By the displayed formula we see that $\Omega_{D/S,x} \otimes \kappa(x)$ can be generated by at most $\dim_x(U_s) - 1$ elements. Note that $\dim_x(D_s) = \dim_x(U_s) - 1$ for example because $\dim(\mathcal{O}_{D_s,x}) = \dim(\mathcal{O}_{U_s,x}) - 1$ by Algebra, Lemma 7.57.11 (also $D_s \subset U_s$ is effective Cartier, see Divisors, Lemma 26.10.1) and then using Morphisms, Lemma 24.27.1. Thus we conclude that $\Omega_{D/S,x} \otimes \kappa(x)$ can be generated by at most $\dim_x(D_s)$ elements and we conclude that $D \to S$ is smooth at x by Morphisms, Lemma 24.33.14 again. After shrinking U we get that $D \to S$ is smooth and we win. □

Lemma 33.26.2. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Assume*

(1) *f is smooth at x, and*
(2) *the map*
$$\Omega_{X_s/s,x} \otimes_{\mathcal{O}_{X_s,x}} \kappa(x) \longrightarrow \Omega_{\kappa(x)/\kappa(s)}$$
has a nonzero kernel.

Then there exists an affine open neighbourhood $U \subset X$ of x and an effective Cartier divisor $D \subset U$ containing x such that $D \to S$ is smooth.

Proof. Write $k = \kappa(s)$ and $R = \mathcal{O}_{X_s,x}$. Denote $\mathfrak{m}$ the maximal ideal of R and $\kappa = R/\mathfrak{m}$ so that $\kappa = \kappa(x)$. As formation of modules of differentials commutes with localization (see Algebra, Lemma 7.122.8) we have $\Omega_{X_s/s,x} = \Omega_{R/k}$. By Algebra, Lemma 7.122.9 there is an exact sequence
$$\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\text{d}} \Omega_{R/k} \otimes_R \kappa \to \Omega_{\kappa/k} \to 0.$$
Hence if (2) holds, there exists an element $\overline{h} \in \mathfrak{m}$ such that $\text{d}\overline{h}$ is nonzero. Choose a lift $h \in \mathcal{O}_{X,x}$ of $\overline{h}$ and apply Lemma 33.26.1. □

Remark 33.26.3. The second condition in Lemma 33.26.2 is necessary even if x is a closed point of a positive dimensional fibre. An example is the following: Let k be a field of characteristic $p > 0$ which is imperfect. Let $a \in k$ be an element which is not a pth power. Let $\mathfrak{m} = (x, y^p - a) \subset k[x, y]$. This corresponds to a closed point w of $X = \mathbf{A}^2_k$. Set $S = \mathbf{A}^1_k$ and let $f : X \to S$ be the morphism corresponding to $k[x] \to k[x, y]$. Then there does not exist any commutative diagram

$$\begin{tikzcd} S' \arrow[rr, "h"] \arrow[dr, "g"'] & & X \arrow[dl, "f"] \\ & S & \end{tikzcd}$$

with g étale and w in the image of h. This is clear as the residue field extension $\kappa(f(w)) \subset \kappa(w)$ is purely inseparable, but for any $s' \in S'$ with $g(s') = f(w)$ the extension $\kappa(f(w)) \subset \kappa(s')$ would be separable.

If you assume the residue field extension is separble then the phenomenon of Remark 33.26.3 does not happen. Here is the precise result.

Lemma 33.26.4. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Assume*

(1) *f is smooth at x,*
(2) *the residue field extension $\kappa(s) \subset \kappa(x)$ is separable, and*
(3) *x is not a generic point of X_s.*

Then there exists an affine open neighbourhood $U \subset X$ of x and an effective Cartier divisor $D \subset U$ containing x such that $D \to S$ is smooth.

Proof. Write $k = \kappa(s)$ and $R = \mathcal{O}_{X_s,x}$. Denote $\mathfrak{m}$ the maximal ideal of R and $\kappa = R/\mathfrak{m}$ so that $\kappa = \kappa(x)$. As formation of modules of differentials commutes with localization (see Algebra, Lemma 7.122.8) we have $\Omega_{X_s/s,x} = \Omega_{R/k}$. By assumption (2) and Algebra, Lemma 7.129.4 the map

$$\mathrm{d} : \mathfrak{m}/\mathfrak{m}^2 \longrightarrow \Omega_{R/k} \otimes_R \kappa(\mathfrak{m})$$

is injective. Assumption (3) implies that $\mathfrak{m}/\mathfrak{m}^2 \neq 0$. Thus there exists an element $\overline{h} \in \mathfrak{m}$ such that $\mathrm{d}\overline{h}$ is nonzero. Choose a lift $h \in \mathcal{O}_{X,x}$ of $\overline{h}$ and apply Lemma 33.26.1. □

The subscheme Z constructed in the following lemma is really a complete intersection in an affine open neighbourhood of x. If we ever need this we will explicitly formulate a separate lemma stating this fact.

Lemma 33.26.5. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Assume*

1. *f is smooth at x, and*
2. *x is a closed point of X_s and $\kappa(s) \subset \kappa(x)$ is separable.*

Then there exists an immersion $Z \to X$ containing x such that

1. *$Z \to S$ is étale, and*
2. *$Z_s = \{x\}$ set theoretically.*

Proof. We may and do replace S by an affine open neighbourhoof of s. We may and do replace X by an affine open neighbourhood of x such that $X \to S$ is smooth. We will prove the lemma for smooth morphisms of affines by induction on $d = \dim_x(X_s)$.

The case $d = 0$. In this case we show that we may take Z to be an open neighbourhood of x. Namely, if $d = 0$, then $X \to S$ is quasi-finite at x, see Morphisms, Lemma 24.28.5. Hence there exists an affine open neighbourhood $U \subset X$ such that $U \to S$ is quasi-finite, see Morphisms, Lemma 24.47.2. Thus after replacing X by U we see that X is quasi-finite and smooth over S, hence smooth of relative dimension 0 over S, hence étale over S. Moreover, the fibre X_s is a finite discrete set. Hence after replacing X by a further affine open neigbourhood of X we see that that $f^{-1}(\{s\}) = \{x\}$ (because the topology on X_s is induced from the topology on X, see Schemes, Lemma 21.18.5). This proves the lemma in this case.

Next, assume $d > 0$. Note that because x is a closed point of its fibre the extension $\kappa(s) \subset \kappa(x)$ is finite (by the Hilbert Nullstellensatz, see Morphisms, Lemma 24.19.3). Thus we see $\Omega_{\kappa(x)/\kappa(s)} = 0$ as this holds for algebraic separable field extensions. Thus we may apply Lemma 33.26.2 to find a diagram

$$\begin{tikzcd} D \arrow[r] \arrow[drr] & U \arrow[r] \arrow[dr] & X \arrow[d] \\ & & S \end{tikzcd}$$

with $x \in D$. Note that $\dim_x(D_s) = \dim_x(X_s) - 1$ for example because $\dim(\mathcal{O}_{D_s,x}) = \dim(\mathcal{O}_{X_s,x}) - 1$ by Algebra, Lemma 7.57.11 (also $D_s \subset X_s$ is effective Cartier, see Divisors, Lemma 26.10.1) and then using Morphisms, Lemma 24.27.1. Thus the morphism $D \to S$ is smooth with $\dim_x(D_s) = \dim_x(X_s) - 1 = d - 1$. By induction hypothesis we can find an immersion $Z \to D$ as desired, which finishes the proof. □

Lemma 33.26.6. *Let $f : X \to S$ be a smooth morphism of schemes. Let $s \in S$ be a point in the image of f. Then there exists an étale neighbourhood $(S', s') \to (S, s)$ and a S-morphism $S' \to X$.*

First proof of Lemma 33.26.6. By assumption $X_s \neq \emptyset$. By Varieties, Lemma 28.15.6 there exists a closed point $x \in X_s$ such that $\kappa(x)$ is a finite separable field extension of $\kappa(s)$. Hence by Lemma 33.26.5 there exists an immersion $Z \to X$ such that $Z \to S$ is étale and such that $x \in Z$. Take $(S', s') = (Z, x)$. $\square$

Second proof of Lemma 33.26.6. Pick an point $x \in X$ with $f(x) = s$. Choose a diagram

$$\begin{tikzcd} X \arrow[d] & U \arrow[l] \arrow[r, "\pi"] \arrow[d] & \mathbf{A}^d_V \arrow[dl] \\ Y & V \arrow[l] & \end{tikzcd}$$

with π étale, $x \in U$ and $V = Spec(R)$ affine, see Morphisms, Lemma 24.35.20. In particular $s \in V$. The morphism $\pi : U \to \mathbf{A}^d_V$ is open, see Morphisms, Lemma 24.35.13. Thus $W = \pi(V) \cap \mathbf{A}^d_s$ is a nonempty open subset of $\mathbf{A}^d_s$. Let $w \in W$ be a point with $\kappa(s) \subset \kappa(w)$ finite separable, see Varieties, Lemma 28.15.5. By Algebra, Lemma 7.105.1 there exist d elements $\overline{f}_1, \ldots, \overline{f}_d \in \kappa(s)[x_1, \ldots, x_d]$ which generate the maximal ideal corresponding to w in $\kappa(s)[x_1, \ldots, x_n]$. After replacing R by a principal localization we may assume there are $f_1, \ldots, f_d \in R[x_1, \ldots, x_d]$ which map to $\overline{f}_1, \ldots, \overline{f}_d \in \kappa(s)[x_1, \ldots, x_d]$. Consider the R-algebra

$$R' = R[x_1, \ldots, x_d]/(f_1, \ldots, f_d)$$

and set $S' = Spec(R')$. By construction we have a closed immersion $j : S' \to \mathbf{A}^d_V$ over V. By construction the fibre of $S' \to V$ over s is a single point s' whose residue field is finite separable over $\kappa(s)$. Let $\mathfrak{q}' \subset R'$ be the corresponding prime. By Algebra, Lemma 7.125.11 we see that $(R')_g$ is a relative global complete intersection over R for some $g \in R'$, $g \not\in \mathfrak{q}$. Thus $S' \to V$ is flat and of finite presentation in a neighbourhood of s', see Algebra, Lemma 7.125.14. By construction the scheme theoretic fibre of $S' \to V$ over s is $Spec(\kappa(s'))$. Hence it follows from Morphisms, Lemma 24.35.15 that $S' \to S$ is étale at s'. Set

$$S'' = U \times_{\pi, \mathbf{A}^d_V, j} S'.$$

By construction there exists a point $s'' \in S''$ which maps to s' via the projection $p : S'' \to S'$. Note that p is étale as the base change of the étale morphism π, see Morphisms, Lemma 24.35.4. Choose a small affine neighbourhood $S''' \subset S''$ of s'' which maps into the nonempty open neighbourhood of $s' \in S'$ where the morphism $S' \to S$ is étale. Then the étale neighbourhood $(S''', s'') \to (S, s)$ is a solution to the problem posed by the lemma. $\square$

The following lemma shows that sheaves for the smooth topology are the same thing as sheaves for the étale topology.

Lemma 33.26.7. *Let S be a scheme. Let $\mathcal{U} = \{S_i \to S\}_{i \in I}$ be a smooth covering of S, see Topologies, Definition 30.5.1. Then there exists an étale covering $\mathcal{V} = \{T_j \to S\}_{j \in J}$ (see Topologies, Definition 30.4.1) which refines (see Sites, Definition 9.8.1) $\mathcal{U}$.*

Proof. For every $s \in S$ there exists an $i \in I$ such that s is in the image of $S_i \to S$. By Lemma 33.26.6 we can find an étale morphism $g_s : T_s \to S$ such that $s \in g_s(T)_s$ and such that g_s factors through $S_i \to S$. Hence $\{T_s \to S\}$ is an étale covering of S that refines $\mathcal{U}$. $\square$

33.27. Finite free locally dominates étale

In this section we explain a result that roughly states that étale coverings of a scheme S can be refined by Zariski coverings of finite locally free covers of S.

Lemma 33.27.1. *Let S be a scheme. Let $s \in S$. Let $f : (U, u) \to (S, s)$ be an étale neighbourhood. There exists an affine open neighbourhood $s \in V \subset S$ and a surjective, finite locally free morphism $\pi : T \to V$ such that for every $t \in \pi^{-1}(s)$ there exists an open neighbourhood $t \in W_t \subset T$ and a commutative diagram*

$$\begin{tikzcd} T \arrow[d, "\pi"] & W_t \arrow[l] \arrow[rr, "h_t"] \arrow[rd] & & U \arrow[ld] \\ V \arrow[rr] & & S & \end{tikzcd}$$

with $h_t(t) = u$.

Proof. The problem is local on S hence we may replace S by any open neighbourhood of s. We may also replace U by an open neighbourhood of u. Hence, by Morphisms, Lemma 24.35.14 we may assume that $U \to S$ is a standard étale morphism of affine schemes. In this case the lemma (with $V = S$) follows from Algebra, Lemma 7.132.17. □

Lemma 33.27.2. *Let $f : U \to S$ be a surjective étale morphism of affine schemes. There exists a surjective, finite locally free morphism $\pi : T \to S$ and a finite open covering $T = T_1 \cup \ldots \cup T_n$ such that each $T_i \to S$ factors through $U \to S$. Diagram:*

$$\begin{tikzcd} & \coprod T_i \arrow[ld] \arrow[rd] & \\ T \arrow[rd, "\pi"] & & U \arrow[ld, "f"'] \\ & S & \end{tikzcd}$$

where the south-west arrow is a Zariski-covering.

Proof. This is a restatement of Algebra, Lemma 7.132.18. □

Remark 33.27.3. In terms of topologies the lemmas above mean the following. Let S be any scheme. Let $\{f_i : U_i \to S\}$ be an étale covering of S. There exists a Zariski open covering $S = \bigcup V_j$, for each j a finite locally free, surjective morphism $W_j \to V_j$, and for each j a Zariski open covering $\{W_{j,k} \to W_j\}$ such that the family $\{W_{j,k} \to S\}$ refines the given étale covering $\{f_i : U_i \to S\}$. What does this mean in practice? Well, for example, suppose we have a descend problem which we know how to solve for Zariski coverings and for fppf coverings of the form $\{\pi : T \to S\}$ with π finite locally free and surjective. Then this descend problem has an affirmative answer for étale coverings as well. This trick was used by Gabber in his proof that $\mathrm{Br}(X) = \mathrm{Br}'(X)$ for an affine scheme X, see [**Hoo82**].

33.28. Étale localization of quasi-finite morphisms

Now we come to a series of lemmas around the theme ``quasi-finite morphisms become finite after étale localization''. The general idea is the following. Suppose given a morphism of schemes $f : X \to S$ and a point $s \in S$. Let $\varphi : (U, u) \to (S, s)$ be an étale neighbourhood of s in S. Consider the fibre product $X_U = U \times_S X$ and the basic diagram

(33.28.0.1)
$$\begin{tikzcd} V \arrow[r] \arrow[dr] & X_U \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ & U \arrow[r, "\varphi"] & S \end{tikzcd}$$

where $V \subset X_U$ is open. Is there some standard model for the morphism $f_U : X_U \to U$, or for the morphism $V \to U$ for suitable opens V? Of course the answer is no in general. But for quasi-finite morphisms we can say something.

Lemma 33.28.1. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. Set $s = f(x)$. Assume that*

(1) *f is locally of finite type, and*
(2) *$x \in X_s$ is isolated*[2].

Then there exist

(a) *an elementary étale neighbourhood $(U, u) \to (S, s)$,*
(b) *an open subscheme $V \subset X_U$ (see 33.28.0.1)*

such that

(i) *$V \to U$ is a finite morphism,*
(ii) *there is a unique point v of V mapping to u in U, and*
(iii) *the point v maps to x under the morphism $X_U \to X$, inducing $\kappa(x) = \kappa(v)$.*

Moreover, for any elementary étale neighbourhood $(U', u') \to (U, u)$ setting $V' = U' \times_U V \subset X_{U'}$ the triple (U', u', V') satisfies the properties (i), (ii), and (iii) as well.

Proof. Let $Y \subset X$, $W \subset S$ be affine opens such that $f(Y) \subset W$ and such that $x \in Y$. Note that x is also an isolated point of the fibre of the morphism $f|_Y : Y \to W$. If we can prove the theorem for $f|_Y : Y \to W$, then the theorem follows for f. Hence we reduce to the case where f is a morphism of affine schemes. This case is Algebra, Lemma 7.132.21. □

In the preceding and following lemma we do not assume that the morphism f is separated. This means that the opens V, V_i created in them are not necessarily closed in X_U. Moreover, if we choose the neighbourhood U to be affine, then each V_i is affine, but the intersections $V_i \cap V_j$ need not be affine (in the nonseparated case).

Lemma 33.28.2. *Let $f : X \to S$ be a morphism of schemes. Let $x_1, \ldots, x_n \in X$ be points having the same image s in S. Assume that*

(1) *f is locally of finite type, and*
(2) *$x_i \in X_s$ is isolated for $i = 1, \ldots, n$.*

Then there exist

(a) *an elementary étale neighbourhood $(U, u) \to (S, s)$,*
(b) *for each i an open subscheme $V_i \subset X_U$,*

such that for each i we have

(i) *$V_i \to U$ is a finite morphism,*
(ii) *there is a unique point v_i of V_i mapping to u in U, and*
(iii) *the point v_i maps to x_i in X and $\kappa(x_i) = \kappa(v_i)$.*

[2] In the presence of (1) this means that f is quasi-finite at x, see Morphisms, Lemma 24.19.6.

Proof. We will use induction on n. Namely, suppose $(U, u) \to (S, s)$ and $V_i \subset X_U$, $i = 1, \ldots, n-1$ work for $x_1, \ldots, x_{n-1}$. Since $\kappa(s) = \kappa(u)$ the fibre $(X_U)_u = X_s$. Hence there exists a unique point $x'_n \in X_u \subset X_U$ corresponding to $x_n \in X_s$. Also x'_n is isolated in X_u. Hence by Lemma 33.28.1 there exists an elementary étale neighbourhood $(U', u') \to (U, u)$ and an open $V_n \subset X_{U'}$ which works for x'_n and hence for x_n. By the final assertion of Lemma 33.28.1 the open subschemes $V'_i = U' \times_U V_i$ for $i = 1, \ldots, n-1$ still work with respect to $x_1, \ldots, x_{n-1}$. Hence we win. □

If we allow a nontrivial field extension $\kappa(s) \subset \kappa(u)$, i.e., general étale neighbourhoods, then we can split the points as follows.

Lemma 33.28.3. *Let $f : X \to S$ be a morphism of schemes. Let $x_1, \ldots, x_n \in X$ be points having the same image s in S. Assume that*

(1) *f is locally of finite type, and*
(2) *$x_i \in X_s$ is isolated for $i = 1, \ldots, n$.*

Then there exist

(a) *an étale neighbourhood $(U, u) \to (S, s)$,*
(b) *for each i an integer m_i and open subschemes $V_{i,j} \subset X_U$, $j = 1, \ldots, m_i$*

such that we have

(i) *each $V_{i,j} \to U$ is a finite morphism,*
(ii) *there is a unique point $v_{i,j}$ of $V_{i,j}$ mapping to u in U with $\kappa(u) \subset \kappa(v_{i,j})$ finite purely inseparable,*
(iv) *if $v_{i,j} = v_{i',j'}$, then $i = i'$ and $j = j'$, and*
(iii) *the points $v_{i,j}$ map to x_i in X and no other points of $(X_U)_u$ map to x_i.*

Proof. This proof is a variant of the proof of Algebra, Lemma 7.132.23 in the language of schemes. By Morphisms, Lemma 24.19.6 the morphism f is quasi-finite at each of the points x_i. Hence $\kappa(s) \subset \kappa(x_i)$ is finite for each i (Morphisms, Lemma 24.19.5). For each i, let $\kappa(s) \subset L_i \subset \kappa(x_i)$ be the subfield such that $L_i/\kappa(s)$ is separable, and $\kappa(x_i)/L_i$ is purely inseparable. Choose a finite Galois extension $\kappa(s) \subset L$ such that there exist $\kappa(s)$-embeddings $L_i \to L$ for $i = 1, \ldots, n$. Choose an étale neighbourhood $(U, u) \to (S, s)$ such that $L \cong \kappa(u)$ as $\kappa(s)$-extensions (Lemma 33.25.2).

Let $y_{i,j}$, $j = 1, \ldots, m_i$ be the points of X_U lying over $x_i \in X$ and $u \in U$. By Schemes, Lemma 21.17.5 these points $y_{i,j}$ correspond exactly to the primes in the rings $\kappa(u) \otimes_{\kappa(s)} \kappa(x_i)$. This also explains why there are finitely many; in fact $m_i = [L_i : \kappa(s)]$ but we do not need this. By our choice of L (and elementary field theory) we see that $\kappa(u) \subset \kappa(y_{i,j})$ is finite purely inseparable for each pair i, j. Also, by Morphisms, Lemma 24.19.13 for example, the morphism $X_U \to U$ is quasi-finite at the points $y_{i,j}$ for all i, j.

Apply Lemma 33.28.2 to the morphism $X_U \to U$, the point $u \in U$ and the points $y_{i,j} \in (X_U)_u$. This gives an étale neighbourhood $(U', u') \to (U, u)$ with $\kappa(u) = \kappa(u')$ and opens $V_{i,j} \subset X_{U'}$ with the properties (i), (ii), and (iii) of that lemma. We claim that the étale neighbourhood $(U', u') \to (S, s)$ and the opens $V_{i,j} \subset X_{U'}$ are a solution to the problem posed by the lemma. We omit the verifications. □

Lemma 33.28.4. *Let $f : X \to S$ be a morphism of schemes. Let $s \in S$. Let $x_1, \ldots, x_n \in X_s$. Assume that*

(1) *f is locally of finite type,*
(2) *f is separated, and*
(3) *$x_1, \ldots, x_n$ are pairwise distinct isolated points of X_s.*

Then there exists an elementary étale neighbourhood $(U, u) \to (S, s)$ and a decomposition

$$U \times_S X = W \coprod V_1 \coprod \ldots \coprod V_n$$

into open and closed subschemes such that the morphisms $V_i \to U$ are finite, the fibres of $V_i \to U$ over u are singletons $\{v_i\}$, each v_i maps to x_i with $\kappa(x_i) = \kappa(v_i)$, and the fibre of $W \to U$ over u contains no points mapping to any of the x_i.

Proof. Choose $(U, u) \to (S, s)$ and $V_i \subset X_U$ as in Lemma 33.28.2. Since $X_U \to U$ is separated (Schemes, Lemma 21.21.13) and $V_i \to U$ is finite hence proper (Morphisms, Lemma 24.42.10) we see that $V_i \subset X_U$ is closed by Morphisms, Lemma 24.40.7. Hence $V_i \cap V_j$ is a closed subset of V_i which does not contain v_i. Hence the image of $V_i \cap V_j$ in U is a closed set (because $V_i \to U$ proper) not containing u. After shrinking U we may therefore assume that $V_i \cap V_j = \emptyset$ for all i, j. This gives the decomposition as in the lemma. $\square$

Here is the variant where we reduce to purely inseparable field extensions.

Lemma 33.28.5. *Let $f : X \to S$ be a morphism of schemes. Let $s \in S$. Let $x_1, \ldots, x_n \in X_s$. Assume that*

1. *f is locally of finite type,*
2. *f is separated, and*
3. *$x_1, \ldots, x_n$ are pairwise distinct isolated points of X_s.*

Then there exists an étale neighbourhood $(U, u) \to (S, s)$ and a decomposition

$$U \times_S X = W \coprod \coprod\nolimits_{i=1,\ldots,n} \coprod\nolimits_{j=1,\ldots,m_i} V_{i,j}$$

into open and closed subschemes such that the morphisms $V_{i,j} \to U$ are finite, the fibres of $V_{i,j} \to U$ over u are singletons $\{v_{i,j}\}$, each $v_{i,j}$ maps to x_i, $\kappa(u) \subset \kappa(v_{i,j})$ is purely inseparable, and the fibre of $W \to U$ over u contains no points mapping to any of the x_i.

Proof. This is proved in exactly the same way as the proof of Lemma 33.28.4 except that it uses Lemma 33.28.3 instead of Lemma 33.28.2. $\square$

The following version may be a little easier to parse.

Lemma 33.28.6. *Let $f : X \to S$ be a morphism of schemes. Let $s \in S$. Assume that*

1. *f is locally of finite type,*
2. *f is separated, and*
3. *X_s has at most finitely many isolated points.*

Then there exists an elementary étale neighbourhood $(U, u) \to (S, s)$ and a decomposition

$$U \times_S X = W \coprod V$$

into open and closed subschemes such that the morphisms $V \to U$ is finite, and the fibre W_u of the morphism $W \to U$ contains no isolated points. In particular, if $f^{-1}(s)$ is a finite set, then $W_u = \emptyset$.

Proof. This is clear from Lemma 33.28.4 by choosing $x_1, \ldots, x_n$ the complete set of isolated points of X_s and setting $V = \bigcup V_i$. $\square$

33.29. Application to the structure of quasi-finite morphisms

We can use the existence of good étale neighbourhoods to prove some fundamental facts about quasi-finite morphisms.

Lemma 33.29.1. *(Normalization commutes with smooth base change.) Let*

$$\begin{tikzcd} Y_U \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ U \arrow[r, "\varphi"] & X \end{tikzcd}$$

be a fibre square in the category of schemes. Assume f is quasi-compact and quasi-separated, and $\varphi : U \to X$ is a smooth morphism. Let $Y \to X' \to X$ be the normalization of X in Y. Let $Y_U \to (X_U)' \to U$ be the normalization of U in Y_U. Then $(X_U)' \cong U \times_X X'$.

Proof. Denote $f_U : Y_U \to U$ the base change of f. By definition we have $X' = \underline{Spec}_X(\mathscr{A})$ and $(X_U)' = \underline{Spec}_U(\mathscr{A}')$, where $\mathscr{A} \subset f_*\mathcal{O}_Y$ is the integral closure of $\mathcal{O}_X$ and $\mathscr{A}' \subset (f_U)_*\mathcal{O}_{Y_U}$ is the integral closure of $\mathcal{O}_U$. By Coherent, Lemma 25.6.2 we know that $(f_U)_*\mathcal{O}_{Y_U}$ is the same as $\varphi^*(f_*\mathcal{O}_Y)$. Let $Spec(C) \subset U$, $Spec(R) \subset X$ be affine opens with $\varphi(Spec(C)) \subset Spec(R)$. Hence $R \to C$ is a smooth ring map, see Morphisms, Lemma 24.33.2. Write

$$f_*\mathcal{O}_Y|_{Spec(R)} = \widetilde{B} \quad \text{and} \quad (f_U)_*\mathcal{O}_{Y_U}|_{Spec(C)} = \widetilde{B'}.$$

By the above we have $B' = C \otimes_R B$. Let $A \subset B$ be the integral closure of R in B and let $A' \subset B'$ be the integral closure of C in B'. Then we have

$$\mathscr{A}|_{Spec(R)} = \widetilde{A} \quad \text{and} \quad \mathscr{A}'|_{Spec(C)} = \widetilde{A'},$$

see Morphisms, Lemma 24.46.1. Hence the lemma is reduced to proving that $C \otimes_R A \cong A'$. This is the content of Algebra, Lemma 7.134.4. $\square$

Lemma 33.29.2. *Let $f : X \to S$ be a morphism of schemes. Assume f is of finite type and separated. Let S' be the normalization of S in X, see Morphisms, Definition 24.46.3. Picture:*

$$\begin{tikzcd} X \arrow[rr, "f'"] \arrow[dr, "f"'] & & S' \arrow[dl, "\nu"] \\ & S & \end{tikzcd}$$

Then there exists an open subscheme $U' \subset S'$ such that

1. *$(f')^{-1}(U') \to U'$ is an isomorphism, and*
2. *$(f')^{-1}(U') \subset X$ is the set of points at which f is quasi-finite.*

Proof. By Morphisms, Lemma 24.47.2 the subset $U \subset X$ of points where f is quasi-finite is open, and $U \to S$ is locally quasi-finite. Let $x \in U$. We want to show that

(a) there exists an open neighbourhood $V'' \subset S'$ of $f'(x)$ such that the morphism $f'|_{(f')^{-1}(V'')} : (f')^{-1}(V'') \to V''$ is an isomorphism.

This will prove the lemma since it will imply that $U' = f(U)$ is open, $f^{-1}(U') = U$ and that $f'|_U : U \to U'$ is an isomorphism.

Let $s = f(x)$. Choose an elementary étale neighbourhood $(T, t) \to (S, s)$ and a decomposition

$$X_T = V \coprod W$$

into open and closed subschemes where $V \to T$ is finite, and such that V has a unique point $v \in V$ in the fibre over t which maps to x, and the fibre of $W \to T$ over t contains no point mapping to x. We can do this according to Lemma 33.28.4. Denote $f_T : X_T \to T$ (resp. f'_T) the base change of f (resp. f'). According to Lemma 33.29.1 the factorization

$$X_T \xrightarrow{f'_T} T \times_S S' \longrightarrow T$$

is the normalization of T in X_T. On the other hand, since X_T is a disjoint union of two schemes over T, we see that the normalization of T in X_T is the morphism

$$X_T = V \coprod W \longrightarrow V' \coprod W' \longrightarrow T$$

where V' is the normalization of T in V and W' is the normalization of T in W (Morphisms, Lemma 24.46.7). However, since $V \to T$ is finite we see that $V \to V'$ is an isomorphism (Morphisms, Lemmas 24.42.4 and 24.46.8). Also, $(f'_T)^{-1}(V') = V$. In other words, we have shown the following

(α) there exists an open neighbourhood $V' \subset X'_T$ of $f'_T(v)$ such that the restriction $(f'_T)^{-1}(V') \to V'$ is an isomorphism.

We will show that property (α) implies property (a) above. Since $T \to S$ is étale we see that $X'_T \to X'$ is étale (Morphisms, Lemma 24.35.4). Hence also $V' \to X'$ is étale, in particular open (Morphisms, Lemmas 24.24.9, 24.35.11 and 24.35.12). Denote $V'' \subset X'$ the image. Note that

$$(f'_T)^{-1}(V') = V' \times_{X'} X = V' \times_{V''} (f')^{-1}(V'')$$

Hence the restriction $f'|_{(f')^{-1}(V'')} : (f')^{-1}(V'') \to V''$ is a morphism whose base change to V' is an isomorphism. Since $\{V' \to V''\}$ is an étale covering, we see that $f'|_{(f')^{-1}(V'')} : (f')^{-1}(V'') \to V''$ is an isomorphism also, by Descent, Lemma 31.19.15. This proves (a) and we are done. $\square$

Lemma 33.29.3. *Let $f : X \to S$ be a morphism of schemes. Assume f is quasi-finite and separated. Let S' be the normalization of S in X, see Morphisms, Definition 24.46.3. Picture:*

$$\begin{tikzcd} X \arrow[rr, "f'"] \arrow[dr, "f"'] & & S' \arrow[dl, "\nu"] \\ & S & \end{tikzcd}$$

Then f' is a quasi-compact open immersion and ν is integral. In particular f is quasi-affine.

Proof. This follows from Lemma 33.29.2. Namely, by that lemma there exists an open suscheme $U' \subset S'$ such that $(f')^{-1}(U') = X$ (!) and $X \to U'$ is an isomorphism! In other words, f' is an open immersion. Note that f' is quasi-compact as f is quasi-compact and $\nu : S' \to S$ is separated (Schemes, Lemma 21.21.15). It follows that f is quasi-affine by Morphisms, Lemma 24.12.3. $\square$

Lemma 33.29.4. *Let $f : X \to S$ be a morphism of schemes. Assume f is quasi-finite and separated and assume that S is quasi-compact and quasi-separated. Then there exists a factorization*

$$\begin{tikzcd} X \arrow[rr, "j"] \arrow[dr, "f"'] & & T \arrow[dl, "\pi"] \\ & S & \end{tikzcd}$$

where j is a quasi-compact open immersion and π is finite.

Proof. Let $X \to S' \to S$ be as in the conclusion of Lemma 33.29.3. By Properties, Lemma 23.20.10 we can write $\nu_*\mathcal{O}_{S'} = colim_\lambda \mathcal{A}_\lambda$ as a directed colimit of quasi-coherent $\mathcal{O}_X$-sub algebras $\mathcal{A}_\lambda$ of finite type. Set $T_\lambda = \underline{Spec}_S(\mathcal{A}_\lambda)$. Since $\mathcal{A}_\lambda$ is a quasi-coherent $\mathcal{O}_X$-algebra of finite type contained in the integral $\mathcal{O}_X$-algebra $\nu_*\mathcal{O}_{S'}$, we see that in fact $\mathcal{A}_\lambda$ is finite as an $\mathcal{O}_X$-module, see Algebra, Lemma 7.32.5. Hence $\pi_\lambda : T_\lambda \to S$ is a finite morphism for each λ. Note that the transition morphisms $T_\lambda \to T_\lambda$ are affine and that $S' = lim_\lambda T_\lambda$.

As S is quasi-compact we may choose a finite affine open covering $S = \bigcup_{i=1,\ldots,n} V_i$. As f' is quasi-compact, we can for each i choose a finite number of elements $h_{ij} \in \Gamma(\nu^{-1}(V_i), \mathcal{O}_{S'})$ such that

$$f'(X) \cap \nu^{-1}(V_i) = \bigcup D(h_{ij}).$$

Let $X_{ij} \subset X$ denote the affine open subscheme mapping isomorphically to $D(h_{ij})$. Since $X \to S$ is of finite type we see that

$$\Gamma(V_i, \mathcal{O}_S) \to \Gamma(X_{ij}, \mathcal{O}_X) = \Gamma(D(h_{ij}), \mathcal{O}_{S'}) = \Gamma(\nu^{-1}(V_i), \mathcal{O}_{S'})_{h_{ij}}$$

is a finite type ring map, see Morphisms, Lemma 24.14.2. Choose finitely many $a_{ijk} \in \Gamma(\nu^{-1}(V_i), \mathcal{O}_{S'})$ which together with h_{ij}^{-1} generate $\Gamma(X_{ij}, \mathcal{O}_X)$ as an $\Gamma(V_i, \mathcal{O}_S)$-algebra. Now, pick λ so large that there exist

$$A_{ijk}, H_{ij} \in \Gamma(\pi_\lambda^{-1}(V_i), \mathcal{O}_{T_\lambda})$$

mapping to the elements a_{ijk}, h_{ij} chosen above. Let $U_\lambda \subset T_\lambda$ be the union of the standard affine opens $D(H_{ij})$ determined by the H_{ij} inside $\pi_\lambda^{-1}(V_i)$. By construction the morphism $X \to T_\lambda$ factors through U_λ. By construction the morphism $X \to T_\lambda$ is a closed immersion, because the ring maps on the affine opens $X_{ij} \to D(H_{ij})$ are surjective by construction. Hence $X \to T_\lambda$ is a locally closed immersion. (In fact this morphism will be an open immersion for sufficiently large λ but we don't need this.) By Morphisms, Lemma 24.2.8 we can factor this as $X \to T \to T_\lambda$ where the first arrow is an open immersion and the second a closed immersion. Thus we win. □

Lemma 33.29.5. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *f is finite,*
(2) *f is proper with finite fibres.*
(3) *f is universally closed, separated, locally of finite type and has finite fibres.*

Proof. We have (1) implies (2) by Morphisms, Lemmas 24.42.10, 24.19.10, and 24.42.9. By definition (2) implies (3).

Assume (3). Pick $s \in S$. By Morphisms, Lemma 24.19.7 we see that all the finitely many points of X_s are isolated in X_s. Choose an elementary étale neighbourhood $(U, u) \to (S, s)$ and decomposition $X_U = V \coprod W$ as in Lemma 33.28.6. Note that $W_u = \emptyset$ because all points of X_s are isolated. Since f is universally closed we see that the image of W in U is a closed set not containing u. After shrinking U we may assume that $W = \emptyset$. In other words we see that $X_U = V$ is finite over U. Since $s \in S$ was arbitrary this means there exists a family $\{U_i \to S\}$ of étale morphisms whose images cover S such that the base changes $X_{U_i} \to U_i$ are finite. Note that $\{U_i \to S\}$ is an étale covering, see Topologies, Definition 30.4.1. Hence it is an fpqc covering, see Topologies, Lemma 30.8.6. Hence we conclude f is finite by Descent, Lemma 31.19.21. □

As a consequence we have the following two useful results.

Lemma 33.29.6. *Let $f : X \to S$ be a morphism of schemes. Let $s \in S$. Assume that f is proper and $f^{-1}(\{s\})$ is a finite set. Then there exists an open neighbourhood $V \subset S$ of s such that $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is finite.*

Proof. The morphism f is quasi-finite at all the points of $f^{-1}(\{s\})$ by Morphisms, Lemma 24.19.7. By Morphisms, Lemma 24.47.2 the set of points at which f is quasi-finite is an open $U \subset X$. Let $Z = X \setminus U$. Then $s \notin f(Z)$. Since f is proper the set $f(Z) \subset S$ is closed. Choose any open neighbourhood $V \subset S$ of s with $Z \cap V = \emptyset$. Then $f^{-1}(V) \to V$ is locally quasi-finite and proper. Hence it is quasi-finite (Morphisms, Lemma 24.19.9), hence has finite fibres (Morphisms, Lemma 24.19.10), hence is finite by Lemma 33.29.5. □

Lemma 33.29.7. *Let $f : Y \to X$ be a quasi-finite morphism. There exists a dense open $U \subset X$ such that $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is finite.*

Proof. If $U', U'' \subset X$ are opens such that the restrictions $f|_{f^{-1}(U')} : f^{-1}(U') \to U'$ and $f|_{f^{-1}(U'')} : f^{-1}(U'') \to U''$ are finite, then for $U = U' \cup U''$ the restriction $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is finite, see Morphisms, Lemma 24.42.3. Thus the problem is local on X and we may assume that X is affine.

Assume X is affine. Write $Y = \bigcup_{j=1,\ldots,m} V_j$ with V_j affine. This is possible since f is quasi-finite and hence in particular quasi-compact. Each $V_j \to X$ is quasi-finite and separated. Let $\eta \in X$ be a generic point of an irreducible component of X. We see from Morphisms, Lemmas 24.19.10 and 24.45.1 that there exists an open neighbourhood $\eta \in U_\eta$ such that $f^{-1}(U_\eta) \cap V_j \to U_\eta$ is finite. We may choose U_η such that it works for each $j = 1, \ldots, m$. Note that the collection of generic points of X is dense in X. Thus we see there exists a dense open $W = \bigcup_\eta U_\eta$ such that each $f^{-1}(W) \cap V_j \to W$ is finite. It suffices to show that there exists a dense open $U \subset W$ such that $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is finite. Thus we may replace X by an affine open subscheme of W and assume that each $V_j \to X$ is finite.

Assume X is affine, $Y = \bigcup_{j=1,\ldots,m} V_j$ with V_j affine, and the restrictions $f|_{V_j} : V_j \to X$ are finite. Set

$$\Delta_{ij} = \left(\overline{V_i \cap V_j} \setminus V_i \cap V_j\right) \cap V_j.$$

This is a nowhere dense closed subset of V_j because it is the boundary of the open subset $V_i \cap V_j$ in V_j. By Morphisms, Lemma 24.44.7 the image $f(\Delta_{ij})$ is a nowhere dense closed subset of X. By Topology, Lemma 5.17.2 the union $T = \bigcup f(\Delta_{ij})$ is a nowhere dense closed subset of X. Thus $U = X \setminus T$ is a dense open subset of X. We claim that $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is finite. To see this let $U' \subset U$ be an affine open. Set $Y' = f^{-1}(U') = U' \times_X Y$, $V'_j = Y' \cap V_j = U' \times_X V_j$. Consider the restriction

$$f' = f|_{Y'} : Y' \longrightarrow U'$$

of f. This morphism now has the property that $Y' = \bigcup_{j=1,\ldots,m} V'_j$ is an affine open covering, each $V'_j \to U'$ is finite, and $V'_i \cap V'_j$ is (open and) closed both in V'_i and V'_j. Hence $V'_i \cap V'_j$ is affine, and the map

$$\mathcal{O}(V'_i) \otimes_{\mathbf{Z}} \mathcal{O}(V'_j) \longrightarrow \mathcal{O}(V'_i \cap V'_j)$$

is surjective. This implies that Y' is separated, see Schemes, Lemma 21.21.8. Finally, consider the commutative diagram

$$\begin{tikzcd} \coprod_{j=1,\ldots,m} V'_j \arrow[rr] \arrow[dr] & & Y' \arrow[dl] \\ & U' & \end{tikzcd}$$

The south-east arrow is finite, hence proper, the horizontal arrow is surjective, and the south-west arrow is separated. Hence by Morphisms, Lemma 24.40.8 we conclude that $Y' \to U'$ is proper. Since it is also quasi-finite, we see that it is finite by Lemma 33.29.5, and we win. □

33.30. Application to morphisms with connected fibres

In this section we prove some lemmas that produce morphisms all of whose fibres are geometrically connected or geometrically integral. This will be useful in our study of the local structure of morphisms of finite type later.

Lemma 33.30.1. *Consider a diagram of morphisms of schemes*

$$\begin{tikzcd} Z \arrow[r, "\sigma"'] \arrow[dr] & X \arrow[d] \\ & Y \end{tikzcd}$$

an a point $y \in Y$. Assume

(1) *$X \to Y$ is of finite presentation and flat,*
(2) *$Z \to Y$ is finite locally free,*
(3) *$Z_y \neq \emptyset$,*
(4) *all fibres of $X \to Y$ are geometrically reduced, and*
(5) *X_y is geometrically connected over $\kappa(y)$.*

Then there exists an open $X^0 \subset X$ such that $X^0_y = X_y$ and such that all fibres of $X^0 \to Y$ are geometrically connected.

Proof. In this proof we will use that flat, finite presentation, finite locally free are properties that are preserved under base change and composition. We will also use that a finite locally free morphism is both open and closed. You can find these facts as Morphisms, Lemmas 24.24.7, 24.20.4, 24.44.4, 24.24.5, 24.20.3, 24.44.3, 24.24.9, and 24.42.10.

Note that $X_Z \to Z$ is flat morphism of finite presentation which has a section s coming from σ. Let X^0_Z denote the subset of X_Z defined in Situation 33.22.1. By Lemma 33.22.6 it is an open subset of X_Z.

The pullback $X_{Z\times_Y Z}$ of X to $Z \times_Y Z$ comes equipped with two sections s_0, s_1, namely the base changes of s by $\mathrm{pr}_0, \mathrm{pr}_1 : Z \times_Y Z \to Z$. The construction of Situation 33.22.1 gives two subsets $(X_{Z\times_Y Z})^0_{s_0}$ and $(X_{Z\times_Y Z})^0_{s_1}$. By Lemma 33.22.2 these are the inverse images of X^0_Z under the morphisms $1_X \times \mathrm{pr}_0, 1_X \times \mathrm{pr}_1 : X_{Z\times_Y Z} \to X_Z$. In particular these subsets are open.

Let $(Z \times_Y Z)_y = \{z_1, \ldots, z_n\}$. As X_y is geometrically connected, we see that the fibres of $(X_{Z\times_Y Z})^0_{s_0}$ and $(X_{Z\times_Y Z})^0_{s_1}$ over each z_i agree (being equal to the whole fibre). Another way to say this is that

$$s_0(z_i) \in (X_{Z\times_Y Z})^0_{s_1} \quad \text{and} \quad s_1(z_i) \in (X_{Z\times_Y Z})^0_{s_0}.$$

Since the sets $(X_{Z\times_Y Z})^0_{s_0}$ and $(X_{Z\times_Y Z})^0_{s_1}$ are open in $X_{Z\times_Y Z}$ there exists an open neighbourhood $W \subset Z \times_Y Z$ of $(Z \times_Y Z)_y$ such that

$$s_0(W) \subset (X_{Z\times_Y Z})^0_{s_1} \quad \text{and} \quad s_1(W) \subset (X_{Z\times_Y Z})^0_{s_0}.$$

Then it follows directly from the construction in Situation 33.22.1 that

$$p^{-1}(W) \cap (X_{Z\times_Y Z})^0_{s_0} = p^{-1}(W) \cap (X_{Z\times_Y Z})^0_{s_1}$$

where $p : X_{Z\times_Y Z} \to Z \times_W Z$ is the projection. Because $Z \times_Y Z \to Y$ is finite locally free, hence open and closed, there exists an open neighbourhood $V \subset Y$ of y such that $q^{-1}(V) \subset W$, where $q : Z \times_Y Z \to Y$ is the structure morphism. To prove the lemma we may replace Y by V (because an empty topological space is connected). After we do this we see that $X^0_Z \subset Y_Z$ is an open such that

$$(1_X \times \text{pr}_0)^{-1}(X^0_Z) = (1_X \times \text{pr}_1)^{-1}(X^0_Z).$$

This means that the image $X^0 \subset X$ of X^0_Z is an open such that $(X_Z \to X)^{-1}(X^0) = X^0_Z$, see Descent, Lemma 31.9.2. At this point it is clear that X^0 is the desired open subscheme. □

Lemma 33.30.2. *Let $h : Y \to S$ be a morphism of schemes. Let $s \in S$ be a point. Let $T \subset Y_s$ be an open subscheme. Assume*

(1) *h is flat and of finite presentation,*
(2) *all fibres of h are geometrically reduced, and*
(3) *T is geometrically connected over $\kappa(s)$.*

Then we can find an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open $V \subset Y_{S'}$ such that

(a) *all fibres of $V \to S'$ are geometrically connected,*
(b) *$V_{s'} = T \times_s s'$.*

Proof. The problem is clearly local on S, hence we may replace S by an affine open neighbourhood of s. The topology on Y_s is induced from the topology on X, see Schemes, Lemma 21.18.5. Hence we can find a quasi-compact open $V \subset Y$ such that $V_s = T$. The restriction of h to V is quasi-compact (as S affine and V quasi-compact), quasi-separated, locally of finite presentation, and flat hence flat of finite presentation. Thus after replacing Y by V we may assume, in addition to (1) and (2) that $Y_s = T$ and S affine.

Pick a point $y \in Y_s$ such that h is Cohen-Macaulay at y, see Lemma 33.15.4. By Lemma 33.16.4 there exists a diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[dr] & Y \arrow[d] \\ & S \end{tikzcd}$$

such that $Z \to S$ is flat, locally of finite presentation, locally quasi-finite with $Z_s = \{z\}$. Apply Lemma 33.28.1 to find an elementary neighbourhood $(S', s') \to (S, s)$ and an open $Z' \subset Z_{S'} = S' \times_S Z$ with $Z' \to S'$ finite with a unique point $z' \in Z'$ lying over s. Note that $Z' \to S'$ is also locally of finite presentation and flat (as an open of the base change of $Z \to S$), hence $Z' \to S'$ is finite locally free, see Morphisms, Lemma 24.44.2. Note that $Y_{S'} \to S'$ is flat and of finite presentation with geometrically reduced fibres as a base change of h. Also $Y_{s'} = Y_s$ is geometrically connected. To finish the proof apply Lemma 33.30.1 to $Z' \to Y_{S'}$ over S'. $\square$

Lemma 33.30.3. *Let $h : Y \to S$ be a morphism of schemes. Let $s \in S$ be a point. Let $T \subset Y_s$ be an open subscheme. Assume*

(1) *h is of finite presentation,*
(2) *h is normal, and*
(3) *T is geometrically irreducible over $\kappa(s)$.*

Then we can find an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open $V \subset Y_{S'}$ such that

(a) *all fibres of $V \to S'$ are geometrically integral,*
(b) *$V_{s'} = T \times_s s'$.*

Proof. Apply Lemma 33.30.2 to find an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open $V \subset Y_{S'}$ such that all fibres of $V \to S'$ are geometrically integral and $V_{s'} = T \times_s s'$. Note that $V \to S'$ is open, see Morphisms, Lemma 24.24.9 Hence after replacing S' by the image of $V \to S'$ we see that all fibres of $V \to S'$ are nonempty. As V is an open of the base change of h all fibres of $V \to S'$ are geometrically normal, see Lemma 33.13.2. In particular, they are geometrically reduced. To finish the proof we have to show they are geometrically irreducible. But, if $t \in S'$ then V_t is of finite type over $\kappa(t)$ and hence $V_t \times_{\kappa(t)} \overline{\kappa(t)}$ is of finite type over $\overline{\kappa(t)}$ hence Noetherian. By choice of $S' \to S$ the scheme $V_t \times_{\kappa(t)} \overline{\kappa(t)}$ is connected. Hence $V_t \times_{\kappa(t)} \overline{\kappa(t)}$ is irreducible by Properties, Lemma 23.7.6 and we win. $\square$

33.31. Application to the structure of finite type morphisms

The result in this section can be found in [**GR71**]. Loosely stated it says that a finite type morphism is étale locally on the source and target the composition of a finite morphism by a smooth morphism with geometrically connected fibres of relative dimension equal to the fibre dimension of the original morphism.

Lemma 33.31.1. *Let $f : X \to S$ be a morphism. Let $x \in X$ and set $s = f(x)$. Assume that f is locally of finite type and that $n = \dim_x(X_s)$. Then there exists a commutative diagram*

$$\begin{tikzcd} X \arrow[dd] & X' \arrow[l, "g"'] \arrow[d, "\pi"] \\ & Y \arrow[d, "h"] \\ S \arrow[r, equal] & S \end{tikzcd} \qquad \begin{tikzcd} x \arrow[dd, mapsto] & x' \arrow[l, mapsto] \arrow[d, mapsto] \\ & y \arrow[d, mapsto] \\ s \arrow[r, equal] & s \end{tikzcd}$$

and a point $x' \in X'$ with $g(x') = x$ such that with $y = \pi(x')$ we have

(1) *$h : Y \to S$ is smooth of relative dimension n,*
(2) *$g : (X', x') \to (X, x)$ is an elementary étale neighbourhood,*
(3) *π is finite, and $\pi^{-1}(\{y\}) = \{x'\}$, and*
(4) *$\kappa(y)$ is a purely transcendental extension of $\kappa(s)$.*

Moreover, if f is locally of finite presentation then π is of finite presentation.

Proof. The problem is local on X and S, hence we may assume that X and S are affine. By Algebra, Lemma 7.116.3 after replacing X by a standard open neighbourhood of x in X we may assume there is a factorization

$$X \xrightarrow{\pi} \mathbf{A}^n_S \longrightarrow S$$

such that π is quasi-finite and such that $\kappa(\pi(x))$ is purely transcendental over $\kappa(s)$. By Lemma 33.28.1 there exists an elementary étale neighbourhood

$$(Y, y) \to (\mathbf{A}^n_S, \pi(x))$$

and an open $X' \subset X \times_{\mathbf{A}^n_S} Y$ which contains a unique point x' lying over y such that $X' \to Y$ is finite. This proves (1) -- (4) hold. For the final assertion, use Morphisms, Lemma 24.20.11. $\square$

Lemma 33.31.2. *Let $f : X \to S$ be a morphism. Let $x \in X$ and set $s = f(x)$. Assume that f is locally of finite type and that $n = \dim_x(X_s)$. Then there exists a commutative diagram*

$$\begin{tikzcd} X \arrow[dd] & X' \arrow[l, "g"'] \arrow[d, "\pi"] \\ & Y' \arrow[d, "h"] \\ S & S' \arrow[l, "e"'] \end{tikzcd} \qquad \begin{tikzcd} x \arrow[dd, mapsto] & x' \arrow[l, mapsto] \arrow[d, mapsto] \\ & y' \arrow[d, mapsto] \\ s & s' \arrow[l, mapsto] \end{tikzcd}$$

and a point $x' \in X'$ with $g(x') = x$ such that with $y' = \pi(x')$, $s' = h(y')$ we have

(1) *$h : Y' \to S'$ is smooth of relative dimension n,*
(2) *all fibres of $Y' \to S'$ are geometrically integral,*
(3) *$g : (X', x') \to (X, x)$ is an elementary étale neighbourhood,*
(4) *π is finite, and $\pi^{-1}(\{y'\}) = \{x'\}$,*
(5) *$\kappa(y')$ is a purely transcendental extension of $\kappa(s')$, and*
(6) *$e : (S', s') \to (S, s)$ is an elementary étale neighbourhood.*

Moreover, if f is locally of finite presentation, then π is of finite presentation.

Proof. The question is local on S, hence we may replace S by an affine open neighbourhood of s. Next, we apply Lemma 33.31.1 to get a commutative diagram

$$\begin{tikzcd} X \arrow[dd] & X' \arrow[l, "g"'] \arrow[d, "\pi"] \\ & Y \arrow[d, "h"] \\ S \arrow[r, equal] & S \end{tikzcd} \qquad \begin{tikzcd} x \arrow[dd, mapsto] & x' \arrow[l, mapsto] \arrow[d, mapsto] \\ & y \arrow[d, mapsto] \\ s \arrow[r, equal] & s \end{tikzcd}$$

where h is smooth of relative dimension n and $\kappa(y)$ is a purely transcendental extension of $\kappa(s)$. Since the question is local on X also, we may replace Y by an affine neighbourhood of y (and X' by the inverse image of this under π). As S is affine this guarantees that $Y \to S$ is quasi-compact, separated and smooth, in particular of finite presentation. Let T be the connected component of Y_s containing y. As Y_s is Noetherian we see that T is open. We also see that T is geometrically connected over $\kappa(s)$ by Varieties, Lemma 28.5.14. Since T is also smooth over $\kappa(s)$ it is geometrically normal, see Varieties, Lemma 28.15.4. We conclude that T is geometrically irreducible over $\kappa(s)$ (as a connected Noetherian normal scheme is irreducible, see Properties, Lemma 23.7.6). Finally, note that the smooth morphism h is normal by Lemma 33.13.3. At this point we have verified all assumption of Lemma 33.30.3 hold for the morphism $h : Y \to S$ and open $T \subset Y_s$. As a result of applying Lemma 33.30.3 we obtain $e : S' \to S$, $s' \in S'$, Y' as in the commutative diagram

$$\begin{tikzcd} X \arrow[dd] & X' \arrow[l, "g"'] \arrow[d, "\pi"] & X' \times_Y Y' \arrow[l] \arrow[d] \\ & Y \arrow[d, "h"] & Y' \arrow[l] \arrow[d] \\ S \arrow[r, equal] & S & S' \arrow[l, "e"'] \end{tikzcd} \qquad \begin{tikzcd} x \arrow[dd, maps to] & x' \arrow[l, maps to] \arrow[d, maps to] & (x', s') \arrow[l, maps to] \arrow[d, maps to] \\ & y \arrow[d, maps to] & (y, s') \arrow[l, maps to] \arrow[d, maps to] \\ s \arrow[r, equal] & s & s' \arrow[l, maps to] \end{tikzcd}$$

where $e : (S', s') \to (S, s)$ is an elementary étale neighbourhood, and where $Y' \subset Y_{S'}$ is an open neighbourhood all of whose fibres over S' are geometrically irreducible, such that $Y'_{s'} = T$ via the identification $Y_s = Y_{S',s'}$. Let $(y, s') \in Y'$ be the point corresponding to $y \in T$; this is also the unique point of $Y \times_S S'$ lying over y with residue field equal to $\kappa(y)$ which maps to s' in S'. Similarly, let $(x', s') \in X' \times_Y Y' \subset X' \times_S S'$ be the unique point over x' with residue field equal to $\kappa(x')$ lying over s'. Then the outer part of this diagram is a solution to the problem posed in the lemma. Some minor details omitted. $\square$

Lemma 33.31.3. *Assumption and notation as in Lemma 33.31.2. In addition to properties (1) -- (6) we may also arrange it so that*

(7) *S', Y', X' are affine.*

Proof. Note that if Y' is affine, then X' is affine as π is finite. Choose an affine open neighbourhood $U' \subset S'$ of s'. Choose an affine open neighbourhood $V' \subset h^{-1}(U')$ of y'. Let $W' = h(V')$. This is an open neighbourhood of s' in S', see Morphisms, Lemma 24.33.10, contained in U'. Choose an affine open neighbourhood $U'' \subset W'$ of s'. Then $h^{-1}(U'') \cap V'$ is affine because it is equal to $U'' \times_{U'} V'$. By construction $h^{-1}(U'') \cap V' \to U''$ is a surjective smooth morphism whose fibres are (nonempty) open subschemes of geometrically integral fibres of $Y' \to S'$, and hence geometrically integral. Thus we may replace S' by U'' and Y' by $h^{-1}(U'') \cap V'$. $\square$

The significance of the property $\pi^{-1}(\{y'\}) = \{x'\}$ is partially explained by the following lemma.

Lemma 33.31.4. *Let $\pi : X \to Y$ be a finite morphism. Let $x \in X$ with $y = \pi(x)$ such that $\pi^{-1}(\{y\}) = \{x\}$. Then*

(1) *For every neighbourhood $U \subset X$ of x in X, there exists a neighbourhood $V \subset Y$ of y such that $\pi^{-1}(V) \subset U$.*
(2) *The ring map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is finite.*
(3) *If π is of finite presentation, then $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is of finite presentation.*
(4) *For any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have $\mathcal{F}_x = \pi_*\mathcal{F}_y$ as $\mathcal{O}_{Y,y}$-modules.*

Proof. The first assertion is purely topological; use that π is a continuous and closed map such that $\pi^{-1}(\{y\}) = \{x\}$. To prove the second and third parts we may assume $X = Spec(B)$ and $Y = Spec(A)$. Then $A \to B$ is a finite ring map and y corresponds to a prime $\mathfrak{p}$ of A such that there exists a unique prime $\mathfrak{q}$ of B lying over $\mathfrak{p}$. Then $B_\mathfrak{q} = B_\mathfrak{p}$, see Algebra, Lemma 7.36.11. In other words, the map $A_\mathfrak{p} \to B_\mathfrak{q}$ is equal to the map $A_\mathfrak{p} \to B_\mathfrak{p}$ you get from localizing $A \to B$ at $\mathfrak{p}$. Thus (2) and (3) follow from simple properties of localization (some details omitted). For the final statement, suppose that $\mathcal{F} = \widetilde{M}$ for some B-module M. Then $\mathcal{F} = M_{\mathfrak{q}}$ and $\pi_*\mathcal{F}_y = M_{\mathfrak{p}}$. By the above these localizations agree. Alternatively you can use part (1) and the definition of stalks to see that $\mathcal{F}_x = \pi_*\mathcal{F}_y$ directly. $\square$

33.32. Application to the fppf topology

We can use the above étale localization techniques to prove the following result describing the fppf topology as being equal to the topology ``generated by'' Zariski coverings and by coverings of the form $\{f : T \to S\}$ where f is surjective finite locally free.

Lemma 33.32.1. *Let S be a scheme. Let $\{S_i \to S\}_{i \in I}$ be an fppf covering. Then there exist*

(1) *a Zariski open covering $S = \bigcup U_j$,*
(2) *surjective finite locally free morphisms $W_j \to U_j$,*
(3) *Zariski open coverings $W_j = \bigcup_k W_{j,k}$,*
(4) *surjective finite locally free morphisms $T_{j,k} \to W_{j,k}$*

such that the fppf covering $\{T_{j,k} \to S\}$ refines the given covering $\{S_i \to S\}$.

Proof. We may assume that each $S_i \to S$ is locally quasi-finite, see Lemma 33.16.6.

Fix a point $s \in S$. Pick an $i \in I$ and a point $s_i \in S_i$ mapping to s. Choose an elementary étale neighbourhood $(S', s) \to (S, s)$ such that there exists an open

$$S_i \times_S S' \supset V$$

which contains a unique point $v \in V$ mapping to $s \in S'$ and such that $V \to S'$ is finite, see Lemma 33.28.1. Then $V \to S'$ is finite locally free, because it is finite and because $S_i \times_S S' \to S'$ is flat and locally of finite presentation as a base change of the morphism $S_i \to S$, see Morphisms, Lemmas 24.20.4, 24.24.7, and 24.44.2. Hence $V \to S'$ is open, and after shrinking S' we may assume that $V \to S'$ is surjective finite locally free. Since we can do this for every point of S we conclude that $\{S_i \to S\}$ can be refined by a covering of the form $\{V_a \to S\}_{a \in A}$ where each $V_a \to S$ factors as $V_a \to S'_a \to S$ with $S'_a \to S$ étale and $V_a \to S'_a$ surjective finite locally free.

By Remark 33.27.3 there exists a Zariski open covering $S = \bigcup U_j$, for each j a finite locally free, surjective morphism $W_j \to U_j$, and for each j a Zariski open covering $\{W_{j,k} \to W_j\}$ such that the family $\{W_{j,k} \to S\}$ refines the étale covering $\{S'_a \to S\}$, i.e., for each pair j, k there exists an $a(j,k)$ and a factorization $W_{j,K} \to S'_a \to S$ of the morphism $W_{j,K} \to S$. Set $T_{j,k} = W_{j,k} \times_{S'_a} V_a$ and everything is clear. $\square$

33.33. Closed points in fibres

Some of the material in this section is taken from the preprint [**OP10**].

Lemma 33.33.1. *Let $f : X \to S$ be a morphism of schemes. Let $Z \subset X$ be a closed subscheme. Let $s \in S$. Assume*

(1) *S is irreducible with generic point η,*
(2) *X is irreducible,*
(3) *f is dominant,*
(4) *f is locally of finite type,*
(5) *$\dim(X_s) \leq \dim(X_\eta)$,*
(6) *Z is locally principal in X, and*

(7) $Z_\eta = \emptyset$.

Then the fibre Z_s is (set theoretically) a union of irreducible components of X_s.

Proof. Let X_{red} denote the reduction of X. Then $Z \cap X_{red}$ is a locally principal closed subscheme of X_{red}, see Divisors, Lemma 26.9.7. Hence we may assume that X is reduced. In other words X is integral, see Properties, Lemma 23.3.4. In this case the morphism $X \to S$ factors through S_{red}, see Schemes, Lemma 21.12.6. Thus we may replace S by S_{red} and assume that S is integral too.

The assertion that f is dominant signifies that the generic point of X is mapped to η, see Morphisms, Lemma 24.6.5. Moreover, the scheme X_η is an integral scheme which is locally of finite type over the field $\kappa(\eta)$. Hence $d = \dim(X_\eta) \geq 0$ is equal to $\dim_\xi(X_\eta)$ for every point ξ of X_η, see Algebra, Lemmas 7.105.4 and 7.105.5. In view of Morphisms, Lemma 24.27.4 and condition (5) we conclude that $\dim_x(X_s) = d$ for every $x \in X_s$.

In the Noetherian case the assertion can be proved as follows. If the lemma does not holds there exists $x \in Z_s$ which is a generic point of an irreducible component of Z_s but not a generic point of any irreducible component of X_s. Then we see that $\dim_x(Z_s) \leq d - 1$, because $\dim_x(X_s) = d$ and in a neighbourhood of x in X_s the closed subscheme Z_s does not contain any of the irreducible components of X_s. Hence after replacing X by an open neighbourhood of x we may assume that $\dim_z(Z_{f(z)}) \leq d-1$ for all $z \in Z$, see Morphisms, Lemma 24.27.4. Let $\xi' \in Z$ be a generic point of an irreducible component of Z and set $s' = f(\xi)$. As $Z \neq X$ is locally principal we see that $\dim(\mathcal{O}_{X,\xi}) = 1$, see Algebra, Lemma 7.57.10 (this is where we use X is Noetherian). Let $\xi \in X$ be the generic point of X and let ξ_1 be a generic point of any irreducible component of $X_{s'}$ which contains ξ'. Then we see that we have the specializations

$$\xi \rightsquigarrow \xi_1 \rightsquigarrow \xi'.$$

As $\dim(\mathcal{O}_{X,\xi}) = 1$ one of the two specializations has to be an equality. By assumption $s' \neq \eta$, hence the first specialization is not an equality. Hence $\xi' = \xi_1$ is a generic point of an irreducible component of $X_{s'}$. Applying Morphisms, Lemma 24.27.4 one more time this implies $\dim_{\xi'}(Z_{s'}) = \dim_{\xi'}(X_{s'}) \geq \dim(X_\eta) = d$ which gives the desired contradiction.

In the general case we reduce to the Noetherian case as follows. If the lemma is false then there exists a point $x \in X$ lying over s such that x is a generic point of an irreducible component of Z_s, but not a generic point of any of the irreducible components of X_s. Let $U \subset S$ be an affine neighbourhood of s and let $V \subset X$ be an affine neighbourhood of x with $f(V) \subset U$. Write $U = Spec(A)$ and $V = Spec(B)$ so that $f|_V$ is given by a ring map $A \to B$. Let $\mathfrak{q} \subset B$, resp. $\mathfrak{p} \subset A$ be the prime corresponding to x, resp. s. After possibly shrinking V we may assume $Z \cap V$ is cut out by some element $g \in B$. Denote $K = f.f.(A)$. What we know at this point is the following:

(1) $A \subset B$ is a finitely generated extension of domains,
(2) the element $g \otimes 1$ is invertible in $B \otimes_A K$,
(3) $d = \dim(B \otimes_A K) = \dim(B \otimes_A \kappa(\mathfrak{p}))$,
(4) $g \otimes 1$ is not a unit of $B \otimes_A \kappa(\mathfrak{p})$, and
(5) $g \otimes 1$ is not in any of the minimal primes of $B \otimes_A \kappa(\mathfrak{p})$.

We are seeking a contradiction.

Pick elements $x_1, \ldots, x_n \in B$ which generate B over A. For a finitely generated $\mathbf{Z}$-algebra $A_0 \subset A$ let $B_0 \subset B$ be the A_0-subalgebra generated by $x_1, \ldots, x_n$, denote $K_0 = f.f.(A_0)$, and set $\mathfrak{p}_0 = A_0 \cap \mathfrak{p}$. We claim that when A_0 is large enough then (1) -- (5) also hold for the system $(A_0 \subset B_0, g, \mathfrak{p}_0)$.

We prove each of the conditions in turn. Part (1) holds by construction. For part (2) write $(g \otimes 1)h = 1$ for some $h \otimes 1/a \in B \otimes_A K$. Write $g = \sum a_I x^I$, $h = \sum a'_I x^I$ (multi-index notation) for some coefficients $a_I, a'_I \in A$. As soon as A_0 contains a and the a_I, a'_I then (2) holds because $B_0 \otimes_{A_0} K_0 \subset B \otimes_A K$ (as localizations of the injective map $B_0 \to B$). To achieve (3) consider the exact sequence

$$0 \to I \to A[X_1, \ldots, X_n] \to B \to 0$$

which defines I where the second map sends X_i to x_i. Since $\otimes$ is right exact we see that $I \otimes_A K$, respectively $I \otimes_A \kappa(\mathfrak{p})$ is the kernel of the surjection $K[X_1, \ldots, X_n] \to B \otimes_A K$, respectively $\kappa(\mathfrak{p})[X_1, \ldots, X_n] \to B \otimes_A \kappa(\mathfrak{p})$. As a polynomial ring over a field is Noetherian there exist finitely many elements $h_j \in I$, $j = 1, \ldots, m$ which generate $I \otimes_A K$ and $I \otimes_A \kappa(\mathfrak{p})$. Write $h_j = \sum a_{j,I} X^I$. As soon as A_0 contains all $a_{j,I}$ we get to the situation where

$$B_0 \otimes_{A_0} K_0 \otimes_{K_0} K = B \otimes_A K \quad \text{and} \quad B_0 \otimes_{A_0} \kappa(\mathfrak{p}_0) \otimes_{\kappa(\mathfrak{p}_0)} \kappa(\mathfrak{p}) = B \otimes_A \kappa(\mathfrak{p}).$$

By either Morphisms, Lemma 24.27.3 or Algebra, Lemma 7.107.5 we see that the dimension equalities of (3) are satisfied. Part (4) is immediate. As $B_0 \otimes_{A_0} \kappa(\mathfrak{p}_0) \subset B \otimes_A \kappa(\mathfrak{p})$ each minimal prime of $B_0 \otimes_{A_0} \kappa(\mathfrak{p}_0)$ lies under a minimal prime of $B \otimes_A \kappa(\mathfrak{p})$ by Algebra, Lemma 7.27.6. This implies that (5) holds. In this way we reduce the problem to the Noetherian case which we have dealt with above. □

Here is an algebraic application of the lemma above. The fourth assumption of the lemma holds if $A \to B$ is flat, see Lemma 33.33.3.

Lemma 33.33.2. *Let $A \to B$ be a local homomorphism of local rings, and $g \in \mathfrak{m}_B$. Assume*

1. *A and B are domains and $A \subset B$,*
2. *B is essentially of finite type over A,*
3. *g is not contained in any minimal prime over $\mathfrak{m}_A B$, and*
4. $\dim(B/\mathfrak{m}_A B) + trdeg_{\kappa(\mathfrak{m}_A)}(\kappa(\mathfrak{m}_B)) = trdeg_A(B)$.

Then $A \subset B/gB$, i.e., the generic point of $Spec(A)$ is in the image of the morphism $Spec(B/gB) \to Spec(A)$.

Proof. Note that the two assertions are equivalent by Algebra, Lemma 7.27.6. To start the proof let C be an A-algebra of finite type and $\mathfrak{q}$ a prime of C such that $B = C_\mathfrak{q}$. Of course we may assume that C is a domain and that $g \in C$. After replacing C by a localization we see that $\dim(C/\mathfrak{m}_A C) = \dim(B/\mathfrak{m}_A B) + \text{trdeg}_{\kappa(\mathfrak{m}_A)}(\kappa(\mathfrak{m}_B))$, see Morphisms, Lemma 24.27.1. Setting $K = f.f.(A)$ we see by the same reference that $\dim(C \otimes_A K) = \text{trdeg}_A(B)$. Hence assumption (4) means that the generic and closed fibres of the morphism $Spec(C) \to Spec(A)$ have the same dimension.

Suppose that the lemma is false. Then $(B/gB) \otimes_A K = 0$. This means that $g \otimes 1$ is invertible in $B \otimes_A K = C_\mathfrak{q} \otimes_A K$. As $C_\mathfrak{q}$ is a limit of principal localizations we conclude that $g \otimes 1$ is invertible in $C_h \otimes_A K$ for some $h \in C$, $h \not\in \mathfrak{q}$. Thus after replacing C by C_h we may assume that $(C/gC) \otimes_A K = 0$. We do one more replacement of C to make sure that the minimal primes of $C/\mathfrak{m}_A C$ correspond one-to-one with the minimal primes of $B/\mathfrak{m}_A B$. At this point we apply Lemma 33.33.1 to $X = Spec(C) \to Spec(A) = S$ and the locally closed subscheme $Z = Spec(C/gC)$. Since $Z_K = \emptyset$ we see that $Z \otimes \kappa(\mathfrak{m}_A)$ has to contain an irreducible component of $X \otimes \kappa(\mathfrak{m}_A) = Spec(C/\mathfrak{m}_A C)$. But this contradicts the assumption that g is not contained in any prime minimal over $\mathfrak{m}_A B$. The lemma follows. $\square$

Lemma 33.33.3. *Let $A \to B$ be a local homomorphism of local rings. Assume*

(1) *A and B are domains and $A \subset B$,*
(2) *B is essentially of finite type over A, and*
(3) *B is flat over A.*

Then we have

$$\dim(B/\mathfrak{m}_A B) + trdeg_{\kappa(\mathfrak{m}_A)}(\kappa(\mathfrak{m}_B)) = trdeg_A(B).$$

Proof. Let C be an A-algebra of finite type and $\mathfrak{q}$ a prime of C such that $B = C_\mathfrak{q}$. We may assume C is a domain. We have $\dim_\mathfrak{q}(C/\mathfrak{m}_A C) = \dim(B/\mathfrak{m}_A B) + \text{trdeg}_{\kappa(\mathfrak{m}_A)}(\kappa(\mathfrak{m}_B))$, see Morphisms, Lemma 24.27.1. Setting $K = f.f.(A)$ we see by the same reference that $\dim(C \otimes_A K) = \text{trdeg}_A(B)$. Thus we are really trying to prove that $\dim_\mathfrak{q}(C/\mathfrak{m}_A C) = \dim(C \otimes_A K)$. Choose a valuation ring A' in K dominating A, see Algebra, Lemma 7.46.2. Set $C' = C \otimes_A A'$. Choose a prime $\mathfrak{q}'$ of C' lying over $\mathfrak{q}$; such a prime exists because

$$C'/\mathfrak{m}_{A'}C' = C/\mathfrak{m}_A C \otimes_{\kappa(\mathfrak{m}_A)} \kappa(\mathfrak{m}_{A'})$$

which proves that $C/\mathfrak{m}_A C \to C'/\mathfrak{m}_{A'}C'$ is faithfully flat. This also proves that $\dim_\mathfrak{q}(C/\mathfrak{m}_A C) = \dim_{\mathfrak{q}'}(C'/\mathfrak{m}_{A'}C')$, see Algebra, Lemma 7.107.6. Note that $B' = C'_{\mathfrak{q}'}$ is a localization of $B \otimes_A A'$. Hence B' is flat over A'. The generic fibre $B' \otimes_{A'} K$ is a localization of $B \otimes_A K$. Hence B' is a domain. If we prove the lemma for $A' \subset B'$, then we get the equality $\dim_{\mathfrak{q}'}(C'/\mathfrak{m}_{A'}C') = \dim(C' \otimes_{A'} K)$ which implies the desired equality $\dim_\mathfrak{q}(C/\mathfrak{m}_A C) = \dim(C \otimes_A K)$ by what was said above. This reduces the lemma to the case where A is a valuation ring.

Let $A \subset B$ be as in the lemma with A a valuation ring. As before write $B = C_\mathfrak{q}$ for some domain C of finite type over A. By Algebra, Lemma 7.116.9 we obtain $\dim(C/\mathfrak{m}_A C) = \dim(C \otimes_A K)$ and we win. $\square$

Lemma 33.33.4. *Let $f : X \to S$ be a morphism of schemes. Let $x \rightsquigarrow x'$ be a specialization of points in X. Set $s = f(x)$ and $s' = f(x')$. Assume*

(1) *x' is a closed point of $X_{s'}$, and*
(2) *f is locally of finite type.*

Then the set

$$\{x_1 \in X \text{ such that } f(x_1) = s \text{ and } x_1 \text{ is closed in } X_s \text{ and } x \rightsquigarrow x_1 \rightsquigarrow x'\}$$

is dense in the closure of x in X_s.

Proof. We apply Schemes, Lemma 21.20.4 to the specialization $x \rightsquigarrow x'$. This produces a morphism $\varphi : Spec(B) \to X$ where B is a valuation ring such that φ maps the generic point to x and the closed point to x'. We may also assume that $\kappa(x) = f.f.(B)$. Let $A = B \cap \kappa(s)$. Note that this is a valuation ring (see Algebra, Lemma 7.46.5) which dominates the image of $\mathcal{O}_{S,s'} \to \kappa(s)$. Consider the commutative diagram

$$\begin{tikzcd} Spec(B) \arrow[r] \arrow[rd] & X_A \arrow[r] \arrow[d] & X \arrow[d] \\ & Spec(A) \arrow[r] & S \end{tikzcd}$$

The generic (resp. closed) point of B maps to a point x_A (resp. x'_A) of X_A lying over the generic (resp. closed) point of $Spec(A)$. Note that x'_A is a closed point of the special fibre of X_A by Morphisms, Lemma 24.19.4. Note that the generic fibre of $X_A \to Spec(A)$ is isomorphic to X_s. Thus we have reduced the lemma to the case where S is the spectrum of a valuation ring, $s = \eta \in S$ is the generic point, and $s' \in S$ is the closed point.

We will prove the lemma by induction on $\dim_x(X_\eta)$. If $\dim_x(X_\eta) = 0$, then there are no other points of X_η specializing to x and x is closed in its fibre, see Morphisms, Lemma 24.19.6, and the result holds. Assume $\dim_x(X_\eta) > 0$.

Let $X' \subset X$ be the reduced induced scheme structure on the irreducible closed subscheme $\overline{\{x\}}$ of X, see Schemes, Definition 21.12.5. To prove the lemma we may replace X by X' as this only decreases $\dim_x(X_\eta)$. Hence we may also assume that X is an integral scheme and that x is its generic point. In addition, we may replace X by an affine neighbourhood of x'. Thus we have $X = Spec(B)$ where $A \subset B$ is a finite type extension of domains. Note that in this case $\dim_x(X_\eta) = \dim(X_\eta) = \dim(X_{s'})$, and that in fact $X_{s'}$ is equidimensional, see Algebra, Lemma 7.116.9.

Let $W \subset X_\eta$ be a proper closed subset (this is the subset we want to ``avoid"). As X_s is of finite type over a field we see that W has finitely many irreducible components $W = W_1 \cup \ldots \cup W_n$. Let $\mathfrak{q}_j \subset B$, $j = 1, \ldots, r$ be the corresponding prime ideals. Let $\mathfrak{q} \subset B$ be the maximal ideal corresponding to the point x'. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s \subset B$ be the minimal primes lying over $\mathfrak{m}_A B$. There are finitely many as these correspond to the irreducible components of the Noetherian scheme $X_{s'}$. Moreover, each of these irreducible components has dimension > 0 (see above) hence we see that $\mathfrak{p}_i \neq \mathfrak{q}$ for all i. Now, pick an element $g \in \mathfrak{q}$ such that $g \notin \mathfrak{q}_j$ for all j and $g \notin \mathfrak{p}_i$ for all i, see Algebra, Lemma 7.14.3. Denote $Z \subset X$ the locally principal closed subscheme defined by h. Let $Z_\eta = Z_{1,\eta} \cup \ldots \cup Z_{n,\eta}$, $n \geq 0$ be the decomposition of the generic fibre of Z into irreducible components (finitely many as the generic fibre is Noetherian). Denote $Z_i \subset X$ the closure of $Z_{i,\eta}$. After replacing X by a smaller affine neighbourhood we may assume that $x \in Z_i$ for each $i = 1, \ldots, n$. By construction $Z \cap X_{s'}$ does not contain any irreducible component of $X_{s'}$. Hence by Lemma 33.33.1 we conclude that $Z_\eta \neq \emptyset$! In other words $n \geq 1$. Letting $x_1 \in Z_1$ be the generic point we see that $x_1 \rightsquigarrow x'$ and $f(x_1) = \eta$. Also, by construction $Z_{1,\eta} \cap W_j \subset W_j$ is a proper closed subset. Hence every irreducible component of $Z_{1,\eta} \cap W_j$ has codimension ≥ 2 in X_η whereas $\text{codim}(Z_{1,\eta}, X_\eta) = 1$ by Algebra, Lemma 7.57.10. Thus $W \cap Z_{1,\eta}$ is a proper closed subset. At this point we see that the induction hypothesis applies to $Z_1 \to S$ and the specialization $x_1 \rightsquigarrow x'$. This produces a closed point x_2 of $Z_{1,\eta}$ not contained in W which specializes to x'. Thus we obtain $x \rightsquigarrow x_2 \rightsquigarrow x'$, the point x_2 is closed in X_η, and $x_2 \notin W$ as desired. □

Remark 33.33.5. The proof of Lemma 33.33.4 actually shows that there exists a sequence of specializations

$$x \rightsquigarrow x_1 \rightsquigarrow x_2 \rightsquigarrow \ldots \rightsquigarrow x_d \rightsquigarrow x'$$

where all x_i are in the fibre X_s, each specialization is immediate, and x_d is a closed point of X_s. The integer $d = \text{trdeg}_{\kappa(s)}(\kappa(x)) = \dim(\overline{\{x\}})$ where the closure is taken in X_s. Moreover, the points x_i can be chosen to avoid any closed subset of X_s which does not contain the point x.

Examples, Section 64.20 shows that the following lemma is false if A is not assumed Noetherian.

Lemma 33.33.6. *Let $\varphi : A \to B$ be a local ring map of local rings. Let $V \subset Spec(B)$ be an open subscheme which contains at least one prime not lying over $\mathfrak{m}_A$. Assume A is Noetherian, φ essentially of finite type, and $A/\mathfrak{m}_A \subset B/\mathfrak{m}_B$ is finite. Then there exists a $\mathfrak{q} \in V$, $\mathfrak{m}_A \neq \mathfrak{q} \cap A$ such that $A \to B/\mathfrak{q}$ is the localization of a quasi-finite ring map.*

Proof. Since A is Noetherian and $A \to B$ is essentially of finite type, we know that B is Noetherian too. By Properties, Lemma 23.6.4 the topological space $Spec(B) \setminus \{\mathfrak{m}_B\}$ is Jacobson. Hence we can choose a closed point $\mathfrak{q}$ which is contained in the nonempty open

$$V \setminus \{\mathfrak{q} \subset B \mid \mathfrak{m}_A = \mathfrak{q} \cap A\}.$$

(Nonempty by assumption, open because $\{\mathfrak{m}_A\}$ is a closed subset of $Spec(A)$.) Then $Spec(B/\mathfrak{q})$ has two points, namely $\mathfrak{m}_B$ and $\mathfrak{q}$ and $\mathfrak{q}$ does not lie over $\mathfrak{m}_A$. Write $B/\mathfrak{q} = C_\mathfrak{m}$ for some finite type A-algebra C and prime ideal $\mathfrak{m}$. Then $A \to C$ is quasi-finite at $\mathfrak{m}$ by Algebra, Lemma 7.113.2 (2). Hence by Algebra, Lemma 7.114.14 we see that after replacing C by a principal localization the ring map $A \to C$ is quasi-finite. □

Lemma 33.33.7. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ with image $s \in S$. Let $U \subset X$ be an open subscheme. Assume f locally of finite type, S locally Noetherian, x a closed point of X_s, and assume there exists a point $x' \in U$ with $x' \rightsquigarrow x$ and $f(x') \neq s$. Then there exists a closed subscheme $Z \subset X$ such that (a) $x \in Z$, (b) $f|_Z : Z \to S$ is quasi-finite at x, and (c) there exists a $z \in Z$, $z \in U$, $z \rightsquigarrow x$ and $f(z) \neq s$.*

Proof. This is a reformulation of Lemma 33.33.6. Namely, set $A = \mathcal{O}_{S,s}$ and $B = \mathcal{O}_{X,x}$. Denote $V \subset Spec(B)$ the inverse image of U. The ring map $f^\sharp : A \to B$ is essentially of finite type. By assumption there exists at least one point of V which does not map to the closed point of $Spec(A)$. Hence all the assumptions of Lemma 33.33.6 hold and we obtain a prime $\mathfrak{q} \subset B$ which does not lie over $\mathfrak{m}_A$ and such that $A \to B/\mathfrak{q}$ is the localization of a quasi-finite ring map. Let $z \in X$ be the image of the point $\mathfrak{q}$ under the canonical morphism $Spec(B) \to X$. Set $Z = \overline{\{z\}}$ with the induced reduced scheme structure. As $z \rightsquigarrow x$ we see that $x \in Z$ and $\mathcal{O}_{Z,x} = B/\mathfrak{q}$. By construction $Z \to S$ is quasi-finite at x. □

Remark 33.33.8. We can use Lemma 33.33.6 or its variant Lemma 33.33.7 to give an alternative proof of Lemma 33.33.4 in case S is locally Noetherian. Here is a rough sketch. Namely, first replace S by the spectrum of the local ring at s'. Then we may use induction on $\dim(S)$. The case $\dim(S) = 0$ is trivial because then $s' = s$. Replace X by the reduced induced scheme structure on $\overline{\{x\}}$. Apply Lemma 33.33.7 to $X \to S$ and $x' \mapsto s'$ and any nonempty open $U \subset X$ containing x. This gives us a closed subscheme $x' \in Z \subset X$ a point $z \in Z$ such that $Z \to S$ is quasi-finite at x' and such that $f(z) \neq s'$. Then z is a closed point of $X_{f(z)}$, and $z \rightsquigarrow x'$. As $f(z) \neq s'$ we see $\dim(\mathcal{O}_{S,f(z)}) < \dim(S)$. Since x is the generic point of X we see $x \rightsquigarrow z$, hence $s = f(x) \rightsquigarrow f(z)$. Apply the induction hypothesis to $s \rightsquigarrow f(z)$ and $z \mapsto f(z)$ to win.

Lemma 33.33.9. *Suppose that $f : X \to S$ is locally of finite type, S locally Noetherian, $x \in X$ a closed point of its fibre X_s, and $U \subset X$ an open subscheme such that $U \cap X_s = \emptyset$ and $x \in \overline{U}$, then the conclusions of Lemma 33.33.7 hold.*

Proof. Namely, we can reduce this to the cited lemma as follows: First we replace X and S by affine neighbourhoods of x and s. Then X is Noetherian, in particular U is quasi-compact (see Morphisms, Lemma 24.14.6 and Topology, Lemmas 5.6.2 and 5.9.9). Hence there exists a specialization $x' \rightsquigarrow x$ with $x' \in U$ (see Morphisms, Lemma 24.4.5). Note that $f(x') \neq s$. Thus we see all hypotheses of the lemma are satisfied and we win. □

33.34. Stein factorization

Stein factorization is the statement that a proper morphism $f : X \to S$ with $f_*\mathcal{O}_X = \mathcal{O}_S$ has connected fibres.

Lemma 33.34.1. *Let S be a scheme. Let $f : X \to S$ be a universally closed, quasi-compact and quasi-separated morphism. There exists a factorization*

$$\begin{tikzcd} X \arrow[rr, "f'"] \arrow[rd, "f"'] & & S' \arrow[ld, "\pi"] \\ & S & \end{tikzcd}$$

with the following properties:

(1) *the morphism f' is universally closed, quasi-compact, quasi-separated and surjective,*
(2) *the morphism $\pi : S' \to S$ is integral,*
(3) *we have $f'_*\mathcal{O}_X = \mathcal{O}_{S'}$,*
(4) *we have $S' = \underline{Spec}_S(f_*\mathcal{O}_X)$, and*
(5) *S' is the normalization of S in X, see Morphisms, Definition 24.46.3.*

Proof. We just define S' as the normalization of S in X, so (5) and (2) hold automatically. By Morphisms, Lemma 24.46.9 we see that (4) holds. The morphism f' is universally closed by Morphisms, Lemma 24.40.7. It is quasi-compact by Schemes, Lemma 21.21.15 and quasi-separated by Schemes, Lemma 21.21.14.

To show the remaining statements we may assume the base scheme S is affine, say $S = Spec(R)$. Then $S' = Spec(A)$ with $A = \Gamma(X, \mathcal{O}_X)$ an integral R-algebra. Thus it is clear that $f'_*\mathcal{O}_X$ is $\mathcal{O}_{S'}$ (because $f'_*\mathcal{O}_X$ is quasi-coherent, by Schemes, Lemma 21.24.1, and hence equal to $\widetilde{A}$). This proves (3).

Let us show that f' is surjective. As f' is universally closed (see above) the image of f' is a closed subset $V(I) \subset S' = Spec(A)$. Pick $h \in I$. Then $h|_X = f^\sharp(h)$ is a global section of the structure sheaf of X which vanishes at every point. As X is quasi-compact this means that $h|_X$ is a nilpotent section, i.e., $h^n|X = 0$ for some $n > 0$. But $A = \Gamma(X, \mathcal{O}_X)$, hence $h^n = 0$. In other words I is contained in the radical ideal of A and we conclude that $V(I) = S'$ as desired. $\square$

Lemma 33.34.2. *Let $f : X \to S$ be a morphism of schemes. Let $s \in S$. Then X_s is geometrically connected, if and only if for every étale neighbourhood $(U, u) \to (S, s)$ the base change $X_U \to U$ as connected fibre X_u.*

Proof. If X_s is geometrically connected, then any base change of it is connected. On the other hand, suppose that X_s is not geometrically connected. Then by Varieties, Lemma 28.5.11 we see that $X_s \times_{Spec(\kappa(s))} Spec(k)$ is disconnected for some finite separable field extension $\kappa(s) \subset k$. By Lemma 33.25.2 there exists an affine étale neighbourhood $(U, u) \to (S, s)$ such that $\kappa(s) \subset \kappa(u)$ is identified with $\kappa(s) \subset k$. In this case X_u is disconnected. $\square$

Theorem 33.34.3. *(Stein factorization -- Noetherian case) Let S be a locally Noetherian scheme. Let $f : X \to S$ be a proper morphism. There exists a factorization*

$$\begin{tikzcd} X \arrow[rr, "f'"] \arrow[dr, "f"'] & & S' \arrow[dl, "\pi"] \\ & S & \end{tikzcd}$$

with the following properties:

(1) *the morphism f' is proper, surjective with geometrically connected fibres,*
(2) *the morphism $\pi : S' \to S$ is finite,*
(3) *we have $f'_*\mathcal{O}_X = \mathcal{O}_{S'}$,*
(4) *we have $S' = \underline{Spec}_S(f_*\mathcal{O}_X)$, and*
(5) *S' is the normalization of S in X, see Morphisms, Definition 24.46.3.*

Proof. Let $f = \pi \circ f'$ be the factorization of Lemma 33.34.1. Note that besides the conclusions of Lemma 33.34.1 we also have that f' is separated (Schemes, Lemma 21.21.14) and finite type (Morphisms, Lemma 24.14.8). Hence f' is proper. By Coherent, Lemma 25.18.2 we see that $f_*\mathcal{O}_X$ is a coherent $\mathcal{O}_S$-module. Hence we see that π is finite, i.e., (2) holds.

This proves all but the most interesting assertion, namely that all the fibres of f' are geometrically connected. It is clear from the discussion above that we may replace S by S', and we may therefore assume that S is Noetherian, affine, $f : X \to S$ is proper, and $f_*\mathcal{O}_X = \mathcal{O}_S$. Let $s \in S$ be a point of S. We have to show that X_s is geometrically connected. By Lemma 33.34.2 we see that it suffices to show X_u is connected for every étale neighbourhood $(U, u) \to (S, s)$. We may assume U is affine. Thus U is Noetherian (Morphisms, Lemma 24.14.6), the base change $f_U : X_U \to U$ is proper (Morphisms, Lemma 24.40.5), and that also $(f_U)_*\mathcal{O}_{X_U} = \mathcal{O}_U$ (Coherent, Lemma 25.6.2). Hence after replacing $(f : X \to S, s)$ by the base change $(f_U : X_U \to U, u)$ it suffices to prove that the fibre X_s is connected.

At this point we apply the theorem on formal functions, more precisely Coherent, Lemma 25.19.6. It tells us that

$$\mathcal{O}^\wedge_{S,s} = lim_n\, H^0(X_n, \mathcal{O}_{X_n})$$

where X_n is the nth infinitesimal neighbourhood of X_s. Since the underlying topological space of X_n is equal to that of X_s we see that if $X_s = T_1 \coprod T_2$ is a disjoint union of nonempty open and closed subschemes, then similarly $X_n = T_{1,n} \coprod T_{2,n}$ for all n. And this in turn means $H^0(X_n, \mathcal{O}_{X_n})$ contains a nontrivial idempotent $e_{1,n}$, namely the function which is identically 1 on $T_{1,n}$ and identically 0 on $T_{2,n}$. It is clear that $e_{1,n+1}$ restricts to $e_{1,n}$ on X_n. Hence $e_1 = lim\, e_{1,n}$ is a nontrivial idempotent of the limit. This contradicts the fact that $\mathcal{O}^\wedge_{S,s}$ is a local ring. Thus the assumption was wrong, i.e., X_s is connected, and we win. $\square$

Lemma 33.34.4. *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Let $X \subset \mathbf{P}^n_R$ be a closed subscheme. Assume that $R = \Gamma(X, \mathcal{O}_X)$. Then the special fibre X_k is geometrically connected.*

Proof. Let $R \to R'$ be a flat local ring map so that the residue field of R' is algebraically closed, see Algebra, Lemma 7.142.1. By Coherent, Lemma 25.6.2 we have $\Gamma(X_{R'}, \mathcal{O}_{X_{R'}}) = R'$. Hence we may assume that the residue field of R is algebraically closed. This reduces us to just proving that X_k is connected. (We could also have used Lemma 33.34.2 for this.)

Suppose, to get a contradiction, that $X_k = T_1 \coprod T_2$ for some closed and open subschemes $T_i \subset X_k$. By Constructions, Lemma 22.13.6 we can write

$$X = \mathrm{Proj}(R[T_0, \ldots, T_n]/I)$$

for some graded ideal $I \subset R[T_0, \ldots, T_n]$. We may write $R = colim_\alpha R_\alpha$ as a directed colimit of Noetherian local rings R_α, see Algebra, Lemma 7.118.7. Let k_α be the residue field of R_α. Let $I_\alpha = R_\alpha[T_0, \ldots, T_n] \cap I$ (or more precisely the inverse image). Then $I = colim_\alpha I_\alpha$. Let $X_\alpha = \mathrm{Proj}(R_\alpha[T_0, \ldots, T_n]/I_\alpha)$. Warning: because I may not be finitely generated it may be that the natural morphism

$$X \longrightarrow X_\alpha \times_{Spec(R_\alpha)} Spec(R)$$

is never an isomorphism! The image $\overline{I} \subset k[T_0, \ldots, T_n]$ of I is finitely generated as $k[T_0, \ldots, T_n]$ is a Noetherian ring. Hence for all large enough α the image $\overline{I}_\alpha \subset k_\alpha[T_0, \ldots, T_n]$ of I_α is such that $\overline{I}_\alpha k[T_0, \ldots, T_n] = \overline{I}$. Let $J_1, J_2 \subset k[T_0, \ldots, T_n]$ be graded ideals such that $T_i = \mathrm{Proj}(k[T_0, \ldots, T_n]/J_i)$, see Constructions, Lemma 22.13.6 again. Since $R = colim_\alpha R_\alpha$, also $k = colim_\alpha k_\alpha$. Thus for all large enough α there exist ideals $J_{\alpha,1}, J_{\alpha,2} \subset k_\alpha[T_0, \ldots, T_n]$ such that $J_{\alpha,i} k[T_0, \ldots, T_n] = J_i$. Combining these observations, we see that there exists an α such that

(1) the morphism $X_\alpha \to Spec(R_\alpha)$ has the property that

$$X_\alpha \times_{Spec(R_\alpha)} Spec(k) = (X_\alpha)_{k_\alpha} \times_{Spec(k_\alpha)} Spec(k) = X_k,$$

and

(2) there exists a decomposition $(X_\alpha)_{k_\alpha} = T_{\alpha,1} \coprod T_{\alpha,2}$ such that $(T_{\alpha,i})_k = T_i$.

By the Noetherian case (Theorem 33.34.3) this means there exists a factorization

$$X_\alpha \longrightarrow Spec(R') \longrightarrow Spec(R_\alpha)$$

with $R_\alpha \to R'$ finite and $X_\alpha \to Spec(R')$ having geometrically connected fibres. Let $t_i \in T_i$ be a point, let $t_{\alpha,i} \in T_{\alpha,i}$ be the image points, and let $\mathfrak{m}_i \subset R'$ be the corresponding maximal ideals. Then $\mathfrak{m}_1 \neq \mathfrak{m}_2$ by the connectedness of the fibres. This implies that $X \to Spec(R)$ factors as

$$X \longrightarrow Spec(R \otimes_{R_\alpha} R') \longrightarrow Spec(R)$$

Because t_1 and t_2 map to distinct points in $Spec(R')$ we see that t_1 and t_2 must also map to distinct points in $Spec(R \otimes_{R_\alpha} R')$. Hence there exists an element $f \in R \otimes_{R_\alpha} R'$ such that $f|_X$ is zero in t_1 and not in t_2 (or vice versa). This clearly contradicts the assumption that $R = \Gamma(X, \mathcal{O}_X)$ and we win. □

Theorem 33.34.5. *(Stein factorization -- general case) Let S be a scheme. Let $f : X \to S$ be a proper morphism. There exists a factorization*

$$\begin{tikzcd} X \arrow[rr, "f'"] \arrow[dr, "f"'] & & S' \arrow[dl, "\pi"] \\ & S & \end{tikzcd}$$

with the following properties:

(1) *the morphism f' is proper, surjective with geometrically connected fibres,*
(2) *the morphism $\pi : S' \to S$ is integral,*
(3) *we have $f'_*\mathcal{O}_X = \mathcal{O}_{S'}$,*
(4) *we have $S' = \underline{Spec}_S(f_*\mathcal{O}_X)$, and*
(5) *S' is the normalization of S in X, see Morphisms, Definition 24.46.3.*

Proof. We may apply Lemma 33.34.1 to get the morphism $f' : X \to S'$. Note that besides the conclusions of Lemma 33.34.1 we also have that f' is separated (Schemes, Lemma 21.21.14) and finite type (Morphisms, Lemma 24.14.8). Hence f' is proper. At this point we have proved all of the statements except for the statement that f' has geometrically connected fibres.

To prove this we may assume that $S = Spec(R)$ is affine. Use Limits, Lemma 27.8.1 to choose a diagram

$$\begin{tikzcd} X \arrow[dr, "f"'] & X' \arrow[l, "\pi"'] \arrow[r] \arrow[d] & \mathbf{P}^n_R \arrow[dl] \\ & S = Spec(R) & \end{tikzcd}$$

where $X' \to \mathbf{P}^n_S$ is an immersion, and $\pi : X' \to X$ is proper and surjective. Thus $X' \to S$ is proper, hence $X' \to \mathbf{P}^n_R$ is a closed immersion. (See Morphisms, Lemmas 24.40.4 and 24.40.7 and Schemes, Lemma 21.10.4). Set $A = \Gamma(X, \mathcal{O}_X)$, and $A' = \Gamma(X', \mathcal{O}_{X'})$. Then $S' = Spec(A)$. Consider the diagram

(33.34.5.1)
$$\begin{tikzcd} X \arrow[d, "f'"'] & X' \arrow[l, "\pi"'] \arrow[r] \arrow[d, "g"'] & \mathbf{P}^n_{A'} \arrow[dl] \\ Spec(A) & Spec(A') \arrow[l] & \end{tikzcd}$$

Here π is surjective and proper, the vertical arrows are proper and surjective, the right horizontal arrow is a closed immersion, and $Spec(A') \to Spec(A)$ is integral (see arguments above). Let $\mathfrak{p} \subset A$ be a prime, corresponding to a point $p \in Spec(A)$. Let X_p be the fibre. We have to show that X_p is geometrically connected. By Lemma 33.34.2 it suffices to show that for every étale ring map $A \to B$ and a prime $\mathfrak{q}$ of B lying over $\mathfrak{p}$ the fibre of X_B over $\mathfrak{q}$ is connected. As an étale ring map is flat, we see from Coherent, Lemma 25.6.2 that we have

$$\Gamma(X_B, \mathcal{O}_{X_B}) = B,$$

and similarly

$$\Gamma(X' \times_{Spec(A)} Spec(B), \mathcal{O}) = \Gamma(X' \times_{Spec(A')} Spec(B \otimes_A A'), \mathcal{O}) = B \otimes_A A'.$$

This means that everything we said above about the diagram (33.34.5.1) also holds for that diagram base changed to B (some verifications omitted). Hence we may replace A by B and we reduce to proving that X_p is connected.

Consider the scheme $X'_p = \pi^{-1}(f')^{-1}(p)$. It is proper over $\kappa(p)$, hence Noetherian, and hence has finitely many connected components. The morphism g is surjective, hence any point p' of $Spec(A')$ lying over p is the image of a point of X'_p. On the other hand, there are no specializations among the points of $Spec(A')$ lying over p, see Morphisms, Lemma 24.42.8. Hence the map

$$X'_p \longrightarrow \{p' \in Spec(A') \mid p' \text{ lies over } p\}$$

is surjective and constant on connected components. Thus we see there are finitely many points $p'_1, \ldots, p'_n \in Spec(A')$ of $Spec(A')$ lying over p. Let $\mathfrak{p}'_1, \ldots, \mathfrak{p}'_n$ be the corresponding primes of A', i.e., those lying over $\mathfrak{p}$. Let $A'' \subset A'$ be a finitely generated A-subalgebra such that the primes $A'' \cap \mathfrak{p}'_i$ are pairwise distinct. Such an $A'' \subset A'$ exists; argument omitted. As $A \subset A'$ is integral, this implies that A'' is finite over A, see Algebra, Lemma 7.32.5. Note that that $\mathfrak{p}'_1 \cap A'', \ldots, \mathfrak{p}'_n \cap A''$ are the only primes of A'' lying over $\mathfrak{p}$ as

$Spec(A') \to Spec(A'')$ is surjective, see Algebra, Lemma 7.32.15. By Algebra, Lemma 7.132.22 there exists an étale ring map $A \to B$ and a prime $\mathfrak{q}$ lying over $\mathfrak{p}$ such that $\kappa(\mathfrak{p}) = \kappa(\mathfrak{q})$ and $B \otimes_A A'' = B''_1 \times \ldots \times B''_n$ decomposes into algebras B''_i finite over B each with a single prime lying over $\mathfrak{q}$. Hence also $B \otimes_A A' = B'_1 \times \ldots \times B'_n$ decomposes into algebras B'_i integral over B each with a single prime lying over $\mathfrak{q}$ (namely by taking $B'_i = B''_i \otimes_{A''} A'$). After base changing the sitatuation to B as above, we see that we may assume $Spec(A') = V_1 \coprod \ldots \coprod V_n$ each with a single point $p'_i \in V_i$ lying over p.

Let $X'_i \subset X'$ be the inverse image of V_i. Note that

$$X'_p = \coprod X'_{i,p} = \coprod X'_{p'_i}.$$

Since $\Gamma(X', \mathcal{O}_{X'}) = A'$, and since X' is a closed subscheme of $\mathbf{P}^n_{A'}$ we may apply Lemma 33.34.4 to we see that $g : X' \to Spec(A')$ has geometrically connected fibres. Hence each $X'_{i,p} = X'_{p'_i}$ is connected! Hence, if $T \subset X_p$ is open and closed, then $\pi^{-1}(T) \subset X'_p$ is a disjoint union $\pi^{-1}(T) = \coprod_{i \in I} X'_{i,p}$ for some subset $I \subset \{1, \ldots, n\}$. Let $J = I^c \subset \{1, \ldots, n\}$ be the complement. Set

$$X_I = \bigcup\nolimits_{i \in I} \pi(X'_i), \quad \text{and} \quad X_J = \bigcup\nolimits_{j \in J} \pi(X'_j).$$

These are closed subsets whose union is X and which do not meet in the special fibre X_p. Since $f' : X \to Spec(A)$ is proper hence closed we see that $f'(X_I \cap X_J)$ is a closed subset of $Spec(A)$ which does not meet p. Hence after replacing A by A_g for some $g \in A$, $g \not\in \mathfrak{p}$ (i.e., doing a base change with $B = A_g$ as above) we see that $X_I \cap X_J = \emptyset$. Thus we conclude that X_I and X_J are open and closed in X, and

$$\Gamma(X, \mathcal{O}_X) = \Gamma(X_I, \mathcal{O}_{X_I}) \times \Gamma(X_J, \mathcal{O}_{X_J}).$$

If I and J are both nonempty then we see that $\Gamma(X, \mathcal{O}_X)$ contains an idempotent which cannot be the image of an idempotent in A! This contradicts the assumption that $A = \Gamma(X, \mathcal{O}_X)$, hence either $I = \emptyset$ or $J = \emptyset$. In other words, either $T = X_p$ or $T = \emptyset$, i.e., X_p is connected as desired. ☐

33.35. Descending separated locally quasi-finite morphisms

In this section we show that ``separated locally quasi-finite morphisms satisfy descent for fppf-coverings''. See Descent, Definition 31.32.1 for terminology. This is in the marvellous (for many reasons) paper by Raynaud and Gruson hidden in the proof of [**GR71**, Lemma 5.7.1]. It can also be found in [**Mur95**], and [**ABD⁺66**, Exposé X, Lemma 5.4] under the additional hypothesis that the morphism is locally of finite presentation. Here is the formal statement.

Lemma 33.35.1. *Let S be a scheme. Let $\{X_i \to S\}_{i \in I}$ be an fppf covering, see Topologies, Definition 30.7.1. Let $(V_i/X_i, \varphi_{ij})$ be a descent datum relative to $\{X_i \to S\}$. If each morphism $V_i \to X_i$ is separated and locally quasi-finite, then the descent datum is effective.*

Proof. Being separated and being locally quasi-finite are properties of morphisms of schemes which are preserved under any base change, see Schemes, Lemma 21.21.13 and Morphisms, Lemma 24.19.13. Hence Descent, Lemma 31.32.2 applies and it suffices to prove the statement of the lemma in case the fppf-covering is given by a single $\{X \to S\}$ flat surjective morphism of finite presentation of affines. Say $X = Spec(A)$ and $S = Spec(R)$ so that $R \to A$ is a faithfully flat ring map. Let (V, φ) be a descent datum relative to X over S and assume that $\pi : V \to X$ is separated and locally quasi-finite.

Let $W^1 \subset V$ be any affine open. Consider $W = \mathrm{pr}_1(\varphi(W^1 \times_S X)) \subset V$. Here is a picture

$$
\begin{tikzcd}
W^1 \times_S X \arrow[rrrrr] \arrow[dr] \arrow[ddd] & & & & & \varphi(W^1 \times_S X) \arrow[dl] \arrow[ddd] \\
& V \times_S X \arrow[rrr, "\varphi"] \arrow[dr] \arrow[dd] & & & X \times_S V \arrow[dl] \arrow[dd] & \\
& & X \times_S X \arrow[r, "1"] \arrow[d, "\mathrm{pr}_0"'] & X \times_S X \arrow[d, "\mathrm{pr}_1"] & & \\
W^1 \arrow[r] & V \arrow[r] & X & X & V \arrow[l] & W \arrow[l]
\end{tikzcd}
$$

Ok, and now since $X \to S$ is flat and of finite presentation it is universally open (Morphisms, Lemma 24.24.9). Hence we conclude that W is open. Moreover, it is also clearly the case that W is quasi-compact, and $W^1 \subset W$. Moreover, we note that $\varphi(W \times_S X) = X \times_S W$ by the cocycle condition for φ. Hence we obtain a new descent datum (W, φ') by restricting φ to $W \times_S X$. Note that the morphism $W \to X$ is quasi-compact, separated and locally quasi-finite. This implies that it is separated and quasi-finite by definition. Hence it is quasi-affine by Lemma 33.29.3. Thus by Descent, Lemma 31.34.1 we see that the descent datum (W, φ') is effective.

In other words, we find that there exists an open covering $V = \bigcup W_i$ by quasi-compact opens W_i which are stable for the descent morphism φ. Moreover, for each such quasi-compact open $W \subset V$ the corresponding descent data (W, φ') is effective. It is an exercise to show this means the original descent datum is effective by glueing the schemes obtained from descending the opens W_i (details omitted). □

33.36. Pseudo-coherent morphisms

Avoid reading this section at all cost. If you need some of this material, first take a look at the corresponding algebra sections, see More on Algebra, Sections 12.40, 12.45, and 12.46. For now the only thing you need to know is that a ring map $A \to B$ is pseudo-coherent if and only if $B = A[x_1, \ldots, x_n]/I$ and B as an $A[x_1, \ldots, x_n]$-module has a resolution by finite free $A[x_1, \ldots, x_n]$-modules.

Lemma 33.36.1. *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. The following are equivalent*

1. *there exist an affine open covering $S = \bigcup V_j$ and for each j an affine open covering $f^{-1}(V_j) = \bigcup U_{ji}$ such that $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_{ij})$ is a pseudo-coherent ring map, and*
2. *for every pair of affine opens $U \subset X$, $V \subset S$ such that $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is pseudo-coherent.*

Proof. To see this it suffices to check conditions (1)(a), (b), (c) of Morphisms, Definition 24.13.1 for the property of being a pseudo-coherent ring map. These properties follow (using localization is flat) from More on Algebra, Lemmas 12.45.12, 12.45.11, and 12.45.16. □

Definition 33.36.2. A morphism of schemes $f : X \to S$ is called *pseudo-coherent* if the equivalent conditions of Lemma 33.36.1 are satisfied. In this case we also say that X is pseudo-coherent over S.

Beware that a base change of a pseudo-coherent morphism is not pseudo-coherent in general.

Lemma 33.36.3. *A flat base change of a pseudo-coherent morphism is pseudo-coherent.*

Proof. This translates into the following algebra result: Let $A \to B$ be a pseudo-coherent ring map. Let $A \to A'$ be flat. Then $A' \to B \otimes_A A'$ is pseudo-coherent. This follows from the more general More on Algebra, Lemma 12.45.12. $\square$

Lemma 33.36.4. *A composition of pseudo-coherent morphisms of schemes is pseudo-coherent.*

Proof. This translates into the following algebra result: If $A \to B \to C$ are composable pseudo-coherent ring maps then $A \to C$ is pseudo-coherent. This follows from either More on Algebra, Lemma 12.45.13 or More on Algebra, Lemma 12.45.15. $\square$

Lemma 33.36.5. *A pseudo-coherent morphism is locally of finite presentation.*

Proof. Immediate from the definitions. $\square$

Lemma 33.36.6. *A flat morphism which is locally of finite presentation is pseudo-coherent.*

Proof. This follows from the fact that a flat ring map of finite presentation is pseudo-coherent (and even perfect), see More on Algebra, Lemma 12.46.4. $\square$

Lemma 33.36.7. *Let $f : X \to Y$ be a morphism of schemes pseudo-coherent over a base scheme S. Then f is pseudo-coherent.*

Proof. This translates into the following algebra result: If $R \to A \to B$ are composable ring maps and $R \to A$, $R \to B$ pseudo-coherent, then $R \to B$ is pseudo-coherent. This follows from More on Algebra, Lemma 12.45.15. $\square$

Lemma 33.36.8. *Let $f : X \to S$ be a morphism of schemes. If S is locally Noetherian, then f is pseudo-coherent if and only if f is locally of finite type.*

Proof. This translates into the following algebra result: If $R \to A$ is a finite type ring map with R Noetherian, then $R \to A$ is pseudo-coherent if and only if $R \to A$ is of finite type. To see this, note that a pseudo-coherent ring map is of finite type by definition. Conversely, if $R \to A$ is of finite type, then we can write $A = R[x_1, \ldots, x_n]/I$ and it follows from More on Algebra, Lemma 12.40.16 that A is pseudo-coherent as an $R[x_1, \ldots, x_n]$-module, i.e., $R \to A$ is a pseudo-coherent ring map. $\square$

Lemma 33.36.9. *The property $\mathcal{P}(f) =$``f is pseudo-coherent'' is fpqc local on the base.*

Proof. We will use the criterion of Descent, Lemma 31.18.4 to prove this. By Definition 33.36.2 being pseudo-coherent is Zariski local on the base. By Lemma 33.36.3 being pseudo-coherent is preserved under flat base change. The final hypothesis (3) of Descent, Lemma 31.18.4 translates into the following algebra statement: Let $A \to B$ be a faithfully flat ring map. Let $C = A[x_1, \ldots, x_n]/I$ be an A-algebra. If $C \otimes_A B$ is pseudo-coherent as an $B[x_1, \ldots, x_n]$-module, then C is pseudo-coherent as a $A[x_1, \ldots, x_n]$-module. This is More on Algebra, Lemma 12.40.15. $\square$

Lemma 33.36.10. *Let $A \to B$ be a flat ring map of finite presentation. Let $I \subset B$ be an ideal. Then $A \to B/I$ is pseudo-coherent if and only if I is pseudo-coherent as a B-module.*

Proof. Choose a presentation $B = A[x_1, \ldots, x_n]/J$. Note that B is pseudo-coherent as an $A[x_1, \ldots, x_n]$-module because $A \to B$ is a pseudo-coherent ring map by Lemma 33.36.6. Note that $A \to B/I$ is pseudo-coherent if and only if B/I is pseudo-coherent as an $A[x_1, \ldots, x_n]$-module. By More on Algebra, Lemma 12.40.11 we see this is equivalent to the condition that B/I is pseudo-coherent as an B-module. This proves the lemma as the short exact sequence $0 \to I \to B \to B/I \to 0$ shows that I is pseudo-coherent if and only if B/I is (see More on Algebra, Lemma 12.40.6). $\square$

The following lemma will be obsoleted by the stronger Lemma 33.36.12.

Lemma 33.36.11. *The property $\mathcal{P}(f)$ =``f is pseudo-coherent'' is syntomic local on the source.*

Proof. We will use the criterion of Descent, Lemma 31.22.3 to prove this. It follows from Lemmas 33.36.6 and 33.36.4 that being pseudo-coherent is preserved under precomposing with flat morphisms locally of finite presentation, in particular under precomposing with syntomic morphisms (see Morphisms, Lemmas 24.30.7 and 24.30.6). It is clear from Definition 33.36.2 that being pseudo-coherent is Zariski local on the source and target. Hence, according to the aforementioned Descent, Lemma 31.22.3 it suffices to prove the following: Suppose $X' \to X \to Y$ are morphisms of affine schemes with $X' \to X$ syntomic and $X' \to Y$ pseudo-coherent. Then $X \to Y$ is pseudo-coherent. To see this, note that in any case $X \to Y$ is of finite presentation by Descent, Lemma 31.10.1. Choose a closed immersion $X \to \mathbf{A}^n_Y$. By Algebra, Lemma 7.125.19 we can find an affine open covering $X' = \bigcup_{i=1,\ldots,n} X'_i$ and syntomic morphisms $W_i \to \mathbf{A}^n_Y$ lifting the morphisms $X'_i \to X$, i.e., such that there are fibre product diagrams

$$\begin{tikzcd} X'_i \arrow[r] \arrow[d] & W_i \arrow[d] \\ X \arrow[r] & \mathbf{A}^n_Y \end{tikzcd}$$

After replacing X' by $\coprod X'_i$ and setting $W = \coprod W_i$ we obtain a fibre product diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & W \arrow[d, "h"] \\ X \arrow[r] & \mathbf{A}^n_Y \end{tikzcd}$$

with $W \to \mathbf{A}^n_Y$ flat and of finite presentation and $X' \to Y$ still pseudo-coherent. Since $W \to \mathbf{A}^n_Y$ is open (see Morphisms, Lemma 24.24.9) and $X' \to X$ is surjective we can find $f \in \Gamma(\mathbf{A}^n_Y, \mathcal{O})$ such that $X \subset D(f) \subset \mathrm{Im}(h)$. Write $Y = Spec(R)$, $X = Spec(A)$, $X' = Spec(A')$ and $W = Spec(B)$, $A = R[x_1, \ldots, x_n]/I$ and $A' = B/IB$. Then $R \to A'$ is pseudo-coherent. Picture

$$\begin{tikzcd} A' = B/IB & B \arrow[l] \\ A = R[x_1, \ldots, x_n]/I \arrow[u] & R[x_1, \ldots, x_n] \arrow[l] \arrow[u] \end{tikzcd}$$

By Lemma 33.36.10 we see that IB is pseudo-coherent as a B-module. The ring map $R[x_1, \ldots, x_n]_f \to B_f$ is faithfully flat by our choice of f above. This implies that $I_f \subset$

$R[x_1, \ldots, x_n]_f$ is pseudo-coherent, see More on Algebra, Lemma 12.40.15. Applying Lemma 33.36.10 one more time we see that $R \to A$ is pseudo-coherent. $\square$

Lemma 33.36.12. *The property $\mathcal{P}(f) =$ "f is pseudo-coherent" is fppf local on the source.*

Proof. Let $f : X \to S$ be a morphism of schemes. Let $\{g_i : X_i \to X\}$ be an fppf covering such that each composition $f \circ g_i$ is pseudo-coherent. According to Lemma 33.32.1 there exist

(1) a Zariski open covering $X = \bigcup U_j$,
(2) surjective finite locally free morphisms $W_j \to U_j$,
(3) Zariski open coverings $W_j = \bigcup_k W_{j,k}$,
(4) surjective finite locally free morphisms $T_{j,k} \to W_{j,k}$

such that the fppf covering $\{h_{j,k} : T_{j,k} \to X\}$ refines the given covering $\{X_i \to X\}$. Denote $\psi_{j,k} : T_{j,k} \to X_{\alpha(j,k)}$ the morphisms that witness the fact that $\{T_{j,k} \to X\}$ refines the given covering $\{X_i \to X\}$. Note that $T_{j,k} \to X$ is a flat, locally finitely presented morphism, so both X_i and $T_{j,k}$ are pseudo-coherent over X by Lemma 33.36.6. Hence $\psi_{j,k} : T_{j,k} \to X_i$ is pseudo-coherent, see Lemma 33.36.7. Hence $T_{j,k} \to S$ is pseudo coherent as the composition of $\psi_{j,k}$ and $f \circ g_{\alpha(j,k)}$, see Lemma 33.36.4. Thus we see we have reduced the lemma to the case of a Zariski open covering (which is OK) and the case of a covering given by a single surjective finite locally free morphism which we deal with in the following paragraph.

Assume that $X' \to X \to S$ is a sequence of morphisms of schemes with $X' \to X$ surjective finite locally free and $X' \to Y$ pseudo-coherent. Our goal is to show that $X \to S$ is pseudo-coherent. Note that by Descent, Lemma 31.10.3 the morphism $X \to S$ is locally of finite presentation. It is clear that the problem reduces to the case that X', X and S are affine and $X' \to X$ is free of some rank $r > 0$. The corresponding algebra problem is the following: Suppose $R \to A \to A'$ are ring maps such that $R \to A'$ is pseudo-coherent, $R \to A$ is of finite presentation, and $A' \cong A^{\oplus r}$ as an A-module. Goal: Show $R \to A$ is pseudo-coherent. The assumption that $R \to A'$ is pseudo-coherent means that A' as an A'-module is pseudo-coherent relative to R. By More on Algebra, Lemma 12.45.5 this implies that A' as an A-module is pseudo-coherent relative to R. Since $A' \cong A^{\oplus r}$ as an A-module we see that A as an A-module is pseudo-coherent relative to R, see More on Algebra, Lemma 12.45.8. This by definition means that $R \to A$ is pseudo-coherent and we win. $\square$

33.37. Perfect morphisms

In order to understand the material in this section you have to understand the material of the section on pseudo-coherent morphisms just a little bit. For now the only thing you need to know is that a ring map $A \to B$ is perfect if and only if it is pseudo-coherent and B has finite tor dimension as an A-module.

Lemma 33.37.1. *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. The following are equivalent*

(1) *there exist an affine open covering $S = \bigcup V_j$ and for each j an affine open covering $f^{-1}(V_j) = \bigcup U_{ji}$ such that $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_{ij})$ is a perfect ring map, and*
(2) *for every pair of affine opens $U \subset X$, $V \subset S$ such that $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is perfect.*

Proof. Assume (1) and let U, V be as in (2). It follows from Lemma 33.36.1 that $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is pseudo-coherent. Hence it suffices to prove that the property of a ring map being "of finite tor dimension" satisfies conditions (1)(a), (b), (c) of Morphisms, Definition 24.13.1. These properties follow from More on Algebra, Lemmas 12.41.9, 12.41.12, and 12.41.13. Some details omitted. □

Definition 33.37.2. A morphism of schemes $f : X \to S$ is called *perfect* if the equivalent conditions of Lemma 33.36.1 are satisfied. In this case we also say that X is perfect over S.

Note that a perfect morphism is in particular pseudo-coherent, hence locally of finite presentation. Beware that a base change of a perfect morphism is not perfect in general.

Lemma 33.37.3. *A flat base change of a perfect morphism is perfect.*

Proof. This translates into the following algebra result: Let $A \to B$ be a perfect ring map. Let $A \to A'$ be flat. Then $A' \to B \otimes_A A'$ is perfect. This result for pseudo-coherent ring maps we have seen in Lemma 33.36.3. The corresponding fact for finite tor dimension follows from More on Algebra, Lemma 12.41.12. □

Lemma 33.37.4. *A composition of perfect morphisms of schemes is perfect.*

Proof. This translates into the following algebra result: If $A \to B \to C$ are composable perfect ring maps then $A \to C$ is perfect. We have seen this is the case for pseudo-coherent in Lemma 33.36.4 and its proof. By assumption there exist integers n, m such that B has tor dimension $\leq n$ over A and C has tor dimension $\leq m$ over B. Then for any A-module M we have

$$M \otimes_A^{\mathbf{L}} C = (M \otimes_A^{\mathbf{L}} B) \otimes_B^{\mathbf{L}} C$$

and the spectral sequence of More on Algebra, Example 12.6.4 shows that $\text{Tor}_p^A(M, C) = 0$ for $p > n + m$ as desired. □

Lemma 33.37.5. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *f is flat and perfect, and*
(2) *f is flat and locally of finite presentation.*

Proof. The implication (2) $\Rightarrow$ (1) is More on Algebra, Lemma 12.46.4. The converse follows from the fact that a pseudo-coherent morphism is locally of finite presentation, see Lemma 33.36.5. □

Lemma 33.37.6. *Let $f : X \to S$ be a morphism of schemes. Assume S is regular and f is locally of finite type. Then f is perfect.*

Proof. See More on Algebra, Lemma 12.46.5. □

Lemma 33.37.7. *A regular immersion of schemes is perfect. A Koszul-regular immersion of schemes is perfect.*

Proof. Since a regular immersion is a Koszul-regular immersion, see Divisors, Lemma 26.13.2, it suffices to prove the second statement. This translates into the following algebraic statement: Suppose that $I \subset A$ is an ideal generated by a Koszul-regular sequence $f_1, \ldots, f_r$ of A. Then $A \to A/I$ is a perfect ring map. Since $A \to A/I$ is surjective this is a presentation of A/I by a polynomial algebra over A. Hence it suffices to see that A/I is pseudo-coherent as an A-module and has finite tor dimension. By definition of a Koszul sequence the Koszul complex $K(A, f_1, \ldots, f_r)$ is a finite free resolution of A/I. Hence A/I is a perfect complex of A-modules and we win. □

Lemma 33.37.8. *Let*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume $Y \to S$ smooth and $X \to S$ perfect. Then $f : X \to Y$ is perfect.

Proof. We can factor f as the composition

$$X \longrightarrow X \times_S Y \longrightarrow Y$$

where the first morphism is the map $i = (1, f)$ and the second morphism is the projection. Since $Y \to S$ is flat, see Morphisms, Lemma 24.33.9, we see that $X \times_S Y \to Y$ is perfect by Lemma 33.37.3. As $Y \to S$ is smooth, also $X \times_S Y \to X$ is smooth, see Morphisms, Lemma 24.33.5. Hence i is a section of a smooth morphism, therefore i is a regular immersion, see Divisors, Lemma 26.14.7. This implies that i is perfect, see Lemma 33.37.7. We conclude that f is perfect because the composition of perfect morphisms is perfect, see Lemma 33.37.4. □

Remark 33.37.9. It is not true that a morphism between schemes X, Y perfect over a base S is perfect. An example is $S = Spec(k)$, $X = Spec(k)$, $Y = Spec(k[x]/(x^2)$ and $X \to Y$ the unique S-morphism.

Lemma 33.37.10. *The property $\mathcal{P}(f) =$``f is perfect'' is fpqc local on the base.*

Proof. We will use the criterion of Descent, Lemma 31.18.4 to prove this. By Definition 33.37.2 being perfect is Zariski local on the base. By Lemma 33.37.3 being perfect is preserved under flat base change. The final hypothesis (3) of Descent, Lemma 31.18.4 translates into the following algebra statement: Let $A \to B$ be a faithfully flat ring map. Let $C = A[x_1, \ldots, x_n]/I$ be an A-algebra. If $C \otimes_A B$ is perfect as an $B[x_1, \ldots, x_n]$-module, then C is perfect as a $A[x_1, \ldots, x_n]$-module. This is More on Algebra, Lemma 12.42.12. □

Lemma 33.37.11. *Let $f : X \to S$ be a pseudo-coherent morphism of schemes. Then f is perfect if and only if for every $x \in X$ the ring $\mathcal{O}_{X,x}$ has finite tor dimension as an $\mathcal{O}_{S,f(x)}$-module.*

Proof. This translates into the following algebra problem. Suppose that $A \to B$ is a pseudo-coherent ring map. Write $B = A[x_1, \ldots, x_n]/I$. Then the following are equivalent

(1) $B_\mathfrak{q}$ has finite tor dimension over $A_\mathfrak{p}$ for all $\mathfrak{q}$ (with $\mathfrak{p} = A \cap \mathfrak{q}$), and
(2) B is perfect as an $A[x_1, \ldots, x_n]$-module.

The implication (2) ⇒ (1) is clear. For the converse, consider a prime $\mathfrak{q}$ of B lying over $\mathfrak{p}$ as in (1). Let $\mathfrak{q}'$ be the prime of $A[x_1, \ldots, x_n]$ corresponding to $\mathfrak{q}$. By More on Algebra, Lemma 12.42.17 applied to $A_\mathfrak{p} \to A[x_1, \ldots, x_n]_{\mathfrak{q}'}$ we see that $B_\mathfrak{q}$ is a perfect $A[x_1, \ldots, x_n]_{\mathfrak{q}'}$-module. Hence B is a perfect $A[x_1, \ldots, x_n]$-module by More on Algebra, Lemma 12.42.16. Some details omitted. □

Lemma 33.37.12. *The property $\mathcal{P}(f) =$``f is perfect'' is fppf local on the source.*

Proof. Let $\{g_i : X_i \to X\}_{i \in I}$ be an fppf covering of schemes and let $f : X \to S$ be a morphism such that each $f \circ g_i$ is perfect. By Lemma 33.36.12 we conclude that f is pseudo-coherent. Hence by Lemma 33.37.11 it suffices to check that $\mathcal{O}_{X,x}$ is an $\mathcal{O}_{S,f(x)}$-module of finite tor dimension for all $x \in X$. Pick $i \in I$ and and $x_i \in X_i$ mapping to x. Then we see that $\mathcal{O}_{X_i,x_i}$ has finite tor dimension over $\mathcal{O}_{S,f(x)}$ and that $\mathcal{O}_{X,x} \to \mathcal{O}_{X_i,x_i}$ is faithfully flat. The desired conclusion follows from More on Algebra, Lemma 12.41.14. □

33.38. Local complete intersection morphisms

In Divisors, Section 26.13 we have defined 4 different types of regular immersions: regular, Koszul-regular, H_1-regular, and quasi-regular. In this section we consider morphisms $f : X \to S$ which locally on X factors as

$$\begin{tikzcd} X \arrow[rr, "i"] \arrow[dr] & & \mathbf{A}^n_S \arrow[dl] \\ & S & \end{tikzcd}$$

where i is a $*$-regular immersion for $* \in \{\emptyset, Koszul, H_1, quasi\}$. However, we don't know how to prove that this condition is independent of the factorization if $* = \emptyset$, i.e., when we require i to be a regular immersion. On the other hand, we want a local complete intersection morphism to be perfect, which is only going to be true if $* = Koszul$ or $* = \emptyset$. Hence we will define a *local complete intersection morphism* or *Koszul morphism* to be a morphism of schemes $f : X \to S$ that locally on X has a factorization as above with i a Koszul-regular immersion. To see that this works we first prove this is independent of the chosen factorizations.

Lemma 33.38.1. *Let S be a scheme. Let U, P, P' be schemes over S. Let $u \in U$. Let $i : U \to P$, $i' : U \to P'$ be immersions over S. Assume P and P' smooth over S. Then the following are equivalent*

(1) *i is a Koszul-regular immersion in a neighbourhood of x, and*
(2) *i' is a Koszul-regular immersion in a neighbourhood of x.*

Proof. Assume i is a Koszul-regular immersion in a neighbourhood of x. Consider the morphism $j = (i, i') : U \to P \times_S P' = P''$. Since $P'' = P \times_S P' \to P$ is smooth, it follows from Divisors, Lemma 26.14.8 that j is a Koszul-regular immersion, whereupon it follows from Divisors, Lemma 26.14.11 that i' is a Koszul-regular immersion. □

Before we state the definition, let us make the following simple remark. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $x \in X$. Then there exist an open neighbourhood $U \subset X$ and a factorization of $f|_U$ as the composition of an immersion $i : U \to \mathbf{A}^n_S$ followed by the projection $\mathbf{A}^n_S \to S$ which is smooth. Picture

$$\begin{tikzcd} X \arrow[dr] & U \arrow[l] \arrow[r, "i"] \arrow[d] & \mathbf{A}^n_S = P \arrow[dl, "\pi"] \\ & S & \end{tikzcd}$$

In fact you can do this with any affine open neighbourhood U of x in X, see Morphisms, Lemma 24.38.2.

Definition 33.38.2. Let $f : X \to S$ be a morphism of schemes.

(1) Let $x \in X$. We say that f is *Koszul at x* if f is of finite type at x and there exists an open neighbourhood and a factorization of $f|_U$ as $\pi \circ i$ where $i : U \to P$ is a Koszul-regular immersion and $\pi : P \to S$ is smooth.

(2) We say f is a *Koszul morphism*, or that f is a *local complete intersection morphism* if f is Koszul at every point.

We have seen above that the choice of the factorization $f|_U = \pi \circ i$ is irrelevant, i.e., given a factorization of $f|_U$ as an immersion i followed by a smooth morphism π, whether or not i is Koszul regular in a neighbourhood of x is an intrinsic property of f at x. Let us record this here explicitly as a lemma so that we can refer to it

Lemma 33.38.3. *Let $f : X \to S$ be a local complete intersection morphism. Let P be a scheme smooth over S. Let $U \subset X$ be an open subscheme and $i : U \to P$ an immersion of schemes over S. Then i is a Koszul-regular immersion.*

Proof. This is the defining property of a local complete intersection morphism. See discussion above. □

It seems like a good idea to collect here some properties in common with all Koszul morphisms.

Lemma 33.38.4. *Let $f : X \to S$ be a local complete intersection morphism. Then*

(1) *f is locally of finite presentation,*
(2) *f is pseudo-coherent, and*
(3) *f is perfect.*

Proof. Since a perfect morphism is pseudo-coherent (because a perfect ring map is pseudo-coherent) and a pseudo-coherent morphism is locally of finite presentation (because a pseudo-coherent ring map is of finite presentation) it suffices to prove the last statement. Being perfect is a local property, hence we may assume that f factors as $\pi \circ i$ where π is smooth and i is a Koszul-regular immersion. A Koszul-regular immersion is perfect, see Lemma 33.37.7. A smooth morphism is perfect as it is flat and locally of finite presentation, see Lemma 33.37.5. Finally a composition of perfect morphisms is perfect, see Lemma 33.37.4. □

Lemma 33.38.5. *Let $f : X = Spec(B) \to S = Spec(A)$ be a morphism of affine schemes. Then f is a local complete intersection morphism if and only if $A \to B$ is a local complete intersection homomorphism, see More on Algebra, Definition 12.24.2.*

Proof. Follows immediately from the definitions. □

Beware that a base change of a Koszul morphism is not Koszul in general.

Lemma 33.38.6. *A flat base change of a local complete intersection morphism is a local complete intersection morphism.*

Proof. Omitted. Hint: This is true because a base change of a smooth morphism is smooth and a flat base change of a Koszul-regular immersion is a Koszul-regular immersion, see Divisors, Lemma 26.13.3. □

Lemma 33.38.7. *A composition of local complete intersection morphisms is a local complete intersection morphism.*

Proof. Let $g : Y \to S$ and $f : X \to Y$ be local complete intersection morphisms. Let $x \in X$ and set $y = f(x)$. Choose an open neighbourhood $V \subset Y$ of y and a factorization $g|_V = \pi \circ i$ for some Koszul-regular immersion $i : V \to P$ and smooth morphism $\pi : P \to S$. Next choose an open neighbourhood U of $x \in X$ and a factorization $f|_U = \pi' \circ i'$ for some Koszul-regular immersion $i' : U \to P'$ and smooth morphism $\pi' : P' \to Y$. In fact, we may assume that $P' = \mathbf{A}^n_V$, see discussion preceding and following Definition 33.38.2. Picture:

$$\begin{tikzcd}
X \arrow[d] & U \arrow[l] \arrow[r, "i'"'] & P' = \mathbf{A}^n_V \arrow[d] & \\
Y \arrow[d] & & V \arrow[ll] \arrow[r, "i"'] & P \arrow[d] \\
S & & & S \arrow[lll]
\end{tikzcd}$$

Set $P'' = \mathbf{A}^n_P$. Then $U \to P' \to P''$ is a Koszul-regular immersion as a composition of Koszul-regular immersions, namely i' and the flat base change of i via $P'' \to P$, see Divisors, Lemma 26.13.3 and Divisors, Lemma 26.13.7. Also $P'' \to P \to S$ is smooth as a composition of smooth morphisms, see Morphisms, Lemma 24.33.4. Hence we conclude that $X \to S$ is Koszul at x as desired. □

Lemma 33.38.8. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *f is flat and a local complete intersection morphism, and*
(2) *f is syntomic.*

Proof. Assume (2). By Morphisms, Lemma 24.30.10 for every point x of X there exist affine open neighbourhoods U of x and V of $f(x)$ such that $f|_U : U \to V$ is standard syntomic. This means that $U = Spec(R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)) \to V = Spec(R)$ where $R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a relative global complete intersection over R. By Algebra, Lemma 7.125.13 the sequence $f_1, \ldots, f_c$ is a regular sequence in each local ring $R[x_1, \ldots, x_n]_\mathfrak{q}$ for every prime $\mathfrak{q} \supset (f_1, \ldots, f_c)$. Consider the Koszul complex $K_\bullet = K_\bullet(R[x_1, \ldots, x_n], f_1, \ldots, f_c)$ with homology groups $H_i = H_i(K_\bullet)$. By More on Algebra, Lemma 12.22.2 we see that $(H_i)_\mathfrak{q} = 0$, $i > 0$ for every $\mathfrak{q}$ as above. On the other hand, by More on Algebra, Lemma 12.21.6 we see that H_i is annihilated by $(f_1, \ldots, f_c)$. Hence we see that $H_i = 0$, $i > 0$ and $f_1, \ldots, f_c$ is a Koszul-regular sequence. This proves that $U \to V$ factors as a Koszul-regular immersion $U \to \mathbf{A}^n_V$ followed by a smooth morphism as desired.

Assume (1). Then f is a flat and locally of finite presentation (Lemma 33.38.4). Hence, according to Morphisms, Lemma 24.30.10 it suffices to show that the local rings $\mathcal{O}_{X_s,x}$ are local complete intersection rings. Choose, locally on X, a factorization $f = \pi \circ i$ for some Koszul-regular immersion $i : X \to P$ and smooth morphism $\pi : P \to S$. Note that $X \to P$ is a relative quasi-regular immersion over S, see Divisors, Definition 26.14.2. Hence according to Divisors, Lemma 26.14.4 we see that $X \to P$ is a regular immersion and the same remains true after any base change. Thus each fibre is a regular immersion, whence all the local rings of all the fibres of X are local complete intersections. □

Lemma 33.38.9. *A regular immersion of schemes is a local complete intersection morphism. A Koszul-regular immersion of schemes is a local complete intersection morphism.*

Proof. Since a regular immersion is a Koszul-regular immersion, see Divisors, Lemma 26.13.2, it suffices to prove the second statement. The second statement follows immediately from the definition. □

Lemma 33.38.10. *Let*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume $Y \to S$ smooth and $X \to S$ is a local complete intersection morphism. Then $f : X \to Y$ is a local complete intersection morphism.

Proof. Immediate from the definitions. □

Lemma 33.38.11. *The property $\mathcal{P}(f)$ =``f is a local complete intersection morphism'' is fpqc local on the base.*

Proof. Let $f : X \to S$ be a morphism of schemes. Let $\{S_i \to S\}$ be an fpqc covering of S. Assume that each base change $f_i : X_i \to S_i$ of f is a local complete intersection morphism. Note that this implies in particular that f is locally of finite type, see Lemma 33.38.4 and Descent, Lemma 31.19.8. Let $x \in X$. Choose an open neighbourhood U of x and an immersion $j : U \to \mathbf{A}^n_S$ over S (see discussion preceding Definition 33.38.2). We have to show that j is a Koszul-regular immersion. Since f_i is a local complete intersection morphism, we see that the base change $j_i : U \times_S S_i \to \mathbf{A}^n_{S_i}$ is a Koszul-regular immersion, see Lemma 33.38.3. Because $\{\mathbf{A}^n_{S_i} \to \mathbf{A}^n_S\}$ is a fpqc covering we see from Descent, Lemma 31.19.30 that j is a Koszul-regular immersion as desired. □

Lemma 33.38.12. *The property $\mathcal{P}(f)$ =``f is a local complete intersection morphism'' is syntomic local on the source.*

Proof. We will use the criterion of Descent, Lemma 31.22.3 to prove this. It follows from Lemmas 33.38.8 and 33.38.7 that being a local complete intersection morphism is preserved under precomposing with syntomic morphisms. It is clear from Definition 33.38.2 that being a local complete intersection morphism is Zariski local on the source and target. Hence, according to the aforementioned Descent, Lemma 31.22.3 it suffices to prove the following: Suppose $X' \to X \to Y$ are morphisms of affine schemes with $X' \to X$ syntomic and $X' \to Y$ a local complete intersection morphism. Then $X \to Y$ is a local complete intersection morphism. To see this, note that in any case $X \to Y$ is of finite presentation by Descent, Lemma 31.10.1. Choose a closed immersion $X \to \mathbf{A}^n_Y$. By Algebra, Lemma 7.125.19 we can find an affine open covering $X' = \bigcup_{i=1,\ldots,n} X'_i$ and syntomic morphisms $W_i \to \mathbf{A}^n_Y$ lifting the morphisms $X'_i \to X$, i.e., such that there are fibre product diagrams

$$\begin{tikzcd} X'_i \arrow[r] \arrow[d] & W_i \arrow[d] \\ X \arrow[r] & \mathbf{A}^n_Y \end{tikzcd}$$

After replacing X' by $\coprod X'_i$ and setting $W = \coprod W_i$ we obtain a fibre product diagram of affine schemes

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & W \arrow[d, "h"] \\ X \arrow[r] & \mathbf{A}^n_Y \end{tikzcd}$$

with $h : W \to \mathbf{A}^n_Y$ syntomic and $X' \to Y$ still a local complete intersection morphism. Since $W \to \mathbf{A}^n_Y$ is open (see Morphisms, Lemma 24.24.9) and $X' \to X$ is surjective we see that X is contained in the image of $W \to \mathbf{A}^n_Y$. Choose a closed immersion $W \to \mathbf{A}^{n+m}_Y$ over $\mathbf{A}^n_Y$. Now the diagram looks like

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & W \arrow[r] \arrow[d, "h"] & \mathbf{A}^{n+m}_Y \arrow[dl] \\ X \arrow[r] & \mathbf{A}^n_Y & \end{tikzcd}$$

Because h is syntomic and hence a local complete intersection morphism (see above) the morphism $W \to \mathbf{A}^{n+m}_Y$ is a Koszul-regular immersion. Because $X' \to Y$ is a local complete intersection morphism the morphism $X' \to \mathbf{A}^{n+m}_Y$ is a Koszul-regular immersion. We conclude from Divisors, Lemma 26.13.8 that $X' \to W$ is a Koszul-regular immersion. Hence, since being a Koszul-regular immersion is fpqc local on the target (see Descent, Lemma 31.19.30) we conclude that $X \to \mathbf{A}^n_Y$ is a Koszul-regular immersion which is what we had to show. □

Lemma 33.38.13. *Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes over S. Assume both X and Y are flat and locally of finite presentation over S. Then the set*

$$\{x \in X \mid f \text{ Koszul at } x\}.$$

is open in X and its formation commutes with arbitrary base change $S' \to S$.

Proof. The set is open by definition (see Definition 33.38.2). Let $S' \to S$ be a morphism of schemes. Set $X' = S' \times_S X$, $Y' = S' \times_S Y$, and denote $f' : X' \to Y'$ the base change of f. Let $x' \in X'$ be a point such that f' is Koszul at x'. Denote $s' \in S'$, $x \in X$, $y' \in Y'$, $y \in Y$, $s \in S$ the image of x'. Note that f is locally of finite presentation, see Morphisms, Lemma 24.20.11. Hence we may choose an affine neighbourhood $U \subset X$ of x and an immersion $i : U \to \mathbf{A}^n_Y$. Denote $U' = S' \times_S U$ and $i' : U' \to \mathbf{A}^n_{Y'}$ the base change of i. The assumption that f' is Koszul at x' implies that i' is a Koszul-regular immersion in a neighbourhood of x', see Lemma 33.38.3. The scheme X' is flat and locally of finite presentation over S' as a base change of X (see Morphisms, Lemmas 24.24.7 and 24.20.4). Hence i' is a relative H_1-regular immersion over S' in a neighbourhood of x' (see Divisors, Definition 26.14.2). Thus the base change $i'_{s'} : U'_{s'} \to \mathbf{A}^n_{Y'_{s'}}$ is a H_1-regular immersion in an open neighbourhood of x', see Divisors, Lemma 26.14.1 and the discussion following Divisors, Definition 26.14.2. Since $s' = Spec(\kappa(s')) \to Spec(\kappa(s)) = s$ is a surjective flat universally open morphism (see Morphisms, Lemma 24.22.4) we conclude that the base change $i_s : U_s \to \mathbf{A}^n_{Y_s}$ is an H_1-regular immersion in a neighbourhood of x, see Descent, Lemma 31.19.30. Finally, note that $\mathbf{A}^n_Y$ is flat and locally of finite presentation over S, hence Divisors, Lemma 26.14.6 implies that i is a (Koszul-)regular immersion in a neighbourhood of x as desired. □

Lemma 33.38.14. *Let $f : X \to Y$ be a local complete intersection morphism of schemes. Then f is unramified if and only if f is formally unramified and in this case the conormal sheaf $\mathcal{C}_{X/Y}$ is finite locally free on X.*

Proof. The first assertion follows immediately from Lemma 33.4.8 and the fact that a local complete intersection morphism is locally of finite type. To compute the conormal sheaf of f we choose, locally on X, a factorization of f as $f = p \circ i$ where $i : X \to V$ is a

Koszul-regular immersion and $V \to Y$ is smooth. By Lemma 33.9.11 we see that $\mathcal{C}_{X/Y}$ is a locally direct summand of $\mathcal{C}_{X/V}$ which is finite locally free as i is a Koszul-regular (hence quasi-regular) immersion, see Divisors, Lemma 26.13.5. $\square$

Lemma 33.38.15. *Let $Z \to Y \to X$ be formally unramified morphisms of schemes. Assume that $Z \to Y$ is a local complete intersection morphism. The exact sequence*

$$0 \to i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

of Lemma 33.5.12 is short exact.

Proof. The question is local on Z hence we may assume there exists a factorization $Z \to \mathbf{A}^n_Y \to Y$ of the morphism $Z \to Y$. Then we get a commutative diagram

$$\begin{tikzcd} Z \arrow[r, "i'"'] \arrow[d, equal] & \mathbf{A}^n_Y \arrow[r] \arrow[d] & \mathbf{A}^n_X \arrow[d] \\ Z \arrow[r, "i"] & Y \arrow[r] & X \end{tikzcd}$$

As $Z \to Y$ is a local complete intersection morphism, we see that $Z \to \mathbf{A}^n_Y$ is a Koszul-regular immersion. Hence by Divisors, Lemma 26.13.6 the sequence

$$0 \to (i')^*\mathcal{C}_{\mathbf{A}^n_Y/\mathbf{A}^n_X} \to \mathcal{C}_{Z/\mathbf{A}^n_X} \to \mathcal{C}_{Z/\mathbf{A}^n_Y} \to 0$$

is exact and locally split. Note that $i^*\mathcal{C}_{Y/X} = (i')^*\mathcal{C}_{\mathbf{A}^n_Y/\mathbf{A}^n_X}$ by Lemma 33.5.7 and note that the diagram

$$\begin{tikzcd} (i')^*\mathcal{C}_{\mathbf{A}^n_Y/\mathbf{A}^n_X} \arrow[r] & \mathcal{C}_{Z/\mathbf{A}^n_X} \\ i^*\mathcal{C}_{Y/X} \arrow[r] \arrow[u, "\cong"] & \mathcal{C}_{Z/X} \arrow[u] \end{tikzcd}$$

is commutative. Hence the lower horizontal arrow is a locally split injection. This proves the lemma. $\square$

33.39. Exact sequences of differentials and conormal sheaves

In this section we collect some results on exact sequences of conormal sheaves and sheaves of differentials. In some sense these are all realizations of the triangle of cotangent complexes associated to a pair of composable morphisms of schemes.

In the sequences below each of the maps are as constructed in either Morphisms, Lemma 24.32.9 or Lemma 33.5.5. Let $g : Z \to Y$ and $f : Y \to X$ be morphisms of schemes.

(1) There is a canonical exact sequence

$$g^*\Omega_{Y/X} \to \Omega_{Z/X} \to \Omega_{Z/Y} \to 0,$$

see Morphisms, Lemma 24.32.11. If $g : Z \to Y$ is formally smooth, then this sequence is a short exact sequence, see Lemma 33.9.9.

(2) If g is formally unramified, then there is a canonical exact sequence

$$\mathcal{C}_{Z/Y} \to g^*\Omega_{Y/X} \to \Omega_{Z/X} \to 0,$$

see Lemma 33.5.10. If $f \circ g : Z \to X$ is formally smooth, then this sequence is a short exact sequence, see Lemma 33.9.10.

(3) If g and $f \circ g$ are formally unramified, then there is a canonical exact sequence
$$\mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to g^*\Omega_{Y/X} \to 0,$$
see Lemma 33.5.11. If $f : Y \to X$ is formally smooth, then this sequence is a short exact sequence, see Lemma 33.9.11.

(4) If g and f are formally unramified, then there is a canonical exact sequence
$$g^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0.$$
see Lemma 33.5.12. If $g : Z \to Y$ is a local complete intersection morphism, then this sequence is a short exact sequence, see Lemma 33.38.15.

33.40. Other chapters

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CHAPTER 34

More on flatness

34.1. Introduction

In this chapter, we discuss some advanced results on flat modules and flat morphisms of schemes. Most of these results can be found in the paper [**GR71**] by Raynaud and Gruson.

Before reading this chapter we advise the reader to take a look at the following results (this list also serves as a pointer to previous results):

(1) General discussion on flat modules in Algebra, Section 7.35.
(2) The relationship between Tor-groups and flatness, see Algebra, Section 7.69.
(3) The sections on flatness criteria, namely, Algebra, Section 7.91 (Noetherian case), Algebra, Section 7.93 (Artinian case), Algebra, Section 7.119 (non-Noetherian case), and finally More on Morphisms, Section 33.12.
(4) Generic flatness, see Algebra, Section 7.109 and Morphisms, Section 24.26.
(5) Openness of the flat locus, see Algebra, Section 7.120 and More on Morphisms, Section 33.11.
(6) Flattening stratification, see More on Algebra, Section 12.11.
(7) Additional algebraic results in More on Algebra, Sections 12.16, 12.17, 12.18, and 12.19.

34.2. A remark on finite type versus finite presentation

Let $R \to A$ be a finite type ring map. Let M be an A-module. In More on Algebra, Section 12.44 we defined what it means for M to be finitely presented relative to R. We also proved this notion has good localization properties and glues. Hence we can define the corresponding global notion as follows.

Definition 34.2.1. Let $f : X \to S$ be locally of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Then we say *$\mathcal{F}$ is locally finitely presented relative to S* if there exists an affine open covering $S = \bigcup V_i$ and $f^{-1}(V_i) = \bigcup_j U_{ij}$ such that $\mathcal{F}(U_{ij})$ is a $\mathcal{O}_X(U_{ij})$-module of finite presentation relative to $\mathcal{O}_S(V_i)$.

In this way we can make sense of when a sheaf of modules on X is locally of finite presentation over S even if X is not locally of finite presentation over S. And of course, $X \to S$ is locally of finite presentation if and only if $\mathcal{O}_X$ is locally of finite presentation relative to S.

34.3. Lemmas on étale localization

In this section we list some lemmas on étale localization which will be useful later in this chapter. Please skip this section on a first reading.

Lemma 34.3.1. *Let $i : Z \to X$ be a closed immersion of affine schemes. Let $Z' \to Z$ be an étale morphism with Z' affine. Then there exists an étale morphism $X' \to X$ with X' affine such that $Z' \cong Z \times_X X'$ as schemes over Z.*

Proof. See Algebra, Lemma 7.132.10. □

Lemma 34.3.2. *Let*

$$\begin{tikzcd} X & X' \arrow[l] \arrow[d] \\ S \arrow[from=u] & S' \arrow[l] \end{tikzcd}$$

be a commutative diagram of schemes with $X' \to X$ and $S' \to S$ étale. Let $s' \in S'$ be a point. Then

$$X' \times_{S'} Spec(\mathcal{O}_{S',s'}) \longrightarrow X \times_S Spec(\mathcal{O}_{S',s'})$$

is étale.

Proof. This is true because $X' \to X_{S'}$ is étale as a morphism of schemes étale over X, see Morphisms, Lemma 24.35.18 and the base change of an étale morphism is étale, see Morphisms, Lemma 24.35.4. □

Lemma 34.3.3. *Let $X \to T \to S$ be morphisms of schemes with $T \to S$ étale. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x \in X$ be a point. Then*

$$\mathcal{F} \text{ flat over } S \text{ at } x \Leftrightarrow \mathcal{F} \text{ flat over } T \text{ at } x$$

In particular $\mathcal{F}$ is flat over S if and only if $\mathcal{F}$ is flat over T.

Proof. As an étale morphism is a flat morphism (see Morphisms, Lemma 24.35.12) the implication "$\Leftarrow$" follows from Algebra, Lemma 7.35.3. For the converse assume that $\mathcal{F}$ is flat at x over S. Denote $\tilde{x} \in X \times_S T$ the point lying over x in X and over the image of x in T in T. Then $(X \times_S T \to X)^*\mathcal{F}$ is flat at $\tilde{x}$ over T via $\text{pr}_2 : X \times_S T \to T$, see Morphisms, Lemma 24.24.6. The diagonal $\Delta_{T/S} : T \to T \times_S T$ is an open immersion; combine Morphisms, Lemmas 24.34.13 and 24.35.5. So X is identified with open subscheme of $X \times_S T$, the restriction of pr_2 to this open is the given morphism $X \to T$, the point $\tilde{x}$ corresponds to the point x in this open, and $(X \times_S T \to X)^*\mathcal{F}$ restricted to this open is $\mathcal{F}$. Whence we see that $\mathcal{F}$ is flat at x over T. □

Lemma 34.3.4. *Let $T \to S$ be an étale morphism. Let $t \in T$ with image $s \in S$. Let M be a $\mathcal{O}_{T,t}$-module. Then*

$$M \text{ flat over } \mathcal{O}_{S,s} \Leftrightarrow M \text{ flat over } \mathcal{O}_{T,t}.$$

Proof. We may replace S by an affine neighbourhood of s and after that T by an affine neighbourhood of t. Set $\mathcal{F} = (Spec(\mathcal{O}_{T,t}) \to T)_*\widetilde{M}$. This is a quasi-coherent sheaf (see Schemes, Lemma 21.24.1 or argue directly) on T whose stalk at t is M (details omitted). Apply Lemma 34.3.3. □

Lemma 34.3.5. *Let S be a scheme and $s \in S$ a point. Denote $\mathcal{O}_{S,s}^h$ (resp. $\mathcal{O}_{S,s}^{sh}$) the henselization (resp. strict henselization), see Algebra, Definition 7.139.14. Let M^{sh} be a $\mathcal{O}_{S,s}^{sh}$-module. The following are equivalent*

1. *M^{sh} is flat over $\mathcal{O}_{S,s}$,*
2. *M^{sh} is flat over $\mathcal{O}_{S,s}^h$, and*
3. *M^{sh} is flat over $\mathcal{O}_{S,s}^{sh}$.*

If $M^{sh} = M^h \otimes_{\mathcal{O}^h_{S,s}} \mathcal{O}^{sh}_{S,s}$ this is also equivalent to

(4) *M^h is flat over $\mathcal{O}_{S,s}$, and*
(5) *M^h is flat over $\mathcal{O}^h_{S,s}$.*

If $M^h = M \otimes_{\mathcal{O}_{S,s}} \mathcal{O}^h_{S,s}$ this is also equivalent to

(6) *M is flat over $\mathcal{O}_{S,s}$.*

Proof. We may assume that S is an affine scheme. It is shown in Algebra, Lemmas 7.139.23 and 7.139.24 that $\mathcal{O}^h_{S,s}$ and $\mathcal{O}^{sh}_{S,s}$ are filtered colimits of the rings $\mathcal{O}_{T,t}$ where $T \to S$ is étale and affine. Hence the local ring maps $\mathcal{O}_{S,s} \to \mathcal{O}^h_{S,s} \to \mathcal{O}^{sh}_{S,s}$ are flat as directed colimits of étale ring maps, see Algebra, Lemma 7.35.2. Hence (3) ⇒ (2) ⇒ (1) and (5) ⇒ (4) follow from Algebra, Lemma 7.35.3. Of course these maps are faithfully flat, see Algebra, Lemma 7.35.16. Hence the equivalences (6) ⇔ (5) and (5) ⇔ (3) follow from Algebra, Lemma 7.35.7. Thus it suffices to show that (1) ⇒ (2) ⇒ (3) and (4) ⇒ (5).

Assume (1). By Lemma 34.3.4 we see that M^{sh} is flat over $\mathcal{O}_{T,t}$ for any étale neighbourhood $(T, t) \to (S, s)$. Since $\mathcal{O}^h_{S,s}$ and $\mathcal{O}^{sh}_{S,s}$ are directed colimits of local rings of the form $\mathcal{O}_{T,t}$ (see above) we conclude that M^{sh} is flat over $\mathcal{O}^h_{S,s}$ and $\mathcal{O}^{sh}_{S,s}$ by Algebra, Lemma 7.35.5. Thus (1) implies (2) and (3). Of course this implies also (2) ⇒ (3) by replacing $\mathcal{O}_{S,s}$ by $\mathcal{O}^h_{S,s}$. The same argument applies to prove (4) ⇒ (5). □

Lemma 34.3.6. *Let $g : T \to S$ be a finite flat morphism of schemes. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_S$-module. Let $t \in T$ be a point with image $s \in S$. Then*

$$t \in WeakAss(g^*\mathcal{G}) \Leftrightarrow s \in WeakAss(\mathcal{G})$$

Proof. The implication "⇐" follows immediately from Divisors, Lemma 26.6.4. Assume $t \in \text{WeakAss}(g^*\mathcal{G})$. Let $Spec(A) \subset S$ be an affine open neighbourhood of s. Let $\mathcal{G}$ be the quasi-coherent sheaf associated to the A-module M. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to s. As g is finite flat we have $g^{-1}(Spec(A)) = Spec(B)$ for some finite flat A-algebra B. Note that $g^*\mathcal{G}$ is the quasi-coherent $\mathcal{O}_{Spec(B)}$-module associated to the B-module $M \otimes_A B$ and $g_*g^*\mathcal{G}$ is the quasi-coherent $\mathcal{O}_{Spec(A)}$-module associated to the A-module $M \otimes_A B$. By Algebra, Lemma 7.72.4 we have $B_\mathfrak{p} \cong A_\mathfrak{p}^{\oplus n}$ for some integer $n \geq 0$. Note that $n \geq 1$ as we assumed there exists at least one point of T lying over s. Hence we see by looking at stalks that

$$s \in \text{WeakAss}(\mathcal{G}) \Leftrightarrow s \in \text{WeakAss}(g_*g^*\mathcal{G})$$

Now the assumption that $t \in \text{WeakAss}(g^*\mathcal{G})$ implies that $s \in \text{WeakAss}(g_*g^*\mathcal{G})$ by Divisors, Lemma 26.6.3 and hence by the above $s \in \text{WeakAss}(\mathcal{G})$. □

Lemma 34.3.7. *Let $h : U \to S$ be an étale morphism of schemes. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_S$-module. Let $u \in U$ be a point with image $s \in S$. Then*

$$u \in WeakAss(h^*\mathcal{G}) \Leftrightarrow s \in WeakAss(\mathcal{G})$$

Proof. After replacing S and U by affine neighbourhoods of s and u we may assume that g is a standard étale morphism of affines, see Morphisms, Lemma 24.35.14. Thus we may assume $S = Spec(A)$ and $X = Spec(A[x, 1/g]/(f))$, where f is monic and f' is invertible in $A[x, 1/g]$. Note that $A[x, 1/g]/(f) = (A[x]/(f))_g$ is also the localization of the finite free A-algebra $A[x]/(f)$. Hence we may think of U as an open subscheme of the scheme $T = Spec(A[x]/(f))$ which is finite locally free over S. This reduces us to Lemma 34.3.6 above. □

34.4. The local structure of a finite type module

The key technical lemma that makes a lot of the arguments in this chapter work is the geometric Lemma 34.4.2.

Lemma 34.4.1. *Let $f : X \to S$ be a finite type morphism of affine schemes. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $x \in X$ with image $s = f(x)$ in S. Set $\mathcal{F}_s = \mathcal{F}|_{X_s}$. Then there exist a closed immersion $i : Z \to X$ of finite presentation, and a quasi-coherent finite type $\mathcal{O}_Z$-module $\mathcal{G}$ such that $i_*\mathcal{G} = \mathcal{F}$ and $Z_s = Supp(\mathcal{F}_s)$.*

Proof. Say the morphism $f : X \to S$ is given by the ring map $A \to B$ and that $\mathcal{F}$ is the quasi-coherent sheaf associated to the B-module M. By Morphisms, Lemma 24.14.2 we know that $A \to B$ is a finite type ring map, and by Properties, Lemma 23.16.1 we know that M is a finite B-module. In particular the support of $\mathcal{F}$ is the closed subscheme of $Spec(B)$ cut out by the annihilator $I = \{x \in B \mid xm = 0\ \forall m \in M\}$ of M, see Algebra, Lemma 7.59.4. Let $\mathfrak{q} \subset B$ be the prime ideal corresponding to x and let $\mathfrak{p} \subset A$ be the prime ideal corresponding to s. Note that $X_s = Spec(B \otimes_A \kappa(\mathfrak{p}))$ and that $\mathcal{F}_s$ is the quasi-coherent sheaf associated to the $B \otimes_A \kappa(\mathfrak{p})$ module $M \otimes_A \kappa(\mathfrak{p})$. By Coherent, Lemma 25.10.3 the support of $\mathcal{F}_s$ is equal to $V(I(B \otimes_A \kappa(\mathfrak{p})))$. Since $B \otimes_A \kappa(\mathfrak{p})$ is of finite type over $\kappa(\mathfrak{p})$ there exist finitely many elements $f_1, \ldots, f_m \in I$ such that

$$I(B \otimes_A \kappa(\mathfrak{p})) = (f_1, \ldots, f_n)(B \otimes_A \kappa(\mathfrak{p})).$$

Denote $i : Z \to X$ the closed subscheme cut out by $(f_1, \ldots, f_m)$, in a formula $Z = Spec(B/(f_1, \ldots, f_m))$. Since M is annihilated by I we can think of M as an $B/(f_1, \ldots, f_m)$-module. In other words, $\mathcal{F}$ is the pushforward of a finite type module on Z. As $Z_s = \mathrm{Supp}(\mathcal{F}_s)$ by construction, this proves the lemma. □

Lemma 34.4.2. *Let $f : X \to S$ be morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $x \in X$ with image $s = f(x)$ in S. Set $\mathcal{F}_s = \mathcal{F}|_{X_s}$ and $n = \dim_x(Supp(\mathcal{F}_s))$. Then we can construct*

(1) *elementary étale neighbourhoods $g : (X', x') \to (X, x)$, $e : (S', s') \to (S, s)$,*
(2) *a commutative diagram*

$$\begin{tikzcd} X & X' \arrow[l, "g"'] \arrow[d] & Z' \arrow[l, "i"'] \arrow[d, "\pi"] \\ & & Y' \arrow[d, "h"] \\ S \arrow[from=uu, "f"'] & S' \arrow[l, "e"'] \arrow[from=u] & S' \arrow[l, equal] \end{tikzcd}$$

(3) *a point $z' \in Z'$ with $i(z') = x'$, $y' = \pi(z')$, $h(y') = s'$,*
(4) *a finite type quasi-coherent $\mathcal{O}_{Z'}$-module $\mathcal{G}$,*

such that the following properties hold

(1) *X', Z', Y', S' are affine schemes,*
(2) *i is a closed immersion of finite presentation,*
(3) *$i_*(\mathcal{G}) \cong g^*\mathcal{F}$,*
(4) *π is finite and $\pi^{-1}(\{y'\}) = \{z'\}$,*
(5) *the extension $\kappa(s') \subset \kappa(y')$ is purely transcendental,*
(6) *h is smooth of relative dimension n with geometrically integral fibres.*

Proof. Let $V \subset S$ be an affine neighbourhood of s. Let $U \subset f^{-1}(V)$ be an affine neighbourhood of x. Then it suffices to prove the lemma for $f|_U : U \to V$ and $\mathcal{F}|_U$. Hence in the rest of the proof we assume that X and S are affine.

First, suppose that $X_s = \mathrm{Supp}(\mathcal{F}_s)$, in particular $n = \dim_x(X_s)$. Apply More on Morphisms, Lemmas 33.31.2 and 33.31.3. This gives us a commutative diagram

$$\begin{tikzcd} X \arrow[dd] & X' \arrow[l, "g"'] \arrow[d, "\pi"] \\ & Y' \arrow[d, "h"] \\ S & S' \arrow[l, "e"] \end{tikzcd}$$

and point $x' \in X'$. We set $Z' = X'$, $i = \mathrm{id}$, and $\mathcal{G} = g^*\mathcal{F}$ to obtain a solution in this case.

In general choose a closed immersion $Z \to X$ and a sheaf $\mathcal{G}$ on Z as in Lemma 34.4.1. Applying the result of the previous paragraph to $Z \to S$ and $\mathcal{G}$ we obtain a diagram

$$\begin{tikzcd} X \arrow[dd, "f"'] & Z \arrow[l] \arrow[dd, "f|_Z"'] & Z' \arrow[l, "g"'] \arrow[d, "\pi"] \\ & & Y' \arrow[d, "h"] \\ S \arrow[r, equal] & S & S' \arrow[l, "e"] \end{tikzcd}$$

and point $z' \in Z'$ satisfying all the required properties. We will use Lemma 34.3.1 to embed Z' into a scheme étale over X. We cannot apply the lemma directly as we want X' to be a scheme over S'. Instead we consider the morphisms

$$Z' \longrightarrow Z \times_S S' \longrightarrow X \times_S S'$$

The first morphism is étale by Morphisms, Lemma 24.35.18. The second is a closed immersion as a base change of a closed immersion. Finally, as X, S, S', Z, Z' are all affine we may apply Lemma 34.3.1 to get an étale morphism of affine schemes $X' \to X \times_S S'$ such that

$$Z' = (Z \times_S S') \times_{(X \times_S S')} X' = Z \times_X X'.$$

As $Z \to X$ is a closed immersion of finite presentation, so is $Z' \to X'$. Let $x' \in X'$ be the point corresponding to $z' \in Z'$. Then the completed diagram

$$\begin{tikzcd} X \arrow[dd] & X' \arrow[l] \arrow[dd] & Z' \arrow[l, "i"'] \arrow[d, "\pi"] \\ & & Y' \arrow[d, "h"] \\ S & S' \arrow[l, "e"] \arrow[r, equal] & S' \end{tikzcd}$$

is a solution of the original problem. □

Lemma 34.4.3. *Assumptions and notation as in Lemma 34.4.2. If f is locally of finite presentation then π is of finite presentation. In this case the following are equivalent*

(1) *$\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation in a neighbourhood of x,*
(2) *$\mathcal{G}$ is an $\mathcal{O}_{Z'}$-module of finite presentation in a neighbourhood of z', and*
(3) *$\pi_*\mathcal{G}$ is an $\mathcal{O}_{Y'}$-module of finite presentation in a neighbourhood of y'.*

Still assuming f locally of finite presentation the following are equivalent to each other

(a) *$\mathcal{F}_x$ is an $\mathcal{O}_{X,x}$-module of finite presentation,*
(b) *$\mathcal{G}_{z'}$ is an $\mathcal{O}_{Z',z'}$-module of finite presentation, and*
(c) *$(\pi_*\mathcal{G})_{y'}$ is an $\mathcal{O}_{Y',y'}$-module of finite presentation.*

Proof. Assume f locally of finite presentation. Then $Z' \to S$ is locally of finite presentation as a composition of such, see Morphisms, Lemma 24.20.3. Note that $Y' \to S$ is also locally of finite presentation as a composition of a smooth and an étale morphism. Hence Morphisms, Lemma 24.20.11 implies π is locally of finite presentation. Since π is finite we conclude that it is also separated and quasi-compact, hence π is actually of finite presentation.

To prove the equivalence of (1), (2), and (3) we also consider: (4) $g^*\mathcal{F}$ is a $\mathcal{O}_{X'}$-module of finite presentation in a neighbourhood of x'. The pull back of a module of finite presentation is of finite presentation, see Modules, Lemma 15.11.4. Hence (1) $\Rightarrow$ (4). The étale morphism g is open, see Morphisms, Lemma 24.35.13. Hence for any open neighbourhood $U' \subset X'$ of x', the image $g(U')$ is an open neighbourhood of x and the map $\{U' \to g(U')\}$ is an étale covering. Thus (4) $\Rightarrow$ (1) by Descent, Lemma 31.5.2. Using Descent, Lemma 31.5.8 and some easy topological arguments (see More on Morphisms, Lemma 33.31.4) we see that (4) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).

To prove the equivalence of (a), (b), (c) consider the ring maps

$$\mathcal{O}_{X,x} \to \mathcal{O}_{X',x'} \to \mathcal{O}_{Z',z'} \leftarrow \mathcal{O}_{Y',y'}$$

The first ring map is faithfully flat. Hence $\mathcal{F}_x$ is of finite presentation over $\mathcal{O}_{X,x}$ if and only if $g^*\mathcal{F}_{x'}$ is of finite presentation over $\mathcal{O}_{X',x'}$, see Algebra, Lemma 7.77.2. The second ring map is surjective (hence finite) and finitely presented by assumption, hence $g^*\mathcal{F}_{x'}$ is of finite presentation over $\mathcal{O}_{X',x'}$ if and only if $\mathcal{G}_{z'}$ is of finite presentation over $\mathcal{O}_{Z',z'}$, see Algebra, Lemma 7.7.4. Because π is finite, of finite presentation, and $\pi^{-1}(\{y'\}) = \{x'\}$ the ring homomorphism $\mathcal{O}_{Y',y'} \leftarrow \mathcal{O}_{Z',z'}$ is finite and of finite presentation, see More on Morphisms, Lemma 33.31.4. Hence $\mathcal{G}_{z'}$ is of finite presentation over $\mathcal{O}_{Z',z'}$ if and only if $\pi_*\mathcal{G}_{y'}$ is of finite presentation over $\mathcal{O}_{Y',y'}$, see Algebra, Lemma 7.7.4. □

Lemma 34.4.4. *Assumptions and notation as in Lemma 34.4.2. The following are equivalent*

(1) *$\mathcal{F}$ is flat over S in a neighbourhood of x,*
(2) *$\mathcal{G}$ is flat over S' in a neighbourhood of z', and*
(3) *$\pi_*\mathcal{G}$ is flat over S' in a neighbourhood of y'.*

The following are equivalent also

(a) *$\mathcal{F}_x$ is flat over $\mathcal{O}_{S,s}$,*
(b) *$\mathcal{G}_{z'}$ is flat over $\mathcal{O}_{S',s'}$, and*
(c) *$(\pi_*\mathcal{G})_{y'}$ is flat over $\mathcal{O}_{S',s'}$.*

Proof. To prove the equivalence of (1), (2), and (3) we also consider: (4) $g^*\mathcal{F}$ is flat over S in a neighbourhood of x'. We will use Lemma 34.3.3 to equate flatness over S and S' without further mention. The étale morphism g is flat and open, see Morphisms, Lemma 24.35.13. Hence for any open neighbourhood $U' \subset X'$ of x', the image $g(U')$ is an open neighbourhood of x and the map $U' \to g(U')$ is surjective and flat. Thus (4) $\Leftrightarrow$ (1) by Morphisms, Lemma 24.24.11. Note that

$$\Gamma(X', g^*\mathcal{F}) = \Gamma(Z', \mathcal{G}) = \Gamma(Y', \pi_*\mathcal{G})$$

Hence the flatness of $g^*\mathcal{F}$, $\mathcal{G}$ and $\pi_*\mathcal{G}$ over S' are all equivalent (this uses that X', Z', Y', and S' are all affine). Some omitted topological arguments (compare More on Morphisms, Lemma 33.31.4) regarding affine neighbourhoods now show that (4) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).

To prove the equivalence of (a), (b), (c) consider the commutative diagram of local ring maps

$$\begin{tikzcd}
\mathcal{O}_{X',x'} \arrow[r, "\iota"'] & \mathcal{O}_{Z',z'} & \mathcal{O}_{Y',y'} \arrow[l, "\alpha"'] & \mathcal{O}_{S',s'} \arrow[l, "\beta"'] \\
\mathcal{O}_{X,x} \arrow[u, "\gamma"] & & & \mathcal{O}_{S,s} \arrow[lll, "\varphi"'] \arrow[u, "\epsilon"']
\end{tikzcd}$$

We will use Lemma 34.3.4 to equate flatness over $\mathcal{O}_{S,s}$ and $\mathcal{O}_{S',s'}$ without further mention. The map γ is faithfully flat. Hence $\mathcal{F}_x$ is flat over $\mathcal{O}_{S,s}$ if and only if $g^*\mathcal{F}_{x'}$ is flat over $\mathcal{O}_{S',s'}$, see Algebra, Lemma 7.35.8. As $\mathcal{O}_{S',s'}$-modules the modules $g^*\mathcal{F}_{x'}$, $\mathcal{G}_{z'}$, and $\pi_*\mathcal{G}_{y'}$ are all isomorphic, see More on Morphisms, Lemma 33.31.4. This finishes the proof. □

34.5. One step dévissage

In this section we explain what is a one step dévissage of a module. A one step dévissage exist étale locally on base and target. We discuss base change, Zariski shrinking and étale localization of a one step dévissage.

Definition 34.5.1. Let S be a scheme. Let X be locally of finite type over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $s \in S$ be a point. A *one step dévissage of $\mathcal{F}/X/S$ over s* is given by morphisms of schemes over S

$$X \xleftarrow{i} Z \xrightarrow{\pi} Y$$

and a quasi-coherent $\mathcal{O}_Z$-module $\mathcal{G}$ of finite type such that

1. X, S, Z and Y are affine,
2. i is a closed immersion of finite presentation,
3. $\mathcal{F} \cong i_*\mathcal{G}$,
4. π is finite, and
5. the structure morphism $Y \to S$ is smooth with geometrically irreducible fibres of dimension $\dim(\text{Supp}(\mathcal{F}_s))$.

In this case we say $(Z, Y, i, \pi, \mathcal{G})$ is a one step dévissage of $\mathcal{F}/X/S$ over s.

Note that such a one step dévissage can only exist if X and S are affine. In the definition above we only require X to be (locally) of finite type over S and we continue working in this setting below. In [**GR71**] the authors use consistently the setup where $X \to S$ is locally of finite presentation and $\mathcal{F}$ quasi-coherent $\mathcal{O}_X$-module of finite type. The advantage of this choice is that it ``makes sense'' to ask for $\mathcal{F}$ to be of finite presentation as an $\mathcal{O}_X$-module, whereas in our setting it ``does not make sense''. Please see Section 34.2 for a discussion; the observations made there show that in our setup we may consider the condition of $\mathcal{F}$ being ``locally of finite presentation relative to S'', and we could work consistently with this notion. Instead however, we will rely on the results of Lemma 34.4.3 and the observations in Remark 34.7.3 to deal with this issue in an ad hoc fashion whenever it comes up.

Definition 34.5.2. Let S be a scheme. Let X be locally of finite type over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $x \in X$ be a point with image s in S. A *one step dévissage of $\mathcal{F}/X/S$ at x* is a system $(Z, Y, i, \pi, \mathcal{G}, z, y)$, where $(Z, Y, i, \pi, \mathcal{G})$ is a one step dévissage of $\mathcal{F}/X/S$ over s and

1. $\dim_x(\text{Supp}(\mathcal{F}_s)) = \dim(\text{Supp}(\mathcal{F}_s))$,
2. $z \in Z$ is a point with $i(z) = x$ and $\pi(z) = y$,
3. we have $\pi^{-1}(\{y\}) = \{z\}$,
4. the extension $\kappa(s) \subset \kappa(y)$ is purely transcendental.

A one step dévissage of $\mathcal{F}/X/S$ at x can only exist if X and S are affine. Condition (1) assures us that $Y \to S$ has relative dimension equal to $\dim_x(\text{Supp}(\mathcal{F}_s))$ via condition (5) of Definition 34.5.1.

Lemma 34.5.3 (Reformulation of Lemma 34.4.2)**.** *Let $f : X \to S$ be morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $x \in X$ with image $s = f(x)$ in S. Then there exists a commutative diagram of pointed schemes*

$$\begin{tikzcd} (X, x) \arrow[d, "f"'] & (X', x') \arrow[l, "g"'] \arrow[d] \\ (S, s) & (S', s') \arrow[l] \end{tikzcd}$$

such that $(S', s') \to (S, s)$ and $(X', x') \to (X, x)$ are elementary étale neighbourhoods, and such that $g^*\mathcal{F}/X'/S'$ has a one step dévissage at x'.

Proof. This is immediate from Definition 34.5.2 and Lemma 34.4.2. □

Lemma 34.5.4. *Let S, X, $\mathcal{F}$, s be as in Definition 34.5.1. Let $(Z, Y, i, \pi, \mathcal{G})$ be a one step dévissage of $\mathcal{F}/X/S$ over s. Let $(S', s') \to (S, s)$ be any morphism of pointed schemes. Given this data let X', Z', Y', i', π' be the base changes of X, Z, Y, i, π via $S' \to S$. Let $\mathcal{F}'$ be the pullback of $\mathcal{F}$ to X' and let $\mathcal{G}'$ be the pullback of $\mathcal{G}$ to Z'. If S' is affine, then $(Z', Y', i', \pi', \mathcal{G}')$ is a one step dévissage of $\mathcal{F}'/X'/S'$ over s'.*

Proof. Fibre products of affines are affine, see Schemes, Lemma 21.17.2. Base change preserves closed immersions, morphisms of finite presentation, finite morphisms, smooth morphisms, morphisms with geometrically irreducible fibres, and morphisms of relative dimension n, see Morphisms, Lemmas 24.2.4, 24.20.4, 24.42.6, 24.33.5, 24.28.2, and More on Morphisms, Lemma 33.20.2. We have $i'_*\mathcal{G}' \cong \mathcal{F}'$ because pushforward along the finite morphism i commutes with base change, see Coherent, Lemma 25.6.1. We have $\dim(\text{Supp}(\mathcal{F}_s)) = \dim(\text{Supp}(\mathcal{F}'_{s'}))$ by Morphisms, Lemma 24.27.3 because

$$\text{Supp}(\mathcal{F}_s) \times_s s' = \text{Supp}(\mathcal{F}'_{s'}).$$

This proves the lemma. □

Lemma 34.5.5. *Let S, X, $\mathcal{F}$, x, s be as in Definition 34.5.2. Let $(Z, Y, i, \pi, \mathcal{G}, z, y)$ be a one step dévissage of $\mathcal{F}/X/S$ at x. Let $(S', s') \to (S, s)$ be a morphism of pointed schemes which induces an isomorphism $\kappa(s) = \kappa(s')$. Let $(Z', Y', i', \pi', \mathcal{G}')$ be as constructed in Lemma 34.5.4 and let $x' \in X'$ (resp. $z' \in Z'$, $y' \in Y'$) be the unique point mapping to both $x \in X$ (resp. $z \in Z$, $y \in Y$) and $s' \in S'$. If S' is affine, then $(Z', Y', i', \pi', \mathcal{G}', z', y')$ is a one step dévissage of $\mathcal{F}'/X'/S'$ at x'.*

Proof. By Lemma 34.5.4 $(Z', Y', i', \pi', \mathcal{G}')$ is a one step dévissage of $\mathcal{F}'/X'/S'$ over s'. Properties (1) -- (4) of Definition 34.5.2 hold for $(Z', Y', i', \pi', \mathcal{G}', z', y')$ as the assumption that $\kappa(s) = \kappa(s')$ insures that the fibres $X'_{s'}$, $Z'_{s'}$, and $Y'_{s'}$ are isomorphic to X_s, Z_s, and Y_s. ☐

Definition 34.5.6. Let S, X, $\mathcal{F}$, x, s be as in Definition 34.5.2. Let $(Z, Y, i, \pi, \mathcal{G}, z, y)$ be a one step dévissage of $\mathcal{F}/X/S$ at x. Let us define a *standard shrinking* of this situation to be given by standard opens $S' \subset S$, $X' \subset X$, $Z' \subset Z$, and $Y' \subset Y$ such that $s \in S'$, $x \in X'$, $z \in Z'$, and $y \in Y'$ and such that

$$(Z', Y', i|_{Z'}, \pi|_{Z'}, \mathcal{G}|_{Z'}, z, y)$$

is a one step dévissage of $\mathcal{F}|_{X'}/X'/S'$ at x.

Lemma 34.5.7. *With assumption and notation as in Definition 34.5.6 we have:*

1. *If $S' \subset S$ is a standard open neighbourhood of s, then setting $X' = X_{S'}$, $Z' = Z_{S'}$ and $Y' = Y_{S'}$ we obtain a standard shrinking.*
2. *Let $W \subset Y$ be a standard open neighbourhood of y. Then there exists a standard shrinking with $Y' = W \times_S S'$.*
3. *Let $U \subset X$ be an open neighbourhood of x. Then there exists a standard shrinking with $X' \subset U$.*

Proof. Part (1) is immediate from Lemma 34.5.5 and the fact that the inverse image of a standard open under a morphism of affine schemes is a standard open, see Algebra, Lemma 7.16.4.

Let $W \subset Y$ as in (2). Because $Y \to S$ is smooth it is open, see Morphisms, Lemma 24.33.10. Hence we can find a standard open neighbourhood S' of s contained in the image of W. Then the fibres of $W_{S'} \to S'$ are nonempty open subschemes of the fibres of $Y \to S$ over S' and hence geometrically irreducible too. Setting $Y' = W_{S'}$ and $Z' = \pi^{-1}(Y')$ we see that $Z' \subset Z$ is a standard open neighbourhood of z. Let $\overline{h} \in \Gamma(Z, \mathcal{O}_Z)$ be a function such that $Z' = D(\overline{h})$. As $i : Z \to X$ is a closed immersion, we can find a function $h \in \Gamma(X, \mathcal{O}_X)$ such that $i^\sharp(h) = \overline{h}$. Take $X' = D(h) \subset X$. In this way we obtain a standard shrinking as in (2).

Let $U \subset X$ be as in (3). We may after shrinking U assume that U is a standard open. By More on Morphisms, Lemma 33.31.4 there exists a standard open $W \subset Y$ neighbourhood of y such that $\pi^{-1}(W) \subset i^{-1}(U)$. Apply (2) to get a standard shrinking X', S', Z', Y' with $Y' = W_{S'}$. Since $Z' \subset \pi^{-1}(W) \subset i^{-1}(U)$ we may replace X' by $X' \cap U$ (still a standard open as U is also standard open) without violating any of the conditions defining a standard shrinking. Hence we win. ☐

Lemma 34.5.8. *Let S, X, $\mathcal{F}$, x, s be as in Definition 34.5.2. Let $(Z, Y, i, \pi, \mathcal{G}, z, y)$ be a one step dévissage of $\mathcal{F}/X/S$ at x. Let*

$$\begin{tikzcd} (Y, y) \arrow[d] & (Y', y') \arrow[l] \arrow[d] \\ (S, s) & (S', s') \arrow[l] \end{tikzcd}$$

be a commutative diagram of pointed schemes such that the horizontal arrows are elementary étale neighbourhoods. Then there exists a commutative diagram

$$
\begin{tikzcd}
 & & (X'', x'') \arrow[dll] \arrow[d] & (Z'', z'') \arrow[l] \arrow[dll] \arrow[d] \\
(X, x) \arrow[d] & (Z, z) \arrow[l] \arrow[d] & (S'', s'') \arrow[dll] & (Y'', y'') \arrow[l] \arrow[dll] \\
(S, s) & (Y, y) \arrow[l] & &
\end{tikzcd}
$$

of pointed schemes with the following properties:

(1) *$(S'', s'') \to (S', s')$ is an elementary étale neighbourhood and the morphism $S'' \to S$ is the composition $S'' \to S' \to S$,*
(2) *Y'' is an open subscheme of $Y' \times_{S'} S''$,*
(3) *$Z'' = Z \times_Y Y''$,*
(4) *$(X'', x'') \to (X, x)$ is an elementary étale neighbourhood, and*
(5) *$(Z'', Y'', i'', \pi'', \mathcal{G}'', z'', y'')$ is a one step dévissage at x'' of the sheaf $\mathcal{F}''$.*

Here $\mathcal{F}''$ (resp. $\mathcal{G}''$) is the pullback of $\mathcal{F}$ (resp. $\mathcal{G}$) via the morphism $X'' \to X$ (resp. $Z'' \to Z$) and $i'' : Z'' \to X''$ and $\pi'' : Z'' \to Y''$ are as in the diagram.

Proof. Let $(S'', s'') \to (S', s')$ be any elementary étale neighbourhood with S'' affine. Let $Y'' \subset Y' \times_{S'} S''$ be any affine open neighbourhood containing the point $y'' = (y', s'')$. Then we obtain an affine (Z'', z'') by (3). Moreover $Z_{S''} \to X_{S''}$ is a closed immersion and $Z'' \to Z_{S''}$ is an étale morphism. Hence Lemma 34.3.1 applies and we can find an étale morphism $X'' \to X_{S'}$ of affines such that $Z'' \cong X'' \times_{X_{S'}} Z_{S'}$. Denote $i'' : Z'' \to X''$ the corresponding closed immersion. Setting $x'' = i''(z'')$ we obtain a commutative diagram as in the lemma. Properties (1), (2), (3), and (4) hold by construction. Thus it suffices to show that (5) holds for a suitable choice of $(S'', s'') \to (S', s')$ and Y''.

We first list those properties which hold for any choice of $(S'', s'') \to (S', s')$ and Y'' as in the first paragraph. As we have $Z'' = X'' \times_X Z$ by construction we see that $i''_* \mathcal{G}'' = \mathcal{F}''$ (with notation as in the statement of the lemma), see Coherent, Lemma 25.6.1. Set $n = \dim(\text{Supp}(\mathcal{F}_s)) = \dim_x(\text{Supp}(\mathcal{F}_s))$. The morphism $Y'' \to S''$ is smooth of relative dimension n (because $Y' \to S'$ is smooth of relative dimension n as the composition $Y' \to Y_{S'} \to S'$ of an étale and smooth morphism of relative dimension n and because base change preserves smooth morphisms of relative dimension n). We have $\kappa(y'') = \kappa(y)$ and $\kappa(s) = \kappa(s'')$ hence $\kappa(y'')$ is a purely transcendental extension of $\kappa(s'')$. The morphism of fibres $X''_{s''} \to X_s$ is an étale morphism of affine schemes over $\kappa(s) = \kappa(s'')$ mapping the point x'' to the point x and pulling back $\mathcal{F}_s$ to $\mathcal{F}''_{s''}$. Hence

$$\dim(\text{Supp}(\mathcal{F}''_{s''})) = \dim(\text{Supp}(\mathcal{F}_s)) = n = \dim_x(\text{Supp}(\mathcal{F}_s)) = \dim_{x''}(\text{Supp}(\mathcal{F}''_{s''}))$$

because dimension is invariant under étale localization, see Descent, Lemma 31.17.2. As $\pi'' : Z'' \to Y''$ is the base change of π we see that π'' is finite and as $\kappa(y) = \kappa(y'')$ we see that $\pi^{-1}(\{y''\}) = \{z''\}$.

At this point we have verified all the conditions of Definition 34.5.1 except we have not verfied that $Y'' \to S''$ has geometrically irreducible fibres. Of course in general this is not going to be true, and it is at this point that we will use that $\kappa(s) \subset \kappa(y)$ is purely transcendental. Namely, let $T \subset Y'_{s'}$ be the irreducible component of $Y'_{s'}$ containing $y' = (y, s')$. Note that T is an open subscheme of $Y'_{s'}$ as this is a smooth scheme over $\kappa(s')$. By Varieties, Lemma 28.5.14 we see that T is geometrically connected because $\kappa(s') = \kappa(s)$ is algebraically closed in $\kappa(y') = \kappa(y)$. As T is smooth we see that T is geometrically irreducible. Hence More on Morphisms, Lemma 33.30.3 applies and we can find an elementary étale morphism $(S'', s'') \to (S', s')$ and an affine open $Y'' \subset Y'_{S''}$ such that all fibres of $Y'' \to S''$ are geometrically irreducible and such that $T = Y''_{s''}$. After shrinking (first Y'' and then S'') we may assume that both Y'' and S'' are affine. This finishes the proof of the lemma. $\square$

Lemma 34.5.9. *Let S, X, $\mathcal{F}$, s be as in Definition 34.5.1. Let $(Z, Y, i, \pi, \mathcal{G})$ be a one step dévissage of $\mathcal{F}/X/S$ over s. Let $\xi \in Y_s$ be the (unique) generic point. Then there exists an integer $r > 0$ and an $\mathcal{O}_Y$-module map*

$$\alpha : \mathcal{O}_Y^{\oplus r} \longrightarrow \pi_* \mathcal{G}$$

such that

$$\alpha : \kappa(\xi)^{\oplus r} \longrightarrow (\pi_* \mathcal{G})_\xi \otimes_{\mathcal{O}_{Y,\xi}} \kappa(\xi)$$

is an isomorphism. Moreover, in this case we have

$$\dim(Supp(Coker(\alpha)_s)) < \dim(Supp(\mathcal{F}_s)).$$

Proof. By assumption the schemes S and Y are affine. Write $S = Spec(A)$ and $Y = Spec(B)$. As π is finite the $\mathcal{O}_Y$-module $\pi_*\mathcal{G}$ is a finite type quasi-coherent $\mathcal{O}_Y$-module. Hence $\pi_*\mathcal{G} = \widetilde{N}$ for some finite B-module N. Let $\mathfrak{p} \subset B$ be the prime ideal corresponding to ξ. To obtain α set $r = \dim_{\kappa(\mathfrak{p})} N \otimes_B \kappa(\mathfrak{p})$ and pick $x_1, \ldots, x_r \in N$ which form a basis of $N \otimes_B \kappa(\mathfrak{p})$. Take $\alpha : B^{\oplus r} \to N$ to be the map given by the formula $\alpha(b_1, \ldots, b_r) = \sum b_i x_i$. It is clear that $\alpha : \kappa(\mathfrak{p})^{\oplus r} \to N \otimes_B \kappa(\mathfrak{p})$ is an isomorphism as desired. Finally, suppose α is any map with this property. Then $N' = \text{Coker}(\alpha)$ is a finite B-module such that $N' \otimes \kappa(\mathfrak{p}) = 0$. By Nakayama's lemma (Algebra, Lemma 7.14.5) we see that $N'_\mathfrak{p} = 0$. Since the fibre Y_s is geometrically irreducible of dimension n with generic point ξ and since we have just seen that ξ is not in the support of $\text{Coker}(\alpha)$ the last assertion of the lemma holds. $\square$

34.6. Complete dévissage

In this section we explain what is a complete dévissage of a module and prove that such exist. The material in this section is mainly bookkeeping.

Definition 34.6.1. Let S be a scheme. Let X be locally of finite type over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $s \in S$ be a point. A *complete dévissage of*

$\mathcal{F}/X/S$ over s is given by a diagram

$$
\begin{tikzcd}
X & Z_1 \arrow[l, "i_1"'] \arrow[d, "\pi_1"] & & & \\
& Y_1 & Z_2 \arrow[l, "i_2"'] \arrow[d, "\pi_2"] & & \\
& & Y_2 & Z_3 \arrow[l] \arrow[d] & \\
& & & \dots & \dots \arrow[l] \arrow[d] \\
& & & & Y_n
\end{tikzcd}
$$

of schemes over S, finite type quasi-coherent $\mathcal{O}_{Z_k}$-modules $\mathcal{G}_k$, and $\mathcal{O}_{Y_k}$-module maps

$$\alpha_k : \mathcal{O}_{Y_k}^{\oplus r_k} \longrightarrow \pi_{k,*}\mathcal{G}_k, \quad k = 1, \ldots, n$$

satisfying the following properties:

(1) $(Z_1, Y_1, i_1, \pi_1, \mathcal{G}_1)$ is a one step dévissage of $\mathcal{F}/X/S$ over s,
(2) the map α_k induces an isomorphism
$$\kappa(\xi_k)^{\oplus r_k} \longrightarrow (\pi_{k,*}\mathcal{G}_k)_{\xi_k} \otimes_{\mathcal{O}_{Y_k,\xi_k}} \kappa(\xi_k)$$
where $\xi_k \in (Y_k)_s$ is the unique generic point,
(3) for $k = 2, \ldots, n$ the system $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k)$ is a one step dévissage of $\mathrm{Coker}(\alpha_{k-1})/Y_{k-1}/S$ over s,
(4) $\mathrm{Coker}(\alpha_n) = 0$.

In this case we say that $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k)_{k=1,\ldots,n}$ is a complete dévissage of $\mathcal{F}/X/S$ over s.

Definition 34.6.2. Let S be a scheme. Let X be locally of finite type over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $x \in X$ be a point with image $s \in S$. A *complete dévissage of $\mathcal{F}/X/S$ at x* is given by a system

$$(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k, z_k, y_k)_{k=1,\ldots,n}$$

such that $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k)$ is a complete dévissage of $\mathcal{F}/X/S$ over s, and such that

(1) $(Z_1, Y_1, i_1, \pi_1, \mathcal{G}_1, z_1, y_1)$ is a one step dévissage of $\mathcal{F}/X/S$ at x,
(2) for $k = 2, \ldots, n$ the system $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, z_k, y_k)$ is a one step dévissage of $\mathrm{Coker}(\alpha_{k-1})/Y_{k-1}/S$ at y_{k-1}.

Again we remark that a complete dévissage can only exist if X and S are affine.

Lemma 34.6.3. *Let S, X, $\mathcal{F}$, s be as in Definition 34.6.1. Let $(S', s') \to (S, s)$ be any morphism of pointed schemes. Let $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k)_{k=1,\ldots,n}$ be a complete dévissage of $\mathcal{F}/X/S$ over s. Given this data let $X', Z'_k, Y'_k, i'_k, \pi'_k$ be the base changes of X, Z_k, Y_k, i_k, π_k via $S' \to S$. Let $\mathcal{F}'$ be the pullback of $\mathcal{F}$ to X' and let $\mathcal{G}'_k$ be the pullback of $\mathcal{G}_k$ to Z'_k. Let α'_k be the pullback of α_k to Y'_k. If S' is affine, then $(Z'_k, Y'_k, i'_k, \pi'_k, \mathcal{G}'_k, \alpha'_k)_{k=1,\ldots,n}$ is a complete dévissage of $\mathcal{F}'/X'/S'$ over s'.*

Proof. By Lemma 34.5.4 we know that the base change of a one step dévissage is a one step dévissage. Hence it suffices to prove that formation of $\mathrm{Coker}(\alpha_k)$ commutes with base change and that condition (2) of Definition 34.6.1 is preserved by base change. The first is true as $\pi'_{k,*}\mathcal{G}'_k$ is the pullback of $\pi_{k,*}\mathcal{G}_k$ (by Coherent, Lemma 25.6.1) and because $\otimes$ is right exact. The second because by the same token we have

$$(\pi_{k,*}\mathcal{G}_k)_{\xi_k} \otimes_{\mathcal{O}_{Y_k,\xi_k}} \kappa(\xi_k) \otimes_{\kappa(\xi_k)} \kappa(\xi'_k) \cong (\pi'_{k,*}\mathcal{G}'_k)_{\xi'_k} \otimes_{\mathcal{O}_{Y'_k,\xi'_k}} \kappa(\xi'_k)$$

with obvious notation. □

Lemma 34.6.4. *Let S, X, $\mathcal{F}$, x, s be as in Definition 34.6.2. Let $(S', s') \to (S, s)$ be a morphism of pointed schemes which induces an isomorphism $\kappa(s) = \kappa(s')$. Let $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k, z_k, y_k)_{k=1,\ldots,n}$ be a complete dévissage of $\mathcal{F}/X/S$ at x. Let $(Z'_k, Y'_k, i'_k, \pi'_k, \mathcal{G}'_k, \alpha'_k)_{k=1,\ldots,n}$ be as constructed in Lemma 34.6.3 and let $x' \in X'$ (resp. $z'_k \in Z'$, $y'_k \in Y'$) be the unique point mapping to both $x \in X$ (resp. $z_k \in Z_k$, $y_k \in Y_k$) and $s' \in S'$. If S' is affine, then $(Z'_k, Y'_k, i'_k, \pi'_k, \mathcal{G}'_k, \alpha'_k, z'_k, y'_k)_{k=1,\ldots,n}$ is a complete dévissage of $\mathcal{F}'/X'/S'$ at x'.*

Proof. Combine Lemma 34.6.3 and Lemma 34.5.5. □

Definition 34.6.5. Let S, X, $\mathcal{F}$, x, s be as in Definition 34.6.2. Consider a complete dévissage $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k, z_k, y_k)_{k=1,\ldots,n}$ of $\mathcal{F}/X/S$ at x. Let us define a *standard shrinking* of this situation to be given by standard opens $S' \subset S$, $X' \subset X$, $Z'_k \subset Z_k$, and $Y'_k \subset Y_k$ such that $s_k \in S'$, $x_k \in X'$, $z_k \in Z'$, and $y_k \in Y'$ and such that

$$(Z'_k, Y'_k, i'_k, \pi'_k, \mathcal{G}'_k, \alpha'_k, z_k, y_k)_{k=1,\ldots,n}$$

is a one step dévissage of $\mathcal{F}'/X'/S'$ at x where $\mathcal{G}'_k = \mathcal{G}_k|_{Z'_k}$ and $\mathcal{F}' = \mathcal{F}|_{X'}$.

Lemma 34.6.6. *With assumption and notation as in Definition 34.6.5 we have:*

1. *If $S' \subset S$ is a standard open neighbourhood of s, then setting $X' = X_{S'}$, $Z'_k = Z_{S'}$ and $Y'_k = Y_{S'}$ we obtain a standard shrinking.*
2. *Let $W \subset Y_n$ be a standard open neighbourhood of y. Then there exists a standard shrinking with $Y'_n = W \times_S S'$.*
3. *Let $U \subset X$ be an open neighbourhood of x. Then there exists a standard shrinking with $X' \subset U$.*

Proof. Part (1) is immediate from Lemmas 34.6.4 and 34.5.7.

Proof of (2). For convenience denote $X = Y_0$. We apply Lemma 34.5.7 (2) to find a standard shrinking S', Y'_{n-1}, Z'_n, Y'_n of the one step dévissage of $\mathrm{Coker}(\alpha_{n-1})/Y_{n-1}/S$ at y_{n-1} with $Y'_n = W \times_S S'$. We may repeat this procedure and find a standard shrinking $S'', Y''_{n-2}, Z''_{n-1}, Y''_{n-1}$ of the one step dévissage of $\mathrm{Coker}(\alpha_{n-2})/Y_{n-2}/S$ at y_{n-2} with $Y''_{n-1} = Y'_{n-1} \times_S S''$. We may continue in this manner until we obtain $S^{(n)}, Y^{(n)}_0, Z^{(n)}_1, Y^{(n)}_1$. At this point it is clear that we obtain our desired standard shrinking by taking $S^{(n)}$, $X^{(n)}$, $Z^{(n-k)}_k \times_S S^{(n)}$, and $Y^{(n-k)}_k \times_S S^{(n)}$ with the desired property.

Proof of (3). We use induction on the length of the complete dévissage. First we apply Lemma 34.5.7 (3) to find a standard shrinking S', X', Z'_1, Y'_1 of the one step dévissage of $\mathcal{F}/X/S$ at x with $X' \subset U$. If $n = 1$, then we are done. If $n > 1$, then by induction we can find a standard shrinking S'', Y''_1, Z''_k, and Y''_k of the complete dévissage $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k, z_k, y_k)_{k=2,\ldots,n}$ of $\mathrm{Coker}(\alpha_1)/Y_1/S$ at x such that $Y''_1 \subset Y'_1$. Using

Lemma 34.5.7 (2) we can find $S''' \subset S'$, $X''' \subset X'$, Z_1''' and $Y_1''' = Y_1'' \times_S S'''$ which is a standard shrinking. The solution to our problem is to take

$$S''', X''', Z_1''', Y_1''', Z_2'' \times_S S''', Y_2'' \times_S S''', \ldots, Z_n'' \times_S S''', Y_n'' \times_S S'''$$

This ends the proof of the lemma. □

Proposition 34.6.7. *Let S be a scheme. Let X be locally of finite type over S. Let $x \in X$ be a point with image $s \in S$. There exists a commutative diagram*

$$\begin{tikzcd} (X, x) \arrow[d] & (X', x') \arrow[l, "g"'] \arrow[d] \\ (S, s) & (S', s') \arrow[l] \end{tikzcd}$$

of pointed schemes such that the horizontal arrows are elementary étale neighbourhoods and such that $g^*\mathcal{F}/X'/S'$ has a complete dévissage at x.

Proof. We prove this by induction on the integer $d = \dim_x(\text{Supp}(\mathcal{F}_s))$. By Lemma 34.5.3 there exists a diagram

$$\begin{tikzcd} (X, x) \arrow[d] & (X', x') \arrow[l, "g"'] \arrow[d] \\ (S, s) & (S', s') \arrow[l] \end{tikzcd}$$

of pointed schemes such that the horizontal arrows are elementary étale neighbourhoods and such that $g^*\mathcal{F}/X'/S'$ has a one step dévissage at x'. The local nature of the problem implies that we may replace $(X, x) \to (S, s)$ by $(X', x') \to (S', s')$. Thus after doing so we may assume that there exists a one step dévissage $(Z_1, Y_1, i_1, \pi_1, \mathcal{G}_1)$ of $\mathcal{F}/X/S$ at x.

We apply Lemma 34.5.9 to find a map

$$\alpha_1 : \mathcal{O}_{Y_1}^{\oplus r_1} \longrightarrow \pi_{1,*}\mathcal{G}_1$$

which induces an isomorphism of vector spaces over $\kappa(\xi_1)$ where $\xi_1 \in Y_1$ is the unique generic point of the fibre of Y_1 over s. Moreover $\dim_{y_1}(\text{Supp}(\text{Coker}(\alpha_1)_s)) < d$. It may happen that the stalk of $\text{Coker}(\alpha_1)_s$ at y_1 is zero. In this case we may shrink Y_1 by Lemma 34.5.7 (2) and assume that $\text{Coker}(\alpha_1) = 0$ so we obtain a complete dévissage of length zero.

Assume now that the stalk of $\text{Coker}(\alpha_1)_s$ at y_1 is not zero. In this case, by induction, there exists a commutative diagram

(34.6.7.1)
$$\begin{tikzcd} (Y_1, y_1) \arrow[d] & (Y_1', y_1') \arrow[l, "h"'] \arrow[d] \\ (S, s) & (S', s') \arrow[l] \end{tikzcd}$$

of pointed schemes such that the horizontal arrows are elementary étale neighbourhoods and such that $h^*\text{Coker}(\alpha_1)/Y_1'/S'$ has a complete dévissage

$$(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k, z_k, y_k)_{k=2,\ldots,n}$$

at y_1'. (In particular $i_2 : Z_2 \to Y_1'$ is a closed immersion into Y_2'.) At this point we apply Lemma 34.5.8 to $S, X, \mathcal{F}, x, s$, the system $(Z_1, Y_1, i_1, \pi_1, \mathcal{G}_1)$ and diagram (34.6.7.1). We obtain a diagram

$$\begin{tikzcd}[row sep=small, column sep=small]
 & & (X'', x'') \arrow[dll] \arrow[d] & (Z''_1, z''_1) \arrow[l] \arrow[dll] \arrow[d] \\
(X, x) \arrow[d] & (Z_1, z_1) \arrow[l] \arrow[d] & (S'', s'') \arrow[dll] & (Y''_1, y''_1) \arrow[l] \arrow[dll] \\
(S, s) & (Y_1, y_1) \arrow[l] & &
\end{tikzcd}$$

with all the properties as listed in the referenced lemma. In particular $Y''_1 \subset Y'_1 \times_{S'} S''$. Set $X_1 = Y'_1 \times_{S'} S''$ and let $\mathcal{F}_1$ denote the pullback of $\text{Coker}(\alpha_1)$. By Lemma 34.6.4 the system

$$(Z_k \times_{S'} S'', Y_k \times_{S'} S'', i''_k, \pi''_k, \mathcal{G}''_k, \alpha''_k, z''_k, y''_k)_{k=2,\ldots,n} \tag{34.6.7.2}$$

is a complete dévissage of $\mathcal{F}_1$ to X_1. Again, the nature of the problem allows us to replace $(X, x) \to (S, s)$ by $(X'', x'') \to (S'', s'')$. In this we see that we may assume:

(a) There exists a one step dévissage $(Z_1, Y_1, i_1, \pi_1, \mathcal{G}_1)$ of $\mathcal{F}/X/S$ at x,
(b) there exists an $\alpha_1 : \mathcal{O}_{Y_1}^{\oplus r_1} \to \pi_{1,*}\mathcal{G}_1$ such that $\alpha \otimes \kappa(\xi_1)$ is an isomorphism,
(c) $Y_1 \subset X_1$ is open, $y_1 = x_1$, and $\mathcal{F}_1|_{Y_1} \cong \text{Coker}(\alpha_1)$, and
(d) there exists a complete dévissage $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k, z_k, y_k)_{k=2,\ldots,n}$ of $\mathcal{F}_1/X_1/S$ at x_1.

To finish the proof all we have to do is shrink the one step dévissage and the complete dévissage such that they fit together to a complete dévissage. (We suggest the reader do this on their own using Lemmas 34.5.7 and 34.6.6 instead of reading the proof that follows.) Since $Y_1 \subset X_1$ is an open neighbourhood of x_1 we may apply Lemma 34.6.6 (3) to find a standard shrinking $S', X'_1, Z'_2, Y'_2, \ldots, Y'_n$ of the datum (d) so that $X'_1 \subset Y_1$. Note that X'_1 is also a standard open of the affine scheme Y_1. Next, we shrink the datum (a) as follows: first we shrink the base S to S', see Lemma 34.5.7 (1) and then we shrink the result to S'', X'', Z''_1, Y''_1 using Lemma 34.5.7 (2) such that eventually $Y''_1 = X'_1 \times_S S''$ and $S'' \subset S'$. Then we see that

$$Z''_1, Y''_1, Z'_2 \times_{S'} S'', Y'_2 \times_{S'} S'', \ldots, Y'_n \times_{S'} S''$$

gives the complete dévissage we were looking for. □

Some more bookkeeping gives the following consequence.

Lemma 34.6.8. *Let $X \to S$ be a finite type morphism of schemes. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $s \in S$ be a point. There exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and étale morphisms $h_i : Y_i \to X_{S'}$, $i = 1, \ldots, n$ such that for each i there exists a complete dévissage of $\mathcal{F}_i/Y_i/S'$ over s', where $\mathcal{F}_i$ is the pullback of $\mathcal{F}$ to Y_i and such that $X_s = (X_{S'})_{s'} \subset \bigcup h_i(Y_i)$.*

Proof. For every point $x \in X_s$ we can find a diagram

$$\begin{tikzcd}
(X, x) \arrow[d] & (X', x') \arrow[l, "g"'] \arrow[d] \\
(S, s) & (S', s') \arrow[l]
\end{tikzcd}$$

of pointed schemes such that the horizontal arrows are elementary étale neighbourhoods and such that $g^*\mathcal{F}/X'/S'$ has a complete dévissage at x'. As $X \to S$ is of finite type the fibre X_s is quasi-compact, and since each $g : X' \to X$ as above is open we can cover X_s by a finite union of $g(X'_{s'})$. Thus we can find a finite family of such diagrams

$$\begin{tikzcd} (X, x) \arrow[d] & (X'_i, x'_i) \arrow[l, "g_i"'] \arrow[d] \\ (S, s) & (S'_i, s'_i) \arrow[l] \end{tikzcd} \quad i = 1, \ldots, n$$

such that $X_s = \bigcup g_i(X'_i)$. Set $S' = S'_1 \times_S \ldots \times_S S'_n$ and let $Y_i = X_i \times_{S'_i} S'$ be the base change of X'_i to S'. By Lemma 34.6.3 we see that the pullback of $\mathcal{F}$ to Y_i has a complete dévissage over s and we win. $\square$

34.7. Translation into algebra

It may be useful to spell out algebraically what it means to have a complete dévissage. We introduce the following notion (which is not that useful so we give it an impossibly long name).

Definition 34.7.1. Let $R \to S$ be a ring map. Let $\mathfrak{q}$ be a prime of S lying over the prime $\mathfrak{p}$ of R. A *elementary étale localization of the ring map $R \to S$ at $\mathfrak{q}$* is given by a commutative diagram of rings and accompanying primes

$$\begin{tikzcd} S \arrow[r] & S' \\ R \arrow[u] \arrow[r] & R' \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{q} \arrow[r, dash] \arrow[d, dash] & \mathfrak{q}' \arrow[d, dash] \\ \mathfrak{p} \arrow[r, dash] & \mathfrak{p}' \end{tikzcd}$$

such that $R \to R'$ and $S \to S'$ are étale ring maps and $\kappa(\mathfrak{p}) = \kappa(\mathfrak{p}')$ and $\kappa(\mathfrak{q}) = \kappa(\mathfrak{q}')$.

Definition 34.7.2. Let $R \to S$ be a finite type ring map. Let $\mathfrak{r}$ be a prime of R. Let N be a finite S-module. A *complete dévissage of $N/S/R$ over $\mathfrak{r}$* is given by R-algebra maps

$$\begin{tikzcd}[column sep=small] & A_1 & & A_2 & & \ldots & & A_n & \\ S \arrow[ur] & & B_1 \arrow[ul] \arrow[ur] & & \ldots \arrow[ul] \arrow[ur] & & \ldots \arrow[ul] \arrow[ur] & & B_n \arrow[ul] \end{tikzcd}$$

finite A_i-modules M_i and B_i-module maps $\alpha_i : B_i^{\oplus r_i} \to M_i$ such that

(1) $S \to A_1$ is surjective and of finite presentation,
(2) $B_i \to A_{i+1}$ is surjective and of finite presentation,
(3) $B_i \to A_i$ is finite,
(4) $R \to B_i$ is smooth with geometrically irreducible fibres,
(5) $N \cong M_1$ as S-modules,
(6) $\mathrm{Coker}(\alpha_i) \cong M_{i+1}$ as B_i-modules,
(7) $\alpha_i : \kappa(\mathfrak{p}_i)^{\oplus r_i} \to M_i \otimes_{B_i} \kappa(\mathfrak{p}_i)$ is an isomorphism where $\mathfrak{p}_i = \mathfrak{r}B_i$, and
(8) $\mathrm{Coker}(\alpha_n) = 0$.

In this situation we say that $(A_i, B_i, M_i, \alpha_i)_{i=1,\ldots,n}$ is a complete dévissage of $N/S/R$ over $\mathfrak{r}$.

Remark 34.7.3. Note that the R-algebras B_i for all i and A_i for $i \geq 2$ are of finite presentation over R. If S is of finite presentation over R, then it is also the case that A_1 is of finite presentation over R. In this case all the ring maps in the complete dévissage are of finite presentation. See Algebra, Lemma 7.6.2. Still assuming S of finite presentation over R the following are equivalent

(1) M is of finite presentation over S,
(2) M_1 is of finite presentation over A_1,
(3) M_1 is of finite presentation over B_1,
(4) each M_i is of finite presentation both as an A_i-module and as a B_i-module.

The equivalences (1) $\Leftrightarrow$ (2) and (2) $\Leftrightarrow$ (3) follow from Algebra, Lemma 7.7.4. If M_1 is finitely presented, so is $\text{Coker}(\alpha_1)$ (see Algebra, Lemma 7.5.4) and hence M_2, etc.

Definition 34.7.4. Let $R \to S$ be a finite type ring map. Let $\mathfrak{q}$ be a prime of S lying over the prime $\mathfrak{r}$ of R. Let N be a finite S-module. A *complete dévissage of $N/S/R$ at* $\mathfrak{q}$ is given by a complete dévissage $(A_i, B_i, M_i, \alpha_i)_{i=1,\ldots,n}$ of $N/S/R$ over $\mathfrak{r}$ and prime ideals $\mathfrak{q}_i \subset B_i$ lying over $\mathfrak{r}$ such that

(1) $\kappa(\mathfrak{r}) \subset \kappa(\mathfrak{q}_i)$ is purely transcendental,
(2) there is a unique prime $\mathfrak{q}_i' \subset A_i$ lying over $\mathfrak{q}_i \subset B_i$,
(3) $\mathfrak{q} = \mathfrak{q}_1' \cap S$ and $\mathfrak{q}_i = \mathfrak{q}_{i+1}' \cap A_i$,
(4) $R \to B_i$ has relative dimension $\dim_{\mathfrak{q}_i}(\text{Supp}(M_i \otimes_R \kappa(\mathfrak{r})))$.

Remark 34.7.5. Let $A \to B$ be a finite type ring map and let N be a finite B-module. Let $\mathfrak{q}$ be a prime of B lying over the prime $\mathfrak{r}$ of A. Set $X = Spec(B)$, $S = Spec(A)$ and $\mathcal{F} = \widetilde{N}$ on X. Let x be the point corresponding to $\mathfrak{q}$ and let $s \in S$ be the point corresponding to $\mathfrak{p}$. Then

(1) if there exists a complete dévissage of $\mathcal{F}/X/S$ over s then there exists a complete dévissage of $N/B/A$ over $\mathfrak{p}$, and
(2) there exists a complete dévissage of $\mathcal{F}/X/S$ at x if and only if there exists a complete dévissage of $N/B/A$ at $\mathfrak{q}$.

There is just a small twist in that we omitted the condition on the relative dimension in the formulation of ``a complete dévissage of $N/B/A$ over $\mathfrak{p}$'' which is why the implication in (1) only goes in one direction. The notion of a complete dévissage at $\mathfrak{q}$ does have this condition built in. In any case we will only use that existence for $\mathcal{F}/X/S$ implies the existence for $N/B/A$.

Lemma 34.7.6. *Let $R \to S$ be a finite type ring map. Let M be a finite S-module. Let $\mathfrak{q}$ be a prime ideal of S. There exists an elementary étale localization $R' \to S', \mathfrak{q}', \mathfrak{p}'$ of the ring map $R \to S$ at $\mathfrak{q}$ such that there exists a complete dévissage of $(M \otimes_S S')/S'/R'$ at $\mathfrak{q}'$.*

Proof. This is a reformulation of Proposition 34.6.7 via Remark 34.7.5 $\square$

34.8. Localization and universally injective maps

Lemma 34.8.1. *Let $R \to S$ be a ring map. Let N be a S-module. Assume*

(1) *R is a local ring with maximal ideal $\mathfrak{m}$,*
(2) *$\overline{S} = S/\mathfrak{m}S$ is Noetherian, and*
(3) *$\overline{N} = N/\mathfrak{m}_R N$ is a finite $\overline{S}$-module.*

Let $\Sigma \subset S$ be the multiplicative subset of elements which are not a zero divisor on $\overline{N}$. Then $\Sigma^{-1}S$ is a semi-local ring whose spectrum consists of primes $\mathfrak{q} \subset S$ contained in an element of $\text{Ass}_S(\overline{N})$. Moreover, any maximal ideal of $\Sigma^{-1}S$ corresponds to an associated prime of $\overline{N}$ over $\overline{S}$.

Proof. Note that $\text{Ass}_S(\overline{N}) = \text{Ass}_{\overline{S}}(\overline{N})$, see Algebra, Lemma 7.60.13. This is a finite set by Algebra, Lemma 7.60.5. Say $\{\mathfrak{q}_1, \ldots, \mathfrak{q}_r\} = \text{Ass}_S(\overline{N})$. We have $\Sigma = S \setminus (\bigcup \mathfrak{q}_i)$ by Algebra, Lemma 7.60.9. By the description of $Spec(\Sigma^{-1}S)$ in Algebra, Lemma 7.16.5 we see that the primes of $\Sigma^{-1}S$ correspond to the primes of S contained in one of the $\mathfrak{q}_i$. Hence the maximal ideals of $\Sigma^{-1}S$ correspond one-to-one with the maximal (w.r.t. inclusion) elements of the set $\{\mathfrak{q}_1, \ldots, \mathfrak{q}_r\}$. This proves the lemma. □

Lemma 34.8.2. *Assumption and notation as in Lemma 34.8.1. Assume moreover that*

1. *S is local and $R \to S$ is a local homomorphism,*
2. *S is essentially of finite presentation over R,*
3. *N is finitely presented over S, and*
4. *N is flat over R.*

Then each $s \in \Sigma$ defines a universally injective R-module map $s : N \to N$, and the map $N \to \Sigma^{-1}N$ is R-universally injective.

Proof. By Algebra, Lemma 7.119.4 the sequence $0 \to N \to N \to N/sN \to 0$ is exact and N/sN is flat over R. This implies that $s : N \to N$ is universally injective, see Algebra, Lemma 7.35.11. The map $N \to \Sigma^{-1}N$ is universally injective as the directed colimit of the maps $s : N \to N$. □

Lemma 34.8.3. *Let $R \to S$ be a ring map. Let N be an S-module. Let $S \to S'$ be a ring map. Assume*

1. *$R \to S$ is a local homomorphism of local rings*
2. *S is essentially of finite presentation over R,*
3. *N is of finite presentation over S,*
4. *N is flat over R,*
5. *$S \to S'$ is flat, and*
6. *the image of $Spec(S') \to Spec(S)$ contains all primes $\mathfrak{q}$ of S lying over $\mathfrak{m}_R$ such that $\mathfrak{q}$ is an associated prime of $N/\mathfrak{m}_R N$.*

Then $N \to N \otimes_R S'$ is R-universally injective.

Proof. Set $N' = N \otimes_R S'$. Consider the commutative diagram

$$\begin{tikzcd} N \arrow[r] \arrow[d] & N' \arrow[d] \\ \Sigma^{-1}N \arrow[r] & \Sigma^{-1}N' \end{tikzcd}$$

where $\Sigma \subset S$ is the set of elements which are not a zero divisor on $N/\mathfrak{m}_R N$. If we can show that the map $N \to \Sigma^{-1}N'$ is universally injective, then $N \to N'$ is too (see Algebra, Lemma 7.76.10).

By Lemma 34.8.1 the ring $\Sigma^{-1}S$ is a semi-local ring whose maximal ideals correspond to associated primes of $N/\mathfrak{m}_R N$. Hence the image of $Spec(\Sigma^{-1}S') \to Spec(\Sigma^{-1}S)$ contains all these maximal ideals by assumption. By Algebra, Lemma 7.35.15 the ring map $\Sigma^{-1}S \to \Sigma^{-1}S'$ is faithfully flat. Hence $\Sigma^{-1}N \to \Sigma^{-1}N'$, which is the map

$$N \otimes_S \Sigma^{-1}S \longrightarrow N \otimes_S \Sigma^{-1}S'$$

is universally injective, see Algebra, Lemmas 7.76.11 and 7.76.8. Finally, we apply Lemma 34.8.2 to see that $N \to \Sigma^{-1}N$ is universally injective. As the composition of universally injective module maps is universally injective (see Algebra, Lemma 7.76.9) we conclude that $N \to \Sigma^{-1}N'$ is universally injective and we win. □

Lemma 34.8.4. *Let $R \to S$ be a ring map. Let N be an S-module. Let $S \to S'$ be a ring map. Assume*

1. *$R \to S$ is of finite presentation and N is of finite presentation over S,*
2. *N is flat over R,*
3. *$S \to S'$ is flat, and*
4. *the image of $Spec(S') \to Spec(S)$ contains all primes $\mathfrak{q}$ such that $\mathfrak{q}$ is an associated prime of $N \otimes_R \kappa(\mathfrak{p})$ where $\mathfrak{p}$ is the inverse image of $\mathfrak{q}$ in R.*

Then $N \to N \otimes_R S'$ is R-universally injective.

Proof. By Algebra, Lemma 7.76.12 it suffices to show that $N_{\mathfrak{q}} \to (N \otimes_R S')_{\mathfrak{q}}$ is a $R_{\mathfrak{p}}$-universally injective for any prime $\mathfrak{q}$ of S lying over $\mathfrak{p}$ in R. Thus we may apply Lemma 34.8.3 to the ring maps $R_{\mathfrak{p}} \to S_{\mathfrak{q}} \to S'_{\mathfrak{q}}$ and the module $N_{\mathfrak{q}}$. □

The reader may want to compare the following lemma to Algebra, Lemma 7.91.1 and Lemma 7.119.4. In each case the conclusion is that the map $u : M \to N$ is universally injective with flat cokernel.

Lemma 34.8.5. *Let $(R, \mathfrak{m})$ be a local ring. Let $u : M \to N$ be an R-module map. If M is a projective R-module, N is a flat R-module, and $\overline{u} : M/\mathfrak{m}M \to N/\mathfrak{m}N$ is injective then u is universally injective.*

Proof. By Algebra, Theorem 7.79.4 the module M is free. If we show the result holds for every finitely generated direct summand of M, then the lemma follows. Hence we may assume that M is finite free. Write $N = colim_i N_i$ as a directed colimit of finite free modules, see Algebra, Theorem 7.75.4. Note that $u : M \to N$ factors through N_i for some i (as M is finite free). Denote $u_i : M \to N_i$ the corresponding R-module map. As $\overline{u}$ is injective we see that $\overline{u_i} : M/\mathfrak{m}M \to N_i/\mathfrak{m}N_i$ is injective and remains injective on composing with the maps $N_i/\mathfrak{m}N_i \to N_{i'}/\mathfrak{m}N_{i'}$ for all $i' \geq i$. As M and $N_{i'}$ are finite free over the local ring R this implies that $M \to N_{i'}$ is a split injection for all $i' \geq i$. Hence for any R-module Q we see that $M \otimes_R Q \to N_{i'} \otimes_R Q$ is injective for all $i' \geq i$. As $- \otimes_R Q$ commutes with colimits we conclude that $M \otimes_R Q \to N_{i'} \otimes_R Q$ is injective as desired. □

Lemma 34.8.6. *Assumption and notation as in Lemma 34.8.1. Assume moreover that N is projective as an R-module. Then each $s \in \Sigma$ defines a universally injective R-module map $s : N \to N$, and the map $N \to \Sigma^{-1}N$ is R-universally injective.*

Proof. Pick $s \in \Sigma$. By Lemma 34.8.5 the map $s : N \to N$ is universally injective. The map $N \to \Sigma^{-1}N$ is universally injective as the directed colimit of the maps $s : N \to N$. □

34.9. Completion and Mittag-Leffler modules

Lemma 34.9.1. *Let R be a ring. Let $I \subset R$ be an ideal. Let A be a set. Assume R is Noetherian and complete with respect to I. The completion $(\bigoplus_{\alpha \in A} R)^\wedge$ is flat and Mittag-Leffler.*

Proof. By More on Algebra, Lemma 12.20.1 the map $(\bigoplus_{\alpha \in A} R)^\wedge \to \prod_{\alpha \in A} R$ is universally injective. Thus, by Algebra, Lemmas 7.76.7 and 7.83.6 it suffices to show that $\prod_{\alpha \in A} R$ is flat and Mittag-Leffler. By Algebra, Proposition 7.84.5 (and Algebra, Lemma 7.84.4) we see that $\prod_{\alpha \in A} R$ is flat. Thus we conclude because a product of copies of R is Mittag-Leffler, see Algebra, Lemma 7.85.3. □

Lemma 34.9.2. *Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module. Assume*

(1) *R is Noetherian and I-adically complete,*
(2) *M is flat over R, and*
(3) *M/IM is a projective R/I-module.*

Then the I-adic completion $M^\wedge$ is a flat Mittag-Leffler R-module.

Proof. Choose a surjection $F \to M$ where F is a free R-module. By Algebra, Lemma 7.90.18 the module $M^\wedge$ is a direct summand of the module $F^\wedge$. Hence it suffices to prove the lemma for F. In this case the lemma follows from Lemma 34.9.1. $\square$

In Lemmas 34.9.3 and 34.9.4 the assumption that S be Noetherian holds if $R \to S$ is of finite type, see Algebra, Lemma 7.28.1.

Lemma 34.9.3. *Let R be a ring. Let $I \subset R$ be an ideal. Let $R \to S$ be a ring map, and N an S-module. Assume*

(1) *R is a Noetherian ring,*
(2) *S is a Noetherian ring,*
(3) *N is a finite S-module, and*
(4) *for any finite R-module Q, any $\mathfrak{q} \in Ass_S(Q \otimes_R N)$ satisfies $IS + \mathfrak{q} \neq S$.*

Then the map $N \to N^\wedge$ of N into the I-adic completion of N is universally injective as a map of R-modules.

Proof. We have to show that for any finite R-module Q the map $Q \otimes_R N \to Q \otimes_R N^\wedge$ is injective, see Algebra, Theorem 7.76.3. As there is a canonical map $Q \otimes_R N^\wedge \to (Q \otimes_R N)^\wedge$ it suffices to prove that the canonical map $Q \otimes_R N \to (Q \otimes_R N)^\wedge$ is injective. Hence we may replace N by $Q \otimes_R N$ and it suffices to prove the injectivity for the map $N \to N^\wedge$.

Let $K = \mathrm{Ker}(N \to N^\wedge)$. It suffices to show that $K_\mathfrak{q} = 0$ for $\mathfrak{q} \in \mathrm{Ass}(N)$ as N is a submodule of $\prod_{\mathfrak{q} \in \mathrm{Ass}(N)} N_\mathfrak{q}$, see Algebra, Lemma 7.60.18. Pick $\mathfrak{q} \in \mathrm{Ass}(N)$. By the last assumption we see that there exists a prime $\mathfrak{q}' \supset IS + \mathfrak{q}$. Since $K_\mathfrak{q}$ is a localization of $K_{\mathfrak{q}'}$ it suffices to prove the vanishing of $K_{\mathfrak{q}'}$. Note that $K = \bigcap I^n N$, hence $K_{\mathfrak{q}'} \subset \bigcap I^n N_{\mathfrak{q}'}$. Hence $K_{\mathfrak{q}'} = 0$ by Algebra, Lemma 7.47.6. $\square$

Lemma 34.9.4. *Let R be a ring. Let $I \subset R$ be an ideal. Let $R \to S$ be a ring map, and N an S-module. Assume*

(1) *R is a Noetherian ring,*
(2) *S is a Noetherian ring,*
(3) *N is a finite S-module,*
(4) *N is flat over R, and*
(5) *for any prime $\mathfrak{q} \subset S$ which is an associated prime of $N \otimes_R \kappa(\mathfrak{p})$ where $\mathfrak{p} = R \cap \mathfrak{q}$ we have $IS + \mathfrak{q} \neq S$.*

Then the map $N \to N^\wedge$ of N into the I-adic completion of N is universally injective as a map of R-modules.

Proof. This follows from Lemma 34.9.3 because Algebra, Lemma 7.62.5 and Remark 7.62.6 guarantee that the set of associated primes of tensor products $N \otimes_R Q$ are contained in the set of associated primes of the modules $N \otimes_R \kappa(\mathfrak{p})$. $\square$

34.10. Projective modules

The following lemma can be used to prove projectivity by Noetherian induction on the base, see Lemma 34.10.2.

Lemma 34.10.1. *Let R be a ring. Let $I \subset R$ be an ideal. Let $R \to S$ be a ring map, and N an S-module. Assume*

(1) *R is Noetherian and I-adically complete,*
(2) *$R \to S$ is of finite type,*
(3) *N is a finite S-module,*
(4) *N is flat over R,*
(5) *N/IN is projective as a R/I-module, and*
(6) *for any prime $\mathfrak{q} \subset S$ which is an associated prime of $N \otimes_R \kappa(\mathfrak{p})$ where $\mathfrak{p} = R \cap \mathfrak{q}$ we have $IS + \mathfrak{q} \neq S$.*

Then N is projective as an R-module.

Proof. By Lemma 34.9.4 the map $N \to N^\wedge$ is universally injective. By Lemma 34.9.2 the module $N^\wedge$ is Mittag-Leffler. By Algebra, Lemma 7.83.6 we conclude that N is Mittag-Leffler. Hence N is countably generated, flat and Mittag-Leffler as an R-module, whence projective by Algebra, Lemma 7.87.1. $\square$

Lemma 34.10.2. *Let R be a ring. Let $R \to S$ be a ring map. Assume*

(1) *R is Noetherian,*
(2) *$R \to S$ is of finite type and flat, and*
(3) *every fibre ring $S \otimes_R \kappa(\mathfrak{p})$ is geometrically integral over $\kappa(\mathfrak{p})$.*

Then S is projective as an R-module.

Proof. Consider the set

$$\{I \subset R \mid S/IS \text{ not projective as } R/I\text{-module}\}$$

We have to show this set is empty. To get a contradiction assume it is nonempty. Then it contains a maximal element I. Let $J = \sqrt{I}$ be its radical. If $I \neq J$, then S/JS is projective as a R/J-module, and S/IS is flat over R/I and J/I is a nilpotent ideal in R/I. Applying Algebra, Lemma 7.71.5 we see that S/IS is a projective R/I-module, which is a contradiction. Hence we may assume that I is a radical ideal. In other words we are reduced to proving the lemma in case R is a reduced ring and S/IS is a projective R/I-module for every nonzero ideal I of R.

Assume R is a reduced ring and S/IS is a projective R/I-module for every nonzero ideal I of R. By generic flatness, Algebra, Lemma 7.109.1 (applied to a localization R_g which is a domain) or the more general Algebra, Lemma 7.109.7 there exists a nonzero $f \in R$ such that S_f is free as an R_f-module. Denote $R^\wedge = lim\, R/(f^n)$ the (f)-adic completion of R. Note that the ring map

$$R \longrightarrow R_f \times R^\wedge$$

is a faithfully flat ring map, see Algebra, Lemma 7.90.3. Hence by faithfully flat descent of projectivity, see Algebra, Theorem 7.89.5 it suffices to prove that $S \otimes_R R^\wedge$ is a projective $R^\wedge$-module. To see this we will use the criterion of Lemma 34.10.1. First of all, note that $S/fS = (S \otimes_R R^\wedge)/f(S \otimes_R R^\wedge)$ is a projective $R/(f)$-module and that $S \otimes_R R^\wedge$ is flat and of finite type over $R^\wedge$ as a base change of such. Next, suppose that $\mathfrak{p}^\wedge$ is a prime ideal of $R^\wedge$. Let $\mathfrak{p} \subset R$ be the corresponding prime of R. As $R \to S$ has geometrically integral fibre rings, the same is true for the fibre rings of any base change. Hence $\mathfrak{q}^\wedge = \mathfrak{p}^\wedge(S \otimes_R R^\wedge)$, is a prime ideals lying over $\mathfrak{p}^\wedge$ and it is the unique associated prime of $S \otimes_R \kappa(\mathfrak{p}^\wedge)$. Thus we win if $f(S \otimes_R R^\wedge) + \mathfrak{q}^\wedge \neq S \otimes_R R^\wedge$. This is true because $\mathfrak{p}^\wedge + fR^\wedge \neq R^\wedge$ as f lies in the radical of the f-adically complete ring $R^\wedge$ and because $R^\wedge \to S \otimes_R R^\wedge$ is surjective on spectra as its fibres are nonempty (irreducible spaces are nonempty). $\square$

Lemma 34.10.3. *Let R be a ring. Let $R \to S$ be a ring map. Assume*

(1) *$R \to S$ is of finite presentation and flat, and*
(2) *every fibre ring $S \otimes_R \kappa(\mathfrak{p})$ is geometrically integral over $\kappa(\mathfrak{p})$.*

Then S is projective as an R-module.

Proof. We can find a cocartesian diagram of rings

$$\begin{tikzcd} S_0 \arrow[r] & S \\ R_0 \arrow[u] \arrow[r] & R \arrow[u] \end{tikzcd}$$

such that R_0 is of finite type over $\mathbf{Z}$, the map $R_0 \to S_0$ is of finite type and flat with geometrically integral fibres, see More on Morphisms, Lemmas 33.24.4, 33.24.6, 33.24.7, and 33.24.11. By Lemma 34.10.2 we see that S_0 is a projective R_0-module. Hence $S = S_0 \otimes_{R_0} R$ is a projective R-module, see Algebra, Lemma 7.88.1. $\square$

Remark 34.10.4. Lemma 34.10.3 is a key step in the development of results in this chapter. The analogue of this lemma in [**GR71**] is [**GR71**, I Proposition 3.3.1]: If $R \to S$ is smooth with geometrically integral fibres, then S is projective as an R-module. This is a special case of Lemma 34.10.3, but as we will later improve on this lemma anyway, we do not gain much from having a stronger result at this point. We briefly sketch the proof of this as it is given in [**GR71**].

(1) First reduce to the case where R is Noetherian as above.
(2) Since projectivity descends through faithfully flat ring maps, see Algebra, Theorem 7.89.5 we may work locally in the fppf topology on R, hence we may assume that $R \to S$ has a section $\sigma : S \to R$. (Just by the usual trick of base changing to S.) Set $I = \mathrm{Ker}(S \to R)$.
(3) Localizing a bit more on R we may assume that I/I^2 is a free R-module and that the completion $S^\wedge$ of S with respect to I is isomorphic to $R[[t_1, \ldots, t_n]]$, see Morphisms, Lemma 24.33.20. Here we are using that $R \to S$ is smooth.
(4) To prove that S is projective as an R-module, it suffices to prove that S is flat, countably generated and Mittag-Leffler as an R-module, see Algebra, Lemma 7.87.1. The first two properties are evident. Thus it suffices to prove that S is Mittag-Leffler as an R-module. By Algebra, Lemma 7.85.4 the module $R[[t_1, \ldots, t_n]]$ is Mittag-Leffler over R. Hence Algebra, Lemma 7.83.6 shows that it suffices to show that the $S \to S^\wedge$ is universally injective as a map of R-modules.
(5) Apply Lemma 34.8.4 to see that $S \to S^\wedge$ is R-universally injective. Namely, as $R \to S$ has geometrically integral fibres, any associated point of any fibre ring is just the generic point of the fibre ring which is in the image of $Spec(S^\wedge) \to Spec(S)$.

There is an analogy between the proof as sketched just now, and the development of the arguments leading to the proof of Lemma 34.10.3. In both a completion plays an essential role, and both times the assumption of having geometrically integral fibres assures one that the map from S to the completion of S is R-universally injective.

34.11. Flat finite type modules, Part I

In some cases given a ring map $R \to S$ of finite presentation and a finite S-module N the flatness of N over R implies that N is of finite presentation. In this section we prove this is true ``pointwise''. We remark that the first proof of Proposition 34.11.3 uses the geometric results of Section 34.4 but not the existence of a complete dévissage.

Lemma 34.11.1. *Let $(R, \mathfrak{m})$ be a local ring. Let $R \to S$ be a finitely presented flat ring map with geometrically integral fibres. Write $\mathfrak{p} = \mathfrak{m}S$. Let $\mathfrak{q} \subset S$ be a prime ideal lying over $\mathfrak{m}$. Let N be a finite S-module. There exists $r \geq 0$ and an S-module map*

$$\alpha : S^{\oplus r} \longrightarrow N$$

such that $\alpha : \kappa(\mathfrak{p})^{\oplus r} \to N \otimes_S \kappa(\mathfrak{p})$ is an isomorphism. For any such α the following are equivalent:

(1) *$N_\mathfrak{q}$ is R-flat,*
(2) *α is R-universally injective and $Coker(\alpha)_\mathfrak{q}$ is R-flat,*
(3) *α is injective and $Coker(\alpha)_\mathfrak{q}$ is R-flat,*
(4) *$\alpha_\mathfrak{p}$ is an isomorphism and $Coker(\alpha)_\mathfrak{q}$ is R-flat, and*
(5) *$\alpha_\mathfrak{q}$ is injective and $Coker(\alpha)_\mathfrak{q}$ is R-flat.*

Proof. To obtain α set $r = \dim_{\kappa(\mathfrak{p})} N \otimes_S \kappa(\mathfrak{p})$ and pick $x_1, \ldots, x_r \in N$ which form a basis of $N \otimes_S \kappa(\mathfrak{p})$. Define $\alpha(s_1, \ldots, s_r) = \sum s_i x_i$. This proves the existence.

Fix an α. The most interesting implication is (1) $\Rightarrow$ (2) which we prove first. Assume (1). Because $S/\mathfrak{m}S$ is a domain with fraction field $\kappa(\mathfrak{p})$ we see that $(S/\mathfrak{m}S)^{\oplus r} \to N_\mathfrak{p}/\mathfrak{m}N_\mathfrak{p} = N \otimes_S \kappa(\mathfrak{p})$ is injective. Hence by Lemmas 34.8.5 and 34.10.3. the map $S^{\oplus r} \to N_\mathfrak{p}$ is R-universally injective. It follows that $S^{\oplus r} \to N$ is R-universally injective, see Algebra, Lemma 7.76.10. Then also the localization $\alpha_\mathfrak{q}$ is R-universally injective, see Algebra, Lemma 7.76.13. We conclude that $\mathrm{Coker}(\alpha)_\mathfrak{q}$ is R-flat by Algebra, Lemma 7.76.7.

The implication (2) $\Rightarrow$ (3) is immediate. If (3) holds, then $\alpha_\mathfrak{p}$ is injective as a localization of an injective module map. By Nakayama's lemma (Algebra, Lemma 7.14.5) $\alpha_\mathfrak{p}$ is surjective too. Hence (3) $\Rightarrow$ (4). If (4) holds, then $\alpha_\mathfrak{p}$ is an isomorphism, so α is injective as $S_\mathfrak{q} \to S_\mathfrak{p}$ is injective. Namely, elements of $S \setminus \mathfrak{p}$ are nonzero divisors on S by a combination of Lemmas 34.8.6 and 34.10.3. Hence (4) $\Rightarrow$ (5). Finally, if (5) holds, then $N_\mathfrak{q}$ is R-flat as an extension of flat modules, see Algebra, Lemma 7.35.12. Hence (5) $\Rightarrow$ (1) and the proof is finished. □

Lemma 34.11.2. *Let $(R, \mathfrak{m})$ be a local ring. Let $R \to S$ be a ring map of finite presentation. Let N be a finite S-module. Let $\mathfrak{q}$ be a prime of S lying over $\mathfrak{m}$. Assume that $N_\mathfrak{q}$ is flat over R, and assume there exists a complete dévissage of $N/S/R$ at $\mathfrak{q}$. Then N is a finitely presented S-module, free as an R-module, and there exists an isomorphism*

$$N \cong B_1^{\oplus r_1} \oplus \ldots \oplus B_n^{\oplus r_n}$$

as R-modules where each B_i is a smooth R-algebra with geometrically irreducible fibres.

Proof. Let $(A_i, B_i, M_i, \alpha_i, \mathfrak{q}_i)_{i=1,\ldots,n}$ be the given complete dévissage. We prove the lemma by induction on n. Note that N is finitely presented as an S-module if and only if M_1 is finitely presented as an B_1-module, see Remark 34.7.3. Note that $N_\mathfrak{q} \cong (M_1)_{\mathfrak{q}_1}$ as R-modules because (a) $N_\mathfrak{q} \cong (M_1)_{\mathfrak{q}_1'}$ where $\mathfrak{q}_1'$ is the unique prime in A_1 lying over $\mathfrak{q}_1$ and (b) $(A_1)_{\mathfrak{q}_1'} = (A_1)_{\mathfrak{q}_1}$ by Algebra, Lemma 7.36.11, so (c) $(M_1)_{\mathfrak{q}_1'} \cong (M_1)_{\mathfrak{q}_1}$. Hence $(M_1)_{\mathfrak{q}_1}$ is a flat R-module. Thus we may replace (S, N) by (B_1, M_1) in order to prove the lemma. By Lemma 34.11.1 the map $\alpha_1 : B_1^{\oplus r_1} \to M_1$ is R-universally injective and $\mathrm{Coker}(\alpha_1)_{\mathfrak q}$ is R-flat. Note that $(A_i, B_i, M_i, \alpha_i, \mathfrak{q}_i)_{i=2,\ldots,n}$ is a complete dévissage of $\mathrm{Coker}(\alpha_1)/B_1/R$ at $\mathfrak{q}_1$. Hence the induction hypothesis implies that $\mathrm{Coker}(\alpha_1)$ is finitely presented as a B_1-module, free as an R-module, and has a decomposition as in the lemma. This implies that M_1 is finitely presented as a B_1-module, see Algebra, Lemma 7.5.4. It further implies that $M_1 \cong B_1^{\oplus r_1} \oplus \mathrm{Coker}(\alpha_1)$ as R-modules, hence a decomposition as in the lemma. Finally, B_1 is projective as an R-module by Lemma 34.10.3 hence free as an R-module by Algebra, Theorem 7.79.4. This finishes the proof. $\square$

Proposition 34.11.3. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $x \in X$ with image $s \in S$. Assume that*

1. *f is locally of finite presentation,*
2. *$\mathcal{F}$ is of finite type, and*
3. *$\mathcal{F}$ is flat at x over S.*

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open subscheme

$$V \subset X \times_S Spec(\mathcal{O}_{S',s'})$$

which contains the unique point of $X \times_S Spec(\mathcal{O}_{S',s'})$ mapping to x such that the pullback of $\mathcal{F}$ to V is an $\mathcal{O}_V$-module of finite presentation and flat over $\mathcal{O}_{S',s'}$.

First proof. This proof is longer but does not use the existence of a complete dévissage. The problem is local around x and s, hence we may assume that X and S are affine. During the proof we will finitely many times replace S by an elementary étale neighbourhood of (S, s). The goal is then to find (after such a replacement) an open $V \subset X \times_S Spec(\mathcal{O}_{S,s})$ containing x such that $\mathcal{F}|_V$ is flat over S and finitely presented. Of course we may also replace S by $Spec(\mathcal{O}_{S,s})$ at any point of the proof, i.e., we may assume S is a local scheme. We will prove the lemma by induction on the integer $n = \dim_x(\mathrm{Supp}(\mathcal{F}_s))$.

We can choose

1. elementary étale neighbourhoods $g : (X', x') \to (X, x)$, $e : (S', s') \to (S, s)$,
2. a commutative diagram

$$\begin{tikzcd}
X \arrow[d, "f"'] & X' \arrow[l, "g"'] \arrow[dd] & Z' \arrow[l, "i"'] \arrow[d, "\pi"] \\
 & & Y' \arrow[d, "h"] \\
S & S' \arrow[l, "e"'] \arrow[r, equal] & S'
\end{tikzcd}$$

3. a point $z' \in Z'$ with $i(z') = x'$, $y' = \pi(z')$, $h(y') = s'$,
4. a finite type quasi-coherent $\mathcal{O}_{Z'}$-module $\mathcal{G}$,

as in Lemma 34.4.2. We are going to replace S by $Spec(\mathcal{O}_{S',s'})$, see remarks in first paragraph of the proof. Consider the diagram

$$\begin{tikzcd} X_{\mathcal{O}_{S',s'}} \arrow[ddr, "f"'] & X'_{\mathcal{O}_{S',s'}} \arrow[l, "g"'] \arrow[dd] & Z'_{\mathcal{O}_{S',s'}} \arrow[l, "i"'] \arrow[d, "\pi"] \\ & & Y'_{\mathcal{O}_{S',s'}} \arrow[dl, "h"] \\ & Spec(\mathcal{O}_{S',s'}) & \end{tikzcd}$$

Here we have base changed the schemes X', Z', Y' over S' via $Spec(\mathcal{O}_{S',s'}) \to S'$ and the scheme X over S via $Spec(\mathcal{O}_{S',s'}) \to S$. It is still the case that g is étale, see Lemma 34.3.2. After replacing X by $X_{\mathcal{O}_{S',s'}}$, X' by $X'_{\mathcal{O}_{S',s'}}$, Z' by $Z'_{\mathcal{O}_{S',s'}}$, and Y' by $Y'_{\mathcal{O}_{S',s'}}$ we may assume we have a diagram as Lemma 34.4.2 where in addition $S = S'$ is a local scheme with closed point s. By Lemmas 34.4.3 and 34.4.4 the result for $Y' \to S$, the sheaf $\pi_*\mathcal{G}$, and the point y' implies the result for $X \to S$, $\mathcal{F}$ and x. Hence we may assume that S is local and $X \to S$ is a smooth morphism of affines with geometrically irreducible fibres of dimension n.

The base case of the induction: $n = 0$. As $X \to S$ is smooth with geometrically irredible fibres of dimension 0 we see that $X \to S$ is an open immersion, see Descent, Lemma 31.21.2. As S is local and the closed point is in the image of $X \to S$ we conclude that $X = S$. Thus we see that $\mathcal{F}$ corresponds to a finite flat $\mathcal{O}_{S,s}$ module. In this case the result follows from Algebra, Lemma 7.72.4 which tells us that $\mathcal{F}$ is in fact finite free.

The induction step. Assume the result holds whenever the dimension of the support in the closed fibre is $< n$. Write $S = Spec(A)$, $X = Spec(B)$ and $\mathcal{F} = \widetilde{N}$ for some B-module N. Note that A is a local ring; denote its maximal ideal $\mathfrak{m}$. Then $\mathfrak{p} = \mathfrak{m}B$ is the unique minimal prime lying over $\mathfrak{m}$ as $X \to S$ has geometrically irreducible fibres. Finally, let $\mathfrak{q} \subset B$ be the prime corresponding to x. By Lemma 34.11.1 we can choose a map

$$\alpha : B^{\oplus r} \to N$$

such that $\kappa(\mathfrak{p})^{\oplus r} \to N \otimes_B \kappa(\mathfrak{p})$ is an isomorphism. Moreover, as $N_\mathfrak{q}$ is A-flat the lemma also shows that α is injective and that $\mathrm{Coker}(\alpha)_\mathfrak{q}$ is A-flat. Set $Q = \mathrm{Coker}(\alpha)$. Note that the support of $Q/\mathfrak{m}Q$ does not contain $\mathfrak{p}$. Hence it is certainly the case that $\dim_\mathfrak{q}(\mathrm{Supp}(Q/\mathfrak{m}Q)) < n$. Combining everything we know about Q we see that the induction hypothesis applies to Q. It follows that there exists an elementary étale morphism $(S', s) \to (S, s)$ such that the conclusion holds for $Q \otimes_A A'$ over $B \otimes_A A'$ where $A' = \mathcal{O}_{S',s'}$. After replacing A by A' we have an exact sequence

$$0 \to B^{\oplus r} \to N \to Q \to 0$$

(here we use that α is injective as mentioned above) of finite B-modules and we also get an element $g \in B$, $g \notin \mathfrak{q}$ such that Q_g is finitely presented over B_g and flat over A. Since localization is exact we see that

$$0 \to B_g^{\oplus r} \to N_g \to Q_g \to 0$$

is still exact. As B_g and Q_g are flat over A we conlude that N_g is flat over A, see Algebra, Lemma 7.35.12, and as B_g and Q_g are finitely presented over B_g the same holds for N_g, see Algebra, Lemma 7.5.4. □

Second proof. We apply Proposition 34.6.7 to find a commutative diagram

$$\begin{tikzcd} (X, x) \arrow[d] & (X', x') \arrow[l, "g"] \arrow[d] \\ (S, s) & (S', s') \arrow[l] \end{tikzcd}$$

of pointed schemes such that the horizontal arrows are elementary étale neighbourhoods and such that $g^*\mathcal{F}/X'/S'$ has a complete dévissage at x. (In particular S' and X' are affine.) By Morphisms, Lemma 24.24.11 we see that $g^*\mathcal{F}$ is flat at x' over S and by Lemma 34.3.3 we see that it is flat at x' over S'. Via Remark 34.7.5 we deduce that

$$\Gamma(X', g^*\mathcal{F})/\Gamma(X', \mathcal{O}_{X'})/\Gamma(S', \mathcal{O}_{S'})$$

has a complete dévisage at the prime of $\Gamma(X', \mathcal{O}_{X'})$ corresponding to x'. We may base change this complete dévissage to the local ring $\mathcal{O}_{S',s'}$ of $\Gamma(S', \mathcal{O}_{S'})$ at the prime corresponding to s'. Thus Lemma 34.11.2 implies that

$$\Gamma(X', \mathcal{F}') \otimes_{\Gamma(S', \mathcal{O}_{S'})} \mathcal{O}_{S',s'}$$

is flat over $\mathcal{O}_{S',s'}$ and of finite presentation over $\Gamma(X', \mathcal{O}_{X'}) \otimes_{\Gamma(S',\mathcal{O}_{S'})} \mathcal{O}_{S',s'}$. In other words, the restriction of $\mathcal{F}$ to $X' \times_{S'} Spec(\mathcal{O}_{S',s'})$ is of finite presentation and flat over $\mathcal{O}_{S',s'}$. Since the morphism $X' \times_{S'} Spec(\mathcal{O}_{S',s'}) \to X \times_S Spec(\mathcal{O}_{S',s'})$ is étale (Lemma 34.3.2) its image $V \subset X \times_S Spec(\mathcal{O}_{S',s'})$ is an open subscheme, and by étale descent the restriction of $\mathcal{F}$ to V is of finite presentation and flat over $\mathcal{O}_{S',s'}$. (Results used: Morphisms, Lemma 24.35.13, Descent, Lemma 31.5.2, and Morphisms, Lemma 24.24.11.) □

Lemma 34.11.4. *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $s \in S$. Then the set*

$$\{x \in X_s \mid \mathcal{F} \text{ flat over } S \text{ at } x\}$$

is open in the fibre X_s.

Proof. Suppose $x \in U$. Choose an elementary étale neighbourhood $(S', s') \to (S, s)$ and open $V \subset X \times_S Spec(\mathcal{O}_{S',s'})$ as in Proposition 34.11.3. Note that $X_{s'} = X_s$ as $\kappa(s) = \kappa(s')$. If $x' \in V \cap X_{s'}$, then the pullback of $\mathcal{F}$ to $X \times_S S'$ is flat over S' at x'. Hence $\mathcal{F}$ is flat at x' over S, see Morphisms, Lemma 24.24.11. In other words $X_s \cap V \subset U$ is an open neighbourhood of x in U. □

Lemma 34.11.5. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $x \in X$ with image $s \in S$. Assume that*

(1) *f is locally of finite type,*
(2) *$\mathcal{F}$ is of finite type, and*
(3) *$\mathcal{F}$ is flat at x over S.*

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open subscheme

$$V \subset X \times_S Spec(\mathcal{O}_{S',s'})$$

which contains the unique point of $X \times_S Spec(\mathcal{O}_{S',s'})$ mapping to x such that the pullback of $\mathcal{F}$ to V is flat over $\mathcal{O}_{S',s'}$.

Proof. (The only difference between this and Proposition 34.11.3 is that we do not assume f is of finite presentation.) The question is local on X and S, hence we may assume X and S are affine. Write $X = Spec(B)$, $S = Spec(A)$ and write $B = A[x_1, \ldots, x_n]/I$. In other words we obtain a closed immersion $i : X \to \mathbf{A}^n_S$. Denote $t = i(x) \in \mathbf{A}^n_S$. We may apply Proposition 34.11.3 to $\mathbf{A}^n_S \to S$, the sheaf $i_*\mathcal{F}$ and the point t. We obtain an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open subscheme

$$W \subset \mathbf{A}^n_{\mathcal{O}_{S',s'}}$$

such that the pull back of $i_*\mathcal{F}$ to W is flat over $\mathcal{O}_{S',s'}$. This means that $V := W \cap \big(X \times_S Spec(\mathcal{O}_{S',s'})\big)$ is the desired open subscheme. □

Lemma 34.11.6. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $s \in S$. Assume that*

(1) *f is of finite presentation,*
(2) *$\mathcal{F}$ is of finite type, and*
(3) *$\mathcal{F}$ is flat over S at every point of the fibre X_s.*

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open subscheme

$$V \subset X \times_S Spec(\mathcal{O}_{S',s'})$$

which contains the fibre $X_s = X \times_S s'$ such that the pullback of $\mathcal{F}$ to V is an $\mathcal{O}_V$-module of finite presentation and flat over $\mathcal{O}_{S',s'}$.

Proof. For every point $x \in X_s$ we can use Proposition 34.11.3 to find an elementary étale neighbourhood $(S_x, s_x) \to (S, s)$ and an open $V_x \subset X \times_S Spec(\mathcal{O}_{S_x,s_x})$ such that $x \in X_s = X \times_S s_x$ is contained in V_x and such that the pullback of $\mathcal{F}$ to V_x is an $\mathcal{O}_{V_x}$-module of finite presentation and flat over $\mathcal{O}_{S_x,s_x}$. In particular we may view the fibre $(V_x)_{s_x}$ as an open neighbourhood of x in X_s. Because X_s is quasi-compact we can find a finite number of points $x_1, \ldots, x_n \in X_s$ such that X_s is the union of the $(V_{x_i})_{s_{x_i}}$. Choose an elementary étale neighbourhood $(S', s') \to (S, s)$ which dominates each of the neighbourhoods (S_{x_i}, s_{x_i}), see More on Morphisms, Lemma 33.25.4. Set $V = \bigcup V_i$ where V_i is the inverse images of the open V_{x_i} via the morphism

$$X \times_S Spec(\mathcal{O}_{S',s'}) \longrightarrow X \times_S Spec(\mathcal{O}_{S_{x_i},s_{x_i}})$$

By construction V contains X_s and by construction the pullback of $\mathcal{F}$ to V is an $\mathcal{O}_V$-module of finite presentation and flat over $\mathcal{O}_{S',s'}$. □

Lemma 34.11.7. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $s \in S$. Assume that*

(1) *f is of finite type,*
(2) *$\mathcal{F}$ is of finite type, and*
(3) *$\mathcal{F}$ is flat over S at every point of the fibre X_s.*

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open subscheme

$$V \subset X \times_S Spec(\mathcal{O}_{S',s'})$$

which contains the fibre $X_s = X \times_S s'$ such that the pullback of $\mathcal{F}$ to V is flat over $\mathcal{O}_{S',s'}$.

Proof. (The only difference between this and Lemma 34.11.6 is that we do not assume f is of finite presentation.) For every point $x \in X_s$ we can use Lemma 34.11.5 to find an elementary étale neighbourhood $(S_x, s_x) \to (S, s)$ and an open $V_x \subset X \times_S Spec(\mathcal{O}_{S_x,s_x})$ such that $x \in X_s = X \times_S s_x$ is contained in V_x and such that the pullback of $\mathcal{F}$ to V_x is flat over $\mathcal{O}_{S_x,s_x}$. In particular we may view the fibre $(V_x)_{s_x}$ as an open neighbourhood of x in X_s. Because X_s is quasi-compact we can find a finite number of points $x_1, \ldots, x_n \in X_s$ such that

X_s is the union of the $(V_{x_i})_{s_{x_i}}$. Choose an elementary étale neighbourhood $(S', s') \to (S, s)$ which dominates each of the neighbourhoods (S_{x_i}, s_{x_i}), see More on Morphisms, Lemma 33.25.4. Set $V = \bigcup V_i$ where V_i is the inverse images of the open V_{x_i} via the morphism

$$X \times_S Spec(\mathcal{O}_{S',s'}) \longrightarrow X \times_S Spec(\mathcal{O}_{S_{x_i},s_{x_i}})$$

By construction V contains X_s and by construction the pullback of $\mathcal{F}$ to V is flat over $\mathcal{O}_{S',s'}$. □

Lemma 34.11.8. *Let S be a scheme. Let X be locally of finite type over S. Let $x \in X$ with image $s \in S$. If X is flat at x over S, then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open subscheme*

$$V \subset X \times_S Spec(\mathcal{O}_{S',s'})$$

which contains the unique point of $X \times_S Spec(\mathcal{O}_{S',s'})$ mapping to x such that $V \to Spec(\mathcal{O}_{S',s'})$ is flat and of finite presentation.

Proof. The question is local on X and S, hence we may assume X and S are affine. Write $X = Spec(B)$, $S = Spec(A)$ and write $B = A[x_1, \ldots, x_n]/I$. In other words we obtain a closed immersion $i : X \to \mathbf{A}^n_S$. Denote $t = i(x) \in \mathbf{A}^n_S$. We may apply Proposition 34.11.3 to $\mathbf{A}^n_S \to S$, the sheaf $\mathcal{F} = i_*\mathcal{O}_X$ and the point t. We obtain an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open subscheme

$$W \subset \mathbf{A}^n_{\mathcal{O}_{S',s'}}$$

such that the pull back of $i_*\mathcal{O}_X$ is flat and of finite presentation. This means that $V := W \cap \left(X \times_S Spec(\mathcal{O}_{S',s'})\right)$ is the desired open subscheme. □

Lemma 34.11.9. *Let $f : X \to S$ be a morphism which is locally of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. If $x \in X$ and $\mathcal{F}$ is flat at x over S, then $\mathcal{F}_x$ is an $\mathcal{O}_{X,x}$-module of finite presentation.*

Proof. Let $s = f(x)$. By Proposition 34.11.3 there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ such that the pullback of $\mathcal{F}$ to $X \times_S Spec(\mathcal{O}_{S',s'})$ is of finite presentation in a neighbourhood of the point $x' \in X_{s'} = X_s$ corresponding to x. The ring map

$$\mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X \times_S Spec(\mathcal{O}_{S',s'}),x'} = \mathcal{O}_{X \times_S S',x'}$$

is flat and local as a localization of an étale ring map. Hence $\mathcal{F}_x$ is of finite presentation over $\mathcal{O}_{X,x}$ by descent, see Algebra, Lemma 7.77.2 (and also that a flat local ring map is faithfully flat, see Algebra, Lemma 7.35.16). □

Lemma 34.11.10. *Let $f : X \to S$ be a morphism which is locally of finite type. Let $x \in X$ with image $s \in S$. If f is flat at x over S, then $\mathcal{O}_{X,x}$ is essentially of finite presentation over $\mathcal{O}_{S,s}$.*

Proof. We may assume X and S affine. Write $X = Spec(B)$, $S = Spec(A)$ and write $B = A[x_1, \ldots, x_n]/I$. In other words we obtain a closed immersion $i : X \to \mathbf{A}^n_S$. Denote $t = i(x) \in \mathbf{A}^n_S$. We may apply Lemma 34.11.9 to $\mathbf{A}^n_S \to S$, the sheaf $\mathcal{F} = i_*\mathcal{O}_X$ and the point t. We conclude that $\mathcal{O}_{X,x}$ is of finite presentation over $\mathcal{O}_{\mathbf{A}^n_S,t}$ which implies what we want. □

34.12. Flat finitely presented modules

In some cases given a ring map $R \to S$ of finite presentation and a finitely presented S-module N the flatness of N over R implies that N is projective as an R-module, at least after replacing S by an étale extension. In this section we collect a some results of this nature.

Lemma 34.12.1. *Let R be a ring. Let $R \to S$ be a finitely presented flat ring map with geometrically integral fibres. Let $\mathfrak{q} \subset S$ be a prime ideal lying over the prime $\mathfrak{r} \subset R$. Set $\mathfrak{p} = \mathfrak{r}S$. Let N be a finitely presented S-module. There exists $r \geq 0$ and an S-module map*

$$\alpha : S^{\oplus r} \longrightarrow N$$

such that $\alpha : \kappa(\mathfrak{p})^{\oplus r} \to N \otimes_S \kappa(\mathfrak{p})$ is an isomorphism. For any such α the following are equivalent:

1. *$N_{\mathfrak{q}}$ is R-flat,*
2. *there exists an $f \in R$, $f \notin \mathfrak{r}$ such that $\alpha_f : S_f^{\oplus r} \to N_f$ is R_f-universally injective and a $g \in S$, $g \notin \mathfrak{q}$ such that $Coker(\alpha)_g$ is R-flat,*
3. *$\alpha_{\mathfrak{r}}$ is $R_{\mathfrak{r}}$-universally injective and $Coker(\alpha)_{\mathfrak{q}}$ is R-flat*
4. *$\alpha_{\mathfrak{r}}$ is injective and $Coker(\alpha)_{\mathfrak{q}}$ is R-flat,*
5. *$\alpha_{\mathfrak{p}}$ is an isomorphism and $Coker(\alpha)_{\mathfrak{q}}$ is R-flat, and*
6. *$\alpha_{\mathfrak{q}}$ is injective and $Coker(\alpha)_{\mathfrak{q}}$ is R-flat.*

Proof. To obtain α set $r = \dim_{\kappa(\mathfrak{p})} N \otimes_S \kappa(\mathfrak{p})$ and pick $x_1, \ldots, x_r \in N$ which form a basis of $N \otimes_S \kappa(\mathfrak{p})$. Define $\alpha(s_1, \ldots, s_r) = \sum s_i x_i$. This proves the existence.

Fix a choice of α. We may apply Lemma 34.11.1 to the map $\alpha_{\mathfrak{r}} : S_{\mathfrak{r}}^{\oplus r} \to N_{\mathfrak{r}}$. Hence we see that (1), (3), (4), (5), and (6) are all equivalent. Since it is also clear that (2) implies (3) we see that all we have to do is show that (1) implies (2).

Assume (1). By openness of flatness, see Algebra, Theorem 7.120.4, the set

$$U_1 = \{\mathfrak{q}' \subset S \mid N_{\mathfrak{q}'} \text{ is flat over } R\}$$

is open in $Spec(S)$. It contains $\mathfrak{q}$ by assumption and hence $\mathfrak{p}$. Because $S^{\oplus r}$ and N are finitely presented S-modules the set

$$U_2 = \{\mathfrak{q}' \subset S \mid \alpha_{\mathfrak{q}'} \text{ is an isomorphism}\}$$

is open in $Spec(S)$, see Algebra, Lemma 7.73.2. It contains $\mathfrak{p}$ by (5). As $R \to S$ is finitely presented and flat the map $\Phi : Spec(S) \to Spec(R)$ is open, see Algebra, Proposition 7.36.8. For any prime $\mathfrak{r}' \in \Phi(U_1 \cap U_2)$ we see that there exists a prime $\mathfrak{q}'$ lying over $\mathfrak{r}'$ such that $N_{\mathfrak{q}'}$ is flat and such that $\alpha_{\mathfrak{q}'}$ is an isomorphism, which implies that $\alpha \otimes \kappa(\mathfrak{p}')$ is an isomorphism where $\mathfrak{p}' = \mathfrak{r}'S$. Thus $\alpha_{\mathfrak{r}'}$ is $R_{\mathfrak{r}'}$-universally injective by the implication (1) $\Rightarrow$ (3). Hence if we pick $f \in R$, $f \notin \mathfrak{r}$ such that $D(f) \subset \Phi(U_1 \cap U_2)$ then we conclude that α_f is R_f-universally injective, see Algebra, Lemma 7.76.12. The same reasoning also shows that for any $\mathfrak{q}' \in U_1 \cap \Phi^{-1}(\Phi(U_1 \cap U_2))$ the module $\mathrm{Coker}(\alpha)_{\mathfrak{q}'}$ is R-flat. Note that $\mathfrak{q} \in U_1 \cap \Phi^{-1}(\Phi(U_1 \cap U_2))$. Hence we can find a $g \in S$, $g \notin \mathfrak{q}$ such that $D(g) \subset U_1 \cap \Phi^{-1}(\Phi(U_1 \cap U_2))$ and we win. $\square$

Lemma 34.12.2. *Let $R \to S$ be a ring map of finite presentation. Let N be a finitely presented S-module flat over R. Let $\mathfrak{r} \subset R$ be a prime ideal. Assume there exists a complete dévissage of $N/S/R$ over $\mathfrak{r}$. Then there exists an $f \in R$, $f \notin \mathfrak{r}$ such that*

$$N_f \cong B_1^{\oplus r_1} \oplus \ldots \oplus B_n^{\oplus r_n}$$

as R-modules where each B_i is a smooth R_f-algebra with geometrically irreducible fibres. Moreover, N_f is projective as an R_f-module.

Proof. Let $(A_i, B_i, M_i, \alpha_i)_{i=1,\ldots,n}$ be the given complete dévissage. We prove the lemma by induction on n. Note that the assertions of the lemma are entirely about the structure of N as an R-module. Hence we may replace N by M_1, and we may think of M_1 as a B_1-module. See Remark 34.7.3 in order to see why M_1 is of finite presentation as a B_1-module. By Lemma 34.12.1 we may, after replacing R by R_f for some $f \in R$, $f \not\in \mathfrak r$, assume the map $\alpha_1 : B_1^{\oplus r_1} \to M_1$ is R-universally injective. Since M_1 and $B_1^{\oplus r_1}$ are R-flat and finitely presented as B_1-modules we see that $\text{Coker}(\alpha_1)$ is R-flat (Algebra, Lemma 7.76.7) and finitely presented as a B_1-module. Note that $(A_i, B_i, M_i, \alpha_i)_{i=2,\ldots,n}$ is a complete dévissage of $\text{Coker}(\alpha_1)$. Hence the induction hypothesis implies that, after replacing R by R_f for some $f \in R$, $f \not\in \mathfrak r$, we may assume that $\text{Coker}(\alpha_1)$ has a decomposition as in the lemma and is projective. In particular $M_1 = B_1^{\oplus r_1} \oplus \text{Coker}(\alpha_1)$. This proves the statement regarding the decomposition. The statement on projectivity follows as B_1 is projective as an R-module by Lemma 34.10.3. $\square$

Remark 34.12.3. There is a variant of Lemma 34.12.2 where we weaken the flatness condition by assuming only that N is flat at some given prime $\mathfrak q$ lying over $\mathfrak r$ but where we strengthen the dévissage condition by assuming the existence of a complete dévissage *at* $\mathfrak q$. Compare with Lemma 34.11.2.

The following is the main result of this section.

Proposition 34.12.4. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $x \in X$ with image $s \in S$. Assume that*

(1) *f is locally of finite presentation,*
(2) *$\mathcal{F}$ is of finite presentation, and*
(3) *$\mathcal{F}$ is flat at x over S.*

Then there exists a commutative diagram of pointed schemes

$$\begin{tikzcd} (X, x) \arrow[d] & (X', x') \arrow[l, "g"'] \arrow[d] \\ (S, s) & (S', s') \arrow[l] \end{tikzcd}$$

whose horizontal arrows are elementary étale neighbourhoods such that X', S' are affine and such that $\Gamma(X', g^*\mathcal{F})$ is a projective $\Gamma(S', \mathcal{O}_{S'})$-module.

Proof. By openness of flatness, see More on Morphisms, Theorem 33.11.1 we may replace X by an open neighbourhood of x and assume that $\mathcal{F}$ is flat over S. Next, we apply Proposition 34.6.7 to find a diagram as in the statement of the proposition such that $g^*\mathcal{F}/X'/S'$ has a complete dévissage over s'. (In particular S' and X' are affine.) By Morphisms, Lemma 24.24.11 we see that $g^*\mathcal{F}$ is flat over S and by Lemma 34.3.3 we see that it is flat over S'. Via Remark 34.7.5 we deduce that

$$\Gamma(X', g^*\mathcal{F})/\Gamma(X', \mathcal{O}_{X'})/\Gamma(S', \mathcal{O}_{S'})$$

has a complete dévisage over the prime of $\Gamma(S', \mathcal{O}_{S'})$ corresponding to s'. Thus Lemma 34.12.2 implies that the result of the proposition holds after replacing S' by a standard open neighbourhood of s'. $\square$

In the rest of this section we prove a number of variants on this result. The first is a ``global'' version.

Lemma 34.12.5. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $s \in S$. Assume that*

(1) *f is of finite presentation,*
(2) *$\mathcal{F}$ is of finite presentation, and*
(3) *$\mathcal{F}$ is flat over S at every point of the fibre X_s.*

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram of schemes

$$\begin{tikzcd} X \arrow[d] & X' \arrow[l, "g"'] \arrow[d] \\ S & S' \arrow[l] \end{tikzcd}$$

such that g is étale, $X_s \subset g(X')$, the schemes X', S' are affine, and such that $\Gamma(X', g^*\mathcal{F})$ is a projective $\Gamma(S', \mathcal{O}_{S'})$-module.

Proof. For every point $x \in X_s$ we can use Proposition 34.12.4 to find a commutative diagram

$$\begin{tikzcd} (X, x) \arrow[d] & (Y_x, y_x) \arrow[l, "g_x"'] \arrow[d] \\ (S, s) & (S_x, s_x) \arrow[l] \end{tikzcd}$$

whose horizontal arrows are elementary étale neighbourhoods such that Y_x, S_x are affine and such that $\Gamma(Y_x, g_x^*\mathcal{F})$ is a projective $\Gamma(S_x, \mathcal{O}_{S_x})$-module. In particular $g_x(Y_x) \cap X_s$ is an open neighbourhood of x in X_s. Because X_s is quasi-compact we can find a finite number of points $x_1, \ldots, x_n \in X_s$ such that X_s is the union of the $g_{x_i}(Y_{x_i}) \cap X_s$. Choose an elementary étale neighbourhood $(S', s') \to (S, s)$ which dominates each of the neighbourhoods (S_{x_i}, s_{x_i}), see More on Morphisms, Lemma 33.25.4. We may also assume that S' is affine. Set $X' = \coprod Y_{x_i} \times_{S_{x_i}} S'$ and endow it with the obvious morphism $g : X' \to X$. By construction $g(X')$ contains X_s and

$$\Gamma(X', g^*\mathcal{F}) = \bigoplus \Gamma(Y_{x_i}, g_{x_i}^*\mathcal{F}) \otimes_{\Gamma(S_{x_i}, \mathcal{O}_{S_{x_i}})} \Gamma(S', \mathcal{O}_{S'}).$$

This is a projective $\Gamma(S', \mathcal{O}_{S'})$-module, see Algebra, Lemma 7.88.1. □

The following two lemmas are reformulations of the results above in case $\mathcal{F} = \mathcal{O}_X$.

Lemma 34.12.6. *Let $f : X \to S$ be locally of finite presentation. Let $x \in X$ with image $s \in S$. If f is flat at x over S, then there exists a commutative diagram of pointed schemes*

$$\begin{tikzcd} (X, x) \arrow[d] & (X', x') \arrow[l, "g"'] \arrow[d] \\ (S, s) & (S', s') \arrow[l] \end{tikzcd}$$

whose horizontal arrows are elementary étale neighbourhoods such that X', S' are affine and such that $\Gamma(X', \mathcal{O}_{X'})$ is a projective $\Gamma(S', \mathcal{O}_{S'})$-module.

Proof. This is a special case of Proposition 34.12.4. □

Lemma 34.12.7. *Let $f : X \to S$ be of finite presentation. Let $s \in S$. If X is flat over S at all points of X_s, then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram of schemes*

$$\begin{tikzcd} X \arrow[d] & X' \arrow[l, "g"'] \arrow[d] \\ S & S' \arrow[l] \end{tikzcd}$$

with g étale, $X_s \subset g(X')$, such that X', S' are affine, and such that $\Gamma(X', \mathcal{O}_{X'})$ is a projective $\Gamma(S', \mathcal{O}_{S'})$-module.

Proof. This is a special case of Lemma 34.12.5. □

The following lemmas explain consequences of Proposition 34.12.4 in case we only assume the morphism and the sheaf are of finite type (and not necessarily of finite presentation).

Lemma 34.12.8. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $x \in X$ with image $s \in S$. Assume that*

(1) *f is locally of finite presentation,*
(2) *$\mathcal{F}$ is of finite type, and*
(3) *$\mathcal{F}$ is flat at x over S.*

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram of pointed schemes

$$\begin{tikzcd} (X, x) \arrow[d] & (X', x') \arrow[l, "g"'] \arrow[d] \\ (S, s) & (Spec(\mathcal{O}_{S',s'}), s') \arrow[l] \end{tikzcd}$$

such that $X' \to X \times_S Spec(\mathcal{O}_{S',s'})$ is étale, $\kappa(x) = \kappa(x')$, the scheme X' is affine of finite presentation over $\mathcal{O}_{S',s'}$, the sheaf $g^*\mathcal{F}$ is of finite presentation over $\mathcal{O}_{X'}$, and such that $\Gamma(X', g^*\mathcal{F})$ is a free $\mathcal{O}_{S',s'}$-module.

Proof. To prove the lemma we may replace (S, s) by any elementary étale neighbourhood, and we may also replace S by $Spec(\mathcal{O}_{S,s})$. Hence by Proposition 34.11.3 we may assume that $\mathcal{F}$ is finitely presented and flat over S in a neighbourhood of x. In this case the result follows from Proposition 34.12.4 because Algebra, Theorem 7.79.4 assures us that projective = free over a local ring. □

Lemma 34.12.9. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $x \in X$ with image $s \in S$. Assume that*

(1) *f is locally of finite type,*
(2) *$\mathcal{F}$ is of finite type, and*
(3) *$\mathcal{F}$ is flat at x over S.*

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram of pointed schemes

$$\begin{tikzcd} (X, x) \arrow[d] & (X', x') \arrow[l, "g"'] \arrow[d] \\ (S, s) & (Spec(\mathcal{O}_{S',s'}), s') \arrow[l] \end{tikzcd}$$

such that $X' \to X \times_S Spec(\mathcal{O}_{S',s'})$ is étale, $\kappa(x) = \kappa(x')$, the scheme X' is affine, and such that $\Gamma(X', g^*\mathcal{F})$ is a free $\mathcal{O}_{S',s'}$-module.

Proof. (The only difference with Lemma 34.12.8 is that we do not assume f is of finite presentation.) The problem is local on X and S. Hence we may assume X and S are affine, say $X = Spec(B)$ and $S = Spec(A)$. Since B is a finite type A-algebra we can find a surjection $A[x_1, \ldots, x_n] \to B$. In other words, we can choose a closed immersion $i : X \to \mathbf{A}^n_S$. Set $t = i(x)$ and $\mathcal{G} = i_*\mathcal{F}$. Note that $\mathcal{G}_t \cong \mathcal{F}_x$ are $\mathcal{O}_{S,s}$-modules. Hence $\mathcal{G}$ is flat over S at t. We apply Lemma 34.12.8 to the morphism $\mathbf{A}^n_S \to S$, the point t, and the sheaf $\mathcal{G}$. Thus we can find an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram of pointed schemes

$$\begin{tikzcd}
(\mathbf{A}^n_S, t) \arrow[d] & (Y, y) \arrow[l, "h"'] \arrow[d] \\
(S, s) & (Spec(\mathcal{O}_{S',s'}), s') \arrow[l]
\end{tikzcd}$$

such that $Y \to \mathbf{A}^n_{\mathcal{O}_{S',s'}}$ is étale, $\kappa(t) = \kappa(y)$, the scheme Y is affine, and such that $\Gamma(Y, h^*\mathcal{G})$ is a projective $\mathcal{O}_{S',s'}$-module. Then a solution to the orginal problem is given by the closed subscheme $X' = Y \times_{\mathbf{A}^n_S} X$ of Y. $\square$

Lemma 34.12.10. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $s \in S$. Assume that*

1. *f is of finite presentation,*
2. *$\mathcal{F}$ is of finite type, and*
3. *$\mathcal{F}$ is flat over S at all points of X_s.*

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram of schemes

$$\begin{tikzcd}
X \arrow[d] & X' \arrow[l, "g"'] \arrow[d] \\
S & Spec(\mathcal{O}_{S',s'}) \arrow[l]
\end{tikzcd}$$

such that $X' \to X \times_S Spec(\mathcal{O}_{S',s'})$ is étale, $X_s = g((X')_{s'})$, the scheme X' is affine of finite presentation over $\mathcal{O}_{S',s'}$, the sheaf $g^*\mathcal{F}$ is of finite presentation over $\mathcal{O}_{X'}$, and such that $\Gamma(X', g^*\mathcal{F})$ is a free $\mathcal{O}_{S',s'}$-module.

Proof. For every point $x \in X_s$ we can use Lemma 34.12.8 to find an elementary étale neighbourhood $(S_x, s_x) \to (S, s)$ and a commutative diagram

$$\begin{tikzcd}
(X, x) \arrow[d] & (Y_x, y_x) \arrow[l, "g_x"'] \arrow[d] \\
(S, s) & (Spec(\mathcal{O}_{S_x,s_x}), s_x) \arrow[l]
\end{tikzcd}$$

such that $Y_x \to X \times_S Spec(\mathcal{O}_{S_x,s_x})$ is étale, $\kappa(x) = \kappa(y_x)$, the scheme Y_x is affine of finite presentation over $\mathcal{O}_{S_x,s_x}$, the sheaf $g_x^*\mathcal{F}$ is of finite presentation over $\mathcal{O}_{Y_x}$, and such that $\Gamma(Y_x, g_x^*\mathcal{F})$ is a free $\mathcal{O}_{S_x,s_x}$-module. In particular $g_x((Y_x)_{s_x})$ is an open neighbourhood of x in X_s. Because X_s is quasi-compact we can find a finite number of points $x_1, \ldots, x_n \in X_s$ such that X_s is the union of the $g_{x_i}((Y_{x_i})_{s_{x_i}})$. Choose an elementary étale neighbourhood

$(S', s') \to (S, s)$ which dominates each of the neighbourhoods (S_{x_i}, s_{x_i}), see More on Morphisms, Lemma 33.25.4. Set

$$X' = \coprod Y_{x_i} \times_{Spec(\mathcal{O}_{S_{x_i}, s_{x_i}})} Spec(\mathcal{O}_{S', s'})$$

and endow it with the obvious morphism $g : X' \to X$. By construction $X_s = g(X'_{s'})$ and

$$\Gamma(X', g^*\mathcal{F}) = \bigoplus \Gamma(Y_{x_i}, g_{x_i}^*\mathcal{F}) \otimes_{\mathcal{O}_{S_{x_i}, s_{x_i}}} \mathcal{O}_{S', s'}.$$

This is a free $\mathcal{O}_{S', s'}$-module as a direct sum of base changes of free modules. Some minor details omitted. □

Lemma 34.12.11. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $s \in S$. Assume that*

1. *f is of finite type,*
2. *$\mathcal{F}$ is of finite type, and*
3. *$\mathcal{F}$ is flat over S at all points of X_s.*

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram of schemes

$$\begin{tikzcd} X \arrow[d] & X' \arrow[l, "g"'] \arrow[d] \\ S & Spec(\mathcal{O}_{S', s'}) \arrow[l] \end{tikzcd}$$

such that $X' \to X \times_S Spec(\mathcal{O}_{S', s'})$ is étale, $X_s = g((X')_{s'})$, the scheme X' is affine, and such that $\Gamma(X', g^*\mathcal{F})$ is a free $\mathcal{O}_{S', s'}$-module.

Proof. (The only difference with Lemma 34.12.10 is that we do not assume f is of finite presentation.) For every point $x \in X_s$ we can use Lemma 34.12.9 to find an elementary étale neighbourhood $(S_x, s_x) \to (S, s)$ and a commutative diagram

$$\begin{tikzcd} (X, x) \arrow[d] & (Y_x, y_x) \arrow[l, "g_x"'] \arrow[d] \\ (S, s) & (Spec(\mathcal{O}_{S_x, s_x}), s_x) \arrow[l] \end{tikzcd}$$

such that $Y_x \to X \times_S Spec(\mathcal{O}_{S_x, s_x})$ is étale, $\kappa(x) = \kappa(y_x)$, the scheme Y_x is affine, and such that $\Gamma(Y_x, g_x^*\mathcal{F})$ is a free $\mathcal{O}_{S_x, s_x}$-module. In particular $g_x((Y_x)_{s_x})$ is an open neighbourhood of x in X_s. Because X_s is quasi-compact we can find a finite number of points $x_1, \ldots, x_n \in X_s$ such that X_s is the union of the $g_{x_i}((Y_{x_i})_{s_{x_i}})$. Choose an elementary étale neighbourhood $(S', s') \to (S, s)$ which dominates each of the neighbourhoods (S_{x_i}, s_{x_i}), see More on Morphisms, Lemma 33.25.4. Set

$$X' = \coprod Y_{x_i} \times_{Spec(\mathcal{O}_{S_{x_i}, s_{x_i}})} Spec(\mathcal{O}_{S', s'})$$

and endow it with the obvious morphism $g : X' \to X$. By construction $X_s = g(X'_{s'})$ and

$$\Gamma(X', g^*\mathcal{F}) = \bigoplus \Gamma(Y_{x_i}, g_{x_i}^*\mathcal{F}) \otimes_{\mathcal{O}_{S_{x_i}, s_{x_i}}} \mathcal{O}_{S', s'}.$$

This is a free $\mathcal{O}_{S', s'}$-module as a direct sum of base changes of free modules. □

34.13. Flat finite type modules, Part II

The following lemma will be superseded by the stronger Lemma 34.13.3 below.

Lemma 34.13.1. *Let $(R, \mathfrak{m})$ be a local ring. Let $R \to S$ be of finite presentation. Let N be a finitely presented S-module which is free as an R-module. Let M be an R-module. Let $\mathfrak{q}$ be a prime of S lying over $\mathfrak{m}$. Then*

(1) *if $\mathfrak{q} \in WeakAss_S(M \otimes_R N)$ then $\mathfrak{m} \in WeakAss_R(M)$ and $\overline{\mathfrak{q}} \in Ass_{\overline{S}}(\overline{N})$,*
(2) *if $\mathfrak{m} \in WeakAss_R(M)$ and $\overline{\mathfrak{q}} \in Ass_{\overline{S}}(\overline{N})$ is a maximal element then $\mathfrak{q} \in WeakAss_S(M \otimes_R N)$.*

Here $\overline{S} = S/\mathfrak{m}S$, $\overline{\mathfrak{q}} = \mathfrak{q}\overline{S}$, and $\overline{N} = N/\mathfrak{m}N$.

Proof. Suppose that $\overline{\mathfrak{q}} \not\in \text{Ass}_{\overline{S}}(\overline{N})$. By Algebra, Lemmas 7.60.9, 7.60.5, and 7.14.3 there exists an element $\overline{g} \in \overline{\mathfrak{q}}$ which is not a zero divisor on $\overline{N}$. Let $g \in \mathfrak{q}$ be an element which maps to $\overline{g}$ in $\overline{\mathfrak{q}}$. By Lemma 34.8.6 the map $g : N \to N$ is R-universally injective. In particular we see that $g : M \otimes_R N \to M \otimes_R N$ is injective. Clearly this implies that $\mathfrak{q} \not\in \text{WeakAss}_S(M \otimes_R N)$. We conclude that $\mathfrak{q} \in \text{WeakAss}_S(M \otimes_R N)$ implies $\overline{\mathfrak{q}} \in \text{Ass}_{\overline{S}}(\overline{N})$.

Assume $\mathfrak{q} \in \text{WeakAss}_S(M \otimes_R N)$. Let $z \in M \otimes_R N$ be an element whose annihilator in S has radical $\mathfrak{q}$. As N is a free R-module, we can find a finite free direct summand $F \subset N$ such that $z \in M \otimes_R F$. The radical of the annihilator of $z \in M \otimes_R F$ in R is $\mathfrak{m}$ (by our assumption on z and because $\mathfrak{q}$ lies over $\mathfrak{m}$). Hence we see that $\mathfrak{m} \in \text{WeakAss}(M \otimes_R F)$ which implies that $\mathfrak{m} \in \text{WeakAss}(M)$ by Algebra, Lemma 7.63.3. This finishes the proof of (1).

Assume that $\mathfrak{m} \in \text{WeakAss}_R(M)$ and $\overline{\mathfrak{q}} \in \text{Ass}_{\overline{S}}(\overline{N})$ is a maximal element. Let $y \in M$ be an element whose annihilator $I = \text{Ann}_R(y)$ has radical $\mathfrak{m}$. Then $R/I \subset M$ and by flatness of N over R we get $N/IN = R/I \otimes_R N \subset M \otimes_R N$. Hence it is enough to show that $\mathfrak{q} \in \text{WeakAss}(N/IN)$. Write $\overline{\mathfrak{q}} = (\overline{g}_1, \ldots, \overline{g}_n)$ for some $\overline{g}_i \in \overline{S}$. Choose lifts $g_i \in \mathfrak{q}$. Consider the map

$$\Psi : N/IN \longrightarrow N/IN^{\oplus n}, \quad z \longmapsto (g_1 z, \ldots, g_n z).$$

We may think of this as a map of free R/I-modules. As the ring R/I is auto-associated (since $\mathfrak{m}/I$ is locally nilpotent) and since $\Psi \otimes R/\mathfrak{m}$ isn't injective (since $\overline{\mathfrak{q}} \in \text{Ass}(\overline{N})$) we see by More on Algebra, Lemma 12.10.4 that Ψ isn't injective. Pick $z \in N/IN$ nonzero in the kernel of Ψ. The annihilator of z contains I and g_i, whence its radical $J = \sqrt{\text{Ann}_S(z)}$ contains $\mathfrak{q}$. Let $\mathfrak{q}' \supset J$ be a minimal prime over J. Then $\mathfrak{q}' \in \text{WeakAss}(M \otimes_R N)$ (by definition) and by (1) we see that $\overline{\mathfrak{q}}' \in \text{Ass}(\overline{N})$. Then since $\mathfrak{q} \subset \mathfrak{q}'$ by construction the maximality of $\overline{\mathfrak{q}}$ implies $\mathfrak{q} = \mathfrak{q}'$ whence $\mathfrak{q} \in \text{WeakAss}(M \otimes_R N)$. This proves part (2) of the lemma. □

Lemma 34.13.2. *Let S be a scheme. Let $f : X \to S$ be locally of finite type. Let $x \in X$ with image $s \in S$. Let $\mathcal{F}$ be a finite type quasi-coherent sheaf on X. Let $\mathcal{G}$ be a quasi-coherent sheaf on Y. If $\mathcal{F}$ is flat at x over S, then*

$$x \in WeakAss_X(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}) \Leftrightarrow s \in WeakAss_S(\mathcal{G}) \text{ and } x \in Ass_{X_s}(\mathcal{F}_s).$$

Proof. The question is local on X and S, hence we may assume X and S are affine. Write $X = Spec(B)$, $S = Spec(A)$ and write $B = A[x_1, \ldots, x_n]/I$. In other words we obtain a closed immersion $i : X \to \mathbf{A}^n_S$ over S. Denote $t = i(x) \in \mathbf{A}^n_S$. Note that $i_*\mathcal{F}$ is a finite type quasi-coherent sheaf on $\mathbf{A}^n_S$ which is flat at t over S and note that

$$i_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}) = i_*\mathcal{F} \otimes_{\mathcal{O}_{\mathbf{A}^n_S}} p^*\mathcal{G}$$

where $p : \mathbf{A}^n_S \to S$ is the projection. Note that t is a weakly associated point of $i_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G})$ if and only if x is a weakly associated popint of $\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}$, see Divisors, Lemma 26.6.3. Similarly $x \in \text{Ass}_{X_s}(\mathcal{F}_s)$ if and only if $t \in \text{Ass}_{\mathbf{A}^n_s}((i_*\mathcal{F})_s)$ (see Algebra, Lemma 7.60.13). Hence it suffices to prove the the lemma in case $X = \mathbf{A}^n_S$. In particular we may assume that $X \to S$ is of finite presentation.

Recall that $\text{Ass}_{X_s}(\mathcal{F}_s)$ is a locally finite subset of the locally Noetherian scheme X_s, see Divisors, Lemma 26.2.5. After replacing X by a suitable affine neighbourhood of x we may assume that

(∗) if $x' \in \text{Ass}_{X_s}(\mathcal{F}_s)$ and $x \rightsquigarrow x'$ then $x = x'$.

(Proof omitted. Hint: using Algebra, Lemma 7.14.3 invert a function which does not vanish at x but does vanish in all the finitely many points of $\text{Ass}_{X_s}(\mathcal{F}_s)$ which are specializations of x but not equal to x.) In words, no point of $\text{Ass}_{X_s}(\mathcal{F}_s)$ is a proper specialization of x.

Suppose given a commutative diagram

$$\begin{tikzcd} (X, x) \arrow[d] & (X', x') \arrow[l, "g"'] \arrow[d] \\ (S, s) & (S', s') \arrow[l, "e"'] \end{tikzcd}$$

of pointed schemes whose horizontal arrows are elementary étale neighbourhoods. Then it suffices to prove the statement for x', s', $g^*\mathcal{F}$ and $e^*\mathcal{G}$, see Lemma 34.3.7. Note that property (∗) is preserved by such an étale localization by the same lemma (if there is a proper specialization $x' \rightsquigarrow x''$ on $X'_{s'}$ then this maps to a proper specialization on X_s because the fibres of an étale morphism are discrete). We may also replace S by the spectrum of its local ring as the condition of being an associated point of a quasi-coherent sheaf depends only on the stalk of the sheaf. Again property (∗) is preserved by this as well. Thus we may first apply Proposition 34.11.3 to reduce to the case where $\mathcal{F}$ is of finite presentation and flat over S, whereupon we may use Proposition 34.12.4 to reduce to the case that $X \to S$ is a morphism of affines and $\Gamma(X, \mathcal{F})$ is a finitely presented $\Gamma(X, \mathcal{O}_X)$-module which is projective as a $\Gamma(S, \mathcal{O}_S)$-module. Localizing S once more we may assume that $\Gamma(S, \mathcal{O}_S)$ is a local ring such that s corresponds to the maximal ideal. In this case Algebra, Theorem 7.79.4 guarantees that $\Gamma(X, \mathcal{F})$ is free as an $\Gamma(S, \mathcal{O}_S)$-module. The implication $x \in \text{WeakAss}_X(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}) \Rightarrow s \in \text{WeakAss}_S(\mathcal{G})$ and $x \in \text{Ass}_{X_s}(\mathcal{F}_s)$ follows from part (1) of Lemma 34.13.1. The converse implication follows from part (2) of Lemma 34.13.1 as property (∗) insures that the prime corresponding to x gives rise to a maximal element of $\text{Ass}_{\overline{S}}(\overline{N})$ exactly as in the statement of part (2) of Lemma 34.13.1. □

Lemma 34.13.3. *Let $R \to S$ be a ring map which is essentially of finite type. Let N be a localization of a finite S-module flat over R. Let M be an R-module. Then*

$$WeakAss_S(M \otimes_R N) = \bigcup\nolimits_{\mathfrak{p} \in WeakAss_R(M)} Ass_{S \otimes_R \kappa(\mathfrak{p})}(N \otimes_R \kappa(\mathfrak{p}))$$

Proof. This lemma is a translation of Lemma 34.13.2 into algebra. Details of translation omitted. □

Lemma 34.13.4. *Let $f : X \to S$ be a morphism which is locally of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent sheaf on X which is flat over S. Let $\mathcal{G}$ be a quasi-coherent sheaf on S. Then we have*

$$WeakAss_X(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}) = \bigcup\nolimits_{s \in WeakAss_S(\mathcal{G})} Ass_{X_s}(\mathcal{F}_s)$$

Proof. Immediate consequence of Lemma 34.13.2. □

Theorem 34.13.5. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume*

1. *$X \to S$ is locally of finite presentation,*
2. *$\mathcal{F}$ is an $\mathcal{O}_X$-module of finite type, and*
3. *the set of weakly associated points of S is locally finite in S.*

Then $U = \{x \in X \mid \mathcal{F}$ flat at x over $S\}$ is open in X and $\mathcal{F}|_U$ is an $\mathcal{O}_U$-module of finite presentation and flat over S.

Proof. Let $x \in X$ be such that $\mathcal{F}$ is flat at x over S. We have to find an open neighbourhood of x such that $\mathcal{F}$ restricts to a S-flat finitely presented module on this neighbourhood. The problem is local on X and S, hence we may assume that X and S are affine. As $\mathcal{F}_x$ is a finitely presented $\mathcal{O}_{X,x}$-module by Lemma 34.11.9 we conclude from Algebra, Lemma 7.117.5 there exists a finitely presented $\mathcal{O}_X$-module $\mathcal{F}'$ and a map $\varphi : \mathcal{F}' \to \mathcal{F}$ which induces an isomorphism $\varphi_x : \mathcal{F}'_x \to \mathcal{F}_x$. In particular we see that $\mathcal{F}'$ is flat over S at x, hence by openness of flatness More on Morphisms, Theorem 33.11.1 we see that after shrinking X we may assume that $\mathcal{F}'$ is flat over S. As $\mathcal{F}$ is of finite type after shrinking X we may assume that φ is surjective, see Modules, Lemma 15.9.4 or alternatively use Nakayama's lemma (Algebra, Lemma 7.14.5). By Lemma 34.13.4 we have

$$\mathrm{WeakAss}_X(\mathcal{F}') \subset \bigcup\nolimits_{s \in \mathrm{WeakAss}(S)} \mathrm{Ass}_{X_s}(\mathcal{F}'_s)$$

As $\mathrm{WeakAss}(S)$ is finite by assumption and since $\mathrm{Ass}_{X_s}(\mathcal{F}'_s)$ is finite by Divisors, Lemma 26.2.5 we conclude that $\mathrm{WeakAss}_X(\mathcal{F}')$ is finite. Using Algebra, Lemma 7.14.3 we may, after shrinking X once more, assume that $\mathrm{WeakAss}_X(\mathcal{F}')$ is contained in the generalization of x. Now consider $\mathcal{K} = \mathrm{Ker}(\varphi)$. We have $\mathrm{WeakAss}_X(\mathcal{K}) \subset \mathrm{WeakAss}_X(\mathcal{F}')$ (by Divisors, Lemma 26.5.4) but on the other hand, φ_x is an isomorphism, also $\varphi_{x'}$ is an isomorphism for all $x' \rightsquigarrow x$. We conclude that $\mathrm{WeakAss}_X(\mathcal{K}) = \emptyset$ whence $\mathcal{K} = 0$ by Divisors, Lemma 26.5.5. □

Lemma 34.13.6 (Algebra version of Theorem 34.13.5)**.** *Let $R \to S$ be a ring map of finite presentation. Let M be a finite S-module. Assume $WeakAss_S(S)$ is finite. Then*

$$U = \{\mathfrak{q} \subset S \mid M_{\mathfrak{q}} \text{ flat over } R\}$$

is open in $Spec(S)$ and for every $g \in S$ such that $D(g) \subset U$ the localization M_g is a finitely presented S_g-module flat over R.

Proof. Follows immediately from Theorem 34.13.5. □

Lemma 34.13.7. *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Assume the set of weakly associated points of S is locally finite in S. Then the set of points $x \in X$ where f is flat is an open subscheme $U \subset X$ and $U \to S$ is flat and locally of finite presentation.*

Proof. The problem is local on X and S, hence we may assume that X and S are affine. Then $X \to S$ corresponds to a finite type ring map $A \to B$. Choose a surjection $A[x_1, \ldots, x_n] \to B$ and consider B as an $A[x_1, \ldots, x_n]$-module. An application of Lemma 34.13.6 finishes the proof. □

Lemma 34.13.8. *Let $f : X \to S$ be a morphism of schemes which is locally of finite type and flat. If S is integral, then f is locally of finite presentation.*

Proof. Special case of Lemma 34.13.7. □

Lemma 34.13.9. *Let $A \to B$ be a finite type, flat ring map with A an integral domain. Then B is a finitely presented A-algebra.*

Proof. Special case of Lemma 34.13.8. It is also a consequence of More on Algebra, Proposition 12.18.8. □

Remark 34.13.10 (Finite type version of Theorem 34.13.5)**.** Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume

(1) $X \to S$ is locally of finite type,
(2) $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite type, and
(3) the set of weakly associated points of S is locally finite in S.

Then $U = \{x \in X \mid \mathcal{F} \text{ flat at } x \text{ over } S\}$ is open in X and $\mathcal{F}|_U$ is flat over S and locally finitely presented relative to S (see Definition 34.2.1). If we ever need this result in the stacks project we will convert this remark into a lemma with a proof.

Remark 34.13.11 (Algebra version of Remark 34.13.10)**.** Let $R \to S$ be a ring map of finite type. Let M be a finite S-module. Assume $\text{WeakAss}_S(S)$ is finite. Then

$$U = \{\mathfrak{q} \subset S \mid M_{\mathfrak{q}} \text{ flat over } R\}$$

is open in $Spec(S)$ and for every $g \in S$ such that $D(g) \subset U$ the localization M_g is flat over R and an S_g-module finitely presented relative to R (see More on Algebra, Definition 12.44.2). If we ever need this result in the stacks project we will convert this remark into a lemma with a proof.

34.14. Examples of relatively pure modules

In the short section we discuss some examples of results that will serve as motivation for the notion of a *relatively pure module* and the concept of an *impurity* which we will introduce later. Each of the examples is stated as a lemma. Note the similarity with the condition on associated primes to the conditions appearing in Lemmas 34.8.4, 34.9.3, 34.9.4, and 34.10.1. See also Algebra, Lemma 7.62.1 for a discussion.

Lemma 34.14.1. *Let R be a local ring with maximal ideal $\mathfrak{m}$. Let $R \to S$ be a ring map. Let N be an S-module. Assume*

(1) N is projective as an R-module, and
(2) $S/\mathfrak{m}S$ is Noetherian and $N/\mathfrak{m}N$ is a finite $S/\mathfrak{m}S$-module.

Then for any prime $\mathfrak{q} \subset S$ which is an associated prime of $N \otimes_R \kappa(\mathfrak{p})$ where $\mathfrak{p} = R \cap \mathfrak{q}$ we have $\mathfrak{q} + \mathfrak{m}S \neq S$.

Proof. Note that the hypotheses of Lemmas 34.8.1 and 34.8.6 are satisfied. We will use the conclusions of these lemmas without further mention. Let $\Sigma \subset S$ be the multiplicative set of elements which are not zero divisors on $N/\mathfrak{m}N$. The map $N \to \Sigma^{-1}N$ is R-universally injective. Hence we see that any $\mathfrak{q} \subset S$ which is an associated prime of $N \otimes_R \kappa(\mathfrak{p})$ is also an associated prime of $\Sigma^{-1}N \otimes_R \kappa(\mathfrak p)$. Clearly this implies that $\mathfrak q$ corresponds to a prime of $\Sigma^{-1}S$. Thus $\mathfrak q \subset \mathfrak q'$ where $\mathfrak q'$ corresponds to an associated prime of $N/\mathfrak m N$ and we win. $\square$

The following lemma gives another (slightly silly) example of this phenomenon.

Lemma 34.14.2. *Let R be a ring. Let $I \subset R$ be an ideal. Let $R \to S$ be a ring map. Let N be an S-module. If N is I-adically complete, then for any R-module M and for any prime $\mathfrak q \subset S$ which is an associated prime of $N \otimes_R M$ we have $\mathfrak q + IS \neq S$.*

Proof. Let $S^\wedge$ denote the I-adic completion of S. Note that N is an $S^\wedge$-module, hence also $N \otimes_R M$ is an $S^\wedge$-module. Let $z \in N \otimes_R M$ be an element such that $\mathfrak q = \text{Ann}_S(z)$. Since $z \neq 0$ we see that $\text{Ann}_{S^\wedge}(z) \neq S^\wedge$. Hence $\mathfrak q S^\wedge \neq S^\wedge$. Hence there exists a maximal ideal $\mathfrak m \subset S^\wedge$ with $\mathfrak q S^\wedge \subset \mathfrak m$. Since $IS^\wedge \subset \mathfrak m$ by Algebra, Lemma 7.90.11 we win. $\square$

Note that the following lemma gives an alternative proof of Lemma 34.14.1 as a projective module over a local ring is free, see Algebra, Theorem 7.79.4.

Lemma 34.14.3. *Let R be a local ring with maximal ideal $\mathfrak m$. Let $R \to S$ be a ring map. Let N be an S-module. Assume N is isomorphic as an R-module to a direct sum of finite R-modules. Then for any R-module M and for any prime $\mathfrak q \subset S$ which is an associated prime of $N \otimes_R M$ we have $\mathfrak q + \mathfrak m S \neq S$.*

Proof. Write $N = \bigoplus_{i \in I} M_i$ with each M_i a finite R-module. Let M be an R-module and let $\mathfrak q \subset S$ be an associated prime of $N \otimes_R M$ such that $\mathfrak q + \mathfrak m S = S$. Let $z \in N \otimes_R M$ be an element with $\mathfrak q = \text{Ann}_S(z)$. After modifying the direct sum decomposition a little bit we may assume that $z \in M_1 \otimes_R M$ for some element $1 \in I$. Write $1 = f + \sum x_j g_j$ for some $f \in \mathfrak q$, $x_j \in \mathfrak m$, and $g_j \in S$. For any $g \in S$ denote g' the R-linear map

$$M_1 \to N \xrightarrow{g} N \to M_1$$

where the first arrow is the inclusion map, the second arrow is multiplication by g and the third arrow is the projection map. Because each $x_j \in R$ we obtain the equality

$$f' + \sum x_j g'_j = \text{id}_{M_1} \in \text{End}_R(M_1)$$

By Nakayama's lemma (Algebra, Lemma 7.14.5) we see that f' is surjective, hence by Algebra, Lemma 7.15.4 we see that f' is an isomorphism. In particular the map

$$M_1 \otimes_R M \to N \otimes_R M \xrightarrow{f} N \otimes_R M \to M_1 \otimes_R M$$

is an isomorphism. This contradicts the assumption that $fz = 0$. $\square$

Lemma 34.14.4. *Let R be a henselian local ring with maximal ideal $\mathfrak m$. Let $R \to S$ be a ring map. Let N be an S-module. Assume N is countably generated and Mittag-Leffler as an R-module. Then for any R-module M and for any prime $\mathfrak q \subset S$ which is an associated prime of $N \otimes_R M$ we have $\mathfrak q + \mathfrak m S \neq S$.*

Proof. This lemma reduces to Lemma 34.14.3 by Algebra, Lemma 7.139.26. $\square$

Suppose $f : X \to S$ is a morphism of schemes and $\mathcal F$ is a quasi-coherent module on X. Let $\xi \in \text{Ass}_{X/S}(\mathcal F)$ and let $Z = \overline{\{\xi\}}$. Picture

$$\begin{tikzcd} \xi \arrow[d, mapsto] & Z \arrow[r] \arrow[dr] & X \arrow[d, "f"] \\ f(\xi) & & S \end{tikzcd}$$

Note that $f(Z) \subset \overline{\{f(\xi)\}}$ and that $f(Z)$ is closed if and only if equality holds, i.e., $f(Z) = \overline{\{f(\xi)\}}$. It follows from Lemma 34.14.1 that if S, X are affine, the fibres X_s are Noetherian, $\mathcal{F}$ is of finite type, and $\Gamma(X, \mathcal{F})$ is a projective $\Gamma(S, \mathcal{O}_S)$-module, then $f(Z) = \overline{\{f(\xi)\}}$ is a closed subset. Slightly different analogous statements holds for the cases described in Lemmas 34.14.2, 34.14.3, and 34.14.4.

34.15. Impurities

We want to formalize the phenomenon of which we gave examples in Section 34.14 in terms of specializations of points of $\text{Ass}_{X/S}(\mathcal{F})$. We also want to work locally around a point $s \in S$. In order to do so we make the following definitions.

Situation 34.15.1. Here S, X are schemes and $f : X \to S$ is a finite type morphism. Also, $\mathcal{F}$ is a finite type quasi-coherent $\mathcal{O}_X$-module. Finally s is a point of S.

In this situation consider a morphism $g : T \to S$, a point $t \in T$ with $g(t) = s$, a specialization $t' \rightsquigarrow t$, and a point $\xi \in X_T$ in the base change of X lying over t'. Picture

(34.15.1.1)
$$\begin{tikzcd} \xi \arrow[d, mapsto] & & X_T \arrow[r] \arrow[d] & X \arrow[d] \\ t' \arrow[r, rightsquigarrow] & t \arrow[d, mapsto] & T \arrow[r, "g"] \arrow[d, "g"] & S \\ & s & S & \end{tikzcd}$$

Moreover, denote $\mathcal{F}_T$ the pullback of $\mathcal{F}$ to X_T.

Definition 34.15.2. In Situation 34.15.1 we say a diagram (34.15.1.1) defines an *impurity of $\mathcal{F}$ above s* if $\xi \in \text{Ass}_{X_T/T}(\mathcal{F}_T)$ and $\overline{\{\xi\}} \cap X_t = \emptyset$. We will indicate this by saying ``let $(g : T \to S, t' \rightsquigarrow t, \xi)$ be an impurity of $\mathcal{F}$ above s''.

Lemma 34.15.3. *In Situation 34.15.1. If there exists an impurity of $\mathcal{F}$ above s, then there exists an impurity $(g : T \to S, t' \rightsquigarrow t, \xi)$ of $\mathcal{F}$ above s such that g is locally of finite presentation and t a closed point of the fibre of g above s.*

Proof. Let $(g : T \to S, t' \rightsquigarrow t, \xi)$ be any impurity of $\mathcal{F}$ above s. We apply Limits, Lemma 27.10.1 to $t \in T$ and $Z = \overline{\{\xi\}}$ to obtain an open neighbourhood $V \subset T$ of t, a commutative diagram

$$\begin{tikzcd} V \arrow[r, "a"] \arrow[d] & T' \arrow[d, "b"] \\ T \arrow[r, "g"] & S, \end{tikzcd}$$

and a closed subscheme $Z' \subset X_{T'}$ such that

(1) the morphism $b : T' \to S$ is locally of finite presentation,
(2) we have $Z' \cap X_{a(t)} = \emptyset$, and
(3) $Z \cap X_V$ maps into Z' via the morphism $X_V \to X_{T'}$.

As t' specializes to t we may replace T by the open neighbourhood V of t. Thus we have a commutative diagram

$$\begin{tikzcd} X_T \arrow[r] \arrow[d] & X_{T'} \arrow[r] \arrow[d] & X \arrow[d] \\ T \arrow[r, "a"] & T' \arrow[r, "b"] & S \end{tikzcd}$$

where $b \circ a = g$. Let $\xi' \in X_{T'}$ denote the image of ξ. By Divisors, Lemma 26.7.2 we see that $\xi' \in \text{Ass}_{X_{T'}/T'}(\mathcal{F}_{T'})$. Moreover, by construction the closure of $\overline{\{\xi'\}}$ is contained in the closed subset Z' which avoids the fibre $X_{a(t)}$. In this way we see that $(T' \to S, a(t') \rightsquigarrow a(t), \xi')$ is an impurity of $\mathcal{F}$ above s.

Thus we may assume that $g : T \to S$ is locally of finite presentation. Let $Z = \overline{\{\xi\}}$. By assumption $Z_t = \emptyset$. By More on Morphisms, Lemma 33.17.1 this means that $Z_{t''} = \emptyset$ for t'' in an open subset of $\overline{\{t\}}$. Since the fibre of $T \to S$ over s is a Jacobson scheme, see Morphisms, Lemma 24.15.10 we find that there exist a closed point $t'' \in \overline{\{t\}}$ such that $Z_{t''} = \emptyset$. Then $(g : T \to S, t' \rightsquigarrow t'', \xi)$ is the desired impurity. □

Lemma 34.15.4. *In Situation 34.15.1. Let $(g : T \to S, t' \rightsquigarrow t, \xi)$ be an impurity of $\mathcal{F}$ above s. Assume S is affine and that T is written $T = lim_{i \in I} T_i$ as a directed colimit of affine schemes over S. Then for some i the triple $(T_i \to S, t_i' \rightsquigarrow t_i, \xi_i)$ is an impurity of $\mathcal{F}$ above s.*

Proof. The notation in the statement means this: Let $f_i : T \to T_i$ be the projection morphisms, let $t_i = f_i(t)$ and $t_i' = f_i(t')$. Finally $\xi_i \in X_{T_i}$ is the image of ξ. By Divisors, Lemma 26.7.2 it is true that ξ_i is a point of the relative assassin of $\mathcal{F}_{T_i}$ over T_i. Thus the only point is to show that $\overline{\{\xi_i\}} \cap X_{t_i} = \emptyset$ for some i. Set $Z = \overline{\{\xi\}}$. Apply Limits, Lemma 27.10.1 to this situation to obtain an open neighbourhood $V \subset T$ of t, a commutative diagram

$$\begin{tikzcd} V \arrow[r, "a"] \arrow[d] & T' \arrow[d, "b"] \\ T \arrow[r, "g"] & S, \end{tikzcd}$$

and a closed subscheme $Z' \subset X_{T'}$ such that

(1) the morphism $b : T' \to S$ is locally of finite presentation,
(2) we have $Z' \cap X_{a(t)} = \emptyset$, and
(3) $Z \cap X_V$ maps into Z' via the morphism $X_V \to X_{T'}$.

We may assume V is an affine open of T, hence by Limits, Lemmas 27.3.5 and 27.3.7 we can find an i and an affine open $V_i \subset T_i$ with $V = f_i^{-1}(V_i)$. By Limits, Proposition 27.4.1 after possibly increasing i a bit we can find a morphism $a_i : V_i \to T'$ such that $a = a_i \circ f_i|_V$. The induced morphism $X_{T_i} \to X_{T'}$ maps ξ_i into Z'. As $Z' \cap X_{a(t)} = \emptyset$ we conclude that $(T_i \to S, t_i' \rightsquigarrow t_i, \xi_i)$ is an impurity of $\mathcal{F}$ above s. □

Lemma 34.15.5. *In Situation 34.15.1. If there exists an impurity $(g : T \to S, t' \rightsquigarrow t, \xi)$ of $\mathcal{F}$ above s with g quasi-finite at t, then there exists an impurity $(g : T \to S, t' \rightsquigarrow t, \xi)$ such that $(T, t) \to (S, s)$ is an elementary étale neighbourhood.*

Proof. Let $(g : T \to S, t' \rightsquigarrow t, \xi)$ be an impurity of $\mathcal{F}$ above s such that g is quasi-finite at t. After shrinking T we may assume that g is locally of finite type. Apply More on

Morphisms, Lemma 33.28.1 to $T \to S$ and $t \mapsto s$. This gives us a diagram

$$\begin{tikzcd} T \arrow[d] & T \times_S U \arrow[l] \arrow[d] & V \arrow[l] \arrow[dl] \\ S & U \arrow[l] & \end{tikzcd}$$

where $(U, u) \to (S, s)$ is an elementary étale neighbourhood and $V \subset T \times_S U$ is an open neighbouhood of $v = (t, u)$ such that $V \to U$ is finite and such that v is the unique point of V lying over u. Since the morphism $V \to T$ is étale hence flat we see that there exists a specialization $v' \rightsquigarrow v$ such that $v' \mapsto t'$. Note that $\kappa(t') \subset \kappa(v')$ is finite separable. Pick any point $\zeta \in X_{v'}$ mapping to $\xi \in X_{t'}$. By Divisors, Lemma 26.7.2 we see that $\zeta \in \text{Ass}_{X_V/V}(\mathcal{F}_V)$. Moreover, the closure $\overline{\{\zeta\}}$ does not meet the fibre X_v as by assumption the closure $\overline{\{\xi\}}$ does not meet X_t. In other words $(V \to S, v' \rightsquigarrow v, \zeta)$ is an impurity of $\mathcal{F}$ above S.

Next, let $u' \in U'$ be the image of v' and let $\theta \in X_U$ be the image of ζ. Then $\theta \mapsto u'$ and $u' \rightsquigarrow u$. By Divisors, Lemma 26.7.2 we see that $\theta \in \text{Ass}_{X_U/U}(\mathcal{F})$. Moreover, as $\pi : X_V \to X_U$ is finite we see that $\pi\big(\overline{\{\zeta\}}\big) = \overline{\{\pi(\zeta)\}}$. Since v is the unique point of V lying over u we see that $X_u \cap \overline{\{\pi(\zeta)\}} = \emptyset$ because $X_v \cap \overline{\{\zeta\}} = \emptyset$. In this way we conclude that $(U \to S, u' \rightsquigarrow u, \theta)$ is an impurity of $\mathcal{F}$ above s and we win. $\square$

Lemma 34.15.6. *In Situation 34.15.1. Assume that S is locally Noetherian. If there exists an impurity of $\mathcal{F}$ above s, then there exists an impurity $(g : T \to S, t' \rightsquigarrow t, \xi)$ of $\mathcal{F}$ above s such that g is quasi-finite at t.*

Proof. We may replace S by an affine neighbourhood of s. By Lemma 34.15.3 we may assume that we have an impurity $(g : T \to S, t' \rightsquigarrow t, \xi)$ of such that g is locally of finite type and t a closed point of the fibre of g above s. We may replace T by the reduced induced scheme structure on $\overline{\{t'\}}$. Let $Z = \overline{\{\xi\}} \subset X_T$. By assumption $Z_t = \emptyset$ and the image of $Z \to T$ contains t'. By More on Morphisms, Lemma 33.18.1 there exists a nonempty open $V \subset Z$ such that for any $w \in f(V)$ any generic point ξ' of V_w is in $\text{Ass}_{X_T/T}(\mathcal{F}_T)$. By More on Morphisms, Lemma 33.17.2 there exists a nonempty open $W \subset T$ with $W \subset f(V)$. By More on Morphisms, Lemma 33.33.7 there exists a closed subscheme $T' \subset T$ such that $t \in T'$, $T' \to S$ is quasi-finite at t, and there exists a point $z \in T' \cap W$, $z \rightsquigarrow t$ which does not map to s. Choose any generic point ξ' of the nonempty scheme V_z. Then $(T' \to S, z \rightsquigarrow t, \xi')$ is the desired impurity. $\square$

In the following we will use the henselization $S^h = Spec(\mathcal{O}^h_{S,s})$ of S at s, see Étale Cohomology, Definition 38.33.2. Since $S^h \to S$ maps to closed point of S^h to s and induces an isomorphism of residue fields, we will indicate $s \in S^h$ this closed point also. Thus $(S^h, s) \to (S, s)$ is a morphism of pointed schemes.

Lemma 34.15.7. *In Situation 34.15.1. If there exists an impurity $(S^h \to S, s' \rightsquigarrow s, \xi)$ of $\mathcal{F}$ above s then there exists an impurity $(T \to S, t' \rightsquigarrow t, \xi)$ of $\mathcal{F}$ above s where $(T, t) \to (S, s)$ is an elementary étale neighbourhood.*

Proof. We may replace S by an affine neighbourhood of s. Say $S = Spec(A)$ and s corresponds to the prime $\mathfrak{p} \subset A$. Then $\mathcal{O}^h_{S,s} = colim_{(T,t)} \Gamma(T, \mathcal{O}_T)$ where the limit is over the opposite of the cofiltered category of affine elementary étale neighbourhoods (T, t) of (S, s), see More on Morphisms, Lemma 33.25.5 and its proof. Hence $S^h = lim_i T_i$ and we win by Lemma 34.15.4. $\square$

Lemma 34.15.8. *In Situation 34.15.1 the following are equivalent*

(1) *there exists an impurity* $(S^h \to S, s' \rightsquigarrow s, \xi)$ *of* $\mathcal{F}$ *above* s *where* S^h *is the henselization of* S *at* s*,*
(2) *there exists an impurity* $(T \to S, t' \rightsquigarrow t, \xi)$ *of* $\mathcal{F}$ *above* s *such that* $(T, t) \to (S, s)$ *is an elementary étale neighbourhood, and*
(3) *there exists an impurity* $(T \to S, t' \rightsquigarrow t, \xi)$ *of* $\mathcal{F}$ *above* s *such that* $T \to S$ *is quasi-finite at* t*.*

Proof. As an étale morphism is locally quasi-finite it is clear that (2) implies (3). We have seen that (3) implies (2) in Lemma 34.15.5. We have seen that (1) implies (2) in Lemma 34.15.7. Finally, if $(T \to S, t' \rightsquigarrow t, \xi)$ is an impurity of $\mathcal{F}$ above s such that $(T, t) \to (S, s)$ is an elementary étale neighbourhood, then we can choose a factorization $S^h \to T \to S$ of the structure morphism $S^h \to S$. Choose any point $s' \in S^h$ mapping to t' and choose any $\xi' \in X_{s'}$ mapping to $\xi \in X_{t'}$. Then $(S^h \to S, s' \rightsquigarrow s, \xi')$ is an impurity of $\mathcal{F}$ above s. We omit the details. □

34.16. Relatively pure modules

The notion of a module pure relative to a base was introduced in [**GR71**].

Definition 34.16.1. Let $f : X \to S$ be a morphism of schemes which is of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module.

(1) Let $s \in S$. We say $\mathcal{F}$ is *pure along* X_s if there is no impurity $(g : T \to S, t' \rightsquigarrow t, \xi)$ of $\mathcal{F}$ above s with $(T, t) \to (S, s)$ an elementary étale neighbourhood.
(2) We say $\mathcal{F}$ is *universally pure along* X_s if there does not exist any impurity of $\mathcal{F}$ above s.
(3) We say that X is *pure along* X_s if $\mathcal{O}_X$ is pure along X_s.
(4) We say $\mathcal{F}$ is *universally* S*-pure*, or *universally pure relative to* S if $\mathcal{F}$ is universally pure along X_s for every $s \in S$.
(5) We say $\mathcal{F}$ is S*-pure*, or *pure relative to* S if $\mathcal{F}$ is pure along X_s for every $s \in S$.
(6) We say that X is S*-pure* or *pure relative to* S if $\mathcal{O}_X$ is pure relative to S.

We intentionally restrict ourselves here to morphisms which are of finite type and not just morphisms which are locally of finite type, see Remark 34.16.2 for a discussion. In the situation of the definition Lemma 34.15.8 tells us that the following are equivalent

(1) $\mathcal{F}$ is pure along X_s,
(2) there is no impurity $(g : T \to S, t' \rightsquigarrow t, \xi)$ with g quasi-finite at t,
(3) there does not exist any impurity of the form $(S^h \to S, s' \rightsquigarrow s, \xi)$, where S^h is the henselization of S at s.

If we denote $X^h = X \times_S S^h$ and $\mathcal{F}^h$ the pullback of $\mathcal{F}$ to X^h, then we can formulate the last condition in the following more positive way:

(4) All points of $\mathrm{Ass}_{X^h/S^h}(\mathcal{F}^h)$ specialize to points of X_s.

In particular, it is clear that $\mathcal{F}$ is pure along X_s if and only if the pullback of $\mathcal{F}$ to $X \times_S Spec(\mathcal{O}_{S,s})$ is pure along X_s.

Remark 34.16.2. Let $f : X \to S$ be a morphism which is locally of finite type and $\mathcal{F}$ a quasi-coherent finite type $\mathcal{O}_X$-module. In this case it is still true that (1) and (2) above are equivalent because the proof of Lemma 34.15.5 does not use that f is quasi-compact. It is also clear that (3) and (4) are equivalent. However, we don't know if (1) and (3) are equivalent. In this case it may sometimes be more convenient to define purity using the equivalent conditions (3) and (4) as is done in [**GR71**]. On the other hand, for many applications it seems that the correct notion is really that of being universally pure.

A natural question to ask is if the propery of being pure relative to the base is preserved by base change, i.e., if being pure is the same thing as being universally pure. It turns out that this is true over Noetherian base schemes (see Lemma 34.16.5), or if the sheaf is flat (see Lemmas 34.18.3 and 34.18.4). It is not true in general, even if the morphism and the sheaf are of finite presentation, see Examples, Section 64.21 for a counter example. First we match our usage of ``universally'' to the usual notion.

Lemma 34.16.3. *Let $f : X \to S$ be a morphism of schemes which is of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $s \in S$. The following are equivalent*

1. *$\mathcal{F}$ is universally pure along X_s, and*
2. *for every morphism of pointed schemes $(S', s') \to (S, s)$ the pullback $\mathcal{F}_{S'}$ is pure along $X_{s'}$.*

In particular, $\mathcal{F}$ is universally pure relative to S if and only if every base change $\mathcal{F}_{S'}$ of $\mathcal{F}$ is pure relative to S'.

Proof. This is formal. $\square$

Lemma 34.16.4. *Let $f : X \to S$ be a morphism of schemes which is of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $s \in S$. Let $(S', s') \to (S, s)$ be a morphism of pointed schemes. If $S' \to S$ is quasi-finite at s' and $\mathcal{F}$ is pure along X_s, then $\mathcal{F}_{S'}$ is pure along $X_{s'}$.*

Proof. It $(T \to S', t' \rightsquigarrow t, \xi)$ is an impurity of $\mathcal{F}_{S'}$ above s' with $T \to S'$ quasi-finite at t, then $(T \to S, t' \to t, \xi)$ is an impurity of $\mathcal{F}$ above s with $T \to S$ quasi-finite at t, see Morphisms, Lemma 24.19.12. Hence the lemma follows immediately from the characterization (2) of purity given following Definition 34.16.1. $\square$

Lemma 34.16.5. *Let $f : X \to S$ be a morphism of schemes which is of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $s \in S$. If $\mathcal{O}_{S,s}$ is Noetherian then $\mathcal{F}$ is pure along X_s if and only if $\mathcal{F}$ is universally pure along X_s.*

Proof. First we may replace S by $Spec(\mathcal{O}_{S,s})$, i.e., we may assume that S is Noetherian. Next, use Lemma 34.15.6 and characterization (2) of purity given in discussion following Definition 34.16.1 to conclude. $\square$

Purity satisfies flat descent.

Lemma 34.16.6. *Let $f : X \to S$ be a morphism of schemes which is of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $s \in S$. Let $(S', s') \to (S, s)$ be a morphism of pointed schemes. Assume $S' \to S$ is flat at s'.*

1. *If $\mathcal{F}_{S'}$ is pure along $X_{s'}$, then $\mathcal{F}$ is pure along X_s.*
2. *If $\mathcal{F}_{S'}$ is universally pure along $X_{s'}$, then $\mathcal{F}$ is universally pure along X_s.*

Proof. Let $(T \to S, t' \rightsquigarrow t, \xi)$ be an impurity of $\mathcal{F}$ above s. Set $T_1 = T \times_S S'$, and let t_1 be the unique point of T_1 mapping to t and s'. Since $T_1 \to T$ is flat at t_1, see Morphisms, Lemma 24.24.7, there exists a specialization $t'_1 \rightsquigarrow t_1$ lying over $t' \rightsquigarrow t$, see Algebra, Section 7.36. Choose a point $\xi_1 \in X_{t'_1}$ which corresponds to a generic point of $Spec(\kappa(t'_1) \otimes_{\kappa(t')} \kappa(\xi))$, see Schemes, Lemma 21.17.5. By Divisors, Lemma 26.7.2 we see that $\xi_1 \in \text{Ass}_{X_{T_1}/T_1}(\mathcal{F}_{T_1})$. As the Zariski closure of $\{\xi_1\}$ in X_{T_1} maps into the Zariski closure of $\{\xi\}$ in X_T we conclude that this closure is disjoint from X_{t_1}. Hence

$(T_1 \to S', t_1' \rightsquigarrow t_1, \xi_1)$ is an impurity of $\mathcal{F}_{S'}$ above s'. In other words we have proved the contrapositive to part (2) of the lemma. Finally, if $(T, t) \to (S, s)$ is an elementary étale neighbourhood, then $(T_1, t_1) \to (S', s')$ is an elementary étale neighbourhood too, and in this way we see that (1) holds. □

Lemma 34.16.7. *Let $i : Z \to X$ be a closed immersion of schemes of finite type over a scheme S. Let $s \in S$. Let $\mathcal{F}$ be a finite type, quasi-coherent sheaf on Z. Then $\mathcal{F}$ is (universally) pure along Z_s if and only if $i_*\mathcal{F}$ is (universally) pure along X_s.*

Proof. Omitted. □

34.17. Examples of relatively pure sheaves

Here are some example cases where it is possible to see what purity means.

Lemma 34.17.1. *Let $f : X \to S$ be a proper morphism of schemes. Then every finite type, quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is universally pure relative to S. In particular X is universally pure relative to S.*

Proof. Let $(g : T \to S, t' \rightsquigarrow t, \xi)$ be an impurity of $\mathcal{F}$ above $s \in S$. Since f is proper, it is universally closed. Hence $f_T : X_T \to T$ is closed. Since $f_T(\xi) = t'$ this implies that $t \in f(\overline{\{\xi\}})$ which is a contradiction. □

Lemma 34.17.2. *Let $f : X \to S$ be a separated, finite type morphism of schemes. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_X$-module. Assume that $\text{Supp}(\mathcal{F}_s)$ is finite for every $s \in S$. Then the following are equivalent*

(1) *$\mathcal{F}$ is pure relative to S,*
(2) *the scheme theoretic support of $\mathcal{F}$ is finite over S, and*
(3) *$\mathcal{F}$ is universally pure relative to S.*

In particular, given a quasi-finite separated morphism $X \to S$ we see that X is pure relative to S if and only if $X \to S$ is finite.

Proof. Let $Z \subset X$ be the scheme theoretic support of $\mathcal{F}$, see Coherent, Definition 25.10.5. Then $Z \to S$ is a separated, finite type morphism of schemes with finite fibres. Hence it is separated and quasi-finite, see Morphisms, Lemma 24.19.10. By Lemma 34.16.7 it suffices to prove the lemma for $Z \to S$ and the sheaf $\mathcal{F}$ viewed as a finite type quasi-coherent module on Z. Hence we may assume that $X \to S$ is separated and quasi-finite and that $\text{Supp}(\mathcal{F}) = X$.

It follows from Lemma 34.17.1 and Morphisms, Lemma 24.42.10 that (2) implies (3). Trivially (3) implies (1). Assume (1) holds. We will prove that (2) holds. It is clear that we may assume S is affine. By More on Morphisms, Lemma 33.29.4 we can find a diagram

$$\begin{tikzcd} X \arrow[rr, "j"] \arrow[dr, "f"'] & & T \arrow[dl, "\pi"] \\ & S & \end{tikzcd}$$

with π finite and j a quasi-compact open immersion. If we show that j is closed, then j is a closed immersion and we conclude that $f = \pi \circ j$ is finite. To show that j is closed it suffices to show that specializations lift along j, see Schemes, Lemma 21.19.8. Let $x \in X$, set $t' = j(x)$ and let $t' \rightsquigarrow t$ be a specialization. We have to show $t \in j(X)$. Set $s' = f(x)$ and

$s = \pi(t)$ so $s' \rightsquigarrow s$. By More on Morphisms, Lemma 33.28.4 we can find an elementary étale neighbourhood $(U, u) \to (S, s)$ and a decomposition

$$T_U = T \times_S U = V \amalg W$$

into open and closed subschemes, such that $V \to U$ is finite and there exists a unique point v of V mapping to u, and such that v maps to t in T. As $V \to T$ is étale, we can lift generalizations, see Morphisms, Lemmas 24.24.8 and 24.35.12. Hence there exists a specialization $v' \rightsquigarrow v$ such that v' maps to $t' \in T$. In particular we see that $v' \in X_U \subset T_U$. Denote $u' \in U$ the image of t'. Note that $v' \in \text{Ass}_{X_U/U}(\mathcal{F})$ because $X_{u'}$ is a finite discrete set and $X_{u'} = \text{Supp}(\mathcal{F}_{u'})$. As $\mathcal{F}$ is pure relative to S we see that v' must specialize to a point in X_u. Since v is the only point of V lying over u (and since no point of W can be a specialization of v') we see that $v \in X_u$. Hence $t \in X$. $\square$

Lemma 34.17.3. *Let $f : X \to S$ be a finite type, flat morphism of schemes with geometrically integral fibres. Then X is universally pure over S.*

Proof. Let $\xi \in X$ with $s' = f(\xi)$ and $s' \rightsquigarrow s$ a specialization of S. If ξ is an associated point of $X_{s'}$, then ξ is the unique generic point because $X_{s'}$ is an integral scheme. Let ξ_0 be the unique generic point of X_s. As $X \to S$ is flat we can lift $s' \rightsquigarrow s$ to a specialization $\xi' \rightsquigarrow \xi_0$ in X, see Morphisms, Lemma 24.24.8. The $\xi \rightsquigarrow \xi'$ because ξ is the generic point of $X_{s'}$ hence $\xi \rightsquigarrow \xi_0$. This means that $(\text{id}_S, s' \to s, \xi)$ is not an impurity of $\mathcal{O}_X$ above s. Since the assumption that f is finite type, flat with geometrically integral fibres is preserved under base change, we see that there doesn't exist an impurity after any base change. In this way we see that X is universally S-pure. $\square$

Lemma 34.17.4. *Let $f : X \to S$ be a finite type, affine morphism of schemes. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module such that $f_*\mathcal{F}$ is locally projective on S, see Properties, Definition 23.19.1. Then $\mathcal{F}$ is universally pure over S.*

Proof. After reducing to the case where S is the spectrum of a henselian local ring this follows from Lemma 34.14.1. $\square$

34.18. A criterion for purity

We first prove that given a flat family of finite type quasi-coherent sheaves the points in the relative assassin specialize to points in the relative assassins of nearby fibres (if they specialize at all).

Lemma 34.18.1. *Let $f : X \to S$ be a morphism of schemes of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $s \in S$. Assume that $\mathcal{F}$ is flat over S at all points of X_s. Let $x' \in \text{Ass}_{X/S}(\mathcal{F})$ with $f(x') = s'$ such that $s' \rightsquigarrow s$ is a specialization in S. If x' specializes to a point of X_s, then $x' \rightsquigarrow x$ with $x \in \text{Ass}_{X_s}(\mathcal{F}_s)$.*

Proof. Let $x' \rightsquigarrow t$ be a specialization with $t \in X_s$. We may replace X by an affine neighbourhood of t and S by an affine neighbourhood of s. Choose a closed immersion $i : X \to \mathbf{A}^n_S$. Then it suffices to prove the lemma for the module $i_*\mathcal{F}$ on $\mathbf{A}^n_S$ and the point $i(x')$. Hence we may assume $X \to S$ is of finite presentation.

Let $x' \rightsquigarrow t$ be a specialization with $t \in X_s$. Set $A = \mathcal{O}_{S,s}$, $B = \mathcal{O}_{X,t}$, and $N = \mathcal{F}_t$. Note that B is essentially of finite presentation over A and that N is a finite B-module flat over A. Also N is a finitely presented B-module by Lemma 34.11.9. Let $\mathfrak{q}' \subset B$ be the prime ideal corresponding to x' and let $\mathfrak{p}' \subset A$ be the prime ideal corresponding to s'. The assumption $x' \in \text{Ass}_{X/S}(\mathcal{F})$ means that $\mathfrak{q}'$ is an associated prime of $N \otimes_A \kappa(\mathfrak{p}')$. Let

$\Sigma \subset B$ be the multiplicative subset of elements which are not zero divisors on $N/\mathfrak{m}_A N$. By Lemma 34.8.2 the map $N \to \Sigma^{-1}N$ is universally injective. In particular, we see that $N \otimes_A \kappa(\mathfrak{p}') \to \Sigma^{-1}N \otimes_A \kappa(\mathfrak{p}')$ is injective which implies that $\mathfrak{q}'$ is an associated prime of $\Sigma^{-1}N \otimes_A \kappa(\mathfrak{p}')$ and hence $\mathfrak{q}'$ is in the image of $Spec(\Sigma^{-1}B) \to Spec(B)$. Thus Lemma 34.8.1 implies that $\mathfrak{q}' \subset \mathfrak{q}$ for some prime $\mathfrak{q} \in \text{Ass}_B(N/\mathfrak{m}_A N)$ (which in particular implies that $\mathfrak{m}_A = A \cap \mathfrak{q}$). If $x \in X_s$ denotes the point corresponding to $\mathfrak{q}$, then $x \in \text{Ass}_{X_s}(\mathcal{F}_s)$ and $x' \rightsquigarrow x$ as desired. □

Lemma 34.18.2. *Let $f : X \to S$ be a morphism of schemes of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $s \in S$. Let $(S', s') \to (S, s)$ be an elementary étale neighbourhood and let*

$$\begin{tikzcd} X \arrow[d] & X' \arrow[l, "g"'] \arrow[d] \\ S & S' \arrow[l] \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume

(1) *$\mathcal{F}$ is flat over S at all points of X_s,*
(2) *$X' \to S'$ is of finite type,*
(3) *$g^*\mathcal{F}$ is pure along $X'_{s'}$,*
(4) *$g : X' \to X$ is étale, and*
(5) *$g(X')$ contains $\text{Ass}_{X_s}(\mathcal{F}_s)$.*

In this situation $\mathcal{F}$ is pure along X_s if and only if the image of $X' \to X \times_S S'$ contains the points $\text{Ass}_{X \times_S S'/S'}(\mathcal{F} \times_S S')$ lying over points in S' which specialize to s'.

Proof. Since the morphism $S' \to S$ is étale, we see that if $\mathcal{F}$ is pure along X_s, then $\mathcal{F} \times_S S'$ is pure along X_s, see Lemma 34.16.4. Since purity satisfies flat descent, see Lemma 34.16.6, we see that if $\mathcal{F} \times_S S'$ is pure along $X_{s'}$, then $\mathcal{F}$ is pure along X_s. Hence we may replace S by S' and assume that $S = S'$ so that $g : X' \to X$ is an étale morphism between schemes of finite type over S. Moreover, we may replace S by $Spec(\mathcal{O}_{S,s})$ and assume that S is local.

First, assume that $\mathcal{F}$ is pure along X_s. In this case every point of $\text{Ass}_{X/S}(\mathcal{F})$ specializes to a point of X_s by purity. Hence by Lemma 34.18.1 we see that every point of $\text{Ass}_{X/S}(\mathcal{F})$ specializes to a point of $\text{Ass}_{X_s}(\mathcal{F}_s)$. Thus every point of $\text{Ass}_{X/S}(\mathcal{F})$ is in the image of g (as the image is open and contains $\text{Ass}_{X_s}(\mathcal{F}_s)$).

Conversely, assume that $g(X')$ contains $\text{Ass}_{X/S}(\mathcal{F})$. Let $S^h = Spec(\mathcal{O}_{S,s}^h)$ be the henselization of S at s. Denote $g^h : (X')^h \to X^h$ the base change of g by $S^h \to S$, and denote $\mathcal{F}^h$ the pullback of $\mathcal{F}$ to X^h. By Divisors, Lemma 26.7.2 and Remark 26.7.3 the relative assassin $\text{Ass}_{X^h/S^h}(\mathcal{F}^h)$ is the inverse image of $\text{Ass}_{X/S}(\mathcal{F})$ via the projection $X^h \to X$. As we have assumed that $g(X')$ contains $\text{Ass}_{X/S}(\mathcal{F})$ we conclude that the base change $g^h((X')^h) = g(X') \times_S S^h$ contains $\text{Ass}_{X^h/S^h}(\mathcal{F}^h)$. In this way we reduce to the case where S is the spectrum of a henselian local ring. Let $x \in \text{Ass}_{X/S}(\mathcal{F})$. To finish the proof of the lemma we have to show that x specializes to a point of X_s, see criterion (4) for purity in discussion following Definition 34.16.1. By assumption there exists a $x' \in X'$ such that $g(x') = x$. As $g : X' \to X$ is étale, we see that $x' \in \text{Ass}_{X'/S}(g^*\mathcal{F})$, see Lemma 34.3.7 (applied to the morphism of fibres $X'_w \to X_w$ where $w \in S$ is the image of x'). Since $g^*\mathcal{F}$ is pure along X'_s we see that $x' \rightsquigarrow y$ for some $y \in X'_s$. Hence $x = g(x') \rightsquigarrow g(y)$ and $g(y) \in X_s$ as desired. □

Lemma 34.18.3. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $s \in S$. Assume*

(1) *f is of finite type,*
(2) *$\mathcal{F}$ is of finite type,*
(3) *$\mathcal{F}$ is flat over S at all points of X_s, and*
(4) *$\mathcal{F}$ is pure along X_s.*

Then $\mathcal{F}$ is universally pure along X_s.

Proof. We first make a preliminary remark. Suppose that $(S', s') \to (S, s)$ is an elementary étale neighbourhood. Denote $\mathcal{F}'$ the pullback of $\mathcal{F}$ to $X' = X \times_S S'$. By the discussion following Definition 34.16.1 we see that $\mathcal{F}'$ is pure along $X'_{s'}$. Moreover, $\mathcal{F}'$ is flat over S' along $X'_{s'}$. Then it suffices to prove that $\mathcal{F}'$ is universally pure along $X'_{s'}$. Namely, given any morphism $(T, t) \to (S, s)$ of pointed schemes the fibre product $(T', t') = (T \times_S S', (t, s'))$ is flat over (T, t) and hence if $\mathcal{F}_{T'}$ is pure along $X_{t'}$ then $\mathcal{F}_T$ is pure along X_t by Lemma 34.16.6. Thus during the proof we may always replace (s, S) by an elementary étale neighbourhood. We may also replace S by $Spec(\mathcal{O}_{S,s})$ due to the local nature of the problem.

Choose an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram

$$\begin{tikzcd} X \arrow[d] & X' \arrow[l, "g"'] \arrow[d] \\ S & Spec(\mathcal{O}_{S',s'}) \arrow[l] \end{tikzcd}$$

such that $X' \to X \times_S Spec(\mathcal{O}_{S',s'})$ is étale, $X_s = g((X')_{s'})$, the scheme X' is affine, and such that $\Gamma(X', g^*\mathcal{F})$ is a free $\mathcal{O}_{S',s'}$-module, see Lemma 34.12.11. Note that $X' \to Spec(\mathcal{O}_{S',s'})$ is of finite type (as a quasi-compact morphism which is the composition of an étale morphism and the base change of a finite type morphism). By our preliminary remarks in the first paragraph of the proof we may replace S by $Spec(\mathcal{O}_{S',s'})$. Hence we may assume there exists a commutative diagram

$$\begin{tikzcd} X \arrow[dr] & & X' \arrow[ll, "g"'] \arrow[dl] \\ & S & \end{tikzcd}$$

of schemes of finite type over S, where g is étale, $X_s \subset g(X')$, with S local with closed point s, with X' affine, and with $\Gamma(X', g^*\mathcal{F})$ a free $\Gamma(S, \mathcal{O}_S)$-module. Note that in this case $g^*\mathcal{F}$ is universally pure over S, see Lemma 34.17.4.

In this situation we apply Lemma 34.18.2 to deduce that $\text{Ass}_{X/S}(\mathcal{F}) \subset g(X')$ from our assumption that $\mathcal{F}$ is pure along X_s and flat over S along X_s. By Divisors, Lemma 26.7.2 and Remark 26.7.3 we see that for any morphism of pointed schemes $(T, t) \to (S, s)$ we have

$$\text{Ass}_{X_T/T}(\mathcal{F}_T) \subset (X_T \to X)^{-1}(\text{Ass}_{X/S}(\mathcal{F})) \subset g(X') \times_S T = g_T(X'_T).$$

Hence by Lemma 34.18.2 applied to the base change of our displayed diagram to (T, t) we conclude that $\mathcal{F}_T$ is pure along X_t as desired. $\square$

Lemma 34.18.4. *Let $f : X \to S$ be a finite type morphism of schemes. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Assume $\mathcal{F}$ is flat over S. In this case $\mathcal{F}$ is pure relative to S if and only if $\mathcal{F}$ is universally pure relative to S.*

Proof. Immediate consequence of Lemma 34.18.3 and the definitions. □

Lemma 34.18.5. *Let I be a directed partially ordered set. Let $(S_i, g_{ii'})$ be an inverse system of affine schemes over I. Set $S = lim_i S_i$ and $s \in S$. Denote $g_i : S \to S_i$ the projections and set $s_i = g_i(s)$. Suppose that $f : X \to S$ is a morphism of finite presentation, $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module of finite presentation which is pure along X_s and flat over S at all points of X_s. Then there exists an $i \in I$, a morphism of finite presentation $X_i \to S_i$, a quasi-coherent $\mathcal{O}_{X_i}$-module $\mathcal{F}_i$ of finite presentation which is pure along $(X_i)_{s_i}$ and flat over S_i at all points of $(X_i)_{s_i}$ such that $X \cong X_i \times_{S_i} S$ and such that the pullback of $\mathcal{F}_i$ to X is isomorphic to $\mathcal{F}$.*

Proof. Let $U \subset X$ be the set of points where $\mathcal{F}$ is flat over S. By More on Morphisms, Theorem 33.11.1 this is an open subscheme of X. By assumption $X_s \subset U$. As X_s is quasi-compact, we can find a quasi-compact open $U' \subset U$ with $X_s \subset U'$. By Limits, Lemma 27.6.1 we can find an $i \in I$ and a morphism of finite presentation $f_i : X_i \to S_i$ whose base change to S is isomorphic to f_i. Fix such a choice and set $X_{i'} = X_i \times_{S_i} S_{i'}$. Then $X = lim_{i'} X_{i'}$ with affine transition morphisms. By Limits, Lemma 27.6.8 we can, after possible increasing i assume there exists a quasi-coherent $\mathcal{O}_{X_i}$-module $\mathcal{F}_i$ of finite presentation whose base change to S is isomorphic to $\mathcal{F}$. By Limits, Lemma 27.3.5 after possibly increasing i we may assume there exists an open $U'_i \subset X_i$ whose inverse image in X is U'. Note that in particular $(X_i)_{s_i} \subset U'_i$. By Limits, Lemma 27.6.9 (after increasing i once more) we may assume that $\mathcal{F}_i$ is flat on U'_i. In particular we see that $\mathcal{F}_i$ is flat along $(X_i)_{s_i}$.

Next, we use Lemma 34.12.5 to choose an elementary étale neighbourhood $(S'_i, s'_i) \to (S_i, s_i)$ and a commutative diagram of schemes

$$\begin{tikzcd} X_i \arrow[d] & X'_i \arrow[l, "g_i"'] \arrow[d] \\ S_i & S'_i \arrow[l] \end{tikzcd}$$

such that g_i is étale, $(X_i)_{s_i} \subset g_i(X'_i)$, the schemes X'_i, S'_i are affine, and such that $\Gamma(X'_i, g_i^*\mathcal{F}_i)$ is a projective $\Gamma(S'_i, \mathcal{O}_{S'_i})$-module. Note that $g_i^*\mathcal{F}_i$ is universally pure over S'_i, see Lemma 34.17.4. We may base change the diagram above to a diagram with morphisms $(S'_{i'}, s'_{i'}) \to (S_{i'}, s_{i'})$ and $g_{i'} : X'_{i'} \to X_{i'}$ over $S_{i'}$ for any $i' \geq i$ and we may base change the diagram to a diagram with morphisms $(S', s') \to (S, s)$ and $g : X' \to X$ over S.

At this point we can use our criterion for purity. Set $W'_i \subset X_i \times_{S_i} S'_i$ equal to the image of the étale morphism $X'_i \to X_i \times_{S_i} S'_i$. For every $i' \geq i$ we have similarly the image $W'_{i'} \subset X_{i'} \times_{S_{i'}} S'_{i'}$ and we have the image $W' \subset X \times_S S'$. Taking images commutes with base change, hence $W'_{i'} = W'_i \times_{S'_i} S'_{i'}$ and $W' = W_i \times_{S'_i} S'$. Because $\mathcal{F}$ is pure along X_s the Lemma 34.18.2 implies that

$$f^{-1}(Spec(\mathcal{O}_{S',s'})) \cap \mathrm{Ass}_{X \times_S S'/S'}(\mathcal{F} \times_S S') \subset W' \tag{34.18.5.1}$$

By More on Morphisms, Lemma 33.18.5 we see that

$$E = \{t \in S' \mid \mathrm{Ass}_{X_t}(\mathcal{F}_t) \subset W'\} \quad \text{and} \quad E_{i'} = \{t \in S'_{i'} \mid \mathrm{Ass}_{X_t}(\mathcal{F}_{i',t}) \subset W'_{i'}\}$$

are locally constructible subsets of S' and $S'_{i'}$. By More on Morphisms, Lemma 33.18.4 we see that $E_{i'}$ is the inverse image of E_i under the morphism $S'_{i'} \to S'_i$ and that E is the inverse image of E_i under the morphism $S' \to S'_i$. Thus Equation (34.18.5.1) is equivalent to the assertion that $Spec(\mathcal{O}_{S',s'})$ maps into E_i. As $\mathcal{O}_{S',s'} = colim_{i' \geq i} \mathcal{O}_{S'_{i'},s'_{i'}}$ we see that $Spec(\mathcal{O}_{S'_{i'},s'_{i'}})$ maps into E_i for some $i' \geq i$, see Limits, Lemma 27.3.4. Then, applying Lemma 34.18.2 to the situation over $S_{i'}$, we conclude that $\mathcal{F}_{i'}$ is pure along $(X_{i'})_{s_{i'}}$. $\square$

Lemma 34.18.6. *Let $f : X \to S$ be a morphism of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite presentation flat over S. Then the set*

$$U = \{s \in S \mid \mathcal{F} \text{ is pure along } X_s\}$$

is open in S.

Proof. Let $s \in U$. Using Lemma 34.12.5 we can find an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram

$$\begin{tikzcd} X \arrow[d] & X' \arrow[l, "g"'] \arrow[d] \\ S & S' \arrow[l] \end{tikzcd}$$

such that g is étale, $X_s \subset g(X')$, the schemes X', S' are affine, and such that $\Gamma(X', g^*\mathcal{F})$ is a projective $\Gamma(S', \mathcal{O}_{S'})$-module. Note that $g^*\mathcal{F}$ is universally pure over S', see Lemma 34.17.4. Set $W' \subset X \times_S S'$ equal to the image of the étale morphism $X' \to X \times_S S'$. Note that W is open and quasi-compact over S'. Set

$$E = \{t \in S' \mid \text{Ass}_{X_t}(\mathcal{F}_t) \subset W'\}.$$

By More on Morphisms, Lemma 33.18.5 E is a constructible subset of S'. By Lemma 34.18.2 we see that $Spec(\mathcal{O}_{S',s'}) \subset E$. By Morphisms, Lemma 24.21.4 we see that E contains an open neighbourhood V' of s'. Applying Lemma 34.18.2 once more we see that for any point s_1 in the image of V' in S the sheaf $\mathcal{F}$ is pure along X_{s_1}. Since $S' \to S$ is étale the image of V' in S is open and we win. $\square$

34.19. How purity is used

Here are some examples of how purity can be used. The first lemma actually uses a slightly weaker form of purity.

Lemma 34.19.1. *Let $f : X \to S$ be a morphism of finite type. Let $\mathcal{F}$ be a quasi-coherent sheaf of finite type on X. Assume S is local with closed point s. Assume $\mathcal{F}$ is pure along X_s and that $\mathcal{F}$ is flat over S. Let $\varphi : \mathcal{F} \to \mathcal{G}$ of quasi-coherent $\mathcal{O}_X$-modules. Then the following are equivalent*

1. *the map on stalks φ_x is injective for all $x \in Ass_{X_s}(\mathcal{F}_s)$, and*
2. *φ is injective.*

Proof. Let $\mathcal{K} = \text{Ker}(\varphi)$. Our goal is to prove that $\mathcal{K} = 0$. In order to do this it suffices to prove that $\text{WeakAss}_X(\mathcal{K}) = \emptyset$, see Divisors, Lemma 26.5.5. We have $\text{WeakAss}_X(\mathcal{K}) \subset \text{WeakAss}_X(\mathcal{F})$, see Divisors, Lemma 26.5.4. As $\mathcal{F}$ is flat we see from Lemma 34.13.4 that $\text{WeakAss}_X(\mathcal{F}) \subset \text{Ass}_{X/S}(\mathcal{F})$. By purity any point x' of $\text{Ass}_{X/S}(\mathcal{F})$ is a generalization of a point of X_s, and hence is the specialization of a point $x \in \text{Ass}_{X_s}(\mathcal{F}_s)$, by Lemma 34.18.1. Hence the injectivity of φ_x implies the injectivity of $\varphi_{x'}$, whence $\mathcal{K}_{x'} = 0$. $\square$

Proposition 34.19.2. *Let $f : X \to S$ be an affine, finitely presented morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite presentation, flat over S. Then the following are equivalent*

(1) *$f_*\mathcal{F}$ is locally projective on S, and*
(2) *$\mathcal{F}$ is pure relative to S.*

In particular, given a ring map $A \to B$ of finite presentation and a finitely presented B-module N flat over A we have: N is projective as an A-module if and only if $\widetilde{N}$ on $Spec(B)$ is pure relative to $Spec(A)$.

Proof. The implication (1) $\Rightarrow$ (2) is Lemma 34.17.4. Assume $\mathcal{F}$ is pure relative to S. Note that by Lemma 34.18.3 this implies $\mathcal{F}$ remains pure after any base change. By Descent, Lemma 31.5.5 it suffices to prove $f_*\mathcal{F}$ is fpqc locally projective on S. Pick $s \in S$. We will prove that the restriction of $f_*\mathcal{F}$ to an étale neighbourhood of s is locally projective. Namely, by Lemma 34.12.5, after replacing S by an affine elementary étale neighbourhood of s, we may assume there exists a diagram

$$\begin{tikzcd} X \arrow[dr] & & X' \arrow[ll, "g"'] \arrow[dl] \\ & S & \end{tikzcd}$$

of schemes affine and of finite presentation over S, where g is étale, $X_s \subset g(X')$, and with $\Gamma(X', g^*\mathcal{F})$ a projective $\Gamma(S, \mathcal{O}_S)$-module. Note that in this case $g^*\mathcal{F}$ is universally pure over S, see Lemma 34.17.4. Hence by Lemma 34.18.2 we see that the open $g(X')$ contains the points of $\text{Ass}_{X/S}(\mathcal{F})$ lying over $Spec(\mathcal{O}_{S,s})$. Set

$$E = \{t \in S \mid \text{Ass}_{X_t}(\mathcal{F}_t) \subset g(X')\}.$$

By More on Morphisms, Lemma 33.18.5 E is a constructible subset of S. We have seen that $Spec(\mathcal{O}_{S,s}) \subset E$. By Morphisms, Lemma 24.21.4 we see that E contains an open neighbourhood of s. Hence after replacing S by an affine neighbourhood of s we may assume that $\text{Ass}_{X/S}(\mathcal{F}) \subset g(X')$. By Lemma 34.8.4 this means that

$$\Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X', g^*\mathcal{F})$$

is $\Gamma(S, \mathcal{O}_S)$-universally injective. By Algebra, Lemma 7.83.6 we conclude that $\Gamma(X, \mathcal{F})$ is Mittag-Leffler as an $\Gamma(S, \mathcal{O}_S)$-module. Since $\Gamma(X, \mathcal{F})$ is countably generated and flat as a $\Gamma(S, \mathcal{O}_S)$-module, we conclude it is projective by Algebra, Lemma 7.87.1. ☐

We can use the proposition to improve some of our earlier results. The following lemma is an improvement of Proposition 34.12.4.

Lemma 34.19.3. *Let $f : X \to S$ be a morphism which is locally of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module which is of finite presentation. Let $x \in X$ with $s = f(x) \in S$. If $\mathcal{F}$ is flat at x over S there exists an affine elementary étale neighbourhood $(S', s') \to (S, s)$ and an affine open $U' \subset X \times_S S'$ which contains $x' = (x, s')$ such that $\Gamma(U', \mathcal{F}|_{U'})$ is a projective $\Gamma(S', \mathcal{O}_{S'})$-module.*

Proof. During the proof we may replace X by an open neighbourhood of x and we may replace S by an elementary étale neighbourhood of s. Hence, by openness of flatness (see More on Morphisms, Theorem 33.11.1) we may assume that $\mathcal{F}$ is flat over S. We may assume S and X are affine. After shrinking X some more we may assume that any point of $\text{Ass}_{X_s}(\mathcal{F}_s)$ is a generalization of x. This property is preserved on replacing (S, s) by an elementary étale neighbourhood. Hence we may apply Lemma 34.12.5 to arrive at the situation where there exists a diagram

$$\begin{tikzcd} X & & X' \arrow[ll, "g"'] \arrow[dl] \\ & S \arrow[from=ul, crossing over] & \end{tikzcd}$$

of schemes affine and of finite presentation over S, where g is étale, $X_s \subset g(X')$, and with $\Gamma(X', g^*\mathcal{F})$ a projective $\Gamma(S, \mathcal{O}_S)$-module. Note that in this case $g^*\mathcal{F}$ is universally pure over S, see Lemma 34.17.4.

Let $U \subset g(X')$ be an affine open neighbourhood of x. We claim that $\mathcal{F}|_U$ is pure along U_s. If we prove this, then the lemma follows because $\mathcal{F}|_U$ will be pure relative to S after shrinking S, see Lemma 34.18.6, whereupon the projectivity follows from Proposition 34.19.2. To prove the claim we have to show, after replacing (S, s) by an arbitrary elementary étale neighbourhood, that any point ξ of $\text{Ass}_{U/S}(\mathcal{F}|_U)$ lying over some $s' \in S$, $s' \rightsquigarrow s$ specializes to a point of U_s. Since $U \subset g(X')$ we can find a $\xi' \in X'$ with $g(\xi') = \xi$. Because $g^*\mathcal{F}$ is pure over S, using Lemma 34.18.1, we see there exists a specialization $\xi' \rightsquigarrow x'$ with $x' \in \text{Ass}_{X'_s}(g^*\mathcal{F}_s)$. Then $g(x') \in \text{Ass}_{X_s}(\mathcal{F}_s)$ (see for example Lemma 34.3.7 applied to the étale morphism $X'_s \to X_s$ of Noetherian schemes) and hence $g(x') \rightsquigarrow x$ by our choice of X above! Since $x \in U$ we conclude that $g(x') \in U$. Thus $\xi = g(\xi') \rightsquigarrow g(x') \in U_s$ as desired. □

The following lemma is an improvement of Lemma 34.12.9.

Lemma 34.19.4. *Let $f : X \to S$ be a morphism which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module which is of finite type. Let $x \in X$ with $s = f(x) \in S$. If $\mathcal{F}$ is flat at x over S there exists an affine elementary étale neighbourhood $(S', s') \to (S, s)$ and an affine open $U' \subset X \times_S Spec(\mathcal{O}_{S',s'})$ which contains $x' = (x, s')$ such that $\Gamma(U', \mathcal{F}|_{U'})$ is a free $\mathcal{O}_{S',s'}$-module.*

Proof. The question is Zariski local on X and S. Hence we may assume that X and S are affine. Then we can find a closed immersion $i : X \to \mathbf{A}^n_S$ over S. It is clear that it suffices to prove the lemma for the sheaf $i_*\mathcal{F}$ on $\mathbf{A}^n_S$ and the point $i(x)$. In this way we reduce to the case where $X \to S$ is of finite presentation. After replacing S by $Spec(\mathcal{O}_{S',s'})$ and X by an open of $X \times_S Spec(\mathcal{O}_{S',s'})$ we may assume that $\mathcal{F}$ is of finite presentation, see Proposition 34.11.3. In this case we may appeal to Lemma 34.19.3 and Algebra, Theorem 7.79.4 to conclude. □

Lemma 34.19.5. *Let $A \to B$ be a local ring map of local rings which is essentially of finite type. Let N be a finite B-module which is flat as an A-module. If A is henselian, then N is a filtered colimit*

$$N = colim_i\, F_i$$

of free A-modules F_i such that all transition maps $u_i : F_i \to F_{i'}$ of the system induce injective maps $\overline{u}_i : F_i/\mathfrak{m}_A F_i \to F_{i'}/\mathfrak{m}_A F_{i'}$.

Proof. We can find a morphism of finite type $X \to S = Spec(A)$ and a point $x \in X$ lying over the closed point s of S and a finite type quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ such that $\mathcal{F}_x \cong N$ as an A-module. After shrinking X we may assume that each point of $\text{Ass}_{X_s}(\mathcal{F}_s)$ specializes to x. By Lemma 34.19.4 we see that there exists a fundamental system of affine open neighbourds $U_i \subset X$ of x such that $\Gamma(U_i, \mathcal{F})$ is a free A-module F_i. Note that if $U_{i'} \subset U_i$, then

$$F_i/\mathfrak{m}_A F_i = \Gamma(U_{i,s}, \mathcal{F}_s) \longrightarrow \Gamma(U_{i',s}, \mathcal{F}_s) = F_{i'}/\mathfrak{m}_A F_{i'}$$

is injective because a section of the kernel would be supported at a closed subset of X_s not meeting x which is a contradiction to our choice of X above. □

34.20. Flattening functors

Let S be a scheme. Recall that a functor $F : (Sch/S)^{opp} \to Sets$ is called limit preserving if for every directed inverse system $\{T_i\}_{i\in I}$ of affine schemes with limit T we have $F(T) = colim_i\, F(T_i)$.

Situation 34.20.1. Let $f : X \to S$ be a morphism of schemes. Let $u : \mathcal{F} \to \mathcal{G}$ be a homomorphismm of quasi-coherent $\mathcal{O}_X$-modules. For any scheme T over S we will denote $u_T : \mathcal{F}_T \to \mathcal{G}_T$ the base change of u to T, in other words, u_T is the pullback of u via the projection morphism $X_T = X \times_S T \to X$. In this situation we can consider the functor

$$(34.20.1.1) \quad F_{iso} : (Sch/S)^{opp} \longrightarrow Sets, \quad T \longrightarrow \begin{cases} \{*\} & \text{if} \quad u_T \text{ is an isomorphism,} \\ \emptyset & \text{else.} \end{cases}$$

There are variants F_{inj}, F_{surj} where we ask that u_T is injective, resp. surjective.

Lemma 34.20.2. *In Situation 34.20.1.*

(1) *Each of the functors F_{iso}, F_{inj}, F_{surj} satisfies the sheaf property for the fpqc topology.*
(2) *If f is quasi-compact and $\mathcal{G}$ is of finite type, then F_{surj} is limit preserving.*
(3) *If f is quasi-compact, $\mathcal{F}$ is of finite type, and $\mathcal{G}$ is of finite presentation, then F_{iso} is limit preserving.*

Proof. Let $\{T_i \to T\}_{i\in I}$ be an fpqc covering of schemes over S. Set $X_i = X_{T_i} = X \times_S T_i$ and $u_i = u_{T_i}$. Note that $\{X_i \to X_T\}_{i\in I}$ is an fpqc covering of X_T, see Topologies, Lemma 30.8.7. In particular, for every $x \in X_T$ there exists an $i \in I$ and an $x_i \in X_i$ mapping to x. Since $\mathcal{O}_{X_T,x} \to \mathcal{O}_{X_i,x_i}$ is flat, hence faithfully flat (see Algebra, Lemma 7.35.16) we conclude that $(u_i)_{x_i}$ is injective, surjective, or bijective if and only if $(u_T)_x$ is injective, surjective, or bijective. Whence part (1) of the lemma.

Assume f quasi-compact and $\mathcal{G}$ of finite type. Let $T = lim_{i\in I}\, T_i$ be a directed limit of affine S-schemes and assume that u_T is surjective. Set $X_i = X_{T_i} = X \times_S T_i$ and $u_i = u_{T_i} : \mathcal{F}_i = \mathcal{F}_{T_i} \to \mathcal{G}_i = \mathcal{G}_{T_i}$. To prove part (2) we have to show that u_i is surjective for some i. Pick $i_0 \in I$ and replace I by $\{i \mid i \geq i_0\}$. Since f is quasi-compact each the scheme X_{i_0} is quasi-compact. Hence we may choose affine opens $W_1, \ldots, W_m \subset X$ and an affine open covering $X_{i_0} = U_{1,i_0} \cup \ldots \cup U_{m,i_0}$ such that U_{j,i_0} maps into W_j under the projection morphism $X_{i_0} \to X$. For any $i \in I$ let $U_{j,i}$ be the inverse image of U_{j,i_0}. Setting $U_j = lim_i\, U_{j,i}$ we see that $X_T = U_1 \cup \ldots \cup U_m$ is an affine open covering of X_T. Now it suffices to show, for a given $j \in \{1, \ldots, m\}$ that $u_i|_{U_{j,i}}$ is surjective for some $i = i(j) \in I$. Using Properties, Lemma 23.16.1 this translates into the following algebra problem: Let A be a ring and let $u : M \to N$ be an A-module map. Suppose that $R = colim_{i\in I}\, R_i$ is a directed colimit of A-algebras. If N is a finite A-module and if $u \otimes 1 : M \otimes_A R \to N \otimes_A R$ is surjective, then for some i the map $u \otimes 1 : M \otimes_A R_i \to N \otimes_A R_i$ is surjective. This is Algebra, Lemma 7.118.3 part (2).

Assume f quasi-compact and $\mathcal{F}, \mathcal{G}$ of finite presentation. Arguing in exactly the same manner as in the previous paragraph (using in addition also Properties, Lemma 23.16.2) part (3) translates into the following algebra statement: Let A be a ring and let $u : M \to N$ be an A-module map. Suppose that $R = colim_{i\in I} R_i$ is a directed colimit of A-algebras. Assume M is a finite A-module, N is a finitely presented A-module, and $u \otimes 1 : M \otimes_A R \to N \otimes_A R$ is an isomorphism. Then for some i the map $u \otimes 1 : M \otimes_A R_i \to N \otimes_A R_i$ is an isomorphism. This is Algebra, Lemma 7.118.3 part (3). □

Situation 34.20.3. Let $(A, \mathfrak{m}_A)$ be a local ring. Denote $\mathcal{C}$ the category whose objects are A-algebras A' which are local rings such that the algebra structure $A \to A'$ is a local homomorphism of local rings. A morphism between objects A', A'' of $\mathcal{C}$ is a local homomorphism $A' \to A''$ of A-algebras. Let $A \to B$ be a local ring map of local rings and let M be a B-module. If A' is an object of $\mathcal{C}$ we set $B' = B \otimes_A A'$ and we set $M' = M \otimes_A A'$ as a B'-module. Given $A' \in Ob(\mathcal{C})$, consider the condition

$$\forall \mathfrak{q} \in V(\mathfrak{m}_{A'} B' + \mathfrak{m}_B B') \subset Spec(B') : M'_{\mathfrak{q}} \text{ is flat over } A'. \tag{34.20.3.1}$$

Note the similarity with More on Algebra, Equation (12.14.1.1). In particular, if $A' \to A''$ is a morphism of $\mathcal{C}$ and (34.20.3.1) holds for A', then it holds for A'', see More on Algebra, Lemma 12.14.2. Hence we obtain a functor

$$F_{lf} : \mathcal{C} \longrightarrow Sets, \quad A' \longrightarrow \begin{cases} \{*\} & \text{if (34.20.3.1) holds,} \\ \emptyset & \text{else.} \end{cases} \tag{34.20.3.2}$$

Lemma 34.20.4. *In Situation 34.20.3.*

1. *If $A' \to A''$ is a flat morphism in $\mathcal{C}$ then $F_{fl}(A') = F_{lf}(A'')$.*
2. *If $A \to B$ is essentially of finite presentation and M is a B-module of finite presentation, then F_{fl} is limit preserving: If $\{A_i\}_{i\in I}$ is a directed system of objects of $\mathcal{C}$, then $F_{fl}(colim_i A_i) = colim_i F_{fl}(A_i)$.*

Proof. Part (1) is a special case of More on Algebra, Lemma 12.14.3. Part (2) is a special case of More on Algebra, Lemma 12.14.4. □

Lemma 34.20.5. *In Situation 34.20.3 suppose that $B \to C$ is a local map of local A-algebras and that $M \cong N$ as B-modules. Denote $F'_{lf} : \mathcal{C} \to Sets$ the functor associated to the pair (C, N). If $B \to C$ is finite, then $F_{lf} = F'_{lf}$.*

Proof. Let A' be an object of $\mathcal{C}$. Set $C' = C \otimes_A A'$ and $N' = N \otimes_A A'$ similarly to the definitions of B', M' in Situation 34.20.3. Note that $M' \cong N'$ as B'-modules. The assumption that $B \to C$ is finite has two consequences: (a) $\mathfrak{m}_C = \sqrt{\mathfrak{m}_B C}$ and (b) $B' \to C'$ is finite. Consequence (a) implies that

$$V(\mathfrak{m}_{A'} C' + \mathfrak{m}_C C') = \left(Spec(C') \to Spec(B')\right)^{-1} V(\mathfrak{m}_{A'} B' + \mathfrak{m}_B B').$$

Suppose $\mathfrak{q} \subset V(\mathfrak{m}_{A'} B' + \mathfrak{m}_B B')$. Then $M'_{\mathfrak{q}}$ is flat over A' if and only if the $C'_{\mathfrak{q}}$-module $N'_{\mathfrak{q}}$ is flat over A' (because these are isomorphic as A'-modules) if and only if for every maximal ideal $\mathfrak{r}$ of $C'_{\mathfrak{q}}$ the module $N'_{\mathfrak{r}}$ is flat over A' (see Algebra, Lemma 7.35.19). As $B'_{\mathfrak{q}} \to C'_{\mathfrak{q}}$ is finite by (b), the maximal ideals of $C'_{\mathfrak{q}}$ correspond exactly to the primes of C' lying over $\mathfrak{q}$ (see Algebra, Lemma 7.32.20) and these primes are all contained in $V(\mathfrak{m}_{A'} C' + \mathfrak{m}_C C')$ by the displayed equation above. Thus the result of the lemma holds. □

Lemma 34.20.6. *In Situation 34.20.3 suppose that $B \to C$ is a flat local homomorphism of local rings. Set $N = M \otimes_B C$. Denote $F'_{lf} : \mathcal{C} \to Sets$ the functor associated to the pair (C, N). Then $F_{lf} = F'_{lf}$.*

Proof. Let A' be an object of $\mathcal{C}$. Set $C' = C \otimes_A A'$ and $N' = N \otimes_A A' = M' \otimes_{B'} C'$ similarly to the defintions of B', M' in Situation 34.20.3. Note that

$$V(\mathfrak{m}_{A'}B' + \mathfrak{m}_B B') = Spec(\kappa(\mathfrak{m}_B) \otimes_A \kappa(\mathfrak{m}_{A'}))$$

and similarly for $V(\mathfrak{m}_{A'}C' + \mathfrak{m}_C C')$. The ring map

$$\kappa(\mathfrak{m}_B) \otimes_A \kappa(\mathfrak{m}_{A'}) \longrightarrow \kappa(\mathfrak{m}_C) \otimes_A \kappa(\mathfrak{m}_{A'})$$

is faithfully flat, hence $V(\mathfrak{m}_{A'}C' + \mathfrak{m}_C C') \to V(\mathfrak{m}_{A'}B' + \mathfrak{m}_B B')$ is surjective. Finally, if $\mathfrak{r} \in V(\mathfrak{m}_{A'}C' + \mathfrak{m}_C C')$ maps to $\mathfrak{q} \in V(\mathfrak{m}_{A'}B' + \mathfrak{m}_B B')$, then $M'_\mathfrak{q}$ is flat over A' if and only if $N'_\mathfrak{r}$ is flat over A' because $B' \to C'$ is flat, see Algebra, Lemma 7.35.8. The lemma follows formally from these remarks. □

Situation 34.20.7. Let $f : X \to S$ be a smooth morphism with geometrically irreducible fibres. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. For any scheme T over S we will denote $\mathcal{F}_T$ the base change of $\mathcal{F}$ to T, in other words, $\mathcal{F}_T$ is the pullback of $\mathcal{F}$ via the projection morphism $X_T = X \times_S T \to X$. Note that $X_T \to T$ is smooth with geometrically irreducible fibres, see Morphisms, Lemma 24.33.5 and More on Morphisms, Lemma 33.20.2. Let $p \geq 0$ be an integer. Given a point $t \in T$ consider the condition

(34.20.7.1) $\qquad \mathcal{F}_T$ is free of rank p in a neighbourhood of ξ_t

where ξ_t is the generic point of the fibre X_t. This condition for all $t \in T$ is stable under base change, and hence we obtain a functor

(34.20.7.2)

$$H_p : (Sch/S)^{opp} \longrightarrow Sets, \quad T \longrightarrow \begin{cases} \{*\} & \text{if } \mathcal{F}_T \text{ satisfies (34.20.7.1) } \forall t \in T, \\ \emptyset & \text{else.} \end{cases}$$

Lemma 34.20.8. *In Situation 34.20.7.*

(1) *The functor H_p satisfies the sheaf property for the fpqc topology.*
(2) *If $\mathcal{F}$ is of finite presentation, then functor H_p is limit preserving.*

Proof. Let $\{T_i \to T\}_{i \in I}$ be an fpqc[1] covering of schemes over S. Set $X_i = X_{T_i} = X \times_S T_i$ and denote $\mathcal{F}_i$ the pullback of $\mathcal{F}$ to X_i. Assume that $\mathcal{F}_i$ satisfies (34.20.7.1) for all i. Pick $t \in T$ and let $\xi_t \in X_T$ denote the generic point of X_t. We have to show that $\mathcal{F}$ is free in a neighbourhood of ξ_t. For some $i \in I$ we can find a $t_i \in T_i$ mapping to t. Let $\xi_i \in X_i$ denote the generic point of X_{t_i}, so that ξ_i maps to ξ_t. The fact that $\mathcal{F}_i$ is free of rank p in a neighbourhood of ξ_i implies that $(\mathcal{F}_i)_{x_i} \cong \mathcal{O}_{X_i,x_i}^{\oplus p}$ which implies that $\mathcal{F}_{T,\xi_t} \cong \mathcal{O}_{X_T,\xi_t}^{\oplus p}$ as $\mathcal{O}_{X_T,\xi_t} \to \mathcal{O}_{X_i,x_i}$ is flat, see for example Algebra, Lemma 7.72.5. Thus there exists an affine neighbourhood U of ξ_t in X_T and a surjection $\mathcal{O}_U^{\oplus p} \to \mathcal{F}_U = \mathcal{F}_T|_U$, see Modules, Lemma 15.9.4. After shrinking T we may assume that $U \to T$ is surjective. Hence $U \to T$ is a smooth morphism of affines with geometrically irreducible fibres. Moreover, for every $t' \in T$ we see that the induced map

$$\alpha : \mathcal{O}_{U,\xi_{t'}}^{\oplus p} \longrightarrow \mathcal{F}_{U,\xi_{t'}}$$

is an isomorphism (since by the same argument as before the module on the right is free of rank p). It follows from Lemma 34.11.1 that

$$\Gamma(U, \mathcal{O}_U^{\oplus p}) \otimes_{\Gamma(T,\mathcal{O}_T)} \mathcal{O}_{T,t'} \longrightarrow \Gamma(U, \mathcal{F}_U) \otimes_{\Gamma(T,\mathcal{O}_T)} \mathcal{O}_{T,t'}$$

[1] It is quite easy to show that H_p is a sheaf for the fppf topology using that flat morphisms of finite presentation are open. This is all we really need later on. But it is kind of fun to prove directly that it also satisfies the sheaf condition for the fpqc topology.

is injective for every $t' \in T$. Hence we see the surjection α is an isomorphism. This finishes the proof of (1).

Assume that $\mathcal{F}$ is of finite presentation. Let $T = lim_{i \in I} T_i$ be a directed limit of affine S-schemes and assume that $\mathcal{F}_T$ satisfies (34.20.7.1). Set $X_i = X_{T_i} = X \times_S T_i$ and denote $\mathcal{F}_i$ the pullback of $\mathcal{F}$ to X_i. Let $U \subset X_T$ denote the open subscheme of points where $\mathcal{F}_T$ is flat over T, see More on Morphisms, Theorem 33.11.1. By assumption every generic point of every fibre is a point of U, i.e., $U \to T$ is a smooth surjective morphism with geometrically irreducible fibres. We may shrink U a bit and assume that U is quasi-compact. Using Limits, Lemma 27.3.5 we can find an $i \in I$ and a quasi-compact open $U_i \subset X_i$ whose inverse image in X_T is U. After increasing i we may assume that $\mathcal{F}_i|_{U_i}$ is flat over T_i, see Limits, Lemma 27.6.9. In particular, $\mathcal{F}_i|_{U_i}$ is finite locally free hence defines a locally constant rank function $\rho : U_i \to \{0, 1, 2, \ldots\}$. Let $(U_i)_p \subset U_i$ denote the open and closed subset where ρ has value p. Let $V_i \subset T_i$ be the image of $(U_i)_p$; note that V_i is open and quasi-compact. By assumption the image of $T \to T_i$ is contained in V_i. Hence there exists an $i' \geq i$ such that $T_{i'} \to T_i$ factors through V_i by Limits, Lemma 27.3.5. Then $\mathcal{F}_{i'}$ satisfies (34.20.7.1) as desired. Some details omitted. □

Situation 34.20.9. Let $f : X \to S$ be a morphism of schemes which is of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. For any scheme T over S we will denote $\mathcal{F}_T$ the base change of $\mathcal{F}$ to T, in other words, $\mathcal{F}_T$ is the pullback of $\mathcal{F}$ via the projection morphism $X_T = X \times_S T \to X$. Note that $X_T \to T$ is of finite type and that $\mathcal{F}_T$ is an $\mathcal{O}_{X_T}$-module of finite type, see Morphisms, Lemma 24.14.4 and Modules, Lemma 15.9.2. Let $n \geq 0$. We say that *$\mathcal{F}_T$ is flat over T in dimensions $\geq n$* if for every $t \in T$ the closed subset $Z \subset X_t$ of points where $\mathcal{F}_T$ is not flat over T (see Lemma 34.11.4) satisfies $\dim(Z) < n$ for all $t \in T$. Note that if this is the case, and if $T' \to T$ is a morphism, then $\mathcal{F}_{T'}$ is also flat in dimensions $\geq n$ over T', see Morphisms, Lemmas 24.24.6 and 24.27.3. Hence we obtain a functor

$$(34.20.9.1) \quad F_n : (Sch/S)^{opp} \longrightarrow Sets, \quad T \longrightarrow \begin{cases} \{*\} & \text{if } \mathcal{F}_T \text{ is flat over } T \text{ in } \dim \geq n, \\ \emptyset & \text{else.} \end{cases}$$

Lemma 34.20.10. *In Situation 34.20.9.*

(1) *The functor F_n satisfies the sheaf property for the fpqc topology.*
(2) *If f is quasi-compact and locally of finite presentation and $\mathcal{F}$ is of finite presentation, then the functor F_n is limit preserving.*

Proof. Let $\{T_i \to T\}_{i \in I}$ be an fpqc covering of schemes over S. Set $X_i = X_{T_i} = X \times_S T_i$ and denote $\mathcal{F}_i$ the pullback of $\mathcal{F}$ to X_i. Assume that $\mathcal{F}_i$ is flat over T_i in dimensions $\geq n$ for all i. Let $t \in T$. Choose an index i and a point $t_i \in T_i$ mapping to t. Consider the cartesian diagram

$$\begin{tikzcd} X_{Spec(\mathcal{O}_{T,t})} \arrow[d] & X_{Spec(\mathcal{O}_{T_i,t_i})} \arrow[l] \arrow[d] \\ Spec(\mathcal{O}_{T,t}) & Spec(\mathcal{O}_{T_i,t_i}) \arrow[l] \end{tikzcd}$$

As the lower horizontal morphism is flat we see from More on Morphisms, Lemma 33.11.2 that the set $Z_i \subset X_{t_i}$ where $\mathcal{F}_i$ is not flat over T_i and the set $Z \subset X_t$ where $\mathcal{F}_T$ is not flat over T are related by the rule $Z_i = Z_{\kappa(t_i)}$. Hence we see that $\mathcal{F}_T$ is flat over T in dimensions $\geq n$ by Morphisms, Lemma 24.27.3.

Assume that f is quasi-compact and locally of finite presentation and that $\mathcal{F}$ is of finite presentation. In this paragraph we first reduce the proof of (2) to the case where f is of finite presentation. Let $T = lim_{i\in I} T_i$ be a directed limit of affine S-schemes and assume that $\mathcal{F}_T$ is flat in dimensions $\geq n$. Set $X_i = X_{T_i} = X \times_S T_i$ and denote $\mathcal{F}_i$ the pullback of $\mathcal{F}$ to X_i. We have to show that $\mathcal{F}_i$ is flat in dimensions $\geq n$ for some i. Pick $i_0 \in I$ and replace I by $\{i \mid i \geq i_0\}$. Since T_{i_0} is affine (hence quasi-compact) there exist finitely many affine opens $W_j \subset S$, $j = 1, \ldots, m$ and an affine open overing $T_{i_0} = \bigcup_{j=1,\ldots,m} V_{j,i_0}$ such that $T_{i_0} \to S$ maps V_{j,i_0} into W_j. For $i \geq i_0$ denote $V_{j,i}$ the inverse image of V_{j,i_0} in T_i. If we can show, for each j, that there exists an i such that $\mathcal{F}_{V_{j,i_0}}$ is flat in dimensions $\geq n$, then we win. In this way we reduce to the case that S is affine. In this case X is quasi-compact and we can choose a finite affine open covering $X = W_1 \cup \ldots \cup W_m$. In this case the result for $(X \to S, \mathcal{F})$ is equivalent to the result for $(\coprod W_j, \coprod \mathcal{F}|_{W_j})$. Hence we may assume that f is of finite presentation.

Assume f is of finite presentation and $\mathcal{F}$ is of finite presentation. Let $U \subset X_T$ denote the open subscheme of points where $\mathcal{F}_T$ is flat over T, see More on Morphisms, Theorem 33.11.1. By assumption the dimension of every fibre of $Z = X_T \setminus U$ over T has dimension $\leq n$. By Limits, Lemma 27.11.2 we can find a closed subscheme $Z \subset Z' \subset X_T$ such that $\dim(Z'_t) < n$ for all $t \in T$ and such that $Z' \to X_T$ is of finite presentation. By Limits, Lemmas 27.6.1 and 27.6.5 there exists an $i \in I$ and a closed subscheme $Z'_i \subset X_i$ of finite presentation whose base change to T is Z'. By Limits, Lemma 27.11.1 we may assume all fibres of $Z'_i \to T_i$ have dimension $< n$. By Limits, Lemma 27.6.9 we may assume that $\mathcal{F}_i|_{X_i \setminus T'_i}$ is flat over T_i. This implies that $\mathcal{F}_i$ is flat in dimensions $\geq n$; here we use that $Z' \to X_T$ is of finite presentation, and hence the complement $X_T \setminus Z'$ is quasi-compact! Thus part (2) is proved and the proof of the lemma is complete. □

Situation 34.20.11. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. For any scheme T over S we will denote $\mathcal{F}_T$ the base change of $\mathcal{F}$ to T, in other words, $\mathcal{F}_T$ is the pullback of $\mathcal{F}$ via the projection morphism $X_T = X \times_S T \to X$. Since the base change of a flat module is flat we obtain a functor

$$F_{flat} : (Sch/S)^{opp} \longrightarrow Sets, \quad T \longrightarrow \begin{cases} \{*\} & \text{if } \mathcal{F}_T \text{ is flat over } T, \\ \emptyset & \text{else.} \end{cases} \tag{34.20.11.1}$$

Lemma 34.20.12. *In Situation 34.20.11.*

(1) *The functor F_{flat} satisfies the sheaf property for the fpqc topology.*
(2) *If f is quasi-compact and locally of finite presentation and $\mathcal{F}$ is of finite presentation, then the functor F_{flat} is limit preserving.*

Proof. Part (1) follows from the following statement: If $T' \to T$ is a surjective flat morphism of schemes over S, then $\mathcal{F}_{T'}$ is flat over T' if and only if $\mathcal{F}_T$ is flat over T, see More on Morphisms, Lemma 33.11.2. Part (2) follows from Limits, Lemma 27.6.9 after reducing to the case where X and S are affine (compare with the proof of Lemma 34.20.10). □

34.21. Flattening stratifications

Just the definitions and an important baby case.

Definition 34.21.1. Let $X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. We say that the *universal flattening of $\mathcal{F}$ exists* if the functor F_{flat} defined in Situation 34.20.11 is representable by a scheme S' over S. We say that the *universal flattening of X exists* if the universal flattening of $\mathcal{O}_X$ exists.

Note that if the universal flattening S'[2] of $\mathcal{F}$ exists, then the morphism $S' \to S$ is a monomorphism of schemes such that $\mathcal{F}_{S'}$ is flat over S' and such that a morphism $T \to S$ factors through S' if and only if $\mathcal{F}_T$ is flat over T.

A *stratification $\{S_i\}_{i \in I}$ of S by locally closed subschemes* is given by locally closed subschemes $S_i \subset S$ such that every point of S is contained in a unique S_i. In this case we obtain a monomorphism

$$S' = \coprod\nolimits_{i \in I} S_i \longrightarrow S.$$

We will call this the *monomorphism associated to the stratification*. With this terminology we can define what it means to have a flattening stratification.

Definition 34.21.2. Let $X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. We say that $\mathcal{F}$ has a *flattening stratification* if the functor F_{flat} defined in Situation 34.20.11 is representable by a monomorphism $S' \to S$ associated to a stratification of S by locally closed subschemes. We say that X has a *flattening stratification* if $\mathcal{O}_X$ has a flattening stratification.

Of course in this situation it is important to understand the index set for the strata in the stratification. This often has to do with ranks of modules, as in the baby case below.

Lemma 34.21.3. *Let S be a scheme. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_S$-module. Let $r \geq 0$. The functor*

$$F_r : (Sch/S)^{opp} \to Sets, \quad T \mapsto \begin{cases} \{*\} & \text{if} \quad \mathcal{F}_T \text{ locally free rank } r \\ \emptyset & \text{else.} \end{cases}$$

is representable by a monomorphism $S_r \to S$ of schemes. If $\mathcal{F}$ is of finite presentation, then $S_r \to S$ is of finite presentation.

Proof. We refer to the chapter on exercises for more information on *fitting ideals*. Let

$$\mathcal{I}_0 \subset \mathcal{I}_1 \subset \mathcal{I}_2 \subset \ldots \subset \mathcal{O}_S$$

be the fitting ideals of $\mathcal{F}$ as an $\mathcal{O}_S$-module. If $U \subset X$ is open, and

$$\bigoplus\nolimits_{i \in I} \mathcal{O}_U \to \mathcal{O}_U^{\oplus n} \to \mathcal{F}|_U \to 0$$

is a presentation of $\mathcal{F}$ over U, then $\mathcal{I}_k|_U$ is generated by the $(n-k) \times (n-k)$-minors of the matrix defining the first arrow of the presentation. In particular, $\mathcal{I}_k$ is locally generated by sections, whence quasi-coherent. For any morphism $g : T \to S$ we see that $\mathcal{F}_T$ is locally free of rank r if and only if $\mathcal{I}_r \cdot \mathcal{O}_T = \mathcal{O}_T$ and $\mathcal{I}_{r-1} \cdot \mathcal{O}_T = 0$. Hence, letting $Z_k \subset S$ denote the closed subscheme defined by $\mathcal{I}_k$ we see that $S_r = Z_r \setminus Z_{r-1}$ works. If $\mathcal{F}$ is of finite presentation, then each of the morphisms $Z_k \to X$ is of finite presentation as $\mathcal{I}_k$ is locally generated by finitely many minors. This implies that $Z_k \setminus Z_{r-1}$ is a retrocompact open in Z_k and hence the morphism $S_r \to Z_r$ is of finite presentation as well. $\square$

Lemma 34.21.4. *Let S be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_S$-module of finite presentation. There exists a flattening stratification $S' = \coprod_{r \geq 0} S_r$ for $\mathcal{F}$ (relative to $id_S : S \to S$) such that $\mathcal{F}|_{S_r}$ is locally free of rank r. Moreover, each $S_r \to S$ is of finite presentation.*

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[2]The scheme S' is sometimes called the *universal flatificator*. In [**GR71**] it is called the *platificateur universel*. Existence of the universal flattening should not be confused with the type of results discussed in More on Algebra, Section 12.19.

Proof. Suppose that $g : T \to S$ is a morphism of schemes such that the pullback $\mathcal{F}_T = g^*\mathcal{F}$ is flat. Then $\mathcal{F}_T$ is a flat $\mathcal{O}_T$-module of finite presentation. Hence $\mathcal{F}_T$ is finite locally free, see Properties, Lemma 23.18.2. Thus $T = \coprod_{r \geq 0} T_r$, where $\mathcal{F}_T|_{T_r}$ is locally free of rank r. This implies that

$$F_{flat} = \coprod\nolimits_{r \geq 0} F_r$$

in the category of Zariski sheaves on Sch/S. Hence it follows that F_{flat} is represented by $\coprod_{r \geq 0} S_r$ where S_r is as in Lemma 34.21.3. □

34.22. Flattening stratification over an Artinian ring

A flatting stratification exists when the base scheme is the spectrum of an Artinian ring.

Lemma 34.22.1. *Let S be the spectrum of an Artinian ring. For any scheme X over S, and any quasi-coherent $\mathcal{O}_X$-module there exists a universal flattening. In fact the universal flattening is given by a closed immersion $S' \to S$, and hence is a flattening stratification for $\mathcal{F}$ as well.*

Proof. Choose an affine open covering $X = \bigcup U_i$. Then F_{flat} is the product of the functors associated to each of the pairs $(U_i, \mathcal{F}|_{U_i})$. Hence it suffices to prove the result for each $(U_i, \mathcal{F}|_{U_i})$. In the affine case the lemma follows immediately from More on Algebra, Lemma 12.12.2. □

34.23. Flattening a map

Theorem 34.23.3 is the key to further flattening statements.

Lemma 34.23.1. *Let S be a scheme. Let $g : X' \to X$ be a flat morphism of schemes over S with X locally of finite type over S. Let $\mathcal{F}$ be a finite type $\mathcal{O}_X$-module which is flat over S. If $Ass_{X/S}(\mathcal{F}) \subset g(X')$ then the canonical map*

$$\mathcal{F} \longrightarrow g_*g^*\mathcal{F}$$

is injective, and remains injective after any base change.

Proof. The final assertion means that $\mathcal{F}_T \to (g_T)_*g_T^*\mathcal{F}_T$ is injective for any morphism $T \to S$. The assumption $\text{Ass}_{X/S}(\mathcal{F}) \subset g(X')$ is preserved by base change, see Divisors, Lemma 26.7.2 and Remark 26.7.3. The same holds for the assumption of flatness and finite type. Hence it suffices to prove the injectivity of the displayed arrow. Let $\mathcal{K} = \text{Ker}(\mathcal{F} \to g_*g^*\mathcal{F})$. Our goal is to prove that $\mathcal{K} = 0$. In order to do this it suffices to prove that $\text{WeakAss}_X(\mathcal{K}) = \emptyset$, see Divisors, Lemma 26.5.5. We have $\text{WeakAss}_X(\mathcal{K}) \subset \text{WeakAss}_X(\mathcal{F})$, see Divisors, Lemma 26.5.4. As $\mathcal{F}$ is flat we see from Lemma 34.13.4 that $\text{WeakAss}_X(\mathcal{F}) \subset \text{Ass}_{X/S}(\mathcal{F})$. By assumption any point x of $\text{Ass}_{X/S}(\mathcal{F})$ is the image of some $x' \in X'$. Since g is flat the local ring map $\mathcal{O}_{X,x} \to \mathcal{O}_{X',x'}$ is faithfully flat, hence the map

$$\mathcal{F}_x \longrightarrow g^*\mathcal{F}_{x'} = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X',x'}$$

is injective (see Algebra, Lemma 7.76.11). This implies that $\mathcal{K}_x = 0$ as desired. □

Lemma 34.23.2. *Let A be a ring. Let $u : M \to N$ be a surjective map of A-modules. If M is projective as an A-module, then there exists an ideal $I \subset A$ such that for any ring map $\varphi : A \to B$ the following are equivalent*

(1) *$u \otimes 1 : M \otimes_A B \to N \otimes_A B$ is an isomorphism, and*
(2) *$\varphi(I) = 0$.*

Proof. As M is projective we can find a projective A-module C such that $F = M \oplus C$ is a free R-module. By replacing u by $u \oplus 1 : F = M \oplus C \to N \oplus C$ we see that we may assume M is free. In this case let I be the ideal of A generated by coefficients of all the elements of $\text{Ker}(u)$ with respect to some (fixed) basis of M. The reason this works is that, since u is surjective and $\otimes_A B$ is right exact, $\text{Ker}(u \otimes 1)$ is the image of $\text{Ker}(u) \otimes_A B$ in $M \otimes_A B$. □

Theorem 34.23.3. *In Situation 34.20.1 assume*

(1) *f is of finite presentation,*
(2) *$\mathcal{F}$ is of finite presentation, flat over S, and pure relative to S, and*
(3) *u is surjective.*

Then F_{iso} is representable by a closed immersion $Z \to S$. Moreover $Z \to S$ is of finite presentation if $\mathcal{G}$ is of finite presentation.

Proof. We will use without further mention that $\mathcal{F}$ is universally pure over S, see Lemma 34.18.3. By Lemma 34.20.2 and Descent, Lemma 31.33.2 and 31.35.1 the question is local for the étale topology on S. Hence it suffices to prove, given $s \in S$, that there exists an étale neighbourhood of (S, s) so that the theorem holds.

Using Lemma 34.12.5 and after replacing S by an elementary étale neighbourhood of s we may asume there exists a commutative diagram

$$\begin{tikzcd} X \arrow[dr] & & X' \arrow[ll, "g"'] \arrow[dl] \\ & S & \end{tikzcd}$$

of schemes of finite presentation over S, where g is étale, $X_s \subset g(X')$, the schemes X' and S are affine, $\Gamma(X', g^*\mathcal{F})$ a projective $\Gamma(S, \mathcal{O}_S)$-module. Note that $g^*\mathcal{F}$ is universally pure over S, see Lemma 34.17.4. Hence by Lemma 34.18.2 we see that the open $g(X')$ contains the points of $\text{Ass}_{X/S}(\mathcal{F})$ lying over $Spec(\mathcal{O}_{S,s})$. Set

$$E = \{t \in S \mid \text{Ass}_{X_t}(\mathcal{F}_t) \subset g(X')\}.$$

By More on Morphisms, Lemma 33.18.5 E is a constructible subset of S. We have seen that $Spec(\mathcal{O}_{S,s}) \subset E$. By Morphisms, Lemma 24.21.4 we see that E contains an open neighbourhood of s. Hence after replacing S by a smaller affine neighbourhood of s we may assume that $\text{Ass}_{X/S}(\mathcal{F}) \subset g(X')$.

Since we have assumed that u is surjective we have $F_{iso} = F_{inj}$. From Lemma 34.23.1 it follows that $u : \mathcal{F} \to \mathcal{G}$ is injective if and only if $g^*u : g^*\mathcal{F} \to g^*\mathcal{G}$ is injective, and the same remains true after any base change. Hence we have reduced to the case where, in addition to the assumptions in the theorem, $X \to S$ is a morphism of affine schemes and $\Gamma(X, \mathcal{F})$ is a projective $\Gamma(S, \mathcal{O}_S)$-module. This case follows immediately from Lemma 34.23.2.

To see that Z is of finite presentation if $\mathcal{G}$ is of finite presentation, combine Lemma 34.20.2 part (3) with Limits, Remark 27.4.2. □

Lemma 34.23.4. *Let $f : X \to S$ be a morphism of schemes which is of finite presentation, flat, and pure. Let Y be a closed subscheme of X. Let $F = f_*Y$ be the Weil restriction functor of Y along f, defined by*

$$F : (Sch/S)^{opp} \to Sets, \quad T \mapsto \begin{cases} \{*\} & \text{if} \quad Y_T \to X_T \text{ is an isomorphism,} \\ \emptyset & \text{else.} \end{cases}$$

Then F is representable by a closed immersion $Z \to S$. Moreover $Z \to S$ is of finite presentation if $Y \to S$ is.

Proof. Let $\mathcal{I}$ be the ideal sheaf defining Y in X and let $u : \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}$ be the surjection. Then for an S-scheme T, the closed immersion $Y_T \to X_T$ is an isomorphism if and only if u_T is an isomorphism. Hence the result follows from Theorem 34.23.3. $\square$

34.24. Flattening in the local case

In this section we start applying the earlier material to obtain a shadow of the flattening stratification.

Theorem 34.24.1. *In Situation 34.20.3 assume that B is essentially of finite type over A and that M is a finite B-module. Then there exists an ideal $I \subset A$ such that A/I corepresents the functor F_{lf} on the category $\mathcal{C}$. In other words given a local homomorphism of local rings $\varphi : A \to A'$ with $B' = B \otimes_A A'$ and $M' = M \otimes_A A'$ the following are equivalent:*

(1) $\forall \mathfrak{q} \in V(\mathfrak{m}_{A'}B' + \mathfrak{m}_B B') \subset Spec(B') : M'_{\mathfrak{q}}$ *is flat over A', and*
(2) $\varphi(I) = 0$.

If B is essentially of finite presentation over A and M of finite presentation over B, then I is a finitely generated ideal.

Proof. Choose a finite type ring map $A \to C$ and a finite C-module N and a prime $\mathfrak{q}$ of C such that $B = C_{\mathfrak{q}}$ and $M = N_{\mathfrak{q}}$. In the following, when we say ``the theorem holds for $(N/C/A, \mathfrak{q})$ we mean that it holds for $(A \to B, M)$ where $B = C_{\mathfrak{q}}$ and $M = N_{\mathfrak{q}}$. By Lemma 34.20.6 the functor F_{lf} is unchanged if we replace B by a local ring flat over B. Hence, since A is henselian, we may apply Lemma 34.7.6 and assume that there exists a complete dévissage of $N/C/A$ at $\mathfrak{q}$.

Let $(A_i, B_i, M_i, \alpha_i, \mathfrak{q}_i)_{i=1,\ldots,n}$ be such a complete dévissage of $N/C/A$ at $\mathfrak{q}$. Let $\mathfrak{q}'_i \subset A_i$ be the unique prime lying over $\mathfrak{q}_i \subset B_i$ as in Definition 34.7.4. Since $C \to A_1$ is surjective and $N \cong M_1$ as C-modules, we see by Lemma 34.20.5 it suffices to prove the theorem holds for $(M_1/A_1/A, \mathfrak{q}'_1)$. Since $B_1 \to A_1$ is finite and $\mathfrak{q}_1$ is the only prime of B_1 over $\mathfrak{q}'_1$ we see that $(A_1)_{\mathfrak{q}'_1} \to (B_1)_{\mathfrak{q}_1}$ is finite (see Algebra, Lemma 7.36.11 or More on Morphisms, Lemma 33.31.4). Hence by Lemma 34.20.5 it suffices to prove the theorem holds for $(M_1/B_1/A, \mathfrak{q}_1)$.

At this point we may assume, by induction on the length n of the dévissage, that the theorem holds for $(M_2/B_2/A, \mathfrak{q}_2)$. (If $n = 1$, then $M_2 = 0$ which is flat over A.) Reversing the last couple of steps of the previous paragraph, using that $M_2 \cong \mathrm{Coker}(\alpha_2)$ as B_1-modules, we see that the theorem holds for $(\mathrm{Coker}(\alpha_1)/B_1/A, \mathfrak{q}_1)$.

Let A' be an object of $\mathcal{C}$. At this point we use Lemma 34.11.1 to see that if $(M_1 \otimes_A A')_{\mathfrak{q}'}$ is flat over A' for a prime $\mathfrak{q}'$ of $B_1 \otimes_A A'$ lying over $\mathfrak{m}_{A'}$, then $(\mathrm{Coker}(\alpha_1) \otimes_A A')_{\mathfrak{q}'}$ is flat over A'. Hence we conclude that F_{lf} is a subfunctor of the functor F'_{lf} associated to the module $\mathrm{Coker}(\alpha_1)_{\mathfrak{q}_1}$ over $(B_1)_{\mathfrak{q}_1}$. By the previous paragraph we know F'_{lf} is corepresented by A/J for some ideal $J \subset A$. Hence we may replace A by A/J and assume that $\mathrm{Coker}(\alpha_1)_{\mathfrak{q}_1}$ is flat over A.

Since $\mathrm{Coker}(\alpha_1)$ is a B_1-module for which there exist a complete dévissage of $N_1/B_1/A$ at $\mathfrak{q}_1$ and since $\mathrm{Coker}(\alpha_1)_{\mathfrak{q}_1}$ is flat over A by Lemma 34.11.2 we see that $\mathrm{Coker}(\alpha_1)$ is free as an A-module, in particular flat as an A-module. Hence Lemma 34.11.1 implies $F_{lf}(A')$ is nonempty if and only if $\alpha \otimes 1_{A'}$ is injective. Let $N_1 = \mathrm{Im}(\alpha_1) \subset M_1$ so that we have exact sequences

$$0 \to N_1 \to M_1 \to \mathrm{Coker}(\alpha_1) \to 0 \quad \text{and} \quad B_1^{\oplus r_1} \to N_1 \to 0$$

The flatness of $\text{Coker}(\alpha_1)$ implies the first sequence is universally exact (see Algebra, Lemma 7.76.5). Hence $\alpha \otimes 1_{A'}$ is injective if and only if $B_1^{\oplus r_1} \otimes_A A' \to N_1 \otimes_A A'$ is an isomorphism. Finally, Theorem 34.23.3 applies to show this functor is corepresentable by A/I for some ideal I and we conclude F_{lf} is corepresentable by A/I also.

To prove the final statement, suppose that $A \to B$ is essentially of finite presentation and M of finite presentation over B. Let $I \subset A$ be the ideal such that F_{lf} is corepresented by A/I. Write $I = \bigcup I_\lambda$ where I_λ ranges over the finitely generated ideals contained in I. Then, since $F_{lf}(A/I) = \{*\}$ we see that $F_{lf}(A/I_\lambda) = \{*\}$ for some λ, see Lemma 34.20.4 part (2). Clearly this implies that $I = I_\lambda$. □

Remark 34.24.2. Here is a scheme theoretic reformulation of Theorem 34.24.1. Let $(X, x) \to (S, s)$ be a morphism of pointed schemes which is locally of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Assume S henselian local with closed point s. There exists a closed subscheme $Z \subset S$ with the following property: for any morphism of pointed schemes $(T, t) \to (S, s)$ the following are equivalent

1. $\mathcal{F}_T$ is flat over T at all points of the fibre X_t which map to $x \in X_s$, and
2. $Spec(\mathcal{O}_{T,t}) \to S$ factors through Z.

Moreover, if $X \to S$ is of finite presentation at x and $\mathcal{F}_x$ of finite presentation over $\mathcal{O}_{X,x}$, then $Z \to S$ is of finite presentation.

At this point we can obtain some very general results completely for free from the result above. Note that perhaps the most interesting case is when $T = X_s$!

Lemma 34.24.3. *Let S be the spectrum of a henselian local ring with closed point s. Let $X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $T \subset X_s$ be a subset. There exists a closed subscheme $Z \subset S$ with the following property: for any morphism of pointed schemes $(T, t) \to (S, s)$ the following are equivalent*

1. *$\mathcal{F}_T$ is flat over T at all points of the fibre X_t which map to a point of $T \subset X_s$, and*
2. *$Spec(\mathcal{O}_{T,t}) \to S$ factors through Z.*

Moreover, if $X \to S$ is locally of finite presentation, $\mathcal{F}$ is of finite presentation, and $T \subset X_s$ is closed and quasi-compact, then $Z \to S$ is of finite presentation.

Proof. For $x \in X_s$ denote $Z_x \subset S$ the closed subscheme we found in Remark 34.24.2. Then it is clear that $Z = \bigcap_{x \in T} Z_x$ works!

To prove the final statement assume X locally of finite presentation, $\mathcal{F}$ of finite presentation and Z closed and quasi-compact. First, choose finitely many affine opens $W_j \subset X$ such that $T \subset \bigcup W_j$. It clearly suffices to prove the result for each morphism $W_j \to S$ with sheaf $\mathcal{F}|_{X_j}$ and closed subset $T \cap W_j$. Hence we may assume X is affine. In this case, More on Algebra, Lemma 12.14.4 shows that the functor defined by (1) is ``limit preserving''. Hence we can show that $Z \to S$ is of finite presentation exactly as in the last part of the proof of Theorem 34.24.1. □

Remark 34.24.4. Tracing the proof of Lemma 34.24.3 to its origins we find a long and winding road. But if we assume that

1. f is of finite type,
2. $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module,
3. $T = X_s$, and
4. S is the spectrum of a Noetherian complete local ring.

then there is a proof relying completely on more elementary algebra as follows: first we reduce to the case where X is affine by taking a finite affine open cover. In this case Z exists by More on Algebra, Lemma 12.15.3. The key step in this proof is constructing the closed subscheme Z step by step inside the truncations $Spec(\mathcal{O}_{S,s}/\mathfrak{m}_s^n)$. This relies on the fact that flattening stratifications always exist when the base is Artinian, and the fact that $\mathcal{O}_{S,s} = lim\, \mathcal{O}_{S,s}/\mathfrak{m}_s^n$.

34.25. Flat finite type modules, Part III

The following lemma improves Algebra, Lemma 7.119.4.

Lemma 34.25.1. *Let $\varphi : A \to B$ be a local ring homomorphism of local rings which is essentially of finite type. Let M be a flat A-module, N a finite B-module and $u : N \to M$ an A-module map such that $\overline{u} : N/\mathfrak{m}_A N \to M/\mathfrak{m}_A M$ is injective. Then u is A-universally injective, N is of finite presentation over B, and N is flat over A.*

Proof. Let $A \to A^h$ be the henselization of A. Let B' be the localization of $B \otimes_A A^h$ at the maximal ideal $\mathfrak{m}_B \otimes A^h + B \otimes \mathfrak{m}_{A^h}$. Since $B \to B'$ is flat (hence faithfully flat, see Algebra, Lemma 7.35.16), we may replace $A \to B$ with $A^h \to B'$, M by $M \otimes_A A^h$, and N by $N \otimes_B B'$, see Algebra, Lemmas 7.77.2 and 7.35.8. Thus we may and do assume that A is a henselian local ring.

Write $B = C/I$ where C is the localization of a polynomial algebra over A at a prime. If we can show that N is finitely presented as a C-module, then a fortiori this shows that N is finitely presented as a B-module (see discussion in Section 34.2; more precisely, see Algebra, Lemma 7.6.4). Hence we may assume that B is essentially of finite presentation over A (even the localization of a polynomial algebra). Next, write $N = B^{\oplus n}/K$ for some submodule $K \subset B^{\oplus n}$. Since $B/\mathfrak{m}_A B$ is Noetherian (as it is essentially of finite type over a field), there exist finitely many elements $k_1, \ldots, k_s \in K$ such that for $K' = \sum Bk_i$ and $N' = B^{\oplus n}/K'$ the canonical surjection $N' \to N$ induces an isomorphism $N'/\mathfrak{m}_A N' \cong N/\mathfrak{m}_A N$. Thus, if we can prove the lemma for the composition $u' : N' \to M$, then u' is injective, hence $N' = N$ and N is of finite presentation. In this way we reduce to the case where N is of finite presentation over B!

Assume A is a henselian local ring, B is essentially of finite presentation over A, N of finite presentation over B and let us temporarily make the additional assumption that N is flat over A. Then N is a filtered colimit $N = colim_i\, F_i$ of free A-modules F_i such that the transition maps $u_{ii'} : F_i \to F_{i'}$ are injective modulo $\mathfrak{m}_A$, see Lemma 34.19.5. Each of the compositions $u_i : F_i \to M$ is A-universally injective by Lemma 34.8.5 wherefore $u = colim\, u_i$ is A-universally injective as desired.

Assume A is a henselian local ring, B is essentially of finite presentation over A, N of finite presentation over B. By Theorem 34.24.1 there exists a finitely generated ideal $I \subset A$ such that N/IN is flat over A/I and such that N/I^2N is not flat over A/I^2 unless $I = 0$. The result of the previous paragraph shows that the lemma holds for u mod $I : N/IN \to M/IM$ over A/I. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M \otimes_A I/I^2 \arrow[r] & M/I^2M \arrow[r] & M/IM \arrow[r] & 0 \\
& N \otimes_A I/I^2 \arrow[r] \arrow[u, "u"] & N/I^2N \arrow[r] \arrow[u, "u"] & N/IN \arrow[r] \arrow[u, "u"] & 0
\end{tikzcd}$$

whose rows are exact by right exactness of $\otimes$ and the fact that M is flat over A. Note that the left vertical arrow is the map $N/IN \otimes_{A/I} I/I^2 \to M/IM \otimes_{A/I} I/I^2$, hence is injective. A diagram chase shows that the lower left arrow is injective, i.e., $\mathrm{Tor}^1_{A/I^2}(I/I^2, M/I^2) = 0$ see Algebra, Remark 7.69.8. Hence N/I^2N is flat over A/I^2 by Algebra, Lemma 7.91.8 a contradiction unless $I = 0$. □

Theorem 34.25.2. *Let $f : X \to S$ be locally of finite type. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite type. Let $x \in X$ with image $s \in S$. The following are equivalent*

1. *$\mathcal{F}$ is flat at x over S, and*
2. *for every $x' \in Ass_{X_s}(\mathcal{F}_s)$ which specializes to x we have that $\mathcal{F}$ is flat at x' over S.*

Proof. It is clear that (1) implies (2) as $\mathcal{F}_{x'}$ is a localization of $\mathcal{F}_x$ for every point which specializes to x. Set $A = \mathcal{O}_{S,s}$, $B = \mathcal{O}_{X,x}$ and $N = \mathcal{F}_x$. Let $\Sigma \subset B$ be the multiplicative subset of B of elements which act as nonzero divisors on $N/\mathfrak{m}_AN$. Assumption (2) implies that $\Sigma^{-1}N$ is A-flat by the description of $Spec(\Sigma^{-1}N)$ in Lemma 34.8.1. On the other hand, the map $N \to \Sigma^{-1}N$ is injective modulo $\mathfrak{m}_A$ by construction. Hence applying Lemma 34.25.1 we win. □

Now we apply this directly to obtain the following useful results.

Lemma 34.25.3. *Let S be a local scheme with closed point s. Let $f : X \to S$ be locally of finite type. Let $\mathcal{F}$ be a finite type $\mathcal{O}_X$-module. Assume that*

1. *every point of $\mathrm{Ass}_{X/S}(\mathcal{F})$ specializes to a point of the closed fibre X_s[3],*
2. *$\mathcal{F}$ is flat over S at every point of X_s.*

Then $\mathcal{F}$ is flat over S.

Proof. This is immediate from the fact that it suffices to check for flatness at points of the relative assassin of $\mathcal{F}$ over S by Theorem 34.25.2. □

34.26. Universal flattening

If $f : X \to S$ is a proper, finitely presented morphism of schemes then one can find a universal flattening of f. In this section we discuss this and some of its variants.

Lemma 34.26.1. *In Situation 34.20.7. For each $p \geq 0$ the functor H_p (34.20.7.2) is representable by a locally closed immersion $S_p \to S$. If $\mathcal{F}$ is of finite presentation, then $S_p \to S$ is of finite presentation.*

Proof. For each S we will prove the statement for all $p \geq 0$ concurrently. The functor H_p is a sheaf for the fppf topology by Lemma 34.20.8. Hence combining Descent, Lemma 31.35.1, More on Morphisms, Lemma 33.35.1 , and Descent, Lemma 31.20.1 we see that the question is local for the étale topology on S. In particular, the question is Zariski local on S.

For $s \in S$ denote ξ_s the unique generic point of the fibre X_s. Note that for every $s \in S$ the restriction $\mathcal{F}_s$ of $\mathcal{F}$ is locally free of some rank $p(s) \geq 0$ in some neighbourhood of ξ_s. (As X_s is irreducible and smooth this follows from generic flatness for $\mathcal{F}_s$ over X_s, see Algebra, Lemma 7.109.1 allthough this is overkill.) For future reference we note that

$$p(s) = \dim_{\kappa(\xi_s)}(\mathcal{F}_{\xi_s} \otimes_{\mathcal{O}_{X,\xi_s}} \kappa(\xi_s)).$$

[3]For example this holds if f is finite type and $\mathcal{F}$ is pure along X_s, or if f is proper.

In particular $H_{p(s)}(s)$ is nonempty and $H_q(s)$ is empty if $q \neq p(s)$.

Let $U \subset X$ be an open subscheme. As $f : X \to S$ is smooth, it is open. It is immediate from (34.20.7.2) that the functor H_p for the pair $(f|_U : U \to f(U), \mathcal{F}|_U)$ and the functor H_p for the pair $(f|_{f^{-1}(f(U))}, \mathcal{F}|_{f^{-1}(f(U))})$ are the same. Hence to prove the existence of S_p over $f(U)$ we may always replace X by U.

Pick $s \in S$. There exists an affine open neighbourhood U of ξ_s such that $\mathcal{F}|_U$ can be generated by at most $p(s)$ elements. By the arguments above we see that in order to prove the statement for $H_{p(s)}$ in an neighbourhood of s we may assume that $\mathcal{F}$ is generated by $p(s)$ elements, i.e., that there exists a surjection

$$u : \mathcal{O}_X^{\oplus p(s)} \longrightarrow \mathcal{F}$$

In this case it is clear that $H_{p(s)}$ is equal to F_{iso} (34.20.1.1) for the map u (this follows immediately from Lemma 34.19.1 but also from Lemma 34.12.1 after shrinking a bit more so that both S and X are affine.) Thus we may apply Theorem 34.23.3 to see that $H_{p(s)}$ is representable by a closed immersion in a neighbourhood of s.

The result follows formally from the above. Namely, the arguments above show that locally on S the function $s \mapsto p(s)$ is bounded. Hence we may use induction on $p = \max_{s \in S} p(s)$. The functor H_p is representable by a closed immersion $S_p \to S$ by the above. Replace S by $S \setminus S_p$ which drops the maximum by at least one and we win by induction hypothesis.

To see that $S_p \to S$ is of finite presentation if $\mathcal{F}$ is of finite presentation combine Lemma 34.20.8 part (2) with Limits, Remark 27.4.2. □

Lemma 34.26.2. *In Situation 34.20.9. Let $h : X' \to X$ be an étale morphism. Set $\mathcal{F}' = h^*\mathcal{F}$ and $f' = f \circ h$. Let F'_n be (34.20.9.1) associated to $(f' : X' \to S, \mathcal{F}')$. Then F_n is a subfunctor of F'_n and if $h(X') \supset Ass_{X/S}(\mathcal{F})$, then $F_n = F'_n$.*

Proof. Let $T \to S$ be any morphism. Then $h_T : X'_T \to X_T$ is étale as a base change of the étale morphism g. For $t \in T$ denote $Z \subset X_t$ the set of points where $\mathcal{F}_T$ is not flat over T, and similarly denote $Z' \subset X'_t$ the set of points where $\mathcal{F}'_T$ is not flat over T. As $\mathcal{F}'_T = h_T^*\mathcal{F}_T$ we see that $Z' = h_t^{-1}(Z)$, see Morphisms, Lemma 24.24.11. Hence $Z' \to Z$ is an étale morphism, so $\dim(Z') \leq \dim(Z)$ (for example by Descent, Lemma 31.17.2 or just because an étale morphism is smooth of relative dimension 0). This implies that $F_n \subset F'_n$.

Finally, suppose that $h(X') \supset \mathrm{Ass}_{X/S}(\mathcal{F})$ and that $T \to S$ is a morphism such that $F'_n(T)$ is nonempty, i.e., such that $\mathcal{F}'_T$ is flat in dimensions $\geq n$ over T. Pick a point $t \in T$ and let $Z \subset X_t$ and $Z' \subset X'_t$ be as above. To get a contradiction assume that $\dim(Z) \geq n$. Pick a generic point $\xi \in Z$ corresponding to a component of dimension $\geq n$. Let $x \in \mathrm{Ass}_{X_t}(\mathcal{F}_t)$ be a generalization of ξ. Then x maps to a point of $\mathrm{Ass}_{X/S}(\mathcal{F})$ by Divisors, Lemma 26.7.2 and Remark 26.7.3. Thus we see that x is in the image of h_T, say $x = h_T(x')$ for some $x' \in X'_T$. But $x' \notin Z'$ as $x \rightsquigarrow \xi$ and $\dim(Z') < n$. Hence $\mathcal{F}'_T$ is flat over T at x' which implies that $\mathcal{F}_T$ is flat at x over T (by Morphisms, Lemma 24.24.11). Since this holds for every such x we conclude that $\mathcal{F}_T$ is flat over T at ξ by Theorem 34.25.2 which is the desired contradiction. □

Lemma 34.26.3. *Assume that $X \to S$ is a smooth morphism of affine schemes with geometrically irreducible fibres of dimension d and that $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module of finite presentation. Then $F_d = \coprod_{p=0,\ldots,c} H_p$ for some $c \geq 0$ with F_d as in (34.20.9.1) and H_p as in (34.20.7.2).*

Proof. As X is affine and $\mathcal{F}$ is quasi-coherent of finite presentation we know that $\mathcal{F}$ can be generated by $c \geq 0$ elements. Then $\dim_{\kappa(x)}(\mathcal{F}_x \otimes \kappa(x))$ in any point $x \in X$ never exceeds c. In particular $H_p = \emptyset$ for $p > c$. Moreover, note that there certainly is an inclusion $\coprod H_p \to F_d$. Having said this the content of the lemma is that, if a base change $\mathcal{F}_T$ is flat in dimensions $\geq d$ over T and if $t \in T$, then $\mathcal{F}_T$ is free of some rank r in an open neighbourhood $U \subset X_T$ of the unique generic point of X_t. (Namely, it then follows that H_r contains an open neighbourhood of t.) To prove this we may replace T by S. Let $s \in S$ and let $\xi \in X$ be the unique generic point of X_s. The assumption that $\mathcal{F}$ is flat in dimensions $\geq d$ over S means that $\mathcal{F}_\xi$ is flat over $\mathcal{O}_{S,s}$. Pick $\alpha : \mathcal{O}_X^{\oplus r} \to \mathcal{F}$ which induces an isomorphism of fibres $\kappa(\xi)^{\oplus r} \to \mathcal{F}_\xi \otimes \kappa(\xi)$ at ξ, see Lemma 34.12.1. The same lemma implies, since $\mathcal{F}_\xi$ is flat over $\mathcal{O}_{S,s}$, that α is an isomorphism in an open neighbourhood of ξ and we win. $\square$

Lemma 34.26.4. *In Situation 34.20.9. Let $s \in S$ let $d \geq 0$. Assume*

(1) *there exists a complete dévisage of $\mathcal{F}/X/S$ over some point $s \in S$,*
(2) *X is of finite presentation over S,*
(3) *$\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation, and*
(4) *$\mathcal{F}$ is flat in dimensions $\geq d + 1$ over S.*

Then after possibly replacing S by an open neighbourhood of s the functor F_d (34.20.9.1) is representable by a monomorphism $Z_d \to S$ of finite presentation.

Proof. A preliminary remark is that X, S are affine schemes and that it suffices to prove F_d is representable by a closed subscheme on the category of affine schemes over S. Hence throughout the proof of the lemma we work in the category of affine schemes over S.

Let $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k)_{k=1,\ldots,n}$ be a complete dévissage of $\mathcal{F}/X/S$ over s, see Definition 34.6.1. We will use induction on the length n of the dévissage. Recall that $Y_k \to S$ is smooth with geometrically irreducible fibres, see Definition 34.5.1. Let d_k be the relative dimension of Y_k over S. Recall that $i_{k,*}\mathcal{G}_k = \mathrm{Coker}(\alpha_k)$ and that i_k is a closed immersion. By the definitions referenced above we have $d_1 = \dim(\mathrm{Supp}(\mathcal{F}_s))$ and

$$d_k = \dim(\mathrm{Supp}(\mathrm{Coker}(\alpha_{k-1})_s)) = \dim(\mathrm{Supp}(\mathcal{G}_{k,s}))$$

for $k = 2, \ldots, n$. It follows that $d_1 > d_2 > \ldots > d_n \geq 0$ because α_k is an isomorphism in the generic point of $(Y_k)_s$.

Note that i_1 is a closed immersion and $\mathcal{F} = i_{1,*}\mathcal{G}_1$. Hence for any morphism of schemes $T \to S$ with T affine, we have $\mathcal{F}_T = i_{1,T,*}\mathcal{G}_{1,T}$ and $i_{1,T}$ is still a closed immersion of schemes over T. Thus $\mathcal{F}_T$ is flat in dimensions $\geq d$ over T if and only if $\mathcal{G}_{1,T}$ is flat in dimensions $\geq d$ over T. Because $\pi_1 : Z_1 \to Y_1$ is finite we see in the same manner that $\mathcal{G}_{1,T}$ is flat in dimensions $\geq d$ over T if and only if $\pi_{1,T,*}\mathcal{G}_{1,T}$ is flat in dimensions $\geq d$ over T. The same arguments work for ``flat in dimensions $\geq d + 1$'' and we conclude in particular that $\pi_{1,*}\mathcal{G}_1$ is flat over S in dimensions $\geq d + 1$ by our assumption on $\mathcal{F}$.

Suppose that $d_1 > d$. It follows from the discussion above that in particular $\pi_{1,*}\mathcal{G}_1$ is flat over S at the generic point of $(Y_1)_s$. By Lemma 34.12.1 we may replace S by an affine neighbourhood of s and assume that α_1 is S-universally injective. Because α_1 is S-universally injective, for any morphism $T \to S$ with T affine, we have a short exact sequence

$$0 \to \mathcal{O}_{Y_{1,T}}^{\oplus r_1} \to \pi_{1,T,*}\mathcal{G}_{1,T} \to \mathrm{Coker}(\alpha_1)_T \to 0$$

and still the first arrow is T-unversally injective. Hence the set of points of $(Y_1)_T$ where $\pi_{1,T,*}\mathcal{G}_{1,T}$ is flat over T is the same as the set of points of $(Y_1)_T$ where $\mathrm{Coker}(\alpha_1)_T$ is flat over

S. In this way the question reduces to the sheaf $\text{Coker}(\alpha_1)$ which has a complete dévissage of length $n-1$ and we win by induction.

If $d_1 < d$ then F_d is represented by S and we win.

The last case is the case $d_1 = d$. This case follows from a combination of Lemma 34.26.3 and Lemma 34.26.1. □

Theorem 34.26.5. *In Situation 34.20.9. Assume moreover that f is of finite presentation, that $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation, and that $\mathcal{F}$ is pure relative to S. Then F_n is representable by a monomorphism $Z_n \to S$ of finite presentation.*

Proof. The functor F_n is a sheaf for the fppf topology by Lemma 34.20.10. Hence combining Descent, Lemma 31.35.1, More on Morphisms, Lemma 33.35.1 , and Descent, Lemmas 31.19.29 and 31.19.11 we see that the question is local for the étale topology on S.

In particular the situation is local for the Zariski toplogy on S and we may assume that S is affine. In this case the dimension of the fibres of f is bounded above, hence we see that F_n is representable for n large enough. Thus we may use descending induction on n. Suppose that we know F_{n+1} is representable by a monomorphism $Z_{n+1} \to S$ of finite presentation. Consider the base change $X_{n+1} = Z_{n+1} \times_S X$ and the pullback $\mathcal{F}_{n+1}$ of $\mathcal{F}$ to X_{n+1}. The morphism $Z_{n+1} \to S$ is quasi-finite as it is a monomorphism of finite presentation, hence Lemma 34.16.4 implies that $\mathcal{F}_{n+1}$ is pure relative to Z_{n+1}. Since F_n is a subfunctor of F_{n+1} we conclude that in order to prove the result for F_n it suffices to prove the result for the corresponding functor for the situation $\mathcal{F}_{n+1}/X_{n+1}/Z_{n+1}$. In this way we reduce to proving the result for F_n in case $S_{n+1} = S$, i.e., we may assume that $\mathcal{F}$ is flat in dimensions $\geq n+1$ over S.

Fix n and assume $\mathcal{F}$ is flat in dimensions $\geq n+1$ over S. To finish the proof we have to show that F_n is representable by a monomorphism $Z_n \to S$ of finite presentation. Since the question is local in the étale topology on S it suffices to show that for every $s \in S$ there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ such that the result holds after base change to S'. Thus by Lemma 34.6.8 we may assume there exist étale morphisms $h_j : Y_j \to X$, $j = 1, \ldots, m$ such that for each i there exists a complete dévissage of $\mathcal{F}_j/Y_j/S$ over s, where $\mathcal{F}_j$ is the pullback of $\mathcal{F}$ to Y_j and such that $X_s \subset \bigcup h_j(Y_j)$. Note that by Lemma 34.26.2 the sheaves $\mathcal{F}_j$ are still flat over in dimensions $\geq n+1$ over S. Set $W = \bigcup h_j(Y_j)$, which is a quasi-compact open of X. As $\mathcal{F}$ is pure along X_s we see that

$$E = \{t \in S \mid \text{Ass}_{X_t}(\mathcal{F}_t) \subset W\}.$$

contains all generalizations of s. By More on Morphisms, Lemma 33.18.5 E is a constructible subset of S. We have seen that $Spec(\mathcal{O}_{S,s}) \subset E$. By Morphisms, Lemma 24.21.4 we see that E contains an open neighbourhood of s. Hence after shrinking S we may assume that $E = S$. It follows from Lemma 34.26.2 that it suffices to prove the lemma for the functor F_n associated to $X = \coprod Y_j$ and $\mathcal{F} = \coprod \mathcal{F}_j$. If $F_{j,n}$ denotes the functor for $Y_j \to S$ and the sheaf $\mathcal{F}_i$ we see that $F_n = \prod F_{j,n}$. Hence it suffices to prove each $F_{j,n}$ is representable by some monomorphism $Z_{j,n} \to S$ of finite presentation, since then

$$Z_n = Z_{1,n} \times_S \ldots \times_S Z_{m,n}$$

Thus we have reduced the theorem to the special case handled in Lemma 34.26.4. □

We make explicit what the theorem means in terms of universal flattenings in the following lemma.

Lemma 34.26.6. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module.*

(1) *If f is of finite presentation, $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation, and $\mathcal{F}$ is pure relative to S, then there exists a universal flattening $S' \to S$ of $\mathcal{F}$. Moreover $S' \to S$ is a monomorphism of finite presentation.*
(2) *If f is of finite presentation and X is pure relative to S, then there exists a universal flattening $S' \to S$ of X. Moreover $S' \to S$ is a monomorphism of finite presentation.*
(3) *If f is proper and of finite presentation and $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation, then there exists a universal flattening $S' \to S$ of $\mathcal{F}$. Moreover $S' \to S$ is a monomorphism of finite presentation.*
(4) *If f is proper and of finite presentation then there exists a universal flattening $S' \to S$ of X.*

Proof. These statements follow immediately from Theorem 34.26.5 applied to $F_0 = F_{flat}$ and the fact that if f is proper then $\mathcal{F}$ is automatically pure over the base, see Lemma 34.17.1. □

34.27. Other chapters

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CHAPTER 35

Groupoid Schemes

35.1. Introduction

This chapter is devoted to generalities concering groupoid schemes. See for example the beautiful paper [**KM97a**] by Keel and Mori.

35.2. Notation

Let S be a scheme. If U, T are schemes over S we denote $U(T)$ for the set of T-valued points of U *over* S. In a formula: $U(T) = Mor_S(T, U)$. We try to reserve the letter T to denote a ``test scheme'' over S, as in the discussion that follows. Suppose we are given schemes X, Y over S and a morphism of schemes $f : X \to Y$ over S. For any scheme T over S we get an induced map of sets

$$f : X(T) \longrightarrow Y(T)$$

which as indicated we denote by f also. In fact this construction is functorial in the scheme T/S. Yoneda's Lemma, see Categories, Lemma 4.3.5, says that f determines and is determined by this transformation of functors $f : h_X \to h_Y$. More generally, we use the same notation for maps between fibre products. For example, if X, Y, Z are schemes over S, and if $m : X \times_S Y \to Z \times_S Z$ is a morphism of schemes over S, then we think of m as corresponding to a collection of maps between T-valued points

$$X(T) \times Y(T) \longrightarrow Z(T) \times Z(T).$$

And so on and so forth.

We continue our convention to label projection maps starting with index 0, so we have $\mathrm{pr}_0 : X \times_S Y \to X$ and $\mathrm{pr}_1 : X \times_S Y \to Y$.

35.3. Equivalence relations

Recall that a *relation* R on a set A is just a subset of $R \subset A \times A$.We usually write aRb to indicate $(a, b) \in R$. We say the relation is *transitive* if $aRb, bRc \Rightarrow aRc$. We say the relation is *reflexive* if aRa for all $a \in A$. We say the relation is *symmetric* if $aRb \Rightarrow bRa$. A relation is called an *equivalence relation* if it is transitive, reflexive and symmetric.

In the setting of schemes we are going to relax the notion of a relation a little bit and just require $R \to A \times A$ to be a map. Here is the definition.

Definition 35.3.1. Let S be a scheme. Let U be a scheme over S.

1. A *pre-relation* on U over S is any morphism $j : R \to U \times_S U$. In this case we set $t = \mathrm{pr}_0 \circ j$ and $s = \mathrm{pr}_1 \circ j$, so that $j = (t, s)$.
2. A *relation* on U over S is a monomorphism $j : R \to U \times_S U$.
3. A *pre-equivalence relation* is a pre-relation $j : R \to U \times_S U$ such that the image of $j : R(T) \to U(T) \times U(T)$ is an equivalence relation for all T/S.

(4) We say a morphism $R \to U \times_S U$ is an *equivalence relation on U over S* if and only if for every T/S the T-valued points of R define an equivalence relation on the set of T-valued points of U.

In other words, an equivalence relation is a pre-equivalence relation such that j is a relation.

Lemma 35.3.2. *Let S be a scheme. Let U be a scheme over S. Let $j : R \to U \times_S U$ be a pre-relation. Let $g : U' \to U$ be a morphism of schemes. Finally, set*

$$R' = (U' \times_S U') \times_{U \times_S U} R \xrightarrow{j'} U' \times_S U'$$

Then j' is a pre-relation on U' over S. If j is a relation, then j' is a relation. If j is a pre-equivalence relation, then j' is a pre-equivalence relation. If j is an equivalence relation, then j' is an equivalence relation.

Proof. Omitted. □

Definition 35.3.3. Let S be a scheme. Let U be a scheme over S. Let $j : R \to U \times_S U$ be a pre-relation. Let $g : U' \to U$ be a morphism of schemes. The pre-relation $j' : R' \to U' \times_S U'$ is called the *restriction*, or *pullback* of the pre-relation j to U'. In this situation we sometimes write $R' = R|_{U'}$.

Lemma 35.3.4. *Let $j : R \to U \times_S U$ be a pre-relation. Consider the relation on points of the scheme U defined by the rule*

$$x \sim y \Leftrightarrow \exists\, r \in R : t(r) = x, s(r) = y.$$

If j is a pre-equivalence relation then this is an equivalence relation.

Proof. Suppose that $x \sim y$ and $y \sim z$. Pick $r \in R$ with $t(r) = x$, $s(r) = y$ and pick $r' \in R$ with $t(r') = y$, $s(r') = z$. Pick a field K fitting into the following commutative diagram

$$\begin{tikzcd} \kappa(r) \arrow[r] & K \\ \kappa(y) \arrow[u] \arrow[r] & \kappa(r') \arrow[u] \end{tikzcd}$$

Denote $x_K, y_K, z_K : Spec(K) \to U$ the morphisms

$$\begin{aligned} Spec(K) \to Spec(\kappa(r)) \to Spec(\kappa(x)) \to U \\ Spec(K) \to Spec(\kappa(r)) \to Spec(\kappa(y)) \to U \\ Spec(K) \to Spec(\kappa(r')) \to Spec(\kappa(z)) \to U \end{aligned}$$

By construction $(x_K, y_K) \in j(R(K))$ and $(y_K, z_K) \in j(R(K))$. Since j is a pre-equivalence relation we see that also $(x_K, z_K) \in j(R(K))$. This clearly implies that $x \sim z$.

The proof that $\sim$ is reflexive and symmetric is omitted. □

35.4. Group schemes

Let us recall that a *group* is a pair (G, m) where G is a set, and $m : G \times G \to G$ is a map of sets with the following properties:

(1) (associativity) $m(g, m(g', g'')) = m(m(g, g'), g'')$ for all $g, g', g'' \in G$,
(2) (identity) there exists a unique element $e \in G$ (called the *identity*, *unit*, or 1 of G) such that $m(g, e) = m(e, g) = g$ for all $g \in G$, and
(3) (inverse) for all $g \in G$ there exists a $i(g) \in G$ such that $m(g, i(g)) = m(i(g), g) = e$, where e is the identity.

Thus we obtain a map $e : \{*\} \to G$ and a map $i : G \to G$ so that the quadruple (G, m, e, i) satisfies the axioms listed above.

A *homomorphism of groups* $\psi : (G, m) \to (G', m')$ is a map of sets $\psi : G \to G'$ such that $m'(\psi(g), \psi(g')) = \psi(m(g, g'))$. This automatically insures that $\psi(e) = e'$ and $i'(\psi(g)) = \psi(i(g))$. (Obvious notation.) We will use this below.

Definition 35.4.1. Let S be a scheme.

(1) A *group scheme over* S is a pair (G, m), where G is a scheme over S and $m : G \times_S G \to G$ is a morphism of schemes over S with the following property: For every scheme T over S the pair $(G(T), m)$ is a group.
(2) A *morphism* $\psi : (G, m) \to (G', m')$ *of group schemes over* S is a morphism $\psi : G \to G'$ of schemes over S such that for every T/S the induced map $\psi : G(T) \to G'(T)$ is a homomorphism of groups.

Let (G, m) be a group scheme over the scheme S. By the discussion above (and the discussion in Section 35.2) we obtain morphisms of schemes over S: (identity) $e : S \to G$ and (inverse) $i : G \to G$ such that for every T the quadruple $(G(T), m, e, i)$ satsifies the axioms of a group listed above.

Let (G, m), (G', m') be group schemes over S. Let $f : G \to G'$ be a morphism of schemes over S. It follows from the definition that f is a morphism of group schemes over S if and only if the following diagram is commutative:

$$\begin{tikzcd}
G \times_S G \arrow[r, "f \times f"'] \arrow[d, "m"'] & G' \times_S G' \arrow[d, "m"] \\
G \arrow[r, "f"] & G'
\end{tikzcd}$$

Lemma 35.4.2. *Let (G, m) be a group scheme over S. Let $S' \to S$ be a morphism of schemes. The pullback $(G_{S'}, m_{S'})$ is a group scheme over S'.*

Proof. Omitted. $\square$

Definition 35.4.3. Let S be a scheme. Let (G, m) be a group scheme over S.

(1) A *closed subgroup scheme* of G is a closed subscheme $H \subset G$ such that $m|_{H \times_S H}$ factors through H and induces a group scheme structure on H over S.
(2) An *open subgroup scheme* of G is an open subscheme $G' \subset G$ such that $m|_{G' \times_S G'}$ factors through G' and induces a group scheme structure on G' over S.

Alternatively, we could say that H is a closed subgroup scheme of G if it is a group scheme over S endowed with a morphism of group schemes $i : H \to G$ over S which identifies H with a closed subscheme of G.

Definition 35.4.4. Let S be a scheme. Let (G, m) be a group scheme over S.

(1) We say G is a *smooth group scheme* if the structure morphism $G \to S$ is smooth.
(2) We say G is a *flat group scheme* if the structure morphism $G \to S$ is flat.
(3) We say G is a *separated group scheme* if the structure morphism $G \to S$ is separated.

Add more as needed.

35.5. Examples of group schemes

Example 35.5.1. (Multiplicative group scheme.) Consider the functor which associates to any scheme T the group $\Gamma(T, \mathcal{O}_T^*)$ of units in the global sections of the structure sheaf. This is representable by the scheme

$$\mathbf{G}_m = Spec(\mathbf{Z}[x, x^{-1}])$$

The morphism giving the group structure is the morphism

$$\begin{matrix} \mathbf{G}_m \times \mathbf{G}_m & \to & \mathbf{G}_m \\ Spec(\mathbf{Z}[x, x^{-1}] \otimes_{\mathbf{Z}} \mathbf{Z}[x, x^{-1}]) & \to & Spec(\mathbf{Z}[x, x^{-1}]) \\ \mathbf{Z}[x, x^{-1}] \otimes_{\mathbf{Z}} \mathbf{Z}[x, x^{-1}] & \leftarrow & \mathbf{Z}[x, x^{-1}] \\ x \otimes x & \leftarrow & x \end{matrix}$$

Hence we see that $\mathbf{G}_m$ is a group scheme over $\mathbf{Z}$. For any scheme S the base change $\mathbf{G}_{m,S}$ is a group scheme over S whose functor of points is

$$T/S \longmapsto \mathbf{G}_{m,S}(T) = \mathbf{G}_m(T) = \Gamma(T, \mathcal{O}_T^*)$$

as before.

Example 35.5.2. (Roots of unity.) Let $n \in \mathbf{N}$. Consider the functor which associates to any scheme T the subgroup of $\Gamma(T, \mathcal{O}_T^*)$ consisting of nth roots of unity. This is representable by the scheme

$$\mu_n = Spec(\mathbf{Z}[x]/(x^n - 1)).$$

The morphism giving the group structure is the morphism

$$\begin{matrix} \mu_n \times \mu_n & \to & \mu_n \\ Spec(\mathbf{Z}[x]/(x^n - 1) \otimes_{\mathbf{Z}} \mathbf{Z}[x]/(x^n - 1)) & \to & Spec(\mathbf{Z}[x]/(x^n - 1)) \\ \mathbf{Z}[x]/(x^n - 1) \otimes_{\mathbf{Z}} \mathbf{Z}[x]/(x^n - 1) & \leftarrow & \mathbf{Z}[x]/(x^n - 1) \\ x \otimes x & \leftarrow & x \end{matrix}$$

Hence we see that μ_n is a group scheme over $\mathbf{Z}$. For any scheme S the base change $\mu_{n,S}$ is a group scheme over S whose functor of points is

$$T/S \longmapsto \mu_{n,S}(T) = \mu_n(T) = \{f \in \Gamma(T, \mathcal{O}_T^*) \mid f^n = 1\}$$

as before.

Example 35.5.3. (Additive group scheme.) Consider the functor which associates to any scheme T the group $\Gamma(T, \mathcal{O}_T)$ of global sections of the structure sheaf. This is representable by the scheme

$$\mathbf{G}_a = Spec(\mathbf{Z}[x])$$

The morphism giving the group structure is the morphism

$$\begin{matrix} \mathbf{G}_a \times \mathbf{G}_a & \to & \mathbf{G}_a \\ Spec(\mathbf{Z}[x] \otimes_{\mathbf{Z}} \mathbf{Z}[x]) & \to & Spec(\mathbf{Z}[x]) \\ \mathbf{Z}[x] \otimes_{\mathbf{Z}} \mathbf{Z}[x] & \leftarrow & \mathbf{Z}[x] \\ x \otimes 1 + 1 \otimes x & \leftarrow & x \end{matrix}$$

Hence we see that $\mathbf{G}_a$ is a group scheme over $\mathbf{Z}$. For any scheme S the base change $\mathbf{G}_{a,S}$ is a group scheme over S whose functor of points is

$$T/S \longmapsto \mathbf{G}_{a,S}(T) = \mathbf{G}_a(T) = \Gamma(T, \mathcal{O}_T)$$

as before.

Example 35.5.4. (General linear group scheme.) Let $n \geq 1$. Consider the functor which associates to any scheme T the group

$$\mathrm{GL}_n(\Gamma(T, \mathcal{O}_T))$$

of invertible $n \times n$ matrices over the global sections of the structure sheaf. This is representable by the scheme

$$\mathrm{GL}_n = Spec(\mathbf{Z}[\{x_{ij}\}_{1 \leq i,j \leq n}][1/d])$$

where $d = \det((x_{ij}))$ with (x_{ij}) the $n \times n$ matrix with entry x_{ij} in the (i, j)-spot. The morphism giving the group structure is the morphism

$$\begin{array}{rcl} \mathrm{GL}_n \times \mathrm{GL}_n & \to & \mathrm{GL}_n \\ Spec(\mathbf{Z}[x_{ij}, 1/d] \otimes_{\mathbf{Z}} \mathbf{Z}[x_{ij}, 1/d]) & \to & Spec(\mathbf{Z}[x_{ij}, 1/d]) \\ \mathbf{Z}[x_{ij}, 1/d] \otimes_{\mathbf{Z}} \mathbf{Z}[x_{ij}, 1/d] & \leftarrow & \mathbf{Z}[x_{ij}, 1/d] \\ \sum x_{ik} \otimes x_{kj} & \leftarrow & x_{ij} \end{array}$$

Hence we see that GL_n is a group scheme over $\mathbf{Z}$. For any scheme S the base change $\mathrm{GL}_{n,S}$ is a group scheme over S whose functor of points is

$$T/S \longmapsto \mathrm{GL}_{n,S}(T) = \mathrm{GL}_n(T) = \mathrm{GL}_n(\Gamma(T, \mathcal{O}_T))$$

as before.

Example 35.5.5. The determinant defines a morphisms of group schemes

$$\det : \mathrm{GL}_n \longrightarrow \mathbf{G}_m$$

over $\mathbf{Z}$. By base change it gives a morphism of group schemes $\mathrm{GL}_{n,S} \to \mathbf{G}_{m,S}$ over any base scheme S.

Example 35.5.6. (Constant group.) Let G be an abstract group. Consider the functor which associates to any scheme T the group of locally constant maps $T \to G$ (where T has the Zariski topology and G the discrete topology). This is representable by the scheme

$$G_{Spec(\mathbf{Z})} = \coprod\nolimits_{g \in G} Spec(\mathbf{Z}).$$

The morphism giving the group structure is the morphism

$$G_{Spec(\mathbf{Z})} \times_{Spec(\mathbf{Z})} G_{Spec(\mathbf{Z})} \longrightarrow G_{Spec(\mathbf{Z})}$$

which maps the component corresponding to the pair (g, g') to the component corresponding to gg'. For any scheme S the base change G_S is a group scheme over S whose functor of points is

$$T/S \longmapsto G_S(T) = \{f : T \to G \text{ locally constant}\}$$

as before.

35.6. Properties of group schemes

In this section we collect some simple properties of group schemes which hold over any base.

Lemma 35.6.1. *Let S be a scheme. Let G be a group scheme over S. Then $G \to S$ is separated (resp. quasi-separated) if and only if the identity morphism $e : S \to G$ is a closed immersion (resp. quasi-compact).*

Proof. We recall that by Schemes, Lemma 21.21.12 we have that e is an immersion which is a closed immersion (resp. quasi-compact) if $G \to S$ is separated (resp. quasi-separated). For the converse, consider the diagram

$$\begin{tikzcd} G \arrow[r, "\Delta_{G/S}"'] \arrow[d] & G \times_S G \arrow[d, "(g,g') \mapsto m(i(g),g')"] \\ S \arrow[r, "e"] & G \end{tikzcd}$$

It is an exercise in the functorial point of view in algebraic geometry to show that this diagram is cartesian. In other words, we see that $\Delta_{G/S}$ is a base change of e. Hence if e is a closed immersion (resp. quasi-compact) so is $\Delta_{G/S}$, see Schemes, Lemma 21.18.2 (resp. Schemes, Lemma 21.19.3). □

Lemma 35.6.2. *Let S be a scheme. Let G be a group scheme over S. Let T be a scheme over S and let $\psi : T \to G$ be a morphism over S. If T is flat over S, then the morphism*

$$T \times_S G \longrightarrow G, \quad (t, g) \longmapsto m(\psi(t), g)$$

is flat. In particular, if G is flat over S, then $m : G \times_S G \to G$ is flat.

Proof. Consider the diagram

$$\begin{tikzcd} T \times_S G \arrow[r, "(t,g) \mapsto (t, m(\psi(t), g))"'] & T \times_S G \arrow[r, "\text{pr}"'] \arrow[d] & G \arrow[d] \\ & T \arrow[r] & S \end{tikzcd}$$

The left top horizontal arrow is an isomorphism and the square is cartesian. Hence the lemma follows from Morphisms, Lemma 24.24.7. □

Lemma 35.6.3. *Let (G, m, e, i) be a group scheme over the scheme S. Denote $f : G \to S$ the structure morphism. Assume f is flat. Then there exist canonical isomorphisms*

$$\Omega_{G/S} \cong f^*\mathcal{C}_{S/G} \cong f^*e^*\Omega_{G/S}$$

where $\mathcal{C}_{S/G}$ denotes the conormal sheaf of the immersion e. In particular, if S is the spectrum of a field, then $\Omega_{G/S}$ is a free $\mathcal{O}_G$-module.

Proof. In Morphisms, Section 24.32 we defined $\Omega_{G/S}$ as the conormal sheaf of the diagonal morphism $\Delta_{G/S}$. In the proof of Lemma 35.6.1 we showed that $\Delta_{G/S}$ is a base change of the immersion e by the morphism $(g, g') \mapsto m(i(g), g')$. This morphism is isomorphic to the morphism $(g, g') \mapsto m(g, g')$ hence is flat by Lemma 35.6.2. Hence we get the first isomorphism by Morphisms, Lemma 24.31.4. By Morphisms, Lemma 24.32.18 we have $\mathcal{C}_{S/G} \cong e^*\Omega_{G/S}$.

If S is the spetrum of a field, then $G \to S$ is flat, and any $\mathcal{O}_S$-module on S is free. □

35.7. Properties of group schemes over a field

In this section we collect some simple properties of group schemes over a field.

Lemma 35.7.1. *If (G, m) is a group scheme over a field k, then the multiplication map $m : G \times_k G \to G$ is open.*

Proof. The multiplication map is isomorphic to the projection map $\text{pr}_0 : G \times_k G \to G$ because the diagram

$$\begin{tikzcd} G \times_k G \arrow[r, "(g,g')\mapsto(m(g,g'),g')"'] \arrow[d, "m"] & G \times_k G \arrow[d, "(g,g')\mapsto g"] \\ G \arrow[r, "\text{id}"] & G \end{tikzcd}$$

is commutative with isomorphisms as horizontal arrows. The projection is open by Morphisms, Lemma 24.22.4. □

Lemma 35.7.2. *Let G be a group scheme over a field. Then G is a separated scheme.*

Proof. Say $S = Spec(k)$ with k a field, and let G be a group scheme over S. By Lemma 35.6.1 we have to show that $e : S \to G$ is a closed immersion. By Morphisms, Lemma 24.19.2 the image of $e : S \to G$ is a closed point of G. It is clear that $\mathcal{O}_G \to e_*\mathcal{O}_S$ is surjective, since $e_*\mathcal{O}_S$ is a skyscraper sheaf supported at the neutral element of G with value k. We conclude that e is a closed immersion by Schemes, Lemma 21.24.2. □

Lemma 35.7.3. *Let G be a group scheme over a field k. Then*

(1) *every local ring $\mathcal{O}_{G,g}$ of G has a unique minimal prime ideal,*
(2) *there is exactly one irreducible component Z of G passing through e, and*
(3) *Z is geometrically irreducible over k.*

Proof. For any point $g \in G$ there exists a field extension $k \subset K$ and a K-valued point $g' \in G(K)$ mapping to g. If we think of g' as a K-rational point of the group scheme G_K, then we see that $\mathcal{O}_{G,g} \to \mathcal{O}_{G_K,g'}$ is a faithfully flat local ring map (as $G_K \to G$ is flat, and a local flat ring map is faithfully flat, see Algebra, Lemma 7.35.16). The result for $\mathcal{O}_{G_K,g'}$ implies the result for $\mathcal{O}_{G,g}$, see Algebra, Lemma 7.27.5. Hence in order to prove (1) it suffices to prove it for k-rational points g of G. In this case translation by g defines an automorphism $G \to G$ which maps e to g. Hence $\mathcal{O}_{G,g} \cong \mathcal{O}_{G,e}$. In this way we see that (2) implies (1), since irreducible components passing through e correspond one to one with minimal prime ideals of $\mathcal{O}_{G,e}$.

In order to prove (2) and (3) it suffices to prove (2) when k is algebraically closed. In this case, let Z_1, Z_2 be two irreducible components of G passing through e. Since k is algebraically closed the closed subscheme $Z_1 \times_k Z_2 \subset G \times_k G$ is irreducible too, see Varieties, Lemma 28.6.4. Hence $m(Z_1 \times_k Z_2)$ is contained in an irreducible component of G. On the other hand it contains Z_1 and Z_2 since $m|_{e\times G} = \text{id}_G$ and $m|_{G\times e} = \text{id}_G$. We conclude $Z_1 = Z_2$ as desired. □

Remark 35.7.4. Warning: The result of Lemma 35.7.3 does not mean that every irreducible component of G/k is geometrically irreducible. For example the group scheme $\mu_{3,\mathbf{Q}} = Spec(\mathbf{Q}[x]/(x^3 - 1))$ over $\mathbf{Q}$ has two irreducible components corresponding to the factorization $x^3 - 1 = (x - 1)(x^2 + x + 1)$. The first factor corresponds to the irreducible component passing through the identity, and the second irreducible component is not geometrically irreducible over $Spec(\mathbf{Q})$.

Lemma 35.7.5. *Let G be a group scheme which is locally of finite type over a field k. Then G is equidimensional and* $\dim(G) = \dim_g(G)$ *for all $g \in G$. For any closed point $g \in G$ we have* $\dim(G) = \dim(\mathcal{O}_{G,g})$.

Proof. Let us first prove that $\dim_g(G) = \dim_{g'}(G)$ for any pair of points $g, g' \in G$. By Morphisms, Lemma 24.27.3 we may extend the ground field at will. Hence we may assume that both g and g' are defined over k. Hence there exists an automorphism of G mapping g to g', whence the equality. By Morphisms, Lemma 24.27.1 we have $\dim_g(G) = \dim(\mathcal{O}_{G,g}) + \text{trdeg}_k(\kappa(g))$. On the other hand, the dimension of G (or any open subset of G) is the supremum of the dimensions of the local rings of of G, see Properties, Lemma 23.11.4. Clearly this is maximal for closed points g in which case $\text{trdeg}_k(\kappa(g)) = 0$ (by the Hilbert Nullstellensatz, see Morphisms, Section 24.15). Hence the lemma follows. □

The following result is sometimes referred to as Cartier's theorem.

Lemma 35.7.6. *Let G be a group scheme which is locally of finite type over a field k of characteristic zero. Then the structure morphism $G \to Spec(k)$ is smooth, i.e., G is a smooth group scheme.*

Proof. By Lemma 35.6.3 the module of differentials of G over k is free. Hence smoothness follows from Varieties, Lemma 28.15.1. □

Remark 35.7.7. Any group scheme over a field of characteristic 0 is reduced, see [**Per75**, I, Theorem 1.1 and I, Corollary 3.9, and II, Theorem 2.4] and also [**Per76**, Proposition 4.2.8]. This was a question raised in [**Oor66**, page 80]. We have seen in Lemma 35.7.6 that this holds when the group scheme is locally of finite type.

Lemma 35.7.8. *Let G be a group scheme which is locally of finite type over a perfect field k of characteristic $p > 0$ (see Lemma 35.7.6 for the characteristic zero case). If G is reduced then the structure morphism $G \to Spec(k)$ is smooth, i.e., G is a smooth group scheme.*

Proof. By Lemma 35.6.3 the sheaf $\Omega_{G/k}$ is free. Hence the lemma follows from Varieties, Lemma 28.15.2. □

Remark 35.7.9. Let k be a field of characteristic $p > 0$. Let $\alpha \in k$ be an element which is not a pth power. The closed subgroup scheme

$$G = V(x^p + \alpha y^p) \subset \mathbf{G}_{a,k}^2$$

is reduced and irreducible but not smooth (not even normal).

Lemma 35.7.10. *Let G be a group scheme over a perfect field k. Then the reduction G_{red} of G is a closed subgroup scheme of G.*

Proof. Omitted. Hint: Use that $G_{red} \times_k G_{red}$ is reduced by Varieties, Lemmas 28.4.3 and 28.4.7. □

The next lemma will be generalized slightly in More on Groupoids, Lemma 36.10.2. Namely, if $G' \to G$ is a morphism of group schemes over a field whose image is open, then its image is closed.

Lemma 35.7.11. *Let G be group scheme over a field k. Let $G' \subset G$ be an open subgroup scheme. Then G' is open and closed in G.*

Proof. Suppose that $k \subset K$ is a field extension such that $G'_K \subset G_K$ is closed. Then it follows from Morphisms, Lemma 24.24.10 that G' is closed (as $G_K \to G$ is flat, quasi-compact and surjective). Hence it suffices to prove the lemma after replacing k by some extension. Choose K to be an algebraically closed field extension of very large cardinality. Then by Varieties, Lemma 28.12.2, we see that G_K is a Jacobson scheme all of whose closed points have residue field equal to K. In other words we may assume G is a Jacobson scheme all of whose closed points have residue field k.

Let $Z = G \setminus G'$. We have to show that Z is open. Because G is Jacobson and Z is closed the closed points of Z are dense in Z. Moreover any closed point $z \in Z$ is a k-rational point and hence we translation by z defines an automorphism $L_z : G \to G$, $g \mapsto m(z, g)$ with $e \mapsto z$. As G' is a subgroup scheme we conclude that $L_z(G') \subset Z$. Alltogether we see that

$$Z = \bigcup\nolimits_{z \in Z(k)} L_z(G')$$

is a union of open subsets, and hence open as desired. □

Lemma 35.7.12. *Let $i : G' \to G$ be an immersion of group schemes over a field k. Then i is a closed immersion, i.e., $i(G')$ is a closed subgroup scheme of G.*

Proof. To show that i is a closed immersion it suffices to show that $i(G')$ is a closed subset of G. Let $k \subset k'$ be a perfect extension of k. If $i(G'_{k'}) \subset G_{k'}$ is closed, then $i(G') \subset G$ is closed by Morphisms, Lemma 24.24.10 (as $G_{k'} \to G$ is flat, quasi-compact and surjective). Hence we may and do assume k is perfect. We will use without further mention that products of reduced schemes over k are reduced. We may replace G' and G by their reductions, see Lemma 35.7.10. Let $\overline{G'} \subset G$ be the closure of $i(G')$ viewed as a reduced closed subscheme. By Varieties, Lemma 28.14.1 we conclude that $\overline{G'} \times_k \overline{G'}$ is the closure of the image of $G' \times_k G' \to G \times_k G$. Hence

$$m\left(\overline{G'} \times_k \overline{G'}\right) \subset \overline{G'}$$

as m is continuous. It follows that $\overline{G'} \subset G$ is a (reduced) closed subgroup scheme. By Lemma 35.7.11 we see that $i(G') \subset \overline{G'}$ is also closed which implies that $i(G') = \overline{G'}$ as desired. □

Lemma 35.7.13. *Let G be a group scheme over a field. There exists an open and closed subscheme $G' \subset G$ which is a countable union of affines.*

Proof. Let $e \in U(k)$ be a quasi-compact open neighbourhood of the identity element. By replacing U by $U \cap i(U)$ we may assume that U is invariant under the inverse map. As G is separated this is still a quasi-compact set. Set

$$G' = \bigcup\nolimits_{n \geq 1} m_n(U \times_k \ldots \times_k U)$$

where $m_n : G \times_k \ldots \times_k G \to G$ is the n-slot multiplication map $(g_1, \ldots, g_n) \mapsto m(m(\ldots(m(g_1, g_2), g_3), \ldots), g_n)$. Each of these maps are open (see Lemma 35.7.1) hence G' is an open subgroup scheme. By Lemma 35.7.11 it is also a closed subgroup scheme. □

Remark 35.7.14. If G is a group scheme over a field, is there always a quasi-compact open and closed subgroup scheme? Or is there a counter example?

35.8. Actions of group schemes

Let (G, m) be a group and let V be a set. Recall that a *(left) action* of G on V is given by a map $a : G \times V \to V$ such that

(1) (associativity) $a(m(g, g'), v) = a(g, a(g', v))$ for all $g, g' \in G$ and $v \in V$, and
(2) (identity) $a(e, v) = v$ for all $v \in V$.

We also say that V is a *G-set* (this usually means we drop the a from the notation -- which is abuse of notation). A *map of G-sets* $\psi : V \to V'$ is any set map such that $\psi(a(g, v)) = a(g, \psi(v))$ for all $v \in V$.

Definition 35.8.1. Let S be a scheme. Let (G, m) be a group scheme over S.

(1) An *action of G on the scheme X/S* is a morphism $a : G \times_S X \to X$ over S such that for every T/S the map $a : G(T) \times X(T) \to X(T)$ defines the structure of a $G(T)$-set on $X(T)$.
(2) Suppose that X, Y are schemes over S each endowed with an action of G. An *equivariant* or more precisely a *G-equivariant* morphism $\psi : X \to Y$ is a morphism of schemes over S such that for every T/S the map $\psi : X(T) \to Y(T)$ is a morphism of $G(T)$-sets.

In situation (1) this means that the diagrams

$$\begin{tikzcd} G \times_S G \times_S X \arrow[r, "1_G \times a"'] \arrow[d, "m \times 1_X"'] & G \times_S X \arrow[d, "a"] \\ G \times_S X \arrow[r, "a"] & X \end{tikzcd} \qquad \begin{tikzcd} G \times_S X \arrow[r, "a"'] & X \\ X \arrow[u, "e \times 1_X"] \arrow[ur, "1_X"'] & \end{tikzcd} \tag{35.8.1.1}$$

are commutative. In situation (2) this just means that the diagram

$$\begin{tikzcd} G \times_S X \arrow[r, "\mathrm{id} \times f"'] \arrow[d, "a"'] & G \times_S Y \arrow[d, "a"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

commutes.

35.9. Principal homogeneous spaces

In Cohomology on Sites, Definition 19.5.1 we have defined a torsor for a sheaf of groups on a site. Suppose $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$ is a topology and (G, m) is a group scheme over S. Since τ is stronger than the canonical topology (see Descent, Lemma 31.9.3) we see that $\underline{G}$ (see Sites, Definition 9.12.3) is a sheaf of groups on $(Sch/S)_\tau$. Hence we already know what it means to have a torsor for $\underline{G}$ on $(Sch/S)_\tau$. A special situation arises if this sheaf is representable. In the following definitions we define directly what it means for the representing scheme to be a G-torsor.

Definition 35.9.1. Let S be a scheme. Let (G, m) be a group scheme over S. Let X be a scheme over S, and let $a : G \times_S X \to X$ be an action of G on X.

(1) We say X is a *pseudo G-torsor* or that X is *formally principally homogeneous under G* if the induced morphism of schemes $G \times_S X \to X \times_S X$, $(g, x) \mapsto (a(g, x), x)$ is an isomorphism of schemes over S.
(2) A pseudo G-torsor X is called *trivial* if there exists an G-equivariant isomorphism $G \to X$ over S where G acts on G by left multiplication.

It is clear that if $S' \to S$ is a morphism of schemes then the pullback $X_{S'}$ of a pseudo G-torsor over S is a pseudo $G_{S'}$-torsor over S'.

Lemma 35.9.2. *In the situation of Definition 35.9.1.*

(1) *The scheme X is a pseudo G-torsor if and only if for every scheme T over S the set $X(T)$ is either empty or the action of the group $G(T)$ on $X(T)$ is simply transitive.*

(2) *A pseudo G-torsor X is trivial if and only if the morphism $X \to S$ has a section.*

Proof. Omitted. □

Definition 35.9.3. Let S be a scheme. Let (G, m) be a group scheme over S. Let X be a pseudo G-torsor over S.

(1) We say X is a *principal homogeneous space* or a *G-torsor* if there exists a fpqc covering[1] $\{S_i \to S\}_{i \in I}$ such that each $X_{S_i} \to S_i$ has a section (i.e., is a trivial pseudo G_{S_i}-torsor).
(2) Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. We say X is a *G-torsor in the τ topology*, or a *τ G-torsor*, or simply a *τ torsor* if there exists a τ covering $\{S_i \to S\}_{i \in I}$ such that each $X_{S_i} \to S_i$ has a section.
(3) If X is a G-torsor, then we say that it is *quasi-isotrivial* if it is a torsor for the étale topology.
(4) If X is a G-torsor, then we say that it is *locally trivial* if it is a torsor for the Zariski topology.

We sometimes say ``let X be a G-torsor over S'' to indicate that X is a scheme over S equippend with an action of G which turns it into a principal homogeneous space over S. Next we show that this agrees with the notation introduced earlier when both apply.

Lemma 35.9.4. *Let S be a scheme. Let (G, m) be a group scheme over S. Let X be a scheme over S, and let $a : G \times_S X \to X$ be an action of G on X. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Then X is a G-torsor in the τ-topology if and only if $\underline{X}$ is a $\underline{G}$-torsor on $(Sch/S)_\tau$.*

Proof. Omitted. □

Remark 35.9.5. Let (G, m) be a group scheme over the scheme S. In this situation we have the following natural types of questions:

(1) If $X \to S$ is a pseudo G-torsor and $X \to S$ is surjective, then is X necessarily a G-torsor?
(2) Is every $\underline{G}$-torsor on $(Sch/S)_{fppf}$ representable? In other words, does every $\underline{G}$-torsor come from a fppf G-torsor?
(3) Is every G-torsor an fppf (resp. smooth, resp. étale, resp. Zariski) torsor?

In general the answers to these questions is no. To get a positive answer we need to impose additional conditions on $G \to S$. For example: If S is the spectrum of a field, then the answer to (1) is yes because then $\{X \to S\}$ is a fpqc covering trivializing X. If $G \to S$ is affine, then the answer to (2) is yes (insert future reference here). If $G = \mathrm{GL}_{n,S}$ then the answer to (3) is yes and in fact any $\mathrm{GL}_{n,S}$-torsor is locally trivial (insert future reference here).

35.10. Equivariant quasi-coherent sheaves

We think of ``functions'' as dual to ``space''. Thus for a morphism of spaces the map on functions goes the other way. Moreover, we think of the sections of a sheaf of modules as ``functions''. This leads us naturally to the direction of the arrows chosen in the following definition.

[1] This means that the default type of torsor is a pseudo torsor which is trivial on an fpqc covering. This is the definition in [**ABD+66**, Exposé IV, 6.5]. It is a little bit inconvenient for us as we most often work in the fppf topology.

Definition 35.10.1. Let S be a scheme, let (G, m) be a group scheme over S, and let $a : G \times_S X \to X$ be an action of the group scheme G on X/S. An *G-equivariant quasi-coherent $\mathcal{O}_X$-module*, or simply a *equivariant quasi-coherent $\mathcal{O}_X$-module*, is a pair $(\mathcal{F}, \alpha)$, where $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module, and α is a $\mathcal{O}_{G\times_S X}$-module map

$$\alpha : a^*\mathcal{F} \longrightarrow \mathrm{pr}_1^*\mathcal{F}$$

where $\mathrm{pr}_1 : G \times_S X \to X$ is the projection such that

(1) the diagram

$$\begin{tikzcd}
(1_G \times a)^*\mathrm{pr}_2^*\mathcal{F} \arrow[r, "\mathrm{pr}_{12}^*\alpha"'] & \mathrm{pr}_2^*\mathcal{F} \\
(1_G \times a)^*a^*\mathcal{F} \arrow[u, "(1_G\times a)^*\alpha"] \arrow[r, equal] & (m \times 1_X)^*a^*\mathcal{F} \arrow[u, "(m\times 1_X)^*\alpha"']
\end{tikzcd}$$

is a commutative in the category of $\mathcal{O}_{G\times_S G\times_S X}$-modules, and

(2) the pullback

$$(e \times 1_X)^*\alpha : \mathcal{F} \longrightarrow \mathcal{F}$$

is the identity map.

For explanation compare with the relevant diagrams of Equation (35.8.1.1).

Note that the commutativity of the first diagram garantees that $(e \times 1_X)^*\alpha$ is an idempotent operator on $\mathcal{F}$, and hence condition (2) is just the condition that it is an isomorphism.

Lemma 35.10.2. *Let S be a scheme. Let G be a group scheme over S. Let $f : X \to Y$ be a G-equivariant morphism between S-schemes endowed with G-actions. Then pullback f^* given by $(\mathcal{F}, \alpha) \mapsto (f^*\mathcal{F}, (1_G \times f)^*\alpha)$ defines a functor from the category of G-equivariant sheaves on X to the category of quasi-coherent G-equivariant sheaves on Y.*

Proof. Omitted. □

35.11. Groupoids

Recall that a groupoid is a category in which every morphism is an isomorphism, see Categories, Definition 4.2.5. Hence a groupoid has a set of objects Ob, a set of arrows Arrows, a *source* and *target* map s, t : Arrows $\to$ Ob, and a *composition law* c : Arrows $\times_{s,\mathrm{Ob},t}$ Arrows $\to$ Arrows. These maps satisfy exactly the following axioms

(1) (associativity) $c\circ(1, c) = c\circ(c, 1)$ as maps Arrows$\times_{s,\mathrm{Ob},t}$Arrows$\times_{s,\mathrm{Ob},t}$Arrows $\to$ Arrows,
(2) (identity) there exists a map e : Ob $\to$ Arrows such that
  (a) $s \circ e = t \circ e = \mathrm{id}$ as maps Ob $\to$ Ob,
  (b) $c \circ (1, e \circ s) = c \circ (e \circ t, 1) = 1$ as maps Arrows $\to$ Arrows,
(3) (inverse) there exists a map i : Arrows $\to$ Arrows such that
  (a) $s \circ i = t$, $t \circ i = s$ as maps Arrows $\to$ Ob, and
  (b) $c \circ (1, i) = e \circ t$ and $c \circ (i, 1) = e \circ s$ as maps Arrows $\to$ Arrows.

If this is the case the maps e and i are uniquely determined and i is a bijection. Note that if (Ob′, Arrows′, s', t', c') is a second groupoid category, then a functor f : (Ob, Arrows, s, t, c) $\to$ (Ob′, Arrows′, s', t', c') is given by a pair of set maps f : Ob $\to$ Ob′ and f : Arrows $\to$ Arrows′ such that $s' \circ f = f \circ s$, $t' \circ f = f \circ t$, and $c' \circ (f, f) = f \circ c$. The compatibility with identity and inverse is automatic. We will use this below. (Warning: The compatibility with identity has to be imposed in the case of general categories.)

Definition 35.11.1. Let S be a scheme.

(1) A *groupoid scheme over S*, or simply a *groupoid over S* is a quintuple (U, R, s, t, c) where U and R are schemes over S, and $s, t : R \to U$ and $c : R \times_{s,U,t} R \to R$ are morphisms of schemes over S with the following property: For any scheme T over S the quintuple
$$(U(T), R(T), s, t, c)$$
is a groupoid category in the sense described above.

(2) A *morphism $f : (U, R, s, t, c) \to (U', R', s', t', c')$ of groupoid schemes over S* is given by morphisms of schemes $f : U \to U'$ and $f : R \to R'$ with the following property: For any scheme T over S the maps f define a functor from the groupoid category $(U(T), R(T), s, t, c)$ to the groupoid category $(U'(T), R'(T), s', t', c')$.

Let (U, R, s, t, c) be a groupoid over S. Note that, by the remarks preceding the definition and the Yoneda lemma, there are unique morphisms of schemes $e : U \to R$ and $i : R \to R$ over S such that for every scheme T over S the induced map $e : U(T) \to R(T)$ is the identity, and $i : R(T) \to R(T)$ is the inverse of the groupoid category. The septuple (U, R, s, t, c, e, i) satisfies commutative diagrams corresponding to each of the axioms (1), (2)(a), (2)(b), (3)(a) and (3)(b) above, and conversely given a septuple with this property the quintuple (U, R, s, t, c) is a groupoid scheme. Note that i is an isomorphism, and e is a section of both s and t. Moreover, given a groupoid scheme over S we denote
$$j = (t, s) : R \longrightarrow U \times_S U$$
which is compatible with our conventions in Section 35.3 above. We sometimes say ``let (U, R, s, t, c, e, i) be a groupoid over S'' to stress the existence of identity and inverse.

Lemma 35.11.2. *Given a groupoid scheme (U, R, s, t, c) over S the morphism $j : R \to U \times_S U$ is a pre-equivalence relation.*

Proof. Omitted. This is a nice exercise in the definitions. ☐

Lemma 35.11.3. *Given an equivalence relation $j : R \to U$ over S there is a unique way to extend it to a groupoid (U, R, s, t, c) over S.*

Proof. Omitted. This is a nice exercise in the definitions. ☐

Lemma 35.11.4. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid over S. In the commutative diagram*

$$\begin{tikzcd} & U & \\ R \arrow[ur, "t"] \arrow[d, "s"'] & R \times_{s,U,t} R \arrow[l, "pr_0"] \arrow[r, "c"'] \arrow[d, "pr_1"] & R \arrow[ul, "t"'] \arrow[d, "s"] \\ U & R \arrow[l, "t"] \arrow[r, "s"'] & U \end{tikzcd}$$

the two lower squares are fibre product squares. Moreover, the triangle on top (which is really a square) is also cartesian.

Proof. Omitted. Exercise in the definitions and the functorial point of view in algebraic geometry. ☐

Lemma 35.11.5. *Let S be a scheme. Let (U, R, s, t, c, e, i) be a groupoid over S. The diagram*

(35.11.5.1)
$$\begin{tikzcd} R \times_{t,U,t} R \arrow[r, shift left, "pr_1"] \arrow[r, shift right, "pr_0"'] \arrow[d, "pr_0 \times c\circ(i,1)"'] & R \arrow[r, "t"] \arrow[d, "id_R"] & U \arrow[d, "id_U"] \\ R \times_{s,U,t} R \arrow[r, shift left, "c"] \arrow[r, shift right, "pr_0"'] \arrow[d, "pr_1"'] & R \arrow[r, "t"] \arrow[d, "s"] & U \\ R \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & U & \end{tikzcd}$$

is commutative. The two top rows are isomorphic via the vertical maps given. The two lower left squares are cartesian.

Proof. The commutativity of the diagram follows from the axioms of a groupoid. Note that, in terms of groupoids, the top left vertical arrow assigns to a pair of morphisms (α, β) with the same target, the pair of morphisms $(\alpha, \alpha^{-1} \circ \beta)$. In any groupoid this defines a bijection between Arrows $\times_{t,\mathrm{Ob},t}$ Arrows and Arrows $\times_{s,\mathrm{Ob},t}$ Arrows. Hence the second assertion of the lemma. The last assertion follows from Lemma 35.11.4. $\square$

35.12. Quasi-coherent sheaves on groupoids

See the introduction of Section 35.10 for our choices in direction of arrows.

Definition 35.12.1. Let S be a scheme, let (U, R, s, t, c) be a groupoid scheme over S. A *quasi-coherent module on* (U, R, s, t, c) is a pair $(\mathcal{F}, \alpha)$, where $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_U$-module, and α is a $\mathcal{O}_R$-module map

$$\alpha : t^*\mathcal{F} \longrightarrow s^*\mathcal{F}$$

such that

(1) the diagram

$$\begin{tikzcd} & \mathrm{pr}_1^* t^*\mathcal{F} \arrow[r, "\mathrm{pr}_1^*\alpha"'] \arrow[dl, equal] & \mathrm{pr}_1^* s^*\mathcal{F} \arrow[dr, equal] & \\ \mathrm{pr}_0^* s^*\mathcal{F} & & & c^*s^*\mathcal{F} \\ & \mathrm{pr}_0^* t^*\mathcal{F} \arrow[ul, "\mathrm{pr}_0^*\alpha"] \arrow[r, equal] & c^*t^*\mathcal{F} \arrow[ur, "c^*\alpha"'] & \end{tikzcd}$$

is a commutative in the category of $\mathcal{O}_{R\times_{s,U,t}R}$-modules, and

(2) the pullback

$$e^*\alpha : \mathcal{F} \longrightarrow \mathcal{F}$$

is the identity map.

Compare with the commutative diagrams of Lemma 35.11.4.

The commutativity of the first diagram forces the operator $e^*\alpha$ to be idempotent. Hence the second condition can be reformulated as saying that $e^*\alpha$ is an isomorphism. In fact, the condition implies that α is an isomorphism.

Lemma 35.12.2. *Let S be a scheme, let (U, R, s, t, c) be a groupoid scheme over S. If $(\mathcal{F}, \alpha)$ is a quasi-coherent module on (U, R, s, t, c) then α is an isomorphism.*

Proof. Pull back the commutative diagram of Definition 35.12.1 by the morphism $(i, 1) : R \to R \times_{s,U,t} R$. Then we see that $i^*\alpha \circ \alpha = s^*e^*\alpha$. Pulling back by the morphism $(1, i)$ we obtain the relation $\alpha \circ i^*\alpha = t^*e^*\alpha$. By the second assumption these morphisms are the identity. Hence $i^*\alpha$ is an inverse of α. $\square$

Lemma 35.12.3. *Let S be a scheme. Consider a morphism $f : (U, R, s, t, c) \to (U', R', s', t', c')$ of groupoid schemes over S. Then pullback f^* given by*

$$(\mathcal{F}, \alpha) \mapsto (f^*\mathcal{F}, f^*\alpha)$$

defines a functor from the category of quasi-coherent sheaves on (U', R', s', t', c') to the category of quasi-coherent sheaves on (U, R, s, t, c).

Proof. Omitted. $\square$

Lemma 35.12.4. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. The category of quasi-coherent modules on (U, R, s, t, c) has colimits.*

Proof. Let $i \mapsto (\mathcal{F}_i, \alpha_i)$ be a diagram over the index category $\mathcal{I}$. We can form the colimit $\mathcal{F} = colim\, \mathcal{F}_i$ which is a quasi-coherent sheaf on U, see Schemes, Section 21.24. Since colimits commute with pullback we see that $s^*\mathcal{F} = colim\, s^*\mathcal{F}_i$ and similarly $t^*\mathcal{F} = colim\, t^*\mathcal{F}_i$. Hence we can set $\alpha = colim\, \alpha_i$. We omit the proof that $(\mathcal{F}, \alpha)$ is the colimit of the diagram in the category of quasi-coherent modules on (U, R, s, t, c). $\square$

Lemma 35.12.5. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. If s, t are flat, then the category of quasi-coherent modules on (U, R, s, t, c) is abelian.*

Proof. Let $\varphi : (\mathcal{F}, \alpha) \to (\mathcal{G}, \beta)$ be a homomorphism of quasi-coherent modules on (U, R, s, t, c). Since s is flat we see that

$$0 \to s^*\text{Ker}(\varphi) \to s^*\mathcal{F} \to s^*\mathcal{G} \to s^*\text{Coker}(\varphi) \to 0$$

is exact and similarly for pullback by t. Hence α and β induce isomorphisms $\kappa : t^*\text{Ker}(\varphi) \to s^*\text{Ker}(\varphi)$ and $\lambda : t^*\text{Coker}(\varphi) \to s^*\text{Coker}(\varphi)$ which satisfy the cocycle condition. Then it is straightforward to verify that $(\text{Ker}(\varphi), \kappa)$ and $(\text{Coker}(\varphi), \lambda)$ are a kernel and cokernel in the category of quasi-coherent modules on (U, R, s, t, c). Moreover, the condition $\text{Coim}(\varphi) = \text{Im}(\varphi)$ follows because it holds over U. $\square$

Let S be a scheme. Let (U, R, s, t, c) be a groupoid in schemes over S. Let κ be a cardinal. In the following we will say that a quasi-coherent sheaf $(\mathcal{F}, \alpha)$ on (U, R, s, t, c) is κ-generated if $\mathcal{F}$ is a κ-generated $\mathcal{O}_U$-module, see Properties, Definition 23.21.1.

Lemma 35.12.6. *Let (U, R, s, t, c) be a groupoid scheme over S. Let κ be a cardinal. There exists a set T and a family $(\mathcal{F}_t, \alpha_t)_{t \in T}$ of κ-generated quasi-coherent modules on (U, R, s, t, c) such that every κ-generated quasi-coherent module on (U, R, s, t, c) is isomorphic to one of the $(\mathcal{F}_t, \alpha_t)$.*

Proof. For each quasi-coherent module $\mathcal{F}$ on U there is a (possibly empty) set of maps $\alpha : t^*\mathcal{F} \to s^*\mathcal{F}$ such that $(\mathcal{F}, \alpha)$ is a quasi-coherent modules on (U, R, s, t, c). By Properties, Lemma 23.21.2 there exists a set of isomorphism classes of κ-generated quasi-coherent $\mathcal{O}_U$-modules. $\square$

Lemma 35.12.7. *Let (U, R, s, t, c) be a groupoid scheme over S. Assume that s, t are flat. There exists a cardinal κ such that every quasi-coherent module $(\mathcal{F}, \alpha)$ on (U, R, s, t, c) is the directed colimit of its κ-generated quasi-coherent submodules.*

Proof. In the statement of the lemma and in this proof a *submodule* of a quasi-coherent module $(\mathcal{F}, \alpha)$ is a quasi-coherent submodule $\mathcal{G} \subset \mathcal{F}$ such that $\alpha(t^*\mathcal{G}) = s^*\mathcal{G}$ as subsheaves of $s^*\mathcal{F}$. This makes sense because since s, t are flat the pullbacks s^* and t^* are exact, i.e., preserve subsheaves. The proof will be a repeat of the proof of Properties, Lemma 23.21.3. We urge the reader to read that proof first.

Choose an affine open covering $U = \bigcup_{i \in I} U_i$. For each pair i, j choose affine open coverings

$$U_i \cap U_j = \bigcup\nolimits_{k \in I_{ij}} U_{ijk} \quad \text{and} \quad s^{-1}(U_i) \cap t^{-1}(U_j) = \bigcup\nolimits_{k \in J_{ij}} W_{ijk}.$$

Write $U_i = \mathrm{Spec}(A_i)$, $U_{ijk} = \mathrm{Spec}(A_{ijk})$, $W_{ijk} = \mathrm{Spec}(B_{ijk})$. Let κ be any infinite cardinal $\geq$ than the cardinality of any of the sets I, I_{ij}, J_{ij}.

Let $(\mathcal{F}, \alpha)$ be a quasi-coherent module on (U, R, s, t, c). Set $M_i = \mathcal{F}(U_i)$, $M_{ijk} = \mathcal{F}(U_{ijk})$. Note that

$$M_i \otimes_{A_i} A_{ijk} = M_{ijk} = M_j \otimes_{A_j} A_{ijk}$$

and that α gives isomorphisms

$$\alpha|_{W_{ijk}} : M_i \otimes_{A_i, t} B_{ijk} \longrightarrow M_j \otimes_{A_j, s} B_{ijk}$$

see Schemes, Lemma 21.7.3. Using the axiom of choice we choose a map

$$(i, j, k, m) \mapsto S(i, j, k, m)$$

which associates to every $i, j \in I$, $k \in I_{ij}$ or $k \in J_{ij}$ and $m \in M_i$ a finite subset $S(i, j, k, m) \subset M_j$ such that we have

$$m \otimes 1 = \sum\nolimits_{m' \in S(i,j,k,m)} m' \otimes a_{m'} \quad \text{or} \quad \alpha(m \otimes 1) = \sum\nolimits_{m' \in S(i,j,k,m)} m' \otimes b_{m'}$$

in M_{ijk} for some $a_{m'} \in A_{ijk}$ or $b_{m'} \in B_{ijk}$. Moreover, let's agree that $S(i, i, k, m) = \{m\}$ for all $i, j = i, k, m$ when $k \in I_{ij}$. Fix such a collection $S(i, j, k, m)$

Given a family $\mathcal{S} = (S_i)_{i \in I}$ of subsets $S_i \subset M_i$ of cardinality at most κ we set $\mathcal{S}' = (S'_i)$ where

$$S'_j = \bigcup\nolimits_{(i,j,k,m) \text{ such that } m \in S_i} S(i, j, k, m)$$

Note that $S_i \subset S'_i$. Note that S'_i has cardinality at most κ because it is a union over a set of cardinality at most κ of finite sets. Set $\mathcal{S}^{(0)} = \mathcal{S}$, $\mathcal{S}^{(1)} = \mathcal{S}'$ and by induction $\mathcal{S}^{(n+1)} = (\mathcal{S}^{(n)})'$. Then set $\mathcal{S}^{(\infty)} = \bigcup_{n \geq 0} \mathcal{S}^{(n)}$. Writing $\mathcal{S}^{(\infty)} = (S_i^{(\infty)})$ we see that for any element $m \in S_i^{(\infty)}$ the image of m in M_{ijk} can be written as a finite sum $\sum m' \otimes a_{m'}$ with $m' \in S_j^{(\infty)}$. In this way we see that setting

$$N_i = A_i\text{-submodule of } M_i \text{ generated by } S_i^{(\infty)}$$

we have

$$N_i \otimes_{A_i} A_{ijk} = N_j \otimes_{A_j} A_{ijk} \quad \text{and} \quad \alpha(N_i \otimes_{A_i, t} B_{ijk}) = N_j \otimes_{A_j, s} B_{ijk}$$

as submodules of M_{ijk} or $M_j \otimes_{A_j, s} B_{ijk}$. Thus there exists a quasi-coherent submodule $\mathcal{G} \subset \mathcal{F}$ with $\mathcal{G}(U_i) = N_i$ such that $\alpha(t^*\mathcal{G}) = s^*\mathcal{G}$ as submodules of $s^*\mathcal{F}$. In other words, $(\mathcal{G}, \alpha|_{t^*\mathcal{G}})$ is a submodule of $(\mathcal{F}, \alpha)$. Moreover, by construction $\mathcal{G}$ is κ-generated.

Let $\{(\mathcal{G}_t, \alpha_t)\}_{t\in T}$ be the set of κ-generated quasi-coherent submodules of $(\mathcal{F}, \alpha)$. If $t, t' \in T$ then $\mathcal{G}_t + \mathcal{G}_{t'}$ is also a κ-generated quasi-coherent submodule as it is the image of the map $\mathcal{G}_t \oplus \mathcal{G}_{t'} \to \mathcal{F}$. Hence the system (ordered by inclusion) is directed. The arguments above show that every section of $\mathcal{F}$ over U_i is in one of the $\mathcal{G}_t$ (because we can start with $\mathcal{S}$ such that the given section is an element of S_i). Hence $colim_t \mathcal{G}_t \to \mathcal{F}$ is both injective and surjective as desired. □

35.13. Groupoids and group schemes

There are many ways to construct a groupoid out of an action a of a group G on a set V. We choose the one where we think of an element $g \in G$ as an arrow with source v and target $a(g, v)$. This leads to the following construction for group actions of schemes.

Lemma 35.13.1. *Let S be a scheme. Let Y be a scheme over S. Let (G, m) be a group scheme over Y with identity e_G and inverse i_G. Let X/Y be a scheme over Y and let $a : G \times_Y X \to X$ be an action of G on X/Y. Then we get a groupoid scheme (U, R, s, t, c, e, i) over S in the following manner:*

1. *We set $U = X$, and $R = G \times_Y X$.*
2. *We set $s : R \to U$ equal to $(g, x) \mapsto x$.*
3. *We set $t : R \to U$ equal to $(g, x) \mapsto a(g, x)$.*
4. *We set $c : R \times_{s,U,t} R \to R$ equal to $((g, x), (g', x')) \mapsto (m(g, g'), x')$.*
5. *We set $e : U \to R$ equal to $x \mapsto (e_G(x), x)$.*
6. *We set $i : R \to R$ equal to $(g, x) \mapsto (i_G(g), a(g, x))$.*

Proof. Omitted. Hint: It is enough to show that this works on the set level. For this use the description above the lemma describing g as an arrow from v to $a(g, v)$. □

Lemma 35.13.2. *Let S be a scheme. Let Y be a scheme over S. Let (G, m) be a group scheme over Y. Let X be a scheme over Y and let $a : G \times_Y X \to X$ be an action of G on X over Y. Let (U, R, s, t, c) be the groupoid scheme constructed in Lemma 35.13.1. The rule $(\mathcal{F}, \alpha) \mapsto (\mathcal{F}, \alpha)$ defines an equivalence of categories between G-equivariant $\mathcal{O}_X$-modules and the category of quasi-coherent modules on (U, R, s, t, c).*

Proof. The assertion makes sense because $t = a$ and $s = \mathrm{pr}_1$ as morphisms $R = G \times_Y X \to X$, see Definitions 35.10.1 and 35.12.1. Using the translation in Lemma 35.13.1 the commutativity requirements of the two definitions match up exactly. □

35.14. The stabilizer group scheme

Given a groupoid scheme we get a group scheme as follows.

Lemma 35.14.1. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid over S. The scheme G defined by the cartesian square*

$$\begin{tikzcd} G \arrow[r] \arrow[d] & R \arrow[d, "j=(t,s)"] \\ U \arrow[r, "\Delta"] & U \times_S U \end{tikzcd}$$

is a group scheme over U with compostion law m induced by the composition law c.

Proof. This is true because in a groupoid category the set of self maps of any object forms a group. □

Since Δ is an immersion we see that $G = j^{-1}(\Delta_{U/S})$ is a locally closed subscheme of R. Thinking of it in this way, the structure morphism $j^{-1}(\Delta_{U/S}) \to U$ is induced by either s or t (it is the same), and m is induced by c.

Definition 35.14.2. Let S be a scheme. Let (U, R, s, t, c) be a groupoid over S. The group scheme $j^{-1}(\Delta_{U/S}) \to U$ is called the *stabilizer of the groupoid scheme* (U, R, s, t, c).

In the literature the stabilizer group scheme is often denoted S (because the word stabilizer starts with an ``s'' presumably); we cannot do this since we have already used S for the base scheme.

Lemma 35.14.3. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid over S, and let G/U be its stabilizer. Denote R_t/U the scheme R seen as a scheme over U via the morphism $t : R \to U$. There is a canonical left action*

$$a : G \times_U R_t \longrightarrow R_t$$

induced by the composition law c.

Proof. In terms of points over T/S we define $a(g, r) = c(g, r)$. □

Lemma 35.14.4. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let G be the stabilizer group scheme of R. Let*

$$G_0 = G \times_{U, pr_0} (U \times_S U) = G \times_S U$$

as a group scheme over $U \times_S U$. The action of G on R of Lemma 35.14.3 induces an action of G_0 on R over $U \times_S U$ which turns R into a pseudo G_0-torsor over $U \times_S U$.

Proof. This is true because in a groupoid category $\mathcal{C}$ the set $Mor_{\mathcal{C}}(x, y)$ is a principal homogeneous set under the group $Mor_{\mathcal{C}}(y, y)$. □

Lemma 35.14.5. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $p \in U \times_S U$ be a point. Denote R_p the scheme theoretic fibre of $j = (t, s) : R \to U \times_S U$. If $R_p \neq \emptyset$, then the action*

$$G_{0,\kappa(p)} \times_{\kappa(p)} R_p \longrightarrow R_p$$

(see Lemma 35.14.4) which turns R_p into a $G_{\kappa(p)}$-torsor over $\kappa(p)$.

Proof. The action is a pseudo-torsor by the lemma cited in the statement. And if R_p is not the empty scheme, then $\{R_p \to p\}$ is an fpqc covering which trivializs the pseudo-torsor. □

35.15. Restricting groupoids

Consider a (usual) groupoid $\mathcal{C} = (\text{Ob}, \text{Arrows}, s, t, c)$. Suppose we have a map of sets $g : \text{Ob}' \to \text{Ob}$. Then we can construct a groupoid $\mathcal{C}' = (\text{Ob}', \text{Arrows}', s', t', c')$ by thinking of a morphism between elements x', y' of Ob' as a morphisms in $\mathcal{C}$ between $g(x'), g(y')$. In other words we set

$$\text{Arrows}' = \text{Ob}' \times_{g,\text{Ob},t} \text{Arrows} \times_{s,\text{Ob},g} \text{Ob}'.$$

with obvious choices for s', t', and c'. There is a canonical functor $\mathcal{C}' \to \mathcal{C}$ which is fully faithful, but not necessarily essentially surjective. This groupoid $\mathcal{C}'$ endowed with the functor $\mathcal{C}' \to \mathcal{C}$ is called the *restriction* of the groupoid $\mathcal{C}$ to Ob'.

Lemma 35.15.1. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $g : U' \to U$ be a morphism of schemes. Consider the following diagram*

$$\begin{tikzcd}
R' \arrow[r] \arrow[d] \arrow[rr, bend left, "s'"] \arrow[dd, bend right, "t'"'] & R \times_{s,U} U' \arrow[r] \arrow[d] & U' \arrow[d, "g"] \\
U' \times_{U,t} R \arrow[r] \arrow[d] & R \arrow[r, "s"] \arrow[d, "t"] & U \\
U' \arrow[r, "g"] & U &
\end{tikzcd}$$

where all the squares are fibre product squares. Then there is a canonical composition law $c' : R' \times_{s',U',t'} R' \to R'$ such that (U', R', s', t', c') is a groupoid scheme over S and such that $U' \to U$, $R' \to R$ defines a morphism $(U', R', s', t', c') \to (U, R, s, t, c)$ of groupoid schemes over S. Moreover, for any scheme T over S the functor of groupoids

$$(U'(T), R'(T), s', t', c') \to (U(T), R(T), s, t, c)$$

is the restriction (see above) of $(U(T), R(T), s, t, c)$ via the map $U'(T) \to U(T)$.

Proof. Omitted. $\square$

Definition 35.15.2. Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $g : U' \to U$ be a morphism of schemes. The morphism of groupoids $(U', R', s', t', c') \to (U, R, s, t, c)$ constructed in Lemma 35.15.1 is called the *restriction of (U, R, s, t, c) to U'*. We sometime use the notation $R' = R|_{U'}$ in this case.

Lemma 35.15.3. *The notions of restricting groupoids and (pre-)equivalence relations defined in Definitions 35.15.2 and 35.3.3 agree via the constructions of Lemmas 35.11.2 and 35.11.3.*

Proof. What we are saying here is that R' of Lemma 35.15.1 is also equal to

$$R' = (U' \times_S U') \times_{U \times_S U} R \longrightarrow U' \times_S U'$$

In fact this might have been a clearer way to state that lemma. $\square$

Lemma 35.15.4. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $g : U' \to U$ be a morphism of schemes. Let (U', R', s', t', c') be the restriction of (U, R, s, t, c) via g. Let G be the stabilizer of (U, R, s, t, c) and let G' be the stabilizer of (U', R', s', t', c'). Then G' is the base change of G by g, i.e., there is a canonical identification $G' = U' \times_{g,U} G$.*

Proof. Omitted. $\square$

35.16. Invariant subschemes

In this section we discuss briefly the notion of an invariant subscheme.

Definition 35.16.1. Let (U, R, s, t, c) be a groupoid scheme over the base scheme S.

1. We say an open $W \subset U$ is *R-invariant* if $t(s^{-1}(W)) \subset W$.
2. A closed subscheme $Z \subset U$ is called *R-invariant* if $t^{-1}(Z) = s^{-1}(Z)$. Here we use the scheme theoretic inverse image, see Schemes, Definition 21.17.7.
3. A monomorphism of schemes $T \to U$ is *R-invariant* if $T \times_{U,t} R = R \times_{s,U} T$ as schemes over R.

For an open subscheme $W \subset U$ the R-invariance is also equivalent to requiring that $s^{-1}(W) = t^{-1}(W)$. If $W \subset U$ is R-equivariant then the restriction of R to W is just $R_W = s^{-1}(W) = t^{-1}(W)$. Similarly, if $Z \subset U$ is an R-invariant closed subscheme, then the restriction of R to Z is just $R_Z = s^{-1}(Z) = t^{-1}(Z)$.

Lemma 35.16.2. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S.*

(1) *If s and t are open, then for every open $W \subset U$ the open $s(t^{-1}(W))$ is R-invariant.*
(2) *If s and t are open and quasi-compact, then U has an open covering consisting of R-invariant quasi-compact open subschemes.*

Proof. Assume s and t open and $W \subset U$ open. Since s is open the set $W' = s(t^{-1}(W))$ is an open subset of U. Now it is quite easy to using the functorial point of view that this is an R-invariant open subset of U, but we are going to argue this directly by some diagrams, since we think it is instructive. Note that $t^{-1}(W')$ is the image of the morphism

$$A := t^{-1}(W) \times_{s|_{t^{-1}(W)}, U, t} R \xrightarrow{\mathrm{pr}_1} R$$

and that $s^{-1}(W')$ is the image of the morphism

$$B := R \times_{s, U, s|_{t^{-1}(W)}} t^{-1}(W) \xrightarrow{\mathrm{pr}_0} R.$$

The schemes A, B on the left of the arrows above are open subschemes of $R \times_{s,U,t} R$ and $R \times_{s,U,s} R$ respectively. By Lemma 35.11.4 the diagram

$$\begin{tikzcd} R \times_{s,U,t} R \arrow[rr, "(\mathrm{pr}_1, c)"'] \arrow[dr, "\mathrm{pr}_1"'] & & R \times_{s,U,s} R \arrow[dl, "\mathrm{pr}_0"] \\ & R & \end{tikzcd}$$

is commutative, and the horizontal arrow is an isomorphism. Moreover, it is clear that $(\mathrm{pr}_1, c)(A) = B$. Hence we conclude $s^{-1}(W') = t^{-1}(W')$, and W' is R-invariant. This proves (1).

Assume now that s, t are both open and quasi-compact. Then, if $W \subset U$ is a quasi-compact open, then also $W' = s(t^{-1}(W))$ is a quasi-compact open, and invariant by the discussion above. Letting W range over all affine opens of U we see (2). $\square$

35.17. Quotient sheaves

Let $\tau \in \{Zariski, étale, fppf, smooth, syntomic\}$. Let S be a scheme. Let $j : R \to U \times_S U$ be a pre-relation over S. Say U, R, S are objects of a τ-site Sch_τ (see Topologies, Section 30.2). Then we can consider the functors

$$h_U, h_R : (Sch/S)_\tau^{opp} \longrightarrow Sets.$$

These are sheaves, see Descent, Lemma 31.9.3. The morphism j induces a map $j : h_R \to h_U \times h_U$. For each object $T \in Ob((Sch/S)_\tau)$ we can take the equivalence relation $\sim_T$ generated by $j(T) : R(T) \to U(T) \times U(T)$ and consider the quotient. Hence we get a presheaf

$$(Sch/S)_\tau^{opp} \longrightarrow Sets, \quad T \longmapsto U(T)/\sim_T \qquad (35.17.0.1)$$

Definition 35.17.1. Let τ, S, and the pre-relation $j : R \to U \times_S U$ be as above. In this setting the *quotient sheaf* U/R associated to j is the sheafification of the presheaf (35.17.0.1) in the τ-topology. If $j : R \to U \times_S U$ comes from the action of a group scheme G/S on U as in Lemma 35.13.1 then we sometimes denote the quotient sheaf U/G.

This means exactly that the diagram

$$h_R \rightrightarrows h_U \longrightarrow U/R$$

is a coequalizer diagram in the category of sheaves of sets on $(Sch/S)_\tau$. Using the Yoneda embedding we may view $(Sch/S)_\tau$ as a full subcategory of sheaves on $(Sch/S)_\tau$ and hence identify schemes with representable functors. Using this abuse of notation we will often depict the diagram above simply

$$\begin{tikzcd} R \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & U \arrow[r] & U/R \end{tikzcd}$$

We will mostly work with the fppf topology when considering quotient sheaves of groupoids/equivalence relations.

Definition 35.17.2. In the situation of Definition 35.17.1. We say that the pre-relation j has a *representable quotient* if the sheaf U/R is representable. We will say a groupoid (U, R, s, t, c) has a *representable quotient* if the quotient U/R with $j = (t, s)$ is representable.

The following lemma characterizes schemes M representing the quotient. It applies for example if $\tau = fppf$, $U \to M$ is flat, of finite presentation and surjective, and $R \cong U \times_M U$.

Lemma 35.17.3. *In the situation of Definition 35.17.1. Assume there is a scheme M, and a morphism $U \to M$ such that*

(1) *the morphism $U \to M$ equalizes s, t,*
(2) *the morphism $U \to M$ induces a surjection of sheaves $h_U \to h_M$ in the τ-topology, and*
(3) *the induced map $(t, s) : R \to U \times_M U$ induces a surjection of sheaves $h_R \to h_{U\times_M U}$ in the τ-topology.*

In this case M represents the quotient sheaf U/R.

Proof. Condition (1) says that $h_U \to h_M$ factors through U/R. Condition (2) says that $U/R \to h_M$ is surjective as a map of sheaves. Condition (3) says that $U/R \to h_M$ is injective as a map of sheaves. Hence the lemma follows. $\square$

The following lemma is wrong if we do not require j to be a pre-equivalence relation (but just a pre-relation say).

Lemma 35.17.4. *Let $\tau \in \{Zariski, étale, fppf, smooth, syntomic\}$. Let S be a scheme. Let $j : R \to U \times_S U$ be a pre-equivalence relation over S. Assume U, R, S are objects of a τ-site Sch_τ. For $T \in Ob((Sch/S)_\tau)$ and $a, b \in U(T)$ the following are equivalent:*

(1) *a and b map to the same element of $(U/R)(T)$, and*
(2) *there exists a τ-covering $\{f_i : T_i \to T\}$ of T and morphisms $r_i : T_i \to R$ such that $a \circ f_i = s \circ r_i$ and $b \circ f_i = t \circ r_i$.*

In other words, in this case the map of τ-sheaves

$$h_R \longrightarrow h_U \times_{U/R} h_U$$

is surjective.

Proof. Omitted. Hint: The reason this works is that the presheaf (35.17.0.1) in this case is really given by $T \mapsto U(T)/j(R(T))$ as $j(R(T)) \subset U(T) \times U(T)$ is an equivalence relation, see Definition 35.3.1. $\square$

Lemma 35.17.5. *Let $\tau \in \{Zariski, étale, fppf, smooth, syntomic\}$. Let S be a scheme. Let $j : R \to U \times_S U$ be a pre-equivalence relation over S and $g : U' \to U$ a morphism of schemes over S. Let $j' : R' \to U' \times_S U'$ be the restriction of j to U'. Assume U, U', R, S are objects of a τ-site Sch_τ. The map of quotient sheaves*

$$U'/R' \longrightarrow U/R$$

is injective. If g defines a surjection $h_{U'} \to h_U$ of sheaves in the τ-topology (for example if $\{g : U' \to U\}$ is a τ-covering), then $U'/R' \to U/R$ is an isomorphism.

Proof. Suppose $\xi, \xi' \in (U'/R')(T)$ are sections which map to the same section of U/R. Then we can find a τ-covering $\mathcal{T} = \{T_i \to T\}$ of T such that $\xi|_{T_i}, \xi'|_{T_i}$ are given by $a_i, a_i' \in U'(T_i)$. By Lemma 35.17.4 and the axioms of a site we may after refining $\mathcal{T}$ assume there exist morphisms $r_i : T_i \to R$ such that $g \circ a_i = s \circ r_i$, $g \circ a_i' = t \circ r_i$. Since by construction $R' = R \times_{U \times_S U} (U' \times_S U')$ we see that $(r_i, (a_i, a_i')) \in R'(T_i)$ and this shows that a_i and a_i' define the same section of U'/R' over T_i. By the sheaf condition this implies $\xi = \xi'$.

If $h_{U'} \to h_U$ is a surjection of sheaves, then of course $U'/R' \to U/R$ is surjective also. If $\{g : U' \to U\}$ is a τ-covering, then the map of sheaves $h_{U'} \to h_U$ is surjective, see Sites, Lemma 9.12.5. Hence $U'/R' \to U/R$ is surjective also in this case. $\square$

Lemma 35.17.6. *Let $\tau \in \{Zariski, étale, fppf, smooth, syntomic\}$. Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $g : U' \to U$ a morphism of schemes over S. Let (U', R', s', t', c') be the restriction of (U, R, s, t, c) to U'. Assume U, U', R, S are objects of a τ-site Sch_τ. The map of quotient sheaves*

$$U'/R' \longrightarrow U/R$$

is injective. If the composition

$$\begin{tikzcd} U' \times_{g,U,t} R \arrow[r, "pr_1"'] \arrow[rr, bend left, "h"] & R \arrow[r, "s"'] & U \end{tikzcd}$$

defines a surjection of sheaves in the τ-topology then the map is bijective. This holds for example if $\{h : U' \times_{g,U,t} R \to U\}$ is a τ-covering, or if $U' \to U$ defines a surjection of sheaves in the τ-topology, or if $\{g : U' \to U\}$ is a covering in the τ-topology.

Proof. Injectivity follows on combining Lemmas 35.11.2 and 35.17.5. To see surjectivity (see Sites, Section 9.11 for a characterization of surjective maps of sheaves) we argue as follows. Suppose that T is a scheme and $\sigma \in U/R(T)$. There exists a covering $\{T_i \to T\}$ such that $\sigma|_{T_i}$ is the image of some element $f_i \in U(T_i)$. Hence we may assume that σ if the image of $f \in U(T)$. By the assumption that h is a surjection of sheaves, we can find a τ-covering $\{\varphi_i : T_i \to T\}$ and morphisms $f_i : T_i \to U' \times_{g,U,t} R$ such that $f \circ \varphi_i = h \circ f_i$. Denote $f_i' = \text{pr}_0 \circ f_i : T_i \to U'$. Then we see that $f_i' \in U'(T_i)$ maps to $g \circ f_i' \in U(T_i)$ and that $g \circ f_i' \sim_{T_i} h \circ f_i = f \circ \varphi_i$ notation as in (35.17.0.1). Namely, the element of $R(T_i)$ giving the relation is $\text{pr}_1 \circ f_i$. This means that the restriction of σ to T_i is in the image of $U'/R'(T_i) \to U/R(T_i)$ as desired.

If $\{h\}$ is a τ-covering, then it induces a surjection of sheaves, see Sites, Lemma 9.12.5. If $U' \to U$ is surjective, then also h is surjective as s has a section (namely the neutral element e of the groupoid scheme). $\square$

35.18. Separation conditions

This really means conditions on the morphism $j : R \to U \times_S U$ when given a groupoid (U, R, s, t, c) over S. As in the previous section we first formulate the corresponding diagram.

Lemma 35.18.1. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid over S. Let $G \to U$ be the stabilizer group scheme. The commutative diagram*

$$\begin{tikzcd} R \arrow[rr, "f \mapsto (f, s(f))"'] \arrow[d, "\Delta_{R/U \times_S U}"] & & R \times_{s,U} U \arrow[r] \arrow[d] & U \arrow[d] \\ R \times_{(U \times_S U)} R \arrow[rr, "(f,g) \mapsto (f, f^{-1} \circ g)"] & & R \times_{s,U} G \arrow[r] & G \end{tikzcd}$$

the two left horizontal arrows are isomorphisms and the right square is a fibre product square.

Proof. Omitted. Exercise in the definitions and the functorial point of view in algebraic geometry. $\square$

Lemma 35.18.2. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid over S. Let $G \to U$ be the stabilizer group scheme.*

(1) *The following are equivalent*
   (a) *$j : R \to U \times_S U$ is separated,*
   (b) *$G \to U$ is separated, and*
   (c) *$e : U \to G$ is a closed immersion.*
(2) *The following are equivalent*
   (a) *$j : R \to U \times_S U$ is quasi-separated,*
   (b) *$G \to U$ is quasi-separated, and*
   (c) *$e : U \to G$ is quasi-compact.*

Proof. The group scheme $G \to U$ is the base change of $R \to U \times_S U$ by the diagonal morphism $U \to U \times_S U$, see Lemma 35.14.1. Hence if j is separated (resp. quasi-separated), then $G \to U$ is separated (resp. quasi-separated). (See Schemes, Lemma 21.21.13). Thus (a) $\Rightarrow$ (b) in both (1) and (2).

If $G \to U$ is separated (resp. quasi-separated), then the morphism $U \to G$, as a section of the structure morphism $G \to U$ is a closed immersion (resp. quasi-compact), see Schemes, Lemma 21.21.12. Thus (b) $\Rightarrow$ (a) in both (1) and (2).

By the result of Lemma 35.18.1 (and Schemes, Lemmas 21.18.2 and 21.19.3) we see that if e is a closed immersion (resp. quasi-compact) $\Delta_{R/U \times_S U}$ is a closed immersion (resp. quasi-compact). Thus (c) $\Rightarrow$ (a) in both (1) and (2). $\square$

35.19. Finite flat groupoids, affine case

Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = Spec(A)$, and $R = Spec(B)$ are affine. In this case we get two ring maps $s^\sharp, t^\sharp : A \longrightarrow B$. Let C be the equalizer of $s^\sharp$ and $t^\sharp$. In a formula

$$C = \{a \in A \mid t^\sharp(a) = s^\sharp(a)\}. \tag{35.19.0.1}$$

We will sometimes call this the *ring of R-invariant functions on U*. What properties does $M = Spec(C)$ have? The first observation is that the diagram

$$\begin{tikzcd} R \arrow[r, "s"] \arrow[d, "t"'] & U \arrow[d] \\ U \arrow[r] & M \end{tikzcd}$$

is commutative, i.e., the morphism $U \to M$ equalizes s, t. Moreover, if T is any affine scheme, and if $U \to T$ is a morphism which equalizes s, t, then $U \to T$ factors through $U \to M$. In other words, $U \to M$ is a coequalizer in the category of affine schemes.

We would like to find conditions that garantee the morphism $U \to M$ is really a ``quotient" in the category of schemes. We will discuss this at length elsewhere (insert future reference here); here we just discuss some special cases. Namely, we will focus on the case where s, t are finite locally free.

Example 35.19.1. Let k be a field. Let $U = \mathrm{GL}_{2,k}$. Let $B \subset \mathrm{GL}_2$ be the closed subgroup scheme of upper triangular matrices. Then the quotient sheaf $\mathrm{GL}_{2,k}/B$ (in the Zariski, étale or fppf topology, see Definition 35.17.1) is representable by the projective line: $\mathbf{P}^1 = \mathrm{GL}_{2,k}/B$. (Details omitted.) On the other hand, the ring of invariant functions in this case is just k. Note that in this case the morphisms $s, t : R = \mathrm{GL}_{2,k} \times_k B \to \mathrm{GL}_{2,k} = U$ are smooth of relative dimension 3.

Recall that in Exercises, Exercises 65.15.6 and 65.15.7 we have defined the determinant and the norm for finitely locally free modules and finite locally free ring extensions. If $\varphi : A \to B$ is a finite locally free ring map, then we will denote $\mathrm{Norm}_\varphi(b) \in A$ the norm of $b \in B$.

Lemma 35.19.2. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = Spec(A)$, and $R = Spec(B)$ are affine, and $s, t : R \to U$ finite locally free. Let C be as in (35.19.0.1). Let $f \in A$. Then $Norm_{s^\sharp}(t^\sharp(f)) \in C$.*

Proof. Consider the commutative diagram

$$\begin{tikzcd} & U & \\ R \arrow[ur, "t"] \arrow[d, "s"'] & R \times_{s,U,t} R \arrow[l, "\mathrm{pr}_0"] \arrow[r, "c"'] \arrow[d, "\mathrm{pr}_1"] & R \arrow[ul, "t"'] \arrow[d, "s"] \\ U & R \arrow[l, "t"'] \arrow[r, "s"] & U \end{tikzcd}$$

of Lemma 35.11.4. Think of $f \in \Gamma(U, \mathcal{O}_U)$. The commutativity of the top part of the diagram shows that $pr_0^\sharp(t^\sharp(f)) = c^\sharp(t^\sharp(f))$ as elements of $\Gamma(R \times_{S,U,t} R, \mathcal{O})$. Looking at the right lower cartesian square the compatibility of the norm construction with base change shows that $s^\sharp(\mathrm{Norm}_{s^\sharp}(t^\sharp(f))) = \mathrm{Norm}_{\mathrm{pr}_1}(c^\sharp(t^\sharp(f)))$. Similarly we get $t^\sharp(\mathrm{Norm}_{s^\sharp}(t^\sharp(f))) = \mathrm{Norm}_{\mathrm{pr}_1}(\mathrm{pr}_0^\sharp(t^\sharp(f)))$. Hence by the first equality of this proof we see that $s^\sharp(\mathrm{Norm}_{s^\sharp}(t^\sharp(f))) = t^\sharp(\mathrm{Norm}_{s^\sharp}(t^\sharp(f)))$ as desired. $\square$

Lemma 35.19.3. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $s, t : R \to U$ finite locally free. Then*

$$U = \coprod\nolimits_{r \geq 1} U_r$$

is a disjoint union of R-invariant opens such that the restriction R_r of R to U_r has the property that $s, t : R_r \to U_r$ are finite locally free of rank 1.

Proof. By Morphisms, Lemma 24.44.5 there exists a decomposition $U = \coprod_{r \geq 0} U_r$ such that $s : s^{-1}(U_r) \to U_r$ is finite locally free of rank r. As s is surjective we see that $U_0 = \emptyset$. Note that $u \in U_r \Leftrightarrow$ the scheme theoretic fibre $s^{-1}(u)$ has degree r over $\kappa(u)$. Now, if $z \in R$ with $s(z) = u$ and $t(z) = u'$ then $\text{pr}_1^{-1}(z)$ see diagram of Lemma 35.11.4 is a scheme over $\kappa(z)$ which is the base change of both $s^{-1}(u)$ and $s^{-1}(u')$ via $\kappa(u) \to \kappa(z)$ and $\kappa(u') \to \kappa(z)$ by the properties of that diagram. Hence we see that the open subsets U_r are R-invariant. In particular the restricton of R to U_r is just $s^{-1}(U_r)$ and $s : R_r \to U_r$ is finite locally free of rank r. As $t : R_r \to U_r$ is isomorphic to s by the inverse of R_r we see that it has also rank r. □

Lemma 35.19.4. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = Spec(A)$, and $R = Spec(B)$ are affine, and $s, t : R \to U$ finite locally free. Let $C \subset A$ be as in (35.19.0.1). Then A is integral over C.*

Proof. First, by Lemma 35.19.3 we know that (U, R, s, t, c) is a disjoint union of groupoid schemes (U_r, R_r, s, t, c) such that each $s, t : R_r \to U_r$ has constant rank r. As U is quasi-compact, we have $U_r =$ for almost all r. It suffices to prove the lemma for each (U_r, R_r, s, t, c) and hence we may assume that s, t are finite locally free of rank r.

Assume that s, t are finite locally free of rank r. Let $f \in A$. Consider the element $x - f \in A[x]$, where we think of x as the coordinate on $\mathbf{A}^1$. Since

$$(U \times \mathbf{A}^1, R \times \mathbf{A}^1, s \times \text{id}_{\mathbf{A}^1}, t \times \text{id}_{\mathbf{A}^1}, c \times \text{id}_{\mathbf{A}^1})$$

is also a groupoid scheme with finite source and target, we may apply Lemma 35.19.2 to it and we see that $P(x) = \text{Norm}_{s^\sharp}(t^\sharp(x - f))$ is an element of $C[x]$. Because $s^\sharp : A \to B$ is finite locally free of rank r we see that P is monic of degree r. Moreover $P(f) = 0$ by Cayley-Hamilton (Algebra, Lemma 7.15.1). □

Lemma 35.19.5. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = Spec(A)$, and $R = Spec(B)$ are affine, and $s, t : R \to U$ finite locally free. Let $C \subset A$ be as in (35.19.0.1). Let $C \to C'$ be a ring map, and set $U' = Spec(A \otimes_C C')$, $R' = Spec(B \otimes_C C')$. Then*

1. *the maps s, t, c induce maps s', t', c' such that (U', R', s', t', c') is a groupoid scheme, and*
2. *there is a canonical map $\varphi : C' \to C^1$ of C' into the R'-invariant functions C^1 on U' with the properties*
   (a) *for every $f \in C^1$ there exists an $n > 0$ such that f^n is in the image of φ, and*
   (b) *for every $f \in Ker(\varphi)$ there exists an $n > 0$ such that $f^n = 0$.*
3. *if $C \to C'$ is flat then φ is an isomorphism.*

Proof. The proof of part (1) is omitted. Let us denote $A' = A \otimes_C C'$ and $B' = B \otimes_C C'$. Then we have

$$C^1 = \{x \in A' \mid (t')^\sharp(x) = (s')^\sharp(x)\} = \{a \in A \otimes_C C' \mid t^\sharp \otimes 1(x) = s^\sharp \otimes 1(x)\}.$$

In other words, C^1 is the kernel of the difference map $(t^\sharp - s^\sharp) \otimes 1$ which is just the base change of the C-linear map $t^\sharp - s^\sharp : A \to B$ by $C \to C'$. Hence (3) follows.

Proof of part (2)(b). Since $C \to A$ is integral (Lemma 35.19.4) and injective we see that $Spec(A) \to Spec(C)$ is surjective, see Algebra, Lemma 7.32.15. Thus also $Spec(A') \to$

$Spec(C')$ is surjective as a base change of a surjective morphism (Morphisms, Lemma 24.9.4). Hence $Spec(C^1) \to Spec(C')$ is surjective also. This implies that the kernel of φ is contained in the radical of the ring C', i.e., (2)(b) holds.

Proof of part (2)(a). By Lemma 35.19.3 we know that A is a finite product of rings A_r and B is a finite product of rings B_r such that the groupoid scheme decomposes accordingly (see the proof of Lemma 35.19.4). Then also C is a product of rings C_r and correspondingly C' decomposes as a product. Hence we may and do assume that the ring maps $s^\sharp, t^\sharp : A \to B$ are finite locally free of a fixed rank r. Let $f \in C^1 \subset A' = A \otimes_C C'$. We may replace C' by a finitely generated C-subalgebra of C' and hence we may assume that $C' = C[X_1, \ldots, X_n]/I$ for some ideal I. Choose a lift $\tilde{f} \in A \otimes_C C[X_i] = A[X_i]$ of the element f. Note that $f^r = \mathrm{Norm}_{(s')^\sharp}((t')^\sharp(f))$ in A as $t^\sharp(f) = s^\sharp(f)$. Hence we see that

$$h = \mathrm{Norm}_{s^\sharp \otimes 1}(t^\sharp \otimes 1(f)) \in A[X_i]$$

is invariant according to Lemma 35.19.2 and maps to f^r in A'. Since $C \to C[X_i]$ is flat we see from (3) that $h \in C[X_i]$. Hence it follows that f^r is in the image of φ. □

Lemma 35.19.6. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = Spec(A)$, and $R = Spec(B)$ are affine, and $s, t : R \to U$ finite locally free. Let $C \subset A$ be as in (35.19.0.1). Then $U \to M = Spec(C)$ has the following properties:*

1. *the map on points $|U| \to |M|$ is surjective and $u_0, u_1 \in |U|$ map to the same point if and only if there exists a $r \in |R|$ with $t(r) = u_0$ and $s(r) = u_1$, in a formula*
$$|M| = |U|/|R|$$
2. *for any algebraically closed field k we have*
$$M(k) = U(k)/R(k)$$

Proof. Let k be an algebraically closed field. Since $C \to A$ is integral (Lemma 35.19.4) and injective we see that $Spec(A) \to Spec(C)$ is surjective, see Algebra, Lemma 7.32.15. Thus $|M| \to |U|$ is surjective. Let $C \to k$ be a ring map. Since surjective morphisms are preserved under base change (Morphisms, Lemma 24.9.4) we see that $A \otimes_C k$ is not zero. Now $k \subset A \otimes_C k$ is a nonzero integral extension. Hence any residue field of $A \otimes_C k$ is an algebraic extension of k, hence equal to k. Thus we see that $U(k) \to M(k)$ is surjective.

Let $a_0, a_1 : A \to k$ be ring maps. If there exists a ring map $b : B \to k$ such that $a_0 = b \circ t^\sharp$ and $a_1 = b \circ s^\sharp$ then we see that $a_0|_C = a_1|_C$ by definition. Conversely, suppose that $a_0|_C = a_1|_C$. Let us name this algebra map $c : C \to k$. Consider the diagram

$$\begin{tikzcd}
 & B \arrow[dl, dashed] \\
k & A \arrow[l, shift right, "a_0"'] \arrow[l, shift left, "a_1"] \arrow[u, shift left] \arrow[u, shift right] \\
 & C \arrow[ul, "c"] \arrow[u]
\end{tikzcd}$$

We are trying to construct the dotted arrow, and if we do then part (2) follows, which in turn implies part (1). Since $A \to B$ is finite and faithfully flat there exist finitely many ring maps $b_1, \ldots, b_n : B \to k$ such that $b_i \circ s^\sharp = a_1$. If the dotted arrow does not exist, then we see that none of the $a'_i = b_i \circ t^\sharp$, $i = 1, \ldots, n$ is equal to a_0. Hence the maximal ideals

$$\mathfrak{m}'_i = \mathrm{Ker}(a'_i \otimes 1 : A \otimes_C k \to k)$$

of $A \otimes_C k$ are distinct from $\mathfrak{m} = \mathrm{Ker}(a_0 \otimes 1 : A \otimes_C k \to k)$. By Algebra, Lemma 7.14.3 we would get an element $f \in A \otimes_C k$ with $f \in \mathfrak{m}$, but $f \not\in \mathfrak{m}'_i$ for $i = 1, \ldots, n$. Consider the norm

$$g = \mathrm{Norm}_{s^\sharp \otimes 1}(t^\sharp \otimes 1(f)) \in A \otimes_C k$$

By Lemma 35.19.2 this lies in the invariants $C^1 \subset A \otimes_C k$ of the base change groupoid (base change via the map $c : C \to k$). On the one hand, $a_1(g) \in k^*$ since the value of $t^\sharp(f)$ at all the points (which correspond to $b_1, \ldots, b_n$) lying over a_1 is invertible (insert future reference on property determinant here). On the other hand, since $f \in \mathfrak{m}$, we see that f is not a unit, hence $t^\sharp(f)$ is not a unit (as $t^\sharp \otimes 1$ is faithfully flat), hence its norm is not a unit (insert future reference on property determinant here). We conclude that C^1 contains an element which is not nilpotent and not a unit. We will now show that this leads to a contradiction. Namely, apply Lemma 35.19.5 to the map $c : C \to C' = k$, then we see that the map of k into the invariants C^1 is injective and moreover, that for any element $x \in C^1$ there exists an integer $n > 0$ such that $x^n \in k$. Hence every element of C^1 is either a unit or nilpotent. $\square$

Lemma 35.19.7. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume*

(1) *$U = Spec(A)$, and $R = Spec(B)$ are affine, and*
(2) *there exist elements $x_i \in A$, $i \in I$ such that $B = \bigoplus_{i \in I} s^\sharp(A)t^\sharp(x_i)$.*

Then $A = \bigoplus_{i \in I} Cx_i$, and $B \cong A \otimes_C A$ where $C \subset A$ is the R-invariant functions on U as in (35.19.0.1).

Proof. During this proof we will write $s, t : A \to B$ instead of $s^\sharp, t^\sharp$, and similarly $c : B \to B \otimes_{s,A,t} B$. We write $p_0 : B \to B \otimes_{s,A,t} B$, $b \mapsto b \otimes 1$ and $p_1 : B \to B \otimes_{s,A,t} B$, $b \mapsto 1 \otimes b$. By Lemma 35.11.5 and the definition of C we have the following commutative diagram

$$\begin{tikzcd}
B \otimes_{s,A,t} B & B \arrow[l, shift right, "c"'] \arrow[l, shift left, "p_0"] & A \arrow[l, "t"] \\
B \arrow[u, "p_1"] & A \arrow[l, shift right, "s"'] \arrow[l, shift left, "t"] \arrow[u, "s"] & C \arrow[l] \arrow[u]
\end{tikzcd}$$

Moreover the tow left squares are cocartesian in the category of rings, and the top row is isomorphic to the diagram

$$\begin{tikzcd}
B \otimes_{t,A,t} B & B \arrow[l, shift right, "p_1"'] \arrow[l, shift left, "p_0"] & A \arrow[l, "t"]
\end{tikzcd}$$

which is an equalizer diagram according to Descent, Lemma 31.3.6 because condition (2) implies in particular that s (and hence also then isomorphic arrow t) is faithfully flat. The lower row is an equalizer diagram by definition of C. We can use the x_i and get a commutative diagram

$$\begin{tikzcd}
B \otimes_{s,A,t} B & B \arrow[l, shift right, "c"'] \arrow[l, shift left, "p_0"] & A \arrow[l, "t"] \\
\bigoplus_{i \in I} Bx_i \arrow[u, "p_1"] & \bigoplus_{i \in I} Ax_i \arrow[l, shift right, "s"'] \arrow[l, shift left, "t"] \arrow[u, "s"] & \bigoplus_{i \in I} Cx_i \arrow[l] \arrow[u]
\end{tikzcd}$$

where in the right vertical arrow we map x_i to x_i, in the middle vertical arrow we map x_i to $t(x_i)$ and in the left vertical arrow we map x_i to $c(t(x_i)) = t(x_i) \otimes 1 = p_0(t(x_i))$

(equality by the commutativity of the top part of the diagram in Lemma 35.11.4). Then the diagram commutes. Moreover the middle vertical arrow is an isomorphism by assumption. Since the left two squares are cocartesian we conclude that also the left vertical arrow is an isomorphism. On the other hand, the horizontal rows are exact (i.e., they are equalizers). Hence we conclude that also the right vertical arrow is an isomorphism. □

Proposition 35.19.8. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume*

(1) *$U = Spec(A)$, and $R = Spec(B)$ are affine,*
(2) *$s, t : R \to U$ finite locally free, and*
(3) *$j = (t, s)$ is an equivalence.*

In this case, let $C \subset A$ be as in (35.19.0.1). Then $U \to M = Spec(C)$ is finite locally free and $R = U \times_M U$. Moreover, M represents the quotient sheaf U/R in the fppf topology (see Definition 35.17.1).

Proof. During this proof we use the notation $s, t : A \to B$ instead of the notation $s^\sharp, t^\sharp$. By Lemma 35.17.3 it suffices to show that $C \to A$ is finite locally free and that the map

$$t \otimes s : A \otimes_C A \longrightarrow B$$

is an isomorphism. First, note that j is a monomorphism, and also finite (since already s and t are finite). Hence we see that j is a closed immersion by Morphisms, Lemma 24.42.13. Hence $A \otimes_C A \to B$ is surjective.

We will perform base change by flat ring maps $C \to C'$ as in Lemma 35.19.5, and we will use that formation of invariants commutes with flat base change, see part (3) of the lemma cited. We will show below that for every prime $\mathfrak{p} \subset C$, there exists a local flat ring map $C_\mathfrak{p} \to C'_\mathfrak{p}$ such that the result holds after a base change to $C'_\mathfrak{p}$. This implies immediately that $A \otimes_C A \to B$ is injective (use Algebra, Lemma 7.21.1). It also implies that $C \to A$ is flat, by combining Algebra, Lemmas 7.35.16, 7.35.19, and 7.35.7. Then since $U \to Spec(C)$ is surjective also (Lemma 35.19.6) we conclude that $C \to A$ is faithfully flat. Then the isomorphism $B \cong A \otimes_C A$ implies that A is a finitely presented C-module, see Algebra, Lemma 7.77.2. Hence A is finite locally free over C, see Algebra, Lemma 7.72.2.

By Lemma 35.19.3 we know that A is a finite product of rings A_r and B is a finite product of rings B_r such that the groupoid scheme decomposes accordingly (see the proof of Lemma 35.19.4). Then also C is a product of rings C_r and correspondingly C' decomposes as a product. Hence we may and do assume that the ring maps $s, t : A \to B$ are finite locally free of a fixed rank r.

The local ring maps $C_\mathfrak{p} \to C'_\mathfrak{p}$ we are going to use are any local flat ring maps such that the residue field of $C'_\mathfrak{p}$ is infinite. By Algebra, Lemma 7.142.1 such local ring maps exist.

Assume C is a local ring with maximal ideal $\mathfrak{m}$ and infinite residue field, and assume that $s, t : A \to B$ is finite locally free of constant rank $r > 0$. Since $C \subset A$ is integral (Lemma 35.19.4) all primes lying over $\mathfrak{m}$ are maximal, and all maximal ideals of A lie over $\mathfrak{m}$. Similarly for $C \subset B$. Pick a maximal ideal $\mathfrak{m}'$ of A lying over $\mathfrak{m}$ (exists by Lemma 35.19.6). Since $t : A \to B$ is finite locally free there exist at most finitely many maximal ideals of B lying over $\mathfrak{m}'$. Hence we conclude (by Lemma 35.19.6 again) that A has finitely many maximal ideals, i.e., A is semi-local. This in turn implies that B is semi-local as well. OK, and now, because $t \otimes s : A \otimes_C A \to B$ is surjective, we can apply Algebra, Lemma 7.72.7 to the ring map $C \to A$, the A-module $M = B$ (seen as an A-module via t) and the C-submodule $s(A) \subset B$. This lemma implies that there exist

$x_1, \ldots, x_r \in A$ such that M is free over A on the basis $s(x_1), \ldots, s(x_r)$. Hence we conlude that $C \to A$ is finite free and $B \cong A \otimes_C A$ by applying Lemma 35.19.7. □

35.20. Finite flat groupoids

Lemma 35.20.1. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume s, t are finite locally free. Let $u \in U$ be a point such that $t(s^{-1}(\{u\}))$ is contained in an affine open of U. Then there exists an R-invariant affine open neighbourhood of u in U.*

Proof. Since s is finite locally free it has finite fibres. Hence $t(s^{-1}(\{u\})) = \{u_1, \ldots, u_n\}$ is a finite set. Note that $u \in \{u_1, \ldots, u_n\}$. Let $W \subset U$ be an affine open containing $\{u_1, \ldots, u_n\}$, in particular $u \in W$. Consider $Z = R \setminus s^{-1}(W) \cap t^{-1}(W)$. This is a closed subset of R. The image $t(Z)$ is a closed subset of U which can be loosely described as the set of points of U which are not R-equivalent to a point of W. Hence $W' = U \setminus t(Z)$ is an R-invariant, open subscheme of U contained in W, and $\{u_1, \ldots, u_n\} \subset W'$. Picture

$$\{u_1, \ldots, u_n\} \subset W' \subset W \subset U.$$

Let $f \in \Gamma(W, \mathcal{O}_W)$ be an element such that $\{u_1, \ldots, u_n\} \subset D(f) \subset W'$. Such an f exists by Algebra, Lemma 7.14.3. By our choice of W' we have $s^{-1}(W') \subset t^{-1}(W)$, and hence we get a diagram

$$\begin{tikzcd} s^{-1}(W') \arrow[r, "t"'] \arrow[d, "s"'] & W \\ W' & \end{tikzcd}$$

The vertical arrow is finite locally free by assumption. Set

$$g = \mathrm{Norm}_s(t^\sharp f) \in \Gamma(W', \mathcal{O}_{W'})$$

By construction g is a function on W' which is nonzero in u, as $t^\sharp(f)$ is nonzero in each of the points of R lying over u, since f is nonzero in $u_1, \ldots, u_n$. Similarly, $D(g) \subset W'$ is equal to the set of points w such that f is not zero in any of the points equivalent to w. This means that $D(g)$ is an R-invariant affine open of W'. The final picture is

$$\{u_1, \ldots, u_n\} \subset D(g) \subset D(f) \subset W' \subset W \subset U$$

and hence we win. □

35.21. Descent data give equivalence relations

In Descent, Section 31.36 we saw how descent data relative to $X \to S$ can be formulated in terms of cartesian simplicial schemes over $(X/S)_\bullet$. Here we link this to equivalence relations as follows.

Lemma 35.21.1. *Let $f : X \to S$ be a morphism of schemes. Let $\pi : V_\bullet \to (X/S)_\bullet$ be a cartesian morphism, see Descent, Definition 31.36.1. Then the morphism*

$$j = (d^1_1, d^1_0) : V_1 \to V_0 \times_S V_0$$

defines an equivalence relation on V_0 over S, see Definition 35.3.1.

Proof. Note that j is a monomorphism. Namely the composition $V_1 \to V_0 \times_S V_0 \to V_0 \times_S X$ is an isomorphism as π is cartesian.

Consider the morphism

$$(d^2_2, d^2_0) : V_2 \to V_1 \times_{d^1_0, V_0, d^1_1} V_1.$$

This works because $d_0 \circ d_2 = d_1 \circ d_0$, see Simplicial, Remark 14.3.3. Also, it is a morphism over $(X/S)_2$. It is an isomorphism because $V_\bullet \to (X/S)_\bullet$ is cartesian. Note for example that the right hand side is isomorphic to $V_0 \times_{\pi_0, X, \mathrm{pr}_1} (X \times_S X \times_S X) = X \times_S V_0 \times_S X$ because π is cartesian. Details omitted.

As usual, see Definition 35.3.1 we denote $t = \mathrm{pr}_0 \circ j = d^1_1$ and $s = \mathrm{pr}_1 \circ j = d^1_0$. The isomorphism above, combined with the morphism $d^2_1 : V_2 \to V_1$ give us a composition morphism

$$c : V_1 \times_{s, V_0, t} V_1 \longrightarrow V_1$$

over $V_0 \times_S V_0$. This immediately implies that for any scheme T/S the relation $V_1(T) \subset V_0(T) \times V_0(T)$ is transitive.

Reflexivity follows from the fact that the restriction of the morphism j to the diagonal $\Delta : X \to X \times_S X$ is an isomorphism (again use the cartesian property of π).

To see symmetry we consider the morphism

$$(d^2_2, d^2_1) : V_2 \to V_1 \times_{d^1_1, V_0, d^1_1} V_1.$$

This works because $d_1 \circ d_2 = d_1 \circ d_1$, see Simplicial, Remark 14.3.3. It is an isomorphism because $V_\bullet \to (X/S)_\bullet$ is cartesian. Note for example that the right hand side is isomorphic to $V_0 \times_{\pi_0, X, \mathrm{pr}_0} (X \times_S X \times_S X) = V_0 \times_S X \times_S X$ because π is cartesian. Details omitted.

Let T/S be a scheme. Let $a \sim b$ for $a, b \in V_0(T)$ be synonymous with $(a, b) \in V_1(T)$. The isomorphism (d^2_2, d^2_1) above implies that if $a \sim b$ and $a \sim c$, then $b \sim c$. Combined with reflexivity this shows that $\sim$ is an equivalence relation. □

35.22. An example case

In this section we show that disjoint unions of spectra of Artinian rings can be descended along a quasi-compact surjective flat morphism of schemes.

Lemma 35.22.1. *Let $X \to S$ be a morphism of schemes. Suppose $V_\bullet \to (X/S)_\bullet$ is cartesian. For $v \in V_0$ a point define*

$$T_v = \{v' \in V \mid \exists\, v_1 \in V_1 : d^1_1(v_1) = v, d^1_0(v_1) = v'\}$$

as a subset of V_0. Then $v \in T_v$ and $T_v \cap T_{v'} \neq \emptyset \Rightarrow T_v = T_{v'}$.

Proof. Combine Lemmas 35.21.1 and 35.3.4. □

Lemma 35.22.2. *Let $X \to S$ be a morphism of schemes. Suppose $V_\bullet \to (X/S)_\bullet$ is cartesian. Let $v \in V_0$ be a point. If $X \to S$ is quasi-compact, then*

$$T_v = \{v' \in V \mid \exists\, v_1 \in V_1 : d^1_1(v_1) = v, d^1_0(v_1) = v'\}$$

is a quasi-compact subset of V_0.

Proof. Let F_v be the scheme theoretic fibre of $d^1_1 : V_1 \to V_0$ at v. Then we see that T_v is the image of the morphism

$$\begin{tikzcd} F_v \arrow[r] \arrow[d] & V_1 \arrow[r, "d^1_0"] \arrow[d, "d^1_1"] & V_0 \\ v \arrow[r] & V_0 & \end{tikzcd}$$

Note that F_v is quasi-compact. This proves the lemma. □

Lemma 35.22.3. *Let $X \to S$ be a quasi-compact flat surjective morphism. Let (V, φ) be a descent datum relative to $X \to S$. If V is a disjoint union of spectra of Artinian rings, then (V, φ) is effective.*

Proof. We may write $V = \coprod_{i \in I} Spec(A_i)$ with each A_i local Artinian. Moreover, let $v_i \in V$ be the unique closed point of $Spec(A_i)$ for all $i \in I$. Write $i \sim j$ if and only if $v_i \in T_{v_j}$ with notation as in Lemma 35.22.1 above. By Lemmas 35.22.1 and 35.22.2 this is an equivalence relation with finite equivalence classes. Let $\overline{I} = I/\sim$. Then we can write $V = \coprod_{\overline{i} \in \overline{I}} V_{\overline{i}}$ with $V_{\overline{i}} = \coprod_{i \in \overline{i}} Spec(A_i)$. By construction we see that $\varphi : V \times_S X \to X \times_S V$ maps the open and closed subspaces $V_{\overline{i}} \times_S X$ into the open and closed subspaces $X \times_S V_{\overline{i}}$. In other words, we get descent data $(V_{\overline{i}}, \varphi_{\overline{i}})$, and (V, φ) is the coproduct of them in the category of descent data. Since each of the $V_{\overline{i}}$ is a finite union of spectra of Artinian local rings the morphism $V_{\overline{i}} \to X$ is affine, see Morphisms, Lemma 24.11.13. Since $\{X \to S\}$ is an fpqc covering we see that all the descent data $(V_{\overline{i}}, \varphi_{\overline{i}})$ are effective by Descent, Lemma 31.33.1. Hence we win. ☐

To be sure, the lemma above has very limited applicability!

35.23. Other chapters

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CHAPTER 36

More on Groupoid Schemes

36.1. Introduction

This chapter is devoted to advanced topics on groupoid schemes. Even though the results are stated in terms of groupoid schemes, the reader should keep in mind the 2-cartesian diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d] & U \arrow[d] \\ U \arrow[r] & {[U/R]} \end{tikzcd} \tag{36.1.0.1}$$

where $[U/R]$ is the quotient stack, see Groupoids in Spaces, Remark 52.19.4. Many of the results are motivated by thinking about this diagram. See for example the beautiful paper [**KM97a**] by Keel and Mori.

36.2. Notation

We continue to abide by the conventions and notation introduced in Groupoids, Section 35.2.

36.3. Useful diagrams

We briefly restate the results of Groupoids, Lemmas 35.11.4 and 35.11.5 for easy reference in this chapter. Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. In the commutative diagram

$$\begin{tikzcd} & U & \\ R \arrow[ru, "t"] \arrow[d, "s"'] & R \times_{s,U,t} R \arrow[l, "\mathrm{pr}_0"] \arrow[r, "c"'] \arrow[d, "\mathrm{pr}_1"] & R \arrow[lu, "t"'] \arrow[d, "s"] \\ U & R \arrow[l, "t"'] \arrow[r, "s"] & U \end{tikzcd} \tag{36.3.0.2}$$

the two lower squares are fibre product squares. Moreover, the triangle on top (which is really a square) is also cartesian.

The diagram

$$
\begin{tikzcd}
R \times_{t,U,t} R \arrow[r, shift left, "\mathrm{pr}_1"] \arrow[r, shift right, "\mathrm{pr}_0"'] \arrow[d, "\mathrm{pr}_0 \times c \circ (i,1)"'] & R \arrow[r, "t"] \arrow[d, "\mathrm{id}_R"] & U \arrow[d, "\mathrm{id}_U"] \\
R \times_{s,U,t} R \arrow[r, shift left, "c"] \arrow[r, shift right, "\mathrm{pr}_0"'] \arrow[d, "\mathrm{pr}_1"'] & R \arrow[r, "t"] \arrow[d, "s"] & U \\
R \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & U &
\end{tikzcd}
\tag{36.3.0.3}
$$

is commutative. The two top rows are isomorphic via the vertical maps given. The two lower left squares are cartesian.

36.4. Sheaf of differentials

The following lemma is the analogue of Groupoids, Lemma 35.6.3.

Lemma 36.4.1. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. The sheaf of differentials of R seen as a scheme over U via t is a quotient of the pullback via t of the conormal sheaf of the immersion $e : U \to R$. In a formula: there is a canonical surjection $t^*\mathcal{C}_{U/R} \to \Omega_{R/U}$. If s is flat, then this map is an isomorphism.*

Proof. Note that $e : U \to R$ is an immersion as it is a section of the morphism s, see Schemes, Lemma 21.21.12. Consider the following diagram

$$
\begin{tikzcd}
R \arrow[r, "(1,i)"'] \arrow[d, "t"'] & R \times_{s,U,t} R \arrow[r, "(\mathrm{pr}_0, i \circ \mathrm{pr}_1)"'] \arrow[d, "c"] & R \times_{t,U,t} R \\
U \arrow[r, "e"] & R &
\end{tikzcd}
$$

The square on the left is cartesian, because if $a \circ b = e$, then $b = i(a)$. The composition of the horizontal maps is the diagonal morphism of $t : R \to U$. The right top horizontal arrow is an isomorphism. Hence since $\Omega_{R/U}$ is the conormal sheaf of the composition it is isomorphic to the conormal sheaf of $(1, i)$. By Morphisms, Lemma 24.31.4 we get the surjection $t^*\mathcal{C}_{U/R} \to \Omega_{R/U}$ and if c is flat, then this is an isomorphism. Since c is a base change of s by the properties of Diagram (36.3.0.3) we conclude that if s is flat, then c is flat, see Morphisms, Lemma 24.24.7. □

36.5. Properties of groupoids

Let (U, R, s, t, c) be a groupoid scheme. The idea behind the results in this section is that $s : R \to U$ is a base changes of the morphism $U \to [U/R]$ (see Diagram (36.1.0.1). Hence the local properties of $s : R \to U$ should reflect local properties of the morphism $U \to [U/R]$. This doesn't work, because $[U/R]$ is not always an algebraic stack, and hence we cannot speak of geometric or algebraic properties of $U \to [U/R]$. But it turns out that we can make some of it work without even referring to the quotient stack at all.

Here is a first example of such a result. The open $W \subset U'$ found in the lemma is roughly speaking the locus where the morphism $U' \to [U/R]$ has property $\mathcal{P}$.

Lemma 36.5.1. *Let S be a scheme. Let (U, R, s, t, c, e, i) be a groupoid over S. Let $g : U' \to U$ be a morphism of schemes. Denote h the composition*

$$h : U' \times_{g,U,t} R \xrightarrow[pr_1]{} R \xrightarrow[s]{} U.$$

Let $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ be properties of morphisms of schemes. Assume

(1) *$\mathcal{R} \Rightarrow \mathcal{Q}$,*
(2) *$\mathcal{Q}$ is preserved under base change and composition,*
(3) *for any morphism $f : X \to Y$ which has $\mathcal{Q}$ there exists a largest open $W(\mathcal{P}, f) \subset X$ such that $f|_{W(\mathcal{P},f)}$ has $\mathcal{P}$, and*
(4) *for any morphism $f : X \to Y$ which has $\mathcal{Q}$, and any morphism $Y' \to Y$ which has $\mathcal{R}$ we have $Y' \times_Y W(\mathcal{P}, f) = W(\mathcal{P}, f')$, where $f' : X_{Y'} \to Y'$ is the base change of f.*

If s, t have $\mathcal{R}$ and g has $\mathcal{Q}$, then there exists an open subscheme $W \subset U'$ such that $W \times_{g,U,t} R = W(\mathcal{P}, h)$.

Proof. Note that the following diagram is commutative

$$\begin{tikzcd} U' \times_{g,U,t} R \times_{t,U,t} R \arrow[r, "\mathrm{pr}_{12}"'] \arrow[d, shift right, "\mathrm{pr}_{01}"'] \arrow[d, shift left, "\mathrm{pr}_{02}"] & R \times_{t,U,t} R \arrow[d, shift right, "\mathrm{pr}_0"'] \arrow[d, shift left, "\mathrm{pr}_1"] \\ U' \times_{g,U,t} R \arrow[r, "\mathrm{pr}_1"] & R \end{tikzcd}$$

with both squares cartesian (this uses that the two maps $t \circ \mathrm{pr}_i : R \times_{t,U,t} R \to U$ are equal). Combining this with the properties of diagram (36.3.0.3) we get a commutative diagram

$$\begin{tikzcd} U' \times_{g,U,t} R \times_{t,U,t} R \arrow[r, "c\circ(i,1)"'] \arrow[d, shift right, "\mathrm{pr}_{01}"'] \arrow[d, shift left, "\mathrm{pr}_{02}"] & R \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\ U' \times_{g,U,t} R \arrow[r, "h"] & U \end{tikzcd}$$

where both squares are cartesian.

Assume s, t have $\mathcal{R}$ and g has $\mathcal{Q}$. Then h has $\mathcal{Q}$ as a composition of s (which has $\mathcal{R}$ hence $\mathcal{Q}$) and a base change of g (which has $\mathcal{Q}$). Thus $W(\mathcal{P}, h) \subset U' \times_{g,U,t} R$ exists. By our assumptions we have $\mathrm{pr}_{01}^{-1}(W(\mathcal{P}, h)) = \mathrm{pr}_{02}^{-1}(W(\mathcal{P}, h))$ since both are the largest open on which $c \circ (i, 1)$ has $\mathcal{P}$. Note that the projection $U' \times_{g,U,t} R \to U'$ has a section, namely $\sigma : U' \to U' \times_{g,U,t} R$, $u' \mapsto (u', e(g(u')))$. Also via the isomorphism

$$(U' \times_{g,U,t} R) \times_{U'} (U' \times_{g,U,t} R) = U' \times_{g,U,t} R \times_{t,U,t} R$$

the two projections of the left hand side to $U' \times_{g,U,t} R$ agree with the morphisms pr_{01} and pr_{02} on the right hand side. Since $\mathrm{pr}_{01}^{-1}(W(\mathcal{P}, h)) = \mathrm{pr}_{02}^{-1}(W(\mathcal{P}, h))$ we conclude that $W(\mathcal{P}, h)$ is the inverse image of a subset of U, which is necessarily the open set $W = \sigma^{-1}(W(\mathcal{P}, h))$. $\square$

Remark 36.5.2. Warning: Lemma 36.5.1 should be used with care. For example, it applies to $\mathcal{P}$=``flat", $\mathcal{Q}$ =``empty", and $\mathcal{R}$ =``flat and locally of finite presentation". But given a morphism of schemes $f : X \to Y$ the largest open $W \subset X$ such that $f|_W$ is flat is *not* the set of points where f is flat!

Remark 36.5.3. Notwithstanding the warning in Remark 36.5.2 there are some cases where Lemma 36.5.1 can be used without causing too much ambiguity. We give a list. In each case we omit the verification of assumptions (1) and (2) and we give references which imply (3) and (4). Here is the list:

(1) $\mathcal{Q} = \mathcal{R} =$``locally of finite type", and $\mathcal{P}$=``relative dimension $\leq d$". See Morphisms, Definition 24.28.1 and Morphisms, Lemmas 24.27.4 and 24.27.3.
(2) $\mathcal{Q} = \mathcal{R} =$``locally of finite type", and $\mathcal{P}$=``locally quasi-finite". This is the case $d = 0$ of the previous item, see Morphisms, Lemma 24.28.5.
(3) $\mathcal{Q} = \mathcal{R} =$``locally of finite type", and $\mathcal{P}$=``unramified". See Morphisms, Lemmas 24.34.3 and 24.34.15.

What is interesting about the cases listed above is that we do not need to assume that s, t are flat to get a conclusion about the locus where the morphism h has property $\mathcal{P}$. We continue the list:

(4) $\mathcal{Q}$ =``locally of finite presentation", $\mathcal{R}$ =``flat and locally of finite presentation", and $\mathcal{P}$=``flat". See More on Morphisms, Theorem 33.11.1 and Lemma 33.11.2.
(5) $\mathcal{Q}$ =``locally of finite presentation", $\mathcal{R}$ =``flat and locally of finite presentation", and $\mathcal{P}$ =``Cohen-Macaulay". See More on Morphisms, Definition 33.15.1 and More on Morphisms, Lemmas 33.15.3 and 33.15.4.
(6) $\mathcal{Q}$ =``locally of finite presentation", $\mathcal{R}$ =``flat and locally of finite presentation", and $\mathcal{P}$=``syntomic" use Morphisms, Lemma 24.30.12 (the locus is automically open).
(7) $\mathcal{Q}$ =``locally of finite presentation", $\mathcal{R}$ =``flat and locally of finite presentation", and $\mathcal{P}$=``smooth". See Morphisms, Lemma 24.33.15 (the locus is automically open).
(8) $\mathcal{Q}$ =``locally of finite presentation", $\mathcal{R}$ =``flat and locally of finite presentation", and $\mathcal{P}$ =``étale". See Morphisms, Lemma 24.35.17 (the locus is automically open).

Here is the second result. The R-invariant open $W \subset U$ should be thought of as the inverse image of the largest open of $[U/R]$ over which the morphism $U \to [U/R]$ has property $\mathcal{P}$.

Lemma 36.5.4. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid over S. Let $\tau \in \{Zariski, fppf, étale, smooth, syntomic\}$[1]. Let $\mathcal{P}$ be a property of morphisms of schemes which is τ-local on the target (Descent, Definition 31.18.1). Assume $\{s : R \to U\}$ and $\{t : R \to U\}$ are coverings for the τ-topology. Let $W \subset U$ be the maximal open subscheme such that $s|_{s^{-1}(W)} : s^{-1}(W) \to W$ has property $\mathcal{P}$. Then W is R-invariant, see Groupoids, Definition 35.16.1.*

Proof. The existence and properties of the open $W \subset U$ are described in Descent, Lemma 31.18.3. In Diagram (36.3.0.2) let $W_1 \subset R$ be the maximal open subscheme over which the morphism $\text{pr}_1 : R \times_{s,U,t} R \to R$ has property $\mathcal{P}$. It follows from the aforementioned Descent, Lemma 31.18.3 and the assumption that $\{s : R \to U\}$ and $\{t : R \to U\}$ are coverings for the τ-topology that $t^{-1}(W) = W_1 = s^{-1}(W)$ as desired. □

Lemma 36.5.5. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid over S. Let $G \to U$ be its stabilizer group scheme. Let $\tau \in \{fppf, étale, smooth, syntomic\}$. Let $\mathcal{P}$ be a property of morphisms which is τ-local on the target. Assume $\{s : R \to U\}$ and $\{t : R \to U\}$ are coverings for the τ-topology. Let $W \subset U$ be the maximal open subscheme such that $G_W \to W$ has property $\mathcal{P}$. Then W is R-invariant (see Groupoids, Definition 35.16.1).*

[1] The fact that $fpqc$ is missing is not a typo.

Proof. The existence and properties of the open $W \subset U$ are described in Descent, Lemma 31.18.3. The morphism

$$G \times_{U,t} R \longrightarrow R \times_{s,U} G, \quad (g, r) \longmapsto (r, r^{-1} \circ g \circ r)$$

is an isomorphism over R (where $\circ$ denotes composition in the groupoid). Hence $s^{-1}(W) = t^{-1}(W)$ by the properties of W proved in the aforementioned Descent, Lemma 31.18.3. $\square$

36.6. Comparing fibres

Let (U, R, s, t, c, e, i) be a groupoid scheme over S. Diagram (36.3.0.2) gives us a way to compare the fibres of the map $s : R \to U$ in a groupoid. For a point $u \in U$ we will denote $F_u = s^{-1}(u)$ the scheme theoretic fibre of $s : R \to U$ over u. For example the diagram implies that if $u, u' \in U$ are points such that $s(r) = u$ and $t(r) = u'$, then $(F_u)_{\kappa(r)} \cong (F_{u'})_{\kappa(r)}$. This is a special case of the more general and more precise Lemma 36.6.1 below. To see this take $r' = i(r)$.

A pair (X, x) consisting of a scheme X and a point $x \in X$ is sometimes called the *germ of X at x*. A *morphism of germs* $f : (X, x) \to (S, s)$ is a morphism $f : U \to S$ defined on an open neighbourhood of x with $f(x) = s$. Two such f, f' are said to give the same morphism of germs if and only if f and f' agree in some open neighbourhood of x. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. We temporarily introduce the following concept: We say that two morphisms of germs $f : (X, x) \to (S, s)$ and $f' : (X', x') \to (S', s')$ are *isomorphic locally on the base in the τ-topology*, if there exists a pointed scheme (S'', s'') and morphisms of germs $g : (S'', s'') \to (S, s)$, and $g' : (S'', s'') \to (S', s')$ such that

(1) g and g' are an open immersion (resp. étale, smooth, syntomic, flat and locally of finite presentation) at s'',
(2) there exists an isomorphism

$$(S'' \times_{g,S,f} X, \tilde{x}) \cong (S'' \times_{g',S',f'} X', \tilde{x}')$$

of germs over the germ (S'', s'') for some choice of points $\tilde{x}$ and $\tilde{x}'$ lying over (s'', x) and (s'', x').

Finally, we simply say that the maps of germs $f : (X, x) \to (S, s)$ and $f' : (X', x') \to (S', s')$ are *flat locally on the base isomorphic* if there exist S'', s'', g, g' as above but with (1) replaced by the condition that g and g' are flat at s'' (this is much weaker than any of the τ conditions above as a flat morphism need not be open).

Lemma 36.6.1. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid over S. Let $r, r' \in R$ with $t(r) = t(r')$ in U. Set $u = s(r)$, $u' = s(r')$. Denote $F_u = s^{-1}(u)$ and $F_{u'} = s^{-1}(u')$ the scheme theoretic fibres.*

(1) *There exists a common field extension $\kappa(u) \subset k$, $\kappa(u') \subset k$ and an isomorphism $(F_u)_k \cong (F_{u'})_k$.*
(2) *We may choose the isomorphism of (1) such that a point lying over r maps to a point lying over r'.*
(3) *If the morphisms s, t are flat then the morphisms of germs $s : (R, r) \to (U, u)$ and $s : (R, r') \to (U, u')$ are flat locally on the base isomorphic.*
(4) *If the morphisms s, t are étale (resp. smooth, syntomic, or flat and locally of finite presentation) then the morphisms of germs $s : (R, r) \to (U, u)$ and $s : (R, r') \to (U, u')$ are locally on the base isomorphic in the étale (resp. smooth, syntomic, or fppf) topology.*

Proof. We repeatedly use the properties and the existence of diagram (36.3.0.2). By the properties of the diagram (and Schemes, Lemma 21.17.5) there exists a point ξ of $R \times_{s,U,t} R$ with $\mathrm{pr}_0(\xi) = r$ and $c(\xi) = r'$. Let $\tilde{r} = \mathrm{pr}_1(\xi) \in R$.

Proof of (1). Set $k = \kappa(\tilde{r})$. Since $t(\tilde{r}) = u$ and $s(\tilde{r}) = u'$ we see that k is a common extension of both $\kappa(u)$ and $\kappa(u')$ and in fact that both $(F_u)_k$ and $(F_{u'})_k$ are isomorphic to the fibre of $\mathrm{pr}_1 : R \times_{s,U,t} R \to R$ over $\tilde{r}$. Hence (1) is proved.

Part (2) follows since the point ξ maps to r, resp. r'.

Part (3) is clear from the above (using the point ξ for $\tilde{u}$ and $\tilde{u}'$) and the definitions.

If s and t are flat and of finite presentation, then they are open morphisms (Morphisms, Lemma 24.24.9). Hence the image of some affine open neighbourhood V'' of $\tilde{r}$ will cover an open neighbourhood V of u, resp. V' of u'. These can be used to show that properties (1) and (2) of the definition of ``locally on the base isomorphic in the τ-topology". ☐

36.7. Cohen-Macaulay presentations

Given any groupoid (U, R, s, t, c) with s, t flat and locally of finite presentation there exists an ``equivalent" groupoid $(U', R', s', t', c')$ such that $s'$ and $t'$ are Cohen-Macaulay morphisms (and locally of finite presentation). See More on Morphisms, Section 33.15 for more information on Cohen-Macaulay morphisms. Here ``equivalent" can be taken to mean that the quotient stacks $[U/R]$ and $[U'/R']$ are equivalent stacks, see Groupoids in Spaces, Section 52.19 and Section 52.24.

Lemma 36.7.1. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid over S. Assume s and t are flat and locally of finite presentation. Then there exists an open $U' \subset U$ such that*

1. *$t^{-1}(U') \subset R$ is the largest open subscheme of R on which the morphism s is Cohen-Macaulay,*
2. *$s^{-1}(U') \subset R$ is the largest open subscheme of R on which the morphism t is Cohen-Macaulay,*
3. *the morphism $t|_{s^{-1}(U')} : s^{-1}(U') \to U$ is surjective,*
4. *the morphism $s|_{t^{-1}(U')} : t^{-1}(U') \to U$ is surjective, and*
5. *the restriction $R' = s^{-1}(U') \cap t^{-1}(U')$ of R to U' defines a groupoid (U', R', s', t', c') which has the property that the morphisms s' and t' are Cohen-Macaulay and locally of finite presentation.*

Proof. Apply Lemma 36.5.1 with $g = \mathrm{id}$ and $\mathcal{Q} =$``locally of finite presentation", $\mathcal{R} =$``flat and locally of finite presentation", and $\mathcal{P} =$``Cohen-Macaulay", see Remark 36.5.3. This gives us an open $U' \subset U$ such that Let $t^{-1}(U') \subset R$ is the largest open subscheme of R on which the morphism s is Cohen-Macaulay. This proves (1). Let $i : R \to R$ be the inverse of the groupoid. Since i is an isomorphism, and $s \circ i = t$ and $t \circ i = s$ we see that $s^{-1}(U')$ is also the largest open of R on which t is Cohen-Macaulay. This proves (2). By More on Morphisms, Lemma 33.15.4 the open subset $t^{-1}(U')$ is dense in every fibre of $s : R \to U$. This proves (3). Same argument for (4). Part (5) is a formal consequence of (1) and (2) and the discussion of restrictions in Groupoids, Section 35.15. ☐

36.8. Restricting groupoids

In this section we collect a bunch of lemmas on properties of groupoids which are inherited by restrictions. Most of these lemmas can be proved by contemplating the defining diagram

(36.8.0.1)
$$\begin{tikzcd}
R' \arrow[r] \arrow[rr, bend left, "s'"] \arrow[d] \arrow[dd, bend right, "t'"'] & R \times_{s,U} U' \arrow[r] \arrow[d] & U' \arrow[d, "g"] \\
U' \times_{U,t} R \arrow[r] \arrow[d] & R \arrow[r, "s"] \arrow[d, "t"] & U \\
U' \arrow[r, "g"] & U &
\end{tikzcd}$$

of a restriction. See Groupoids, Lemma 35.15.1.

Lemma 36.8.1. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $g : U' \to U$ be a morphism of schemes. Let (U', R', s', t', c') be the restriction of (U, R, s, t, c) via g.*

(1) *If s, t are locally of finite type and g is locally of finite type, then s', t' are locally of finite type.*
(2) *If s, t are locally of finite presentation and g is locally of finite presentation, then s', t' are locally of finite presentation.*
(3) *If s, t are flat and g is flat, then s', t' are flat.*
(4) *Add more here.*

Proof. The property of being locally of finite type is stable under composition and arbitrary base change, see Morphisms, Lemmas 24.14.3 and 24.14.4. Hence (1) is clear from Diagram (36.8.0.1). For the other cases, see Morphisms, Lemmas 24.20.3, 24.20.4, 24.24.5, and 24.24.7. □

The following lemma could have been used to prove the results of the preceding lemma in a more uniform way.

Lemma 36.8.2. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $g : U' \to U$ be a morphism of schemes. Let (U', R', s', t', c') be the restriction of (U, R, s, t, c) via g, and let $h = s \circ pr_1 : U' \times_{g,U,t} R \to U$. If $\mathcal{P}$ is a property of morphisms of schemes such that*

(1) *h has property $\mathcal{P}$, and*
(2) *$\mathcal{P}$ is preserved under base change,*

then s', t' have property $\mathcal{P}$.

Proof. This is clear as s' is the base change of h by Diagram (36.8.0.1) and t' is isomorphic to s' as a morphism of schemes. □

Lemma 36.8.3. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $g : U' \to U$ and $g' : U'' \to U'$ be morphisms of schemes. Set $g'' = g \circ g'$. Let (U', R', s', t', c') be the restriction of R to U'. Let $h = s \circ pr_1 : U' \times_{g,U,t} R \to U$, let $h' = s' \circ pr_1 : U'' \times_{g',U',t} R \to U'$, and let $h'' = s \circ pr_1 : U'' \times_{g'',U,t} R \to U$. The following diagram is* *commutative*

$$\begin{tikzcd} U'' \times_{g',U',t} R' \arrow[d, "h'"] & (U' \times_{g,U,t} R) \times_U (U'' \times_{g'',U,t} R) \arrow[l] \arrow[r] \arrow[d] & U'' \times_{g'',U,t} R \arrow[d, "h''"] \\ U' & U' \times_{g,U,t} R \arrow[l, "pr_0"] \arrow[r, "h"] & U \end{tikzcd}$$

with both squares cartesian where the left upper horizontal arrow is given by the rule

$$\begin{matrix} (U' \times_{g,U,t} R) \times_U (U'' \times_{g'',U,t} R) & \longrightarrow & U'' \times_{g',U',t} R' \\ ((u', r_0), (u'', r_1)) & \longmapsto & (u'', (c(r_1, i(r_0)), (g'(u''), u'))) \end{matrix}$$

with notation as explained in the proof.

Proof. We work this out by exploiting the functorial point of view and reducing the lemma to a statement on arrows in restrictions of a groupoid category. In the last formula of the lemma the notation $((u', r_0), (u'', r_1))$ indicates a T-valued point of $(U' \times_{g,U,t} R) \times_U (U'' \times_{g'',U,t} R)$. This means that u', u'', r_0, r_1 are T-valued points of U', U'', R, R and that $g(u') = t(r_0)$, $g(g'(u'')) = g''(u'') = t(r_1)$, and $s(r_0) = s(r_1)$. It would be more correct here to write $g \circ u' = t \circ r_0$ and so on but this makes the notation even more unreadable. If we think of r_1 and r_0 as arrows in a groupoid category then we can represent this by the picture

$$t(r_0) = g(u') \xleftarrow{r_0} s(r_0) = s(r_1) \xrightarrow{r_1} t(r_1) = g(g'(u''))$$

This diagram in particular demonstrates that the composition $c(r_1, i(r_0))$ makes sense. Recall that

$$R' = R \times_{(t,s),U \times_S U, g \times g} U' \times_S U'$$

hence a T-valued point of R' looks like $(r, (u'_0, u'_1))$ with $t(r) = g(u'_0)$ and $s(r) = g(u'_1)$. In particular given $((u', r_0), (u'', r_1))$ as above we get the T-valued point $(c(r_1, i(r_0)), (g'(u''), u'))$ of R' because we have $t(c(r_1, i(r_0))) = t(r_1) = g(g'(u''))$ and $s(c(r_1, i(r_0))) = s(i(r_0)) = t(r_0) = g(u')$. We leave it to the reader to show that the left square commutes with this definition.

To show that the left square is cartesian, suppose we are given (v'', p') and (v', p) which are T-valued points of $U'' \times_{g',U',t} R'$ and $U' \times_{g,U,t} R$ with $v' = s'(p')$. This also means that $g'(v'') = t'(p')$ and $g(v') = t(p)$. By the discussion above we know that we can write $p' = (r, (u'_0, u'_1))$ with $t(r) = g(u'_0)$ and $s(r) = g(u'_1)$. Using this notation we see that $v' = s'(p') = u'_1$ and $g'(v'') = t'(p') = u'_0$. Here is a picture

$$s(p) \xrightarrow{p} g(v') = g(u'_1) \xrightarrow{r} g(u'_0) = g(g'(v''))$$

What we have to show is that there exists a unique T-valued point $((u', r_0), (u'', r_1))$ as above such that $v' = u'$, $p = r_0$, $v'' = u''$ and $p' = (c(r_1, i(r_0)), (g'(u''), u'))$. Comparing the two diagrams above it is clear that we have no choice but to take

$$((u', r_0), (u'', r_1)) = ((v', p), (v'', c(r, p)))$$

Some details omitted. □

Lemma 36.8.4. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $g : U' \to U$ and $g' : U'' \to U'$ be morphisms of schemes. Set $g'' = g \circ g'$. Let (U', R', s', t', c') be the restriction of R to U'. Let $h = s \circ pr_1 : U' \times_{g,U,t} R \to U$, let $h' = s' \circ pr_1 : U'' \times_{g',U',t} R \to U'$, and let $h'' = s \circ pr_1 : U'' \times_{g'',U,t} R \to U$. Let $\tau \in \{Zariski,$*

étale, smooth, syntomic, fppf, fpqc}. Let $\mathcal{P}$ be a property of morphisms of schemes which is preserved under base change, and which is local on the target for the τ-topology. If

(1) *$h(U' \times_U R)$ is open in U,*
(2) *$\{h : U' \times_U R \to h(U' \times_U R)\}$ is a τ-covering,*
(3) *h' has property $\mathcal{P}$,*

then h'' has property $\mathcal{P}$. Conversely, if

(a) *$\{t : R \to U\}$ is a τ-covering,*
(d) *h'' has property $\mathcal{P}$,*

then h' has property $\mathcal{P}$.

Proof. This follows formally from the properties of the diagram of Lemma 36.8.3. In the first case, note that the image of the morphism h'' is contained in the image of h, as $g'' = g \circ g'$. Hence we may replace the U in the lower right corner of the diagram by $h(U' \times_U R)$. This explains the significance of conditions (1) and (2) in the lemma. In the second case, note that $\{\text{pr}_0 : U' \times_{g,U,t} R \to U'\}$ is a τ-covering as a base change of τ and condition (a). $\square$

36.9. Properties of groupoids on fields

A ``groupoid on a field'' indicates a groupoid scheme (U, R, s, t, c) where U is the spectrum of a field. It does **not** mean that (U, R, s, t, c) is defined over a field, more precisely, it does **not** mean that the morphisms $s, t : R \to U$ are equal. Given any field k, an abstract group G and a group homomorphism $\varphi : G \to \text{Aut}(k)$ we obtain a groupoid scheme (U, R, s, t, c) over $\mathbf{Z}$ by setting

$$
\begin{aligned}
U &= Spec(k) \\
R &= \coprod\nolimits_{g \in G} Spec(k) \\
s &= \coprod\nolimits_{g \in G} Spec(\text{id}_k) \\
t &= \coprod\nolimits_{g \in G} Spec(\varphi(g)) \\
c &= \text{composition in } G
\end{aligned}
$$

This example still is a groupoid scheme over $Spec(k^G)$. Hence, if G is finite, then $U = Spec(k)$ is finite over $Spec(k^G)$. In some sense our goal in this section is to show that suitable finiteness conditions on s, t force any groupoid on a field to be defined over a finite index subfield $k' \subset k$.

If k is a field and (G, m) is a group scheme over k with structure morphism $p : G \to Spec(k)$, then $(Spec(k), G, p, p, m)$ is an example of a groupoid on a field (and in this case of course the whole structure is defined over a field). Hence this section can be viewed as the analogue of Groupoids, Section 35.7.

Lemma 36.9.1. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. If U is the spectrum of a field, then the composition morphism $c : R \times_{s,U,t} R \to R$ is open.*

Proof. The composition is isomorphic to the projection map $\text{pr}_1 : R \times_{t,U,t} R \to R$ by Diagram (36.3.0.3). The projection is open by Morphisms, Lemma 24.22.4. $\square$

Lemma 36.9.2. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. If U is the spectrum of a field, then R is a separated scheme.*

Proof. By Groupoids, Lemma 35.7.2 the stabilizer group scheme $G \to U$ is separated. By Groupoids, Lemma 35.18.2 the morphism $j = (t, s) : R \to U \times_S U$ is separated. As U is the spectrum of a field the scheme $U \times_S U$ is affine (by the construction of fibre products in Schemes, Section 21.17). Hence R is a separated scheme, see Schemes, Lemma 21.21.13. $\square$

Lemma 36.9.3. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = Spec(k)$ with k a field. For any points $r, r' \in R$ there exists a field extension $k \subset k'$ and points $r_1, r_2 \in R \times_{s,Spec(k)} Spec(k')$ and a diagram*

$$R \xleftarrow{pr_0} R \times_{s,Spec(k)} Spec(k') \xrightarrow{\varphi} R \times_{s,Spec(k)} Spec(k') \xrightarrow{pr_0} R$$

such that φ is an isomorphism of schemes over $Spec(k')$, we have $\varphi(r_1) = r_2$, $pr_0(r_1) = r$, and $pr_0(r_2) = r'$.

Proof. This is a special case of Lemma 36.6.1 parts (1) and (2). $\square$

Lemma 36.9.4. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = Spec(k)$ with k a field. Let $k \subset k'$ be a field extension, $U' = Spec(k')$ and let (U', R', s', t', c') be the restriction of (U, R, s, t, c) via $U' \to U$. In the defining diagram*

$$\begin{tikzcd}
R' \arrow[r] \arrow[rr, bend left, "s'"] \arrow[d] \arrow[dr, dotted] \arrow[dd, bend right, "t'"'] & R \times_{s,U} U' \arrow[r] \arrow[d] & U' \arrow[d] \\
U' \times_{U,t} R \arrow[r] \arrow[d] & R \arrow[r, "s"] \arrow[d, "t"] & U \\
U' \arrow[r] & U &
\end{tikzcd}$$

all the morphisms are surjective, flat, and universally open. The dotted arrow $R' \to R$ is in addition affine.

Proof. The morphism $U' \to U$ equals $Spec(k') \to Spec(k)$, hence is affine, surjective and flat. The morphisms $s, t : R \to U$ and the morphism $U' \to U$ are universally open by Morphisms, Lemma 24.22.4. Since R is not empty and U is the spectrum of a field the morphisms $s, t : R \to U$ are surjective and flat. Then you conclude by using Morphisms, Lemmas 24.9.4, 24.9.2, 24.22.3, 24.11.8, 24.11.7, 24.24.7, and 24.24.5. $\square$

Lemma 36.9.5. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = Spec(k)$ with k a field. For any point $r \in R$ there exist*

(1) *a field extension $k \subset k'$ with k' algebraically closed,*
(2) *a point $r' \in R'$ where (U', R', s', t', c') is the restriction of (U, R, s, t, c) via $Spec(k') \to Spec(k)$*

such that

(1) *the point r' maps to r under the morphism $R' \to R$, and*
(2) *the maps $s', t' : R' \to Spec(k')$ induce isomorphisms $k' \to \kappa(r')$.*

Proof. Translating the geometric statement into a statement on fields, this means that we can find a diagram

$$\begin{tikzcd}
k' & k' \arrow[l, "1"'] & \\
k' \arrow[u, "\tau"] & \kappa(r) \arrow[ul, "\sigma"] & k \arrow[l, "s"] \arrow[ul, "i"'] \\
& k \arrow[u, "t"'] \arrow[ul, "i"] &
\end{tikzcd}$$

where $i : k \to k'$ is the embedding of k into k', the maps $s, t : k \to \kappa(r)$ are induced by $s, t : R \to U$, and the map $\tau : k' \to k'$ is an automorphism. To produce such a diagram we may proceed in the following way:

(1) Pick $i : k \to k'$ a field map with k' algebraically closed of very large transcendence degree over k.
(2) Pick an embedding $\sigma : \kappa(r) \to k'$ such that $\sigma \circ s = i$. Such a σ exists because we can just choose a transcendence basis $\{x_\alpha\}_{\alpha \in A}$ of $\kappa(r)$ over k and find $y_\alpha \in k'$, $\alpha \in A$ which are algebraically independent over $i(k)$, and map $s(k)(\{x_\alpha\})$ into k' by the rules $s(\lambda) \mapsto i(\lambda)$ for $\lambda \in k$ and $x_\alpha \mapsto y_\alpha$ for $\alpha \in A$. Then extend to $\tau : \kappa(\alpha) \to k'$ using that k' is algebraically closed.
(3) Pick an automorphism $\tau : k' \to k'$ such that $\tau \circ i = \sigma \circ t$. To do this pick a transcendence basis $\{x_\alpha\}_{\alpha \in A}$ of k over its prime field. On the one hand, extend $\{i(x_\alpha)\}$ to a transcedence basis of k' by adding $\{y_\beta\}_{\beta \in B}$ and extend $\{\sigma(t(x_\alpha))\}$ to a transcendence basis of k' by adding $\{z_\gamma\}_{\gamma \in C}$. As k' is algebraically closed we can extend the isomorphism $\sigma \circ t \circ i^{-1} : i(k) \to \sigma(t(k))$ to an isomorphism $\tau' : \overline{i(k)} \to \overline{\sigma(t(k))}$ of their algebraic closures in k'. As k' has large transcendence degree we see that the sets B and C have the same cardinality. Thus we can use a bijection $B \to C$ to extend τ' to an isomorphism
$$\overline{i(k)}(\{y_\beta\}) \longrightarrow \overline{\sigma(t(k))}(\{z_\gamma\})$$
and then since k' is the algebraic closure of both sides we see that this extends to an automorphism $\tau : k' \to k'$ as desired.

This proves the lemma. □

Lemma 36.9.6. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = Spec(k)$ with k a field. If $r \in R$ is a point such that s, t induce isomorphisms $k \to \kappa(r)$, then the map*
$$R \longrightarrow R, \quad x \longmapsto c(r, x)$$
(see proof for precise notation) is an automorphism $R \to R$ which maps e to r.

Proof. This is completely obvious if you think about groupoids in a functorial way. But we will also spell it out completely. Denote $a : U \to R$ the morphism with image r such that $s \circ a = \mathrm{id}_U$ which exists by the hypothesis that $s : k \to \kappa(r)$ is an isomorphism. Similarly, denote $b : U \to R$ the morphism with image r such that $t \circ b = \mathrm{id}_U$. Note that $b = a \circ (t \circ a)^{-1}$, in particular $a \circ s \circ b = b$.

Consider the morphism $\Psi : R \to R$ given on T-valued points by
$$(f : T \to R) \longmapsto (c(a \circ t \circ f, f) : T \to R)$$

To see this is defined we have to check that $s \circ a \circ t \circ f = t \circ f$ which is obvious as $s \circ a = 1$. Note that $\Phi(e) = a$, so that in order to prove the lemma it suffices to show that Φ is an automorphism of R. Let $\Phi : R \to R$ be the morphism given on T-valued points by

$$(g : T \to R) \longmapsto (c(i \circ b \circ t \circ g, g) : T \to R).$$

This is defined because $s \circ i \circ b \circ t \circ g = t \circ b \circ t \circ g = t \circ g$. We claim that Φ and Ψ are inverse to each other. To see this we compute

$$\begin{aligned} & c(a \circ t \circ c(i \circ b \circ t \circ g, g), c(i \circ b \circ t \circ g, g)) \\ & = c(a \circ t \circ i \circ b \circ t \circ g, c(i \circ b \circ t \circ g, g)) \\ & = c(a \circ s \circ b \circ t \circ g, c(i \circ b \circ t \circ g, g)) \\ & = c(b \circ t \circ g, c(i \circ b \circ t \circ g, g)) \\ & = c(c(b \circ t \circ g, i \circ b \circ t \circ g), g)) \\ & = c(e, g) \\ & = g \end{aligned}$$

where we have used the relation $a \circ s \circ b = b$ shown above. In the other direction we have

$$\begin{aligned} & c(i \circ b \circ t \circ c(a \circ t \circ f, f), c(a \circ t \circ f, f)) \\ & = c(i \circ b \circ t \circ a \circ t \circ f, c(a \circ t \circ f, f)) \\ & = c(i \circ a \circ (t \circ a)^{-1} \circ t \circ a \circ t \circ f, c(a \circ t \circ f, f)) \\ & = c(i \circ a \circ t \circ f, c(a \circ t \circ f, f)) \\ & = c(c(i \circ a \circ t \circ f, a \circ t \circ f), f) \\ & = c(e, f) \\ & = f \end{aligned}$$

The lemma is proved. $\square$

Lemma 36.9.7. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = Spec(k)$ with k a field. By abuse of notation denote $e \in R$ the image of the identity morphism $e : U \to R$. Then*

1. *every local ring $\mathcal{O}_{R,r}$ of R has a unique minimal prime ideal,*
2. *there is exactly one irreducible component Z of R passing through e, and*
3. *Z is geometrically irreducible over k via either s or t.*

Proof. Let $r \in R$ be a point. In this proof we will use the correspondence between irreducible components of R passing through a point r and minimal primes of the local ring $\mathcal{O}_{R,r}$ without further mention. Choose $k \subset k'$ and $r' \in R'$ as in Lemma 36.9.5. Note that $\mathcal{O}_{R,r} \to \mathcal{O}_{R',r'}$ is faithfully flat and local, see Lemma 36.9.4. Hence the result for $r' \in R'$ implies the result for $r \in R$. In other words we may assume that $s, t : k \to \kappa(r)$ are isomorphisms. By Lemma 36.9.6 there exists an automorphism moving e to r. Hence we may assume $r = e$, i.e., part (1) follows from part (2).

We first prove (2) in case k is separably algebraically closed. Namely, let $X, Y \subset R$ be irreducible components passing through e. Then by Varieties, Lemma 28.6.4 and 28.6.3 the scheme $X \times_{s,U,t} Y$ is irreducible as well. Hence $c(X \times_{s,U,t} Y) \subset R$ is an irreducible subset. We claim it contains both X and Y (as subsets of R). Namely, let T be the spectrum of a field. If $x : T \to X$ is a T-valued point of X, then $c(x, e \circ s \circ x) = x$ and $e \circ s \circ x$ factors through Y as $e \in Y$. Similarly for points of Y. This clearly implies that $X = Y$, i.e., there is a unique irreducible component of R passing through e.

Proof of (2) and (3) in general. Let $k \subset k'$ be a separable algebraic closure, and let (U', R', s', t', c') be the restriction of (U, R, s, t, c) via $Spec(k') \to Spec(k)$. By the previous paragraph there is exactly one irreducible component Z' of R' passing through e'. Denote $e'' \in R \times_{s,U} U'$ the base change of e. As $R' \to R \times_{s,U} U'$ is faithfully flat, see Lemma 36.9.4, and $e' \mapsto e''$ we see that there is exactly one irreducible component Z'' of $R \times_{s,k} k'$ passing through e''. This implies, as $R \times_k k' \to R$ is faithfully flat, that there is exactly one irreducible component Z of R passing through e. This proves (2).

To prove (3) let $Z''' \subset R \times_k k'$ be an arbitrary irreducible component of $Z \times_k k'$. By Varieties, Lemma 28.6.12 we see that $Z''' = \sigma(Z'')$ for some $\sigma \in \mathrm{Gal}(k'/k)$. Since $\sigma(e'') = e''$ we see that $e'' \in Z'''$ and hence $Z''' = Z''$. This means that Z is geometrically irreducible over $Spec(k)$ via the morphism s. The same argument implies that Z is geometrically irreducible over $Spec(k)$ via the morphism t. $\square$

Lemma 36.9.8. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = Spec(k)$ with k a field. Assume s, t are locally of finite type. Then*

(1) *R is equidimensional,*
(2) $\dim(R) = \dim_r(R)$ *for all $r \in R$,*
(3) *for any $r \in R$ we have $trdeg_{s(k)}(\kappa(r)) = trdeg_{t(k)}(\kappa(r))$, and*
(4) *for any closed point $r \in R$ we have* $\dim(R) = \dim(\mathcal{O}_{R,r})$.

Proof. Let $r, r' \in R$. Then $\dim_r(R) = \dim_{r'}(R)$ by Lemma 36.9.3 and Morphisms, Lemma 24.27.3. By Morphisms, Lemma 24.27.1 we have

$$\dim_r(R) = \dim(\mathcal{O}_{R,r}) + \mathrm{trdeg}_{s(k)}(\kappa(r)) = \dim(\mathcal{O}_{R,r}) + \mathrm{trdeg}_{t(k)}(\kappa(r)).$$

On the other hand, the dimension of R (or any open subset of R) is the supremum of the dimensions of the local rings of of R, see Properties, Lemma 23.11.4. Clearly this is maximal for closed points r in which case $\mathrm{trdeg}_k(\kappa(r)) = 0$ (by the Hilbert Nullstellensatz, see Morphisms, Section 24.15). Hence the lemma follows. $\square$

Lemma 36.9.9. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = Spec(k)$ with k a field. Assume s, t are locally of finite type. Then* $\dim(R) = \dim(G)$ *where G is the stabilizer group scheme of R.*

Proof. Let $Z \subset R$ be the irreducible component passing through e (see Lemma 36.9.7) thought of as an integral closed subscheme of R. Let k'_s, resp. k'_t be the integral closure of $s(k)$, resp. $t(k)$ in $\Gamma(Z, \mathcal{O}_Z)$. Recall that k'_s and k'_t are fields, see Varieties, Lemma 28.17.4. By Varieties, Proposition 28.18.1 we have $k'_s = k'_t$ as subrings of $\Gamma(Z, \mathcal{O}_Z)$. As e factors through Z we obtain a commutative diagram

$$\begin{tikzcd}
k \arrow[dr, "t"'] \arrow[drr, "1"] & & \\
& \Gamma(Z, \mathcal{O}_Z) \arrow[r, "e"] & k \\
k \arrow[ur, "s"] \arrow[urr, "1"'] & &
\end{tikzcd}$$

This on the one hand shows that $k'_s = s(k)$, $k'_t = t(k)$, so $s(k) = t(k)$, which combined with the diagram above implies that $s = t$! In other words, we conclude that Z is a closed subscheme of $G = R \times_{(t,s), U \times_S U, \Delta} U$. The lemma follows as both G and R are equidimensional, see Lemma 36.9.8 and Groupoids, Lemma 35.7.5. $\square$

Remark 36.9.10. Warning: Lemma 36.9.9 is wrong without the condition that s and t are locally of finite type. An easy example is to start with the action

$$\mathbf{G}_{m,\mathbf{Q}} \times_{\mathbf{Q}} \mathbf{A}^1_{\mathbf{Q}} \to \mathbf{A}^1_{\mathbf{Q}}$$

and restrict the corresponding groupoid scheme to the generic point of $\mathbf{A}^1_{\mathbf{Q}}$. In other words restrict via the morphism $Spec(\mathbf{Q}(x)) \to Spec(\mathbf{Q}[x]) = \mathbf{A}^1_{\mathbf{Q}}$. Then you get a groupoid scheme (U, R, s, t, c) with $U = Spec(\mathbf{Q}(x))$ and

$$R = Spec\left(\mathbf{Q}(x)[y]\left[\frac{1}{P(xy)}, P \in \mathbf{Q}[T], P \neq 0\right]\right)$$

In this case $\dim(R) = 1$ and $\dim(G) = 0$.

Lemma 36.9.11. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume*

(1) *$U = Spec(k)$ with k a field,*
(2) *s, t are locally of finite type, and*
(3) *the characteristic of k is zero.*

Then $s, t : R \to U$ are smooth.

Proof. By Lemma 36.4.1 the sheaf of differentials of $R \to U$ is free. Hence smoothness follows from Varieties, Lemma 28.15.1. ☐

Lemma 36.9.12. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume*

(1) *$U = Spec(k)$ with k a field,*
(2) *s, t are locally of finite type,*
(3) *R is reduced, and*
(4) *k is perfect.*

Then $s, t : R \to U$ are smooth.

Proof. By Lemma 36.4.1 the sheaf $\Omega_{R/U}$ is free. Hence the lemma follows from Varieties, Lemma 28.15.2. ☐

36.10. Morphisms of groupoids on fields

This section studies morphisms between groupoids on fields. This is slightly more general, but very akin to, studying morphisms of groupschemes over a field.

Situation 36.10.1. Let S be a scheme. Let $U = Spec(k)$ be a scheme over S with k a field. Let (U, R_1, s_1, t_1, c_1), (U, R_2, s_2, t_2, c_2) be groupoid schemes over S with identical first component. Let $a : R_1 \to R_2$ be a morphism such that (id_U, a) defines a morphism of groupoid schemes over S, see Groupoids, Definition 35.11.1. In particular, the following diagrams commute

$$\begin{tikzcd} R_2 \arrow[dr, "a"] \arrow[drr, "t_2"] \arrow[dd, "s_2"'] & & \\ & R_1 \arrow[r, "s_1"'] \arrow[d, "t_1"] & U \\ & U & \end{tikzcd} \qquad \begin{tikzcd} R_1 \times_{s_1, U, t_1} R_1 \arrow[r, "c_1"'] \arrow[d, "a \times a"'] & R_1 \arrow[d, "a"] \\ R_2 \times_{s_2, U, t_2} R_2 \arrow[r, "c_2"] & R_2 \end{tikzcd}$$

The following lemma is a generalization of Groupoids, Lemma 35.7.11.

Lemma 36.10.2. *Notation and assumptions as in Situation 36.10.1. If $a(R_1)$ is open in R_2, then $a(R_1)$ is closed in R_2.*

Proof. Let $r_2 \in R_2$ be a point in the closure of $a(R_1)$. We want to show $r_2 \in a(R_1)$. Pick $k \subset k'$ and $r_2' \in R_2'$ adapted to (U, R_2, s_2, t_2, c_2) and r_2 as in Lemma 36.9.5. Let R_i' be the restriction of R_i via the morphism $U' = Spec(k') \to U = Spec(k)$. Let $a' : R_1' \to R_2'$ be the base change of a. The diagram

$$\begin{array}{ccc} R_1' & \xrightarrow{a'} & R_2' \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ R_1 & \xrightarrow{a} & R_2 \end{array}$$

is a fibre square. Hence the image of a' is the inverse image of the image of a via the morphism $p_2 : R_2' \to R_2$. By Lemma 36.9.4 the map p_2 is surjective and open. Hence by Topology, Lemma 5.15.2 we see that r_2' is in the closure of $a'(R_1')$. This means that we may assume that $r_2 \in R_2$ has the property that the maps $k \to \kappa(r_2)$ induced by s_2 and t_2 are isomorphisms.

In this case we can use Lemma 36.9.6. This lemma implies $c(r_2, a(R_1))$ is an open neighbourhood of r_2. Hence $a(R_1) \cap c(r_2, a(R_1)) \neq \emptyset$ as we assumed that r_2 was a point of the closure of $a(R_1)$. Using the inverse of R_2 and R_1 we see this means $c_2(a(R_1), a(R_1))$ contains r_2. As $c_2(a(R_1), a(R_1)) \subset a(c_1(R_1, R_1)) = a(R_1)$ we conclude $r_2 \in a(R_1)$ as desired. $\square$

Lemma 36.10.3. *Notation and assumptions as in Situation 36.10.1. Let $Z \subset R_2$ be the reduced closed subscheme (see Schemes, Definition 21.12.5) whose underlying topological space is the closure of the image of $a : R_1 \to R_2$. Then $c_2(Z \times_{s_2, U, t_2} Z) \subset Z$ set theoretically.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} R_1 \times_{s_1, U, t_1} R_1 & \longrightarrow & R_1 \\ \downarrow & & \downarrow \\ R_2 \times_{s_2, U, t_2} R_2 & \longrightarrow & R_2 \end{array}$$

By Varieties, Lemma 28.14.2 the closure of the image of the left vertical arrow is (set theoretically) $Z \times_{s_2, U, t_2} Z$. Hence the result follows. $\square$

Lemma 36.10.4. *Notation and assumptions as in Situation 36.10.1. Assume that k is perfect. Let $Z \subset R_2$ be the reduced closed subscheme (see Schemes, Definition 21.12.5) whose underlying topological space is the closure of the image of $a : R_1 \to R_2$. Then*

$$(U, Z, s_2|_Z, t_2|_Z, c_2|_Z)$$

is a groupoid scheme over S.

Proof. We first explain why the statement makes sense. Since U is the spectrum of a perfect field k, the scheme Z is geometrically reduced over k (via either projection), see Varieties, Lemma 28.4.3. Hence the scheme $Z \times_{s_2, U, t_2} Z \subset Z$ is reduced, see Varieties, Lemma 28.4.7. Hence by Lemma 36.10.3 we see that c induces a morphism $Z \times_{s_2, U, t_2} Z \to Z$. Finally, it is clear that e_2 factors through Z and that the map $i_2 : R_2 \to R_2$ preserves Z. Since the morphisms of the septtuple $(U, R_2, s_2, t_2, c_2, e_2, i_2)$ satisfies the axioms of a groupoid, it follows that after restricting to Z they satisfy the axioms. $\square$

Lemma 36.10.5. *Notation and assumptions as in Situation 36.10.1. If the image $a(R_1)$ is a locally closed subset of R_2 then it is a closed subset.*

Proof. Let $k \subset k'$ be a perfect closure of the field k. Let R_i' be the restriction of R_i via the morphism $U' = Spec(k') \to Spec(k)$. Note that the morphisms $R_i' \to R_i$ are universal homeomorphisms as compositions of base changes of the universal homeomorphism $U' \to U$ (see diagram in statement of Lemma 36.9.4). Hence it suffices to prove that $a'(R_1')$ is closed in R_2'. In other words, we may assume that k is perfect.

If k is perfect, then the closure of the image is a groupoid scheme $Z \subset R_2$, by Lemma 36.10.4. By the same lemma applied to $\mathrm{id}_{R_1} : R_1 \to R_1$ we see that $(R_2)_{red}$ is a groupoid scheme. Thus we may apply Lemma 36.10.2 to the morphism $a|_{(R_2)_{red}} : (R_2)_{red} \to Z$ to conclude that Z equals the image of a. □

Lemma 36.10.6. *Notation and assumptions as in Situation 36.10.1. Assume that $a : R_1 \to R_2$ is a quasi-compact morphism. Let $Z \subset R_2$ be the scheme theoretic image (see Morphisms, Definition 24.4.2) of $a : R_1 \to R_2$. Then*

$$(U, Z, s_2|_Z, t_2|_Z, c_2|_Z)$$

is a groupoid scheme over S.

Proof. The main difficulty is to show that $c_2|_{Z\times_{s_2,U,t_2}Z}$ maps into Z. Consider the commutative diagram

$$\begin{tikzcd} R_1 \times_{s_1,U,t_1} R_1 \arrow[r] \arrow[d, "a\times a"] & R_1 \arrow[d] \\ R_2 \times_{s_2,U,t_2} R_2 \arrow[r] & R_2 \end{tikzcd}$$

By Varieties, Lemma 28.14.3 we see that the scheme theoretic image of $a\times a$ is $Z\times_{s_2,U,t_2} Z$. By the commutativity of the diagram we conclude that $Z \times_{s_2,U,t_2} Z$ maps into Z by the bottom horizontal arrow. As in the proof of Lemma 36.10.4 it is also true that $i_2(Z) \subset Z$ and that e_2 factors through Z. Hence we conclude as in the proof of that lemma. □

Lemma 36.10.7. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume U is the spectrum of a field. Let $Z \subset U \times_S U$ be the reduced closed subscheme (see Schemes, Definition 21.12.5) whose underlying topological space is the closure of the image of $j = (t, s) : R \to U \times_S U$. Then $pr_{02}(Z \times_{pr_1,U,pr_0} Z) \subset Z$ set theoretically.*

Proof. As $(U, U \times_S U, \mathrm{pr}_1, \mathrm{pr}_0, \mathrm{pr}_{02})$ is a groupoid scheme over S this is a special case of Lemma 36.10.3. But we can also prove it directly as follows.

Write $U = Spec(k)$. Denote R_s (resp. Z_s, resp. U_s^2) the scheme R (resp. Z, resp. $U \times_S U$) viewed as a scheme over k via s (resp. $\mathrm{pr}_1|_Z$, resp. pr_1). Simlarly, denote ${}_tR$ (resp. ${}_tZ$, resp. ${}_tU^2$) the scheme R (resp. Z, resp. $U\times_S U$) viewed as a scheme over k via t (resp. $\mathrm{pr}_0|_Z$, resp. pr_0). The morphism j induces morphisms of schemes $j_s : R_s \to U_s^2$ and ${}_tj : {}_tR \to {}_tU^2$ over k. Consider the commutative diagram

$$\begin{tikzcd} R_s \times_k {}_tR \arrow[r, "c"] \arrow[d, "j_s\times_t j"'] & R \arrow[d, "j"] \\ U_s^2 \times_k {}_tU^2 \arrow[r] & U \times_S U \end{tikzcd}$$

By Varieties, Lemma 28.14.2 we see that the closure of the image of $j_s \times_t j$ is $Z_s \times_{k\ t} Z$. By the commutativity of the diagram we conclude that $Z_s \times_{k\ t} Z$ maps into Z by the bottom horizontal arrow. $\square$

Lemma 36.10.8. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume U is the spectrum of a perfect field. Let $Z \subset U \times_S U$ be the reduced closed subscheme (see Schemes, Definition 21.12.5) whose underlying topological space is the closure of the image of $j = (t, s) : R \to U \times_S U$. Then*

$$(U, Z, pr_0|_Z, pr_1|_Z, pr_{02}|_{Z \times_{pr_1, U, pr_0} Z})$$

is a groupoid scheme over S.

Proof. As $(U, U \times_S U, \mathrm{pr}_1, \mathrm{pr}_0, \mathrm{pr}_{02})$ is a groupoid scheme over S this is a special case of Lemma 36.10.4. But we can also prove it directly as follows.

We first explain why the statement makes sense. Since U is the spectrum of a perfect field k, the scheme Z is geometrically reduced over k (via either projection), see Varieties, Lemma 28.4.3. Hence the scheme $Z \times_{\mathrm{pr}_1, U, \mathrm{pr}_0} Z \subset Z$ is reduced, see Varieties, Lemma 28.4.7. Hence by Lemma 36.10.7 we see that pr_{02} induces a morphism $Z \times_{\mathrm{pr}_1, U, \mathrm{pr}_0} Z \to Z$. Finally, it is clear that $\Delta_{U/S}$ factors through Z and that the map $\sigma : U \times_S U \to U \times_S U$, $(x, y) \mapsto (y, x)$ preserves Z. Since $(U, U \times_S U, \mathrm{pr}_0, \mathrm{pr}_1, \mathrm{pr}_{02}, \Delta_{U/S}, \sigma)$ satisfies the axioms of a groupoid, it follows that after restricting to Z they satisfy the axioms. $\square$

Lemma 36.10.9. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume U is the spectrum of a field and assume R is quasi-compact (equivalently s, t are quasi-compact). Let $Z \subset U \times_S U$ be the scheme theoretic image (see Morphisms, Definition 24.4.2) of $j = (t, s) : R \to U \times_S U$. Then*

$$(U, Z, pr_0|_Z, pr_1|_Z, pr_{02}|_{Z \times_{pr_1, U, pr_0} Z})$$

is a groupoid scheme over S.

Proof. As $(U, U \times_S U, \mathrm{pr}_1, \mathrm{pr}_0, \mathrm{pr}_{02})$ is a groupoid scheme over S this is a special case of Lemma 36.10.6. But we can also prove it directly as follows.

The main difficulty is to show that $\mathrm{pr}_{02}|_{Z \times_{\mathrm{pr}_1, U, \mathrm{pr}_0} Z}$ maps into Z. Write $U = Spec(k)$. Denote R_s (resp. Z_s, resp. U_s^2) the scheme R (resp. Z, resp. $U \times_S U$) viewed as a scheme over k via s (resp. $\mathrm{pr}_1|_Z$, resp. pr_1). Simlarly, denote ${}_tR$ (resp. ${}_tZ$, resp. ${}_tU^2$) the scheme R (resp. Z, resp. $U \times_S U$) viewed as a scheme over k via t (resp. $\mathrm{pr}_0|_Z$, resp. pr_0). The morphism j induces morphisms of schemes $j_s : R_s \to U_s^2$ and ${}_tj : {}_tR \to {}_tU^2$ over k. Consider the commutative diagram

$$\begin{tikzcd} R_s \times_{k\ t} R \arrow[r, "c"] \arrow[d, "j_s \times_t j"'] & R \arrow[d, "j"] \\ U_s^2 \times_{k\ t} U^2 \arrow[r] & U \times_S U \end{tikzcd}$$

By Varieties, Lemma 28.14.3 we see that the scheme theoretic image of $j_s \times_t j$ is $Z_s \times_{k\ t} Z$. By the commutativity of the diagram we conclude that $Z_s \times_{k\ t} Z$ maps into Z by the bottom horizontal arrow. As in the proof of Lemma 36.10.8 it is also true that $\sigma(Z) \subset Z$ and that $\Delta_{U/S}$ factors through Z. Hence we conclude as in the proof of that lemma. $\square$

36.11. Slicing groupoids

The following lemma shows that we may slice a Cohen-Macaulay groupoid scheme in order to reduce the dimension of the fibres, provided that the dimension of the stabilizer is small. This is an essential step in the process of improving a given presentation of a quotient stack.

Situation 36.11.1. Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $g : U' \to U$ be a morphism of schemes. Let $u \in U$ be a point, and let $u' \in U'$ be a point such that $g(u') = u$. Given these data, denote (U', R', s', t', c') the restriction of (U, R, s, t, c) via the morphism g. Denote $G \to U$ the stabilizer group scheme of R, which is a locally closed subscheme of R. Denote h the composition

$$h = s \circ \mathrm{pr}_1 : U' \times_{g,U,t} R \longrightarrow U.$$

Denote $F_u = s^{-1}(u)$ (scheme theoretic fibre), and G_u the scheme theoretic fibre of G over u. Similarly for R' we denote $F'_{u'} = (s')^{-1}(u')$. Because $g(u') = u$ we have

$$F'_{u'} = h^{-1}(u) \times_{Spec(\kappa(u))} Spec(\kappa(u')).$$

The point $e(u) \in R$ may be viewed as a point on G_u and F_u also, and $e'(u')$ is a point of R' (resp. $G'_{u'}$, resp. $F'_{u'}$) which maps to $e(u)$ in R (resp. G_u, resp. F_u).

Lemma 36.11.2. *Let S be a scheme. Let (U, R, s, t, c, e, i) be a groupoid scheme over S. Let $G \to U$ be the stabilizer group scheme. Assume s and t are Cohen-Macaulay and locally of finite presentation. Let $u \in U$ be a finite type point of the scheme U, see Morphisms, Definition 24.15.3. With notation as in Situation 36.11.1, set*

$$d_1 = \dim(G_u), \quad d_2 = \dim_{e(u)}(F_u).$$

If $d_2 > d_1$, then there exist an affine scheme U' and a morphism $g : U' \to U$ such that (with notation as in Situation 36.11.1)

1. *g is an immersion*
2. *$u \in U'$,*
3. *g is locally of finite presentation,*
4. *the morphism $h : U' \times_{g,U,t} R \longrightarrow U$ is Cohen-Macaulay at $(u, e(u))$, and*
5. *we have $\dim_{e'(u)}(F'_u) = d_2 - 1$.*

Proof. Let $Spec(A) \subset U$ be an affine neighbourhood of u such that u corresponds to a closed point of U, see Morphisms, Lemma 24.15.4. Let $Spec(B) \subset R$ be an affine neighbourhood of $e(u)$ which maps via j into the open $Spec(A) \times_S Spec(A) \subset U \times_S U$. Let $\mathfrak{m} \subset A$ be the maximal ideal corresponding to u. Let $\mathfrak{q} \subset B$ be the prime ideal corresponding to $e(u)$. Pictures:

$$\begin{tikzcd} B & A \arrow[l, "s"'] \\ A \arrow[u, "t"] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} B_{\mathfrak{q}} & A_{\mathfrak{m}} \arrow[l, "s"'] \\ A_{\mathfrak{m}} \arrow[u, "t"] & \end{tikzcd}$$

Note that the two induced maps $s, t : \kappa(\mathfrak{m}) \to \kappa(\mathfrak{q})$ are equal and isomorphisms as $s \circ e = t \circ e = \mathrm{id}_U$. In particular we see that $\mathfrak{q}$ is a maximal ideal as well. The ring maps $s, t : A \to B$ are of finite presentation and flat. By assumption the ring

$$\mathcal{O}_{F_u, e(u)} = B_{\mathfrak{q}}/s(\mathfrak{m})B_{\mathfrak{q}}$$

is Cohen-Macaulay of dimension d_2. The equality of dimension holds by Morphisms, Lemma 24.27.1.

Let R'' be the restriction of R to $u = Spec(\kappa(u))$ via the morphism $Spec(\kappa(u)) \to U$. As $u \to U$ is locally of finite type, we see that $(Spec(\kappa(u)), R'', s'', t'', c'')$ is a groupoid scheme with s'', t'' locally of finite type, see Lemma 36.8.1. By Lemma 36.9.9 this implies that $\dim(G'') = \dim(R'')$. We also have $\dim(R'') = \dim_{e''}(R'') = \dim(\mathcal{O}_{R'',e''})$, see Lemma 36.9.8. By Groupoids, Lemma 35.15.4 we have $G'' = G_u$. Hence we conclude that $\dim(\mathcal{O}_{R'',e''}) = d_1$.

As a scheme R'' is

$$R'' = R \times_{(U \times_S U)} \Big(Spec(\kappa(\mathfrak{m})) \times_S Spec(\kappa(\mathfrak{m})) \Big)$$

Hence an affine open neighbourhood of e'' is the spectrum of the ring

$$B \otimes_{(A \otimes A)} (\kappa(\mathfrak{m}) \otimes \kappa(\mathfrak{m})) = B/s(\mathfrak{m})B + t(\mathfrak{m})B$$

We conclude that

$$\mathcal{O}_{R'',e''} = B_{\mathfrak{q}}/s(\mathfrak{m})B_{\mathfrak{q}} + t(\mathfrak{m})B_{\mathfrak{q}}$$

and so now we know that this ring has dimension d_1.

We claim this implies we can find an element $f \in \mathfrak{m}$ such that

$$\dim(B_{\mathfrak{q}}/(s(\mathfrak{m})B_{\mathfrak{q}} + fB_{\mathfrak{q}}) < d_2$$

Namely, suppose $\mathfrak{n}_j \supset s(\mathfrak{m})B_{\mathfrak{q}}$, $j = 1, \ldots, m$ correspond to the minimal primes of the local ring $B_{\mathfrak{q}}/s(\mathfrak{m})B_{\mathfrak{q}}$. There are finitely many as this ring is Noetherian (since it is essentially of finite type over a field -- but also because a Cohen-Macaulay ring is Noetherian). By the Cohen-Macaulay condition we have $\dim(B_{\mathfrak{q}}/\mathfrak{n}_j) = d_2$, for example by Algebra, Lemma 7.96.4. Note that $\dim(B_{\mathfrak{q}}/(\mathfrak{n}_j + t(\mathfrak{m})B_{\mathfrak{q}})) \leq d_1$ as it is a quotient of the ring $\mathcal{O}_{R'',e''} = B_{\mathfrak{q}}/s(\mathfrak{m})B_{\mathfrak{q}} + t(\mathfrak{m})B_{\mathfrak{q}}$ which has dimension d_1. As $d_1 < d_2$ this implies that $\mathfrak{m} \not\subset t^{-1}(\mathfrak{n}_i)$. By prime avoidence, see Algebra, Lemma 7.14.3, we can find $f \in \mathfrak{m}$ with $t(f) \not\in \mathfrak{n}_j$ for $j = 1, \ldots, m$. For this choice of f we have the displayed inequality above, see Algebra, Lemma 7.57.11.

Set $A' = A/fA$ and $U' = Spec(A')$. Then it is clear that $U' \to U$ is an immersion, locally of finite presentation and that $u \in U'$. Thus (1), (2) and (3) of the lemma hold. The morphism

$$U' \times_{g,U,t} R \longrightarrow U$$

factors through $Spec(A)$ and corresponds to the ring map

$$B/t(f)B = A/(f) \otimes_{A,t} B \xleftarrow{s} A$$

Now, we see $t(f)$ is not a zero divisor on $B_{\mathfrak{q}}/s(\mathfrak{m})B_{\mathfrak{q}}$ as this is a Cohen-Macaulay ring of positive dimension and f is not contained in any minimal prime, see for example Algebra, Lemma 7.96.2. Hence by Algebra, Lemma 7.119.5 we conclude that $s : A_{\mathfrak{m}} \to B_{\mathfrak{q}}/t(f)B_{\mathfrak{q}}$ is flat with fibre ring $B_{\mathfrak{q}}/(s(\mathfrak{m})B_{\mathfrak{q}} + t(f)B_{\mathfrak{q}})$ which is Cohen-Macaulay by Algebra, Lemma 7.96.2 again. This implies part (4) of the lemma. To see part (5) note that by Diagram (36.8.0.1) the fibre F'_u is equal to the fibre of h over u. Hence $\dim_{e'(u)}(F'_u) = \dim(B_{\mathfrak{q}}/(s(\mathfrak{m})B_{\mathfrak{q}} + t(f)B_{\mathfrak{q}}))$ by Morphisms, Lemma 24.27.1 and the dimension of this ring is $d_2 - 1$ by Algebra, Lemma 7.96.2 once more. This proves the final assertion of the lemma and we win. ☐

Now that we know how to slice we can combine it with the preceding material to get the following ``optimal" result. It is optimal in the sense that since G_u is a locally closed subscheme of F_u one always has the inequality $\dim(G_u) = \dim_{e(u)}(G_u) \leq \dim_{e(u)}(F_u)$ so it is not possible to slice more than in the lemma.

Lemma 36.11.3. *Let S be a scheme. Let (U, R, s, t, c, e, i) be a groupoid scheme over S. Let $G \to U$ be the stabilizer group scheme. Assume s and t are Cohen-Macaulay and locally of finite presentation. Let $u \in U$ be a finite type point of the scheme U, see Morphisms, Definition 24.15.3. With notation as in Situation 36.11.1 there exist an affine scheme U' and a morphism $g : U' \to U$ such that*

(1) *g is an immersion,*
(2) *$u \in U'$,*
(3) *g is locally of finite presentation,*
(4) *the morphism $h : U' \times_{g,U,t} R \longrightarrow U$ is Cohen-Macaulay and locally of finite presentation,*
(5) *the morphisms $s', t' : R' \to U'$ are Cohen-Macaulay and locally of finite presentation, and*
(6) $\dim_{e(u)}(F'_u) = \dim(G'_u)$.

Proof. As s is locally of finite presentation the scheme F_u is locally of finite type over $\kappa(u)$. Hence $\dim_{e(u)}(F_u) < \infty$ and we may argue by induction on $\dim_{e(u)}(F_u)$.

If $\dim_{e(u)}(F_u) = \dim(G_u)$ there is nothing to prove. Assume $\dim_{e(u)}(F_u) > \dim(G_u)$. This means that Lemma 36.11.2 applies and we find a morphism $g : U' \to U$ which has properties (1), (2), (3), instead of (6) we have $\dim_{e(u)}(F'_u) < \dim_{e(u)}(F_u)$, and instead of (4) and (5) we have that the composition

$$h = s \circ \mathrm{pr}_1 : U' \times_{g,U,t} R \longrightarrow U$$

is Cohen-Macaulay at the point $(u, e(u))$. We apply Remark 36.5.3 and we obtain an open subscheme $U'' \subset U'$ such that $U'' \times_{g,U,t} R \subset U' \times_{g,U,t} R$ is the largest open subscheme on which h is Cohen-Macaulay. Since $(u, e(u)) \in U'' \times_{g,U,t} R$ we see that $u \in U''$. Hence we may replace U' by U'' and assume that in fact h is Cohen-Macaulay everywhere! By Lemma 36.8.2 we conclude that s', t' are locally of finite presentation and Cohen-Macaulay (use Morphisms, Lemma 24.20.4 and More on Morphisms, Lemma 33.15.3).

By construction $\dim_{e'(u)}(F'_u) < \dim_{e(u)}(F_u)$, so we may apply the induction hypothesis to (U', R', s', t', c') and the point $u \in U'$. Note that u is also a finite type point of U' (for example you can see this using the characterization of finite type points from Morphisms, Lemma 24.15.4). Let $g' : U'' \to U'$ and $(U'', R'', s'', t'', c'')$ be the solution of the corresponding problem starting with (U', R', s', t', c') and the point $u \in U'$. We claim that the composition

$$g'' = g \circ g' : U'' \longrightarrow U$$

is a solution for the original problem. Properties (1), (2), (3), (5), and (6) are immediate. To see (4) note that the morphism

$$h'' = s \circ \mathrm{pr}_1 : U'' \times_{g'',U,t} R \longrightarrow U$$

is locally of finite presentation and Cohen-Macaulay by an application of Lemma 36.8.4 (use More on Morphisms, Lemma 33.15.7 to see that Cohen-Macaulay morphisms are fppf local on the target). □

In case the stabilizer group scheme has fibres of dimension 0 this leads to the following slicing lemma.

Lemma 36.11.4. *Let S be a scheme. Let (U, R, s, t, c, e, i) be a groupoid scheme over S. Let $G \to U$ be the stabilizer group scheme. Assume s and t are Cohen-Macaulay and locally of finite presentation. Let $u \in U$ be a finite type point of the scheme U, see Morphisms,* *Definition 24.15.3. Assume that $G \to U$ is locally quasi-finite. With notation as in Situation 36.11.1 there exist an affine scheme U' and a morphism $g : U' \to U$ such that*

(1) *g is an immersion,*
(2) *$u \in U'$,*
(3) *g is locally of finite presentation,*
(4) *the morphism $h : U' \times_{g,U,t} R \longrightarrow U$ is flat, locally of finite presentation, and locally quasi-finite, and*
(5) *the morphisms $s', t' : R' \to U'$ are flat, locally of finite presentation, and locally quasi-finite.*

Proof. Take $g : U' \to U$ as in Lemma 36.11.3. Since $h^{-1}(u) = F'_u$ we see that h has relative dimension ≤ 0 at $(u, e(u))$. Hence, by Remark 36.5.3, we obtain an open subscheme $U'' \subset U'$ such that $u \in U''$ and $U'' \times_{g,U,t} R$ is the maximal open subscheme of $U' \times_{g,U,t} R$ on which h has relative dimension ≤ 0. After replacing U' by U'' we see that h has relative dimension ≤ 0. This implies that h is locally quasi-finite by Morphisms, Lemma 24.28.5. Since it is still locally of finite presentation and Cohen-Macaulay we see that it is flat, locally of finite presentation and locally quasi-finite, i.e., (4) above holds. This implies that s' is flat, locally of finite presentation and locally quasi-finite as a base change of h, see Lemma 36.8.2. □

36.12. Étale localization of groupoids

In this section we begin applying the étale localization techniques of More on Morphisms, Section 33.28 to groupoid schemes. More advanced material of this kind can be found in More on Groupoids in Spaces, Section 53.11. Lemma 36.12.2 will be used to prove results on algebraic spaces separated and quasi-finite over a scheme, namely Morphisms of Spaces, Proposition 42.39.2 and its corollary Morphisms of Spaces, Lemma 42.40.1.

Lemma 36.12.1. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $p \in S$ be a point, and let $u \in U$ be a point lying over p. Assume that*

(1) *$U \to S$ is locally of finite type,*
(2) *$U \to S$ is quasi-finite at u,*
(3) *$U \to S$ is separated,*
(4) *$R \to S$ is separated,*
(5) *s, t are flat and locally of finite presentation, and*
(6) *$s^{-1}(\{u\})$ is finite.*

Then there exists an étale neighbourhood $(S', p') \to (S, p)$ with $\kappa(p) = \kappa(p')$ and a base change diagram

$$\begin{tikzcd}
R' \coprod W' \arrow[r, equal] & S' \times_S R \arrow[r] \arrow[d, shift right, "t'"'] \arrow[d, shift left, "s'"] & R \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\
U' \coprod W \arrow[r, equal] & S' \times_S U \arrow[r] \arrow[d] & U \arrow[d] \\
 & S' \arrow[r] & S
\end{tikzcd}$$

where the equal signs are decompositions into open and closed subschemes such that

(a) *there exists a point u' of U' mapping to u in U,*
(b) *the fibre $(U')_{p'}$ equals $t'\left((s')^{-1}(\{u'\})\right)$ set theoretically,*
(c) *the fibre $(R')_{p'}$ equals $(s')^{-1}\left((U')_{p'}\right)$ set theoretically,*

(d) *the schemes U' and R' are finite over S',*
(e) *we have $s'(R') \subset U'$ and $t'(R') \subset U'$,*
(f) *we have $c'(R' \times_{s',U',t'} R') \subset R'$ where c' is the base change of c, and*
(g) *the morphisms s', t', c' determine a groupoid structure by taking the system $(U', R', s'|_{R'}, t'|_{R'}, c'|_{R' \times_{s',U',t'} R'})$.*

Proof. Let us denote $f : U \to S$ the structure morphism of U. By assumption (6) we can write $s^{-1}(\{u\}) = \{r_1, \ldots, r_n\}$. Since this set is finite, we see that s is quasi-finite at each of these finitely many inverse images, see Morphisms, Lemma 24.19.7. Hence we see that $f \circ s : R \to S$ is quasi-finite at each r_i (Morphisms, Lemma 24.19.12). Hence r_i is isolated in the fibre R_p, see Morphisms, Lemma 24.19.6. Write $t(\{r_1, \ldots, r_n\}) = \{u_1, \ldots, u_m\}$. Note that it may happen that $m < n$ and note that $u \in \{u_1, \ldots, u_m\}$. Since t is flat and locally of finite presentation, the morphism of fibres $t_p : R_p \to U_p$ is flat and locally of finite presentation (Morphisms, Lemmas 24.24.7 and 24.20.4), hence open (Morphisms, Lemma 24.24.9). The fact that each r_i is isolated in R_p implies that each $u_j = t(r_i)$ is isolated in U_p. Using Morphisms, Lemma 24.19.6 again, we see that f is quasi-finite at $u_1, \ldots, u_m$.

Denote $F_u = s^{-1}(u)$ and $F_{u_j} = s^{-1}(u_j)$ the scheme theoretic fibres. Note that F_u is finite over $\kappa(u)$ as it is locally of finite type over $\kappa(u)$ with finitely many points (for example it follows from the much more general Morphisms, Lemma 24.48.8). By Lemma 36.6.1 we see that F_u and F_{u_j} become isomorphic over a common field extension of $\kappa(u)$ and $\kappa(u_j)$. Hence we see that F_{u_j} is finite over $\kappa(u_j)$. In particular we see $s^{-1}(\{u_j\})$ is a finite set for each $j = 1, \ldots, m$. Thus we see that assumptions (2) and (6) hold for each u_j also (above we saw that $U \to S$ is quasi-finite at u_j). Hence the argument of the first paragraph applies to each u_j and we see that $R \to U$ is quasi-finite at each of the points of

$$\{r_1, \ldots, r_N\} = s^{-1}(\{u_1, \ldots, u_m\})$$

Note that $t(\{r_1, \ldots, r_N\}) = \{u_1, \ldots, u_m\}$ and $t^{-1}(\{u_1, \ldots, u_m\}) = \{r_1, \ldots, r_N\}$ since R is a groupoid[2]. Moreover, we have $\mathrm{pr}_0(c^{-1}(\{r_1, \ldots, r_N\})) = \{r_1, \ldots, r_N\}$ and $\mathrm{pr}_1(c^{-1}(\{r_1, \ldots, r_N\})) = \{r_1, \ldots, r_N\}$. Similarly we get $e(\{u_1, \ldots, u_m\}) \subset \{r_1, \ldots, r_N\}$ and $i(\{r_1, \ldots, r_N\}) = \{r_1, \ldots, r_N\}$.

We may apply More on Morphisms, Lemma 33.28.4 to the pairs $(U \to S, \{u_1, \ldots, u_m\})$ and $(R \to S, \{r_1, \ldots, r_N\})$ to get an étale neighbourhood $(S', p') \to (S, p)$ which induces an identification $\kappa(p) = \kappa(p')$ such that $S' \times_S U$ and $S' \times_S R$ decompose as

$$S' \times_S U = U' \coprod W, \quad S' \times_S R = R' \coprod W'$$

with $U' \to S'$ finite and $(U')_{p'}$ mapping bijectively to $\{u_1, \ldots, u_m\}$, and $R' \to S'$ finite and $(R')_{p'}$ mapping bijectively to $\{r_1, \ldots, r_N\}$. Moreover, no point of $W_{p'}$ (resp. $(W')_{p'}$) maps to any of the points u_j (resp. r_i). At this point (a), (b), (c), and (d) of the lemma are satisfied. Moreover, the inclusions of (e) and (f) hold on fibres over p', i.e., $s'((R')_{p'}) \subset (U')_{p'}$, $t'((R')_{p'}) \subset (U')_{p'}$, and $c'((R' \times_{s',U',t'} R')_{p'}) \subset (R')_{p'}$.

We claim that we can replace S' by a Zariski open neighbourhood of p' so that the inclusions of (e) and (f) hold. For example, consider the set $E = (s'|_{R'})^{-1}(W)$. This is open and closed in R' and does not contain any points of R' lying over p'. Since $R' \to S'$ is closed, after replacing S' by $S' \setminus (R' \to S')(E)$ we reach a situation where E is empty. In other words s' maps R' into U'. Note that this property is preserved under further shrinking S'. Similarly, we can arrange it so that t' maps R' into U'. At this point (e) holds. In the same

[2]Explanation in groupoid language: The original set $\{r_1, \ldots, r_n\}$ was the set of arrows with source u. The set $\{u_1, \ldots, u_m\}$ was the set of objects isomorphic to u. And $\{r_1, \ldots, r_N\}$ is the set of all arrows between all the objects equivalent to u.

manner, consider the set $E = (c'|_{R' \times_{s',U',t'} R'})^{-1}(W')$. It is open and closed in the scheme $R' \times_{s',U',t'} R'$ which is finite over S', and does not contain any points lying over p'. Hence after replacing S' by $S' \setminus (R' \times_{s',U',t'} R' \to S')(E)$ we reach a situation where E is empty. In other words we obtain the inclusion in (f). We may repeat the argument also with the identity $e' : S' \times_S U \to S' \times_S R$ and the inverse $i' : S' \times_S R \to S' \times_S R$ so that we may assume (after shrinking S' some more) that $(e'|_{U'})^{-1}(W') = \emptyset$ and $(i'|_{R'})^{-1}(W') = \emptyset$.

At this point we see that we may consider the structure

$$(U', R', s'|_{R'}, t'|_{R'}, c'|_{R' \times_{t',U',s'} R'}, e'|_{U'}, i'|_{R'}).$$

The axioms of a groupoid scheme over S' hold because they hold for the groupoid scheme $(S' \times_S U, S' \times_S R, s', t', c', e', i')$. □

Lemma 36.12.2. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $p \in S$ be a point, and let $u \in U$ be a point lying over p. Assume assumptions (1) -- (6) of Lemma 36.12.1 hold as well as*

(7) *$j : R \to U \times_S U$ is universally closed*[3].

Then we can choose $(S', p') \to (S, p)$ and decompositions $S' \times_S U = U' \amalg W$ and $S' \times_S R = R' \amalg W'$ and $u' \in U'$ such that (a) -- (g) of Lemma 36.12.1 hold as well as

(h) *R' is the restriction of $S' \times_S R$ to U'.*

Proof. We apply Lemma 36.12.1 for the groupoid (U, R, s, t, c) over the scheme S with points p and u. Hence we get an étale neighbourhood $(S', p') \to (S, p)$ and disjoint union decompositions

$$S' \times_S U = U' \amalg W, \quad S' \times_S R = R' \amalg W'$$

and $u' \in U'$ satisfying conclusions (a), (b), (c), (d), (e), (f), and (g). We may shrink S' to a smaller neighbourhood of p' without affecting the conclusions (a) -- (g). We will show that for a suitable shrinking conclusion (h) holds as well. Let us denote j' the base change of j to S'. By conclusion (e) it is clear that

$$j'^{-1}(U' \times_{S'} U') = R' \amalg Rest$$

for some open and closed $Rest$ piece. Since $U' \to S'$ is finite by conclusion (d) we see that $U' \times_{S'} U'$ is finite over S'. Since j is universally closed, also j' is universally closed, and hence $j'|_{Rest}$ is universally closed too. By conclusions (b) and (c) we see that the fibre of

$$(U' \times_{S'} U' \to S') \circ j'|_{Rest} : Rest \longrightarrow S'$$

over p' is empty. Hence, since $Rest \to S'$ is closed as a composition of closed morphisms, after replacing S' by $S' \setminus \mathrm{Im}(Rest \to S')$, we may assume that $Rest = \emptyset$. And this is exactly the condition that R' is the restriction of $S' \times_S R$ to the open subscheme $U' \subset S' \times_S U$, see Groupoids, Lemma 35.15.3 and its proof. □

36.13. Other chapters

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Commutative Algebra
(8) Brauer Groups
(9) Sites and Sheaves
(10) Homological Algebra
(11) Derived Categories
(12) More on Algebra

[3]In view of the other conditions this is equivalent to requiring j to be proper.

(13) Smoothing Ring Maps
(14) Simplicial Methods
(15) Sheaves of Modules
(16) Modules on Sites
(17) Injectives
(18) Cohomology of Sheaves
(19) Cohomology on Sites
(20) Hypercoverings
(21) Schemes
(22) Constructions of Schemes
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(24) Morphisms of Schemes
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(30) Topologies on Schemes
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(36) More on Groupoid Schemes
(37) Étale Morphisms of Schemes
(38) Étale Cohomology
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CHAPTER 37

Étale Morphisms of Schemes

37.1. Introduction

In this Chapter, we discuss étale morphisms of schemes. We illustrate some of the more important concepts by working with the Noetherian case. Our principal goal is to collect for the reader with enough commutative algebra results to start reading a treatise on étale cohomology. An auxiliary goal is to provide enough evidence to ensure that the reader stops calling the phrase ``the étale topology of schemes'' an exercise in general nonsense, if (s)he does indulge in such blasphemy.

We will refer to the other chapters of the stacks project for standard results in algebraic geometry (on schemes and commutative algebra). We will provide detailed proofs of the new results that we state here.

37.2. Conventions

In this chapter, frequently schemes will be assumed locally Noetherian and frequently rings will be assumed Noetherian. But in all the statements we will reiterate this when necessary, and make sure we list all the hypotheses! On the other hand, here are some general facts that we will use often and are useful to keep in mind:

(1) A ring homomorphism $A \to B$ of finite type with A Noetherian is of finite presentation. See Algebra, Lemma 7.28.4.
(2) A morphism (locally) of finite type between locally Noetherian schemes is automatically (locally) of finite presentation. See Morphisms, Lemma 24.20.9.
(3) Add more like this here.

37.3. Unramified morphisms

We first define the notion of unramified morphisms for local rings, and then globalise it to get one for arbitrary schemes.

Definition 37.3.1. Let A, B be Noetherian local rings. A local homomorphism $A \to B$ is said to be *unramified homomorphism of local rings* if

(1) $\mathfrak{m}_A B = \mathfrak{m}_B$,
(2) $\kappa(\mathfrak{m}_A)$ is a finite separable extension of $\kappa(\mathfrak{m}_B)$, and
(3) B is essentially of finite type over A (this means that B is the localization of a finite type A-algebra at a prime).

This is the local version of the definition in Algebra, Section 7.138. In that section a ring map $R \to S$ is defined to be unramified if and only if it is of finite type, and $\Omega_{S/R} = 0$. It is shown in Algebra, Lemmas 7.138.5 and 7.138.7 that given a ring map $R \to S$ of finite type, and a prime $\mathfrak{q}$ of S lying over $\mathfrak{p} \subset R$, then we have

$$R \to S \text{ is unramified at } \mathfrak{q} \Leftrightarrow \mathfrak{p}S_{\mathfrak{q}} = \mathfrak{q}S_{\mathfrak{q}} \text{ and } \kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q}) \text{ finite separable}$$

Thus we see that for a local homomorphism of local rings the properties of our definition above are closely related to the question of being unramified. In fact, we have proved the following lemma.

Lemma 37.3.2. *Let $A \to B$ be of finite type with A a Noetherian ring. Let $\mathfrak{q}$ be a prime of B lying over $\mathfrak{p} \subset A$. Then $A \to B$ is unramified at $\mathfrak{q}$ if and only if $A_{\mathfrak{p}} \to B_{\mathfrak{q}}$ is an unramified homomorphism of local rings.*

Proof. See discussion above. □

We will characterize the property of being unramified in terms of completions. For a Noetherian local ring A we denote $A^\wedge$ the completion of A with respect to the maximal ideal. It is also a Noetherian local ring, see Algebra, Lemma 7.90.10.

Lemma 37.3.3. *Let A, B be Noetherian local rings. Let $A \to B$ be a local homomorphism.*

(1) *if $A \to B$ is an unramified homomorphism of local rings, then $B^\wedge$ is a finite $A^\wedge$ module,*
(2) *if $A \to B$ is an unramified homomorphism of local rings and $\kappa(\mathfrak{m}_A) = \kappa(\mathfrak{m}_B)$, then $A^\wedge \to B^\wedge$ is surjective,*
(3) *if $A \to B$ is an unramified homomorphism of local rings and $\kappa(\mathfrak{m}_A)$ is separably closed, then $A^\wedge \to B^\wedge$ is surjective,*
(4) *if A and B are complete discrete valuation rings, then $A \to B$ is an unramified homomorphism of local rings if and only the uniformizer for A maps to a uniformizer for B, and the residue field extension is finite separable (and B is essentially of finite type over A).*

Proof. Part (1) is a special case of Algebra, Lemma 7.90.16. For part (2), note that the $\kappa(\mathfrak{m}_A)$-vector space $B^\wedge/\mathfrak{m}_{A^\wedge}B^\wedge$ is generated by 1. Hence by Nakayama's lemma (Algebra, Lemma 7.14.5) the map $A^\wedge \to B^\wedge$ is surjective. Part (3) is a special case of part (2). Part (4) is immediate from the definitions. □

Lemma 37.3.4. *Let A, B be Noetherian local rings. Let $A \to B$ be a local homomorphism such that B is essentially of finite type over A. The following are equivalent*

(1) *$A \to B$ is an unramified homomorphism of local rings*
(2) *$A^\wedge \to B^\wedge$ is an unramified homomorphism of local rings, and*
(3) *$A^\wedge \to B^\wedge$ is unramified.*

Proof. The equivalence of (1) and (2) follows from the fact that $\mathfrak{m}_A A^\wedge$ is the maximal ideal of $A^\wedge$ (and similarly for B) and faithful flatness of $B \to B^\wedge$. For example if $A^\wedge \to B^\wedge$ is unramified, then $\mathfrak{m}_A B^\wedge = (\mathfrak{m}_A B)B^\wedge = \mathfrak{m}_B B^\wedge$ and hence $\mathfrak{m}_A B = \mathfrak{m}_B$.

Assume the equivalent conditions (1) and (2). By Lemma 37.3.3 we see that $A^\wedge \to B^\wedge$ is finite. Hence $A^\wedge \to B^\wedge$ is of finite presentation, and by Algebra, Lemma 7.138.7 we conclude that $A^\wedge \to B^\wedge$ is unramified at $\mathfrak{m}_{B^\wedge}$. Since $B^\wedge$ is local we conclude that $A^\wedge \to B^\wedge$ is unramified.

Assume (3). By Algebra, Lemma 7.138.5 we conclude that $A^\wedge \to B^\wedge$ is an unramified homomorphism of local rings, i.e., (2) holds. □

Definition 37.3.5. (See Morphisms, Definition 24.34.1 for the definition in the general case.) Let Y be a locally Noetherian scheme. Let $f : X \to Y$ be locally of finite type. Let $x \in X$.

(1) We say f is *unramified at x* if $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is an unramified homomorphism of local rings.

(2) The morphism $f : X \to Y$ is said to be *unramified* if it is unramified at all points of X.

Let us prove that this definition agrees with the definition in the chapter on morphisms of schemes. This in particular garantees that the set of points where a morphism is unramified is open.

Lemma 37.3.6. *Let Y be a locally Noetherian scheme. Let $f : X \to Y$ be locally of finite type. Let $x \in X$. The morphism f is unramified at x in the sense of Definition 37.3.5 if and only if it is unramified in the sense of Morphisms, Definition 24.34.1.*

Proof. This follows from Lemma 37.3.2 and the definitions. □

Here are some results on unramified morphisms. The formulations as given in this list apply only to morphisms locally of finite type between locally Noetherian schemes. In each case we give a reference to the general result as proved earlier in the project, but in some cases one can prove the result more easily in the Noetherian case. Here is the list:

(1) Unramifiedness is local on the source and the target in the Zariski topology.
(2) Unramified morphisms are stable under base change and composition. See Morphisms, Lemmas 24.34.5 and 24.34.4.
(3) Unramified morphisms of schemes are locally quasi-finite and quasi-compact unramified morphisms are quasi-finite. See Morphisms, Lemma 24.34.10
(4) Unramified morphisms have relative dimension 0. See Morphisms, Definition 24.28.1 and Morphisms, Lemma 24.28.5.
(5) A morphism is unramified if and only if all its fibres are unramified. That is, unramifiedness can be checked on the scheme theoretic fibres. See Morphisms, Lemma 24.34.12.
(6) Let X and Y be unramified over a base scheme S. Any S-morphism from X to Y is unramified. See Morphisms, Lemma 24.34.16.

37.4. Three other characterizations of unramified morphisms

The following theorem gives three equivalent notions of being unramified at a point. See Morphisms, Lemma 24.34.14 for (part of) the statement for general schemes.

Theorem 37.4.1. *Let Y be a locally Noetherian scheme. Let $f : X \to Y$ be a morphism of schemes which is locally of finite type. Let x be a point of X. The following are equivalent*

(1) *f is unramified at x,*
(2) *the stalk $\Omega_{X/Y,x}$ of the module of relative differentials at x is trivial,*
(3) *there exist open neighbourhoods U of x and V of $f(x)$, and a commutative diagram*

$$\begin{tikzcd} U \arrow[rr, "i"] \arrow[dr] & & \mathbf{A}^n_V \arrow[dl] \\ & V & \end{tikzcd}$$

where i is a closed immersion defined by a quasi-coherent sheaf of ideals $\mathcal{I}$ such that the differentials dg for $g \in \mathcal{I}_{i(x)}$ generate $\Omega_{\mathbf{A}^n_V/V,i(x)}$, and
(4) *the diagonal $\Delta_{X/Y} : X \to X \times_Y X$ is a local isomorphism at x.*

Proof. The equivalence of (1) and (2) is proved in Morphisms, Lemma 24.34.14.

If f is unramified at x, then f is unramified in an open neighbourhood of x; this does not follow immediately from Definition 37.3.5 of this chapter but it does follow from Morphisms, Definition 24.34.1 which we proved to be equivalent in Lemma 37.3.6. Choose affine opens $V \subset Y$, $U \subset X$ with $f(U) \subset V$ and $x \in U$, such that f is unramified on U, i.e., $f|_U : U \to V$ is unramified. By Morphisms, Lemma 24.34.13 the morphism $U \to U \times_V U$ is an open immersion. This proves that (1) implies (4).

If $\Delta_{X/Y}$ is a local isomorphism at x, then $\Omega_{X/Y,x} = 0$ by construction of the sheaf of relative differentials (see Morphisms, Definition 24.32.4). Hence we see that (4) implies (2). At this point we know that (1), (2) and (4) are all equivalent.

Assume (3). The assumption on the diagram combined with Morphisms, Lemma 24.32.17 show that $\Omega_{U/V,x} = 0$. Since $\Omega_{U/V,x} = \Omega_{X/Y,x}$ we conclude (2) holds.

Finally, assume that (2) holds. To prove (3) we may localize on X and Y and assume that X and Y are affine. Say $X = Spec(B)$ and $Y = Spec(A)$. The point $x \in X$ corresponds to a prime $\mathfrak{q} \subset B$. Our assumption is that $\Omega_{B/A,\mathfrak{q}} = 0$ (see Morphisms, Lemma 24.32.7 for the relationship between differentials on schemes and modules of differentials in commutative algebra). Since Y is locally Noetherian and f locally of finite type we see that A is Noetherian and $B \cong A[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$, see Properties, Lemma 23.5.2 and Morphisms, Lemma 24.14.2. In particular, $\Omega_{B/A}$ is a finite B-module. Hence we can find a single $g \in B$, $g \notin \mathfrak{q}$ such that the principal localization $(\Omega_{B/A})_g$ is zero. Hence after replacing B by B_g we see that $\Omega_{B/A} = 0$ (formation of modules of differentials commutes with localization, see Algebra, Lemma 7.122.8). This means that $d(f_j)$ generate the kernel of the canonical map $\Omega_{A[x_1,\ldots,x_n]/A} \otimes_A B \to \Omega_{B/A}$. Thus the surjection $A[x_1, \ldots, x_n] \to B$ of A-algebras gives the commutative diagram of (3), and the theorem is proved. $\square$

How can we use this theorem? Well, here are a few remarks:

(1) Suppose that $f : X \to Y$ and $g : Y \to Z$ are two morphisms locally of finite type between locally Noetherian schemes. There is a canonical short exact sequence
$$f^*(\Omega_{Y/Z}) \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$$
see Morphisms, Lemma 24.32.11. The theorem therefore implies that if $g \circ f$ is unramified, then so is f. This is Morphisms, Lemma 24.34.16.

(2) The definition of $\Omega_{X/Y}$ as the pullback $\Delta^*(\mathcal{J}/\mathcal{J}^2)$ of the conormal sheaf of the diagonal morphism (see Morphisms, Definition 24.32.4) allows us to conclude that if $X \to Y$ is a monomorphism of locally Noetherian schemes and locally of finite type, then $X \to Y$ is unramified. In particular, open and closed immersions of locally Noetherian schemes are unramified. See Morphisms, Lemmas 24.34.7 and 24.34.8.

(3) The theorem also implies that the set of points where a morphism $f : X \to Y$ (locally of finite type of locally Noetherian schemes) is not unramified is the support of the coherent sheaf $\Omega_{X/Y}$. This allows one to give a scheme theoretic definition to the ``ramification locus''.

37.5. The functorial characterization of unramified morphisms

In basic algebraic geometry we learn that some classes of morphisms can be characterised functorially, and that such descriptions are quite useful. Unramified morphisms too have such a characterisation.

Theorem 37.5.1. *Let $f : X \to S$ be a morphism of schemes. Assume S is a locally Noetherian scheme, and f is locally of finite type. Then the following are equivalent:*

(1) *f is unramified,*
(2) *the morphism f is formally unramified: for any affine S-scheme T and subscheme T_0 of T defined by a square-zero ideal, the natural map*

$$Hom_S(T, X) \longrightarrow Hom_S(T_0, X)$$

is injective.

Proof. See More on Morphisms, Lemma 33.4.8 for a more general statement and proof. What follows is a sketch of the proof in the current case.

Firstly, one checks both properties are local on the source and the target. This we may assume that S and X are affine. Say $X = Spec(B)$ and $S = Spec(R)$. Say $T = Spec(C)$. Let J be the square-zero ideal of C with $T_0 = Spec(C/J)$. Assume that we are given the diagram

$$\begin{tikzcd} & B \arrow[d, "\phi"] \arrow[dr, "\bar{\phi}"] & \\ R \arrow[ur] \arrow[r] & C \arrow[r] & C/J \end{tikzcd}$$

Secondly, one checks that the association $\phi' \mapsto \phi' - \phi$ gives a bijection between the set of liftings of $\bar{\phi}$ and the module $\mathrm{Der}_R(B, J)$. Thus, we obtain the implication (1) $\Rightarrow$ (2) via the description of unramified morphisms having trivial module of differentials, see Theorem 37.4.1.

To obtain the reverse implication, consider the surjection $q : C = (B \otimes_R B)/I^2 \to B = C/J$ defined by the square zero ideal $J = I/I^2$ where I is the kernel of the multiplication map $B \otimes_R B \to B$. We already have a lifting $B \to C$ defined by, say, $b \mapsto b \otimes 1$. Thus, by the same reasoning as above, we obtain a bijective correspondence between liftings of $\mathrm{id} : B \to C/J$ and $\mathrm{Der}_R(B, J)$. The hypothesis therefore implies that the latter module is trivial. But we know that $J \cong \Omega_{B/R}$. Thus, B/R is unramified. □

37.6. Topological properties of unramified morphisms

The first topological result that will be of utility to us is one which says that unramified and separated morphisms have ``nice'' sections. The material in this section does not require any Noetherian hypotheses.

Proposition 37.6.1. *(Sections of unramified morphisms.)*

(1) *Any section of an unramified morphism is an open immersion.*
(2) *Any section of a separated morphism is a closed immersion.*
(3) *Any section of an unramified separated morphism is open and closed.*

Proof. Fix a base scheme S. If $f : X' \to X$ is any S-morphism, then the graph $\Gamma_f : X' \to X' \times_S X$ is obtained as the base change of the diagonal $\Delta_{X/S} : X \to X \times_S X$ via the projection $X' \times_S X \to X \times_S X$. If $g : X \to S$ is separated (resp. unramified) then the diagonal is a closed immersion (resp. open immersion) by Schemes, Definition 21.21.3 (resp. Morphisms, Lemma 24.34.13). Hence so is the graph as a base change (by Schemes, Lemma 21.18.2). In the special case $X' = S$, we obtain (1), resp. (2). Part (3) follows on combining (1) and (2). □

We can now explicitly describe the sections of unramified morphisms.

Theorem 37.6.2. *Let Y be a connected scheme. Let $f : X \to Y$ be unramified and separated. Every section of f is an isomorphism onto a connected component. There exists a bijective correspondence*

$$\text{sections of } f \leftrightarrow \left\{ \begin{matrix} \text{connected components } X' \text{ of } X \text{ such that} \\ \text{the induced map } X' \to Y \text{ is an isomorphism} \end{matrix} \right\}$$

In particular, given $x \in X$ there is at most one section passing through x.

Proof. Direct from Proposition 37.6.1 part (3). □

The preceding theorem gives us some idea of the ``rigidity'' of unramified morphisms. Further indication is provided by the following proposition which, besides being intrinsically interesting, is also useful in the theory of the algebraic fundamental group (see [**Gro71**, Exposé V]). See also the more general Morphisms, Lemma 24.34.17.

Proposition 37.6.3. *Let S is be a scheme. Let $\pi : X \to S$ be unramified and separated. Let Y be an S-scheme and $y \in Y$ a point. Let $f, g : Y \to X$ be two S-morphisms. Assume*

1. *Y is connected*
2. *$x = f(y) = g(y)$, and*
3. *the induced maps $f^\sharp, g^\sharp : \kappa(x) \to \kappa(y)$ on residue fields are equal.*

Then $f = g$.

Proof. The maps $f, g : Y \to X$ define maps $f', g' : Y \to X_Y = Y \times_S X$ which are sections of the structure map $X_Y \to Y$. Note that $f = g$ if and only if $f' = g'$. The structure map $X_Y \to Y$ is the base change of π and hence unramified and separated also (see Morphisms, Lemmas 24.34.5 and Schemes, Lemma 21.21.13). Thus according to Theorem 37.6.2 it suffices to prove that f' and g' pass through the same point of X_Y. And this is exactly what the hypotheses (2) and (3) garantee, namely $f'(y) = g'(y) \in X_Y$. □

37.7. Universally injective, unramified morphisms

Recall that a morphism of schemes $f : X \to Y$ is universally injective if any base change of f is injective (on underlying topological spaces), see Morphisms, Definition 24.10.1. Universally injective and unramified morphisms can be characterized as follows.

Lemma 37.7.1. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

1. *f is unramified and a monomorphism,*
2. *f is unramified and universally injective,*
3. *f is locally of finite type and a monomorphism,*
4. *f is universally injective, locally of finite type, and formally unramified,*
5. *f is locally of finite type and X_y is either empty or $X_y \to y$ is an isomorphism for all $y \in Y$.*

Proof. We have seen in More on Morphisms, Lemma 33.4.8 that being formally unramified and locally of finite type is the same thing as being unramified. Hence (4) is equivalent to (2). A monomorphism is certainly universally injective and formally unramified hence (3) implies (4). It is clear that (1) implies (3). Finally, if (2) holds, then $\Delta : X \to X \times_S X$ is both an open immersion (Morphisms, Lemma 24.34.13) and surjective (Morphisms, Lemma 24.10.2) hence an isomorphism, i.e., f is a monomorphism. In this way we see that (2) implies (1).

Condition (3) implies (5) because monomorphisms are preserved under base change (Schemes, Lemma 21.23.5) and because of the description of monomorphisms towards the spectra of fields in Schemes, Lemma 21.23.10. Condition (5) implies (4) by Morphisms, Lemmas 24.10.2 and 24.34.12. □

This leads to the following useful characterization of closed immersions.

Lemma 37.7.2. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

1. *f is a closed immersion,*
2. *f is a proper monomorphism,*
3. *f is proper, unramified, and universally injective,*
4. *f is universally closed, unramified, and a monomorphism,*
5. *f is universally closed, unramified, and universally injective,*
6. *f is universally closed, locally of finite type, and a monomorphism,*
7. *f is universally closed, universally injective, locally of finite type, and formally unramified.*

Proof. The equivalence of (4) -- (7) follows immediately from Lemma 37.7.1.

Let $f : X \to S$ satisfy (6). Then f is separated, see Schemes, Lemma 21.23.3 and has finite fibres. Hence More on Morphisms, Lemma 33.29.5 shows f is finite. Then Morphisms, Lemma 24.42.13 implies f is a closed immersion, i.e., (1) holds.

Note that (1) $\Rightarrow$ (2) because a closed immersion is proper and a monomorphism (Morphisms, Lemma 24.40.6 and Schemes, Lemma 21.23.7). By Lemma 37.7.1 we see that (2) implies (3). It is clear that (3) implies (5). □

Here is another result of a similar flavor.

Lemma 37.7.3. *Let $\pi : X \to S$ be a morphism of schemes. Let $s \in S$. Assume that*

1. *π is finite,*
2. *π is unramified,*
3. *$\pi^{-1}(\{s\}) = \{x\}$, and*
4. *$\kappa(s) \subset \kappa(x)$ is purely inseparable*[1].

Then there exists an open neighbourhood U of s such that $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is a closed immersion.

Proof. The question is local on S. Hence we may assume that $S = Spec(A)$. By definition of a finite morphism this implies $X = Spec(B)$. Note that the ring map $\varphi : A \to B$ defining π is a finite unramified ring map. Let $\mathfrak{p} \subset A$ be the prime corresponding to s. Let $\mathfrak{q} \subset B$ be the prime corresponding to x. By Conditions (2), (3) and (4) imply that $B_{\mathfrak{q}}/\mathfrak{p}B_{\mathfrak{q}} = \kappa(\mathfrak{p})$. Algebra, Lemma 7.36.11 we have $B_{\mathfrak{q}} = B_{\mathfrak{p}}$ (note that a finite ring map satisfies going up, see Algebra, Section 7.36.) Hence we see that $B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}} = \kappa(\mathfrak{p})$. As B is a finite A-module we see from Nakayama's lemma (see Algebra, Lemma 7.14.5) that $B_{\mathfrak{p}} = \varphi(A_{\mathfrak{p}})$. Hence (using the finiteness of B as an A-module again) there exists a $f \in A$, $f \notin \mathfrak{p}$ such that $B_f = \varphi(A_f)$ as desired. □

The topological results presented above will be used to give a functorial characterisation of étale morphisms similar to Theorem 37.5.1.

[1] In view of condition (2) this is equivalent to $\kappa(s) = \kappa(x)$.

37.8. Examples of unramified morphisms

Here are a few examples.

Example 37.8.1. Let k be a field. Unramified quasi-compact morphisms $X \to Spec(k)$ are affine. This is true because X has dimension 0 and is Noetherian, hence is a finite discrete set, and each point gives an affine open, so X is a finite disjoint union of affines hence affine. Noether normalisation forces X to be the spectrum of a finite k-algebra A. This algebra is a product of finite separable field extensions of k. Thus, an unramified quasi-compact morphism to $Spec(k)$ corresponds to a finite number of finite separable field extensions of k. In particular, an unramified morphism with a connected source and a one point target is forced to be a finite separable field extension. As we will see later, $X \to Spec(k)$ is étale if and only if it is unramified. Thus, in this case at least, we obtain a very easy description of the étale topology of a scheme. Of course, the cohomology of this topology is another story.

Example 37.8.2. Property (3) in Theorem 37.4.1 gives us a canonical source of examples for unramified morphisms. Fix a ring R and an integer n. Let $I = (g_1, \cdots, g_m)$ be an ideal in $R[x_1, \cdots, x_n]$. Let $\mathfrak q \subset R[x_1, \ldots, x_n]$ be a prime. Assume $I \subset \mathfrak q$ and that the matrix

$$\left(\frac{\partial g_i}{\partial x_j}\right) \bmod \mathfrak q \quad \in \quad \mathrm{Mat}(n \times m, \kappa(\mathfrak q))$$

has rank n. Then the morphism $f : Z = Spec(R[x_1, \cdots, x_n]/I) \to Spec(R)$ is unramified at the point $x \in Z \subset \mathbf{A}^n_R$ corresponding to $\mathfrak q$. Clearly we must have $m \geq n$. In the extreme case $m = n$, i.e., the differential of the map $\mathbf{A}^n_R \to \mathbf{A}^n_R$ defined by the g_i's is an isomorphism of the tangent spaces, then f is also flat x and, hence, is an étale map (see Algebra, Definition 7.126.6, Lemma 7.126.7 and Example 7.126.8).

Example 37.8.3. Fix an extension of number fields L/K with rings of integers $\mathcal{O}_L$ and $\mathcal{O}_K$. The injection $K \to L$ defines a morphism $f : Spec(\mathcal{O}_L) \to Spec(\mathcal{O}_K)$. As discussed above, the points where f is unramified in our sense correspond to the set of points where f is unramified in the conventional sense. In the conventional sense, the locus of ramification in $Spec(\mathcal{O}_L)$ can be defined by vanishing set of the different; this is an ideal in $\mathcal{O}_L$. In fact, the different is nothing but the annihilator of the module $\Omega_{\mathcal{O}_L/\mathcal{O}_K}$. Similarly, the discriminant is an ideal in $\mathcal{O}_K$, namely it is the norm of the different. The vanishing set of the discriminant is precisely the set of points of K which ramify in L. Thus, denoting by X the complement of the closed subset defined by the different in $Spec(\mathcal{O}_L)$, we obtain a morphism $X \to Spec(\mathcal{O}_L)$ which is unramified. Furthermore, this morphism is also flat, as any local homomorphism of discrete valuation rings is flat, and hence this morphism is actually étale. If L/K is Galois, then denoting by Y the complement of the closed subset defined by the discriminant in $Spec(\mathcal{O}_K)$, we see that we get even a finite étale morphism $X \to Y$. Thus, this is an example of a finite étale covering.

37.9. Flat morphisms

This section simply exists to summarise the properties of flatness that will be useful to us. Thus, we will be content with stating the theorems precisely and giving references for the proofs.

After briefly recalling the necessary facts about flat modules over Noetherian rings, we state a theorem of Grothendieck which gives sufficient conditions for ``hyperplane sections'' of certain modules to be flat.

Definition 37.9.1. Flatness of modules and rings.

(1) A module N over a ring A is said to be *flat* if the functor $M \mapsto M \otimes_A N$ is exact.
(2) If this functor is also faithful, we say that N is *faithfully flat* over A.
(3) A morphism of rings $f : A \to B$ is said to be *flat (resp. faithfully flat)* if the functor $M \mapsto M \otimes_A B$ is exact (resp. faithful and exact).

Here is a list of facts with references to the algebra chapter.

(1) Free and projective modules are flat. This is clear for free modules and follows for projective modules as they are direct summands of free modules and $\otimes$ commutes with direct sums.
(2) Flatness is a local property, that is, M is flat over A if and only if $M_{\mathfrak{p}}$ is flat over $A_{\mathfrak{p}}$ for all $\mathfrak{p} \in Spec(A)$. See Algebra, Lemma 7.35.19.
(3) If M is a flat A-module and $A \to B$ is a ring map, then $M \otimes_A B$ is a flat B-module. See Algebra, Lemma 7.35.6.
(4) Finite flat modules over local rings are free. See Algebra, Lemma 7.72.4.
(5) If $f : A \to B$ is a morphism of arbitrary rings, f is flat if and only if the induced maps $A_{f^{-1}(\mathfrak{q})} \to B_{\mathfrak{q}}$ are flat for all $\mathfrak{q} \in Spec(B)$. See Algebra, Lemma 7.35.19
(6) If $f : A \to B$ is a local homomorphism of local rings, f is flat if and only if it is faithfully flat. See Algebra, Lemma 7.35.16.
(7) A map $A \to B$ of rings is faithfully flat if and only if it is flat and the induced map on spectra is surjective. See Algebra, Lemma 7.35.15.
(8) If A is a noetherian local ring, the completion $A^\wedge$ is faithfully flat over A. See Algebra, Lemma 7.90.4.
(9) Let A be a Noetherian local ring and M an A-module. Then M is flat over A if and only if $M \otimes_A A^\wedge$ is flat over $A^\wedge$. (Combine the previous statement with Algebra, Lemma 7.35.7.)

Before we move on to the geometric category, we present Grothendieck's theorem, which provides a convenient recipe for producing flat modules.

Theorem 37.9.2. *Let A, B be Noetherian local rings. Let $f : A \to B$ be a local homomorphism. If M is a finite B-module that is flat as an A-module, and $t \in \mathfrak{m}_B$ is an element such that multiplication by t is injective on $M/\mathfrak{m}_A M$, then M/tM is also A-flat.*

Proof. See Algebra, Lemma 7.91.1. See also [**Mat70**, Section 20]. ☐

Definition 37.9.3. (See Morphisms, Definition 24.24.1). Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module.

(1) Let $x \in X$. We say $\mathcal{F}$ is *flat over Y at $x \in X$* if $\mathcal{F}_x$ is a flat $\mathcal{O}_{Y,f(x)}$-module. This uses the map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ to think of $\mathcal{F}_x$ as a $\mathcal{O}_{Y,f(x)}$-module.
(2) Let $x \in X$. We say f is *flat at $x \in X$* if $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat.
(3) We say f is *flat* if it is flat at all points of X.
(4) A morphism $f : X \to Y$ that is flat and surjective is sometimes said to be *faithfully flat*.

Once again, here is a list of results:

(1) The property (of a morphism) of being flat is, by fiat, local in the Zariski topology on the source and the target.
(2) Open immersions are flat. (This is clear because it induces isomorphisms on local rings.)

(3) Flat morphisms are stable under base change and composition. Morphisms, Lemmas 24.24.7 and 24.24.5.
(4) If $f : X \to Y$ is flat, then the pullback functor $QCoh(\mathcal{O}_Y) \to QCoh(\mathcal{O}_X)$ is exact. This is immediate by looking at stalks.
(5) Let $f : X \to Y$ be a morphism of schemes, and assume Y is quasi-compact and quasi-separated. In this case if the functor f^* is exact then f is flat. (Proof omitted. Hint: Use Properties, Lemma 23.20.1 to see that Y has ``enough'' ideal sheaves and use the characterization of flatness in Algebra, Lemma 7.35.4.)

37.10. Topological properties of flat morphisms

We ``recall'' below some openness properties that flat morphisms enjoy.

Theorem 37.10.1. *Let Y be a locally Noetherian scheme. Let $f : X \to Y$ be a morphism which is locally of finite type. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The set of points in X where $\mathcal{F}$ is flat over S is an open set. In particular the set of points where f is flat is open in X.*

Proof. See More on Morphisms, Theorem 33.11.1. □

Theorem 37.10.2. *Let Y be a locally Noetherian scheme. Let $f : X \to Y$ be a morphism which is flat and locally of finite type. Then f is (universally) open.*

Proof. See Morphisms, Lemma 24.24.9. □

Theorem 37.10.3. *A faithfully flat quasi-compact morphism is a quotient map for the Zariski topology.*

Proof. See Morphisms, Lemma 24.24.10. □

An important reason to study flat morphisms is that they provide the adequate framework for capturing the notion of a family of schemes parametrised by the points of another scheme. Naively one may think that any morphism $f : X \to S$ should be thought of as a family parametrized by the points of S. However, without a flatness restriction on f, really bizarre things can happen in this so-called family. For instance, we aren't guaranteed that relative dimension (dimension of the fibres) is constant in a family. Other numerical invariants, such as the Hilbert polynomial, too may change from fibre to fibre. Flatness prevents such things from happening and, therefore, provides some ``continuity'' to the fibres.

37.11. Étale morphisms

In this section, we will define étale morphisms and prove a number of important properties about them. The most important one, no doubt, is the functorial characterisation presented in Theorem 37.16.1. Following this, we will also discuss a few properties of rings which are insensitive to an étale extension (properties which hold for a ring if and only if they hold for all its étale extensions) to motivate the basic tenet of étale cohomology -- étale morphisms are the algebraic analogue of local isomorphisms.

As the title suggests, we will define the class of étale morphisms -- the class of morphisms (whose surjective families) we shall deem to be coverings in the category of schemes over a base scheme S in order to define the étale site $S_{étale}$. Intuitively, an étale morphism is supposed to capture the idea of a covering space and, therefore, should be close to a local isomorphism. If we're working with varieties over algebraically closed fields, this last statement can be made into a definition provided we replace ``local isomorphism'' with ``formal local isomorphism'' (isomorphism after completion). One can then give a definition over any base field by asking that the base change to the algebraic closure be étale (in the aforementioned sense). But, rather than proceeding via such aesthetically displeasing constructions, we will adopt a cleaner, albeit slightly more abstract, algebraic approach.

Definition 37.11.1. Let A, B be Noetherian local rings. A local homomorphism $f : A \to B$ is said to be a *étale homomorphism of local rings* if it is flat and unramified homomorphism of local rings (please see Definition 37.3.1).

This is the local version of the definition of an étale ring map in Algebra, Section 7.132. The exact definition given in that section is that it is a smooth ring map of relative dimension 0. It is shown (in Algebra, Lemma 7.132.2 after some work) that an étale R-algebra S always has a presentation

$$S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$$

such that

$$g = \det \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_2/\partial x_1 & \ldots & \partial f_n/\partial x_1 \\ \partial f_1/\partial x_2 & \partial f_2/\partial x_2 & \ldots & \partial f_n/\partial x_2 \\ \ldots & \ldots & \ldots & \ldots \\ \partial f_1/\partial x_n & \partial f_2/\partial x_n & \ldots & \partial f_n/\partial x_n \end{pmatrix}$$

maps to an invertible element in S. The following two lemmas link the two notions.

Lemma 37.11.2. *Let $A \to B$ be of finite type with A a Noetherian ring. Let $\mathfrak{q}$ be a prime of B lying over $\mathfrak{p} \subset A$. Then $A \to B$ is étale at $\mathfrak{q}$ if and only if $A_{\mathfrak{p}} \to B_{\mathfrak{q}}$ is an étale homomorphism of local rings.*

Proof. See Algebra, Lemmas 7.132.3 (flatness of étale maps), 7.132.5 (étale maps are unramified) and 7.132.7 (flat and unramified maps are étale). □

Lemma 37.11.3. *Let A, B be Noetherian local rings. Let $A \to B$ be a local homomorphism such that B is essentially of finite type over A. The following are equivalent*

(1) *$A \to B$ is an étale homomorphism of local rings*
(2) *$A^\wedge \to B^\wedge$ is an étale homomorphism of local rings, and*
(3) *$A^\wedge \to B^\wedge$ is étale.*

Moreover, in this case $B^\wedge \cong (A^\wedge)^{\oplus n}$ as $A^\wedge$-modules for some $n \geq 1$.

Proof. To see the equivalences of (1), (2) and (3), as we have the corresponding results for unramified ring maps (Lemma 37.3.4) it suffices to prove that $A \to B$ is flat if and only if $A^\wedge \to B^\wedge$ is flat. This is clear from our lists of properties of flat maps since the ring maps $A \to A^\wedge$ and $B \to B^\wedge$ are faithfully flat. For the final statement, by Lemma 37.3.3 we see that $B^\wedge$ is a finite flat $A^\wedge$ module. Hence it is finite free by our list of properties on flat modules in Section 37.9. □

The integer n which occurs in the lemma above is nothing other than the degree $[\kappa(\mathfrak{m}_B) : \kappa(\mathfrak{m}_A)]$ of the residue field extension. In particular, if $\kappa(\mathfrak{m}_A)$ is separably closed, we see that $A^\wedge \to B^\wedge$ is an isomorphism, which vindicates our earlier claims.

Definition 37.11.4. (See Morphisms, Definition 24.35.1.) Let Y be a locally Noetherian scheme. Let $f : X \to Y$ be a morphism of schemes which is locally of finite type.

(1) Let $x \in X$. We say f is *étale at $x \in X$* if $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is an étale homomorphism of local rings.
(2) The morphism is said to be *étale* if it is étale at all its points.

Let us prove that this definition agrees with the definition in the chapter on morphisms of schemes. This in particular guarantees that the set of points where a morphism is étale is open.

Lemma 37.11.5. *Let Y be a locally Noetherian scheme. Let $f : X \to Y$ be locally of finite type. Let $x \in X$. The morphism f is étale at x in the sense of Definition 37.11.4 if and only if it is unramified at x in the sense of Morphisms, Definition 24.35.1.*

Proof. This follows from Lemma 37.11.2 and the definitions. □

Here are some results on étale morphisms. The formulations as given in this list apply only to morphisms locally of finite type between locally Noetherian schemes. In each case we give a reference to the general result as proved earlier in the project, but in some cases one can prove the result more easily in the Noetherian case. Here is the list:

1. An étale morphism is unramified. (Clear from our definitions.)
2. Étaleness is local on the source and the target in the Zariski topology.
3. Étale morphisms are stable under base change and composition. See Morphisms, Lemmas 24.35.4 and 24.35.3.
4. Étale morphisms of schemes are locally quasi-finite and quasi-compact étale morphisms are quasi-finite. (This is true because it holds for unramified morphisms as seen earlier.)
5. Étale morphisms have relative dimension 0. See Morphisms, Definition 24.28.1 and Morphisms, Lemma 24.28.5.
6. A morphism is étale if and only if it is flat and all its fibres are étale. See Morphisms, Lemma 24.35.8.
7. Étale morphisms are open. This is true because an étale morphism is flat, and Theorem 37.10.2.
8. Let X and Y be étale over a base scheme S. Any S-morphism from X to Y is étale. See Morphisms, Lemma 24.35.18.

37.12. The structure theorem

We present a theorem which describes the local structure of étale and unramified morphisms. Besides its obvious independent importance, this theorem also allows us to make the transition to another definition of étale morphisms that captures the geometric intuition better than the one we've used so far.

To state it we need the notion of a *standard étale ring map*, see Algebra, Definition 7.132.13. Namely, suppose that R is a ring and $f, g \in R[t]$ are polynomials such that

(a) f is a monic polynomial, and
(b) $f' = \mathrm{d}f/\mathrm{d}t$ is invertible in the localization $R[t]_g$.

Then the map

$$R \longrightarrow R[t]_g/(f) = R[t, 1/g]/(f)$$

is a standard étale algebra, and any standard étale algebra is isomorphic to one of these. It is a pleasant exercise to prove that such a ring map is flat, and unramified and hence étale (as expected of course). A special case of a standard étale ring map is any ring map

$$R \longrightarrow R[t]_{f'}/(f) = R[t, 1/f']/(f)$$

with f a monic polynomial, and any standard étale algebra is (isomorphic to) a principal localization of one of these.

Theorem 37.12.1. *Let $f : A \to B$ be an étale homomorphism of local rings. Then there exist $f, g \in A[t]$ such that*

(1) *$B' = A[t]_g/(f)$ is standard étale -- see (a) and (b) above, and*
(2) *B is isomorphic to a localization of B' at a prime.*

Proof. Write $B = B'_{\mathfrak{q}}$ for some finite type A-algebra B' (we can do this because B is essentially of finite type over A). By Lemma 37.11.2 we see that $A \to B'$ is étale at $\mathfrak{q}$. Hence we may apply Algebra, Proposition 7.132.16 to see that a principal localization of B' is standard étale. □

Here is the version for unramified homomorphisms of local rings.

Theorem 37.12.2. *Let $f : A \to B$ be an unramified morphism of local rings. Then there exist $f, g \in A[t]$ such that*

(1) *$B' = A[t]_g/(f)$ is standard étale -- see (a) and (b) above, and*
(2) *B is isomorphic to a quotient of a localization of B' at a prime.*

Proof. Write $B = B'_{\mathfrak{q}}$ for some finite type A-algebra B' (we can do this because B is essentially of finite type over A). By Lemma 37.3.2 we see that $A \to B'$ is unramified at $\mathfrak{q}$. Hence we may apply Algebra, Proposition 7.138.8 to see that a principal localization of B' is a quotient of a standard étale A-algebra. □

Via standard lifting arguments, one then obtains the following geometric statement which will be of essential use to us.

Theorem 37.12.3. *Let $\varphi : X \to Y$ be a morphism of schemes. Let $x \in X$. If φ is étale at x, then there exist exist affine opens $V \subset Y$ and $U \subset X$ with $x \in U$ and $\varphi(U) \subset V$ such that we have the following diagram*

$$\begin{tikzcd} X \arrow[d] & U \arrow[l] \arrow[r, "j"'] \arrow[d] & Spec(R[t]_{f'}/(f)) \arrow[d] \\ Y & V \arrow[l] \arrow[r, equal] & Spec(R) \end{tikzcd}$$

where j is an open immersion, and $f \in R[t]$ is monic.

Proof. This is equivalent to Morphisms, Lemma 24.35.14 allthough the statements differ slightly. □

37.13. Étale and smooth morphisms

An étale morphism is smooth of relative dimension zero. The projection $\mathbf{A}^n_S \to S$ is a standard example of a smooth morphism of relative dimension n. It turns out that any smooth morphism is étale locally of this form. Here is the precise statement.

Theorem 37.13.1. *Let $\varphi : X \to Y$ be a morphism of schemes. Let $x \in X$. If φ is smooth at x, then there exist exist and integer $n \geq 0$ and affine opens $V \subset Y$ and $U \subset X$ with $x \in U$ and $\varphi(U) \subset V$ such that there exists a commutative diagram*

$$\begin{tikzcd} X \arrow[d] & U \arrow[l] \arrow[r, "\pi"'] \arrow[d] & \mathbf{A}^n_R \arrow[r, equal] \arrow[d] & Spec(R[x_1, \ldots, x_n]) \arrow[dl] \\ Y & V \arrow[l] \arrow[r, equal] & Spec(R) & \end{tikzcd}$$

where π is étale.

Proof. See Morphisms, Lemma 24.35.20. □

37.14. Topological properties of étale morphisms

We present a few of the topological properties of étale and unramified morphisms. First, we give what Grothendieck calls the *fundamental property of étale morphisms*, see [**Gro71**, Exposé I.5].

Theorem 37.14.1. *Let $f : X \to Y$ be a morphism of schemes The following are equivalent:*

(1) *f is an open immersion,*
(2) *f is universally injective and étale, and*
(3) *f is a flat monomorphism, locally of finite presentation.*

Proof. An open immersion is universally injective since any base change of an open immersion is an open immersion. Moreover, it is étale by Morphisms, Lemma 24.35.9. Hence (1) implies (2).

Assume f is universally injective and étale. Since f is étale it is flat and locally of finite presentation, see Morphisms, Lemmas 24.35.12 and 24.35.11. By Lemma 37.7.1 we see that f is a monomorphism. Hence (2) implies (3).

Assume f is flat, locally of finite presentation, and a monomorphism. Then f is open, see Morphisms, Lemma 24.24.9. Thus we may replace Y by $f(X)$ and we may assume f is surjective. Then f is open and bijective hence a homeomorphism. Hence f is quasi-compact. Hence Descent, Lemma 31.21.1 shows that f is an isomorphism and we win. □

Here is another result of a similar flavor.

Lemma 37.14.2. *Let $\pi : X \to S$ be a morphism of schemes. Let $s \in S$. Assume that*

(1) *π is finite,*
(2) *π is étale,*
(3) *$\pi^{-1}(\{s\}) = \{x\}$, and*
(4) *$\kappa(s) \subset \kappa(x)$ is purely inseparable*[2].

Then there exists an open neighbourhood U of s such that $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is an isomorphism.

Proof. By Lemma 37.7.3 there exists an open neighbourhood U of s such that $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is a closed immersion. But a morphism which is étale and a closed immersion is an open immersion (for example by Theorem 37.14.1). Hence after shrinking U we obtain an isomorphism. □

37.15. Topological invariance of the étale topology

Next, we present an extremely crucial theorem which, roughly speaking, says that étaleness is a topological property.

Theorem 37.15.1. *Let X and Y be two schemes over a base scheme S. Let S_0 be a closed subscheme of S whose ideal sheaf has square zero. Denote X_0 (resp. Y_0) the base change $S_0 \times_S X$ (resp. $S_0 \times_S Y$). If X is étale over S, then the map*

$$Mor_S(Y, X) \longrightarrow Mor_{S_0}(Y_0, X_0)$$

is bijective.

[2] In view of condition (2) this is equivalent to $\kappa(s) = \kappa(x)$.

Proof. After base changing via $Y \to S$, we may assume that $Y = S$. In this case the theorem states that any S-morphism $\sigma_0 : S_0 \to X$ actually factors uniquely through a section $S \to X$ of the étale structure morphism $X \to S$.

Existence. Since we have equality of underlying topological spaces $|S_0| = |S|$ and $|X_0| = |X|$, by Theorem 37.6.2, the section σ_0 is uniquely determined by a connected component X' of X such that the base change $X'_0 = S_0 \times_S X'$ maps isomorphically to S_0. In particular, $X' \to S$ is a universal homeomorphism and therefore universally injective. Since $X' \to S$ is étale, it follows from Theorem 37.14.1 that $X' \to S$ is an isomorphism and, therefore, it has an inverse σ which is the required section.

Uniqueness. This follows from Theorem 37.5.1, or directly from Theorem 37.6.2, or, if one carefully observes, from our proof itself. □

From the proof of preceeding theorem, we also obtain one direction of the promised functorial characterisation of étale morphisms. The following theorem will be strengthened in Étale Cohomology, Theorem 38.45.1.

Theorem 37.15.2. *(Une equivalence remarquable de catégories.) Let S be a scheme. Let $S_0 \subset S$ be a closed subscheme defined by an ideal with square zero. The functor*

$$X \longmapsto X_0 = S_0 \times_S X$$

defines an equivalence of categories

$$\{\textit{schemes } X \textit{ étale over } S\} \leftrightarrow \{\textit{schemes } X_0 \textit{ étale over } S_0\}$$

Proof. By Theorem 37.15.1 we see that this functor is fully faithful. It remains to show that the functor is essentially surjective. Let $Y \to S_0$ be an étale morphism of schemes.

Suppose that the result holds if S and Y are affine. In that case, we choose an affine open covering $Y = \bigcup V_j$ such that each V_j maps into an affine open of S. By assumption (affine case) we can find étale morphisms $W_j \to S$ such that $W_{j,0} \cong V_j$ (as schemes over S_0). Let $W_{j,j'} \subset W_j$ be the open subscheme whose underlying topological space corresponds to $V_j \cap V_{j'}$. Because we have isomorphisms

$$W_{j,j',0} \cong V_j \cap V_{j'} \cong W_{j',j,0}$$

as schemes over S_0 we see by fully faithfulness that we obtain isomorphisms $\theta_{j,j'} : W_{j,j'} \to W_{j',j}$ of schemes over S. We omit the verification that these isomorphisms satisfy the cocyle condition of Schemes, Section 21.14. Applying Schemes, Lemma 21.14.2 we obtain a scheme $X \to S$ by glueing the schemes W_j along the identifications $\theta_{j,j'}$. It is clear that $X \to S$ is étale and $X_0 \cong Y$ by construction.

Thus it suffices to show the lemma in case S and Y are affine. Say $S = Spec(R)$ and $S_0 = Spec(R/I)$ with $I^2 = 0$. By Algebra, Lemma 7.132.2 we know that Y is the spectrum of a ring $\overline{A}$ with

$$\overline{A} = (R/I)[x_1, \ldots, x_n]/(\overline{f}_1, \ldots, \overline{f}_n)$$

such that

$$\overline{g} = \det \begin{pmatrix} \partial\overline{f}_1/\partial x_1 & \partial\overline{f}_2/\partial x_1 & \ldots & \partial\overline{f}_n/\partial x_1 \\ \partial\overline{f}_1/\partial x_2 & \partial\overline{f}_2/\partial x_2 & \ldots & \partial\overline{f}_n/\partial x_2 \\ \ldots & \ldots & \ldots & \ldots \\ \partial\overline{f}_1/\partial x_n & \partial\overline{f}_2/\partial x_n & \ldots & \partial\overline{f}_n/\partial x_n \end{pmatrix}$$

maps to an invertible element in A. Choose any lifts $f_i \in R[x_1, \ldots, x_n]$. Since I is nilpotent it follows that the determinant of the matrix of partials of the f_i is invertible in the algebra A defined by

$$A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$$

Hence $R \to A$ is étale and $(R/I) \otimes_R A \cong \overline{A}$. To prove the general case one argues with glueing affine pieces. □

37.16. The functorial characterization

We finally present the promised functorial characterisation. Thus there are four ways to think about étale morphisms of schemes:

(1) as a smooth morphism of relative dimension 0,
(2) as locally finitely presented, flat, and unramified morphisms,
(3) using the structure theorem, and
(4) using the functorial characterisation.

Theorem 37.16.1. *Let $f : X \to S$ be a morphism that is locally of finite presentation. The following are equivalent*

(1) *f is étale,*
(2) *for all affine S-schemes Y, and closed subschemes $Y_0 \subset Y$ defined by square-zero ideals, the natural map*

$$Mor_S(Y, X) \longrightarrow Mor_S(Y_0, X)$$

is bijective.

Proof. This is More on Morphisms, Lemma 33.6.9. □

This characterisation says that solutions to the equations defining X can be lifted uniquely through nilpotent thickenings.

37.17. Étale local structure of unramified morphisms

In the chapter More on Morphisms, Section 33.28 the reader can find some results on the étale local structure of quasi-finite morphisms. In this section we want to combine this with the topological properties of unramified morphisms we have seen in this chapter. The basic overall picture to keep in mind is

$$\begin{tikzcd} V \arrow[r] \arrow[dr] & X_U \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ & U \arrow[r] & S \end{tikzcd}$$

see More on Morphisms, Equation (33.28.0.1). We start with a very general case.

Lemma 37.17.1. *Let $f : X \to S$ be a morphism of schemes. Let $x_1, \ldots, x_n \in X$ be points having the same image s in S. Assume f is unramified at each x_i. Then there exists an étale neighbourhood $(U, u) \to (S, s)$ and opens $V_{i,j} \subset X_U$, $i = 1, \ldots, n$, $j = 1, \ldots, m_i$ such that*

(1) *$V_{i,j} \to U$ is a closed immersion passing through u,*
(2) *u is not in the image of $V_{i,j} \cap V_{i',j'}$ unless $i = i'$ and $j = j'$, and*
(3) *any point of $(X_U)_u$ mapping to x_i is in some $V_{i,j}$.*

Proof. By Morphisms, Definition 24.34.1 there exists an open neighbourhood of each x_i which is locally of finite type over S. Replacing X by an open neighbourhood of $\{x_1, \ldots, x_n\}$ we may assume f is locally of finite type. Apply More on Morphisms, Lemma 33.28.3 to get the étale neighbourhood (U, u) and the opens $V_{i,j}$ finite over U. By Lemma 37.7.3 after possibly shrinking U we get that $V_{i,j} \to U$ is a closed immersion. □

Lemma 37.17.2. *Let $f : X \to S$ be a morphism of schemes. Let $x_1, \ldots, x_n \in X$ be points having the same image s in S. Assume f is separated and f is unramified at each x_i. Then there exists an étale neighbourhood $(U, u) \to (S, s)$ and a disjoint union decomposition*

$$X_U = W \amalg \coprod\nolimits_{i,j} V_{i,j}$$

such that

1. *$V_{i,j} \to U$ is a closed immersion passing through u,*
2. *the fibre W_u contains no point mapping to any x_i.*

In particular, if $f^{-1}(\{s\}) = \{x_1, \ldots, x_n\}$, then the fibre W_u is empty.

Proof. Apply Lemma 37.17.1. We may assume U is affine, so X_U is separated. Then $V_{i,j} \to X_U$ is a closed map, see Morphisms, Lemma 24.40.7. Suppose $(i,j) \neq (i', j')$. Then $V_{i,j} \cap V_{i',j'}$ is closed in $V_{i,j}$ and its image in U does not contain u. Hence after shrinking U we may assume that $V_{i,j} \cap V_{i',j'} = \emptyset$. Moreover, $\bigcup V_{i,j}$ is a closed and open subscheme of X_U and hence has an open and closed complement W. This finishes the proof. □

The following lemma is in some sense much weaker than the preceding one but it may be useful to state it explicitly here. It says that a finite uramified morphism is étale locally on the base a closed immersion.

Lemma 37.17.3. *Let $f : X \to S$ be a finite unramified morphism of schemes. Let $s \in S$. There exists an étale neighbourhood $(U, u) \to (S, s)$ and a disjoint union decomposition*

$$X_U = \coprod\nolimits_j V_j$$

such that each $V_j \to U$ is a closed immersion.

Proof. Since $X \to S$ is finite the fibre over S is a finite set $\{x_1, \ldots, x_n\}$ of points of X. Apply Lemma 37.17.2 to this set (a finite morphism is separated, see Morphisms, Section 24.42). The image of W in U is a closed subset (as $X_U \to U$ is finite, hence proper) which does not contain u. After removing this from U we see that $W = \emptyset$ as desired. □

37.18. Étale local structure of étale morphisms

This is a bit silly, but perhaps helps form intuition about étale morphisms. We simply copy over the results of Section 37.17 and change ``closed immersion'' into ``isomorphism''.

Lemma 37.18.1. *Let $f : X \to S$ be a morphism of schemes. Let $x_1, \ldots, x_n \in X$ be points having the same image s in S. Assume f is étale at each x_i. Then there exists an étale neighbourhood $(U, u) \to (S, s)$ and opens $V_{i,j} \subset X_U$, $i = 1, \ldots, n$, $j = 1, \ldots, m_i$ such that*

1. *$V_{i,j} \to U$ is an isomorphism,*
2. *u is not in the image of $V_{i,j} \cap V_{i',j'}$ unless $i = i'$ and $j = j'$, and*
3. *any point of $(X_U)_u$ mapping to x_i is in some $V_{i,j}$.*

Proof. An étale morphism is unramified, hence we may apply Lemma 37.17.1. Now $V_{i,j} \to U$ is a closed immersion and étale. Hence it is an open immersion, for example by Theorem 37.14.1. Replace U by the intersection of the images of $V_{i,j} \to U$ to get the lemma. □

Lemma 37.18.2. *Let $f : X \to S$ be a morphism of schemes. Let $x_1, \ldots, x_n \in X$ be points having the same image s in S. Assume f is separated and f is étale at each x_i. Then there exists an étale neighbourhood $(U, u) \to (S, s)$ and a disjoint union decomposition*

$$X_U = W \amalg \coprod\nolimits_{i,j} V_{i,j}$$

such that

(1) *$V_{i,j} \to U$ is an isomorphism,*
(2) *the fibre W_u contains no point mapping to any x_i.*

In particular, if $f^{-1}(\{s\}) = \{x_1, \ldots, x_n\}$, then the fibre W_u is empty.

Proof. An étale morphism is unramified, hence we may apply Lemma 37.17.2. As in the proof of Lemma 37.18.1 the morphisms $V_{i,j} \to U$ are open immersions and we win after replacing U by the intersection of their images. □

The following lemma is in some sense much weaker than the preceding one but it may be useful to state it explicitly here. It says that a finite étale morphism is étale locally on the base a ``topological covering space'', i.e., a finite product of copies of the base.

Lemma 37.18.3. *Let $f : X \to S$ be a finite étale morphism of schemes. Let $s \in S$. There exists an étale neighbourhood $(U, u) \to (S, s)$ and a disjoint union decomposition*

$$X_U = \coprod\nolimits_j V_j$$

such that each $V_j \to U$ is an isomorphism.

Proof. An étale morphism is unramified, hence we may apply Lemma 37.17.3. As in the proof of Lemma 37.18.1 we see that $V_{i,j} \to U$ is an open immersion and we win after replacing U by the intersection of their images. □

37.19. Permanence properties

In what follows, we present a few ``permanence'' properties of étale homomorphisms of Noetherian local rings (as defined in Definition 37.11.1). See More on Algebra, Section 12.33 for the analogue of this material for the completion of a Noetherian local ring.

Lemma 37.19.1. *Let A, B be Noetherian local rings. Let $A \to B$ be a étale homomorphism of local rings. Then* $\dim(A) = \dim(B)$.

Proof. See for example Algebra, Lemma 7.103.7. □

Proposition 37.19.2. *Let A, B be Noetherian local rings. Let $f : A \to B$ be an étale homomorphism of local rings. Then $depth(A) = depth(B)$*

Proof. See Algebra, Lemma 7.145.1. □

Proposition 37.19.3. *Let A, B be Noetherian local rings. Let $f : A \to B$ be an étale homomorphism of local rings. Then A is Cohen-Macaulay if and only if B is so.*

Proof. A local ring A is Cohen-Macaulay if and only $\dim(A) = \text{depth}(A)$. As both of these invariants is preserved under an étale extension, the claim follows. □

Proposition 37.19.4. *Let A, B be Noetherian local rings. Let $f : A \to B$ be an étale homomorphism of local rings. Then A is regular if and only if B is so.*

Proof. If B is regular, then A is regular by Algebra, Lemma 7.102.8. Assume A is regular. Let $\mathfrak m$ be the maximal ideal of A. Then $\dim_{\kappa(\mathfrak m)} \mathfrak m/\mathfrak m^2 = \dim(A) = \dim(B)$ (see Lemma 37.19.1). On the other hand, $\mathfrak m B$ is the maximal ideal of B and hence $\mathfrak m_B/\mathfrak m_B = \mathfrak m B/\mathfrak m^2 B$ is generated by at most $\dim(B)$ elements. Thus B is regular. (You can also use the slightly more general Algebra, Lemma 7.103.8.) □

Proposition 37.19.5. *Let A, B be Noetherian local rings. Let $f : A \to B$ be an étale homomorphism of local rings. Then A is reduced if and only if B is so.*

Proof. It is clear from the faithful flatness of $A \to B$ that if B is reduced, so is A. See also Algebra, Lemma 7.146.2. Conversely, assume A is reduced. By assumption B is a localization of a finite type A-algebra B' at some prime $\mathfrak q$. After replacing B' by a localization we may assume that B' is étale over A, see Lemma 37.11.2. Then we see that Algebra, Lemma 7.145.6 applies to $A \to B'$ and B' is reduced. Hence B is reduced. □

Remark 37.19.6. The result on ``reducedness" does not hold with a weaker definition of étale local ring maps $A \to B$ where one drops the assumption that B is essentially of finite type over A. Namely, it can happen that a Noetherian local domain A has nonreduced completion $A^\wedge$, see Examples, Section 64.8. But the ring map $A \to A^\wedge$ is flat, and $\mathfrak m_A A^\wedge$ is the maximal ideal of $A^\wedge$ and of course A and $A^\wedge$ have the same residue fields. This is why it is important to consider this notion only for ring extensions which are essentially of finite type (or essentially of finite presentation if A is not Noetherian).

Proposition 37.19.7. *Let A, B be Noetherian local rings. Let $f : A \to B$ be an étale homomorphism of local rings. Then A is a normal domain if and only if B is so.*

Proof. See Algebra, Lemma 7.146.3 for descending normality. Conversely, assume A is normal. By assumption B is a localization of a finite type A-algebra B' at some prime $\mathfrak q$. After replacing B' by a localization we may assume that B' is étale over A, see Lemma 37.11.2. Then we see that Algebra, Lemma 7.145.7 applies to $A \to B'$ and we conclude that B' is normal. Hence B is a normal domain. □

The preceeding propositions give some indication as to why we'd like to think of étale maps as ``local isomorphisms". Another property that gives an excellent indication that we have the ``right" definition is the fact that for $\mathbf{C}$-schemes of finite type, a morphism is étale if and only if the associated morphism on analytic spaces (the $\mathbf{C}$-valued points given the complex topology) is a local isomorphism in the analytic sense (open embedding locally on the source). This fact can be proven with the aid of the structure theorem and the fact that the analytification commutes with the formation of the completed local rings -- the details are left to the reader.

37.20. Other chapters

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(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Commutative Algebra
(8) Brauer Groups
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CHAPTER 38

Étale Cohomology

38.1. Introduction

These are the notes of a course on étale cohomology taught by Johan de Jong at Columbia University in the Fall of 2009. The original note takers were Thibaut Pugin, Zachary Maddock and Min Lee. Over time we will add references to background material in the rest of the stacks project and provide rigorous proofs of all the statements.

38.2. Which sections to skip on a first reading?

We want to use the material in this chapter for the development of theory related to algebraic spaces, Deligne-Mumford stacks, algebraic stacks, etc. Thus we have added some pretty technical material to the original exposition of étale cohomology for schemes. The reader can recognize this material by the frequency of the word ``topos", or by discussions related to set theory, or by proofs dealing with very general properties of morphisms of schemes. Some of these discussions can be skipped on a first reading.

In particular, we suggest that the reader skip the following sections:

(1) Comparing big and small topoi, Section 38.39.
(2) Recovering morphisms, Section 38.40.
(3) Push and pull, Section 38.41.
(4) Property (A), Section 38.42.
(5) Property (B), Section 38.43.
(6) Property (C), Section 38.44.
(7) Topological invariance of the small étale site, Section 38.45.
(8) Integral universally injective morphisms, Section 38.47.
(9) Big sites and pushforward, Section 38.48.
(10) Exactness of big lower shriek, Section 38.49.

Besides these sections there are some sporadic results that may be skipped that the reader can recognize by the keywords given above.

38.3. Prologue

These lectures are about another cohomology theory. The first thing to remark is that the Zariski topology is not entirely satisfactory. One of the main reasons that it fails to give the results that we would want is that if X is a complex variety and $\mathcal{F}$ is a constant sheaf then

$$H^i(X, \mathcal{F}) = 0, \quad \text{for all } i > 0.$$

The reason for that is the following. In an irreducible scheme (a variety in particular), any two nonempty open subsets meet, and so the restriction mappings of a constant sheaf are surjective. We say that the sheaf is *flasque*. In this case, all higher Čech cohomology groups vanish, and so do all higher Zariski cohomology groups. In other words, there are ``not enough" open sets in the Zariski topology to detect this higher cohomology.

On the other hand, if X is a smooth projective complex variety, then

$$H^{2\dim X}_{Betti}(X(\mathbf{C}),\Lambda)=\Lambda \quad \text{for } \Lambda=\mathbf{Z},\ \mathbf{Z}/n\mathbf{Z},$$

where $X(\mathbf{C})$ means the set of complex points of X. This is a feature that would be nice to replicate in algebraic geometry. In positive characteristic in particular.

38.4. The étale topology

It is very hard to simply ``add'' extra open sets to refine the Zariski topology. One efficient way to define a topology is to consider not only open sets, but also some schemes that lie over them. To define the étale topology, one considers all morphisms $\varphi : U \to X$ which are étale. If X is a smooth projective variety over $\mathbf{C}$, then this means

(1) U is a disjoint union of smooth varieties ; and
(2) φ is (analytically) locally an isomorphism.

The word ``analytically'' refers to the usual (transcendental) topology over $\mathbf{C}$. So the second condition means that the derivative of φ has full rank everywhere (and in particular all the components of U have the same dimension as X).

A double cover -- loosely defined as a finite degree 2 map between varieties -- for example

$$Spec(\mathbf{C}[t]) \longrightarrow Spec(\mathbf{C}[t]), \quad t \longmapsto t^2$$

will not be an étale morphism if it has a fibre consisting of a single point. In the example this happens when $t=0$. For a finite map between varieties over $\mathbf{C}$ to be étale all the fibers should have the same number of points. Removing the point $t=0$ from the source of the map in the example will make the morphism étale. But we can remove other points from the source of the morphism also, and the morphism will still be étale. To consider the étale topology, we have to look at all such morphisms. Unlike the Zariski topology, these need not be merely be open subsets of X, even though their images always are.

Definition 38.4.1. A family of morphisms $\{\varphi_i : U_i \to X\}_{i\in I}$ is called an *étale covering* if each φ_i is an étale morphism and their images cover X, i.e., $X=\bigcup_{i\in I}\varphi_i(U_i)$.

This ``defines'' the étale topology. In other words, we can now say what the sheaves are. An *étale sheaf* $\mathcal{F}$ of sets (resp. abelian groups, vector spaces, etc) on X is the data:

(1) for each étale morphism $\varphi : U \to X$ a set (resp. abelian group, vector space, etc) $\mathcal{F}(U)$,
(2) for each pair U, U' of étale schemes over X, and each morphism $U \to U'$ over X (which is automatically étale) a restriction map $\rho^U_{U'} : \mathcal{F}(U) \to \mathcal{F}(U')$

These data have to satisfy the following *sheaf axiom*:

(*) for every étale covering $\{\varphi_i : U_i \to X\}_{i\in I}$, the diagram

$$\emptyset \longrightarrow \mathcal{F}(U) \longrightarrow \Pi_{i\in I}\mathcal{F}(U_i) \rightrightarrows \Pi_{i,j\in I}\mathcal{F}(U_i\times_U U_j)$$

is exact in the category of sets (resp. abelian groups, vector spaces, etc).

Remark 38.4.2. In the last statement, it is essential not to forget the case where $i=j$ which is in general a highly nontrivial condition (unlike in the Zariski topology). In fact, frequently important coverings have only one element.

Since the identity is an étale morphism, we can compute the global sections of an étale sheaf, and cohomology will simply be the corresponding right-derived functors. In other words, once more theory has been developed and statements have been made precise, there will be no obstacle to defining cohomology.

38.5. Feats of the étale topology

For a natural number $n \in \mathbf{N} = \{1, 2, 3, 4, \ldots\}$ it is true that

$$H^2_{et}(\mathbf{P}^1_{\mathbf{C}}, \mathbf{Z}/n\mathbf{Z}) = \mathbf{Z}/n\mathbf{Z}.$$

More generally, if X is a complex variety, then its étale Betti numbers with coefficients in a finite field agree with the usual Betti numbers of $X(\mathbf{C})$, i.e.,

$$\dim_{\mathbf{F}_q} H^{2i}_{et}(X, \mathbf{F}_q) = \dim_{\mathbf{F}_q} H^{2i}_{Betti}(X(\mathbf{C}), \mathbf{F}_q).$$

This is extremely satisfactory. However, these equalities only hold for torsion coefficients, not in general. For integer coefficients, one has

$$H^2_{et}(\mathbf{P}^1_{\mathbf{C}}, \mathbf{Z}) = 0.$$

There are ways to get back to nontorsion coefficients from torsion ones by a limit procedure which we will come to shortly.

38.6. A computation

How do we compute the cohomology of $\mathbf{P}^1_{\mathbf{C}}$ with coefficients $\Lambda = \mathbf{Z}/n\mathbf{Z}$? We use Čech cohomology. A covering of $\mathbf{P}^1_{\mathbf{C}}$ is given by the two standard opens U_0, U_1, which are both isomorphic to $\mathbf{A}^1_{\mathbf{C}}$, and which intersection is isomorphic to $\mathbf{A}^1_{\mathbf{C}} \setminus \{0\} = \mathbf{G}_{m,\mathbf{C}}$. It turns out that the Mayer-Vietoris sequence holds in étale cohomology. This gives an exact sequence

$$H^{i-1}_{et}(U_0 \cap U_1, \Lambda) \to H^i_{et}(\mathbf{P}^1_{\mathbf{C}}, \Lambda) \to H^i_{et}(U_0, \Lambda) \oplus H^i_{et}(U_1, \Lambda) \to H^i_{et}(U_0 \cap U_1, \Lambda).$$

To get the answer we expect, we would need to show that the direct sum in the third term vanishes. In fact, it is true that, as for the usual topology,

$$H^q_{et}(\mathbf{A}^1_{\mathbf{C}}, \Lambda) = 0 \quad \text{for } q \geq 1,$$

and

$$H^q_{et}(\mathbf{A}^1_{\mathbf{C}} \setminus \{0\}, \Lambda) = \begin{cases} \Lambda & \text{if } q = 1, \text{ and} \\ 0 & \text{for } q \geq 2. \end{cases}$$

These results are already quite hard (what is an elementary proof?). Let us explain how we would compute this once the machinery of étale cohomology is at our disposal.

Higher cohomology. This is taken care of by the following general fact: if X is an affine curve over $\mathbf{C}$, then

$$H^q_{et}(X, \mathbf{Z}/n\mathbf{Z}) = 0 \quad \text{for } q \geq 2.$$

This is proved by considering the generic point of the curve and doing some Galois cohomology. So we only have to worry about the cohomology in degree 1.

Cohomology in degree 1. We use the following identifications:

$$\begin{aligned} H^1_{et}(X, \mathbf{Z}/n\mathbf{Z}) &= \left\{ \begin{array}{c} \text{sheaves of sets } \mathcal{F} \text{ on the étale site } X_{\acute{e}t} \text{ endowed with an} \\ \text{action } \mathbf{Z}/n\mathbf{Z} \times \mathcal{F} \to \mathcal{F} \text{ such that } \mathcal{F} \text{ is a } \mathbf{Z}/n\mathbf{Z}\text{-torsor.} \end{array} \right\} \Big/ \cong \\ &= \left\{ \begin{array}{c} \text{morphisms } Y \to X \text{ which are finite étale together} \\ \text{with a free } \mathbf{Z}/n\mathbf{Z} \text{ action such that } X = Y/(\mathbf{Z}/n\mathbf{Z}). \end{array} \right\} \Big/ \cong . \end{aligned}$$

The first identification is very general (it is true for any cohomology theory on a site) and has nothing to do with the étale topology. The second identification is a consequence of descent theory. The last set describes a collection of geometric objects on which we can get our hands.

The curve $\mathbf{A}^1_{\mathbf{C}}$ has no nontrivial finite étale covering and hence $H^1_{et}(\mathbf{A}^1_{\mathbf{C}}, \mathbf{Z}/n\mathbf{Z}) = 0$. This can be seen either topologically or by using the argument in the next paragraph.

Let us describe the finite étale coverings $\varphi : Y \to \mathbf{A}^1_{\mathbf{C}} \setminus \{0\}$. It suffices to consider the case where Y is connected, which we assume. We are going to find out what Y can be by applying the Riemann-Hurwitz formula (of course this is a bit silly, and you can go ahead and skip the the next section if you like). Say that this morphism is n to 1, and consider a projective compactification

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[d, "\varphi"] & \bar{Y} \arrow[d, "\bar{\varphi}"] \\ \mathbf{A}^1_{\mathbf{C}} \setminus \{0\} \arrow[r, hook] & \mathbf{P}^1_{\mathbf{C}} \end{tikzcd}$$

Even though φ is étale and does not ramify, $\bar{\varphi}$ may ramify at 0 and ∞. Say that the preimages of 0 are the points $y_1, \ldots, y_r$ with indices of ramification $e_1, \ldots e_r$, and that the preimages of ∞ are the points $y'_1, \ldots, y'_s$ with indices of ramification $d_1, \ldots d_s$. In particular, $\sum e_i = n = \sum d_j$. Applying the Riemann-Hurwitz formula, we get

$$2g_Y - 2 = -2n + \sum (e_i - 1) + \sum (d_j - 1)$$

and therefore $g_Y = 0$, $r = s = 1$ and $e_1 = d_1 = n$. Hence $Y \cong \mathbf{A}^1_{\mathbf{C}} \setminus \{0\}$, and it is easy to see that $\varphi(z) = \lambda z^n$ for some $\lambda \in \mathbf{C}^*$. After reparametrizing Y we may assume $\lambda = 1$. Thus our covering is given by taking the nth root of the coordinate on $\mathbf{A}^1_{\mathbf{C}} \setminus \{0\}$.

Remember that we need to classify the coverings of $\mathbf{A}^1_{\mathbf{C}} \setminus \{0\}$ together with free $\mathbf{Z}/n\mathbf{Z}$-actions on them. In our case any such action corresponds to an automorphism of Y sending z to $\zeta_n z$, where ζ_n is a primitive nth root of unity. There are $\phi(n)$ such actions (here $\phi(n)$ means the Euler function). Thus there are exactly $\phi(n)$ connected finite étale coverings with a given free $\mathbf{Z}/n\mathbf{Z}$-action, each corresponding to a primitive nth root of unity. We leave it to the reader to see that the disconnected finite étale degree n coverings of $\mathbf{A}^1_{\mathbf{C}} \setminus \{0\}$ with a given free $\mathbf{Z}/n\mathbf{Z}$-action correspond one-to-one with nth roots of 1 which are not primitive. In other words, this computation shows that

$$H^1_{et}(\mathbf{A}^1_{\mathbf{C}} \setminus \{0\}, \mathbf{Z}/n\mathbf{Z}) = \mu_n(\mathbf{C}) \cong \mathbf{Z}/n\mathbf{Z}.$$

The first identification is canonical, the second isn't. We remark that since the proof of Riemann-Hurwitz does not use this fact, the above actually constitutes a proof (provided we fill in the details on vanishing, etc).

38.7. Nontorsion coefficients

To study nontorsion coefficients, one makes the following definition:

$$H^i_{et}(X, \mathbf{Q}_\ell) := \left(lim_n H^i_{et}(X, \mathbf{Z}/\ell^n\mathbf{Z})\right) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell.$$

The symbol lim_n denote the *limit* of the system of cohomology groups $H^i_{et}(X, \mathbf{Z}/\ell^n\mathbf{Z})$ indexed by n, see Categories, Section 4.19. Thus we will need to study systems of sheaves satisfying some compatibility conditions.

38.8. Sheaf theory

At this point we start talking about sites and sheaves in earnest. There is an amazing amount of useful abstract material that could fit in the next few sections. Some of this material is worked out in earlier chapters, such as the chapter on sites, modules on sites, and cohomology on sites. We try to refrain from adding to much material here, just enough so the material later in this chapter makes sense.

38.9. Presheaves

A reference for this section is Sites, Section 9.2.

Definition 38.9.1. Let $\mathcal{C}$ be a category. A *presheaf of sets* (respectively, an *abelian presheaf*) on $\mathcal{C}$ is a functor $\mathcal{C}^{opp} \to Sets$ (resp. Ab).

Terminology. If $U \in Ob(\mathcal{C})$, then elements of $\mathcal{F}(U)$ are called *sections* of $\mathcal{F}$ on U; for $\varphi : V \to U$ in $\mathcal{C}$, the map $\mathcal{F}(\varphi) : \mathcal{F}(V) \to \mathcal{F}(U)$ is denoted $s \mapsto \mathcal{F}(\varphi)(s) = \varphi^*(s) = s|_V$ and called *restriction mapping*. This last notation is ambiguous since the restriction map depends on φ, but it is a standard abuse of notation. We also use the notation $\Gamma(U, \mathcal{F}) = \mathcal{F}(U)$.

Saying that $\mathcal{F}$ is a functor means that if $W \to V \to U$ are morphisms in $\mathcal{C}$ and $s \in \Gamma(U, \mathcal{F})$ then $(s|_V)|_W = s|_W$, with the abuse of notation just seen. Moreover, the restriction mappings corresponding to the identity morphisms $\text{id}_U : U \to U$ are the identity.

The category of presheaves of sets (respectively of abelian presheaves) on $\mathcal{C}$ is denoted $PSh(\mathcal{C})$ (resp. $PAb(\mathcal{C})$). It is the category of functors from $\mathcal{C}^{opp}$ to $Sets$ (resp. Ab), which is to say that the morphisms of presheaves are natural transformations of functors. We only consider the categories $PSh(\mathcal{C})$ and $PAb(\mathcal{C})$ when the category $\mathcal{C}$ is small. (Our convention is that a category is small unless otherwise mentioned, and if it isn't small it should be listed in Categories, Remark 4.2.2.)

Example 38.9.2. Given an object $X \in Ob(\mathcal{C})$, we consider the functor

$$\begin{array}{cccc} h_X : & \mathcal{C}^{opp} & \longrightarrow & Sets \\ & U & \longmapsto & h_X(U) = Mor_{\mathcal{C}}(U, X) \\ & V \xrightarrow{\varphi} U & \longmapsto & \varphi \circ - : h_X(U) \to h_X(V). \end{array}$$

It is a presheaf, called the *representable presheaf associated to* X. It is not true that representable presheaves are sheaves in every topology on every site.

Lemma 38.9.3. *(Yoneda) Let $\mathcal{C}$ be a category, and $X, Y \in Ob(\mathcal{C})$. There is a natural bijection*

$$\begin{array}{ccc} Mor_{\mathcal{C}}(X, Y) & \longrightarrow & Mor_{PSh(\mathcal{C})}(h_X, h_Y) \\ \psi & \longmapsto & h_\psi = \psi \circ - : h_X \to h_Y. \end{array}$$

Proof. See Categories, Lemma 4.3.5. □

38.10. Sites

Definition 38.10.1. Let $\mathcal{C}$ be a category. A *family of morphisms with fixed target* $\mathcal{U} = \{\varphi_i : U_i \to U\}_{i \in I}$ is the data of

(1) an object $U \in \mathcal{C}$;
(2) a set I (possibly empty) ; and
(3) for all $i \in I$, a morphism $\varphi_i : U_i \to U$ of $\mathcal{C}$ with target U.

There is a notion of a *morphism of families of morphisms with fixed target*. A special case of that is the notion of a *refinement*. A reference for this material is Sites, Section 9.8.

Definition 38.10.2. A *site*[1] consists of a category $\mathscr{C}$ and a set $\mathrm{Cov}(\mathscr{C})$ consising of families of morphisms with fixed target called *coverings*, such that

(1) (isomorphism) if $\varphi : V \to U$ is an isomorphism in $\mathscr{C}$, then $\{\varphi : V \to U\}$ is a covering,
(2) (locality) if $\{\varphi_i : U_i \to U\}_{i\in I}$ is a covering and for all $i \in I$ we are given a covering $\{\psi_{ij} : U_{ij} \to U_i\}_{j\in I_i}$, then
$$\{\varphi_i \circ \psi_{ij} : U_{ij} \to U\}_{(i,j)\in\prod_{i\in I}\{i\}\times I_i}$$
is also a covering, and
(3) (base change) if $\{U_i \to U\}_{i\in I}$ is a covering and $V \to U$ is a morphism in $\mathscr{C}$, then
   (a) for all $i \in I$ the fibre product $U_i \times_U V$ exists in $\mathscr{C}$, and
   (b) $\{U_i \times_U V \to V\}_{i\in I}$ is a covering.

For us the category underlying a site is aways ``small'', i.e., its collection of objects form a set, and the collection of coverings of a site is a set as well (as in the definition above). We will mostly, in this chapter, leave out the arguments that cut down the collection of objects and coverings to a set. For further discussion, see Sites, Remark 9.6.3.

Example 38.10.3. If X is a topological space, then it has an associated site $\mathscr{T}_X$ defined as follows: the objects of $\mathscr{T}_X$ are the open subsets of X, the morphisms between these are the inclusion mappings, and the coverings are the usual topological (surjective) coverings. Observe that if $U, V \subset W \subset X$ are open subsets then $U \times_W V = U \cap V$ exists: this category has fiber products. All the verifications are trivial and everything works as expected.

38.11. Sheaves

Definition 38.11.1. A presheaf $\mathscr{F}$ of sets (resp. abelian presheaf) on a site $\mathscr{C}$ is said to be a *separated presheaf* if for all coverings $\{\varphi_i : U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathscr{C})$ the map
$$\mathscr{F}(U) \longrightarrow \prod\nolimits_{i\in I} \mathscr{F}(U_i)$$
is injective. Here the map is $s \mapsto (s|_{U_i})_{i\in I}$. The presheaf $\mathscr{F}$ is a *sheaf* if for all coverings $\{\varphi_i : U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathscr{C})$, the diagram

$$\mathscr{F}(U) \longrightarrow \prod\nolimits_{i\in I} \mathscr{F}(U_i) \rightrightarrows \prod\nolimits_{i,j\in I} \mathscr{F}(U_i \times_U U_j), \tag{38.11.1.1}$$

where the first map is $s \mapsto (s|_{U_i})_{i\in I}$ and the two maps on the right are $(s_i)_{i\in I} \mapsto (s_i|_{U_i\times_U U_j})$ and $(s_i)_{i\in I} \mapsto (s_j|_{U_i\times_U U_j})$, is an equalizer diagram in the category of sets (resp. abelian groups).

Remark 38.11.2. For the empty covering (where $I = \emptyset$), this implies that $\mathscr{F}(\emptyset)$ is an empty product, which is a final object in the corresponding category (a singleton, for both *Sets* and *Ab*).

Example 38.11.3. Working this out for the site $\mathscr{T}_X$ associated to a topological space, see Example 38.10.3, gives the usual notion of sheaves.

Definition 38.11.4. We denote $Sh(\mathscr{C})$ (resp. $Ab(\mathscr{C})$) the full subcategory of $PSh(\mathscr{C})$ (resp. $PAb(\mathscr{C})$) whose objects are sheaves. This is the *category of sheaves of sets* (resp. *abelian sheaves*) on $\mathscr{C}$.

[1]What we call a site is a called a category endowed with a pretopology in [**MA71**, Exposé II, Définition 1.3]. In [**Art62**] it is called a category with a Grothendieck topology.

38.12. The example of G-sets

Let G be a group and define a site $\mathcal{T}_G$ as follows: the underlying category is the category of G-sets, i.e., its objects are sets endowed with a left G-action and the morphisms are equivariant maps; and the coverings of $\mathcal{T}_G$ are the families $\{\varphi_i : U_i \to U\}_{i\in I}$ satisfying $U = \bigcup_{i\in I} \varphi_i(U_i)$.

There is a special object in the site $\mathcal{T}_G$, namely the G-set G endowed with its natural action by left translations. We denote it ${}_GG$. Observe that there is a natural group isomorphism

$$\begin{array}{rccc} \rho : & G^{opp} & \longrightarrow & \mathrm{Aut}_{G\text{-}Sets}({}_GG) \\ & g & \longmapsto & (h \mapsto hg). \end{array}$$

In particular, for any presheaf $\mathcal{F}$, the set $\mathcal{F}({}_GG)$ inherits a G-action via ρ. (Note that by contravariance of $\mathcal{F}$, the set $\mathcal{F}({}_GG)$ is again a left G-set.) In fact, the functor

$$\begin{array}{ccc} Sh(\mathcal{T}_G) & \longrightarrow & G\text{-}Sets \\ \mathcal{F} & \longmapsto & \mathcal{F}({}_GG) \end{array}$$

is an equivalence of categories. Its quasi-inverse is the functor $X \mapsto h_X$. Without giving the complete proof (which can be found in Sites, Section 9.9) let us try to explain why this is true.

(1) If S is a G-set, we can decompose it into orbits $S = \coprod_{i\in I} O_i$. The sheaf axiom for the covering $\{O_i \to S\}_{i\in I}$ says that

$$\mathcal{F}(S) \longrightarrow \prod\nolimits_{i\in I} \mathcal{F}(O_i) \rightrightarrows \prod\nolimits_{i,j\in I} \mathcal{F}(O_i \times_S O_j)$$

is an equalizer. Observing that fibered products in G-$Sets$ are induced from fibered products in $Sets$, and using the fact that $\mathcal{F}(\emptyset)$ is a G-singleton, we get that

$$\prod_{i,j\in I} \mathcal{F}(O_i \times_S O_j) = \prod_{i\in I} \mathcal{F}(O_i)$$

and the two maps above are in fact the same. Therefore the sheaf axiom merely says that $\mathcal{F}(S) = \prod_{i\in I} \mathcal{F}(O_i)$.

(2) If S is the G-set $S = G/H$ and $\mathcal{F}$ is a sheaf on $\mathcal{T}_G$, then we claim that

$$\mathcal{F}(G/H) = \mathcal{F}({}_GG)^H$$

and in particular $\mathcal{F}(\{*\}) = \mathcal{F}({}_GG)^G$. To see this, let's use the sheaf axiom for the covering $\{{}_GG \to G/H\}$ of S. We have

$$\begin{array}{ccc} {}_GG \times_{G/H} {}_GG & \cong & G \times H \\ (g_1, g_2) & \longmapsto & (g_1, g_1g_2^{-1}) \end{array}$$

is a disjoint union of copies of ${}_GG$ (as a G-set). Hence the sheaf axiom reads

$$\mathcal{F}(G/H) \longrightarrow \mathcal{F}({}_GG) \rightrightarrows \prod\nolimits_{h\in H} \mathcal{F}({}_GG)$$

where the two maps on the right are $s \mapsto (s)_{h\in H}$ and $s \mapsto (hs)_{h\in H}$. Therefore $\mathcal{F}(G/H) = \mathcal{F}({}_GG)^H$ as claimed.

This doesn't quite prove the claimed equivalence of categories, but it shows at least that a sheaf $\mathcal{F}$ is entirely determined by its sections over ${}_GG$. Details (and set theoretical remarks) can be found in Sites, Section 9.9.

38.13. Sheafification

Definition 38.13.1. Let $\mathscr{F}$ be a presheaf on the site $\mathscr{C}$ and $\mathscr{U} = \{U_i \to U\} \in \mathrm{Cov}(\mathscr{C})$. We define the *zeroth Cech cohomology group* of $\mathscr{F}$ with respect to $\mathscr{U}$ by

$$\check{H}^0(\mathscr{U}, \mathscr{F}) = \left\{ (s_i)_{i \in I} \in \prod_{i \in I} \mathscr{F}(U_i) \text{ such that } s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j} \right\}.$$

There is a canonical map $\mathscr{F}(U) \to \check{H}^0(\mathscr{U}, \mathscr{F})$, $s \mapsto (s|_{U_i})_{i \in I}$. We say that a *morphism of coverings* from a covering $\mathscr{V} = \{V_j \to V\}_{j \in J}$ to $\mathscr{U}$ is a triple (χ, α, χ_j), where $\chi : \mathscr{V} \to \mathscr{U}$ is a morphism, $\alpha : J \to I$ is a map of sets, and for all $j \in J$ the morphism χ_j fits into a commutative diagram

$$\begin{tikzcd} V_j \arrow[r, "\chi_j"'] \arrow[d] & U_{\alpha(j)} \arrow[d] \\ V \arrow[r, "\chi"] & U. \end{tikzcd}$$

Given the data $\chi, \alpha, \{\chi_j\}_{i \in J}$ we define

$$\begin{array}{ccc} \check{H}^0(\mathscr{U}, \mathscr{F}) & \longrightarrow & \check{H}^0(\mathscr{V}, \mathscr{F}) \\ (s_i)_{i \in I} & \longmapsto & \left(\chi_j^* \left(s_{\alpha(j)} \right) \right)_{j \in J}. \end{array}$$

We then claim that

(1) the map is well-defined, and
(2) depends only on χ and is independent of the choice of $\alpha, \{\chi_j\}_{i \in J}$.

We omit the proof of the first fact. To see part (2), consider another triple (ψ, β, ψ_j) with $\chi = \psi$. Then we have the commutative diagram

$$\begin{tikzcd} V_j \arrow[rr, "(\chi_j, \psi_j)"] \arrow[ddd] & & U_{\alpha(j)} \times_U U_{\beta(j)} \arrow[dl] \arrow[dr] & \\ & U_{\alpha(j)} \arrow[ddr] & & U_{\beta(j)} \arrow[ddl] \\ & & & \\ V \arrow[rr, "\chi = \psi"] & & U. & \end{tikzcd}$$

Given a section $s \in \mathscr{F}(\mathscr{U})$, its image in $\mathscr{F}(V_j)$ under the map given by $(\chi, \alpha, \{\chi_j\}_{i \in J})$ is $\chi_j^* s_{\alpha(j)}$, and its image under the map given by $(\psi, \beta, \{\psi_j\}_{i \in J})$ is $\psi_j^* s_{\beta(j)}$. These two are equal since by assumption $s \in \check{H}(\mathscr{U}, \mathscr{F})$ and hence both are equal to the pullback of the common value

$$s_{\alpha(j)}|_{U_{\alpha(j)} \times_U U_{\beta(j)}} = s_{\beta(j)}|_{U_{\alpha(j)} \times_U U_{\beta(j)}}$$

pulled back by the map (χ_j, ψ_j) in the diagram.

Theorem 38.13.2. *Let $\mathscr{C}$ be a site and $\mathscr{F}$ a presheaf on $\mathscr{C}$.*

(1) *The rule*

$$U \mapsto \mathscr{F}^+(U) := colim_{\mathscr{U} \text{ covering of } U} \check{H}^0(\mathscr{U}, \mathscr{F})$$

is a presheaf. And the colimit is a directed one.

(2) *There is a canonical map of presheaves $\mathscr{F} \to \mathscr{F}^+$.*
(3) *If $\mathscr{F}$ is a separated presheaf then $\mathscr{F}^+$ is a sheaf and the map in (2) is injective.*
(4) *$\mathscr{F}^+$ is a separated presheaf.*

(5) *$\mathscr{F}^\# = (\mathscr{F}^+)^+$ is a sheaf, and the canonical map induces a functorial isomorphism*

$$Hom_{PSh(\mathscr{C})}(\mathscr{F}, \mathscr{G}) = Hom_{Sh(\mathscr{C})}(\mathscr{F}^\#, \mathscr{G})$$

for any $\mathscr{G} \in Sh(\mathscr{C})$.

Proof. See Sites, Theorem 9.10.10. □

In other words, this means that the natural map $\mathscr{F} \to \mathscr{F}^\#$ is a left adjoint to the forgetful functor $Sh(\mathscr{C}) \to PSh(\mathscr{C})$.

38.14. Cohomology

The following is the basic result that makes it possible to define cohomology for abelian sheaves on sites.

Theorem 38.14.1. *The category of abelian sheaves on a site is an abelian category which has enough injectives.*

Proof. See Modules on Sites, Lemma 16.3.1 and Injectives, Theorem 17.11.4. □

So we can define cohomology as the right-derived functors of the sections functor: if $U \in Ob(\mathscr{C})$ and $\mathscr{F} \in Ab(\mathscr{C})$,

$$H^p(U, \mathscr{F}) := R^p\Gamma(U, \mathscr{F}) = H^p(\Gamma(U, \mathscr{I}^\bullet))$$

where $\mathscr{F} \to \mathscr{I}^\bullet$ is an injective resolution. To do this, we should check that the functor $\Gamma(U, -)$ is left exact. This is true and is part of why the category $Ab(\mathscr{C})$ is abelian, see Modules on Sites, Lemma 16.3.1. For more general discussion of cohomology on sites (including the global sections functor and its right derived functors), see Cohomology on Sites, Section 19.3.

38.15. The fpqc topology

Before doing étale cohomology we study a bit the fpqc topology, since it works well for quasi-coherent sheaves.

Definition 38.15.1. Let T be a scheme. An *fpqc covering* of T is a family $\{\varphi_i : T_i \to T\}_{i \in I}$ such that

(1) each φ_i is a flat morphism and $\bigcup_{i \in I} \varphi_i(T_i) = T$, and
(2) for each affine open $U \subset T$ there exists a finite set K, a map $\mathbf{i} : K \to I$ and affine opens $U_{\mathbf{i}(k)} \subset T_{\mathbf{i}(k)}$ such that $U = \bigcup_{k \in K} \varphi_{\mathbf{i}(k)}(U_{\mathbf{i}(k)})$.

Remark 38.15.2. The first condition corresponds to fp, which stands for *fidèlement plat*, faithfully flat in french, and the second to qc, *quasi-compact*. The second part of the first condition is unnecessary when the second condition holds.

Example 38.15.3. Examples of fpqc coverings.

(1) Any Zariski open covering of T is an fpqc covering.
(2) A family $\{Spec(B) \to Spec(A)\}$ is an fpqc covering if and only if $A \to B$ is a faithfully flat ring map.
(3) If $f : X \to Y$ is flat, surjective and quasi-compact, then $\{f : X \to Y\}$ is an fpqc covering.
(4) The morphism $\varphi : \coprod_{x \in \mathbf{A}^1_k} Spec(\mathscr{O}_{\mathbf{A}^1_k, x}) \to \mathbf{A}^1_k$, where k is a field, is flat and surjective. It is not quasi-compact, and in fact the family $\{\varphi\}$ is not an fpqc covering.

(5) Write $\mathbf{A}^2_k = Spec(k[x, y])$. Denote $i_x : D(x) \to \mathbf{A}^2_k$ and $i_y : D(y) \hookrightarrow \mathbf{A}^2_k$ the standard opens. Then the families $\{i_x, i_y, Spec(k[[x, y]]) \to \mathbf{A}^2_k\}$ and $\{i_x, i_y, Spec(\mathcal{O}_{\mathbf{A}^2_k,0}) \to \mathbf{A}^2_k\}$ are fpqc coverings.

Lemma 38.15.4. *The collection of fpqc coverings on the category of schemes satisfies the axioms of site.*

Proof. See Topologies, Lemma 30.8.7. □

It seems that this lemma allows us to define the fpqc site of the category of schemes. However, there is a set theoretical problem that comes up when considering the fpqc topology, see Topologies, Section 38.15. It comes from our requirement that sites are ``small", but that no small category of schemes can contain a cofinal system of fpqc coverings of a given nonempty scheme. Allthough this does not strictly speaking prevent us from defining ``partial" fpqc sites, it does not seem prudent to do so. The work-around is to allow the notion of a sheaf for the fpqc topology (see below) but to prohibit considering the category of all fpqc sheaves.

Definition 38.15.5. Let S be a scheme. The category of schemes over S is denoted Sch/S. Consider a functor $\mathcal{F} : (Sch/S)^{opp} \to Sets$, in other words a presheaf of sets. We say $\mathcal{F}$ *satisfies the sheaf property for the fpqc topology* if for every fpqc covering $\{U_i \to U\}_{i\in I}$ of schemes over S the diagram (38.11.1.1) is an equalizer diagram.

We similarly say that $\mathcal{F}$ *satisfies the sheaf property for the Zariski topology* if for every open covering $U = \bigcup_{i\in I} U_i$ the diagram (38.11.1.1) is an equalizer diagram. See Schemes, Definition 21.15.3. Clearly, this is equivalent to saying that for every scheme T over S the restriction of $\mathcal{F}$ to the opens of T is a (usual) sheaf.

Lemma 38.15.6. *Let $\mathcal{F}$ be a presheaf on Sch/S. Then $\mathcal{F}$ satisfies the sheaf property for the fpqc topology if and only if*

(1) *$\mathcal{F}$ satisfies the sheaf property with respect to the Zariski topology, and*
(2) *for every faithfully flat morphism $Spec(B) \to Spec(A)$ of affine schemes over S, the sheaf axiom holds for the covering $\{Spec(B) \to Spec(A)\}$. Namely, this means that*

$$\mathcal{F}(Spec(A)) \longrightarrow \mathcal{F}(Spec(B)) \rightrightarrows \mathcal{F}(Spec(B \otimes_A B))$$

is an equalizer diagram.

Proof. See Topologies, Lemma 30.8.13. □

An alternative way to think of a presheaf $\mathcal{F}$ on Sch/S which satisfies the sheaf condition for the fpqc topology is as the following data:

(1) for each T/S, a usual (i.e., Zariski) sheaf $\mathcal{F}_T$ on T_{Zar},
(2) for every map $f : T' \to T$ over S, a restriction mapping $f^{-1}\mathcal{F}_T \to \mathcal{F}_{T'}$

such that

(a) the restriction mappings are functorial,
(b) if $f : T' \to T$ is an open immersion then the restriction mapping $f^{-1}\mathcal{F}_T \to \mathcal{F}_{T'}$ is an isomorphism, and
(c) for every faithfully flat morphism $Spec(B) \to Spec(A)$ over S, the diagram

$$\mathcal{F}_{Spec(A)}(Spec(A)) \longrightarrow \mathcal{F}_{Spec(B)}(Spec(B)) \rightrightarrows \mathcal{F}_{Spec(B\otimes_A B)}(Spec(B \otimes_A B))$$

is an equalizer.

Data (1) and (2) and conditions (a), (b) give the data of a presheaf on *Sch*/S satisfying the sheaf condition for the Zariski topology. By Lemma 38.15.6 condition (c) then suffices to get the sheaf condition for the fpqc topology.

Example 38.15.7. Consider the presheaf

$$\begin{array}{rccc} \mathcal{F} : & (Sch/S)^{opp} & \longrightarrow & Ab \\ & T/S & \longmapsto & \Gamma(T, \Omega_{T/S}). \end{array}$$

The compatibility of differentials with localization implies that $\mathcal{F}$ is a sheaf on the Zariski site. However, it does not satisfy the sheaf condition for the fpqc topology. Namely, consider the case $S = Spec(\mathbf{F}_p)$ and the morphism

$$\varphi : V = Spec(\mathbf{F}_p[v]) \to U = Spec(\mathbf{F}_p[u])$$

given by mapping u to v^p. The family $\{\varphi\}$ is an fpqc covering, yet the restriction mapping $\mathcal{F}(U) \to \mathcal{F}(V)$ sends the generator $\mathrm{d}u$ to $\mathrm{d}(v^p) = 0$, so it is the zero map, and the diagram

$$\mathcal{F}(U) \xrightarrow{0} \mathcal{F}(V) \rightrightarrows \mathcal{F}(V \times_U V)$$

is not an equalizer. We will see later that $\mathcal{F}$ does in fact give rise to a sheaf on the étale and smooth sites.

Lemma 38.15.8. *Any representable presheaf on Sch/S satisfies the sheaf condition for the fpqc topology.*

Proof. See Descent, Lemma 31.9.3. □

We will return to this later, since the proof of this fact uses descent for quasi-coherent sheaves, which we will discuss in the next section. A fancy way of expressing the lemma is to say that *the fpqc topology is weaker than the canonical topology*, or that the fpqc topology is *subcanonical*. In the setting of sites this is discussed in Sites, Section 9.12.

Remark 38.15.9. The fpqc is the finest topology that we will see. Hence any presheaf satisfying the sheaf condition for the fpqc topology will be a sheaf in the subsequent sites (étale, smooth, etc). In particular representable presheaves will be sheaves on the étale site of a scheme for example.

Example 38.15.10. Let S be a scheme. Consider the additive group scheme $\mathbf{G}_{a,S} = \mathbf{A}^1_S$ over S, see Groupoids, Example 35.5.3. The associated representable presheaf is given by

$$h_{\mathbf{G}_{a,S}}(T) = Mor_S(T, \mathbf{G}_{a,S}) = \Gamma(T, \mathcal{O}_T).$$

By the above we now know that this is a presheaf of sets which satisfies the sheaf condition for the fpqc topology. On the other hand, it is clearly a presheaf of rings as well. Hence we can think of this as a functor

$$\begin{array}{rccc} \mathcal{O} : & (Sch/S)^{opp} & \longrightarrow & Rings \\ & T/S & \longmapsto & \Gamma(T, \mathcal{O}_T) \end{array}$$

which satisfies the sheaf condition for the fpqc topology. Correspondingly there is a notion of $\mathcal{O}$-module, and so on and so forth.

38.16. Faithfully flat descent

Definition 38.16.1. Let $\mathcal{U} = \{t_i : T_i \to T\}_{i \in I}$ be a family of morphisms of schemes with fixed target. A *descent datum* for quasi-coherent sheaves with respect to $\mathcal{U}$ is a family $(\mathcal{F}_i, \varphi_{ij})_{i,j \in I}$ where

(1) for all i, $\mathcal{F}_i$ is a quasi-coherent sheaf on T_i ; and
(2) for all $i, j \in I$ the map $\varphi_{ij} : \mathrm{pr}_0^* \mathcal{F}_i \cong \mathrm{pr}_1^* \mathcal{F}_j$ is an isomorphism on $T_i \times_T T_j$ such that the diagrams

$$\begin{tikzcd}
\mathrm{pr}_0^*\mathcal{F}_i \arrow[rr, "\mathrm{pr}_{01}^*\varphi_{ij}"] \arrow[dr, "\mathrm{pr}_{02}^*\varphi_{ik}"'] & & \mathrm{pr}_1^*\mathcal{F}_j \arrow[dl, "\mathrm{pr}_{12}^*\varphi_{jk}"] \\
& \mathrm{pr}_2^*\mathcal{F}_k &
\end{tikzcd}$$

commute on $T_i \times_T T_j \times_T T_k$.

This descent datum is called *effective* if there exist a quasi-coherent sheaf $\mathcal{F}$ over T and $\mathcal{O}_{T_i}$-module isomorphisms $\varphi_i : t_i^* \mathcal{F} \cong \mathcal{F}_i$ satisfying the cocycle condition, namely

$$\varphi_{ij} = \mathrm{pr}_1^*(\varphi_j) \circ \mathrm{pr}_0^*(\varphi_i)^{-1}.$$

In this and the next section we discuss some ingredients of the proof of the following theorem, as well as some related material.

Theorem 38.16.2. *If $\mathcal{V} = \{T_i \to T\}_{i \in I}$ is an fpqc covering, then all descent data for quasi-coherent sheaves with respect to $\mathcal{V}$ are effective.*

Proof. See Descent, Proposition 31.4.2. □

In other words, the fibered category of quasi-coherent sheaves is a stack on the fpqc site. The proof of the theorem is in two steps. The first one is to realize that for Zariski coverings this is easy (or well-known) using standard glueing of sheaves (see Sheaves, Section 6.33) and the locality of quasi-coherence. The second step is the case of an fpqc covering of the form $\{Spec(B) \to Spec(A)\}$ where $A \to B$ is a faithfully flat ring map. This is a lemma in algebra, which we now present.

Descent of modules. If $A \to B$ is a ring map, we consider the complex

$$(B/A)_\bullet : \qquad B \to B \otimes_A B \to B \otimes_A B \otimes_A B \to \cdots$$

where B is in degree 0, $B \otimes_A B$ in degree 1, etc, and the maps are given by

$$\begin{aligned}
b &\mapsto 1 \otimes b - b \otimes 1, \\
b_0 \otimes b_1 &\mapsto 1 \otimes b_0 \otimes b_1 - b_0 \otimes 1 \otimes b_1 + b_0 \otimes b_1 \otimes 1, \\
&\text{etc.}
\end{aligned}$$

Lemma 38.16.3. *If $A \to B$ is faithfully flat, then the complex $(B/A)_\bullet$ is exact in positive degrees, and $H^0((B/A)_\bullet) = A$.*

Proof. See Descent, Lemma 31.3.6. □

Grothendieck proves this in three steps. Firstly, he assumes that the map $A \to B$ has a section, and constructs an explicit homotopy to the complex where A is the only nonzero term, in degree 0. Secondly, he observes that to prove the result, it suffices to do so after a faithfully flat base change $A \to A'$, replacing B with $B' = B \otimes_A A'$. Thirdly, he applies the faithfully flat base change $A \to A' = B$ and remarks that the map $A' = B \to B' = B \otimes_A B$ has a natural section.

The same strategy proves the following lemma.

Lemma 38.16.4. *If $A \to B$ is faithfully flat and M is an A-module, then the complex $(B/A)_\bullet \otimes_A M$ is exact in positive degrees, and $H^0((B/A)_\bullet \otimes_A M) = M$.*

Proof. See Descent, Lemma 31.3.6. □

Definition 38.16.5. Let $A \to B$ be a ring map and N a B-module. A *descent datum* for N with respect to $A \to B$ is an isomorphism $\varphi : N \otimes_A B \cong B \otimes_A N$ of $B \otimes_A B$-modules such that the diagram of $B \otimes_A B \otimes_A B$-modules

$$\begin{tikzcd} N \otimes_A B \otimes_A B \arrow[rr, "\varphi_{02}"] \arrow[dr, "\varphi_{01}"'] & & B \otimes_A N \otimes_A B \arrow[dl, "\varphi_{12}"] \\ & B \otimes_A B \otimes_A N & \end{tikzcd}$$

commutes.

If $N' = B \otimes_A M$ for some A-module M, then it has a canonical descent datum given by the map

$$\begin{array}{cccc} \varphi_{\mathrm{can}} : & N' \otimes_A B & \to & B \otimes_A N' \\ & b_0 \otimes m \otimes b_1 & \mapsto & b_0 \otimes b_1 \otimes m. \end{array}$$

Definition 38.16.6. A descent datum (N, φ) is called *effective* if there exists an A-module M such that $(N, \varphi) \cong (B \otimes_A M, \varphi_{\mathrm{can}})$, with the obvious notion of isomorphism of descent data.

Theorem 38.16.2 is a consequence the following result.

Theorem 38.16.7. *If $A \to B$ is faithfully flat then all descent data with respect to $A \to B$ is effective.*

Proof. See Descent, Proposition 31.3.9. See also Descent, Remark 31.3.11 for an alternative view of the proof. □

Remarks 38.16.8. The results on descent of modules have several applications:

(1) The exactness of the Čech complex in positive degrees for the covering $\{Spec(B) \to Spec(A)\}$ where $A \to B$ is faithfully flat. This will give some vanishing of cohomology.
(2) If (N, φ) is a descent datum with respect to a faithfully flat map $A \to B$, then the corresponding A-module is given by

$$M = \ker \begin{pmatrix} N & \longrightarrow & B \otimes_A N \\ n & \longmapsto & 1 \otimes n - \varphi(n \otimes 1) \end{pmatrix}.$$

See Descent, Proposition 31.3.9.

38.17. Quasi-coherent sheaves

We can apply the descent of modules to study quasi-coherent sheaves.

Proposition 38.17.1. *For any quasi-coherent sheaf $\mathcal{F}$ on S the presheaf*

$$\begin{array}{rccc} \mathcal{F}^a : & Sch/S & \to & Ab \\ & (f : T \to S) & \mapsto & \Gamma(T, f^*\mathcal{F}) \end{array}$$

is an $\mathcal{O}$-module which satisfies the sheaf condition for the fpqc topology.

Proof. This is proved in Descent, Lemma 31.6.1. We indicate the proof here. As established in Lemma 38.15.6, it is enough to check the sheaf property on Zariski coverings and faithfully flat morphisms of affine schemes. The sheaf property for Zariski coverings is standard scheme theory, since $\Gamma(U, i^*\mathcal{F}) = \mathcal{F}(U)$ when $i : U \hookrightarrow S$ is an open immersion.

For $\{Spec(B) \to Spec(A)\}$ with $A \to B$ faithfully flat and $\mathcal{F}|_{Spec(A)} = \widetilde{M}$ this corresponds to the fact that $M = H^0\left((B/A)_\bullet \otimes_A M\right)$, i.e., that

$$0 \to M \to B \otimes_A M \to B \otimes_A B \otimes_A M$$

is exact by Lemma 38.16.4. □

There is an abstract notion of a quasi-coherent sheaf on a ringed site. We briefly introduce this here. For more information please consult Modules on Sites, Section 16.23. Let $\mathcal{C}$ be a category, and let U be an object of $\mathcal{C}$. Then $\mathcal{C}/U$ indicates the category of objects over U, see Categories, Example 4.2.13. If $\mathcal{C}$ is a site, then $\mathcal{C}/U$ is a site as well, namely the coverings of V/U are families $\{V_i/U \to V/U\}$ of morphisms of $\mathcal{C}/U$ with fixed target such that $\{V_i \to V\}$ is a covering of $\mathcal{C}$. Moreover, given any sheaf $\mathcal{F}$ on $\mathcal{C}$ the *restriction* $\mathcal{F}|_{\mathcal{C}/U}$ (defined in the obvious manner) is a sheaf as well. See Sites, Section 9.21 for details.

Definition 38.17.2. Let $\mathcal{C}$ be a *ringed site*, i.e., a site endowed with a sheaf of rings $\mathcal{O}$. A sheaf of $\mathcal{O}$-modules $\mathcal{F}$ on $\mathcal{C}$ is called *quasi-coherent* if for all $U \in Ob(\mathcal{C})$ there exists a covering $\{U_i \to U\}_{i\in I}$ of $\mathcal{C}$ such that the restriction $\mathcal{F}|_{\mathcal{C}/U_i}$ is isomorphic to the cokernel of an $\mathcal{O}$-linear map of free $\mathcal{O}$-modules

$$\bigoplus\nolimits_{k\in K} \mathcal{O}|_{\mathcal{C}/U_i} \longrightarrow \bigoplus\nolimits_{l\in L} \mathcal{O}|_{\mathcal{C}/U_i}.$$

The direct sum over K is the sheaf associated to the presheaf $V \mapsto \bigoplus_{k\in K} \mathcal{O}(V)$ and similary for the other.

Allthough it is useful to be able to give a general definition as above this notion is not well behaved in general.

Remark 38.17.3. In the case where $\mathcal{C}$ has a final object, e.g. S, it suffices to check the condition of the definition for $U = S$ in the above statement. See Modules on Sites, Lemma 16.23.3.

Theorem 38.17.4. *(Meta theorem on quasi-coherent sheaves.) Let S be a scheme. Let $\mathcal{C}$ be a site. Assume that*

(1) *the underlying category $\mathcal{C}$ is a full subcategory of Sch/S,*
(2) *any Zariski covering of $T \in Ob(\mathcal{C})$ can be refined by a covering of $\mathcal{C}$,*
(3) *S/S is an object of $\mathcal{C}$,*
(4) *every covering of $\mathcal{C}$ is an fpqc covering of schemes.*

Then the presheaf $\mathcal{O}$ is a sheaf on $\mathcal{C}$ and any quasi-coherent $\mathcal{O}$-module on $(\mathcal{C}, \mathcal{O})$ is of the form $\mathcal{F}^a$ for some quasi-coherent sheaf $\mathcal{F}$ on S.

Proof. After some formal arguments this is exactly Theorem 38.16.2. Details omitted. In Descent, Proposition 31.6.11 we prove a more precise version of the theorem for the big Zariski, fppf, étale, smooth, and syntomic sites of S, as well as the small Zariski and étale sites of S. □

In other words, there is no difference between quasi-coherent modules on the scheme S and quasi-coherent $\mathcal{O}$-modules on sites $\mathcal{C}$ as in the theorem. More precise statements for the big and small sites $(Sch/S)_{fppf}$, $S_{étale}$, etc can be found in Descent, Section 31.6. In this chapter we will sometimes refer to a ``site as in Theorem 38.17.4" in order to conveniently state results which hold in any of those situations.

38.18. Cech cohomology

Our next goal is to use descent theory to show that $H^i(\mathcal{C}, \mathcal{F}^a) = H^i_{Zar}(S, \mathcal{F})$ for all quasi-coherent sheaves $\mathcal{F}$ on S, and any site $\mathcal{C}$ as in Theorem 38.17.4. To this end, we introduce Čech cohomology on sites. See [**Art62**] and Cohomology on Sites, Sections 19.9, 19.10 and 19.11 for more details.

Definition 38.18.1. Let $\mathcal{C}$ be a category, $\mathcal{U} = \{U_i \to U\}_{i\in I}$ a family of morphisms of $\mathcal{C}$ with fixed target, and $\mathcal{F} \in PAb(\mathcal{C})$ an abelian presheaf. We define the *Cech complex* $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ by

$$\prod_{i_0\in I} \mathcal{F}(U_{i_0}) \to \prod_{i_0,i_1\in I} \mathcal{F}(U_{i_0} \times_U U_{i_1}) \to \prod_{i_0,i_1,i_2\in I} \mathcal{F}(U_{i_0} \times_U U_{i_1} \times_U U_{i_2}) \to \cdots$$

where the first term is in degree 0, and the maps are the usual ones. Again, it is essential to allow the case $i_0 = i_1$ etc. The *Cech cohomology groups* are defined by

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})).$$

Lemma 38.18.2. *The functor $\check{\mathcal{C}}^\bullet(\mathcal{U}, -)$ is exact on the category $PAb(\mathcal{C})$.*

In other words, if $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is a short exact sequence of presheaves of abelian groups, then

$$0 \to \check{\mathcal{C}}^\bullet\left(\mathcal{U}, \mathcal{F}_1\right) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}_2) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}_3) \to 0$$

is a short exact sequence of complexes.

Proof. This follows at once from the definition of a short exact sequence of presheaves. Namely, as the category of abelian presheaves is the category of functors on some category with values in Ab, it is automatically an abelian category: a sequence $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3$ is exact in PAb if and only if for all $U \in Ob(\mathcal{C})$, the sequence $\mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \mathcal{F}_3(U)$ is exact in Ab. So the complex above is merely a product of short exact sequences in each degree. See also Cohomology on Sites, Lemma 19.10.1. □

This shows that $\check{H}^\bullet(\mathcal{U}, -)$ is a δ-functor. We now proceed to show that it is a universal δ-functor. We thus need to show that it is an *effaceable* functor. We start by recalling the Yoneda lemma.

Lemma 38.18.3. *(Yoneda Lemma) For any presheaf $\mathcal{F}$ on a category $\mathcal{C}$ there is a functorial isomophism*

$$Hom_{PSh(\mathcal{C})}(h_U, \mathcal{F}) = \mathcal{F}(U).$$

Proof. See Categories, Lemma 4.3.5. □

Given a set E we denote (in this section) $\mathbf{Z}[E]$ the free abelian group on E. In a formula $\mathbf{Z}[E] = \bigoplus_{e\in E} \mathbf{Z}$, i.e., $\mathbf{Z}[E]$ is a free $\mathbf{Z}$-module having a basis consisting of the elements of E. Using this notation we introduce the free abelian presheaf on a presheaf of sets.

Definition 38.18.4. Let $\mathcal{C}$ be a category. Given a presheaf of sets $\mathcal{G}$, we define the *free abelian presheaf on* $\mathcal{G}$, denoted $\mathbf{Z}_\mathcal{G}$, by the rule

$$\mathbf{Z}_\mathcal{G}(U) = \mathbf{Z}[\mathcal{G}(U)]$$

for $U \in Ob(\mathcal{C})$ with restriction maps induced by the restriction maps of $\mathcal{G}$. In the special case $\mathcal{G} = h_U$ we write simply $\mathbf{Z}_U = \mathbf{Z}_{h_U}$.

The functor $\mathcal{G} \mapsto \mathbf{Z}_\mathcal{G}$ is left adjoint to the forgetful functor $PAb(\mathcal{C}) \to PSh(\mathcal{C})$. Thus, for any presheaf $\mathcal{F}$, there is a canonical isomorphism

$$Hom_{PAb(\mathcal{C})}(\mathbf{Z}_U, \mathcal{F}) = Hom_{PSh(\mathcal{C})}(h_U, \mathcal{F}) = \mathcal{F}(U)$$

the last equality by the Yoneda lemma. In particular, we have the following result.

Lemma 38.18.5. *The Cech complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ can be described explicitly as follows*

$$\begin{aligned}\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) &= \left(\prod_{i_0\in I} Hom_{PAb(\mathcal{C})}(\mathbf{Z}_{U_{i_0}}, \mathcal{F}) \to \prod_{i_0,i_1\in I} Hom_{PAb(\mathcal{C})}(\mathbf{Z}_{U_{i_0}\times_U U_{i_1}}, \mathcal{F}) \to \ldots\right) \\ &= Hom_{PAb(\mathcal{C})}\left(\left(\bigoplus_{i_0\in I}\mathbf{Z}_{U_{i_0}} \leftarrow \bigoplus_{i_0,i_1\in I}\mathbf{Z}_{U_{i_0}\times_U U_{i_1}} \leftarrow \ldots\right), \mathcal{F}\right)\end{aligned}$$

Proof. This follows from the formula above. See Cohomology on Sites, Lemma 19.10.3. $\square$

This reduces us to studying only the complex in the first argument of the last *Hom*.

Lemma 38.18.6. *The complex of abelian presheaves*

$$\mathbf{Z}^\bullet_\mathcal{U} \quad : \quad \bigoplus_{i_0\in I}\mathbf{Z}_{U_{i_0}} \leftarrow \bigoplus_{i_0,i_1\in I}\mathbf{Z}_{U_{i_0}\times_U U_{i_1}} \leftarrow \bigoplus_{i_0,i_1,i_2\in I}\mathbf{Z}_{U_{i_0}\times_U U_{i_1}\times_U U_{i_2}} \leftarrow \ldots$$

is exact in all degrees except 0 *in $PAb(\mathcal{C})$.*

Proof. For any $V \in Ob(\mathcal{C})$ the complex of abelian groups $\mathbf{Z}^\bullet_\mathcal{U}(V)$ is

$$\begin{aligned} &\mathbf{Z}\left[\coprod_{i_0\in I} Mor_\mathcal{C}(V, U_{i_0})\right] \leftarrow \mathbf{Z}\left[\coprod_{i_0,i_1\in I} Mor_\mathcal{C}(V, U_{i_0}\times_U U_{i_1})\right] \leftarrow \ldots = \\ \bigoplus_{\varphi : V\to U}&\left(\mathbf{Z}\left[\coprod_{i_0\in I} Mor_\varphi(V, U_{i_0})\right] \leftarrow \mathbf{Z}\left[\coprod_{i_0,i_1\in I} Mor_\varphi(V, U_{i_0}) \times Mor_\varphi(V, U_{i_1})\right] \leftarrow \ldots\right)\end{aligned}$$

where

$$Mor_\varphi(V, U_i) = \{V \to U_i \text{ such that } V \to U_i \to U \text{ equals } \varphi\}.$$

Set $S_\varphi = \coprod_{i\in I} Mor_\varphi(V, U_i)$, so that

$$\mathbf{Z}^\bullet_\mathcal{U}(V) = \bigoplus_{\varphi : V\to U}\left(\mathbf{Z}[S_\varphi] \leftarrow \mathbf{Z}[S_\varphi\times S_\varphi] \leftarrow \mathbf{Z}[S_\varphi\times S_\varphi\times S_\varphi] \leftarrow \ldots\right).$$

Thus it suffices to show that for each $S = S_\varphi$, the complex

$$\mathbf{Z}[S] \leftarrow \mathbf{Z}[S\times S] \leftarrow \mathbf{Z}[S\times S\times S] \leftarrow \ldots$$

is exact in negative degrees. To see this, we can give an explicit homotopy. Fix $s \in S$ and define $K : n_{(s_0,\ldots,s_p)} \mapsto n_{(s,s_0,\ldots,s_p)}$. One easily checks that K is a nullhomotopy for the operator

$$\delta : \eta_{(s_0,\ldots,s_p)} \mapsto \sum\nolimits_{i=0}^p (-1)^p \eta_{(s_0,\ldots,\hat{s}_i,\ldots,s_p)}.$$

See Cohomology on Sites, Lemma 19.10.4 for more details. $\square$

Lemma 38.18.7. *Let $\mathcal{C}$ be a category. If $\mathcal{I}$ is an injective object of $PAb(\mathcal{C})$ and $\mathcal{U}$ is a family of morphisms with fixed target in $\mathcal{C}$, then $\check{H}^p(\mathcal{U}, \mathcal{I}) = 0$ for all $p > 0$.*

Proof. The Čech complex is the result of applying the functor $Hom_{PAb(\mathcal{C})}(-, \mathcal{I})$ to the complex $\mathbf{Z}^\bullet_{\mathcal{U}}$, i.e.,

$$\check{H}^p(\mathcal{U}; \mathcal{I}) = H^p(Hom_{PAb(\mathcal{C})}(\mathbf{Z}^\bullet_{\mathcal{U}}, \mathcal{I})).$$

But we have just seen that $\mathbf{Z}^\bullet_{\mathcal{U}}$ is exact in negative degrees, and the functor $Hom_{PAb(\mathcal{C})}(-, \mathcal{I})$ is exact, hence $Hom_{PAb(\mathcal{C})}(\mathbf{Z}^\bullet_{\mathcal{U}}, \mathcal{I})$ is exact in positive degrees. ☐

Theorem 38.18.8. *On $PAb(\mathcal{C})$ the functors $\check{H}^p(\mathcal{U}, -)$ are the right derived functors of $\check{H}^0(\mathcal{U}, -)$.*

Proof. By the Lemma 38.18.7, the functors $\check{H}^p(\mathcal{U}, -)$ are universal δ-functors since they are effaceable. So are the right derived functors of $\check{H}^0(\mathcal{U}, -)$. Since they agree in degree 0, they agree by the universal property of universal δ-functors. For more details see Cohomology on Sites, Lemma 19.10.6. ☐

Remark 38.18.9. Observe that all of the preceding statements are about presheaves so we haven't made use of the topology yet.

38.19. The Cech-to-cohomology spectral sequence

This spectral sequence is fundamental in proving foundational results on cohomology of sheaves.

Lemma 38.19.1. *The forgetful functor $Ab(\mathcal{C}) \to PAb(\mathcal{C})$ transforms injectives into injectives.*

Proof. This is formal using the fact that the forgetful functor has a left adjoint, namely sheafification, which is an exact functor. For more details see Cohomology on Sites, Lemma 19.11.1. ☐

Theorem 38.19.2. *Let $\mathcal{C}$ be a site. For any covering $\mathcal{U} = \{U_i \to U\}_{i \in I}$ of $U \in Ob(\mathcal{C})$ and any abelian sheaf $\mathcal{F}$ on $\mathcal{C}$ there is a spectral sequence*

$$E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F})) \Rightarrow H^{p+q}(U, \mathcal{F}),$$

where $\underline{H}^q(\mathcal{F})$ is the abelian presheaf $V \mapsto H^q(V, \mathcal{F})$.

Proof. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in $Ab(\mathcal{C})$, and consider the double complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)$ and the maps

$$\begin{tikzcd} \Gamma(U, I^\bullet) \arrow[r] & \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet) \\ & \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \arrow[u] \end{tikzcd}$$

Here the horizontal map is the natural map $\Gamma(U, I^\bullet) \to \check{\mathcal{C}}^0(\mathcal{U}, \mathcal{I}^\bullet)$ to the left column, and the vertical map is induced by $\mathcal{F} \to \mathcal{I}^0$ and lands in the bottom row. By assumption, $\mathcal{I}^\bullet$ is a complex of injectives in $Ab(\mathcal{C})$, hence by Lemma 38.19.1, it is a complex of injectives in $PAb(\mathcal{C})$. Thus, the rows of the double complex are exact in positive degrees, and the kernel of the horizontal map is equal to $\Gamma(U, \mathcal{I}^\bullet)$, since $\mathcal{I}^\bullet$ is a complex of sheaves. In particular, the cohomology of the total complex is the standard cohomology of the global sections functor $H^0(U, \mathcal{F})$.

For the vertical direction, the qth cohomology group of the pth column is

$$\prod_{i_0,\ldots,i_p} H^q(U_{i_0} \times_U \ldots \times_U U_{i_p}, \mathcal{F}) = \prod_{i_0,\ldots,i_p} \underline{H}^q(\mathcal{F})(U_{i_0} \times_U \ldots \times_U U_{i_p})$$

in the entry $E_1^{p,q}$. So this is a standard double complex spectral sequence, and the E_2-page is as prescribed. For more details see Cohomology on Sites, Lemma 19.11.5. □

Remark 38.19.3. This is a Grothendieck spectral sequence for the composition of functors

$$Ab(\mathcal{C}) \longrightarrow PAb(\mathcal{C}) \xrightarrow{\check{H}^0} Ab.$$

38.20. Big and small sites of schemes

Let S be a scheme. Let τ be one of the topologies we will be discussing. Thus $\tau \in \{fppf, syntomic, smooth, étale, Zariski\}$. Of course if you are only interested in the étale topology, then you can simply assume $\tau = étale$ throughout. Moreover, we will discuss étale morphisms, étale coverings, and étale sites in more detail starting in Section 38.25. In order to proceed with the discussion of cohomology of quasi-coherent sheaves it is convenient to introduce the big τ-site and in case $\tau \in \{étale, Zariski\}$, the small τ-site of S. In order to do this we first introduce the notion of a τ-covering.

Definition 38.20.1. (See Topologies, Definitions 30.7.1, 30.6.1, 30.5.1, 30.4.1, and 30.3.1.) Let $\tau \in \{fppf, syntomic, smooth, étale, Zariski\}$. A family of morphisms of schemes $\{f_i : T_i \to T\}_{i\in I}$ with fixed target is called a *τ-covering* if and only if each f_i is flat of finite presentation, syntomic, smooth, étale, resp. an open immersion, and we have $\bigcup f_i(T_i) = T$.

It turns out that the class of all τ-coverings satisfies the axioms (1), (2) and (3) of Definition 38.10.2 (our definition of a site), see Topologies, Lemmas 30.7.3, 30.6.3, 30.5.3, 30.4.3, and 30.3.2. In order to be able to compare any of these new topologies to the fpqc topology and to use the preceding results on descent on modules we single out a special class of τ-coverings of affine schemes called standard coverings.

Definition 38.20.2. (See Topologies, Definitions 30.7.5, 30.6.5, 30.5.5, 30.4.5, and 30.3.4.) Let $\tau \in \{fppf, syntomic, smooth, étale, Zariski\}$. Let T be an affine scheme. A *standard τ-covering* of T is a family $\{f_j : U_j \to T\}_{j=1,\ldots,m}$ with each U_j is affine, and each f_j flat and of finite presentation, standard syntomic, standard smooth, étale, resp. the immersion of a standard principal open in T and $T = \bigcup f_j(U_j)$.

Lemma 38.20.3. *Let $\tau \in \{fppf, syntomic, smooth, étale, Zariski\}$. Any τ-covering of an affine scheme can be refined by a standard τ-covering.*

Proof. See Topologies, Lemmas 30.7.4, 30.6.4, 30.5.4, 30.4.4, and 30.3.3. □

Finally, we come to our definition of the sites we will be working with. This is actually somewhat involved since, contrary to what happens in [**MA71**], we do not want to choose a universe. Instead we pick a ``partial universe'' (which just means a suitably large set), and consider all schemes contained in this set. Of course we make sure that our favorite base scheme S is contained in the partial universe. Having picked the underlying category we pick a suitably large set of τ-coverings which turns this into a site. The details are in the chapter on topologies on schemes; there is a lot of freedom in the choices made, but in the end the actual choices made will not affect the étale (or other) cohomology of S (just as in [**MA71**] the actual choice of universe doesn't matter at the end). Moreover, the way the material is written the reader who is happy using strongly inaccessible cardinals (i.e., universes) can do so as a substitute.

Definition 38.20.4. Let S be a scheme. Let $\tau \in \{fppf, syntomic, smooth, étale, Zariski\}$.

(1) A *big τ-site of S* is any of the sites $(Sch/S)_\tau$ constructed as explained above and in more detail in Topologies, Definitions 30.7.8, 30.6.8, 30.5.8, 30.4.8, and 30.3.7.
(2) If $\tau \in \{étale, Zariski\}$, then the *small τ-site of S* is the full subcategory S_τ of $(Sch/S)_\tau$ whose objects are schemes T over S whose structure morphism $T \to S$ is étale, resp. an open immersion. A covering in S_τ is a covering $\{U_i \to U\}$ in $(Sch/S)_\tau$ such that U is an object of S_τ.

The underlying category of the site $(Sch/S)_\tau$ has reasonable ``closure'' properties, i.e., given a scheme T in it any locally closed subscheme of T is isomorphic to an object of $(Sch/S)_\tau$. Other such closure properties are: closed under fibre products of schemes, taking countable disjoint unions, taking finite type schemes over a given scheme, given an affine scheme $Spec(R)$ one can complete, localize, or take the quotient of R by an ideal while staying inside the category, etc. On the other hand, for example arbitrary disjoint unions of schemes in $(Sch/S)_\tau$ will take you outside of it. Also note that, given an object T of $(Sch/S)_\tau$ there will exist τ-coverings $\{T_i \to T\}_{i\in I}$ (as in Definition 38.20.1) which are not coverings in $(Sch/S)_\tau$ for example because the schemes T_i are not objects of the category $(Sch/S)_\tau$. But our choice of the sites $(Sch/S)_\tau$ is such that there always does exist a covering $\{U_j \to T\}_{j\in J}$ of $(Sch/S)_\tau$ which refines the covering $\{T_i \to T\}_{i\in I}$, see Topologies, Lemmas 30.7.7, 30.6.7, 30.5.7, 30.4.7, and 30.3.6. We will mostly ignore these issues in this chapter.

If $\mathcal{F}$ is a sheaf on $(Sch/S)_\tau$ or S_τ, then we denote

$$H^p_\tau(U, \mathcal{F}), \text{ in particular } H^p_\tau(S, \mathcal{F})$$

the cohomology groups of $\mathcal{F}$ over the object U of the site, see Section 38.14. Thus we have $H^p_{fppf}(S, \mathcal{F})$, $H^p_{syntomic}(S, \mathcal{F})$, $H^p_{smooth}(S, \mathcal{F})$, $H^p_{étale}(S, \mathcal{F})$[2], and $H^p_{Zar}(S, \mathcal{F})$. The last two are potentially ambiguous since they might refer to either the big or small étale or Zariski site. However, this ambiguity is harmless by the following lemma.

Lemma 38.20.5. *Let $\tau \in \{étale, Zariski\}$. If $\mathcal{F}$ is an abelian sheaf defined on $(Sch/S)_\tau$, then the cohomology groups of $\mathcal{F}$ over S agree with the cohomology groups of $\mathcal{F}|_{S_\tau}$ over S.*

Proof. By Topologies, Lemmas 30.3.13 and 30.4.13 the functors $S_\tau \to (Sch/S)_\tau$ satisfy the hypotheses of Sites, Lemma 9.19.8. Hence our lemma follows from Cohomology on Sites, Lemma 19.8.2. □

For completeness we state and prove the invariance under choice of partial universe of the cohomology groups we are considering. For notation and terminology used in this lemma we refer to Topologies, Section 30.10.

Lemma 38.20.6. *Let $\tau \in \{fppf, syntomic, smooth, étale, Zariski\}$. Let S be a scheme. Let $(Sch/S)_\tau$ and $(Sch'/S)_\tau$ be two big τ-sites of S, and assume that the first is contained in the second. In this case for any abelian sheaf $\mathcal{F}'$ defined on $(Sch'/S)_\tau$ and any object U of $(Sch/S)_\tau$ we have*

$$H^p_\tau(U, \mathcal{F}'|_{(Sch/S)_\tau}) = H^p_\tau(U, \mathcal{F}')$$

In words: the cohomology of $\mathcal{F}'$ over U computed in the bigger site agrees with the cohomology of $\mathcal{F}'$ restricted to the smaller site over U.

[2]We will sometimes abbreviate this to $H^p_{et}(S, \mathcal{F})$.

Proof. By Topologies, Lemma 30.10.2 the inclusion functor $(Sch/S)_\tau \to (Sch'/S)_\tau$ satisfies the assumptions of Sites, Lemma 9.19.8. Hence our lemma follows from Cohomology on Sites, Lemma 19.8.2. □

38.21. The étale topos

A *topos* is the category of sheaves of sets on a site, see Sites, Definition 9.15.1. Hence it is customary to refer to the use the phrase ``étale topos of a scheme'' to refer to the category of sheaves on the small étale site of a scheme. Here is the formal definition.

Definition 38.21.1. Let S be a scheme.

(1) The *étale topos*, or the *small étale topos* of S is the category $Sh(S_{étale})$ of sheaves of sets on the small étale site of S.
(2) The *Zariski topos*, or the *small Zariski topos* of S is the category $Sh(S_{Zar})$ of sheaves of sets on the small Zariski site of S.
(3) For $\tau \in \{fppf, syntomic, smooth, étale, Zariski\}$ a *big τ-topos* is the category of sheaves of set on a big τ-topos of S.

Note that the small Zariski topos of S is simply the category of sheaves of sets on the underlying topological space of S, see Topologies, Lemma 30.3.11. Whereas the small étale topos does not depend on the choices made in the construction of the small étale site, in general the big topoi do depend on those choices.

Here is a lemma, which is one of many possible lemmas expressing the fact that it doesn't matter too much which site we choose to define the small étale topos of a scheme.

Lemma 38.21.2. *Let S be a scheme. Let $S_{affine,étale}$ denote the full subcategory of $S_{étale}$ whose objects are those $U/S \in Ob(S_{étale})$ with U affine. A covering of $S_{affine,étale}$ will be a standard étale covering, see Topologies, Definition 30.4.5. Then restriction*

$$\mathcal{F} \longmapsto \mathcal{F}|_{S_{affine,étale}}$$

defines an equivalence of topoi $Sh(S_{étale}) \cong Sh(S_{affine,étale})$.

Proof. This you can show directly from the definitions, and is a good exercise. But it also follows immediately from Sites, Lemma 9.25.1 by checking that the inclusion functor $S_{affine,étale} \to S_{étale}$ is a special cocontinuous functor (see Sites, Definition 9.25.2). □

38.22. Cohomology of quasi-coherent sheaves

We start with a simple lemma (which holds in greater generality than stated). It says that the Čech complex of a standard covering is equal to the Čech complex of an fpqc covering of the form $\{Spec(B) \to Spec(A)\}$ with $A \to B$ faithfully flat.

Lemma 38.22.1. *Let $\tau \in \{fppf, syntomic, smooth, étale, Zariski\}$. Let S be a scheme. Let $\mathcal{F}$ be an abelian sheaf on $(Sch/S)_\tau$, or on S_τ in case $\tau = étale$, and let $\mathcal{U} = \{U_i \to U\}_{i\in I}$ be a standard τ-covering of this site. Let $V = \coprod_{i\in I} U_i$. Then*

(1) *V is an affine scheme,*
(2) *$\mathcal{V} = \{V \to U\}$ is an fpqc covering.*
(3) *the Cech complexes $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ and $\check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F})$ agree.*

Proof. As the covering is a standard τ-covering each of the schemes U_i is affine and I is a finite set. Hence V is an affine scheme. It is clear that $V \to U$ is flat and surjective, hence $\mathcal{V}$ is an fpqc covering, see Example 38.15.3. Note that $\mathcal{U}$ is a refinement of $\mathcal{V}$ and hence there is a map of Čech complexes $\check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$, see Cohomology on Sites, Equation

(19.9.2.1). Next, we observe that if $T = \coprod_{j\in J} T_j$ is a disjoint union of schemes in the site on which $\mathcal{F}$ is defined then the family of morphisms with fixed target $\{T_j \to T\}_{j\in J}$ is a Zariski covering, and so

$$\mathcal{F}(T) = \mathcal{F}(\coprod\nolimits_{j\in J} T_j) = \prod\nolimits_{j\in J} \mathcal{F}(T_j) \tag{38.22.1.1}$$

by the sheaf condition of $\mathcal{F}$. This implies the map of Čech complexes above is an isomorphism in each degree because

$$V \times_U \ldots \times_U V = \prod\nolimits_{i_0,\ldots i_p} U_{i_0} \times_U \ldots \times_U U_{i_p}$$

as schemes. □

Note that Equality (38.22.1.1) is false for a general presheaf. Even for sheaves it does not hold on any site, since coproducts may not lead to coverings, and may not be disjoint. But it does for all the usual ones (at least all the ones we will study).

Remark 38.22.2. In the statement of Lemma 38.22.1, $\mathcal{U}$ is a refinement of $\mathcal{V}$, so this does not mean that it suffices to look at coverings with a single morphism to compute Čech cohomology $\check{H}^n(U, \mathcal{F})$ (which is defined as the colimit over all coverings $\mathcal{U}$ of U of the Cech cohomology groups of $\mathcal{F}$ with respect to $\mathcal{U}$).

Lemma 38.22.3. *(Locality of cohomology) Let $\mathcal{C}$ be a site, $\mathcal{F}$ an abelian sheaf on $\mathcal{C}$, U an object of $\mathcal{C}$, $p > 0$ an integer and $\xi \in H^p(U, \mathcal{F})$. Then there exists a covering $\mathcal{U} = \{U_i \to U\}_{i\in I}$ of U in $\mathcal{C}$ such that $\xi|_{U_i} = 0$ for all $i \in I$.*

Proof. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. Then ξ is represented by a cocycle $\tilde{\xi} \in \mathcal{I}^p(U)$ with $d^p(\tilde{\xi}) = 0$. By assumption, the sequence $\mathcal{I}^{p-1} \to \mathcal{I}^p \to \mathcal{I}^{p+1}$ in exact in $Ab(\mathcal{C})$, which means that there exists a covering $\mathcal{U} = \{U_i \to U\}_{i\in I}$ such that $\tilde{\xi}|_{U_i} = d^{p-1}(\xi_i)$ for some $\xi_i \in \mathcal{I}^{p-1}(U_i)$. Since the cohomology class $\xi|_{U_i}$ is represented by the cocycle $\tilde{\xi}|_{U_i}$ which is a coboundary, it vanishes. For more details see Cohomology on Sites, Lemma 19.8.3. □

Theorem 38.22.4. *Let S be a scheme and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_S$-module. Let $\mathcal{C}$ be either $(Sch/S)_\tau$ for $\tau \in \{fppf, syntomic, smooth, étale, Zariski\}$ or $S_{étale}$. Then*

$$H^p(S, \mathcal{F}) = H^p_\tau(S, \mathcal{F}^a)$$

for all $p \geq 0$ where

1. *the left hand side indicates the usual cohomology of the sheaf $\mathcal{F}$ on the underlying topological space of the scheme S, and*
2. *the right hand side indicates cohomology of the abelian sheaf $\mathcal{F}^a$ (see Proposition 38.17.1) on the site $\mathcal{C}$.*

Remark 38.22.5. Since S is a final object in the category $\mathcal{C}$, the cohomology groups on the right-hand side are merely the right derived functors of the global sections functor. In fact the proof will show that $H^p(U, f^*\mathcal{F}) = H^p_\tau(U, \mathcal{F}^a)$ for any object $f : U \to S$ of the site $\mathcal{C}$.

Proof. We are going to show that $H^p(U, f^*\mathcal{F}) = H^p_\tau(U, \mathcal{F}^a)$ for any object $f : U \to S$ of the site $\mathcal{C}$. The result is true for $p = 0$ by the sheaf property.

Assume that U is affine. Then we want to prove that $H^p_\tau(U, \mathcal{F}^a) = 0$ for all $p > 0$. We use induction on p.

$p = 1$ Pick $\xi \in H^1_\tau(U, \mathcal{F}^a)$. By Lemma 38.22.3, there exists an fpqc covering $\mathcal{U} = \{U_i \to U\}_{i\in I}$ such that $\xi|_{U_i} = 0$ for all $i \in I$. Up to refining $\mathcal{U}$, we may assume that $\mathcal{U}$ is a standard τ-covering. Applying the spectral sequence of Theorem 38.19.2, we see that ξ comes from a cohomology class $\check{\xi} \in \check{H}^1(\mathcal{U}, \mathcal{F}^a)$. Consider the covering $\mathcal{V} = \{\coprod_{i\in I} U_i \to U\}$. By Lemma 38.22.1, $\check{H}^\bullet(\mathcal{U}, \mathcal{F}^a) = \check{H}^\bullet(\mathcal{V}, \mathcal{F}^a)$. On the other hand, since $\mathcal{V}$ is a covering of the form $\{Spec(B) \to Spec(A)\}$ and $f^*\mathcal{F} = \widetilde{M}$ for some A-module M, we see the Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F})$ is none other than the complex $(B/A)_\bullet \otimes_A M$. Now by Lemma 38.16.4, $H^p((B/A)_\bullet \otimes_A M) = 0$ for $p > 0$, hence $\check{\xi} = 0$ and so $\xi = 0$.

$p > 1$ Pick $\xi \in H^p_\tau(U, \mathcal{F}^a)$. By Lemma 38.22.3, there exists an fpqc covering $\mathcal{U} = \{U_i \to U\}_{i\in I}$ such that $\xi|_{U_i} = 0$ for all $i \in I$. Up to refining $\mathcal{U}$, we may assume that $\mathcal{U}$ is a standard τ-covering. We apply the spectral sequence of Theorem 38.19.2. Observe that the intersections $U_{i_0} \times_U \ldots \times_U U_{i_p}$ are affine, so that by induction hypothesis the cohomology groups

$$E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F}^a))$$

vanish for all $0 < q < p$. We see that ξ must come from a $\check{\xi} \in \check{H}^p(\mathcal{U}, \mathcal{F}^a)$. Replacing $\mathcal{U}$ with the covering $\mathcal{V}$ containing only one morphism and using Lemma 38.16.4 again, we see that the Čech cohomology class $\check{\xi}$ must be zero, hence $\xi = 0$.

Next, assume that U is separated. Choose an affine open covering $U = \bigcup_{i\in I} U_i$ of U. The family $\mathcal{U} = \{U_i \to U\}_{i\in I}$ is then an fpqc covering, and all the intersections $U_{i_0} \times_S \cdots \times_S U_{i_p}$ are affine since U is separated. So all rows of the spectral sequence of Theorem 38.19.2 are zero, except the zeroth row. Therefore

$$H^p_\tau(S, \mathcal{F}^a) = \check{H}^p(\mathcal{U}, \mathcal{F}^a) = \check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(S, \mathcal{F})$$

where the last equality results from standard scheme theory, see Coherent, Lemma 25.2.4.

The general case is technical and (to extend the proof as given here) requires a discussion about maps of spectral sequences, so we won't treat it. It follows from Descent, Proposition 31.6.10 (whose proof takes a slightly different approach) combined with Cohomology on Sites, Lemma 19.8.1. ☐

38.23. Examples of sheaves

Let S and τ be as in Section 38.20. We have already seen that any representable presheaf is a sheaf on $(Sch/S)_\tau$ or S_τ, see Lemma 38.15.8 and Remark 38.15.9. Here are some special cases.

Definition 38.23.1. On any of the sites $(Sch/S)_\tau$ or S_τ of Section 38.20.

(1) The sheaf $T \mapsto \Gamma(T, \mathcal{O}_T)$ is denoted $\mathcal{O}_S$, or $\mathbf{G}_a$, or $\mathbf{G}_{a,S}$ if we want to indicate the base scheme.
(2) Similarly, the sheaf $T \mapsto \Gamma(T, \mathcal{O}_T^*)$ is denoted $\mathcal{O}_S^*$, or $\mathbf{G}_m$, or $\mathbf{G}_{m,S}$ if we want to indicate the base scheme.
(3) The *constant sheaf* $\underline{\mathbf{Z}/n\mathbf{Z}}$ on any site is the sheafification of the constant presheaf $U \mapsto \mathbf{Z}/n\mathbf{Z}$.

The first is a sheaf by Theorem 38.17.4 for example. The second is a sub presheaf of the first, which is easily seen to be a sheaf itself. The third is a sheaf by definition. Note that each of these sheaves is representable. The first and second by the schemes $\mathbf{G}_{a,S}$ and $\mathbf{G}_{m,S}$, see Groupoids, Section 35.4. The third by the finite étale group scheme $\mathbf{Z}/n\mathbf{Z}_S$ sometimes denoted $(\mathbf{Z}/n\mathbf{Z})_S$ which is just n copies of S endowed with the obvious group scheme structure over S, see Groupoids, Example 35.5.6 and the following remark.

Remark 38.23.2. Let G be an abstract group. On any of the sites $(Sch/S)_\tau$ or S_τ of Section 38.20 the sheafification $\underline{G}$ of the constant presheaf associated to G in the *Zariski topology* of the site already gives

$$\Gamma(U, \underline{G}) = \{\text{Zariski locally constant maps } U \to G\}$$

This Zariski sheaf is representable by the group scheme G_S according to Groupoids, Example 35.5.6. By Lemma 38.15.8 any representable presheaf satisfies the sheaf condition for the τ-topology as well, and hence we conclude that the Zariski sheafification $\underline{G}$ above is also the τ-sheafification.

Definition 38.23.3. Let S be a scheme. The *structure sheaf* of S is the sheaf of rings $\mathcal{O}_S$ on any of the sites S_{Zar}, $S_{étale}$, or $(Sch/S)_\tau$ discussed above.

If there is some possible confusion as to which site we are working on then we will indicate this by using indices. For example we may use $\mathcal{O}_{S_{étale}}$ to stress the fact that we are working on the small étale site of S.

Remark 38.23.4. In the terminology introduced above a special case of Theorem 38.22.4 is

$$H^p_{fppf}(X, \mathbf{G}_a) = H^p_{étale}(X, \mathbf{G}_a) = H^p_{Zar}(X, \mathbf{G}_a) = H^p(X, \mathcal{O}_X)$$

for all $p \geq 0$. Moreover, we could use the notation $H^p_{fppf}(X, \mathcal{O}_X)$ to indicate the cohomology of the structure sheaf on the big fppf site of X.

38.24. Picard groups

The following theorem is sometimes called ``Hilbert 90''.

Theorem 38.24.1. *For any scheme X we have canonical identifications*

$$\begin{aligned} H^1_{fppf}(X, \mathbf{G}_m) &= H^1_{syntomic}(X, \mathbf{G}_m) \\ &= H^1_{smooth}(X, \mathbf{G}_m) \\ &= H^1_{étale}(X, \mathbf{G}_m) \\ &= H^1_{Zar}(X, \mathbf{G}_m) \\ &= Pic(X) \\ &= H^1(X, \mathcal{O}^*_X) \end{aligned}$$

Proof. Let τ be one of the topologies considered in Section 38.20. By Cohomology on Sites, Lemma 19.7.1 we see that $H^1_\tau(X, \mathbf{G}_m) = H^1_\tau(X, \mathcal{O}^*_\tau) = \text{Pic}(\mathcal{O}_\tau)$ where $\mathcal{O}_\tau$ is the structure sheaf of the site $(Sch/X)_\tau$. Now an invertible $\mathcal{O}_\tau$-module is a quasi-coherent $\mathcal{O}_\tau$-module. By Theorem 38.17.4 or the more precise Descent, Proposition 31.6.11 we see that $\text{Pic}(\mathcal{O}_\tau) = \text{Pic}(X)$. The last equality is proved in the same way. $\square$

38.25. The étale site

At this point we start exploring the étale site of a scheme in more detail. As a first step we discuss a little the notion of an étale morphism.

38.26. Étale morphisms

For more details, see Morphisms, Section 24.35 for the formal definition and Étale Morphisms, Sections 37.11, 37.12, 37.13, 37.14, 37.16, and 37.19 for a survey of interesting properties of étale morphisms.

Recall that an algebra A over an algebraically closed field k is *smooth* if it is of finite type and the module of differentials $\Omega_{A/k}$ is finite locally free of rank equal to the dimension. A scheme X over k is *smooth* over k if it is locally of finite type and each affine open is the spectrum of a smooth k-algebra. If k is not algebraically closed then an A-algebra is said to be a smooth k-algebra if $A \otimes_k \overline{k}$ is a smooth $\overline{k}$-algebra. A ring map $A \to B$ is smooth if it is flat, finitely presented, and for all primes $\mathfrak{p} \subset A$ the fibre ring $\kappa(\mathfrak{p}) \otimes_A B$ is smooth over the residue field $\kappa(\mathfrak{p})$. More generally, a morphism of schemes is *smooth* if it is flat, finitely presented, and the geometric fibers are smooth.

For these facts please see Morphisms, Section 24.33. Using this we may define an étale morphism as follows.

Definition 38.26.1. A morphism of schemes is *étale* if it is smooth of relative dimension 0.

In particular, a morphism of schemes $X \to S$ is étale if it is smooth and $\Omega_{X/S} = 0$.

Proposition 38.26.2. *(Facts on étale morphisms)*

(1) *Let k be a field. A morphism of schemes $U \to Spec(k)$ is étale if and only if $U \cong \coprod_{i\in I} Spec(k_i)$ such that for each $i \in I$ the ring k_i is a field which is a finite separable extension of k.*
(2) *Let $\varphi : U \to S$ be a morphism of schemes. The following conditions are equivalent:*
   (a) *φ is étale,*
   (b) *φ is locally finitely presented, flat, and all its fibres are étale,*
   (c) *φ is flat, unramified and locally of finite presentation.*
(3) *A ring map $A \to B$ is étale if and only if $B \cong A[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ such that $\Delta = \det\left(\frac{\partial f_i}{\partial x_j}\right)$ is invertible in B.*
(4) *The base change of an étale morphism is étale.*
(5) *Compositions of étale morphisms are étale.*
(6) *Fibre products and products of étale morphisms are étale.*
(7) *An étale morphism has relative dimension 0.*
(8) *Let $Y \to X$ be an étale morphism. If X is reduced (respectively regular) then so is Y.*
(9) *Étale morphims are open.*
(10) *If $X \to S$ and $Y \to S$ are étale, then any S-morphism $X \to Y$ is also étale.*

Proof. We have proved these facts (and more) in the preceding chapters. Here is a list of references: (1) Morphisms, Lemma 24.35.7. (2) Morphisms, Lemmas 24.35.8 and 24.35.16. (3) Algebra, Lemma 7.132.2. (4) Morphisms, Lemma 24.35.4. (5) Morphisms, Lemma 24.35.3. (6) Follows formally from (4) and (5). (7) Morphisms, Lemmas 24.35.6 and 24.28.5. (8) See Algebra, Lemmas 7.145.6 and 7.145.5, see also more results of this kind in Étale Morphisms, Section 37.19. (9) See Morphisms, Lemma 24.24.9 and 24.35.12. (10) See Morphisms, Lemma 24.35.18. ☐

Definition 38.26.3. A ring map $A \to B$ is called *standard étale* if $B \cong (A[t]/(f))_g$ with $f, g \in A[t]$, with f monic, and $\mathrm{d}f/\mathrm{d}t$ invertible in B.

It is true that a standard étale ring map is étale. Namely, suppose that $B = (A[t]/(f))_g$ with $f, g \in A[t]$, with f monic, and $\mathrm{d}f/\mathrm{d}t$ invertible in B. Then $A[t]/(f)$ is a finite free A-module of rank equal to the degree of the monic polynomial f. Hence B, as a localization of this free algebra is finitely presented and flat over A. To finish the proof that B is étale it suffices to show that the fibre rings

$$\kappa(\mathfrak{p}) \otimes_A B \cong \kappa(\mathfrak{p}) \otimes_A (A[t]/(f))_g \cong \kappa(\mathfrak{p})[t, 1/\overline{g}]/(\overline{f})$$

are finite products of finite separable field extensions. Here $\overline{f}, \overline{g} \in \kappa(\mathfrak{p})[t]$ are the images of f and g. Let

$$\overline{f} = \overline{f}_1 \dots \overline{f}_a \overline{f}_{a+1}^{e_1} \dots \overline{f}_{a+b}^{e_b}$$

be the factorization of $\overline{f}$ into powers of pairwise distinct irreducible monic factors $\overline{f}_i$ with $e_1, \dots, e_b > 0$. By assumption $\mathrm{d}\overline{f}/\mathrm{d}t$ is invertible in $\kappa(\mathfrak{p})[t, 1/\overline{g}]$. Hence we see that at least all the $\overline{f}_i$, $i > a$ are invertible. We conclude that

$$\kappa(\mathfrak{p})[t, 1/\overline{g}]/(\overline{f}) \cong \prod\nolimits_{i \in I} \kappa(\mathfrak{p})[t]/(\overline{f}_i)$$

where $I \subset \{1, \dots, a\}$ is the subset of indices i such that $\overline{f}_i$ does not divide $\overline{g}$. Moreover, the image of $\mathrm{d}\overline{f}/\mathrm{d}t$ in the factor $\kappa(\mathfrak{p})[t]/(\overline{f}_i)$ is clearly equal to a unit times $\mathrm{d}\overline{f}_i/\mathrm{d}t$. Hence we conclude that $\kappa_i = \kappa(\mathfrak{p})[t]/(\overline{f}_i)$ is a finite field extension of $\kappa(\mathfrak{p})$ generated by one element whose minimal polynomial is separable, i.e., the field extension $\kappa(\mathfrak{p}) \subset \kappa_i$ is finite separable as desired.

It turns out that any étale ring map is locally standard étale. To formulate this we introduce the following notation. A ring map $A \to B$ is *étale at a prime* $\mathfrak{q}$ of B if there exists $h \in B$, $h \notin \mathfrak{q}$ such that $A \to B_h$ is étale. Here is the result.

Theorem 38.26.4. *A ring map $A \to B$ is étale at a prime $\mathfrak{q}$ if and only if there exists $g \in B$, $g \notin \mathfrak{q}$ such that B_g is standard étale over A.*

Proof. See Algebra, Proposition 7.132.16. □

38.27. Étale coverings

We recall the definition.

Definition 38.27.1. An *étale covering* of a scheme U is a family of morphisms of schemes $\{\varphi_i : U_i \to U\}_{i \in I}$ such that

(1) each φ_i is an étale morphism ;
(2) the U_i cover U, i.e., $U = \bigcup_{i \in I} \varphi_i(U_i)$.

Lemma 38.27.2. *Any étale covering is an fpqc covering.*

Proof. (See also Topologies, Lemma 30.8.6.) Let $\{\varphi_i : U_i \to U\}_{i \in I}$ be an étale covering. Since an étale morphism is flat, and the elements of the covering should cover its target, the property fp (faithfully flat) is satisfied. To check the property qc (quasi-compact), let $V \subset U$ be an affine open, and write $\varphi_i^{-1} = \bigcup_{j \in J_i} V_{ij}$ for some affine opens $V_{ij} \subset U_i$. Since φ_i is open (as étale morphisms are open), we see that $V = \bigcup_{i \in I} \bigcup_{j \in J_i} \varphi_i(V_{ij})$ is an open covering of U. Further, since V is quasi-compact, this covering has a finite refinement. □

So any statement which is true for fpqc coverings remains true *a fortiori* for étale coverings. For instance, the étale site is subcanonical.

Definition 38.27.3. (For more details see Section 38.20, or Topologies, Section 30.4.) Let S be a scheme. The *big étale site over S* is the site $(Sch/S)_{étale}$, see Definition 38.20.4. The *small étale site over S* is the site $S_{étale}$, see Definition 38.20.4. We define similarly the *big* and *small Zariski sites* on S, denoted $(Sch/S)_{Zar}$ and S_{Zar}.

Loosely speaking the big étale site of S is made up out of schemes over S and coverings the étale coverings. The small étale site of S is made up out of schemes étale over S with coverings the étale coverings. Actually any morphism between objects of $S_{étale}$ is étale, in virtue of Proposition 38.26.2, hence to check that $\{U_i \to U\}_{i\in I}$ in $S_{étale}$ is a covering it suffices to check that $\coprod U_i \to U$ is surjective.

The small étale site has fewer objects than the big étale site, it contains only the ``opens" of the étale topology on S. It is a full subcategory of the big étale site, and its topology is induced from the topology on the big site. Hence it is true that the restriction functor from the big étale site to the small one is exact and maps injectives to injectives. This has the following consequence.

Proposition 38.27.4. *Let S be a scheme and $\mathcal{F}$ an abelian sheaf on $(Sch/S)_{étale}$. Then $\mathcal{F}|_{S_{étale}}$ is a sheaf on $S_{étale}$ and*

$$H^p_{étale}(S, \mathcal{F}|_{S_{étale}}) = H^p_{étale}(S, \mathcal{F})$$

for all $p \geq 0$.

Proof. This is a special case of Lemma 38.20.5. $\square$

In accordance with the general notation introduced in Section 38.20 we write $H^p_{étale}(S, \mathcal{F})$ for the above cohomology group.

38.28. Kummer theory

Let $n \in \mathbf{N}$ and consider the functor μ_n defined by

$$\begin{array}{ccc} Sch^{opp} & \longrightarrow & Ab \\ S & \longmapsto & \mu_n(T) = \{t \in \Gamma(S, \mathcal{O}_S^*) \mid t^n = 1\}. \end{array}$$

By Groupoids, Example 35.5.2 this is a representable functor, and the scheme representing it is denoted μ_n also. By Lemma 38.15.8 this functor satisfies the sheaf condition for the fpqc topology (in particular, it is also satisfies the sheaf condition for the étale, Zariski, etc topology).

Lemma 38.28.1. *If $n \in \mathcal{O}_S^*$ then*

$$0 \to \mu_{n,S} \to \mathbf{G}_{m,S} \xrightarrow{(\cdot)^n} \mathbf{G}_{m,S} \to 0$$

is a short exact sequence of sheaves on both the small and big étale site of S.

Remark 38.28.2. This lemma is false when ``étale" is replaced with ``Zariski". Since the étale topology is coarser than the smooth topology, see Topologies, Lemma 30.5.2 it follows that the sequence is also exact in the smooth topology.

Proof. By definition the sheaf $\mu_{n,S}$ is the kernel of the map $(\cdot)^n$. Hence it suffices to show that the last map is surjective. Let U be a scheme over S. Let $f \in \mathbf{G}_m(U) = \Gamma(U, \mathcal{O}_U^*)$. We need to show that we can find an étale cover of U over the members of which the restriction of f is an nth power. Set

$$U' = \underline{Spec}_U(\mathcal{O}_U[T]/(T^n - f)) \xrightarrow{\pi} U.$$

(See Constructions, Section 22.3 or 22.4 for a discussion of the relative spectrum.) Let $Spec(A) \subset U$ be an affine open, and say $f|_U$ corresponds to the unit $a \in A^*$. Then $\pi^{-1}(U) = Spec(B)$ with $B = A[T]/(T^n - a)$. The ring map $A \to B$ is finite free of rank n, hence it is faithfully flat, and hence we conclude that $Spec(B) \to Spec(A)$ is surjective. Since this holds for every affine open in U we conclude that π is surjective. In addition, n and T^{n-1} are invertible in B, so $nT^{n-1} \in B^*$ and the ring map $A \to B$ is standard étale, in particular étale. Since this holds for every affine open of U we conclude that π is étale. Hence $\mathcal{U} = \{\pi : U' \to U\}$ is an étale covering. Moreover, $f|_{U'} = (f')^n$ where f' is the class of T in $\Gamma(U', \mathcal{O}_{U'}^*)$, so $\mathcal{U}$ has the desired property. □

By Theorem 38.24.1 and Lemma 38.28.1 and general properties of cohomology we obtain the long exact cohomology sequence

$$\begin{tikzcd}
0 \arrow[r] & H^0_{étale}(S, \mu_{n,S}) \arrow[r] & \Gamma(S, \mathcal{O}_S^*) \arrow[r, "(\cdot)^n"] & \Gamma(S, \mathcal{O}_S^*) \arrow[dll, out=-90, in=90] \\
 & H^1_{étale}(S, \mu_{n,S}) \arrow[r] & \mathrm{Pic}(S) \arrow[r, "(\cdot)^n"] & \mathrm{Pic}(S) \arrow[dll, out=-90, in=90] \\
 & H^2_{étale}(S, \mu_{n,S}) \arrow[r] & \cdots &
\end{tikzcd}$$

at least if n is invertible on S. When n is not invertible on S we can apply the following lemma.

Lemma 38.28.3. *For any $n \in \mathbf{N}$ the sequence*

$$0 \to \mu_{n,S} \to \mathbf{G}_{m,S} \xrightarrow{(\cdot)^n} \mathbf{G}_{m,S} \to 0$$

is a short exact sequence of sheaves on the site $(Sch/S)_{fppf}$ and $(Sch/S)_{syntomic}$.

Remark 38.28.4. This lemma is false for the smooth, étale, or Zariski topology.

Proof. By definition the sheaf $\mu_{n,S}$ is the kernel of the map $(\cdot)^n$. Hence it suffices to show that the last map is surjective. Since the syntomic topology is stronger than the fppf topology, see Topologies, Lemma 30.7.2, it suffices to prove this for the syntomic topology. Let U be a scheme over S. Let $f \in \mathbf{G}_m(U) = \Gamma(U, \mathcal{O}_U^*)$. We need to show that we can find a synomtic cover of U over the members of which the restriction of f is an nth power. Set

$$U' = \underline{Spec}_U(\mathcal{O}_U[T]/(T^n - f)) \xrightarrow{\pi} U.$$

(See Constructions, Section 22.3 or 22.4 for a discussion of the relative spectrum.) Let $Spec(A) \subset U$ be an affine open, and say $f|_U$ corresponds to the unit $a \in A^*$. Then $\pi^{-1}(U) = Spec(B)$ with $B = A[T]/(T^n - a)$. The ring map $A \to B$ is finite free of rank n, hence it is faithfully flat, and hence we conclude that $Spec(B) \to Spec(A)$ is surjective. Since this holds for every affine open in U we conclude that π is surjective. In addition, B is a global relative complete intersection over A, so the ring map $A \to B$ is standard syntomic, in particular syntomic. Since this holds for every affine open of U we conclude that π is syntomic. Hence $\mathcal{U} = \{\pi : U' \to U\}$ is a syntomic covering. Moreover, $f|_{U'} = (f')^n$ where f' is the class of T in $\Gamma(U', \mathcal{O}_{U'}^*)$, so $\mathcal{U}$ has the desired property. □

By Theorem 38.24.1 and Lemma 38.28.3 and general properties of cohomology we obtain the long exact cohomology sequence

$$\begin{tikzcd} 0 \arrow[r] & H^0_{fppf}(S, \mu_{n,S}) \arrow[r] & \Gamma(S, \mathcal{O}_S^*) \arrow[r, "(\cdot)^n"] & \Gamma(S, \mathcal{O}_S^*) \arrow[dll, out=-90, in=90] \\ & H^1_{fppf}(S, \mu_{n,S}) \arrow[r] & \mathrm{Pic}(S) \arrow[r, "(\cdot)^n"] & \mathrm{Pic}(S) \arrow[dll, out=-90, in=90] \\ & H^2_{fppf}(S, \mu_{n,S}) \arrow[r] & \ldots \end{tikzcd}$$

for any scheme S and any integer n. Of course there is a similar sequence with syntomic cohomology.

Let $n \in \mathbf{N}$ and let S be any scheme. There is another more direct way to describe the first cohomology group with values in μ_n. Consider pairs $(\mathcal{L}, \alpha)$ where $\mathcal{L}$ is an invertible sheaf on S and $\alpha : \mathcal{L}^{\otimes n} \to \mathcal{O}_S$ is a trivialization of the nth tensor power of $\mathcal{L}$. Let $(\mathcal{L}', \alpha')$ be a second such pair. An isomorphism $\varphi : (\mathcal{L}, \alpha) \to (\mathcal{L}', \alpha')$ is an isomorphism $\varphi : \mathcal{L} \to \mathcal{L}'$ of invertible sheaves such that the diagram

$$\begin{tikzcd} \mathcal{L}^{\otimes n} \arrow[r, "\alpha"'] \arrow[d, "\varphi^{\otimes n}"'] & \mathcal{O}_S \arrow[d, "1"] \\ (\mathcal{L}')^{\otimes n} \arrow[r, "\alpha'"] & \mathcal{O}_S \end{tikzcd}$$

commutes. Thus we have

$$Isom_S((\mathcal{L}, \alpha), (\mathcal{L}', \alpha')) = \begin{cases} \emptyset & \text{if} \quad \text{they are not isomorphic} \\ H^0(S, \mu_{n,S}) \cdot \varphi & \text{if} \quad \varphi \text{ isomorphism of pairs} \end{cases} \tag{38.28.4.1}$$

Moreover, given two pairs $(\mathcal{L}, \alpha)$, $(\mathcal{L}', \alpha')$ the tensor product

$$(\mathcal{L}, \alpha) \otimes (\mathcal{L}', \alpha') = (\mathcal{L} \otimes \mathcal{L}', \alpha \otimes \alpha')$$

is another pair. The pair $(\mathcal{O}_S, 1)$ is an identity for this tensor product operation, and an inverse is given by

$$(\mathcal{L}, \alpha)^{-1} = (\mathcal{L}^{\otimes -1}, \alpha^{\otimes -1}).$$

Hence the collection of isomorphism classes of pairs forms an abelian group. Note that

$$(\mathcal{L}, \alpha)^{\otimes n} = (\mathcal{L}^{\otimes n}, \alpha^{\otimes n}) \xrightarrow{\alpha} (\mathcal{O}_S, 1)$$

hence every element of this group has order dividing n. We warn the reader that this group is in general **not** the n-torsion in $\mathrm{Pic}(S)$.

Lemma 38.28.5. *Let S be a scheme. There is a canonical identification*

$$H^1_{étale}(S, \mu_n) = \text{group of pairs } (\mathcal{L}, \alpha) \text{ up to isomorphism as above}$$

if n is invertible on S. In general we have

$$H^1_{fppf}(S, \mu_n) = \text{group of pairs } (\mathcal{L}, \alpha) \text{ up to isomorphism as above.}$$

The same result holds with fppf replaced by syntomic.

Proof. We first prove the second isomorphism. Let $(\mathcal{L}, \alpha)$ be a pair as above. Choose an affine open covering $S = \bigcup U_i$ such that $\mathcal{L}|_{U_i} \cong \mathcal{O}_{U_i}$. Say $s_i \in \mathcal{L}(U_i)$ is a generator. Then $\alpha(s_i^{\otimes n}) = f_i \in \mathcal{O}_S^*(U_i)$. Writing $U_i = Spec(A_i)$ we see there exists a global relative complete intersection $A_i \to B_i = A_i[T]/(T^n - f_i)$ such that f_i maps to an nth power in B_i. In other words, setting $V_i = Spec(B_i)$ we obtain a syntomic covering $\mathcal{V} = \{V_i \to S\}_{i \in I}$ and trivializations $\varphi_i : (\mathcal{L}, \alpha)|_{V_i} \to (\mathcal{O}_{V_i}, 1)$.

We will use this result (the existence of the covering $\mathcal{V}$) to associate to this pair a cohomology class in $H^1_{syntomic}(S, \mu_{n,S})$. We give two (equivalent) constructions.

First construction: using Čech cohomology. Over the double overlaps $V_i \times_S V_j$ we have the isomorphism

$$(\mathcal{O}_{V_i \times_S V_j}, 1) \xrightarrow{\mathrm{pr}_0^* \varphi_i^{-1}} (\mathcal{L}|_{V_i \times_S V_j}, \alpha|_{V_i \times_S V_j}) \xrightarrow{\mathrm{pr}_1^* \varphi_j} (\mathcal{O}_{V_i \times_S V_j}, 1)$$

of pairs. By (38.28.4.1) this is given by an element $\zeta_{ij} \in \mu_n(V_i \times_S V_j)$. We omit the verification that these ζ_{ij}'s give a 1-cocycle, i.e., give an element $(\zeta_{i_0 i_1}) \in \check{C}(\mathcal{V}, \mu_n)$ with $d(\zeta_{i_0 i_1}) = 0$. Thus its class is an element in $\check{H}^1(\mathcal{V}, \mu_n)$ and by Theorem 38.19.2 it maps to a cohomology class in $H^1_{syntomic}(S, \mu_{n,S})$.

Second construction: Using torsors. Consider the presheaf

$$\mu_n(\mathcal{L}, \alpha) : U \longmapsto Isom_U((\mathcal{O}_U, 1), (\mathcal{L}, \alpha)|_U))$$

on $(Sch/S)_{syntomic}$. We may view this as a subpresheaf of $\mathcal{H}om_{\mathcal{O}}(\mathcal{O}, \mathcal{L})$ (internal hom sheaf, see Modules on Sites, Section 16.25). Since the conditions defining this subpresheaf are local, we see that it is a sheaf. By (38.28.4.1) this sheaf has a free action of the sheaf $\mu_{n,S}$. Hence the only thing we have to check is that it locally has sections. This is true because of the existence of the trivializing cover $\mathcal{V}$. Hence $\mu_n(\mathcal{L}, \alpha)$ is a $\mu_{n,S}$-torsor and by Cohomology on Sites, Lemma 19.5.3 we obtain a corresponding element of $H^1_{syntomic}(S, \mu_{n,S})$.

Ok, now we have to still show the following

1. The two constructions give the same cohomology class.
2. Isomorphic pairs give rise to the same cohomology class.
3. The cohomology class of $(\mathcal{L}, \alpha) \otimes (\mathcal{L}', \alpha')$ is the sum of the cohomology classes of $\mathcal{L}, \alpha)$ and $(\mathcal{L}', \alpha')$.
4. If the cohomology class is trivial, then the pair is trivial.
5. Any element of $H^1_{syntomic}(S, \mu_{n,S})$ is the cohomology class of a pair.

We omit the proof of (1). Part (2) is clear from the second construction, since isomorphic torsors give the same cohomology classes. Part (3) is clear from the first construction, since the resulting Cech classes add up. Part (4) is clear from the second construction since a torsor is trivial if and only if it has a global section, see Cohomology on Sites, Lemma 19.5.2.

Part (5) can be seen as follows (allthough a direct proof would be preferable). Suppose $\xi \in H^1_{syntomic}(S, \mu_{n,S})$. Then ξ maps to an element $\overline{\xi} \in H^1_{syntomic}(S, \mathbf{G}_{m,S})$ with $n\overline{\xi} = 0$. By Theorem 38.24.1 we see that $\overline{\xi}$ corresponds to an invertible sheaf $\mathcal{L}$ whose nth tensor power is isomorphic to $\mathcal{O}_S$. Hence there exists a pair $(\mathcal{L}, \alpha')$ whose cohomology class ξ' has the same image $\overline{\xi'}$ in $H^1_{syntomic}(S, \mathbf{G}_{m,S})$. Thus it suffices to show that $\xi - \xi'$ is the class of a pair. By construction, and the long exact cohomology sequence above, we see that $\xi - \xi' = \partial(f)$ for some $f \in H^0(S, \mathcal{O}_S^*)$. Consider the pair $(\mathcal{O}_S, f)$. We omit the verification that the cohomology class of this pair is $\partial(f)$, which finishes the proof of the first identification (with fppf replaced with synomtic).

To see the first, note that if n is invertible on S, then the covering $\mathcal{V}$ constructed in the first part of the proof is actually an étale covering (compare with the proof of Lemma 38.28.1). The rest of the proof is independent of the topology, apart from the very last argument which uses that the Kummer sequence is exact, i.e., uses Lemma 38.28.1. □

38.29. Neighborhoods, stalks and points

We can associate to any geometric point of S a stalk functor which is exact. A map of sheaves on $S_{étale}$ is an isomorphism if and only if it is an isomorphism on all these stalks. A complex of abelian sheaves is exact if and only if the complex of stalks is exact at all geometric points. Alltogether this means that the small étale site of a scheme S has enough points. It also turns out that any point of the small étale topos of S (an abstract notion) is given by a geometric point. Thus in some sense the small étale topos of S can be understood in terms of geometric points and neighbourhoods.

Definition 38.29.1. Let S be a scheme.

(1) A *geometric point* of S is a morphism $Spec(k) \to S$ where k is algebraically closed. Such a point is usually denoted $\overline{s}$, i.e., by an overlined small case letter. We often use $\overline{s}$ to denote the scheme $Spec(k)$ as well as the morphism, and we use $\kappa(\overline{s})$ to denote k.
(2) We say $\overline{s}$ *lies over* s to indicate that $s \in S$ is the image of $\overline{s}$.
(3) An *étale neighborhood* of a geometric point $\overline{s}$ of S is a commutative diagram

$$\begin{tikzcd} & U \arrow[d, "\varphi"] \\ \overline{s} \arrow[ur, "\bar{u}"] \arrow[r, "\overline{s}"'] & S \end{tikzcd}$$

where φ is an étale morphism of schemes. We write $(U, \overline{u}) \to (S, \overline{s})$.
(4) A *morphism of étale neighborhoods* $(U, \overline{u}) \to (U', \overline{u}')$ is an S-morphism $h : U \to U'$ such that $\overline{u}' = h \circ \overline{u}$.

Remark 38.29.2. Since U and U' are étale over S, any S-morphism between them is also étale, see Proposition 38.26.2. In particular all morphisms of étale neighborhoods are étale.

Remark 38.29.3. Let S be a scheme and $s \in S$ a point. In More on Morphisms, Definition 33.25.1 we defined the notion of an étale neighbourhood $(U, u) \to (S, s)$ of (S, s). If $\overline{s}$ is a geometric point of S lying over s, then any étale neighbourhood $(U, \overline{u}) \to (S, \overline{s})$ gives rise to an étale neighbourhood (U, u) of (S, s) by taking $u \in U$ to be the unique point of U such that $\overline{u}$ lies over u. Conversely, given an étale neighbourhood (U, u) of (S, s) the residue field extension $\kappa(s) \subset \kappa(u)$ is finite separable (see Proposition 38.26.2) and hence we can find an embedding $\kappa(u) \subset \kappa(\overline{s})$ over $\kappa(s)$. In other words, we can find a geometric point $\overline{u}$ of U lying over u such that $(U, \overline{u})$ is an étale neighbourhood of $(S, \overline{s})$. We will use these observations to go between the two types of étale neighbourhoods.

Lemma 38.29.4. *Let S be a scheme, and let $\overline{s}$ be a geometric point of S. The category of étale neighborhoods is cofiltered. More precisely:*

(1) *Let $(U_i, \overline{u}_i)_{i=1,2}$ be two étale neighborhoods of $\overline{s}$ in S. Then there exists a third étale neighborhood $(U, \overline{u})$ and morphisms $(U, \overline{u}) \to (U_i, \overline{u}_i)$, $i = 1, 2$.*

(2) *Let $h_1, h_2 : (U, \overline{u}) \to (U', \overline{u}')$ be two morphisms between étale neighborhoods of $\overline{s}$. Then there exist an étale neighborhood $(U'', \overline{u}'')$ and a morphism $h : (U'', \overline{u}'') \to (U, \overline{u})$ which equalizes h_1 and h_2, i.e., such that $h_1 \circ h = h_2 \circ h$.*

Proof. For part (1), consider the fibre product $U = U_1 \times_S U_2$. It is étale over both U_1 and U_2 because étale morphisms are preserved under base change, see Proposition 38.26.2. The map $\overline{s} \to U$ defined by $(\overline{u}_1, \overline{u}_2)$ gives it the structure of an étale neighborhood mapping to both U_1 and U_2. For part (2), define U'' as the fibre product

$$\begin{tikzcd} U'' \arrow[r] \arrow[d] & U \arrow[d, "{(h_1,h_2)}"] \\ U' \arrow[r, "\Delta"] & U' \times_S U'. \end{tikzcd}$$

Since $\overline{u}$ and $\overline{u}'$ agree over S with $\overline{s}$, we see that $\overline{u}'' = (\overline{u}, \overline{u}')$ is a geometric point of U''. In particular $U'' \neq \emptyset$. Moreover, since U' is étale over S, so is the fibre product $U' \times_S U'$ (see Proposition 38.26.2). Hence the vertical arrow (h_1, h_2) is étale by Remark 38.29.2 above. Therefore U'' is étale over U' by base change, and hence also étale over S (because compositions of étale morphisms are étale). Thus $(U'', \overline{u}'')$ is a solution to the problem. $\square$

Lemma 38.29.5. *Let S be a scheme. Let $\overline{s}$ be a geometric point of S. Let $(U, \overline{u})$ an étale neighborhood of $\overline{s}$. Let $\mathcal{U} = \{\varphi_i : U_i \to U\}_{i \in I}$ be an étale covering. Then there exist $i \in I$ and $\overline{u}_i : \overline{s} \to U_i$ such that $\varphi_i : (U_i, \overline{u}_i) \to (U, \overline{u})$ is a morphism of étale neighborhoods.*

Proof. As $U = \bigcup_{i \in I} \varphi_i(U_i)$, the fibre product $\overline{s} \times_{\overline{u}, U, \varphi_i} U_i$ is not empty for some i. Then look at the cartesian diagram

$$\begin{tikzcd} \overline{s} \times_{\overline{u}, U, \varphi_i} U_i \arrow[r, "\mathrm{pr}_2"'] \arrow[d, "\mathrm{pr}_1"] & U_i \arrow[d, "\varphi_i"] \\ Spec(k) = \overline{s} \arrow[r, "\overline{u}"] \arrow[u, bend left, "\sigma"] & U \end{tikzcd}$$

The projection pr_1 is the base change of an étale morphisms so it is étale, see Proposition 38.26.2. Therefore, $\overline{s} \times_{\overline{u}, U, \varphi_i} U_i$ is a disjoint union of finite separable extensions of k, by Proposition 38.26.2. Here $\overline{s} = Spec(k)$. But k is algebraically closed, so all these extensions are trivial, and there exists a section σ of pr_1. The composition $\mathrm{pr}_2 \circ \sigma$ gives a map compatible with $\overline{u}$. $\square$

Definition 38.29.6. Let S be a scheme. Let $\mathcal{F}$ be a presheaf on $S_{étale}$. Let $\overline{s}$ be a geometric point of S. The *stalk* of $\mathcal{F}$ at $\overline{s}$ is

$$\mathcal{F}_{\overline{s}} = colim_{(U, \overline{u})} \mathcal{F}(U)$$

where $(U, \overline{u})$ runs over all étale neighborhoods of $\overline{s}$ in S.

By Lemma 38.29.4, this colimit is over a filtered index category, namely the opposite of the category of étale neighbourhoods. In other words, an element of $\mathcal{F}_{\overline{s}}$ can be thought of as a triple $(U, \overline{u}, \sigma)$ where $\sigma \in \mathcal{F}(U)$. Two triples $(U, \overline{u}, \sigma)$, $(U', \overline{u}', \sigma')$ define the same element of the stalk if there exists a third étale neighbourhood $(U'', \overline{u}'')$ and morphisms of étale neighbourhoods $h : (U'', \overline{u}'') \to (U, \overline{u})$, $h' : (U'', \overline{u}'') \to (U', \overline{u}')$ such that $h^*\sigma = (h')^*\sigma'$ in $\mathcal{F}(U'')$. See Categories, Section 4.17.

Lemma 38.29.7. *Let S be a scheme. Let $\overline{s}$ be a geometric point of S. Consider the functor*

$$\begin{aligned} u : S_{\acute{e}tale} &\longrightarrow Sets, \\ U &\longmapsto |U_{\overline{s}}| = \{\overline{u} \text{ such that } (U, \overline{u}) \text{ is an étale neighbourhood of } \overline{s}\}. \end{aligned}$$

Here $|U_{\overline{s}}|$ denotes the underlying set of the geometric fibre. Then u defines a point p of the site $S_{\acute{e}tale}$ (Sites, Definition 9.28.2) and its associated stalk functor $\mathcal{F} \mapsto \mathcal{F}_p$ (Sites, Equation 9.28.1.1) is the functor $\mathcal{F} \mapsto \mathcal{F}_{\overline{s}}$ defined above.

Proof. In the proof of Lemma 38.29.5 we have seen that the scheme $U_{\overline{s}}$ is a disjoint union of schemes isomorphic to $\overline{s}$. Thus we can also think of $|U_{\overline{s}}|$ as the set of geometric points of U lying over $\overline{s}$, i.e., as the collection of morphisms $\overline{u} : \overline{s} \to U$ fitting into the diagram of Definition 38.29.1. From this it follows that $u(S)$ is a singleton, and that $u(U \times_V W) = u(U) \times_{u(V)} u(W)$ whenever $U \to V$ and $W \to V$ are morphisms in $S_{\acute{e}tale}$. And, given a covering $\{U_i \to U\}_{i \in I}$ in $S_{\acute{e}tale}$ we see that $\coprod u(U_i) \to u(U)$ is surjective by Lemma 38.29.5. Hence Sites, Proposition 9.29.2 applies, so p is a point of the site $S_{\acute{e}tale}$. Finally, the our functor $\mathcal{F} \mapsto \mathcal{F}_{\overline{s}}$ is given by exactly the same colimit as the functor $\mathcal{F} \mapsto \mathcal{F}_p$ associated to p in Sites, Equation 9.28.1.1 which proves the final assertion. $\square$

Remark 38.29.8. Let S be a scheme and let $\overline{s} : Spec(k) \to S$ and $\overline{s}' : Spec(k') \to S$ be two geometric points of S. A *morphism $a : \overline{s} \to \overline{s}'$ of geometric points* is simply a morphism $a : Spec(k) \to Spec(k')$ such that $a \circ \overline{s}' = \overline{s}$. Given such a morphism we obtain a functor from the category of étale neighbourhoods of $\overline{s}'$ to the category of étale neighbourhoods of $\overline{s}$ by the rule $(U, \overline{u}') \mapsto (U, a \circ \overline{u}')$. Hence we obtain a canonical map

$$\mathcal{F}_{\overline{s}'} = colim_{(U, \overline{u}')} \mathcal{F}(U) \longrightarrow colim_{(U, \overline{u})} \mathcal{F}(U) = \mathcal{F}_{\overline{s}}$$

from Categories, Lemma 4.13.7. Using the description of elements of stalks as triples this maps the element of $\mathcal{F}_{\overline{s}'}$ represented by the triple $(U, \overline{u}', \sigma)$ to the element of $\mathcal{F}_{\overline{s}}$ represented by the triple $(U, a \circ \overline{u}', \sigma)$. Since the functor above is clearly an equivalence we conclude that this canonical map is an isomorphism of stalk functors.

Let us make sure we have the map of stalks corresponding to a pointing in the correct direction. Note that the above means, according to Sites, Definition 9.33.2, that a defines a morphism $a : p \to p'$ between the points p, p' of the site $S_{\acute{e}tale}$ associated to $\overline{s}, \overline{s}'$ by Lemma 38.29.7. There are more general morphisms of points (corresponding to specializations of points of S) which we will describe later, and which will not be isomorphisms (insert future reference here).

Lemma 38.29.9. *Let S be a scheme. Let $\overline{s}$ be a geometric point of S.*

(1) *The stalk functor $PAb(S_{\acute{e}tale}) \to Ab$, $\mathcal{F} \mapsto \mathcal{F}_{\overline{s}}$ is exact.*
(2) *We have $(\mathcal{F}^{\#})_{\overline{s}} = \mathcal{F}_{\overline{s}}$ for any presheaf of sets $\mathcal{F}$ on $S_{\acute{e}tale}$.*
(3) *The functor $Ab(S_{\acute{e}tale}) \to Ab$, $\mathcal{F} \mapsto \mathcal{F}_{\overline{s}}$ is exact.*
(4) *Similarly the functors $PSh(S_{\acute{e}tale}) \to Sets$ and $Sh(S_{\acute{e}tale}) \to Sets$ given by the stalk functor $\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$ are exact (see Categories, Definition 4.21.1) and commute with arbitrary colimits.*

Proof. Before we indicate how to prove this by direct arguments we note that the result follows from the general material in Modules on Sites, Section 16.30. This is true because $\mathcal{F} \mapsto \mathcal{F}_{\overline{s}}$ comes from a point of the small étale site of S, see Lemma 38.29.7. We will only give a direct proof of (1), (2) and (3), and omit a dirext proof of (4).

Exactness as a functor on $PAb(S_{étale})$ is formal from the fact that directed colimits commute with all colimits and with finite limits. The identification of the stalks in (2) is via the map

$$\kappa : \mathcal{F}_{\overline{s}} \longrightarrow (\mathcal{F}^\#)_{\overline{s}}$$

induced by the natural morphism $\mathcal{F} \to \mathcal{F}^\#$, see Theorem 38.13.2. We claim that this map is an isomorphism of abelian groups. We will show injectivity and omit the proof of surjectivity.

Let $\sigma \in \mathcal{F}_{\overline{s}}$. There exists an étale neighborhood $(U, \overline{u}) \to (S, \overline{s})$ such that σ is the image of some section $s \in \mathcal{F}(U)$. If $\kappa(\sigma) = 0$ in $(\mathcal{F}^\#)_{\overline{s}}$ then there exists a morphism of étale neighborhoods $(U', \overline{u}') \to (U, \overline{u})$ such that $s|_{U'}$ is zero in $\mathcal{F}^\#(U')$. It follows there exists an étale covering $\{U'_i \to U'\}_{i\in I}$ such that $s|_{U'_i} = 0$ in $\mathcal{F}(U'_i)$ for all i. By Lemma 38.29.5 there exist $i \in I$ and a morphism $\overline{u}'_i : \overline{s} \to U'_i$ such that $(U'_i, \overline{u}'_i) \to (U', \overline{u}') \to (U, \overline{u})$ are morphisms of étale neighborhoods. Hence $\sigma = 0$ since $(U'_i, \overline{u}'_i) \to (U, \overline{u})$ is a morphism of étale neighbourhoods such that we have $s|_{U'_i} = 0$. This proves κ is injective.

To show that the functor $Ab(S_{étale}) \to Ab$ is exact, consider any short exact sequence in $Ab(S_{étale})$: $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$. This gives us the exact sequence of presheaves

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to \mathcal{H}/^p\mathcal{G} \to 0,$$

where $/^p$ denotes the quotient in $PAb(S_{étale})$. Taking stalks at $\overline{s}$, we see that $(\mathcal{H}/^p\mathcal{G})_{\overline{s}} = (\mathcal{H}/\mathcal{G})_{\overline{s}} = 0$, since the sheafification of $\mathcal{H}/^p\mathcal{G}$ is 0. Therefore,

$$0 \to \mathcal{F}_{\overline{s}} \to \mathcal{G}_{\overline{s}} \to \mathcal{H}_{\overline{s}} \to 0 = (\mathcal{H}/^p\mathcal{G})_{\overline{s}}$$

is exact, since taking stalks is exact as a functor from presheaves. □

Theorem 38.29.10. *Let S be a scheme. A map $a : \mathcal{F} \to \mathcal{G}$ of sheaves of sets is injective (resp. surjective) if and only if the map on stalks $a_{\overline{s}} : \mathcal{F}_{\overline{s}} \to \mathcal{G}_{\overline{s}}$ is injective (resp. surjective) for all geometric points of S. A sequence of abelian sheaves on $S_{étale}$ is exact if and only if it is exact on all stalks at geometric points of S.*

Proof. The necessity of exactness on stalks follows from Lemma 38.29.9. For the converse, it suffices to show that a map of sheaves is surjective (respectively injective) if and only if it is surjective (respectively injective) on all stalks. We prove this in the case of surjectivity, and omit the proof in the case of injectivity.

Let $\alpha : \mathcal{F} \to \mathcal{G}$ be a map of abelian sheaves such that $\mathcal{F}_{\overline{s}} \to \mathcal{G}_{\overline{s}}$ is surjective for all geometric points. Fix $U \in Ob(S_{étale})$ and $s \in \mathcal{G}(U)$. For every $u \in U$ choose some $\overline{u} \to U$ lying over u and an étale neighborhood $(V_u, \overline{v}_u) \to (U, \overline{u})$ such that $s|_{V_u} = \alpha(s_{V_u})$ for some $s_{V_u} \in \mathcal{F}(V_u)$. This is possible since α is surjective on stalks. Then $\{V_u \to U\}_{u\in U}$ is an étale covering on which the restrictions of s are in the image of the map α. Thus, α is surjective, see Sites, Section 9.11. □

Remarks 38.29.11. On points of the geometric sites.

1. In the terminology of Sites, Definition 9.34.1 the proof of Theorem 38.29.10 shows that the small étale site of S has enough points.
2. Suppose $\mathcal{F}$ is a sheaf on the big étale site of S. Let $T \to S$ be an object of the big étale site of S, and let $\overline{t}$ be a geometric point of T. Then we define $\mathcal{F}_{\overline{t}}$ as the stalk of the restriction $\mathcal{F}|_{T_{étale}}$ of $\mathcal{F}$ to the small étale site of T. In other words, we can define the stalk of $\mathcal{F}$ at any geometric point of any scheme $T/S \in Ob((Sch/S)_{étale})$.
3. The big étale site of S also has enough points, by considering all geometric points of all objects of this site, see (2).

The following lemma should be skipped on a first reading.

Lemma 38.29.12. *Let S be a scheme.*

(1) *Let p be a point of the small étale site $S_{étale}$ of S given by a functor $u : S_{étale} \to Sets$. Then there exists a geometric point $\overline{s}$ of S such that p is isomorphic to the point of $S_{étale}$ associated to $\overline{s}$ in Lemma 38.29.7.*
(2) *Let $p : Sh(pt) \to Sh(S_{étale})$ be a point of the small étale topos of S. Then p comes from a geometric point of S, i.e., the stalk functor $\mathcal{F} \mapsto \mathcal{F}_p$ is isomorphic to a stalk functor as defined in Definition 38.29.6.*

Proof. By Sites, Lemma 9.28.7 there is a one to one correspondence between points of the site and points of the associated topos, hence it suffices to prove (1). By Sites, Proposition 9.29.2 the functor u has the following properties: (a) $u(S) = \{*\}$, (b) $u(U \times_V W) = u(U) \times_{u(V)} u(W)$, and (c) if $\{U_i \to U\}$ is an étale covering, then $\coprod u(U_i) \to u(U)$ is surjective. In particular, if $U' \subset U$ is an open subscheme, then $u(U') \subset u(U)$. Moreover, by Sites, Lemma 9.28.7 we can write $u(U) = p^{-1}(h_U^\#)$, in other words $u(U)$ is the stalk of the representable sheaf h_U. If $U = V \amalg W$, then we see that $h_U = (h_V \amalg h_W)^\#$ and we get $u(U) = u(V) \amalg u(W)$ since p^{-1} is exact.

Consider the restriction of u to S_{Zar}. By Sites, Examples 9.29.4 and 9.29.5 there exists a unique point $s \in S$ such that for $S' \subset S$ open we have $u(S') = \{*\}$ if $s \in S'$ and $u(S') = \emptyset$ if $s \not\in S'$. Note that if $\varphi : U \to S$ is an object of $S_{étale}$ then $\varphi(U) \subset S$ is open (see Proposition 38.26.2) and $\{U \to \varphi(U)\}$ is an étale covering. Hence we conclude that $u(U) = \emptyset \Leftrightarrow s \in \varphi(U)$.

Pick a geometric point $\overline{s} : \overline{s} \to S$ lying over s, see Definition 38.29.1 for customary abuse of notation. Suppose that $\varphi : U \to S$ is an object of $S_{étale}$ with U affine. Note that φ is separated, and that the fibre U_s of φ over s is an affine scheme over $Spec(\kappa(s))$ which is the spectrum of a finite product of finite separable extensions k_i of $\kappa(s)$. Hence we may apply Étale Morphisms, Lemma 37.18.2 to get an étale neighbourhood $(V, \overline{v})$ of $(S, \overline{s})$ such that

$$U \times_S V = U_1 \amalg \ldots \amalg U_n \amalg W$$

with $U_i \to V$ an isomorphism and W having no point lying over $\overline{v}$. Thus we conclude that

$$u(U) \times u(V) = u(U \times_S V) = u(U_1) \amalg \ldots \amalg u(U_n) \amalg u(W)$$

and of course also $u(U_i) = u(V)$. After shrinking V a bit we can assume that V has exactly one point lying over s, and hence W has no point lying over s. By the above this then gives $u(W) = \emptyset$. Hence we obtain

$$u(U) \times u(V) = u(U_1) \amalg \ldots \amalg u(U_n) = \coprod\nolimits_{i=1,\ldots,n} u(V)$$

Note that $u(V) \neq \emptyset$ as s is in the image of $V \to S$. In particular, we see that in this situation $u(U)$ is a finite set with n elements.

Consider the limit

$$lim_{(V,\overline{v})} u(V)$$

over the category of étale neighbourhoods $(V, \overline{v})$ of $\overline{s}$. It is clear that we get the same value when taking the limit over the subcategory of $(V, \overline{v})$ with V affine. By the previous paragraph (applied with the roles of V and U switched) we see that in this case $u(V)$ is always a finite nonempty set. Moreover, the limit is cofiltered, see Lemma 38.29.4. Hence by Categories, Section 4.18 the limit is nonempty. Pick an element x from this limit. This means we obtain a $x_{V,\overline{v}} \in u(V)$ for every étale neighbourhood $(V, \overline{v})$ of $(S, \overline{s})$ such that for every morphism of étale neighbourhoods $\varphi : (V', \overline{v}') \to (V, \overline{v})$ we have $u(\varphi)(x_{V',\overline{v}'}) = x_{V,\overline{v}}$.

We will use the choice of x to construct a functorial bijective map

$$c : |U_{\overline{s}}| \longrightarrow u(U)$$

for $U \in Ob(S_{\acute{e}tale})$ which will conclude the proof. See Lemma 38.29.7 and its proof for a description of $|U_{\overline{s}}|$. First we claim that it suffices to construct the map for U affine. We omit the proof of this claim. Assume $U \to S$ in $S_{\acute{e}tale}$ with U affine, and let $\overline{u} : \overline{s} \to U$ be an element of $|U_{\overline{s}}|$. Choose a $(V, \overline{v})$ such that $U \times_S V$ decomposes as in the third paragraph of the proof. Then the pair $(\overline{u}, \overline{v})$ gives a geometric point of $U \times_S V$ lying over $\overline{v}$ and determines one of the components U_i of $U \times_S V$. More precisely, there exists a section $\sigma : V \to U \times_S V$ of the projection pr_U such that $(\overline{u}, \overline{v}) = \sigma \circ \overline{v}$. Set $c(\overline{u}) = u(\text{pr}_U)(u(\sigma)(x_{V,\overline{v}})) \in u(U)$. We have to check this is independent of the choice of $(V, \overline{v})$. By Lemma 38.29.4 the category of étale neighbourhoods is cofiltered. Hence it suffice to show that given a morphism of étale neighbourhood $\varphi : (V', \overline{v}') \to (V, \overline{v})$ and a choice of a section $\sigma' : V' \to U \times_S V'$ of the projection such that $(\overline{u}, \overline{v'}) = \sigma' \circ \overline{v}'$ we have $u(\sigma')(x_{V',\overline{v}'}) = u(\sigma)(x_{V,\overline{v}})$. Consider the diagram

$$\begin{tikzcd} V' \arrow[r, "\varphi"'] \arrow[d, "\sigma'"] & V \arrow[d, "\sigma"] \\ U \times_S V' \arrow[r, "1\times\varphi"] & U \times_S V \end{tikzcd}$$

Now, it may not be the case that this diagram commutes. The reason is that the schemes V' and V may not be connected, and hence the decompositions used to construct σ' and σ above may not be unique. But we do know that $\sigma \circ \varphi \circ \overline{v}' = (1 \times \varphi) \circ \sigma' \circ \overline{v}'$ by construction. Hence, since $U \times_S V$ is étale over S, there exists an open neighbourhood $V'' \subset V'$ of $\overline{v'}$ such that the diagram does commute when restricted to V'', see Morphisms, Lemma 24.34.17. This means we may extend the diagram above to

$$\begin{tikzcd} V'' \arrow[r] \arrow[d, "\sigma'|_{V''}"] & V' \arrow[r, "\varphi"'] \arrow[d, "\sigma'"] & V \arrow[d, "\sigma"] \\ U \times_S V'' \arrow[r] & U \times_S V' \arrow[r, "1\times\varphi"] & U \times_S V \end{tikzcd}$$

such that the left square and the outer rectangle commute. Since u is a functor this implies that $x_{V'',\overline{v}'}$ maps to the same element in $u(U \times_S V)$ no matter which route we take through the diagram. On the other hand, it maps to the elements $x_{V',\overline{v}'}$ and $x_{V,\overline{v}}$ in $u(V')$ and $u(V)$. This implies the desired equality $u(\sigma')(x_{V',\overline{v}'}) = u(\sigma)(x_{V,\overline{v}})$.

In a similar manner one proves that the construction $c : |U_{\overline{s}}| \to u(U)$ is functorial in U; details omitted. And finally, by the results of the third paragraph it is clear that the map c is bijective which ends the proof of the lemma. □

38.30. Points in other topologies

In this section we briefly discuss the existence of points for some sites other than the étale site of a scheme. We refer to Sites, Section 9.34 and Topologies, Section 30.2 ff for the terminology used in this section. All of the geometric sites have enough points.

Lemma 38.30.1. *Let S be a scheme. All of the following sites have enough points S_{Zar}, $S_{étale}$, $(Sch/S)_{Zar}$, $(Aff/S)_{Zar}$, $(Sch/S)_{étale}$, $(Aff/S)_{étale}$, $(Sch/S)_{smooth}$, $(Aff/S)_{smooth}$, $(Sch/S)_{syntomic}$, $(Aff/S)_{syntomic}$, $(Sch/S)_{fppf}$, and $(Aff/S)_{fppf}$.*

Proof. For each of the big sites the associated topos is equivalent to the topos defined by the site $(Aff/S)_\tau$, see Topologies, Lemmas 30.3.10, 30.4.11, 30.5.9, 30.6.9, and 30.7.11. The result for the sites $(Aff/S)_\tau$ follows immediately from Deligne's result Sites, Proposition 9.35.3.

The result for S_{Zar} is clear. The result for $S_{étale}$ either follows from (the proof of) Theorem 38.29.10 or from Lemma 38.21.2 and Deligne's result applied to $S_{affine,étale}$. □

The lemma above guarantees the existence of points, but it doesn't tell us what these points look like. We can explicitly construct *some* points as follows. Suppose $\overline{s} : Spec(k) \to S$ is a geometric point with k algebraically closed. Consider the functor

$$u : (Sch/S)_{fppf} \longrightarrow Sets, \quad u(U) = U(k) = Mor_S(Spec(k), U).$$

Note that $U \mapsto U(k)$ commutes with direct limits as $S(k) = \{\overline{s}\}$ and $(U_1 \times_U U_2)(k) = U_1(k) \times_{U(k)} U_2(k)$. Moreover, if $\{U_i \to U\}$ is an fppf covering, then $\coprod U_i(k) \to U(k)$ is surjective. By Sites, Proposition 9.29.2 we see that u defines a point p of $(Sch/S)_{fppf}$ with stalks

$$\mathcal{F}_p = colim_{(U,x)} \mathcal{F}(U)$$

where the colimit is over pairs $U \to S$, $x \in U(k)$ as usual. But... this category has an initial object, namely $(Spec(k), \text{id})$, hence we see that

$$\mathcal{F}_p = \mathcal{F}(Spec(k))$$

which isn't terribly interesting! In fact, in general these points won't form a conservative family of points. A more intereseting type of point is described in the following remark.

Remark 38.30.2. Let $S = Spec(A)$ be an affine scheme. Let (p, u) be a point of the site $(Aff/S)_{fppf}$. Let $B = \mathcal{O}_p$ be the stalk of the structure sheaf at the point p. Since $A = \Gamma(S, \mathcal{O})$ we see that B is an A-algebra. Ignoring set theoretical difficulties, we see that $Spec(B)$ is an object of $(Aff/S)_{fppf}$. Recall that

$$B = colim_{(U,x)} \mathcal{O}(U) = colim_{(Spec(C),x)} C$$

where $x \in u(Spec(C))$. Hence there are canonical maps $C \to B$ and we see that the system has an initial object $(Spec(B), x_B)$ for a suitable $x_B \in u(Spec(B))$. It follows that $\mathcal{F}_p = \mathcal{F}(Spec(B))$ for any sheaf $\mathcal{F}$ on $(Aff/S)_{fppf}$. In other words, every stalk functor is representable. It is straightforward to see that if $\mathcal{F} \mapsto \mathcal{F}(B)$ comes from a point, then B has to be a local A-algebra such that for every faithfully flat, finitely presented ring map $B \to B'$ there is a section $B' \to B$. Conversely, any such A-algebra B gives rise to a point. Details omitted.

38.31. Supports of abelian sheaves

First we talk about supports of local sections.

Lemma 38.31.1. *Let S be a scheme. Let $\mathcal{F}$ be a subsheaf of the final object of the étale topos of S (see Sites, Example 9.10.2). Then there exists a unique open $W \subset S$ such that $\mathcal{F} = h_W$.*

Proof. The condition means that $\mathcal{F}(U)$ is a singleton or empty for all $\varphi : U \to S$ in $Ob(S_{étale})$. In particular local sections always glue. If $\mathcal{F}(U) \neq \emptyset$, then $\mathcal{F}(\varphi(U)) \neq \emptyset$ because $\{\varphi : U \to \varphi(U)\}$ is a covering. Hence we can take $W = \bigcup_{\varphi : U \to S, \mathcal{F}(U) \neq \emptyset} \varphi(U)$. □

Lemma 38.31.2. *Let S be a scheme. Let $\mathcal{F}$ be an abelian sheaf on $S_{étale}$. Let $\sigma \in \mathcal{F}(U)$ be a local section. There exists an open subset $W \subset U$ such that*

(1) *$W \subset U$ is the largest Zariski open subset of U such that $\sigma|_W = 0$,*
(2) *for every $\varphi : V \to U$ in $S_{étale}$ we have*
$$\sigma|_V = 0 \Leftrightarrow \varphi(V) \subset W,$$
(3) *for every geometric point $\overline{u}$ of U we have*
$$(U, \overline{u}, \sigma) = 0 \text{ in } \mathcal{F}_{\overline{s}} \Leftrightarrow \overline{u} \in W$$
where $\overline{s} = (U \to S) \circ \overline{u}$.

Proof. Since $\mathcal{F}$ is a sheaf in the étale topology the restriction of $\mathcal{F}$ to U_{Zar} is a sheaf on U in the Zariski topology. Hence there exists a Zariski open W having property (1), see Modules, Lemma 15.5.2. Let $\varphi : V \to U$ be an arrow of $S_{étale}$. Note that $\varphi(V) \subset U$ is an open subset and that $\{V \to \varphi(V)\}$ is an étale covering. Hence if $\sigma|_V = 0$, then by the sheaf condition for $\mathcal{F}$ we see that $\sigma|_{\varphi(V)} = 0$. This proves (2). To prove (3) we have to show that if $(U, \overline{u}, \sigma)$ defines the zero element of $\mathcal{F}_{\overline{s}}$, then $\overline{u} \in W$. This is true because the assumption means there exists a morphism of étale neighbourhoods $(V, \overline{v}) \to (U, \overline{u})$ such that $\sigma|_V = 0$. Hence by (2) we see that $V \to U$ maps into W, and hence $\overline{u} \in W$. □

Let S be a scheme. Let $s \in S$. Let $\mathcal{F}$ be a sheaf on $S_{étale}$. By Remark 38.29.8 the isomorphism class of the stalk of the sheaf $\mathcal{F}$ at a geometric points lying over s is well defined.

Definition 38.31.3. Let S be a scheme. Let $\mathcal{F}$ be an abelian sheaf on $S_{étale}$.

(1) The *support of* $\mathcal{F}$ is the set of points $s \in S$ such that $\mathcal{F}_{\overline{s}} \neq 0$ for any (some) geometric point $\overline{s}$ lying over s.
(2) Let $\sigma \in \mathcal{F}(U)$ be a section. The *support of* σ is the closed subset $U \setminus W$, where $W \subset U$ is the largest open subset of U on which σ restricts to zero (see Lemma 38.31.2).

In general the support of an abelian sheaf is not closed. For example, suppose that $S = Spec(\mathbf{A}^1_{\mathbf{C}})$. Let $i_t : Spec(\mathbf{C}) \to S$ be the inclusion of the point $t \in \mathbf{C}$. We will see later that $\mathbf{F}_t = i_{t,*}(\mathbf{Z}/2\mathbf{Z})$ is an abelian sheaf whose support is exactly $\{t\}$, see Section 38.46. Then
$$\bigoplus\nolimits_{n \in \mathbf{N}} \mathcal{F}_n$$
is an abelian sheaf with support $\{1, 2, 3, \ldots\} \subset S$. This is true because taking stalks commutes with colimits, see Lemma 38.29.9. Thus an example of an abelian sheaf whose support is not closed. Here are some basic facts on supports of sheaves and sections.

Lemma 38.31.4. *Let S be a scheme. Let $\mathcal{F}$ be an abelian sheaf on $S_{étale}$. Let $U \in Ob(S_{étale})$ and $\sigma \in \mathcal{F}(U)$.*

(1) *The support of σ is closed in U.*
(2) *The support of $\sigma + \sigma'$ is contained in the union of the supports of $\sigma, \sigma' \in \mathcal{F}(U)$.*
(3) *If $\varphi : \mathcal{F} \to \mathcal{G}$ is a map of abelian sheaves on $S_{étale}$, then the support of $\varphi(\sigma)$ is contained in the support of $\sigma \in \mathcal{F}(U)$.*
(4) *The support of $\mathcal{F}$ is the union of the images of the supports of all local sections of $\mathcal{F}$.*

(5) *If $\mathcal{F} \to \mathcal{G}$ is surjective then the support of $\mathcal{G}$ is a subset of the support of $\mathcal{F}$.*
(6) *If $\mathcal{F} \to \mathcal{G}$ is injective then the support of $\mathcal{F}$ is a subset of the support of $\mathcal{G}$.*

Proof. Part (1) holds by definition. Parts (2) and (3) hold because they holds for the restriction of $\mathcal{F}$ and $\mathcal{G}$ to U_{Zar}, see Modules, Lemma 15.5.2. Part (4) is a direct consequence of Lemma 38.31.2 part (3). Parts (5) and (6) follow from the other parts. □

Lemma 38.31.5. *The support of a sheaf of rings on $S_{étale}$ is closed.*

Proof. This is true because (according to our conventions) a ring is 0 if and only if $1 = 0$, and hence the support of a sheaf of rings is the support of the unit section. □

38.32. Henselian rings

We begin by stating a theorem which has already been used many times in the stacks project. There are many versions of this resut; here we just state the algebraic version.

Theorem 38.32.1. *Let $A \to B$ be finite type ring map and $\mathfrak{p} \subset A$ a prime ideal. Then there exist an étale ring map $A \to A'$ and a prime $\mathfrak{p}' \subset A'$ lying over $\mathfrak{p}$ such that*

(1) $\kappa(\mathfrak{p}) = \kappa(\mathfrak{p}')$,
(2) $B \otimes_A A' = B_1 \times \cdots \times B_r \times C$,
(3) *$A' \to B_i$ is finite and there exists a unique prime $q_i \subset B_i$ lying over $\mathfrak{p}'$, and*
(4) *all irreducible components of the fibre $Spec(C \otimes_{A'} \kappa(\mathfrak{p}'))$ of C over $\mathfrak{p}'$ have dimension at least 1.*

Proof. See Algebra, Lemma 7.132.22, or see [**GD67**, Théorème 18.12.1]. For a slew of versions in terms of morphisms of schemes, see More on Morphisms, Section 33.28. □

Recall Hensel's lemma. There are many versions of this lemma. Here are two:

(f) if $f \in \mathbf{Z}_p[T]$ monic and f mod $p = g_0 h_0$ with $gcd(g_0, h_0) = 1$ then f factors as $f = gh$ with $\bar{g} = g_0$ and $\bar{h} = h_0$,
(r) if $f \in \mathbf{Z}_p[T]$, monic $a_0 \in \mathbf{F}_p$, $\bar{f}(a_0) = 0$ but $\bar{f}'(a_0) \neq 0$ then there exists $a \in \mathbf{Z}_p$ with $f(a) = 0$ and $\bar{a} = a_0$.

Both versions are true (we will see this later). The first version asks for lifts of factorizations into coprime parts, and the second version asks for lifts of simple roots modulo the maximal ideal. It turns out that requiring these conditions for a general local ring are equivalent, and are equivalent to many other conditions. We use the root lifting property as the definition of a henselian local ring as it is often the easiest one to check.

Definition 38.32.2. (See Algebra, Definition 7.139.1.) A local ring $(R, \mathfrak{m}, \kappa)$ is called *henselian* if for all $f \in R[T]$ monic, for all $a_0 \in \kappa$ such that $\bar{f}(a_0) = 0$ and $\bar{f}'(a_0) \neq 0$, there exists an $a \in R$ such that $f(a) = 0$ and a mod $\mathfrak{m} = a_0$.

A good example of henselian local rings to keep in mind is complete local rings. Recall (Algebra, Definition 7.143.1) that a complete local ring is a local ring $(R, \mathfrak{m})$ such that $R \cong lim_n\, R/\mathfrak{m}^n$, i.e., it is complete and separated for the $\mathfrak{m}$-adic topology.

Theorem 38.32.3. *Complete local rings are henselian.*

Proof. Newton's method. See Algebra, Lemma 7.139.10. □

Theorem 38.32.4. *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. The following are equivalent:*

(1) *R is henselian,*

(2) *for any $f \in R[T]$ and any factorization $\bar{f} = g_0 h_0$ in $\kappa[T]$ with* $\gcd(g_0, h_0) = 1$, *there exists a factorization $f = gh$ in $R[T]$ with $\bar{g} = g_0$ and $\bar{h} = h_0$,*
(3) *any finite R-algebra S is isomorphic to a finite product of finite local rings,*
(4) *any finite type R-algebra A is isomorphic to a product $A \cong A' \times C$ where $A' \cong A_1 \times \ldots \times A_r$ is a product of finite local R-algebras and all the irreducible components of $C \otimes_R \kappa$ have dimension at least 1,*
(5) *if A is an étale R-algebra and $\mathfrak{n}$ is a maximal ideal of A lying over $\mathfrak{m}$ such that $\kappa \cong A/\mathfrak{n}$, then there exists an isomorphism $\varphi : A \cong R \times A'$ such that $\varphi(\mathfrak{n}) = \mathfrak{m} \times A' \subset R \times A'$.*

Proof. This is just a subset of the results from Algebra, Lemma 7.139.3. Note that part (5) above corresponds to part (8) of Algebra, Lemma 7.139.3 but is formulated slightly differently. □

Lemma 38.32.5. *If R is henselian and A is a finite R-algebra, then A is a finite product of henselian local rings.*

Proof. See Algebra, Lemma 7.139.4. □

Definition 38.32.6. A local ring R is called *strictly henselian* if it is henselian and its residue field is separably closed.

Example 38.32.7. In the case $R = \mathbf{C}[[t]]$, the étale R-algebras are finite products of the trivial extension $R \to R$ and the extensions $R \to R[X, X^{-1}]/(X^n - t)$. The latter ones factor through the open $D(t) \subset Spec(R)$, so any étale covering can be refined by the covering $\{\mathrm{id} : Spec(R) \to Spec(R)\}$. We will see below that this is a somewhat general fact on étale coverings of spectra of henselian rings. This will show that higher étale cohomology of the spectrum of a strictly henselian ring is zero.

Theorem 38.32.8. *Let $(R, \mathfrak{m}, \kappa)$ be a local ring and $\kappa \subset \kappa^{sep}$ a separable algebraic closure. There exist canonical flat local ring maps $R \to R^h \to R^{sh}$ where*

(1) *R^h, R^{sh} are filtered colimits of étale R-algebras,*
(2) *R^h is henselian, R^{sh} is strictly henselian,*
(3) *$\mathfrak{m}R^h$ (resp. $\mathfrak{m}R^{sh}$) is the maximal ideal of R^h (resp. R^{sh}), and*
(4) *$\kappa = R^h/\mathfrak{m}R^h$, and $\kappa^{sep} = R^{sh}/\mathfrak{m}R^{sh}$ as extensions of κ.*

Proof. The structure of R^h and R^{sh} is described in Algebra, Lemmas 7.139.12 and 7.139.13. □

The rings constructed in Theorem 38.32.8 are called respectively the *henselization* and the *strict henselization* of the local ring R, see Algebra, Definition 7.139.14. Many of the properties of R are reflected in its (strict) henselization.

Lemma 38.32.9. *Let R be a local ring. The following are equivalent: R is reduced, the henselization R^h of R is reduced, and the strict henselization R^{sh} of R is reduced.*

Proof. The ring maps $R \to R^h \to R^{sh}$ are faithfully flat. Hence one direction of the implications follows from Algebra, Lemma 7.146.2. Conversely, assume R is reduced. Since R^h and R^{sh} are filtered colimits of étale, hence smooth R-algebras, the result follows from Algebra, Lemma 7.145.6. □

Lemma 38.32.10. *Let R be a local ring. The following are equivalent: R is a normal domain, the henselization R^h of R is a normal domain, and the strict henselization R^{sh} of R is a normal domain.*

Proof. A preliminary remark is that a local ring is normal if and only if it is a normal domain (see Algebra, Definition 7.33.10). The ring maps $R \to R^h \to R^{sh}$ are faithfully flat. Hence one direction of the implications follows from Algebra, Lemma 7.146.3. Conversely, assume R is reduced. Since R^h and R^{sh} are filtered colimits of étale, hence smooth R-algebras, the result follows from Algebra, Lemma 7.145.7. □

Lemma 38.32.11. *Let R be a local ring. The following are equivalent: R is Noetherian, the henselization R^h of R is Noetherian, and the strict henselization R^{sh} of R is Noetherian. In this case the map of completions $R^\wedge \to (R^h)^\wedge$ is an isomorphism.*

Proof. Since $R \to R^h \to R^{sh}$ are flat local ring maps, we see that R^h or R^{sh} being Noetherian implies that R is Noetherian, see Algebra, Lemma 7.146.1. In the rest of the proof we assume R is Noetherian.

Denote $\mathfrak m \subset R$ the maximal ideal. We have seen that $\mathfrak m R^h$ is the maximal ideal of R, $R/\mathfrak m = R^h/\mathfrak m R^h$, and $R \to R^h$ is flat, see Theorem 38.32.8. Hence we see that $R^h/\mathfrak m^n R^h \cong R/\mathfrak m^n$ because $R^h/\mathfrak m^n R^h$ is flat hence free (see Algebra, Lemma 7.93.2) of rank 1 over $R/\mathfrak m^n$. This implies that $R^\wedge \to (R^h)^\wedge$ is an isomorphism.

Next we prove that $(R^h)^\wedge$ is flat over R^h. Write $R^h = colim_i R_i$ as a directed colimit of localizations of étale R-algebras. By Algebra, Lemma 7.35.5 it suffices to show that $(R^h)^\wedge$ is flat over each R_i. This holds because $R_i^\wedge = (R^h)^\wedge$ (by first paragraph proof) is flat over R_i (see Algebra, Lemma 7.90.4). It follows from the flatness of $R^h \to (R^h)^\wedge$ and the Noetherianness of $R^\wedge = (R^h)^\wedge$ (see Algebra, Lemma 7.90.10) that R^h is Noetherian, see Algebra, Lemma 7.146.1.

By Algebra, Lemma 7.90.9 the completion $(R^{sh})^\wedge$ is a Noetherian local ring (as we are completing with respect to the finitely generated maximal ideal $\mathfrak m R^{sh}$). Hence if we can show that $(R^{sh})^\wedge$ is flat over R^{sh} then R^{sh} is Noetherian by Algebra, Lemma 7.146.1 again. Using the limit argument of the second paragraph above we see that it suffices to show that $(R^{sh})^\wedge$ is flat over R.

Pick a separable algebraic extension $\kappa \subset \kappa^{sep}$ which is separably algebraically closed. Pick a basis $\{\overline{x}_j\}_{j\in J}$ of κ^{sep} over κ. Let R^{sh} be the strict henselization of R with respect to $\kappa \subset \kappa^{sep}$. Choose $x_j \in R^{sh}$ mapping to $\overline{x}_j$ in in the residue field $\kappa^{sh} = R^{sh}/\mathfrak m R^{sh}$ of the strict henselization. Arguing as in the first paragraph of the proof, $R^{sh}/\mathfrak m^n R^{sh}$ is free on the elements x_j as a module over $R/\mathfrak m^n$, see Algebra, Lemma 7.93.1. It follows that $(R^{sh})^\wedge$ is isomorphic to the $\mathfrak m$-adic completion of $\bigoplus_{j\in J} R$ as an R-module. By More on Algebra, Lemma 12.20.2 we see this is flat over R and we win. □

Lemma 38.32.12. *Given any local ring R we have* $\dim(R) = \dim(R^h) = \dim(R^{sh})$.

Proof. To see this note that $R \to R^{sh}$ is a flat local ring homomorphism (see Algebra, Section 7.139) and hence $\dim(R^{sh}) \geq \dim(R)$ by going down, see Algebra, Section 7.36. For the converse, we write $R^{sh} = colim\, R_i$ as a directed colimit of local rings R_i each of which is a localization of an étale R-algebra. We can do this by the construction of the strict henselization in Algebra, Section 7.139. Now if $\mathfrak q_0 \subset \mathfrak q_1 \subset \ldots \subset \mathfrak q_n$ is a chain of prime ideals in R^{sh}, then for some sufficiently large i the sequence

$$R_i \cap \mathfrak q_0 \subset R_i \cap \mathfrak q_1 \subset \ldots \subset R_i \cap \mathfrak q_n$$

is a chain of primes in R_i. Thus we see that $\dim(R^{sh}) \leq \sup_i \dim(R_i)$. But by the result of Descent, Lemma 31.17.3 we have $\dim(R_i) = \dim(R)$ for each i and we win. □

Lemma 38.32.13. *Given a Noetherian local ring R we have depth(R) = depth(R^h) = depth(R^{sh}).*

Proof. By Lemma 38.32.11 we know that R^h and R^{sh} are Noetherian. Hence the lemma follows from Algebra, Lemma 7.145.1. $\square$

Lemma 38.32.14. *Let R be a Noetherian local ring. The following are equivalent: R is Cohen-Macaulay, the henselization R^h of R is Cohen-Macaulay, and the strict henselization R^{sh} of R is Cohen-Macaulay.*

Proof. By Lemma 38.32.11 we know that R^h and R^{sh} are Noetherian, hence the lemma makes sense. Since we have $\text{depth}(R) = \text{depth}(R^h) = \text{depth}(R^{sh})$ and $\dim(R) = \dim(R^h) = \dim(R^{sh})$ by Lemmas 38.32.13 and 38.32.12 we conclude. $\square$

Lemma 38.32.15. *Let R be a Noetherian local ring. The following are equivalent: R is a regular local ring, the henselization R^h of R is a regular local ring, and the strict henselization R^{sh} of R is a regular local ring.*

Proof. By Lemma 38.32.11 we know that R^h and R^{sh} are Noetherian, hence the lemma makes sense. Let $\mathfrak{m}$ be the maximal ideal of R. Let $x_1, \ldots, x_t \in \mathfrak{m}$ be a minimal system of generators of $\mathfrak{m}$, i.e., such that the images in $\mathfrak{m}/\mathfrak{m}^2$ form a basis over $\kappa = R/\mathfrak{m}$. Because $R \to R^h$ and $R \to R^{sh}$ are faithfully flat, it follows that the images $x_1^h, \ldots, x_t^h$ in R^h, resp. $x_1^{sh}, \ldots, x_t^{sh}$ in R^{sh} are a minimal system of generators for $\mathfrak{m}^h = \mathfrak{m}R^h$, resp. $\mathfrak{m}^{sh} = \mathfrak{m}R^{sh}$. Regularity of R by definition means $t = \dim(R)$ and similarly for R^h and R^{sh}. Hence the lemma follows from the equality of dimensions $\dim(R) = \dim(R^h) = \dim(R^{sh})$ of Lemma 38.32.12 $\square$

38.33. Stalks of the structure sheaf

In this section we identify the stalk of the structure sheaf at a geometric point with the strict henselization of the local ring at the corresponding ``usual" point.

Lemma 38.33.1. *Let S be a scheme. Let $\overline{s}$ be a geometric point of S lying over $s \in S$. Let $\kappa = \kappa(s)$ and let $\kappa \subset \kappa^{sep} \subset \kappa(\overline{s})$ denote the separable algebraic closure of κ in $\kappa(\overline{s})$. Then there is a canonical identification*

$$(\mathcal{O}_{S,s})^{sh} \cong \mathcal{O}_{S,\overline{s}}$$

where the left hand side is the strict henselization of the local ring $\mathcal{O}_{S,s}$ as described in Theorem 38.32.8 and right hand side is the stalk of the structure sheaf $\mathcal{O}_S$ on $S_{étale}$ at the geometric point $\overline{s}$.

Proof. Let $Spec(A) \subset S$ be an affine neighbourhood of s. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to s. With these choices we have canonical isomorphisms $\mathcal{O}_{S,s} = A_{\mathfrak{p}}$ and $\kappa(s) = \kappa(\mathfrak{p})$. Thus we have $\kappa(\mathfrak{p}) \subset \kappa^{sep} \subset \kappa(\overline{s})$. Recall that

$$\mathcal{O}_{S,\overline{s}} = colim_{(U,\overline{u})}\, \mathcal{O}(U)$$

where the limit is over the étale neighbourhoods of $(S, \overline{s})$. A cofinal system is given by those étale neighbourhoods $(U, \overline{u})$ such that U is affine and $U \to S$ factors through $Spec(A)$. In other words, we see that

$$\mathcal{O}_{S,\overline{s}} = colim_{(B,\mathfrak{q},\phi)}\, B$$

where the colimit is over étale A-algebras B endowed with a prime $\mathfrak{q}$ lying over $\mathfrak{p}$ and a $\kappa(\mathfrak{p})$-algebra map $\phi : \kappa(\mathfrak{q}) \to \kappa(\overline{s})$. Note that since $\kappa(\mathfrak{q})$ is finite separable over $\kappa(\mathfrak{p})$ the image of ϕ is contained in κ^{sep}. Via these translations the result of the lemma is equivalent to the result of Algebra, Lemma 7.139.24. ☐

Definition 38.33.2. Let S be a scheme. Let $\overline{s}$ be a geometric point of S lying over the point $s \in S$.

(1) The *étale local ring of S at $\overline{s}$* is the stalk of the structure sheaf $\mathcal{O}_S$ on $S_{étale}$ at $\overline{s}$. We sometimes call this the *strict henselization of $\mathcal{O}_{S,s}$* relative to the geometric point $\overline{s}$. Notation used: $\mathcal{O}_{S,\overline{s}} = \mathcal{O}_{S,s}^{sh}$.
(2) The *henselization of $\mathcal{O}_{S,s}$* is the henselization of the local ring of S at s. See Algebra, Definition 7.139.14, and Theorem 38.32.8. Notation: $\mathcal{O}_{S,s}^h$.
(3) The *strict henselization of S at $\overline{s}$* is the scheme $Spec(\mathcal{O}_{S,s}^{sh})$.
(4) The *henselization of S at s* is the scheme $Spec(\mathcal{O}_{S,s}^h)$.

Lemma 38.33.3. *Let S be a scheme. Let $s \in S$. Then we have*

$$\mathcal{O}_{S,s}^h = colim_{(U,u)} \mathcal{O}(U)$$

where the colimit is over the filtered category of étale neighbourhoods (U, u) of (S, s) such that $\kappa(s) = \kappa(u)$.

Proof. This lemma is a copy of More on Morphisms, Lemma 33.25.5. ☐

Remark 38.33.4. Let S be a scheme. Let $s \in S$. If S is locally noetherian then $\mathcal{O}_{S,s}^h$ is also noetherian and it has the same completion:

$$\widehat{\mathcal{O}_{S,s}} \cong \widehat{\mathcal{O}_{S,s}^h}.$$

In particular, $\mathcal{O}_{S,s} \subset \mathcal{O}_{S,s}^h \subset \widehat{\mathcal{O}_{S,s}}$. The henselization of $\mathcal{O}_{S,s}$ is in general much smaller than its completion and inherits many of its properties. For example, if $\mathcal{O}_{S,s}$ is reduced, then so is $\mathcal{O}_{S,s}^h$, but this is not true for the completion in general. Insert future references here.

Lemma 38.33.5. *Let S be a scheme. The small étale site $S_{étale}$ endowed with its structure sheaf $\mathcal{O}_S$ is a locally ringed site, see Modules on Sites, Definition 16.34.4.*

Proof. This follows because the stalks $\mathcal{O}_{S,s}^{sh} = \mathcal{O}_{S,\overline{s}}$ are local, and because $S_{étale}$ has enough points, see Lemma 38.33.1, Theorem 38.29.10, and Remarks 38.29.11. See Modules on Sites, Lemmas 16.34.2 and 16.34.3 for the fact that this implies the small étale site is locally ringed. ☐

38.34. Functoriality of small étale topos

Sofar we haven't yet discussed the functoriality of the étale site, in other words what happens when given a morphism of schemes. A precise formal discussion can be found in Topologies, Section 30.4. In this and the next sections we discuss this material briefly specifically in the setting of small étale sites.

Let $f : X \to Y$ be a morphism of schemes. We obtain a functor

$$u : Y_{étale} \longrightarrow X_{étale}, \quad V/Y \longmapsto X \times_Y V/X. \tag{38.34.0.1}$$

This functor has the following important properties

(1) $u(\text{fi}) = \text{fi}$,
(2) u preserves fibre products,
(3) if $\{V_j \to V\}$ is a covering in $Y_{étale}$, then $\{u(V_j) \to u(V)\}$ is a covering in $X_{étale}$.

Each of these is easy to check (omitted). As a consequence we obtain what is called a *morphism of sites*

$$f_{small} : X_{étale} \longrightarrow Y_{étale},$$

see Sites, Definition 9.14.1 and Sites, Proposition 9.14.6. It is not necessary to know about the abstract notion in detail in order to work with étale sheaves and étale cohomology. It usually suffices to know that there are functors $f_{small,*}$ (pushforward) and f_{small}^{-1} (pullback) on étale sheaves, and to know some of their simple properties. We will discuss these properties in the next sections, but we will sometimes refer to the more abstract material for proofs since that is often the natural setting to prove them.

38.35. Direct images

Let us define the pushforward of a presheaf.

Definition 38.35.1. Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ a presheaf of sets on $X_{étale}$. The *direct image*, or *pushforward* of $\mathcal{F}$ (under f) is

$$f_*\mathcal{F} : Y_{étale}^{opp} \longrightarrow Sets, \quad (V/Y) \longmapsto \mathcal{F}(X \times_Y V/X).$$

We sometimes write $f_* = f_{small,*}$ to distinguish from other direct image functors (such as usual Zariski pushforward or $f_{big,*}$).

This is a well-defined étale presheaf since the base change of an étale morphism is again étale. A more categorical way of saying this is that $f_*\mathcal{F}$ is the composition of functors $\mathcal{F} \circ u$ where u is as in Equation (38.34.0.1). This makes it clear that the construction is functorial in the presheaf $\mathcal{F}$ and hence we obtain a functor

$$f_* = f_{small,*} : PSh(X_{étale}) \longrightarrow PSh(Y_{étale})$$

Note that if $\mathcal{F}$ is a presheaf of abelian groups, then $f_*\mathcal{F}$ is also a presheaf of abelian groups and we obtain

$$f_* = f_{small,*} : PAb(X_{étale}) \longrightarrow PAb(Y_{étale})$$

as before (i.e., defined by exactly the same rule).

Remark 38.35.2. We claim that the direct image of a sheaf is a sheaf. Namely, if $\{V_j \to V\}$ is an étale covering in $Y_{étale}$ then $\{X \times_Y V_j \to X \times_Y V\}$ is an étale covering in $X_{étale}$. Hence the sheaf condition for $\mathcal{F}$ with respect to $\{X \times_Y V_i \to X \times_Y V\}$ is equivalent to the sheaf condition for $f_*\mathcal{F}$ with respect to $\{V_i \to V\}$. Thus if $\mathcal{F}$ is a sheaf, so is $f_*\mathcal{F}$.

Definition 38.35.3. Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ a sheaf of sets on $X_{étale}$. The *direct image*, or *pushforward* of $\mathcal{F}$ (under f) is

$$f_*\mathcal{F} : Y_{étale}^{opp} \longrightarrow Sets, \quad (V/Y) \longmapsto \mathcal{F}(X \times_Y V/X)$$

which is a sheaf by Remark 38.35.2. We sometimes write $f_* = f_{small,*}$ to distinguish from other direct image functors (such as usual Zariski pushforward or $f_{big,*}$).

The exact same discussion as above applies and we obtain functors

$$f_* = f_{small,*} : Sh(X_{étale}) \longrightarrow Sh(Y_{étale})$$

and

$$f_* = f_{small,*} : Ab(X_{étale}) \longrightarrow Ab(Y_{étale})$$

called *direct image* again.

The functor f_* on abelian sheaves is left exact. (See Homology, Section 10.5 for what it means for a functor between abelian categories to be left exact.) Namely, if $0 \to \mathcal{F}_1 \to$

$\mathcal{F}_2 \to \mathcal{F}_3$ is exact on $X_{étale}$, then for every $U/X \in Ob(X_{étale})$ the sequence of abelian groups $0 \to \mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \mathcal{F}_3(U)$ is exact. Hence for every $V/Y \in Ob(Y_{étale})$ the sequence of abelian groups $0 \to f_*\mathcal{F}_1(V) \to f_*\mathcal{F}_2(V) \to f_*\mathcal{F}_3(V)$ is exact, because this is the previous sequence with $U = X \times_Y V$.

Definition 38.35.4. Let $f : X \to Y$ be a morphism of schemes. The right derived functors $\{R^p f_*\}_{p\geq 1}$ of $f_* : Ab(X_{étale}) \to Ab(Y_{étale})$ are called *higher direct images*.

The higher direct images and their derived category variants are discussed in more detail in (insert future reference here).

38.36. Inverse image

In this section we briefly discuss pullback of sheaves on the small étale sites. The precise construction of this is in Topologies, Section 30.4.

Definition 38.36.1. Let $f : X \to Y$ be a morphism of schemes. The *inverse image*, or *pullback*[3] functors are the functors

$$f^{-1} = f^{-1}_{small} : Sh(Y_{étale}) \longrightarrow Sh(X_{étale})$$

and

$$f^{-1} = f^{-1}_{small} : Ab(Y_{étale}) \longrightarrow Ab(X_{étale})$$

which are left adjoint to $f_* = f_{small,*}$. Thus f^{-1} thus characterized by the fact that

$$Hom_{Sh(X_{étale})}(f^{-1}\mathcal{G}, \mathcal{F}) = Hom_{Sh(Y_{étale})}(\mathcal{G}, f_*\mathcal{F})$$

functorially, for any $\mathcal{F} \in Sh(X_{étale})$ and $\mathcal{G} \in Sh(Y_{étale})$. We similarly have

$$Hom_{Ab(X_{étale})}(f^{-1}\mathcal{G}, \mathcal{F}) = Hom_{Ab(Y_{étale})}(\mathcal{G}, f_*\mathcal{F})$$

for $\mathcal{F} \in Ab(X_{étale})$ and $\mathcal{G} \in Ab(Y_{étale})$.

It is not trivial that such an adjoint exists. On the other hand, it exists in a fairly general setting, see Remark 38.36.3 below. The general machinery shows that $f^{-1}\mathcal{G}$ is the sheaf associated to the presheaf

$$U/X \longmapsto colim_{U\to X\times_Y V}\mathcal{G}(V/Y) \tag{38.36.1.1}$$

where the colimit is over the category of pairs $(V/Y, \varphi : U/X \to X\times_Y V/X)$. To see this apply Sites, Proposition 9.14.6 to the functor u of Equation (38.34.0.1) and use the description of $u_s = (u_p\ )^\#$ in Sites, Sections 9.13 and 9.5. We will occasionally use this formula for the pullback in order to prove some of its basic properties.

Lemma 38.36.2. *Let $f : X \to Y$ be a morphism of schemes.*

(1) *The functor $f^{-1} : Ab(Y_{étale}) \to Ab(X_{étale})$ is exact.*
(2) *The functor $f^{-1} : Sh(Y_{étale}) \to Sh(X_{étale})$ is exact, i.e., it commutes with finite limits and colimits, see Categories, Definition 4.21.1.*
(3) *Let $\overline{x} \to X$ be a geometric point. Let $\mathcal{G}$ be a sheaf on $Y_{étale}$. Then there is a canonical identification*

$$(f^{-1}\mathcal{G})_{\overline{x}} = \mathcal{G}_{\overline{y}}.$$

where $\overline{y} = f \circ \overline{x}$.
(4) *For any $V \to Y$ étale we have $f^{-1}h_V = h_{X\times_Y V}$.*

[3] We use the notation f^{-1} for pullbacks of sheaves of sets or sheaves of abelian groups, and we reserve f^* for pullbacks of sheaves of modules via a morphism of ringed sites/topoi.

Proof. The exactness of f^{-1} on sheaves of sets is a consequence of Sites, Proposition 9.14.6 applied to our functor u of Equation (38.34.0.1). In fact the exactness of pullback is part of the definition of of a morphism of topoi (or sites if you like). Thus we see (2) holds. It implies part (1) since given an abelian sheaf $\mathcal{G}$ on $Y_{étale}$ the underlying sheaf of sets of $f^{-1}\mathcal{F}$ is the same as f^{-1} of the underlying sheaf of sets of $\mathcal{F}$, see Sites, Section 9.38. See also Modules on Sites, Lemma 16.27.2. In the literature (1) and (2) are sometimes deduced from (3) via Theorem 38.29.10.

Part (3) is a general fact about stalks of pullbacks, see Sites, Lemma 9.30.1. We will also prove (3) directly as follows. Note that by Lemma 38.29.9 taking stalks commutes with sheafification. Now recall that $f^{-1}\mathcal{G}$ is the sheaf associated to the presheaf

$$U \longrightarrow colim_{U \to X \times_Y V} \mathcal{G}(V),$$

see Equation (38.36.1.1). Thus we have

$$\begin{aligned}
(f^{-1}\mathcal{G})_{\overline{x}} &= colim_{(U,\overline{u})}\, f^{-1}\mathcal{G}(U) \\
&= colim_{(U,\overline{u})}\, colim_{a : U \to X \times_Y V} \mathcal{G}(V) \\
&= colim_{(V,\overline{v})}\, \mathcal{G}(V) \\
&= \mathcal{G}_{\overline{y}}
\end{aligned}$$

in the third equality the pair $(U, \overline{u})$ and the map $a : U \to X \times_Y V$ corresponds to the pair $(V, a \circ \overline{u})$.

Part (4) can be proved in a similar manner by identfying the colimits which define $f^{-1}h_V$. Or you can use Yoneda's lemma (Categories, Lemma 4.3.5) and the functorial equalities

$$Mor_{Sh(X_{étale})}(f^{-1}h_V, \mathcal{F}) = Mor_{Sh(Y_{étale})}(h_V, f_*\mathcal{F}) = f_*\mathcal{F}(V) = \mathcal{F}(X \times_Y V)$$

combined with the fact that representable presheaves are sheaves. See also Sites, Lemma 9.13.5 for a completely general result. □

The pair of functors (f_*, f^{-1}) define a morphism of small étale topoi

$$f_{small} : Sh(X_{étale}) \longrightarrow Sh(Y_{étale})$$

Many generalities on cohomology of sheaves hold for topoi and morphisms of topoi. We will try to point out when results are general and when they are specific to the étale topos.

Remark 38.36.3. More generally, let $\mathcal{C}_1, \mathcal{C}_2$ be sites, and assume they have final objects and fibre products. Let $u : \mathcal{C}_2 \to \mathcal{C}_1$ be a functor satisfying:

(1) if $\{V_i \to V\}$ is a covering of $\mathcal{C}_2$, then $\{u(V_i) \to V_i\}$ is a covering of $\mathcal{C}_1$ (we say that u is *continuous*), and
(2) u commutes with finite limits (i.e., u is left exact, i.e., u preserves fibre products and final objects).

Then one can define $f_* : Sh(\mathcal{C}_1) \to Sh(\mathcal{C}_2)$ by $f_*\mathcal{F}(V) = \mathcal{F}(u(V))$. Moreover, there exists an exact functor f^{-1} which is left adjoint to f_*, see Sites, Definition 9.14.1 and Proposition 9.14.6. Warning: It is not enough to require simply that u is continuous and commutes with fibre products in order to get a morphism of topoi.

38.37. Functoriality of big topoi

Given a morphism of schemes $f : X \to Y$ there are a whole host of morphisms of topoi associated to f, see Topologies, Section 30.9 for a list. Perhaps the most used ones are the morphisms of topoi

$$f_{big} = f_{big,\tau} : Sh((Sch/X)_\tau) \longrightarrow Sh((Sch/Y)_\tau)$$

where $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. These each correspond to a continuous functor

$$(Sch/Y)_\tau \longrightarrow (Sch/X)_\tau, \quad V/Y \longmapsto X \times_Y V/X$$

which preserves final objects, fibre products and covering, and hence defines a morphism of sites

$$f_{big} : (Sch/X)_\tau \longrightarrow (Sch/Y)_\tau.$$

See Topologies, Sections 30.3, 30.4, 30.5, 30.6, and 30.7. In particular, pushforward along f_{big} is given by the rule

$$(f_{big,*}\mathcal{F})(V/Y) = \mathcal{F}(X \times_Y V/X)$$

It turns out that these morphisms of topoi have an inverse image functor f_{big}^{-1} which is very easy to describe. Namely, we have

$$(f_{big}^{-1}\mathcal{G})(U/X) = \mathcal{G}(U/Y)$$

where the structure morphism of U/Y is the composition of the structure morphism $U \to X$ with f, see Topologies, Lemmas 30.3.15, 30.4.15, 30.5.10, 30.6.10, and 30.7.12.

38.38. Functoriality and sheaves of modules

In this section we are going to reformulate some of the material explained in Descent, Section 31.6 in the setting of étale topologies. Let $f : X \to Y$ be a morphism of schemes. We have seen above, see Sections 38.34, 38.35, and 38.36 that this induces a morphism f_{small} of small étale sites. In Descent, Remark 31.6.4 we have seen that f also induces a natural map

$$f_{small}^\sharp : \mathcal{O}_{Y_{étale}} \longrightarrow f_{small,*}\mathcal{O}_{X_{étale}}$$

of sheaves of rings on $Y_{étale}$ such that $(f_{small}, f_{small}^\sharp)$ is a morphism of ringed sites. See Modules on Sites, Definition 16.6.1 for the definition of a morphism of ringed sites. Let us just recall here that $f_{small}^\sharp$ is defined by the compatible system of maps

$$\mathrm{pr}_V^\sharp : \mathcal{O}(V) \longrightarrow \mathcal{O}(X \times_Y V)$$

for V varying over the objects of $Y_{étale}$.

It is clear that this construction is compatible with compositions of morphisms of schemes. More precisely, if $f : X \to Y$ and $g : Y \to Z$ are morphisms of schemes, then we have

$$(g_{small}, g_{small}^\sharp) \circ (f_{small}, f_{small}^\sharp) = ((g \circ f)_{small}, (g \circ f)_{small}^\sharp)$$

as morphisms of ringed topoi. Moreover, by Modules on Sites, Definition 16.13.1 we see that given a morphism $f : X \to Y$ of schemes we get well defined pullback and direct image functors

$$f_{small}^* : Mod(\mathcal{O}_{Y_{étale}}) \longrightarrow Mod(\mathcal{O}_{X_{étale}}),$$
$$f_{small,*} : Mod(\mathcal{O}_{X_{étale}}) \longrightarrow Mod(\mathcal{O}_{Y_{étale}})$$

which are adjoint in the usual way. If $g : Y \to Z$ is another morphism of schemes, then we have $(g \circ f)^*_{small} = f^*_{small} \circ g^*_{small}$ and $(g \circ f)_{small,*} = g_{small,*} \circ f_{small,*}$ because of what we said about compositions.

There is quite a bit of difference between the category of all $\mathcal{O}_X$ modules on X and the category between all $\mathcal{O}_{X_{étale}}$-modules on $X_{étale}$. But the results of Descent, Section 31.6 tell us that there is not much difference between considering quasi-coherent modules on S and quasi-coherent modules on $S_{étale}$. (We have already seen this in Theorem 38.17.4 for example.) In particular, if $f : X \to Y$ is any morphism of schemes, then the pullback functors f^*_{small} and f^* match for quasi-coherent sheaves, see Descent, Proposition 31.6.14. Moreover, the same is true for pushforward provided f is quasi-compact and quasi-separated, see Descent, Lemma 31.6.15.

A few words about functoriality of the structure sheaf on big sites. Let $f : X \to Y$ be a morphism of schemes. Choose any of the topologies $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Then the morphism $f_{big} : (Sch/X)_\tau \to (Sch/Y)_\tau$ becomes a morphism of ringed sites by a map

$$f^\sharp_{big} : \mathcal{O}_Y \longrightarrow f_{big,*}\mathcal{O}_X$$

see Descent, Remark 31.6.4. In fact it is given by the same construction as in the case of small sites explained above.

38.39. Comparing big and small topoi

Let X be a scheme. In Topologies, Lemma 30.4.13 we have introduced comparison morphisms $\pi_X : (Sch/X)_{étale} \to X_{étale}$ and $i_X : Sh(X_{étale}) \to Sh((Sch/X)_{étale})$ with $\pi_X \circ i_X = \text{id}$ and $\pi_{X,*} = i_X^{-1}$. In Descent, Remark 31.6.4 we have extended these to a morphism of ringed sites

$$\pi_X : ((Sch/X)_{étale}, \mathcal{O}) \to (X_{étale}, \mathcal{O}_X)$$

and a morphism of ringed topoi

$$i_X : (Sh(X_{étale}), \mathcal{O}_X) \to (Sh((Sch/X)_{étale}), \mathcal{O})$$

Note that the restriction $i_X^{-1} = \pi_{X,*}$ (see Topologies, Definition 30.4.14) transforms $\mathcal{O}$ into $\mathcal{O}_X$. Hence $i_X^*\mathcal{F} = i_X^{-1}\mathcal{F}$ for any $\mathcal{O}$-module $\mathcal{F}$ on $(Sch/X)_{étale}$. In particular i_X^* is exact. This functor is often denoted $\mathcal{F} \mapsto \mathcal{F}|_{X_{étale}}$.

Lemma 38.39.1. *Let X be a scheme.*

(1) *$\mathcal{I}|_{X_{étale}}$ is injective in $Ab(X_{étale})$ for $\mathcal{I}$ injective in $Ab((Sch/X)_{étale})$, and*
(2) *$\mathcal{I}|_{X_{étale}}$ is injective in $Mod(X_{étale}, \mathcal{O}_X)$ for $\mathcal{I}$ injective in $Mod((Sch/X)_{étale}, \mathcal{O})$.*

Proof. This follows formally from the fact that the restriction functor $\pi_{X,*} = i_X^{-1}$ is an exact left adjoint of $i_{X,*}$, see Homology, Lemma 10.22.1. □

Let $f : X \to Y$ be a morphism of schemes. The commutative diagram of Topologies, Lemma 30.4.16 (3) leads to a commutative diagram of ringed sites

$$\begin{tikzcd}
(T_{étale}, \mathcal{O}_T) \arrow[d, "f_{small}"'] & ((Sch/T)_{étale}, \mathcal{O}) \arrow[l, "\pi_T"'] \arrow[d, "f_{big}"] \\
(S_{étale}, \mathcal{O}_S) & ((Sch/S)_{étale}, \mathcal{O}) \arrow[l, "\pi_S"']
\end{tikzcd}$$

as one easily sees by writing out the definitions of $f^\sharp_{small}$, $f^\sharp_{big}$, $\pi^\sharp_S$, and $\pi^\sharp_T$. In particular this means that

$$(f_{big,*}\mathcal{F})|_{Y_{\acute{e}tale}} = f_{small,*}(\mathcal{F}|_{X_{\acute{e}tale}}) \tag{38.39.1.1}$$

for any sheaf $\mathcal{F}$ on $(Sch/X)_{\acute{e}tale}$ and if $\mathcal{F}$ is a sheaf of $\mathcal{O}$-modules, then (38.39.1.1) is an isomorphism of $\mathcal{O}_Y$-modules on $Y_{\acute{e}tale}$.

Lemma 38.39.2. *Let $f : X \to Y$ be a morphism of schemes.*

(1) *For any $\mathcal{F} \in Ab((Sch/X)_{\acute{e}tale})$ we have*
$$(Rf_{big,*}\mathcal{F})|_{Y_{\acute{e}tale}} = Rf_{small,*}(\mathcal{F}|_{X_{\acute{e}tale}}).$$
in $D(Y_{\acute{e}tale})$.

(2) *For any object $\mathcal{F}$ of $Mod((Sch/X)_{\acute{e}tale}, \mathcal{O})$ we have*
$$(Rf_{big,*}\mathcal{F})|_{Y_{\acute{e}tale}} = Rf_{small,*}(\mathcal{F}|_{X_{\acute{e}tale}}).$$
in $D(Mod(Y_{\acute{e}tale}, \mathcal{O}_Y))$.

Proof. Follows immediately from Lemma 38.39.1 and (38.39.1.1) on choosing an injective resolution of $\mathcal{F}$. $\square$

38.40. Recovering morphisms

In this section we prove that the rule which associates to a scheme its locally ringed small étale topos is fully faithful in a suitable sense, see Theorem 38.40.5.

Lemma 38.40.1. *Let $f : X \to Y$ be a morphism of schemes. The morphism of ringed sites $(f_{small}, f^\sharp_{small})$ associated to f is a morphism of locally ringed sites, see Modules on Sites, Definition 16.34.8.*

Proof. Note that the assertion makes sense since we have seen that $(X_{\acute{e}tale}, \mathcal{O}_{X_{\acute{e}tale}})$ and $(Y_{\acute{e}tale}, \mathcal{O}_{Y_{\acute{e}tale}})$ are locally ringed sites, see Lemma 38.33.5. Moreover, we know that $X_{\acute{e}tale}$ has enough points, see Theorem 38.29.10 and Remarks 38.29.11. Hence it suffices to prove that $(f_{small}, f^\sharp_{small})$ satisfies condition (3) of Modules on Sites, Lemma 16.34.7. To see this take a point p of $X_{\acute{e}tale}$. By Lemma 38.29.12 p corresponds to a geometric point $\overline{x}$ of X. By Lemma 38.36.2 the point $q = f_{small} \circ p$ corresponds to the geometric point $\overline{y} = f \circ \overline{x}$ of Y. Hence the assertion we have to prove is that the induced map of stalks
$$\mathcal{O}_{Y,\overline{y}} \longrightarrow \mathcal{O}_{X,\overline{x}}$$
is a local ring map. Suppose that $a \in \mathcal{O}_{Y,\overline{y}}$ is an element of the left hand side which maps to an element of the maximal ideal of the right hand side. Suppose that a is the equivalence class of a triple $(V, \overline{v}, a)$ with $V \to Y$ étale, $\overline{v} : \overline{x} \to V$ over Y, and $a \in \mathcal{O}(V)$. It maps to the equivalence class of $(X \times_Y V, \overline{x} \times \overline{v}, \mathrm{pr}_V^\sharp(a))$ in the local ring $\mathcal{O}_{X,\overline{x}}$. But it is clear that being in the maximal ideal means that pulling back $\mathrm{pr}_V^\sharp(a)$ to an element of $\kappa(\overline{x})$ gives zero. Hence also pulling back a to $\kappa(\overline{x})$ is zero. Which means that a lies in the maximal ideal of $\mathcal{O}_{Y,\overline{y}}$. $\square$

Lemma 38.40.2. *Let X, Y be schemes. Let $f : X \to Y$ be a morphism of schemes. Let t be a 2-morphism from $(f_{small}, f^\sharp_{small})$ to itself, see Modules on Sites, Definition 16.8.1. Then $t = id$.*

Proof. This means that $t : f_{small}^{-1} \to f_{small}^{-1}$ is a transformation of functors such that the diagram

$$\begin{tikzcd} f_{small}^{-1}\mathcal{O}_Y \arrow[dr, "f_{small}^\sharp"'] & & f_{small}^{-1}\mathcal{O}_Y \arrow[ll, "t"] \arrow[dl, "f_{small}^\sharp"] \\ & \mathcal{O}_X & \end{tikzcd}$$

is commutative. Suppose $V \to Y$ is étale with V affine. By Morphisms, Lemma 24.38.2 we may choose an immersion $i : V \to \mathbf{A}^n_Y$ over Y. In terms of sheaves this means that i induces an injection $h_i : h_V \to \prod_{j=1,\ldots,n} \mathcal{O}_Y$ of sheaves. The base change i' of i to X is an immersion (Schemes, Lemma 21.18.2). Hence $i' : X \times_Y V \to \mathbf{A}^n_X$ is an immersion, which in turn means that $h_{i'} : h_{X \times_Y V} \to \prod_{j=1,\ldots,n} \mathcal{O}_X$ is an injection of sheaves. Via the identification $f_{small}^{-1}h_V = h_{X\times_Y V}$ of Lemma 38.36.2 the map $h_{i'}$ is equal to

$$f_{small}^{-1}h_V \xrightarrow{f^{-1}h_i} \prod\nolimits_{j=1,\ldots,n} f_{small}^{-1}\mathcal{O}_Y \xrightarrow{\prod f^\sharp} \prod\nolimits_{j=1,\ldots,n} \mathcal{O}_X$$

(verification omitted). This means that the map $t : f_{small}^{-1}h_V \to f_{small}^{-1}h_V$ fits into the commutative diagram

$$\begin{tikzcd} f_{small}^{-1}h_V \arrow[r, "f^{-1}h_i"] \arrow[d, "t"] & \prod_{j=1,\ldots,n} f_{small}^{-1}\mathcal{O}_Y \arrow[r, "\prod f^\sharp"] \arrow[d, "\prod t"] & \prod_{j=1,\ldots,n} \mathcal{O}_X \arrow[d, "\mathrm{id}"] \\ f_{small}^{-1}h_V \arrow[r, "f^{-1}h_i"] & \prod_{j=1,\ldots,n} f_{small}^{-1}\mathcal{O}_Y \arrow[r, "\prod f^\sharp"] & \prod_{j=1,\ldots,n} \mathcal{O}_X \end{tikzcd}$$

The commutativity of the right square holds by our assumption on t explained above. Since the composition of the horizontal arrows is injective by the discussion above we conclude that the left vertical arrow is the identity map as well. Any sheaf of sets on $Y_{étale}$ admits a surjection from a (huge) coproduct of sheaves of the form h_V with V affine (combine Lemma 38.21.2 with Sites, Lemma 9.12.4). Thus we conclude that $t : f_{small}^{-1} \to f_{small}^{-1}$ is the identity transformation as desired. $\square$

Lemma 38.40.3. *Let X, Y be schemes. Any two morphisms $a, b : X \to Y$ of schemes for which there exists a 2-isomorphism $(a_{small}, a_{small}^\sharp) \cong (b_{small}, b_{small}^\sharp)$ in the 2-category of ringed topoi are equal.*

Proof. Let us argue this carefully since it is a bit confusing. Let $t : a_{small}^{-1} \to b_{small}^{-1}$ be the 2-isomorphism. Consider any open $V \subset Y$. Note that h_V is a subsheaf of the final sheaf $*$. Thus both $a_{small}^{-1}h_V = h_{a^{-1}(V)}$ and $b_{small}^{-1}h_V = h_{b^{-1}(V)}$ are subsheaves of the final sheaf. Thus the isomorphism

$$t : a_{small}^{-1}h_V = h_{a^{-1}(V)} \to b_{small}^{-1}h_V = h_{b^{-1}(V)}$$

has to be the identity, and $a^{-1}(V) = b^{-1}(V)$. It follows that a and b are equal on underlying topological spaces. Next, take a section $f \in \mathcal{O}_Y(V)$. This determines and is determined by a map of sheaves of sets $f : h_V \to \mathcal{O}_Y$. Pull this back and apply t to get a commutative

diagram

$$\begin{tikzcd} h_{b^{-1}(V)} \arrow[r, equal] & b_{small}^{-1} h_V \arrow[d, "b_{small}^{-1}(f)"] & & a_{small}^{-1} h_V \arrow[ll, "t"'] \arrow[d, "a_{small}^{-1}(f)"] \arrow[r, equal] & h_{a^{-1}(V)} \\ & b_{small}^{-1}\mathcal{O}_Y \arrow[dr, "b^\sharp"'] & & a_{small}^{-1}\mathcal{O}_Y \arrow[ll, "t"'] \arrow[dl, "a^\sharp"] & \\ & & \mathcal{O}_X & & \end{tikzcd}$$

where the triangle is commutative by definition of a 2-isomorphism in Modules on Sites, Section 16.8. Above we have seen that the composition of the top horizontal arrows comes from the identity $a^{-1}(V) = b^{-1}(V)$. Thus the commutativity of the diagram tells us that $a_{small}^\sharp(f) = b_{small}^\sharp(f)$ in $\mathcal{O}_X(a^{-1}(V)) = \mathcal{O}_X(b^{-1}(V))$. Since this holds for every open V and every $f \in \mathcal{O}_Y(V)$ we conclude that $a = b$ as morphisms of schemes. $\square$

Lemma 38.40.4. *Let X, Y be affine schemes. Let*

$$(g, g^\#) : (Sh(X_{étale}), \mathcal{O}_X) \longrightarrow (Sh(Y_{étale}), \mathcal{O}_Y)$$

be a morphism of locally ringed topoi. Then there exists a unique morphism of schemes $f : X \to Y$ such that $(g, g^\#)$ is 2-isomorphic to $(f_{small}, f_{small}^\sharp)$, see Modules on Sites, Definition 16.8.1.

Proof. In this proof we write $\mathcal{O}_X$ for the structure sheaf of the small étale site $X_{étale}$, and similarly for $\mathcal{O}_Y$. Say $Y = Spec(B)$ and $X = Spec(A)$. Since $B = \Gamma(Y_{étale}, \mathcal{O}_Y)$, $A = \Gamma(X_{étale}, \mathcal{O}_X)$ we see that $g^\sharp$ induces a ring map $\varphi : B \to A$. Let $f = Spec(\varphi) : X \to Y$ be the corresponding morphism of affine schemes. We will show this f does the job.

Let $V \to Y$ be an affine scheme étale over Y. Thus we may write $V = Spec(C)$ with C an étale B-algebra. We can write

$$C = B[x_1, \ldots, x_n]/(P_1, \ldots, P_n)$$

with P_i polynomials such that $\Delta = \det(\partial P_i/\partial x_j)$ is invertible in C, see for example Algebra, Lemma 7.132.2. If T is a scheme over Y, then a T-valued point of V is given by n sections of $\Gamma(T, \mathcal{O}_T)$ which satisfy the polynomial equations $P_1 = 0, \ldots, P_n = 0$. In other words, the sheaf h_V on $Y_{étale}$ is the equalizer of the two maps

$$\begin{tikzcd} \prod_{i=1,\ldots,n} \mathcal{O}_Y \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & \prod_{j=1,\ldots,n} \mathcal{O}_Y \end{tikzcd}$$

where $b(h_1, \ldots, h_n) = 0$ and $a(h_1, \ldots, h_n) = (P_1(h_1, \ldots, h_n), \ldots, P_n(h_1, \ldots, h_n))$. Since g^{-1} is exact we conclude that the top row of the following solid commutative diagram is an equalizer diagram as well:

$$\begin{tikzcd} g^{-1}h_V \arrow[r] \arrow[d, dashed] & \prod_{i=1,\ldots,n} g^{-1}\mathcal{O}_Y \arrow[r, shift left, "g^{-1}a"] \arrow[r, shift right, "g^{-1}b"'] \arrow[d, "\prod g^\sharp"] & \prod_{j=1,\ldots,n} g^{-1}\mathcal{O}_Y \arrow[d, "\prod g^\sharp"] \\ h_{X \times_Y V} \arrow[r] & \prod_{i=1,\ldots,n} \mathcal{O}_X \arrow[r, shift left, "a'"] \arrow[r, shift right, "b'"'] & \prod_{j=1,\ldots,n} \mathcal{O}_X \end{tikzcd}$$

Here b' is the zero map and a' is the map defined by the images $P'_i = \varphi(P_i) \in A[x_1, \ldots, x_n]$ via the same rule $a'(h_1, \ldots, h_n) = (P'_1(h_1, \ldots, h_n), \ldots, P'_n(h_1, \ldots, h_n))$. that a was defined by. The commutativity of the diagram follows from the fact that $\varphi = g^\sharp$ on global sections. The lower row is an equalizer diagram also, by exactly the same arguments as before since $X \times_Y V$ is the affine scheme $Spec(A \otimes_B C)$ and $A \otimes_B C = A[x_1, \ldots, x_n]/(P'_1, \ldots, P'_n)$. Thus we obtain a unique dotted arrow $g^{-1}h_V \to h_{X \times_Y V}$ fitting into the diagram

We claim that the map of sheaves $g^{-1}h_V \to h_{X \times_Y V}$ is an isomorphism. Since the small étale site of X has enough points (Theorem 38.29.10) it suffices to prove this on stalks. Hence let $\overline{x}$ be a geometric point of X, and denote p the associate point of the small étale topos of X. Set $q = g \circ p$. This is a point of the small étale topos of Y. By Lemma 38.29.12 we see that q corresponds to a geometric point $\overline{y}$ of Y. Consider the map of stalks

$$(g^\sharp)_p : \mathcal{O}_{Y,\overline{y}} = \mathcal{O}_{Y,q} = (g^{-1}\mathcal{O}_Y)_p \longrightarrow \mathcal{O}_{X,p} = \mathcal{O}_{X,\overline{x}}$$

Since $(g, g^\sharp)$ is a morphism of *locally* ringed topoi $(g^\sharp)_p$ is a local ring homomorphism of strictly henselian local rings. Applying localization to the big commutative diagram above and Algebra, Lemma 7.139.25 we conclude that $(g^{-1}h_V)_p \to (h_{X \times_Y V})_p$ is an isomorphism as desired.

We claim that the isomorphisms $g^{-1}h_V \to h_{X \times_Y V}$ are functorial. Namely, suppose that $V_1 \to V_2$ is a morphism of affine schemes étale over Y. Write $V_i = Spec(C_i)$ with

$$C_i = B[x_{i,1}, \ldots, x_{i,n_i}]/(P_{i,1}, \ldots, P_{i,n_i})$$

The morphism $V_1 \to V_2$ is given by a B-algebra map $C_2 \to C_1$ which in turn is given by some polynomials $Q_j \in B[x_{1,1}, \ldots, x_{1,n_1}]$ for $j = 1, \ldots, n_2$. Then it is an easy matter to show that the diagram of sheaves

$$\begin{tikzcd}
h_{V_1} \arrow[r] \arrow[d] & \prod_{i=1,\ldots,n_1} \mathcal{O}_Y \arrow[d, "Q_1,\ldots,Q_{n_2}"] \\
h_{V_2} \arrow[r] & \prod_{i=1,\ldots,n_2} \mathcal{O}_Y
\end{tikzcd}$$

is commutative, and pulling back to $X_{\acute{e}tale}$ we obtain the solid commutative diagram

$$\begin{tikzcd}
g^{-1}h_{V_1} \arrow[r] \arrow[drr] \arrow[dd, dotted] & \prod_{i=1,\ldots,n_1} g^{-1}\mathcal{O}_Y \arrow[drr, "Q_1,\ldots,Q_{n_2}"] \arrow[dd, "g^\sharp"] & & \\
& & g^{-1}h_{V_2} \arrow[r] \arrow[dd, dotted] & \prod_{i=1,\ldots,n_2} g^{-1}\mathcal{O}_Y \arrow[dd, "g^\sharp"] \\
h_{X \times_Y V_1} \arrow[r] \arrow[drr] & \prod_{i=1,\ldots,n_1} \mathcal{O}_X \arrow[drr, "Q'_1,\ldots,Q'_{n_2}"] & & \\
& & h_{X \times_Y V_2} \arrow[r] & \prod_{i=1,\ldots,n_2} \mathcal{O}_X
\end{tikzcd}$$

where $Q'_j \in A[x_{1,1}, \ldots, x_{1,n_1}]$ is the image of Q_j via φ. Since the dotted arrows exist, make the two squares commute, and the horizontal arrows are injective we see that the whole diagram commutes. This proves functoriality (and also that the construction of $g^{-1}h_V \to$

$h_{X\times_Y V}$ is independent of the choice of the presentation, allthough we strictly speaking do not need to show this).

At this point we are able to show that $f_{small,*} \cong g_*$. Namely, let $\mathcal{F}$ be a sheaf on $X_{étale}$. For every $V \in Ob(X_{étale})$ affine we have

$$\begin{aligned}(g_*\mathcal{F})(V) &= Mor_{Sh(Y_{étale})}(h_V, g_*\mathcal{F}) \\ &= Mor_{Sh(X_{étale})}(g^{-1}h_V, \mathcal{F}) \\ &= Mor_{Sh(X_{étale})}(h_{X\times_Y V}, \mathcal{F}) \\ &= \mathcal{F}(X \times_Y V) \\ &= f_{small,*}\mathcal{F}(V)\end{aligned}$$

where in the third equality we use the isomorphism $g^{-1}h_V \cong h_{X\times_Y V}$ constructed above. These isomorphisms are clearly functorial in $\mathcal{F}$ and functorial in V as the isomorphisms $g^{-1}h_V \cong h_{X\times_Y V}$ are functorial. Now any sheaf on $Y_{étale}$ is determined by the restriction to the subcategory of affine schemes (Lemma 38.21.2), and hence we obtain an isomorphism of functors $f_{small,*} \cong g_*$ as desired.

Finally, we have to check that, via the isomorphism $f_{small,*} \cong g_*$ above, the maps $f_{small}^\sharp$ and $g^\sharp$ agree. By construction this is already the case for the global sections of $\mathcal{O}_Y$, i.e., for the elements of B. We only need to check the result on sections over an affine V étale over Y (by Lemma 38.21.2 again). Writing $V = Spec(C)$, $C = B[x_i]/(P_j)$ as before it suffices to check that the coordinate functions x_i are mapped to the same sections of $\mathcal{O}_X$ over $X \times_Y V$. And this is exactly what it means that the diagram

$$\begin{tikzcd} g^{-1}h_V \arrow[r] \arrow[d, dotted] & \prod_{i=1,\ldots,n} g^{-1}\mathcal{O}_Y \arrow[d, "\prod g^\sharp"] \\ h_{X\times_Y V} \arrow[r] & \prod_{i=1,\ldots,n} \mathcal{O}_X \end{tikzcd}$$

commutes. Thus the lemma is proved. ☐

Here is a version for general schemes.

Theorem 38.40.5. *Let X, Y be schemes. Let*

$$(g, g^\#) : (Sh(X_{étale}), \mathcal{O}_X) \longrightarrow (Sh(Y_{étale}), \mathcal{O}_Y)$$

be a morphism of locally ringed topoi. Then there exists a unique morphism of schemes $f : X \to Y$ such that $(g, g^\#)$ is isomorphic to $(f_{small}, f_{small}^\sharp)$. In other words, the construction

$$Sch \longrightarrow Locally\ ringed\ topoi, \quad X \longrightarrow (X_{étale}, \mathcal{O}_X)$$

is fully faithful (morphisms up to 2-isomorphisms on the right hand side).

Proof. You can prove this theorem by carefully adjusting the arguments of the proof of Lemma 38.40.4 to the global setting. However, we want to indicate how we can glue the result of that lemma to get a global morphism due to the rigidity provided by the result of Lemma 38.40.2. Unfortunately, this is a bit messy.

Let us prove existence when Y is affine. In this case choose an affine open covering $X = \bigcup U_i$. For each i the inclusion morphism $j_i : U_i \to X$ induces a morphism of locally ringed topoi $(j_{i,small}, j_{i,small}^\sharp) : (Sh(U_{i,étale}), \mathcal{O}_{U_i}) \to (Sh(X_{étale}), \mathcal{O}_X)$ by Lemma 38.40.1. We can compose this with $(g, g^\sharp)$ to obtain a morphism of locally ringed topoi

$$(g, g^\sharp) \circ (j_{i,small}, j_{i,small}^\sharp) : (Sh(U_{i,étale}), \mathcal{O}_{U_i}) \to (Sh(X_{étale}), \mathcal{O}_X)$$

see Modules on Sites, Lemma 16.34.9. By Lemma 38.40.4 there exists a unique morphism of schemes $f_i : U_i \to Y$ and a 2-isomorphism

$$t_i : (f_{i,small}, f_{i,small}^\sharp) \longrightarrow (g, g^\sharp) \circ (j_{i,small}, j_{i,small}^\sharp).$$

Set $U_{i,i'} = U_i \cap U_{i'}$, and denote $j_{i,i'} : U_{i,i'} \to U_i$ the inclusion morphism. Since we have $j_i \circ j_{i,i'} = j_{i'} \circ j_{i',i}$ we see that

$$\begin{aligned}(g, g^\sharp) \circ (j_{i,small}, j_{i,small}^\sharp) \circ (j_{i,i',small}, j_{i,i',small}^\sharp) = \\ (g, g^\sharp) \circ (j_{i',small}, j_{i',small}^\sharp) \circ (j_{i',i,small}, j_{i',i,small}^\sharp)\end{aligned}$$

Hence by uniqueness (see Lemma 38.40.3) we conclude that $f_i \circ j_{i,i'} = f_{i'} \circ j_{i',i}$, in other words the morphisms of schemes $f_i = f \circ j_i$ are the restrictions of a global morphism of schemes $f : X \to Y$. Consider the diagram of 2-isomorphisms (where we drop the components $^\sharp$ to ease the notation)

$$\begin{tikzcd} g \circ j_{i,small} \circ j_{i,i',small} \arrow[r, "t_i \star \mathrm{id}_{j_{i,i',small}}"] \arrow[d, equal] & f_{small} \circ j_{i,small} \circ j_{i,i',small} \arrow[d, equal] \\ g \circ j_{i',small} \circ j_{i',i,small} \arrow[r, "t_{i'} \star \mathrm{id}_{j_{i',i,small}}"] & f_{small} \circ j_{i',small} \circ j_{i',i,small} \end{tikzcd}$$

The notation $\star$ indicates horizontal composition, see Categories, Definition 4.26.1 in general and Sites, Section 9.32 for our particular case. By the result of Lemma 38.40.2 this diagram commutes. Hence for any sheaf $\mathcal{G}$ on $Y_{étale}$ the isomorphisms $t_i : f_{small}^{-1}\mathcal{G}|_{U_i} \to g^{-1}\mathcal{G}|_{U_i}$ agree over $U_{i,i'}$ and we obtain a global isomorphism $t : f_{small}^{-1}\mathcal{G} \to g^{-1}\mathcal{G}$. It is clear that this isomorphism is functorial in $\mathcal{G}$ and is compatible with the maps $f_{small}^\sharp$ and $g^\sharp$ (because it is compatible with these maps locally). This proves the theorem in case Y is affine.

In the general case, let $V \subset Y$ be an affine open. Then h_V is a subsheaf of the final sheaf $*$ on $Y_{étale}$. As g is exact we see that $g^{-1}h_V$ is a subsheaf of the final sheaf on $X_{étale}$. Hence by Lemma 38.31.1 there exists an open subscheme $W \subset X$ such that $g^{-1}h_V = h_W$. By Modules on Sites, Lemma 16.34.11 there exists a commutative diagram of morphisms of locally ringed topoi

$$\begin{tikzcd} (Sh(W_{étale}), \mathcal{O}_W) \arrow[r] \arrow[d, "g'"'] & (Sh(X_{étale}), \mathcal{O}_X) \arrow[d, "g"] \\ (Sh(V_{étale}), \mathcal{O}_V) \arrow[r] & (Sh(Y_{étale}), \mathcal{O}_Y) \end{tikzcd}$$

where the horizontal arrows are the localization morphisms (induced by the inclusion morphisms $V \to Y$ and $W \to X$) and where g' is induced from g. By the result of the preceding paragraph we obtain a morphism of schemes $f' : W \to V$ and a 2-isomorphism $t : (f'_{small}, (f'_{small})^\sharp) \to (g', (g')^\sharp)$. Exactly as before these morphisms f' (for varying affine opens $V \subset Y$) agree on overlaps by uniqueness, so we get a morphism $f : X \to Y$. Moreover, the 2-isomorphisms t are compatible on overlaps by Lemma 38.40.2 again and we obtain a global 2-isomorphism $(f_{small}, (f_{small})^\sharp) \to (g, (g)^\sharp)$. as desired. Some details omitted. $\square$

38.41. Push and pull

Let $f : X \to Y$ be a morphism of schemes. Here is a list of conditions we will consider in the following:

(A) For every étale morphism $U \to X$ and $u \in U$ there exist an étale morphism $V \to Y$ and a disjoint union decomposition $X \times_Y V = W \amalg W'$ and a morphism $h : W \to U$ over X with u in the image of h.
(B) For every $V \to Y$ étale, and every étale covering $\{U_i \to X \times_Y V\}$ there exists an étale covering $\{V_j \to V\}$ such that for each j we have $X \times_Y V_j = \coprod W_{ji}$ where $W_{ij} \to X \times_Y V$ factors through $U_i \to X \times_Y V$ for some i.
(C) For every $U \to X$ étale, there exists a $V \to Y$ étale and a surjective morphism $X \times_Y V \to U$ over X.

It turns out that each of these properties has meaning in terms of the behaviour of the functor $f_{small,*}$. We will work this out in the next few sections.

38.42. Property (A)

Lemma 38.42.1. *Let $f : X \to Y$ be a morphism of schemes. Assume (A).*

1. *$f_{small,*} : Ab(X_{étale}) \to Ab(Y_{étale})$ reflects injections and surjections,*
2. *$f_{small}^{-1} f_{small,*}\mathcal{F} \to \mathcal{F}$ is surjective for any abelian sheaf $\mathcal{F}$ on $X_{étale}$,*
3. *$f_{small,*} : Ab(X_{étale}) \to Ab(Y_{étale})$ is faithful.*

Proof. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. Let U be an object of $X_{étale}$. By assumption we can find a covering $\{W_i \to U\}$ in $X_{étale}$ such that each W_i is an open and closed subscheme of $X \times_Y V_i$ for some object V_i of $Y_{étale}$. The sheaf condition shows that

$$\mathcal{F}(U) \subset \prod \mathcal{F}(W_i)$$

and that $\mathcal{F}(W_i)$ is a direct summand of $\mathcal{F}(X \times_Y V_i) = f_{small,*}\mathcal{F}(V_i)$. Hence it is clear that $f_{small,*}$ reflects injections.

Next, suppose that $a : \mathcal{G} \to \mathcal{F}$ is a map of abelian sheaves such that $f_{small,*}a$ is surjective. Let $s \in \mathcal{F}(U)$ with U as above. With W_i, V_i as above we see that it suffices to show that $s|_{W_i}$ is étale locally the image of a section of $\mathcal{G}$ under a. Since $\mathcal{F}(W_i)$ is a direct summand of $\mathcal{F}(X \times_Y V_i)$ it suffices to show that for any $V \in Ob(Y_{étale})$ any element $s \in \mathcal{F}(X \times_Y V)$ is étale locally on $X \times_Y V$ the image of a section of $\mathcal{G}$ under a. Since $\mathcal{F}(X \times_Y V) = f_{small,*}\mathcal{F}(V)$ we see by assumption that there exists a covering $\{V_j \to V\}$ such that s is the image of $s_j \in f_{small,*}\mathcal{G}(V_j) = \mathcal{G}(X \times_Y V_j)$. This proves $f_{small,*}$ reflects surjections.

Parts (2), (3) follow formally from part (1), see Modules on Sites, Lemma 16.15.1. $\square$

Lemma 38.42.2. *Let $f : X \to Y$ be a separated locally quasi-finite morphism of schemes. Then property (A) above holds.*

Proof. Let $U \to X$ be an étale morphism and $u \in U$. The geometric statement (A) reduces directly to the case where U and Y are affine schemes. Denote $x \in X$ and $y \in Y$ the images of u. Since $X \to Y$ is locally quasi-finite, and $U \to X$ is locally quasi-finite (see Morphisms, Lemma 24.35.6) we see that $U \to Y$ is locally quasi-finite (see Morphisms, Lemma 24.19.12). Moreover both $X \to Y$ and $U \to Y$ are separated. Thus More on

Morphisms, Lemma 33.28.5 applies to both morphisms. This means we may pick an étale neighbourhood $(V, v) \to (Y, y)$ such that

$$X \times_Y V = W \amalg R, \quad U \times_Y V = W' \amalg R'$$

and points $w \in W$, $w' \in W'$ such that

(1) W, R are open and closed in $X \times_Y V$,
(2) W', R' are open and closed in $U \times_Y V$,
(3) $W \to V$ and $W' \to V$ are finite,
(4) w, w' map to v,
(5) $\kappa(v) \subset \kappa(w)$ and $\kappa(v) \subset \kappa(w')$ are purely inseparable, and
(6) no other point of W or W' maps to v.

Here is a commutative diagram

$$\begin{tikzcd}
U & U \times_Y V \arrow[l] \arrow[d] & W' \amalg R' \arrow[l] \arrow[d] \\
X \arrow[d] & X \times_Y V \arrow[l] \arrow[d] & W \amalg R \arrow[l] \\
Y & V \arrow[l]
\arrow[from=1-1, to=2-1]
\end{tikzcd}$$

After shrinking V we may assume that W' maps into W: just remove the image the inverse image of R in W'; this is a closed set (as $W' \to V$ is finite) not containing v. Then $W' \to W$ is finite because both $W \to V$ and $W' \to V$ are finite. Hence $W' \to W$ is finite étale, and there is exactly one point in the fibre over w with $\kappa(w) = \kappa(w')$. Hence $W' \to W$ is an isomorphism in an open neighbourhood W° of w, see Étale Morphisms, Lemma 37.14.2. Since $W \to V$ is finite the image of $W \setminus W^\circ$ is a closed subset T of V not containing v. Thus after replacing V by $V \setminus T$ we may assume that $W' \to W$ is an isomorphism. Now the decomposition $X \times_Y V = W \amalg R$ and the morphism $W \to U$ are as desired and we win. $\square$

Lemma 38.42.3. *Let $f : X \to Y$ be an integral morphism of schemes. Then property (A) holds.*

Proof. Let $U \to X$ be étale, and let $u \in U$ be a point. We have to find $V \to Y$ étale, a disjoint union decomposition $X \times_Y V = W \amalg W'$ and an X-morphism $W \to U$ with u in the image. We may shrink U and Y and assume U and Y are affine. In this case also X is affine, since an integral morphism is affine by definition. Write $Y = Spec(A)$, $X = Spec(B)$ and $U = Spec(C)$. Then $A \to B$ is an integral ring map, and $B \to C$ is an étale ring map. By Algebra, Lemma 7.132.3 we can find a finite A-subalgebra $B' \subset B$ and an étale ring map $B' \to C'$ such that $C = B \otimes_{B'} C'$. Thus the question reduces to the étale morphism $U' = Spec(C') \to X' = Spec(B')$ over the finite morphism $X' \to Y$. In this case the result follows from Lemma 38.42.2. $\square$

Lemma 38.42.4. *Let $f : X \to Y$ be a morphism of schemes. Denote $f_{small} : Sh(X_{étale}) \to Sh(Y_{étale})$ the associated morphism of small étale topoi. Assume at least one of the following*

(1) *f is integral, or*
(2) *f is separated and locally quasi-finite.*

Then the functor $f_{small,*} : Ab(X_{étale}) \to Ab(Y_{étale})$ has the following properties

(1) *the map $f_{small}^{-1} f_{small,*}\mathcal{F} \to \mathcal{F}$ is always surjective,*
(2) *$f_{small,*}$ is faithful, and*

(3) *$f_{small,*}$ reflects injections and surjections.*

Proof. Combine Lemmas 38.42.2, 38.42.3, and 38.42.1. □

38.43. Property (B)

Lemma 38.43.1. *Let $f : X \to Y$ be a morphism of schemes. Assume (B) holds. Then the functor $f_{small,*} : Sh(X_{étale}) \to Sh(Y_{étale})$ transforms surjections into surjections.*

Proof. This follows from Sites, Lemma 9.36.2. □

Lemma 38.43.2. *Let $f : X \to Y$ be a morphism of schemes. Suppose*

(1) *$V \to Y$ is an étale morphism of schemes,*
(2) *$\{U_i \to X \times_Y V\}$ is an étale covering, and*
(3) *$v \in V$ is a point.*

Assume that for any such data there exists an étale neighbourhood $(V', v') \to (V, v)$, a disjoint union decomposition $X \times_Y V' = \coprod W_i'$, and morphisms $W_i' \to U_i$ over $X \times_Y V$. Then property (B) holds.

Proof. Omitted. □

Lemma 38.43.3. *Let $f : X \to Y$ be a finite morphism of schemes. Then property (B) holds.*

Proof. Consider $V \to Y$ étale, $\{U_i \to X \times_Y V\}$ an étale covering, and $v \in V$. We have to find a $V' \to V$ and decomposition and maps as in Lemma 38.43.2. We may shrink V and Y, hence we may assume that V and Y are affine. Since X is finite over Y, this also implies that X is affine. During the proof we may (finitely often) replace (V, v) by an étale neighbourhood (V', v') and correspondingly the covering $\{U_i \to X \times_Y V\}$ by $\{V' \times_V U_i \to X \times_Y V'\}$.

Since $X \times_Y V \to V$ is finite there exist finitely many (pairwise distinct) points $x_1, \ldots, x_n \in X \times_Y V$ mapping to v. We may apply More on Morphisms, Lemma 33.28.5 to $X \times_Y V \to V$ and the points $x_1, \ldots, x_n$ lying over v and find an étale neighbourhood $(V', v') \to (V, v)$ such that

$$X \times_Y V' = R \amalg \coprod T_a$$

with $T_a \to V'$ finite with exactly one point p_a lying over v' and moreover $\kappa(v') \subset \kappa(p_a)$ purely inseparable, and such that $R \to V'$ has empty fibre over v'. Because $X \to Y$ is finite, also $R \to V'$ is finite. Hence after shrinking V' we may assume that $R = \emptyset$. Thus we may assume that $X \times_Y V = X_1 \amalg \ldots \amalg X_n$ with exactly one point $x_l \in X_l$ lying over v with moreover $\kappa(v) \subset \kappa(x_l)$ purely inseparable. Note that this property is preserved under refinement of the étale neighbourhood (V, v).

For each l choose an i_l and a point $u_l \in U_{i_l}$ mapping to x_l. Now we apply property (A) for the finite morphism $X \times_Y V \to V$ and the étale morphisms $U_{i_l} \to X \times_Y V$ and the points u_l. This is permissible by Lemma 38.42.3 This gives produces an étale neighbourhood $(V', v') \to (V, v)$ and decompositions

$$X \times_Y V' = W_l \amalg R_l$$

and X-morphisms $a_l : W_l \to U_{i_l}$ whose image contains u_{i_l}. Here is a picture:

$$
\begin{tikzcd}
 & & & U_{i_l} \arrow[d] & \\
W_l \arrow[r] \arrow[urrr] & W_l \amalg R_l \arrow[r, equal] & X \times_Y V' \arrow[r] \arrow[d] & X \times_Y V \arrow[r] \arrow[d] & X \arrow[d] \\
 & & V' \arrow[r] & V \arrow[r] & Y
\end{tikzcd}
$$

After replacing (V, v) by (V', v') we conclude that each x_l is contained in an open and closed neighbourhood W_l such that the inclusion morphism $W_l \to X \times_Y V$ factors through $U_i \to X \times_Y V$ for some i. Replacing W_l by $W_l \cap X_l$ we see that these open and closed sets are disjoint and moreover that $\{x_1, \ldots, x_n\} \subset W_1 \cup \ldots \cup W_n$. Since $X \times_Y V \to V$ is finite we may shrink V and assume that $X \times_Y V = W_1 \amalg \ldots \amalg W_n$ as desired. $\square$

Lemma 38.43.4. *Let $f : X \to Y$ be an integral morphism of schemes. Then property (B) holds.*

Proof. Consider $V \to Y$ étale, $\{U_i \to X \times_Y V\}$ an étale covering, and $v \in V$. We have to find a $V' \to V$ and decomposition and maps as in Lemma 38.43.2. We may shrink V and Y, hence we may assume that V and Y are affine. Since X is integral over Y, this also implies that X and $X \times_Y V$ are affine. We may refine the covering $\{U_i \to X \times_Y V\}$, and hence we may assume that $\{U_i \to X \times_Y V\}_{i=1,\ldots,n}$ is a standard étale covering. Write $Y = Spec(A)$, $X = Spec(B)$, $V = Spec(C)$, and $U_i = Spec(B_i)$. Then $A \to B$ is an integral ring map, and $B \otimes_A C \to B_i$ are étale ring maps. By Algebra, Lemma 7.132.3 we can find a finite A-subalgebra $B' \subset B$ and an étale ring map $B' \otimes_A C \to B'_i$ for $i = 1, \ldots, n$ such that $B_i = B \otimes_{B'} B'_i$. Thus the question reduces to the étale covering $\{Spec(B'_i) \to X' \times_Y V\}_{i=1,\ldots,n}$ with $X' = Spec(B')$ finite over Y. In this case the result follows from Lemma 38.43.3. $\square$

Lemma 38.43.5. *Let $f : X \to Y$ be a morphism of schemes. Assume f is integral (for example finite). Then*

(1) *$f_{small,*}$ transforms surjections into surjections (on sheaves of sets and on abelian sheaves),*
(2) *$f_{small}^{-1} f_{small,*}\mathcal{F} \to \mathcal{F}$ is surjective for any abelian sheaf $\mathcal{F}$ on $X_{étale}$,*
(3) *$f_{small,*} : Ab(X_{étale}) \to Ab(Y_{étale})$ is faithful and reflects injections and surjections, and*
(4) *$f_{small,*} : Ab(X_{étale}) \to Ab(Y_{étale})$ is exact.*

Proof. Parts (2), (3) we have seen in Lemma 38.42.4. Part (1) follows from Lemmas 38.43.4 and 38.43.1. Part (4) is a consequence of part (1), see Modules on Sites, Lemma 16.15.2. $\square$

38.44. Property (C)

Lemma 38.44.1. *Let $f : X \to Y$ be a morphism of schemes. Assume (C) holds. Then the functor $f_{small,*} : Sh(X_{étale}) \to Sh(Y_{étale})$ reflects injections and surjections.*

Proof. Follows from Sites, Lemma 9.36.4. We omit the verification that property (C) implies that the functor $Y_{étale} \to X_{étale}$, $V \mapsto X \times_Y V$ satisfies the assumption of Sites, Lemma 9.36.4. $\square$

Remark 38.44.2. Property (C) appears to be a very strong property, but the following example shows that it does not imply that $f_{small,*}$ is exact. Let K be an algebraic closure of $k(x, y)$ where k is a field. Let R be the integral closure of $k[x, y]$ in K. Set $Y = Spec(R)$ and $X = Y \setminus \{0\}$. Then property (C) holds for the morphism $j : X \to Y$, as every étale morphism $U \to X$ is a local isomorphism. But $j_{small,*}$ is not exact on $Ab(X_{\acute{e}tale})$. Details omitted. Hint: In this example étale sheaves are the same thing as Zariski sheaves.

Lemma 38.44.3. *Let $f : X \to Y$ be a morphism of schemes. Assume that for any $V \to Y$ étale we have that*

(1) *$X \times_Y V \to V$ has property (C), and*
(2) *$X \times_Y V \to V$ is closed.*

Then the functor $Y_{\acute{e}tale} \to X_{\acute{e}tale}$, $V \mapsto X \times_Y V$ is almost cocontinuous, see Sites, Definition 9.37.3.

Proof. Let $V \to Y$ be an object of $Y_{\acute{e}tale}$ and let $\{U_i \to X \times_Y V\}_{i \in I}$ be a covering of $X_{\acute{e}tale}$. By assumption (1) for each i we can find an étale morphism $h_i : V_i \to V$ and a surjective morphism $X \times_Y V_i \to U_i$ over $X \times_Y V$. Note that $\bigcup h_i(V_i) \subset V$ is an open set containing the closed set $Z = \mathrm{Im}(X \times_Y V \to V)$. Let $h_0 : V_0 = V \setminus Z \to V$ be the open immersion. It is clear that $\{V_i \to V\}_{i \in I \cup \{0\}}$ is an étale covering such that for each $i \in I \cup \{0\}$ we have either $V_i \times_Y X = \emptyset$ (namely if $i = 0$), or $V_i \times_Y X \to V \times_Y X$ factors through $U_i \to X \times_Y V$ (if $i \neq 0$). Hence the functor $Y_{\acute{e}tale} \to X_{\acute{e}tale}$ is almost cocontinuous. $\square$

Lemma 38.44.4. *Let $f : X \to Y$ be an integral morphism of schemes which defines a homeomorphism of X with a closed subset of Y. Then property (C) holds.*

Proof. Let $g : U \to X$ be an étale morphism. We need to find an object $V \to Y$ of $Y_{\acute{e}tale}$ and a surjective morphism $X \times_Y V \to U$ over X. Suppose that for every $u \in U$ we can find an object $V_u \to Y$ of $Y_{\acute{e}tale}$ and a morphism $h_u : X \times_Y V_u \to U$ over X with $u \in \mathrm{Im}(h_u)$. Then we can take $V = \coprod V_u$ and $h = \coprod h_u$ and we win. Hence given a point $u \in U$ we find a pair (V_u, h_u) as above. To do this we may shrink U and assume that U is affine. In this case $g : U \to X$ is locally quasi-finite. Let $g^{-1}(g(\{u\})) = \{u, u_2, \ldots, u_n\}$. Since there are no specializations $u_i \leadsto u$ we may replace U by an affine neighbourhood so that $g^{-1}(g(\{u\})) = \{u\}$.

The image $g(U) \subset X$ is open, hence $f(g(U))$ is locally closed in Y. Choose an open $V \subset Y$ such that $f(g(U)) = f(X) \cap V$. It follows that g factors through $X \times_Y V$ and that the resulting $\{U \to X \times_Y V\}$ is an étale covering. Since f has property (B) , see Lemma 38.43.4, we see that there exists an étale covering $\{V_j \to V\}$ such that $X \times_Y V_j \to X \times_Y V$ factor through U. This implies that $V' = \coprod V_j$ is étale over Y and that there is a morphism $h : X \times_Y V' \to U$ whose image surjects onto $g(U)$. Since u is the only point in its fibre it must be in the image of h and we win. $\square$

We urge the reader to think of the following lemma as a way station[4] on the journey towards the ultimate truth regarding $f_{small,*}$ for integral universally injective morphisms.

Lemma 38.44.5. *Let $f : X \to Y$ be a morphism of schemes. Assume that f is universally injective and integral (for example a closed immersion). Then*

(1) *$f_{small,*} : Sh(X_{\acute{e}tale}) \to Sh(Y_{\acute{e}tale})$ reflects injections and surjections,*
(2) *$f_{small,*} : Sh(X_{\acute{e}tale}) \to Sh(Y_{\acute{e}tale})$ commutes with pushouts and coequalizers (and more generally finite, nonempty, connected colimits),*

[4] A way station is a place where people stop to eat and rest when they are on a long journey.

(3) *$f_{small,*}$ transforms surjections into surjections (on sheaves of sets and on abelian sheaves),*
(4) *the map $f_{small}^{-1} f_{small,*}\mathcal{F} \to \mathcal{F}$ is surjective for any sheaf (of sets or of abelian groups) $\mathcal{F}$ on $X_{\acute{e}tale}$,*
(5) *the functor $f_{small,*}$ is faithful (on sheaves of sets and on abelian sheaves),*
(6) *$f_{small,*} : Ab(X_{\acute{e}tale}) \to Ab(Y_{\acute{e}tale})$ is exact, and*
(7) *the functor $Y_{\acute{e}tale} \to X_{\acute{e}tale}$, $V \mapsto X \times_Y V$ is almost cocontinuous.*

Proof. By Lemmas 38.42.3, 38.43.4 and 38.44.4 we know that the morphism f has properties (A), (B), and (C). Moreover, by Lemma 38.44.3 we know that the functor $Y_{\acute{e}tale} \to X_{\acute{e}tale}$ is almost cocontinuous. Now we have

(1) property (C) implies (1) by Lemma 38.44.1,
(2) almost continuous implies (2) by Sites, Lemma 9.37.6,
(3) property (B) implies (3) by Lemma 38.43.1.

Properties (4), (5), and (6) follow formally from the first three, see Sites, Lemma 9.36.1 and Modules on Sites, Lemma 16.15.2. Property (7) we saw above. □

38.45. Topological invariance of the small étale site

In the following theorem we show that the small étale site is a topological invariant in the following sense: If $f : X \to Y$ is a morphism of schemes which is a universal homeomorphism, then $X_{\acute{e}tale} \cong Y_{\acute{e}tale}$ as sites. This improves the result of Étale Morphisms, Theorem 37.15.2.

Theorem 38.45.1. *Let $f : X \to Y$ be a morphism of schemes. Assume f is integral, universally injective and surjective (i.e., f is a universal homeomorphism, see Morphisms, Lemma 24.43.3). The functor*

$$V \longmapsto V_X = X \times_Y V$$

defines an equivalence of categories

$$\{\text{schemes } V \text{ étale over } Y\} \leftrightarrow \{\text{schemes } U \text{ étale over } X\}$$

Proof. We claim that it suffices to prove that the functor defines an equivalence

$$\{\text{affine schemes } V \text{ étale over } Y\} \leftrightarrow \{\text{affine schemes } U \text{ étale over } X\} \quad (38.45.1.1)$$

when X and Y are affine. We omit the proof of this claim.

Assume X and Y affine. Let us prove (38.45.1.1) is fully faithful. Suppose that V, V' are affine schemes étale over Y, and that $\varphi : V_X \to V'_X$ is a morphism over X. To prove that $\varphi = \psi_X$ for some $\psi : V \to V'$ over Y we may work locally on V. The graph

$$\Gamma_\varphi \subset (V \times_Y V')_X$$

of φ is an open and closed subscheme, see Étale Morphisms, Proposition 37.6.1. Since f is a universal homeomorphism we see that there exists an open and closed subscheme $\Gamma \subset V \times_Y V'$ with $\Gamma_X = \Gamma_\varphi$. We see that Γ is an affine scheme endowed with an étale, universally injective, and surjective morphism $\Gamma \to V$. This implies that $\Gamma \to V$ is an isomorphism (see Étale Morphisms, Theorem 37.14.1), and hence Γ is the graph of a morphism $\psi : V \to V'$ over Y as desired.

Let us prove (38.45.1.1) is essentially surjective. Let $U \to X$ be an affine scheme étale over X. We have to find $V \to Y$ étale (and affine) such that $X \times_Y V$ is isomorphic to U over X. Note that an étale morphism of affines has universally bounded fibres, see Morphisms, Lemma 24.35.6 and Lemma 24.48.8. Hence we can do induction on the integer n bounding the degree of the fibres of $U \to X$. See Morphisms, Lemma 24.48.7 for a description of this integer in the case of an étale morphism. If $n = 1$, then $U \to X$ is an open immersion (see Étale Morphisms, Theorem 37.14.1), and the result is clear. Assume $n > 1$.

By Lemma 38.44.4 there exists an étale morphism of schemes $W \to Y$ and a surjective morphism $W_X \to U$ over X. As U is quasi-compact we may replace W by a disjoint union of finitely many affine opens of W, hence we may assume that W is affine as well. Here is a diagram

$$\begin{tikzcd} U \arrow[d] & U \times_Y W \arrow[l] \arrow[d] \arrow[r, equal] & W_X \amalg R \\ X \arrow[d] & W_X \arrow[l] \arrow[d] & \\ Y & W \arrow[l] & \end{tikzcd}$$

The disjoint union decomposition arises because by construction the étale morphism of affine schemes $U \times_Y W \to W_X$ has a section. OK, and now we see that the morphism $R \to X \times_Y W$ is an étale morphism of affine schemes whose fibres have degree universally bounded by $n - 1$. Hence by induction assumption there exists a scheme $V' \to W$ étale such that $R \cong W_X \times_W V'$. Taking $V'' = W \amalg V'$ we find a scheme V'' étale over W whose base change to W_X is isomorphic to $U \times_Y W$ over $X \times_Y W$.

At this point we can use descent to find V over Y whose base change to X is isomorphic to U over X. Namely, by the fully faithfulness of the functor (38.45.1.1) corresponding to the universal homeomorphism $X \times_Y (W \times_Y W) \to (W \times_Y W)$ there exists a unique isomorphism $\varphi : V'' \times_Y W \to W \times_Y V''$ whose base change to $X \times_Y (W \times_Y W)$ is the canonical descent datum for $U \times_Y W$ over $X \times_Y W$. In particular φ satisfies the cocycle condition. Hence by Descent, Lemma 31.33.1 we see that φ is effective (recall that all schemes above are affine). Thus we obtain $V \to Y$ and an isomorphism $V'' \cong W \times_Y V$ such that the canonical descent datum on $W \times_Y V/W/Y$ agrees with φ. Note that $V \to Y$ is étale, by Descent, Lemma 31.19.27. Moreover, there is an isomorphism $V_X \cong U$ which comes from descending the isomorphism

$$V_X \times_X W_X = X \times_Y V \times_Y W = (X \times_Y W) \times_W (W \times_Y V) \cong W_X \times_W V'' \cong U \times_Y W$$

which we have by construction. Some details omitted. $\square$

Remark 38.45.2. In the situation of Theorem 38.45.1 it is also true that $V \mapsto V_X$ induces an equivalence between those étale morphisms $V \to Y$ with V affine and those étale morphisms $U \to X$ with U affine. This follows for example from Limits, Proposition 27.7.2.

38.46. Closed immersions and pushforward

Before stating and proving Proposition 38.46.4 in its correct generality we briefly state and prove it for closed immersions. Namely, some of the preceding arguments are quite a bit easier to follow in the case of a closed immersion and so we repeat them here in their simplified form.

In the rest of this section $i : Z \to X$ is a closed immersion. The functor

$$Sch/X \longrightarrow Sch/Z, \quad U \longmapsto U_Z = Z \times_X U$$

will be denoted $U \mapsto U_Z$ as indicated. Since being a closed immersion is preserved under arbitrary base change the scheme U_Z is a closed subscheme of U.

Lemma 38.46.1. *Let $i : Z \to X$ be a closed immersion of schemes. Let U, U' be schemes étale over X. Let $h : U_Z \to U'_Z$ be a morphism over Z. Then there exists a diagram*

$$U \xleftarrow{a} W \xrightarrow{b} U'$$

such that $a_Z : W_Z \to U_Z$ is an isomorphism and $h = b_Z \circ (a_Z)^{-1}$.

Proof. Consider the scheme $M = U \times_Y U'$. The graph $\Gamma_h \subset M_Z$ of h is open. This is true for example as Γ_h is the image of a section of the étale morphism $\mathrm{pr}_{1,Z} : M_Z \to U_Z$, see Étale Morphisms, Proposition 37.6.1. Hence there exists an open subscheme $W \subset M$ whose intersection with the closed subset M_Z is Γ_h. Set $a = \mathrm{pr}_1|_W$ and $b = \mathrm{pr}_2|_W$. □

Lemma 38.46.2. *Let $i : Z \to X$ be a closed immersion of schemes. Let $V \to Z$ be an étale morphism of schemes. There exist étale morphisms $U_i \to X$ and morphisms $U_{i,Z} \to V$ such that $\{U_{i,Z} \to V\}$ is a Zariski covering of V.*

Proof. Since we only have to find a Zariski covering of V consisting of schemes of the form U_Z with U étale over X, we may Zariski localize on X and V. Hence we may assume X and V affine. In the affine case this is Algebra, Lemma 7.132.10. □

If $\overline{x} : Spec(k) \to X$ is a geometric point of X, then either $\overline{x}$ factors (uniquely) through the closed subscheme Z, or $Z_{\overline{x}} = \emptyset$. If $\overline{x}$ factors through Z we say that $\overline{x}$ is a geometric point of Z (because it is) and we use the notation "$\overline{x} \in Z$" to indicate this.

Lemma 38.46.3. *Let $i : Z \to X$ be a closed immersion of schemes. Let $\mathcal{G}$ be a sheaf of sets on $Z_{étale}$. Let $\overline{x}$ be a geometric point of X. Then*

$$(i_{small,*}\mathcal{G})_{\overline{x}} = \begin{cases} * & if \quad \overline{x} \not\in Z \\ \mathcal{F}_{\overline{x}} & if \quad \overline{x} \in Z \end{cases}$$

where $*$ denotes a singleton set.

Proof. Note that $i_{small,*}\mathcal{G}|_{U_{étale}} = *$ is the final object in the category of étale sheaves on U, i.e., the sheaf which associates a singleton set to each scheme étale over U. This explains the value of $(i_{small,*}\mathcal{G})_{\overline{x}}$ if $\overline{x} \not\in Z$.

Next, suppose that $\overline{x} \in Z$. Note that

$$(i_{small,*}\mathcal{G})_{\overline{x}} = colim_{(U,\overline{u})}\, \mathcal{G}(U_Z)$$

and on the other hand

$$\mathcal{G}_{\overline{x}} = colim_{(V,\overline{v})}\, \mathcal{G}(V).$$

Let $\mathcal{C}_1 = \{(U, \overline{u})\}^{opp}$ be the opposite of the category of étale neighbourhoods of $\overline{x}$ in X, and let $\mathcal{C}_2 = \{(V, \overline{v})\}^{opp}$ be the opposite of the category of étale neighbourhoods of $\overline{x}$ in Z. The canonical map

$$\mathcal{G}_{\overline{x}} \longrightarrow (i_{small,*}\mathcal{G})_{\overline{x}}$$

corresponds to the functor $F : \mathcal{C}_1 \to \mathcal{C}_2$, $F(U, \overline{u}) = (U_Z, \overline{x})$. Now Lemmas 38.46.2 and 38.46.1 imply that $\mathcal{C}_1$ is cofinal in $\mathcal{C}_2$, see Categories, Definition 4.17.5. Hence it follows that the displayed arrow is an isomorphism, see Categories, Lemma 4.17.6. □

Proposition 38.46.4. *Let $i : Z \to X$ be a closed immersion of schemes.*

(1) *The functor*

$$i_{small,*} : Sh(Z_{étale}) \longrightarrow Sh(X_{étale})$$

is fully faithful and its essential image is those sheaves of sets $\mathcal{F}$ on $X_{étale}$ whose restriction to $X \setminus Z$ is isomorphic to $*$, and

(2) *the functor*
$$i_{small,*} : Ab(Z_{étale}) \longrightarrow Ab(X_{étale})$$
is fully faithful and its essential image is those abelian sheaves on $X_{étale}$ whose support is contained in Z.

In both cases i_{small}^{-1} is a left inverse to the functor $i_{small,*}$.

Proof. Let's discuss the case of sheaves of sets. For any sheaf $\mathcal{G}$ on Z the morphism $i_{small}^{-1} i_{small,*}\mathcal{G} \to \mathcal{G}$ is an isomorphism by Lemma 38.46.3 (and Theorem 38.29.10). This implies formally that $i_{small,*}$ is fully faithful, see Sites, Lemma 9.36.1. It is clear that $i_{small,*}\mathcal{G}|_{U_{étale}} \cong *$ where $U = X \setminus Z$. Conversely, suppose that $\mathcal{F}$ is a sheaf of sets on X such that $\mathcal{F}|_{U_{étale}} \cong *$. Consider the adjunction mapping
$$\mathcal{F} \longrightarrow i_{small,*} i_{small}^{-1}\mathcal{F}$$
Combining Lemmas 38.46.3 and 38.36.2 we see that it is an isomorphism. This finishes the proof of (1). The proof of (2) is identical. □

38.47. Integral universally injective morphisms

Here is the general version of Proposition 38.46.4.

Proposition 38.47.1. *Let $f : X \to Y$ be a morphism of schemes which is integral and universally injective.*

(1) *The functor*
$$f_{small,*} : Sh(X_{étale}) \longrightarrow Sh(Y_{étale})$$
is fully faithful and its essential image is those sheaves of sets $\mathcal{F}$ on $Y_{étale}$ whose restriction to $Y \setminus f(X)$ is isomorphic to $*$, and

(2) *the functor*
$$f_{small,*} : Ab(X_{étale}) \longrightarrow Ab(Y_{étale})$$
is fully faithful and its essential image is those abelian sheaves on $Y_{étale}$ whose support is contained in $f(X)$.

In both cases f_{small}^{-1} is a left inverse to the functor $f_{small,*}$.

Proof. We may factor f as
$$X \xrightarrow{h} Z \xrightarrow{i} Y$$
where h is integral, universally injective and surjective and $i : Z \to Y$ is a closed immersion. Apply Proposition 38.46.4 to i and apply Theorem 38.45.1 to h. □

38.48. Big sites and pushforward

In this section we prove some technical results on $f_{big,*}$ for certain types of morphisms of schemes.

Lemma 38.48.1. *Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Let $f : X \to Y$ be a monomorphism of schemes. Then the canonical map $f_{big}^{-1} f_{big,*}\mathcal{F} \to \mathcal{F}$ is an isomorphism for any sheaf $\mathcal{F}$ on $(Sch/X)_\tau$.*

Proof. In this case the functor $(Sch/X)_\tau \to (Sch/Y)_\tau$ is continuous, cocontinuous and fully faithful. Hence the result follows from Sites, Lemma 9.19.7. □

Remark 38.48.2. In the situation of Lemma 38.48.1 it is true that the canonical map $\mathcal{F} \to f_{big}^{-1} f_{big!}\mathcal{F}$ is an isomorphism for any sheaf of sets $\mathcal{F}$ on $(Sch/X)_\tau$. The proof is the same. This also holds for sheaves of abelian groups. However, note that the functor $f_{big!}$ for sheaves of abelian groups is defined in Modules on Sites, Section 16.16 and is in general different from $f_{big!}$ on sheaves of sets. The result for sheaves of abelian groups follows from Modules on Sites, Lemma 16.16.4.

Lemma 38.48.3. *Let $f : X \to Y$ be a closed immersion of schemes. Let $U \to X$ be a syntomic (resp. smooth, resp. étale) morphism. Then there exist syntomic (resp. smooth, resp. étale) morphisms $V_i \to Y$ and morphisms $V_i \times_Y X \to U$ such that $\{V_i \times_Y X \to U\}$ is a Zariski covering of U.*

Proof. Let us prove the lemma when $\tau = syntomic$. The question is local on U. Thus we may assume that U is an affine scheme mapping into an affine of Y. Hence we reduce to proving the following case: $Y = Spec(A)$, $X = Spec(A/I)$, and $U = Spec(\overline{B})$, where $A/I \to \overline{B}$ be a syntomic ring map. By Algebra, Lemma 7.125.19 we can find elements $\overline{g}_i \in \overline{B}$ such that $\overline{B}_{\overline{g}_i} = A_i/IA_i$ for certain syntomic ring maps $A \to A_i$. This proves the lemma in the syntomic case. The proof of the smooth case is the same except it uses Algebra, Lemma 7.126.19. In the étale case use Algebra, Lemma 7.132.10. □

Lemma 38.48.4. *Let $f : X \to Y$ be a closed immersion of schemes. Let $\{U_i \to X\}$ be a syntomic (resp. smooth, resp. étale) covering. There exists a syntomic (resp. smooth, resp. étale) covering $\{V_j \to Y\}$ such that for each j, either $V_j \times_Y X = \emptyset$, or the morphism $V_j \times_Y X \to X$ factors through U_i for some i.*

Proof. For each i we can choose syntomic (resp. smooth, resp. étale) morphisms $g_{ij} : V_{ij} \to Y$ and morphisms $V_{ij} \times_Y X \to U_i$ over X, such that $\{V_{ij} \times_Y X \to U_i\}$ are Zariski coverings, see Lemma 38.48.3. This in particular implies that $\bigcup_{ij} g_{ij}(V_{ij})$ contains the closed subset $f(X)$. Hence the family of syntomic (resp. smooth, resp. étale) maps g_{ij} together with the open immersion $Y \setminus f(X) \to Y$ forms the desired syntomic (resp. smooth, resp. étale) covering of Y. □

Lemma 38.48.5. *Let $f : X \to Y$ be a closed immersion of schemes. Let $\tau \in \{syntomic, smooth, étale\}$. The functor $V \mapsto X \times_Y V$ defines an almost cocontinuous functor (see Sites, Definition 9.37.3) $(Sch/Y)_\tau \to (Sch/X)_\tau$ between big τ sites.*

Proof. We have to show the following: given a morphism $V \to Y$ and any syntomic (resp. smooth, resp. étale) covering $\{U_i \to X \times_Y V\}$, there exists a smooth (resp. smooth, resp. étale) covering $\{V_j \to V\}$ such that for each j, either $X \times_Y V_j$ is empty, or $X \times_Y V_j \to Z \times_Y V$ factors through one of the U_i. This follows on applying Lemma 38.48.4 above to the closed immersion $X \times_Y V \to V$. □

Lemma 38.48.6. *Let $f : X \to Y$ be a closed immersion of schemes. Let $\tau \in \{syntomic, smooth, étale\}$.*

1. *The pushforward $f_{big,*} : Sh((Sch/X)_\tau) \to Sh((Sch/Y)_\tau)$ commutes with coequalizers and pushouts.*
2. *The pushforward $f_{big,*} : Ab((Sch/X)_\tau) \to Ab((Sch/Y)_\tau)$ is exact.*

Proof. This follows from Sites, Lemma 9.37.6, Modules on Sites, Lemma 16.15.3, and Lemma 38.48.5 above. □

Remark 38.48.7. In Lemma 38.48.6 the case $\tau = fppf$ is missing. The reason is that given a ring A, an ideal I and a faithfully flat, finitely presented ring map $A/I \to \overline{B}$, there is no reason to think that one can find *any* flat finitely presented ring map $A \to B$ with $B/IB \neq 0$ such that $A/I \to B/IB$ factors through $\overline{B}$. Hence the proof of Lemma 38.48.5 does not work for the fppf topology. In fact it is likely false that $f_{big,*} : Ab((Sch/X)_{fppf}) \to Ab((Sch/Y)_{fppf})$ is exact when f is a closed immersion. If you know an example, please email stacks.project@gmail.com.

38.49. Exactness of big lower shriek

This is just the following technical result. Note that the functor $f_{big!}$ has nothing whatsoever to do with cohomology with compact support in general.

Lemma 38.49.1. *Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Let $f : X \to Y$ be a morphism of schemes. Let*

$$f_{big} : Sh((Sch/X)_\tau) \longrightarrow Sh((Sch/Y)_\tau)$$

be the corresponding morphism of topoi as in Topologies, Lemma 30.3.15, 30.4.15, 30.5.10, 30.6.10, or 30.7.12.

(1) *The functor $f_{big}^{-1} : Ab((Sch/Y)_\tau) \to Ab((Sch/X)_\tau)$ has a left adjoint*
$$f_{big!} : Ab((Sch/X)_\tau) \to Ab((Sch/Y)_\tau)$$
which is exact.

(2) *The functor $f_{big}^* : Mod((Sch/Y)_\tau, \mathcal{O}) \to Mod((Sch/X)_\tau, \mathcal{O})$ has a left adjoint*
$$f_{big!} : Mod((Sch/X)_\tau, \mathcal{O}) \to Mod((Sch/Y)_\tau, \mathcal{O})$$
which is exact.

Moreover, the two functors $f_{big!}$ agree on underlying sheaves of abelian groups.

Proof. Recall that f_{big} is the morphism of topoi associated to the continuous and cocontinuous functor $u : (Sch/X)_\tau \to (Sch/Y)_\tau$, $U/X \mapsto U/Y$. Moreover, we have $f_{big}^{-1}\mathcal{O} = \mathcal{O}$. Hence the existence of $f_{big!}$ follows from Modules on Sites, Lemma 16.16.2, respectively Modules on Sites, Lemma 16.35.1. Note that if U is an object of $(Sch/X)_\tau$ then the functor u induces an equivalence of categories

$$u' : (Sch/X)_\tau/U \longrightarrow (Sch/Y)_\tau/U$$

because both sides of the arrow are equal to $(Sch/U)_\tau$. Hence the agreement of $f_{big!}$ on underlying abelian sheaves follows from the discussion in Modules on Sites, Remark 16.35.2. The exactness of $f_{big!}$ follows from Modules on Sites, Lemma 16.16.3 as the functor u above which commutes with fibre products and equalizers. $\square$

Next, we prove a technical lemma that will be useful later when comparing sheaves of modules on different sites associated to algebraic stacks.

Lemma 38.49.2. *Let X be a scheme. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Let $\mathcal{C}_1 \subset \mathcal{C}_2 \subset (Sch/X)_\tau$ be full subcategories with the following properties:*

(1) *For an object U/X of $\mathcal{C}_t$,*
   (a) *if $\{U_i \to U\}$ is a covering of $(Sch/X)_\tau$, then U_i/X is an object of $\mathcal{C}_t$,*
   (b) *$U \times \mathbf{A}^1/X$ is an object of $\mathcal{C}_t$.*
(2) *X/X is an object of $\mathcal{C}_t$.*

We endow $\mathcal{C}_t$ with the structure of a site whose coverings are exactly those coverings $\{U_i \to U\}$ of $(Sch/X)_\tau$ with $U \in Ob(\mathcal{C}_t)$. Then

(i) *The functor $\mathcal{C}_1 \to \mathcal{C}_2$ is fully faithful, continuous, and cocontinuous.*

Denote $g : Sh(\mathcal{C}_1) \to Sh(\mathcal{C}_2)$ the corresponding morphism of topoi. Denote $\mathcal{O}_t$ the restriction of $\mathcal{O}$ to $\mathcal{C}_t$. Denote $g_!$ the functor of Modules on Sites, Definition 16.16.1.

(ii) *The canonical map $g_!\mathcal{O}_1 \to \mathcal{O}_2$ is an isomorphism.*

Proof. Assertion (i) is immediate from the definitions. In this proof all schemes are schemes over X and all morphisms of schemes are morphisms of schemes over X. Note that g^{-1} is given by restriction, so that for an object U of $\mathcal{C}_1$ we have $\mathcal{O}_1(U) = \mathcal{O}_2(U) = \mathcal{O}(U)$. Recall that $g_!\mathcal{O}_1$ is the sheaf associated to the presheaf $g_{p!}\mathcal{O}_1$ which associates to V in $\mathcal{C}_2$ the group

$$colim_{V \to U}\, \mathcal{O}(U)$$

where U runs over the objects of $\mathcal{C}_1$ and the colimit is taken in the category of abelian groups. Below we will use frequently that if

$$V \to U \to U'$$

are morphisms with $U, U' \in Ob(\mathcal{C}_1)$ and if $f' \in \mathcal{O}(U')$ restricts to $f \in \mathcal{O}(U)$, then $(V \to U, f)$ and $(V \to U', f')$ define the same element of the colimit. Also, $g_!\mathcal{O}_1 \to \mathcal{O}_2$ maps the element $(V \to U, f)$ simply to the pullback of f to V.

Surjectivity. Let V be a scheme and let $h \in \mathcal{O}(V)$. Then we obtain a morphism $V \to X \times \mathbf{A}^1$ induced by h and the structure morphism $V \to X$. Writing $\mathbf{A}^1 = Spec(\mathbf{Z}[x])$ we see the element $x \in \mathcal{O}(X \times \mathbf{A}^1)$ pulls back to h. Since $X \times \mathbf{A}^1$ is an object of $\mathcal{C}_1$ by assumptions (1)(b) and (2) we obtain the desired surjectivity.

Injectivity. Let V be a scheme. Let $s = \sum_{i=1,\ldots,n}(V \to U_i, f_i)$ be an element of the colimit displayed above. For any i we can use the morphism $f_i : U_i \to X \times \mathbf{A}^1$ to see that $(V \to U_i, f_i)$ defines the same element of the colimit as $(f_i : V \to X \times \mathbf{A}^1, x)$. Then we can consider

$$f_1 \times \ldots \times f_n : V \to X \times \mathbf{A}^n$$

and we see that s is equivalent in the colimit to

$$\sum\nolimits_{i=1,\ldots,n}(f_1 \times \ldots \times f_n : V \to X \times \mathbf{A}^n, x_i) = (f_1 \times \ldots \times f_n : V \to X \times \mathbf{A}^n, x_1 + \ldots + x_n)$$

Now, if $x_1 + \ldots + x_n$ restricts to zero on V, then we see that $f_1 \times \ldots \times f_n$ factors through $X \times \mathbf{A}^{n-1} = V(x_1 + \ldots + x_n)$. Hence we see that s is equivalent to zero in the colimit. $\square$

38.50. Étale cohomology

In the following sections we prove some basic results on étale cohomology.

38.51. Colimits

Let us start by recalling that if $(\mathcal{F}_i, \varphi_{ii'})$ is a diagram of sheaves on a topological space X its colimit (in the category of sheaves) is the sheafification of the presheaf $U \mapsto colim_i\, \mathcal{F}_i(U)$. See Sheaves, Section 6.28. In the case where X is Noetherian and the system is directed, the sheafification is superfluous: See [**Har77**, Chapter II, Exercise 1.11] for a special case, see Sheaves, Lemma 6.29.1 for a general result. See Cohomology, Lemma 18.15.1 for a result dealing with higher cohomology groups of colimits of abelian sheaves. Finally, see Modules, Lemma 15.11.6 for a result on Hom sheaves of $\mathcal{O}_X$-modules when X is a ringed space.

Theorem 38.51.1. *Let X be a quasi-compact and quasi-separated scheme. Let $\left(\mathcal{F}_i, \varphi_{ij}\right)$ be a system of abelian sheaves on $X_{étale}$ over the partially ordered set I. If I is directed then*

$$colim_{i\in I} H^p_{et}(X, \mathcal{F}_i) = H^p_{et}(X, colim_{i\in I} \mathcal{F}_i).$$

Sketch of proof. This is proven for all X at the same time, by induction on p.

(1) For any quasi-compact and quasi-separated scheme X and any étale covering $\mathcal{U}$ of X, show that there exists a refinement $\mathcal{V} = \left\{\mathcal{V}_j \to X\right\}_{j\in J}$ with J finite and each V_j quasi-compact and quasi-separated such that all the $\mathcal{V}_{j_0} \times_X \cdots \times_X \mathcal{V}_{j_p}$ are also quasi-compact and quasi-separated.
(2) Using the previous step and the definition of colimits in the category of sheaves, show that the theorem holds for $p = 0$, all X. (Exercise.)
(3) Using the locality of cohomology (Lemma 38.22.3), the Čech-to-cohomology spectral sequence (Theorem 38.19.2) and the fact that the induction hypothesis applies to all $\mathcal{V}_{j_0} \times_X \cdots \times_X \mathcal{V}_{j_p}$ in the above situation, prove the induction step $p \to p + 1$.

□

Theorem 38.51.2. *Let A be a ring, $(I, \leq)$ a directed poset and (B_i, φ_{ij}) a system of A-algebras. Set $B = colim_{i\in I} B_i$. Let $X \to Spec(A)$ be a quasi-compact and quasi-separated morphism of schemes and $\mathcal{F}$ an abelian sheaf on $X_{étale}$. Denote $X_i = X \times_{Spec(A)} Spec(B_i)$, $Y = X \times_{Spec(A)} Spec(B)$, $\mathcal{F}_i = (X_i \to X)^{-1}\mathcal{F}$ and $\mathcal{G} = (Y \to X)^{-1}\mathcal{F}$. Then*

$$H^p_{et}(Y, \mathcal{G}) = colim_{i\in I} H^p_{et}((X_i), \mathcal{F}_i).$$

Sketch of proof. The proof proceeds along the following steps.

(1) Given $\mathcal{V} \to Y$ étale with $\mathcal{V}$ quasi-compact and quasi-separated, there exist $i \in I$ and $\mathcal{U}_i \to X_i$ such that $\mathcal{V} = \mathcal{U}_i \times_{X_i} Y$.

If all the schemes considered were affine, this would correspond to the following algebra statement: if $B = colim\, B_i$ and $B \to C$ is étale, then there exist $i \in I$ and $B_i \to C_i$ étale such that $C \cong B \otimes_{B_i} C_i$.

This is proven as follows: write $C \cong B\left[x_1, \ldots, x_n\right]/(f_1, \ldots, f_n)$ with $\det(f_j(x_k)) \in C^*$ and pick $i \in I$ large enough so that all the coefficients of the f_js lie in B_i, and let $C_i = B_i\left[x_1, \ldots, x_n\right]/(f_1, \ldots, f_n)$. This makes sense by the assumption. After further increasing i, $\det(f_j(x_k))$ will be invertible in C_i, and C_i will be étale over B_i.

(2) By (1), we see that for every étale covering $\mathcal{V} = \left\{\mathcal{V}_j \to Y\right\}_{j\in J}$ with J finite and the $\mathcal{V}_j$s quasi-compact and quasi-separated, there exists $i \in I$ and an étale covering $\mathcal{V}_i = \left\{\mathcal{V}_{ij} \to X_i\right\}_{j\in J}$ such that $\mathcal{V} \cong \mathcal{V}_i \times_{X_i} Y$.
(3) Show that (2) implies

$$\check{H}^*(\mathcal{V}, \mathcal{G}) = colim_{i\in I} \check{H}^*(\mathcal{V}_i, \mathcal{F}_i).$$

This is not clear, as we have not explained how to deal with $\mathcal{F}_i$ and $\mathcal{G}$, in particular with the dual.
(4) Use the Čech-to-cohomology spectral sequence (Theorem 38.19.2).

□

38.52. Stalks of higher direct images

Lemma 38.52.1. *Let $f : X \to Y$ be a morphism of schemes and $\mathscr{F} \in Ab(X_{étale})$. Then $R^p f_* \mathscr{F}$ is the sheaf associated to the presheaf*

$$(V \to Y) \longmapsto H^0_{et}\left(X \times_Y V, \mathscr{F}|_{X \times_Y V}\right).$$

This lemma is valid for topological spaces, and the proof in this case is the same.

Theorem 38.52.2. *Let $f : X \to S$ be a quasi-compact and quasi-separated morphism of schemes, $\mathscr{F}$ an abelian sheaf on $X_{étale}$, and $\bar{s}$ a geometric point of S. Then*

$$\left(R^p f_* \mathscr{F}\right)_{\bar{s}} = H^p_{et}\left(X \times_S Spec(\mathscr{O}^{\mathrm{sh}}_{S,\bar{s}}), pr^{-1}\mathscr{F}\right)$$

where pr is the projection $X \times_S Spec(\mathscr{O}^{\mathrm{sh}}_{S,\bar{s}}) \to X$.

Proof. Let $\mathscr{I}$ be the category opposite to the category of étale neighborhoods of $\bar{s}$ on S. By Lemma 38.52.1 we have

$$\left(R^p f_* \mathscr{F}\right)_{\bar{s}} = colim_{(\mathscr{V},\bar{v}) \in \mathscr{I}} H^p(X \times_S \mathscr{V}, \mathscr{F}|_{X \times_S \mathscr{V}}).$$

On the other hand,

$$\mathscr{O}^{\mathrm{sh}}_{S,\bar{s}} = colim_{(\mathscr{V},\bar{v}) \in \mathscr{I}} \Gamma(\mathscr{V}, \mathscr{O}_{\mathscr{V}}).$$

Replacing $\mathscr{I}$ with its cofinal subset $\mathscr{I}^{\mathrm{aff}}$ consisting of affine étale neighborhoods $\mathscr{V}_i = Spec(B_i)$ of $\bar{s}$ mapping into some fixed affine open $Spec(A) \subset S$, we get

$$\mathscr{O}^{\mathrm{sh}}_{S,\bar{s}} = colim_{i \in \mathscr{I}^{\mathrm{aff}}} B_i,$$

and the result follows from Theorem 38.51.2. □

38.53. The Leray spectral sequence

Lemma 38.53.1. *Let $f : X \to Y$ be a morphism and $\mathscr{I}$ an injective sheaf in $Ab(X_{étale})$. Then*

1. *for any $\mathscr{V} \in Ob(Y_{étale})$ and any étale covering $\mathscr{V} = \left\{\mathscr{V}_j \to \mathscr{V}\right\}_{j \in J}$ we have $\check{H}^p(\mathscr{V}, f_*\mathscr{I}) = 0$ for all $p > 0$;*
2. *$f_*\mathscr{I}$ is acyclic for the functors $\Gamma(Y, -)$ and $\Gamma(\mathscr{V}, -)$; and*
3. *if $g : Y \to Z$, then $f_*\mathscr{I}$ is acyclic for g_*.*

Proof. Observe that $\check{\mathscr{C}}^\bullet(\mathscr{V}, f_*\mathscr{I}) = \check{\mathscr{C}}^\bullet(\mathscr{V} \times_Y X, \mathscr{I})$ which has no cohomology by Lemma 38.18.7, which proves *i*. The second statement is a great exercise in using the Čech-to-cohomology spectral sequence. See (insert future reference) for more details. Part *iii* is a consequence of *ii* and the description of $R^p g_*$ from Theorem 38.52.2. □

Using the formalism of Grothendieck spectral sequences, this gives the following.

Proposition 38.53.2. *(Leray spectral sequence) Let $f : X \to Y$ be a morphism of schemes and $\mathscr{F}$ an étale sheaf on X. Then there is a spectral sequence*

$$E_2^{p,q} = H^p_{et}(Y, R^q f_* \mathscr{F}) \Rightarrow H^{p+q}_{et}(X, \mathscr{F}).$$

38.54. Vanishing of finite higher direct images

The next goal is to prove that the higher direct images of a finite morphism of schemes vanish.

Lemma 38.54.1. *Let R be a strictly henselian ring and $S = Spec(R)$. Then the global sections functor $\Gamma(S, -) : Ab(S_{étale}) \to Ab$ is exact. In particular*

$$\forall p \geq 1, \quad H^p_{et}(S, \mathcal{F}) = 0$$

for all $\mathcal{F} \in Ab(S_{étale})$.

Proof. Let $\mathcal{U} = \left\{ f_i : \mathcal{U}_i \to S \right\}_{i \in I}$ be an étale covering, and denote s the closed point of S. Then $s = f_i(u_i)$ for some $i \in I$ and some $u_i \in U_i$ by Lemma 38.29.5. Pick an affine open neighborhood $Spec(A)$ of u_i in $\mathcal{U}_i$. Then there is a commutative diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d] & A \arrow[d] \\ \kappa(s) \arrow[r] & \kappa(u_i) \end{tikzcd}$$

where $\kappa(s)$ is separably closed, and the residue extension is finite separable. Therefore, $\kappa(s) \cong \kappa(u_i)$, and using part v of Theorem 38.32.4, we see that $A \cong R \times A'$ and we get a section

$$\begin{tikzcd} Spec(A) \arrow[r, hook] \arrow[dr] & \mathcal{U}_i \arrow[d] \\ & S. \arrow[ul, bend left] \end{tikzcd}$$

In particular, the covering $\{\mathrm{id} : S \to S\}$ refines $\mathcal{U}$. This implies that if

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \xrightarrow{\alpha} \mathcal{F}_3 \to 0$$

is a short exact sequence in $Ab(S_{étale})$, then the sequence

$$0 \to \Gamma(S_{étale}, \mathcal{F}_1) \to \Gamma(S_{étale}, \mathcal{F}_2) \to \Gamma(S_{étale}, \mathcal{F}_3) \to 0$$

is also exact. Indeed, exactness is clear except possibly at the last step. But given a section $s \in \Gamma(S_{étale}, \mathcal{F}_3)$, we know that there exist a covering $\mathcal{U}$ and local sections s_i such that $\alpha(s_i) = s|_{\mathcal{U}_i}$. But since this covering can be refined by the identity, the s_i must agree locally with s, hence they glue to a global section of $\mathcal{F}_2$. □

Proposition 38.54.2. *Let $f : X \to Y$ be a finite morphism of schemes. Then for all $q \geq 1$ and all $\mathcal{F} \in Ab(X_{étale})$, $R^q f_* \mathcal{F} = 0$.*

Proof. Let $X^{sh}_{\bar{y}}$ denote the fiber product $X \times_Y Spec(\mathcal{O}^{sh}_{Y,\bar{y}})$. It suffices to show that for all $q \geq 1$, $H^q_{et}(X^{sh}_{\bar{y}}, \mathcal{G}) = 0$. Since f is finite, $X^{sh}_{\bar{y}}$ is finite over $Spec(\mathcal{O}^{sh}_{Y,\bar{y}})$, thus $X^{sh}_{\bar{y}} = Spec(A)$ for some ring A finite over $\mathcal{O}^{sh}_{Y,\bar{y}}$. Since the latter is strictly henselian, Lemma 38.32.5 implies that A is henselian and therefore splits as a product of henselian local rings $A_1 \times \cdots \times A_r$. Furthermore, $\kappa(\mathcal{O}^{sh}_{Y,\bar{y}})$ is separably closed and for each i, the residue field extension $\kappa(\mathcal{O}^{sh}_{Y,\bar{y}}) \subset \kappa(A_i)$ is finite, hence $\kappa(A_i)$ is separably closed and A_i is strictly henselian. This implies that $Spec(A) = \coprod_{i=1}^r Spec(A_i)$, and we can apply Lemma 38.54.1 to get the result. □

38.55. Schemes étale over a point

In this section we describe schemes étale over the spectrum of a field. Before we state the result we introduce the category of G-sets for a topological group G.

Definition 38.55.1. Let G be a topological group. A *G-set*, sometime called a *discrete G-set*, is a set X endowed with a left action $a : G \times X \to X$ such that a is continuous when X is given the discrete topology and $G \times X$ the product topology. A *morphism of G-sets* $f : X \to Y$ is simply any G-equivariant map from X to Y. The category of G-sets is denoted *G-Sets*.

The condition that $a : G \times X \to X$ is continuous signifies simply that the stabilizer of any $x \in X$ is open in G. If G is an abstract group G (i.e., a group but not a topological group) then this agrees with our preceding definition (see for example Sites, Example 9.6.5) provided we endow G with the discrete topology.

Recall that if $K \subset L$ is an infinite Galois extension the Galois group $G = \text{Gal}(L/K)$ comes endowed with a canonical topology. Namely the open subgroups are the subgroups of the form $\text{Gal}(L/K') \subset G$ where K'/K is a finite subextension of L/K. The index of an open subgroup is always finite. We say that G is a profinite (topological) group.

Lemma 38.55.2. *Let K be a field. Let K^{sep} a separable closure of K. Consider the profinite group*

$$G = Aut_{Spec(K)}(Spec(K^{sep}))^{opp} = Gal(K^{sep}/K)$$

The functor

$$\begin{matrix} \textit{schemes étale over } K & \longrightarrow & G\textit{-Sets} \\ X/K & \longmapsto & Mor_{Spec(K)}(Spec(K^{sep}), X) \end{matrix}$$

is an equivalence of categories.

Proof. A scheme X over K is étale over K if and only if $X \cong \coprod_{i \in I} Spec(K_i)$ with each K_i a finite separable extension of K. The functor of the lemma associates to X the G-set

$$\coprod\nolimits_i Hom_K(K_i, K^{sep})$$

with its natural left G-action. Each element has an open stabilizer by definition of the topology on G. Conversely, any G-set S is a disjoint union of its orbits. Say $S = \coprod S_i$. Pick $s_i \in S_i$ and denote $G_i \subset G$ its open stabilizer. By Galois theory the fields $(K^{sep})^{G_i}$ are finite separable field extensions of K, and hence the scheme

$$\coprod\nolimits_i Spec((K^{sep})^{G_i})$$

is étale over K. This gives an inverse to the functor of the lemma. Some details omitted. □

Remark 38.55.3. Under the correspondence of the lemma, the coverings in the small étale site $Spec(K)_{étale}$ of K correspond to surjective families of maps in *G-Sets*.

38.56. Galois action on stalks

In this section we define an action of the absolute Galois group of a residue field of a point s of S on the stalk functor at any geometric point lying over s.

Galois action on stalks. Let S be a scheme. Let $\overline{s}$ be a geometric point of S. Let $\sigma \in \text{Aut}(\kappa(\overline{s})/\kappa(s))$. Define an action of σ on the stalk $\mathcal{F}_{\overline{s}}$ of a sheaf $\mathcal{F}$ as follows

$$\begin{matrix} \mathcal{F}_{\overline{s}} & \longrightarrow & \mathcal{F}_{\overline{s}} \\ (U, \overline{u}, t) & \longmapsto & (U, \overline{u} \circ Spec(\sigma), t). \end{matrix} \tag{38.56.0.1}$$

where we use the description of elements of the stalk in terms of triples as in the discussion following Definition 38.29.6. This is a left action, since if $\sigma_i \in \text{Aut}(\kappa(\overline{s})/\kappa(s))$ then

$$\begin{aligned}\sigma_1 \cdot (\sigma_2 \cdot (U, \overline{u}, t)) &= \sigma_1 \cdot (U, \overline{u} \circ Spec(\sigma_2), t) \\ &= (U, \overline{u} \circ Spec(\sigma_2) \circ Spec(\sigma_1), t) \\ &= (U, \overline{u} \circ Spec(\sigma_1 \circ \sigma_2), t) \\ &= (\sigma_1 \circ \sigma_2) \cdot (U, \overline{u}, t)\end{aligned}$$

It is clear that this action is functorial in the sheaf $\mathcal{F}$. We note that we could have defined this action by referring directly to Remark 38.29.8.

Definition 38.56.1. Let S be a scheme. Let $\overline{s}$ be a geometric point lying over the point s of S. Let $\kappa(s) \subset \kappa(s)^{sep} \subset \kappa(\overline{s})$ denote the separable algebraic closure of $\kappa(s)$ in the algebraically closed field $\kappa(\overline{s})$.

(1) In this situation the *absolute Galois group* of $\kappa(s)$ is $\text{Gal}(\kappa(s)^{sep}/\kappa(s))$. It is sometimes denoted $\text{Gal}_{\kappa(s)}$.
(2) The geometric point $\overline{s}$ is called *algebraic* if $\kappa(s) \subset \kappa(\overline{s})$ is an algebraic closure of $\kappa(s)$.

Example 38.56.2. The geometric point $Spec(\mathbf{C}) \to Spec(\mathbf{Q})$ is not algebraic.

Let $\kappa(s) \subset \kappa(s)^{sep} \subset \kappa(\overline{s})$ be as in the definition. Note that as $\kappa(\overline{s})$ is algebraically closed the map

$$\text{Aut}(\kappa(\overline{s})/\kappa(s)) \longrightarrow \text{Gal}(\kappa(s)^{sep}/\kappa(s)) = \text{Gal}_{\kappa(s)}$$

is surjective. Suppose $(U, \overline{u})$ is an étale neighbourhood of $\overline{s}$, and say $\overline{u}$ lies over the point u of U. Since $U \to S$ is étale, the residue field extension $\kappa(s) \subset \kappa(u)$ is finite separable. This implies the following

(1) If $\sigma \in \text{Aut}(\kappa(\overline{s})/\kappa(s)^{sep})$ then σ acts trivially on $\mathcal{F}_{\overline{s}}$.
(2) More precisely, the action of $\text{Aut}(\kappa(\overline{s})/\kappa(s))$ determines and is determined by an action of the absolute Galois group $\text{Gal}_{\kappa(s)}$ on $\mathcal{F}_{\overline{s}}$.
(3) Given $(U, \overline{u}, t)$ representing an element ξ of $\mathcal{F}_{\overline{s}}$ any element of $\text{Gal}(\kappa(s)^{sep}/K)$ acts trivially, where $\kappa(s) \subset K \subset \kappa(s)^{sep}$ is the image of $\overline{u}^\sharp : \kappa(u) \to \kappa(\overline{s})$.

Alltogether we see that $\mathcal{F}_{\overline{s}}$ becomes a $\text{Gal}_{\kappa(s)}$-set (see Definition 38.55.1). Hence we may think of the stalk functor as a functor

$$Sh(S_{étale}) \longrightarrow \text{Gal}_{\kappa(s)}\text{-}Sets, \quad \mathcal{F} \longmapsto \mathcal{F}_{\overline{s}}$$

and from now on we usually do think about the stalk functor in this way.

Theorem 38.56.3. *Let $S = Spec(K)$ with K a field. Let $\overline{s}$ be a geometric point of S. Let $G = Gal_{\kappa(s)}$ denote the absolute Galois group. Then the functor above induces an equivalence of categories*

$$Sh(S_{étale}) \longrightarrow G\text{-}Sets, \quad \mathcal{F} \longmapsto \mathcal{F}_{\overline{s}}.$$

Proof. Let us construct the inverse to this functor. In Lemma 38.55.2 we have seen that given a G-set M there exists an étale morphism $X \to Spec(K)$ such that $Mor_K(Spec(K^{sep}), X)$ is isomorphic to M as a G-set. Consider the sheaf $\mathcal{F}$ on $Spec(K)_{étale}$ defined by the rule $U \mapsto Mor_K(U, X)$. This is a sheaf as the étale topology is subcanonical. Then we see that $\mathcal{F}_{\overline{s}} = Mor_K(Spec(K^{sep}), X) = M$ as G-sets (details omitted). This gives the inverse of the functor and we win. □

Remark 38.56.4. Another way to state the conclusions of Lemmas 38.55.2 and Theorem 38.56.3 is to say that every sheaf on $Spec(K)_{étale}$ is representable by a scheme X étale over $Spec(K)$. This does not mean that every sheaf is representable in the sense of Sites, Definition 9.12.3. The reason is that in our construction of $Spec(K)_{étale}$ we chose a sufficiently large set of schemes étale over $Spec(K)$, whereas sheaves on $Spec(K)_{étale}$ form a proper class.

Lemma 38.56.5. *Assumptions and notations as in Theorem 38.56.3. There is a functorial bijection*

$$\Gamma(S, \mathcal{F}) = (\mathcal{F}_{\overline{s}})^G$$

Proof. We can prove this using formal arguments and the result of Theorem 38.56.3 as follows. Given a sheaf $\mathcal{F}$ corresponding to the G-set $M = \mathcal{F}_{\overline{s}}$ we have

$$\begin{aligned} \Gamma(S, \mathcal{F}) &= Mor_{Sh(S_{étale})}(h_{Spec(K)}, \mathcal{F}) \\ &= Mor_{G\text{-}Sets)}(\{*\}, M) \\ &= M^G \end{aligned}$$

Here the first identification is explained in Sites, Sections 9.2 and 9.12, the second results from Theorem 38.56.3 and the third is clear. We will also give a direct proof[5].

Suppose that $t \in \Gamma(S, \mathcal{F})$ is a global section. Then the triple $(S, \overline{s}, t)$ defines an element of $\mathcal{F}_{\overline{s}}$ which is clearly invariant under the action of G. Conversely, suppose that $(U, \overline{u}, t)$ defines an element of $\mathcal{F}_{\overline{s}}$ which is invariant. Then we may shrink U and assume $U = Spec(L)$ for some finite separable field extension of K, see Proposition 38.26.2. In this case the map $\mathcal{F}(U) \to \mathcal{F}_{\overline{s}}$ is injective, because for any morphism of étale neighbourhoods $(U', \overline{u}') \to (U, \overline{u})$ the restriction map $\mathcal{F}(U) \to \mathcal{F}(U')$ is injective since $U' \to U$ is a covering of $S_{étale}$. After enlarging L a bit we may assume $K \subset L$ is a finite Galois extension. At this point we use that

$$Spec(L) \times_{Spec(K)} Spec(L) = \coprod\nolimits_{\sigma \in \mathrm{Gal}(L/K)} Spec(L)$$

where the maps $Spec(L \otimes_K L) \to Spec(L)$ come from the ring maps $a \otimes b \mapsto a\sigma(b)$. Hence we see that the condition that $(U, \overline{u}, t)$ is invariant under all of G implies that $t \in \mathcal{F}(Spec(L))$ maps to the same element of $\mathcal{F}(Spec(L) \times_{Spec(K)} Spec(L))$ via restriction by either projection (this uses the injectivity mentioned above; details omitted). Hence the sheaf condition of $\mathcal{F}$ for the étale covering $\{Spec(L) \to Spec(K)\}$ kicks in and we conclude that t comes from a unique section of $\mathcal{F}$ over $Spec(K)$. □

Remark 38.56.6. Let S be a scheme and let $\overline{s} : Spec(k) \to S$ be a geometric point of S. By definition this means that k is algebraically closed. In particular the absolute Galois group of k is trivial. Hence by Theorem 38.56.3 the category of sheaves on $Spec(k)_{étale}$ is equivalent to the category of sets. The equivalence is given by taking sections over $Spec(k)$. This finally provides us with an alternative definition of the stalk functor. Namely, the functor

$$Sh(S_{étale}) \longrightarrow Sets, \quad \mathcal{F} \longmapsto \mathcal{F}_{\overline{s}}$$

is isomorphic to the functor

$$Sh(S_{étale}) \longrightarrow Sh(Spec(k)_{étale}) = Sets, \quad \mathcal{F} \longmapsto \overline{s}^*\mathcal{F}$$

To prove this rigorously one can use Lemma 38.36.2 part (3) with $f = \overline{s}$. Moreover, having said this the general case of Lemma 38.36.2 part (3) follows from functoriality of pullbacks.

[5]For the doubting Thomases out there.

38.57. Cohomology of a point

As a consequence of the discussion in the preceding two sections we obtain the equivalence of étale cohomology of the spectrum of a field with Galois cohomology.

Definition 38.57.1. Let G be a topological group. A *G-module*, sometime called a *discrete G-module*, is an abelian group M endowed with a left action $a : G \times M \to M$ by group homomorphisms such that a is continuous when M is given the discrete topology and $G \times M$ the product topology. A *morphism of G-modules* $f : M \to N$ is simply any G-equivariant homomorphism from M to N. The category of G-modules is denoted Mod_G.

The condition that $a : G \times M \to M$ is continuous signifies simply that the stabilizer of any $x \in M$ is open in G. If G is an abstract group G (i.e., a group but not a topological group) then this corresponds to the notion of an abelian group endowed with a G-action provided we endow G with the discrete topology.

Lemma 38.57.2. *Let $S = Spec(K)$ with K a field. Let $\overline{s}$ be a geometric point of S. Let $G = Gal_{\kappa(s)}$ denote the absolute Galois group. The stalk functor induces an equivalence of categories*

$$Ab(S_{étale}) \longrightarrow Mod_G, \quad \mathcal{F} \longmapsto \mathcal{F}_{\overline{s}}.$$

Proof. In Theorem 38.56.3 we have seen the equivalence between sheaves of sets and G-sets. The current lemma follows formally from this as an abelian sheaf is just a sheaf of sets endowed with a commutative group law, and a G-module is just a G-set endowed with a commutative group law. □

The category Mod_G has enough injectives, see Injectives, Lemma 17.7.1. Consider the left exact functor

$$Mod_G \longrightarrow Ab, \quad M \longmapsto M^G = \{x \in M \mid g \cdot x = x \ \forall g \in G\}$$

We sometimes denote $M^G = H^0(G, M)$ and sometimes we write $M^G = \Gamma_G(M)$. This functor has a total right derived functor $R\Gamma_G(M)$ and ith right derived functor $R^i\Gamma_G(M) = H^i(G, M)$ for any $i \geq 0$.

Definition 38.57.3. Let G be a topological group.

(1) The right derived functors $H^i(G, M)$ are called the *continuous group cohomology groups* of M.
(2) If G is an abstract group endowed with the discrete topology then the $H^i(G, M)$ are called the *group cohomology groups* of M.
(3) If G is a Galois group, then the groups $H^i(G, M)$ are called the *Galois cohomology groups* of M.
(4) If G is the absolute Galois group of a field K, then the groups $H^i(G, M)$ are sometimes called the *Galois cohomology groups of K with coefficients in M*.

Lemma 38.57.4. *Notation and assumptions as in Lemma 38.57.2. Let $\mathcal{F}$ be an abelian sheaf on $Spec(K)_{étale}$ which corresponds to the G-module M. Then*

(1) *in $D(Ab)$ we have a canonical isomorphism $R\Gamma(S, \mathcal{F}) = R\Gamma_G(M)$,*
(2) *$H^0_{et}(S, \mathcal{F}) = M^G$, and*
(3) *$H^q_{et}(S, \mathcal{F}) = H^q(G, M)$.*

Proof. Combine Lemma 38.57.2 with Lemma 38.56.5. □

Example 38.57.5. Sheaves on $Spec(K)_{étale}$. Let $G = \mathrm{Gal}(K^{sep}/K)$ be the absolute Galois group of K.

(1) The constant sheaf $\underline{\mathbf{Z}/n\mathbf{Z}}$ corresponds to the module $\mathbf{Z}/n\mathbf{Z}$ with trivial G-action,
(2) the sheaf $\mathbf{G}_m|_{Spec(K)_{étale}}$ corresponds to $(K^{sep})^*$ with its G-action,
(3) the sheaf $\mathbf{G}_a|_{Spec(K^{sep})}$ corresponds to $(K^{sep}, +)$ with its G-action, and
(4) the sheaf $\mu_n|_{Spec(K^{sep})}$ corresponds to $\mu_n(K^{sep})$ with its G-action.

By Remark 38.23.4 and Theorem 38.24.1 we have the following identifications for cohomology groups:

$$H^0_{et}(S_{étale}, \mathbf{G}_m) = \Gamma(S, \mathcal{O}_S^*)$$
$$H^1_{et}(S_{étale}, \mathbf{G}_m) = H^1_{Zar}(S, \mathcal{O}_S^*) = \mathrm{Pic}(S)$$
$$H^i_{et}(S_{étale}, \mathbf{G}_a) = H^i_{Zar}(S, \mathcal{O}_S)$$

Also, for any quasi-coherent sheaf $\mathcal{F}$ on $S_{étale}$ we have

$$H^i(S_{étale}, \mathcal{F}) = H^i_{Zar}(S, \mathcal{F}),$$

see Theorem 38.22.4. In particular, this gives the following sequence of equalities

$$0 = \mathrm{Pic}(Spec(K)) = H^1_{et}(Spec(K)_{étale}, \mathbf{G}_m) = H^1(G, (K^{sep})^*)$$

which is none other than Hilbert's 90 theorem. Similarly, for $i \geq 1$,

$$0 = H^i(Spec(K), \mathcal{O}) = H^i_{et}(Spec(K)_{étale}, \mathbf{G}_a) = H^i(G, K^{sep})$$

where the K^{sep} indicates K^{sep} as a Galois module with addition as group law. In this way we may consider the work we have done sofar as a complicated way of computing Galois cohomology groups.

38.58. Cohomology of curves

The next task at hand is to compute the étale cohomology of a smooth curve with torsion coefficients, and in particular show that it vanishes in degree at least 3. To prove this, we will compute cohomology at the generic point, which amounts to some Galois cohomology. We now review without proofs. the relevant facts about Brauer groups. For references, see [**Ser62**], [**Ser97**] or [**Wei48**].

38.59. Brauer groups

Brauer groups of fields, defined using finite central simple algebras, are discussed in the chapter Brauer Groups, Section 8.1. Here we give a synopsis.

Theorem 38.59.1. *Let K be a field. For a unital, associative (not necessarily commutative) K-algebra A the following are equivalent*

(1) *A is finite central simple K-algebra,*
(2) *A is a finite dimensional K-vector space, K is the center of A, and A has no nontrivial two-sided ideal,*
(3) *there exists $d \geq 1$ such that $A \otimes_K \bar{K} \cong Mat(d \times d, \bar{K})$,*
(4) *there exists $d \geq 1$ such that $A \otimes_K K^{sep} \cong Mat(d \times d, K^{sep})$,*
(5) *there exist $d \geq 1$ and a finite Galois extension $K \subset K'$ such that $A \otimes_{K'} K' \cong Mat(d \times d, K')$,*
(6) *there exist $n \geq 1$ and a finite central skew field D over K such that $A \cong Mat(n \times n, D)$.*

The integer d is called the degree of A.

Proof. This is a copy of Brauer Groups, Lemma 8.8.6. ☐

Lemma 38.59.2. *Let A be a finite central simple algebra over K. Then*

$$\begin{array}{ccc} A \otimes_K A^{opp} & \longrightarrow & End_K(A) \\ a \otimes a' & \longmapsto & (x \mapsto axa') \end{array}$$

is an isomorphism of algebras over K.

Proof. See Brauer Groups, Lemma 8.4.10. □

Definition 38.59.3. Two finite central simple algebras A_1 and A_2 over K are called *similar*, or *equivalent* if there exist $m, n \geq 1$ such that $\mathrm{Mat}(n \times n, A_1) \cong \mathrm{Mat}(m \times m, A_2)$. We write $A_1 \sim A_2$.

Definition 38.59.4. Let K be a field. The *Brauer group* of K is the set $\mathrm{Br}(K)$ of similarity classes of finite central simple algebras over K, endowed with the group law induced by tensor product (over K). The class of A in $\mathrm{Br}(K)$ is denoted by $[A]$. The neutral element is $[K] = [\mathrm{Mat}(d \times d, K)]$ for any $d \geq 1$.

The previous lemma thus mean that inverses exist, and that $-[A] = [A^{opp}]$. The Brauer group is always torsion, but not finitely generated in general. We will see that $A^{\otimes \deg A} \sim K$ for any finite central simple algebra A (insert future reference here).

Lemma 38.59.5. *Let K be a field and $\mathcal{G} = Gal(K^{sep}|K)$). Then the set of isomorphism classes of central simple algebras of degree d over K is in bijection with the non-abelian cohomology $H^1_{cont}(\mathcal{G}, PGL_d(K^{sep}))$.*

Sketch of proof. The Skolem-Noether theorem (see Brauer Groups, Theorem 8.6.1) implies that for any field L the group $\mathrm{Aut}_{L\text{-Algebras}}(\mathrm{Mat}_d(L))$ equals $\mathrm{PGL}_d(L)$. By Theorem 38.59.1, we see that central simple algebras of degree d correspond to forms of the K-algebra $\mathrm{Mat}_d(K)$, which in turn correspond to $H^1_{cont}(\mathcal{G}, \mathrm{PGL}_d(K^{sep}))$. For more details on twisting, see for example [**Sil86**]. □

If A is a finite central simple algebra over K, we denote ξ_A the corresponding cohomology class in $H^1_{cont}(\mathcal{G}, \mathrm{PGL}_{\deg A}(K^{sep}))$. Consider now the short exact sequence

$$1 \to (K^{sep})^* \to \mathrm{GL}_d(K^{sep}) \to \mathrm{PGL}_d(K^{sep}) \to 1,$$

which gives rise to a long exact cohomology sequence (up to degree 2) with coboundary map

$$\delta_d : H^1_{cont}(\mathcal{G}, \mathrm{PGL}_d(K^{sep})) \to H^2(\mathcal{G}, (K^{sep})^*).$$

Explicitly, this is given as follows: if ξ is a cohomology class represented by the 1-cocyle (g_σ), then $\delta_d(\xi)$ is the class of the 2-cocycle $((g_\sigma^\tau)^{-1} g_{\sigma\tau} g_\tau^{-1})$.

Theorem 38.59.6. *The map*

$$\begin{array}{cccc} \delta : & Br(K) & \longrightarrow & H^2(\mathcal{G}, (K^{sep})^*) \\ & [A] & \longmapsto & \delta_{\deg A}(\xi_A) \end{array}$$

is a group isomorphism.

Proof. Omitted. Hints: In the abelian case ($d = 1$), one has the identification

$$H^1(\mathcal{G}, \mathrm{GL}_d(K^{sep})) = H^1_{et}(Spec(K), \mathrm{GL}_d(\mathcal{O}))$$

the latter of which is trivial by fpqc descent. If this were true in the non-abelian case, this would readily imply injectivity of δ. (See [**Del77**].) Rather, to prove this, one can reinterpret $\delta([A])$ as the obstruction to the existence of a K-vector space V with a left A-module structure and such that $\dim_K V = \deg A$. In the case where V exists, one has $A \cong \mathrm{End}_K(V)$. For surjectivity, pick a cohomology class $\xi \in H^2(\mathcal{G}, (K^{sep})^*)$, then there exists a finite Galois extension $K \subset K' \subset K^{sep}$ such that ξ is the image of some $\xi' \in H^2_{cont}(\mathrm{Gal}(K'|K), (K')^*)$. Then write down an explicit central simple algebra over K using the data K', ξ'. □

The Brauer group of a scheme. Let S be a scheme. An $\mathcal{O}_S$-algebra $\mathcal{A}$ is called *Azumaya* if it is étale locally a matrix algebra, i.e., if there exists an étale covering $\mathcal{U} = \{\varphi_i : \mathcal{U}_i \to S\}_{i\in I}$ such that $\varphi_i^*\mathcal{A} \cong \mathrm{Mat}_{d_i}(\mathcal{O}_{\mathcal{U}_i})$ for some $d_i \geq 1$. Two such $\mathcal{A}$ and $\mathcal{B}$ are called *equivalent* if there exist finite locally free sheaves $\mathcal{F}$ and $\mathcal{G}$ on S such that $\mathcal{A} \otimes_{\mathcal{O}_S} \mathrm{End}(\mathcal{F}) \cong \mathcal{B} \otimes_{\mathcal{O}_S} \mathrm{End}(\mathcal{G})$. The *Brauer group* of S is the set $\mathrm{Br}(S)$ of equivalence classes of Azumaya $\mathcal{O}_S$-algebras with the operation induced by tensor product (over $\mathcal{O}_S$).

In this setting, the analogue of the isomorphism δ of Theorem 38.59.6 is a map

$$\delta_S : \mathrm{Br}(S) \to H^2_{et}(S, \mathbf{G}_m).$$

It is true that δ_S is injective (the previous argument still works). If S is quasi-compact or connected, then $\mathrm{Br}(S)$ is a torsion group, so in this case the image of δ_S is contained in the *cohomological Brauer group* of S

$$\mathrm{Br}'(S) := H^2_{et}(S, \mathbf{G}_m)_{\mathrm{torsion}}.$$

So if S is quasi-compact or connected, there is an inclusion $\mathrm{Br}(S) \subset \mathrm{Br}'(S)$. This is not always an equality: there exists a nonseparated singular surface S for which $\mathrm{Br}(S) \subset \mathrm{Br}'(S)$ is a strict inclusion. If S is quasi-projective, then $\mathrm{Br}(S) = \mathrm{Br}'(S)$. However, it is not known whether this holds for a smooth proper variety over $\mathbf{C}$, say.

Proposition 38.59.7. *Let K be a field, $\mathcal{G} = Gal(K^{sep}|K)$ and suppose that for any finite extension K' of K, $Br(K') = 0$. Then*

(1) *for all $q \geq 1$, $H^q(\mathcal{G}, (K^{sep})^*) = 0$; and*
(2) *for any torsion $\mathcal{G}$-module M and any $q \geq 2$, $H^q_{cont}(\mathcal{G}, M) = 0$.*

See [**Ser97**] for proofs.

Definition 38.59.8. A field K is called C_r if for every $0 < d^r < n$ and every $f \in K[T_1, \ldots, T_n]$ homogeneous of degree d, there exist $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \in K$ not all zero, such that $f(\alpha) = 0$. Such an α is called a *nontrivial solution* of f.

Example 38.59.9. An algebraically closed field is C_r.

In fact, we have the following simple lemma.

Lemma 38.59.10. *Let k be an algebraically closed field. Let $f_1, \ldots, f_s \in k[T_1, \ldots, T_n]$ be homogeneous polynomials of degree $d_1, \ldots, d_s$ with $d_i > 0$. If $s < n$, then $f_1 = \ldots = f_s = 0$ have a common nontrivial solution.*

Proof. Omitted. □

The following result computes the Brauer group of C_1 fields.

Theorem 38.59.11. *Let K be a C_1 field. Then $Br(K) = 0$.*

Proof. Let D be a finite dimensional division algebra over K with center K. We have seen that

$$D \otimes_K K^{sep} \cong \mathrm{Mat}_d(K^{sep})$$

uniquely up to inner isomorphism. Hence the determinant $\det : \mathrm{Mat}_d(K^{sep}) \to K^{sep}$ is Galois invariant and descends to a homogeneous degree d map

$$\det = N_{\mathrm{red}} : D \longrightarrow K$$

called the *reduced norm*. Since K is C_1, if $d > 1$, then there exists a nonzero $x \in D$ with $N_{\text{red}}(x) = 0$. This clearly implies that x is not invertible, which is a contradiction. Hence $\text{Br}(K) = 0$. □

Theorem 38.59.12. *(Tsen) The function field of a variety of dimension r over an algebraically closed field k is C_r.*

Proof.
(1) Projective space. The field $k(x_1, \dots, x_r)$ is C_r (exercise).
(2) General case. Without loss of generality, we may assume X to be projective. Let $f \in K[T_1, \dots, T_n]_d$ with $0 < d^r < n$. Say the coefficients of f are in $\Gamma(X, \mathcal{O}_X(H))$ for some ample $H \subset X$. Let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$ with $\alpha_i \in \Gamma(X, \mathcal{O}_X(eH))$. Then $f(\boldsymbol{\alpha}) \in \Gamma(X, \mathcal{O}_X((de+1)H))$. Consider the system of equations $f(\boldsymbol{\alpha}) = 0$. Then by asymptotic Riemann-Roch,
   - the number of variables is $n \dim_K \Gamma(X, \mathcal{O}_X(eH)) \sim n\frac{e^r}{r!}(H^r)$; and
   - the number of equations is $\dim_K \Gamma(X, \mathcal{O}_X((de+1)H)) \sim \frac{(de+1)^r}{r!}(H^r)$.

   Since $n > d^r$, there are more variables than equations, and since there is a trivial solution, there are also nontrivial solutions.

□

Definition 38.59.13. We call *variety* a separated, geometrically irreducible and geometrically reduced scheme of finite type over a field, and *curve* a variety of dimension 1.

Lemma 38.59.14. *Let C be a curve over an algebraically closed field k. Then the Brauer group of the function field of C is zero: $Br(k(C)) = 0$.*

Proof. This is clear from Tsen's theorem, Theorem 38.59.12. □

Lemma 38.59.15. *Let k be an algebraically closed field and $k \subset K$ a field extension of transcendence degree 1. Then for all $q \geq 1$, $H^q_{et}(Spec(K), \mathbf{G}_m) = 0$.*

Proof. It suffices to show that if $K \subset K'$ is a finite field extension, then $\text{Br}(K') = 0$. Now observe that $K' = colim\, K''$, where K'' runs over the finitely generated subextensions of k contained in K' of transcendence degree 1. By some result in [**Har77**], each K'' is the function field of a curve, hence has trivial Brauer group by Lemma 38.59.14. It now suffices to observe that $\text{Br}(K') = colim\, \text{Br}(K'')$. □

38.60. Higher vanishing for the multiplicative group

In this section, we fix an algebraically closed field k and a smooth curve X over k. We denote $i_x : x \hookrightarrow X$ the inclusion of a closed point of X and $j : \eta \hookrightarrow X$ the inclusion of the generic point. We also denote X^0 the set of closed points of X.

Theorem 38.60.1. *(The Fundamental Exact Sequence) There is a short exact sequence of étale sheaves on X*

$$0 \longrightarrow \mathbf{G}_{m,X} \longrightarrow j_*\mathbf{G}_{m,\eta} \xrightarrow{\div} \bigoplus_{x \in X^0} i_{x*}\underline{\mathbf{Z}} \longrightarrow 0.$$

Proof. Let $\varphi : \mathcal{U} \to X$ be an étale morphism. Then by properties v and vi of étale morphisms (Proposition 38.26.2), $\mathcal{U} = \coprod_i \mathcal{U}_i$ where each $\mathcal{U}_i$ is a smooth curve mapping to X. The above sequence for X is a product of the corresponding sequences for each $\mathcal{U}_i$, so it suffices to treat the case where $\mathcal{U}$ is connected, hence irreducible. In this case, there is a well known exact sequence (see [**Har77**])

$$1 \longrightarrow \Gamma(\mathcal{U}, \mathcal{O}_{\mathcal{U}}^*) \longrightarrow k(\mathcal{U})^* \xrightarrow{\div} \bigoplus_{y \in \mathcal{U}^0} \mathbf{Z}_y.$$

This amounts to a sequence

$$\Gamma(\mathcal{U}, \mathcal{O}_{\mathcal{U}}^*) \longrightarrow \Gamma(\eta \times_X \mathcal{U}, \mathcal{O}_{\eta \times_X \mathcal{U}}^*) \xrightarrow{\div} \bigoplus_{x \in X^0} \Gamma(x \times_X \mathcal{U}, \underline{\mathbf{Z}})$$

which, unfolding definitions, is nothing but a sequence

$$\mathbf{G}_m(\mathcal{U}) \longrightarrow j_*\mathbf{G}_{m,\eta}(\mathcal{U}) \xrightarrow{\div} \bigoplus_{x \in X^0} i_{x*}\underline{\mathbf{Z}}(\mathcal{U}).$$

This defines the maps in the Fundamental Exact Sequence and shows it is exact except possibly at the last step. To see surjectivity, let us recall (from [**Har77**] again) that if C is a nonsingular curve and D is a divisor on C, then there exists a Zariski open covering $\{\mathcal{V}_j \to C\}$ of C such that $D|_{\mathcal{V}_j} = \div(f_j)$ for some $f_j \in k(C)^*$. $\square$

Lemma 38.60.2. *For any $q \geq 1$, $R^q j_* \mathbf{G}_{m,\eta} = 0$.*

Proof. We need to show that $(R^q j_* \mathbf{G}_{m,\eta})_{\bar{x}} = 0$ for every geometric point $\bar{x}$ of X.

(1) Assume that $\bar{x}$ lies over a closed point x of X. Let $Spec(A)$ be an open neighborhood of x in X, and K the fraction field of A, so that

$$Spec(\mathcal{O}_{X,\bar{x}}^{sh}) \times_X \eta = Spec(\mathcal{O}_{X,\bar{x}}^{sh} \otimes_A K).$$

The ring $\mathcal{O}_{X,\bar{x}}^{sh} \otimes_A K$ is a localization of the discrete valuation ring $\mathcal{O}_{X,\bar{x}}^{sh}$, so it is either $\mathcal{O}_{X,\bar{x}}^{sh}$ again, or its fraction field $K_{\bar{x}}^{sh}$. But since some local uniformizer gets inverted, it must be the latter. Hence

$$(R^q j_* \mathbf{G}_{m,\eta})_{(X,\bar{x})} = H_{et}^q(Spec\, K_{\bar{x}}^{sh}, \mathbf{G}_m).$$

Now recall that $\mathcal{O}_{X,\bar{x}}^{sh} = colim_{(\mathcal{U},\bar{u}) \to \bar{x}}\, \mathcal{O}(\mathcal{U}) = colim_{A \subset B}\, B$ where $A \to B$ is étale, hence $K_{\bar{x}}^{sh}$ is an algebraic extension of $k(X)$, and we may apply Lemma 38.59.15 to get the vanishing.

(2) Assume that $\bar{x} = \bar{\eta}$ lies over the generic point η of X (in fact, this case is superfluous). Then $\mathcal{O}_{X,\bar{\eta}} = \kappa(\eta)^{sep}$ and thus

$$\begin{aligned} (R^q j_* \mathbf{G}_{m,\eta})_{\bar{\eta}} &= H_{et}^q(Spec(\kappa(\eta)^{sep}) \times_X \eta, \mathbf{G}_m) \\ &= H_{et}^q(Spec(\kappa(\eta)^{sep}), \mathbf{G}_m) \\ &= 0 \quad \text{for } q \geq 1 \end{aligned}$$

since the corresponding Galois group is trivial.

$\square$

Lemma 38.60.3. *For all $p \geq 1$, $H_{et}^p(X, j_* \mathbf{G}_{m,\eta}) = 0$.*

Proof. The Leray spectral sequence reads

$$E_2^{p,q} = H_{et}^p(X, R^q j_* \mathbf{G}_{m,\eta}) \Rightarrow H_{et}^{p+q}(\eta, \mathbf{G}_{m,\eta}),$$

which vanishes for $p + q \geq 1$ by Lemma 38.59.15. Taking $q = 0$, we get the desired vanishing. $\square$

Lemma 38.60.4. *For all $q \geq 1$, $H_{et}^q(X, \bigoplus_{x \in X^0} i_{x*}\underline{\mathbf{Z}}) = 0$.*

Proof. For X quasi-compact and quasi-separated, cohomology commutes with colimits, so it suffices to show the vanishing of $H^q_{et}(X, i_{x*}\underline{\mathbf{Z}})$. But then the inclusion i_x of a closed point is finite so $R^p i_{x*}\underline{\mathbf{Z}} = 0$ for all $p \geq 1$ by Proposition 38.54.2. Applying the Leray spectral sequence, we see that $H^q_{et}(X, i_{x*}\underline{\mathbf{Z}}) = H^q_{et}(x, \underline{\mathbf{Z}})$. Finally, since x is the spectrum of an algebraically closed field, all higher cohomology on x vanishes. □

Concluding this series of lemmata, we get the following result.

Theorem 38.60.5. *Let X be a smooth curve over an algebraically closed field. Then*

$$H^q_{et}(X, \mathbf{G}_m) = 0 \quad \textit{for all } q \geq 2.$$

We also get the cohomology long exact sequence

$$0 \to H^0_{et}(X, \mathbf{G}_m) \to H^0_{et}(X, j_*\mathbf{G}_{m\eta}) \xrightarrow{\div} H^0_{et}(X, \bigoplus i_{x*}\underline{\mathbf{Z}}) \to H^1_{et}(X, \mathbf{G}_m) \to 0$$

although this is the familiar

$$0 \to H^0_{Zar}(X, \mathcal{O}_X^*) \to k(X)^* \xrightarrow{\div} \mathrm{Div}(X) \to \mathrm{Pic}(X) \to 0.$$

We would like to use the Kummer sequence to deduce some information about the cohomology group of a curve with finite coefficients. In order to get vanishing in the long exact sequence, we review some facts about Picard groups.

38.61. Picards groups of curves

Let X be a smooth projective curve over an algebraically closed field k. There exists a short exact sequence

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \xrightarrow{\deg} \mathbf{Z} \to 0.$$

The abelian group $\mathrm{Pic}^0(X)$ can be identified with $\mathrm{Pic}^0(X) = \underline{\mathrm{Pic}}^0_{X/k}(k)$, i.e., the k-valued points of an abelian variety $\underline{\mathrm{Pic}}^0_{X/k}$ of dimension $g = g(X)$ over k.

Definition 38.61.1. An *abelian variety* over k is a proper smooth connected group scheme over k (i.e., a proper group variety over k).

Proposition 38.61.2. *Let A be an abelian variety over an algebraically closed field k. Then*

(1) *A is projective over k;*
(2) *A is a commutative group scheme;*
(3) *the morphism $[n] : A \to A$ is surjective for all $n \geq 1$, in other words $A(k)$ is a divisible abelian group;*
(4) *$A[n] = Ker(A \xrightarrow{[n]} A)$ is a finite flat group scheme of rank $n^{2\dim A}$ over k. It is reduced if and only if $n \in k^*$;*
(5) *if $n \in k^*$ then $A(k)[n] = A[n](k) \cong (\mathbf{Z}/n\mathbf{Z})^{2\dim(A)}$.*

Consequently, if $n \in k^*$ then $\mathrm{Pic}^0(X)[n] \cong (\mathbf{Z}/n\mathbf{Z})^{2g}$ as abelian groups.

Lemma 38.61.3. *Let X be a smooth projective of genus g over an algebraically closed field k and $n \geq 1$, $n \in k^*$. Then there are canonical identifications*

$$H^q_{et}(X, \mu_n) = \begin{cases} \mu_n(k) & \textit{if } q = 0\,; \\ Pic^0(X)[n] & \textit{if } q = 1\,; \\ \mathbf{Z}/n\mathbf{Z} & \textit{if } q = 2\,; \\ 0 & \textit{if } q \geq 3. \end{cases}$$

Since $\mu_n \cong \underline{\mathbf{Z}/n\mathbf{Z}}$, this gives (noncanonical) identifications

$$H^q_{et}(X, \underline{\mathbf{Z}/n\mathbf{Z}}) \cong \begin{cases} \mathbf{Z}/n\mathbf{Z} & \text{if } q = 0\text{ ;} \\ (\mathbf{Z}/n\mathbf{Z})^{2g} & \text{if } q = 1\text{ ;} \\ \mathbf{Z}/n\mathbf{Z} & \text{if } q = 2\text{ ;} \\ 0 & \text{if } q \geq 3. \end{cases}$$

Proof. The Kummer sequence $0 \to \mu_{n,X} \to \mathbf{G}_{m,X} \xrightarrow{(\cdot)^n} \mathbf{G}_{m,X} \to 0$ give the long exact cohomology sequence

$$\begin{tikzcd}
0 \arrow[r] & \mu_n(k) \arrow[r] & k^* \arrow[r, "(\cdot)^n"] & k^* \arrow[dll, out=-90, in=90] \\
& H^1_{et}(X, \mu_n) \arrow[r] & \mathrm{Pic}(X) \arrow[r, "(\cdot)^n"] & \mathrm{Pic}(X) \arrow[dll, out=-90, in=90] \\
& H^2_{et}(X, \mu_n) \arrow[r] & 0 \arrow[r] & 0 \cdots
\end{tikzcd}$$

The n power map $k^* \to k^*$ is surjective since k is algebraically closed. So we need to compute the kernel and cokernel of the map $\mathrm{Pic}(X) \xrightarrow{(\cdot)^n} \mathrm{Pic}(X)$. Consider the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Pic}^0(X) \arrow[r] \arrow[d, two heads, "(\cdot)^n"] & \mathrm{Pic}(X) \arrow[r, "\deg"] \arrow[d, "(\cdot)^n"] & \mathbf{Z} \arrow[r] \arrow[d, hook, "n"] & 0 \\
0 \arrow[r] & \mathrm{Pic}^0(X) \arrow[r] & \mathrm{Pic}(X) \arrow[r, "\deg"] & \mathbf{Z} \arrow[r] & 0
\end{tikzcd}$$

where the left vertical map is surjective by Proposition 38.61.2 (3). Applying the snake lemma gives the desired identifications. $\square$

Lemma 38.61.4. *Let X be an affine smooth curve over an algebraically closed field k and $n \in k^*$. Then*

(1) $H^0_{et}(X, \mu_n) = \mu_n(k)$*;*
(2) $H^1_{et}(X, \mu_n) \cong (\mathbf{Z}/n\mathbf{Z})^{2g+r-1}$*, where r is the number of points in $\bar{X} - X$ for some smooth projective compactification $\bar{X}$ of X ; and*
(3) *for all $q \geq 2$,* $H^q_{et}(X, \mu_n) = 0$.

Proof. Write $X = \bar{X} - \{x_1, \ldots, x_r\}$. Then $\mathrm{Pic}(X) = \mathrm{Pic}(\bar{X})/R$, where R is the subgroup generated by $\mathcal{O}_{\bar{X}}(x_i)$, $1 \leq i \leq r$. Since $r \geq 1$, we see that $\mathrm{Pic}^0(X) \to \mathrm{Pic}(X)$ is surjective, hence $\mathrm{Pic}(X)$ is divisible. Applying the Kummer sequence, we get *i* and *iii*. For *ii*, recall that

$$\begin{aligned} H^1_{et}(X, \mu_n) &= \left\{(\mathcal{L}, \alpha) | \mathcal{L} \in \mathrm{Pic}(X), \alpha : \mathcal{L}^{\otimes n} \cong \mathcal{O}_X\right\} \Big/ \cong \\ &= \left\{(\bar{\mathcal{L}},\ D,\ \bar{\alpha})\right\} \Big/ \tilde{R} \end{aligned}$$

where $\bar{\mathcal{L}} \in \mathrm{Pic}^0(\bar{X})$, D is a divisor on $\bar{X}$ supported on $\left\{x_1, \cdots, x_r\right\}$ and $\bar{\alpha} : \bar{\mathcal{L}}^{\otimes n} \cong \mathcal{O}_{\bar{X}}(D)$ is an isomorphism. Note that D must have degree 0. Further $\tilde{R}$ is the subgroup of triples of the form $(\mathcal{O}_{\bar{X}}(D'), nD', 1^{\otimes n})$ where D' is supported on $\{x_1, \cdots, x_r\}$ and has degree 0. Thus, we get an exact sequence

$$0 \longrightarrow H^1_{et}(\bar{X}, \mu_n) \longrightarrow H^1_{et}(X, \mu_n) \longrightarrow \bigoplus_{i=1}^{r} \mathbf{Z}/n\mathbf{Z} \xrightarrow{\Sigma} \mathbf{Z}/n\mathbf{Z} \longrightarrow 0$$

where the middle map sends the class of a triple $(\bar{\mathcal{L}}, D, \bar{\alpha})$ with $D = \sum_{i=1}^{r} a_i(x_i)$ to the r-tuple $(a_i)_{i=1}^{r}$. It now suffices to use Lemma 38.61.3 to count ranks. □

Remark 38.61.5. The ``natural'' way to prove the previous corollary is to excise X from $\bar{X}$. This is possible, we just haven't developed that theory.

Our main goal is to prove the following result.

Theorem 38.61.6. *Let X be a separated, finite type, dimension 1 scheme over an algebraically closed field k and $\mathcal{F}$ a torsion sheaf on $X_{étale}$. Then*

$$H^q_{et}(X, \mathcal{F}) = 0, \quad \forall q \geq 3.$$

If X affine then also $H^2_{et}(X, \mathcal{F}) = 0$.

Recall that an abelian sheaf is called a *torsion sheaf* if all of its stalks are torsion groups. We have computed the cohomology of constant sheaves. We now generalize the latter notion to get all the way to torsion sheaves.

38.62. Constructible sheaves

Definition 38.62.1. Let X be a scheme and $\mathcal{F}$ an abelian sheaf on $X_{étale}$. We say that $\mathcal{F}$ is *finite locally constant* if it is represented by a finite étale morphism to X.

Lemma 38.62.2. *Let X be a scheme and $\mathcal{F}$ an abelian sheaf on $X_{étale}$. Then the following are equivalent*

(1) *$\mathcal{F}$ is finite locally constant ;*
(2) *there exists an étale covering $\{\mathcal{U}_i \to X\}_{i \in I}$ such that $\mathcal{F}|_{\mathcal{U}_i} \cong \underline{A_i}$ for some finite abelian group A_i.*

For a proof, see [**Del77**].

Definition 38.62.3. Let X be a quasi-compact and quasi-separated scheme. A sheaf $\mathcal{F}$ on $X_{étale}$ is *constructible* if there exists a finite decomposition of X into locally closed subsets $X = \coprod_i X_i$ such that $\mathcal{F}|_{X_i}$ is finite locally constant for all i.

Lemma 38.62.4. *The kernel and cokernel of a map of finite locally constant sheaves are finite locally constant.*

Proof. Let $\mathcal{U}$ be a connected scheme, A and B finite abelian groups. Then

$$Hom_{Ab(\mathcal{U}_{étale})}\left(\underline{A}_{\mathcal{U}}, \underline{B}_{\mathcal{U}}\right) = Hom_{Ab}(A, B),$$

so $\mathrm{Ker}\left(\underline{A}_{\mathcal{U}} \xrightarrow{\varphi} \underline{B}_{\mathcal{U}}\right) = \underline{\mathrm{Ker}(\varphi)}_{\mathcal{U}}$ and similarly for the cokernel. □

Remark 38.62.5. If X is noetherian, then (with out definitions) any constructible sheaf on $X_{étale}$ is a torsion sheaf.

Lemma 38.62.6. *Let X be a noetherian scheme. Then:*

(1) *the category of constructible sheaves is abelian ;*
(2) *it is a full exact subcategory of $Ab(X_{étale})$;*

(3) *any extension of constructible sheaves is constructible ; and*
(4) *the image of a map from a constructible sheaf to any other sheaf is constructible.*

Proof. Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a map of constructible sheaves. By assumption, there exists a stratification $X = \coprod X_i$ such that $\mathscr{F}|_{X_i}$ and $\mathscr{G}|_{X_i}$ are finite locally constant. Since pullback if exact, we thus have $\operatorname{Ker}(\varphi|_{X_i}) = \operatorname{Ker}(\mathscr{F}|_{X_i} \xrightarrow{\varphi} \mathscr{G}|_{X_i})$ which is finite locally constant by Lemma 38.62.4. Statement (4) means that if $\varphi : \mathscr{F} \to \mathscr{G}$ is a map in $Ab(X_{étale})$ and $\mathscr{F}$ is constructible then $\operatorname{Im}(\varphi)$ is constructible. It is proven in [**Del77**]. □

Lemma 38.62.7. *Let $\varphi : \mathscr{U} \to X$ be an étale morphism of noetherian schemes. Then there exists a stratification $X = \coprod_i X_i$ such that for all i, $X_i \times_X \mathscr{U} \to X_i$ is finite étale.*

Proof. By noetherian induction it suffices to find some nonempty open $\mathscr{V} \subset X$ such that $\varphi^{-1}(\mathscr{V}) \to \mathscr{V}$ is finite. This follows from the following very general lemma. □

Lemma 38.62.8. *(Morphisms, Lemma 24.45.1). Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of schemes and η a generic point of Y such that $f^{-1}(\eta)$ is finite. Then there exists an open $\mathscr{V} \subset Y$ containing η such that $f^{-1}(\mathscr{V}) \to \mathscr{V}$ is finite.*

38.63. Extension by zero

Definition 38.63.1. Let $j : \mathscr{U} \to X$ be an étale morphism of schemes. The restriction functor j^{-1} is right exact, so it has a left adjoint, denoted $j_! : Ab(\mathscr{U}_{étale}) \to Ab(X_{étale})$ and called *extension by zero*. Thus it is characterized by the functorial isomorphism

$$Hom_X(j_!\mathscr{F}, \mathscr{G}) = Hom_{\mathscr{U}}(\mathscr{F}, j^{-1}\mathscr{G})$$

for all $\mathscr{F} \in Ab(\mathscr{U}_{étale})$ and $\mathscr{G} \in Ab(X_{étale})$.

To describe it more explicitly, recall that j^{-1} is just the restriction functor $\mathscr{U}_{étale} \to X_{étale}$, that is,

$$j^{-1}\mathscr{G}(\mathscr{U}' \to \mathscr{U}) = \mathscr{G}\left(\mathscr{U}' \to \mathscr{U} \xrightarrow{j} X\right).$$

For $\mathscr{F} \in Ab(\mathscr{U}_{étale})$ we consider the presheaf

$$\begin{array}{cccc} j_!^{PSh}\mathscr{F} : & X_{étale} & \longrightarrow & Ab \\ & (\mathscr{V} \to X) & \longmapsto & \displaystyle\bigoplus_{\mathscr{V} \xrightarrow{\varphi} \mathscr{U} \text{ over } X} \mathscr{F}(\mathscr{V} \xrightarrow{\varphi} \mathscr{U}), \end{array}$$

then $j_!\mathscr{F}$ is the sheafification $\left(j_!^{PSh}\mathscr{F}\right)^\#$.

Exercise 38.63.2. Prove directly that $j_!$ is left adjoint to j^{-1} and that j_* is right adjoint to j^{-1}.

Proposition 38.63.3. *Let $j : \mathscr{U} \to X$ be an étale morphism of schemes. Then*
(1) *the functors j^{-1} and $j_!$ are exact ;*
(2) *j^{-1} transforms injectives into injectives ;*
(3) *$H^p_{et}(\mathscr{U}, \mathscr{G}) = H^p_{et}(\mathscr{U}, j^{-1}\mathscr{G})$ for any $\mathscr{G} \in Ab(X_{étale})$*
(4) *if $\bar{x}$ is a geometric point of X, then $\left(j_!\mathscr{F}\right)_{\bar{x}} = \bigoplus_{(\mathscr{U},\bar{u}) \to (X,x)} \mathscr{F}_{\bar{u}}$.*

Proof. The functor j^{-1} has both a right and a left adjoint, so it is exact. The functor $j_!$ has a right adjoint, so it is right exact. To see that it is left exact, use the description above and the fact that sheafification is exact. Property *ii* is standard general nonsense. In part *iii*, the left-hand side refers (as it should) to the right derived functors of $\mathscr{G} \mapsto \mathscr{G}(\mathscr{U})$

on $Ab(X_{étale})$, and the right-hand side refers to global cohomology on $Ab(\mathcal{U}_{étale})$. It is a formal consequence of *ii*. Part *iv* is again a consequence of the above description. □

Lemma 38.63.4. *Extension by zero commutes with base change. More precisely, let $f : Y \to X$ be a morphism of schemes, $j : \mathcal{V} \to X$ be an étale morphism and $\mathcal{F}$ a sheaf on $\mathcal{V}_{étale}$. Consider the cartesian diagram*

$$\begin{tikzcd} \mathcal{V}' = Y \times_X \mathcal{V} \arrow[r, "j'"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ \mathcal{V} \arrow[r, "j"] & X \end{tikzcd}$$

then $j'_! f'^{-1}\mathcal{F} = f^{-1} j_! \mathcal{F}$.

Sketch of proof. By general nonsense, there exists a map $j'_! \circ f'^{-1} \to f^{-1} \circ j_!$. We merely verify that they agree on stalks. We have

$$\left(j'_! f'^{-1}\mathcal{F}\right)_{\bar{y}} = \bigoplus_{\bar{v}' \to \bar{y}} (f'^{-1}\mathcal{F})_{\bar{v}'} = \bigoplus_{\bar{v} \to f(\bar{y})} \mathcal{F}_{\bar{v}} = (j_!\mathcal{F})_{f(\bar{y})} = (f^{-1} j_! \mathcal{F})_{\bar{y}}.$$

□

Lemma 38.63.5. *Let $j : \mathcal{V} \to X$ be finite and étale. Then $j_! = j_*$.*

Sketch of proof. In this situation, one can again construct a map $j_! \to j_*$ although in this case it is not just by general nonsense and uses the assumptions on j. Again, we only check that the stalks agree. We have on the one hand

$$(j_!\mathcal{F})_{\bar{x}} = \bigoplus_{\bar{v} \to \bar{x}} \mathcal{F}_{\bar{v}},$$

and on the other hand

$$\left(j_*\mathcal{F}\right)_{\bar{x}} = H^0_{et}(Spec(\mathcal{O}^{sh}_{X,\bar{x}}) \times_X \mathcal{V}, \mathcal{F}).$$

But j is finite and $\mathcal{O}_{X,\bar{x}}$ is strictly henselian, hence $Spec(\mathcal{O}^{sh}_{X,\bar{x}}) \times_X \mathcal{V}$ splits completely into spectra of strictly henselian local rings

$$Spec(\mathcal{O}^{sh}_{X,\bar{x}}) \times_X \mathcal{V} = \coprod_{\bar{v} \to \bar{x}} Spec(\mathcal{O}^{sh}_{X,\bar{x}})$$

and so $\left(j_*\mathcal{F}\right)_{\bar{x}} = \prod_{\bar{v} \to \bar{x}} \mathcal{F}_{\bar{v}}$ by Lemma 38.63.4. Since finite products and finite coproducts agree, we get the result. Note that this last step fails if we take infinite colimits, and indeed the result is not true anymore for ind-morphisms, say. □

Lemma 38.63.6. *Let X be a noetherian scheme and $j : \mathcal{U} \to X$ an étale, quasi-compact morphism. Then $j_! \underline{\mathbf{Z}/n\mathbf{Z}}$ is constructible on X.*

Proof. By Lemma 38.62.7, X has a stratification $\coprod_i X_i$ such that $\pi_i : j^{-1}(X_i) \to X_i$ is finite étale, hence

$$j_!(\underline{\mathbf{Z}/n\mathbf{Z}})|_{X_i} = \pi_{i!}(\underline{\mathbf{Z}/n\mathbf{Z}}) = \pi_{i*}(\underline{\mathbf{Z}/n\mathbf{Z}})$$

by Lemma 38.63.5. Thus it suffices to show that for $\pi : Y \to X$ finite étale, $\pi_*(\underline{\mathbf{Z}/n\mathbf{Z}})$ is finite locally constant. This is clear because it is the sheaf represented by $Y \times \mathbf{Z}/n\mathbf{Z}$. □

Remark 38.63.7. Using the alternative definition of finite locally constant (as in Lemma 38.62.2), the last step is replaced by considering a Galois closure of Y.

Lemma 38.63.8. *Let X be a noetherian scheme and $\mathcal{F}$ a torsion sheaf on $X_{\acute{e}tale}$. Then $\mathcal{F}$ is a directed (filtered) colimit of constructible sheaves.*

Sketch of proof. Let $j : \mathcal{U} \to X$ in $X_{\acute{e}tale}$ and $s \in \mathcal{F}(\mathcal{U})$ for some $\mathcal{U}$ noetherian. Then $ns = 0$ for some $n > 0$. Hence we get a map $\underline{\mathbf{Z}/n\mathbf{Z}}_{\mathcal{U}} \to \mathcal{F}|_{\mathcal{U}}$, by sending $\bar{1}$ to s. By adjointness, this gives a map $\varphi : j_!(\underline{\mathbf{Z}/n\mathbf{Z}}) \to \mathcal{F}$ whose image contains s. There is an element $1_{\mathrm{id}_{\mathcal{U}}} \in \Gamma(\mathcal{U}, j_!\underline{\mathbf{Z}/n\mathbf{Z}})$ which maps to s. Thus, $\mathrm{Im}(\varphi) \subset \mathcal{F}$ is a constructible subsheaf and $s \in \mathrm{Im}(\varphi)(\mathcal{U})$. A similar argument applies for a finite collection of sections, and the result follows by taking colimits. $\square$

38.64. Higher vanishing for torsion sheaves

The goal of this section is to prove the result that follows now.

Theorem 38.64.1. *Let X be an affine curve over an algebraically closed field k and $\mathcal{F}$ a torsion sheaf on $X_{\acute{e}tale}$. Then $H^q_{et}(X, \mathcal{F}) = 0$ for all $q \geq 2$.*

We begin by reducing the proof to a more simpler statement.

(1) *If suffices to prove the vanishing when $\mathcal{F}$ is a constructible sheaf.*

Using the compatibility of étale cohomology with colimits and Lemma 38.63.8, we have $colim\, H^q_{et}(X, \mathcal{F}) = H^q_{et}(X, colim\, \mathcal{F}_i)$ for some constructible sheaves $\mathcal{F}_i$, whence the result.

(2) *It suffices to assume that $\mathcal{F} = j_!\mathcal{G}$ where $\mathcal{U} \subset X$ is open, $\mathcal{G}$ is finite locally constant on $\mathcal{U}$ smooth.*

Choose a nonempty open $\mathcal{U} \subset X$ such that $\mathcal{F}|_{\mathcal{U}}$ is finite locally constant, and consider the exact sequence

$$0 \to j_!(\mathcal{F}|_{\mathcal{U}}) \to \mathcal{F} \to Q \to 0.$$

By looking at stalks we get $Q_{\bar{x}} = 0$ unless $\bar{x} \in X - U$. It follows that $Q = \bigoplus_{x \in X-U} i_{x*}(Q_x)$ which has no higher cohomology.

(3) *It suffices to assume that X is smooth and affine (over k), $\mathcal{G}$ is a finite locally constant sheaf on a open $\mathcal{U}$ of X and $\mathcal{F} = j_!\mathcal{G}$.*

Let $\mathcal{U}$, X and $\mathcal{G}$ be as in the step 2, and consider the commutative diagram

$$\begin{tikzcd} & X^\nu \arrow[d, "\nu"] \\ \mathcal{U} \arrow[ru, "j^\nu"] \arrow[r, "j"] & X \end{tikzcd}$$

where $\nu : X^\nu \to X$ is the normalization of X. Since ν is finite, $H^*_{et}(X, j_!\mathcal{G}) = H^*_{et}(X^\nu, j^\nu_!\mathcal{G})$, which implies that $\nu_*((j^\nu)_!\mathcal{G}) = j_!\mathcal{G}$ by looking at stalks. We are thus reduced to proving the following lemma.

Lemma 38.64.2. *Let X be a smooth affine curve over an algebraically closed field k, $j : \mathcal{U} \hookrightarrow X$ an open immersion and $\mathcal{F}$ a finite locally constant sheaf on $\mathcal{U}_{\acute{e}tale}$. Then for all $q \geq 2$, $H^q_{et}(X, j_!\mathcal{F}) = 0$.*

The proof of this follows the ``méthode de la trace" as explained in [**MA71**, Exposé IX, §5].

Definition 38.64.3. Let $f : Y \to X$ be a finite étale morphism. There are pairs of adjoint functors $(f_!, f^{-1})$ and (f^{-1}, f_*) on $Ab(X_{\acute{e}tale})$. The adjunction map $\mathrm{id} \to f_*f^{-1}$ is called *restriction*. Since f is finite, $f_! = f_*$ and the adjunction map $f_*f^{-1} = f_!f^{-1} \to \mathrm{id}$ is called the *trace*.

The trace map is characterized by the following two properties:

(1) it commutes with étale localization ; and
(2) if $f : Y = \coprod_{i=1}^{d} X \to X$ then the trace map is just the sum map $f_* f^{-1}\mathcal{F} = \mathcal{F}^{\oplus d} \to \mathcal{F}$.

It follows that if f has constant degree d, then the composition $\mathcal{F} \xrightarrow{res} f_* f^{-1}\mathcal{F} \xrightarrow{trace} \mathcal{F}$ is multiplication by d. The ``méthode" then essentially consits in the following observation: if $\mathcal{F}$ is an abelian sheaf on $X_{étale}$ such that multiplication by d is an isomorphism $\mathcal{F} \cong \mathcal{F}$, and if furthermore $H^q_{et}(Y, f^{-1}\mathcal{F}) = 0$ then $H^q_{et}(X, \mathcal{F}) = 0$ as well. Indeed, multiplication by d induces an isomorphism on $H^q_{et}(X, \mathcal{F})$ which factors through $H^q_{et}(Y, f^{-1}\mathcal{F}) = 0$.

Using this method, we further reduce the proof of Lemma 38.64.2 to a yet simpler statement.

(4) *We may assume that $\mathcal{F}$ is killed by a prime ℓ.*

Writing $\mathcal{F} = \mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_r$ where $\mathcal{F}_i$ is ℓ_i-primary for some prime ℓ_i, we may assume that ℓ^n kills $\mathcal{F}$ for some prime ℓ. Now consider the exact sequence

$$0 \to \mathcal{F}[\ell] \to \mathcal{F} \to \mathcal{F}/\mathcal{F}[\ell] \to 0.$$

Applying the exact functor $j_!$ and looking at the long exact cohomology sequence, we see that it suffices to assume that $\mathcal{F}$ is ℓ-torsion, which we do.

(5) *There exists a finite étale morphism $f : \mathcal{V} \to \mathcal{U}$ of degree prime to ℓ such that $f^{-1}\mathcal{F}$ has a filtration*

$$0 \subset \mathcal{G}_1 \subset \mathcal{G}_2 \subset \cdots \subset \mathcal{G}_s = f^{-1}\mathcal{F}$$

with $\mathcal{G}_i/\mathcal{G}_{i-1} \cong \underline{\mathbf{Z}/\ell\mathbf{Z}}_{\mathcal{V}}$ for all $i \leq s$.

Since $\mathcal{F}$ is finite locally constant, there exists a finite étale Galois cover $h : \mathcal{U}' \to \mathcal{U}$ such that $h^{-1}\mathcal{F} \cong \underline{A}_{\mathcal{U}'}$ for some finite abelian group A. Note that $A \cong (\mathbf{Z}/\ell\mathbf{Z})^{\oplus m}$ for some m. Saying that the cover is *Galois* means that the finite group $G = \text{Aut}(\mathcal{U}'|\mathcal{U})$ has (maximal) cardinality $\#G = \deg h$. Now let $H \subset G$ be the ℓ-Sylow, and set

$$\mathcal{U}' \xrightarrow{\pi} \mathcal{V} = \mathcal{U}'/H \xrightarrow{f} \mathcal{U}.$$

The quotient exists by taking invariants (schemes are affine). By construction, $\deg f = \#G/\#H$ is prime to ℓ. The sheaf $\mathcal{G} = f^{-1}\mathcal{F}$ is then a finite locally constant sheaf on $\mathcal{V}$ and

$$\pi^{-1}\mathcal{G} = h^{-1}\mathcal{F} \cong \underline{(\mathbf{Z}/\ell\mathbf{Z})}^{\oplus m}_{\mathcal{U}'}.$$

Moreover,

$$H^0_{et}(\mathcal{V}, \mathcal{G}) = H^0_{et}(\mathcal{U}', \pi^{-1}\mathcal{G})^H = \left((\mathbf{Z}/\ell\mathbf{Z})^{\oplus m}\right)^H \neq 0,$$

where the first equality follows from writing out the sheaf condition for $\mathcal{G}$ (again, schemes are affine), and the last inequality is an exercise in linear algebra over $\mathbf{F}_\ell$. Following, we have found a subsheaf $\underline{\mathbf{Z}/\ell\mathbf{Z}}_{\mathcal{V}} \hookrightarrow \mathcal{G}$. Repeating the argument for the quotient $\mathcal{G}/\underline{\mathbf{Z}/\ell\mathbf{Z}}_{\mathcal{V}}$ if necessary, we eventually get a subsheaf of $\mathcal{G}$ with quotient $\underline{\mathbf{Z}/\ell\mathbf{Z}}_{\mathcal{V}}$. This is the first step of the filtration.

Exercise 38.64.4. Let $f : X \to Y$ be a finite étale morphism with Y noetherian, and X, Y irreducible. Then there exists a finite étale Galois morphism $X' \to Y$ which dominates X over Y.

(6) *We consider the normalization Y of X in $\mathcal{V}$, that is, we have the commutative diagram*

$$\begin{tikzcd} \mathcal{V} \arrow[r, hook, "j'"] \arrow[d, "f"] & Y \arrow[d, "f'"] \\ \mathcal{U} \arrow[r, hook, "j"] & X. \end{tikzcd}$$

Then there is an injection $H^q_{et}(X, j_!\mathcal{F}) \hookrightarrow H^q_{et}(Y, j'_! f^{-1}\mathcal{F})$ for all q.

We have seen that the composition $\mathcal{F} \xrightarrow{res} f_* f^{-1}\mathcal{F} \xrightarrow{trace} \mathcal{F}$ is multiplication by the degree of f, which is prime to ℓ. On the other hand,

$$j_! f_* f^{-1}\mathcal{F} = j_! f_! f^{-1}\mathcal{F} = f'_* j'_! f^{-1}\mathcal{F}$$

since f and f' are both finite and the above diagram is commutative. Hence applying $j_!$ to the previous sequence gives a sequence

$$j_!\mathcal{F} \longrightarrow f'^* j'_! f^{-1}\mathcal{F} \longrightarrow j_!\mathcal{F}.$$

Taking cohomology, we see that $H^q_{et}(X, j_!\mathcal{F})$ injects into $H^q_{et}(X, f'^* j'_! f^{-1}\mathcal{F})$. But since f' is finite, this is merely $H^q_{et}(Y, j'_! f^{-1}\mathcal{F})$, as desired.

(7) *It suffices to prove $H^q_{et}(Y, j'_!\underline{\mathbf{Z}/\ell\mathbf{Z}}) = 0$.*

By Step 3, it suffices to show vanishing of $H^q_{et}(Y, j'_! f^{-1}\mathcal{F})$. But then by Step 2, we may assume that $f^{-1}\mathcal{F}$ has a finite filtration with quotients isomorphic to $\underline{\mathbf{Z}/n\mathbf{Z}}$, whence the claim.

Finally, we are reduced to proving the following lemma.

Lemma 38.64.5. *Let X be a smooth affine curve over an algebraically closed field, $j : \mathcal{U} \hookrightarrow X$ an open immersion and ℓ a prime number. Then for all $q \geq 2$, $H^q_{et}(X, j_!\underline{\mathbf{Z}/\ell\mathbf{Z}}) = 0$.*

Proof. Consider the short exact sequence

$$0 \longrightarrow j_!\underline{\mathbf{Z}/\ell\mathbf{Z}}_{\mathcal{U}} \longrightarrow \underline{\mathbf{Z}/\ell\mathbf{Z}}_X \longrightarrow \bigoplus_{x \in X - \mathcal{U}} i_{x*}(\underline{\mathbf{Z}/\ell\mathbf{Z}}) \longrightarrow 0.$$

We know that the cohomology of the middle sheaf vanishes in degree at least 2 by Lemma 38.61.4 and that of the skyscraper sheaf on the right vanishes in degree at least 1. Thus applying the long exact cohomology sequence, we get the vanishing of $j_!\underline{\mathbf{Z}/\ell\mathbf{Z}}_{\mathcal{U}}$ in degree at least 2. This finishes the proof of the lemma, hence of Lemma 38.64.2, hence of Theorem 38.64.1. □

Remarks 38.64.6. Here are some remarks about what happened above.

- This method is very general. For instance, it applies in Galois cohomology, and this is essentially how Proposition 38.59.7 is proved.
- In fact, we have overlooked the case where ℓ is the characteristic of the field k, since the Kummer sequence is not exact then and we cannot use Lemma 38.61.4 anymore. The result is still true, as shown by considering the *Artin-Schreier* exact sequence for a scheme S of characteristic $p > 0$, namely

$$0 \longrightarrow \underline{\mathbf{Z}/p\mathbf{Z}}_S \longrightarrow \mathbf{G}_{a,S} \xrightarrow{F-1} \mathbf{G}_{a,S} \longrightarrow 0$$

where $F - 1$ is the map $x \mapsto x^p - x$. Using this, it can be shown that is S is affine then $H^q_{et}(S, \underline{\mathbf{Z}/p\mathbf{Z}}) = 0$ for all $q \geq 2$. In fact, if X is projective over k, then $H^q_{et}(X, \underline{\mathbf{Z}/p\mathbf{Z}}) = 0$ for all $q \geq \dim X + 2$.

- If X is a projective curve over an algebraically closed field then $H^q_{et}(X, \mathcal{F}) = 0$ for all $q \geq 3$ and all torsion sheaves $\mathcal{F}$ on $X_{étale}$. This can be shown using Serre's Mayer Vietoris argument, thereby proving Theorem 38.61.6.
- We can prove using the same methods vanishing of higher cohomology on 1-dimensional schemes of finite type over an algebraically closed field. However, it is easier to reduce to the case of a curve by using the topological invariance of étale cohomology as stated below.

Proposition 38.64.7. *(Topological invariance of étale cohomology) Let X be a scheme and $X_0 \hookrightarrow X$ a closed immersion defined by a nilpotent sheaf of ideals. Then the étale sites $X_{étale}$ and $(X_0)_{étale}$ are isomorphic. In particular, for any sheaf $\mathcal{F}$ on $X_{étale}$, $H^q(X, \mathcal{F}) = H^q(X_0, \mathcal{F}|_{X_0})$ for all q.*

38.65. The trace formula

A typical course in étale cohomology would normally state and prove the proper and smooth base change theorems, purity and Poincaré duality. All of these can be found in [**Del77**, Arcata]. Instead, we are going to study the trace formula for the frobenius, following the account of Deligne in [**Del77**, Rapport]. We will only look at dimension 1, but using proper base change this is enough for the general case. Since all the cohomology groups considered will be étale, we drop the subscript $_{étale}$. Let us now describe the formula we are after. Let X be a finite type scheme of dimension 1 over a finite field k, ℓ a prime number and $\mathcal{F}$ a constructible, flat $\mathbf{Z}/\ell^n\mathbf{Z}$ sheaf. Then

$$\sum\nolimits_{x \in X(k)} \mathrm{Tr}(\mathrm{Frob}|\mathcal{F}_{\bar{x}}) = \sum\nolimits_{i=0}^{2} (-1)^i \mathrm{Tr}(\pi_X^* | H^i_c(X \otimes_k \bar{k}, \mathcal{F})) \tag{38.65.0.1}$$

as elements of $\mathbf{Z}/\ell^n\mathbf{Z}$. As we will see, this formulation is slightly wrong as stated. Let us nevertheless describe the symbols that occur therein.

38.66. Frobenii

Throughout this section, X will denote a scheme of finite type over a finite field k with $q = p^f$ elements. Let $\alpha : X \to Spec(k)$ denote the structural morphism, $\bar{k}$ a fixed algebraic closure of k and $G_k = \mathrm{Gal}(\bar{k}|k)$ the absolute Galois group of k.

Definition 38.66.1. The *absolute frobenius* of X is the morphism $F = F_X : X \to X$ which is the identity on the induced topological space, and which takes a section to its pth power. That is, $F^\sharp : \mathcal{O}_X \to \mathcal{O}_X$ is given by $g \mapsto g^p$. It is clear that this induces the identity on the topological space indeed.

Theorem 38.66.2. *(The Baffling Theorem) Let X be a scheme in characteristic $p > 0$. Then the absolute frobenius induces (by pullback) the trivial map on cohomology, i.e., for all integers $j \geq 0$,*

$$F_X^* : H^j(X, \underline{\mathbf{Z}/n\mathbf{Z}}) \longrightarrow H^j(X, \underline{\mathbf{Z}/n\mathbf{Z}})$$

is the identity.

This theorem is purely formal. It is a good idea, however, to review how to compute the pullback of a cohomology class. Let us simply say that in the case where cohomology agrees with Čech cohomology, it suffices to pull back (using the fiber products on a site) the Čech cocycles. The general case is quite technical and can be found in (insert future reference here). A topological analogue of the baffling theorem is the following.

Exercise 38.66.3. Let X be a topological space and $g : X \to X$ a continuous map such that $g^{-1}(U) = U$ for all opens U of X. Then g induces the identity on cohomology on X (for any coefficients).

We now turn to the statement for the étale site.

Lemma 38.66.4. *Let X be a scheme and $g : X \to X$ a morphism. Assume that for all $\varphi : \mathcal{U} \to X$ étale, there is a functorial isomorphism*

$$\begin{tikzcd} \mathcal{U} \arrow[rr, "\sim"] \arrow[dr, "\varphi"'] & & \mathcal{U} \times_{\varphi, X, g} X \arrow[dl, "pr_2"] \\ & X, & \end{tikzcd}$$

then g induces the identity on cohomology (for any sheaf).

The proof is formal and without difficulty. To prove the theorem, we merely verify that the assumption of the lemma holds for the frobenius.

Proof of Theorem 38.66.2. We need to verify the existence of a functorial isomorphism as above. For an étale morphism $\varphi : \mathcal{U} \to S$, consider the diagram

$$\begin{tikzcd} \mathcal{U} \arrow[dr, dashed] \arrow[drr, bend left, "F_{\mathcal{U}}"] \arrow[ddr, bend right, "\varphi"'] & & \\ & \mathcal{U} \times_{\varphi, X, F_X} X \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"] & \mathcal{U} \arrow[d, "\varphi"] \\ & X \arrow[r, "F_X"] & X. \end{tikzcd}$$

The dotted arrow is an étale morphism which induces an isomorphism on the underlying topological spaces, so it is an isomorphism. $\square$

Definition 38.66.5. The *geometric frobenius* of X is the morphism $\pi_X : X \to X$ over $Spec(k)$ which equals F_X^f. We can base change it to any scheme over k, and in particular to $X_{\bar{k}} = Spec(\bar{k}) \times_{Spec(k)} X$ to get the morphism $\mathrm{id}_{Spec(\bar{k})} \times \pi_X : X_{\bar{k}} \to X_{\bar{k}}$ which we denote π_X again. This should not be ambiguous, as $X_{\bar{k}}$ does not have a geometric frobenius of its own.

Lemma 38.66.6. *Let $\mathcal{F}$ be a sheaf on $X_{étale}$. Then there are canonical isomorphisms $\pi_X^{-1}\mathcal{F} \cong \mathcal{F}$ and $\mathcal{F} \cong \pi_{X*}\mathcal{F}$.*

This is false for the fppf site.

Proof. Let $\varphi : \mathcal{U} \to X$ be étale. Recall that $\pi_{X*}\mathcal{F}(\mathcal{U}) = \mathcal{F}(\mathcal{U} \times \varphi, X, \pi_X X)$. Since $\pi_X = F_X^f$, by Lemma 38.66.4 that there is a functorial isomorphism

$$\begin{tikzcd} \mathcal{U} \arrow[rr, "\sim", "\gamma_{\mathcal{U}}"'] \arrow[dr, "\varphi"'] & & \mathcal{U} \times_{\varphi, X, \pi_X} X \arrow[dl, "\mathrm{pr}_2"] \\ & X & \end{tikzcd}$$

where $\gamma_{\mathcal{U}} = (\varphi, F_{\mathcal{U}}^f)$. Now we define an isomorphism

$$\mathcal{F}(\mathcal{U}) \longrightarrow \pi_{X*}\mathcal{F}(\mathcal{U}) = \mathcal{F}(\mathcal{U} \times_{\varphi, X, \pi_X} X)$$

by taking the restriction map of $\mathcal{F}$ along $\gamma_{\mathcal{U}}^{-1}$. The other isomorphism is analogous. □

Remark 38.66.7. It may or may not be the case that $F^f_{\mathcal{U}}$ equals $\pi_{\mathcal{U}}$.

Let $\mathcal{F}$ be an abelian sheaf on $X_{\acute{e}tale}$. Consider the cohomology group $H^j(X_{\bar{k}}, \mathcal{F}|_{X_{\bar{k}}})$ as a left G_k-module as follows: if $\sigma \in G_k$, the diagram

$$\begin{tikzcd} X_{\bar{k}} \arrow[rr, "Spec(\sigma)\times \mathrm{id}_X"] \arrow[dr] & & X_{\bar{k}} \arrow[dl] \\ & X & \end{tikzcd}$$

commutes. Thus we can set, for $\xi \in H^j(X_{\bar{k}}, \mathcal{F}|_{X_{\bar{k}}})$

$$\sigma \cdot \xi := (Spec(\sigma) \times \mathrm{id}_X)^*\xi \in H^j(X_{\bar{k}}, (Spec(\sigma) \times \mathrm{id}_X)^{-1}\mathcal{F}|X_{\bar{k}}) = H^j(X_{\bar{k}}, \mathcal{F}|_{X_{\bar{k}}}),$$

where the last equality follows from the commutativity of the previous diagram. This endows the latter group with the structure of a G_k-module.

Lemma 38.66.8. *Let $\mathcal{F}$ be an abelian sheaf on $X_{\acute{e}tale}$. Consider $(R^j\alpha_*\mathcal{F})_{Spec(\bar{k})}$ endowed with its natural Galois action as in Section 38.56. Then the identification*

$$(R^j\alpha_*\mathcal{F})_{Spec(\bar{k})} \cong H^j(X_{\bar{k}}, \mathcal{F}|_{X_{\bar{k}}})$$

from Theorem 38.52.2 is an isomorphism of G_k-modules.

A similar result holds comparing $(R^j\alpha_!\mathcal{F})_{Spec(\bar{k})}$ with $H^j_c(X_{\bar{k}}, \mathcal{F}|_{X_{\bar{k}}})$. We omit the proof.

Definition 38.66.9. The *arithmetic frobenius* is the map $\mathrm{frob}_k : \bar{k} \to \bar{k}$, $x \mapsto x^q$ of G_k.

Theorem 38.66.10. *Let $\mathcal{F}$ be an abelian sheaf on $X_{\acute{e}tale}$. Then for all $j \geq 0$, $frob_k$ acts on the cohomology group $H^j(X_{\bar{k}}, \mathcal{F}|_{X_{\bar{k}}})$ as the inverse of the map π_X^*.*

The map π_X^* is defined by the composition

$$H^j(X_{\bar{k}}, \mathcal{F}|_{X_{\bar{k}}}) \xrightarrow{\pi_{X_{\bar{k}}}^*} H^j(X_{\bar{k}}, (\pi_X^{-1}\mathcal{F})|_{X_{\bar{k}}}) \cong H^j(X_{\bar{k}}, \mathcal{F}|_{X_{\bar{k}}}).$$

where the last isomorphism comes from the canonical isomorphism $\pi_X^{-1}\mathcal{F} \cong \mathcal{F}$ of Lemma 38.66.6.

Proof. The composition $X_{\bar{k}} \xrightarrow{Spec(\mathrm{frob}_k)} X_{\bar{k}} \xrightarrow{\pi_X} X_{\bar{k}}$ is equal to $F^f_{X_{\bar{k}}}$, hence the result follows from the baffling theorem suitably generalized to nontrivial coefficients. Note that the previous composition commutes in the sense that $F^f_{X_{\bar{k}}} = \pi_X \circ Spec(\mathrm{frob}_k) = Spec(\mathrm{frob}_k) \circ \pi_X$. □

Definition 38.66.11. If $x \in X(k)$ is a rational point and $\bar{x} : Spec(\bar{k}) \to X$ the geometric point lying over x, we let $\pi_x : \mathcal{F}_{\bar{x}} \to \mathcal{F}_{\bar{x}}$ denote the action by frob_k^{-1} and call it the *geometric frobenius*[6]

[6] This notation is not standard. This operator is denoted F_x in [**Del77**]. We will likely change this notation in the future.

We can now make a more precise statement (albeit a false one) of the trace formula (38.65.0.1). Let X be a finite type scheme of dimension 1 over a finite field k, ℓ a prime number and $\mathcal{F}$ a constructible, flat $\mathbf{Z}/\ell^n\mathbf{Z}$ sheaf. Then

$$\sum\nolimits_{x\in X(k)} \mathrm{Tr}(\pi_X|\mathcal{F}_{\bar{x}}) = \sum\nolimits_{i=0}^{2}(-1)^i \mathrm{Tr}(\pi_X^*|H_c^i(X_{\bar{k}}, \mathcal{F})) \tag{38.66.11.1}$$

as elements of $\mathbf{Z}/\ell^n\mathbf{Z}$. The reason this equation is wrong is that the trace in the right-hand side does not make sense for the kind of sheaves considered. Before addressing this issue, we try to motivate the appearance of the geometric frobenius (apart from the fact that it is a natural morphism!).

Let us consider the case where $X = \mathbf{P}^1_k$ and $\mathcal{F} = \underline{\mathbf{Z}/\ell\mathbf{Z}}$. For any point, the Galois module $\mathcal{F}_{\bar{x}}$ is trivial, hence for any morphism φ acting on $\mathcal{F}_{\bar{x}}$, the left-hand side is

$$\sum\nolimits_{x\in X(k)} \mathrm{Tr}(\varphi|\mathcal{F}_{\bar{x}}) = \#\mathbf{P}^1_k(k) = q + 1.$$

Now $\mathbf{P}^1_k$ is proper, so compactly supported cohomology equals standard cohomology, and so for a morphism $\pi : \mathbf{P}^1_k \to \mathbf{P}^1_k$, the right-hand side equals

$$\mathrm{Tr}(\pi^*|H^0(\mathbf{P}^1_{\bar{k}}, \underline{\mathbf{Z}/\ell\mathbf{Z}})) + \mathrm{Tr}(\pi^*|H^2(\mathbf{P}^1_{\bar{k}}, \underline{\mathbf{Z}/\ell\mathbf{Z}})).$$

The Galois module $H^0(\mathbf{P}^1_{\bar{k}}, \underline{\mathbf{Z}/\ell\mathbf{Z}}) = \mathbf{Z}/\ell\mathbf{Z}$ is trivial, since the pullback of the identity is the identity. Hence the first trace is 1, regardless of π. For the second trace, we need to compute the pullback of a map $\pi : \mathbf{P}^1_{\bar{k}} \to \mathbf{P}^1_{\bar{k}}$ on $H^2(\mathbf{P}^1_{\bar{k}}, \underline{\mathbf{Z}/\ell\mathbf{Z}}))$. This is a good exercise and the answer is multiplication by the degree of π. In other words, this works as in the familiar situation of complex cohomology. In particular, if π is the geometric frobenius we get

$$\mathrm{Tr}(\pi_X^*|H^2(\mathbf{P}^1_{\bar{k}}, \underline{\mathbf{Z}/\ell\mathbf{Z}})) = q$$

and if π is the arithmetic frobenius then we get

$$\mathrm{Tr}(\mathrm{frob}_k^*|H^2(\mathbf{P}^1_{\bar{k}}, \underline{\mathbf{Z}/\ell\mathbf{Z}})) = q^{-1}.$$

The latter option is clearly wrong.

Remark 38.66.12. The computation of the degrees can be done by lifting (in some obvious sense) to characteristic 0 and considering the situation with complex coefficients. This method almost never works, since lifting is in general impossible for schemes which are not projective space.

The question remains as to why we have to consider compactly supported cohomology. In fact, in view of Poincaré duality, it is not strictly necessary for smooth varieties, but it involves adding in certain powers of q. For example, let us consider the case where $X = \mathbf{A}^1_k$ and $\mathcal{F} = \underline{\mathbf{Z}/\ell\mathbf{Z}}$. The action on stalks is again trivial, so we only need look at the action on cohomology. But then π_X^* acts as the identity on $H^0(\mathbf{A}^1_{\bar{k}}, \underline{\mathbf{Z}/\ell\mathbf{Z}})$ and as multiplication by q on $H^2_c(\mathbf{A}^1_{\bar{k}}, \underline{\mathbf{Z}/\ell\mathbf{Z}})$.

38.67. Traces

We now explain how to take the trace of an endomorphism of a module over a noncommutative ring. Fix a finite ring Λ with cardinality prime to p. Typically, Λ is the group ring $(\mathbf{Z}/\ell^n\mathbf{Z})[G]$ for some finite group G. By convention, all the Λ-modules considered will be left Λ-modules.

We introduce the following notation: We set $\Lambda^\natural$ to be the quotient of Λ by its additive subgroup generated by the commutators (i.e., the elements of the form $ab - ba$, $a, b \in \Lambda$). Note that $\Lambda^\natural$ is not a ring.

For instance, the module $(\mathbf{Z}/\ell^n\mathbf{Z})[G]^\natural$ is the dual of the class functions, so

$$(\mathbf{Z}/\ell^n\mathbf{Z})[G]^\natural = \bigoplus_{\substack{conjugacy\\ classesofG}} \mathbf{Z}/\ell^n\mathbf{Z}.$$

For a free Λ-module, we have $\mathrm{End}_\Lambda(\Lambda^{\oplus m}) = \mathrm{Mat}_n(\Lambda)$. Note that since the modules are left modules, representation of endomorphism by matrices is a right action: if $a \in \mathrm{End}(\Lambda^{\oplus m})$ has matrix A and $v \in \Lambda$, then $a(v) = vA$.

Definition 38.67.1. The *trace* of the endomorphism a is the sum of the diagonal entries of a matrix representing it. This defines an additive map $\mathrm{Tr} : \mathrm{End}_\Lambda(\Lambda^{\oplus m}) \to \Lambda^\natural$.

Exercise 38.67.2. Given maps

$$\Lambda^{\oplus n} \xrightarrow{a} \Lambda^{\oplus n} \xrightarrow{b} \Lambda^{\oplus m}$$

show that $\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$.

We extend the definition of the trace to a finite projective Λ-module $P$and an endomorphism φ of P as follows. Write P as the summand of a free Λ-module, i.e., consider maps $P \xrightarrow{a} \Lambda^{\oplus n} \xrightarrow{b} P$ with

(1) $\Lambda^{\oplus n} = \mathrm{Im}(a) \oplus \ker(b)$; and
(2) $b \circ a = \mathrm{id}_P$.

Then we set $\mathrm{Tr}(\varphi) = \mathrm{Tr}(a\varphi b)$. It is easy to check that this is well-defined, using the previous exercise.

38.68. Why derived categories?

With this definition of the trace, let us now discuss another issue with the formula as stated. Let C be a smooth projective curve over k. Then there is a correspondence between finite locally constant sheaves $\mathcal{F}$ on $C_{étale}$ which stalks are isomorphic to $(\mathbf{Z}/\ell^n\mathbf{Z})^{\oplus m}$ on the one hand, and continuous representations $\rho : \pi_1(C, \bar{c}) \to \mathrm{GL}_m(\mathbf{Z}/\ell^n\mathbf{Z}))$ (for some fixed choice of $\bar{c}$) on the other hand. We denote $\mathcal{F}_\rho$ the sheaf corresponding to ρ. Then $H^2(C_{\bar{k}}, \mathcal{F}_\rho)$ is the group of coinvariants for the action of $\rho(\pi_1(C, \bar{c}))$ on $(\mathbf{Z}/\ell^n\mathbf{Z})^{\oplus m}$, and there is a short exact sequence

$$0 \longrightarrow \pi_1(C_{\bar{k}}, \bar{c}) \longrightarrow \pi_1(C, \bar{c}) \longrightarrow G_k \longrightarrow 0.$$

For instance, let $\mathbf{Z} = \mathbf{Z}\sigma$ act on $\mathbf{Z}/\ell^2\mathbf{Z}$ via $\sigma(x) = (1+\ell)x$. The coinvariants are $(\mathbf{Z}/\ell^2\mathbf{Z})_\sigma = \mathbf{Z}/\ell\mathbf{Z}$, which is not a flat $\mathbf{Z}/\ell\mathbf{Z}$-module. Hence we cannot take the trace of some action on $H^2(C_{\bar{k}}, \mathcal{F}_\rho)$, at least not in the sense of the previous section.

In fact, our goal is to consider a trace formula for ℓ-adic coefficients. But $\mathbf{Q}_\ell = \mathbf{Z}_\ell[1/\ell]$ and $\mathbf{Z}_\ell = lim\, \mathbf{Z}/\ell^n\mathbf{Z}$, and even for a flat $\mathbf{Z}/\ell^n\mathbf{Z}$ sheaf, the individual cohomology groups may not be flat, so we cannot compute traces. One possible remedy is consider the total derived complex $R\Gamma(C_{\bar{k}}, \mathcal{F}_\rho)$ in the derived category $D(\mathbf{Z}/\ell^n\mathbf{Z})$ and show that it is a perfect object, which means that it is quasi-isomorphic to a finite complex of finite free module. For such complexes, we can define the trace, but this will require an account of derived categories.

38.69. Derived categories

To set up notation, let $\mathscr{A}$ be an abelian category. Let $\mathrm{Comp}(\mathscr{A})$ be the abelian category of complexes in $\mathscr{A}$. Let $K(\mathscr{A})$ be the category of complexes up to homotopy, with objects equal to complexes in $\mathscr{A}$ and objects equal to homotopy classes of morphisms of complexes. This is not an abelian category. Loosely speaking, $D(A)$ is defined to be the category obtained by inverting all quasi-isomorphisms in $\mathrm{Comp}(\mathscr{A})$ or, equivalently, in $K(\mathscr{A})$. Moreover, we can define $\mathrm{Comp}^+(\mathscr{A})$, $K^+(\mathscr{A})$, $D^+(\mathscr{A})$ analogously using only bounded below complexes. Similarly, we can define $\mathrm{Comp}^-(\mathscr{A})$, $K^-(\mathscr{A})$, $D^-(\mathscr{A})$ using bounded above complexes, and we can define $\mathrm{Comp}^b(\mathscr{A})$, $K^b(\mathscr{A})$, $D^b(\mathscr{A})$ using bounded complexes.

Remark 38.69.1. Notes on derived categories.

(1) There are some set-theoretical problems when $\mathscr{A}$ is somewhat arbitrary, which we will happily disregard.
(2) The categories $K(A)$ and $D(A)$ may be endowed with the structure of triangulated category, but we will not need these structures in the following discussion.
(3) The categories $\mathrm{Comp}(\mathscr{A})$ and $K(\mathscr{A})$ can also be defined when $\mathscr{A}$ is an additive category.

The homology functor $H^i : \mathrm{Comp}(\mathscr{A}) \to \mathscr{A}$ taking a complex $K^\bullet \mapsto H^i(K^\bullet)$ extends to functors $H^i : K(\mathscr{A}) \to \mathscr{A}$ and $H^i : D(\mathscr{A}) \to \mathscr{A}$.

Lemma 38.69.2. *An object E of $D(\mathscr{A})$ is contained in $D^+(\mathscr{A})$ if and only if $H^i(E) = 0$ for all $i \ll 0$. Similar statements hold for D^- and D^+.*

The proof uses truncation functors.

Lemma 38.69.3. *Morphisms between objects in the derived category.*

(1) *Let $I^\bullet$ be a complex in $\mathscr{A}$ with I^n injective for all $n \in \mathbf{Z}$. Then*
$$Hom_{D(\mathscr{A})}(K^\bullet, I^\bullet) = Hom_{K(\mathscr{A})}(K^\bullet, I^\bullet).$$
(2) *Let $P^\bullet \in Comp^-(\mathscr{A})$ with P^n is projective for all $n \in \mathbf{Z}$. Then*
$$Hom_{D(\mathscr{A})}(P^\bullet, K^\bullet) = Hom_{K(\mathscr{A})}(P^\bullet, K^\bullet).$$
(3) *If $\mathscr{A}$ has enough injectives and $\mathscr{I} \subset \mathscr{A}$ is the additive subcategory of injectives, then $D^+(\mathscr{A}) \cong K^+(\mathscr{I})$ (as triangulated categories).*
(4) *If $\mathscr{A}$ has enough projectives and $\mathscr{P} \subset \mathscr{A}$ is the additive subcategory of projectives, then $D^-(\mathscr{A}) \cong K^-(\mathscr{P})$.*

Proof. Omitted. $\square$

Definition 38.69.4. Let $F : \mathscr{A} \to \mathscr{B}$ be a left exact functor and assume that $\mathscr{A}$ has enough injectives. We define the *total right derived functor of F* as the functor $RF : D^+(\mathscr{A}) \to D^+(\mathscr{B})$ fitting into the diagram

$$\begin{tikzcd} D^+(\mathscr{A}) \arrow[r, "RF"] & D^+(\mathscr{B}) \\ K^+(\mathscr{I}) \arrow[u] \arrow[r, "F"] & K^+(\mathscr{B}). \arrow[u] \end{tikzcd}$$

This is possible since the left vertical arrow is invertible by the previous lemma. Similarly, let $G : \mathscr{A} \to \mathscr{B}$ be a right exact functor and assume that $\mathscr{A}$ has enough projectives. We define the *total right derived functor of G* as the functor $LG : D^-(\mathscr{A}) \to D^-(\mathscr{B})$ fitting into the diagram

$$\begin{tikzcd} D^-(\mathscr{A}) \arrow[r, "LG"] & D^-(\mathscr{B}) \\ K^-(\mathscr{P}) \arrow[u] \arrow[r, "G"] & K^-(\mathscr{B}). \arrow[u] \end{tikzcd}$$

This is possible since the left vertical arrow is invertible by the previous lemma.

Remark 38.69.5. In these cases, it is true that $R^iF(K^\bullet) = H^i(RF(K^\bullet))$, where the left hand side is defined to be ith homology of the complex $F(K^\bullet)$.

38.70. Filtered derived category

It turns out we have to do it all again and build the filtered derived category also.

Definition 38.70.1. Let $\mathscr{A}$ be an abelian category. Let $\mathrm{Fil}(\mathscr{A})$ be the category of filtered objects (A, F) of $\mathscr{A}$, where F is a filtration of the form

$$A \supseteq \cdots \supseteq F^nA \supseteq F^{n+1}A \supseteq \cdots \supseteq 0.$$

This is an additive category. We denote $\mathrm{Fil}^f(\mathscr{A})$ the full subcategory of $\mathrm{Fil}(\mathscr{A})$ whose objects (A, F) have finite filtration. This is also an additive category. An object $I \in \mathrm{Fil}^f(\mathscr{A})$ is called *filtered injective* (respectively *projective*) provided that $\mathrm{gr}^p(I) = \mathrm{gr}^p_F(I) = F^pI/F^{p+1}I$ is injective (resp. projective) in $\mathscr{A}$ for all p. The categories $\mathrm{Comp}(\mathrm{Fil}^f(\mathscr{A})) \supseteq \mathrm{Comp}^+(\mathrm{Fil}^f(\mathscr{A}))$ and $K(\mathrm{Fil}^f(\mathscr{A})) \supseteq K^+(\mathrm{Fil}^f(\mathscr{A}))$ are defined as before.
A morphism $\alpha : K^\bullet \to L^\bullet$ of complexes in $\mathrm{Comp}(\mathrm{Fil}^f(\mathscr{A}))$ is called a *filtered quasi-isomorphism* provided that

$$\mathrm{gr}^p(\alpha) : \mathrm{gr}^p(K^\bullet) \to \mathrm{gr}^p(L^\bullet)$$

is a quasi-isomorphism for all $p \in \mathbf{Z}$. Finally, we define $DF(\mathscr{A})$ (resp. $DF^+(\mathscr{A})$) by inverting the filtered quasi-isomorphisms in $K(\mathrm{Fil}^f(\mathscr{A}))$ (resp. $K^+(\mathrm{Fil}^f(\mathscr{A}))$).

Lemma 38.70.2. *If $\mathscr{A}$ has enough injectives, then $DF^+(\mathscr{A}) \cong K^+(\mathscr{I})$, where $\mathscr{I}$ is the full additive subcategory of $\mathrm{Fil}^f(\mathscr{A})$ consisting of filtered injective objects. Similarly, if $\mathscr{A}$ has enough projectives, then $DF^-(\mathscr{A}) \cong K^+(\mathscr{P})$, where $\mathscr{P}$ is the full additive subcategory of $\mathrm{Fil}^f(\mathscr{A})$ consisting of filtered projective objects.*

Proof. Omitted. □

38.71. Filtered derived functors

And then there are the filetered derived functors.

Definition 38.71.1. Let $T : \mathscr{A} \to \mathscr{B}$ be a left exact functor and assume that $\mathscr{A}$ has enough injectives. Define $RT : DF^+(\mathscr{A}) \to DF^+(\mathscr{B})$ to fit in the diagram

$$\begin{tikzcd} DF^+(\mathscr{A}) \arrow[r, "RT"] & DF^+(\mathscr{B}) \\ K^+(\mathscr{I}) \arrow[u] \arrow[r, "T"] & K^+(\mathrm{Fil}^f(\mathscr{B})). \arrow[u] \end{tikzcd}$$

This is well-defined by the previous lemma. Let $G : \mathscr{A} \to \mathscr{B}$ be a right exact functor and assume that $\mathscr{A}$ has enough projectives. Define $LG : DF^{+}(\mathscr{A}) \to DF^{+}(\mathscr{B})$ to fit in the diagram

$$\begin{tikzcd} DF^{-}(\mathscr{A}) \arrow[r, "LG"] & DF^{-}(\mathscr{B}) \\ K^{-}(\mathscr{P}) \arrow[u] \arrow[r, "G"] & K^{-}(\mathrm{Fil}^{f}(\mathscr{B})). \arrow[u] \end{tikzcd}$$

Again, this is well-defined by the previous lemma. The functors RT, resp. LG, are called the *filtered derived functor* of T, resp. G.

Proposition 38.71.2. *In the situation above, we have*

$$\mathrm{gr}^{p} \circ RT = RT \circ \mathrm{gr}^{p}$$

where the RT on the left is the filtered derived functor while the one on the right is the total derived functor. That is, there is a commuting diagram

$$\begin{tikzcd} DF^{+}(\mathscr{A}) \arrow[r, "RT"] \arrow[d, "gr^{p}"'] & DF^{+}(\mathscr{B}) \arrow[d, "gr^{p}"] \\ D^{+}(\mathscr{A}) \arrow[r, "RT"] & D^{+}(\mathscr{B}). \end{tikzcd}$$

Proof. Omitted. □

Given $K^{\bullet} \in DF^{+}(\mathscr{B})$, we get a spectral sequence

$$E_{1}^{p,q} = H^{p+q}(\mathrm{gr}^{p} K^{\bullet}) \Rightarrow H^{p+q}(\text{forget filt}(K^{\bullet})).$$

38.72. Application of filtered complexes

Let $\mathscr{A}$ be an abelian category with enough injectives, and $0 \to L \to M \to N \to 0$ a short exact sequence in $\mathscr{A}$. Consider $\widetilde{M} \in \mathrm{Fil}^{f}(\mathscr{A})$ to be M along with the filtration defined by

$$F^{1} M = L,\ F^{n} M = M \text{ for } n \leq 0, \text{ and } F^{n} M = 0 \text{ for } n \geq 2.$$

By definition, we have

$$\text{forget filt}(\widetilde{M}) = M, \quad \mathrm{gr}^{0}(\widetilde{M}) = N, \quad \mathrm{gr}^{1}(\widetilde{M}) = L$$

and $\mathrm{gr}^{n}(\widetilde{M}) = 0$ for all other $n \neq 0, 1$. Let $T : \mathscr{A} \to \mathscr{B}$ be a left exact functor. Assume that $\mathscr{A}$ has enough injectives. Then $RT(\widetilde{M}) \in DF^{+}(\mathscr{B})$ is a filtered complex with

$$\mathrm{gr}^{p}(RT(\widetilde{M})) \overset{\text{qis}}{=} \begin{cases} 0 & \text{if} \quad p \neq 0, 1, \\ RT(N) & \text{if} \quad p = 0, \\ RT(L) & \text{if} \quad p = 1. \end{cases}$$

and forget filt$(RT(\widetilde{M})) \overset{\text{qis}}{=} RT(M)$. The spectral sequence applied to $RT(\widetilde{M})$ gives

$$E_{1}^{p,q} = R^{p+q}T(\mathrm{gr}^{p}(\widetilde{M})) \Rightarrow R^{p+q}T(\text{forget filt}(\widetilde{M})).$$

Unwinding the spectral sequence gives us the long exact sequence

$$\begin{tikzcd} 0 \arrow[r] & T(L) \arrow[r] & T(M) \arrow[r] & T(N) \arrow[dll, out=-90, in=90] \\ & R^{1}T(L) \arrow[r] & R^{1}T(M) \arrow[r] & \cdots \end{tikzcd}$$

This will be used as follows. Let X/k be a scheme of finite type. Let $\mathcal{F}$ be a flat constructible $\mathbf{Z}/\ell^n\mathbf{Z}$-module. Then we want to show that the trace

$$\mathrm{Tr}(\pi_X^* | R\Gamma_c(X_{\bar{k}}, \mathcal{F})) \in \mathbf{Z}/\ell^n\mathbf{Z}$$

is additive on short exact sequences. To see this, it will not be enough to work with $R\Gamma_c(X_{\bar{k}}, -) \in D^+(\mathbf{Z}/\ell^n\mathbf{Z})$, but we will have to use the filtered derived category.

38.73. Perfectness

Let Λ be a (possibly noncommutative) ring, Mod_Λ the category of left Λ-modules, $K(\Lambda) = K(\mathrm{Mod}_\Lambda)$ its homotopy category, and $D(\Lambda) = D(\mathrm{Mod}_\Lambda)$ the derived category.

Definition 38.73.1. We denote by $K_{perf}(\Lambda)$ the category whose objects are bounded complexes of finite projective Λ-modules, and whose morphisms are morphisms of complexes up to homotopy. The functor $K_{perf}(\Lambda) \to D(\Lambda)$ is fully faithful, and we denote $D_{perf}(\Lambda)$ its essential image. An object of $D(\Lambda)$ is called *perfect* if it is in $D_{perf}(\Lambda)$.

Proposition 38.73.2. *Let $K \in D_{perf}(\Lambda)$ and $f \in End_{D(\Lambda)}(K)$. Then the trace $Tr(f) \in \Lambda^\natural$ is well defined.*

Proof. Let $P^\bullet$ be a bounded complex of finite projective Λ-modules and $\alpha : P^\bullet \cong K$ be an isomorphism in $D(\Lambda)$. Then $\alpha^{-1} \circ f \circ \alpha$ is the class of some morphism of complexes $f^\bullet : P^\bullet \to P^\bullet$ by (insert reference here). Set

$$\mathrm{Tr}(f) = \sum_i (-1)^i \mathrm{Tr}(f^i : P^i \to P^i) \in \Lambda^\natural.$$

Given $P^\bullet$ and α, this is independent of the choice of $f^\bullet$: any other choice is of the form $\tilde{f}^\bullet = f^\bullet + dh + hd$ for some $h^i : P^i \to P^{i-1} (i \in \mathbf{Z})$. But

$$\begin{aligned} \mathrm{Tr}(dh) &= \sum_i (-1)^i \mathrm{Tr}(P^i \xrightarrow{dh} P^i) \\ &= \sum_i (-1)^i \mathrm{Tr}(P^{i-1} \xrightarrow{hd} P^{i-1}) \\ &= -\sum_i (-1)^{i-1} \mathrm{Tr}(P^{i-1} \xrightarrow{hd} P^{i-1}) \\ &= -\mathrm{Tr}(hd) \end{aligned}$$

and so $\sum_i (-1)^i \mathrm{Tr}((dh + hd)|_{P^i}) = 0$. Furthermore, this is independent of the choice of $(P^\bullet, \alpha)$: suppose $(Q^\bullet, \beta)$ is another choice. Then by ???, the compositions

$$Q^\bullet \xrightarrow{\beta} K \xrightarrow{\alpha^{-1}} P^\bullet \quad \text{and} \quad P^\bullet \xrightarrow{\alpha} K \xrightarrow{\beta^{-1}} Q^\bullet$$

are representable by morphisms of complexes $\gamma_1^\bullet$ and $\gamma_2^\bullet$ respectively, such that $\gamma_1^\bullet \circ \gamma_2^\bullet$ is homotopic to the identity. Thus, the morphism of complexes $\gamma_2^\bullet \circ f^\bullet \circ \gamma_1^\bullet : Q^\bullet \to Q^\bullet$ represents the morphism $\beta^{-1} \circ f \circ \beta$ in $D(\Lambda)$. Now

$$\begin{aligned} \mathrm{Tr}(\gamma_2^\bullet \circ f^\bullet \circ \gamma_1^\bullet|_{Q^\bullet}) &= \mathrm{Tr}(\gamma_1^\bullet \circ \gamma_2^\bullet \circ f^\bullet|_{P^\bullet}) \\ &= \mathrm{Tr}(f^\bullet|_{P^\bullet}) \end{aligned}$$

by the fact that $\gamma_1^\bullet \circ \gamma_2^\bullet$ is homotopic to the identity and the independence from $(P^\bullet, \alpha)$ already proved. $\square$

38.74. Filtrations and perfect complexes

We now present a filtered version of the category of perfect complexes. An object (M, F) of $\mathrm{Fil}^f(\mathrm{Mod}_\Lambda)$ is called *filtered finite projective* if for all p, $\mathrm{gr}^p_F(M)$ is finite and projective. We then consider the homotopy category $K\mathrm{F}_{\mathrm{perf}}(\Lambda)$ of bounded complexes of filtered finite projective objects of $\mathrm{Fil}^f(\mathrm{Mod}_\Lambda)$. We have a diagram of categories

$$\begin{array}{ccc} K\mathrm{F}(\Lambda) & \supseteq & K\mathrm{F}_{\mathrm{perf}}(\Lambda) \\ \downarrow & & \downarrow \\ D\mathrm{F}(\Lambda) & \supseteq & D\mathrm{F}_{\mathrm{perf}}(\Lambda) \end{array}$$

where the vertical functor on the right is fully faithful and the category $D\mathrm{F}_{\mathrm{perf}}(\Lambda)$ is its essential image, as before.

Lemma 38.74.1. *(Additivity) Let $K \in D\mathrm{F}_{perf}(\Lambda)$ and $f \in End_{D\mathrm{F}}(K)$. Then*

$$Tr(f|_K) = \sum\nolimits_{p \in \mathbf{Z}} Tr(f|_{gr^p K}).$$

Proof. By Proposition 38.73.2, we may assume we have a bounded complex $P^\bullet$ of filtered finite projectives of $\mathrm{Fil}^f(\mathrm{Mod}_\Lambda)$ and a map $f^\bullet : P^\bullet \to P^\bullet$ in $\mathrm{Comp}(\mathrm{Fil}^f(\mathrm{Mod}_\Lambda))$. So the lemma follows from the following result, which proof is left to the reader. □

Lemma 38.74.2. *Let $P \in Fil^f(Mod_\Lambda)$ be filtered finite projective, and $f : P \to P$ an endomorphism in $Fil^f(Mod_\Lambda)$. Then*

$$Tr(f|_P) = \sum\nolimits_p Tr(f|_{gr^p(P)}).$$

Proof. Omitted. □

38.75. Characterizing perfect objects

Definition 38.75.1. An object $K \in D^-(\Lambda)$ is said to have *finite Tor-dimension* if there exists $r \in \mathbf{Z}$ such that for any right Λ-module N, $H^i(N \otimes^{\mathbf{L}}_\Lambda K) = 0$ for all $i \leq r$ (in other words, $\mathrm{Tor}^i_\Lambda(N, K) = 0$). Recall that $N \otimes^{\mathbf{L}}_\Lambda K$ is the total left derived functor of the functor $\mathrm{Mod}_\Lambda \to Ab$, $M \mapsto N \otimes_\Lambda M$. It is thus a complex of abelian groups.

Lemma 38.75.2. *Let Λ be a left noetherian ring and $K \in D^-(\Lambda)$. Then K is perfect if and only if the two following conditions hold:*

(1) *K has finite Tor-dimension ; and*
(2) *for all $i \in \mathbf{Z}$, $H^i(K)$ is a finite Λ-module.*

The reader is strongly urged to try and prove this. The proof relies on the fact that a finite module on a finitely left-presented ring is flat if and only if it is projective.

Remark 38.75.3. A common variant of this lemma is to consider instead a noetherian scheme X and the category $D_{perf}(\mathcal{O}_X)$ of complexes which are locally quasi-isomorphic to a finite complex of finite locally free $\mathcal{O}_X$-modules.

Notation: Let Λ be a finite ring, X a noetherian scheme, $K(X, \Lambda)$ the homotopy category of sheaves of Λ-modules on $X_{étale}$, and $D(X, \Lambda)$ the corresponding derived category. We denote by D^b (respectively D^+, D^-) the full subcategory of bounded (resp. above, below) complexes in $D(X, \Lambda)$.

Definition 38.75.4. With notation as above, consider the full subcategory $D^b_{ctf}(X, \Lambda)$ of $D^-(X, \Lambda)$ consisting of objects which are quasi-isomorphic to a bounded complex of constructible flat Λ-modules. Its objects are abusively called *perfect complexes*.

Remark 38.75.5. In fact, for a bounded complex $K^\bullet$ of constructible flat Λ-modules each stalk $K^p_{\bar{x}}$ is a finite projective Λ-module.

Remark 38.75.6. This construction differs from the common variant mentioned above. It can happen that a complex of $\mathcal{O}_X$-modules is locally quasi-isomorphic to a finite complex of finite locally free $\mathcal{O}_X$-modules, without being globally quasi-isomorphic to a bounded complex of locally free $\mathcal{O}_X$-modules. This does not happen in the étale site for constructible sheaves.

In this framework, Lemma 38.75.2 reads as follows.

Lemma 38.75.7. *Let $K \in D^-(X, \Lambda)$. Then $K \in D^b_{ctf}(X, \Lambda)$ if and only if*

(1) *K has finite Tor-dimension ; and*
(2) *for all $i \in \mathbf{Z}$, $\underline{H}^i(K)$ is constructible.*

The first condition can be checked on stalks (provided that the bounds are uniform).

Remark 38.75.8. This lemma is used to prove that if $f : X \to Y$ is a separated, finite type morphism of schemes and Y is noetherian, then $Rf_!$ induces a functor $D^b_{ctf}(X, \Lambda) \to D^b_{ctf}(Y, \Lambda)$. We only need this fact in the case where Y is the spectrum of a field and X is a curve.

Proposition 38.75.9. *Let X be a projective curve over a field k, Λ a finite ring and $K \in D^b_{ctf}(X, \Lambda)$. Then $R\Gamma(X_{\bar{k}}, K) \in D_{perf}(\Lambda)$.*

Sketch of proof. The first step is to show:

(1) *The cohomology of $R\Gamma(X_{\bar{k}}, K)$ is bounded.*

Consider the spectral sequence

$$H^i(X_{\bar{k}}, \underline{H}^j(K)) \Rightarrow H^{i+j}(R\Gamma(X_{\bar{k}}, K)).$$

Since K is bounded and Λ is finite, the sheaves $\underline{H}^j(K)$ are torsion. Moreover, $X_{\bar{k}}$ has finite cohomological dimension, so the left-hand side is nonzero for finitely many i and j only. Therefore, so is the right-hand side.

(2) *The cohomology groups $H^{i+j}(R\Gamma(X_{\bar{k}}, K))$ are finite.*

Since the sheaves $\underline{H}^j(K)$ are constructible, the groups $H^i(X_{\bar{k}}, \underline{H}^j(K))$ are finite, [7] so it follows by the spectral sequence again.

(3) *$R\Gamma(X_{\bar{k}}, K)$ has finite Tor-dimension.*

Let N be a right Λ-module (in fact, since Λ is finite, it suffices to assume that N is finite). By the projection formula (change of module),

$$N \otimes^{\mathbf{L}}_\Lambda R\Gamma(X_{\bar{k}}, K) = R\Gamma(X_{\bar{k}}, N \otimes^{\mathbf{L}}_\Lambda K).$$

Therefore,

$$H^i(N \otimes^{\mathbf{L}}_\Lambda R\Gamma(X_{\bar{k}}, K)) = H^i(R\Gamma(X_{\bar{k}}, N \otimes^{\mathbf{L}}_\Lambda K)).$$

Now consider the spectral sequence

$$H^i(X_{\bar{k}}, \underline{H}^j(N \otimes^{\mathbf{L}}_\Lambda K)) \Rightarrow H^{i+j}(R\Gamma(X_{\bar{k}}, N \otimes^{\mathbf{L}}_\Lambda K)).$$

Since K has finite Tor-dimension, $\underline{H}^j(N \otimes^{\mathbf{L}}_\Lambda K)$ vanishes universally for j small enough, and the left-hand side vanishes whenever $i < 0$. Therefore $R\Gamma(X_{\bar{k}}, K)$ has finite Tor-dimension, as claimed. So it is a perfect complex by Lemma 38.75.2. □

[7] In Section 38.64 where we proved vanishing of cohomology, we should have proved -- using the exact same arguments -- that étale cohomology with values in a torsion sheaf is finite. Maybe that section should be updated. It's flabbergasting that we even forgot to mention it.

38.76. Lefschetz numbers

The fact that the total cohomology of a constructible complex with finite coefficients is a perfect complex is the key technical reason why cohomology behaves well, and allows us to define rigorously the traces occurring in the trace formula.

Definition 38.76.1. Let Λ be a finite ring, X a projective curve over a finite field k and $K \in D_{ctf}^b(X, \Lambda)$ (for instance $K = \underline{\Lambda}$). There is a canonical map $c_K : \pi_X^{-1}K \to K$, and its base change $c_K|_{X_{\bar{k}}}$ induces an action denoted π_X^* on the perfect complex $R\Gamma(X_{\bar{k}}, K|_{X_{\bar{k}}})$. The *global Lefschetz number* of K is the trace $\text{Tr}(\pi_X^*\big|_{R\Gamma(X_{\bar{k}},K)})$ of that action. It is an element of $\Lambda^\natural$.

Definition 38.76.2. With Λ, X, k, K as in Definition 38.76.1. Since $K \in D_{ctf}^b(X, \Lambda)$, for any geometric point $\bar{x}$ of X, the complex $K_{\bar{x}}$ is a perfect complex (in $D_{perf}(\Lambda)$). As we have seen in Section 38.66, the Frobenius π_X acts on $K_{\bar{x}}$. The *local Lefschetz number* of K is the sum

$$\sum_{x \in X(k)} \text{Tr}(\pi_X \big|_{K_{\bar{x}}})$$

which is again an element of $\Lambda^\natural$.

At last, we can formulate precisely the trace formula.

Theorem 38.76.3. *(Lefschetz Trace Formula) Let X be a projective curve over a finite field k, Λ a finite ring and $K \in D_{ctf}^b(X, \Lambda)$. Then the global and local Lefschetz numbers of K are equal, i.e.,*

$$Tr(\pi_X^*\big|_{R\Gamma(X_{\bar{k}},K)}) = \sum\nolimits_{x \in X(k)} Tr(\pi_X\big|_{K_{\bar{x}}}) \tag{38.76.3.1}$$

in $\Lambda^\natural$.

We will use, rather than prove, the trace formula. Nevertheless, we will give quite a few details of the proof of the theorem as given in [**Del77**] (some of the things that are not adequately explained are listed in Section 38.83).

We only stated the formula for curves, and in some weak sense it is a consequence of the following result.

Theorem 38.76.4. *(Weil) Let C be a nonsingular projective curve over an algebraically closed field k, and $\varphi : C \to C$ a k-endomorphism of C distinct from the identity. Let $V(\varphi) = \Delta_C \cdot \Gamma_\varphi$, where Δ_C is the diagonal, Γ_φ is the graph of φ, and the intersection number is taken on $C \times C$. Let $J = \underline{Pic}^0_{C/k}$ be the jacobian of C and denote $\varphi^* : J \to J$ the action induced by φ by taking pullbacks. Then*

$$V(\varphi) = 1 - Tr_J(\varphi^*) + \deg \varphi.$$

The number $V(\varphi)$ is the number of fixed points of φ, it is equal to

$$V(\varphi) = \sum_{c \in |C| : \varphi(c) = c} m_{\text{Fix}(\varphi)}(c)$$

where $m_{\text{Fix}(\varphi)}(c)$ is the multiplicity of c as a fixed point of φ, namely the order or vanishing of the image of a local uniformizer under $\varphi - \text{id}_C$. Proofs of this theorem can be found in [**Lan02, Wei48**].

Example 38.76.5. Let $C = E$ be an elliptic curve and $\varphi = [n]$ be multiplication by n. Then $\varphi^* = \varphi^t$ is multiplication by n on the jacobian, so it has trace $2n$ and degree n^2. On the other hand, the fixed points of φ are the points $p \in E$ such that $np = p$, which is the $(n-1)$-torsion, which has cardinality $(n-1)^2$. So the theorem reads

$$(n-1)^2 = 1 - 2n + n^2.$$

Jacobians. We now discuss without proofs the correspondence between a curve and its jacobian which is used in Weil's proof. Let C be a nonsingular projective curve over an algebraically closed field k and choose a base point $c_0 \in C(k)$. Denote by $A^1(C \times C)$ (or $\text{Pic}(C \times C)$, or $\text{CaCl}(C \times C)$) the abelian group of codimension 1 divisors of $C \times C$. Then

$$A^1(C \times C) = \text{pr}_1^*(A^1(C)) \oplus \text{pr}_2^*(A^1(C)) \oplus R$$

where

$$R = \{Z \in A^1(C \times C) \mid Z|_{C \times \{c_0\}} \sim_{\text{rat}} 0 \text{ and } Z|_{\{c_0\} \times C} \sim_{\text{rat}} 0\}.$$

In other words, R is the subgroup of line bundles which pull back to the trivial one under either projection. Then there is a canonical isomorphism of abelian groups $R \cong \text{End}(J)$ which maps a divisor Z in R to the endomorphism

$$\begin{matrix} J & \to & J \\ \left[\mathcal{O}_C(D)\right] & \mapsto & (\text{pr}_1|_Z)_*(\text{pr}_2|_Z)^*(D). \end{matrix}$$

The aforementioned correspondence is the following. We denote by σ the automorphism of $C \times C$ that switches the factors.

$\text{End}(J)$	R
composition of α, β	$\text{pr}_{13*}(\text{pr}_{12}{}^*(\alpha) \circ \text{pr}_{23}{}^*(\beta))$
id_J	$\Delta_C - \{c_0\} \times C - C \times \{c_0\}$
φ^*	$\Gamma_\varphi - C \times \{\varphi(c_0)\} - \sum_{\varphi(c)=c_0} \{c\} \times C$
the trace form $\alpha, \beta \mapsto \text{Tr}(\alpha\beta)$	$\alpha, \beta \mapsto -\int_{C \times C} \alpha.\sigma^*\beta$
the Rosati involution $\alpha \mapsto \alpha^\dagger$	$\alpha \mapsto \sigma^*\alpha$
positivity of Rosati $\text{Tr}(\alpha\alpha^\dagger) > 0$	Hodge index theorem on $C \times C$ $-\int_{C \times C} \alpha\sigma^*\alpha > 0.$

In fact, in light of the Kunneth formula, the subgroup R corresponds to the 1, 1 hodge classes in $H^1(C) \otimes H^1(C)$.

Weil's proof. Using this correspondence, we can prove the trace formula. We have

$$\begin{aligned} V(\varphi) &= \int_{C\times C} \Gamma_\varphi.\Delta \\ &= \int_{C\times C} \Gamma_\varphi.\left(\Delta_C - \{c_0\}\times C - C\times\{c_0\}\right) + \int_{C\times C} \Gamma_\varphi.\left(\{c_0\}\times C + C\times\{c_0\}\right). \end{aligned}$$

Now, on the one hand

$$\int_{C\times C} \Gamma_\varphi.\left(\{c_0\}\times C + C\times\{c_0\}\right) = 1 + \deg\varphi$$

and on the other hand, since R is the the orthogonal of the ample divisor $\{c_0\}\times C + C\times\{c_0\}$,

$$\begin{aligned} &\int_{C\times C} \Gamma_\varphi.\left(\Delta_C - \{c_0\}\times C - C\times\{c_0\}\right) \\ = &\int_{C\times C}\left(\Gamma_\varphi - C\times\{\varphi(c_0)\} - \sum_{\varphi(c)=c_0}\{c\}\times C\right).\left(\Delta_C - \{c_0\}\times C - C\times\{c_0\}\right) \\ = &-\mathrm{Tr}_J(\varphi^*\circ \mathrm{id}_J). \end{aligned}$$

Recapitulating, we have

$$V(\varphi) = 1 - \mathrm{Tr}_J(\varphi^*) + \deg\varphi$$

which is the trace formula.

Lemma 38.76.6. *Consider the situation of Theorem 38.76.4 and let ℓ be a prime number invertible in k. Then*

$$\sum\nolimits_{i=0}^{2}(-1)^i Tr(\varphi^*|_{H^i(C,\underline{\mathbf{Z}/\ell^n\mathbf{Z}})}) = V(\varphi) \mod \ell^n.$$

Sketch of proof. Observe first that the assumption makes sense because $H^i(C,\underline{\mathbf{Z}/\ell^n\mathbf{Z}})$ is a free $\mathbf{Z}/\ell^n\mathbf{Z}$-module for all i. The trace of φ^* on the 0th degree cohomology is 1. The choice of a primitive ℓ^nth root of unity in k gives an isomorphism

$$H^i(C,\underline{\mathbf{Z}/\ell^n\mathbf{Z}}) \cong H^i(C,\mu_{\ell^n})$$

compatibly with the action of the geometric Frobenius. On the other hand, $H^1(C,\mu_{\ell^n}) = J[\ell^n]$. Therefore,

$$\begin{aligned} \mathrm{Tr}(\varphi^*|_{H^1(C,\underline{\mathbf{Z}/\ell^n\mathbf{Z}})}) &= \mathrm{Tr}_J(\varphi^*) \mod \ell^n \\ &= \mathrm{Tr}_{\mathbf{Z}/\ell^n\mathbf{Z}}(\varphi^* : J[\ell^n]\to J[\ell^n]). \end{aligned}$$

Moreover, $H^2(C,\mu_{\ell^n}) = \mathrm{Pic}(C)/\ell^n\mathrm{Pic}(C) \cong \mathbf{Z}/\ell^n\mathbf{Z}$ where φ^* is multiplication by $\deg\varphi$. Hence

$$\mathrm{Tr}(\varphi^*|_{H^2(C,\underline{\mathbf{Z}/\ell^n\mathbf{Z}})}) = \deg\varphi.$$

Thus we have

$$\sum_{i=0}^{2}(-1)^i\mathrm{Tr}(\varphi^*|_{H^i(C,\underline{\mathbf{Z}/\ell^n\mathbf{Z}})}) = 1 - \mathrm{Tr}_J(\varphi^*) + \deg\varphi \mod \ell^n$$

and the corollary follows from Theorem 38.76.4. □

An alternative way to prove this corollary is to show that

$$X \mapsto H^*(X,\mathbf{Q}_\ell) = \mathbf{Q}_\ell \otimes lim_n\, H^*(X,\mathbf{Z}/\ell^n\mathbf{Z})$$

defines a Weil cohomology theory on smooth projective varieties over k. Then the trace formula

$$V(\varphi) = \sum_{i=0}^{2}(-1)^i \mathrm{Tr}(\varphi^*|_{H^i(C,\mathbf{Q}_\ell)})$$

is a formal consequence of the axioms (it's an exercise in linear algebra, the proof is the same as in the topological case).

38.77. Preliminaries and sorites

Notation: We fix the notation for this section. We denote by A a commutative ring, Λ a (possibly noncommutative) ring with a ring map $A \to \Lambda$ which image lies in the center of Λ. We let G be a finite group, Γ a *monoid extension of G by* $\mathbf{N}$, meaning that there is an exact sequence

$$1 \to G \to \tilde{\Gamma} \to \mathbf{Z} \to 1$$

and Γ consists of those elements of $\tilde{\Gamma}$ which image is nonnegative. Finally, we let P be an $A[\Gamma]$-module which is finite and projective as an $A[G]$-module, and M a $\Lambda[\Gamma]$-module which is finite and projective as a Λ-module.

Our goal is to compute the trace of $1 \in \mathbf{N}$ acting over Λ on the coinvariants of G on $P \otimes_A M$, that is, the number

$$\mathrm{Tr}_\Lambda \left(1; \left(P \otimes_A M\right)_G\right) \in \Lambda^\natural.$$

The element $1 \in \mathbf{N}$ will correspond to the Frobenius.

Lemma 38.77.1. *Let $e \in G$ denote the neutral element. The map*

$$\begin{array}{ccc} \Lambda[G] & \longrightarrow & \Lambda^\natural \\ \sum \lambda_g \cdot g & \longmapsto & \lambda_e \end{array}$$

factors through $\Lambda[G]^\natural$. We denote $\varepsilon : \Lambda[G]^\natural \to \Lambda^\natural$ the induced map.

Proof. We have to show the map annihilates commutators. One has

$$\left(\sum \lambda_g g\right)\left(\sum \mu_g g\right) - \left(\sum \mu_g g\right)\left(\sum \lambda_g g\right) = \sum_g \left(\sum_{g_1 g_2 = g} \lambda_{g_1}\mu_{g_2} - \mu_{g_1}\lambda_{g_2}\right) g$$

The coefficient of e is

$$\sum_g \left(\lambda_g \mu_{g^{-1}} - \mu_g \lambda_{g^{-1}}\right) = \sum_g \left(\lambda_g \mu_{g^{-1}} - \mu_{g^{-1}} \lambda_g\right)$$

which is a sum of commutators, hence it it zero in $\Lambda^\natural$. □

Definition 38.77.2. Let $f : P \to P$ be an endomorphism of a finite projective $\Lambda[G]$-module P. We define

$$\mathrm{Tr}^G_\Lambda(f; P) := \varepsilon\left(\mathrm{Tr}_{\Lambda[G]}(f; P)\right)$$

to be the *G-trace of f on P*.

Lemma 38.77.3. *Let $f : P \to P$ be an endomorphism of the finite projective $\Lambda[G]$-module P. Then*

$$Tr_\Lambda(f; P) = \#G \cdot Tr^G_\Lambda(f; P).$$

Proof. By additivity, reduce to the case $P = \Lambda[G]$. In that case, f is given by right multiplication by some element $\sum \lambda_g \cdot g$ of $\Lambda[G]$. In the basis $(g)_{g \in G}$, the matrix of f has coefficient $\lambda_{g_2^{-1} g_1}$ in the (g_1, g_2) position. In particular, all diagonal coefficients are λ_e, and there are $\#G$ such coefficients. □

Lemma 38.77.4. *The map $A \to \Lambda$ defines an A-module structure on $\Lambda^\natural$.*

This is clear.

Lemma 38.77.5. *Let P be a finite projective $A[G]$-module and M a $\Lambda[G]$-module, finite projective as a Λ-module. Then $P \otimes_A M$ is a finite projective $\Lambda[G]$-module, for the structure induced by the diagonal action of G.*

Note that $P \otimes_A M$ is naturally a Λ-module since M is. Explictly, together with the diagonal action this reads

$$\left(\sum \lambda_g g\right)(p \otimes m) = \sum gp \otimes \lambda_g gm.$$

Proof. For any $\Lambda[G]$-module N one has

$$Hom_{\Lambda[G]}\left(P \otimes_A M, N\right) = Hom_{A[G]}\left(P, Hom_\Lambda(M, N)\right)$$

where the G-action on $Hom_\Lambda(M, N)$ is given by $(g \cdot \varphi)(m) = g\varphi(g^{-1}m)$. Now it suffices to observe that the right-hand side is a composition of exact functors, because of the projectivity of P and M. $\square$

Lemma 38.77.6. *With assumptions as in Lemma 38.77.5, let $u \in End_{A[G]}(P)$ and $v \in End_{\Lambda[G]}(M)$. Then*

$$Tr_\Lambda^G\left(u \otimes v; P \otimes_A M\right) = Tr_A^G(u; P) \cdot Tr_\Lambda(v; M).$$

Sketch of proof. Reduce to the case $P = A[G]$. In that case, u is right multiplication by some element $a = \sum a_g g$ of $A[G]$, which we write $u = R_a$. There is an isomorphism of $\Lambda[G]$-modules

$$\begin{array}{cccc} \varphi: & A[G] \otimes_A M & \cong & \left(A[G] \otimes_A M\right)' \\ & g \otimes m & \longmapsto & g \otimes g^{-1}m \end{array}$$

where $\left(A[G] \otimes_A M\right)'$ has the module structure given by the left G-action, together with the Λ-linearity on M. This transport of structure changes $u \otimes v$ into $\sum_g a_g R_g \otimes g^{-1}v$. In other words,

$$\varphi \circ (u \otimes v) \circ \varphi^{-1} = \sum_g a_g R_g \otimes g^{-1}v.$$

Working out explicitly both sides of the equation, we have to show

$$\mathrm{Tr}_\Lambda^G\left(\sum_g a_g R_g \otimes g^{-1}v\right) = a_e \cdot \mathrm{Tr}_\Lambda(v; M).$$

This is done by showing that

$$\mathrm{Tr}_\Lambda^G\left(a_g R_g \otimes g^{-1}v\right) = \begin{cases} 0 & \text{if } g \neq e \\ a_e \mathrm{Tr}_\Lambda(v; M) & \text{if } g = e \end{cases}$$

by reducing to $M = \Lambda$. $\square$

Notation: Consider the monoid extension $1 \to G \to \Gamma \to \mathbf{N} \to 1$ and let $\gamma \in \Gamma$. Then we write $Z_\gamma = \{g \in G \mid g\gamma = \gamma g\}$.

Lemma 38.77.7. *Let P be a $\Lambda[\Gamma]$-module, finite and projective as a $\Lambda[G]$-module, and $\gamma \in \Gamma$. Then*

$$Tr_\Lambda(\gamma; P) = \# Z_\gamma \cdot Tr_\Lambda^{Z_\gamma}(\gamma; P).$$

Proof. This follows readily from Lemma 38.77.3. $\square$

Lemma 38.77.8. *Let P be an $A[\Gamma]$-module, finite projective as $A[G]$-module. Let M be a $\Lambda[\Gamma]$-module, finite projective as a Λ-module. Then*

$$Tr_{\Lambda}^{Z_\gamma}(\gamma; P \otimes_A M) = Tr_A^{Z_\gamma}(\gamma; P) \cdot Tr_{\Lambda}(\gamma; M).$$

Proof. This follows directly from Lemma 38.77.6. □

Lemma 38.77.9. *Let P be a $\Lambda[\Gamma]$-module, finite projective as $\Lambda[G]$-module. Then the coinvariants $P_G = \Lambda \otimes_{\Lambda[G]} P$ form a finite projective Λ-module, endowed with an action of $\Gamma/G = \mathbf{N}$. Moreover, we have*

$$Tr_{\Lambda}(1; P_G) = \sum\nolimits'_{\gamma \mapsto 1} Tr_{\Lambda}^{Z_\gamma}(\gamma; P)$$

where $\sum'_{\gamma\mapsto 1}$ means taking the sum over the G-conjugacy classes in Γ.

Sketch of proof. We first prove this after multiplying by $\#G$.

$$\#G \cdot \mathrm{Tr}_{\Lambda}(1; P_G) = \mathrm{Tr}_{\Lambda}\Big(\sum_{\gamma\mapsto 1} \gamma; P_G\Big) = \mathrm{Tr}_{\Lambda}\Big(\sum_{\gamma\mapsto 1}\gamma; P\Big)$$

where the second equality follows by considering the commutative triangle

$$\begin{tikzcd} P^G \arrow[r, hook, "a"] & P \arrow[r, two heads, "b"] & P_G \arrow[ll, bend left, "c"] \end{tikzcd}$$

where a is the canonical inclusion, b the canonical surjection and $c = \sum_{\gamma\mapsto 1}\gamma$. Then we have

$$\Big(\sum_{\gamma\mapsto 1}\gamma\Big)\Big|_P = a \circ c \circ b \qquad \text{and} \qquad \Big(\sum_{\gamma\mapsto 1}\gamma\Big)\Big|_{P_G} = b \circ a \circ c$$

hence they have the same trace. We then have

$$\#G \cdot \mathrm{Tr}_{\Lambda}(1; P_G) = {\sum_{\gamma\mapsto 1}}' \frac{\#G}{\#Z_\gamma}\mathrm{Tr}_{\Lambda}(\gamma; P) = \#G {\sum_{\gamma\mapsto 1}}' \mathrm{Tr}_{\Lambda}^{Z_\gamma}(\gamma; P).$$

To finish the proof, reduce to case Λ torsion-free by some universality argument. See [**Del77**] for details. □

Remark 38.77.10. Let us try to illustrate the content of the formula of Lemma 38.77.8. Suppose that Λ, viewed as a trivial Γ-module, admits a finite resolution $0 \to P_r \to \ldots \to P_1 \to P_0 \to \Lambda \to 0$ by some $\Lambda[\Gamma]$-modules P_i which are finite and projective as $\Lambda[G]$-modules. In that case

$$H_*\left(\left(P_\bullet\right)_G\right) = \mathrm{Tor}_*^{\Lambda[G]}(\Lambda, \Lambda) = H_*(G, \Lambda);$$

and

$$\mathrm{Tr}_{\Lambda}^{Z_\gamma}\left(\gamma; P_\bullet\right) = \frac{1}{\#Z_\gamma}\mathrm{Tr}_{\Lambda}(\gamma; P_\bullet) = \frac{1}{\#Z_\gamma}\mathrm{Tr}(\gamma; \Lambda) = \frac{1}{\#Z_\gamma}.$$

Therefore, Lemma 38.77.8 says

$$\mathrm{Tr}_{\Lambda}(1; P_G) = \mathrm{Tr}\left(1 \Big|_{H_*(G,\Lambda)}\right) = {\sum_{\gamma\mapsto 1}}' \frac{1}{\#Z_\gamma}.$$

This can be interpreted as a point count on the stack BG. If $\Lambda = \mathbf{F}_\ell$ with ℓ prime to $\#G$, then $H_*(G, \Lambda)$ is $\mathbf{F}_\ell$ in degree 0 (and 0 in other degrees) and the formula reads

$$1 = \sum_{\substack{\sigma\text{-conjugacy} \\ \text{classes}\langle\gamma\rangle}} \frac{1}{\#Z_\gamma} \mod \ell.$$

This is in some sense a ``trivial" trace formula for G. Later we will see that (38.76.3.1) can in some cases be viewed as a highly nontrivial trace formula for a certain type of group, see Section 38.92.

38.78. Proof of the trace formula

Theorem 38.78.1. *Let k be a finite field and X a finite type, separated scheme of dimension at most 1 over k. Let Λ be a finite ring whose cardinality is prime to that of k, and $K \in D^b_{ctf}(X, \Lambda)$. Then*

$$Tr\left(\pi_X^*\big|_{R\Gamma_c(X_{\bar k}, K)}\right) = \sum\nolimits_{x \in X(k)} Tr\left(\pi_x\big|_{K_{\bar x}}\right) \tag{38.78.1.1}$$

in $\Lambda^\natural$.

Remark 38.78.2. Remarks on the formulation above.

(1) This formula holds in any dimension. By a dévissage lemma (which uses proper base change etc.) it reduces to the current statement -- in that generality.
(2) The complex $R\Gamma_c(X_{\bar k}, K)$ is defined by choosing an open immersion $j : X \hookrightarrow \bar X$ with $\bar X$ projective over k of dimension at most 1 and setting
$$R\Gamma_c(X_{\bar k}, K) := R\Gamma(\bar X_{\bar k}, j_! K).$$
That this is independent of the choice made follows from (the missing section).

Notation: For short, we write
$$T'(X, K) = \sum\nolimits_{x \in X(k)} \mathrm{Tr}\left(\pi_x\big|_{K_{\bar x}}\right)$$
for the right-hand side of (38.78.1.1) and
$$T''(X, K) = \mathrm{Tr}\left(\pi_x^*\big|_{R\Gamma_c(X_{\bar k}, K)}\right)$$
for the left-hand side.

Proof of Theorem 38.78.1. The proof proceeds in a number of steps.

(1) *Let $j : \mathcal{U} \hookrightarrow X$ be an open immersion with complement $Y = X - \mathcal{U}$ and $i : Y \hookrightarrow X$. Then $T''(X, K) = T''(\mathcal{U}, j^{-1}K) + T''(Y, i^{-1}K)$ and $T'(X, K) = T'(\mathcal{U}, j^{-1}K) + T'(Y, i^{-1}K)$.*

This is clear for T'. For T'' use the exact sequence
$$0 \to j_! j^{-1} K \to K \to i_* i^{-1} K \to 0$$
to get a filtration on K. This gives rise to an object $\widetilde{K} \in D\mathrm{F}(X, \Lambda)$ whose graded pieces are $j_! j^{-1} K$ and $i_* i^{-1} K$, both of which lie in $D^b_{ctf}(X, \Lambda)$. Then, by filtered derived abstract nonsense (INSERT REFERENCE), $R\Gamma_c(X_{\bar k}, K) \in D\mathrm{F}_{perf}(\Lambda)$, and it comes equipped with π_x^* in $D\mathrm{F}_{perf}(\Lambda)$. By the discussion of traces on filtered complexes (INSERT REFERENCE) we get
$$\begin{aligned} \mathrm{Tr}\left(\pi_X^*\big|_{R\Gamma_c(X_{\bar k}, K)}\right) &= \mathrm{Tr}\left(\pi_X^*\big|_{R\Gamma_c(X_{\bar k}, j_! j^{-1} K)}\right) + \mathrm{Tr}\left(\pi_X^*\big|_{R\Gamma_c(X_{\bar k}, i_* i^{-1} K)}\right) \\ &= T''(U, i^{-1}K) + T''(Y, i^{-1}K). \end{aligned}$$

(2) *The theorem holds if* $\dim X \leq 0$.

Indeed, in that case

$$R\Gamma_c(X_{\bar k}, K) = R\Gamma(X_{\bar k}, K) = \Gamma(X_{\bar k}, K) = \bigoplus_{\bar x \in X_{\bar k}} K_{\bar x} \leftarrow \pi_X * .$$

Since the fixed points of $\pi_X : X_{\bar k} \to X_{\bar k}$ are exactly the points $\bar x \in X_{\bar k}$ which lie over a k-rational point $x \in X(k)$ we get

$$\mathrm{Tr}\big(\pi_X^*|_{R\Gamma_c(X_{\bar k}, K)}\big) = \sum\nolimits_{x \in X(k)} \mathrm{Tr}(\pi_{\bar x}|_{K_{\bar x}}).$$

(3) *It suffices to prove the equality $T'(\mathcal{U}, \mathcal{F}) = T''(\mathcal{U}, \mathcal{F})$ in the case where*
- *$\mathcal{U}$ is a smooth irreducible affine curve over k ;*
- *$\mathcal{U}(k) = \emptyset$;*
- *$K = \mathcal{F}$ is a finite locally constant sheaf of Λ-modules on $\mathcal{U}$ whose stalk(s) are finite projective Λ-modules ; and*
- *Λ is killed by a power of a prime ℓ and $\ell \in k^*$.*

Indeed, because of Step 2, we can throw out any finite set of points. But we have only finitely many rational points, so we may assume there are none[8]. We may assume that $\mathcal{U}$ is smooth irreducible and affine by passing to irreducible components and throwing away the bad points if necessary. The assumptions of $\mathcal{F}$ come from unwinding the definition of $D^b_{ctf}(X, \Lambda)$ and those on Λ from considering its primary decomposition.

For the remainder of the proof, we consider the situation

$$\begin{tikzcd} \mathcal{V} \arrow[r, hook] \arrow[d, "f"'] & Y \arrow[d, "\bar f"] \\ \mathcal{U} \arrow[r, hook] & X \end{tikzcd}$$

where $\mathcal{U}$ is as above, f is a finite étale Galois covering, $\mathcal{V}$ is connected and the horizontal arrows are projective completions. Denoting $G = \mathrm{Aut}(\mathcal{V}|\mathcal{U})$, we also assume (as we may) that $f^{-1}\mathcal{F} = \underline{M}$ is constant, where the module $M = \Gamma(\mathcal{V}, f^{-1}\mathcal{F})$ is a $\Lambda[G]$-module which is finite and projective over Λ. This corresponds to the trivial monoid extension

$$1 \to G \to \Gamma = G \times \mathbf{N} \to \mathbf{N} \to 1.$$

In that context, using the reductions above, we need to show that $T''(\mathcal{U}, \mathcal{F}) = 0$. We now present a series of lemmata in order to complete the proof.

(A) *There is a natural action of G on $f_*f^{-1}\mathcal{F}$ and the trace map $f_*f^{-1}\mathcal{F} \to \mathcal{F}$ defines an isomorphism*

$$(f_*f^{-1}\mathcal{F}) \otimes_{\Lambda[G]} \Lambda = (f_*f^{-1}\mathcal{F})_G \cong \mathcal{F}.$$

To prove this, simply unwind everything at a geometric point.

(B) *Let $A = \mathbf{Z}/\ell^n\mathbf{Z}$ with $n \gg 0$. Then $f_*f^{-1}\mathcal{F} \cong (f_*\underline{A}) \otimes_{\underline{A}} \underline{M}$ with diagonal G-action.*

(C) *There is a canonical isomorphism $(f_*\underline{A} \otimes_{\underline{A}} \underline{M}) \otimes_{\Lambda[G]} \underline{\Lambda} \cong \mathcal{F}$.*

In fact, this is a derived tensor product, because of the projectivity assumption on $\mathcal{F}$.

(D) *There is a canonical isomorphism*

$$R\Gamma_c(\mathcal{U}_{\bar k}, \mathcal{F}) = (R\Gamma_c(\mathcal{U}_{\bar k}, f_*A) \otimes^{\mathbf{L}}_A M) \otimes^{\mathbf{L}}_{\Lambda[G]} \Lambda,$$

compatible with the action of $\pi_{\mathcal{U}}^*$.

---

[8] At this point, there should be an evil laugh in the background.

This comes from the universal coefficient theorem, i.e., the fact that $R\Gamma_c$ commutes with $\otimes^{\mathbf{L}}$, and the flatness of $\mathcal{F}$ as a Λ-module.

We have

$$\begin{aligned}\mathrm{Tr}(\pi_{\mathcal{U}}^*|_{R\Gamma_c(\mathcal{U}_{\bar{k}},\mathcal{F})}) &= \sum_{g\in G}{}' \mathrm{Tr}_\Lambda^{Z_g}\left((g,\pi_{\mathcal{U}}^*)|_{R\Gamma_c(\mathcal{U}_{\bar{k}},f_*A)\otimes_A^{\mathbf{L}}M}\right) \\ &= \sum_{g\in G}{}' \mathrm{Tr}_A^{Z_g}((g,\pi_{\mathcal{U}}^*)|_{R\Gamma_c(\mathcal{U}_{\bar{k}},f_*A)})\cdot \mathrm{Tr}_\Lambda(g|_M)\end{aligned}$$

where Γ acts on $R\Gamma_c(\mathcal{U}_{\bar{k}},\mathcal{F})$ by G and $(e,1)$ acts via $\pi_{\mathcal{U}}^*$. So the monoidal extension is given by $\Gamma = G\times\mathbf{N}\to\mathbf{N}$, $\gamma\mapsto 1$. The first equality follows from Lemma 38.77.9 and the second from lemma 38.77.8.

(4) *It suffices to show that* $Tr_A^{Z_g}\left((g,\pi_{\mathcal{U}}^*)|_{R\Gamma_c(\mathcal{U}_{\bar{k}},f_*A)}\right)\in A$ *maps to zero in* Λ.

Recall that

$$\begin{aligned}\#Z_g\cdot \mathrm{Tr}_A^{Z_g}\left((g,\pi_{\mathcal{U}}^*)|_{R\Gamma_c(\mathcal{U}_{\bar{k}},f_*A)}\right) &= \mathrm{Tr}_A\left((g,\pi_{\mathcal{U}}^*)|_{R\Gamma_c(\mathcal{U}_{\bar{k}},f_*A)}\right) \\ &= \mathrm{Tr}_A\left((g^{-1}\pi_{\mathcal{V}})^*|_{R\Gamma_c(\mathcal{V}_{\bar{k}},A)}\right).\end{aligned}$$

The first equality is Lemma 38.77.7, the second is the Leray spectral sequence, using the finiteness of f and the fact that we are only taking traces over A. Now since $A=\mathbf{Z}/\ell^n\mathbf{Z}$ with $n\gg 0$ and $\#Z_g=\ell^a$ for some (fixed) a, it suffices to show the following result.

(5) $Tr_A\left((g^{-1}\pi_{\mathcal{V}})^*|_{R\Gamma_c(\mathcal{V},A)}\right)=0$ *in* A.

By additivity again, we have

$$\begin{aligned}\mathrm{Tr}_A\left((g^{-1}\pi_{\mathcal{V}})^*|_{R\Gamma_c(\mathcal{V}_{\bar{k}}A)}\right) &+ \mathrm{Tr}_A\left((g^{-1}\pi_{\mathcal{V}})^*|_{R\Gamma_c(Y-\mathcal{V})_{\bar{k}},A)}\right) \\ &= \mathrm{Tr}_A\left((g^{-1}\pi_Y)^*|_{R\Gamma(Y_{\bar{k}},A)}\right)\end{aligned}$$

The latter trace is the number of fixed points of $g^{-1}\pi_Y$ on Y, by Weil's trace formula Theorem 38.76.4. Moreover, by the 0-dimensional case already proven in step 2,

$$\mathrm{Tr}_A\left((g^{-1}\pi_{\mathcal{V}})^*|_{R\Gamma_c(Y-\mathcal{V})_{\bar{k}},A)}\right)$$

is the number of fixed points of $g^{-1}\pi_Y$ on $(Y-\mathcal{V})_{\bar{k}}$. Therefore,

$$\mathrm{Tr}_A\left((g^{-1}\pi_{\mathcal{V}})^*|_{R\Gamma_c(\mathcal{V}_{\bar{k}},A)}\right)$$

is the number of fixed points of $g^{-1}\pi_Y$ on $\mathcal{V}_{\bar{k}}$. But there are no such points: if $\bar{y}\in Y_{\bar{k}}$ is fixed under $g^{-1}\pi_Y$, then $\bar{f}(\bar{y})\in X_{\bar{k}}$ is fixed under π_X. But $\mathcal{U}$ has no k-rational point, so we must have $\bar{f}(\bar{y})\in(X-\mathcal{U})_{\bar{k}}$ and so $\bar{y}\notin\mathcal{V}_{\bar{k}}$, a contradiction. This finishes the proof. □

Remark 38.78.3. Even though all we did are reductions and mostly algebra, the trace formula Theorem 38.78.1 is much stronger than Weil's geometric trace formula (Theorem 38.76.4) because it applies to coefficient systems (sheaves), not merely constant coefficients.

38.79. Applications

OK, having indicated the proof of the trace formula, let's try to use it for something.

38.80. On l-adic sheaves

Definition 38.80.1. Let X be a noetherian scheme. A $\mathbf{Z}_\ell$-*sheaf* on X, or simply a ℓ-adic sheaf is an inverse system $\left\{\mathcal{F}_n\right\}_{n\geq 1}$ where

(1) $\mathcal{F}_n$ is a constructible $\mathbf{Z}/\ell^n\mathbf{Z}$-module on $X_{étale}$, and
(2) the transition maps $\mathcal{F}_{n+1} \to \mathcal{F}_n$ induce isomorphisms $\mathcal{F}_{n+1} \otimes_{\mathbf{Z}/\ell^{n+1}\mathbf{Z}} \mathbf{Z}/\ell^n\mathbf{Z} \cong \mathcal{F}_n$.

We say that $\mathcal{F}$ is *lisse* if each $\mathcal{F}_n$ is locally constant. A *morphism* of such is merely a morphism of inverse systems.

Lemma 38.80.2. *Let $\{\mathcal{G}_n\}_{n\geq 1}$ be an inverse system of constructible $\mathbf{Z}/\ell^n\mathbf{Z}$-modules. Suppose that for all $k \geq 1$, the maps*

$$\mathcal{G}_{n+1}/\ell^k\mathcal{G}_{n+1} \to \mathcal{G}_n/\ell^k\mathcal{G}_n$$

are isomorphisms for all $n \gg 0$ (where the bound possibly depends on k). In other words, assume that the system $\{\mathcal{G}/\ell^k\mathcal{G}_n\}_{n\geq 1}$ is eventually constant, and call $\mathcal{F}_k$ the corresponding sheaf. Then the system $\left\{\mathcal{F}_k\right\}_{k\geq 1}$ forms a $\mathbf{Z}_\ell$-sheaf on X.

The proof is obvious.

Lemma 38.80.3. *The category of $\mathbf{Z}_\ell$-sheaves on X is abelian.*

Proof. Let $\Phi = \left\{\varphi_n\right\}_{n\geq 1} : \left\{\mathcal{F}_n\right\} \to \left\{\mathcal{G}_n\right\}$ be a morphism of $\mathbf{Z}_\ell$-sheaves. Set

$$\mathrm{Coker}(\Phi) = \left\{\mathrm{Coker}\left(\mathcal{F}_n \xrightarrow{\varphi_n} \mathcal{G}_n\right)\right\}_{n\geq 1}$$

and $\mathrm{Ker}(\Phi)$ is the result of Lemma 38.80.2 applied to the inverse system

$$\left\{\bigcap_{m\geq n} \mathrm{Im}\left(\mathrm{Ker}(\varphi_m) \to \mathrm{Ker}(\varphi_n)\right)\right\}_{n\geq 1}.$$

That this defines an abelian category is left to the reader. $\square$

Example 38.80.4. Let $X = Spec(\mathbf{C})$ and $\Phi : \mathbf{Z}_\ell \to \mathbf{Z}_\ell$ be multiplication by ℓ. More precisely,

$$\Phi = \left\{\mathbf{Z}/\ell^n\mathbf{Z} \xrightarrow{\ell} \mathbf{Z}/\ell^n\mathbf{Z}\right\}_{n\geq 1}.$$

To compute the kernel, we consider the inverse system

$$\cdots \to \mathbf{Z}/\ell\mathbf{Z} \xrightarrow{0} \mathbf{Z}/\ell\mathbf{Z} \xrightarrow{0} \mathbf{Z}/\ell\mathbf{Z}.$$

Since the images are always zero, $\mathrm{Ker}(\Phi)$ is zero as a system.

Remark 38.80.5. If $\mathcal{F} = \left\{\mathcal{F}_n\right\}_{n\geq 1}$ is a $\mathbf{Z}_\ell$-sheaf on X and $\bar{x}$ is a geometric point then $M_n = \left\{\mathcal{F}_{n,\bar{x}}\right\}$ is an inverse system of finite $\mathbf{Z}/\ell^n\mathbf{Z}$-modules such that $M_{n+1} \to M_n$ is surjective and $M_n = M_{n+1}/\ell^n M_{n+1}$. It follows that

$$M = lim_n\, M_n = lim\, \mathcal{F}_{n,\bar{x}}$$

is a finite $\mathbf{Z}_\ell$-module. Indeed, $M/\ell M = M_1$ is finite over $\mathbf{F}_\ell$, so by Nakayama M is finite over $\mathbf{Z}_\ell$. Therefore, $M \cong \mathbf{Z}_\ell^{\oplus r} \oplus \bigoplus_{i=1}^t \mathbf{Z}_\ell/\ell^{e_i}\mathbf{Z}_\ell$ for some $r, t \geq 0$, $e_i \geq 1$. The module $M = \mathcal{F}_{\bar{x}}$ is called the *stalk* of $\mathcal{F}$ at $\bar{x}$.

Definition 38.80.6. A $\mathbf{Z}_\ell$-sheaf $\mathcal{F}$ is *torsion* if $\ell^n : \mathcal{F} \to \mathcal{F}$ is the zero map for some n. The abelian category of $\mathbf{Q}_\ell$-sheaves on X is the quotient of the abelian category of $\mathbf{Z}_\ell$-sheaves by the Serre subcategory of torsion sheaves. In other words, its objects are $\mathbf{Z}_\ell$-sheaves on X, and if $\mathcal{F}, \mathcal{G}$ are two such, then

$$Hom_{\mathbf{Q}_\ell}(\mathcal{F}, \mathcal{G}) = Hom_{\mathbf{Z}_\ell}(\mathcal{F}, \mathcal{G}) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell.$$

We denote by $\mathcal{F} \mapsto \mathcal{F} \otimes \mathbf{Q}_\ell$ the quotient functor (right adjoint to the inclusion). If $\mathcal{F} = \mathcal{F}' \otimes \mathbf{Q}_\ell$ where $\mathcal{F}'$ is a $\mathbf{Z}_\ell$-sheaf and $\bar{x}$ is a geometric point, then the *stalk* of $\mathcal{F}$ at $\bar{x}$ is $\mathcal{F}_{\bar{x}} = \mathcal{F}'_{\bar{x}} \otimes \mathbf{Q}_\ell$.

Remark 38.80.7. Since a $\mathbf{Z}_\ell$-sheaf is only defined on a noetherian scheme, it is torsion if and only if its stalks are torsion.

Definition 38.80.8. If X is a separated scheme of finite type over an algebraically closed field k and $\mathcal{F} = \left\{\mathcal{F}_n\right\}_{n\geq 1}$ is a $\mathbf{Z}_\ell$-sheaf on X, then we define

$$H^i(X, \mathcal{F}) := lim_n\, H^i(X, \mathcal{F}_n) \qquad \text{and} \qquad H^i_c(X, \mathcal{F}) := lim_n\, H^i_c(X, \mathcal{F}_n).$$

If $\mathcal{F} = \mathcal{F}' \otimes \mathbf{Q}_\ell$ for a $\mathbf{Z}_\ell$-sheaf $\mathcal{F}'$ then we set

$$H^i_c(X, \mathcal{F}) := H^i_c(X, \mathcal{F}') \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell.$$

We call these the *ℓ-adic cohomology* of X with coefficients $\mathcal{F}$.

38.81. L-functions

Definition 38.81.1. Let X be a scheme of finite type over a finite field k. Let Λ be a finite ring of order prime to the characteristic of k and $\mathcal{F}$ a constructible flat Λ-module on $X_{\acute{e}tale}$. Then we set

$$L(X, \mathcal{F}) := \prod_{x \in |X|} \det\left(1 - \pi_x^* \, T^{\deg x}\Big|_{\mathcal{F}_{\bar{x}}}\right)^{-1} \in \Lambda[[T]]$$

where $|X|$ is the set of closed points of X, $\deg x = [\kappa(x) : k]$ and $\bar{x}$ is a geometric point lying over x. This definition clearly generalizes to the case where $\mathcal{F}$ is replace by a $K \in D^b_{ctf}(X, \Lambda)$. We call this the *$L$-function of $\mathcal{F}$.*

Remark 38.81.2. Intuitively, T should be thought of as $T = t^f$ where $p^f = \# k$. The definitions are then independent of the size of the ground field.

Definition 38.81.3. Now assume that $\mathcal{F}$ is a $\mathbf{Q}_\ell$-sheaf on X. In this case we define

$$L(X, \mathcal{F}) := \prod_{x \in |X|} \det\left(1 - \pi_x^* \, T^{\deg x}\Big|_{\mathcal{F}_{\bar{x}}}\right)^{-1} \in \mathbf{Q}_\ell[[T]].$$

Note that this product converges since there are finitely many points of a given degree. We call this the *L-function of $\mathcal{F}$.*

38.82. Cohomological interpretation

This is how Grothendieck interpreted the L-function.

Theorem 38.82.1. *(Finite Coefficients) Let X be a scheme of finite type over a finite field k. Let Λ be a finite ring of order prime to the characteristic of k and $\mathcal{F}$ a constructible flat Λ-module on $X_{\acute{e}tale}$. Then*

$$L(X, \mathcal{F}) = \det\left(1 - \pi_X^* \, T\Big|_{R\Gamma_c(X_{\bar{k}}, \mathcal{F})}\right)^{-1} \in \Lambda[[T]].$$

Thus far, we don't even know whether each cohomology group $H^i_c(X_{\bar k}, \mathcal{F})$ is free.

Theorem 38.82.2. *($\mathbf{Q}_\ell$-sheaves) Let X be a scheme of finite type over a finite field k, and $\mathcal{F}$ a $\mathbf{Q}_\ell$-sheaf on X. Then*

$$L(X, \mathcal{F}) = \prod_i \det\left(1 - \pi_X^* \, T\Big|_{H^i_c(X_{\bar k}, \mathcal{F})}\right)^{(-1)^{i+1}} \in \mathbf{Q}_\ell[[T]].$$

Remark 38.82.3. Since we have only developed some theory of traces and not of determinants, Theorem 38.82.1 is harder to prove than Theorem 38.82.2. We will only prove the latter, for the former see [**Del77**]. Observe also that there is no version of this theorem more general for $\mathbf{Z}_\ell$ coefficients since there is no ℓ-torsion.

We reduce the proof of Theorem 38.82.2 to a trace formula. Since $\mathbf{Q}_\ell$ has characteristic 0, it suffices to prove the equality after taking logarithmic derivatives. More precisely, we apply $T\frac{d}{dT}\log$ to both sides. We have on the one hand

$$\begin{aligned} T\frac{d}{dT}\log L(X, \mathcal{F}) &= T\frac{d}{dT}\log \prod_{x\in|X|} \det\left(1 - \pi_x^* \, T^{\deg x}\Big|_{\mathcal{F}_{\bar x}}\right)^{-1} \\ &= \sum_{x\in|X|} T\frac{d}{dT}\log\left(\det\left(1 - \pi_x^* \, T^{\deg x}\Big|_{\mathcal{F}_{\bar x}}\right)^{-1}\right) \\ &= \sum_{x\in|X|} \deg x \sum_{n\geq 1} \mathrm{Tr}\left(\left(\pi_x^n\right)^*\big|_{\mathcal{F}_{\bar x}}\right) T^{n\deg x} \end{aligned}$$

where the last equality results from the formula

$$T\frac{d}{dT}\log\left(\det\left(1 - fT|_M\right)^{-1}\right) = \sum_{n\geq 1}\mathrm{Tr}(f^n|_M)T^n$$

which holds for any commutative ring Λ and any endomorphism f of a finite projective Λ-module M. On the other hand, we have

$$\begin{aligned} &T\frac{d}{dT}\log\left(\prod_i \det\left(1 - \pi_X^* \, T\Big|_{H^i_c(X_{\bar k}, \mathcal{F})}\right)^{(-1)^{i+1}}\right) \\ &\qquad = \sum_i (-1)^i \sum_{n\geq 1}\mathrm{Tr}\left(\left(\pi_X^n\right)^*\big|_{H^i_c(X_{\bar k}, \mathcal{F})}\right) T^n \end{aligned}$$

by the same formula again. Now, comparing powers of T and using the Mobius inversion formula, we see that Theorem 38.82.2 is a consequence of the following equality

$$\sum_{d|n} d \sum_{\substack{x\in|X| \\ \deg x = d}} \mathrm{Tr}\left(\left(\pi_X^{n/d}\right)^*\Big|_{\mathcal{F}_{\bar x}}\right) = \sum_i (-1)^i \mathrm{Tr}\left(\left(\pi_X^n\right)^*\big|_{H^i_c(X_{\bar k}, \mathcal{F})}\right).$$

Writing k_n for the degree n extension of k, $X_n = X \times_{Spec\, k} Spec(k_n)$ and ${}_n\mathcal{F} = \mathcal{F}|_{X_n}$, this boils down to

$$\sum_{x\in X_n(k_n)} \mathrm{Tr}\left(\pi_X^*\Big|_{{}_n\mathcal{F}_{\bar x}}\right) = \sum_i (-1)^i \mathrm{Tr}\left(\left(\pi_X^n\right)^*\big|_{H^i_c((X_n)_{\bar k}, {}_n\mathcal{F})}\right)$$

which is a consequence of the following result.

Theorem 38.82.4. *Let X be a separated scheme of finite type over a finite field k and $\mathcal{F}$ be a $\mathbf{Q}_\ell$-sheaf on X. Then $\dim_{\mathbf{Q}_\ell} H^i_c(X_{\bar k}, \mathcal{F})$ is finite for all i, and is nonzero for $0 \le i \le 2\dim X$ only. Furthermore, we have*

$$\sum_{x\in X(k)} Tr\left(\pi_x \Big|_{\mathcal{F}_{\bar x}}\right) = \sum_i (-1)^i Tr\left(\pi_X^*\Big|_{H^i_c(X_{\bar k},\mathcal{F})}\right).$$

Theorem 38.82.5. *Let X/k be as above, let Λ be a finite ring with $\#\Lambda \in k^*$ and $K \in D^b_{ctf}(X,\Lambda)$. Then $R\Gamma_c(X_{\bar k}, K) \in D_{perf}(\Lambda)$ and*

$$\sum_{x\in X(k)} Tr\left(\pi_x \Big|_{K_{\bar x}}\right) = Tr\left(\pi_X^* \Big|_{R\Gamma_c(X_{\bar k},K)}\right).$$

Note that we have already proved this (REFERENCE) when $\dim X \le 1$. The general case follows easily from that case together with the proper base change theorem. We now explain how to deduce Theorem 38.82.4 from theorem 38.82.5. We first use some étale cohomology arguments to reduce the proof to an algebraic statement which we subsequently prove.

Let $\mathcal{F}$ be as in Theorem 38.82.4. We can write $\mathcal{F}$ as $\mathcal{F}' \otimes \mathbf{Q}_\ell$ where $\mathcal{F}' = \left\{\mathcal{F}'_n\right\}$ is a $\mathbf{Z}_\ell$-sheaf without torsion, i.e., $\ell : \mathcal{F}' \to \mathcal{F}'$ has trivial kernel in the category of $\mathbf{Z}_\ell$-sheaves. Then each $\mathcal{F}'_n$ is a flat constructible $\mathbf{Z}/\ell^n\mathbf{Z}$-module on $X_{étale}$, so $\mathcal{F}'_n \in D^b_{ctf}(X, \mathbf{Z}/\ell^n\mathbf{Z})$ and $\mathcal{F}'_{n+1} \otimes^{\mathbf{L}}_{\mathbf{Z}/\ell^{n+1}\mathbf{Z}} \mathbf{Z}/\ell^n\mathbf{Z} = \mathcal{F}'_n$. Note that the last equality holds also for standard (non-derived) tensor product, since $\mathcal{F}'_n$ is flat (it is the same equality). Therefore,

(1) the complex $K_n = R\Gamma_c\left(X_{\bar k}, \mathcal{F}'_n\right)$ is perfect, and it is endowed with an endomorphism $\pi_n : K_n \to K_n$ in $D(\mathbf{Z}/\ell^n\mathbf{Z})$;
(2) there are identifications
$$K_{n+1} \otimes^{\mathbf{L}}_{\mathbf{Z}/\ell^{n+1}\mathbf{Z}} \mathbf{Z}/\ell^n\mathbf{Z} = K_n$$
in $D_{perf}(\mathbf{Z}/\ell^n\mathbf{Z})$, compatible with the endomorphisms π_{n+1} and π_n (see [**Del77**, Rapport 4.12]) ;
(3) the equality $\mathrm{Tr}\left(\pi_X^*\Big|_{K_n}\right) = \sum_{x\in X(k)} \mathrm{Tr}\left(\pi_x \Big|_{(\mathcal{F}'_n)_{\bar x}}\right)$ holds ; and
(4) for each $x \in X(k)$, the elements $\mathrm{Tr}\left(\pi_x\Big|_{\mathcal{F}'_{n,\bar x}}\right) \in \mathbf{Z}/\ell^n\mathbf{Z}$ form an element of $\mathbf{Z}_\ell$ which is equal to $\mathrm{Tr}\left(\pi_x \Big|_{\mathcal{F}_{\bar x}}\right) \in \mathbf{Q}_\ell$.

It thus suffices to prove the following algebra lemma.

Lemma 38.82.6. *Suppose we have $K_n \in D_{perf}(\mathbf{Z}/\ell^n\mathbf{Z})$, $\pi_n : K_n \to K_n$ and isomorphisms $\varphi_n : K_{n+1} \otimes^{\mathbf{L}}_{\mathbf{Z}/\ell^{n+1}\mathbf{Z}} \mathbf{Z}/\ell^n\mathbf{Z} \cong K_n$ compatible with π_{n+1} and π_n. Then*

(1) *the elements $t_n = Tr(\pi_n \Big|_{K_n}) \in \mathbf{Z}/\ell^n\mathbf{Z}$ form an element $t_\infty = \{t_n\}$ of $\mathbf{Z}_\ell$;*
(2) *the $\mathbf{Z}_\ell$-module $H^i_\infty = lim_n H^i(k_n)$ is finite and is nonzero for finitely many i only ; and*
(3) *the operators $H^i(\pi_n) : H^i(K_n) \to H^i(K_n)$ are compatible and define $\pi^i_\infty : H^i_\infty \to H^i_\infty$ satisfying*
$$\sum (-1)^i Tr\left(\pi^i_\infty\Big|_{H^i_\infty \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell}\right) = t_\infty.$$

Proof. Since $\mathbf{Z}/\ell^n\mathbf{Z}$ is a local ring and K_n is perfect, each K_n can be represented by a finite complex $K_n^\bullet$ of finite free $\mathbf{Z}/\ell^n\mathbf{Z}$-modules such that the map $K_n^p \to K_n^{p+1}$ has image contained in ℓK_n^{p+1}. It is a fact that such a complex is unique up to isomorphism. Moreover

π_n can be represented by a morphism of complexes $\pi_n^\bullet : K_n^\bullet \to K_n^\bullet$ (which is unique up to homotopy). By the same token the isomorphism $\varphi_n : K_{n+1} \otimes^{\mathbf{L}}_{\mathbf{Z}/\ell^{n+1}\mathbf{Z}} \mathbf{Z}/\ell^n\mathbf{Z} \to K_n$ is represented by a map of complexes

$$\varphi_n^\bullet : K_{n+1}^\bullet \otimes_{\mathbf{Z}/\ell^{n+1}\mathbf{Z}} \mathbf{Z}/\ell^n\mathbf{Z} \to K_n^\bullet.$$

In fact, $\varphi_n^\bullet$ is an isomorphism of complexes, thus we see that

- there exist $a, b \in \mathbf{Z}$ independent of n such that $K_n^i = 0$ for all $i \notin [a, b]$; and
- the rank of K_n^i is independent of n.

Therefore, the module $K_\infty^i = lim_n\{K_n^i, \varphi_n^i\}$ is a finite free $\mathbf{Z}_\ell$-module and $K_\infty^\bullet$ is a finite complex of finite free $\mathbf{Z}_\ell$-modules. By induction on the number of nonzero terms, one can prove that $H^i\left(K_\infty^\bullet\right) = lim_n H^i\left(K_n^\bullet\right)$ (this is not true for unbounded complexes). We conclude that $H_\infty^i = H^i\left(K_\infty^\bullet\right)$ is a finite $\mathbf{Z}_\ell$-module. This proves *ii*. To prove the remainder of the lemma, we need to overcome the possible noncommutativity of the diagrams

$$\begin{tikzcd} K_{n+1}^\bullet \arrow[r, "\varphi_n^\bullet"] \arrow[d, "\pi_{n+1}^\bullet"'] & K_n^\bullet \arrow[d, "\pi_n^\bullet"] \\ K_{n+1}^\bullet \arrow[r, "\varphi_n^\bullet"'] & K_n^\bullet. \end{tikzcd}$$

However, this diagram does commute in the derived category, hence it commutes up to homotopy. We inductively replace $\pi_n^\bullet$ for $n \geq 2$ by homotopic maps of complexes making these diagrams commute. Namely, if $h^i : K_{n+1}^i \to K_n^{i-1}$ is a homotopy, i.e.,

$$\pi_n^\bullet \circ \varphi_n^\bullet - \varphi_n^\bullet \circ \pi_{n+1}^\bullet = dh + hd,$$

then we choose $\tilde{h}^i : K_{n+1}^i \to K_{n+1}^{i-1}$ lifting h^i. This is possible because K_{n+1}^i free and $K_{n+1}^{i-1} \to K_n^{i-1}$ is surjective. Then replace $\pi_n^\bullet$ by $\tilde{\pi}_n^\bullet$ defined by

$$\tilde{\pi}_{n+1}^\bullet = \pi_{n+1}^\bullet + d\tilde{h} + \tilde{h}d.$$

With this choice of $\{\pi_n^\bullet\}$, the above diagrams commute, and the maps fit together to define an endomorphism $\pi_\infty^\bullet = lim_n \pi_n^\bullet$ of $K_\infty^\bullet$. Then part *i* is clear: the elements $t_n = \sum(-1)^i \mathrm{Tr}\left(\pi_n^i\Big|_{K_n^i}\right)$ fit into an element t_∞ of $\mathbf{Z}_\ell$. Moreover

$$\begin{aligned} t_\infty &= \sum(-1)^i \mathrm{Tr}_{\mathbf{Z}_\ell}\left(\pi_\infty^i\big|_{K_\infty^i}\right) \\ &= \sum(-1)^i \mathrm{Tr}_{\mathbf{Q}_\ell}\left(\pi_\infty^i\big|_{K_\infty^i \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell}\right) \\ &= \sum(-1)^i \mathrm{Tr}\left(\pi_\infty\big|_{H^i(K_\infty^\bullet \otimes \mathbf{Q}_\ell)}\right) \end{aligned}$$

where the last equality follows from the fact that $\mathbf{Q}_\ell$ is a field, so the complex $K_\infty^\bullet \otimes \mathbf{Q}_\ell$ is quasi-isomorphic to its cohomology $H^i(K_\infty^\bullet \otimes \mathbf{Q}_\ell)$. The latter is also equal to $H^i(K_\infty^\bullet) \otimes_{\mathbf{Z}} \mathbf{Q}_\ell = H_\infty^i \otimes \mathbf{Q}_\ell$, which finishes the proof of the lemma, and also that of Theorem 38.82.4. $\square$

38.83. List of things which we should add above

What did we skip the proof of in the lectures sofar:

(1) curves and their Jacobians,
(2) proper base change theorem,

(3) inadequate discussion of $R\Gamma_c$,
(4) more generally, given $f : X \to S$ finite type, separated S quasi-projective, discussion of $Rf_!$ on étale sheaves.
(5) discussion of $\otimes^{\mathbf{L}}$
(6) discussion of why $R\Gamma_c$ commutes with $\otimes^{\mathbf{L}}$

38.84. Examples of L-functions

We use Theorem 38.82.2 for curves to give examples of L-functions

38.85. Constant sheaves

Let k be a finite field, X a smooth, geometrically irreducible curve over k and $\mathcal{F} = \underline{\mathbf{Q}_\ell}$ the constant sheaf. If $\bar{x}$ is a geometric point of X, the Galois module $\mathcal{F}_{\bar{x}} = \mathbf{Q}_\ell$ is trivial, so

$$\det\left(1 - \pi_x^* T^{\deg x}\Big|_{\mathcal{F}_{\bar{x}}}\right)^{-1} = \frac{1}{1 - T^{\deg x}}.$$

Applying Theorem 38.82.2, we get

$$\begin{aligned} L(X, \mathcal{F}) &= \prod_{i=0}^{2} \det\left(1 - \pi_X^* T\big|_{H_c^i(X_{\bar{k}}, \mathbf{Q}_\ell)}\right)^{(-1)^{i+1}} \\ &= \frac{\det\left(1 - \pi_X^* T\big|_{H_c^1(X_{\bar{k}}, \mathbf{Q}_\ell)}\right)}{\det\left(1 - \pi_X^* T\big|_{H_c^0(X_{\bar{k}}, \mathbf{Q}_\ell)}\right) \cdot \det\left(1 - \pi_X^* T\big|_{H_c^2(X_{\bar{k}}, \mathbf{Q}_\ell)}\right)}. \end{aligned}$$

To compute the latter, we distinguish two cases.

Projective case. Assume that X is projective, so $H_c^i(X_{\bar{k}}, \mathbf{Q}_\ell) = H^i(X_{\bar{k}}, \mathbf{Q}_\ell)$, and we have

$$H^i(X_{\bar{k}}, \mathbf{Q}_\ell) = \begin{cases} \mathbf{Q}_\ell & \text{if } i = 0, \text{ and } \pi_X^* \text{ acts as } 1 \text{ ;} \\ \mathbf{Q}_\ell^{2g} & \text{if } i = 1 \text{ ;} \\ \mathbf{Q}_\ell & \text{if } i = 2, \text{ and } \pi_X^* \text{ acts as multiplication by } q = \deg \pi_X. \end{cases}$$

We do not know much about the action of π_X^* on the degree 1 cohomology. Let us call $\alpha_1, \ldots, \alpha_{2g}$ its eigenvalues in $\bar{\mathbf{Q}}_\ell$. Putting everything together, Theorem 38.82.2 yields the equality

$$\prod_{x \in |X|} \frac{1}{1 - T^{\deg x}} = \frac{\det\left(1 - \pi_X^* T\big|_{H^1(X_{\bar{k}}, \mathbf{Q}_\ell)}\right)}{(1 - T)(1 - qT)}$$

from which we deduce the following result.

Lemma 38.85.1. *Let X be a smooth, projective, geometrically irreducible curve over a finite field k. Then*

(1) *the L-function $L(X, \mathbf{Q}_\ell)$ is a rational funtion ;*
(2) *the eigenvalues $\alpha_1, \ldots, \alpha_{2g}$ of π_X^* on $H^1(X_{\bar{k}}, \mathbf{Q}_\ell)$ are algebraic integers independent of ℓ,*
(3) *the number of rational points of X on k_n, where $[k_n : k] = n$, is*

$$\#X(k_n) = 1 - \sum\nolimits_{i=1}^{2g} \alpha_i^n + q^n,$$

(4) *for each i, $|\alpha_i| < q$.*

Part (3) is Theorem 38.82.4 applied to $\mathscr{F} = \underline{\mathbf{Q}_\ell}$ on $X \otimes k_n$. For part (4), use the following result.

Exercise 38.85.2. Let $\alpha_1, \dots, \alpha_n \in \mathbf{C}$. Then for any conic sector containing the positive real axis of the form $C_\varepsilon = \{z \in \mathbf{C} \mid |\arg z| < \varepsilon\}$ with $\varepsilon > 0$, there exists an integer $k \geq 1$ such that $\alpha_1^k, \dots, \alpha_n^k \in C_\varepsilon$.

Then prove that $|\alpha_i| \leq q$ for all i. Then, use elementary considerations on complex numbers to prove (as in the proof of the prime number theorem) that $|\alpha_i| < q$. In fact, the Riemann hypothesis says that for all $|\alpha_i| = \sqrt{q}$ for all i. We will come back to this later.

Affine case. Assume now that X is affine, say $X = \bar{X} - \left\{x_1, \dots, x_n\right\}$ where $j : X \hookrightarrow \bar{X}$ is a projective nonsingular completion. Then $H_c^0(X_{\bar{k}}, \mathbf{Q}_\ell) = 0$ and $H_c^2(X_{\bar{k}}, \mathbf{Q}_\ell) = H^2(\bar{X}_{\bar{k}}, \mathbf{Q}_\ell)$ so Theorem 38.82.2 reads

$$L(X, \mathbf{Q}_\ell) = \prod_{x \in |X|} \frac{1}{1 - T^{\deg x}} = \frac{\det\left(1 - \pi_X^* \, T\big|_{H_c^1(X_{\bar{k}}, \mathbf{Q}_\ell)}\right)}{1 - qT}.$$

On the other hand, the previous case gives

$$\begin{aligned} L(X, \mathbf{Q}_\ell) &= L(\bar{X}, \mathbf{Q}_\ell) \prod_{i=1}^n \left(1 - T^{\deg x_i}\right) \\ &= \frac{\prod_{i=1}^n (1 - T^{\deg x_i}) \prod_{j=1}^{2g} (1 - \alpha_j T)}{(1 - T)(1 - qT)}. \end{aligned}$$

Therefore, we see that $\dim H_c^1(X_{\bar{k}}, \mathbf{Q}_\ell) = 2g + \sum_{i=1}^n \deg(x_i) - 1$, and the eigenvalues $\alpha_1, \dots, \alpha_{2g}$ of $\pi_{\bar{X}}^*$ acting on the degree 1 cohomology are roots of unity. More precisely, each x_i gives a complete set of $\deg(x_i)$th roots of unity, and one occurrence of 1 is omitted. To see this directly using coherent sheaves, consider the short exact sequence on $\bar{X}$

$$0 \to j_! \mathbf{Q}_\ell \to \mathbf{Q}_\ell \to \bigoplus_{i=1}^n \mathbf{Q}_{\ell, x_i} \to 0.$$

The long exact cohomology sequence reads

$$0 \to \mathbf{Q}_\ell \to \bigoplus_{i=1}^n \mathbf{Q}_\ell^{\oplus \deg x_i} \to H_c^1(X_{\bar{k}}, \mathbf{Q}_\ell) \to H_c^1(\bar{X}_{\bar{k}}, \mathbf{Q}_\ell) \to 0$$

where the action of Frobenius on $\bigoplus_{i=1}^n \mathbf{Q}_\ell^{\oplus \deg x_i}$ is by cyclic permutation of each term; and $H_c^2(X_{\bar{k}}, \mathbf{Q}_\ell) = H_c^2(\bar{X}_{\bar{k}}, \mathbf{Q}_\ell)$.

38.86. The Legendre family

Let k be a finite field of odd characteristic, $X = Spec(k[\lambda, \frac{1}{\lambda(\lambda-1)}])$, and consider the family of elliptic curves $f : E \to X$ on $\mathbf{P}_X^2$ whose affine equation is $y^2 = x(x-1)(x-\lambda)$. We set $\mathscr{F} = Rf_*^1 \mathbf{Q}_\ell = \left\{R^1 f_* \mathbf{Z}/\ell^n \mathbf{Z}\right\}_{n \geq 1} \otimes \mathbf{Q}_\ell$. In this situation, the following is true

- for each $n \geq 1$, the sheaf $R^1 f_*(\mathbf{Z}/\ell^n\mathbf{Z})$ is finite locally constant -- in fact, it is free of rank 2 over $\mathbf{Z}/\ell^n\mathbf{Z}$;
- the system $\{R^1 f_* \mathbf{Z}/\ell^n \mathbf{Z}\}_{n \geq 1}$ is a lisse ℓ-adic sheaf ; and
- for all $x \in |X|$, $\det\left(1 - \pi_x \, T^{\deg x}\big|_{\mathscr{F}_{\bar{x}}}\right) = (1 - \alpha_x T^{\deg x})(1 - \beta_x T^{\deg x})$ where α_x, β_x are the eigenvalues of the geometric frobenius of E_x acting on $H^1(E_{\bar{x}}, \mathbf{Q}_\ell)$.

Note that E_x is only defined over $\kappa(x)$ and not over k. The proof of these facts uses the proper base change theorem and the local acyclicity of smooth morphisms. For details, see [**Del77**]. It follows that

$$L(E/X) := L(X, \mathscr{F}) = \prod_{x \in |X|} \frac{1}{(1 - \alpha_x T^{\deg x})(1 - \beta_x T^{\deg x})}.$$

Applying Theorem 38.82.2 we get

$$L(E/X) = \prod_{i=0}^{2} \det\left(1 - \pi_X^*\, T\Big|_{H_c^i(X_{\bar{k}}, \mathscr{F})}\right)^{(-1)^{i+1}},$$

and we see in particular that this is a rational function. Furthermore, it is relatively easy to show that $H_c^0(X_{\bar{k}}, \mathscr{F}) = H_c^2(X_{\bar{k}}, \mathscr{F}) = 0$, so we merely have

$$L(E/X) = \det\left(1 - \pi_X^* T\Big|_{H_c^1(X, \mathscr{F})}\right).$$

To compute this determinant explicitly, consider the Leray spectral sequence for the proper morphism $f : E \to X$ over $\mathbf{Q}_\ell$, namely

$$H_c^i(X_{\bar{k}}, R^j f_* \mathbf{Q}_\ell) \Rightarrow H_c^{i+j}(E_{\bar{k}}, \mathbf{Q}_\ell)$$

which degenerates. We have $f_* \mathbf{Q}_\ell = \mathbf{Q}_\ell$ and $R^1 f_* \mathbf{Q}_\ell = \mathscr{F}$. The sheaf $R^2 f_* \mathbf{Q}_\ell = \mathbf{Q}_\ell(-1)$ is the *Tate twist* of $\mathbf{Q}_\ell$, i.e., it is the sheaf $\mathbf{Q}_\ell$ where the Galois action is given by multiplication by $\#\kappa(x)$ on the stalk at $\bar{x}$. It follows that, for all $n \geq 1$,

$$\begin{aligned} \#E(k_n) &= \sum (-1)^i \mathrm{Tr}\left(\pi_E^{n\,*}\big|_{H_c^i(E_{\bar{k}}, \mathbf{Q}_\ell)}\right) \\ &= \sum_{i,j} (-1)^{i+j} \mathrm{Tr}\left(\pi_X^{n\,*}\big|_{H_c^i(X_{\bar{k}}, R^j f_* \mathbf{Q}_\ell)}\right) \\ &= (q^n - 2) + \mathrm{Tr}\left(\pi_X^{n\,*}\big|_{H_c^1(X_{\bar{k}}, \mathscr{F})}\right) + q^n(q^n - 2) \\ &= q^{2n} - q^n - 2 + \mathrm{Tr}\left(\pi_X^{n\,*}\big|_{H_c^1(X_{\bar{k}}, \mathscr{F})}\right) \end{aligned}$$

where the first equality follows from Theorem 38.82.4, the second one from the Leray spectral sequence and the third one by writing down the higher direct images of $\mathbf{Q}_\ell$ under f. Alternatively, we could write

$$\#E(k_n) = \sum_{x \in X(k_n)} \#E_x(k_n)$$

and use the trace formula for each curve. We can also find the number of k_n-rational points simply by counting. The zero section contributes $q^n - 2$ points (we omit the points where $\lambda = 0, 1$) hence

$$\#E(k_n) = q^n - 2 + \#\left\{y^2 = x(x-1)(x-\lambda), \lambda \neq 0, 1\right\}.$$

Now we have

$$\#\left\{y^2 = x(x-1)(x-\lambda),\ \lambda \neq 0,1\right\}$$

$$= \#\left\{y^2 = x(x-1)(x-\lambda) \text{ in } \mathbf{A}^3\right\} - \#\left\{y^2 = x^2(x-1)\right\} - \#\left\{y^2 = x(x-1)^2\right\}$$

$$= \#\left\{\lambda = \tfrac{-y^2}{x(x-1)} + x,\ x \neq 0,1\right\} + \#\left\{y^2 = x(x-1)(x-\lambda), x = 0,1\right\} - 2(q^n - \varepsilon_n)$$

$$= q^n(q^n-2) + 2q^n - 2(q^n - \varepsilon_n)$$

$$= q^{2n} - 2q^n + 2\varepsilon_n$$

where $\varepsilon_n = 1$ if -1 is a square in k_n, 0 otherwise, i.e.,

$$\varepsilon_n = \frac{1}{2}\left(1 + \left(\frac{-1}{k_n}\right)\right) = \frac{1}{2}\left(1 + (-1)^{\frac{q^n-1}{2}}\right).$$

Thus $\#E(k_n) = q^{2n} - q^n - 2 + 2\varepsilon_n$. Comparing with the previous formula, we find

$$\mathrm{Tr}\left(\pi_X^{n\,*}\big|_{H^1_c(X_{\bar k},\mathcal{F})}\right) = 2\varepsilon_n = 1 + (-1)^{\frac{q^n-1}{2}},$$

which implies, by elementary algebra of complex numbers, that if -1 is a square in k_n^*, then $\dim H^1_c(X_{\bar k},\mathcal{F}) = 2$ and the eigenvalues are 1 and 1. Therefore, in that case we have

$$L(E/X) = (1-T)^2.$$

38.87. Exponential sums

A standard problem in number theory is to evaluate sums of the form

$$S_{a,b}(p) = \sum_{x \in \mathbf{F}_p - \{0,1\}} e^{\frac{2\pi i x^a (x-1)^b}{p}}.$$

In our context, this can be interpreted as a cohomological sum as follows. Consider the base scheme $S = Spec(\mathbf{F}_p[x, \frac{1}{x(x-1)}])$ and the affine curve $f : X \to \mathbf{P}^1 - \{0,1,\infty\}$ over S given by the equation $y^{p-1} = x^a(x-1)^b$. This is a finite étale Galois cover with group $\mathbf{F}_p^*$ and there is a splitting

$$f_*(\bar{\mathbf{Q}}_\ell^*) = \bigoplus_{\chi : \mathbf{F}_p^* \to \bar{\mathbf{Q}}_\ell^*} \mathcal{F}_\chi$$

where χ varies over the characters of $\mathbf{F}_p^*$ and $\mathcal{F}_\chi$ is a rank 1 lisse $\mathbf{Q}_\ell$-sheaf on which $\mathbf{F}_p^*$ acts via χ on stalks. We get a corresponding decomposition

$$H^1_c(X_{\bar k}, \mathbf{Q}_\ell) = \bigoplus_\chi H^1(\mathbf{P}^1_{\bar k} - \{0,1,\infty\}, \mathcal{F}_\chi)$$

and the cohomological interpretation of the exponential sum is given by the trace formula applied to $\mathcal{F}_\chi$ over $\mathbf{P}^1 - \{0,1,\infty\}$ for some suitable χ. It reads

$$S_{a,b}(p) = -\mathrm{Tr}\left(\pi_X^*\big|_{H^1(\mathbf{P}^1_{\bar k} - \{0,1,\infty\}, \mathcal{F}_\chi)}\right).$$

The general yoga of Weil suggests that there should be some cancellation in the sum. Applying (roughly) the Riemann-Hurwitz formula, we see that

$$2g_X - 2 \approx -2(p-1) + 3(p-2) \approx p$$

so $g_X \approx p/2$, which also suggests that the χ-pieces are small.

38.88. Trace formula in terms of fundamental groups

In the following sections we reformulate the trace formula completely in terms of the fundamental group of a curve, except if the curve happens to be $\mathbf{P}^1$.

38.89. Fundamental groups

X connected scheme $\overline{x} \to X$ geometric point consider the functor

$$\begin{array}{cccc} F_{\overline{x}}: & \begin{array}{c}\text{finite étale}\\ \text{schemes over } X\end{array} & \longrightarrow & \text{finite sets} \\ & Y/X & \longmapsto & F_{\overline{x}}(Y) = \left\{ \begin{array}{c}\text{geom points } \overline{y}\\ \text{of } Y \text{ lying over } \overline{x}\end{array} \right\} = Y_{\overline{x}} \end{array}$$

Set

$$\pi_1(X, \overline{x}) = Aut(F_{\overline{x}}) = \text{ set of automorphisms of the functor } F_{\overline{x}}$$

Note that for every finite étale $Y \to X$ there is an action

$$\pi_1(X, \overline{x}) \times F_{\overline{x}}(Y) \to F_{\overline{x}}(Y)$$

Definition 38.89.1. A subgroup of the form $\mathrm{Stab}(\overline{y} \in F_{\overline{x}}(Y)) \subset \pi_1(X, \overline{x})$ is called *open*.

Theorem 38.89.2. *(Grothendieck, see* **[Gro71]***) X connected*

(1) *there is a topology on $\pi_1(X, \overline{x})$ such that the open subgroups form a fundamental system of open nbhds of $e \in \pi_1(X, \overline{x})$.*
(2) *$\pi_1(X, \overline{x})$ is a profinite group.*
(3) *the functor*

$$\begin{array}{ccc} \begin{array}{c}\textit{schemes finite}\\ \textit{étale over } X\end{array} & \to & \begin{array}{c}\textit{finite discrete continuous}\\ \pi_1(X,\overline{x})\textit{-sets}\end{array} \\ Y/X & \mapsto & F_{\overline{x}}(Y) \textit{ with its natural action} \end{array}$$

is an equivalence of categories.

Proposition 38.89.3. *Let X be an integral normal Netherian scheme. Let $\overline{y} \to X$ be an algebraic geometric point lying over the generic point $\eta \in X$. Then*

$$\pi_x(X, \overline{\eta}) = Gal(M/\kappa(\eta))$$

($\kappa(\eta)$, function field of X) where

$$\kappa(\overline{\eta}) \supset M \supset \kappa(\eta) = k(X)$$

is the max sub-extension such that for every finite sub extension $M \supset L \supset \kappa(\eta)$ the normalization of X in L is finite étale over X.

Change of base point. For any $\overline{x}_1$, $\overline{x}_2$ geom. points of X there exists an isom. of fibre functions

$$\mathcal{F}_{\overline{x}_1} \cong \mathcal{F}_{\overline{x}_2}$$

(This is a path from $\overline{x}_1$ to $\overline{x}_2$.) Conjugation by this path gives isom

$$\pi_1(X, \overline{x}_1) \cong \pi_1(X, \overline{x}_2)$$

well defined up to inner actions.

Functoriality. For any morphism $X_1 \to X_2$ of connected schemes any $\overline{x} \in X_1$ there is a canonical map

$$\pi_1(X_1, \overline{x}) \to \pi_1(X_2, \overline{x})$$

(Why? because the fibre functor ...)

Base field. Let X be a variety over a field k. Then we get

$$\pi_1(X, \overline{x}) \to \pi_1(Spec(k), \overline{x}) =^{\text{prop}} Gal(k^{\text{sep}}/k)$$

This map is surjective iff X is geom. connected over k. So in the geometrically connected case we get s.e.s. of profinite groups

$$1 \to \pi_1(X_{\overline{k}}, \overline{x}) \to \pi_1(X, \overline{x}) \to Gal(k^{\text{sep}}/k) \to 1$$

($\pi_1(X_{\overline{k}}, \overline{x})$: geometric fundamental group of X, $\pi_1(X, \overline{x})$: arithmetic fundamental group of X)

Comparison. If X is a variety over $\mathbf{C}$ then

$$\pi_1(X, \overline{x}) = \text{ profinite completion of } \pi_1(X(\mathbf{C})(\text{ usual topology}), x)$$

(have $x \in X(\mathbf{C})$)

Frobenii. X variety over k, $\sharp k < \infty$. For any $x \in X$ closed point, let

$$F_x \in \pi_1(x, \overline{x}) = \text{Gal}(\kappa(x)^{\text{sep}}/\kappa(x))$$

be the geometric frobenius. Let $\overline{\eta}$ be an alg. geom. gen. pt. Then

$$\pi_1(X, \overline{\eta}) \leftarrow^{\cong} \pi_1(X, \overline{x}) \overset{\text{fundtoriality}}{\leftarrow} \pi_1(x, \overline{x})$$

Easy fact:

$$\begin{array}{ccc} \pi_1(X, \overline{\eta}) & \to^{\deg} \pi_1(Spec(k), \overline{\eta})* & = Gal(k^{sep}/k) \\ & & || \\ & & \widehat{\mathbf{Z}} \cdot F_{Spec(k)} \\ F_x & \mapsto & \deg(x) \cdot F_{Spec(k)} \end{array}$$

Recall: $\deg(x) = [\kappa(x) : k]$

Fundamental groups and lisse sheaves. Let X be a connected scheme, $\overline{x}$ geom. pt. There are equivalences of categories

(Λ finite ring)	fin. loc. const. sheaves of Λ-modules of $X_{étale}$	$\leftrightarrow$	finite(discrete) Λ-modules with continuous $\pi_1(X,\overline{x})$-action
(l a prime)	lisse l-adic sheaves	$\leftrightarrow$	finitely generated $\mathbf{Z}_l$-modules M with continuous $\pi_1(X,\overline{x})$ action where we use l-adic topology on M

In particular lisse $\mathbf{Q}_l$-sheaves correspond to continuous homomorphisms

$$\pi_1(X, \overline{x}) \to GL_r(\mathbf{Q}_l), \quad r \geq 0$$

Notation: A module with action (M, ρ) corresponds to the sheaf $\mathcal{F}_\rho$.

Trace formulas. X variety over k, $\sharp k < \infty$.

(1) Λ finite ring $(\sharp\Lambda, \sharp k) = 1$

$$\rho : \pi_1(X, \overline{x}) \to GL_r(\Lambda)$$

continuous. For every $n \geq 1$ we have

$$\sum_{d|n} d \left(\sum_{\substack{x \in |X|, \\ \deg(x) = d}} \text{Tr}(\rho(F_x^{n/d})) \right) = \text{Tr}\left((\pi_x^n)^* \Big|_{R\Gamma_c(X_{\overline{k}}, \mathcal{F}_\rho)} \right)$$

(2) $l \neq char(k)$ prime, $\rho : \pi_1(X, \overline{x}) \to GL_r(\mathbf{Q}_l)$. For any $n \geq 1$

$$\sum_{d|n} d \left(\sum_{\substack{x \in |X| \\ \deg(x)=d}} \mathrm{Tr}\left(\rho(F_x^{n/d})\right) \right) = \sum_{i=0}^{2\dim X} (-1)^i \mathrm{Tr}\left(\pi_X^* \Big|_{H_c^i(X_{\overline{k}}, \mathcal{F}_\rho)} \right)$$

Weil conjectures. (Deligne-Weil I, 1974) X smooth proj. over k, $\sharp k = q$, then the eigenvalues of π_X^* on $H^i(X_{\overline{k}}, \mathbf{Q}_l)$ are algebraic integers α with $|\alpha| = q^{1/2}$.

Deligne's conjectures. (almost completely proved by Lafforgue + $\cdots$) Let X be a normal variety over k finite

$$\rho : \pi_1(X, \overline{x}) \longrightarrow GL_r(\mathbf{Q}_l)$$

continuous. Assume: ρ irreducible $\det(\rho)$ of finite order. Then

(1) there exists a number field E such that for all $x \in |X|$(closed points) the char. poly of $\rho(F_x)$ has coefficients in E.
(2) for any $x \in |X|$ the eigenvalues $\alpha_{x,i}$, $i = 1, \ldots, r$ of $\rho(F_x)$ have complex absolute value 1. (these are algebraic numbers not necessary integers)
(3) for every finite place λ(not dividing p), of E (maybe after enlarging E a bit) there exists
$$\rho\lambda : \pi_1(X, \overline{x}) \to GL_r(E_\lambda)$$
compatible with ρ. (some char. polys of F_x's)

Theorem 38.89.4. *(Deligne, Weil II -- not the original formulation) For a sheaf $\mathcal{F}_\rho$ with ρ satisfying the conclusions of the conjecture above then the eigenvalues of π_X^* on $H_c^i(X_{\overline{k}}, \mathcal{F}_\rho)$ are algebraic numbers α with absolute values*

$$|\alpha| = q^{w/2}, \text{ for } w \in \mathbf{Z},\ w \leq i$$

Moreover, if X smooth and proj. then $w = i$.

38.90. Profinite groups, cohomology and homology

Let G be a profinite group.

Cohomology. Consider the category of discrete modules with continuous G-action. This category has enough injectives and we can define

$$H^i(G, M) = R^i H^0(G, M) = R^i(M \mapsto M^G)$$

Also there is a derived version $RH^0(G, -)$.

Homology. Consider the category of compact abelian groups with continuous G-action. This category has enough projectives and we can define

$$H_i(G, M) = L_i H_0(G, M) = L_i(M \mapsto M_G)$$

and there is also a derived version.

Trivial duality. The functor $M \mapsto M^\wedge = Hom_{cont}(M, S^1)$ exchanges the categories above and

$$H^i(G, M)^\wedge = H_i(G, M^\wedge)$$

Moreover, this functor maps torsion discrete G-modules to profinite continuous G-modules and vice versa, and if M is either a discrete or profinite continuous G-module, then $M^\wedge = Hom(M, \mathbf{Q}/\mathbf{Z})$.

Notes on Homology.

(1) If we look at Λ-modules for a finite ring Λ then we can identify
$$H_i(G, M) = Tor_i^{\Lambda[[G]]}(M, \Lambda)$$
where $\Lambda[[G]]$ is the limit of the group algebras of the finite quotients of G.
(2) If $G \lhd \Gamma$, and Γ is also profinite then
  - $H^0(G, -)$: discrete Γ-module$\rightarrow$ discrete Γ/G-modules
  - $H_0(G, -)$: compact Γ-modules $\rightarrow$ compact Γ/G-modules

  and hence the profinite group Γ/G acts on the cohomology groups of G with values in a Γ-module. In other words, there are derived functors
$$RH^0(G, -) : D^+(\text{discrete } \Gamma\text{-modules}) \longrightarrow D^+(\text{discrete } \Gamma/G\text{-modules})$$
  and similarly for $LH_0(G, -)$.

38.91. Cohomology of curves, revisited

Let k be a field, X be geometric connected, smooth curve over k. We have the fundamental short exact sequence
$$1 \to \pi_1(X_{\overline{k}}, \overline{\eta}) \to \pi_1(X, \overline{\eta}) \to \text{Gal}(k^{sep}/k) \to 1$$
If Λ is a finite ring with $\#\Lambda \in k^*$ and M a finite Λ-module, and we are given
$$\rho : \pi_1(X, \overline{\eta}) \to \text{Aut}_\Lambda(M)$$
continuous, then $\mathcal{F}_\rho$ denotes the associated sheaf on $X_{\acute{e}tale}$.

Lemma 38.91.1. *There is a canonical isomorphism*
$$H^2_c(X_{\overline{k}}, \mathcal{F}_\rho) = (M)_{\pi_1(X_{\overline{k}}, \overline{\eta})}(-1)$$
as $Gal(k^{sep}/k)$*-modules.*

Here the subscript $_{\pi_1(X_{\overline{k}}, \overline{\eta})}$ indicates co-invariants, and (-1) indicates the Tate twist i.e., $\sigma \in \text{Gal}(k^{sep}/k)$ acts via
$$\chi_{cycl}(\sigma)^{-1}.\sigma \text{ on RHS}$$
where
$$\chi_{cycl} : \text{Gal}(k^{sep}/k) \to \prod\nolimits_{l \neq char(k)} \mathbf{Z}_l^*$$
is the cyclotomic character.

Reformulation (Deligne, Weil II, page 338). For any finite locally constant sheaf $\mathcal{F}$ on X there is a maximal quotient $\mathcal{F} \to \mathcal{F}''$ with $\mathcal{F}''/X_{\overline{k}}$ a constant sheaf, hence
$$\mathcal{F}'' = (X \to Spec(k))^{-1}F''$$
where F'' is a sheaf $Spec(k)$, i.e., a $\text{Gal}(k^{sep}/k)$-module. Then
$$H^2_c(X_{\overline{k}}, \mathcal{F}) \to H^2_c(X_{\overline{k}}, \mathcal{F}'') \to F''(-1)$$
is an isomorphism.

Proof of Lemma 38.91.1. Let $Y \to^{\varphi} X$ be the finite étale Galois covering corresponding to $Ker(\rho) \subset \pi_1(X, \overline{\eta})$. So
$$\text{Aut}(Y/X) = Ind(\rho)$$
is Galois group. Then $\varphi^*\mathcal{F}_\rho = \underline{M}_Y$ and
$$\varphi_*\varphi^*\mathcal{F}_\rho \to \mathcal{F}_\rho$$

which gives

$$\begin{aligned} & H_c^2(X_{\overline{k}}, \varphi_* \varphi^* \mathscr{F}_\rho) \to H_c^2(X_{\overline{k}}, \mathscr{F}_\rho) \\ & = H_c^2(Y_{\overline{k}}, \varphi^* \mathscr{F}_\rho) \\ & = H_c^2(Y_{\overline{k}}, \underline{M}) = \bigoplus_{\substack{\text{irred. comp. of} \\ Y_{\overline{k}}}} M \end{aligned}$$

$$\mathrm{Im}(\rho) \to H_c^2(Y_{\overline{k}}, \underline{M}) = \bigoplus_{\substack{\text{irred. comp. of} \\ Y_{\overline{k}}}} M \to_{Im(\rho) \text{ equivalent}} H_c^2(X_{\overline{k}}, \mathscr{F}_\rho) \leftarrow^{\text{trivial } Im(\rho)}_{\text{action}}$$

irreducible curve $C/\overline{k}$, $H_c^2(C, \underline{M}) = M$.
Since

$$\begin{matrix}\text{set of irreducible} \\ \text{components of } Y_k\end{matrix} = \frac{Im(\rho)}{Im(\rho|_{\pi_1(X_{\overline{k}}, \overline{\eta})})}$$

We conclude that $H_c^2(X_{\overline{k}}, \mathscr{F}_\rho)$ is a quotient of $M_{\pi_1(X_{\overline{k}}, \overline{\eta})}$. On the other hand, there is a surjection

$$\mathscr{F}_\rho \to \mathscr{F}'' = \begin{matrix}\text{sheaf on } X \text{ associated to} \\ (M)_{\pi_1(X_{\overline{k}}, \overline{\eta})} \leftarrow \pi_1(X, \overline{\eta})\end{matrix}$$

$$H_c^2(X_{\overline{k}}, \mathscr{F}_\rho) \to M_{\pi_1(X_{\overline{k}}, \overline{\eta})}$$

The twist in Galois action comes from the fact that $H_c^2(X_{\overline{k}}, \mu_n) =^{\text{can}} \mathbf{Z}/n\mathbf{Z}$. □

Remark 38.91.2. Thus we conclude that if X is also projective then we have functorially in the representation ρ the identifications

$$H^0(X_{\overline{k}}, \mathscr{F}_\rho) = M^{\pi_1(X_{\overline{k}}, \overline{\eta})}$$

and

$$H_c^2(X_{\overline{k}}, \mathscr{F}_\rho) = M_{\pi_1(X_{\overline{k}}, \overline{\eta})}(-1)$$

Of course if X is not projective, then $H_c^0(X_{\overline{k}}, \mathscr{F}_\rho) = 0$.

Proposition 38.91.3. *Let X/k as before but $X_{\overline{k}} \neq \mathbf{P}^1_{\overline{k}}$ The functors $(M, \rho) \mapsto H_c^{2-i}(X_{\overline{k}}, \mathscr{F}_\rho)$ are the left derived functor of $(M, \rho) \mapsto H_c^2(X_{\overline{k}}, \mathscr{F}_\rho)$ so*

$$H_c^{2-i}(X_{\overline{k}}, \mathscr{F}_\rho) = H_i(\pi_1(X_{\overline{k}}, \overline{\eta}), M)(-1)$$

Moreover, there is a derived version, namely

$$R\Gamma_c(X_{\overline{k}}, \mathscr{F}_\rho) = LH_0(\pi_1(X_{\overline{k}}, \overline{\eta}), M(-1)) = M(-1) \otimes^{\mathbf{L}}_{\Lambda[[\pi_1(X_{\overline{k}}, \overline{\eta})]]} \Lambda$$

in $D(\Lambda[[\widehat{\mathbf{Z}}]])$. Similarly, the functors $(M, \rho) \mapsto H^i(X_{\overline{k}}, \mathscr{F}_\rho)$ are the right derived functor of $(M, \rho) \mapsto M^{\pi_1(X_{\overline{k}}, \overline{\eta})}$ so

$$H^i(X_{\overline{k}}, \mathscr{F}_\rho) = H^i(\pi_1(X_{\overline{k}}, \overline{\eta}), M)$$

Moreover, in this case there is a derived version too.

Proof. (Idea) Show both sides are universal δ-functors. □

Remark 38.91.4. By the proposition and Trivial duality then you get

$$H_c^{2-i}(X_{\overline{k}}, \mathscr{F}_\rho) \times H^i(X_{\overline{k}}, \mathscr{F}_\rho^\wedge(1)) \to \mathbf{Q}/\mathbf{Z}$$

a perfect pairing. If X is projective then this is Poincare duality.

38.92. Abstract trace formula

Suppose given an extension of profinite groups,

$$1 \to G \to \Gamma \xrightarrow{\deg} \widehat{\mathbf{Z}} \to 1$$

We say Γ *has an abstract trace formula* if and only if there exist

(1) an integer $q \geq 1$, and
(2) for every $d \geq 1$ a finite set S_d and for each $x \in S_d$ a conjugacy class $F_x \in \Gamma$ with $\deg(F_x) = d$

such that the following hold

(1) for all ℓ not dividing q have $cd_\ell(G) < \infty$, and
(2) for all finite rings Λ with $q \in \Lambda^*$, for all finite projective Λ-modules M with continuous Γ-action, for all $n > 0$ we have

$$\sum\nolimits_{d|n} d\left(\sum\nolimits_{x \in S_d} \mathrm{Tr}(F_x^{n/d}\big|_M)\right) = q^n \mathrm{Tr}(F^n\Big|_{M \otimes^{\mathbf{L}}_{\Lambda[[G]]}\Lambda})$$

in $\Lambda^\natural$.

Here $M \otimes^{\mathbf{L}}_{\Lambda[[G]]} \Lambda = LH_0(G, M)$ denotes derived homology, and $F = 1$ in $\Gamma/G = \widehat{\mathbf{Z}}$.

Remark 38.92.1. Here are some observations concerning this notion.

(1) If modeling projective curves then we can use cohomology and we don't need factor q^n.
(2) The only examples I know are $\Gamma = \pi_1(X, \overline{\eta})$ where X is smooth, geometrically irreducible and $K(\pi, 1)$ over finite field. In this case $q = (\sharp k)^{\dim X}$. Modulo the proposition, we proved this for curves in this course.
(3) Given the integer q then the sets S_d are uniquely determined. (You can multiple q by an integer m and then replace S_d by m^d copies of S_d without changing the formula.)

Example 38.92.2. Fix an integer $q \geq 1$

$$\begin{array}{ccccccccc} 1 & \to & G = \widehat{\mathbf{Z}}^{(q)} & \to & \Gamma & \to & \widehat{\mathbf{Z}} & \to & 1 \\ & & = \prod_{l\nmid q} \mathbf{Z}_l & & F & \mapsto & 1 & & \end{array}$$

with $FxF^{-1} = ux$, $u \in (\widehat{\mathbf{Z}}^{(q)})^*$. Just using the trivial modules $\mathbf{Z}/m\mathbf{Z}$ we see

$$q^n - (qu)^n \equiv \sum_{d|n} d\sharp S_d$$

in $\mathbf{Z}/m\mathbf{Z}$ for all $(m, q) = 1$ (up to $u \to u^{-1}$) this implies $qu = a \in \mathbf{Z}$ and $|a| < q$. The special case $a = 1$ does occur

$$\pi_1^t(\mathbf{G}_{m,\mathbf{F}_p}, \overline{\eta})$$

$$\sharp S_1 = q - 1$$

$$\sharp S_2 = \frac{(q^2 - 1) - (q - 1)}{2}$$

38.93. Automorphic forms and sheaves

References: See especially the amazing papers [**Dri83**], [**Dri84**] and [**Dri80**] by Drinfeld.

Unramified cusp forms. Let k be a finite field of characteristic p. Let X geometrically irreducible projective smooth curve over k. Set $K = k(X)$ equal to the function field of X. Let v be a place of K which is the same thing as a closed point $x \in X$. Let K_v be the completion of K at v, which is the same thing as the fraction field of the completion of the local ring of X at x, i.e., $K_v = f.f.(\widehat{O_{X,x}})$. Denote $O_v \subset K_v$ the ring of integers. We further set

$$O = \prod\nolimits_v O_v \subset \mathbf{A} = \prod_v{}' K_v$$

and we let Λ be any ring with p invertible in Λ.

Definition 38.93.1. An *unramified cusp form on* $GL_2(\mathbf{A})$ *with values in* Λ[9] is a function

$$f : GL_2(\mathbf{A}) \to \Lambda$$

such that

(1) $f(x\gamma) = f(x)$ for all $x \in GL_2(\mathbf{A})$ and all $\gamma \in GL_2(K)$
(2) $f(ux) = f(x)$ for all $x \in GL_2(\mathbf{A})$ and all $u \in GL_2(O)$
(3) for all $x \in GL_2(\mathbf{A})$,
$$\int_{\mathbf{A} \bmod K} f\left(x \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right) dz = 0$$
see [**dJ01**, Section 4.1] for an explanation of how to make sense out of this for a general ring Λ in which p is invertible.

Hecke Operators. For v a place of K and f an unramified cusp form we set

$$T_v(f)(x) = \int_{g \in M_v} f(g^{-1}x)dg,$$

and

$$U_v(f)(x) = f\left(\begin{pmatrix} \pi_v^{-1} & 0 \\ 0 & \pi_v^{-1} \end{pmatrix} x\right)$$

Notations used: here $\pi_v \in O_v$ is a uniformizer

$$M_v = \left\{h \in Mat(2 \times 2,\ O_v) \,\middle|\, \det h = \pi_v O_v^*\right\}$$

and $dg =$ is the Haar measure on $GL_2(K_v)$ with $\int_{GL_2(O_v)} dg = 1$. Explicitly we have

$$T_v(f)(x) = f\left(\begin{pmatrix} \pi_v^{-1} & 0 \\ 0 & 1 \end{pmatrix} x\right) + \sum_{i=1}^{q_v} f\left(\begin{pmatrix} 1 & 0 \\ -\pi_v^{-1}\lambda_i & \pi_v^{-1} \end{pmatrix} x\right)$$

with $\lambda_i \in O_v$ a set of representatives of $O_v/(\pi_v) = \kappa_v$, $q_v = \#\kappa_v$.

Eigenforms. An *eigenform* f is an unramified cusp form such that some value of f is a unit and $T_v f = t_v f$ and $U_v f = u_v f$ for some (uniquely determined) $t_v,\ u_v \in \Lambda$.

---

[9] This is likely nonstandard notation.

Theorem 38.93.2. *(See* [**Dri80**].*) Given an eigenform f with values in $\overline{\mathbf{Q}}_l$ and eigenvalues $u_v \in \overline{\mathbf{Z}}_l^*$ then there exists*

$$\rho : \pi_1(X) \to GL_2(E)$$

continuous, absolutely irreducible where E is a finite extension of $\mathbf{Q}_\ell$ contained in $\overline{\mathbf{Q}}_l$ such that $t_v = Tr(\rho(F_v))$, and $u_v = q_v^{-1} \det\left(\rho(F_v)\right)$ for all places v.

Theorem 38.93.3. *Suppose $\mathbf{Q}_l \subset E$ finite, and*

$$\rho : \pi_1(X) \to GL_2(E)$$

absolutely irreducible, continuous. Then there exists an eigenform f with values in $\overline{\mathbf{Q}}_l$ whose eigenvalues t_v, u_v satisfy the equalities $t_v = Tr(\rho(F_v))$ and $u_v = q_v^{-1}\det(\rho(F_v))$.

Remark 38.93.4. We now have, thanks to Lafforgue and many other mathematiciens, a complete theorems like this two above for GL_n and allowing ramification! In other words, the full global Langlands correspondence for GL_n is known for function fields of curves over finite fields. At the same time this does not mean there aren't a lot of interesting questions left to answer about the fundamental groups of curves over finite fields, as we shall see below.

Central character. If f is an eigenform then

$$\begin{array}{cccc} \chi_f : & O^*\backslash\mathbf{A}^*/K^* & \to & \Lambda^* \\ & (1,\ldots,\pi_v,1,\ldots,1) & \mapsto & u_v^{-1} \end{array}$$

is called the *central character*. If corresponds to the determinant of ρ via normalizations as above. Set

$$C(\Lambda) = \left\{ \begin{array}{c} \text{unr. cusp forms } f \text{ with coefficients in } \Lambda \\ \text{such that } U_v f = \varphi_v^{-1} f \forall v \end{array} \right\}$$

Proposition 38.93.5. *(See* [**dJ01**, Proposition 4.7]*) If Λ is Noetherian then $C(\Lambda)$ is a finitely generated Λ-module. Moreover, if Λ is a field with prime subfield $\mathbf{F} \subset \Lambda$ then*

$$C(\Lambda) = (C(\mathbf{F})) \otimes_{\mathbf{F}} \Lambda$$

compatibly with T_v acting.

This proposition trivially implies the following lemma.

Lemma 38.93.6. *Algebraicity of eigenvalues. If Λ is a field then the eigenvalues t_v for $f \in C(\Lambda)$ are algebraic over the prime subfield $\mathbf{F} \subset \Lambda$.*

Combining all of the above we can do the following very useful trick.

Lemma 38.93.7. *Switching l. Let E be a number field. Start with*

$$\rho : \pi_1(X) \to SL_2(E_\lambda)$$

absolutely irreducible continuous, where λ is a place of E not lying above p. Then for any second place λ' of E not lying above p there exists a finite extension $E'_{\lambda'}$ and a absolutely irreducible continuous representation

$$\rho' : \pi_1(X) \to SL_2(E'_{\lambda'})$$

which is compatible with ρ in the sense that the characteristic polynomials of all Frobenii are the same.

Note how this is an instance of Deligne's conjecture!

Proof. To prove the switching lemma use Theorem 38.93.3 to obtain $f \in C(\overline{\mathbf{Q}}_l)$ eigenform ass. to ρ. Next, use Proposition 38.93.5 to see that we may choose $f \in C(E')$ with $E \subset E'$ finite. Next we may complete E' to see that we get $f \in C(E'_{\lambda'})$ eigenform with $E'_{\lambda'}$ a finite extension of $E_{\lambda'}$. And finally we use Theorem 38.93.2 to obtain $\rho' : \pi_1(X) \to SL_2(E'_{\lambda'})$ abs. irred. and continuous after perhaps enlarging $E'_{\lambda'}$ a bit again. □

Speculation: If for a (topological) ring Λ we have

$$\left(\begin{array}{c} \rho : \pi_1(X) \to SL_2(\Lambda) \\ \text{abs irred} \end{array} \right) \leftrightarrow \text{ eigen forms in } C(\Lambda)$$

then all eigenvalues of $\rho(F_v)$ algebraic (won't work in an easy way if Λ is a finite ring. Based on the speculation that the Langlands correspondence works more generally than just over fields one arrives at the following conjecture.

Conjecture. (See [**dJ01**]) For any continuous

$$\rho : \pi_1(X) \to GL_n(\mathbf{F}_l[[t]])$$

we have $\#\rho(\pi_1(X_{\overline{k}})) < \infty$.

A rephrasing in the language of sheaves: "For any lisse sheaf of $\overline{\mathbf{F}_l((t))}$-modules the geom monodromy is finite. "

Theorem 38.93.8. *(See* [**dJ01**]*) The Conjecture holds if $n \leq 2$.*

Theorem 38.93.9. *(See* [**Gai07**]*) Conjecture holds if $l > 2n$ modulo some unproven things.*

It turns out the conjecture is useful for something. See work of Drinfeld on Kashiwara's conjectures. But there is also the much more down to earth application as follows.

Theorem 38.93.10. *(See* [**dJ01**, Theorem 3.5]*) Suppose*

$$\rho_0 : \pi_1(X) \to GL_n(\mathbf{F}_l)$$

is a continuous, $l \neq p$. Assume

(1) *Conj. holds for X,*
(2) *$\rho_0\big|_{\pi_1(X_{\overline{k}})}$ abs. irred., and*
(3) *l does not divide n.*

Then the universal determination ring R_{univ} of ρ_0 is finite flat over $\mathbf{Z}_l$.

Explanation: There is a representation $\rho_{\text{univ}} : \pi_1(X) \to GL_n(R_{\text{univ}})$ (Univ. Defo ring) R_{univ} loc. complete, residue field $\mathbf{F}_l$ and $(R_{\text{univ}} \to \mathbf{F}_l) \circ \rho_{\text{univ}} \cong \rho_0$. And given any $R \to \mathbf{F}_l$, R local complete and $\rho : \pi_1(X) \to GL_n(R)$ then there exists $\psi : R_{\text{univ}} \to R$ such that $\psi \circ \rho_{\text{univ}} \cong \rho$. The theorem says that the morphism

$$Spec(R_{\text{univ}}) \longrightarrow Spec(\mathbf{Z}_l)$$

is finite and flat. In particular, such a ρ_0 lifts to a $\rho : \pi_1(X) \to GL_n(\overline{\mathbf{Q}}_l)$.

Notes:

(1) The theorem on deformations is easy.
(2) Any result towards the conjecture seems hard.
(3) It would be interesting to have more conjectures on $\pi_1(X)$!

38.94. Counting points

Let X be a smooth, geometrically irreducible, projective curve over k. $q = \sharp k$. Trace formula gives:
there exists algebraic integers $w_1, \ldots, w_{2g}$ such that

$$\sharp X(k_n) = q^n - \sum_{i=1}^{2g_X} w_i^n + 1.$$

If $\sigma \in \mathrm{Aut}(X)$ then for all i, there exists j such that $\sigma(w_i) = w_j$.

Riemann-Hypothesis. For all i we have $|\omega_i| = \sqrt{q}$.

This was formulated by Emil Artin, in 1924, for hyperelliptic curves. Proved by Weil 1940. Weil gave two proofs

- using intersection theory on $X \times X$, using the Hodge index theorem, and
- using the Jacobian of X.

There is another proof whose initial idea is due to Stephanov, and which was given by Bombieri: it uses the function field $k(X)$ and its Frobenius operator (1969). The starting point is that given $f \in k(X)$ one observes that $f^q - f$ is a rational function which vanishes in all the $\mathbf{F}_q$-rational points of X, and that one can try to use this idea to give an upper bound for the number of points.

38.95. Precise form of Chebotarov

As a first application let us prove a precise form of Chebatarov for a finite étale Galois covering of curves. Let $Y \to_G^\varphi X$, Galois covering, finite étale,

$$G = \mathrm{Aut}(\overline{Y}/X) \leftarrow \pi_1(X).$$

$G = Gal(Y/X)$. Assume $Y_{\overline{k}} =$ irreducible.
If $C \subset G$ is a conjugacy class then for all $n > 0$, we have

$$\left| \sharp \left\{ x \in X(k_n) \,\middle|\, F_x \in C \right\} - \frac{\sharp C}{\sharp G} \cdot \sharp X(k_n) \right| \leq (\sharp C)(2g-2)\sqrt{q^n}$$

(Warning: Please check ($\sharp C$) carefully before using.)

Sketch.

$$\varphi_*(\overline{Q}_l) = \oplus_{\pi \in \widehat{G}} \mathcal{F}_\pi$$

where $\widehat{G}$ = set of isom. classes of irred representations of G over $\overline{\mathbf{Q}}_l$. For $\pi \in \widehat{G}$,

$$\chi_\pi : G \to \overline{\mathbf{Q}}_l$$

character of π.

$$H^*(Y_{\overline{k}}, \overline{\mathbf{Q}}_l) = \oplus_{\pi \in \widehat{G}} H^*(Y_{\overline{k}}, \overline{\mathbf{Q}}_l)_\pi =_{(\varphi \text{ finite })} \oplus_{\pi \in \widehat{G}} H^*(X_{\overline{k}}, \mathcal{F}_\pi)$$

If $\pi \neq 1$

$$H^0(X_{\overline{k}}, \mathcal{F}_\pi) = H^2(X_{\overline{k}}, \mathcal{F}_\pi) = 0, \ \dim H^1(X_{\overline{k}}, \mathcal{F}_\pi) = (2g_X - 2)d_\pi^2$$

(can get this from trace formula for acting on ...)

$$\left| \sum_{x \in X(k_n)} \chi_\pi(\mathcal{F}_x) \right| \leq_{\pi \neq 1} (2g_X - 2)d_\pi^2 \sqrt{q^n}$$

Write $1_C = \sum_\pi a_\pi \chi_\pi,\ a_\pi = \langle 1_C, \chi_\pi \rangle, a_1 = \langle 1_C, \chi_1 \rangle = \frac{\sharp C}{\sharp G}$

$$\langle f, h \rangle = \frac{1}{\sharp G} \sum_{g \in G} f(g)\overline{h(g)}$$

$$\frac{\sharp C}{\sharp G} = ||1_C||^2 = \sum |a_\pi|^2$$

Final step:

$$\begin{aligned} \sharp \left\{ x \in X(k_n) \,\middle|\, F_x \in C \right\} &= \sum_{x \in X(k_n)} 1_C(x) = \sum_{x \in X)k_n} \sum_\pi a_\pi \chi_\pi(F_x) \\ &= \underbrace{\frac{\sharp C}{\sharp G} \sharp X(k_n)}_{\text{term for } \pi=1} + \underbrace{\sum_{\pi \neq 1} a_\pi \sum_{x \in X(k_n)} \chi_\pi(F_x)}_{\text{error term (to be bounded by } E)} \end{aligned}$$

$$\begin{aligned} |E| &\leq \sum_{\substack{\pi \in \widehat{G}, \\ \pi \neq 1}} |a_\pi|(2g-2)d_\pi^2 \sqrt{q^n} \\ &\leq \sum_{\pi \neq 1} \frac{\sharp C}{\sharp G}(2g_X - 2)d_\pi^3 \sqrt{q^n} \end{aligned}$$

By Weil's conjecture, $\sharp X(k_n) \sim q^n$. $\square$

38.96. How many primes decompose completely?

This section gives a second application of the Riemann Hypothesis for curves over a finite field. For number theorists it may be nice to look at the paper by Ihara, entitled ``How many primes decompose completely in an infinite unramified Galois extension of a global field?", see [**Iha83**]. Consider the fundamental exact sequence

$$1 \to \pi_1(X_{\overline{k}}) \to \pi_1(X) \xrightarrow{\deg} \widehat{\mathbf{Z}} \to 1$$

Proposition 38.96.1. *There exists a finite set $x_1, \ldots, x_n$ of closed points of X such that that set of all frobenius elements corresponding to these points topologically generate $\pi_1(X)$.*

Another way to state this is: There exist $x_1, \ldots, x_n \in |X|$ such that the smallest normal closed subgroup Γ of $\pi_1(X)$ containing 1 frobenius element for each x_i is all of $\pi_1(X)$. i.e., $\Gamma = \pi_1(X)$.

Proof. Pick $N \gg 0$ and let

$$\left\{ x_1, \ldots, x_n \right\} = \begin{array}{c} \text{set of all closed points of} \\ X \text{ of degree} \leq N \text{ over } k \end{array}$$

Let $\Gamma \subset \pi_1(X)$ be as in variant statement for these points. Assume $\Gamma \neq \pi_1(X)$. We can pick $\Gamma \lhd \pi_1(X)$ with $U \neq \pi_1(X)$. By R.H. for X this set I will have some x_{i_1} of degree N, some x_{i_2} of degree $N-1$. This shows $\Gamma \to^{\deg} \widehat{\mathbf{Z}}$ and z_0 and so also U. This exactly means if $Y \to X$ is the finite étale Galois covering as to U, then $Y_{\overline{k}}$ irreducible.

$$Y \to^G X,\ G = \pi_1(X)/U$$

By construction all points of X of degree $\leq N$, split completely in Y. So, in particular

$$\sharp Y(k_N) \geq (\sharp G)\sharp X(k_N)$$

Use R.H. on both sides. So you get

$$q^N + 1 + 2g_Y q^{N/2} \geq \sharp G \sharp X(k_N) \geq \sharp G(q^N + 1 - 2g_X q^{N/2})$$

Since $2g_Y - 2 = (\sharp G)(2g_X - 2)$,

$$q^N + 1 + (\sharp G)(2g_X - 1) + 1)q^{N/2} \geq \sharp G(q^N + 1 - 2g_X q^{N/2})$$

□

Weird Question. Set $W_X = \deg^{-1}(\mathbf{Z}) \subset \pi_1(X)$. Is it true that for some finite set of closed points $x_1, \ldots, x_n$ of X the set of all frobenii corresponding to these points *algebraically* generate W_X?

By a Baire category argument this translates into the same question for all Frobenii.

38.97. How many points are there really?

If the genus of the curve is large relative to q, then the main term in the formula $\#X(k) = q - \sum \omega_i + 1$ is not q but the second term $\sum \omega_i$ which can (a priori) have size about $2g_X\sqrt{q}$. In the paper [**VD83**] the authors Drinfeld and Vladut show that this maximum is (as predicted by Ihara earlier) actually at most about $g\sqrt{q}$. Fix q, set

$$A(q) = lim\sup_{X/k} \frac{\sharp X(k)}{g_X}$$

(X as behave $k = \mathbf{F}_q$)

$$g_x \to \infty$$

- RH $\Rightarrow$ $A(q) \leq 2\sqrt{q}$
- Ihara $\Rightarrow$ $A(q) \leq \sqrt{2q}$
- DV $A(q) \leq \sqrt{q} - 1$ (actually this is sharp of q is a square)

Proof. $X \to w_1, \ldots, w_{2g}$, $g = g_X$. Set $\alpha_i = \frac{w_i}{\sqrt{q}}$, $|\alpha_i| = 1$ If α_i occurs then $\overline{\alpha}_i = \alpha_i^{-1}$ also occurs. Then

$$N = \sharp X(k) \leq X(k_r) = q^r + 1 - (\sum \alpha_i) q^{r/2}$$

Rewrite:

$$-\sum \alpha_i^r \geq N q^{-r/2} - q^{r/2} - q^{-r/2}$$

$$0 \leq |\alpha_i^n + \alpha_i^{n-1} + \cdots + \alpha_i + 1|^2 = (n+1) + \sum_{j=1}^{M} (n+1-j)(\alpha_i^j + \alpha_i^{-j})$$

So

$$\begin{aligned}
2g(n+1) &\geq -\sum_i \left(\sum_{j=1}^{n} (n+1-j)(\alpha_i^j + \alpha_i^{-j}) \right) \\
&= -\sum_{j=1}^{n} (n+1-j) \left(\sum_i \alpha_i^j + \sum_i \alpha_i^{-j} \right)
\end{aligned}$$

$$\begin{aligned}
g(n+1) &\geq -\sum_{j=1}^{n} (n+1-j)(\sum_i \alpha_i^j) \\
&\geq N \sum_{j=1}^{n} (n+1-j) q^{-j/2} - \sum_{j=1}^{n} (n+1-j)(q^{j/2} + q^{-j/2})
\end{aligned}$$

This gives

$$\frac{N}{g} \leq \left(\sum_{j=1}^{n} \frac{n+1-j}{n+1} q^{-j/2} \right)^{-1} \cdot \left(1 + \frac{1}{g} \sum_{j=1}^{n} \frac{n+1-j}{n+1} (q^{j/2} + q^{-j/2}) \right)$$

Fix n let $g \to \infty$

$$A(q) \leq \left(\sum_{j=1}^{n} \frac{n+1-j}{n+1} q^{-j/2} \right)^{-1}$$

So

$$A(q) \leq lim_{n \to \infty}(\cdots) = \left(\sum_{j=1}^{\infty} q^{-j/2} \right)^{-1} = \sqrt{q} - 1$$

□

38.98. Other chapters

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CHAPTER 39

Crystalline Cohomology

39.1. Introduction

This chapter is based on a lecture series given by Johan de Jong held in 2012 at Columbia University. The goals of this chapter are to give a quick introduction to crystalline cohomology. A reference is the book [**Ber74**].

39.2. Divided powers

In this section we collect some results on divided power rings. We will use the convention $0! = 1$ (as empty products should give 1).

Definition 39.2.1. Let A be a ring. Let I be an ideal of A. A collection of maps $\gamma_n : I \to I$, $n > 0$ is called a *divided power structure* on I if for all $n \geq 0$, $m > 0$, $x, y \in I$, and $a \in A$ we have

(1) $\gamma_1(x) = x$, we also set $\gamma_0(x) = 1$,
(2) $\gamma_n(x)\gamma_m(x) = \frac{(n+m)!}{n!m!}\gamma_{n+m}(x)$,
(3) $\gamma_n(ax) = a^n\gamma_n(x)$,
(4) $\gamma_n(x+y) = \sum_{i=0,\ldots,n}\gamma_i(x)\gamma_{n-i}(y)$,
(5) $\gamma_n(\gamma_m(x)) = \frac{(nm)!}{n!(m!)^n}\gamma_{nm}(x)$.

Note that the rational numbers $\frac{(n+m)!}{n!m!}$ and $\frac{(nm)!}{n!(m!)^n}$ occuring in the definition are in fact integers; the first is the number of ways to choose n out of $n+m$ and the second counts the number of ways to divide a group of nm objects into n groups of m. We make some remarks about the definition which show that $\gamma_n(x)$ is a replacement for $x^n/n!$ in I.

Lemma 39.2.2. *Let A be a ring. Let I be an ideal of A.*

(1) *If γ is a divided power structure on I, then $n!\,\gamma_n(x) = x^n$ for $n \geq 1$, $x \in I$.*

Assume A is torsion free as a $\mathbf{Z}$-module.

(2) *A divided power structure on I, if it exists, is unique.*
(3) *If $\gamma_n : I \to I$ are maps then*
$$\gamma \text{ is a divided power structure} \Leftrightarrow n!\,\gamma_n(x) = x^n\ \forall x \in I, n \geq 1.$$
(4) *The ideal I has a divided power structure if and only if there exists a set of generators x_i of I as an ideal such that for all $n \geq 1$ we have $x_i^n \in (n!)I$.*

Proof. Proof of (1). If γ is a divided power structure, then condition (2) implies that $n\gamma_n(x) = \gamma_1(x)\gamma_{n-1}(x)$. Hence by induction and condition (1) we get $n!\,\gamma_n(x) = x^n$.

Assume A is torsion free as a $\mathbf{Z}$-module. Proof of (2). This is clear from (1).

Proof of (3). Assume that $n!\,\gamma_n(x) = x^n$ for all $x \in I$ and $n \geq 1$. Since $A \subset A \otimes_{\mathbf{Z}} \mathbf{Q}$ it suffices to prove (1) -- (5) in case A is a $\mathbf{Q}$-algebra. In this case $\gamma_n(x) = x^n/n!$ and it is straightforward to verify (1) -- (5), for example (4) corresponds to the binomial formula

$$(x + y)^n = \sum \frac{n!}{i!\,(n-i)!} x^i y^{n-i}$$

We encourage the reader to do the verifications to make sure that we have the coefficients correct.

Proof of (4). Assume we have generators x_i of I as an ideal such that $x_i^n \in (n!\,)I$ for all $n \geq 1$. We claim that for all $x \in I$ we have $x^n \in (n!\,)I$. If the claim holds then we can set $\gamma_n(x) = x^n/n!$ which is a divided power structure by (3). To prove the claim we note that it holds for $x = ax_i$. Hence we see that the claim holds for a set of generators of I as an abelian group. By induction on the length of an expression in terms of these, it suffices to prove the claim for $x+y$ if it holds for x and y. This follows immediately from the binomial theorem. □

Example 39.2.3. Let p be a prime number. Let A be a ring such that every integer n not divisible by p is invertible, i.e., A is a $\mathbf{Z}_{(p)}$-algebra. Then $I = pA$ has a canonical divided power structure. Namely, given $x = pa \in A$ we set

$$\gamma_n(x) = \frac{p^n}{n!} a^n$$

The reader verifies immediately that $p^n/n! \in \mathbf{Z}_{(p)}$ so that the definition makes sense. It is a straightforward exercise to verify that conditions (1) -- (5) of Definition 39.2.1 are satisfied. Alternatively, it is clear that the definition works for $A_0 = \mathbf{Z}_{(p)}$ and then the result follows from Lemma 39.4.2.

Lemma 39.2.4. *Let A be a ring. Let I be an ideal of A. Let $\gamma_n : I \to I$, $n \geq 1$ be a sequence of maps. Assume*

(a) *(1), (3), and (4) of Definition 39.2.1 hold for all $x, y \in I$, and*
(b) *properties (2) and (5) hold for x in set of generators of I as an ideal.*

Then γ is a divided power structure on I.

Proof. The numbers (1), (2), (3), (4), (5) in this proof refer to the conditions listed in Definition 39.2.1. Applying (3) we see that if (2) and (5) hold for x then (2) and (5) hold for ax for all $a \in A$. Hence we see (b) implies (2) and (5) hold for a set of generators of I as an abelian group. Hence, by induction of the length of an expression in terms of these it suffices to prove that, given $x, y \in I$ such that (2) and (5) hold for x and y, then (2) and (5) hold for $x + y$.

Proof of (2) for $x + y$. By (4) we have

$$\gamma_n(x + y)\gamma_m(x + y) = \sum\nolimits_{i+j=n,\ k+l=m} \gamma_i(x)\gamma_k(x)\gamma_j(y)\gamma_l(y)$$

Using (2) for x and y this equals

$$\sum \frac{(i+k)!}{i!\,k!} \frac{(j+l)!}{j!\,l!} \gamma_{i+k}(x)\gamma_{j+l}(y)$$

Comparing this with the expansion

$$\gamma_{n+m}(x + y) = \sum \gamma_a(x)\gamma_b(y)$$

we see that we have to prove that given $a + b = n + m$ we have

$$\sum\nolimits_{i+k=a,\ j+l=b,\ i+j=n,\ k+l=m} \frac{(i+k)!}{i!\,k!} \frac{(j+l)!}{j!\,l!} = \frac{(n+m)!}{n!\,m!}.$$

Instead of arguing this directly, we note that the result is true for the ideal $I = (x, y)$ in the polynomial ring $\mathbf{Q}[x, y]$ because $\gamma_n(f) = f^n/n!$, $f \in I$ defines a divided power structure on I. Hence the equality of rational numbers above is true.

Proof of (5) for $x + y$ given that (1) -- (4) hold and that (5) holds for x and y. We will again reduce the proof to an equality of rational numbers. Namely, using (4) we can write $\gamma_n(\gamma_m(x + y)) = \gamma_n(\sum \gamma_i(x)\gamma_j(y))$. Using (4) we can write $\gamma_n(\gamma_m(x + y))$ as a sum of terms which are products of factors of the form $\gamma_k(\gamma_i(x)\gamma_j(y))$. If $i > 0$ then

$$\begin{aligned}\gamma_k(\gamma_i(x)\gamma_j(y)) &= \gamma_j(y)^k\gamma_k(\gamma_i(x)) \\ &= \frac{(ki)!}{k!\,(i!)^k}\gamma_j(y)^k\gamma_{ki}(x) \\ &= \frac{(ki)!}{k!\,(i!)^k}\frac{(kj)!}{k!\,(j!)^k}\gamma_{ik}(x)\gamma_{kj}(y)\end{aligned}$$

using (3) in the first equality, (5) for x in the second, and (2) exactly k times in the third. Using (5) for y we see the same equality holds when $i = 0$. Continuing like this using all axioms but (5) we see that we can write

$$\gamma_n(\gamma_m(x + y)) = \sum\nolimits_{i+j=nm} c_{ij}\gamma_i(x)\gamma_j(y)$$

for certain universal constants $c_{ij} \in \mathbf{Z}$. Again the fact that the equality is valid in the polynomial ring $\mathbf{Q}[x, y]$ implies that the coefficients c_{ij} are all equal to $(nm)!/n!\,(m!)^n$ as desired. □

Lemma 39.2.5. *Let A be a ring with two ideals $I, J \subset A$. Let γ be a divided power structure on I and let δ be a divided power structure on J. Then*

1. *γ and δ agree on IJ,*
2. *if γ and δ agree on $I \cap J$ then they are the restriction of a unique divided power structure ϵ on $I + J$.*

Proof. Let $x \in I$ and $y \in J$. Then

$$\gamma_n(xy) = y^n\gamma_n(x) = n!\,\delta_n(y)\gamma_n(x) = \delta_n(y)x^n = \delta_n(xy).$$

Hence γ and δ agree on a set of (additive) generators of IJ. By property (4) of Definition 39.2.1 it follows that they agree on all of IJ.

Let $z \in I + J$. Write $z = x + y$ with $x \in I$ and $y \in J$. Then we set

$$\epsilon_n(z) = \sum \gamma_i(x)\delta_{n-i}(y)$$

To see that this is well defined, suppose that $z = x' + y'$ is another representation with $x' \in I$ and $y' \in J$. Then $w = x - x' = y' - y \in I \cap J$. Hence

$$\begin{aligned}\sum\nolimits_{i+j=n}\gamma_i(x)\delta_j(y) &= \sum\nolimits_{i+j=n}\gamma_i(x' + w)\delta_j(y) \\ &= \sum\nolimits_{i'+l+j=n}\gamma_{i'}(x')\gamma_l(w)\delta_j(y) \\ &= \sum\nolimits_{i'+l+j=n}\gamma_{i'}(x')\delta_l(w)\delta_j(y) \\ &= \sum\nolimits_{i'+j'=n}\gamma_{i'}(x')\delta_{j'}(y + w) \\ &= \sum\nolimits_{i'+j'=n}\gamma_{i'}(x')\delta_{j'}(y')\end{aligned}$$

as desired. Next, we prove conditions (1) -- (5) of Definition 39.2.1. Properties (1) and (3) are clear. To see (4), suppose that $z = x + y$ and $z' = x' + y'$ with $x, x' \in I$ and $y, y' \in J$ and compute

$$\begin{aligned} \epsilon_n(z + z') &= \sum\nolimits_{a+b=n} \gamma_i(x + x')\delta_j(y + y') \\ &= \sum\nolimits_{i+i'+j+j'=n} \gamma_i(x)\gamma_{i'}(x')\delta_j(y)\delta_{j'}(y') \\ &= \sum\nolimits_{k=0,\ldots,n} \sum\nolimits_{i+j=k} \gamma_i(x)\delta_j(y) \sum\nolimits_{i'+j'=n-k} \gamma_{i'}(x')\delta_{j'}(y') \\ &= \sum\nolimits_{k=0,\ldots,n} \epsilon_k(z)\epsilon_{n-k}(z') \end{aligned}$$

as desired. Now we see that it suffices to prove (2) and (5) for elements of I or J, see Lemma 39.2.4. This is clear because γ and δ are divided power structures. □

Lemma 39.2.6. *Let p be a prime number. Let A be a ring, let $I \subset A$ be an ideal, and let γ be a divided power structure on I. Assume p is nilpotent in A/I. Then I is locally nilpotent if and only if p is nilpotent in A.*

Proof. If $p^N = 0$ in A, then for $x \in I$ we have $x^{pN} = (pN)!\,\gamma_N(x) = 0$ because $(pN)!$ is divisible by p^N. Conversely, assume I is locally nilpotent. We've also assumed that p is nilpotent in A/I, hence $p^r \in I$ for some r, hence p^r nilpotent, hence p nilpotent. □

The following lemma can be found in [**BO83**].

Lemma 39.2.7. *Let p be a prime number. Let A be a ring such that every integer n not divisible by p is invertible, i.e., A is a $\mathbf{Z}_{(p)}$-algebra. Let $I \subset A$ be an ideal. Two divided power structures γ, γ' on I are equal if and only if $\gamma_p = \gamma'_p$. Moreover, given a map $\delta : I \to I$ such that*

1. *$p!\,\delta(x) = x^p$ for all $x \in I$,*
2. *$\delta(ax) = a^p\delta(x)$ for all $a \in A$, $x \in I$, and*
3. *$\delta(x + y) = \delta(x) + \sum_{i+j=p,i,j\geq 1} \frac{1}{i!j!}x^i y^j + \delta(y)$ for all $x, y \in I$,*

then there exists a unique divided power structure γ on I such that $\gamma_p = \delta$.

Proof. If n is not divisible by p, then $\gamma_n(x) = cx\gamma_{n-1}(x)$ where c is a unit in $\mathbf{Z}_{(p)}$. Moreover,

$$\gamma_{pm}(x) = c\gamma_m(\gamma_p(x))$$

where c is a unit in $\mathbf{Z}_{(p)}$. Thus the first assertion is clear. For the second assertion, we can, working backwards, use these equalities to define all γ_n. More precisely, if $n = a_0 + a_1 p + \ldots + a_e p^e$ with $a_i \in \{0, \ldots, p-1\}$ then we set

$$\gamma_n(x) = c_n x^{a_0}\delta(x)^{a_1} \ldots \delta^e(x)^{a_e}$$

for $c_n \in \mathbf{Z}_{(p)}$ defined by

$$c_n = (p!\,)^{a_1+a_2(1+p)+\ldots+a_e(1+\ldots+p^{e-1})}/n!.$$

Now we have to show the axioms (1) -- (5) of a divided power structure, see Definition 39.2.1. We observe that (1) and (3) are immediate. Verification of (2) and (5) is by a direct calculation which we omit. Let $x, y \in I$. We claim there is a ring map

$$\varphi : \mathbf{Z}_{(p)}\langle x, y\rangle \longrightarrow A$$

which maps $x^{[n]}$ to $\gamma_n(x)$ and $y^{[n]}$ to $\gamma_n(y)$. By construction of $\mathbf{Z}_{(p)}\langle x, y\rangle$ this means we have to check that $\gamma_n(x)\gamma_m(x) = \frac{(n+m)!}{n!m!}\gamma_{n+m}(x)$ and similarly for y, which follows as (2) holds for γ. Let ϵ denote the divided power structure on the ideal $\mathbf{Z}_{(p)}\langle x, y\rangle_+$ of $\mathbf{Z}_{(p)}\langle x, y\rangle$.

Next, we claim that $\varphi(\epsilon_n(f)) = \gamma_n(\varphi(f))$ for $f \in \mathbf{Z}_{(p)}\langle x, y\rangle_+$ and all n. This is clear for $n = 0, 1, \ldots, p-1$. For $n = p$ it suffices to prove it for a set of generators of the ideal $\mathbf{Z}_{(p)}\langle x, y\rangle_+$ because both ϵ_p and $\gamma_p = \delta$ satisfy properties (1) and (3) of the lemma. Hence it suffices to prove that $\gamma_p(\gamma_n(x)) = \frac{(pn)!}{p!(n!)^p}\gamma_{pn}(x)$ and similarly for y, which follows as (5) holds for γ. Now, if $n = a_0 + a_1 p + \ldots + a_e p^e$ is an arbitrary integer written in p-adic expansion as above, then

$$\epsilon_n(f) = c_n f^{a_0}\gamma_p(f)^{a_1}\ldots\gamma_p^e(f)^{a_e}$$

because ϵ is a divided power structure. Hence we see that $\varphi(\epsilon_n(f)) = \gamma_n(\varphi(f))$ holds for all n. Applying this for $f = x + y$ we see that axiom (4) for γ follows from the fact that ϵ is a divided power structure. □

39.3. Divided power rings

There is a category of divided power rings. Here is the definition.

Definition 39.3.1. A *divided power ring* is a triple (A, I, γ) where A is a ring, $I \subset A$ is an ideal, and $\gamma = (\gamma_n)_{n\geq 1}$ is a divided power structure on I. A *homomorphism of divided power rings* $\varphi : (A, I, \gamma) \to (B, J, \delta)$ is a ring homomorphism $\varphi : A \to B$ such that $\varphi(I) \subset J$ and such that $\delta_n(\varphi(x)) = \varphi(\gamma_n(x))$ for all $x \in I$.

We sometimes say ``let (B, J, δ) be a divided power algebra over (A, I, γ)" to indicate that (B, J, δ) is a divided power ring which comes equipped with a homomorphism of divided power rings $(A, I, \gamma) \to (B, J, \delta)$.

Lemma 39.3.2. *The category of divided power rings has all limits and they agree with limits in the category of rings.*

Proof. The empty limit is the zero ring (that's weird but we need it). The product of a collection of divided power rings (A_t, I_t, γ_t), $t \in T$ is given by $(\prod A_t, \prod I_t, \gamma)$ where $\gamma_n((x_t)) = (\gamma_{t,n}(x_t))$. The equalizer of $\alpha, \beta : (A, I, \gamma) \to (B, J, \delta)$ is just $C = \{a \in A \mid \alpha(a) = \beta(a)\}$ with ideal $C \cap I$ and induced divided powers. It follows that all limits exist, see Categories, Lemma 4.13.10. □

The following lemma illustrates a very general category theoretic phenomenon in the case of divided power algebras.

Lemma 39.3.3. *Let $\mathcal{C}$ be the category of divided power rings. Let $F : \mathcal{C} \to Sets$ be a functor. Assume that*

1. *there exists a cardinal κ such that for every $f \in F(A, I, \gamma)$ there exists a morphism $(A', I', \gamma') \to (A, I, \gamma)$ of $\mathcal{C}$ such that f is the image of $f' \in F(A', I', \gamma')$ and $|A'| \leq \kappa$, and*
2. *F commutes with limits.*

Then F is representable, i.e., there exists an object (B, J, δ) of $\mathcal{C}$ such that

$$F(A, I, \gamma) = Hom_{\mathcal{C}}((B, J, \delta), (A, I, \gamma))$$

functorially in (A, I, γ).

Proof. Consider a set of objects $\mathcal{U}$ of $\mathcal{C}$ containing an object isomorphic to every (A, I, γ) with $|A| \leq \kappa$. Let $\mathcal{I}$ be the category of pairs (U, f) where $U \in \mathcal{U}$ and $f \in F(U)$. A morphism $(U, f) \to (U', f')$ of $\mathcal{I}$ is a map $u : U \to U'$ of $\mathcal{C}$ such that $F(u)(f) = f'$. Set

$$(B, J, \delta) = lim_{(U,f)\in\mathcal{I}} U.$$

The limit exists by Lemma 39.3.2. As F commutes with limits we have

$$F(B, J, \delta) = lim_{(U,f)\in\mathcal{I}} F(U).$$

Hence there is a universal element $\xi \in F(B, J, \delta)$ which for $U \in \mathcal{U}$ maps to $f \in F(U)$ under F applied to the projection map $(B, J, \delta) \to U$ of the limit corresponding to f. Using ξ we obtain a transformation of functors

$$\xi : \mathrm{Hom}_{\mathcal{C}}((B, J, \delta), -) \longrightarrow F(-)$$

see Categories, Section 4.3. Let (A, I, γ) be an arbitrary object of $\mathcal{C}$ and let $f \in F(A, I, \gamma)$. Choose $U \to (A, I, \gamma)$ with $U \in \mathcal{U}$ and $f' \in F(U)$ mapping to f which is possible by assumption (1). Then F applied to the maps

$$(B, J, \delta) \longrightarrow U \longrightarrow (A, I, \gamma)$$

(the first being the projection map of the limit defining B) sends ξ to f. Hence the transformation ξ is surjective. Finally, suppose that $a, b : (B, J, \delta) \to (A, I, \gamma)$ are two maps such that $F(a)(\xi) = F(b)(\xi)$. Since F commutes with limits, it commutes with equalizers. This means that ξ comes from an element $\xi' \in F(B', J', \delta')$ where $B' \subset B$ is the equalizer of a and b.

At this point there are two ways to finish the proof. The first is to show that $B' = B$ using compatibility of F with equalizers and the construction of B as a limit over $\mathcal{I}$ above; we omit the details. The second is to replace B by the smallest divided power subring $(B', J', \delta') \subset (B, J, \delta)$ such that ξ comes from an element $\xi' \in F(B', J', \delta')$. Since F commutes with limits F commutes with intersections hence a smallest divided power subring exists. It is clear that the transformation defined by ξ' is still surjective, and the argument above shows that it is also injective. □

Lemma 39.3.4. *The category of divided power rings has all colimits.*

Proof. The empty colimit is $\mathbf{Z}$ with divided power ideal (0). Let's discuss general colimits. Let $\mathcal{C}$ be a category and let $c \mapsto (A_c, I_c, \gamma_c)$ be a diagram. Consider the functor

$$F(B, J, \delta) = lim_{c\in\mathcal{C}} Hom((A_c, I_c, \gamma_c), (B, J, \delta))$$

Note that any $f = (f_c)_{c\in C} \in F(B, J, \delta)$ has the property that all the images $f_c(A_c)$ generate a subring B' of B of bounded cardinality κ and that all the images $f_c(I_c)$ generate a divided power sub ideal J' of B'. And we get a factorization of f as a f' in $F(B')$ followed by the inclusion $B' \to B$. Also, F commutes with limits. Hence we may apply Lemma 39.3.3 to see that F is representable and we win. □

Remark 39.3.5. The forgetful functor $(A, I, \gamma) \mapsto A$ does not commute with colimits. For example, let

$$\begin{tikzcd} (B, J, \delta) \arrow[r] & (B'', J'', \delta'') \\ (A, I, \gamma) \arrow[r] \arrow[u] & (B', J', \delta') \arrow[u] \end{tikzcd}$$

be a push out in the category of divided power rings. Then in general the map $B \otimes_A B' \to B''$ isn't an isomorphism. (It is always surjective.) An explicit example is given by $(A, I, \gamma) = (\mathbf{Z}, (0), \emptyset)$, $(B, J, \delta) = (\mathbf{Z}/4\mathbf{Z}, 2\mathbf{Z}/4\mathbf{Z}, \delta)$, and $(B', J', \delta') = (\mathbf{Z}/4\mathbf{Z}, 2\mathbf{Z}/4\mathbf{Z}, \delta')$ where $\delta_2(2) = 2$ and $\delta_2'(2) = 0$ and all higher divided powers equal to zero. Then $(B'', J'', \delta'') =$

$(\mathbf{F}_2, (0), \emptyset)$ which doesn't agree with the tensor product. However, note that it is always true that

$$B''/J'' = B/J \otimes_{A/I} B'/J'$$

as can be seen from the universal property of the push out by considering maps into divided power algebras of the form $(C, (0), \emptyset)$.

39.4. Extending divided powers

Here is the definition.

Definition 39.4.1. Given a divided power ring (A, I, γ) and a ring map $A \to B$ we say γ *extends* to B if there exists a divided power structure $\bar{\gamma}$ on IB such that $(A, I, \gamma) \to (B, IB, \bar{\gamma})$ is a homomorphism of divided power rings.

Lemma 39.4.2. *Let (A, I, γ) be a divided power ring. Let $A \to B$ be a ring map. If γ extends to B then it extends uniquely. Assume (at least) one of the following conditions holds*

(1) *$IB = 0$,*
(2) *I is principal, or*
(3) *$A \to B$ is flat.*

Then γ extends to B.

Proof. Any element of IB can be written as a finite sum $\sum b_i x_i$ with $b_i \in B$ and $x_i \in I$. If γ extends to $\bar{\gamma}$ on IB then $\bar{\gamma}_n(x_i) = \gamma_n(x_i)$. Thus conditions (3) and (4) imply that

$$\bar{\gamma}_n(\sum b_i x_i) = \sum\nolimits_{n_1+\ldots+n_t=n} \prod\nolimits_{i=1}^{t} b_i^{n_i} \gamma_{n_i}(x_i)$$

Thus we see that $\bar{\gamma}$ is unique if it exists.

If $IB = 0$ then setting $\bar{\gamma}_n(0) = 0$ works. If $I = (x)$ then we define $\bar{\gamma}_n(bx) = b^n \gamma_n(x)$. This is well defined: if $b'x = bx$, i.e., $(b - b')x = 0$ then

$$\begin{aligned} b^n\gamma_n(x) - (b')^n\gamma_n(x) &= (b^n - (b')^n)\gamma_n(x) \\ &= (b^{n-1} + \ldots + (b')^{n-1})(b - b')\gamma_n(x) = 0 \end{aligned}$$

because $\gamma_n(x)$ is divisible by x and hence annihilated by $b - b'$. Next, we prove conditions (1) -- (5) of Definition 39.2.1. Parts (1), (2), (3), (5) are obvious from the construction. For (4) suppose that $y, z \in IB$, say $y = bx$ and $z = cx$. Then $y + z = (b + c)x$ hence

$$\begin{aligned} \bar{\gamma}_n(y + z) &= (b + c)^n\gamma_n(x) \\ &= \sum \frac{n!}{i!\,(n-i)!} b^i c^{n-i}\gamma_n(x) \\ &= \sum b^i c^{n-i}\gamma_i(x)\gamma_{n-i}(x) \\ &= \sum \bar{\gamma}_i(y)\bar{\gamma}_{n-i}(z) \end{aligned}$$

as desired.

Assume $A \to B$ is flat. Suppose that $b_1, \ldots, b_r \in B$ and $x_1, \ldots, x_r \in I$. Then

$$\bar{\gamma}_n(\sum b_i x_i) = \sum b_1^{e_1} \ldots b_r^{e_r} \gamma_{e_1}(x_1) \ldots \gamma_{e_r}(x_r)$$

where the sum is over $e_1 + \ldots + e_r = n$ if $\bar{\gamma}_n$ exists. Next suppose that we have $c_1, \ldots, c_s \in B$ and $a_{ij} \in A$ such that $b_i = \sum a_{ij} c_j$. Setting $y_j = \sum a_{ij} x_i$ we claim that

$$\sum b_1^{e_1} \ldots b_r^{e_r} \gamma_{e_1}(x_1) \ldots \gamma_{e_r}(x_r) = \sum c_1^{d_1} \ldots c_s^{d_s} \gamma_{d_1}(y_1) \ldots \gamma_{d_s}(y_s)$$

in B where on the right hand side we are summing over $d_1 + \ldots + d_s = n$. Namely, using the axioms of a divided power structure we can expand both sides into a sum with coefficients in $\mathbf{Z}[a_{ij}]$ of terms of the form $c_1^{d_1} \ldots c_s^{d_s} \gamma_{e_1}(x_1) \ldots \gamma_{e_r}(x_r)$. To see that the coefficients agree we note that the result is true in $\mathbf{Q}[x_1, \ldots, x_r, c_1, \ldots, c_s, a_{ij}]$ with γ the unique divided power structure on $(x_1, \ldots, x_r)$. By Lazard's theorem (Algebra, Theorem 7.75.4) we can write B as a directed colimit of finite free A-modules. In particular, if $z \in IB$ is written as $z = \sum x_i b_i$ and $z = \sum x'_{i'} b'_{i'}$, then we can find $c_1, \ldots, c_s \in B$ and $a_{ij}, a'_{i'j} \in A$ such that $b_i = \sum a_{ij} c_j$ and $b'_{i'} = \sum a'_{i'j} c_j$ such that $y_j = \sum x_i a_{ij} = \sum x'_{i'} a'_{i'j}$. Hence the procedure above gives a well defined map $\bar{\gamma}_n$ on IB. By construction $\bar{\gamma}$ satisfies conditions (1), (3), and (4). Moreover, for $x \in I$ we have $\bar{\gamma}_n(x) = \gamma_n(x)$. Hence it follows from Lemma 39.2.4 that $\bar{\gamma}$ is a divided power structure on IB. □

Lemma 39.4.3. *Let (A, I, γ) be a divided power ring.*

(1) *If $\varphi : (A, I, \gamma) \to (B, J, \delta)$ is a homomorphism of divided power rings, then $Ker(\varphi) \cap I$ is preserved by γ_n for all $n \geq 1$.*
(2) *Let $\mathfrak{a} \subset A$ be an ideal and set $I' = I \cap \mathfrak{a}$. The following are equivalent*
    (a) *I' is preserved by γ_n for all $n > 0$,*
    (b) *γ extends to $A/\mathfrak{a}$, and*
    (c) *there exist a set of generators x_i of I' as an ideal such that $\gamma_n(x_i) \in I'$ for all $n > 0$.*

Proof. Proof of (1). This is clear. Assume (2)(a). Define $\bar{\gamma}_n(x \bmod I') = \gamma_n(x) \bmod I'$ for $x \in I$. This is well defined since $\gamma_n(x + y) = \gamma_n(x) \bmod I'$ for $y \in I'$ by Definition 39.2.1 (4) and the fact that $\gamma_j(y) \in I'$ by assumption. It is clear that $\bar{\gamma}$ is a divided power structure as γ is one. Hence (2)(b) holds. Also, (2)(b) implies (2)(a) by part (1). It is clear that (2)(a) implies (2)(c). Assume (2)(c). Note that $\gamma_n(x) = a^n \gamma_n(x_i) \in I'$ for $x = ax_i$. Hence we see that $\gamma_n(x) \in I'$ for a set of generators of I' as an abelian group. By induction on the length of an expression in terms of these, it suffices to prove $\forall n : \gamma_n(x + y) \in I'$ if $\forall n : \gamma_n(x), \gamma_n(y) \in I'$. This follows immediately from the fourth axiom of a divided power structure. □

Lemma 39.4.4. *Let (A, I, γ) be a divided power ring. Let $E \subset I$ be a subset. Then the smallest ideal $J \subset I$ preserved by γ and containing all $f \in E$ is the ideal J generated by $\gamma_n(f)$, $n \geq 1$, $f \in E$.*

Proof. Follows immediately from Lemma 39.4.3. □

Lemma 39.4.5. *Let (A, I, γ) be a divided power ring. Let p be a prime. If p is nilpotent in A/I, then*

(1) *the p-adic completion $A^\wedge = lim_e A/p^e A$ surjects onto A/I,*
(2) *the kernel of this map is the p-adic completion $I^\wedge$ of I, and*
(3) *each γ_n is continuous for the p-adic topology and extends to $\gamma_n^\wedge : I^\wedge \to I^\wedge$ defining a divided power structure on $I^\wedge$.*

If moreover A is a $\mathbf{Z}_{(p)}$-algebra, then

(4) *for e large enough the ideal $p^e A \subset I$ is preserved by the divided power structure γ and*
$$(A^\wedge, I^\wedge, \gamma^\wedge) = lim_e(A/p^e A, I/p^e A, \bar{\gamma})$$
in the category of divided power rings.

Proof. Let $t \geq 1$ be an integer such that $p^t A/I = 0$, i.e., $p^t A \subset I$. The map $A^\wedge \to A/I$ is the composition $A^\wedge \to A/p^t A \to A/I$ which is surjective (for example by Algebra, Lemma 7.90.1). As $p^e I \subset p^e A \cap I \subset p^{e-t} I$ for $e \geq t$ we see that the kernel of the composition $A^\wedge \to A/I$ is the p-adic completion of I. The map γ_n is continuous because

$$\gamma_n(x + p^e y) = \sum\nolimits_{i+j=n} p^{je}\gamma_i(x)\gamma_j(y) = \gamma_n(x) \bmod p^e I$$

by the axioms of a divided power structure. It is clear that the axioms for divided power structures are inherited by the maps $\gamma_n^\wedge$ from the maps γ_n. Finally, to see the last statement say $e > t$. Then $p^e A \subset I$ and $\gamma_1(p^e A) \subset p^e A$ and for $n > 1$ we have

$$\gamma_n(p^e a) = p^n \gamma_n(p^{e-1} a) = \frac{p^n}{n!} p^{n(e-1)} a^n \in p^e A$$

as $p^n/n! \in \mathbf{Z}_{(p)}$ and as $n \geq 2$ and $e \geq 2$ so $n(e-1) \geq e$. This proves that γ extends to $A/p^e A$, see Lemma 39.4.3. The statement on limits is clear from the construction of limits in the proof of Lemma 39.3.2. □

39.5. Divided power polynomial algebras

A very useful example is the *divided power polynomial algebra*. Let A be a ring. Let $t \geq 1$. We will denote $A\langle x_1, \ldots, x_t\rangle$ the following A-algebra: As an A-module we set

$$A\langle x_1, \ldots, x_t\rangle = \bigoplus\nolimits_{n_1, \ldots, n_t \geq 0} A x_1^{[n_1]} \ldots x_t^{[n_t]}$$

with multiplication given by

$$x_i^{[n]} x_i^{[m]} = \frac{(n+m)!}{n!\, m!} x_i^{[n+m]}.$$

We also set $x_i = x_i^{[1]}$. Note that $1 = x_1^{[0]} \ldots x_t^{[0]}$. There is a similar construction which gives the divided power polynomial algebra in infinitely many variables. There is an canonical A-algebra map $A\langle x_1, \ldots, x_t\rangle \to A$ sending $x_i^{[n]}$ to zero for $n > 0$. The kernel of this map is denoted $A\langle x_1, \ldots, x_t\rangle_+$.

Lemma 39.5.1. *Let (A, I, γ) be a divided power ring. There exists a unique divided power structure δ on*

$$J = IA\langle x_1, \ldots, x_t\rangle + A\langle x_1, \ldots, x_t\rangle_+$$

such that

(1) *$\delta_n(x_i) = x_i^{[n]}$, and*
(2) *$(A, I, \gamma) \to (A\langle x_1, \ldots, x_t\rangle, J, \delta)$ is a homomorphism of divided power rings.*

Moreover, $(A\langle x_1, \ldots, x_t\rangle, J, \delta)$ has the following universal property: A homomorphism of divided power rings $\varphi : (A\langle x\rangle, J, \delta) \to (C, K, \epsilon)$ is the same thing as a homomorphism of divided power rings $A \to C$ and elements $k_1, \ldots, k_t \in K$.

Proof. We will prove the lemma in case of a divided power polynomial algebra in one variable. The result for the general case can be argued in exactly the same way, or by noting that $A\langle x_1, \ldots, x_t\rangle$ is isomorphic to the ring obtained by adjoining the divided power variables $x_1, \ldots, x_t$ one by one.

Let $A\langle x\rangle_+$ be the ideal generated by $x, x^{[2]}, x^{[3]}, \ldots$. Note that $J = IA\langle x\rangle + A\langle x\rangle_+$ and that

$$IA\langle x\rangle \cap A\langle x\rangle_+ = IA\langle x\rangle \cdot A\langle x\rangle_+$$

Hence by Lemma 39.2.5 it suffices to show that there exist divided power structures on the ideals $IA\langle x\rangle$ and $A\langle x\rangle_+$. The existence of the first follows from Lemma 39.4.2 as

$A \to A\langle x\rangle$ is flat. For the second, note that if A is torsion free, then we can apply Lemma 39.2.2 (4) to see that δ exists. Namely, choosing as generators the elements $x^{[m]}$ we see that $(x^{[m]})^n = \frac{(nm)!}{(m!)^n} x^{[nm]}$ and $n!$ divides the integer $\frac{(nm)!}{(m!)^n}$. In general write $A = R/\mathfrak{a}$ for some torsion free ring R (e.g., a polynomial ring over $\mathbf{Z}$). The kernel of $R\langle x\rangle \to A\langle x\rangle$ is $\bigoplus \mathfrak{a} x^{[m]}$. Applying criterion (2)(c) of Lemma 39.4.3 we see that the divided power structure on $R\langle x\rangle_+$ extends to $A\langle x\rangle$ as desired.

Proof of the universal property. Given a homomorphism $\varphi : A \to C$ of divided power rings and $k_1, \ldots, k_t \in K$ we consider

$$A\langle x_1, \ldots, x_t\rangle \to C, \quad x_1^{[n_1]} \ldots x_t^{[n_t]} \longmapsto \epsilon_{n_1}(k_1) \ldots \epsilon_{n_t}(k_t)$$

using φ on coefficients. The only thing to check is that this is an A-algebra homomorphism (details omitted). The inverse construction is clear. $\square$

Remark 39.5.2. Let (A, I, γ) be a divided power ring. There is a variant of Lemma 39.5.1 for infinitely many variables. First note that if $s < t$ then there is a canonical map

$$A\langle x_1, \ldots, x_s\rangle \to A\langle x_1, \ldots, x_t\rangle$$

Hence if W is any set, then we set

$$A\langle x_w, w \in W\rangle = colim_{E \subset W} A\langle x_e, e \in E\rangle$$

(colimit over E finite subset of W) with transition maps as above. By the definition of a colimit we see that the universal mapping property of $A\langle x_w, w \in W\rangle$ is completely analogous to the mapping property stated in Lemma 39.5.1.

39.6. Divided power envelope

The construction of the following lemma will be dubbed the divided power envelope. It will play an important role later.

Lemma 39.6.1. *Let (A, I, γ) be a divided power ring. Let $A \to B$ be a ring map. Let $J \subset B$ be an ideal with $IB \subset J$. There exists a homomorphism of divided power rings*

$$(A, I, \gamma) \longrightarrow (D, \bar{J}, \bar{\gamma})$$

such that

$$Hom_{(A,I,\gamma)}((D, \bar{J}, \bar{\gamma}), (C, K, \delta)) = Hom_A((B, J), (C, K))$$

functorially in the divided power algebra (C, K, δ) over (A, I, γ).

Proof. Denote $\mathcal{C}$ the category of divided power rings (C, K, δ). Consider the functor $F : \mathcal{C} \longrightarrow Sets$ defined by

$$F(C, K, \delta) = \left\{ (\varphi, \psi) \middle| \begin{array}{l} \varphi : (A, I, \gamma) \to (C, K, \delta) \text{ homomorphism of divided power rings} \\ \psi : (B, J) \to (C, K) \text{ an } A\text{-algebra homomorphism with } \psi(J) \subset K \end{array} \right\}$$

We will show that Lemma 39.3.3 applies to this functor which will prove the lemma. Suppose that $(\varphi, \psi) \in F(C, K, \delta)$. Let $C' \subset C$ be the subring generated by $\varphi(A)$, $\psi(B)$, and $\delta_n(\psi(f))$ for all $f \in J$. Let $K' \subset K \cap C'$ be the ideal of C' generated by $\varphi(I)$ and $\delta_n(\psi(f))$ for $f \in J$. Then $(C', K', \delta|_{K'})$ is a divided power ring and C' has cardinality bounded by the cardinal $\kappa = |A| \otimes |B|^{\aleph_0}$. Moreover, φ factors as $A \to C' \to C$ and ψ factors as $B \to B' \to B$. This proves assumption (1) of Lemma 39.3.3 holds. Assumption (2) is clear as limits in the category of divided power rings commute with the forgetful functor $(C, K, \delta) \mapsto (C, K)$, see Lemma 39.3.2 and its proof. $\square$

Definition 39.6.2. Let (A, I, γ) be a divided power ring. Let $A \to B$ be a ring map. Let $J \subset B$ be an ideal with $IB \subset J$. The divided power algebra $(D, \bar{J}, \bar{\gamma})$ constructed in Lemma 39.6.1 is called the *divided power envelope of J in B relative to* (A, I, γ) and is denoted $D_B(J)$ or $D_{B,\gamma}(J)$.

Let $(A, I, \gamma) \to (C, K, \delta)$ be a homomorphism of divided power rings. The universal property of $D_{B,\gamma}(J) = (D, \bar{J}, \bar{\gamma})$ is

$$\begin{matrix}\text{ring maps } B \to C \\ \text{which map } J \text{ into } K\end{matrix} \longleftrightarrow \begin{matrix}\text{divided power homomorphisms} \\ (D, \bar{J}, \bar{\gamma}) \to (C, K, \delta)\end{matrix}$$

and the correspondence is given by precomposing with the map $B \to D$ which corresponds to id_D. Here are some properties of $(D, \bar{J}, \bar{\gamma})$ which follow directly from the universal property. There are A-algebra maps

$$B \longrightarrow D \longrightarrow B/J \tag{39.6.2.1}$$

The first arrow maps J into $\bar{J}$ and $\bar{J}$ is the kernel of the second arrow. The elements $\bar{\gamma}_n(x)$ where $n > 0$ and x is an element in the image of $J \to D$ generate $\bar{J}$ as an ideal in D and generate D as a B-algebra.

Lemma 39.6.3. *Let (A, I, γ) be a divided power ring. Let $\varphi : B' \to B$ be a surjection of A-algebras with kernel K. Let $IB \subset J \subset B$ be an ideal. Let $J' \subset B'$ be the inverse image of J. Write $D_{B',\gamma}(J') = (D', \bar{J}', \bar{\gamma})$. Then $D_{B,\gamma}(J) = (D'/K', \bar{J}'/K', \bar{\gamma})$ where K' is the ideal generated by the elements $\bar{\gamma}_n(k)$ for $n \geq 1$ and $k \in K$.*

Proof. Write $D_{B,\gamma}(J) = (D, \bar{J}, \bar{\gamma})$. The universal property of D' gives us a homomorphism $D' \to D$ of divided power algebras. As $B' \to B$ and $J' \to J$ are surjective, we see that $D' \to D$ is surjective (see remarks above). It is clear that $\bar{\gamma}_n(k)$ is in the kernel for $n \geq 1$ and $k \in K$, i.e., we obtain a homomorphism $D'/K' \to D$. Conversely, there exists a divided power structure on $\bar{J}'/K' \subset D'/K'$, see Lemma 39.4.3. Hence the universal property of D gives an inverse $D \to D'/K'$ and we win. $\square$

In the situation of Definition 39.6.2 we can choose a surjection $P \to B$ where P is a polynomial algebra over A and let $J' \subset P$ be the inverse image of J. The previous lemma describes $D_{B,\gamma}(J)$ in terms of $D_{P,\gamma}(J')$. Note that γ extends to a divided power structure γ' on IP by Lemma 39.4.2. Hence $D_{P,\gamma}(J') = D_{P,\gamma'}(J')$ is an example of a special case of divided power envelopes we describe in the following lemma.

Lemma 39.6.4. *Let (B, I, γ) be a divided power algebra. Let $I \subset J \subset B$ be an ideal. Let $(D, \bar{J}, \bar{\gamma})$ be the divided power envelope of J relative to γ. Choose elements $f_t \in J$, $t \in T$ such that $J = I + (f_t)$. Then there exists a surjection*

$$\Psi : B\langle x_t \rangle \longrightarrow D$$

of divided power rings mapping x_t to the image of f_t in D. The kernel of Ψ is generated by the elements $x_t - f_t$ and all

$$\delta_n\left(\sum r_t x_t - r_0\right)$$

whenever $\sum r_t f_t = r_0$ in B for some $r_t \in B$, $r_0 \in I$.

Proof. In the statement of the lemma we think of $B\langle x_t \rangle$ as a divided power ring with ideal $J' = IB\langle x_t \rangle + B\langle x_t \rangle_+$, see Remark 39.5.2. The existence of Ψ follows from the universal property of divided power polynomial rings. Surjectivity of Ψ follows from the fact that its image is a divided power subring of D, hence equal to D by the universal property of D. It is clear that $x_t - f_t$ is in the kernel. Set

$$\mathcal{R} = \{(r_0, r_t) \in I \oplus \bigoplus\nolimits_{t \in T} B \mid \sum r_t f_t = r_0 \text{ in } B\}$$

If $(r_0, r_t) \in \mathcal{R}$ then it is clear that $\sum r_t x_t - r_0$ is in the kernel. As Ψ is a homomorphism of divided power rings and $\sum r_t x_t = r_0 \in J'$ it follows that $\delta_n(\sum r_t x_t - r_0)$ is in the kernel as well. Let $K \subset B\langle x_t \rangle$ be the ideal generated by $x_t - f_t$ and the elements $\delta_n(\sum r_t x_t - r_0)$ for $(r_0, r_t) \in \mathcal{R}$. To show that $K = \text{Ker}(\Psi)$ it suffices to show that δ extends to $B\langle x_t \rangle / K$. Namely, if so the universal property of D gives a map $D \to B\langle x_t \rangle / K$ inverse to Ψ. Hence we have to show that $K \cap J'$ is preserved by δ_n, see Lemma 39.4.3. Let $K' \subset B\langle x_t \rangle$ be the ideal generated by the elements

(1) $\delta_m(\sum r_t x_t - r_0)$ where $m > 0$ and $(r_0, r_t) \in \mathcal{R}$,
(2) $x_{t'}^{[m]}(x_t - f_t)$ where $m > 0$ and $t', t \in I$.

We claim that $K' = K \cap J'$. The claim proves that $K \cap J'$ is preserved by δ_n, $n > 0$ by the criterion of Lemma 39.4.3 (2)(c) and a computation of δ_n of the elements listed which we leave to the reader. To prove the claim note that $K' \subset K \cap J'$. Conversely, if $h \in K \cap J'$ then, modulo K' we can write

$$h = \sum r_t(x_t - f_t)$$

for some $r_t \in B$. As $h \in K \cap J' \subset J'$ we see that $r_0 = \sum r_t f_t \in I$. Hence $(r_0, r_t) \in \mathcal{R}$ and we see that

$$h = \sum r_t x_t - r_0$$

is in K' as desired. □

Lemma 39.6.5. *Let (A, I, γ) be a divided power ring. Let B be an A-algebra and $IB \subset J \subset B$ an ideal. Let x_i be a set of variables. Then*

$$D_{B[x_i],\gamma}(JB[x_i] + (x_i)) = D_{B,\gamma}(J)\langle x_i \rangle$$

Proof. One possible proof is to deduce this from Lemma 39.6.4 as any relation between x_i in $B[x_i]$ is trivial. On the other hand, the lemma follows from the universal property of the divided power polynomial algebra and the universal property of divided power envelopes. □

Conditions (1) and (2) of the following lemma hold if $B \to B'$ is flat at all primes of $V(IB') \subset Spec(B')$ and is very closely related to that condition, see Algebra, Lemma 7.91.8. It in particular says that taking the divided power envelope commutes with localization.

Lemma 39.6.6. *Let (A, I, γ) be a divided power ring. Let $B \to B'$ be a homomorphism of A-algebras. Assume that*

(1) *$B/IB \to B'/IB'$ is flat, and*
(2) *$Tor_1^B(B', B/IB) = 0$.*

Then for any ideal $IB \subset J \subset B$ the canonical map

$$D_B(J) \otimes_B B' \longrightarrow D_{B'}(JB')$$

is an isomorphism.

Proof. Set $D = D_B(J)$ and denote $\bar{J} \subset D$ its divided power ideal with divided power structure $\bar{\gamma}$. The universal property of D produces a B-algebra map $D \to D_{B'}(JB')$, whence a map as in the lemma. It suffices to show that the divided powers $\bar{\gamma}$ extend to $D \otimes_B B'$ since then the universal property of $D_{B'}(JB')$ will produce a map $D_{B'}(JB') \to D \otimes_B B'$ inverse to the one in the lemma.

Choose a surjection $P \to B'$ where P is a polynomial algebra over B. In particular $B \to P$ is flat, hence $D \to D \otimes_B P$ is flat by Algebra, Lemma 7.35.6. Then $\bar{\gamma}$ extends to $D \otimes_B P$ by Lemma 39.4.2; we will denote this extension $\bar{\gamma}$ also. Set $\mathfrak{a} = \mathrm{Ker}(P \to B')$ so that we have the short exact sequence

$$0 \to \mathfrak{a} \to P \to B' \to 0$$

Thus $\mathrm{Tor}_1^B(B', B/IB) = 0$ implies that $\mathfrak{a} \cap IP = I\mathfrak{a}$. Now we have the following commutative diagram

$$\begin{tikzcd}
B/J \otimes_B \mathfrak{a} \arrow[r, "\beta"'] & B/J \otimes_B P \arrow[r] & B/J \otimes_B B' \\
D \otimes_B \mathfrak{a} \arrow[r, "\alpha"] \arrow[u] & D \otimes_B P \arrow[r] \arrow[u] & D \otimes_B B' \arrow[u] \\
\bar{J} \otimes_B \mathfrak{a} \arrow[r] \arrow[u] & \bar{J} \otimes_B P \arrow[r] \arrow[u] & \bar{J} \otimes_B B' \arrow[u]
\end{tikzcd}$$

This diagram is exact even with 0's added at the top and the right. We have to show the divided powers on the ideal $\bar{J} \otimes_B P$ preserve the ideal $\mathrm{Im}(\alpha) \cap \bar{J} \otimes_B P$, see Lemma 39.4.3. Consider the exact sequence

$$0 \to \mathfrak{a}/I\mathfrak{a} \to P/IP \to B'/IB' \to 0$$

(which uses that $\mathfrak{a} \cap IP = I\mathfrak{a}$ as seen above). As B'/IB' is flat over B/IB this sequence remains exact after applying $B/J \otimes_{B/IB} -$, see Algebra, Lemma 7.35.11. Hence

$$\mathrm{Ker}(B/J \otimes_{B/IB} \mathfrak{a}/I\mathfrak{a} \to B/J \otimes_{B/IB} P/IP) = \mathrm{Ker}(\mathfrak{a}/J\mathfrak{a} \to P/JP)$$

is zero. Thus β is injective. It follows that $\mathrm{Im}(\alpha) \cap \bar{J} \otimes_B P$ is the image of $\bar{J} \otimes \mathfrak{a}$. Now if $f \in \bar{J}$ and $a \in \mathfrak{a}$, then $\bar{\gamma}_n(f \otimes a) = \bar{\gamma}_n(f) \otimes a^n$ hence the result is clear. □

The following lemma is a special case of [**dJ95**, Proposition 2.1.7] which in turn is a generalization of [**Ber74**, Proposition 2.8.2].

Lemma 39.6.7. *Let $(B, I, \gamma) \to (B', I', \gamma')$ be a homomorphism of divided power rings. Let $I \subset J \subset B$ and $I' \subset J' \subset B'$ be ideals. Assume*

(1) *$B/I \to B'/I'$ is flat, and*
(2) *$J' = JB' + I'$.*

Then the canonical map

$$D_{B,\gamma}(J) \otimes_B B' \longrightarrow D_{B',\gamma'}(J')$$

is an isomorphism.

Proof. Set $D = D_B(J)$ and denote $\bar{J} \subset D$ its divided power ideal with divided power structure $\bar{\gamma}$. The universal property of D produces a homomorphism of divided power rings $D \to D_{B'}(J')$, whence a map as in the lemma. It suffices to show that there exist divided powers on the image of $D \otimes_B I' + \bar{J} \otimes_B B' \to D \otimes_B B'$ compatible with $\bar{\gamma}$ and γ' since then the universal property of $D_{B'}(J')$ will produce a map $D_{B'}(J') \to D \otimes_B B'$ inverse to the one in the lemma.

Choose elements $f_t \in J$ which generate J/I. Set $\mathcal{R} = \{(r_0, r_t) \in I \oplus \bigoplus_{t\in T} B \mid \sum r_t f_t = r_0 \text{ in } B\}$ as in the proof of Lemma 39.6.4. This lemma shows that

$$D = B\langle x_t \rangle / K$$

where K is generated by the elements $x_t - f_t$ and $\delta_n(\sum r_t x_t - r_0)$ for $(r_0, r_t) \in \mathcal{R}$. Thus we see that

$$D \otimes_B B' = B'\langle x_t \rangle / K' \tag{39.6.7.1}$$

where K' is generated by the images in $B'\langle x_t \rangle$ of the generators of K listed above. Let $f'_t \in B'$ be the image of f_t. By assumption (1) we see that the elements $f'_t \in J'$ generate J'/I' and we see that $x_t - f'_t \in K'$. Set

$$\mathcal{R}' = \{(r'_0, r'_t) \in I' \oplus \bigoplus\nolimits_{t\in T} B' \mid \sum r'_t f'_t = r'_0 \text{ in } B'\}$$

To finish the proof we have to show that $\delta'_n(\sum r'_t x_t - r'_0) \in K'$ for $(r'_0, r'_t) \in \mathcal{R}'$, because then the presentation (39.6.7.1) of $D \otimes_B B'$ is identical to the presentation of $D_{B',\gamma'}(J')$ obtain in Lemma 39.6.4 from the generators f'_t. Suppose that $(r'_0, r'_t) \in \mathcal{R}'$. Then $\sum r'_t f'_t = 0$ in B'/I'. As $B/I \to B'/I'$ is flat by assumption (1) we can apply the equational criterion of flatness (Algebra, Lemma 7.35.10) to see that there exist an $m > 0$ and $r_{jt} \in B$ and $c_j \in B'$, $j = 1, \ldots, m$ such that

$$r_{j0} = \sum r_{jt} f_t \in I \text{ for } j = 1, \ldots, m, \quad \text{and} \quad r'_t = \sum c_j r_{jt}.$$

Note that this also implies that $r'_0 = \sum c_j r_{j0}$. Then we have

$$\begin{aligned} \delta'_n(\sum r'_t x_t - r'_0) &= \delta'_n(\sum c_j(\sum r_{jt} x_t - r_{j0})) \\ &= \sum c_1^{n_1} \ldots c_m^{n_m} \delta_{n_1}(\sum r_{1t} x_t - r_{10}) \ldots \delta_{n_m}(\sum r_{mt} x_t - r_{m0}) \end{aligned}$$

where the sum is over $n_1 + \ldots + n_m = n$. This proves what we want. ☐

39.7. Some explicit divided power thickenings

The constructions in this section will help us to define the connection on a crystal in modules on the crystalline site.

Lemma 39.7.1. *Let (A, I, γ) be a divided power ring. Let M be an A-module. Let $B = A \oplus M$ as an A-algebra where M is an ideal of square zero and set $J = I \oplus M$. Set*

$$\delta_n(x + z) = \gamma_n(x) + \gamma_{n-1}(x)z$$

for $x \in I$ and $z \in M$. Then δ is a divided power structure and $A \to B$ is a homomorphism of divided power rings from (A, I, γ) to (B, J, δ).

Proof. We have to check conditions (1) -- (5) of Definition 39.2.1. We will prove this directly for this case, but please see the proof of the next lemma for a method which avoids calculations. Conditions (1) and (3) are clear. Condition (2) follows from

$$\begin{aligned} \delta_n(x + z)\delta_m(x + z) &= (\gamma_n(x) + \gamma_{n-1}(x)z)(\gamma_m(x) + \gamma_{m-1}(x)z) \\ &= \gamma_n(x)\gamma_m(x) + \gamma_n(x)\gamma_{m-1}(x)z + \gamma_{n-1}(x)\gamma_m(x)z \\ &= \frac{(n+m)!}{n!\,m!}\gamma_{n+m}(x) + \left(\frac{(n+m-1)!}{n!\,(m-1)!} + \frac{(n+m-1)!}{(n-1)!\,m!}\right)\gamma_{n+m-1}(x)z \\ &= \frac{(n+m)!}{n!\,m!}\delta_{n+m}(x + z) \end{aligned}$$

Condition (5) follows from

$$\begin{aligned} \delta_n(\delta_m(x+z)) &= \delta_n(\gamma_m(x) + \gamma_{m-1}(x)z) \\ &= \gamma_n(\gamma_m(x)) + \gamma_{n-1}(\gamma_m(x))\gamma_{m-1}(x)z \\ &= \frac{(nm)!}{n!\,(m!\,)^n}\gamma_{nm}(x) + \frac{((n-1)m)!}{(n-1)!\,(m!\,)^{n-1}}\gamma_{(n-1)m}(x)\gamma_{m-1}(x)z \\ &= \frac{(nm)!}{n!\,(m!\,)^n}(\gamma_{nm}(x) + \gamma_{nm-1}(x)z) \end{aligned}$$

by elementary number theory. To prove (4) we have to see that

$$\delta_n(x + x' + z + z') = \gamma_n(x + x') + \gamma_{n-1}(x + x')(z + z')$$

is equal to

$$\sum\nolimits_{i=0}^{n} (\gamma_i(x) + \gamma_{i-1}(x)z)(\gamma_{n-i}(x') + \gamma_{n-i-1}(x')z')$$

This follows easily on collecting the coefficients of 1, z, and z' and using condition (4) for γ. ☐

Lemma 39.7.2. *Let (A, I, γ) be a divided power ring. Let M, N be A-modules. Let $q : M \times M \to N$ be an A-bilinear map. Let $B = A \oplus M \oplus N$ as an A-algebra with multiplication*

$$(x, z, w) \cdot (x', z', w') = (xx', xz' + x'z, xw' + x'w + q(z, z') + q(z', z))$$

and set $J = I \oplus M \oplus N$. Set

$$\delta_n(x, z, w) = (\gamma_n(x), \gamma_{n-1}(x)z, \gamma_{n-1}(z)w + \gamma_{n-2}(x)q(z, z))$$

for $(a, m, n) \in J$. Then δ is a divided power structure and $A \to B$ is a homomorphism of divided power rings from (A, I, γ) to (B, J, δ).

Proof. Suppose we want to prove that property (4) of Definition 39.2.1 is satisfied. Pick (x, z, w) and (x', z', w') in J. Pick a map

$$A_0 = \mathbf{Z}\langle s, s' \rangle \longrightarrow A, \quad s \longmapsto x, s' \longmapsto x'$$

which is possible by the universal property of divided power polynomial rings. Set $M_0 = A_0 \oplus A_0$ and $N_0 = A_0 \oplus A_0 \oplus M_0 \otimes_{A_0} M_0$. Let $q_0 : M_0 \times M_0 \to N_0$ be the obvious map. Define $M_0 \to M$ as the A_0-linear map which sends the basis vectors of M_0 to z and z'. Define $N_0 \to N$ as the A_0 linear map which sends the first two basis vectors of N_0 to w and w' and uses $M_0 \otimes_{A_0} M_0 \to M \otimes_A M \xrightarrow{q} N$ on the last summand. Then we see that it suffices to prove the identitity (4) for the situation (A_0, M_0, N_0, q_0). Similarly for the other identities. This reduces us to the case of a $\mathbf{Z}$-torsion free ring and $\mathbf{A}$-torsion free modules. In this case all we have to do is show that

$$n!\,\delta_n(x, z, w) = (x, z, w)^n$$

in the ring A, see Lemma 39.2.2. To see this note that

$$(x, z, w)^2 = (x^2, 2xz, 2xw + 2q(z, z))$$

and by induction

$$(x, z, w)^n = (x^n, nx^{n-1}z, nx^{n-1}w + n(n-1)x^{n-2}q(z, z))$$

On the other hand,

$$n!\,\delta_n(x, z, w) = (n!\,\gamma_n(x), n!\,\gamma_{n-1}(x)z, n!\,\gamma_{n-1}(x)w + n!\,\gamma_{n-2}(x)q(z, z))$$

which matches. This finishes the proof. ☐

39.8. Compatibility

This section isn't required reading; it explains how our discussion fits with that of [**Ber74**]. Consider the following technical notion.

Definition 39.8.1. Let (A, I, γ) and (B, J, δ) be divided power rings. Let $A \to B$ be a ring map. We say δ *is compatible with* γ if there exists a divided power structure $\bar{\gamma}$ on $J + IB$ such that

$$(A, I, \gamma) \to (B, J + IB, \bar{\gamma}) \quad \text{and} \quad (B, J, \delta) \to (B, J + IB, \bar{\gamma})$$

are homomorphisms of divided power rings.

Let p be a prime number. Let (A, I, γ) be a divided power ring. Let $A \to C$ be a ring map with p nilpotent in C. Assume that γ extends to IC (see Lemma 39.4.2). In this situation, the (big affine) crystalline site of $Spec(C)$ over $Spec(A)$ as defined in [**Ber74**] is the opposite of the category of systems

$$(B, J, \delta, A \to B, C \to B/J)$$

where

1. (B, J, δ) is a divided power ring with p nilpotent in B,
2. δ is compatible with γ, and
3. the diagram
$$\begin{tikzcd} B \arrow[r] & B/J \\ A \arrow[u] \arrow[r] & C \arrow[u] \end{tikzcd}$$
is commutative.

The conditions ``γ extends to C and δ compatible with γ" are used in [**Ber74**] to insure that the crystalline cohomology of $Spec(C)$ is the same as the crystalline cohomology of $Spec(C/IC)$. We will avoid this issue by working exclusively with C such that $IC = 0$[1]. In this case, for a system $(B, J, \delta, A \to B, C \to B/J)$ as above, the commutativity of the displayed diagram above implies $IB \subset J$ and compatibility is equivalent to the condition that $(A, I, \gamma) \to (B, J, \delta)$ is a homomorphism of divided power rings.

39.9. Affine crystalline site

In this section we discuss the algebraic variant of the crystalline site. Our basic situation in which we discuss this material will be as follows.

Situation 39.9.1. Here p is a prime number, (A, I, γ) is a divided power ring such that A is a $\mathbf{Z}_{(p)}$-algebra, and $A \to C$ is a ring map such that $IC = 0$ and such that p is nilpotent in C.

Usually the prime number p will be contained in the divided power ideal I.

Definition 39.9.2. In Situation 39.9.1.

1. A *divided power thickening* of C over (A, I, γ) is a homomorphism of divided power algebras $(A, I, \gamma) \to (B, J, \delta)$ such that p is nilpotent in B and a ring map

[1] Of course there will be a price to pay.

$C \to B/J$ such that

$$\begin{tikzcd} B \arrow[r] & B/J \\ & C \arrow[u] \\ A \arrow[uu] \arrow[r] & A/I \arrow[u] \end{tikzcd}$$

is commutative.

(2) A *homomorphism of divided power thickenings*

$$(B, J, \delta, C \to B/J) \longrightarrow (B', J', \delta', C \to B'/J')$$

is a homomorphism $\varphi : B \to B'$ of divided power A-algebras such that $C \to B/J \to B'/J'$ is the given map $C \to B'/J'$.

(3) We denote $\text{CRIS}(C/A, I, \gamma)$ or simply $\text{CRIS}(C/A)$ the category of divided power thickenings of C over (A, I, γ).

(4) We denote $\text{Cris}(C/A, I, \gamma)$ or simply $\text{Cris}(C/A)$ the full subcategory consisting of $(B, J, \delta, C \to B/J)$ such that $C \to B/J$ is an isomorphism. We often denote such an object $(B \to C, \delta)$ with $J = \text{Ker}(B \to C)$ being understood.

Note that for a divided power thickening (B, J, δ) as above the ideal J is locally nilpotent, see Lemma 39.2.6. There is a canonical functor

$$\text{CRIS}(C/A) \longrightarrow C\text{-algebras}, \quad (B, J, \delta) \longmapsto B/J \tag{39.9.2.1}$$

This category does not have equalizers or fibre products in general. It also doesn't have an initial object (= empty colimit) in general.

Lemma 39.9.3. *In Situation 39.9.1.*

(1) *$CRIS(C/A)$ has products,*
(2) *$CRIS(C/A)$ has all finite nonempty colimits and (39.9.2.1) commutes with these, and*
(3) *$Cris(C/A)$ has all finite nonempty colimits and $Cris(C/A) \to CRIS(C/A)$ commutes with them.*

Proof. The empty product is $(C, 0, \emptyset)$. If (B_t, J_t, δ_t) is a family of objects of $\text{CRIS}(C/A)$ then we can form the product $(\prod B_t, \prod J_t, \prod \delta_t)$ as in Lemma 39.3.4. The map $C \to \prod B_t/\prod J_t = \prod B_t/J_t$ is clear.

Given two objects (B, J, γ) and (B', J', γ') of $\text{CRIS}(C/A)$ we can form a cocartesian diagram

$$\begin{tikzcd} (B, J, \delta) \arrow[r] & (B'', J'', \delta'') \\ (A, I, \gamma) \arrow[u] \arrow[r] & (B', J', \delta') \arrow[u] \end{tikzcd}$$

in the category of divided power rings. Then we see that we have

$$B''/J'' = B/J \otimes_{A/I} B'/J' \longleftarrow C \otimes_{A/I} C$$

see Remark 39.3.5. Denote $J'' \subset K \subset B''$ the ideal such that

$$\begin{tikzcd} B''/J'' \arrow[r] & B''/K \\ C \otimes_{A/I} C \arrow[u] \arrow[r] & C \arrow[u] \end{tikzcd}$$

is a pushout, i.e., $B''/K \cong B/J \otimes_C B'/J'$. Let $D_{B''}(K) = (D, \bar{K}, \bar{\delta})$ be the divided power envelope of K in B'' relative to (B'', J'', δ''). Then it is easily verified that $(D, \bar{K}, \bar{\delta})$ is a coproduct of (B, J, δ) and (B', J', δ') in CRIS(C/A).

Next, we come to coequalizers. Let $\alpha, \beta : (B, J, \delta) \to (B', J', \delta')$ be morphisms of CRIS(C/A). Consider $B'' = B'/(\alpha(b) - \beta(b))$. Let $J'' \subset B''$ be the image of J'. Let $D_{B''}(J'') = (D, \bar{J}, \bar{\delta})$ be the divided power envelope of J'' in B'' relative to (B', J', δ'). Then it is easily verified that $(D, \bar{J}, \bar{\delta})$ is the coequalizer of (B, J, δ) and (B', J', δ') in CRIS(C/A).

By Categories, Lemma 4.16.6 we have all finite nonempty colimits in CRIS(C/A). The constructions above shows that (39.9.2.1) commutes with them. This formally implies part (3) as Cris(C/A) is the fibre category of (39.9.2.1) over C. ☐

Remark 39.9.4. In Situation 39.9.1 we denote Cris^(C/A) the category whose objects are pairs $(B \to C, \delta)$ such that

(1) B is a p-adically complete A-algebra,
(2) $B \to C$ is a surjection of A-algebras,
(3) δ is a divided power structure on Ker$(B \to C)$,
(4) $A \to B$ is a homomorphism of divided power rings.

Morphisms are defined as in Definition 39.9.2. Then Cris$(C/A) \subset$ Cris^(C/A) is the full subcategory consisting of those B such that p is nilpotent in B. Conversely, any object $(B \to C, \delta)$ of Cris^(C/A) is equal to the limit

$$(B \to C, \delta) = lim_e (B/p^e B \to C, \delta)$$

where for $e \gg 0$ the object $(B/p^e B \to C, \delta)$ lies in Cris(C/A), see Lemma 39.4.5. In particular, we see that Cris^(C/A) is a full subcategory of the category of pro-objects of Cris(C/A), see Categories, Remark 4.20.4.

Lemma 39.9.5. *In Situation 39.9.1. Let $P \to C$ be a surjection of A-algebras with kernel J. Write $D_{P,\gamma}(J) = (D, \bar{J}, \bar{\gamma})$. Let $(D^\wedge, J^\wedge, \bar{\gamma}^\wedge)$ be the p-adic completion of D, see Lemma 39.4.5. For every $e \geq 1$ set $P_e = P/p^e P$ and $J_e \subset P_e$ the image of J and write $D_{P_e,\gamma}(J_e) = (D_e, \bar{J}_e, \bar{\gamma})$. Then for all e large enough we have*

(1) *$p^e D \subset \bar{J}$ and $p^e D^\wedge \subset \bar{J}^\wedge$ are preserved by divided powers,*
(2) *$D^\wedge/p^e D^\wedge = D/p^e D = D_e$ as divided power rings,*
(3) *$(D_e, \bar{J}_e, \bar{\gamma})$ is an object of Cris(C/A),*
(4) *$(D^\wedge, \bar{J}^\wedge, \bar{\gamma}^\wedge)$ is equal to $lim_e (D_e, \bar{J}_e, \bar{\gamma})$, and*
(5) *$(D^\wedge, \bar{J}^\wedge, \bar{\gamma}^\wedge)$ is an object of Cris^(C/A).*

Proof. Part (1) follows from Lemma 39.4.5. It is a general property of p-adic completion that $D/p^e D = D^\wedge/p^e D^\wedge$. Since $D/p^e D$ is a divided power ring and since $P \to D/p^e D$ factors through P_e, the universal property of D_e produces a map $D_e \to D/p^e D$. Conversely, the universal property of D produces a map $D \to D_e$ which factors through $D/p^e D$. We omit the verification that these maps are mutually inverse. This proves (2). If e is large enough, then $p^e C = 0$, hence we see (3) holds. Part (4) follows from Lemma 39.4.5. Part (5) is clear from the definitions. ☐

Lemma 39.9.6. *In Situation 39.9.1. Let P be a polynomial algebra over A and let $P \to C$ be a surjection of A-algebras with kernel J. With $(D_e, \bar{J}_e, \bar{\gamma})$ as in Lemma 39.9.5: for every object (B, J_B, δ) of $CRIS(C/A)$ there exists an e and a morphsm $D_e \to B$ of $CRIS(C/A)$.*

Proof. We can find an A-algebra homomorphism $P \to B$ lifting the map $C \to B/J_B$. By our definition of CRIS(C/A) we see that $p^e B = 0$ for some e hence $P \to B$ factors as $P \to P_e \to B$. By the universal property of the divided power envelope we conclude that $P_e \to B$ factors through D_e. □

Lemma 39.9.7. *In Situation 39.9.1. Let P be a polynomial algebra over A and let $P \to C$ be a surjection of A-algebras with kernel J. Let $(D, \bar{J}, \bar{\gamma})$ be the p-adic completion of $D_{P,\gamma}(J)$. For every object $(B \to C, \delta)$ of $Cris^\wedge(C/A)$ there exists a morphsm $D \to B$ of $Cris^\wedge(C/A)$.*

Proof. We can find an A-algebra homomorphism $P \to B$ compatible with maps to C. By our definition of Cris(C/A) we see that $P \to B$ factors as $P \to D_{P,\gamma}(J) \to B$. As B is p-adically complete we can factor this map through D. □

39.10. Module of differentials

In this section we develop a theory of modules of differentials for divided power rings.

Definition 39.10.1. Let A be a ring. Let (B, J, δ) be a divided power ring. Let $A \to B$ be a ring map. Let M be an B-module. A *divided power A-derivation* into M is a map $\theta : B \to M$ which is additive, annihilates the elements of A, satisfies the Leibniz rule $\theta(bb') = b\theta(b') + b'\theta(b)$ and satisfies

$$\theta(\gamma_n(x)) = \gamma_{n-1}(x)\theta(x)$$

for all $n \geq 1$ and all $x \in J$.

In the situation of the definition, just as in the case of usual derivations, there exists a *universal divided power A-derivation*

$$\mathrm{d}_{B/A,\delta} : B \to \Omega_{B/A,\delta}$$

such that any divided power A-derivation $\theta : B \to M$ is equal to $\theta = \xi \circ d_{B/A,\delta}$ for some B-linear map $\Omega_{B/A,\delta} \to M$. If $(A, I, \gamma) \to (B, J, \delta)$ is a homomorphism of divided power rings, then we can forget the divided powers on A and consider the divided power derivations of B over A. Here are some basic properties of the divided power module of differentials.

Lemma 39.10.2. *Let A be a ring. Let (B, J, δ) be a divided power ring and $A \to B$ a ring map.*

1. *Consider $B[x]$ with divided power ideal $(JB[x], \delta')$ where δ' is the extension of δ to $B[x]$. Then*
$$\Omega_{B[x]/A,\delta'} = \Omega_{B/A,\delta} \otimes_B B[x] \oplus B[x]dx.$$
2. *Consider $B\langle x\rangle$ with divided power ideal $(JB\langle x\rangle + B\langle x\rangle_+, \delta')$. Then*
$$\Omega_{B\langle x\rangle/A,\delta'} = \Omega_{B/A,\delta} \otimes_B B\langle x\rangle \oplus B\langle x\rangle dx.$$
3. *Let $K \subset J$ be an ideal preserved by δ_n for all $n > 0$. Set $B' = B/K$ and denote δ' the induced divided power on J/K. Then $\Omega_{B'/A,\delta'}$ is the quotient of $\Omega_{B/A,\delta} \otimes_B B'$ by the B'-submodule generated by dk for $k \in K$.*

Proof. These are proved directly from the construction of $\Omega_{B/A,\delta}$ as the free B-module on the elements $\mathrm{d}b$ modulo the relations

(1) $\mathrm{d}(b + b') = \mathrm{d}b + \mathrm{d}b'$, $b, b' \in B$,
(2) $\mathrm{d}a = 0$, $a \in A$,
(3) $\mathrm{d}(bb') = b\mathrm{d}b' + b'\mathrm{d}b$, $b, b' \in B$,
(4) $\mathrm{d}\delta_n(f) = \delta_{n-1}(f)\mathrm{d}f$, $f \in J$, $n > 1$.

Note that the last relation explains why we get ``the same" answer for the divided power polynomial algebra and the usual polynomial algebra: in the first case x is an element of the divided power ideal and hence $\mathrm{d}x^{[n]} = x^{[n-1]}\mathrm{d}x$. □

Let (A, I, γ) be a divided power ring. In this setting the correct version of the powers of I is given by the divided powers

$$I^{[n]} = \text{ideal generate by } \gamma_{e_1}(x_1) \ldots \gamma_{e_t}(x_t) \text{ with } \sum e_j \geq n \text{ and } x_j \in I.$$

Of course we have $I^n \subset I^{[n]}$. Note that $I^{[1]} = I$. Sometimes we also set $I^{[0]} = A$.

Lemma 39.10.3. *Let $(A, I, \gamma) \to (B, J, \delta)$ be a homomorphism of divided power rings. Let $(B(1), J(1), \delta(1))$ be the coproduct of (B, J, δ) with itself over (A, I, γ), i.e., such that*

$$\begin{tikzcd}
(B, J, \delta) \arrow[r] & (B(1), J(1), \delta(1)) \\
(A, I, \gamma) \arrow[u] \arrow[r] & (B, J, \delta) \arrow[u]
\end{tikzcd}$$

is cocartesian. Denote $K = Ker(B(1) \to B)$. Then $K \cap J(1) \subset J(1)$ is preserved by the divided power structure and

$$\Omega_{B/A,\delta} = K/\left(K^2 + (K \cap J(1))^{[2]}\right)$$

canonically.

Proof. The fact that $K \cap J(1) \subset J(1)$ is preserved by the divided power structure follows from the fact that $B(1) \to B$ is a homomorphism of divided power rings.

Recall that K/K^2 has a canonical B-module structure. Denote $s_0, s_1 : B \to B(1)$ the two coprojections and consider the map $\mathrm{d} : B \to K/K^2 + (K \cap J(1))^{[2]}$ given by $b \mapsto s_1(b) - s_0(b)$. It is clear that d is additive, annihilates A, and satisfies the Leibniz rule. We claim that d is an A-derivation. Let $x \in J$. Set $y = s_1(x)$ and $z = s_0(x)$. Denote δ the divided power structure on $J(1)$. We have to show that $\delta_n(y) - \delta_n(z) = \delta_{n-1}(y)(y - z)$ modulo $K^2 + (K \cap J(1))^{[2]}$ for $n \geq 1$. We will show this by induction on n. It is true for $n = 1$. Let $n > 1$ and that it holds for all smaller values. Note that

$$\delta_n(z - y) = \sum\nolimits_{i=0}^{n} (-1)^{n-i}\delta_i(z)\delta_{n-i}(y)$$

is an element of $K^2 + (K \cap J(1))^{[2]}$. From this and induction we see that working modulo $K^2 + (K \cap J(1))^{[2]}$ we have

$$\begin{aligned}
&\delta_n(y) - \delta_n(z) \\
&= \delta_n(y) + \sum\nolimits_{i=0}^{n-1} (-1)^{n-i}\delta_i(z)\delta_{n-i}(y) \\
&= \delta_n(y) + (-1)^n\delta_n(y) + \sum\nolimits_{i=1}^{n-1} (-1)^{n-i}(\delta_i(y) - \delta_{i-1}(y)(y - z))\delta_{n-i}(y)
\end{aligned}$$

Using that $\delta_i(y)\delta_{n-i}(y) = \binom{n}{i}\delta_n(y)$ and that $\delta_{i-1}(y)\delta_{n-i}(y) = \binom{n-1}{i}\delta_{n-1}(y)$ the reader easily verifies that this expression comes out to give $\delta_{n-1}(y)(y - z)$ as desired.

Let M be a B-module. Let $\theta : B \to M$ be a divided power A-derivation. Set $D = B \oplus M$ where M is an ideal of square zero. Define a divided power structure on $J \oplus M \subset D$ by setting $\delta_n(x + m) = \delta_n(x) + \delta_{n-1}(x)m$ for $n > 1$, see Lemma 39.7.1. There are two divided power algebra homomorphisms $B \to D$: the first is given by the inclusion and the second by the map $b \mapsto b + \theta(b)$. Hence we get a canonical homomorphism $B(1) \to D$ of divided power algebras over (A, I, γ). This induces a map $K \to M$ which annihilates K^2 (as M is an ideal of square zero) and $(K \cap J(1))^{[2]}$ as $M^{[2]} = 0$. The composition $B \to K/K^2 + (K \cap J(1))^{[2]} \to M$ equals θ by construction. It follows that d is a universal divided power A-derivation and we win. □

Remark 39.10.4. Let $A \to B$ be a ring map and let (J, δ) be a divided power structure on B. The universal module $\Omega_{B/A,\delta}$ comes with a little bit of extra structure, namely the B-submodule N of $\Omega_{B/A,\delta}$ generated by $\mathrm{d}_{B/A,\delta}(J)$. In terms of the isomorphism given in Lemma 39.10.3 this corresponds to the image of $K \cap J(1)$ in $\Omega_{B/A,\delta}$. Consider the A-algebra $D = B \oplus \Omega^1_{B/A,\delta}$ with ideal $\bar{J} = J \oplus N$ and divided powers $\bar{\delta}$ as in the proof of the lemma. Then $(D, \bar{J}, \bar{\delta})$ is a divided power ring and the two maps $B \to D$ given by $b \mapsto b$ and $b \mapsto b + \mathrm{d}_{B/A,\delta}(b)$ are homomorphisms of divided power rings over A. Moreover, N is the smallest submodule of $\Omega_{B/A,\delta}$ such that this is true.

Lemma 39.10.5. *In Situation 39.9.1. Let (B, J, δ) be an object of CRIS(C/A). Let $(B(1), J(1), \delta(1))$ be the coproduct of (B, J, δ) with itself in CRIS(C/A). Denote $K = Ker(B(1) \to B)$. Then $K \cap J(1) \subset J(1)$ is preserved by the divided power structure and*

$$\Omega_{B/A,\delta} = K/\left(K^2 + (K \cap J(1))^{[2]}\right)$$

canonically.

Proof. Word for word the same as the proof of Lemma 39.10.3. The only point that has to be checked is that the divided power ring $D = B \oplus M$ is an object of CRIS(C/A) and that the two maps $B \to C$ are morphisms of CRIS(C/A). Since $D/(J \oplus M) = B/J$ we can use $C \to B/J$ to view D as an object of CRIS(C/A) and the statement on morphisms is clear from the construction. □

Lemma 39.10.6. *Let (A, I, γ) be a divided power ring. Let $A \to B$ be a ring map and let $IB \subset J \subset B$ be an ideal. Let $D_{B,\gamma}(J) = (D, \bar{J}, \bar{\gamma})$ be the divided power envelope. Then we have*

$$\Omega_{D/A,\bar{\gamma}} = \Omega_{B/A} \otimes_B D$$

Proof. We will prove this first when B is flat over A. In this case γ extends to a divided power structure γ' on IB, see Lemma 39.4.2. Hence $D = D_{B',\gamma'}(J)$ is equal to a quotient of the divided power ring (D', J', δ) where $D' = B\langle x_t\rangle$ and $J' = IB\langle x_t\rangle + B\langle x_t\rangle_+$ by the elements $x_t - f_t$ and $\delta_n(\sum r_t x_t - r_0)$, see Lemma 39.6.4 for notation and explanation. Write $\mathrm{d} : D' \to \Omega_{D'/A,\delta}$ for the universal derivation. Note that

$$\Omega_{D'/A,\delta} = \Omega_{B/A} \otimes_B D' \oplus \bigoplus D' \mathrm{d}x_t,$$

see Lemma 39.10.2. We conclude that $\Omega_{D/A,\bar{\gamma}}$ is the quotient of $\Omega_{D'/A,\delta} \otimes_{D'} D$ by the submodule generated by d applied to the generators of the kernel of $D' \to D$ listed above, see Lemma 39.10.2. Since $\mathrm{d}(x_t - f_t) = -\mathrm{d}f_t + \mathrm{d}x_t$ we see that we have $\mathrm{d}x_t = \mathrm{d}f_t$ in the quotient. In particular we see that $\Omega_{B/A} \otimes_B D \to \Omega_{D/A,\gamma}$ is surjective with kernel given by the images of d applied to the elements $\delta_n(\sum r_t x_t - r_0)$. However, given a relation

$\sum r_t f_t - r_0 = 0$ in B with $r_t \in B$ and $r_0 \in IB$ we see that

$$\begin{aligned} \mathrm{d}\delta_n(\sum r_t x_t - r_0) &= \delta_{n-1}(\sum r_t x_t - r_0)\mathrm{d}(\sum r_t x_t - r_0) \\ &= \delta_{n-1}(\sum r_t x_t - r_0)\left(\sum r_t \mathrm{d}(x_t - f_t) + \sum (x_t - f_t)\mathrm{d}r_t\right) \end{aligned}$$

because $\sum r_t f_t - r_0 = 0$ in B. Hence this is already zero in $\Omega_{B/A} \otimes_A D$ and we win in the case that B is flat over A.

In the general case we write B as a quotient of a polynomial ring $P \to B$ and let $J' \subset P$ be the inverse image of J. Then $D = D'/K'$ with notation as in Lemma 39.6.3. By the case handled in the first paragraph of the proof we have $\Omega_{D'/A,\bar\gamma'} = \Omega_{P/A} \otimes_P D'$. Then $\Omega_{D/A,\bar\gamma}$ is the quotient of $\Omega_{P/A} \otimes_P D$ by the submodule generated by $\mathrm{d}\bar\gamma'_n(k)$ where k is an element of the kernel of $P \to B$, see Lemma 39.10.2 and the description of K' from Lemma 39.6.3. Since $\mathrm{d}\bar\gamma'_n(k) = \bar\gamma'_{n-1}(k)\mathrm{d}k$ we see again that it suffices to divided by the submodule generated by $\mathrm{d}k$ with $k \in \mathrm{Ker}(P \to B)$ and since $\Omega_{B/A}$ is the quotient of $\Omega_{P/A} \otimes_A B$ by these elements (Algebra, Lemma 7.122.9) we win. □

Remark 39.10.7. Let B be a ring. Write $\Omega_B = \Omega_{B/\mathbf{Z}}$ for the absolute[2] module of differentials of B. Let $\mathrm{d} : B \to \Omega_B$ denote the universal derivation. Set $\Omega^i_B = \wedge^i_B(\Omega_B)$ as in Algebra, Section 7.12. The absolute *de Rham complex*

$$\Omega^0_B \to \Omega^1_B \to \Omega^2_B \to \ldots$$

Here $\mathrm{d} : \Omega^p_B \to \Omega^{p+1}_B$ is defined by the rule

$$\mathrm{d}\left(b_0\mathrm{d}b_1 \wedge \ldots \wedge \mathrm{d}b_p\right) = \mathrm{d}b_0 \wedge \mathrm{d}b_1 \wedge \ldots \wedge \mathrm{d}b_p$$

which we will show is well defined; note that $\mathrm{d}\circ\mathrm{d} = 0$ so we get a complex. Recall that Ω_B is the B-module generated by elements $\mathrm{d}b$ subject to the relations $\mathrm{d}(a + b) = \mathrm{d}a + \mathrm{d}b$ and $\mathrm{d}(ab) = b\mathrm{d}a + a\mathrm{d}b$ for $a, b \in B$. To prove that our map is well defined for $p = 1$ we have to show that the elements

$$a\mathrm{d}(b + c) - a\mathrm{d}b - a\mathrm{d}c \quad \text{and} \quad a\mathrm{d}(bc) - ac\mathrm{d}b - ab\mathrm{d}c, \quad a, b, c \in B$$

are mapped to zero by our rule. This is clear by direct computation (using the Leibniz rule). Thus we get a map

$$\Omega_B \otimes_{\mathbf{Z}} \ldots \otimes_{\mathbf{Z}} \Omega_B \longrightarrow \Omega^{p+1}_B$$

defined by the formula

$$\omega_1 \otimes \ldots \otimes \omega_p \longmapsto \sum (-1)^{i+1}\omega_1 \wedge \ldots \wedge \mathrm{d}(\omega_i) \wedge \ldots \wedge \omega_p$$

which matches our rule above on elements of the form $b_0\mathrm{d}b_1 \otimes \mathrm{d}b_2 \otimes \ldots \otimes \mathrm{d}b_p$. It is clear that this map is alternating. To finish we have to show that

$$\omega_1 \otimes \ldots \otimes f\omega_i \otimes \ldots \otimes \omega_p \quad \text{and} \quad \omega_1 \otimes \ldots \otimes f\omega_j \otimes \ldots \otimes \omega_p$$

are mapped to the same element. By $\mathbf{Z}$-linearity and the alternating property, it is enough to show this for $p = 2$, $i = 1$, $j = 2$, $\omega_1 = a_1\mathrm{d}b_1$ and $\omega_2 = a_2\mathrm{d}b_2$. Thus we need to show that

$$\begin{aligned} &\mathrm{d}fa_1 \wedge \mathrm{d}b_1 \wedge a_2\mathrm{d}b_2 - fa_1\mathrm{d}b_1 \wedge \mathrm{d}a_2 \wedge \mathrm{d}b_2 \\ &= \mathrm{d}a_1 \wedge \mathrm{d}b_1 \wedge fa_2\mathrm{d}b_2 - a_1\mathrm{d}b_1 \wedge \mathrm{d}fa_2 \wedge \mathrm{d}b_2 \end{aligned}$$

[2] This actually makes sense: if Ω_B is the module of differentials where we only assume the Leibniz rule and not the vanishing of d1, then the Leibniz rule gives $\mathrm{d}1 = \mathrm{d}(1 \cdot 1) = 1\mathrm{d}1 + 1\mathrm{d}1 = 2\mathrm{d}1$ and hence $\mathrm{d}1 = 0$ in Ω_B.

in other words that

$$(a_2 \mathrm{d}fa_1 + fa_1 \mathrm{d}a_2 - fa_2 \mathrm{d}a_1 - a_1 \mathrm{d}fa_2) \wedge \mathrm{d}b_1 \wedge \mathrm{d}b_2 = 0.$$

This follows from the Leibniz rule.

Lemma 39.10.8. *Let B be a ring. Let $\pi : \Omega_B \to \Omega$ be a surjective B-module map. Denote $d : B \to \Omega$ the composition of π with $d_B : B \to \Omega_B$. Set $\Omega^i = \wedge^i_B(\Omega)$. Assume that the kernel of π is generated, as a B-module, by elements $\omega \in \Omega_B$ such that $d_B(\omega) \in \Omega^2_B$ maps to zero in Ω^2. Then there is a de Rham complex*

$$\Omega^0 \to \Omega^1 \to \Omega^2 \to \dots$$

whose differential is defined by the rule

$$d : \Omega^p \to \Omega^{p+1}, \quad d\left(f_0 df_1 \wedge \dots \wedge df_p\right) = df_0 \wedge df_1 \wedge \dots \wedge df_p$$

Proof. We will show that there exists a commutative diagram

$$\begin{tikzcd}
\Omega^0_B \arrow[r, "\mathrm{d}_B"'] \arrow[d] & \Omega^1_B \arrow[r, "\mathrm{d}_B"'] \arrow[d, "\pi"'] & \Omega^2_B \arrow[r, "\mathrm{d}_B"'] \arrow[d, "\wedge^2\pi"'] & \dots \\
\Omega^0 \arrow[r, "\mathrm{d}"] & \Omega^1 \arrow[r, "\mathrm{d}"] & \Omega^2 \arrow[r, "\mathrm{d}"] & \dots
\end{tikzcd}$$

the description of the map d will follow from the construction of d_B in Remark 39.10.7. Since the left most vertical arrow is an isomorphism we have the first square. Because π is surjective, to get the second square it suffices to show that d_B maps the kernel of π into the kernel of $\wedge^2\pi$. We are given that any element of the kernel of π is of the form $\sum b_i\omega_i$ with $\pi(\omega_i) = 0$ and $\wedge^2\pi(\mathrm{d}_B(\omega_i)) = 0$. By the Leibniz rule for d_B we have $\mathrm{d}_B(\sum b_i\omega_i) = \sum b_i \mathrm{d}_B(\omega_i) + \sum \mathrm{d}_B(b_i) \wedge \omega_i$. Hence this maps to zero under $\wedge^2\pi$.

For $i > 1$ we note that $\wedge^i\pi$ is surjective with kernel the image of $\mathrm{Ker}(\pi) \wedge \Omega^{i-1}_B \to \Omega^i_B$. For $\omega_1 \in \mathrm{Ker}(\pi)$ and $\omega_2 \in \Omega^{i-1}_B$ we have

$$\mathrm{d}_B(\omega_1 \wedge \omega_2) = \mathrm{d}_B(\omega_1) \wedge \omega_2 - \omega_1 \wedge \mathrm{d}_B(\omega_2)$$

which is in the kernel of $\wedge^{i+1}\pi$ by what we just proved above. Hence we get the $(i+1)$st square in the diagram above. This concludes the proof. □

Remark 39.10.9. Let $A \to B$ be a ring map and let (J, δ) be a divided power structure on B. Set $\Omega^i_{B/A,\delta} = \wedge^i_B \Omega_{B/A,\delta}$ where $\Omega_{B/A,\delta}$ is the target of the universal divided power A-derivation $\mathrm{d} = \mathrm{d}_{B/A} : B \to \Omega_{B/A,\delta}$. Note that $\Omega_{B/A,\delta}$ is the quotient of Ω_B by the B-submodule generated by the elements $\mathrm{d}a = 0$ for $a \in A$ and $\mathrm{d}\delta_n(x) - \delta_{n-1}(x)\mathrm{d}x$ for $x \in J$. We claim Lemma 39.10.8 applies. To see this it suffices to verify the elements $\mathrm{d}a$ and $\mathrm{d}\delta_n(x) - \delta_{n-1}(x)\mathrm{d}x$ of Ω_B are mapped to zero in $\Omega^2_{B/A,\delta}$. This is clear for the first, and for the last we observe that

$$\mathrm{d}(\delta_{n-1}(x)) \wedge \mathrm{d}x = \delta_{n-2}(x)\mathrm{d}x \wedge \mathrm{d}x = 0$$

in $\Omega^2_{B/A,\delta}$ as desired. Hence we obtain a *divided power de Rham complex*

$$\Omega^0_{B/A,\delta} \to \Omega^1_{B/A,\delta} \to \Omega^2_{B/A,\delta} \to \dots$$

which will play an important role in the sequel.

Remark 39.10.10. Let B be a ring. Let $\Omega_B \to \Omega$ be a quotient satisfying the assumptions of Lemma 39.10.8. Let M be a B-module. A *connection* is an additive map

$$\nabla : M \longrightarrow M \otimes_B \Omega$$

such that $\nabla(bm) = b\nabla(m) + m \otimes \mathrm{d}b$ for $b \in B$ and $m \in M$. In this situation we can define maps

$$\nabla : M \otimes_B \Omega^i \longrightarrow M \otimes_B \Omega^{i+1}$$

by the rule $\nabla(m \otimes \omega) = \nabla(m) \wedge \omega + m \otimes \mathrm{d}\omega$. This works because if $b \in B$, then

$$\begin{aligned}\nabla(bm \otimes \omega) - \nabla(m \otimes b\omega) &= \nabla(bm) \otimes \omega + bm \otimes \mathrm{d}\omega - \nabla(m) \otimes b\omega - m \otimes \mathrm{d}(b\omega) \\ &= b\nabla(m) \otimes \omega + m \otimes \mathrm{d}b \wedge \omega + bm \otimes \mathrm{d}\omega \\ &\quad - b\nabla(m) \otimes \omega - bm \otimes \mathrm{d}(\omega) - m \otimes \mathrm{d}b \wedge \omega = 0\end{aligned}$$

As is customary we say the connection is *integrable* if and only if the composition

$$M \xrightarrow{\nabla} M \otimes_B \Omega^1 \xrightarrow{\nabla} M \otimes_B \Omega^2$$

is zero. In this case we obtain a complex

$$M \xrightarrow{\nabla} M \otimes_B \Omega^1 \xrightarrow{\nabla} M \otimes_B \Omega^2 \xrightarrow{\nabla} M \otimes_B \Omega^3 \xrightarrow{\nabla} M \otimes_B \Omega^4 \to \ldots$$

which is called the de Rham complex of the connection.

Remark 39.10.11. Let $\varphi : B \to B'$ be a ring map. Let $\Omega_B \to \Omega$ and $\Omega_{B'} \to \Omega'$ be quotients satisfying the assumptions of Lemma 39.10.8. Assume that the map $\Omega_B \to \Omega_{B'}$, $b_1 \mathrm{d}b_2 \mapsto \varphi(b_1)\mathrm{d}\varphi(b_2)$ fits into a commutative diagram

$$\begin{tikzcd} B \arrow[r] \arrow[d] & \Omega_B \arrow[r] \arrow[d] & \Omega \arrow[d, "\varphi"] \\ B' \arrow[r] & \Omega_{B'} \arrow[r] & \Omega' \end{tikzcd}$$

In this situation, given any pair (M, ∇) where M is a B-module and $\nabla : M \to M \otimes_B \Omega$ is a connection we obtain a *base change* $(M \otimes_B B', \nabla')$ where

$$\nabla' : M \otimes_B B' \longrightarrow (M \otimes_B B') \otimes_{B'} \Omega' = M \otimes_B \Omega'$$

is defined by the rule

$$\nabla'(m \otimes b') = \sum m_i \otimes b' \mathrm{d}\varphi(b_i) + m \otimes \mathrm{d}b'$$

if $\nabla(m) = \sum m_i \otimes \mathrm{d}b_i$. If ∇ is integrable, then so is ∇', and in this case there is a canonical map of de Rham complexes

$$M \otimes_B \Omega^\bullet \longrightarrow (M \otimes_B B') \otimes_{B'} (\Omega')^\bullet = M \otimes_B (\Omega')^\bullet \tag{39.10.11.1}$$

which maps $m \otimes \eta$ to $m \otimes \varphi(\eta)$.

Lemma 39.10.12. *Let $A \to B$ be a ring map and let (J, δ) be a divided power structure on B. Let p be a prime number. Assume that A is a $\mathbf{Z}_{(p)}$-algebra and that p is nilpotent in B/J. Then we have*

$$lim_e\, \Omega_{B_e/A,\bar\delta} = lim_e\, \Omega_{B/A,\delta}/p^e\Omega_{B/A,\delta} = lim_e\, \Omega_{B^\wedge/A,\delta^\wedge}/p^e\Omega_{B^\wedge/A,\delta^\wedge}$$

see proof for notation and explanation.

Proof. By Lemma 39.4.5 we see that δ extends to $B_e = B/p^e B$ for all sufficiently large e. Hence the first limit make sense. The lemma also produces a divided power structure $\delta^\wedge$ on the completion $B^\wedge = lim_e B_e$, hence the last limit makes sense. By Lemma 39.10.2 and the fact that $\mathrm{d}p^e = 0$ (always) we see that the surjection $\Omega_{B/A,\delta} \to \Omega_{B_e/A,\bar{\delta}}$ has kernel $p^e\Omega_{B/A,\delta}$. Similarly for the kernel of $\Omega_{B^\wedge/A,\delta^\wedge} \to \Omega_{B_e/A,\bar{\delta}}$. Hence the lemma is clear. □

39.11. Divided power schemes

Some remarks on how to globalize the previous notions.

Definition 39.11.1. Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. Let $\mathcal{I} \subset \mathcal{O}$ be a sheaf of ideals. A *divided power structure* γ on $\mathcal{I}$ is a sequence of maps $\gamma_n : \mathcal{I} \to \mathcal{I}$, $n \geq 1$ such that for any object U of $\mathcal{C}$ the triple

$$(\mathcal{O}(U), \mathcal{I}(U), \gamma)$$

is a divided power ring.

To be sure this applies in particular to sheaves of rings on topological spaces. But it's good to be a little bit more general as the structure sheaf of the crystalline site lives on a... site! A triple $(\mathcal{C}, \mathcal{I}, \gamma)$ as in the definition above is sometimes called a *divided power topos* in this chapter. Given a second $(\mathcal{C}', \mathcal{I}', \gamma')$ and given a morphism of ringed topoi $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ we say that $(f, f^\sharp)$ induces a *morphism of divided power topoi* if $f^\sharp(f^{-1}\mathcal{I}') \subset \mathcal{I}$ and the diagrams

$$\begin{tikzcd} f^{-1}\mathcal{I}' \arrow[r, "f^\sharp"'] \arrow[d, "f^{-1}\gamma'_n"'] & \mathcal{I} \arrow[d, "\gamma_n"] \\ f^{-1}\mathcal{I}' \arrow[r, "f^\sharp"] & \mathcal{I} \end{tikzcd}$$

are commutative for all $n \geq 1$. If f comes from a morphism of sites induced by a functor $u : \mathcal{C}' \to \mathcal{C}$ then this just means that

$$(\mathcal{O}'(U'), \mathcal{I}'(U'), \gamma') \longrightarrow (\mathcal{O}(u(U')), \mathcal{I}(u(U')), \gamma)$$

is a homomorphism of divided power rings for all $U' \in Ob(\mathcal{C}')$.

In the case of schemes we require the divided power ideal to be **quasi-coherent**. But apart from this the definition is exactly the same as in the case of topoi. Here it is.

Definition 39.11.2. A *divided power scheme* is a triple $(S, \mathcal{I}, \gamma)$ where S is a scheme, $\mathcal{I}$ is a quasi-coherent sheaf of ideals, and γ is a divided power structure on $\mathcal{I}$. A *morphism of divided power schemes* $(S, \mathcal{I}, \gamma) \to (S', \mathcal{I}', \gamma')$ is a morphism of schemes $f : S \to S'$ such that $f^{-1}\mathcal{I}'\mathcal{O}_S \subset \mathcal{I}$ and such that

$$(\mathcal{O}_S(U'), \mathcal{I}(U'), \gamma) \longrightarrow (\mathcal{O}_{S'}(f^{-1}U'), \mathcal{I}(f^{-1}U'), \gamma)$$

is a homomorphism of divided power rings for all $U' \subset S'$ open.

Recall that there is a 1-to-1 correspondence between quasi-coherent sheaves of ideals and closed immersions, see Morphisms, Section 24.2. Thus given a divided power scheme $(T, \mathcal{J}, \gamma)$ we get a canonical closed immersion $U \to T$ defined by $\mathcal{J}$. Conversely, given a closed immersion $U \to T$ and a divided power structure γ on the sheaf of ideals $\mathcal{J}$ associated to $U \to T$ we obtain a divided power scheme $(T, \mathcal{J}, \gamma)$. In many situations we only want to consider such triples (U, T, γ) when the morphism $U \to T$ is a thickening, see More on Morphisms, Definition 33.2.1.

Definition 39.11.3. A triple (U, T, γ) as above is called a *divided power thickening* if $U \to T$ is a thickening.

Fibre products of divided power schemes exist when one of the three is a divided power thickening. Here is a formal statement.

Lemma 39.11.4. *Let $(U', T', \delta') \to (S'_0, S', \gamma')$ and $(S_0, S, \gamma) \to (S'_0, S', \gamma')$ be morphisms of divided power schemes. If (U', T', δ') is a divided power thickening, then there exists a divided power scheme (T_0, T, δ) and*

$$\begin{tikzcd} T \arrow[r] \arrow[d] & T' \arrow[d] \\ S \arrow[r] & S' \end{tikzcd}$$

which is a cartesian diagram in the category of divided power schemes.

Proof. Omitted. Hints: If T exists, then $T_0 = S_0 \times_{S'_0} U'$ (argue as in Remark 39.3.5). Since T' is a divided power thickening, we see that T (if it exists) will be a divided power thickening too. Hence we can define T as the scheme with underlying topological space the underlying topological space of $T_0 = S_0 \times_{S'_0} U'$ and as structure sheaf on affine pieces the ring given by Lemma 39.9.3. □

We make the following observation. Suppose that (U, T, γ) is triple as above. Assume that T is a scheme over $\mathbf{Z}_{(p)}$ and that p is locally nilpotent on U. Then

(1) p locally nilpotent on $T \Leftrightarrow U \to T$ is a thickening (see Lemma 39.2.6), and
(2) $p^e \mathcal{O}_T$ is locally on T preserved by γ for $e \gg 0$ (see Lemma 39.4.5).

This suggest that good results on divided power thickenings will be available under the following hypotheses.

Situation 39.11.5. Here p is a prime number and $(S, \mathcal{I}, \gamma)$ is a divided power scheme over $\mathbf{Z}_{(p)}$. We set $S_0 = V(\mathcal{I}) \subset S$. Finally, $X \to S_0$ is a morphism of schemes such that p is locally nilpotent on X.

It is in this situation that we will define the big and small crystalline sites.

39.12. The big crystalline site

We first define the big site. Given a divided power scheme $(S, \mathcal{I}, \gamma)$ we say $(T, \mathcal{J}, \delta)$ is a divided power scheme over $(S, \mathcal{I}, \gamma)$ if T comes endowed with a morphism $T \to S$ of divided power schemes. Similarly, we say a divided power thickening (U, T, δ) is a divided power thickening over $(S, \mathcal{I}, \gamma)$ if T comes endowed with a morphism $T \to S$ of divided power schemes.

Definition 39.12.1. In Situation 39.11.5.

(1) A *divided power thickening of X relative to* $(S, \mathcal{I}, \gamma)$ is given by a divided power thickening (U, T, δ) over $(S, \mathcal{I}, \gamma)$ and an S-morphism $U \to X$.
(2) A *morphism of divided power thickenings of X relative to* $(S, \mathcal{I}, \gamma)$ is defined in the obvious manner.

The category of divided power thickenings of X relative to $(S, \mathcal{I}, \gamma)$ is denoted $\text{CRIS}(X/S, \mathcal{I}, \gamma)$ or simply $\text{CRIS}(X/S)$.

For any (U, T, δ) in $\text{CRIS}(X/S)$ we have that p is locally nilpotent on T, see discussion after Definition 39.11.3. A good way to visualize all the data associated to (U, T, δ) is the commutative diagram

$$\begin{tikzcd} T \arrow[dd] & U \arrow[l] \arrow[d] \\ & X \arrow[d] \\ S & S_0 \arrow[l] \end{tikzcd}$$

where $S_0 = V(\mathcal{J}) \subset S$. Morphisms of $\text{CRIS}(X/S)$ can be similarly visualized as huge commutative diagrams. In particular, there is a canonical forgetful functor

$$\text{CRIS}(X/S) \longrightarrow Sch/X, \quad (U, T, \delta) \longmapsto U \tag{39.12.1.1}$$

as well as its one sided inverse (and left adjoint)

$$Sch/X \longrightarrow \text{CRIS}(X/S), \quad U \longmapsto (U, U, \emptyset) \tag{39.12.1.2}$$

which is sometimes useful.

Lemma 39.12.2. *In Situation 39.11.5. The category $CRIS(X/S)$ has all finite nonempty limits, in particular products of pairs and fibre products. The functor (39.12.1.1) commutes with limits.*

Proof. Omitted. Hint: See Lemma 39.9.3 for the affine case. See also Remark 39.3.5. $\square$

Lemma 39.12.3. *In Situation 39.11.5. Let*

$$\begin{tikzcd} (U_3, T_3, \delta_3) \arrow[r] \arrow[d] & (U_2, T_2, \delta_2) \arrow[d] \\ (U_1, T_1, \delta_1) \arrow[r] & (U, T, \delta) \end{tikzcd}$$

be a fibre square in the category of divided power thickenings of X relative to $(S, \mathcal{J}, \gamma)$. If $T_2 \to T$ is flat, then $T_3 = T_1 \times_T T_2$ (as schemes).

Proof. This is true because a divided power structure extends uniquely along a flat ring map. See Lemma 39.4.2. $\square$

The lemma above means that the base change of a flat morphism of divided power thickenings is another flat morphism, and in fact is the ``usual'' base change of the morphism. This implies that the following definition makes sense.

Definition 39.12.4. In Situation 39.11.5.

(1) A family of morphisms $\{(U_i, T_i, \delta_i) \to (U, T, \delta)\}$ of divided power thickenings of X/S is a *Zariski, étale, smooth, syntomic, or fppf covering* if and only if the family of morphisms of schemes $\{T_i \to T\}$ is one.
(2) The *big crystalline site* of X over $(S, \mathcal{J}, \gamma)$, is the category $\text{CRIS}(X/S)$ endowed with the Zariski topology.
(3) The topos of sheaves on $\text{CRIS}(X/S)$ is denoted $(X/S)_{\text{CRIS}}$ or sometimes $(X/S, \mathcal{J}, \gamma)_{\text{CRIS}}$[3].

There are some obvious functorialities concerining these topoi.

[3]This clashes with our convention to denote the topos associated to a site $\mathcal{C}$ by $Sh(\mathcal{C})$.

Remark 39.12.5 (Functoriality)**.** Let p be a prime number. Let $(S, \mathcal{I}, \gamma) \to (S', \mathcal{I}', \gamma')$ be a morphism of divided power schemes over $\mathbf{Z}_{(p)}$. Set $S_0 = V(\mathcal{I})$ and $S'_0 = V(\mathcal{I}')$. Let

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ S_0 \arrow[r] & S'_0 \end{tikzcd}$$

be a commutative diagram of morphisms of schemes and assume p is locally nilpotent on X and Y. Then we get a continuous and cocontinuous functor

$$\mathrm{CRIS}(X/S) \longrightarrow \mathrm{CRIS}(Y/S')$$

by letting (U, T, δ) correspond to (U, T, δ) with $U \to X \to Y$ as the S'-morphism from U to Y. Hence we get a morphism of topoi

$$f_{\mathrm{CRIS}} : (X/S)_{\mathrm{CRIS}} \longrightarrow (Y/S')_{\mathrm{CRIS}}$$

see Sites, Section 9.19.

Remark 39.12.6 (Comparison with Zariski site)**.** In Situation 39.11.5. The functor (39.12.1.1) is continuous, cocontinuous, and commutes with products and fibred products. Hence we obtain a morphism of topoi

$$U_{X/S} : (X/S)_{\mathrm{CRIS}} \longrightarrow Sh((Sch/X)_{Zar})$$

from the big crystalline topos of X/S to the big Zarisk topos of X. See Sites, Section 9.19.

Remark 39.12.7 (Structure morphism)**.** In Situation 39.11.5. Consider the closed subscheme $S_0 = V(\mathcal{I}) \subset S$. If we assume that p is locally nilpotent on S_0 (which is always the case in practice) then we obtain a situation as in Definition 39.12.1 with S_0 instead of X. Hence we get a site $\mathrm{CRIS}(S_0/S)$. If $f : X \to S_0$ is the structure morphism of X over S, then we get a commutative diagram of morphisms of ringed topoi

$$\begin{tikzcd} (X/S)_{\mathrm{CRIS}} \arrow[r, "f_{\mathrm{CRIS}}"] \arrow[d, "U_{X/S}"] & (S_0/S)_{\mathrm{CRIS}} \arrow[d, "U_{S_0/S}"] & \\ Sh((Sch/X)_{Zar}) \arrow[r, "f_{big}"] & Sh((Sch/S_0)_{Zar}) \arrow[dr] & \\ & & Sh((Sch/S)_{Zar}) \end{tikzcd}$$

by Remark 39.12.5. We think of the composition $(X/S)_{\mathrm{CRIS}} \to Sh((Sch/S)_{Zar})$ as the structure morphism of the big crystalline site. Even if p is not locally nilpotent on S_0 the structure morphism

$$(X/S)_{\mathrm{CRIS}} \longrightarrow Sh((Sch/S)_{Zar})$$

is defined as we can take the lower route through the diagram above. Thus it is the morphism of topoi corresponding to the cocontinuous functor $\mathrm{CRIS}(X/S) \to (Sch/S)_{Zar}$ given by the rule $(U, T, \delta)/S \mapsto T/S$, see Sites, Section 9.19.

Remark 39.12.8 (Compatibilities)**.** The morphisms defined above satisfy numerous compatibilities. For example, in the situation of Remark 39.12.5 we obtain a commutative diagram of ringed topoi

$$\begin{tikzcd} (X/S)_{\mathrm{CRIS}} \arrow[r] \arrow[d] & (Y/S')_{\mathrm{CRIS}} \arrow[d] \\ Sh((Sch/S)_{Zar}) \arrow[r] & Sh((Sch/S')_{Zar}) \end{tikzcd}$$

where the vertical arrows are the structure morphisms.

39.13. The crystalline site

Since (39.12.1.1) commutes with products and fibre products, we see that looking at those (U, T, δ) such that $U \to X$ is an open immersion defines a full subcategory preserved under fibre products (and more generally finite nonempty limits). Hence the following definition makes sense.

Definition 39.13.1. In Situation 39.11.5.

(1) The (small) *crystalline site* of X over $(S, \mathcal{I}, \gamma)$, denoted $\mathrm{Cris}(X/S, \mathcal{I}, \gamma)$ or simply $\mathrm{Cris}(X/S)$ is the full subcategory of $\mathrm{CRIS}(X/S)$ consisting of those (U, T, δ) in $\mathrm{CRIS}(X/S)$ such that $U \to X$ is an open immersion. It comes endowed with the Zariski topology.
(2) The topos of sheaves on $\mathrm{Cris}(X/S)$ is denoted $(X/S)_{\mathrm{cris}}$ or sometimes $(X/S, \mathcal{I}, \gamma)_{\mathrm{cris}}$[4].

For any (U, T, δ) in $\mathrm{Cris}(X/S)$ the morphism $U \to X$ defines an object of the small Zariski site X_{Zar} of X. Hence a canonical forgetful functor

$$\mathrm{Cris}(X/S) \longrightarrow X_{Zar}, \quad (U, T, \delta) \longmapsto U \tag{39.13.1.1}$$

and a left adjoint

$$X_{Zar} \longrightarrow \mathrm{Cris}(X/S), \quad U \longmapsto (U, U, \emptyset) \tag{39.13.1.2}$$

which is sometimes useful.

We can compare the small and big crystalline sites, just like we can compare the small and big Zariski sites of a scheme, see Topologies, Lemma 30.3.13.

Lemma 39.13.2. *Assumptions as in Definition 39.12.1. The inclusion functor*

$$Cris(X/S) \to CRIS(X/S)$$

commutes with finite nonempty limits, is fully faithful, continuous, and cocontinuous. There are morphisms of topoi

$$(X/S)_{cris} \xrightarrow{i} (X/S)_{CRIS} \xrightarrow{\pi} (X/S)_{cris}$$

whose composition is the identity and of which the first is induced by the inclusion functor. Moreover, $\pi_* = i^{-1}$.

Proof. For the first assertion see Lemma 39.12.2. This gives us a morphism of topoi $i : (X/S)_{\mathrm{cris}} \to (X/S)_{\mathrm{CRIS}}$ and a left adjoint $i_!$ such that $i^{-1}i_! = i^{-1}i_* = \mathrm{id}$, see Sites, Lemmas 9.19.5, 9.19.6, and 9.19.7. We claim that $i_!$ is exact. If this is true, then we can define π by the rules $\pi^{-1} = i_!$ and $\pi_* = i^{-1}$ and everything is clear. To prove the claim, note that we already know that $i_!$ is right exact and preserves fibre products (see references given). Hence it suffices to show that $i_! * = *$ where $*$ indicates the final object in the category

[4]This clashes with our convention to denote the topos associated to a site $\mathcal{C}$ by $Sh(\mathcal{C})$.

of sheaves of sets. To see this it suffices to produce a set of objects (U_i, T_i, δ_i), $i \in I$ of Cris(X/S) such that

$$\coprod\nolimits_{i \in I} h_{(U_i, T_i, \delta_i)} \to *$$

is surjective in $(X/S)_{\text{CRIS}}$ (details omitted; hint: use that Cris(X/S) has products and that the functor Cris(X/S) → CRIS(X/S) commutes with them). In the affine case this follows from Lemma 39.9.6. We omit the proof in general. □

Remark 39.13.3 (Functoriality)**.** Let p be a prime number. Let $(S, \mathcal{I}, \gamma) \to (S', \mathcal{I}', \gamma')$ be a morphism of divided power schemes over $\mathbf{Z}_{(p)}$. Let

$$\begin{tikzcd} X \arrow[r, "f"'] \arrow[d] & Y \arrow[d] \\ S_0 \arrow[r] & S'_0 \end{tikzcd}$$

be a commutative diagram of morphisms of schemes and assume p is locally nilpotent on X and Y. By analogy with Topologies, Lemma 30.3.16 we define

$$f_{\text{cris}} : (X/S)_{\text{cris}} \longrightarrow (Y/S')_{\text{cris}}$$

by the formula $f_{\text{cris}} = \pi_Y \circ f_{\text{CRIS}} \circ i_X$ where i_X and π_Y are as in Lemma 39.13.2 for X and Y and where f_{CRIS} is as in Remark 39.12.5.

Remark 39.13.4 (Comparison with Zariski site)**.** In Situation 39.11.5. The functor (39.13.1.1) is continuous, cocontinuous, and commutes with products and fibred products. Hence we obtain a morphism of topoi

$$u_{X/S} : (X/S)_{\text{cris}} \longrightarrow Sh(X_{Zar})$$

relating the small crystalline topos of X/S with the small Zarisk topos of X. See Sites, Section 9.19.

Lemma 39.13.5. *In Situation 39.11.5. Let $X' \subset X$ and $S' \subset S$ be open subschemes such that X' maps into S'. Then there is a fully faithful functor $Cris(X'/S') \to Cris(X/S)$ which gives rise to a morphism of topoi fitting into the commutative diagram*

$$\begin{tikzcd} (X'/S')_{cris} \arrow[r] \arrow[d, "u_{X'/S'}"'] & (X/S)_{cris} \arrow[d, "u_{X/S}"] \\ Sh(X'_{Zar}) \arrow[r] & Sh(X_{Zar}) \end{tikzcd}$$

Moreover, this diagram is an example of localization of morphisms of topoi as in Sites, Lemma 9.27.1.

Proof. The fully faithful functor comes from thinking of objects of Cris(X'/S') as divided power thickenings (U, T, δ) of X where $U \to X$ factors through $X' \subset X$ (since then automatically $T \to S$ will factor through S'). This functor is clearly cocontinuous hence we obtain a morphism of topoi as indicated. Let $h_{X'} \in Sh(X_{Zar})$ be the representable sheaf associated to X' viewed as an object of X_{Zar}. It is clear that $Sh(X'_{Zar})$ is the localization $Sh(X_{Zar})/h_{X'}$. On the other hand, the category Cris(X/S)$/u_{X/S}^{-1}h_{X'}$ (see Sites, Lemma 9.26.3) is canonically identified with Cris(X'/S') by the functor above. This finishes the proof. □

Remark 39.13.6 (Structure morphism)**.** In Situation 39.11.5. Consider the closed subscheme $S_0 = V(\mathcal{I}) \subset S$. If we assume that p is locally nilpotent on S_0 (which is always the case in practice) then we obtain a situation as in Definition 39.12.1 with S_0 instead of X. Hence we get a site $\text{Cris}(S_0/S)$. If $f : X \to S_0$ is the structure morphism of X over S, then we get a commutative diagram of ringed topoi

$$\begin{tikzcd}
(X/S)_{\text{cris}} \arrow[r, "f_{\text{cris}}"'] \arrow[d, "u_{X/S}"'] & (S_0/S)_{\text{cris}} \arrow[d, "u_{S_0/S}"] & \\
Sh(X_{Zar}) \arrow[r, "f_{small}"] & Sh(S_{0,Zar}) \arrow[dr] & \\
& & Sh(S_{Zar})
\end{tikzcd}$$

see Remark 39.13.3. We think of the compostion $(X/S)_{\text{cris}} \to Sh(S_{Zar})$ as the structure morphism of the crystalline site. Even if p is not locally nilpotent on S_0 the structure morphism

$$\tau_{X/S} : (X/S)_{\text{cris}} \longrightarrow Sh(S_{Zar})$$

is defined as we can take the lower route through the diagram above.

Remark 39.13.7 (Compatibilities)**.** The morphisms defined above satisfy numerous compatibilities. For example, in the situation of Remark 39.13.3 we obtain a commutative diagram of ringed topoi

$$\begin{tikzcd}
(X/S)_{\text{cris}} \arrow[r] \arrow[d] & (Y/S')_{\text{cris}} \arrow[d] \\
Sh((Sch/S)_{Zar}) \arrow[r] & Sh((Sch/S')_{Zar})
\end{tikzcd}$$

where the vertical arrows are the structure morphisms.

39.14. Sheaves on the crystalline site

Notation and assumptions as in Situation 39.11.5. In order to discuss the small and big crystalline sites of X/S simultaneously in this section we let

$$\mathcal{C} = \text{CRIS}(X/S) \quad \text{or} \quad \mathcal{C} = \text{Cris}(X/S).$$

A sheaf $\mathcal{F}$ on $\mathcal{C}$ gives rise to a *restriction* $\mathcal{F}_T$ for every object (U, T, δ) of $\mathcal{C}$. Namely, $\mathcal{F}_T$ is the Zariski sheaf on the scheme T defined by the rule

$$\mathcal{F}_T(W) = \mathcal{F}(U \cap W, W, \delta|_W)$$

for $W \subset T$ is open. Moreover, if $f : T \to T'$ is a morphism between objects (U, T, δ) and (U', T', δ') of $\mathcal{C}$, then there is a canonical *comparison* map

$$c_f : f^{-1}\mathcal{F}_{T'} \longrightarrow \mathcal{F}_T. \tag{39.14.0.1}$$

Namely, if $W' \subset T'$ is open then f induces a morphism

$$f|_{f^{-1}W'} : (U \cap f^{-1}(W'), f^{-1}W', \delta|_{f^{-1}W'}) \longrightarrow (U' \cap W', W', \delta|_{W'})$$

of $\mathcal{C}$, hence we can use the restriction mapping $(f|_{f^{-1}W'})^*$ of $\mathcal{F}$ to define a map $\mathcal{F}_{T'}(W') \to \mathcal{F}_T(f^{-1}W')$. These maps are clearly compatible with further restriction, hence define an f-map from $\mathcal{F}_{T'}$ to $\mathcal{F}_T$ (see Sheaves, Section 6.21 and especially Sheaves, Definition 6.21.7).

Thus a map c_f as in (39.14.0.1). Note that if f is an open immersion, then c_f is an isomorphism, because in that case $\mathcal{F}_T$ is just the restriction of $\mathcal{F}_{T'}$ to T.

Conversely, given Zariski sheaves $\mathcal{F}_T$ for every object (U, T, δ) of $\mathcal{C}$ and comparion maps c_f as above which (a) are isomorphisms for open immersions, and (b) satisfy a suitable cocycle condition, we obtain a sheaf on $\mathcal{C}$. This is proved exactly as in Topologies, Lemma 30.3.18.

The *structure sheaf* on $\mathcal{C}$ is the sheaf $\mathcal{O}_{X/S}$ defined by the rule

$$\mathcal{O}_{X/S} : (U, T, \delta) \longmapsto \Gamma(T, \mathcal{O}_T)$$

This is a sheaf by the definition of coverings in $\mathcal{C}$. Suppose that $\mathcal{F}$ is a sheaf of $\mathcal{O}_{X/S}$-modules. In this case the comparison mappings (39.14.0.1) define a comparison map

$$c_f : f^*\mathcal{F}_T \longrightarrow \mathcal{F}_{T'} \tag{39.14.0.2}$$

of $\mathcal{O}_{T'}$-modules.

Another type of example comes by starting with a sheaf $\mathcal{G}$ on $(Sch/X)_{Zar}$ or X_{Zar} (depending on whether $\mathcal{C} = \mathrm{CRIS}(X/S)$ or $\mathcal{C} = \mathrm{Cris}(X/S)$). Then $\underline{\mathcal{G}}$ defined by the rule

$$\underline{\mathcal{G}} : (U, T, \delta) \longmapsto \mathcal{G}(U)$$

is a sheaf on $\mathcal{C}$. In particular, if we take $\mathcal{G} = \mathbf{G}_a = \mathcal{O}_X$, then we obtain

$$\underline{\mathbf{G}_a} : (U, T, \delta) \longmapsto \Gamma(U, \mathcal{O}_U)$$

There is a surjective map of sheaves $\mathcal{O}_{X/S} \to \underline{\mathbf{G}_a}$ defined by the canonical maps $\Gamma(T, \mathcal{O}_T) \to \Gamma(U, \mathcal{O}_U)$ for objects (U, T, δ). The kernel of this map is denoted $\mathcal{J}_{X/S}$, hence a short exact sequence

$$0 \to \mathcal{J}_{X/S} \to \mathcal{O}_{X/S} \to \underline{\mathbf{G}_a} \to 0$$

Note that $\mathcal{J}_{X/S}$ comes equipped with a canonical divided power structure. After all, for each object (U, T, δ) the third component δ *is* a divided power structure on the kernel of $\mathcal{O}_T \to \mathcal{O}_U$. Hence the (big) crystalline topos is a divided power topos.

39.15. Crystals in modules

It turns out that a crystal is a very general gadget. However, the definition may be a bit hard to parse, so we first give the definition in the case of modules on the crystalline sites.

Definition 39.15.1. In Situation 39.11.5. Let $\mathcal{C} = \mathrm{CRIS}(X/S)$ or $\mathcal{C} = \mathrm{Cris}(X/S)$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_{X/S}$-modules on $\mathcal{C}$.

(1) We say $\mathcal{F}$ is *locally quasi-coherent* if for every object (U, T, δ) of $\mathcal{C}$ the restriction $\mathcal{F}_T$ is a quasi-coherent $\mathcal{O}_T$-module.
(2) We say $\mathcal{F}$ is *quasi-coherent* if it is quasi-coherent in the sense of Modules on Sites, Definition 16.23.1.
(3) We say $\mathcal{F}$ is a *crystal in $\mathcal{O}_{X/S}$-modules* if all the comparison maps (39.14.0.2) are isomorphisms.

It turns out that we can relate these notions as follows.

Lemma 39.15.2. *With notation $X/S, \mathcal{J}, \gamma, \mathcal{C}, \mathcal{F}$ as in Definition 39.15.1. The following are equivalent*

(1) *$\mathcal{F}$ is quasi-coherent, and*
(2) *$\mathcal{F}$ is locally quasi-coherent and a crystal in $\mathcal{O}_{X/S}$-modules.*

Proof. Assume (1). Let $f : (U', T', \delta') \to (U, T, \delta)$ be an object of $\mathcal{C}$. We have to prove (a) $\mathcal{F}_T$ is a quasi-coherent $\mathcal{O}_T$-module and (b) $c_f : f^*\mathcal{F}_T \to \mathcal{F}_{T'}$ is an isomorphism. The assumption means that we can find a covering $\{(T_i, U_i, \delta_i) \to (T, U, \delta)\}$ and for each i the restriction of $\mathcal{F}$ to $\mathcal{C}/(T_i, U_i, \delta_i)$ has a global presentation. Since it suffices to prove (a) and (b) Zariski locally, we may replace $f : (T', U', \delta') \to (T, U, \delta)$ by the base change to (T_i, U_i, δ_i) and assume that $\mathcal{F}$ restricted to $\mathcal{C}/(T, U, \delta)$ has a global presentation

$$\bigoplus\nolimits_{j\in J} \mathcal{O}_{X/S}|_{\mathcal{C}/(U,T,\delta)} \longrightarrow \bigoplus\nolimits_{i\in I} \mathcal{O}_{X/S}|_{\mathcal{C}/(U,T,\delta)} \longrightarrow \mathcal{F}|_{\mathcal{C}/(U,T,\delta)} \longrightarrow 0$$

It is clear that this gives a presentation

$$\bigoplus\nolimits_{j\in J} \mathcal{O}_T \longrightarrow \bigoplus\nolimits_{i\in I} \mathcal{O}_T \longrightarrow \mathcal{F}_T \longrightarrow 0$$

and hence (a) holds. Moreover, the presentation restricts to T' to give a similar presentation of $\mathcal{F}_{T'}$, whence (b) holds.

Assume (2). Let (U, T, δ) be an object of $\mathcal{C}$. We have to find a covering of (U, T, δ) such that $\mathcal{F}$ has a global presentation when we restrict to the localization of $\mathcal{C}$ at the members of the covering. Thus we may assume that T is affine. In this case we can choose a presentation

$$\bigoplus\nolimits_{j\in J} \mathcal{O}_T \longrightarrow \bigoplus\nolimits_{i\in I} \mathcal{O}_T \longrightarrow \mathcal{F}_T \longrightarrow 0$$

as $\mathcal{F}_T$ is assumed to be a quasi-coherent $\mathcal{O}_T$-module. Then by the crystal property of $\mathcal{F}$ we see that this pulls back to a presentation of $\mathcal{F}_{T'}$ for any morphism $f : (U', T', \delta') \to (U, T, \delta)$ of $\mathcal{C}$. Thus the desired presentation of $\mathcal{F}|_{\mathcal{C}/(U,T,\delta)}$. □

Definition 39.15.3. If $\mathcal{F}$ satisfies the equivalent conditions of Lemma 39.15.2, then we say that $\mathcal{F}$ is a *crystal in quasi-coherent modules*. We say that $\mathcal{F}$ is a *crystal in finite locally free modules* if, in addition, $\mathcal{F}$ is finite locally free.

Of course, as Lemma 39.15.2 shows, this notation is somewhat heavy since a quasi-coherent module is always a crystal. But it is standard terminology in the literature.

Remark 39.15.4. To formulate the general notion of a crystal we use the language of stacks and strongly cartesian morphisms, see Stacks, Definition 50.4.1 and Categories, Definition 4.30.1. In Situation 39.11.5 let $p : \mathcal{C} \to \text{Cris}(X/S)$ be a stack. A *crystal in objects of $\mathcal{C}$ on X relative to S* is a *cartesian section* $\sigma : \text{Cris}(X/S) \to \mathcal{C}$, i.e., a functor σ such that $p \circ \sigma = \text{id}$ and such that $\sigma(f)$ is strongly cartesian for all morphisms f of $\text{Cris}(X/S)$. Similarly for the big crystalline site.

39.16. Sheaf of differentials

In this section we will stick with the (small) crystalline site as it seems more natural. We globalize Definition 39.10.1 as follows.

Definition 39.16.1. In Situation 39.11.5 let $\mathcal{F}$ be a sheaf of $\mathcal{O}_{X/S}$-modules on $\text{Cris}(X/S)$. An *S-derivation* $D : \mathcal{O}_{X/S} \to \mathcal{F}$ is a map of sheaves such that for every object (U, T, δ) of $\text{Cris}(X/S)$ the map

$$D : \Gamma(T, \mathcal{O}_T) \longrightarrow \Gamma(T, \mathcal{F})$$

is a divided power $\Gamma(V, \mathcal{O}_V)$-derivation where $V \subset S$ is any open such that $T \to S$ factors through V.

This means that D is additive, satisfies the Leibniz rule, annihilates functions coming from S, and satisfies $D(f^{[n]}) = f^{[n-1]}D(f)$ for a local section f of the divided power ideal $\mathcal{J}_{X/S}$. This is a special case of a very general notion which we now describe.

Please compare the following discussion with Modules on Sites, Section 16.29. Let $\mathcal{C}$ be a site, let $\mathcal{A} \to \mathcal{B}$ be a map of sheaves of rings on $\mathcal{C}$, let $\mathcal{J} \subset \mathcal{B}$ be a sheaf of ideals, let δ be a divided power structure on $\mathcal{J}$, and let $\mathcal{F}$ be a sheaf of $\mathcal{B}$-modules. Then there is a notion of a *divided power $\mathcal{A}$-derivation* $D : \mathcal{B} \to \mathcal{F}$. This means that D is $\mathcal{A}$-linear, satisfies the Leibnize rule, and satisfies $D(\delta_n(x)) = \delta_{n-1}(x)D(x)$ for local sections x of $\mathcal{J}$. In this situation there exists a *universal divided power $\mathcal{A}$-derivation*

$$\mathrm{d}_{\mathcal{B}/\mathcal{A},\delta} : \mathcal{B} \longrightarrow \Omega_{\mathcal{B}/\mathcal{A},\delta}$$

Moreover, $\mathrm{d}_{\mathcal{B}/\mathcal{A},\delta}$ is the compostion

$$\mathcal{B} \longrightarrow \Omega_{\mathcal{B}/\mathcal{A}} \longrightarrow \Omega_{\mathcal{B}/\mathcal{A},\delta}$$

where the first map is the universal derivation constructed in the proof of Modules on Sites, Lemma 16.29.2 and the second arrow is the quotient by the submodule generated by the local sections $\mathrm{d}_{\mathcal{B}/\mathcal{A}}(\delta_n(x)) - \delta_{n-1}(x)\mathrm{d}_{\mathcal{B}/\mathcal{A}}(x)$.

We translate this into a relative notion as follows. Suppose $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ is a morphism of ringed topoi, $\mathcal{J} \subset \mathcal{O}$ a sheaf of ideals, δ a divided power structure on $\mathcal{J}$, and $\mathcal{F}$ a sheaf of $\mathcal{O}$-modules. In this situation we say $D : \mathcal{O} \to \mathcal{F}$ is a divided power $\mathcal{O}'$-derivation if D is a divided power $f^{-1}\mathcal{O}'$-derivation as defined above. Moreover, we write

$$\Omega_{\mathcal{O}/\mathcal{O}',\delta} = \Omega_{\mathcal{O}/f^{-1}\mathcal{O}',\delta}$$

which is the receptacle of the universal divided power $\mathcal{O}'$-derivation.

Appying this to the structure morphism

$$(X/S)_{\text{Cris}} \longrightarrow Sh(S_{Zar})$$

(see Remark 39.13.6) we recover the notion of Definition 39.16.1 above. In particular, there is a universal divided power derivation

$$d_{X/S} : \mathcal{O}_{X/S} \to \Omega_{X/S}$$

Note that we omit from the notation the decoration indicating the module of differentials is compatible with divided powers (it seems unlikely anybody would ever consider the usual module of differentials of the structure sheaf on the crystalline site).

Lemma 39.16.2. *Let $(T, \mathcal{J}, \delta)$ be a divided power scheme. Let $T \to S$ be a morphism of schemes. The quotient $\Omega_{T/S} \to \Omega_{T/S,\delta}$ described above is a quasi-coherent $\mathcal{O}_T$-module. For $W \subset T$ affine open mapping into $V \subset S$ affine open we have*

$$\Gamma(W, \Omega_{T/S,\delta}) = \Omega_{\Gamma(W,\mathcal{O})/\Gamma(V,\mathcal{O}_V),\delta}$$

where the right hand side is as constructed in Section 39.10.

Proof. Omitted. □

Lemma 39.16.3. *In Situation 39.11.5. For (U, T, δ) in $Cris(X/S)$ the restriction $(\Omega_{X/S})_T$ to T is $\Omega_{T/S,\delta}$ and the restriction $d_{X/S}|_T$ is equal to $d_{T/S,\delta}$.*

Proof. Omitted. □

Lemma 39.16.4. *In Situation 39.11.5. For any affine object (U, T, δ) of $Cris(X/S)$ mapping into an affine open $V \subset S$ we have*

$$\Gamma((U, T, \delta), \Omega_{X/S}) = \Omega_{\Gamma(T,\mathcal{O})/\Gamma(V,\mathcal{O}_V),\delta}$$

where the right hand side is as constructed in Section 39.10.

Proof. Combine Lemmas 39.16.2 and 39.16.3. □

Lemma 39.16.5. *In Situation 39.11.5. Let (U, T, δ) be an object of $Cris(X/S)$. Let*

$$(U(1), T(1), \delta(1)) = (U, T, \delta) \times (U, T, \delta)$$

in $Cris(X/S)$. Let $\mathcal{K} \subset \mathcal{O}_{T(1)}$ be the quasi-coherent sheaf of ideals corresponding to the closed immersion $\Delta : T \to T(1)$. Then $\mathcal{K} \subset \mathcal{J}_{T(1)}$ is preserved by the divided structure on $\mathcal{J}_{T(1)}$ and we have

$$(\Omega_{X/S})_T = \mathcal{K}/\mathcal{K}^{[2]}$$

Proof. Note that $U = U(1)$ as $U \to X$ is an open immersion and as (39.13.1.1) commutes with products. Hence we see that $\mathcal{K} \subset \mathcal{J}_{T(1)}$. Given this fact the lemma follows by working affine locally on T and using Lemmas 39.16.4 and 39.10.5. □

It turns out that $\Omega_{X/S}$ is not a crystal in quasi-coherent $\mathcal{O}_{X/S}$-modules. But it does satisfy two closely related properties (compare with Lemma 39.15.2).

Lemma 39.16.6. *In Situation 39.11.5. The sheaf of differentials $\Omega_{X/S}$ has the following two properties:*

1. *$\Omega_{X/S}$ is locally quasi-coherent, and*
2. *for any morphism $(U, T, \delta) \to (U', T', \delta')$ of $Cris(X/S)$ where $f : T \to T'$ is a closed immersion the map $c_f : f^*(\Omega_{X/S})_{T'} \to (\Omega_{X/S})_T$ is surjective.*

Proof. Part (1) follows from a combination of Lemmas 39.16.2 and 39.16.3. Part (2) follows from the fact that $(\Omega_{X/S})_T = \Omega_{T/S,\delta}$ is a quotient of $\Omega_{T/S}$ and that $f^*\Omega_{T'/S} \to \Omega_{T/S}$ is surjective. □

39.17. Two universal thickenings

The constructions in this section will help us define a connection on a crystal in modules on the crystalline site. In some sense the constructions here are the ``sheafified, universal'' versions of the constructions in Section 39.7.

Remark 39.17.1. In Situation 39.11.5. Let (U, T, δ) be an object of $Cris(X/S)$. Write $\Omega_{T/S,\delta} = (\Omega_{X/S})_T$, see Lemma 39.16.3. We explicitly describe a first order thickening T' of T. Namely, set

$$\mathcal{O}_{T'} = \mathcal{O}_T \oplus \Omega_{T/S,\delta}$$

with algebra structure such that $\Omega_{T/S,\delta}$ is an ideal of square zero. Let $\mathcal{J} \subset \mathcal{O}_T$ be the ideal sheaf of the closed immersion $U \to T$. Set $\mathcal{J}' = \mathcal{J} \oplus \Omega_{T/S,\delta}$. Define a divided power structure on $\mathcal{J}'$ by setting

$$\delta_n'(f, \omega) = (\delta_n(f), \delta_{n-1}(f)\omega),$$

see Lemma 39.7.1. There are two ring maps

$$p_0, p_1 : \mathcal{O}_T \to \mathcal{O}_{T'}$$

The first is given by $f \mapsto (f, 0)$ and the second by $f \mapsto (f, \mathrm{d}_{T/S,\delta}f)$. Note that both are compatible with the divided power structures on $\mathcal{J}$ and $\mathcal{J}'$ and so is the quotient map $\mathcal{O}_{T'} \to \mathcal{O}_T$. Thus we get an object (U, T', δ') of $\mathrm{Cris}(X/S)$ and a commutative diagram

$$\begin{tikzcd} & T \arrow[dl, "\mathrm{id}"'] \arrow[d, "i"] \arrow[dr, "\mathrm{id}"] & \\ T & T' \arrow[l, "p_0"'] \arrow[r, "p_1"] & T \end{tikzcd}$$

of $\mathrm{Cris}(X/S)$ such that i is a first order thickening whose ideal sheaf is identified with $\Omega_{T/S,\delta}$ and such that $p_1^* - p_0^* : \mathcal{O}_T \to \mathcal{O}_{T'}$ is identified with the universal derivation $\mathrm{d}_{T/S,\delta}$ composed with the inclusion $\Omega_{T/S,\delta} \to \mathcal{O}_{T'}$.

Remark 39.17.2. In Situation 39.11.5. Let (U, T, δ) be an object of $\mathrm{Cris}(X/S)$. Write $\Omega_{T/S,\delta} = (\Omega_{X/S})_T$, see Lemma 39.16.3. We also write $\Omega^2_{T/S,\delta}$ for its second exterior power. We explicitly describe a second order thickening T'' of T. Namely, set

$$\mathcal{O}_{T''} = \mathcal{O}_T \oplus \Omega_{T/S,\delta} \oplus \Omega_{T/S,\delta} \oplus \Omega^2_{T/S,\delta}$$

with algebra structure defined in the following way

$$(f, \omega_1, \omega_2, \eta)\cdot(f', \omega_1', \omega_2', \eta') = (ff', f\omega_1' + f'\omega_1, f\omega_2' + f'\omega_2', f\eta' + f'\eta + \omega_1 \wedge \omega_2' + \omega_1' \wedge \omega_2).$$

Let $\mathcal{J} \subset \mathcal{O}_T$ be the ideal sheaf of the closed immersion $U \to T$. Let $\mathcal{J}''$ be the inverse image of $\mathcal{J}$ under the projection $\mathcal{O}_{T''} \to \mathcal{O}_T$. Define a divided power structure on $\mathcal{J}''$ by setting

$$\delta_n''(f, \omega_1, \omega_2, \eta) = (\delta_n(f), \delta_{n-1}(f)\omega_1, \delta_{n-1}(f)\omega_2, \delta_{n-1}(f)\eta + \delta_{n-2}(f)\omega_1 \wedge \omega_2)$$

see Lemma 39.7.2. There are three ring maps $q_0, q_1, q_2 : \mathcal{O}_T \to \mathcal{O}_{T''}$ given by

$$\begin{aligned} q_0(f) &= (f, 0, 0, 0), \\ q_1(f) &= (f, \mathrm{d}f, 0, 0), \\ q_2(f) &= (f, \mathrm{d}f, \mathrm{d}f, 0) \end{aligned}$$

where $\mathrm{d} = \mathrm{d}_{T/S,\delta}$. Note that all three are compatible with the divided power structures on $\mathcal{J}$ and $\mathcal{J}''$. There are three ring maps $q_{01}, q_{12}, q_{02} : \mathcal{O}_{T'} \to \mathcal{O}_{T''}$ where $\mathcal{O}_{T'}$ is as in Remark 39.17.1. Namely, set

$$\begin{aligned} q_{01}(f, \omega) &= (f, \omega, 0, 0), \\ q_{12}(f, \omega) &= (f, \mathrm{d}f, \omega, \mathrm{d}\omega), \\ q_{02}(f, \omega) &= (f, \omega, \omega, 0) \end{aligned}$$

These are also compatible with the given divided power structures. Let's do the verifications for q_{12}: Note that q_{12} is a ring homomorphism as

$$\begin{aligned} q_{12}(f, \omega)q_{12}(g, \eta) &= (f, \mathrm{d}f, \omega, \mathrm{d}\omega)(g, \mathrm{d}g, \eta, \mathrm{d}\eta) \\ &= (fg, f\mathrm{d}g + g\mathrm{d}f, f\eta + g\omega, f\mathrm{d}\eta + g\mathrm{d}\omega + \mathrm{d}f \wedge \eta + \mathrm{d}g \wedge \omega) \\ &= q_{12}(fg, f\eta + g\omega) = q_{12}((f, \omega)(g, \eta)) \end{aligned}$$

Note that q_{12} is compatible with divided powers because

$$\begin{aligned} \delta_n''(q_{12}(f, \omega)) &= \delta_n''((f, \mathrm{d}f, \omega, \mathrm{d}\omega)) \\ &= (\delta_n(f), \delta_{n-1}(f)\mathrm{d}f, \delta_{n-1}(f)\omega, \delta_{n-1}(f)\mathrm{d}\omega + \delta_{n-2}(f)\mathrm{d}(f) \wedge \omega) \\ &= q_{12}((\delta_n(f), \delta_{n-1}(f)\omega)) = q_{12}(\delta_n'(f, \omega)) \end{aligned}$$

The verifications for q_{01} and q_{02} are easier. Note that $q_0 = q_{01} \circ p_0$, $q_1 = q_{01} \circ p_1$, $q_1 = q_{12} \circ p_0$, $q_2 = q_{12} \circ p_1$, $q_0 = q_{02} \circ p_0$, and $q_2 = q_{02} \circ p_1$. Thus (U, T'', δ'') is an object of $\text{Cris}(X/S)$ and we get morphisms

$$T'' \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} T' \substack{\longrightarrow \\ \longrightarrow} T$$

of $\text{Cris}(X/S)$ satisfying the relations described above. In applications we will use $q_i : T'' \to T$ and $q_{ij} : T'' \to T'$ to denote the morphisms associated to the ring maps described above.

39.18. The de Rham complex

In Situation 39.11.5. Working on the (small) crystalline site, we define $\Omega^i_{X/S} = \wedge^i_{\mathcal{O}_{X/S}} \Omega_{X/S}$ for $i \geq 0$. The universal S-derivation $\text{d}_{X/S}$ gives rise to the *de Rham complex*

$$\mathcal{O}_{X/S} \to \Omega^1_{X/S} \to \Omega^2_{X/S} \to \ldots$$

on $\text{Cris}(X/S)$, see Lemma 39.16.4 and Remark 39.10.9.

39.19. Connections

In Situation 39.11.5. Given an $\mathcal{O}_{X/S}$-module $\mathcal{F}$ on $\text{Cris}(X/S)$ a *connection* is a map of abelian sheaves

$$\nabla : \mathcal{F} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega_{X/S}$$

such that $\nabla(fs) = f\nabla(s) + s \otimes \text{d}f$ for local sections s, f of $\mathcal{F}$ and $\mathcal{O}_{X/S}$. Given a connection there are canonical maps $\nabla : \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^i_{X/S} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^{i+1}_{X/S}$ defined by the rule $\nabla(s \otimes \omega) = \nabla(s) \wedge \omega + s \otimes \text{d}\omega$ as in Remark 39.10.10. We say the connection is *integrable* if $\nabla \circ \nabla = 0$. If ∇ is integrable we obtain the *de Rham complex*

$$\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^1_{X/S} \to \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^2_{X/S} \to \ldots$$

on $\text{Cris}(X/S)$. It turns out that any crystal in $\mathcal{O}_{X/S}$-modules comes equipped with a canonical integrable connection.

Lemma 39.19.1. *In Situation 39.11.5. Let $\mathcal{F}$ be a crystal in $\mathcal{O}_{X/S}$-modules on $\text{Cris}(X/S)$. Then $\mathcal{F}$ comes equipped with a canonical integrable connection.*

Proof. Say (U, T, δ) is an object of $\text{Cris}(X/S)$. Let (U, T', δ') be the infinitesimal thickening of T by $(\Omega_{X/S})_T = \Omega_{T/S,\delta}$ constructed in Remark 39.17.1. It comes with projections $p_0, p_1 : T' \to T$ and a diagonal $i : T \to T(1)$. By assumption we get isomorphisms

$$p_0^*\mathcal{F}_T \xrightarrow{c_0} \mathcal{F}_{T'} \xleftarrow{c_1} p_1^*\mathcal{F}_T$$

of $\mathcal{O}_{T'}$-modules. Pulling $c = c_1^{-1} \circ c_0$ back to T by i we obtain the identity map of $\mathcal{F}_T$. Hence if $s \in \Gamma(T, \mathcal{F}_T)$ then $\nabla(s) = p_1^*s - c(p_0^*s)$ is a section of $p_1^*\mathcal{F}_T$ which vanishes on pulling back by Δ. Hence $\nabla(s)$ is a section of

$$\mathcal{F}_T \otimes_{\mathcal{O}_T} \Omega_{T/S,\delta}$$

because this is the kernel of $p_1^*\mathcal{F}_T \to \mathcal{F}_T$ as $\Omega_{T/S,\delta}$ is the kernel of $\mathcal{O}_{T'} \to \mathcal{O}_T$ by construction.

The collection of maps

$$\nabla : \Gamma(T, \mathcal{F}_T) \to \Gamma(T, \mathcal{F}_T \otimes_{\mathcal{O}_T} \Omega_{T/S,\delta})$$

so obtained is functorial in T because the construction of T' is functorial in T. Hence we obtain a connection.

To show that the connection is integrable we consider the object (U, T'', δ'') constructed in Remark 39.17.2. Because $\mathcal{F}$ is a sheaf we see that

$$\begin{tikzcd} q_0^*\mathcal{F}_T \arrow[rr, "q_{01}^*c"] \arrow[dr, "q_{02}^*c"'] & & q_1^*\mathcal{F}_T \arrow[dl, "q_{12}^*c"] \\ & q_2^*\mathcal{F}_T & \end{tikzcd}$$

is a commutative map of $\mathcal{O}_{T''}$-modules. For $s \in \Gamma(T, \mathcal{F}_T)$ we have $c(p_0^*s) = p_1^*s - \nabla(s)$. Write $\nabla(s) = \sum p_1^*s_i \cdot \omega_i$ where s_i is a local section of $\mathcal{F}_T$ and ω_i is a local section of $\Omega_{T/S,\delta}$. We think of ω_i as a local section of the structure sheaf of $\mathcal{O}_{T'}$ and hence we write product instead of tensor product. On the one hand

$$\begin{aligned} q_{12}^*c \circ q_{01}^*c(q_0^*s) &= q_{12}^*c(q_1^*s - \sum q_1^*s_i \cdot q_{01}^*\omega_i) \\ &= q_2^*s - \sum q_2^*s_i \cdot q_{12}^*\omega_i - \sum q_2^*s_i \cdot q_{01}^*\omega_i + \sum q_{12}^*\nabla(s_i) \cdot q_{01}^*\omega_i \end{aligned}$$

and on the other hand

$$q_{02}^*c(q_0^*s) = q_2^*s - \sum q_2^*s_i \cdot q_{02}^*\omega_i.$$

From the formulae of Remark 39.17.2 we see that $q_{01}^*\omega_i + q_{12}^*\omega_i - q_{02}^*\omega_i = \mathrm{d}\omega_i$. Hence the difference of the two expressions above is

$$\sum q_2^*s_i \cdot \mathrm{d}\omega_i - \sum q_{12}^*\nabla(s_i) \cdot q_{01}^*\omega_i$$

Note that $q_{12}^*\omega \cdot q_{01}^*\omega' = \omega' \wedge \omega = -\omega \wedge \omega'$ by the definition of the multiplication on $\mathcal{O}_{T''}$. Thus the expression above is $\nabla^2(s)$ viewed as a section of the subsheaf $\mathcal{F}_T \otimes \Omega^2_{T/S,\delta}$ of $q_2^*\mathcal{F}$. Hence we get the integrability condition. $\square$

39.20. Cosimplicial algebra

This section should be moved somewhere else. A *cosimplicial ring* is a cosimplicial object in the category of rings. Given a ring R, a *cosimplicial R-algebra* is a cosimplicial object in the category of R-algebras. A *cosimplicial ideal* in a cosimplicial ring A_* is given by an ideal $I_n \subset A_n$ for all n such that $A(f)(I_n) \subset I_m$ for all $f : [n] \to [m]$ in Δ.

Let A_* be a cosimplicial ring. Let $\mathcal{C}$ be the category of pairs (A, M) where A is a ring and M is a module over A. A morphism $(A, M) \to (A', M')$ consists of a ring map $A \to A'$ and an A-module map $M \to M'$ where M' is viewed as an A-module via $A \to A'$ and the A'-module structure on M'. Having said this we can define a *cosimplicial module M_* over A_** as a cosimplicial object (A_*, M_*) of $\mathcal{C}$ whose first entry is equal to A_*. A *homomorphism $\varphi_* : M_* \to N_*$ of cosimplicial modules over A_** is a morphism $(A_*, M_*) \to (A_*, N_*)$ of cosimplicial objects in $\mathcal{C}$ whose first component is 1_{A_*}.

A *homotopy* between homomorphisms $\varphi_*, \psi_* : M_* \to N_*$ of cosimplicial modules over A_* is a homotopy between the associated maps $(A_*, M_*) \to (A_*, N_*)$ whose first component is the trival homotopy (dual to Simplicial, Example 14.24.3). We spell out what this means. Such a homotopy is a homotopy

$$h : M_* \longrightarrow \mathrm{Hom}(\Delta[1], N_*)$$

between φ_* and ψ_* as homomorphisms of cosimplicial abelian groups such that for each n the map $h_n : M_n \to \prod_{\alpha \in \Delta[1]_n} N_n$ is A_n-linear. The following lemma is a version of Simplicial, Lemma 14.26.3 for cosimplicial modules.

Lemma 39.20.1. *Let A_* be a cosimplicial ring. Let $\varphi_*, \psi_* : K_* \to M_*$ be homomorphisms of cosimplicial A_*-modules.*

(1) *If φ_* and ψ_* are homotopic, then*
$$\varphi_* \otimes 1, \psi_* \otimes 1 : K_* \otimes_{A_*} L_* \longrightarrow M_* \otimes_{A_*} L_*$$
are homotopic for any cosimplicial A_*-module L_*.

(2) *If φ_* and ψ_* are homotopic, then*
$$\wedge^i(\varphi_*), \wedge^i(\psi_*) : \wedge^i(K_*) \longrightarrow \wedge^i(M_*)$$
are homotopic.

(3) *If φ_* and ψ_* are homotopic, and $A_* \to B_*$ is a homomorphism of cosimplicial rings, then*
$$\varphi_* \otimes 1, \psi_* \otimes 1 : K_* \otimes_{A_*} B_* \longrightarrow M_* \otimes_{A_*} B_*$$
are homotopic as homomorphisms of cosimplicial B_*-modules.

(4) *If $I_* \subset A_*$ is a cosimplicial ideal, then the induced maps*
$$\varphi_*^\wedge, \psi_*^\wedge : K_*^\wedge \longrightarrow M_*^\wedge$$
between completions are homotopic.

(5) *Add more here as needed, for example symmetric powers.*

Proof. Let $h : M_* \longrightarrow \text{Hom}(\Delta[1], N_*)$ be the given homotopy. In degree n we have
$$h_n = (h_{n,\alpha}) : K_n \longrightarrow \prod\nolimits_{\alpha \in \Delta[1]_n} K_n$$
see Simplicial, Section 14.26. In order for a collection of $h_{n,\alpha}$ to form a homotopy, it is necessary and sufficient if for every $f : [n] \to [m]$ we have
$$h_{m,\alpha} \circ M_*(f) = N_*(f) \circ h_{n,\alpha \circ f}$$
see Simplicial, Equation (14.26.1.1). We also should have that $\psi_n = h_{n,0:[n]\to[1]}$ and $\varphi_n = h_{n,1:[n]\to[1]}$.

In each of the cases of the lemma we can produce the corresponding maps. Case (1). We can use the homotopy $h \otimes 1$ defined in degree n by setting
$$(h \otimes 1)_{n,\alpha} = h_{n,\alpha} \otimes 1_{L_n} : K_n \otimes_{A_n} L_n \longrightarrow M_n \otimes_{A_n} L_n.$$
Case (2). We can use the homotopy $\wedge^i h$ defined in degree n by setting
$$\wedge^i(h)_{n,\alpha} = \wedge^i(h_{n,\alpha}) : \wedge_{A_n}(K_n) \longrightarrow \wedge^i_{A_n}(M_n).$$
Case (3). We can use the homotopy $h \otimes 1$ defined in degree n by setting
$$(h \otimes 1)_{n,\alpha} = h_{n,\alpha} \otimes 1 : K_n \otimes_{A_n} B_n \longrightarrow M_n \otimes_{A_n} B_n.$$
Case (4). We can use the homotopy $h^\wedge$ defined in degree n by setting
$$(h^\wedge)_{n,\alpha} = h_{n,\alpha}^\wedge : K_n^\wedge \longrightarrow M_n^\wedge.$$
This works because each $h_{n,\alpha}$ is A_n-linear. $\square$

39.21. Notes on Rlim

This section should be moved somewhere else. We briefly discuss $R^1\, lim$. Consider the category $\mathbf{N}$ whose objects are natural numbers and whose morphisms are unique arrows $i \to j$ if $j \geq i$. Endow $\mathbf{N}$ with the chaotic topology so that a sheaf $\mathcal{F}$ is the same thing as an inverse system

$$\mathcal{F}_1 \leftarrow \mathcal{F}_2 \leftarrow \mathcal{F}_3 \leftarrow \ldots$$

over $\mathbf{N}$. Note that $\Gamma(\mathbf{N}, \mathcal{F}) = lim\, \mathcal{F}_n$. For an inverse system of abelian groups $\mathcal{F}_n$ we define

$$R^p\, lim\, \mathcal{F}_n = H^p(\mathbf{N}, \mathcal{F}).$$

Actually, it turns out that $R^p\, lim = 0$ for $p > 1$. Namely, note that the morphisms $i \to j$ are monomorphisms, which are turned into surjections by an injective sheaf of abelian groups (this is a generality about injective sheaves on any site). In particular, injective modules satisfy the Mittag-Leffler condition (see Homology, Section 10.23). Apply Derived Categories, Lemma 11.15.6 using Homology, Lemma 10.23.3 to the collection of inverse systems of abelian groups having ML, to conclude that $R^p\, lim$, $p > 0$ vanishes on all systems with ML. Applying Homology, Lemma 10.23.3 one more time we see that any inverse system of abelian groups has a two term resolution by systems having ML, which proves that $R^p\, lim = 0$ for $p > 1$.

Next, let's consider the derived functor

$$R\, lim : D(Ab(\mathbf{N})) \longrightarrow D(Ab)$$

as defined in Cohomology on Sites, Section 19.19. Another possible reference for the existence of $R\, lim$ is Derived Categories, Lemma 11.29.2. An object of $D(Ab(\mathbf{N}))$ is a complex of inverse systems of abelian groups. You can also think of this as an inverse system $(K_e^\bullet)$ of complexes. However, this is **not** the same thing as an inverse system of objects of $D(Ab)$; we will come back and explain the difference later.

Lemma 39.21.1. *Let $K = (K_e^\bullet)$ be an object of $D(Ab(\mathbf{N}))$. If for each n the inverse system (K_e^n) satisfies the Mittag-Leffler condition, then $R\, lim\, K$ is represented by the complex whose term in degree n is $lim_e\, K_e^n$.*

Proof. In the case that K is in $D^+(Ab(\mathbf{N}))$ this follows from the fact that each system $(K_e^n)_e$ is acyclic for $R\, lim$ (see above) and Derived Categories, Proposition 11.15.8. In fact, the same result holds for unbounded complexes, see Derived Categories, Lemma 11.29.2. $\square$

The products in the following lemma can be seen as termwise products of complexes or as products in the derived category $D(Ab)$, see Injectives, Remark 17.17.6.

Lemma 39.21.2. *Let $K = (K_e^\bullet)$ be an object of $D(Ab(\mathbf{N}))$. There exists a canonical distinguished triangle*

$$R\, lim\, K \to \prod\nolimits_e K_e^\bullet \to \prod\nolimits_e K_e^\bullet \to R\, lim\, K[1]$$

in $D(Ab)$ where the middle map fits into the commutative diagrams

$$\begin{tikzcd} \prod_e K_e^\bullet \arrow[r] \arrow[d] & \prod_e K_e^\bullet \arrow[d] \\ K_e^\bullet \oplus K_{e+1}^\bullet \arrow[r, "1-\pi"] & K_e^\bullet \end{tikzcd}$$

whose vertical maps are projections and where $\pi : K_{e+1}^\bullet \to K_e^\bullet$ is the transition map of the system.

Proof. Suppose that all the transition maps $K_{e+1}^\bullet \to K_e^\bullet$ are surjective. Then the map of complexes $\prod_e K_e^\bullet \to \prod_e K_e^\bullet$ of the statement of the lemma is surjective with kernel equal to the complex with term $lim_e K_e^n$ in degree n. Since this complex computes $R\,lim\,K$ by Lemma 39.21.1 we see that the lemma holds in this case.

In general one uses that there exists a quasi-isomorphism $K \to K'$ in $D(Ab(\mathbf{N}))$ such that the preceding argument applies to K'. Namely, given any complex in $Ab(\mathbf{N})$ there exists a quasi-isomorphism to a complex whose terms are injective objects of $Ab(\mathbf{N})$, see for example Injectives, Theorem 17.16.6. (We encourage the reader to find a direct argument him/herself in this special case.) Note that $\prod K_e^\bullet$ is quasi-isomorphic to $\prod (K')_e^\bullet$ as products are exact in Ab, whence the result for K' does imply the result for K. □

Lemma 39.21.3. *With notation as in Lemma 39.21.2 the long exact cohomology sequence associated to the distinguished triangle breaks up into short exact sequences*

$$0 \to R^1\,lim_e\, H^{p-1}(K_e^\bullet) \to H^p(R\,lim\,K) \to lim_e\, H^p(K_e^\bullet) \to 0$$

Proof. The long exact sequence of the distinguished triangle is

$$\ldots \to H^p(R\,lim\,K) \to \prod\nolimits_e H^p(K_e^\bullet) \to \prod\nolimits_e H^p(K_e^\bullet) \to H^{p+1}(R\,lim\,K) \to \ldots$$

The map in the middle has kernel $lim_e\, H^p(K_e^\bullet)$ by its explict description given in the lemma. Moreover, the cokernel of this map is $R^1\,lim_e\, H^p(K_e^\bullet)$ by an application of Lemma 39.21.2 to the case of an inverse system of abelian groups (placed in degree 0). The lemma follows. □

A sheaf of rings on $\mathbf{N}$ is just an inverse system of rings (A_e). A sheaf of modules over (A_e) is an inverse system (M_e) of abelian groups such that each M_e is an A_e-module and the transition maps $M_{e+1} \to M_e$ are A_{e+1}-module maps. The results on cohomology above apply to sheaves of modules as it is true in general that cohomology of groups and modules agree, see Cohomology on Sites, Lemma 19.12.4. Alternatively, one can rerun the arguments above for the case of modules. In particular we obtain a derived $R\,lim$ on modules

$$R\,lim : D(Mod(\mathbf{N}, (A_e))) \longrightarrow D(A)$$

where $A = lim\,A_e$ is the global sections of our given sheaf of modules. As in the case of abelian groups an object $M = (M_e^\bullet)$ of $D(Mod(\mathbf{N}, (A_e)))$ is an inverse system of complexes of modules, which is not the same thing as an inverse system of objects in the derived categories. However, it turns out one can still define $R\,lim$ of such a system well defined up to noncanonical isomorphism.

Remark 39.21.4. Suppose that we have an inverse system of rings (A_e) as above. Now suppose that we have an inverse system of objects $K_e^\bullet$ of $D(A_e)$. More precisely, suppose that we are given

(1) for every e an object $K_e^\bullet$ of $D(A_e)$, and
(2) for every e a map $\varphi_e : K_{e+1}^\bullet \to K_e^\bullet$ of $D(A_{e+1})$ where we think of $K_e^\bullet$ as an object of $D(A_{e+1})$ by restriction via the restriction map $A_{e+1} \to A_e$.

To be completely clear, by our definitions each $K_e^\bullet$ is a complex of A_e-modules, but the maps φ_e are in the derived category, hence may not be given by maps of complexes. We claim that there exists an object $M = (M_e^\bullet) \in D(Mod(\mathbf{N}, (A_e)))$ and isomorphisms $\psi_e : M_e^\bullet \to K_e^\bullet$ in

$D(A_e)$ such that the diagrams

$$\begin{tikzcd} M_{e+1}^\bullet \arrow[r, "\psi_{e+1}"'] \arrow[d] & K_{e+1}^\bullet \arrow[d, "\varphi_e"] \\ M_e^\bullet \arrow[r, "\psi_e"] & K_e^\bullet \end{tikzcd}$$

commute in $D(A_{e+1})$. Namely, set $M_1^\bullet = K_1^\bullet$. Suppose we have constructed $M_n^\bullet \to M_{n-1}^\bullet \to \ldots \to M_1^\bullet$ and maps of complexes $\psi_e : M_e^\bullet \to K_e^\bullet$ such that the diagrams above commute for all $e < n$. Then we consider the diagram

$$\begin{tikzcd} & M_n^\bullet \arrow[d, "\psi_n"] \\ K_{n+1}^\bullet \arrow[r, "\varphi_n"] & K_n^\bullet \end{tikzcd}$$

in $D(A_{n+1})$. By the definition of morphisms in $D(A_{n+1})$ we can find a quasi-isomorphism $\psi_{n+1} : M_{n+1}^\bullet \to K_{n+1}^\bullet$ of complexes of A_{n+1}-modules such that there exists a morphism of complexes $M_{n+1}^\bullet \to M_n^\bullet$ of A_{n+1}-modules representing the composition $\psi_n^{-1} \circ \varphi_n \circ \psi_{n+1}$ in $D(A_{n+1})$. Thus the claim holds by induction.

A priori there are many isomorphism classes of objects M of $D(Mod(\mathbf{N}, (A_e)))$ which give rise to the system $(K_e^\bullet, \varphi_e)$ as above. For each such M we can consider the complex $R\,lim\,M \in D(A)$ where $A = lim\,A_e$. By Lemma 39.21.2 there exists a canonical distinguished triangle

$$R\,lim\,M \to \prod\nolimits_e K_e^\bullet \to \prod\nolimits_e K_e^\bullet \to R\,lim\,M[1]$$

in $D(A)$. Hence we see that the isomorphism class of $R\,lim\,M$ in $D(A)$ is independent of the choices made in constructing M, by axiom TR3 of triangulated categories and Derived Categories, Lemma 11.4.3.

39.22. Crystals in quasi-coherent modules

In Situation 39.9.1. Set $X = Spec(C)$ and $S = Spec(A)$. We are going to classify crystals in quasi-coherent modules on Cris(X/S). Before we do so we fix some notation.

Choose a polynomial ring $P = A[x_i]$ over A and a surjection $P \to C$ of A-algebras with kernel $J = \mathrm{Ker}(P \to C)$. Set

(39.22.0.1) $$D = lim_e\, D_{P,\gamma}(J)/p^e D_{P,\gamma}(J)$$

for the p-adically completed divided power envelope. This ring comes with a divided power ideal $\bar{J}$ and divided power structure $\bar{\gamma}$, see Lemma 39.9.5. Set $D_e = D/p^e D$ and denote $\bar{J}_e$ the image of $\bar{J}$ in D_e. We will use the short hand

(39.22.0.2) $$\Omega_D = lim_e\, \Omega_{D_e/A,\bar{\gamma}} = lim_e\, \Omega_{D/A,\bar{\gamma}}/p^e\Omega_{D/A,\bar{\gamma}}$$

for the p-adic completion of the module of divided power differentials, see Lemma 39.10.12. It is also the p-adic completion of $\Omega_{D_{P,\gamma}(J)/A,\bar{\gamma}}$ which is free on $\mathrm{d}x_i$, see Lemma 39.10.6. Hence any element of Ω_D can be written uniquely as a sum $\sum f_i \mathrm{d}x_i$ with for all e only finitely many f_i not in $p^e D$. Moreover, the maps $\mathrm{d}_{D_e/A,\bar{\gamma}} : D_e \to \Omega_{D_e/A,\bar{\gamma}}$ fit together to define a divided power A-derivation

(39.22.0.3) $$\mathrm{d} : D \longrightarrow \Omega_D$$

on p-adic completions.

We will also need the ``products $Spec(D(n))$ of $Spec(D)$", see Proposition 39.26.1 and its proof for an explanation. Formally these are defined as follows. For $n \geq 0$ let $J(n) = \text{Ker}(P \otimes_A \ldots \otimes_A P \to C)$ where the tensor product has $n+1$ factors. We set

$$D(n) = lim_e \, D_{P\otimes_A \ldots \otimes_A P,\gamma}(J(n))/p^e D_{P\otimes_A \ldots \otimes_A P,\gamma}(J(n)) \tag{39.22.0.4}$$

equal to the p-adic completion of the divided power envelope. We denote $\bar{J}(n)$ its divided power ideal and $\bar{\gamma}(n)$ its divided powers. We also introduce $D(n)_e = D(n)/p^e D(n)$ as well as the p-adically completed module of differentials

$$\Omega_{D(n)} = lim_e \, \Omega_{D(n)_e/A,\bar{\gamma}} = lim_e \, \Omega_{D(n)/A,\bar{\gamma}}/p^e \Omega_{D(n)/A,\bar{\gamma}} \tag{39.22.0.5}$$

and derivation

$$\mathrm{d} : D(n) \longrightarrow \Omega_{D(n)} \tag{39.22.0.6}$$

Of course we have $D = D(0)$. Note that the rings $D(0), D(1), D(2), \ldots$ form a cosimplicial object in the category of divided power rings.

Lemma 39.22.1. *Let D and $D(n)$ be as in (39.22.0.1) and (39.22.0.4). The coprojection $P \to P \otimes_A \ldots \otimes_A P$, $f \mapsto f \otimes 1 \otimes \ldots \otimes 1$ induces an isomorphism*

$$D(n) = lim_e \, D\langle \xi_i(j)\rangle / p^e D\langle \xi_i(j)\rangle \tag{39.22.1.1}$$

of algebras over D with

$$\xi_i(j) = x_i \otimes 1 \otimes \ldots \otimes 1 - 1 \otimes \ldots \otimes 1 \otimes x_i \otimes 1 \otimes \ldots \otimes 1$$

for $j = 1, \ldots, n$.

Proof. We have

$$P \otimes_A \ldots \otimes_A P = P[\xi_i(j)]$$

and $J(n)$ is generated by J and the elements $\xi_i(j)$. Hence the lemma follows from Lemma 39.6.5. □

Lemma 39.22.2. *Let D and $D(n)$ be as in (39.22.0.1) and (39.22.0.4). Then $(D, \bar{J}, \bar{\gamma})$ and $(D(n), \bar{J}(n), \bar{\gamma}(n))$ are objects of $Cris^\wedge(C/A)$, see Remark 39.9.4, and*

$$D(n) = \coprod\nolimits_{j=0,\ldots,n} D$$

in $Cris^\wedge(C/A)$.

Proof. The first assertion is clear. For the second, if $(B \to C, \delta)$ is an object of $\text{Cris}^\wedge(C/A)$, then we have

$$\text{Mor}_{\text{Cris}^\wedge(C/A)}(D, B) = \text{Hom}_A((P, J), (B, \text{Ker}(B \to C)))$$

and similarly for $D(n)$ replacing (P, J) by $(P \otimes_A \ldots \otimes_A P, J(n))$. The property on coproducts follows as $P \otimes_A \ldots \otimes_A P$ is a coproduct. □

In the lemma below we will consider pairs (M, ∇) satisfying the following conditions

(1) M is a p-adically complete D-module,
(2) $\nabla : M \to M \otimes^\wedge_D \Omega_D$ is a connection, i.e., $\nabla(fm) = m \otimes \mathrm{d}f + f\nabla(m)$,
(3) ∇ is integrable (see Remark 39.10.10), and
(4) ∇ is *topologically quasi-nilpotent*: If we write $\nabla(m) = \sum \theta_i(m)\mathrm{d}x_i$ for some operators $\theta_i : M \to M$, then for any $m \in M$ there are only finitely many pairs (i, k) such that $\theta_i^k(m) \notin pM$.

The operators θ_i are sometimes denoted $\nabla_{\partial/\partial x_i}$ in the literature. In the following lemma we construct a functor from crystals in quasi-coherent modules on $\text{Cris}(X/S)$ to the category of such pairs. We will show this functor is an equivalent in Proposition 39.22.4.

Lemma 39.22.3. *In the situation above there is a functor*

$$\begin{matrix} \textit{crystals in quasi-coherent} \\ \mathcal{O}_{X/S}\textit{-modules on } \mathit{Cris}(X/S) \end{matrix} \longrightarrow \begin{matrix} \textit{pairs } (M, \nabla) \textit{ satisfying} \\ \textit{(1), (2), (3), and (4)} \end{matrix}$$

Proof. Let $\mathcal{F}$ be a crystal in quasi-coherent modules on X/S. Set $T_e = \mathit{Spec}(D_e)$ so that $(X, T_e, \bar\gamma)$ is an object of $\text{Cris}(X/S)$ for $e \gg 0$. We have morphisms

$$(X, T_e, \bar\gamma) \to (X, T_{e+1}, \bar\gamma) \to \ldots$$

which are closed immersions. We set

$$M = \mathit{lim}_e\, \Gamma((X, T_e, \bar\gamma), \mathcal{F}) = \mathit{lim}_e\, \Gamma(T_e, \mathcal{F}_{T_e}) = \mathit{lim}_e\, M_e$$

Note that since $\mathcal{F}$ is locally quasi-coherent we have $\mathcal{F}_{T_e} = \widetilde{M_e}$. Since $\mathcal{F}$ is a crystal we have $M_e = M_{e+1}/p^e M_{e+1}$. Hence we see that $M_e = M/p^e M$ and that M is p-adically complete.

By Lemma 39.19.1 we know that $\mathcal{F}$ comes endowed with a canonical integrable connection $\nabla : \mathcal{F} \to \mathcal{F} \otimes \Omega_{X/S}$. If we evaluate this connection on the objects T_e constructed above we obtain a canonical integrable connection

$$\nabla : M \longrightarrow M \otimes^\wedge_D \Omega_D$$

To see that this is topologically nilpotent we work out what this means.

Now we can do the same procedure for the rings $D(n)$. This produces a p-adically complete $D(n)$-module $M(n)$. Again using the crystal property of $\mathcal{F}$ we obtain isomorphisms

$$M \otimes^\wedge_{D, p_0} D(1) \to M(1) \leftarrow M \otimes^\wedge_{D, p_1} D(1)$$

compare with the proof of Lemma 39.19.1. Denote c the composition from left to right. Pick $m \in M$. Write $\xi_i = x_i \otimes 1 - 1 \otimes x_i$. Using (39.22.1.1) we can write uniquely

$$c(m \otimes 1) = \sum\nolimits_K \theta_K(m) \otimes \prod \xi_i^{[k_i]}$$

for some $\theta_K(m) \in M$ where the sum is over multi-indices $K = (k_i)$ with $k_i \geq 0$ and $\sum k_i < \infty$. Set $\theta_i = \theta_K$ where K has a 1 in the ith spot and zeros elsewhere. We have

$$\nabla(m) = \sum \theta_i(m) \mathrm{d}x_i.$$

as can be seen by comparing with the definition of ∇. Namely, the defining equation is $p_1^*m = \nabla(m) - c(p_0^*m)$ in Lemma 39.19.1 but the sign works out because in the stacks project we consistently use $\mathrm{d}f = p_1(f) - p_0(f)$ modulo the ideal of the diagonal squared, and hence $\xi_i = x_i \otimes 1 - 1 \otimes x_i$ maps to $-\mathrm{d}x_i$ modulo the ideal of the diagonal squared.

Denote $q_i : D \to D(2)$ and $q_{ij} : D(1) \to D(2)$ the coprojections corresponding to the indices i, j. As in the last paragraph of the proof of Lemma 39.19.1 we see that

$$q_{02}^* c = q_{12}^* c \circ q_{01}^* c.$$

This means that

$$\sum\nolimits_{K''} \theta_{K''}(m) \otimes \prod \zeta_i''^{[k_i'']} = \sum\nolimits_{K', K} \theta_{K'}(\theta_K(m)) \otimes \prod \zeta_i'^{[k_i']} \prod \zeta_i^{[k_i]}$$

in $M \otimes^{\wedge}_{D,q_2} D(2)$ where

$$\zeta_i = x_i \otimes 1 \otimes 1 - 1 \otimes x_i \otimes 1,$$
$$\zeta_i' = 1 \otimes x_i \otimes 1 - 1 \otimes 1 \otimes x_i,$$
$$\zeta_i'' = x_i \otimes 1 \otimes 1 - 1 \otimes 1 \otimes x_i.$$

In particular $\zeta_i'' = \zeta_i + \zeta_i'$ and we have that $D(2)$ is the p-adic completion of the divided power polynomial ring in ζ_i, ζ_i' over $q_2(D)$, see Lemma 39.22.1. Comparing coefficients in the expression above it follows immediately that $\theta_i \circ \theta_j = \theta_j \circ \theta_i$ (this provides an alternative proof of the integrability of ∇) and that

$$\theta_K(m) = (\prod \theta_i^{k_i})(m).$$

In particular, as the sum expressing $c(m \otimes 1)$ above has to converge p-adically we conclude that for each i and each $m \in M$ only a finite number of $\theta_i^k(m)$ are allowed to be nonzero modulo p. □

Proposition 39.22.4. *The functor*

$$\begin{matrix} \textit{crystals in quasi-coherent} \\ \mathcal{O}_{X/S}\textit{-modules on } \mathrm{Cris}(X/S) \end{matrix} \longrightarrow \begin{matrix} \textit{pairs } (M, \nabla) \textit{ satisfying} \\ \textit{(1), (2), (3), and (4)} \end{matrix}$$

of Lemma 39.22.3 is an equivalence of categories.

Proof. Let (M, ∇) be given. We are going to construct a crystal in quasi-coherent modules $\mathcal{F}$. Write $\nabla(m) = \sum \theta_i(m) \mathrm{d}x_i$. Then $\theta_i \circ \theta_j = \theta_j \circ \theta_i$ and we can set $\theta_K(m) = (\prod \theta_i^{k_i})(m)$ for any multi-index $K = (k_i)$ with $k_i \geq 0$ and $\sum k_i < \infty$.

Let (U, T, δ) be any object of $\mathrm{Cris}(X/S)$ with T affine. Say $T = Spec(B)$ and the ideal of $U \to T$ is $J_B \subset B$. By Lemma 39.9.6 there exists an integer e and a morphism

$$f : (U, T, \delta) \longrightarrow (X, T_e, \bar{\gamma})$$

where $T_e = Spec(D_e)$ as in the proof of Lemma 39.22.3. Choose such an e and f; denote $f : D \to B$ also the corresponding divided power A-algebra map. We will set $\mathcal{F}_T$ equal to the quasi-coherent sheaf of $\mathcal{O}_T$-modules associated to the B-module

$$M \otimes_{D,f} B.$$

However, we have to show that this is independent of the choice of f. Suppose that $g : D \to B$ is a second such morphism. Since f and g are morphisms in $\mathrm{Cris}(X/S)$ we see that the image of $f - g : D \to B$ is contained in the divided power ideal J_B. Write $\xi_i = f(x_i) - g(x_i) \in J_B$. By analogy with the proof of Lemma 39.22.3 we define an isomorphism

$$c_{f,g} : M \otimes_{D,f} B \longrightarrow M \otimes_{D,g} B$$

by the formula

$$m \otimes 1 \longmapsto \sum\nolimits_K \theta_K(m) \otimes \prod \xi_i^{[k_i]}$$

which makes sense by our remarks above and the fact that ∇ is topologically quasi-nilpotent (so the sum is finite!). A computation shows that

$$c_{g,h} \circ c_{f,g} = c_{f,h}$$

if given a third morphism $h : (U, T, \delta) \longrightarrow (X, T_e, \bar{\gamma})$. It is also true that $c_{f,f} = 1$. Hence these maps are all isomorphisms and we see that the module $\mathcal{F}_T$ is independent of the choice of f.

If $a : (U', T', \delta') \to (U, T, \delta)$ is a morphism of affine objects of $\text{Cris}(X/S)$, then choosing $f' = f \circ a$ it is clear that there exists a canonical isomorphism $a^*\mathcal{F}_T \to \mathcal{F}_{T'}$. We omit the verification that this map is independent of the choice of f. Using these maps as the restriction maps it is clear that we obtain a crystal in quasi-coherent modules on the full subcategory of $\text{Cris}(X/S)$ consisting of affine objects. We omit the proof that this extends to a crystal on all of $\text{Cris}(X/S)$. We also omit the proof that this procedure is a functor and that it is quasi-inverse to the functor constructed in Lemma 39.22.3. □

Lemma 39.22.5. *In Situation 39.9.1. Let $A \to P' \to C$ be ring maps with $A \to P'$ smooth and $P' \to C$ surjective with kernel J'. Let D' be the p-adic completion of $D_{P',\gamma}(J')$. There are homomorphisms of divided power A-algebras*

$$a : D \longrightarrow D', \quad b : D' \longrightarrow D$$

compatible with the maps $D \to C$ and $D' \to C$ such that $a \circ b = \text{id}_{D'}$. These maps induce an equivalence of categories of pairs (M, ∇) satisfying (1), (2), (3), and (4) over D and pairs (M', ∇') satisfying (1), (2), (3), and (4) over D'. In particular, the equivalence of categories of Proposition 39.22.4 also holds for the corresponding functor towards pairs over D'.

Proof. We can pick the map $P = A[x_i] \to C$ such that it factors through a surjection of A-algebras $P \to P'$ (we may have to increase the number of variables in P to do this). Hence we obtain a surjective map $a : D \to D'$ by functoriality of divided power envelopes and completion. Pick e large enough so that D_e is a divided power thickening of C over A. Then $D_e \to C$ is a surjection whose kernel is locally nilpotent, see Lemma 39.2.6. Setting $D'_e = D'/p^e D'$ we see that the kernel of $D_e \to D'_e$ is locally nilpotent. Hence by Algebra, Lemma 7.127.16 we can find a lift $\beta_e : P' \to D_e$ of the map $P' \to D'_e$. Note that $D_{e+i+1} \to D_{e+i} \times_{D'_{e+i}} D'_{e+i+1}$ is surjective with square zero kernel for any $i \geq 0$ because $p^{e+i}D \to p^{e+i}D'$ is surjective. Applying the usual lifting property (Algebra, Proposition 7.127.13) succesively to the diagrams

$$\begin{tikzcd} P' \arrow[r] & D_{e+i} \times_{D'_{e+i}} D'_{e+i+1} \\ A \arrow[u] \arrow[r] & D_{e+i+1} \arrow[u] \end{tikzcd}$$

we see that we can find an A-algebra map $\beta : P' \to D$ whose composition with a is the given map $P' \to D'$. By the universal property of the divided power envelope we obtain a map $D_{P',\gamma}(J') \to D$. As D is p-adically complete we obtain $b : D' \to D$ such that $a \circ b = \text{id}_{D'}$.

Consider the base change functor

$$(M, \nabla) \longmapsto (M \otimes^\wedge_D D', \nabla')$$

from pairs for D to pairs for D', see Remark 39.10.11. Similarly, we have the base change functor corresponding to the divided power homomorphism $D' \to D$. To finish the proof of the lemma we have to show that the base change for the compositions $b \circ a : D \to D$ and $a \circ b : D' \to D'$ are isomorphic to the identity functor. This is clear for the second as $a \circ b = \text{id}_{D'}$. To prove it for the first, we use the functorial isomorphism

$$c_{\text{id}_D, b\circ a} : M \otimes_{D,\text{id}_D} D \longrightarrow M \otimes_{D,b\circ a} D$$

of the proof of Proposition 39.22.4. The only thing to prove is that these maps are horizontal, which we omit.

The last statement of the proof now follows. □

Remark 39.22.6. The equivalence of Proposition 39.22.4 holds if we start with a surjection $P \to C$ where P/A satisfies the strong lifting property of Algebra, Lemma 7.127.16. To prove this we can argue as in the proof of Lemma 39.22.5. (Details will be added here if we ever need this.) Presumably there is also a direct proof of this result, but the advantage of using polynomial rings is that the rings $D(n)$ are p-adic completions of divided power polynomial rings and the algebra is simplified.

39.23. General remarks on cohomology

In this section we do a bit of work to translate the cohomology of modules on the cristalline site of an affine scheme into an algebraic question.

Lemma 39.23.1. *In Situation 39.11.5. Let $\mathcal{F}$ be a locally quasi-coherent $\mathcal{O}_{X/S}$-module on $Cris(X/S)$. Then we have*

$$H^p((U,T,\delta),\mathcal{F}) = 0$$

for all $p > 0$ and all (U,T,δ) with T or U affine.

Proof. As $U \to T$ is a thickening we see that U is affine if and only if T is affine, see Limits, Lemma 27.7.1. Having said this, let us apply Cohomology on Sites, Lemma 19.11.8 to the collection $\mathcal{B}$ of affine objects (U,T,δ) and the collection Cov of affine open coverings $\mathcal{U} = \{(U_i,T_i,\delta_i) \to (U,T,\delta)\}$. The Čech complex $\check{C}^*(\mathcal{U},\mathcal{F})$ for such a covering is simply the Cech complex of the quasi-coherent $\mathcal{O}_T$-module $\mathcal{F}_T$ (here we are using the assumption that $\mathcal{F}$ is locally quasi-coherent) with respect to the affine open covering $\{T_i \to T\}$ of the affine scheme T. Hence the Čech cohomology is zero by Coherent, Lemma 25.2.4 and 25.2.2. Thus the hypothesis of Cohomology on Sites, Lemma 19.11.8 are satisfied and we win. □

Lemma 39.23.2. *In Situation 39.11.5. Assume moreover X and S are affine schemes. Consider the full subcategory $\mathcal{C} \subset Cris(X/S)$ consisting of divided power thickenings (X,T,δ) endowed with the chaotic topology (see Sites, Example 9.6.6). For any locally quasi-coherent $\mathcal{O}_{X/S}$-module $\mathcal{F}$ we have*

$$R\Gamma(\mathcal{C},\mathcal{F}|_\mathcal{C}) = R\Gamma(Cris(X/S),\mathcal{F})$$

Proof. We will use without further mention that $\mathcal{C}$ and $\text{Cris}(X/S)$ have products and fibre products, see Lemma 39.12.2. Note that the inclusion functor $u : \mathcal{C} \to \text{Cris}(X/S)$ is fully faithful, continuous and commutes with products and fibre products. We claim it defines a morphism of ringed sites

$$f : (\text{Cris}(X/S),\mathcal{O}_{X/S}) \longrightarrow (Sh(\mathcal{C}),\mathcal{O}_{X/S}|_\mathcal{C})$$

To see this we will use Sites, Lemma 9.14.5. Note that $\mathcal{C}$ has fibre products and u commutes with them so the categories $\mathcal{I}^u_{(U,T,\delta)}$ are disjoint unions of directed categories (by Sites, Lemma 9.5.1 and Categories, Lemma 4.17.3). Hence it suffices to show that $\mathcal{I}^u_{(U,T,\delta)}$ is nonempty and directed. Nonempty follows from Lemma 39.9.6 and connected follows from the fact that $\mathcal{C}$ has products and that u commutes with them (compare with the proof of Sites, Lemma 9.5.2).

Note that $f_*\mathcal{F} = \mathcal{F}|_{\mathcal{C}}$. Hence the lemma follows if $R^p f_*\mathcal{F} = 0$ for $p > 0$, see Cohomology on Sites, Lemma 19.14.5. By Cohomology on Sites, Lemma 19.8.4 it suffices to show that $H^p((X, T, \delta), \mathcal{F}) = 0$ for all (X, T, δ). This follows from Lemma 39.23.1. □

Lemma 39.23.3. *In Situation 39.9.1. Set $\mathcal{C} = (Cris(C/A))^{opp}$ and $\mathcal{C}^\wedge = (Cris^\wedge(C/A))^{opp}$ endowed with the chaotic topology, see Remark 39.9.4 for notation. There is a morphism of topoi*

$$g : Sh(\mathcal{C}) \longrightarrow Sh(\mathcal{C}^\wedge)$$

such that if $\mathcal{F}$ is a sheaf of abelian groups on $\mathcal{C}$, then

$$R^p g_*\mathcal{F}(B \to C, \delta) = \begin{cases} lim_e\, \mathcal{F}(B_e \to C, \delta) & \text{if } p = 0 \\ R^1\, lim_e\, \mathcal{F}(B_e \to C, \delta) & \text{if } p = 1 \\ 0 & \text{else} \end{cases}$$

where $B_e = B/p^e B$ for $e \gg 0$.

Proof. Any functor between categories defines a morphism between chaotic topoi in the same direction, for example because such a functor can be considered as a cocontinuous functor between sites, see Sites, Section 9.19. Proof of the description of $g_*\mathcal{F}$ is omitted. Note that in the statement we take $(B_e \to C, \delta)$ is an object of Cris(C/A) only for e large enough. Let $\mathcal{I}$ be an injective abelian sheaf on $\mathcal{C}$. Then the transition maps

$$\mathcal{I}(B_e \to C, \delta) \leftarrow \mathcal{I}(B_{e+1} \to C, \delta)$$

are surjective as the morphisms

$$(B_e \to C, \delta) \longrightarrow (B_{e+1} \to C, \delta)$$

are monomorphisms in the category $\mathcal{C}$. Hence for an injective abelian sheaf both sides of the displayed formula of the lemma agree. Taking an injective resolution of $\mathcal{F}$ one easily obtains the result (sheaves are presheaves, so exactness is measured on the level of groups of sections over objects). □

Lemma 39.23.4. *Let $\mathcal{C}$ be a category endowed with the chaotic topology. Let X be an object of $\mathcal{C}$ such that every object of $\mathcal{C}$ has a morphism towards X. Assume that $\mathcal{C}$ has products. Then for every abelian sheaf $\mathcal{F}$ on $\mathcal{C}$ the total cohomology $R\Gamma(\mathcal{C}, \mathcal{F})$ is represented by the complex*

$$\mathcal{F}(X) \to \mathcal{F}(X \times X) \to \mathcal{F}(X \times X \times X) \to \ldots$$

associated to the cosimplicial abelian group $[n] \mapsto \mathcal{F}(X^n)$.

Proof. Note that $H^q(X^p, \mathcal{F}) = 0$ for all $q > 0$ as sheaves are presheaves on $\mathcal{C}$. The assumption on X is that $h_X \to *$ is surjective. Using that $H^q(X, \mathcal{F}) = H^p(h_X, \mathcal{F})$ and $H^p(\mathcal{C}, \mathcal{F}) = H^p(*, \mathcal{F})$ we see that our statement is a special case of Cohomology on Sites, Lemma 19.13.2. □

39.24. Cosimplicial preparations

In this section we compare crystalline cohomology with de Rham cohomology. We follow [**Bd11**].

Example 39.24.1. Suppose that A_* is any cosimplicial ring. Consider the cosimplicial module M_* defined by the rule

$$M_n = \bigoplus\nolimits_{i=0,\ldots,n} A_n e_i$$

For a map $f : [n] \to [m]$ define $M_*(f) : M_n \to M_m$ to be the unique $A_*(f)$-linear map which maps e_i to $e_{f(i)}$. We claim the identity on M_* is homotopic to 0. Namely, a homotopy is given by a map of cosimplicial modules

$$h : M_* \longrightarrow \mathrm{Hom}(\Delta[1], M_*)$$

see Section 39.20. For $j \in \{0, \ldots, n+1\}$ we let $\alpha^n_j : [n] \to [1]$ be the map defined by $\alpha^n_j(i) = 0 \Leftrightarrow i < j$. Then $\Delta[1]_n = \{\alpha^n_0, \ldots, \alpha^n_{n+1}\}$ and correspondingly $\mathrm{Hom}(\Delta[1], M_*)_n = \prod_{j=0,\ldots,n+1} M_n$, see Simplicial, Sections 14.24 and 14.26. Instead of using this product representation, we think of an element in $\mathrm{Hom}(\Delta[1], M_*)_n$ as a function $\Delta[1]_n \to M_n$. Using this notation, we define h in degree n by the rule

$$h_n(e_i)(\alpha^n_j) = \begin{cases} e_i & \text{if} \quad i < j \\ 0 & \text{else} \end{cases}$$

We first check h is a morphism of cosimplicial modules. Namely, for $f : [n] \to [m]$ we will show that

$$h_m \circ M_*(f) = \mathrm{Hom}(\Delta[1], M_*)(f) \circ h_n \tag{39.24.1.1}$$

The left hand side of (39.24.1.1) evaluted at e_i and then in turn evaluated at α^m_j is

$$h_m(e_{f(i)})(\alpha^m_j) = \begin{cases} e_{f(i)} & \text{if} \quad f(i) < j \\ 0 & \text{else} \end{cases}$$

Note that $\alpha^m_j \circ f = \alpha^n_{j'}$, where $0 \leq j' \leq n+1$ is the unique index such that $f(i) < j$ if and only if $i < j'$. Thus the right hand side of (39.24.1.1) evaluted at e_i and then in turn evaluated at α^m_j is

$$M_*(f)(h_n(e_i)(\alpha^m_j \circ f) = M_*(f)(h_n(e_i)(\alpha^n_{j'})) = \begin{cases} e_{f(i)} & \text{if} \quad i < j' \\ 0 & \text{else} \end{cases}$$

It follows from our description of j' that the two answers are equal. Hence h is a map of cosimplicial modules. Let $0 : \Delta[0] \to \Delta[1]$ and $1 : \Delta[0] \to \Delta[1]$ be the obvious maps, and denote $ev_0, ev_1 : \mathrm{Hom}(\Delta[1], M_*) \to M_*$ the corresponding evaluation maps. The reader verifies readily that the the compositions

$$ev_0 \circ h, ev_1 \circ h : M_* \longrightarrow M_*$$

are 0 and 1 respectively, whence h is the desired homotopy between 0 and 1.

Lemma 39.24.2. *With notation as in (39.22.0.5) the complex*

$$\Omega_{D(0)} \to \Omega_{D(1)} \to \Omega_{D(2)} \to \cdots$$

is homotopic to zero as a $D(*)$-cosimplicial module.

Proof. We are going to use the principle of Simplicial, Lemma 14.26.3 and more specifically Lemma 39.20.1 which tells us that homotopic maps between (co)simplicial objects are transformed by any functor into homotopic maps. The complex of the lemma is equal to the p-adic completion of the base change of the cosimplicial module

$$M_* = \left(\Omega_{P/A} \to \Omega_{P\otimes_A P/A} \to \Omega_{P\otimes_A P\otimes_A P/A} \to \cdots\right)$$

via the cosimplicial ring map $P \otimes_A \ldots \otimes_A P \to D(n)$. This follows from Lemma 39.10.6, see comments following (39.22.0.2). Hence it suffices to show that the cosimplicial module M_* is homotopic to zero (uses base change and p-adic completion). We can even assume

$A = \mathbf{Z}$ and $P = \mathbf{Z}[\{x_i\}_{i \in I}]$ as we can use base change with $\mathbf{Z} \to A$. In this case $P^{\otimes n+1}$ is the polynomial algebra on the elements

$$x_i(e) = 1 \otimes \cdots \otimes x_i \otimes \cdots \otimes 1$$

with x_i in the eth slot. The modules of the complex are free on the generators $\mathrm{d}x_i(e)$. Note that if $f : [n] \to [m]$ is a map then we see that

$$M_*(f)(\mathrm{d}x_i(e)) = \mathrm{d}x_i(f(e))$$

Hence we see that M_* is a direct sum over I of copies of the module studied in Example 39.24.1 and we win. □

Lemma 39.24.3. *With notation as in (39.22.0.4) and (39.22.0.5), given any cosimplicial module M_* over $D(*)$ and $i > 0$ the cosimplicial module*

$$M_0 \otimes^\wedge_{D(0)} \Omega^i_{D(0)} \to M_1 \otimes^\wedge_{D(1)} \Omega^i_{D(1)} \to M_2 \otimes^\wedge_{D(2)} \Omega^i_{D(2)} \to \ldots$$

is homotopic to zero, where $\Omega^i_{D(n)}$ is the p-adic completion of the ith exterior power of $\Omega_{D(n)}$.

Proof. By Lemma 39.24.2 the endomorphisms 0 and 1 of $\Omega_{D(*)}$ are homotopic. If we apply the functor $\wedge^i$ we see that the same is true for the cosimplicial module $\wedge^i\Omega_{D(*)}$, see Lemma 39.20.1. Another application of the same lemma shows the p-adic completion $\Omega^i_{D(*)}$ is homotopy equivalent to zero. Tensoring with M_* we see that $M_* \otimes_{D(*)} \Omega^i_{D(*)}$ is homotopic to zero, see Lemma 39.20.1 again. A final application of the p-adic completion functor finishes the proof. □

39.25. Divided power Poincaré lemma

Just the simplest possible version.

Lemma 39.25.1. *Let A be a ring. Let $P = A\langle x_i\rangle$ be a divided power polynomial ring over A. For any A-module M the complex*

$$0 \to M \to M \otimes_A P \to M \otimes_A \Omega^1_{P/A,\delta} \to M \otimes_A \Omega^2_{P/A,\delta} \to \ldots$$

is exact. Let D be the p-adic completion of P. Let Ω^i_D be the p-adic completion of the ith exterior power of $\Omega_{D/A,\delta}$. For any p-adically complete A-module M the complex

$$0 \to M \to M \otimes^\wedge_A D \to M \otimes^\wedge_A \Omega^1_D \to M \otimes^\wedge_A \Omega^2_D \to \ldots$$

is exact.

Proof. It suffices to show that the complex

$$E : (0 \to A \to P \to \Omega^1_{P/A,\delta} \to \Omega^2_{P/A,\delta} \to \ldots)$$

is homotopy equivalent to zero as a complex of A-modules. For every multi-index $K = (k_i)$ we can consider the subcomplex $E(K)$ which in degree j consists of

$$\bigoplus\nolimits_{I=\{i_1,\ldots,i_j\}\subset \mathrm{Supp}(K)} A \prod\nolimits_{i \not\in I} x_i^{[k_i]} \prod\nolimits_{i\in I} x_i^{[k_i-1]} \mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_j}$$

Since $E = \bigoplus E(K)$ we see that it suffices to prove each of the complexes $E(K)$ is homotopic to zero. If $K = 0$, then $E(K) : (A \to A)$ is homotopic to zero. If K has nonempty (finite) support S, then the complex $E(K)$ is isomorphic to the complex

$$0 \to A \to \bigoplus\nolimits_{s\in S} A \to \wedge^2(\bigoplus\nolimits_{s\in S} A) \to \ldots \to \wedge^{\#S}(\bigoplus\nolimits_{s\in S} A) \to 0$$

which is homotopic to zero, for example by More on Algebra, Lemma 12.21.5. □

An alternative (more direct) approach to the following lemma is explained in Example 39.30.2.

Lemma 39.25.2. *Let A be a ring. Let (B, J, δ) be a divided power ring. Let $P = B\langle x_i\rangle$ be a divided power polynomial ring over B with divided power ideal $J = IP + B\langle x_i\rangle_+$ as usual. Let M be a B-module endowed with an integrable connection $\nabla : M \to M \otimes_B \Omega^1_{B/A,\delta}$. Then the map of de Rham complexes*

$$M \otimes_B \Omega^*_{B/A,\delta} \longrightarrow M \otimes_P \Omega^*_{P/A,\delta}$$

is a quasi-isomorphism. Let D, resp. D' be the p-adic completion of B, resp. P and let Ω^i_D, resp. $\Omega^i_{D'}$ be the p-adic completion of $\Omega^i_{B/A,\delta}$, resp. $\Omega^i_{P/A,\delta}$. Let M be a p-adically complete D-module endowed with an integral connection $\nabla : M \to M \otimes^\wedge_D \Omega^1_D$. Then the map of de Rham complexes

$$M \otimes^\wedge_D \Omega^*_D \longrightarrow M \otimes^\wedge_D \Omega^*_{D'}$$

is a quasi-isomorphism.

Proof. Consider the decreasing filtration F^* on $\Omega^*_{B/A,\delta}$ given by the subcomplexes $F^i(\Omega^*_{B/A,\delta}) = \sigma_{\geq i}\Omega^*_{B/A,\delta}$. See Homology, Section 10.11. This induces a decreasing filtration F^* on $\Omega^*_{P/A,\delta}$ by setting

$$F^i(\Omega^*_{P/A,\delta}) = F^i(\Omega^*_{B/A,\delta}) \wedge \Omega^*_{P/A,\delta}.$$

We have a split short exact sequence

$$0 \to \Omega^1_{B/A,\delta} \otimes_B P \to \Omega^1_{P/A,\delta} \to \Omega^1_{P/B,\delta} \to 0$$

and the last module is free on $\mathrm{d}x_i$. It follows from this that $F^i(\Omega^*_{P/A,\delta}) \to \Omega^*_{P/A,\delta}$ is a termwise split injection and that

$$\mathrm{gr}^i_F(\Omega^*_{B/A,\delta}) = \Omega^i_{B/A,\delta} \otimes_B \Omega^*_{P/B,\delta}$$

as complexes. Thus we can define a filtration F^* on $M \otimes_B \Omega^*_{B/A,\delta}$ by setting

$$F^i(M \otimes_B \Omega^*_{P/A,\delta}) = M \otimes_B F^i(\Omega^*_{P/A,\delta})$$

and we have

$$\mathrm{gr}^i_F(M \otimes_B \Omega^*_{P/A,\delta}) = M \otimes_B \Omega^i_{B/A,\delta} \otimes_B \Omega^*_{P/B,\delta}$$

as complexes. By Lemma 39.25.1 each of these complexes is quasi-isomorphic to $M \otimes_B \Omega^i_{B/A,\delta}$ placed in degree 0. Hence we see that the first displayed map of the lemma is a morphism of filtered complexes which induces a quasi-isomorphism on graded pieces. This implies that it is a quasi-isomorphism, for example by the spectral sequence associated to a filtered complex, see Homology, Section 10.18.

The proof of the second quasi-isomorphism is exactly the same. □

39.26. Cohomology in the affine case

Let's go back to the situation studied in Section 39.22. We start with (A, I, γ) and $A/I \to C$ and set $X = Spec(C)$ and $S = Spec(A)$. Then we choose a polynomial ring P over A and a surjection $P \to C$ with kernel J. We obtain D and $D(n)$ see (39.22.0.1) and (39.22.0.4). Set $T(n)_e = \mathrm{Spec}(D(n)/p^e D(n))$ so that $(X, T(n)_e, \delta(n))$ is an object of $\mathrm{Cris}(X/S)$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_{X/S}$-modules and set

$$M(n) = lim_e\, \Gamma((X, T(n)_e, \delta(n)), \mathcal{F})$$

for $n = 0, 1, 2, 3, \ldots$. This forms a cosimplicial module over the cosimplicial ring $D(0), D(1), D(2), \ldots$.

Proposition 39.26.1. *With notations as above assume that*

(1) *$\mathcal{F}$ is locally quasi-coherent, and*
(2) *for any morphism $(U, T, \delta) \to (U', T', \delta')$ of Cris(X/S) where $f : T \to T'$ is a closed immersion the map $c_f : f^*\mathcal{F}_{T'} \to \mathcal{F}_T$ is surjective.*

Then the complex

$$M(0) \to M(1) \to M(2) \to \ldots$$

computes $R\Gamma(Cris(X/S), \mathcal{F})$.

Proof. Using assumption (1) and Lemma 39.23.2 we see that $R\Gamma(\text{Cris}(X/S), \mathcal{F})$ is isomorphic to $R\Gamma(\mathcal{C}, \mathcal{F})$. Note that the categories $\mathcal{C}$ used in Lemmas 39.23.2 and 39.23.3 agree. Let $f : T \to T'$ be a closed immersion as in (2). Surjectivity of $c_f : f^*\mathcal{F}_{T'} \to \mathcal{F}_T$ is equivalent to surjectivity of $\mathcal{F}_{T'} \to f_*\mathcal{F}_T$. Hence, if $\mathcal{F}$ satisfies (1) and (2), then we obtain a short exact sequence

$$0 \to \mathcal{K} \to \mathcal{F}_{T'} \to f_*\mathcal{F}_T \to 0$$

of quasi-coherent $\mathcal{O}_{T'}$-modules on T', see Schemes, Section 21.24 and in particular Lemma 21.24.1. Thus, if T' is affine, then we conclude that the restriction map $\mathcal{F}(U', T', \delta') \to \mathcal{F}(U, T, \delta)$ is surjective by the vanishing of $H^1(T', \mathcal{K})$, see Coherent, Lemma 25.2.2. Hence the transition maps of the inverse systems in Lemma 39.23.3 are surjective. We conclude that that $R^pg_*(\mathcal{F}|_{\mathcal{C}}) = 0$ for all $p \geq 1$ where g is as in Lemma 39.23.3. The object D of the category $\mathcal{C}^\wedge$ satisfies the assumption of Lemma 39.23.4 by Lemma 39.9.7 with

$$D \times \ldots \times D = D(n)$$

in $\mathcal{C}$ because $D(n)$ is the $n + 1$-fold coproduct of D in Cris$^\wedge(C/A)$, see Lemma 39.22.2. Thus we win. $\square$

Lemma 39.26.2. *Assumptions and notation as in Proposition 39.26.1. Then*

$$H^j(Cris(X/S), \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^i_{X/S}) = 0$$

for all $i > 0$ and all $j \geq 0$.

Proof. Using Lemma 39.16.6 it follows that $\mathcal{H} = \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^i_{X/S}$ also satisfies assumptions (1) and (2) of Proposition 39.26.1. Write $M(n)_e = \Gamma((X, T(n)_e, \delta(n)), \mathcal{F})$ so that $M(n) = lim_e\, M(n)_e$. Then

$$\begin{aligned} lim_e\, \Gamma((X, T(n)_e, \delta(n)), \mathcal{H}) &= lim_e\, M(n)_e \otimes_{D(n)_e} \Omega_{D(n)}/p^e\Omega_{D(n)} \\ &= lim_e\, M(n)_e \otimes_{D(n)} \Omega_{D(n)} \end{aligned}$$

By Lemma 39.24.3 the cosimplicial modules

$$M(0)_e \otimes_{D(0)} \Omega^i_{D(0)} \to M(1)_e \otimes_{D(1)} \Omega^i_{D(1)} \to M(2)_e \otimes_{D(2)} \Omega^i_{D(2)} \to \ldots$$

are homotopic to zero. Because the transition maps $M(n)_{e+1} \to M(n)_e$ are surjective, we see that the inverse limit of the associated complexes are acyclic[5]. Hence the vanshing of cohomology of $\mathcal{H}$ by Proposition 39.26.1. $\square$

[5]Actually, they are even homotopic to zero as the homotopies fit together, but we don't need this. The reason for this roundabout argument is that the limit $lim_e\, M(n)_e \otimes_{D(n)} \Omega^i_{D(n)}$ isn't the p-adic completion of $M(n) \otimes_{D(n)} \Omega^i_{D(n)}$ as with the assumptions of the lemma we don't know that $M(n)_e = M(n)_{e+1}/p^eM(n)_{e+1}$. If $\mathcal{F}$ is a crystal then this does hold.

Proposition 39.26.3. *Assumptions as in Proposition 39.26.1 but now assume that $\mathcal{F}$ is a crystal in quasi-coherent modules. Let (M, ∇) be the corresponding module with connection over D, see Proposition 39.22.4. Then the complex*

$$M \otimes^{\wedge}_D \Omega^*_D$$

computes $R\Gamma(Cris(X/S), \mathcal{F})$.

Proof. We will prove this using the two spectral sequences associated to the double complex $K^{*,*}$ with terms

$$K^{a,b} = M \otimes^{\wedge}_D \Omega^a_{D(b)}$$

What do we know so far? Well, Lemma 39.24.3 tells us that each column $K^{a,*}$, $a > 0$ is acyclic. Proposition 39.26.1 tells us that the first column $K^{0,*}$ is quasi-isomorphic to $R\Gamma(\text{Cris}(X/S), \mathcal{F})$. Hence the first spectral sequence associated to the double complex shows that there is a canonical quasi-isomorphism of $R\Gamma(\text{Cris}(X/S), \mathcal{F})$ with $\text{Tot}(K^{*,*})$.

Next, let's consider the rows $K^{*,b}$. By Lemma 39.22.1 each of the $b+1$ maps $D \to D(b)$ presents $D(b)$ as the p-adic completion of a divided power polynomial algebra over D. Hence Lemma 39.25.2 shows that the map

$$M \otimes^{\wedge}_D \Omega^*_D \longrightarrow M \otimes^{\wedge}_{D(b)} \Omega^*_{D(b)} = K^{*,b}$$

is a quasi-isomorphism. Note that each of these maps defines the *same* map on cohomology (and even the same map in the derived category) as the inverse is given by the co-diagonal map $D(b) \to D$ (corresponding to the multiplication map $P \otimes_A \ldots \otimes_A P \to P$). Hence if we look at the E_1 page of the second spectral sequence we obtain

$$E_1^{a,b} = H^a(M \otimes^{\wedge}_D \Omega^*_D)$$

with differentals

$$E_1^{a,0} \xrightarrow{0} E_1^{a,1} \xrightarrow{1} E_1^{a,2} \xrightarrow{0} E_1^{a,3} \xrightarrow{1} \ldots$$

as each of these is the alternation sum of the given identifications $H^a(M \otimes^{\wedge}_D \Omega^*_D) = E_1^{a,0} = E_1^{a,1} = \ldots$. Thus we see that the E_2 page is equal $H^a(M \otimes^{\wedge}_D \Omega^*_D)$ on the first row and zero elsewhere. It follows that the identification of $M \otimes^{\wedge}_D \Omega^*_D$ with the first row induces a quasi-isomorphism of $M \otimes^{\wedge}_D \Omega^*_D$ with $\text{Tot}(K^{*,*})$. □

Lemma 39.26.4. *Assumptions as in Proposition 39.26.3. Let $A \to P' \to C$ be ring maps with $A \to P'$ smooth and $P' \to C$ surjective with kernel J'. Let D' be the p-adic completion of $D_{P',\gamma}(J')$. Let (M', ∇') be the pair over D' corresponding to $\mathcal{F}$, see Lemma 39.22.5. Then the complex*

$$M' \otimes^{\wedge}_{D'} \Omega^*_{D'}$$

computes $R\Gamma(Cris(X/S), \mathcal{F})$.

Proof. Choose $a : D \to D'$ and $b : D' \to D$ as in Lemma 39.22.5. Note that the base change $M = M' \otimes_{D',b} D$ with its connection ∇ corresponds to $\mathcal{F}$. Hence we know that $M \otimes^{\wedge}_D \Omega^*_D$ computes the crystalline cohomology of $\mathcal{F}$, see Proposition 39.26.3. Hence it suffices to show that the base change maps (induced by a and b)

$$M' \otimes^{\wedge}_{D'} \Omega^*_{D'} \longrightarrow M \otimes^{\wedge}_D \Omega^*_D \quad \text{and} \quad M \otimes^{\wedge}_D \Omega^*_D \longrightarrow M' \otimes^{\wedge}_{D'} \Omega^*_{D'}$$

are quasi-isomorphisms. Since $a \circ b = \text{id}_{D'}$ we see that the composition one way around is the identity on the complex $M' \otimes^{\wedge}_{D'} \Omega^*_{D'}$. Hence it suffices to show that the map

$$M \otimes^{\wedge}_D \Omega^*_D \longrightarrow M \otimes^{\wedge}_D \Omega^*_D$$

induced by $b \circ a : D \to D$ is a quasi-isomorphism. (Note that we have the same complex on both sides as $M = M' \otimes^\wedge_{D',b} D$, hence $M \otimes^\wedge_{D,b\circ a} D = M' \otimes^\wedge_{D',b\circ a\circ b} D = M' \otimes^\wedge_{D',b} D = M$.) In fact, we claim that for any divided power A-algebra homomorphism $\rho : D \to D$ compatible with the augmentation to C the induced map $M \otimes^\wedge_D \Omega^*_D \to M \otimes^\wedge_{D,\rho} \Omega^*_D$ is a quasi-isomorphism.

Write $\rho(x_i) = x_i + z_i$. The elements z_i are in the divided power ideal of D because ρ is compatible with the augmentation to C. Hence we can factor the map ρ as a composition

$$D \xrightarrow{\sigma} D\langle \xi_i \rangle^\wedge \xrightarrow{\tau} D$$

where the first map is given by $x_i \mapsto x_i + \xi_i$ and the second map is the divided power D-algebra map which maps ξ_i to z_i. (This uses the universal properties of polynomial algebra, divided power polynomial algebras, divided power envelopes, and p-adic completion.) Note that there exists an *automorphism* α of $D\langle \xi_i \rangle^\wedge$ with $\alpha(x_i) = x_i - \xi_i$ and $\alpha(\xi_i) = \xi_i$. Applying Lemma 39.25.2 to $\alpha \circ \sigma$ (which maps x_i to x_i) and using that α is an isomorphism we conclude that σ induces a quasi-isomorphism of $M \otimes^\wedge_D \Omega^*_D$ with $M \otimes^\wedge_{D,\sigma} \Omega^*_{D\langle x_i \rangle^\wedge}$. On the other hand the map τ has as a left inverse the map $D \to D\langle x_i \rangle^\wedge$, $x_i \mapsto x_i$ and we conclude (using Lemma 39.25.2 once more) that τ induces a quasi-isomorphism of $M \otimes^\wedge_{D,\sigma} \Omega^*_{D\langle x_i \rangle^\wedge}$ with $M \otimes^\wedge_{D,\tau\circ\sigma} \Omega^*_D$. Composing these two quasi-isomorphisms we obtain that ρ induces a quasi-isomorphism $M \otimes^\wedge_D \Omega^*_D \to M \otimes^\wedge_{D,\rho} \Omega^*_D$ as desired. □

39.27. Two counter examples

Before we turn to some of the successes of crystalline cohomology, let us give two examples which explain why crystalline cohomology does not work very well if the schemes in question are either not proper over the base, or singular. The first example can be found in [**BO83**].

Example 39.27.1. Let $A = \mathbf{Z}_p$ with divided power ideal (p) endowed with its unique divided powers γ. Let $C = \mathbf{F}_p[x, y]/(x^2, xy, y^2)$. We choose the presentation

$$C = P/J = \mathbf{Z}_p[x, y]/(x^2, xy, y^2, p)$$

Let $D = D_{P,\gamma}(J)^\wedge$ with divided power ideal $(\bar{J}, \bar{\gamma})$ as in Section 39.22. We will denote x, y also the images of x and y in D. Consider the element

$$\tau = \bar{\gamma}_p(x^2)\bar{\gamma}_p(y^2) - \bar{\gamma}_p(xy)^2 \in D$$

We note that $p\tau = 0$ as

$$p!\,\bar{\gamma}_p(x^2)\bar{\gamma}_p(y^2) = x^{2p}\bar{\gamma}_p(y^2) = \bar{\gamma}_p(x^2y^2) = x^py^p\bar{\gamma}_p(xy) = p!\,\bar{\gamma}_p(xy)^2$$

in D. We also note that $\mathrm{d}\tau = 0$ in Ω_D as

$$\begin{aligned}
\mathrm{d}(\bar{\gamma}_p(x^2)\bar{\gamma}_p(y^2)) &= \bar{\gamma}_{p-1}(x^2)\bar{\gamma}_p(y^2)\mathrm{d}x^2 + \bar{\gamma}_p(x^2)\bar{\gamma}_{p-1}(y^2)\mathrm{d}y^2 \\
&= 2x\bar{\gamma}_{p-1}(x^2)\bar{\gamma}_p(y^2)\mathrm{d}x + 2y\bar{\gamma}_p(x^2)\bar{\gamma}_{p-1}(y^2)\mathrm{d}y \\
&= 2/(p-1)!\,(x^{2p-1}\bar{\gamma}_p(y^2)\mathrm{d}x + y^{2p-1}\bar{\gamma}_p(x^2)\mathrm{d}y) \\
&= 2/(p-1)!\,(x^{p-1}\bar{\gamma}_p(xy^2)\mathrm{d}x + y^{p-1}\bar{\gamma}_p(x^2y)\mathrm{d}y) \\
&= 2/(p-1)!\,(x^{p-1}y^p\bar{\gamma}_p(xy)\mathrm{d}x + x^py^{p-1}\bar{\gamma}_p(xy)\mathrm{d}y) \\
&= 2\bar{\gamma}_{p-1}(xy)\bar{\gamma}_p(xy)(y\mathrm{d}x + x\mathrm{d}y) \\
&= \mathrm{d}(\bar{\gamma}_p(xy)^2)
\end{aligned}$$

Finally, we claim that $\tau \neq 0$ in D. To see this it suffices to produce an object $(B \to \mathbf{F}_p[x, y]/(x^2, xy, y^2), \delta)$ of $\text{Cris}(C/S)$ such that τ does not map to zero in B. To do this take

$$B = \mathbf{F}_p[x, y, u, v]/(x^3, x^2y, xy^2, y^3, xu, yu, xv, yv, u^2, v^2)$$

with the obvious surjection to C. Let $K = \text{Ker}(B \to C)$ and consider the map

$$\delta_p : K \longrightarrow K, \quad ay^2 + bxy + cy^2 + du + ev + fuv \longmapsto a^p u + c^p v$$

One checks this satisfies the assumptions (1), (2), (3) of Lemma 39.2.7 and hence defines a divided power structure. Moreover, we see that τ maps to uv which is not zero in B. Set $X = Spec(C)$ and $S = Spec(A)$. We draw the following conclusions

(1) $H^0(\text{Cris}(X/S), \mathcal{O}_{X/S})$ has p-torsion, and
(2) pull back by frobenius $F^* : H^0(\text{Cris}(X/S), \mathcal{O}_{X/S}) \to H^0(\text{Cris}(X/S), \mathcal{O}_{X/S})$ is not injective.

Namely, τ defines a nonzero torsion element of $H^0(\text{Cris}(X/S), \mathcal{O}_{X/S})$ by Proposition 39.26.3. Similarly, $F^*(\tau) = \sigma(\tau)$ where $\sigma : D \to D$ is the map induced by any lift of Frobenius on P. If we choose $\sigma(x) = x^p$ and $\sigma(y) = y^p$, then an easy computation shows that $F^*(\tau) = 0$.

The next example shows that even for affine n-space crystalline cohomology does not give the correct thing.

Example 39.27.2. Let $A = \mathbf{Z}_p$ with divided power ideal (p) endowed with its unique divided powers γ. Let $C = \mathbf{F}_p[x_1, \ldots, x_r]$. We choose the presentation

$$C = P/J = P/pP \quad \text{with} \quad P = \mathbf{Z}_p[x_1, \ldots, x_r]$$

Note that pP has divided powers by Lemma 39.4.2. Hence setting $D = P^\wedge$ with divided power ideal (p) we obtain a situation as in Section 39.22. We conclude that $R\Gamma(\text{Cris}(X/S), \mathcal{O}_{X/S})$ is represented by the complex

$$D \to \Omega^1_D \to \Omega^2_D \to \ldots \to \Omega^r_D$$

see Proposition 39.26.3. Assuming $r > 0$ we conclude the following

(1) The cristalline cohomology of the cristalline structure sheaf of $X = \mathbf{A}^r_{\mathbf{F}_p}$ over $S = Spec(\mathbf{Z}_p)$ is zero except in degrees $0, \ldots, r$.
(2) We have $H^0(\text{Cris}(X/S), \mathcal{O}_{X/S}) = \mathbf{Z}_p$.
(3) The cohomology group $H^r(\text{Cris}(X/S), \mathcal{O}_{X/S})$ is infinite and is not a torsion abelian group.
(4) The cohomology group $H^r(\text{Cris}(X/S), \mathcal{O}_{X/S})$ is not separated for the p-adic topology.

While the first two statements are reasonable, parts (3) and (4) are disconcerting! The truth of these statements follows immediately from working out what the complex displayed above looks like. Let's just do this in case $r = 1$. Then we are just looking at the two term complex of p-adically complete modules

$$\text{d} : D = \left(\bigoplus\nolimits_{n \geq 0} \mathbf{Z}_p x^n\right)^\wedge \longrightarrow \Omega^1_D = \left(\bigoplus\nolimits_{n \geq 1} \mathbf{Z}_p x^{n-1} \text{d}x\right)^\wedge$$

The map is given by $\text{diag}(0, 1, 2, 3, 4, \ldots)$ except that the first summand is missing on the right hand side. Now it is clear that $\bigoplus_{n>0} \mathbf{Z}_p/n\mathbf{Z}_p$ is a subgroup of the cokernel, hence the cokernel is infinite. In fact, the element

$$\omega = \sum\nolimits_{e>0} p^e x^{p^{2e}-1} \text{d}x$$

is clearly not a torsion element of the cokernel. But it gets worse. Namely, consider the element

$$\eta = \sum\nolimits_{e>0} p^e x^{p^e - 1} \mathrm{d}x$$

For every $t > 0$ the element η is congruent to $\sum_{e>t} p^e x^{p^e-1}\mathrm{d}x$ modulo the image of d which is divisible by p^t. But η is not in the image of d because it would have to be the image of $a + \sum_{e>0} x^{p^e}$ for some $a \in \mathbf{Z}_p$ which is not an element of the left hand side. In fact, $p^N\eta$ is similarly not in the image of d for any integer N. This implies that η ``generates" a copy of $\mathbf{Q}_p$ inside of $H^1_{\text{cris}}(\mathbf{A}^1_{\mathbf{F}_p}/Spec(\mathbf{Z}_p))$.

39.28. Applications

In this section we collect some applications of the material in the previous sections.

Proposition 39.28.1. *In Situation 39.11.5. Let $\mathcal{F}$ be a crystal in quasi-coherent modules on $Cris(X/S)$. The truncation map of complexes*

$$(\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^1_{X/S} \to \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^2_{X/S} \to \ldots) \longrightarrow \mathcal{F}[0],$$

while not a quasi-isomorphism, becomes a quasi-isomorphism after applying $Ru_{X/S,*}$. In fact, for any $i > 0$, we have

$$Ru_{X/S,*}(\mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^i_{X/S}) = 0.$$

Proof. By Lemma 39.19.1 we get a de Rham complex as indicated in the lemma. We abbreviate $\mathcal{H} = \mathcal{F} \otimes \Omega^i_{X/S}$. Let $X' \subset X$ be an affine open subscheme which maps into an affine open subscheme $S' \subset S$. Then

$$(Ru_{X/S,*}\mathcal{H})|_{X'_{Zar}} = Ru_{X'/S',*}(\mathcal{H}|_{\mathrm{Cris}(X'/S')}),$$

see Lemma 39.13.5. Thus Lemma 39.26.2 shows that $Ru_{X/S,*}\mathcal{H}$ is a complex of sheaves on X_{Zar} whose cohomology on any affine open is trivial. As X has a basis for its topology consisting of affine opens this implies that $Ru_{X/S,*}\mathcal{H}$ is quasi-isomorphic to zero. $\square$

Remark 39.28.2. The proof of Proposition 39.28.1 shows that the conclusion

$$Ru_{X/S,*}(\mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^i_{X/S}) = 0$$

for $i > 0$ is true for any $\mathcal{O}_{X/S}$-module $\mathcal{F}$ which satisfies conditions (1) and (2) of Proposition 39.26.1. This applies to the following non-crystals: $\Omega^i_{X/S}$ for all i, and any sheaf of the form $\underline{\mathcal{F}}$, where $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module. In particular, it applies to the sheaf $\underline{\mathcal{O}_X} = \underline{\mathbf{G}_a}$. But note that we need something like Lemma 39.19.1 to produce a de Rham complex which requires $\mathcal{F}$ to be a crystal. Hence (currently) the collection of sheaves of modules for which the full statement of Proposition 39.28.1 holds is exactly the category of crystals in quasi-coherent modules.

In Situation 39.11.5. Let $\mathcal{F}$ be a crystal in quasi-coherent modules on $\mathrm{Cris}(X/S)$. Let (U, T, δ) be an object of $\mathrm{Cris}(X/S)$. Proposition 39.28.1 allows us to construct a canonical map

$$R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) \longrightarrow R\Gamma(T, \mathcal{F}_T \otimes_{\mathcal{O}_T} \Omega^*_{T/S,\delta}) \tag{39.28.2.1}$$

Namely, we have $R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) = R\Gamma(\mathrm{Cris}(X/S), \mathcal{F} \otimes \Omega^*_{X/S})$, we can restrict global cohomology classes to T, and $\Omega_{X/S}$ restricts to $\Omega_{T/S,\delta}$ by Lemma 39.16.3.

39.29. Some further results

In this section we mention some results whose proof is missing. We will formulate these as a series of remarks and we will convert them into actual lemmas and propositions only when we add detailed proofs.

Remark 39.29.1 (Higher direct images)**.** Let p be a prime number. Let $(S, \mathcal{I}, \gamma) \to (S', \mathcal{I}', \gamma')$ be a morphism of divided power schemes over $\mathbf{Z}_{(p)}$. Let

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & X' \arrow[d] \\ S_0 \arrow[r] & S'_0 \end{tikzcd}$$

be a commutative diagram of morphisms of schemes and assume p is locally nilpotent on X and X'. Let $\mathcal{F}$ be an $\mathcal{O}_{X/S}$-module on $\text{Cris}(X/S)$. Then $Rf_{\text{cris},*}\mathcal{F}$ can be computed as follows.

Given an object (U', T', δ') of $\text{Cris}(X'/S')$ set $U = X \times_{X'} U' = f^{-1}(U')$ (an open subscheme of X). Denote (T_0, T, δ) the divided power scheme over S such that

$$\begin{tikzcd} T \arrow[r] \arrow[d] & T' \arrow[d] \\ S \arrow[r] & S' \end{tikzcd}$$

is cartesian in the category of divided power schemes, see Lemma 39.11.4. There is an induced morphism $U \to T_0$ and we obtain a morphism $(U/T)_{\text{cris}} \to (X/S)_{\text{cris}}$, see Remark 39.13.3. Let $\mathcal{F}_U$ be the pullback of $\mathcal{F}$. Let $\tau_{U/T} : (U/T)_{\text{cris}} \to T_{Zar}$ be the structure morphism. Then we have

$$\left(Rf_{\text{cris},*}\mathcal{F}\right)_{T'} = R(T \to T')_* \left(R\tau_{U/T,*}\mathcal{F}_U\right) \tag{39.29.1.1}$$

where the left hand side is the restriction (see Section 39.14).

Hints: First, show that $\text{Cris}(U/T)$ is the localization (in the sense of Sites, Lemma 9.26.3) of $\text{Cris}(X/S)$ at the sheaf of sets $f_{\text{cris}}^{-1}h_{(U',T',\delta')}$. Next, reduce the statement to the case where $\mathcal{F}$ is an injective module and push forward of modules using that the pullback of an injective $\mathcal{O}_{X/S}$-module is an injective $\mathcal{O}_{U/T}$-module on $\text{Cris}(U/T)$. Finally, check the result holds for plain push forward.

Remark 39.29.2 (Mayer-Vietoris)**.** In the situation of Remark 39.29.1 suppose we have an open covering $X = X' \cup X''$. Denote $X''' = X' \cap X''$. Let f', f'', and f'' be the restriction of f to X', X'', and X'''. Moreover, Let $\mathcal{F}'$, $\mathcal{F}''$, and $\mathcal{F}'''$ be the restriction of $\mathcal{F}$ to the crystalline sites of X', X'', and X'''. Then there exists a distinguished triangle

$$Rf_{\text{cris},*}\mathcal{F} \longrightarrow Rf'_{\text{cris},*}\mathcal{F}' \oplus Rf''_{\text{cris},*}\mathcal{F}'' \longrightarrow Rf'''_{\text{cris},*}\mathcal{F}''' \longrightarrow Rf_{\text{cris},*}\mathcal{F}[1]$$

in $D(\mathcal{O}_{X'/S'})$.

Hints: This is a formal consequence of the fact that the subcategories $\text{Cris}(X'/S)$, $\text{Cris}(X''/S)$, $\text{Cris}(X'''/S)$ correspond to open subobjects of the final sheaf on $\text{Cris}(X/S)$ and that the last is the intersection of the first two.

Remark 39.29.3 (Čech complex)**.** Let p be a prime number. Let (A, I, γ) be a divided power ring with A a $\mathbf{Z}_{(p)}$-algebra. Set $S = Spec(A)$ and $S_0 = Spec(A/I)$. Let X be a separated[6] scheme over S_0 such that p is locally nilpotent on X. Let $\mathcal{F}$ be a crystal in quasi-coherent $\mathcal{O}_{X/S}$-modules.

Choose an affine open covering $X = \bigcup_{\lambda \in \Lambda} U_\lambda$ of X. Write $U_\lambda = Spec(C_\lambda)$. Choose a polynomial algebra P_λ over A and a surjection $P_\lambda \to C_\lambda$. Having fixed these choices we can construct a Čech complex which computes $R\Gamma(\text{Cris}(X/S), \mathcal{F})$.

Given $n \geq 0$ and $\lambda_0, \ldots, \lambda_n \in \Lambda$ write $U_{\lambda_0 \ldots \lambda_n} = U_{\lambda_0} \cap \ldots \cap U_{\lambda_n}$. This is an affine scheme by assumption. Write $U_{\lambda_0 \ldots \lambda_n} = Spec(C_{\lambda_0 \ldots \lambda_n})$. Set

$$P_{\lambda_0 \ldots \lambda_n} = P_{\lambda_0} \otimes_A \ldots \otimes_A P_{\lambda_n}$$

which comes with a canonical surjection onto $C_{\lambda_0 \ldots \lambda_n}$. Denote the kernel $J_{\lambda_0 \ldots \lambda_n}$ and set $D_{\lambda_0 \ldots \lambda_n}$ the p-adically completed divided power envelope of $J_{\lambda_0 \ldots \lambda_n}$ in $P_{\lambda_0 \ldots \lambda_n}$ relative to γ. Denote $\Omega_{\lambda_0 \ldots \lambda_n}$ the p-adically completed module of differentials of $D_{\lambda_0 \ldots \lambda_n}$ over A compatible with the divided power structure. Let $M_{\lambda_0 \ldots \lambda_n}$ be the $P_{\lambda_0 \ldots \lambda_n}$-module corresponding to the restriction of $\mathcal{F}$ to $\text{Cris}(U_{\lambda_0 \ldots \lambda_n}/S)$ via Proposition 39.22.4. By construction we obtain a cosimplicial divided power ring $D(*)$ having in degree n the ring

$$D(n) = \prod\nolimits_{\lambda_0 \ldots \lambda_n} D_{\lambda_0 \ldots \lambda_n}$$

(use that divided power envelopes are functorial and the trivial cosimplicial structure on the ring $P(*)$ defined similarly). Since $M_{\lambda_0 \ldots \lambda_n}$ is the ``value'' of $\mathcal{F}$ on the objects $Spec(D_{\lambda_0 \ldots \lambda_n})$ we see that $M(*)$ defined by the rule

$$M(n) = \prod\nolimits_{\lambda_0 \ldots \lambda_n} M_{\lambda_0 \ldots \lambda_n}$$

forms a cosimplicial $D(*)$-module. Now we claim that we have

$$R\Gamma(\text{Cris}(X/S), \mathcal{F}) = s(M(*))$$

Here $s(-)$ denotes the cochain complex associated to a cosimplicial module (see Simplicial, Section 14.23).

Hints: The proof of this is similar to the proof of Proposition 39.26.1 (in particular the result holds for any module satisfying the assumptions of that proposition).

Remark 39.29.4 (Alternating Čech complex)**.** Let p be a prime number. Let (A, I, γ) be a divided power ring with A a $\mathbf{Z}_{(p)}$-algebra. Set $S = Spec(A)$ and $S_0 = Spec(A/I)$. Let X be a separated quasi-compact scheme over S_0 such that p is locally nilpotent on X. Let $\mathcal{F}$ be a crystal in quasi-coherent $\mathcal{O}_{X/S}$-modules.

Choose a finite affine open covering $X = \bigcup_{\lambda \in \Lambda} U_\lambda$ of X and a total ordering on Λ. Write $U_\lambda = Spec(C_\lambda)$. Choose a polynomial algebra P_λ over A and a surjection $P_\lambda \to C_\lambda$. Having fixed these choices we can construct an alternating Čech complex which computes $R\Gamma(\text{Cris}(X/S), \mathcal{F})$.

We are going to use the notation introduced in Remark 39.29.3. Denote $\Omega_{\lambda_0 \ldots \lambda_n}$ the p-adically completed module of differentials of $D_{\lambda_0 \ldots \lambda_n}$ over A compatible with the divided power

[6] This assumption is not strictly necessary, as using hypercoverings the construction of the remark can be extended to the general case.

structure. Let ∇ be the integrable connection on $M_{\lambda_0\ldots\lambda_n}$ coming from Proposition 39.22.4. Consider the double complex $M^{\bullet,\bullet}$ with terms

$$M^{n,m} = \bigoplus\nolimits_{\lambda_0<\ldots<\lambda_n} M_{\lambda_0\ldots\lambda_n} \otimes^\wedge_{D_{\lambda_0\ldots\lambda_n}} \Omega^m_{D_{\lambda_0\ldots\lambda_n}}.$$

For the differential d_1 (increasing n) we use the usual Čech differential and for the differential d_2 we use the connection, i.e., the differential of the de Rham complex. We claim that

$$R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) = \mathrm{Tot}(M^{\bullet,\bullet})$$

Here $\mathrm{Tot}(-)$ denotes the total complex associated to a double complex, see Homology, Definition 10.19.2.

Hints: We have

$$R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) = R\Gamma(\mathrm{Cris}(X/S), \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^\bullet_{X/S})$$

by Proposition 39.28.1. The right hand side of the formula is simply the alternating Čech complex for the covering $X = \bigcup_{\lambda\in\Lambda} U_\lambda$ (which induces an open covering of the final sheaf of $\mathrm{Cris}(X/S)$) and the complex $\mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^\bullet_{X/S}$, see Proposition 39.26.3. Now the result follows from a general result in cohomology on sites, namely that the alternating Čech complex computes the cohomology provided it gives the correct answer on all the pieces (insert future reference here).

Remark 39.29.5 (Quasi-coherence)**.** In the situation of Remark 39.29.1 assume that $S \to S'$ is quasi-compact and quasi-separated and that $X \to S_0$ is quasi-compact and quasi-separated. Then for a crystal in quasi-coherent $\mathcal{O}_{X/S}$-modules $\mathcal{F}$ the sheaves $R^i f_{\mathrm{cris},*}\mathcal{F}$ are locally quasi-coherent.

Hints: We have to show that the restrictions to T' are quasi-coherent $\mathcal{O}_{T'}$-modules, where (U', T', δ') is any object of $\mathrm{Cris}(X'/S')$. It suffices to do this when T' is affine. We use the formula (39.29.1.1), the fact that $T \to T'$ is quasi-compact and quasi-separated (as T is affine over the base change of T' by $S \to S'$), and Coherent, Lemma 25.5.3 to see that it suffices to show that the sheaves $R^i\tau_{U/T,*}\mathcal{F}_U$ are quasi-coherent. Note that $U \to T_0$ is also quasi-compact and quasi-separated, see Schemes, Lemmas 21.21.15 and 21.21.15.

This reduces us to proving that $R^i\tau_{X/S,*}\mathcal{F}$ is quasi-coherent on S in the case that p locally nilpotent on S. Here $\tau_{X/S}$ is the structure morphism, see Remark 39.13.6. We may work locally on S, hence we may assume S affine (see Lemma 39.13.5). Induction on the number of affines covering X and Mayer-Vietoris (Remark 39.29.2) reduces the question to the case where X is also affine (as in the proof of Coherent, Lemma 25.5.3). Say $X = Spec(C)$ and $S = Spec(A)$ so that (A, I, γ) and $A \to C$ are as in Situation 39.9.1. Choose a polynomial algebra P over A and a surjection $P \to C$ as in Section 39.22. Let (M, ∇) be the module corresponding to $\mathcal{F}$, see Proposition 39.22.4. Applying Proposition 39.26.3 we see that $R\Gamma(\mathrm{Cris}(X/S), \mathcal{F})$ is represented by $M \otimes_D \Omega^*_D$. Note that completion isn't necessary as p is nilpotent in A! We have to show that this is compatible with taking principal opens in $S = Spec(A)$. Suppose that $g \in A$. Then we conclude that similarly $R\Gamma(\mathrm{Cris}(X_g/S_g), \mathcal{F})$ is computed by $M_g \otimes_{D_g} \Omega^*_{D_g}$ (again this uses that p-adic completion isn't necessary). Hence we conclude because localization is an exact functor on A-modules.

Remark 39.29.6 (Boundedness)**.** In the situation of Remark 39.29.1 assume that $S \to S'$ is quasi-compact and quasi-separated and that $X \to S_0$ is of finite type and quasi-separated. Then there exists an integer i_0 such that for any crystal in quasi-coherent $\mathcal{O}_{X/S}$-modules $\mathcal{F}$ we have $R^i f_{\mathrm{cris},*}\mathcal{F} = 0$ for all $i > i_0$.

Hints: Arguing as in Remark 39.29.5 (using Coherent, Lemma 25.5.3) we reduce to proving that $H^i(\text{Cris}(X/S), \mathcal{F}) = 0$ for $i \gg 0$ in the situation of Proposition 39.26.3 when C is a finite type algebra over A. This is clear as we can choose a finite polynomial algebra and we see that $\Omega^i_D = 0$ for $i \gg 0$.

Remark 39.29.7 (Specific boundedness)**.** In Situation 39.11.5 let $\mathcal{F}$ be a crystal in quasi-coherent $\mathcal{O}_{X/S}$-modules. Assume that S_0 has a unique point and that $X \to S_0$ is of finite presentation.

(1) If $\dim X = d$ and X/S_0 has embedding dimension e, then $H^i(\text{Cris}(X/S), \mathcal{F}) = 0$ for $i > d + e$.
(2) If X is separated and can be covered by q affines, and X/S_0 has embedding dimension e, then $H^i(\text{Cris}(X/S), \mathcal{F}) = 0$ for $i > q + e$.

Hints: In case (1) we can use that

$$H^i(\text{Cris}(X/S), \mathcal{F}) = H^i(X_{Zar}, Ru_{X/S,*}\mathcal{F})$$

and that $Ru_{X/S,*}\mathcal{F}$ is locally calculated by a de Rham complex constructed using an embedding of X into a smooth scheme of dimension e over S (see Lemma 39.26.4). These de Rham complexes are zero in all degrees $> e$. Hence (1) follows from Cohomology, Lemma 18.16.5. In case (2) we use the alternating Čech complex (see Remark 39.29.4) to reduce to the case X affine. In the affine case we prove the result using the de Rham complex associated to an embedding of X into a smooth scheme of dimension e over S (it takes some work to construct such a thing).

Remark 39.29.8 (Base change map)**.** In the situation of Remark 39.29.1 assume $S = Spec(A)$ and $S' = Spec(A')$ are affine. Let $\mathcal{F}'$ be an $\mathcal{O}_{X'/S'}$-module. Let $\mathcal{F}$ be the pullback of $\mathcal{F}'$. Then there is a canonical base change map

$$L(S' \to S)^* R\tau_{X'/S',*}\mathcal{F}' \longrightarrow R\tau_{X/S,*}\mathcal{F}$$

where $\tau_{X/S}$ and $\tau_{X'/S'}$ are the structure morphisms, see Remark 39.13.6. On global sections this gives a base change map

$$R\Gamma(\text{Cris}(X'/S'), \mathcal{F}') \otimes^{\mathbf{L}}_{A'} A \longrightarrow R\Gamma(\text{Cris}(X/S), \mathcal{F}) \tag{39.29.8.1}$$

in $D(A)$.

Hint: Compose the very general base change map of Cohomology on Sites, Remark 19.19.2 with the canonical map $Lf^*_{\text{cris}}\mathcal{F}' \to f^*_{\text{cris}}\mathcal{F}' = \mathcal{F}$.

Remark 39.29.9 (Base change isomorphism)**.** The map (39.29.8.1) is an isomorphism provided all of the following conditions are satisfied:

(1) p is nilpotent in A',
(2) $\mathcal{F}'$ is a crystal in quasi-coherent $\mathcal{O}_{X'/S'}$-modules,
(3) $X' \to S'_0$ is a quasi-compact, quasi-separated morphism,
(4) $X = X' \times_{S'_0} S_0$,
(5) $\mathcal{F}'$ is a flat $\mathcal{O}_{X'/S'}$-module,
(6) $X' \to S'_0$ is a local complete intersection morphism (see More on Morphisms, Definition 33.38.2; this holds for example if $X' \to S'_0$ is syntomic or smooth),
(7) X' and S_0 are Tor independent over S'_0 (see More on Algebra, Definition 12.5.1; this holds for example if either $S_0 \to S'_0$ or $X' \to S'_0$ is flat).

Hints: Condition (1) means that in the arguments below p-adic completion does nothing and can be ignored. Using condition (3) and Mayer Vietoris (see Remark 39.29.2) this reduces to the case where X' is affine. In fact by condition (6), after shrinking further, we can assume that $X' = Spec(C')$ and we are given a presentation $C' = A'/I'[x_1, \ldots, x_n]/(\bar{f}'_1, \ldots, \bar{f}'_c)$ where $\bar{f}'_1, \ldots, \bar{f}'_c$ is a Koszul-regular sequence in A'/I'. (This means that smooth locally $\bar{f}'_1, \ldots, \bar{f}'_c$ forms a regular sequence, see More on Algebra, Lemma 12.22.16.) We choose a lift of $\bar{f}'_i$ to an element $f'_i \in A'[x_1, \ldots, x_n]$. By (4) we see that $X = Spec(C)$ with $C = A/I[x_1, \ldots, x_n]/(\bar{f}_1, \ldots, \bar{f}_c)$ where $f_i \in A[x_1, \ldots, x_n]$ is the image of f'_i. By property (7) we see that $\bar{f}_1, \ldots, \bar{f}_c$ is a Koszul-regular sequence in $A/I[x_1, \ldots, x_n]$. The divided power envelope of $I'A'[x_1, \ldots, x_n] + (f'_1, \ldots, f'_c)$ in $A'[x_1, \ldots, x_n]$ relative to γ' is

$$D' = A'[x_1, \ldots, x_n]\langle \xi_1, \ldots, \xi_c\rangle/(\xi_i - f'_i)$$

see Lemma 39.6.4. Then you check that $\xi_1 - f'_1, \ldots, \xi_n - f'_n$ is a Koszul-regular sequence in the ring $A'[x_1, \ldots, x_n]\langle \xi_1, \ldots, \xi_c\rangle$. Similarly the divided power envelope of $IA[x_1, \ldots, x_n] + (f_1, \ldots, f_c)$ in $A[x_1, \ldots, x_n]$ relative to γ is

$$D = A[x_1, \ldots, x_n]\langle \xi_1, \ldots, \xi_c\rangle/(\xi_i - f_i)$$

and $\xi_1 - f_1, \ldots, \xi_n - f_n$ is a Koszul-regular sequence in the ring $A[x_1, \ldots, x_n]\langle \xi_1, \ldots, \xi_c\rangle$. It follows that $D' \otimes^{\mathbf{L}}_{A'} A = D$. Condition (2) implies $\mathcal{F}'$ corresponds to a pair (M', ∇) consisting of a D'-module with connection, see Proposition 39.22.4. Then $M = M' \otimes_{D'} D$ corresponds to the pullback $\mathcal{F}$. By assumption (5) we see that M' is a flat D'-module, hence

$$M = M' \otimes_{D'} D = M' \otimes_{D'} D' \otimes^{\mathbf{L}}_{A'} A = M' \otimes^{\mathbf{L}}_{A'} A$$

Since the modules of differentials $\Omega_{D'}$ and Ω_D (as defined in Section 39.22) are free D'-modules on the same generators we see that

$$M \otimes_D \Omega^\bullet_D = M' \otimes_{D'} \Omega^\bullet_{D'} \otimes_{D'} D = M' \otimes_{D'} \Omega^\bullet_{D'} \otimes^{\mathbf{L}}_{A'} A$$

which proves what we want by Proposition 39.26.3.

Remark 39.29.10 (Rlim)**.** Let p be a prime number. Let (A, I, γ) be a divided power ring with A an algebra over $\mathbf{Z}_{(p)}$ with p nilpotent in A/I. Set $S = Spec(A)$ and $S_0 = Spec(A/I)$. Let X be a scheme over S_0 with p locally nilpotent on X. Let $\mathcal{F}$ be any $\mathcal{O}_{X/S}$-module. For $e \gg 0$ we have $(p^e) \subset I$ is preserved by γ, see Lemma 39.4.5. Set $S_e = Spec(A/p^eA)$ for $e \gg 0$. Then $\text{Cris}(X/S_e)$ is a full subcategory of $\text{Cris}(X/S)$ and we denote $\mathcal{F}_e$ the restriction of $\mathcal{F}$ to $\text{Cris}(X/S_e)$. Then

$$R\Gamma(\text{Cris}(X/S), \mathcal{F}) = R\,lim_e\, R\Gamma(\text{Cris}(X/S_e), \mathcal{F}_e)$$

Hints: Suffices to prove this for $\mathcal{F}$ injective. In this case the sheaves $\mathcal{F}_e$ are injective modules too, the transition maps $\Gamma(\mathcal{F}_{e+1}) \to \Gamma(\mathcal{F}_e)$ are surjective, and we have $\Gamma(\mathcal{F}) = lim_e\, \Gamma(\mathcal{F}_e)$ because any object of $\text{Cris}(X/S)$ is locally an object of one of the categories $\text{Cris}(X/S_e)$ by definition of $\text{Cris}(X/S)$.

Remark 39.29.11 (Comparison)**.** Let p be a prime number. Let (A, I, γ) be a divided power ring with p nilpotent in A. Set $S = Spec(A)$ and $S_0 = Spec(A/I)$. Let Y be a smooth scheme over S and set $X = Y \times_S S_0$. Let $\mathcal{F}$ be a crystal in quasi-coherent $\mathcal{O}_{X/S}$-modules. Then

1. γ extends to a divided power structure on the ideal of X in Y so that (X, Y, γ) is an object of $\text{Cris}(X/S)$,
2. the restriction $\mathcal{F}_Y$ (see Section 39.14) comes endowed with a canonical integrable connection $\nabla : \mathcal{F}_Y \to \mathcal{F}_Y \otimes_{\mathcal{O}_Y} \Omega_{Y/S}$, and

(3) we have
$$R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) = R\Gamma(Y, \mathcal{F}_Y \otimes_{\mathcal{O}_Y} \Omega^\bullet_{Y/S})$$
in $D(A)$.

Hints: See Lemma 39.4.2 for (1). See Lemma 39.19.1 for (2). For Part (3) note that there is a map, see (39.28.2.1). This map is an isomorphism when X is affine, see Lemma 39.26.4. This shows that $Ru_{X/S,*}\mathcal{F}$ and $\mathcal{F}_Y \otimes \Omega^\bullet_{Y/S}$ are quasi-isomorphic as complexes on $Y_{Zar} = X_{Zar}$. Since $R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) = R\Gamma(X_{Zar}, Ru_{X/S,*}\mathcal{F})$ the result follows.

Remark 39.29.12 (Perfectness)**.** Let p be a prime number. Let (A, I, γ) be a divided power ring with p nilpotent in A. Set $S = Spec(A)$ and $S_0 = Spec(A/I)$. Let X be a proper smooth scheme over S_0. Let $\mathcal{F}$ be a crystal in finite locally free quasi-coherent $\mathcal{O}_{X/S}$-modules. Then $R\Gamma(\mathrm{Cris}(X/S), \mathcal{F})$ is a perfect object of $D(A)$.

Hints: By Remark 39.29.9 we have
$$R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) \otimes^{\mathbf{L}}_A A/I \cong R\Gamma(\mathrm{Cris}(X/S_0), \mathcal{F}|_{\mathrm{Cris}(X/S_0)})$$
By Remark 39.29.11 we have
$$R\Gamma(\mathrm{Cris}(X/S_0), \mathcal{F}|_{\mathrm{Cris}(X/S_0)}) = R\Gamma(X, \mathcal{F}_X \otimes \Omega^\bullet_{X/S_0})$$
Using the stupid filtration on the de Rham complex we see that the last displayed complex is perfect in $D(A/I)$ as soon as the complexes
$$R\Gamma(X, \mathcal{F}_X \otimes \Omega^q_{X/S_0})$$
are perfect complexes in $D(A/I)$, see More on Algebra, Lemma 12.42.4. This is true by standard arguments in coherent cohomology using that $\mathcal{F}_X \otimes \Omega^q_{X/S_0}$ is a finite locally free sheaf and $X \to S_0$ is proper and flat (insert future reference here). Applying More on Algebra, Lemma 12.43.4 we see that
$$R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) \otimes^{\mathbf{L}}_A A/I^n$$
is a perfect object of $D(A/I^n)$ for all n. This isn't quite enough unless A is Noetherian. Namely, even though I is locally nilpotent by our assumption that p is nilpotent, see Lemma 39.2.6, we cannot conclude that $I^n = 0$ for some n. A counter example is $\mathbf{F}_p\langle x\rangle$. To prove it in general when $\mathcal{F} = \mathcal{O}_{X/S}$ the argument of `http://math.columbia.edu/~dejong/wordpress/?p=2227` works. When the coefficients $\mathcal{F}$ are non-trivial the argument of [**Fal99**] seems to be as follows. Reduce to the case $pA = 0$ by More on Algebra, Lemma 12.43.4. In this case the Frobenius map $A \to A$, $a \mapsto a^p$ factors as $A \to A/I \xrightarrow{\varphi} A$ (as $x^p = 0$ for $x \in I$). Set $X^{(1)} = X \otimes_{A/I,\varphi} A$. The absolute Frobenius morphism of X factors through a morphism $F_X : X \to X^{(1)}$ (a kind of relative Frobenius). Affine locally if $X = Spec(C)$ then $X^{(1)} = Spec(C \otimes_{A/I,\varphi} A)$ and F_X corresponds to $C \otimes_{A/I,\varphi} A \to C$, $c \otimes a \mapsto c^p a$. This defines morphisms of ringed topoi
$$(X/S)_{\mathrm{cris}} \xrightarrow{(F_X)_{\mathrm{cris}}} (X^{(1)}/S)_{\mathrm{cris}} \xrightarrow{u_{X^{(1)}/S}} Sh(X^{(1)}_{Zar})$$
whose composition is denoted Frob_X. One then shows that $R\mathrm{Frob}_{X,*}\mathcal{F}$ is representable by a perfect complex of $\mathcal{O}_{X^{(1)}}$-modules(!) by a local calculation.

Remark 39.29.13 (Complete perfectness)**.** Let p be a prime number. Let (A, I, γ) be a divided power ring with A a Noetherian p-adically complete ring and p nilpotent in A/I. Set $S = Spec(A)$ and $S_0 = Spec(A/I)$. Let X be a proper smooth scheme over S_0. Let $\mathcal{F}$ be a crystal in finite locally free quasi-coherent $\mathcal{O}_{X/S}$-modules. Then $R\Gamma(\mathrm{Cris}(X/S), \mathcal{F})$ is a perfect object of $D(A)$.

Hints: We know that $K = R\Gamma(\text{Cris}(X/S), \mathcal{F})$ is the derived limit $K = R\,lim\, K_e$ of the cohomologies over A/p^eA, see Remark 39.29.10. Each K_e is a perfect complex of $D(A/p^eA)$ by Remark 39.29.12. Since A is Noetherian and p-adically complete the result follows from (insert future reference here).

Remark 39.29.14 (Complete comparison)**.** Let p be a prime number. Let (A, I, γ) be a divided power ring with A a Noetherian p-adically complete ring and p nilpotent in A/I. Set $S = Spec(A)$ and $S_0 = Spec(A/I)$. Let Y be a proper smooth scheme over S and set $X = Y \times_S S_0$. Let $\mathcal{F}$ be a finite type crystal in quasi-coherent $\mathcal{O}_{X/S}$-modules. Then

(1) there exists a coherent $\mathcal{O}_Y$-module $\mathcal{F}_Y$ endowed with integrable connection
$$\nabla : \mathcal{F}_Y \longrightarrow \mathcal{F}_Y \otimes_{\mathcal{O}_Y} \Omega_{Y/S}$$
such that $\mathcal{F}_Y/p^e\mathcal{F}_Y$ is the module with connection over A/p^eA found in Remark 39.29.11, and

(2) we have
$$R\Gamma(\text{Cris}(X/S), \mathcal{F}) = R\Gamma(Y, \mathcal{F}_Y \otimes_{\mathcal{O}_Y} \Omega^\bullet_{Y/S})$$
in $D(A)$.

Hints: The existence of $\mathcal{F}_Y$ is Grothendieck's existence theorem (insert future reference here). The isomorphism of cohomologies follows as both sides are computed as $R\,lim$ of the versions modulo p^e (see Remark 39.29.10 for the left hand side; use the theorem on formal functions, see Coherent, Theorem 25.19.5 for the right hand side). Each of the versions modulo p^e are isomorphic by Remark 39.29.11.

39.30. Pulling back along α_p-covers

By an α_p-cover we mean a morphism of the form
$$X' = Spec(C[z]/(z^p - c)) \longrightarrow Spec(C) = X$$
where C is an $\mathbf{F}_p$-algebra and $c \in C$. Equivalently, X' is an α_p-torsor over X. An *iterated α_p-cover*[7] is a morphism of schemes in characteristic p which is locally on the target a composition of finitely many α_p-covers. In this section we prove that pullback along such a morphism induces a quasi-isomorphism on crystalline cohomology after inverting the prime p. In fact, we prove a precise version of this result. We being with a preliminary lemma whose formulation need some notation.

Assume we have a ring map $B \to B'$ and quotients $\Omega_B \to \Omega$ and $\Omega_{B'} \to \Omega'$ satisfying the assumptions of Remark 39.10.11. Thus (39.10.11.1) provides a canonical map of complexes
$$c^\bullet_M : M \otimes_B \Omega^\bullet \longrightarrow M \otimes_B (\Omega')^\bullet$$
for all B-modules M endowed with integrable connection $\nabla : M \to M \otimes_B \Omega_B$.

Suppose we have $a \in B$, $z \in B'$, and a map $\theta : B' \to B'$ satisfying the following assumptions

(1) $\mathrm{d}(a) = 0$,
(2) $\Omega' = B' \otimes_B \Omega \oplus B'\mathrm{d}z$; we write $\mathrm{d}(f) = \mathrm{d}_1(f) + \partial_z(f)\mathrm{d}z$ with $\mathrm{d}_1(f) \in B' \otimes \Omega$ and $\partial_z(f) \in N'$ for all $f \in B'$,
(3) $\theta : B' \to B'$ is B-linear,
(4) $\partial_z \circ \theta = a$,
(5) $B \to B'$ is universally injective (and hence $\Omega \to \Omega'$ is injective),

[7] This is nonstandard notation.

(6) $af - \theta(\partial_z(f)) \in B$ for all $f \in B'$,
(7) $(\theta \otimes 1)(\mathrm{d}_1(f)) - \mathrm{d}_1(\theta(f)) \in \Omega$ for all $f \in B'$ where $\theta \otimes 1 : B' \otimes \Omega \to B' \otimes \Omega$

These conditions are not logically independent. For example, assumption (4) implies that $\partial_z(af - \theta(\partial_z(f))) = 0$. Hence if the image of $B \to B'$ is the collection of elements annihilated by ∂_z, then (6) follows. A similar argument can be made for condition (7).

Lemma 39.30.1. *In the situation above there exists a map of complexes*

$$e_M^\bullet : M \otimes_B (\Omega')^\bullet \longrightarrow M \otimes_B \Omega^\bullet$$

such that $c_M^\bullet \circ e_M^\bullet$ and $e_M^\bullet \circ c_M^\bullet$ are homotopic to multiplication by a.

Proof. In this proof all tensor products are over B. Assumption (2) implies that

$$M \otimes (\Omega')^i = (B' \otimes M \otimes \Omega^i) \oplus (B'\mathrm{d}z \otimes M \otimes \Omega^{i-1})$$

for all $i \geq 0$. A collection of additive generators for $M \otimes (\Omega')^i$ is formed by elements of the form $f\omega$ and elements of the form $f\mathrm{d}z \wedge \eta$ where $f \in B'$, $\omega \in M \otimes \Omega^i$, and $\eta \in M \otimes \Omega^{i-1}$.

For $f \in B'$ we write

$$\epsilon(f) = af - \theta(\partial_z(f)) \quad \text{and} \quad \epsilon'(f) = (\theta \otimes 1)(\mathrm{d}_1(f)) - \mathrm{d}_1(\theta(f))$$

so that $\epsilon(f) \in B$ and $\epsilon'(f) \in \Omega$ by assumptions (6) and (7). We define $e_M^\bullet$ by the rules $e_M^i(f\omega) = \epsilon(f)\omega$ and $e_M^i(f\mathrm{d}z \wedge \eta) = \epsilon'(f) \wedge \eta$. We will see below that the collection of maps e_M^i is a map of complexes.

We define

$$h^i : M \otimes_B (\Omega')^i \longrightarrow M \otimes_B (\Omega')^{i-1}$$

by the rules $h^i(f\omega) = 0$ and $h^i(f\mathrm{d}z \wedge \eta) = \theta(f)\eta$ for elements as above. We claim that

$$\mathrm{d} \circ h + h \circ \mathrm{d} = a - c_M^\bullet \circ e_M^\bullet$$

Note that multiplication by a is a map of complexes by (1). Hence, since $c_M^\bullet$ is an injective map of complexes by assumption (5), we conclude that $e_M^\bullet$ is a map of complexes. To prove the claim we compute

$$\begin{aligned}(\mathrm{d} \circ h + h \circ \mathrm{d})(f\omega) &= h\,(\mathrm{d}(f) \wedge \omega + f\nabla(\omega)) \\ &= \theta(\partial_z(f))\omega \\ &= af\omega - \epsilon(f)\omega \\ &= af\omega - c_M^i(e_M^i(f\omega))\end{aligned}$$

The second equality because $\mathrm{d}z$ does not occur in $\nabla(\omega)$ and the third equality by assumption (6). Similarly, we have

$$\begin{aligned}(\mathrm{d} \circ h + h \circ \mathrm{d})(f\mathrm{d}z \wedge \eta) &= \mathrm{d}(\theta(f)\eta) + h\,(\mathrm{d}(f) \wedge \mathrm{d}z \wedge \eta - f\mathrm{d}z \wedge \nabla(\eta)) \\ &= \mathrm{d}(\theta(f)) \wedge \eta + \theta(f)\nabla(\eta) - (\theta \otimes 1)(\mathrm{d}_1(f)) \wedge \eta - \theta(f)\nabla(\eta) \\ &= \mathrm{d}_1(\theta(f)) \wedge \eta + \partial_z(\theta(f))\mathrm{d}z \wedge \eta - (\theta \otimes 1)(\mathrm{d}_1(f)) \wedge \eta \\ &= af\mathrm{d}z \wedge \eta - \epsilon'(f) \wedge \eta \\ &= af\mathrm{d}z \wedge \eta - c_M^i(e_M^i(f\mathrm{d}z \wedge \eta))\end{aligned}$$

The second equality because $\mathrm{d}(f) \wedge \mathrm{d}z \wedge \eta = -\mathrm{d}z \wedge \mathrm{d}_1(f) \wedge \eta$. The fourth equality by assumption (4). On the other hand it is immediate from the definitions that $e_M^i(c_M^i(\omega)) = \epsilon(1)\omega = a\omega$. This proves the lemma. □

Example 39.30.2. A standard example of the situation above occurs when $B' = B\langle z\rangle$ is the divided power polynomial ring over a divided power ring (B, J, δ) with divided powers δ' on $J' = B'_+ + JB' \subset B'$. Namely, we take $\Omega = \Omega_{B,\delta}$ and $\Omega' = \Omega_{B',\delta'}$. In this case we can take $a = 1$ and

$$\theta(\sum b_m z^{[m]}) = \sum b_m z^{[m+1]}$$

Note that

$$f - \theta(\partial_z(f)) = f(0)$$

equals the constant term. It follows that in this case Lemma 39.30.1 recovers the crystalline Poincaré lemma (Lemma 39.25.2).

Lemma 39.30.3. *In Situation 39.9.1. Assume D and Ω_D are as in (39.22.0.1) and (39.22.0.2). Let $\lambda \in D$. Let D' be the p-adic completion of*

$$D[z]\langle\xi\rangle/(\xi - (z^p - \lambda))$$

and let $\Omega_{D'}$ be the p-adic completion of the module of divided power differentials of D' over A. For any pair (M, ∇) over D satisfying (1), (2), (3), and (4) the canonical map of complexes (39.10.11.1)

$$c^\bullet_M : M \otimes^\wedge_D \Omega^\bullet_D \longrightarrow M \otimes^\wedge_D \Omega^\bullet_{D'}$$

has the following property: There exists a map $e^\bullet_M$ in the opposite direction such that both $c^\bullet_M \circ e^\bullet_M$ and $e^\bullet_M \circ c^\bullet_M$ are homotopic to multiplication by p.

Proof. We will prove this using Lemma 39.30.1 with $a = p$. Thus we have to find $\theta : D' \to D'$ and prove (1), (2), (3), (4), (5), (6), (7). We first collect some information about the rings D and D' and the modules Ω_D and $\Omega_{D'}$.

Writing

$$D[z]\langle\xi\rangle/(\xi - (z^p - \lambda)) = D\langle\xi\rangle[z]/(z^p - \xi - \lambda)$$

we see that D' is the p-adic completion of the free D-module

$$\bigoplus\nolimits_{i=0,\ldots,p-1} \bigoplus\nolimits_{n\geq 0} z^i \xi^{[n]} D$$

where $\xi^{[0]} = 1$. It follows that $D \to D'$ has a continuous D-linear section, in particular $D \to D'$ is universally injective, i.e., (5) holds. We think of D' as a divided power algebra over A with divided power ideal $\bar{J}' = \bar{J}D' + (\xi)$. Then D' is also the p-adic completion of the divided power envelope of the ideal generated by $z^p - \lambda$ in D, see Lemma 39.6.4. Hence

$$\Omega_{D'} = \Omega_D \otimes^\wedge_D D' \oplus D'\mathrm{d}z$$

by Lemma 39.10.6. This proves (2). Note that (1) is obvious.

At this point we construct θ. (We wrote a PARI/gp script theta.gp verifying some of the formulas in this proof which can be found in the scripts subdirectory of the stacks project.) Before we do so we compute the derivative of the elements $z^i\xi^{[n]}$. We have $\mathrm{d}z^i = iz^{i-1}\mathrm{d}z$. For $n \geq 1$ we have

$$\mathrm{d}\xi^{[n]} = \xi^{[n-1]}\mathrm{d}\xi = -\xi^{[n-1]}\mathrm{d}\lambda + pz^{p-1}\xi^{[n-1]}\mathrm{d}z$$

because $\xi = z^p - \lambda$. For $0 < i < p$ and $n \geq 1$ we have

$$\begin{aligned} \mathrm{d}(z^i\xi^{[n]}) &= iz^{i-1}\xi^{[n]}\mathrm{d}z + z^i\xi^{[n-1]}\mathrm{d}\xi \\ &= iz^{i-1}\xi^{[n]}\mathrm{d}z + z^i\xi^{[n-1]}\mathrm{d}(z^p - \lambda) \\ &= -z^i\xi^{[n-1]}\mathrm{d}\lambda + (iz^{i-1}\xi^{[n]} + pz^{i+p-1}\xi^{[n-1]})\mathrm{d}z \\ &= -z^i\xi^{[n-1]}\mathrm{d}\lambda + (iz^{i-1}\xi^{[n]} + pz^{i-1}(\xi+\lambda)\xi^{[n-1]})\mathrm{d}z \\ &= -z^i\xi^{[n-1]}\mathrm{d}\lambda + ((i+pn)z^{i-1}\xi^{[n]} + p\lambda z^{i-1}\xi^{[n-1]})\mathrm{d}z \end{aligned}$$

the last equality because $\xi\xi^{[n-1]} = n\xi^{[n]}$. Thus we see that

$$\begin{aligned} \partial_z(z^i) &= iz^{i-1} \\ \partial_z(\xi^{[n]}) &= pz^{p-1}\xi^{[n-1]} \\ \partial_z(z^i\xi^{[n]}) &= (i+pn)z^{i-1}\xi^{[n]} + p\lambda z^{i-1}\xi^{[n-1]} \end{aligned}$$

Motivated by these formulas we define θ by the rules

$$\begin{array}{ccccc} \theta(z^j) & = & p\frac{z^{j+1}}{j+1} & & j = 0, \ldots p-1, \\ \theta(z^{p-1}\xi^{[m]}) & = & \xi^{[m+1]} & & m \geq 1, \\ \theta(z^j\xi^{[m]}) & = & \frac{pz^{j+1}\xi^{[m]} - \theta(p\lambda z^j\xi^{[m-1]})}{(j+1+pm)} & & 0 \leq j < p-1, m \geq 1 \end{array}$$

where in the last line we use induction on m to define our choice of θ. Working this out we get (for $0 \leq j < p-1$ and $1 \leq m$)

$$\theta(z^j\xi^{[m]}) = \frac{pz^{j+1}\xi^{[m]}}{(j+1+pm)} - \frac{p^2\lambda z^{j+1}\xi^{[m-1]}}{(j+1+pm)(j+1+p(m-1))} + \ldots + \frac{(-1)^m p^{m+1}\lambda^m z^{j+1}}{(j+1+pm)\ldots(j+1)}$$

although we will not use this expression below. It is clear that θ extends uniquely to a p-adically continuous D-linear map on D'. By construction we have (3) and (4). It remains to prove (6) and (7).

Proof of (6) and (7). As θ is D-linear and continuous it suffices to prove that $p - \theta \circ \partial_z$, resp. $(\theta \otimes 1) \circ \mathrm{d}_1 - \mathrm{d}_1 \circ \theta$ gives an element of D, resp. Ω_D when evaluated on the elements $z^i\xi^{[n]}$[8]. Set $D_0 = \mathbf{Z}_{(p)}[\lambda]$ and $D_0' = \mathbf{Z}_{(p)}[z, \lambda]\langle\xi\rangle/(\xi - z^p + \lambda)$. Observe that each of the expressions above is an element of D_0' or $\Omega_{D_0'}$. Hence it suffices to prove the result in the case of $D_0 \to D_0'$. Note that D_0 and D_0' are torsion free rings and that $D_0 \otimes \mathbf{Q} = \mathbf{Q}[\lambda]$ and $D_0' \otimes \mathbf{Q} = \mathbf{Q}[z, \lambda]$. Hence $D_0 \subset D_0'$ is the subring of elements annihilated by ∂_z and (6) follows from (4), see the discussion directly preceding Lemma 39.30.1. Similarly, we have $\mathrm{d}_1(f) = \partial_\lambda(f)\mathrm{d}\lambda$ hence

$$\left((\theta \otimes 1) \circ \mathrm{d}_1 - \mathrm{d}_1 \circ \theta\right)(f) = \left(\theta(\partial_\lambda(f)) - \partial_\lambda(\theta(f))\right)\mathrm{d}\lambda$$

Applying ∂_z to the coefficient we obtain

$$\begin{aligned} \partial_z\left(\theta(\partial_\lambda(f)) - \partial_\lambda(\theta(f))\right) &= p\partial_\lambda(f) - \partial_z(\partial_\lambda(\theta(f))) \\ &= p\partial_\lambda(f) - \partial_\lambda(\partial_z(\theta(f))) \\ &= p\partial_\lambda(f) - \partial_\lambda(pf) = 0 \end{aligned}$$

whence the coefficient does not depend on z as desired. This finishes the proof of the lemma. □

[8]This can be done by direct computation: It turns out that $p - \theta \circ \partial_z$ evaluated on $z^i\xi^{[n]}$ gives zero except for 1 which is mapped to p and ξ which is mapped to $-p\lambda$. It turns out that $(\theta \otimes 1) \circ \mathrm{d}_1 - \mathrm{d}_1 \circ \theta$ evaluated on $z^i\xi^{[n]}$ gives zero except for $z^{p-1}\xi$ which is mapped to $-\lambda$.

Note that an interated α_p-cover $X' \to X$ (as defined in the introduction to this section) is finite locally free. Hence if X is connected the degree of $X' \to X$ is constant and is a power of p.

Lemma 39.30.4. *Let p be a prime number. Let $(S, \mathcal{I}, \gamma)$ be a divided power scheme over $\mathbf{Z}_{(p)}$ with $p \in \mathcal{I}$. We set $S_0 = V(\mathcal{I}) \subset S$. Let $f : X' \to X$ be an iterated α_p-cover of schemes over S_0 with constant degree q. Let $\mathcal{F}$ be any crystal in quasi-coherent sheaves on X and set $\mathcal{F}' = f_{cris}^*\mathcal{F}$. In the distinguished triangle*

$$Ru_{X/S,*}\mathcal{F} \longrightarrow f_*Ru_{X'/S,*}\mathcal{F}' \longrightarrow E \longrightarrow Ru_{X/S,*}\mathcal{F}[1]$$

the object E has cohomology sheaves annihilated by q.

Proof. Note that $X' \to X$ is a homeomorphism hence we can identify the underlying topological spaces of X and X'. The question is clearly local on X, hence we may assume X, X', and S affine and $X' \to X$ given as a composition

$$X' = X_n \to X_{n-1} \to X_{n-2} \to \ldots \to X_0 = X$$

where each morphism $X_{i+1} \to X_i$ is an α_p-cover. Denote $\mathcal{F}_i$ the pullback of $\mathcal{F}$ to X_i. It suffices to prove that each of the maps

$$R\Gamma(\text{Cris}(X_i/S), \mathcal{F}_i) \longrightarrow R\Gamma(\text{Cris}(X_{i+1}/S), \mathcal{F}_{i+1})$$

fits into a triangle whose third member has cohomology groups annihilated by p. (This uses axiom TR4 for the triangulated category $D(X)$. Details omitted.)

Hence we may assume that $S = Spec(A)$, $X = Spec(C)$, $X' = Spec(C')$ and $C' = C[z]/(z^p - c)$ for some $c \in C$. Choose a polynomial algebra P over A and a surjection $P \to C$. Let D be the p-adically completed divided power envelop of $\text{Ker}(P \to C)$ in P as in (39.22.0.1). Set $P' = P[z]$ with surjection $P' \to C'$ mapping z to the class of z in C'. Choose a lift $\lambda \in D$ of $c \in C$. Then we see that the p-adically completed divided power envelope D' of $\text{Ker}(P' \to C')$ in P' is isomorphic to the p-adic completion of $D[z]\langle\xi\rangle/(\xi - (z^p - \lambda))$, see Lemma 39.30.3 and its proof. Thus we see that the result follows from this lemma by the computation of cohomology of crystals in quasi-coherent modules in Proposition 39.26.3. $\square$

The bound in the following lemma is probably not optimal.

Lemma 39.30.5. *With notations and assumptions as in Lemma 39.30.4 the map*

$$f^* : H^i(Cris(X/S), \mathcal{F}) \longrightarrow H^i(Cris(X'/S), \mathcal{F}')$$

has kernel and cokernel annihilated by q^{i+1}.

Proof. This follows from the fact that E has nonzero cohomology sheaves in degrees -1 and up, so that the spectral sequence $H^a(\mathcal{H}^b(E)) \Rightarrow H^{a+b}(E)$ coverges. This combined with the long exact cohomology sequence associated to a distinguished triangle gives the bound. $\square$

In Situation 39.11.5 assume that $p \in \mathcal{I}$. Set

$$X^{(1)} = X \times_{S_0, F_{S_0}} S_0.$$

Denote $F_{X/S_0} : X \to X^{(1)}$ the relative Frobenius morphism.

Lemma 39.30.6. *In the situation above, assume that $X \to S_0$ is smooth of relative dimension d. Then F_{X/S_0} is an iterated α_p-cover of degree p^d. Hence Lemmas 39.30.4 and 39.30.5 apply to this situation. In particular, for any crystal in quasi-coherent modules $\mathcal{G}$ on $Cris(X^{(1)}/S)$ the map*

$$F_{X/S_0}^* : H^i(Cris(X^{(1)}/S), \mathcal{G}) \longrightarrow H^i(Cris(X/S), F_{X/S_0,cris}^*\mathcal{G})$$

has kernel and cokernel annihilated by $p^{d(i+1)}$.

Proof. It suffices to prove the first statement. To see this we may assume that X is étale over $\mathbf{A}^d_{S_0}$, see Morphisms, Lemma 24.35.20. Denote $\varphi : X \to \mathbf{A}^d_{S_0}$ this étale morphism. In this case the relative Frobenius of X/S_0 fits into a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X^{(1)} \arrow[d] \\ \mathbf{A}^d_{S_0} \arrow[r] & \mathbf{A}^d_{S_0} \end{tikzcd}$$

where the lower horizontal arrow is the relative frobenius morphism of $\mathbf{A}^d_{S_0}$ over S_0. This is the morphism which raises all the coordinates to the pth power, hence it is an iterated α_p-cover. The proof is finished by observing that the diagram is a fibre square, see the proof of Étale Cohomology, Theorem 38.66.2. $\square$

39.31. Frobenius action on crystalline cohomology

In this section we prove that Frobenius pullback induces a quasi-isomorphism on crystalline cohomology after inverting the prime p. But in order to even formulate this we need to work in a special situation.

Situation 39.31.1. In Situation 39.11.5 assume the following

(1) $S = Spec(A)$ for some divided power ring (A, I, γ) with $p \in I$,
(2) there is given a homomorphism of divided power rings $\sigma : A \to A$ such that $\sigma(x) = x^p \bmod pA$ for all $x \in A$.

In Situation 39.31.1 the morphism $Spec(\sigma) : S \to S$ is a lift of the absolute Frobenius $F_{S_0} : S_0 \to S_0$ and since the diagram

$$\begin{tikzcd} X \arrow[r, "F_X"'] \arrow[d] & X \arrow[d] \\ S_0 \arrow[r, "F_{S_0}"] & S_0 \end{tikzcd}$$

is commutative where $F_X : X \to X$ is the absolute Frobenius morphism of X. Thus we obtain a morphism of crystalline topoi

$$(F_X)_{\text{cris}} : (X/S)_{\text{cris}} \longrightarrow (X/S)_{\text{cris}}$$

see Remark 39.13.3. Here is the terminology concerning F-crystals following the notation of Saavedra, see [**SR72**].

Definition 39.31.2. In Situation 39.31.1 an *F-crystal on X/S (relative to σ)* is a pair $(\mathcal{E}, F_{\mathcal{E}})$ given by a crystal in finite locally free $\mathcal{O}_{X/S}$-modules $\mathcal{E}$ together with a map

$$F_{\mathcal{E}} : (F_X)_{\text{cris}}^*\mathcal{E} \longrightarrow \mathcal{E}$$

An F-crystal is called *nondegenerate* if there exists an integer $i \geq 0$ a map $V : \mathcal{E} \to (F_X)^*_{\text{cris}}\mathcal{E}$ such that $V \circ F_{\mathcal{E}} = p^i\text{id}$.

Remark 39.31.3. Let $(\mathcal{E}, F)$ be an F-crystal as in Definition 39.31.2. In the literature the nondegeneracy condition is often part of the definition of an F-crystal. Moreover, often it is also assumed that $F \circ V = p^n\text{id}$. What is needed for the result below is that there exists an integer $j \geq 0$ such that $\text{Ker}(F)$ and $\text{Coker}(F)$ are killed by p^j. If the rank of $\mathcal{E}$ is bounded (for example if X is quasi-compact), then both of these conditions follow from the nondegeneracy condition as formulated in the definition. Namely, suppose R is a ring, $r \geq 1$ is an integer and $K, L \in \text{Mat}(r \times r, R)$ are matrices with $KL = p^i 1_{r\times r}$. Then $\det(K)\det(L) = p^{ri}$. Let L' be the adjugate matrix of L, i.e., $L'L = LL' = \det(L)$. Set $K' = p^{ri}K$ and $j = ri + i$. Then we have $K'L = p^j 1_{r\times r}$ as $KL = p^i$ and

$$LK' = LK\det(L)\det(M) = LKLL'\det(M) = Lp^iL'\det(M) = p^j 1_{r\times r}$$

It follows that if V is as in Definition 39.31.2 then setting $V' = p^N V$ where $N > i \cdot \text{rank}(\mathcal{E})$ we get $V' \circ F = p^{N+i}$ and $F \circ V' = p^{N+i}$.

Theorem 39.31.4. *In Situation 39.31.1 let $(\mathcal{E}, F_{\mathcal{E}})$ be a nondegenerate F-crystal. Assume A is a p-adically complete Noetherian ring and that $X \to S_0$ is proper smooth. Then the canonical map*

$$F_{\mathcal{E}} \circ (F_X)^*_{cris} : R\Gamma(Cris(X/S), \mathcal{E}) \otimes^{\mathbf{L}}_{A,\sigma} A \longrightarrow R\Gamma(Cris(X/S), \mathcal{E})$$

becomes an isomorphism after inverting p.

Proof. We first write the arrow as a composition of three arrows. Namely, set

$$X^{(1)} = X \times_{S_0, F_{S_0}} S_0$$

and denote $F_{X/S_0} : X \to X^{(1)}$ the relative Frobenius morphism. Denote $\mathcal{E}^{(1)}$ the base change of $\mathcal{E}$ by $Spec(\sigma)$, in other words the pullback of $\mathcal{E}$ to $\text{Cris}(X^{(1)}/S)$ by the morphism of crystalline topoi associated to the commutative diagram

$$\begin{tikzcd} X^{(1)} \arrow[r] \arrow[d] & X \arrow[d] \\ S \arrow[r, "Spec(\sigma)"] & S \end{tikzcd}$$

Then we have the base change map

$$R\Gamma(\text{Cris}(X/S), \mathcal{E}) \otimes^{\mathbf{L}}_{A,\sigma} A \longrightarrow R\Gamma(\text{Cris}(X^{(1)}/S), \mathcal{E}^{(1)}) \tag{39.31.4.1}$$

see Remark 39.29.8. Note that the composition of $F_{X/S_0} : X \to X^{(1)}$ with the projection $X^{(1)} \to X$ is the absolute Frobenius morphism F_X. Hence we see that $F^*_{X/S_0}\mathcal{E}^{(1)} = (F_X)^*_{\text{cris}}\mathcal{E}$. Thus pullback by F_{X/S_0} is a map

$$F^*_{X/S_0} : R\Gamma(\text{Cris}(X^{(1)}/S), \mathcal{E}^{(1)}) \longrightarrow R\Gamma(\text{Cris}(X/S), (F_X)^*_{\text{cris}}\mathcal{E}) \tag{39.31.4.2}$$

Finally we can use $F_{\mathcal{E}}$ to get a map

$$R\Gamma(\text{Cris}(X/S), (F_X)^*_{\text{cris}}\mathcal{E}) \longrightarrow R\Gamma(\text{Cris}(X/S), \mathcal{E}) \tag{39.31.4.3}$$

The map of the theorem is the composition of the three maps (39.31.4.1), (39.31.4.2), and (39.31.4.3) above. The first is a quasi-isomorphism modulo all powers of p by Remark 39.29.9. Hence it is a quasi-isomorphism since the complexes involved are perfect in $D(A)$

see Remark 39.29.13. The third map is a quasi-ismorphism after inverting p simply because $F_{\mathcal{G}}$ has an inverse up to a power of p, see Remark 39.31.3. Finally, the second is an isomorphism after inverting p by Lemma 39.30.6. □

39.32. Other chapters

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CHAPTER 40

Algebraic Spaces

40.1. Introduction

Algebraic spaces were first introduced by Michael Artin, see [**Art69c**], [**Art70a**], [**Art73a**], [**Art71c**], [**Art71a**], [**Art69a**], [**Art69e**], and [**Art74a**]. Some of the foundational material was developed jointly with Knutson, who produced the book [**Knu71a**]. Artin defined (see [**Art69e**, Definition 1.3]) an algebraic space as a sheaf for the étale topology which is locally in the étale topology representable. In most of Artin's work the categories of schemes considered are schemes locally of finite type over a fixed excellent Noetherian base.

Our definition is slightly different. First of all we consider sheaves for the fppf topology. This is just a technical point and scarcely makes any difference. Second, we include the condition that the diagonal is representable.

After defining algebraic spaces we make some foundational observations. The main result in this chapter is that with our definitions an algebraic space is the same thing as an étale equivalence relation, see the discussion in Section 40.9 and Theorem 40.10.5. The analogue of this theorem in Artin's setting is [**Art69e**, Theorem 1.5], or [**Knu71a**, Proposition II.1.7]. In other words, the sheaf defined by an étale equivalence relation has a representable diagonal. It follows that our definition agrees with Artin's original definition in a broad sense. It also means that one can give examples of algebraic spaces by simply writing down an étale equivalence relation.

In Section 40.13 we introduce various separation axioms on algebraic spaces that we have found in the literatur. Finally in Section 40.14 we give some weird and not so weird examples of algebraic spaces.

40.2. General remarks

We work in a suitable big fppf site Sch_{fppf} as in Topologies, Definition 30.7.6. So, if not explicitly stated otherwise all schemes will be objects of Sch_{fppf}. We will record elsewhere what changes if you change the big fppf site (insert future reference here).

We will always work relative to a base S contained in Sch_{fppf}. And we will then work with the big fppf site $(Sch/S)_{fppf}$, see Topologies, Definition 30.7.8. The absolute case can be recovered by taking $S = Spec(\mathbf{Z})$.

If U, T are schemes over S, then we denote $U(T)$ for the set of T-valued points *over* S. In a formula: $U(T) = Mor_S(T, U)$.

Note that any fpqc covering is a universal effective epimorphism, see Descent, Lemma 31.9.3. Hence the topology on Sch_{fppf} is weaker than the canonical topology and all representable presheaves are sheaves.

40.3. Representable morphisms of presheaves

Let S be a scheme contained in Sch_{fppf}. Let $F, G : (Sch/S)_{fppf}^{opp} \to Sets$. Let $a : F \to G$ be a representable transformation of functors, see Categories, Definition 4.8.2. This means that for every $U \in Ob((Sch/S)_{fppf})$ and any $\xi \in G(U)$ the fiber product $h_U \times_{\xi, G} F$ is representable. Choose a representing object V_ξ and an isomorphism $h_{V_\xi} \to h_U \times_G F$. By the Yoneda lemma, see Categories, Lemma 4.3.5, the projection $h_{V_\xi} \to h_U \times_G F \to h_U$ comes from a unique morphism of schemes $a_\xi : V_\xi \to U$. Suggestively we could represent this by the diagram

$$\begin{tikzcd} V_\xi \arrow[r, squiggly] \arrow[d, "a_\xi"'] & h_{V_\xi} \arrow[r] \arrow[d] & F \arrow[d, "a"] \\ U \arrow[r, squiggly] & h_U \arrow[r, "\xi"] & G \end{tikzcd}$$

where the squiggly arrows represent the Yoneda embedding. Here are some lemmas about this notion that work in great generality.

Lemma 40.3.1. *Let S, X, Y be objects of Sch_{fppf}. Let $f : X \to Y$ be a morphism of schemes. Then*

$$h_f : h_X \longrightarrow h_Y$$

is a representable transformation of functors.

Proof. This is formal and relies only on the fact that the category $(Sch/S)_{fppf}$ has fibre products. □

Lemma 40.3.2. *Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)_{fppf}^{opp} \to Sets$. Let $a : F \to G$, $b : G \to H$ be representable transformations of functors. Then*

$$b \circ a : F \longrightarrow H$$

is a representable transformation of functors.

Proof. This is entirely formal and works in any category. □

Lemma 40.3.3. *Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)_{fppf}^{opp} \to Sets$. Let $a : F \to G$ be a representable transformations of functors. Let $b : H \to G$ be any transformation of functors. Consider the fibre product diagram*

$$\begin{tikzcd} H \times_{b,G,a} F \arrow[r, "b'"'] \arrow[d, "a'"'] & F \arrow[d, "a"] \\ H \arrow[r, "b"] & G \end{tikzcd}$$

Then the base change a' is a representable transformation of functors.

Proof. This is entirely formal and works in any category. □

Lemma 40.3.4. *Let S be a scheme contained in Sch_{fppf}. Let $F_i, G_i : (Sch/S)_{fppf}^{opp} \to Sets$, $i = 1, 2$. Let $a_i : F_i \to G_i$, $i = 1, 2$ be representable transformations of functors. Then*

$$a_1 \times a_2 : F_1 \times F_2 \longrightarrow G_1 \times G_2$$

is a representable transformation of functors.

Proof. Write $a_1 \times a_2$ as the composition $F_1 \times F_2 \to G_1 \times F_2 \to G_1 \times G_2$. The first arrow is the base change of a_1 by the map $G_1 \times F_2 \to G_1$, and the second arrow is the base change of a_2 by the map $G_1 \times G_2 \to G_2$. Hence this lemma is a formal consequence of Lemmas 40.3.2 and 40.3.3. □

Lemma 40.3.5. *Let S be a scheme contained in Sch_{fppf}. Let $F, G : (Sch/S)_{fppf}^{opp} \to Sets$. Let $a : F \to G$ be a representable transformation of functors. If G is a sheaf, then so is F.*

Proof. Let $\{\varphi_i : T_i \to T\}$ be a covering of the site $(Sch/S)_{fppf}$. Let $s_i \in F(T_i)$ which satisfy the sheaf condition. Then $\sigma_i = a(s_i) \in G(T_i)$ satisfy the sheaf condition also. Hence there exists a unique $\sigma \in G(T)$ such that $\sigma_i = \sigma|_{T_i}$. By assumption $F' = h_T \times_{\sigma, G, a} F$ is a representable presheaf and hence (see remarks in Section 40.2) a sheaf. Note that $(\varphi_i, s_i) \in F'(T_i)$ satisfy the sheaf condition also, and hence come from some unique $(\mathrm{id}_T, s) \in F'(T)$. Clearly s is the section of F we are looking for. □

Lemma 40.3.6. *Let S be a scheme contained in Sch_{fppf}. Let $F, G : (Sch/S)_{fppf}^{opp} \to Sets$. Let $a : F \to G$ be a representable transformation of functors. Then $\Delta_{F/G} : F \to F \times_G F$ is representable.*

Proof. Let $U \in Ob((Sch/S)_{fppf})$. Let $\xi = (\xi_1, \xi_2) \in (F \times_G F)(U)$. Set $\xi' = a(\xi_1) = a(\xi_2) \in G(U)$. By assumption there exist a scheme V and a morphism $V \to U$ representing the fibre product $h_U \times_{\xi', G} F$. In particular, the elements ξ_1, ξ_2 give morphisms $f_1, f_2 : U \to V$ over U. Because V represents the fibre product $h_U \times_{\xi', G} F$ and because $\xi' = a \circ \xi_1 = a \circ \xi_2$ we see that if $g : U' \to U$ is a morphism then

$$g^*\xi_1 = g^*\xi_2 \Leftrightarrow f_1 \circ g = f_2 \circ g.$$

In other words, we see that $h_U \times_{\xi, F \times_G F} F$ is represented by $V \times_{\Delta, V \times V, (f_1, f_2)} U$ which is a scheme. □

40.4. Lists of useful properties of morphisms of schemes

For ease of reference we list in the following remarks the properties of morphisms which possess some of the properties required of them in later results.

Remark 40.4.1. Here is a list of properties/types of morphisms which are *stable under arbitrary base change*:

(1) closed, open, and locally closed immersions, see Schemes, Lemma 21.18.2,
(2) quasi-compact, see Schemes, Lemma 21.19.3,
(3) universally closed, see Schemes, Definition 21.20.1,
(4) (quasi-)separated, see Schemes, Lemma 21.21.13,
(5) monomorphism, see Schemes, Lemma 21.23.5
(6) surjective, see Morphisms, Lemma 24.9.4,
(7) universally injective, see Morphisms, Lemma 24.10.2,
(8) affine, see Morphisms, Lemma 24.11.8,
(9) quasi-affine, see Morphisms, Lemma 24.12.5,
(10) (locally) of finite type, see Morphisms, Lemma 24.14.4,
(11) (locally) quasi-finite, see Morphisms, Lemma 24.19.13,
(12) (locally) of finite presentation, see Morphisms, Lemma 24.20.4,
(13) locally of finite type of relative dimension d, see Morphisms, Lemma 24.28.2,
(14) universally open, see Morphisms, Definition 24.22.1,
(15) flat, see Morphisms, Lemma 24.24.7,
(16) syntomic, see Morphisms, Lemma 24.30.4,

(17) smooth, see Morphisms, Lemma 24.33.5,
(18) unramified (resp. G-unramified), see Morphisms, Lemma 24.34.5,
(19) étale, see Morphisms, Lemma 24.35.4,
(20) proper, see Morphisms, Lemma 24.40.5,
(21) H-projective, see Morphisms, Lemma 24.41.8,
(22) (locally) projective, see Morphisms, Lemma 24.41.9,
(23) finite or integral, see Morphisms, Lemma 24.42.6,
(24) finite locally free, see Morphisms, Lemma 24.44.4.

Add more as needed.

Remark 40.4.2. Of the properties of morphisms which are stable under base change (as listed in Remark 40.4.1) the following are also *stable under compositions*:

(1) closed, open and locally closed immersions, see Schemes, Lemma 21.24.3,
(2) quasi-compact, see Schemes, Lemma 21.19.4,
(3) universally closed, see Morphisms, Lemma 24.40.4,
(4) (quasi-)separated, see Schemes, Lemma 21.21.13,
(5) monomorphism, see Schemes, Lemma 21.23.4,
(6) surjective, see Morphisms, Lemma 24.9.2,
(7) universally injective, see Morphisms, Lemma 24.10.5,
(8) affine, see Morphisms, Lemma 24.11.7,
(9) quasi-affine, see Morphisms, Lemma 24.12.4,
(10) (locally) of finite type, see Morphisms, Lemma 24.14.3,
(11) (locally) quasi-finite, see Morphisms, Lemma 24.19.12,
(12) (locally) of finite presentation, see Morphisms, Lemma 24.20.3,
(13) universally open, see Morphisms, Lemma 24.22.3,
(14) flat, see Morphisms, Lemma 24.24.5,
(15) syntomic, see Morphisms, Lemma 24.30.3,
(16) smooth, see Morphisms, Lemma 24.33.4,
(17) unramified (resp. G-unramified), see Morphisms, Lemma 24.34.4,
(18) étale, see Morphisms, Lemma 24.35.3,
(19) proper, see Morphisms, Lemma 24.40.4,
(20) H-projective, see Morphisms, Lemma 24.41.7,
(21) finite or integral, see Morphisms, Lemma 24.42.5,
(22) finite locally free, see Morphisms, Lemma 24.44.3.

Add more as needed.

Remark 40.4.3. Of the properties mentioned which are stable under base change (as listed in Remark 40.4.1) the following are also *fpqc local on the base* (and a fortiori fppf local on the base):

(1) for immersions we have this for
    (a) closed immersions, see Descent, Lemma 31.19.17,
    (b) open immersions, see Descent, Lemma 31.19.14, and
    (c) quasi-compact immersions, see Descent, Lemma 31.19.19,
(2) quasi-compact, see Descent, Lemma 31.19.1,
(3) universally closed, see Descent, Lemma 31.19.3,
(4) (quasi-)separated, see Descent, Lemmas 31.19.2, and 31.19.5,
(5) monomorphism, see Descent, Lemma 31.19.29,
(6) surjective, see Descent, Lemma 31.19.6,
(7) universally injective, see Descent, Lemma 31.19.7,

(8) affine, see Descent, Lemma 31.19.16,
(9) quasi-affine, see Descent, Lemma 31.19.18,
(10) (locally) of finite type, see Descent, Lemmas 31.19.8, and 31.19.10,
(11) (locally) quasi-finite, see Descent, Lemma 31.19.22,
(12) (locally) of finite presentation, see Descent, Lemmas 31.19.9, and 31.19.11,
(13) locally of finite type of relative dimension d, see Descent, Lemma 31.19.23,
(14) universally open, see Descent, Lemma 31.19.4,
(15) flat, see Descent, Lemma 31.19.13,
(16) syntomic, see Descent, Lemma 31.19.24,
(17) smooth, see Descent, Lemma 31.19.25,
(18) unramified (resp. G-unramified), see Descent, Lemma 31.19.26,
(19) étale, see Descent, Lemma 31.19.27,
(20) proper, see Descent, Lemma 31.19.12,
(21) finite or integral, see Descent, Lemma 31.19.21,
(22) finite locally free, see Descent, Lemma 31.19.28.

Note that the property of being an ``immersion'' may not be fpqc local on the base, but in Descent, Lemma 31.20.1 we proved that it is fppf local on the base.

40.5. Properties of representable morphisms of presheaves

Here is the definition that makes this work.

Definition 40.5.1. With S, and $a : F \to G$ representable as above. Let $\mathcal{P}$ be a property of morphisms of schemes which

(1) is preserved under any base change, see Schemes, Definition 21.18.3, and
(2) is fppf local on the base, see Descent, Definition 31.18.1.

In this case we say that a has *property $\mathcal{P}$* if for every $U \in Ob((Sch/S)_{fppf})$ and any $\xi \in G(U)$ the resulting morphism of schemes $V_\xi \to U$ has property $\mathcal{P}$.

It is important to note that we will only use this definition for properties of morphisms that are stable under base change, and local in the fppf topology on the base. This is not because the definition doesn't make sense otherwise; rather it is because we may want to give a different definition which is better suited to the property we have in mind.

Remark 40.5.2. Consider the property $\mathcal{P}$=``surjective''. In this case there could be some ambiguity if we say ``let $F \to G$ be a surjective map''. Namely, we could mean the notion defined in Definition 40.5.1 above, or we could mean a surjective map of presheaves, see Sites, Definition 9.3.1, or, if both F and G are sheaves, we could mean a surjective map of sheaves, see Sites, Definition 9.11.1, If not mentioned otherwise when discussing morphisms of algebraic spaces we will always mean the first. See Lemma 40.5.9 for a case where surjectivity implies surjectivity as a map of sheaves.

Here is a sanity check.

Lemma 40.5.3. *Let S, X, Y be objects of Sch_{fppf}. Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{P}$ be as in Definition 40.5.1. Then $h_X \longrightarrow h_Y$ has propery $\mathcal{P}$ if and only if f has property $\mathcal{P}$.*

Proof. Note that the lemma makes sense by Lemma 40.3.1. Proof omitted. □

Lemma 40.5.4. *Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)_{fppf}^{opp} \to Sets$. Let $\mathcal{P}$ be a property as in Definition 40.5.1 which is stable under composition. Let*

$a : F \to G$, $b : G \to H$ be representable transformations of functors. If a and b have property $\mathcal{P}$ so does $b \circ a : F \longrightarrow H$.

Proof. Note that the lemma makes sense by Lemma 40.3.2. Proof omitted. □

Lemma 40.5.5. *Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)_{fppf}^{opp} \to Sets$. Let $\mathcal{P}$ be a property as in Definition 40.5.1. Let $a : F \to G$ be a representable transformations of functors. Let $b : H \to G$ be any transformation of functors. Consider the fibre product diagram*

$$\begin{tikzcd} H \times_{b,G,a} F \arrow[r, "b'"] \arrow[d, "a'"'] & F \arrow[d, "a"] \\ H \arrow[r, "b"] & G \end{tikzcd}$$

If a has property $\mathcal{P}$ then also the base change a' has property $\mathcal{P}$.

Proof. Note that the lemma makes sense by Lemma 40.3.3. Proof omitted. □

Lemma 40.5.6. *Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)_{fppf}^{opp} \to Sets$. Let $\mathcal{P}$ be a property as in Definition 40.5.1. Let $a : F \to G$ be a representable transformations of functors. Let $b : H \to G$ be any transformation of functors. Consider the fibre product diagram*

$$\begin{tikzcd} H \times_{b,G,a} F \arrow[r, "b'"] \arrow[d, "a'"'] & F \arrow[d, "a"] \\ H \arrow[r, "b"] & G \end{tikzcd}$$

Assume that b induces a surjective map of fppf sheaves $H^\# \to G^\#$. In this case, if a' has property $\mathcal{P}$, then also a has property $\mathcal{P}$.

Proof. First we remark that by Lemma 40.3.3 the transformation a' is representable. Let $U \in Ob((Sch/S)_{fppf})$, and let $\xi \in G(U)$. By assumption there exists an fppf covering $\{U_i \to U\}_{i \in I}$ and elements $\xi_i \in H(U_i)$ mapping to $\xi|_U$ via b. From general category theory it follows that for each i we have a fibre product diagram

$$\begin{tikzcd} U_i \times_{\xi_i, H, a'} (H \times_{b,G,a} F) \arrow[r] \arrow[d] & U \times_{\xi, G, a} F \arrow[d] \\ U_i \arrow[r] & U \end{tikzcd}$$

By assumption the left vertical arrow is a morphism of schemes which has property $\mathcal{P}$. Since $\mathcal{P}$ is local in the fppf topology this implies that also the right vertical arrow has property $\mathcal{P}$ as desired. □

Lemma 40.5.7. *Let S be a scheme contained in Sch_{fppf}. Let $F_i, G_i : (Sch/S)_{fppf}^{opp} \to Sets$, $i = 1, 2$. Let $a_i : F_i \to G_i$, $i = 1, 2$ be representable transformations of functors. Let $\mathcal{P}$ be a property as in Definition 40.5.1 which is stable under composition. If a_1 and a_2 have property $\mathcal{P}$ so does $a_1 \times a_2 : F_1 \times F_2 \longrightarrow G_1 \times G_2$.*

Proof. Note that the lemma makes sense by Lemma 40.3.4. Proof omitted. □

Lemma 40.5.8. *Let S be a scheme contained in Sch_{fppf}. Let $F, G : (Sch/S)_{fppf}^{opp} \to Sets$. Let $a : F \to G$ be a representable transformation of functors. Let $\mathcal{P}$, $\mathcal{P}'$ be properties* *as in Definition 40.5.1. Suppose that for any morphism of schemes $f : X \to Y$ we have $\mathcal{P}(f) \Rightarrow \mathcal{P}'(f)$. If a has property $\mathcal{P}$ then a has property $\mathcal{P}'$.*

Proof. Formal. □

Lemma 40.5.9. *Let S be a scheme. Let $F, G : (Sch/S)_{fppf}^{opp} \to Sets$ be sheaves. Let $a : F \to G$ be representable, flat, locally of finite presentation, and surjective. Then $a : F \to G$ is surjective as a map of sheaves.*

Proof. Let T be a scheme over S and let $g : T \to G$ be a T-valued point of G. By assumption $T' = F \times_G T$ is (representable by) a scheme and the morphism $T' \to T$ is a flat, locally of finite presentation, and surjective. Hence $\{T' \to T\}$ is an fppf covering such that $g|_{T'} \in G(T')$ comes from an element of $F(T')$, namely the map $T' \to F$. This proves the map is surjective as a map of sheaves, see Sites, Definition 9.11.1. □

Here is a characterization of those functors for which the diagonal is representable.

Lemma 40.5.10. *Let S be a scheme contained in Sch_{fppf}. Let F be a presheaf of sets on $(Sch/S)_{fppf}$. The following are equivalent:*

(1) *The diagonal $F \to F \times F$ is representable.*
(2) *For every scheme U over S, $U/S \in Ob((Sch/S)_{fppf})$ and any $\xi \in F(U)$ the map $\xi : h_U \to F$ is representable.*

Proof. This is completely formal, see Categories, Lemma 4.8.4. It depends only on the fact that the category $(Sch/S)_{fppf}$ has products of pairs of objects and fibre products, see Topologies, Lemma 30.7.10. □

In the situation of the lemma, for any morphism $\xi : h_U \to F$ as in the lemma, it makes sense to say that ξ has property $\mathcal{P}$, for any property as in Definition 40.5.1. In particular this holds for $\mathcal{P}$ = ``surjective'' and $\mathcal{P}$ = ``étale'', see Remark 40.4.3 above. We will use these in the definition of algebraic spaces below.

40.6. Algebraic spaces

Here is the definition.

Definition 40.6.1. Let S be a scheme contained in Sch_{fppf}. An *algebraic space over S* is a presheaf

$$F : (Sch/S)_{fppf}^{opp} \longrightarrow Sets$$

with the following properties

(1) The presheaf F is a sheaf.
(2) The diagonal morphism $F \to F \times F$ is representable.
(3) There exists a scheme $U \in Ob(Sch_{fppf})$ and a map $h_U \to F$ which is surjective, and étale.

There are two differences with the ``usual'' definition, for example the definition in Knutson's book [**Knu71a**].

The first is that we require F to be a sheaf in the fppf topology. One reason for doing this is that many natural examples of algebraic spaces satisfy the sheaf condition for the fppf coverings (and even for fpqc coverings). Also, one of the reasons that algebraic spaces have been so useful is via Michael Artin's results on algebraic spaces. Built into his method is a condition which guarantees the result is locally of finite presentation over S. Combined it somehow seems to us that the fppf topology is the natural topology to work with. In the end the category of algebraic spaces ends up being the same. See Bootstrap, Section 54.12.

The second is that we only require the diagonal map for F to be representable, whereas in [**Knu71a**] it is required that it also be quasi-compact. If $F = h_U$ for some scheme U over S this corresponds to the condition that S be quasi-separated. Our point of view is to try to prove a certain number of the results that follow only assuming that the diagonal of F be representable, and simply add an addition hypothesis wherever this is necessary. In any case it has the pleasing consquence that the following lemma is true.

Lemma 40.6.2. *A scheme is an algebraic space. More precisely, given a scheme $T \in Ob((Sch/S)_{fppf})$ the representable functor h_T is an algebraic space.*

Proof. The functor h_T is a sheaf by our remarks in Section 40.2. The diagonal $h_T \to h_T \times h_T = h_{T\times T}$ is representable because $(Sch/S)_{fppf}$ has fibre products. The identity map $h_T \to h_T$ is surjective étale. □

Definition 40.6.3. Let F, F' be algebraic spaces over S. A *morphism $f : F \to F'$ of algebraic spaces over S* is a transformation of functors from F to F'.

The category of algebraic spaces over S contains the category $(Sch/S)_{fppf}$ as a full subcategory via the Yoneda embedding $T/S \mapsto h_T$. From now on we no longer distinghuish between a scheme T/S and the algebraic space it represents. Thus when we say ``Let $f : T \to F$ be a morphism from the scheme T to the algebraic space F", we mean that $T \in Ob((Sch/S)_{fppf})$, that F is an algebraic space over S, and that $f : h_T \to F$ is a morphism of algebraic spaces over S.

40.7. Fibre products of algebraic spaces

The category of algebraic spaces over S has both products and fibre products.

Lemma 40.7.1. *Let S be a scheme contained in Sch_{fppf}. Let F, G be algebraic spaces over S. Then $F \times G$ is an algebraic space, and is a product in the category of algebraic spaces over S.*

Proof. It is clear that $H = F\times G$ is a sheaf. The diagonal of H is simply the product of the diagonals of F and G. Hence it is representable by Lemma 40.3.4. Finally, if $U \to F$ and $V \to G$ are surjective étale morphisms, with $U, V \in Ob((Sch/S)_{fppf})$, then $U \times V \to F \times G$ is surjective étale by Lemma 40.5.7. □

Lemma 40.7.2. *Let S be a scheme contained in Sch_{fppf}. Let H be a sheaf on $(Sch/S)_{fppf}$ whose diagonal is representable. Let F, G be algebraic spaces over S. Let $F \to H$, $G \to H$ be maps of sheaves. Then $F \times_H G$ is an algebraic space.*

Proof. We check the 3 conditions of Definition 40.6.1. A fibre product of sheaves is a sheaf, hence $F \times_H G$ is a sheaf. The diagonal of $F \times_H G$ is the left vertical arrow in

$$\begin{tikzcd} F \times_H G \arrow[r] \arrow[d, "\Delta"] & F \times G \arrow[d, "\Delta_F \times \Delta_G"] \\ (F \times F) \times_{(H\times H)} (G \times G) \arrow[r] & (F \times F) \times (G \times G) \end{tikzcd}$$

which is cartesian. Hence Δ is representable as the base change of the morphism on the right which is representable, see Lemmas 40.3.4 and 40.3.3. Finally, let $U, V \in Ob((Sch/S)_{fppf})$

and $a : U \to F$, $b : V \to G$ be surjective and étale. As Δ_H is representable, we see that $U \times_H V$ is a scheme. The morphism

$$U \times_H V \longrightarrow F \times_H G$$

is surjective and étale as a composition of the base changes $U \times_H V \to U \times_H G$ and $U \times_H G \to F \times_H G$ of the étale surjective morphisms $U \to F$ and $V \to G$, see Lemmas 40.3.2 and 40.3.3. This proves the last condition of Definition 40.6.1 holds and we conclude that $F \times_H G$ is an algebraic space. □

Lemma 40.7.3. *Let S be a scheme contained in Sch_{fppf}. Let $F \to H$, $G \to H$ be morphisms of algebraic spaces over S. Then $F \times_H G$ is an algebraic space, and is a fibre product in the category of algebraic spaces over S.*

Proof. It follows from the stronger Lemma 40.7.2 that $F \times_H G$ is an algebraic space. It is clear that $F \times_H G$ is a fibre product in the category of algebraic spaces over S since that is a full subcategory of the category of (pre)sheaves of sets on $(Sch/S)_{fppf}$. □

40.8. Glueing algebraic spaces

In this section we really start abusing notation and not distinguish between schemes and the spaces they represent.

Lemma 40.8.1. *Let $S \in Ob(Sch_{fppf})$. Let $U \in Ob((Sch/S)_{fppf})$. Given a set I and sheaves F_i on $Ob((Sch/S)_{fppf})$, if $U \cong \coprod_{i \in I} F_i$ as sheaves, then each F_i is representable by an open and closed subscheme U_i and $U \cong \coprod U_i$ as schemes.*

Proof. By assumption this means there exists an fppf covering $\{U_j \to U\}_{j \in J}$ such that each $U_j \to U$ factors through $F_{i(j)}$ for some $i(j) \in I$. Denote $V_j = \mathrm{Im}(U_j \to U)$. This is an open of U by Morphisms, Lemma 24.24.9, and $\{U_j \to V_j\}$ is an fppf covering. Hence it follows that $V_j \to U$ factors through $F_{i(j)}$ since $F_{i(j)}$ is a subsheaf. It follows from $F_i \cap F_{i'} = \emptyset$, $i \neq i'$ that $V_j \cap V_{j'} = \emptyset$ unless $i(j) = i(j')$. Hence we can take $U_i = \bigcup_{j,\ i(j)=i} V_j$ and everything is clear. □

Lemma 40.8.2. *Let $S \in Ob(Sch_{fppf})$. Let F be an algebraic space over S. Given a set I and sheaves F_i on $Ob((Sch/S)_{fppf})$, if $F \cong \coprod_{i \in I} F_i$ as sheaves, then each F_i is an algebraic space over S.*

Proof. It follows directly from the representability of $F \to F \times F$ that each diagonal morphism $F_i \to F_i \times F_i$ is representable. Choose a scheme U in $(Sch/S)_{fppf}$ and a surjective étale morphism $U \to \coprod F_i$ (this exist by hypothesis). By considering the inverse image of F_i we get a decomposition of U (as a sheaf) into a coproduct of sheaves. By Lemma 40.8.1 we get correspondingly $U \cong \coprod U_i$. Then it follows easily that $U_i \to F_i$ is surjective and étale (from the corresponding property of $U \to F$). □

The condition on the size of I and the F_i in the following lemma may be ignored by those not worried about set theoretic questions.

Lemma 40.8.3. *Let $S \in Ob(Sch_{fppf})$. Suppose given a set I and algebraic spaces F_i, $i \in I$. Then $F = \coprod_{i \in I} F_i$ is an algebraic space provided I, and the F_i are not too ``large'': for example if we can choose surjective étale morphisms $U_i \to F_i$ such that $\coprod_{i \in I} U_i$ is isomorphic to an object of $(Sch/S)_{fppf}$, then F is an algebraic space.*

Proof. By construction F is a sheaf. We omit the verification that the diagonal morphism of F is representable. Finally, if U is an object of $(Sch/S)_{fppf}$ isomorphic to $\coprod_{i\in I} U_i$ then it is straightforward to verify that the resulting map $U \to \coprod F_i$ is surjective and étale. □

Here is the analogue of Schemes, Lemma 21.15.4.

Lemma 40.8.4. *Let $S \in Ob(Sch_{fppf})$. Let F be a presheaf of sets on $(Sch/S)_{fppf}$. Assume*

1. *F is a sheaf,*
2. *there exists an index set I and subfunctors $F_i \subset F$ such that*
   (a) *each F_i is an algebraic space,*
   (b) *each $F_i \to F$ is a representable,*
   (c) *each $F_i \to F$ is an open immersion (see Definition 40.5.1),*
   (d) *the map of sheaves $\coprod F_i \to F$ is surjective, and*
   (e) *$\coprod F_i$ is an algebraic space (set theoretic condition, see Lemma 40.8.3).*

Then F is an algebraic space.

Proof. Let T, T' be objects of $(Sch/S)_{fppf}$. Let $T \to F$, $T' \to F$ morphisms. The assumptions imply that there exists an open covering $T = \bigcup V_i$ such that $V_i = T \times_F F_i$. Note that this in particular implies that $\coprod F_i \to F$ is surjective in the Zariski topology! Also write similarly $T' = \bigcup V'_i$ with $V'_i = T' \times_F F_i$.

To show that the diagonal $F \to F \times F$ is representable we have to show that $G = T \times_F T'$ is representable. Consider the subfunctors $G_i = G \times_F F_i$. Note that $G_i = V_i \times_{F_i} V'_i$, and hence is representable as F_i is an algebraic space. By the above the G_i form a Zariski covering of F. Hence by Schemes, Lemma 21.15.4 we see G is representable.

Choose a scheme $U \in Ob((Sch/S)_{fppf})$ and a surjective étale morphism $U \to \coprod F_i$ (this exist by hypothesis). We may write $U = \coprod U_i$ with U_i the inverse image of F_i, see Lemma 40.8.1. We claim that $U \to F$ is surjective and étale. Surjectivity follows as $\coprod F_i \to F$ is surjective. Consider the fibre product $U \times_F T$ where $T \to F$ is as above. We have to show that $U \times_F T \to T$ is étale. Since $U \times_F T = \coprod U_i \times_F T$ it suffices to show each $U_i \times_F T \to T$ is étale. Since $U_i \times_F T = U_i \times_{F_i} V_i$ this follows from the fact that $U_i \to F_i$ is étale and $V_i \to T$ is an open immersion (and Morphisms, Lemmas 24.35.9 and 24.35.3). □

40.9. Presentations of algebraic spaces

Given an algebraic space we can find a ``presentation" of it.

Lemma 40.9.1. *Let F be an algebraic space over S. Let $f : U \to F$ be a surjective étale morphism from a scheme to F. Set $R = U \times_F U$. Then*

1. *$j : R \to U \times_S U$ defines an equivalence relation on U over S (see Groupoids, Definition 35.3.1).*
2. *the morphisms $s, t : R \to U$ are étale, and*
3. *the diagram*
$$R \rightrightarrows U \longrightarrow F$$
is a coequalizer diagram in $Sh((Sch/S)_{fppf})$.

Proof. Let T/S be an object of $(Sch/S)_{fppf}$. Then $R(T) = \{(a, b) \in U(T) \times U(T) \mid f \circ a = f \circ b\}$ which is clearly defines an equivalence relation on $U(T)$. The morphisms $s, t : R \to U$ are étale because the morphism $U \to F$ is étale.

To prove (3) we first show that $U \to F$ is a surjection of sheaves, see Sites, Definition 9.11.1. Let $\xi \in F(T)$ with T as above. Let $V = T \times_{\xi, F, f} U$. By assumption V is a scheme and $V \to T$ is surjective étale. Hence $\{V \to T\}$ is a covering for the fppf topology. Since $\xi|_V$ factors through U by construction we conclude $U \to F$ is surjective. To conclude we have to show that given any two morphisms $a, b : T \to U$ such that $f \circ a = f \circ b$ there is a morphism $c : T \to R$ such that $a = \mathrm{pr}_0 \circ c$ and $b = \mathrm{pr}_1 \circ b$. This is clear from the definition of R. $\square$

This lemma suggests the following definitions.

Definition 40.9.2. Let S be a scheme. Let U be a scheme over S. An *étale equivalence relation* on U over S is an equivalence relation $j : R \to U \times_S U$ such that $s, t : R \to U$ are étale morphisms of schemes.

Definition 40.9.3. Let F be an algebraic space over S. A *presentation* of F is given by a scheme U over S and an étale equivalence relation R on U over S, and a surjective étale morphism $U \to F$ such that $R = U \times_F U$.

Equivalently we could ask for the existence of an isomorphism

$$U/R \cong F$$

where the quotient U/R is as defined in Groupoids, Section 35.17. To construct algebraic spaces we will study the converse question, namely, for which equivalence relations the quotient sheaf U/R is an algebraic space. It will finally turn out this is always the case if R is an étale equivalence relation on U over S, see Theorem 40.10.5.

40.10. Algebraic spaces and equivalence relations

Suppose given a scheme U over S and an étale equivalence relation R on U over S. We would like to show this defines an algebraic space. We will produce a series of lemmas that prove the quotient sheaf U/R (see Groupoids, Definition 35.17.1) has all the properties required of it in Definition 40.6.1.

Lemma 40.10.1. *Let S be a scheme. Let U be a scheme over S. Let $j = (s, t) : R \to U \times_S U$ be an étale equivalence relation on U over S. Let $U' \to U$ be an étale morphism. Let R' be the restriction of R to U', see Groupoids, Definition 35.3.3. Then $j' : R' \to U' \times_S U'$ is an étale equivalence relation also.*

Proof. It is clear from the description of s', t' in Groupoids, Lemma 35.15.1 that $s', t' : R' \to U'$ are étale as compositions of base changes of étale morphisms (see Morphisms, Lemma 24.35.4 and 24.35.3). $\square$

We will often use the following lemma to find open subspaces of algebraic spaces. A slight improvement (with more general hypotheses) of this lemma is Bootstrap, Lemma 54.7.1.

Lemma 40.10.2. *Let S be a scheme. Let U be a scheme over S. Let $j = (s, t) : R \to U \times_S U$ be a pre-relation. Let $g : U' \to U$ be a morphism. Assume*

1. *j is an equivalence relation,*
2. *$s, t : R \to U$ are surjective, flat and locally of finite presentation,*
3. *g is flat and locally of finite presentation.*

Let $R' = R|_{U'}$ be the restriction of R to U. Then $R'/U' \to R/U$ is representable, and is an open immersion.

Proof. By Groupoids, Lemma 35.3.2 the morphism $j' = (t', s') : R' \to U' \times_S U'$ defines an equivalence relation. Since g is flat and locally of finite presentation we see that g is universally open as well (Morphisms, Lemma 24.24.9). For the same reason s, t are universally open as well. Let $W^1 = g(U') \subset U$, and let $W = t(s^{-1}(W^1))$. Then W^1 and W are open in U. Moreover, as j is an equivalence relation we have $t(s^{-1}(W)) = W$ (see Groupoids, Lemma 35.16.2 for example).

By Groupoids, Lemma 35.17.5 the map of sheaves $F' = U'/R' \to F = U/R$ is injective. Let $a : T \to F$ be a morphism from a scheme into U/R. We have to show that $T \times_F F'$ is representable by an open subscheme of T.

The morphism a is given by the following data: an fppf covering $\{\varphi_j : T_j \to T\}_{j \in J}$ of T and morphsms $a_j : T_j \to U$ such that the maps

$$a_j \times a_{j'} : T_j \times_T T_{j'} \longrightarrow U \times_S U$$

factor through $j : R \to U \times_S U$ via some (unique) maps $r_{jj'} : T_j \times_T T_{j'} \to R$. The system (a_j) corresponds to a in the sense that the diagrams

$$\begin{tikzcd} T_j \arrow[r, "a_j"'] \arrow[d] & U \arrow[d] \\ T \arrow[r, "a"] & F \end{tikzcd}$$

commute.

Consider the open subsets $W_j = a_j^{-1}(W) \subset T_j$. Since $t(s^{-1}(W)) = W$ we see that

$$W_j \times_T T_{j'} = r_{jj'}^{-1}(t^{-1}(W)) = r_{jj'}^{-1}(s^{-1}(W)) = T_j \times_T W_{j'}.$$

By Descent, Lemma 31.9.2 this means there exists an open $W_T \subset T$ such that $\varphi_j^{-1}(W_T) = W_j$ for all $j \in J$. We claim that $W_T \to T$ represents $T \times_F F' \to T$.

First, let us show that $W_T \to T \to F$ is an element of $F'(W_T)$. Since $\{W_j \to W_T\}_{j \in J}$ is an fppf covering of W_T, it is enough to show that each $W_j \to U \to F$ is an element of $F'(W_j)$ (as F' is a sheaf for the fppf topology). Consider the commutative diagram

$$\begin{tikzcd} W'_j \arrow[rr] \arrow[dr] \arrow[dd] & & U' \arrow[d, "g"] \\ & s^{-1}(W^1) \arrow[r, "s"'] \arrow[d, "t"] & W^1 \arrow[d] \\ W_j \arrow[r, "a_j|_{W_j}"] & W \arrow[r] & F \end{tikzcd}$$

where $W'_j = W_j \times_W s^{-1}(W^1) \times_{W^1} U'$. Since t and g are surjective, flat and locally of finite presentation, so is $W'_j \to W_j$. Hence the restriction of the element $W_j \to U \to F$ to W'_j is an element of F' as desired.

Suppose that $f : T' \to T$ is a morphism of schemes such that $a|_{T'} \in F'(T')$. We have to show that f factors through the open W_T. Since $\{T' \times_T T_j \to T\}$ is an fppf covering of T' it is enough to show each $T' \times_T T_j \to T$ factors through W_T. Hence we may assume f factors as $\varphi_j \circ f_j : T' \to T_j \to T$ for some j. In this case the condition $a|_{T'} \in F'(T')$ means that there exists some fppf covering $\{\psi_i : T'_i \to T'\}_{i \in I}$ and some morphisms $b_i : T'_i \to U'$

such that

$$\begin{tikzcd} T_i' \arrow[r, "b_i"'] \arrow[d, "f_j\circ\psi_i"'] & U' \arrow[r, "g"'] & U \arrow[d] \\ T_j \arrow[r, "a_j"] & U \arrow[r] & F \end{tikzcd}$$

is commutative. This commutativity means that there exists a morphism $r_i' : T_i' \to R$ such that $t \circ r_i' = a_j \circ f_j \circ \psi_i$, and $s \circ r_i' = g \circ b_i$. This implies that $\mathrm{Im}(f_j \circ \psi_i) \subset W_j$ and we win. $\square$

The following lemma is not completely trivial although it looks like it should be trivial.

Lemma 40.10.3. *Let S be a scheme. Let U be a scheme over S. Let $j = (s,t) : R \to U \times_S U$ be an étale equivalence relation on U over S. If the quotient U/R is an algebraic space, then $U \to U/R$ is étale and surjective. Hence $(U, R, U \to U/R)$ is a presentation of the algebraic space U/R.*

Proof. Denote $c : U \to U/R$ the morphism in question. Let T be a scheme and let $a : T \to U/R$ be a morphism. We have to show that the morphism (of schemes) $\pi : T \times_{a,R/U,c} U \to T$ is étale and surjective. The morphism a corresponds to an fppf covering $\{\varphi_i : T_i \to T\}$ and morphisms $a_i : T_i \to U$ such that $a_i \times a_{i'} : T_i \times_T T_{i'} \to U \times_S U$ factors through R, and such that $c \circ a_i = \varphi_i \circ a$. Hence

$$T_i \times_{\varphi_i,T} T \times_{a,R/U,c} U = T_i \times_{c\circ a_i,R/U,c} U = T_i \times_{a_i,U} U \times_{c,R/U,c} U = T_i \times_{a_i,U,t} R.$$

Since t is étale and surjective we conclude that the base change of π to T_i is surjective and étale. Since the property of being surjective and étale is local on the base in the fpqc topology (see Remark 40.4.3) we win. $\square$

Lemma 40.10.4. *Let S be a scheme. Let U be a scheme over S. Let $j = (s,t) : R \to U \times_S U$ be an étale equivalence relation on U over S. Assume that U is affine. Then the quotient $F = U/R$ is an algebraic space, and $U \to F$ is étale and surjective.*

Proof. Since $j : R \to U \times_S U$ is a monomorphism we see that j is separated (see Schemes, Lemma 21.23.3). Since U is affine we see that $U \times_S U$ (which comes equipped with a monomorphism into the affine scheme $U \times U$) is separated. Hence we see that R is separated. In particular the morphisms s, t are separated as well as étale.

Since the compostition $R \to U \times_S U \to U$ is locally of finite type we conclude that j is locally of finite type (see Morphisms, Lemma 24.14.8). As j is also a monomorphism it has finite fibres and we see that j is locally quasi-finite by Morphisms, Lemma 24.19.7. Alltogether we see that j is separated and locally quasi-finite.

Our first step is to show that the quotient map $c : U \to F$ is representable. Consider a scheme T and a morphism $a : T \to F$. We have to show that the sheaf $G = T \times_{a,F,c} U$ is representable. As seen in the proofs of Lemmas 40.10.2 and 40.10.3 there exists an fppf covering $\{\varphi_i : T_i \to T\}_{i \in I}$ and morphisms $a_i : T_i \to U$ such that $a_i \times a_{i'} : T_i \times_T T_{i'} \to U \times_S U$ factors through R, and such that $c \circ a_i = \varphi_i \circ a$. As in the proof of Lemma 40.10.3 we see that

$$\begin{aligned} T_i \times_{\varphi_i,T} G &= T_i \times_{\varphi_i,T} T \times_{a,R/U,c} U \\ &= T_i \times_{c\circ a_i,R/U,c} U \\ &= T_i \times_{a_i,U} U \times_{c,R/U,c} U \\ &= T_i \times_{a_i,U,t} R \end{aligned}$$

Since t is separated and étale, and in particular separated and locally quasi-finite (by Morphisms, Lemmas 24.34.10 and 24.35.16) we see that the restriction of G to each T_i is representable by a morphism of schemes $X_i \to T_i$ which is separated and locally quasi-finite. By Descent, Lemma 31.35.1 we obtain a descent datum $(X_i, \varphi_{ii'})$ relative to the fppf-covering $\{T_i \to T\}$. Since each $X_i \to T_i$ is separated and locally quasi-finite we see by More on Morphisms, Lemma 33.35.1 that this descent datum is effective. Hence by Descent, Lemma 31.35.1 (2) we conclude that G is representable as desired.

The second step of the proof is to show that $U \to F$ is surjective and étale. This is clear from the above since in the first step above we saw that $G = T \times_{a,F,c} U$ is a scheme over T which base changes to schemes $X_i \to T_i$ which are surjective and étale. Thus $G \to T$ is surjective and étale (see Remark 40.4.3). Alternatively one can reread the proof of Lemma 40.10.3 in the current situation.

The third and final step is to show that the diagonal map $F \to F \times F$ is representable. We first observe that the diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d, "j"'] & F \arrow[d, "\Delta"] \\ U \times_S U \arrow[r] & F \times F \end{tikzcd}$$

is a fibre product square. By Lemma 40.3.4 the morphism $U \times_S U \to F \times F$ is representable (note that $h_U \times h_U = h_{U \times_S U}$). Moreover, by Lemma 40.5.7 the morphism $U \times_S U \to F \times F$ is surjective and étale (note also that étale and surjective occur in the lists of Remarks 40.4.3 and 40.4.2). It follows either from Lemma 40.3.3 and the diagram above, or by writing $R \to F$ as $R \to U \to F$ and Lemmas 40.3.1 and 40.3.2 that $R \to F$ is representable as well. Let T be a scheme and let $a : T \to F \times F$ be a morphism. We have to show that $G = T \times_{a,F \times F,\Delta} F$ is representable. By what was said above the morphism (of schemes)

$$T' = (U \times_S U) \times_{F \times F,a} T \longrightarrow T$$

is surjective and étale. Hence $\{T' \to T\}$ is an étale covering of T. Note also that

$$T' \times_T G = T' \times_{U \times_S U,j} R$$

as can be seen contemplating the following cube

$$\begin{tikzcd} & R \arrow[rr] \arrow[dd] & & F \arrow[dd] \\ T' \times_T G \arrow[ur] \arrow[rr] \arrow[dd] & & G \arrow[ur] \arrow[dd] & \\ & U \times_S U \arrow[rr] & & F \times F \\ T' \arrow[ur] \arrow[rr] & & T \arrow[ur] & \end{tikzcd}$$

Hence we see that the restriction of G to T' is representable by a scheme X, and moreover that the morphism $X \to T'$ is a base change of the morphism j. Hence $X \to T'$ is separated and locally quasi-finite (see second paragraph of the proof). By Descent, Lemma 31.35.1 we obtain a descent datum (X, φ) relative to the fppf-covering $\{T' \to T\}$. Since $X \to T$ is separated and locally quasi-finite we see by More on Morphisms, Lemma 33.35.1 that this descent datum is effective. Hence by Descent, Lemma 31.35.1 (2) we conclude that G is representable as desired. □

Theorem 40.10.5. *Let S be a scheme. Let U be a scheme over S. Let $j = (s,t) : R \to U \times_S U$ be an étale equivalence relation on U over S. Then the quotient U/R is an algebraic space, and $U \to U/R$ is étale and surjective, in other words $(U, R, U \to U/R)$ is a presentation of U/R.*

Proof. By Lemma 40.10.3 it suffice to just prove that U/R is an algebraic space. Let $U' \to U$ be a surjective, étale morphism. Then $\{U' \to U\}$ is in particular an fppf covering. Let R' be the restriction of R to U', see Groupoids, Definition 35.3.3. According to Groupoids, Lemma 35.17.6 we see that $U/R \cong U'/R'$. By Lemma 40.10.1 R' is an étale equivalence relation on U'. Thus we may replace U by U'.

We apply the previous remark to $U' = \coprod U_i$, where $U = \bigcup U_i$ is an affine open covering of S. Hence we may and do assume that $U = \coprod U_i$ where each U_i is an affine scheme.

Consider the restriction R_i of R to U_i. By Lemma 40.10.1 this is an étale equivalence relation. Set $F_i = U_i/R_i$ and $F = U/R$. It is clear that $\coprod F_i \to F$ is surjective. By Lemma 40.10.2 each $F_i \to F$ is representable, and an open immersion. By Lemma 40.10.4 applied to (U_i, R_i) we see that F_i is an algebraic space. Then by Lemma 40.10.3 we see that $U_i \to F_i$ is étale and surjective. From Lemma 40.8.3 it follows that $\coprod F_i$ is an algebraic space. Finally, we have verified all hypotheses of Lemma 40.8.4 and it follows that $F = U/R$ is an algebraic space. □

40.11. Algebraic spaces, retrofitted

We start building our arsenal of lemmas dealing with algebraic spaces.

Lemma 40.11.1. *Let S be a scheme contained in Sch_{fppf}. Let F be an algebraic space over S. Let $G \to F$ be a representable transformation of functors. Then G is an algebraic space.*

Proof. By Lemma 40.3.5 we see that G is a sheaf. The diagram

$$\begin{tikzcd}
G \times_F G \arrow[r] \arrow[d] & F \arrow[d, "\Delta_F"] \\
G \times G \arrow[r] & F \times F
\end{tikzcd}$$

is cartesian. Hence we see that $G \times_F G \to G \times G$ is representable by Lemma 40.3.3. By Lemma 40.3.6 we see that $G \to G \times_F G$ is representable. Hence $\Delta_G : G \to G \times G$ is representable as a composition of representable transformations, see Lemma 40.3.2. Finally, let U be an object of $(Sch/S)_{fppf}$ and let $U \to F$ be surjective and étale. By assumption $U \times_F G$ is representable by a scheme U'. By Lemma 40.5.5 the morphism $U' \to G$ is surjective and étale. This verifies the final condition of Definition 40.6.1 and we win. □

Lemma 40.11.2. *Let S be a scheme contained in Sch_{fppf}. Let F, G be algebraic spaces over S. Let $G \to F$ be a representable morphism. Let $U \in Ob((Sch/S)_{fppf})$, and $q : U \to F$ surjective and étale. Set $V = G \times_F U$. Finally, let $\mathcal{P}$ be a property of morphisms of schemes as in Definition 40.5.1. Then $G \to F$ has property $\mathcal{P}$ if and only if $V \to U$ has property $\mathcal{P}$.*

Proof. (This lemma follows from Lemmas 40.5.5 and 40.5.6, but we give a direct proof here also.) It is clear from the definitions that if $G \to F$ has property $\mathcal{P}$, then $V \to U$ has property $\mathcal{P}$. Conversely, assume $V \to U$ has property $\mathcal{P}$. Let $T \to F$ be a morphism from a scheme to F. Let $T' = T \times_F G$ which is a scheme since $G \to F$ is representable. We have to show that $T' \to T$ has property T. Consider the commutative diagram of schemes

$$\begin{tikzcd} V \arrow[d] & T \times_F V \arrow[l] \arrow[r] \arrow[d] & T \times_F G \arrow[d] \arrow[r, equal] & T' \\ U & T \times_F U \arrow[l] \arrow[r] & T & \end{tikzcd}$$

where both squares are fibre product squares. Hence we conclude the middle arrow has property $\mathcal{P}$ as a base change of $V \to U$. Finally, $\{T \times_F U \to T\}$ is a fppf covering as it is surjective étale, and hence we conclude that $T' \to T$ has property $\mathcal{P}$ as it is local on the base in the fppf topology. $\square$

Lemma 40.11.3. *Let S be a scheme contained in Sch_{fppf}. Let $G \to F$ be a transformation of presheaves on $(Sch/S)_{fppf}$. Let $\mathcal{P}$ be a property of morphisms of schemes. Assume*

(1) *$\mathcal{P}$ is preserved under any base change, fppf local on the base, and morphisms of type $\mathcal{P}$ satisfy descent for fppf coverings, see Descent, Definition 31.32.1,*
(2) *G is a sheaf,*
(3) *F is an algebraic space,*
(4) *there exists a $U \in Ob((Sch/S)_{fppf})$ and a surjective étale morphism $U \to F$ such that $V = G \times_F U$ is representable, and*
(5) *$V \to U$ has $\mathcal{P}$.*

Then G is an algebraic space, $G \to F$ is representable and has property $\mathcal{P}$.

Proof. Let $R = U \times_F U$, and denote $t, s : R \to U$ the projection morphisms as usual. Let T be a scheme and let $T \to F$ be a morphism. Then $U \times_F T \to T$ is surjective étale, hence $\{U \times_F T \to T\}$ is a covering for the étale topology. Consider

$$W = G \times_F (U \times_F T) = V \times_F T = V \times_U (U \times_F T).$$

It is a scheme since F is an algebraic space. The morphism $W \to U \times_F T$ has property $\mathcal{P}$ since it is a base change of $V \to U$. There is an isomorphism

$$\begin{aligned} W \times_T (U \times_F T) &= (G \times_F (U \times_F T)) \times_T (U \times_F T) \\ &= (U \times_F T) \times_T (G \times_F (U \times_F T)) \\ &= (U \times_F T) \times_T W \end{aligned}$$

over $(U \times_F T) \times_T (U \times_F T)$. The middle equality maps $((g, (u_1, t)), (u_2, t))$ to $((u_1, t), (g, (u_2, t)))$. This defines a descent datum for $W/U \times_F T/T$, see Descent, Definition 31.30.1. This follows from Descent, Lemma 31.35.1. Namely we have a sheaf $G \times_F T$, whose base change to $U \times_F T$ is represented by W and the isomorphism above is the one from the proof of Descent, Lemma 31.35.1. By assumption on $\mathcal{P}$ the descent datum above is representable. Hence by the last statement of Descent, Lemma 31.35.1 we see that $G \times_F T$ is representable. This proves that $G \to F$ is a representable transformation of functors.

As $G \to F$ is representable, we see that G is an algebraic space by Lemma 40.11.1. The fact that $G \to F$ has property $\mathcal{P}$ now follows from Lemma 40.11.2. $\square$

Lemma 40.11.4. *Let S be a scheme contained in Sch_{fppf}. Let F, G be algebraic spaces over S. Let $a : F \to G$ be a morphism. Given any $V \in Ob((Sch/S)_{fppf})$ and a surjective* *étale morphism $q : V \to G$ there exists a $U \in Ob((Sch/S)_{fppf})$ and a commutative diagram*

$$\begin{tikzcd} U \arrow[r, "\alpha"] \arrow[d, "p"] & V \arrow[d, "q"] \\ F \arrow[r, "a"] & G \end{tikzcd}$$

with p surjective and étale.

Proof. First choose $W \in Ob((Sch/S)_{fppf})$ with surjective étale morphism $W \to F$. Next, put $U = W \times_G V$. Since G is an algebraic space we see that U is isomorphic to an object of $(Sch/S)_{fppf}$. As q is surjective étale, we see that $U \to W$ is surjective étale (see Lemma 40.5.5). Thus $U \to F$ is surjective étale as a composition of surjective étale morphisms (see Lemma 40.5.4). □

40.12. Immersions and Zariski coverings of algebraic spaces

At this point an intersting phenomenon occurs. We have already defined the notion of an open immersion of algebraic spaces (through Definition 40.5.1) but we have yet to define the notion of a *point*[1]. Thus the *Zariski topology* of an algebraic space has already been defined, but there is no space yet!

Perhaps superfluously we formally introduce immersions as follows.

Definition 40.12.1. Let $S \in Ob(Sch_{fppf})$ be a scheme. Let F be an algebraic space over S.

(1) A morphism of algebraic spaces over S is called an *open immersion* if it is representable, and an open immersion in the sense of Definition 40.5.1.
(2) An *open subspace* of F is a subfunctor $F' \subset F$ such that F' is an algebraic space and $F' \to F$ is an open immersion.
(3) A morphism of algebraic spaces over S is called a *closed immersion* if it is representable, and a closed immersion in the sense of Definition 40.5.1.
(4) A *closed subspace* of F is a subfunctor $F' \subset F$ such that F' is an algebraic space and $F' \to F$ is a closed immersion.
(5) A morphism of algebraic spaces over S is called an *immersion* if it is representable, and an immersion in the sense of Definition 40.5.1.
(6) A *locally closed subspace* of F is a subfunctor $F' \subset F$ such that F' is an algebraic space and $F' \to F$ is an immersion.

We note that these definitions make sense since an immersion is in particular a monomorphism (see Schemes, Lemma 21.23.7 and Lemma 40.5.8), and hence the image of an immersion $G \to F$ of algebraic spaces is a subfunctor $F' \subset F$ which is (canonically) isomorphic to G. Thus some of the discussion of Schemes, Section 21.10 carries over to the setting of algebraic spaces.

Lemma 40.12.2. *Let $S \in Ob(Sch_{fppf})$ be a scheme. A composition of (closed, resp. open) immersions of algebraic spaces over S is a (closed, resp. open) immersion of algebraic spaces over S.*

Proof. See Lemma 40.5.4 and Remarks 40.4.3 (see very last line of that remark) and 40.4.2. □

[1] We will associate a topological space to an algebraic space in Properties of Spaces, Section 41.4, and its opens will correspond exactly to the open subspaces defined below.

Lemma 40.12.3. *Let $S \in Ob(Sch_{fppf})$ be a scheme. A base change of a (closed, resp. open) immersion of algebraic spaces over S is a (closed, resp. open) immersion of algebraic spaces over S.*

Proof. See Lemma 40.5.5 and Remark 40.4.3 (see very last line of that remark). □

Lemma 40.12.4. *Let $S \in Ob(Sch_{fppf})$ be a scheme. Let F be an algebraic space over S. Let F_1, F_2 be locally closed subspaces of F. If $F_1 \subset F_2$ as subfunctors of F, then F_1 is a locally closed subspace of F_2. Similarly for closed and open subspaces.*

Proof. Let $T \to F_2$ be a morphism with T a scheme. Since $F_2 \to F$ is a monomorphism, we see that $T \times_{F_2} F_1 = T \times_F F_1$. The lemma follows formally from this. □

Let us formally define the notion of a Zariski open covering of algebraic spaces. Note that in Lemma 40.8.4 we have already encountered such open coverings as a method for constructing algebraic spaces.

Definition 40.12.5. Let $S \in Ob(Sch_{fppf})$ be a scheme. Let F be an algebraic space over S. A *Zariski covering* $\{F_i \subset F\}_{i \in I}$ of F is given by a set I, a collection of open subspaces $F_i \subset F$ such that $\coprod F_i \to F$ is a surjective map of sheaves.

Note that if T is a schemes, and $a : T \to F$ is a morphism, then each of the fibre products $T \times_F F_i$ is identified with an open subscheme $T_i \subset T$. The final condition of the definition signifies exactly that $T = \bigcup_{i \in I} T_i$.

It is clear that the collection $\mathcal{T}$ of open subspaces of F is a set (as $(Sch/S)_{fppf}$ is a site, hence a set). Moreover, we can turn $\mathcal{T}$ into a category by letting the morphisms be inclusions of subfunctors (which are automatically open immersions by Lemma 40.12.4). Finally, Definition 40.12.5 provides the notion of a Zariski covering $\{F_i \to F'\}_{i \in I}$ in the category $\mathcal{T}$. Hence, just as in the case of a topological space (see Sites, Example 9.6.4) by suitably choosing a set of coverings we may obtain a Zariski site of the algebraic space F.

Definition 40.12.6. Let $S \in Ob(Sch_{fppf})$ be a scheme. Let F be an algebraic space over S. A *small Zariski site* F_{Zar} of an algebraic space F is one of the sites $\mathcal{T}$ described above.

Hence this gives a notion of what it means for something to be true Zariski locally on an algebraic space, which is how we will use this notion. In general the Zariski topology is not fine enough for our purposes. For example we can consider the category of Zariski sheaves on an algebraic space. It will turn out that this is not the correct thing to consider, even for quasi-coherent sheaves. One only gets the desired result when using the étale or fppf site of F to define quasi-coherent sheaves.

40.13. Separation conditions on algebraic spaces

A separation condition on an algebraic space F is a condition on the diagonal morphism $F \to F \times F$. Let us first list the properties the diagonal has automatically. Since the diagonal is representable by definition the following lemma makes sense (through Definition 40.5.1).

Lemma 40.13.1. *Let S be a scheme contained in Sch_{fppf}. Let F be an algebraic space over S. Let $\Delta : F \to F \times F$ be the diagonal morphism. Then*

(1) Δ *is locally of finite type,*
(2) Δ *is a monomorphism,*
(3) Δ *is separated, and*
(4) Δ *is locally quasi-finite.*

Proof. Let $F = U/R$ be a presentation of F. As in the proof of Lemma 40.10.4 the diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d, "j"'] & F \arrow[d, "\Delta"] \\ U \times_S U \arrow[r] & F \times F \end{tikzcd}$$

is cartesian. Hence according to Lemma 40.11.2 it suffices to show that j has the properties listed in the lemma. (Note that each of the properties (1) -- (4) occur in the lists of Remarks 40.4.1 and 40.4.3.) Since j is an equivalence relation it is a monomorphism. Hence it is separated by Schemes, Lemma 21.23.3. As R is an étale equivalence relation we see that $s, t : R \to U$ are étale. Hence s, t are locally of finite type. Then it follows from Morphisms, Lemma 24.14.8 that j is locally of finite type. Finally, as it is a monomorphism its fibres are finite. Thus we conclude that it is locally quasi-finite by Morphisms, Lemma 24.19.7. $\square$

Here are some common types of separation conditions, relative to the base scheme S. There is also an absolute notion of these conditions which we will discuss in Properties of Spaces, Section 41.3. Moreover, we will discuss separation conditions for a morphism of algebraic spaces in Morphisms of Spaces, Section 42.5.

Definition 40.13.2. Let S be a scheme contained in Sch_{fppf}. Let F be an algebraic space over S. Let $\Delta : F \to F \times F$ be the diagonal morphism.

(1) We say F is *separated over S* if Δ is a closed immersion.
(2) We say F is *locally separated over S*[2] if Δ is an immersion.
(3) We say F is *quasi-separated over S* if Δ is quasi-compact.
(4) We say F is *Zariski locally quasi-separated over S*[3] if there exists a Zariski covering $F = \bigcup_{i \in I} F_i$ such that each F_i is quasi-separated.

Note that if the diagonal is quasi-compact (when F is separated or quasi-separated) then the diagonal is actually quasi-finite and separated, hence quasi-affine (by More on Morphisms, Lemma 33.29.3).

40.14. Examples of algebraic spaces

In this section we construct some examples of algebraic spaces. Some of these were suggested by B. Conrad. Since we do not yet have a lot of theory at our disposal the discussion is a bit awkward in some places.

Example 40.14.1. Let k be a field. Let $U = \mathbf{A}^1_k$. Set

$$j : R = \Delta \coprod \Gamma \longrightarrow U \times_k U$$

where $\Delta = \{(x, x) \mid x \in \mathbf{A}^1_k\}$ and $\Gamma = \{(x, -x) \mid x \in \mathbf{A}^1_k, x \neq 0\}$. It is clear that $s, t : R \to U$ are étale, and hence j is an étale equivalence relation. The quotient $X = U/R$ is an algebraic space by Theorem 40.10.5. Since R is quasi-compact we see that X is quasi-separated. On the other hand, X is not locally separated because the morphism j is not an immersion.

Example 40.14.2. Let k be a field. Let $k \subset k'$ be a degree 2 Galois extension with $\mathrm{Gal}(k'/k) = \{1, \sigma\}$. Let $S = Spec(k[x])$ and $U = Spec(k'[x])$. Note that

$$U \times_S U = Spec((k' \otimes_k k')[x]) = \Delta(U) \coprod \Delta'(U)$$

[2]In the literature this often refers to quasi-separated and locally separated algebraic spaces.

[3]This definition was suggested by B. Conrad.

where $\Delta' = (1, \sigma) : U \to U \times_S U$. Take

$$R = \Delta(U) \coprod \Delta'(U \setminus \{0_U\})$$

where $0_U \in U$ denotes the k'-rational point whose x-coordinate is zero. It is easy to see that R is an étale equivalence relation on U over S and hence $X = U/R$ is an algebraic space by Theorem 40.10.5. Here are some properties of X (some of which will not make sense until later):

1. $X \to S$ is an isomorphism over $S \setminus \{0_S\}$,
2. the morphism $X \to S$ is étale (see Properties of Spaces, Definition 41.13.2)
3. the fibre 0_X of $X \to S$ over 0_S is isomorphic to $Spec(k') = 0_U$,
4. X is not a scheme (because if it where, then $\mathcal{O}_{X,0_X}$ would be a local domain $(\mathcal{O}, \mathfrak{m}, \kappa)$ with fraction field $k(x)$, with $x \in \mathfrak{m}$ and residue field $\kappa = k'$ which is impossible),
5. the algebraic space X is not separated, but it is locally separated and hence quasi-separated,
6. there exists a surjective, finite, étale morphism $S' \to S$ such that the base change $X' = S' \times_S X$ is a scheme (namely, if we base change to $S' = Spec(k'[x])$ then U splits into two copies of S' and X' becomes isomorphic to the affine line with 0 doubled, see Schemes, Example 21.14.3), and
7. if we think of X as a finite type algebraic space over $Spec(k)$, then similarly the base change $X_{k'}$ is a scheme but X is not a scheme.

In particular, this gives an example of a descent datum for schemes relative to the covering $\{Spec(k') \to Spec(k)\}$ which is not effective.

We will use the following lemma as a convenient way to construct algebraic spaces as quotients of schemes by free group actions.

Lemma 40.14.3. *Let $U \to S$ be a morphism of Sch_{fppf}. Let G be an abstract group. Let $G \to Aut_S(U)$ be a group homomorphism. Assume*

(∗) *if $u \in U$ is a point, and $g(u) = u$ for some non-identity element $g \in G$, then g induces a nontrivial automorphism of $\kappa(u)$.*

Then

$$j : R = \coprod\nolimits_{g \in G} U \longrightarrow U \times_S U, \quad (g, x) \longmapsto (g(x), x)$$

is an étale equivalence relation and hence

$$F = U/R$$

is an algebraic space by Theorem 40.10.5.

Proof. In the statement of the lemma the symbol $\mathrm{Aut}_S(U)$ denotes the group of automorphisms of U over S. Assume (∗) holds. Let us show that

$$j : R = \coprod\nolimits_{g \in G} U \longrightarrow U \times_S U, \quad (g, x) \longmapsto (g(x), x)$$

is a monomorphism. This signifies that if T is a nonempty scheme, and $h : T \to U$ is a T-valued point such that $g \circ h = g' \circ h$ then $g = g'$. Suppose $T \neq \emptyset$, $h : T \to U$ and $g \circ h = g' \circ h$. Let $t \in T$. Consider the composition $Spec(\kappa(t)) \to Spec(\kappa(h(t))) \to U$. Then we conclude that $g' \circ g^{-1}$ fixes $u = h(t)$ and acts as the identity on its residue field. Hence $g = g'$ by (∗).

Thus if (∗) holds we see that j is a relation (see Groupoids, Definition 35.3.1). Moreover, it is an equivalence relation since on T-valued points for a connected scheme T we see that

$R(T) = G \times U(T) \to U(T) \times U(T)$ (recall that we always work over S). Moreover, the morphisms $s, t : R \to U$ are étale since R is a disjoint product of copies of U. This proves that $j : R \to U \times_S U$ is an étale equivalence relation. □

Given a scheme U and an action of a group G on U we say the action of G on U is *free* if condition $(*)$ of Lemma 40.14.3 holds. Thus the lemma says that quotients of schemes by free actions of groups exist in the category of algebraic spaces.

Definition 40.14.4. Notation $U \to S$, G, R as in Lemma 40.14.3. If the action of G on U satisfies $(*)$ we say G *acts freely* on the scheme U. In this case the algebraic space U/R is denoted U/G and is called the *quotient of U by G*.

This notation is consistent with the notation U/G introduced in Groupoids, Definition 35.17.1. We will later make sense of the quotient as an algebraic stack without any assumptions on the action whatsoever; when we do this we will use the notation $[U/G]$. Before we discuss the examples we prove some more lemmas to facilitate the discussion. Here is a lemma discussing the various separation conditions for this quotient when G is finite.

Lemma 40.14.5. *Notation and assumptions as in Lemma 40.14.3. Assume G is finite. Then*

(1) *if $U \to S$ is quasi-separated, then U/G is quasi-separated, and*
(2) *if $U \to S$ is separated, then U/G is separated.*

Proof. In the proof of Lemma 40.13.1 we saw that it suffices to prove the corresponding properties for the morphism $j : R \to U \times_S U$. If $U \to S$ is quasi-separated, then for every affine open $V \subset U$ the opens $g(V) \cap V$ are quasi-compact. It follows that j is quasi-compact. If $U \to S$ is separated, the the diagonal $\Delta_{U/S}$ is a closed immersion. Hence $j : R \to U \times_S U$ is a finite coproduct of closed immersions with disjoint images. Hence j is a closed immersion. □

Lemma 40.14.6. *Notation and assumptions as in Lemma 40.14.3. If $Spec(k) \to U/G$ is a morphism, then there exist*

(1) *a finite Galois extension $k \subset k'$,*
(2) *a finite subgroup $H \subset G$,*
(3) *an isomorphism $H \to Gal(k'/k)$, and*
(4) *an H-equivariant morphism $Spec(k') \to U$.*

Conversely, such data determine a morphism $Spec(k) \to U/G$.

Proof. Consider the fibre product $V = Spec(k) \times_{U/G} U$. Here is a diagram

$$\begin{tikzcd} V \arrow[r] \arrow[d] & U \arrow[d] \\ Spec(k) \arrow[r] & U/G \end{tikzcd}$$

This is a nonempty scheme étale over $Spec(k)$ and hence is a disjoint union of spectra of fields finite separable over k (Morphisms, Lemma 24.35.7). So write $V = \coprod_{i \in I} Spec(k_i)$. The action of G on U induces an action of G on $V = \coprod Spec(k_i)$. Pick an i, and let $H \subset G$ be the stabilizer of i. Since

$$V \times_{Spec(k)} V = Spec(k) \times_{U/G} U \times_{U/G} U = Spec(k) \times_{U/G} U \times G = V \times G$$

we see that (a) the orbit of $Spec(k_i)$ is V and (b) $Spec(k_i \otimes_k k_i) = Spec(k_i) \times H$. Thus H is finite and is the Galois group of k_i/k. We omit the converse construction. □

It follows from this lemma for example that if k'/k is a finite Galois extension, then $Spec(k')/\text{Gal}(k'/k) \cong Spec(k)$. What happens if the extension is infinite? Here is an example.

Example 40.14.7. Let $S = Spec(\mathbf{Q})$. Let $U = Spec(\overline{\mathbf{Q}})$. Let $G = \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ with obvious action on U. Then by construction property $(*)$ of Lemma 40.14.3 holds and we obtain an algebraic space

$$X = Spec(\overline{\mathbf{Q}})/G \longrightarrow S = Spec(\mathbf{Q}).$$

Of course this is totally ridiculus as an approximation of S! Namely, by the Artin-Schreier theorem, see [**Jac64**, Theorem 17, page 316], the only finite subgroups of $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ are $\{1\}$ and the conjugates of the order two group $\text{Gal}(\overline{\mathbf{Q}}/\overline{\mathbf{Q}} \cap \mathbf{R})$. Hence, if $Spec(k) \to X$ is a morphism with k algebraic over $\mathbf{Q}$, then it follows from Lemma 40.14.6 and the theorem just mentioned that either k is $\overline{\mathbf{Q}}$ or isomorphic to $\overline{\mathbf{Q}} \cap \mathbf{R}$.

What is wrong with the example above is that the Galois group comes equipped with a topology, and this should somehow be part of any construction of a quotient of $Spec(\overline{\mathbf{Q}})$. The following example is much more reasonable in my opinion and may actually occur in ``nature''.

Example 40.14.8. Let k be a field of characteristic zero. Let $U = \mathbf{A}^1_k$ and let $G = \mathbf{Z}$. As action we take $n(x) = x + n$, i.e., the action of $\mathbf{Z}$ on the affine line by translation. The only fixed point is the generic point and it is clearly the case that $\mathbf{Z}$ injects into the automorphism group of the field $k(x)$. (This is where we use the characteristic zero assumption.) Consider the morphism

$$\gamma : Spec(k(x)) \longrightarrow X = \mathbf{A}^1_k/\mathbf{Z}$$

of the generic point of the affine line into the quotient. We claim that this morphism does not factor through any monomorphism $Spec(L) \to X$ of the spectrum of a field to X. (Contrary to what happens for schemes, see Schemes, Section 21.13.) In fact, since $\mathbf{Z}$ does not have any finite subgroups we see from Lemma 40.14.6 that for any such factorization $k(x) = L$. Finally, γ is not a monomorphism since

$$Spec(k(x)) \times_{\gamma, X, \gamma} Spec(k(x)) \cong Spec(k(x)) \times \mathbf{Z}.$$

This example suggests that in order to define points of an algebraic space X we should consider equivalence classes of morphisms from spectra of fields into X and not the set of monomorphisms from spectra of fields.

We finish with a truly awful example.

Example 40.14.9. Let k be a field. Let $A = \prod_{n \in \mathbf{N}} k$ be the infinite product. Set $U = Spec(A)$ seen as a scheme over $S = Spec(k)$. Note that the projection maps $\text{pr}_n : A \to k$ define open and closed immersions $f_n : S \to U$. Set

$$R = U \amalg \coprod\nolimits_{(n,m) \in \mathbf{N}^2,\ n \neq m} S$$

with morphism j equal to $\Delta_{U/S}$ on the component U and $j = (f_n, f_m)$ on the component S corresponding to (m, m). It is clear from the remark above that s, t are étale. It is also clear that j is an equivalence relation. Hence we obtain an algebraic space

$$X = U/R.$$

To see what this means we specialize to the case where the field k is finite with q elements. Let us first discuss the topological space $|U|$ associated to the scheme U a little bit. All elements of A satisfy $x^q = x$. Hence every residue field of A is isomorphic to k, and all points of U are closed. But the topology on U isn't the discrete topology. Let $u_n \in |U|$ be the point corresponding to f_n. As mentioned above the points u_n are the open points (and hence isolated). This implies there have to be other points since we know U is quasi-compact, see Algebra, Lemma 7.16.10 (hence not equal to an infinite discrete set). Another way to see this is because the (proper) ideal

$$I = \{x = (x_n) \in A \mid \text{all but a finite number of } x_n \text{ are zero}\}$$

is contained in a maximal ideal. Note also that every element x of A is of the form $x = ue$ where u is a unit and e is an idempotent. Hence a basis for the topology of A consists of open and closed subsets (see Algebra, Lemma 7.18.1.) So the topology on $|U|$ is totally disconnected, but nontrivial. Finally, note that $\{u_n\}$ is dense in $|U|$.

We will later define a topological space $|X|$ associated to X, see Properties of Spaces, Section 41.4. What can we say about $|X|$? It turns out that the map $|U| \to |X|$ is surjective and continuous. All the points u_n map to the same point x_0 of $|X|$, and none of the other points get identified. Since $\{u_n\}$ is dense in $|U|$ we conclude that the closure of x_0 in $|X|$ is $|X|$. In other words $|X|$ is irreducible and x_0 is a generic point of $|X|$. This seems bizarre since also x_0 is the image of a section $S \to X$ of the structure morphism $X \to S$ (and in the case of schemes this would imply it was a closed point, see Morphisms, Lemma 24.19.2).

Whatever you think is actually going on in this example, it certainly shows that some care has to be exercised when defining irreducible components, connectedness, etc of algebraic spaces.

40.15. Change of big site

In this section we briefly discuss what happens when we change big sites. The upshot is that we can always enlarge the big site at will, hence we may assume any set of schemes we want to consider is contained in the big fppf site over which we consider our algebraic space. Here is a precise statement of the result.

Lemma 40.15.1. *Suppose given big sites Sch_{fppf} and Sch'_{fppf}. Assume that Sch_{fppf} is contained in Sch'_{fppf}, see Topologies, Section 30.10. Let S be an object of Sch_{fppf}. Let*

$$g : Sh((Sch/S)_{fppf}) \longrightarrow Sh((Sch'/S)_{fppf}),$$
$$f : Sh((Sch'/S)_{fppf}) \longrightarrow Sh((Sch/S)_{fppf})$$

be the morphisms of topoi of Topologies, Lemma 30.10.2. Let F be a sheaf of sets on $(Sch/S)_{fppf}$. Then

1. *if F is representable by a scheme $X \in Ob((Sch/S)_{fppf})$ over S, then $f^{-1}F$ is representable too, in fact it is representable by the same scheme X, now viewed as an object of $(Sch'/S)_{fppf}$, and*
2. *if F is an algebraic space over S, then $f^{-1}F$ is an algebraic space over S also.*

Proof. Let $X \in Ob((Sch/S)_{fppf})$. Let us write h_X for the representable sheaf on $(Sch/S)_{fppf}$ associated to X, and h'_X for the representable sheaf on $(Sch'/S)_{fppf}$ associated to X. By the description of f^{-1} in Topologies, Section 30.10 we see that $f^{-1}h_X = h'_X$. This proves (1).

Next, suppose that F is an algebraic space over S. By Lemma 40.9.1 this means that $F = h_U/h_R$ for some étale equivalence relation $R \to U \times_S U$ in $(Sch/S)_{fppf}$. Since f^{-1} is an exact functor we conclude that $f^{-1}F = h'_U/h'_R$. Hence $f^{-1}F$ is an algebraic space over S by Theorem 40.10.5. □

Note that this lemma is purely set theoretical and has virtually no content. Moreover, it is not true (in general) that the restriction of an algebraic space over the bigger site is an algebraic space over the smaller site (simply by reasons of cardinality). Hence we can only ever use a simple lemma of this kind to enlarge the base category and never to shrink it.

Lemma 40.15.2. *Suppose Sch_{fppf} is contained in Sch'_{fppf}. Let S be an object of Sch_{fppf}. Denote $Spaces/S$ the category of algebraic spaces over S defined using Sch_{fppf}. Similarly, denote $Spaces'/S$ the category of algebraic spaces over S defined using Sch'_{fppf}. The construction of Lemma 40.15.1 defines a fully faithful functor*

$$Spaces/S \longrightarrow Spaces'/S$$

whose essential image consists of those $X' \in Ob(Spaces'/S)$ such that there exists an object $U \in Ob((Sch/S)_{fppf})$ and a surjective étale morphism $U \to X'$ in the category $Sh((Sch'/S)_{fppf})$.

Proof. In Sites, Lemma 9.19.8 we have seen that the functor $f^{-1} : Sh((Sch/S)_{fppf}) \to Sh((Sch'/S)_{fppf})$ is fully faithful. Hence we see that the displayed functor of the lemma is fully faithful. Suppose that $X' \in Ob(Spaces'/S)$ such that there exists an object $U \in Ob((Sch/S)_{fppf})$ and a surjective étale morphism $h'_U \to X$ (with notation as in the proof of Lemma 40.15.1). Let R be an object of $(Sch'/S)_{fppf}$ representing $h'_U \times_X h'_U$. Note that $R \to U \times_S U$ is a monomorphism of schemes. Hence by Sets, Lemma 3.9.9 the scheme R is isomorphic to an object of $(Sch/S)_{fppf}$ and hence we may (after replacing R by an isomorphic scheme) assume $R \in Ob((Sch/S)_{fppf})$. Now we use Lemma 40.9.1 and Theorem 40.10.5 to see that $X = h_U/h_R$ is an object of $Spaces/S$ such that $f^{-1}X \cong X'$ as desired. □

40.16. Change of base scheme

In this section we briefly discuss what happens when we change base schemes. The upshot is that given a morphism $S \to S'$ of base schemes, any algebraic space over S can be viewed as an algebraic space over S'. And, given an algebraic space F' over S' there is a base change F'_S which is an algebraic space over S. We explain only what happens in case $S \to S'$ is a morphism of the big fppf site under consideration, if only S or S' is contained in the big site, then one first enlarges the big site as in Section 40.15.

Lemma 40.16.1. *Suppose given a big site Sch_{fppf}. Let $g : S \to S'$ be morphism of Sch_{fppf}. Let $j : (Sch/S)_{fppf} \to (Sch/S')_{fppf}$ be the corresponding localization functor. Let F be a sheaf of sets on $(Sch/S)_{fppf}$. Then*

1. *for a scheme T' over S' we have $j_!F(T'/S') = \coprod_{\varphi : T' \to S} F(T' \xrightarrow{\varphi} S)$,*
2. *if F is representable by a scheme $X \in Ob((Sch/S)_{fppf})$, then $j_!F$ is representable by $j(X)$ which is X viewed as a scheme over S', and*
3. *if F is an algebraic space over S, then $j_!F$ is an algebraic space over S', and if $F = U/R$ is a presentation, then $j_!F = j(U)/j(R)$ is a presentation.*

Let F' be a sheaf of sets on $(Sch/S')_{fppf}$. Then

4. *for a scheme T over S we have $j^{-1}F'(T/S) = F'(T/S')$,*
5. *if F' is representable by a scheme $X' \in Ob((Sch/S')_{fppf})$, then $j^{-1}F'$ is representable, namely by $X'_S = S \times_{S'} X'$, and*
6. *if F' is an algebraic space, then $j^{-1}F'$ is an algebraic space, and if $F' = U'/R'$ is a presentation, then $j^{-1}F' = U'_S/R'_S$ is a presentation.*

Proof. The functors $j_!$, j_* and j^{-1} are defined in Sites, Lemma 9.21.7 where it is also shown that $j = j_{S/S'}$ is the localization of $(Sch/S')_{fppf}$ at the object S/S'. Hence all of the material on localization functors is available for j. The formula in (1) is Sites, Lemma 9.23.1. By definition $j_!$ is the left adjoint to restriction j^{-1}, hence $j_!$ is right exact. By Sites, Lemma 9.21.5 it also commutes with fibre products and equalizers. By Sites, Lemma 9.21.3 we see that $j_!h_X = h_{j(X)}$ hence (2) holds. If F is an algebraic space over S, then we can write $F = U/R$ (Lemma 40.9.1) and we get

$$j_!F = j(U)/j(R)$$

because $j_!$ being right exact commutes with coequalizers, and moreover $j(R) = j(U) \times_{j_!F} j(U)$ as $j_!$ commutes with fibre products. Since the morphisms $j(s), j(t) : j(R) \to j(U)$ are simply the morphisms $s, t : R \to U$ (but viewed as morphisms of schemes over S'), they are still étale. Thus $(j(U), j(R), s, t)$ is an étale equivalence relation. Hence by Theorem 40.10.5 we conclude that $j_!F$ is an algebraic space.

Proof of (4), (5), and (6). The description of j^{-1} is in Sites, Section 9.21. The restriction of the representable sheaf associated to X'/S' is the representable sheaf associated to $X'_S = S \times_{S'} Y'$ by Sites, Lemma 9.23.2. The restriction functor j^{-1} is exact, hence $j^{-1}F' = U'_S/R'_S$. Again by exactness the sheaf R'_S is still an equivalence relation on U'_S. Finally the two maps $R'_S \to U'_S$ are étale as base changes of the étale morphisms $R' \to U'$. Hence $j^{-1}F' = U'_S/R'_S$ is an algebraic space by Theorem 40.10.5 and we win. $\square$

Note how the presentation $j_!F = j(U)/j(R)$ is just the presentation of F but viewed as a presentation by schemes over S'. Hence the following definition makes sense.

Definition 40.16.2. Let Sch_{fppf} be a big fppf site. Let $S \to S'$ be a morphism of this site.

(1) If F' is an algebraic space over S', then the *base change of F' to S* is the algebraic space $j^{-1}F'$ described in Lemma 40.16.1. We denote it F'_S.
(2) If F is an algebraic space over S, then *F viewed as an algebraic space over S'* is the algebraic space $j_!F$ over S' described in Lemma 40.16.1. We often simply denote this F; if not then we will write $j_!F$.

The algebraic space $j_!F$ comes equipped with a canonical morphism $j_!F \to S$ of algebraic spaces over S'. This is true simply because the sheaf $j_!F$ maps to h_S (see for example the explicit description in Lemma 40.16.1). In fact, in Sites, Lemma 9.21.4 we have seen that the category of sheaves on $(Sch/S)_{fppf}$ is equivalent to the category of pairs $(\mathcal{F}', \mathcal{F}' \to h_S)$ consisting of a sheaf on $(Sch/S')_{fppf}$ and a map of sheaves $\mathcal{F}' \to h_S$. The equivalence assigns to the sheaf $\mathcal{F}$ the pair $(j_!\mathcal{F}, j_!\mathcal{F} \to h_S)$. This, combined with the above, leads to the following result for categories of algebraic spaces.

Lemma 40.16.3. *Let Sch_{fppf} be a big fppf site. Let $S \to S'$ be a morphism of this site. The construction above give an equivalence of categories*

$$\left\{\begin{matrix}\textit{category of algebraic}\\ \textit{spaces over } S\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\textit{category of pairs } (F', F' \to S) \textit{ consisting}\\ \textit{of an algebraic space } F' \textit{ over } S' \textit{ and a}\\ \textit{morphism } F' \to S \textit{ of algebraic spaces over } S'\end{matrix}\right\}$$

Proof. Let F be an algebraic space over S. The functor from left to right assigns the pair $(j_!F, j_!F \to S)$ ot F which is an object of the right hand side by Lemma 40.16.1. Since this defines an equivalence of categories of sheaves by Sites, Lemma 9.21.4 to finish the proof it suffices to show: if F is a sheaf and $j_!F$ is an algebraic space, then F is an algebraic space. To do this, write $j_!F = U'/R'$ as in Lemma 40.9.1 with $U', R' \in Ob((Sch/S')_{fppf})$. Then the compositions $U' \to j_! F \to S$ and $R' \to j_! F \to S$ are morphisms of schemes over S'. Denote U, R the corresponding objects of $(Sch/S)_{fppf}$. The two morphisms $R' \to U'$ are morphisms over S and hence correspond to morpisms $R \to U$. Since these are simply the same morphisms (but viewed over S) we see that we get an étale equivalence relation over S. As $j_!$ defines an equivalence of categories of sheaves (see reference above) we see that $F = U/R$ and by Theorem 40.10.5 we see that F is an algebraic space. □

The following lemma is a slight rephrasing of the above.

Lemma 40.16.4. *Let Sch_{fppf} be a big fppf site. Let $S \to S'$ be a morphism of this site. Let F' be a sheaf on $(Sch/S')_{fppf}$. The following are equivalent:*

1. *The restriction $F'|_{(Sch/S)_{fppf}}$ is an algebraic space over S, and*
2. *the sheaf $h_S \times F'$ is an algebraic space over S'.*

Proof. The restriction and the product match under the equivalence of categories of Sites, Lemma 9.21.4 so that Lemma 40.16.3 above gives the result. □

We finish this section with a lemma on a compatibility.

Lemma 40.16.5. *Let Sch_{fppf} be a big fppf site. Let $S \to S'$ be a morphism of this site. Let F be an algebraic space over S. Let T be a scheme over S and let $f : T \to F$ be a morphism over S. Let $f' : T' \to F'$ be the morphism over S' we get from f by applying the equivalence of categories described in Lemma 40.16.3. For any property $\mathcal{P}$ as in Definition 40.5.1 we have $\mathcal{P}(f') \Leftrightarrow \mathcal{P}(f)$.*

Proof. Suppose that U is a scheme over S, and $U \to F$ is a surjective étale morphism. Denote U' the scheme U viewed as a scheme over S'. In Lemma 40.16.1 we have seen that $U' \to F'$ is surjective étale. Since

$$j(T \times_{f, F} U) = T' \times_{f', F'} U'$$

the morphism of schemes $T \times_{f,F} U \to U$ is identified with the morphism of schemes $T' \times_{f',F'} U' \to U'$. It is the same morphism, just viewed over different base schemes. Hence the lemma follows from Lemma 40.11.2. □

40.17. Other chapters

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(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Commutative Algebra
(8) Brauer Groups
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(36) More on Groupoid Schemes
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CHAPTER 41

Properties of Algebraic Spaces

41.1. Introduction

Please see Spaces, Section 40.1 for a brief introduction to algebraic spaces, and please read some of that chapter for our basic definitions and conventions concerning algebraic spaces. In this chapter we start introducing some basic notions and properties of algebraic spaces. A fundamental reference for the case of quasi-separated algebraic spaces is [**Knu71a**].

The discussion is somewhat awkward at times since we made the design descision to first talk about properties of algebraic spaces by themselves, and only later about properties of morphisms of algebraic spaces. We make an exception for this rule regarding *étale morphisms* of algebraic spaces, which we introduce in Section 41.13. But until that section whenever we say a morphism has a certain property, it automatically means the source of the morphism is a scheme (or perhaps the morphism is representable).

Some of the material in the chapter (especially regarding points) will be improved upon in the chapter on decent algebraic spaces.

41.2. Conventions

The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $Spec(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$. The reason is that we want to avoid confusion when changing base schemes, as in Spaces, Section 40.16.

41.3. Separation axioms

In this section we collect all the ``absolute" separation conditions of algebraic spaces. Since in our language any algebraic space is an algebraic space over some definite base scheme, any absolute property of X over S corresponds to a conditions imposed on X viewed as an algebraic space over $Spec(\mathbf{Z})$. Here is the precise formulation.

Definition 41.3.1. (Compare Spaces, Definition 40.13.2.) Consider a big fppf site $Sch_{fppf} = (Sch/Spec(\mathbf{Z}))_{fppf}$. Let X be an algebraic space over $Spec(\mathbf{Z})$. Let $\Delta : X \to X \times X$ be the diagonal morphism.

1. We say X is *separated* if Δ is a closed immersion.
2. We say X is *locally separated*[1] if Δ is an immersion.
3. We say X is *quasi-separated* if Δ is quasi-compact.

[1]In the literature this often refers to quasi-separated and locally separated algebraic spaces.

(4) We say X is *Zariski locally quasi-separated*[2] if there exists a Zariski covering $X = \bigcup_{i \in I} X_i$ (see Spaces, Definition 40.12.5) such that each X_i is quasi-separated.

Let S is a scheme contained in Sch_{fppf}, and let X be an algebraic space over S. Then we say X is *separated*, *locally separated*, *quasi-separated*, or *Zariski locally quasi-separated* if X viewed as an algebraic space over $Spec(\mathbf{Z})$ (see Spaces, Definition 40.16.2) has the corresponding property.

It is true that an algebraic space X over S which is separated (in the absolute sense above) is separated over S (and similarly for the other absolute separation properties above). This will be discussed in great detail in Morphisms of Spaces, Section 42.5. We will see in Lemma 41.6.5 that being Zariski locally separated is independent of the base scheme (hence equivalent to the absolute notion).

Lemma 41.3.2. *Let S be a scheme. Let X be an algebraic space over S. We have the following implications among the separation axioms of Definition 41.3.1:*

(1) *separated implies all the others,*
(2) *quasi-separated implies Zariski locally quasi-separated.*

Proof. Omitted. □

41.4. Points of algebraic spaces

As is clear from Spaces, Example 40.14.8 a point of an algebraic space should not be defined as a monomorphism from the spectrum of a field. Instead we define them as equivalence classes of morphisms of specra of fields as equivalence classes of morphisms from spectra of fields. exactly as explained in Schemes, Section 21.13.

Let S be a scheme. Let F be a presheaf on $(Sch/S)_{fppf}$. Let K is a field. Consider a morphism

$$Spec(K) \longrightarrow F.$$

By the Yoneda Lemma this is given by an element $p \in F(Spec(K))$. We say that two such pairs $(Spec(K), p)$ and $(Spec(L), q)$ are *equivalent* if there exists a third field Ω and a commutative diagram

$$\begin{tikzcd} Spec(\Omega) \arrow[r] \arrow[d] & Spec(L) \arrow[d, "q"] \\ Spec(K) \arrow[r, "p"] & F. \end{tikzcd}$$

In other words, there are field extensions $K \to \Omega$ and $L \to \Omega$ such that p and q map to the same element of $F(Spec(\Omega))$. We omit the verification that this defines an equivalence relation.

Definition 41.4.1. Let S be a scheme. Let X be an algebraic space over S. A *point* of X is an equivalence class of morphisms from spectra of fields into X. The set of points of X is denoted $|X|$.

Note that if $f : X \to Y$ is a morphism of algebraic spaces over S, then there is an induced map $|f| : |X| \to |Y|$ which maps a representative $x : Spec(K) \to X$ to the representative $f \circ x : Spec(K) \to Y$.

Lemma 41.4.2. *Let S be a scheme. Let X be a scheme over S. The points of X as a scheme are in canonical 1-1 correspondence with the points of X as an algebraic space.*

[2] This notion was suggested by B. Conrad.

Proof. This is Schemes, Lemma 21.13.3. □

Lemma 41.4.3. *Let S be a scheme. Let*

$$\begin{tikzcd} Z \times_Y X \arrow[r] \arrow[d] & X \arrow[d] \\ Z \arrow[r] & Y \end{tikzcd}$$

be a cartesian diagram of algebraic spaces. Then the map of sets of points

$$|Z \times_Y X| \longrightarrow |Z| \times_{|Y|} |X|$$

is surjective.

Proof. Namely, suppose given fields K, L and morphisms $Spec(K) \to X$, $Spec(L) \to Z$, then the assumption that they agree as elements of $|Y|$ means that there is a common extension $K \subset M$ and $L \subset M$ such that $Spec(M) \to Spec(K) \to X \to Y$ and $Spec(M) \to Spec(L) \to Z \to Y$ agree. And this is exactly the condition which says you get a morphism $Spec(M) \to Z \times_Y X$. □

Lemma 41.4.4. *Let S be a scheme. Let X be an algebraic space over S. Let $f : T \to X$ be a morphism from a scheme to X. The following are equivalent*

(1) *$f : T \to X$ is surjective (according to Spaces, Definition 40.5.1), and*
(2) *$|f| : |T| \to |X|$ is surjective.*

Proof. Assume (1). Let $x : Spec(K) \to X$ be a morphism from the spectrum of a field into X. By assumption the morphism of schemes $Spec(K) \times_X T \to Spec(K)$ is surjective. Hence there exists a field extension $K \subset K'$ and a morphism $Spec(K') \to Spec(K) \times_X T$ such that the left square in the diagram

$$\begin{tikzcd} Spec(K') \arrow[r] \arrow[d] & Spec(K) \times_X T \arrow[r] \arrow[d] & T \arrow[d] \\ Spec(K) \arrow[r, equal] & Spec(K) \arrow[r, "x"] & X \end{tikzcd}$$

is commutative. This shows that $|f| : |T| \to |X|$ is surjective.

Assume (2). Let $Z \to X$ be a morphism where Z is a scheme. We have to show that the morphism of schemes $Z \times_X T \to T$ is surjective, i.e., that $|Z \times_X T| \to |Z|$ is surjective. This follows from (2) and Lemma 41.4.3. □

Lemma 41.4.5. *Let S be a scheme. Let X be an algebraic space over S. Let $X = U/R$ be a presentation of X, see Spaces, Definition 40.9.3. Then the image of $|R| \to |U| \times |U|$ is an equivalence relation and $|X|$ is the quotient of $|U|$ by this equivalence relation.*

Proof. The assumption means that U is a scheme, $p : U \to X$ is a surjective, étale morphism, $R = U \times_X U$ is a scheme and defines an étale equivalence relation on U such that $X = U/R$ as sheaves. By Lemma 41.4.4 we see that $|U| \to |X|$ is surjective. By Lemma 41.4.3 the map

$$|R| \longrightarrow |U| \times_{|X|} |U|$$

is surjective. Hence the image of $|R| \to |U| \times |U|$ is exactly the set of pairs $(u_1, u_2) \in |U| \times |U|$ such that u_1 and u_2 have the same image in $|X|$. Combining these two statements we get the result of the lemma. □

Lemma 41.4.6. *Let S be a scheme. There exists a unique topology on the set of points of algebraic spaces over S with the following properties:*

(1) *for every morphism of algebraic spaces $X \to Y$ over S the map $|X| \to |Y|$ is continuous, and*
(2) *for every étale morphism $U \to X$ with U a scheme the map of topological spaces $|U| \to |X|$ is continuous and open.*

Proof. Let X be an algebraic space over S. Let $p : U \to X$ be a surjective étale morphism where U is a scheme over S. We define $W \subset |X|$ is open if and only if $|p|^{-1}(W)$ is an open subset of $|U|$. This is a topology on $|X|$.

Let us prove that the topology is independent of the choice of the presentation. To do this it suffices to show that if U' is a scheme, and $U' \to X$ is an étale morphism, then the map $|U'| \to |X|$ (with topology on $|X|$ defined using $U \to X$ as above) is open and continuous; which in addition will prove that (2) holds. Set $U'' = U \times_X U'$, so that we have the commutative diagram

$$\begin{tikzcd} U'' \arrow[r] \arrow[d] & U' \arrow[d] \\ U \arrow[r] & X \end{tikzcd}$$

As $U \to X$ and $U' \to X$ are étale we see that both $U'' \to U$ and $U'' \to U'$ are étale morphisms of schemes. Moreover, $U'' \to U'$ is surjective. Hence we get a commutative diagram of maps of sets

$$\begin{tikzcd} |U''| \arrow[r] \arrow[d] & |U'| \arrow[d] \\ |U| \arrow[r] & |X| \end{tikzcd}$$

The lower horizontal arrow is surjective (see Lemma 41.4.4 or Lemma 41.4.5) and continuous by definition of the topology on $|X|$. The top horizontal arrow is surjective, continuous, and open by Morphisms, Lemma 24.35.13. The left vertical arrow is continuous and open (by Morphisms, Lemma 24.35.13 again.) Hence it follows formally that the right vertical arrow is continuous and open.

To finish the proof we prove (1). Let $a : X \to Y$ be a morphism of algebraic spaces. According to Spaces, Lemma 40.11.4 we can find a diagram

$$\begin{tikzcd} U \arrow[r, "\alpha"'] \arrow[d, "p"'] & V \arrow[d, "q"] \\ X \arrow[r, "a"] & Y \end{tikzcd}$$

where U and V are schemes, and p and q are surjective and étale. This gives rise to the diagram

$$\begin{tikzcd} |U| \arrow[r, "\alpha"'] \arrow[d, "p"'] & |V| \arrow[d, "q"] \\ |X| \arrow[r, "a"] & |Y| \end{tikzcd}$$

where all but the lower horizontal arrows are known to be continuous and the two vertical arrows are surjective and open. It follows that the lower horizontal arrow is continuous as desired. $\square$

Definition 41.4.7. Let S be a scheme. Let X be an algebraic space over S. The underlying *topological space* of X is the set of points $|X|$ endowed with the topology constructed in Lemma 41.4.6.

It turns out that this topological space carries the same information as the small Zariski site X_{Zar} of Spaces, Definition 40.12.6.

Lemma 41.4.8. *Let S be a scheme. Let X be an algebraic space over S.*

(1) *The rule $X' \mapsto |X'|$ defines an inclusion preserving bijection between open subspaces X' (see Spaces, Definition 40.12.1) of X, and opens of the topological space $|X|$.*
(2) *A family $\{X_i \subset X\}_{i\in I}$ of open subspaces of X is a Zariski covering (see Spaces, Definition 40.12.5) if and only if $|X| = \bigcup |X_i|$.*

In other words, the small Zariski site X_{Zar} of X is canonically identified with a site associated to the topological space $|X|$ (see Sites, Example 9.6.4).

Proof. In order to prove (1) let us construct the inverse of the rule. Namely, suppose that $W \subset |X|$ is open. Choose a presentation $X = U/R$ corresponding to the surjective étale map $p : U \to X$ and étale maps $s, t : R \to U$. By construction we see that $|p|^{-1}(W)$ is an open of U. Denote $W' \subset U$ the corresponding open subscheme. It is clear that $R' = s^{-1}(W') = t^{-1}(W')$ is a Zariski open of R which defines an étale equivalence relation on W'. By Spaces, Lemma 40.10.2 the morphism $X' = W'/R' \to X$ is an open immersion. Hence X' is an algebraic space by Spaces, Lemma 40.11.1. By construction $|X'| = W$, i.e., X' is a subspace of X corresponding to W. Thus (1) is proved.

To prove (2), note that if $\{X_i \subset X\}_{i\in I}$ is a collection of open subspaces, then it is a Zariski covering if and only if the $U = \bigcup U \times_X X_i$ is an open covering. This follows from the definition of a Zariski covering and the fact that the morphism $U \to X$ is surjective as a map of presheaves on $(Sch/S)_{fppf}$. On the other hand, we see that $|X| = \bigcup |X_i|$ if and only if $U = \bigcup U \times_X X_i$ by Lemma 41.4.5 (and the fact that the projections $U \times_X X_i \to X_i$ are surjective and étale). Thus the equivalence of (2) follows. □

Lemma 41.4.9. *Let S be a scheme. Let X, Y be algebraic spaces over S. Let $X' \subset X$ be an open subspace. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Then f factors through X' if and only if $|f| : |Y| \to |X|$ factors through $|X'| \subset |X|$.*

Proof. By Spaces, Lemma 40.12.3 we see that $Y' = Y \times_X X' \to Y$ is an open immersion. If $|f|(|Y|) \subset |X'|$, then clearly $|Y'| = |Y|$. Hence $Y' = Y$ by Lemma 41.4.8. □

Lemma 41.4.10. *Let S be a scheme. Let X be an algebraic spaces over S. Let U be a scheme and let $f : U \to X$ be an étale morphism. Let $X' \subset X$ be the open subspace corresponding to the open $|f|(|U|) \subset |X|$ via Lemma 41.4.8. Then f factors through a surjective étale morphism $f' : U \to X'$. Moreover, if $R = U \times_X U$, then $R = U \times_{X'} U$ and X' has the presentation $X' = U/R$.*

Proof. The existence of the factorization follows from Lemma 41.4.9. The morphism f' is surjective according to Lemma 41.4.4. To see f' is étale, suppose that $T \to X'$ is a morphism where T is a scheme. Then $T \times_X U = T \times_{X'} U$ as $X" \to X$ is a monomorphism of sheaves. Thus the projection $T \times_{X'} U \to T$ is étale as we assumed f étale. We have $U \times_X U = U \times_{X'} U$ as $X' \to X$ is a monomorphism. Then $X' = U/R$ follows from Spaces, Lemma 40.9.1. □

Lemma 41.4.11. *Let S be a scheme. Let X be an algebraic space over S. Consider the map*

$$\{Spec(k) \to X \text{ monomorphism}\} \longrightarrow |X|$$

This map is injective.

Proof. Suppose that $\varphi_i : Spec(k_i) \to X$ are monomorphisms for $i = 1, 2$. If φ_1 and φ_2 define the same point of $|X|$, then we see that the scheme

$$Y = Spec(k_1) \times_{\varphi_1, X, \varphi_2} Spec(k_2)$$

is nonempty. Since the base change of a monomorphism is a monomorphism this means that the projection morphisms $Y \to Spec(k_i)$ are monomorphisms. Hence $Spec(k_1) = Y = Spec(k_2)$ as schemes over X, see Schemes, Lemma 21.23.10. We conclude that $\varphi_1 = \varphi_2$, which proves the lemma. □

We will see in Decent Spaces, Lemma 43.9.1 that this map is a bijection when X is very reasonable.

41.5. Quasi-compact spaces

Definition 41.5.1. Let S be a scheme. Let X be an algebraic space over S. We say X is *quasi-compact* if there exists a surjective étale morphism $U \to X$ with U quasi-compact.

Lemma 41.5.2. *Let S be a scheme. Let X be an algebraic space over S. Then X is quasi-compact if and only if $|X|$ is quasi-compact.*

Proof. Choose a scheme U and an étale surjective morphism $U \to X$. We will use Lemma 41.4.4. If U is quasi-compact, then since $|U| \to |X|$ is surjective we conclude that $|X|$ is quasi-compact. If $|X|$ is quasi-compact, then since $|U| \to |X|$ is open we see that there exists a quasi-compact open $U' \subset U$ such that $|U'| \to |X|$ is surjective (and still étale). Hence we win. □

Lemma 41.5.3. *A finite disjoint union of quasi-compact algebraic spaces is a quasi-compact algebraic space.*

Proof. This is clear from Lemma 41.5.2 and the corresponding topological fact. □

Example 41.5.4. The space $\mathbf{A}^1_{\mathbf{Q}}/\mathbf{Z}$ is quasi-compact but not very reasonable.

Lemma 41.5.5. *Let S be a scheme. Let X be an algebraic space over S. Every point of $|X|$ has a fundamental system of open quasi-compact neighbourhoods. In particular $|X|$ is locally quasi-compact in the sense of Topology, Definition 5.18.1.*

Proof. This follows formally from the fact that there exists a scheme U and a surjective, open, continuous map $U \to |X|$ of topological spaces. To be a bit more precise, if $u \in U$ maps to $x \in |X|$, then the images of the affine neighbourhoods of u will give a fundamental system of quasi-compact open neighbourhoods of x. □

41.6. Special coverings

In this section we collect some straightforward lemmas on the existence of étale surjective coverings of algebraic spaces.

Lemma 41.6.1. *Let S be a scheme. Let X be an algebraic space over S. There exists a surjective étale morphism $U \to X$ where U is a disjoint union of affine schemes. We may in addition assume each of these affines maps into an affine open of S.*

Proof. Let $V \to X$ be a surjective étale morphism. Let $V = \bigcup_{i \in I} V_i$ be a Zariski open covering such that each V_i maps into an affine open of S. Then set $U = \coprod_{i \in I} V_i$ with induced morphism $U \to V \to X$. This is étale and surjective as a composition of étale and surjective representable transformations of functors (via the general principle Spaces, Lemma 40.5.4 and Morphisms, Lemmas 24.9.2 and 24.35.3). $\square$

Lemma 41.6.2. *Let S be a scheme. Let X be an algebraic space over S. There exists a Zariski covering $X = \bigcup X_i$ such that each algebraic space X_i has a surjective étale covering by an affine scheme. We may in addition assume each X_i maps into an affine open of S.*

Proof. By Lemma 41.6.1 we can find a surjective étale morphism $U = \coprod U_i \to X$, with U_i affine and mapping into an affine open of S. Let $X_i \subset X$ be the open subspace of X such that $U_i \to X$ factors through an étale surjective morphism $U_i \to X_i$, see Lemma 41.4.10. Since $U = \bigcup U_i$ we see that $X = \bigcup X_i$. As $U_i \to X_i$ is surjective it follows that $X_i \to S$ maps into an affine open of S. $\square$

Lemma 41.6.3. *Let S be a scheme. Let X be an algebraic space over S. Then X is quasi-compact if and only if there exists an étale surjective morphism $U \to X$ with U an affine scheme.*

Proof. If there exists an étale surjective morphism $U \to X$ with U affine then X is quasi-compact by Definition 41.5.1. Conversely, if X is quasi-compact, then $|X|$ is quasi-compact. Let $U = \coprod_{i \in I} U_i$ be a disjoint union of affine schemes with an étale and surjective map $\varphi : U \to X$ (Lemma 41.6.1). Then $|X| = \bigcup \varphi(|U_i|)$ and by quasi-compactness there is a finite subset $i_1, \ldots, i_n$ such that $|X| = \bigcup \varphi(|U_{i_j}|)$. Hence $U_{i_1} \cup \ldots \cup U_{i_n}$ is an affine scheme with a finite surjective morphism towards X. $\square$

The following lemma will be obsoleted by the discussion of separated morphisms in the chapter on morphisms of algebraic spaces.

Lemma 41.6.4. *Let S be a scheme. Let X be an algebraic space over S. Let U be a separated scheme and $U \to X$ étale. Then $U \to X$ is separated, and $R = U \times_X U$ is a separated scheme.*

Proof. Let $X' \subset X$ be the open subscheme such that $U \to X$ factors through an étale surjection $U \to X'$, see Lemma 41.4.10. If $U \to X'$ is separated, then so is $U \to X$, see Spaces, Lemma 40.5.4 (as the open immersion $X' \to X$ is separated by Spaces, Lemma 40.5.8 and Schemes, Lemma 21.23.7). Moreover, since $U \times_{X'} U = U \times_X U$ it suffices to prove the result after replacing X by X', i.e., we may assume $U \to X$ surjective. Consider the commutative diagram

$$\begin{tikzcd} R = U \times_X U \arrow[r] \arrow[d] & U \arrow[d] \\ U \arrow[r] & X \end{tikzcd}$$

In the proof of Spaces, Lemma 40.13.1 we have seen that $j : R \to U \times_S U$ is separated. The morphism of schemes $U \to S$ is separated as U is a separated scheme, see Schemes, Lemma 21.21.14. Hence $U \times_S U \to U$ is separated as a base change, see Schemes, Lemma 21.21.13. Hence the scheme $U \times_S U$ is separated (by the same lemma). Since j is separated we see in the same way that R is separated. Hence $R \to U$ is a separated morphism (by Schemes, Lemma 21.21.14 again). Thus by Spaces, Lemma 40.11.2 and the diagram above we conclude that $U \to X$ is separated. $\square$

Lemma 41.6.5. *Let S be a scheme. Let X be an algebraic space over S. The following are equivalent*

(1) *X is Zariski locally quasi-separated over S,*
(2) *X is Zariski locally quasi-separated,*
(3) *there exists a Zariski open covering $X = \bigcup X_i$ such that for each i there exists an affine scheme U_i and a quasi-compact surjective étale morphism $U_i \to X_i$, and*
(4) *there exists a Zariski open covering $X = \bigcup X_i$ such that for each i there exists an affine scheme U_i which maps into an affine open of S and a quasi-compact surjective étale morphism $U_i \to X_i$.*

Proof. Assume $U_i \to X_i \subset X$ are as in (3). To prove (4) choose for each i a finite affine open covering $U_i = U_{i1} \cup \ldots \cup U_{in_i}$ such that each U_{ij} maps into an affine open of S. The compositions $U_{ij} \to U_i \to X_i$ are étale and quasi-compact (see Spaces, Lemma 40.5.4). Let $X_{ij} \subset X_i$ be the open subspace corresponding to the image of $|U_{ij}| \to |X_i|$, see Lemma 41.4.10. Note that $U_{ij} \to X_{ij}$ is quasi-compact as $X_{ij} \subset X_i$ is a monomorphism and as $U_{ij} \to X$ is quasi-compact. Then $X = \bigcup X_{ij}$ is a covering as in (4). The implication (4) $\Rightarrow$ (3) is immediate.

Assume (4). To show that X is Zariski locally quasi-separated over S it suffices to show that X_i is quasi-separated over S. Hence we may assume there exists an affine scheme U mapping into an affine open of S and a quasi-compact surjective étale morphism $U \to X$. Consider the fibre product square

$$\begin{tikzcd} U \times_X U \arrow[r] \arrow[d] & U \times_S U \arrow[d] \\ X \arrow[r, "\Delta_{X/S}"] & X \times_S X \end{tikzcd}$$

The right vertical arrow is surjective étale (see Spaces, Lemma 40.5.7) and $U \times_S U$ is affine (as U maps into an affine open of S, see Schemes, Section 21.17), and $U \times_X U$ is quasi-compact because the projection $U \times_X U \to U$ is quasi-compact as a base change of $U \to X$. It follows from Spaces, Lemma 40.11.2 that $\Delta_{X/S}$ is quasi-compact as desired.

Assume (1). To prove (3) there is an immediate reduction to the case where X is quasi-separated over S. By Lemma 41.6.2 we can find a Zariski open covering $X = \bigcup X_i$ such that each X_i maps into an affine open of S, and such that there exist affine schemes U_i and surjective étale morphisms $U_i \to X_i$. Since $U_i \to S$ maps into an affine open of S we see that $U_i \times_S U_i$ is affine, see Schemes, Section 21.17. As X is quasi-separated over S, the morphisms

$$R_i = U_i \times_{X_i} U_i = U_i \times_X U_i \longrightarrow U_i \times_S U_i$$

as base changes of $\Delta_{X/S}$ are quasi-compact. Hence we conclude that R_i is a quasi-compact scheme. This in turn implies that each projection $R_i \to U_i$ is quasi-compact. Hence, applying Spaces, Lemma 40.11.2 to the covering $U_i \to X_i$ and the morphism $U_i \to X_i$ we conclude that the morphisms $U_i \to X_i$ are quasi-compact as desired.

At this point we see that (1), (3), and (4) are equivalent. Since (3) does not refer to the base scheme we conclude that these are also equivalent with (2). □

41.7. Properties of Spaces defined by properties of schemes

Any étale local property of schemes gives rise to a corresponding property of algebraic spaces via the following lemma.

Lemma 41.7.1. *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{P}$ be a property of schemes which is local in the étale topology, see Descent, Definition 31.11.1. The following are equivalent*

1. *for some scheme U and surjective étale morphism $U \to X$ the scheme U has property $\mathcal{P}$, and*
2. *for every scheme U and every étale morphism $U \to X$ the scheme U has property $\mathcal{P}$.*

If X is representable this is equivalent to $\mathcal{P}(X)$.

Proof. Omitted. □

Definition 41.7.2. Let $\mathcal{P}$ be a property of schemes which is local in the étale topology. Let S be a scheme. Let X be an algebraic space over S. We say *X has property $\mathcal{P}$* if any of the equivalent conditions of Lemma 41.7.1 hold.

Remark 41.7.3. Here is a list of properties which are local for the étale topology (keep in mind that the fpqc, fppf, syntomic, and smooth topologies are stronger than the étale topology):

1. locally Noetherian, see Descent, Lemma 31.12.1,
2. Jacobson, see Descent, Lemma 31.12.2,
3. locally Noetherian and (S_k), see Descent, Lemma 31.13.1,
4. Cohen-Macaulay, see Descent, Lemma 31.13.2,
5. reduced, see Descent, Lemma 31.14.1,
6. normal, see Descent, Lemma 31.14.2,
7. locally Noetherian and (R_k), see Descent, Lemma 31.14.3,
8. regular, see Descent, Lemma 31.14.4,
9. Nagata, see Descent, Lemma 31.14.5.

Any étale local property of germs of schemes gives rise to a corresponding property of algebraic spaces. Here is the obligatory lemma.

Lemma 41.7.4. *Let $\mathcal{P}$ be a property of germs of schemes which is étale local, see Descent, Definition 31.17.1. Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$ be a point of X. Consider étale morphisms $a : U \to X$ where U is a scheme. The following are equivalent*

1. *for any $U \to X$ as above and $u \in U$ with $a(u) = x$ we have $\mathcal{P}(U, u)$, and*
2. *for some $U \to X$ as above and $u \in U$ with $a(u) = x$ we have $\mathcal{P}(U, u)$.*

If X is representable, then this is equivalent to $\mathcal{P}(X, x)$.

Proof. Omitted. □

Definition 41.7.5. Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. Let $\mathcal{P}$ be a property of germs of schemes which is étale local. We say *X has property $\mathcal{P}$ at x* if any of the equivalent conditions of Lemma 41.7.4 hold.

41.8. Dimension at a point

We can use Descent, Lemma 31.17.2 to define the dimension of an algebraic space X at a point x. This will give us a different notion than the topological one (i.e., the dimension of $|X|$ at x).

Definition 41.8.1. Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$ be a point of X. We define the *dimension of X at x* to be the element $\dim_x(X) \in \{0, 1, 2, \ldots, \infty\}$ such that $\dim_x(X) = \dim_u(U)$ for any (equivalently some) pair $(a : U \to X, u)$ consisting of an étale morphism $a : U \to X$ from a scheme to X and a point $u \in U$ with $a(u) = x$. See Definition 41.7.5, Lemma 41.7.4, and Descent, Lemma 31.17.2.

Warning: It is **not** the case that $\dim_x(X) = \dim_x(|X|)$ in general. A counter example is the algebraic space X of Spaces, Example 40.14.9. Namely, in this example we have $\dim_x(X) = 0$ and $\dim_x(|X|) = 1$ (this holds for any $x \in |X|$). In particular, it also means that the dimension of X (as defined below) is different from the dimension of $|X|$.

Definition 41.8.2. Let S be a scheme. Let X be an algebraic space over S. The *dimension* $\dim(X)$ of X is defined by the rule

$$\dim(X) = \sup_{x \in |X|} \dim_x(X)$$

By Properties, Lemma 23.10.2 we see that this is the usual notion if X is a scheme. There is another integer that measures the dimension of a scheme at a point, namely the dimension of the local ring. This invariant is compatible with étale morphisms also, see Section 41.20.

41.9. Reduced spaces

We have already defined reduced algebraic spaces in Section 41.7. Here we just prove some simple lemmas regarding reduced algebraic spaces.

Lemma 41.9.1. *Let S be a scheme. Let X be an algebraic space over S. Let $T \subset |X|$ be a closed subset. There exists a unique closed subspace $Z \subset X$ with the following properties: (a) we have $|Z| = T$, and (b) Z is reduced.*

Proof. Let $U \to X$ be a surjective étale morphism, where U is a scheme. Set $R = U \times_X U$, so that $X = U/R$, see Spaces, Lemma 40.9.1. As usual we denote $s, t : R \to U$ the two projection morphisms. By Lemma 41.4.5 we see that T corresponds to a closed subset $T' \subset |U|$ such that $s^{-1}(T') = t^{-1}(T')$. Let $Z' \subset U$ be the reduced induced scheme structure on T'. In this case the fibre products $Z' \times_{U,t} R$ and $Z' \times_{U,s} R$ are closed subschemes of R (Schemes, Lemma 21.18.2) which are étale over Z' (Morphisms, Lemma 24.35.4), and hence reduced (because being reduced is local in the étale topology, see Remark 41.7.3). Since they have the same underlying topological space (see above) we conclude that $Z' \times_{U,t} R = Z' \times_{U,s} R$. Hence the common value R' is the restriction of R to Z', see Groupoids, Definition 35.15.2. By Spaces, Theorem 40.10.5 we see that $Z = Z'/R'$ is an algebraic space. By Groupoids, Lemma 35.17.6 we see that $Z \to X$ is a monomorphism. By construction we have $U \times_X Z = Z'$, so $U \times_X Z \to Z$ is a closed immersion. This means all the hypotheses of Spaces, Lemma 40.11.3 are satisfied for the transformation $Z \to X$, $\mathcal{P}$=``closed immersion'' (closed immersions satisfy descent for étale coverings, see Descent, Lemma 31.33.2), and the étale surjective morphism $U \to X$. We conclude that $Z \to X$ is representable, a monomorphism and a closed immersion, which is the definition of a closed subspace (see Spaces, Definition 40.12.1). By construction $|Z| = T$ and Z is reduced. This proves existence. We omit the proof of uniqueness. $\square$

Lemma 41.9.2. *Let S be a scheme. Let X, Y be algebraic spaces over S. Let $Z \subset X$ be a closed subspace. Assume Y is reduced. A morphism $f : Y \to X$ factors through Z if and only if $f(|Y|) \subset |Z|$.*

Proof. Assume $f(|Y|) \subset |Z|$. Choose a diagram

$$\begin{tikzcd} V \arrow[r, "h"] \arrow[d, "b"'] & U \arrow[d, "a"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

where U, V are schemes, and the vertical arrows are surjective and étale. The scheme V is reduced, see Lemma 41.7.1. Hence h factors through $a^{-1}(Z)$ by Schemes, Lemma 21.12.6. So $a \circ h$ factors through Z. As $Z \subset X$ is a subsheaf, and $V \to Y$ is a surjection of sheaves on $(Sch/S)_{fppf}$ we conclude that $X \to Y$ factors through Z. □

Definition 41.9.3. Let S be a scheme, and let X be an algebraic space over S. Let $Z \subset |X|$ be a closed subset. An *algebraic space structure on* Z is given by a closed subspace Z' of X with $|Z'|$ equal to Z. The *reduced induced algebraic space structure* on Z is the one constructed in Lemma 41.9.1. The *reduction* X_{red} *of* X is the reduced induced algebraic space structure on $|X|$.

41.10. The schematic locus

Every algebraic space has a largest open subspace which is a scheme; this is more or less clear but we also write out the proof below. Of course this subspace may be empty, for example if $X = \mathbf{A}^1_{\mathbf{Q}}/\mathbf{Z}$ (the universal counter example). On the other hand, if X is very reasonable, then this largest open subscheme is actually dense in X!

Lemma 41.10.1. *Let S be a scheme. Let X be an algebraic space over S. There exists a largest open subspace $X' \subset X$ which is a scheme.*

Proof. Let $U \to X$ be an étale surjective morphism, where U is a scheme. Let $R = U \times_X U$. The open subspaces of X correspond $1-1$ with open subschemes of U which are R-invariant. Hence there is a set of them. Let X_i, $i \in I$ be the set of open subspaces of X which are schemes, i.e., are representable. Consider the open subspace $X' \subset X$ whose underlying set of points is the open $\bigcup |X_i|$ of $|X|$. By Lemma 41.4.4 we see that

$$\coprod X_i \longrightarrow X'$$

is a surjective map of sheaves on $(Sch/S)_{fppf}$. But since each $X_i \to X'$ is representable by open immersions we see that in fact the map is surjective in the Zariski topology. (Because if $T \to X'$ is a morphism from a scheme into X', then $X_i \times'_X T$ is an open subscheme of T.) Hence we can apply Schemes, Lemma 21.15.4 to see that X' is a scheme. □

In the rest of this section we say that an open subspace X' of an algebraic space X is *dense* if the corresponding open subset $|X'| \subset |X|$ is dense.

Lemma 41.10.2. *Let S be a scheme. Let X be an algebraic space over S. If there exists a finite, étale, surjective morphism $U \to X$ where U is a scheme, then there exists a dense open subspace of X which is a scheme.*

Proof. Assume X is an algebraic space, U a scheme, and $U \to X$ is a finite étale surjective morphism. Write $R = U \times_X U$ and denote $s, t : R \to U$ the projections as usual. Note that s, t are surjective, finite and étale. Claim: The union of the R-invariant affine opens of U is topologically dense in U.

Proof of the claim[3]. Let $W \subset U$ be an affine open. Set $W' = t(s^{-1}(W)) \subset U$. Since $s^{-1}(W)$ is affine (hence quasi-compact) we see that $W' \subset U$ is a quasi-compact open. By Properties, Lemma 23.26.3 there exists a dense open $W'' \subset W'$ which is a separated scheme. Set $\Delta' = W' \setminus W''$. This is a nowhere dense closed subset of W''. Since $t|_{s^{-1}(W)} : s^{-1}(W) \to W'$ is open (because it is étale) we see that the inverse image $(t|_{s^{-1}(W)})^{-1}(\Delta') \subset s^{-1}(W)$ is a nowhere dense closed subset (see Topology, Lemma 5.17.6). Hence, by Morphisms, Lemma 24.44.7 we see that

$$\Delta = s\left((t|_{s^{-1}(W)})^{-1}(\Delta')\right)$$

is a nowhere dense closed subset of W. Pick any point $\eta \in W$, $\eta \notin \Delta$ which is a generic point of an irreducible component of W (and hence of U). By our choices above the finite set $t(s^{-1}(\{\eta\})) = \{\eta_1, \ldots, \eta_n\}$ is contained in the separated scheme W''. Note that the fibres of s is are finite discrete spaces, and that generalizations lift along the étale morphism t, see Morphisms, Lemmas 24.35.12 and 24.24.8. In this way we see that each η_i is a generic point of an irreducible component of W''. Thus, by Properties, Lemma 23.26.1 we can find an affine open $V \subset W''$ such that $\{\eta_1, \ldots, \eta_n\} \subset V$. By Groupoids, Lemma 35.20.1 this implies that η is contained in an R-invariant affine open subscheme of U. The claim follows as W was chosen as an arbitrary affine open of U and because the set of generic points of irreducible components of $W \setminus \Delta$ is dense in W.

Using the claim we can finish the proof. Namely, if $W \subset U$ is an R-invariant affine open, then the restriction R_W of R to W equals $R_W = s^{-1}(W) = t^{-1}(W)$ (see Groupoids, Definition 35.16.1 and discussion following it). In particular the maps $R_W \to W$ are finite étale also. It follows in particular that R_W is affine. Thus we see that W/R_W is a scheme, by Groupoids, Proposition 35.19.8. On the other hand, W/R_W is an open subspace of X by Spaces, Lemma 40.10.2. Hence having a dense collection of points contained in R-invariant affine open of U certainly implies that the schematic locus of X (see Lemma 41.10.1) is open dense in X. □

We will improve the following proposition to the case of very reasonable algebraic spaces in Decent Spaces, Proposition 43.8.1.

Proposition 41.10.3. *Let S be a scheme. Let X be an algebraic space over S. If X is Zariski locally quasi-separated (e.g., X is quasi-separated), then there exists a dense open subspace of X which is a scheme.*

Proof. By Lemma 41.10.1 and Lemma 41.6.5 we may assume that there exists an affine scheme U and a surjective, quasi-compact, étale morphism $U \to X$. Set $R = U \times_X U$, and denote $s, t : R \to U$ the projections as usual. Note that s, t are surjective, quasi-compact and étale, hence also quasi-finite (see Étale Morphisms, Section 37.11). By More on Morphisms, Lemma 33.29.7 there exists a dense open subscheme $W \subset U$ such that $s^{-1}(W) \to W$ is finite. By Descent, Lemma 31.19.21 being finite is fpqc (and in particular étale) local on the target. Hence we may apply More on Groupoids, Lemma 36.5.4 which says that the largest open $W \subset U$ over which s is finite is R-invariant. It is still dense of course. The restriction R_W of R to W equals $R_W = s^{-1}(W) = t^{-1}(W)$ (see Groupoids, Definition 35.16.1 and discussion following it). By construction $s_W, t_W : R_W \to W$ are finite étale. If we can show the open subspace $W/R_W \subset X$ (see Spaces, Lemma 40.10.2) contains a dense open subspace which is a scheme, then the proposition follows for X. This reduces us to Lemma 41.10.2. □

[3]The claim is easier to prove if U is assumed quasi-separated, since in that case Properties, Lemma 23.26.1 may be applied immediately to the R-equivalence class of any generic point of U.

41.11. Points on quasi-separated spaces

Points can behave very badly on algebraic spaces in the generality introduced in the stacks project. However, for quasi-separated spaces their behaviour is mostly like the behaviour of points on schemes. We prove a few results on this in this section.

The following lemma is a key lemma which we will use to prove that certain algebraic spaces are isomorphic to the spectrum of a field.

Lemma 41.11.1. *Let S be a scheme. Let k be a field. Let X be an algebraic space over S and assume that there exists a surjective étale morphism $Spec(k) \to X$. If X is quasi-separated, then $X \cong Spec(k')$ where $k' \subset k$ is a finite separable extension.*

Proof. Set $R = Spec(k) \times_X Spec(k)$, so that we have a fibre product diagram

$$\begin{tikzcd} R \arrow[r, "s"'] \arrow[d, "t"'] & Spec(k) \arrow[d] \\ Spec(k) \arrow[r] & X \end{tikzcd}$$

By Spaces, Lemma 40.9.1 we know $X = Spec(k)/R$ is the quotient sheaf. Because $Spec(k) \to X$ is étale, the morphisms s and t are étale. Hence $R = \coprod_{i \in I} Spec(k_i)$ is a disjoint union of spectra of fields, and both s and t induce finite separable field extensions $s, t : k \subset k_i$, see Morphisms, Lemma 24.35.7. Because

$$R = Spec(k) \times_X Spec(k) = (Spec(k) \times_S Spec(k)) \times_{X \times_S X, \Delta} X$$

and since Δ is quasi-compact by assumption we conclude that $R \to Spec(k) \times_S Spec(k)$ is quasi-compact. Hence R is quasi-compact as $Spec(k) \times_S Spec(k)$ is affine. We conclude that I is finite. This implies that s and t are finite locally free morphisms. Hence by Groupoids, Proposition 35.19.8 we conclude that $Spec(k)/R$ is represented by $Spec(k')$, with $k' \subset k$ finite locally free where

$$k' = \{x \in k \mid s_i(x) = t_i(x) \text{ for all } i \in I\}$$

It is easy to see that k' is a field. □

Remark 41.11.2. For improvements of the lemma above, see Decent Spaces, Lemmas 43.9.2 and 43.11.1. It is possible that Lemma 41.11.1 also holds when X is locally separated. To prove this one would have to show that the index set I in the proof of Lemma 41.11.1 is finite, if we only assume that $R \to Spec(k) \times_S Spec(k)$ is an immersion (and an étale equivalence relation of course).

Lemma 41.11.3. *Let S be a scheme. Let X be an algebraic space over S. Let U be a scheme. Let $\varphi : U \to X$ be an étale morphism such that the projections $R = U \times_X U \to U$ are quasi-compact; for example if φ is quasi-compact. Then the fibres of*

$$|U| \to |X| \quad \textit{and} \quad |R| \to |X|$$

are finite.

Proof. Denote $R = U \times_X U$, and $s, t : R \to U$ the projections. Let $u \in U$ be a point, and let $x \in |X|$ be its image. The fibre of $|U| \to |X|$ over x is equal to $s(t^{-1}(\{u\}))$ by Lemma 41.4.3, and the fibre of $|R| \to |X|$ over x is $t^{-1}(s(t^{-1}(\{u\})))$. Since $t : R \to U$ is étale and quasi-compact, it has finite fibres (as its fibres are disjoint unions of spectra of fields by Morphisms, Lemma 24.35.7 and quasi-compact). Hence we win. □

Lemma 41.11.4. *Let S be a scheme. Let X be a Zariski locally quasi-separated algebraic space over S. Then the topological space $|X|$ is sober (see Topology, Definition 5.5.4).*

Proof. Combining Topology, Lemma 5.5.5 and Lemma 41.6.5 we see that we may assume that there exists an affine scheme U and a surjective, quasi-compact, étale morphism $U \to X$. Set $R = U \times_X U$ with projection maps $s, t : R \to U$. Applying Lemma 41.11.3 we see that the fibres of s, t are finite. It follows all the assumptions of Topology, Lemma 5.14.7 are met, and we conclude that $|X|$ is Kolmogorov[4].

It remains to show that every irreducible closed subset $T \subset |X|$ has a generic point. By Lemma 41.9.1 there exists a closed subspace $Z \subset X$ with $|Z| = |T|$. Note that $U \times_X Z \to Z$ is a quasi-compact, surjective, étale morphism from an affine scheme to Z, hence Z is Zariski locally quasi-separated by Lemma 41.6.5. By Proposition 41.10.3 we see that there exists an open dense subspace $Z' \subset Z$ which is a scheme. This means that $|Z'| \subset T$ is open dense. Hence the topological space $|Z'|$ is irreducible, which means that Z' is an irreducible scheme. By Schemes, Lemma 21.11.1 we conclude that $|Z'|$ is the closure of a single point $\eta \in |Z'| \subset T$ and hence also $T = \overline{\{\eta\}}$, and we win. □

41.12. Noetherian spaces

We have already defined locally Noetherian algebraic spaces in Section 41.7.

Definition 41.12.1. Let S be a scheme. Let X be an algebraic space over S. We say X is *Noetherian* if X is quasi-compact, quasi-separated and locally Noetherian.

Note that a Noetherian algebraic space X is not just quasi-compact and locally Noetherian, but also quasi-separated. This does not conflict with the definition of a Noetherian scheme, as a locally Noetherian scheme is quasi-separated, see Properties, Lemma 23.5.4. This does not hold for algebraic spaces. Namely, $X = \mathbf{A}^1_k/\mathbf{Z}$, see Spaces, Example 40.14.8 is locally Noetherian and quasi-compact but not quasi-separated (hence not Noetherian according to our definitions).

A consequence of the choice made above is that an algebraic space of finite type over a Noetherian algebraic space is not automatically Noetherian, i.e., the analogue of Morphisms, Lemma 24.14.6 does not hold. The correct statement is that an algebraic space of finite presentation over a Noetherian algebraic space is Noetherian (see Morphisms of Spaces, Lemma 42.26.6).

A Noetherian algebraic space X is very close to being a scheme. In the rest of this section we collect some lemmas to illustrate this.

Lemma 41.12.2. *Let S be a scheme. Let X be an algebraic space over S.*

(1) *If X is locally Noetherian then $|X|$ is a locally Noetherian topological space.*
(2) *If X is quasi-compact and locally Noetherian, then $|X|$ is a Noetherian toplogical space.*

Proof. Assume X is locally Noetherian. Choose a scheme U and a surjective étale morphism $U \to X$. As X is locally Noetherian we see that U is locally Noetherian. By Properties, Lemma 23.5.5 this means that $|U|$ is a locally Noetherian topological space. Since $|U| \to |X|$ is open and surjective we conclude that $|X|$ is locally Noetherian by Topology,

[4] Actually we use here also Schemes, Lemma 21.11.1 (soberness schemes), Morphisms, Lemmas 24.35.12 and 24.24.8 (generalizations lift along étale morphisms), Lemma 41.4.5 (points on an algebraic space in terms of a presentation), and Lemma 41.4.6 (openness quotient map).

Lemma 5.6.3. This proves (1). If X is quasi-compact and locally Noetherian, then $|X|$ is quasi-compact and locally Noetherian. Hence $|X|$ is Noetherian by Topology, Lemma 5.9.10. □

Lemma 41.12.3. *Let S be a scheme. Let X be an algebraic space over S. If X is Noetherian, then $|X|$ is a sober Noetherian topological space.*

Proof. A quasi-separated algebraic space has an underlying sober topological space, see Lemma 41.11.4. It is Noetherian by Lemma 41.12.2. □

41.13. Étale morphisms of algebraic spaces

This section really belongs in the chapter on morphisms of algebraic spaces, but we need the notion of an algebraic space étale over another in order to define the small étale site of an algebraic space. Thus we need to do some preliminary work on étale morphisms from schemes to algebraic spaces, and étale morphisms between algebraic spaces. For more about étale morphisms of algebraic spaces, see Morphisms of Spaces, Section 42.35.

Lemma 41.13.1. *Let S be a scheme. Let X be an algebraic space over S. Let U, U' be schemes over S.*

1. *If $U \to U'$ is an étale morphism of schemes, and if $U' \to X$ is an étale morphism from U' to X, then the composition $U \to X$ is an étale morphism from U to X.*
2. *If $\varphi : U \to X$ and $\varphi' : U' \to X$ are étale morphisms towards X, and if $\chi : U \to U'$ is a morphism of schemes such that $\varphi = \varphi' \circ \chi$, then χ is an étale morphism of schemes.*

Proof. Recall that our definition of an étale morphism from a scheme into an algebraic space comes from Spaces, Definition 40.5.1 via the fact that any morphism from a scheme into an algebraic space is representable. Part (1) of the lemma follows from this, the fact that étale morphisms are preserved under composition (Morphisms, Lemma 24.35.3) and Spaces, Lemmas 40.5.4 and 40.5.3 (which are formal). To prove part (2) choose a scheme W over S and a surjective étale morphism $W \to X$. Consider the base change $\chi_W : W \times_X U \to W \times_X U'$ of χ. As $W \times_X U$ and $W \times_X U'$ are étale over W, we conclude that χ_W is étale, by Morphisms, Lemma 24.35.19. On the other hand, in the commutative diagram

$$\begin{tikzcd} W \times_X U \arrow[r] \arrow[d] & W \times_X U' \arrow[d] \\ U \arrow[r] & U' \end{tikzcd}$$

the two vertical arrows are étale and surjective. Hence by Descent, Lemma 31.10.4 we conclude that $U \to U'$ is étale. □

Definition 41.13.2. Let S be a scheme. A morphism $f : X \to Y$ between algebraic spaces over S is called *étale* if and only if for every étale morphism $\varphi : U \to X$ where U is a scheme, the composition $\varphi \circ f$ is étale also.

If X and Y are schemes, then this agree with the usual notion of an étale morphism of schemes. In fact, whenever $X \to Y$ is a representable morphism of algebraic spaces, then this agrees with the notion defined via Spaces, Definition 40.5.1. This follows by combining Lemma 41.13.3 below and Spaces, Lemma 40.11.2.

Lemma 41.13.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is étale,*
(2) *there exists a surjective étale morphism $\varphi : U \to X$, where U is a scheme, such that the composition $f \circ \varphi$ is étale (as a morphism of algebraic spaces),*
(3) *there exists a surjective étale morphism $\psi : V \to Y$, where V is a scheme, such that the base change $V \times_X Y \to V$ is étale (as a morphism of algebraic spaces),*
(4) *there exists a commutative diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U, V are schemes, the vertical arrows are étale, and the left vertical arrow is surjective such that the horizontal arrow is étale.

Proof. Let us prove that (4) implies (1). Assume a diagram as in (4) given. Let $W \to X$ be an étale morphism with W a scheme. Then we see that $W \times_X U \to U$ is étale. Hence $W \times_X U \to V$ is étale, and also $W \times_X U \to Y$ is étale by Lemma 41.13.1 (1). Since also the projection $W \times_X U \to W$ is surjective and étale, we conclude from Lemma 41.13.1 (2) that $W \to Y$ is étale.

Let us prove that (1) implies (4). Assume (1). Choose a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where $U \to X$ and $V \to Y$ are surjective and étale, see Spaces, Lemma 40.11.4. By assumption the morphism $U \to Y$ is étale, and hence $U \to V$ is étale by Lemma 41.13.1 (2).

We omit the proof that (2) and (3) are also equivalent to (1). $\square$

Lemma 41.13.4. *The composition of two étale morphisms of algebraic spaces is étale.*

Proof. This is immediate from the definition. $\square$

Lemma 41.13.5. *The base change of an étale morphism of algebraic spaces by any morphism of algebraic spaces is étale.*

Proof. Let $X \to Y$ be an étale morphism of algebraic spaces over S. Let $Z \to Y$ be a morphism of algebraic spaces. Choose a scheme U and a surjective étale morphism $U \to X$. Choose a scheme W and a surjective étale morphism $W \to Z$. Then $U \to Y$ is étale, hence in the diagram

$$\begin{tikzcd} W \times_Y U \arrow[r] \arrow[d] & W \arrow[d] \\ Z \times_Y X \arrow[r] & Z \end{tikzcd}$$

the top horizontal arrow is étale. Moreover, the left vertical arrow is surjective and étale (verification omitted). Hence we conclude that the lower horizontal arrow is étale by Lemma 41.13.3. $\square$

Lemma 41.13.6. *Let S be a scheme. Let X, Y, Z be algebraic spaces. Let $g : X \to Z$, $h : Y \to Z$ be étale morphisms and let $f : X \to Y$ be a morphism such that $h \circ f = g$. Then f is étale.*

Proof. Choose a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\chi"'] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where $U \to X$ and $V \to Y$ are surjective and étale, see Spaces, Lemma 40.11.4. By assumption the morphisms $\varphi : U \to X \to Z$ and $\psi : V \to Y \to Z$ are étale. Moreover, $\psi \circ \chi = \varphi$ by our assumption on f, g, h. Hence $U \to V$ is étale by Lemma 41.13.1 part (2). $\square$

Lemma 41.13.7. *Let S be a scheme. If $X \to Y$ is an étale morphism of algebraic spaces over S, then the associated map $|X| \to |Y|$ of topological spaces is open.*

Proof. This is clear from the diagram in Lemma 41.13.3 and Lemma 41.4.6. $\square$

Finally, here is a fun lemma. It is not true that an algebraic space with an étale morphism towards a scheme is a scheme, see Spaces, Example 40.14.2. But it is true if the target is the spectrum of a field.

Lemma 41.13.8. *Let S be a scheme. Let $X \to Spec(k)$ be étale morphism over S, where k is a field. Then X is a scheme.*

Proof. Let U be an affine scheme, and let $U \to X$ be an étale morphism. By Definition 41.13.2 we see that $U \to Spec(k)$ is an étale morphism. Hence $U = \coprod_{i=1,\ldots,n} Spec(k_i)$ is a finite disjoint union of spectra of finite separable extensions k_i of k, see Morphisms, Lemma 24.35.7. The $R = U \times_X U \to U \times_{Spec(k)} U$ is a monomorphism and $U \times_{Spec(k)} U$ is also a finite disjoint union of spectra of finite separable extensions of k. Hence by Schemes, Lemma 21.23.10 we see that R is similarly a finite disjoint union of spectra of finite separable extensions of k. This U and R are affine and both projections $R \to U$ are finite locally free. Hence U/R is a scheme by Groupoids, Proposition 35.19.8. By Spaces, Lemma 40.10.2 it is also an open subspace of X. By Lemma 41.10.1 we conclude that X is a scheme. $\square$

41.14. Spaces and fpqc coverings

Let S be a scheme. An algebraic space over S is defined as a sheaf in the fppf topology with additional properties. Hence it is not clear that it satisfies the sheaf property for the fpqc topology (see Topologies, Definition 30.8.12). In this section we discuss this question. However, when we say that the algebraic space X satisfies the sheaf property for the fpqc topology we really only consider fpqc coverings $\{f_i : T_i \to T\}_{i \in I}$ such that T, T_i are objects of the big site $(Sch/S)_{fppf}$ (as per our conventions, see Section 41.2). We first address the question as to whether an algebraic space is separated as a presheaf for the fpqc topology.

Lemma 41.14.1. *Let S be a scheme. Let X be an algebraic space over S. Let $\{f_i : T_i \to T\}_{i \in I}$ be a fpqc covering of schemes over S. Then the map*

$$Mor_S(T, X) \longrightarrow \prod\nolimits_{i \in I} Mor_S(T_i, X)$$

is injective.

Proof. Let $a, b : T \to X$ be two morphisms such that $a \circ f_i = b \circ f_i$ for all i. Consider the fibre product

$$T' = X \times_{\Delta_{X/S}, X \times_S X, (a,b)} T.$$

By Spaces, Lemma 40.13.1 the morphism $\Delta_{X/S}$ is a representable monomorphism which is locally of finite type. Hence $T' \to T$ is a monomorphism of schemes which is locally of finite type. Our assumption that $a \circ f_i = b \circ f_i$ implies that each $T_i \to T$ factors (uniquely) through T'. Consider the commutative diagram

$$\begin{tikzcd} T_i \times_T T' \arrow[r] \arrow[d] & T' \arrow[d] \\ T_i \arrow[r] \arrow[ur] \arrow[u, bend left=60] & T \end{tikzcd}$$

Since the projection $T_i \times_T T' \to T_i$ is a monomorphism with a section we conclude it is an isomorphism. Hence we conclude that $T' \to T$ is an isomorphism by Descent, Lemma 31.19.15. This means $a = b$ as desired. $\square$

Lemma 41.14.2. *Let S be a scheme. Let X be an algebraic space over S. Let $X = \bigcup_{j \in J} X_j$ be a Zariski covering, see Spaces, Definition 40.12.5. If each X_j satisfies the sheaf property for the fpqc topology then X satisfies the sheaf property for the fpqc topology.*

Proof. Assume each X_j satisfies the sheaf property for the fpqc topology. Let $\{f_i : T_i \to T\}_{i \in I}$ be a fpqc covering of schemes over S. Let $a_i \in Mor_S(T_i, X)$ be such that $a_i \circ \mathrm{pr}_0 = a_{i'} \circ \mathrm{pr}_1$ as morphisms $T_i \times_T T_{i'} \to X$. We have to prove that these morphisms come from a morphism $a : T \to X$. Consider the open subsets $T_{i,j} = a_i^{-1}(X_j)$. Then it is clear that $\mathrm{pr}_0^{-1}(T_{i,j}) = \mathrm{pr}_1^{-1}(T_{i',j})$ as open subsets of $T_i \times_T T_{i'}$. Hence there exist open subsets $U_j \subset T$ such that $T_{i,j} = f_i^{-1}(U_j)$, see Descent, Lemma 31.9.2. In particular, $\{T_{i,j} \to U_j\}_{i \in I}$ is a fpqc covering of U_j, and the morphisms $a_{i,j} = a_i|_{T_{i,j}}$ are morphisms into X_j. By assumption there exist morphisms $\alpha_j : U_j \to X_j$ such that $T_{i,j} \to U_j \to X_j$ agrees with $a_{i,j}$. By Lemma 41.14.1 we conclude that $\alpha_j|_{U_j \cap U_{j'}}$ agrees with $\alpha_{j'}|_{U_j \cap U_{j'}}$. Hence, since X is a sheaf for the Zariski topology we conclude that the α_j glue to a morphism $a : T \to X$. By construction we have $a \circ f_i|_{T_{i,j}} = a_{i,j} = a_i|_{T_{i,j}}$. Using the sheaf condition for the Zariski topology one more time we conclude that $a \circ f_i = a_i$ as desired. $\square$

Lemma 41.14.3. *Let S be a scheme. Let X be an algebraic space over S. If X is Zariski locally quasi-separated over S, then X satisfies the sheaf condition for the fpqc topology.*

Proof. By Lemmas 41.6.5 and 41.14.2 we may assume there exists an affine scheme U and a surjective étale quasi-compact morphism $\varphi : U \to X$. By Lemma 41.6.4 the morphism $U \to X$ is also separated.

Let $\{f_i : T_i \to T\}_{i \in I}$ be a fpqc covering of schemes over S. Let $a_i \in Mor_S(T_i, X)$ be such that $a_i \circ \mathrm{pr}_0 = a_j \circ \mathrm{pr}_1$ as morphisms $T_i \times_T T_j \to X$. We have to prove that these morphisms come from a morphism $a : T \to X$. Consider the schemes

$$W_i = T_i \times_{a_i, X, \varphi} U.$$

The strucure morphisms $W_i \to T_i$ are surjective, separated, quasi-compact and étale, in particular also quasi-finite (see Morphisms, Lemma 24.35.6). Hence each $W_i \to T_i$ is quasi-affine, see More on Morphisms, Lemma 33.29.3. For each pair of indices $i, j \in I$ we have canonical isomorphisms

$$\begin{aligned} W_i \times_T T_j &= (T_i \times_{a_i, X, \varphi} U) \times_T T_j \\ &= (T_i \times_T T_j) \times_{a_i \circ \mathrm{pr}_0, X, \varphi} U \\ &= (T_i \times_T T_j) \times_{a_j \circ \mathrm{pr}_1, X, \varphi} U \\ &= T_i \times_T (T_j \times_{a_j, X, \varphi} U) \\ &= T_i \times_T W_j. \end{aligned}$$

These isomorphisms satsify the cocycle condition of Descent, Definition 31.30.3 as a_i, a_j and a_k agree over $T_i \times_T T_j \times_T T_k$ (some details omitted). By Descent, Lemma 31.34.1 this descent datum is effective and we conclude that there exists a scheme $W \to T$ and isomorphisms $T_i \times_T W = W_i$ compatible with the canonical descent datum and the one given above. In particular we see that $\{W_i \to W\}_{i \in I}$ is the base change of a fpqc covering, and hence a fpqc covering. Note that by construction the morphisms $b_i : W_i = T_i \times_X U \to U$ have the property $b_i \circ \mathrm{pr}_0 = b_j \circ \mathrm{pr}_1$ as morphisms $W_i \times_W W_j \to U$. Hence by Descent, Lemma 31.9.3 we see that there exists a morphism of schemes $b : W \to U$ which restricts to b_i on W_i for all i.

By Descent, Lemmas 31.19.6 and 31.19.27 the morphism $W \to T$ is surjective and étale. Hence, in order to see that b gives rise to the morphism $a : T \to X$ we are looking for, it suffices to show that $b \circ \mathrm{pr}_0 = b \circ \mathrm{pr}_1$ as morphisms $W \times_T W \to X$. For this we note that we do know that $b_i \circ \mathrm{pr}_0 = b_j \circ \mathrm{pr}_1$ as morphisms $W_i \times_T W_j \to X$, because

$$W_i \times_T W_j = (T_i \times_T T_j) \times_{(a_i, a_j), X \times_S X} U \times_S U$$

and $(a_i, a_j) : T_i \times_T T_j \to X \times_S X$ factors through $\Delta_{X/S}$ by assumption. In other words we conclude that over the members of the fpqc covering $\{W_i \times_T W_j \to W \times_T W\}$ the morphisms $b \circ \mathrm{pr}_0$ and $b \circ \mathrm{pr}_1$ agree, and hence by Lemma 41.14.1 they agree. As X is a sheaf for the fppf topology we obtain a unique morphism $a : T \to X$ whose composition with $W \to T$ agrees with b. We omit the final verification that $a|_{T_i} = a_i$. □

Remark 41.14.4. The proof of Lemma 41.14.3 works for any algebraic space which has a Zariski covering $X = \bigcup X_i$ such that for each i there exists a surjective étale separated quasi-compact morphism $U_i \to X_i$ where U_i is a scheme. This condition is slightly stronger than the condition of being very reasonable, and the current proof does not work even for very reasonable spaces. There are results in the literature, see David Rydh's paper [**Ryd07b**] and its references, to remedy this. On the other hand, it seems that the question for general algebraic spaces as defined in the stacks project is still open. If this is no longer the case, please email stacks.project@gmail.com so we can update this remark.

41.15. The étale site of an algebraic space

In this section we define the small étale site of an algebraic space. This is the analogue of the small étale site $S_{étale}$ of a scheme. Lemma 41.13.1 implies that in the definition below any morphism between objects of the étale site of X is étale, and that any scheme étale over an object of $X_{étale}$ is also an object of $X_{étale}$.

Definition 41.15.1. Let S be a scheme. Let Sch_{fppf} be a big fppf site containing S, and let $Sch_{étale}$ be the corresponding big étale site (i.e., having the same underlying category). Let X be an algebraic space over S. The *small étale site* $X_{étale}$ of X is defined as follows:

(1) An object of $X_{étale}$ is a morphism $\varphi : U \to X$ where $U \in Ob((Sch/S)_{étale})$ is a scheme and φ is an étale morphism,
(2) a morphism $(\varphi : U \to X) \to (\varphi' : U' \to X)$ is given by a morphism of schemes $\chi : U \to U'$ such that $\varphi = \varphi' \circ \chi$, and
(3) a family of morphisms $\{(U_i \to X) \to (U \to X)\}_{i \in I}$ of $X_{étale}$ is a covering if and only if $\{U_i \to U\}_{i \in I}$ is a covering of $(Sch/S)_{étale}$.

A consequence of our choice is that the étale site of an algebraic space in general does not have a final object! On the other hand, if X happens to be a scheme, then the definition above agrees with Topologies, Definition 30.4.8.

There are several other choices we could have made here. For example we could have considered all *algebraic spaces* U which are étale over X, or we could have considered all *affine schemes* U which are étale over X. We decided not to do so, since we like to think of plain old schemes as the fundamental objects of algebraic geometry. On the other hand, we do need these notions also, since the small étale site of an algebraic space is not sufficiently flexible, especially when discussing functoriality of the small étale site, see Lemma 41.15.7 below.

Definition 41.15.2. Let S be a scheme. Let Sch_{fppf} be a big fppf site containing S, and let $Sch_{étale}$ be the corresponding big étale site (i.e., having the same underlying category). Let X be an algebraic space over S. The site $X_{spaces,étale}$ of X is defined as follows:

(1) An object of $X_{spaces,étale}$ is a morphism $\varphi : U \to X$ where U is an algebraic space over S and φ is an étale morphism of algebraic spaces over S,
(2) a morphism $(\varphi : U \to X) \to (\varphi' : U' \to X)$ of $X_{spaces,étale}$ is given by a morphism of algebraic spaces $\chi : U \to U'$ such that $\varphi = \varphi' \circ \chi$, and
(3) a family of morphisms $\{\varphi_i : (U_i \to X) \to (U \to X)\}_{i \in I}$ of $X_{spaces,étale}$ is a covering if and only if $|U| = \bigcup \varphi_i(|U_i|)$.

(As usual we choose a set of coverings of this type, including at least the coverings in $X_{étale}$, as in Sets, Lemma 3.11.1 to turn $X_{spaces,étale}$ into a site.)

Since the identity morphism of X is étale it is clear that $X_{spaces,étale}$ does have a final object. Let us show right away that the corresponding topos equals the small étale topos of X.

Lemma 41.15.3. *The functor*

$$X_{étale} \longrightarrow X_{spaces,étale}, \quad U/X \longmapsto U/X$$

is a special cocontinuous functor (Sites, Definition 9.25.2) and hence induces an equivalence of topoi $Sh(X_{étale}) \to Sh(X_{spaces,étale})$.

Proof. We have to show that the functor satisfies the assumptions (1) -- (5) of Sites, Lemma 9.25.1. It is clear that the functor is continuous and cocontinuous, which proves assumptions (1) and (2). Assumptions (3) and (4) hold simply because the functor is fully faithful. Assumption (5) holds, because an algebraic space by definition has a covering by a scheme. $\square$

Remark 41.15.4. Let us explain the meaning of Lemma 41.15.3. Let S be a scheme, and let X be an algebraic space over S. Let $\mathcal{F}$ be a sheaf on the small étale site $X_{étale}$ of X. The lemma says that there exists a unique sheaf $\mathcal{F}'$ on $X_{spaces,étale}$ which restricts back to $\mathcal{F}$ on the subcategory $X_{étale}$. If $U \to X$ is an étale morphism of algebraic spaces, then how do we compute $\mathcal{F}'(U)$? Well, by definition of an algebraic space there exists a scheme U'

and a surjective étale morphism $U' \to U$. Then $\{U' \to U\}$ is a covering in $X_{spaces,étale}$ and hence we get an equalizer diagram

$$\mathcal{F}'(U) \longrightarrow \mathcal{F}(U') \rightrightarrows \mathcal{F}(U' \times_U U').$$

Note that $U' \times_U U'$ is a scheme, and hence we may write $\mathcal{F}$ and not $\mathcal{F}'$. Thus we see how to compute $\mathcal{F}'$ when given the sheaf $\mathcal{F}$.

Lemma 41.15.5. *Let S be a scheme. Let X be an algebraic space over S. Let $X_{affine,étale}$ denote the full subcategory of $X_{étale}$ whose objects are those $U/X \in Ob(X_{étale})$ with U affine. A covering of $X_{affine,étale}$ will be a standard étale covering, see Topologies, Definition 30.4.5. Then restriction*

$$\mathcal{F} \longmapsto \mathcal{F}|_{X_{affine,étale}}$$

defines an equivalence of topoi $Sh(S_{étale}) \cong Sh(S_{affine,étale})$.

Proof. This you can show directly from the definitions, and is a good exercise. But it also follows immediately from Sites, Lemma 9.25.1 by checking that the inclusion functor $X_{affine,étale} \to X_{étale}$ is a special cocontinuous functor as in Sites, Definition 9.25.2. $\square$

Definition 41.15.6. Let S be a scheme. Let X be an algebraic space over S. The *étale topos* of X, or more precisely the *small étale topos* of X is the category $Sh(X_{étale})$ of sheaves of sets on $X_{étale}$.

By Lemma 41.15.3 we have $Sh(X_{étale}) = Sh(X_{spaces,étale})$, so we can also think of this as the category of sheaves of sets on $X_{spaces,étale}$. Similarly, by Lemma 41.15.5 we see that $Sh(X_{étale}) = Sh(X_{affine,étale})$. It turns out that the topos is functorial with respect to morphisms of algebraic spaces. Here is a precise statement.

Lemma 41.15.7. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.*

(1) *The continuous functor*

$$Y_{spaces,étale} \longrightarrow X_{spaces,étale}, \quad V \longmapsto X \times_Y V$$

induces a morphism of sites

$$f_{spaces,étale} : X_{spaces,étale} \to Y_{spaces,étale}.$$

(2) *The rule $f \mapsto f_{spaces,étale}$ is compatible with compositions, in other words $(f \circ g)_{spaces,étale} = f_{spaces,étale} \circ g_{spaces,étale}$ (see Sites, Definition 9.14.4).*
(3) *The morphism of topoi associated to $f_{spaces,étale}$ induces, via Lemma 41.15.3, a morphism of topoi $f_{small} : Sh(X_{étale}) \to Sh(Y_{étale})$ whose construction is compatible with compositions.*
(4) *If f is a representable morphism of algebraic spaces, then f_{small} comes from a morphism of sites $X_{étale} \to Y_{étale}$, corresponding to the continuous functor $V \mapsto X \times_Y V$.*

Proof. Let us show that the functor described in (1) satisfies the assumptions of Sites, Proposition 9.14.6. Thus we have to show that $Y_{spaces,étale}$ has a final object (namely Y) and that the functor transforms this into a final object in $X_{spaces,étale}$ (namely X). This is clear as $X \times_Y Y = X$ in any category. Next, we have to show that $Y_{spaces,étale}$ has fibre products. This is true since the category of algebraic spaces has fibre products, and since $V \times_Y V'$ is étale over Y if V and V' are étale over Y (see Lemmas 41.13.4 and 41.13.5 above). OK, so the proposition applies and we see that we get a morphism of sites as described in (1).

Part (2) you get by unwinding the definitions. Part (3) is clear by using the equivalences for X and Y from Lemma 41.15.3 above. Part (4) follows, because if f is representable, then the functors above fit into a commutative diagram

$$\begin{tikzcd} X_{étale} \arrow[r] & X_{spaces,étale} \\ Y_{étale} \arrow[r] \arrow[u] & Y_{spaces,étale} \arrow[u] \end{tikzcd}$$

of categories. $\square$

We can do a little bit better than the lemma above in describing the relationship between sheaves on X and sheaves on Y. Namely, we can formulate this in turns of f-maps, compare Sheaves, Definition 6.21.7, as follows.

Definition 41.15.8. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a sheaf of sets on $X_{étale}$ and let $\mathcal{G}$ be a sheaf of sets on $Y_{étale}$. An *f-map* $\varphi : \mathcal{G} \to \mathcal{F}$ is a collection of maps $\varphi_{(U,V,g)} : \mathcal{G}(V) \to \mathcal{F}(U)$ indexed by commutative diagrams

$$\begin{tikzcd} U \arrow[r] \arrow[d, "g"'] & X \arrow[d, "f"] \\ V \arrow[r] & Y \end{tikzcd}$$

where $U \in X_{étale}$, $V \in Y_{étale}$ such that whenever given an extended diagram

$$\begin{tikzcd} U' \arrow[r] \arrow[d, "g'"'] & U \arrow[r] \arrow[d, "g"'] & X \arrow[d, "f"] \\ V' \arrow[r] & V \arrow[r] & Y \end{tikzcd}$$

with $V' \to V$ and $U' \to U$ étale morphisms of schemes the diagram

$$\begin{tikzcd} \mathcal{G}(V) \arrow[r, "\varphi_{(U,V,g)}"'] \arrow[d, "\text{restriction of } \mathcal{G}"'] & \mathcal{F}(U) \arrow[d, "\text{restriction of } \mathcal{F}"] \\ \mathcal{G}(V') \arrow[r, "\varphi_{(U',V',g')}"] & \mathcal{F}(U') \end{tikzcd}$$

commutes.

Lemma 41.15.9. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a sheaf of sets on $X_{étale}$ and let $\mathcal{G}$ be a sheaf of sets on $Y_{étale}$. There are canonical bijections between the following three sets:*

(1) *The set of maps $\mathcal{G} \to f_{small,*}\mathcal{F}$.*
(2) *The set of maps $f_{small}^{-1}\mathcal{G} \to \mathcal{F}$.*
(3) *The set of f-maps $\varphi : \mathcal{G} \to \mathcal{F}$.*

Proof. Note that (1) and (2) are the same because the functors $f_{small,*}$ and f_{small}^{-1} are a pair of adjoint functors. Suppose that $\alpha : f_{small}^{-1}\mathcal{G} \to \mathcal{F}$ is a map of sheaves on $Y_{étale}$. Let a diagram

$$\begin{tikzcd} U \arrow[r, "j_U"'] \arrow[d, "g"'] & X \arrow[d, "f"] \\ V \arrow[r, "j_V"] & Y \end{tikzcd}$$

as in Definition 41.15.8 be given. By the commutativity of the diagram we also get a map $g_{small}^{-1}(j_V)^{-1}\mathcal{G} \to (j_U)^{-1}\mathcal{F}$ (compare Sites, Section 9.21 for the description of the localization functors). Hence we certainly get a map $\varphi_{(V,U,g)} : \mathcal{G}(V) = (j_V)^{-1}\mathcal{G}(V) \to (j_U)^{-1}\mathcal{F}(U) = \mathcal{F}(U)$. We omit the verification that this rule is compatible with further restrictions and defines an f-map from $\mathcal{G}$ to $\mathcal{F}$.

Conversely, suppose that we are given an f-map $\varphi = (\varphi_{(U,V,g)})$. Let $\mathcal{G}'$ (resp. $\mathcal{F}'$) denote the extension of $\mathcal{G}$ (resp. $\mathcal{F}$) to $Y_{spaces,étale}$ (resp. $X_{spaces,étale}$), see Lemma 41.15.3. Then we have to construct a map of sheaves

$$\mathcal{G}' \longrightarrow (f_{spaces,étale})_*\mathcal{F}'$$

To do this, let $V \to Y$ be an étale morphism of algebraic spaces. We have to construct a map of sets

$$\mathcal{G}'(V) \to \mathcal{F}'(X \times_Y V)$$

Choose an étale surjective morphism $V' \to V$ with V' a scheme, and after that choose an étale surjective morphsm $U' \to X \times_U V'$ with U' a scheme. We get a morphism of schemes $g' : U' \to V'$ and also a morphism of schemes

$$g'' : U' \times_{X \times_Y V} U' \longrightarrow V' \times_V V'$$

Consider the following diagram

$$\begin{tikzcd}
\mathcal{F}'(X \times_Y V) \arrow[r] & \mathcal{F}(U') \arrow[r, shift left] \arrow[r, shift right] & \mathcal{F}(U' \times_{X \times_Y V} U') \\
\mathcal{G}'(X \times_Y V) \arrow[r] \arrow[u, dotted] & \mathcal{G}(V') \arrow[r, shift left] \arrow[r, shift right] \arrow[u, "\varphi_{(U',V',g')}"'] & \mathcal{G}(V' \times_V V') \arrow[u, "\varphi_{(U'',V'',g'')}"']
\end{tikzcd}$$

The compatibility of the maps $\varphi_{\ldots}$ with restriction shows that the two right squares commute. The definition of coverings in $X_{spaces,étale}$ shows that the horizontal rows are equalizer diagrams. Hence we get the dotted arrow. We leave it to the reader to show that these arrows are compatible with the restriction mappings. □

If the morphism of algebraic spaces $X \to Y$ is étale, then the morphism of topoi $Sh(X_{étale}) \to Sh(Y_{étale})$ is a localization. Here is a statement.

Lemma 41.15.10. *Let S be a scheme, and let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is étale. In this case there is a functor*

$$j : X_{étale} \to Y_{étale}, \quad (\varphi : U \to X) \mapsto (f \circ \varphi : U \to Y)$$

which is cocontinuous. The morphism of topoi f_{small} is the morphism of topoi associated to j, see Sites, Lemma 9.19.1. Moreover, j is continuous as well, hence Sites, Lemma 9.19.5 applies. In particular $f_{small}^{-1}\mathcal{G}(U) = \mathcal{G}(jU)$ for all sheaves $\mathcal{G}$ on $Y_{étale}$.

Proof. Note that by our very definition of an étale morphism of algebraic spaces (Definition 41.13.2) it is indeed the case that the rule given defines a functor j as indicated. It is clear that j is cocontinuous and continuous, simply because a covering $\{U_i \to U\}$ of $j(\varphi : U \to X)$ in $Y_{étale}$ is the same thing as a covering of $(\varphi : U \to X)$ in $X_{étale}$. It remains to show that j induces the same morphism of topoi as f_{small}. To see this we consider the diagram

$$\begin{tikzcd}
X_{étale} \arrow[r] \arrow[d, "j"] & X_{spaces,étale} \arrow[d, bend right, "j_{spaces}"'] \\
Y_{étale} \arrow[r] & Y_{spaces,étale} \arrow[u, bend right, "v : V \mapsto X \times_Y V"']
\end{tikzcd}$$

of categories. Here the functor j_{spaces} is the obvious extension of j to the category $X_{spaces,étale}$. Thus the inner square is commutative. In fact j_{spaces} can be identified with the localization functor $j_X : Y_{spaces,étale}/X \to Y_{spaces,étale}$ discussed in Sites, Section 9.21. Hence, by Sites, Lemma 9.23.2 the cocontinuous functor j_{spaces} and the functor v of the diagram induce the same morphism of topoi. By Sites, Lemma 9.19.2 the commutativity of the inner square (consisting of cocontinuous functors between sites) gives a commutative diagram of associated morphisms of topoi. Hence, by the construction of f_{small} in Lemma 41.15.7 we win. $\square$

The lemma above says that the pullback of $\mathcal{G}$ via an étale morphism $f : X \to Y$ of algebraic spaces is simply the restriction of $\mathcal{G}$ to the category $X_{étale}$. We will often use the short hand

$$\mathcal{G}|_{X_{étale}} = f_{small}^{-1}\mathcal{G} \tag{41.15.10.1}$$

to indicate this. Note that the functor $j : X_{étale} \to Y_{étale}$ of the lemma in this situation is faithful, but not fully faithful in general. We will discuss this in a more technical fashion in Section 41.24.

Lemma 41.15.11. *Let S be a scheme. Let*

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

be a cartesian square of algebraic spaces over S. Let $\mathcal{F}$ be a sheaf on $X_{étale}$. If g is étale, then

(1) *$f'_{small,*}(\mathcal{F}|_{X'}) = (f_{small,*}\mathcal{F})|_{Y'}$ in $Sh(Y'_{étale})$[5], and*
(2) *if $\mathcal{F}$ is an abelian sheaf, then $R^i f'_{small,*}(\mathcal{F}|_{X'}) = (R^i f_{small,*}\mathcal{F})|_{Y'}$.*

Proof. Consider the following diagram of functors

$$\begin{tikzcd} X'_{spaces,étale} \arrow[r, "j"] & X_{spaces,étale} \\ Y'_{spaces,étale} \arrow[r, "j"] \arrow[u, "V' \mapsto V' \times_{Y'} X'"] & Y_{spaces,étale} \arrow[u, "V \mapsto V \times_Y X"'] \end{tikzcd}$$

The horizontal arrows are localizations and the vertical arrows induce morphisms of sites. Hence the last statement of Sites, Lemma 9.24.1 gives (1). To see (2) apply (1) to an injective resolution of $\mathcal{F}$ and use that restriction is exact and preserves injectives (see Cohomology on Sites, Lemma 19.8.1). $\square$

The following lemma says that you can think of a sheaf on the small étale site of an algebraic space as a compatible collection of sheaves on the small étale sites of schemes étale over the space. Please note that all the comparison mappings c_f in the lemma are isomorphisms, which is compatible with Topologies, Lemma 30.4.18 and the fact that all morphisms between objects of $X_{étale}$ are étale.

Lemma 41.15.12. *Let S be a scheme. Let X be an algebraic space over S. A sheaf $\mathcal{F}$ on $X_{étale}$ is given by the following data:*

(1) *for every $U \in Ob(X_{étale})$ a sheaf $\mathcal{F}_U$ on $U_{étale}$,*
(2) *for every $f : U' \to U$ in $X_{étale}$ an isomorphism $c_f : f_{small}^{-1}\mathcal{F}_U \to \mathcal{F}_{U'}$.*

[5] Also $(f')_{small}^{-1}(\mathcal{G}|_{Y'}) = (f_{small}^{-1}\mathcal{G})|_{X'}$ because of commutativity of the diagram and (41.15.10.1)

These data are subject to the condition that given any $f : U' \to U$ and $g : U'' \to U'$ in $X_{étale}$ the composition $g_{small}^{-1}c_f \circ c_g$ is equal to $c_{f\circ g}$.

Proof. Given a sheaf $\mathcal{F}$ on $X_{étale}$ and an object $\varphi : U \to X$ of $X_{étale}$ we set $\mathcal{F}_U = \varphi_{small}^{-1}\mathcal{F}$. If $\varphi' : U' \to X$ is a second object of $X_{étale}$, and $f : U' \to U$ is a morphism between them, then the isomorphism c_f comes from the fact that $f_{small}^{-1} \circ \varphi_{small}^{-1} = (\varphi')_{small}^{-1}$, see Lemma 41.15.7. The condition on the transitivity of the isomorphisms c_f follows from the functoriality of the small étale sites also; verification omitted.

Conversely, suppose we are given a collection of data $(\mathcal{F}_U, c_f)$ as in the lemma. In this case we simply define $\mathcal{F}$ by the rule $U \mapsto \mathcal{F}_U(U)$. Details omitted. □

Let S be a scheme. Let X be an algebraic space over S. Let $X = U/R$ be a presentation of X coming from any surjective étale morphism $\varphi : U \to X$, see Spaces, Definition 40.9.3. In particular, we obtain a groupoid (U, R, s, t, c, e, i) such that $j = (t, s) : R \to U \times_S U$, see Groupoids, Lemma 35.11.3.

Lemma 41.15.13. *With S, $\varphi : U \to X$, and (U, R, s, t, c, e, i) as above. For any sheaf $\mathcal{F}$ on $X_{étale}$ the sheaf[6] $\mathcal{G} = \varphi^{-1}\mathcal{F}$ comes equipped with a canonical isomorphism*

$$\alpha : t^{-1}\mathcal{G} \longrightarrow s^{-1}\mathcal{G}$$

such that the diagram

$$\begin{tikzcd}
 & pr_1^{-1}t^{-1}\mathcal{G} \arrow[r, "pr_1^{-1}\alpha"'] \arrow[dl, equal] & pr_1^{-1}s^{-1}\mathcal{G} \arrow[dr, equal] & \\
pr_0^{-1}s^{-1}\mathcal{G} & & & c^{-1}s^{-1}\mathcal{G} \\
 & pr_0^{-1}t^{-1}\mathcal{G} \arrow[ul, "pr_0^{-1}\alpha"] \arrow[r, equal] & c^{-1}t^{-1}\mathcal{G} \arrow[ur, "c^{-1}\alpha"'] &
\end{tikzcd}$$

is a commutative. The functor $\mathcal{F} \mapsto (\mathcal{G}, \alpha)$ defines an equivalence of categories between sheaves on $X_{étale}$ and pairs $(\mathcal{G}, \alpha)$ as above.

First proof of Lemma 41.15.13. Let $\mathcal{C} = X_{spaces,étale}$. By Lemma 41.15.10 and its proof we have $U_{spaces,étale} = \mathcal{C}/U$ and the pullback functor φ^{-1} is just the restriction functor. Moreover, $\{U \to X\}$ is a covering of the site $\mathcal{C}$ and $R = U \times_X U$. The isomorphism α is just the canonical identification

$$\left(\mathcal{F}|_{\mathcal{C}/U}\right)|_{\mathcal{C}/U\times_X U} = \left(\mathcal{F}|_{\mathcal{C}/U}\right)|_{\mathcal{C}/U\times_X U}$$

and the commutativity of the diagram is the cocylce condition for glueing data. Hence this lemma is a special case of glueing of sheaves, see Sites, Section 9.22. □

Second proof of Lemma 41.15.13. The existence of α comes from the fact that $\varphi \circ t = \varphi \circ s$ and that pullback is functorial in the morphism, see Lemma 41.15.7. In exacty the same way, i.e., by functoriality of pullback, we see that the isomorphism α fits into the commutative diagram. The construction $\mathcal{F} \mapsto (\varphi^{-1}\mathcal{F}, \alpha)$ is clearly functorial in the sheaf $\mathcal{F}$. Hence we obtain the functor.

Conversely, suppose that $(\mathcal{G}, \alpha)$ is a pair. Let $V \to X$ be an object of $X_{étale}$. In this case the morphism $V' = U \times_X V \to V$ is a surjective étale morphism of schemes, and hence $\{V' \to V\}$

[6]In this lemma and its proof we write simply φ^{-1} instead of φ_{small}^{-1} and similarly for all the other pullbacks.

is an étale covering of V. Set $\mathcal{G}' = (V' \to V)^{-1}\mathcal{G}$. Since $R = U \times_X U$ with $t = \mathrm{pr}_0$ and $s = \mathrm{pr}_0$ we see that $V' \times_V V' = R \times_X V$ with projection maps $s', t' : V' \times_V V' \to V'$ equal to the pullbacks of t and s. Hence α pulls back to an isomorphism $\alpha' : (t')^{-1}\mathcal{G}' \to (s')^{-1}\mathcal{G}'$. Having said this we simply define

$$\mathcal{F}(V) = \!=\!= \mathrm{Equalizer}(\mathcal{G}(V') \rightrightarrows \mathcal{G}(V' \times_V V')).$$

We omit the verification that this defines a sheaf. To see that $\mathcal{G}(V) = \mathcal{F}(V)$ if there exists a morphism $V \to U$ note that in this case the equalizer is $H^0(\{V' \to V\}, \mathcal{G}) = \mathcal{G}(V)$. □

41.16. Points of the small étale site

This section is the analogue of Étale Cohomology, Section 38.29.

Definition 41.16.1. Let S be a scheme. Let X be an algebraic space over S.

(1) A *geometric point* of X is a morphism $\overline{x} : Spec(k) \to X$, where k is an algebraically closed field. We often abuse notation and write $\overline{x} = Spec(k)$.
(2) For every geometric point $\overline{x}$ we have the corresponding ``image'' point $x \in |X|$. We say that $\overline{x}$ is a *geometric point lying over x*.

It turns out that we can take stalks of sheaves on $X_{étale}$ at geometric point exactly in the same way as was done in the case of the small étale site of a scheme. In order to do this we define the notion of an étale neighbourhood as follows.

Definition 41.16.2. Let S be a scheme. Let X be an algebraic space over S. Let $\overline{x}$ be a geometric point of X.

(1) An *étale neighborhood* of $\overline{x}$ of X is a commutative diagram

$$\begin{tikzcd} & U \arrow[d, "\varphi"] \\ \bar{x} \arrow[ur, "\bar{u}"] \arrow[r, "\bar{x}"] & X \end{tikzcd}$$

where φ is an étale morphism of algebraic spaces over S. We will use the notation $\varphi : (U, \overline{u}) \to (X, \overline{x})$ to indicate this situation.
(2) A *morphism of étale neighborhoods* $(U, \overline{u}) \to (U', \overline{u}')$ is an X-morphism $h : U \to U'$ such that $\overline{u}' = h \circ \overline{u}$.

Note that we allow U to be an algebraic space. When we take stalks of a sheaf on $X_{étale}$ we have to restrict to those U which are in $X_{étale}$, and so in this case we will only consider the case where U is a scheme. Alternately we can work with the site $X_{space,étale}$ and consider all étale neighbourhoods. And there won't be any difference because of the last assertion in the following lemma.

Lemma 41.16.3. *Let S be a scheme. Let X be an algebraic space over S. Let $\overline{x}$ be a geometric point of X. The category of étale neighborhoods is cofiltered. More precisely:*

(1) *Let $(U_i, \overline{u}_i)_{i=1,2}$ be two étale neighborhoods of $\overline{x}$ in X. Then there exists a third étale neighborhood $(U, \overline{u})$ and morphisms $(U, \overline{u}) \to (U_i, \overline{u}_i)$, $i = 1, 2$.*
(2) *Let $h_1, h_2 : (U, \overline{u}) \to (U', \overline{u}')$ be two morphisms between étale neighborhoods of $\overline{s}$. Then there exist an étale neighborhood $(U'', \overline{u}'')$ and a morphism $h : (U'', \overline{u}'') \to (U, \overline{u})$ which equalizes h_1 and h_2, i.e., such that $h_1 \circ h = h_2 \circ h$.*

Moreover, given any étale neighbourhood $(U, \overline{u}) \to (X, \overline{x})$ there exists a morphism of étale neighbourhoods $(U', \overline{u}') \to (U, \overline{u})$ where U' is a scheme.

Proof. For part (1), consider the fibre product $U = U_1 \times_X U_2$. It is étale over both U_1 and U_2 because étale morphisms are preserved under base change and composition, see Lemmas 41.13.5 and 41.13.4. The map $\overline{u} \to U$ defined by $(\overline{u}_1, \overline{u}_2)$ gives it the structure of an étale neighborhood mapping to both U_1 and U_2.

For part (2), define U'' as the fibre product

$$\begin{tikzcd} U'' \arrow[r] \arrow[d] & U \arrow[d, "{(h_1,h_2)}"] \\ U' \arrow[r, "\Delta"] & U' \times_X U'. \end{tikzcd}$$

Since $\overline{u}$ and $\overline{u}'$ agree over X with $\overline{x}$, we see that $\overline{u}'' = (\overline{u}, \overline{u}')$ is a geometric point of U''. In particular $U'' \neq \emptyset$. Moreover, since U' is étale over X, so is the fibre product $U' \times_X U'$ (as seen above in the case of $U_1 \times_X U_2$). Hence the vertical arrow (h_1, h_2) is étale by Lemma 41.13.6. Therefore U'' is étale over U' by base change, and hence also étale over X (because compositions of étale morphisms are étale). Thus $(U'', \overline{u}'')$ is a solution to the problem posed by (2).

To see the final assertion, choose any surjective étale morphism $U' \to U$ where U' is a scheme. Then $U' \times_U \overline{u}$ is a scheme surjective and étale over $\overline{u} = Spec(k)$ with k algebraically closed. It follows (see Morphisms, Lemma 24.35.7) that $U' \times_U \overline{u} \to \overline{u}$ has a section which gives us the desired $\overline{u}'$. ☐

Lemma 41.16.4. *Let S be a scheme. Let X be an algebraic space over S. Let $\overline{x} : Spec(k) \to X$ be a geometric point of X lying over $x \in |X|$. Let $\varphi : U \to X$ be an étale morphism of algebraic spaces and let $u \in |U|$ with $\varphi(u) = x$. Then there exists a geometric point $\overline{u} : Spec(k) \to U$ lying over u with $\overline{x} = f \circ \overline{u}$.*

Proof. Choose an affine scheme U' with $u' \in U'$ and an étale morphism $U' \to U$ which maps u' to u. If we can prove the lemma for $(U', u') \to (X, x)$ then the lemma follows. Hence we may assume that U is a scheme, in particular that $U \to X$ is representable. Then look at the cartesian diagram

$$\begin{tikzcd} Spec(k) \times_{\overline{x},X,\varphi} U \arrow[r, "\mathrm{pr}_2"'] \arrow[d, "\mathrm{pr}_1"'] & U \arrow[d, "\varphi"] \\ Spec(k) \arrow[r, "\overline{x}"] & X \end{tikzcd}$$

The projection pr_1 is the base change of an étale morphisms so it is étale, see Lemma 41.13.5. Therefore, the scheme $Spec(k) \times_{\overline{x},X,\varphi} U$ is a disjoint union of finite separable extensions of k, see Morphisms, Lemma 24.35.7. But k is algebraically closed, so all these extensions are trivial, so $Spec(k) \times_{\overline{x},X,\varphi} U$ is a disjoint union of copies of $Spec(k)$ and each of these corresponds to a geometric point $\overline{u}$ with $f \circ \overline{u} = \overline{x}$. By Lemma 41.4.3 the map

$$|Spec(k) \times_{\overline{x},X,\varphi} U| \longrightarrow |Spec(k)| \times_{|X|} |U|$$

is surjective, hence we can pick $\overline{u}$ to lie over u. ☐

Lemma 41.16.5. *Let S be a scheme. Let X be an algebraic space over S. Let $\overline{x}$ be a geometric point of X. Let $(U, \overline{u})$ an étale neighborhood of $\overline{x}$. Let $\{\varphi_i : U_i \to U\}_{i \in I}$ be an étale covering in $X_{spaces,étale}$. Then there exist $i \in I$ and $\overline{u}_i : \overline{x} \to U_i$ such that $\varphi_i : (U_i, \overline{u}_i) \to (U, \overline{u})$ is a morphism of étale neighborhoods.*

Proof. Let $u \in |U|$ be the image of $\overline{u}$. As $|U| = \bigcup_{i \in I} \varphi_i(|U_i|)$ there exists an i and a point $u_i \in U_i$ mapping to x. Apply Lemma 41.16.4 to $(U_i, u_i) \to (U, u)$ and $\overline{u}$ to get the desired geometric point. $\square$

Definition 41.16.6. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a presheaf on $X_{étale}$. Let $\overline{x}$ be a geometric point of X. The *stalk* of $\mathcal{F}$ at $\overline{x}$ is

$$\mathcal{F}_{\overline{x}} = colim_{(U,\overline{u})} \mathcal{F}(U)$$

where $(U, \overline{u})$ runs over all étale neighborhoods of $\overline{x}$ in X with $U \in Ob(X_{étale})$.

By Lemma 41.16.3, this colimit is over a filtered index category, namely the oppsite of the category of étale neighborhoods in $X_{étale}$. More precisely Lemma 41.16.3 says the opposite of the category of all étale neighbourhoods is filtered, and the full subcategory of those which are in $X_{étale}$ is a cofinal subcategory hence also filtered.

This means an element of $\mathcal{F}_{\overline{x}}$ can be thought of as a triple $(U, \overline{u}, \sigma)$ where $U \in Ob(X_{étale})$ and $\sigma \in \mathcal{F}(U)$. Two triples $(U, \overline{u}, \sigma)$, $(U', \overline{u}', \sigma')$ define the same element of the stalk if there exists a third étale neighbourhood $(U'', \overline{u}'')$, $U'' \in Ob(X_{étale})$ and morphisms of étale neighbourhoods $h : (U'', \overline{u}'') \to (U, \overline{u})$, $h' : (U'', \overline{u}'') \to (U', \overline{u}')$ such that $h^*\sigma = (h')^*\sigma'$ in $\mathcal{F}(U'')$. See Categories, Section 4.17.

This also implies that if $\mathcal{F}'$ is the sheaf on $X_{spaces,étale}$ corresponding to $\mathcal{F}$ on $X_{étale}$, then

$$\mathcal{F}_{\overline{x}} = colim_{(U,\overline{u})} \mathcal{F}'(U) \tag{41.16.6.1}$$

where now the colimit is over all the étale neighbourhoods of $\overline{x}$. We will often jump between the point of view of using $X_{étale}$ and $X_{spaces,étale}$ without further mention.

In particular this means that if $\mathcal{F}$ is a presheaf of abelian groups, rings, etc then $\mathcal{F}_{\overline{x}}$ is an abelian group, ring, etc simply by the usual way of defining the group structure on a directed colimit of abelian groups, rings, etc.

Lemma 41.16.7. *Let S be a scheme. Let X be an algebraic space over S. Let $\overline{x}$ be a geometric point of X. Consider the functor*

$$u : X_{étale} \longrightarrow Sets, \quad U \longmapsto |U_{\overline{x}}|$$

Then u defines a point p of the site $X_{étale}$ (Sites, Definition 9.28.2) and its associated stalk functor $\mathcal{F} \mapsto \mathcal{F}_p$ (Sites, Equation 9.28.1.1) is the functor $\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$ defined above.

Proof. In the proof of Lemma 41.16.5 we have seen that the scheme $U_{\overline{x}}$ is a disjoint union of schemes isomorphic to $\overline{x}$. Thus we can also think of $|U_{\overline{x}}|$ as the set of geometric points of U lying over $\overline{x}$, i.e., as the collection of morphisms $\overline{u} : \overline{x} \to U$ fitting into the diagram of Definition 41.16.1. From this it follows that $u(X)$ is a singleton, and that $u(U \times_V W) = u(U) \times_{u(V)} u(W)$ whenever $U \to V$ and $W \to V$ are morphisms in $X_{étale}$. And, given a covering $\{U_i \to U\}_{i \in I}$ in $X_{étale}$ we see that $\coprod u(U_i) \to u(U)$ is surjective by Lemma 41.16.5. Hence Sites, Proposition 9.29.2 applies, so p is a point of the site $X_{étale}$. Finally, the our functor $\mathcal{F} \mapsto \mathcal{F}_{\overline{s}}$ is given by exactly the same colimit as the functor $\mathcal{F} \mapsto \mathcal{F}_p$ associated to p in Sites, Equation 9.28.1.1 which proves the final assertion. $\square$

Lemma 41.16.8. *Let S be a scheme. Let X be an algebraic space over S. Let $\overline{x}$ be a geometric point of X.*

1. *The stalk functor $PAb(X_{étale}) \to Ab$, $\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$ is exact.*
2. *We have $(\mathcal{F}^\#)_{\overline{x}} = \mathcal{F}_{\overline{x}}$ for any presheaf of sets $\mathcal{F}$ on $X_{étale}$.*
3. *The functor $Ab(X_{étale}) \to Ab$, $\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$ is exact.*

(4) *Similarly the functors $PSh(X_{étale}) \to Sets$ and $Sh(X_{étale}) \to Sets$ given by the stalk functor $\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$ are exact (see Categories, Definition 4.21.1) and commute with arbitrary colimits.*

Proof. This result follows from the general material in Modules on Sites, Section 16.30. This is true because $\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$ comes from a point of the small étale site of X, see Lemma 41.16.7. See the proof of Étale Cohomology, Lemma 38.29.9 for a direct proof of some of these statements in the setting of the small étale site of a scheme. □

We will see below that the stalk functor $\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$ is really the pullback along the morphism $\overline{x}$. In that sense the following lemma is a generalization of the lemma above.

Lemma 41.16.9. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.*

(1) *The functor $f_{small}^{-1} : Ab(Y_{étale}) \to Ab(X_{étale})$ is exact.*
(2) *The functor $f_{small}^{-1} : Sh(Y_{étale}) \to Sh(X_{étale})$ is exact, i.e., it commutes with finite limits and colimits, see Categories, Definition 4.21.1.*
(3) *For any étale morphism $V \to Y$ of algebraic spaces we have $f_{small}^{-1} h_V = h_{X \times_Y V}$.*
(4) *Let $\overline{x} \to X$ be a geometric point. Let $\mathcal{G}$ be a sheaf on $Y_{étale}$. Then there is a canonical identification*
$$(f_{small}^{-1}\mathcal{G})_{\overline{x}} = \mathcal{G}_{\overline{y}}.$$
where $\overline{y} = f \circ \overline{x}$.

Proof. Recall that f_{small} is defined via $f_{spaces,small}$ in Lemma 41.15.7. Parts (1), (2) and (3) are general consequences of the fact that $f_{spaces,étale} : X_{spaces,étale} \to Y_{spaces,étale}$ is a morphism of sites, see Sites, Definition 9.14.1 for (2), Modules on Sites, Lemma 16.27.2 for (1), and Sites, Lemma 9.13.5 for (3).

Proof of (4). This statement is a special case of Sites, Lemma 9.30.1 via Lemma 41.16.7. We also provide a direct proof. Note that by Lemma 41.16.8. taking stalks commutes with sheafification. Let $\mathcal{G}'$ be the sheaf on $Y_{spaces,étale}$ whose restriction to $Y_{étale}$ is $\mathcal{G}$. Recall that $f_{spaces,étale}^{-1}\mathcal{G}'$ is the sheaf associated to the presheaf
$$U \longrightarrow colim_{U \to X \times_Y V} \mathcal{G}'(V),$$
see Sites, Sections 9.13 and 9.5. Thus we have
$$\begin{aligned}
(f_{spaces,étale}^{-1}\mathcal{G}')_{\overline{x}} &= colim_{(U,\overline{u})}\, f_{spaces,étale}^{-1}\mathcal{G}'(U) \\
&= colim_{(U,\overline{u})}\, colim_{a : U \to X \times_Y V} \mathcal{G}'(V) \\
&= colim_{(V,\overline{v})}\, \mathcal{G}'(V) \\
&= \mathcal{G}'_{\overline{y}}
\end{aligned}$$
in the third equality the pair $(U, \overline{u})$ and the map $a : U \to X \times_Y V$ corresponds to the pair $(V, a \circ \overline{u})$. Since the stalk of $\mathcal{G}'$ (resp. $f_{spaces,étale}^{-1}\mathcal{G}'$) agrees with the stalk of $\mathcal{G}$ (resp. $f_{small}^{-1}\mathcal{G}$), see Equation (41.16.6.1) the result follows. □

Remark 41.16.10. This remark is the analogue of Étale Cohomology, Remark 38.56.6. Let S be a scheme. Let X be an algebraic space over S. Let $\overline{x} : Spec(k) \to X$ be a geometric point of X. By Étale Cohomology, Theorem 38.56.3 the category of sheaves on

$Spec(k)_{étale}$ is equivalent to the category of sets (by taking a sheaf to its global sections). Hence it follows from Lemma 41.16.9 part (4) applied to the morphism $\overline{x}$ that the functor

$$Sh(X_{étale}) \longrightarrow Sets, \quad \mathcal{F} \longmapsto \mathcal{F}_{\overline{x}}$$

is isomorphic to the functor

$$Sh(X_{étale}) \longrightarrow Sh(Spec(k)_{étale}) = Sets, \quad \mathcal{F} \longmapsto \overline{x}^*\mathcal{F}$$

Hence we may view the stalk functors as pullback functors along geometric morphisms (and not just some abstract morphisms of topoi as in the result of Lemma 41.16.7).

Remark 41.16.11. Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. We claim that for any pair of geometric points $\overline{x}$ and $\overline{x}'$ lying over x the stalk functors are isomorphic. By definition of $|X|$ we can find a third geometric point $\overline{x}''$ so that there exists a commutative diagram

$$\begin{tikzcd} \overline{x}'' \arrow[r] \arrow[d] \arrow[dr, "\overline{x}''"] & \overline{x}' \arrow[d, "\overline{x}'"] \\ \overline{x} \arrow[r, "\overline{x}"] & X. \end{tikzcd}$$

Since the stalk functor $\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$ is given by pullback along the morphism $\overline{x}$ (and similarly for the others) we conclude by functoriality of pullbacks.

The following theorem says that the small étale site of an algebraic space has enough points.

Theorem 41.16.12. *Let S be a scheme. Let X be an algebraic space over S. A map $a : \mathcal{F} \to \mathcal{G}$ of sheaves of sets is injective (resp. surjective) if and only if the map on stalks $a_{\overline{x}} : \mathcal{F}_{\overline{x}} \to \mathcal{G}_{\overline{x}}$ is injective (resp. surjective) for all geometric points of X. A sequence of abelian sheaves on $X_{étale}$ is exact if and only if it is exact on all stalks at geometric points of S.*

Proof. We know the theorem is true if X is a scheme, see Étale Cohomology, Theorem 38.29.10. Choose a surjective étale morphism $f : U \to X$ where U is a scheme. Since $\{U \to X\}$ is a covering (in $X_{spaces,étale}$) we can check whether a map of sheaves is injective, or surjective by restricting to U. Now if $\overline{u} : Spec(k) \to U$ is a geometric point of U, then $(\mathcal{F}|_U)_{\overline{u}} = \mathcal{F}_{\overline{x}}$ where $\overline{x} = f \circ \overline{u}$. (This is clear from the colimits defining the stalks at $\overline{u}$ and $\overline{x}$, but it also follows from Lemma 41.16.9.) Hence the result for U implies the result for X and we win. $\square$

The following lemma should be skipped on a first reading.

Lemma 41.16.13. *Let S be a scheme. Let X be an algebraic space over S. Let $p : Sh(pt) \to Sh(X_{étale})$ be a point of the small étale topos of X. Then there exists a geometric point $\overline{x}$ of X such that the stalk functor $\mathcal{F} \mapsto \mathcal{F}_p$ is isomorphic to the stalk functor $\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$.*

Proof. By Sites, Lemma 9.28.7 there is a one to one correspondence between points of the site and points of the associated topos. Hence we may assume that p is given by a functor $u : X_{étale} \to Sets$ which defines a point of the site $X_{étale}$. Let $U \in Ob(X_{étale})$ be an object whose structure morphism $j : U \to X$ is surjective. Note that h_U is a sheaf which surjects onto the final sheaf. Since taking stalks is exact we see that $(h_U)_p = u(U)$ is not empty (use Sites, Lemma 9.28.3). Pick $x \in u(U)$. By Sites, Lemma 9.31.1 we obtain a point $q : Sh(pt) \to Sh(U_{étale})$ such that $p = j_{small} \circ q$, so that $\mathcal{F}_p = (\mathcal{F}|_U)_q$ functorially. By Étale Cohomology, Lemma 38.29.12 there is a geometric point $\overline{u}$ of U and a functorial isomorphism $\mathcal{G}_q = \mathcal{G}_{\overline{u}}$ for $\mathcal{G} \in Sh(U_{étale})$. Set $\overline{x} = j \circ \overline{u}$. Then we see that $\mathcal{F}_{\overline{x}} \cong (\mathcal{F}|_U)_{\overline{u}}$ functorially in $\mathcal{F}$ on $X_{étale}$ by Lemma 41.16.9 and we win. $\square$

41.17. Supports of abelian sheaves

First we talk about supports of local sections.

Lemma 41.17.1. *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a subsheaf of the final object of the étale topos of X (see Sites, Example 9.10.2). Then there exists a unique open $W \subset X$ such that $\mathcal{F} = h_W$.*

Proof. The condition means that $\mathcal{F}(U)$ is a singleton or empty for all $\varphi : U \to X$ in $Ob(X_{spaces,étale})$. In particular local sections always glue. If $\mathcal{F}(U) \neq \emptyset$, then $\mathcal{F}(\varphi(U)) \neq \emptyset$ because $\varphi(U) \subset X$ is an open subspace (Lemma 41.13.7) and $\{\varphi : U \to \varphi(U)\}$ is a covering in $X_{spaces,étale}$. Take $W = \bigcup_{\varphi : U \to S, \mathcal{F}(U) \neq \emptyset} \varphi(U)$ to conclude. □

Lemma 41.17.2. *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be an abelian sheaf on $X_{spaces,étale}$. Let $\sigma \in \mathcal{F}(U)$ be a local section. There exists an open subspace $W \subset U$ such that*

1. *$W \subset U$ is the largest open subspace of U such that $\sigma|_W = 0$,*
2. *for every $\varphi : V \to U$ in $X_{étale}$ we have*
$$\sigma|_V = 0 \Leftrightarrow \varphi(V) \subset W,$$
3. *for every geometric point $\overline{u}$ of U we have*
$$(U, \overline{u}, \sigma) = 0 \text{ in } \mathcal{F}_{\overline{s}} \Leftrightarrow \overline{u} \in W$$
where $\overline{s} = (U \to S) \circ \overline{u}$.

Proof. Since $\mathcal{F}$ is a sheaf in the étale topology the restriction of $\mathcal{F}$ to U_{Zar} is a sheaf on U in the Zariski topology. Hence there exists a Zariski open W having property (1), see Modules, Lemma 15.5.2. Let $\varphi : V \to U$ be an arrow of $X_{étale}$. Note that $\varphi(V) \subset U$ is an open subspace (Lemma 41.13.7) and that $\{V \to \varphi(V)\}$ is an étale covering. Hence if $\sigma|_V = 0$, then by the sheaf condition for $\mathcal{F}$ we see that $\sigma|_{\varphi(V)} = 0$. This proves (2). To prove (3) we have to show that if $(U, \overline{u}, \sigma)$ defines the zero element of $\mathcal{F}_{\overline{s}}$, then $\overline{u} \in W$. This is true because the assumption means there exists a morphism of étale neighbourhoods $(V, \overline{v}) \to (U, \overline{u})$ such that $\sigma|_V = 0$. Hence by (2) we see that $V \to U$ maps into W, and hence $\overline{u} \in W$. □

Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. Let $\mathcal{F}$ be a sheaf on $X_{étale}$. By Remark 41.16.11 the isomorphism class of the stalk of the sheaf $\mathcal{F}$ at a geometric points lying over x is well defined.

Definition 41.17.3. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$.

1. The *support of $\mathcal{F}$* is the set of points $x \in |X|$ such that $\mathcal{F}_{\overline{x}} \neq 0$ for any (some) geometric point $\overline{x}$ lying over x.
2. Let $\sigma \in \mathcal{F}(U)$ be a section. The *support of σ* is the closed subset $U \setminus W$, where $W \subset U$ is the largest open subset of U on which σ restricts to zero (see Lemma 41.17.2).

Lemma 41.17.4. *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. Let $U \in Ob(X_{étale})$ and $\sigma \in \mathcal{F}(U)$.*

1. *The support of σ is closed in $|X|$.*
2. *The support of $\sigma + \sigma'$ is contained in the union of the supports of $\sigma, \sigma' \in \mathcal{F}(X)$.*
3. *If $\varphi : \mathcal{F} \to \mathcal{G}$ is a map of abelian sheaves on $X_{étale}$, then the support of $\varphi(\sigma)$ is contained in the support of $\sigma \in \mathcal{F}(U)$.*

(4) *The support of $\mathcal{F}$ is the union of the images of the supports of all local sections of $\mathcal{F}$.*
(5) *If $\mathcal{F} \to \mathcal{G}$ is surjective then the support of $\mathcal{G}$ is a subset of the support of $\mathcal{F}$.*
(6) *If $\mathcal{F} \to \mathcal{G}$ is injective then the support of $\mathcal{F}$ is a subset of the support of $\mathcal{G}$.*

Proof. Part (1) holds by definition. Parts (2) and (3) hold because they holds for the restriction of $\mathcal{F}$ and $\mathcal{G}$ to U_{Zar}, see Modules, Lemma 15.5.2. Part (4) is a direct consequence of Lemma 41.17.2 part (3). Parts (5) and (6) follow from the other parts. □

Lemma 41.17.5. *The support of a sheaf of rings on the small étale site of an algebraic space is closed.*

Proof. This is true because (according to our conventions) a ring is 0 if and only if $1 = 0$, and hence the support of a sheaf of rings is the support of the unit section. □

41.18. The structure sheaf of an algebraic space

The structure sheaf of an algebraic space is the sheaf of rings of the following lemma.

Lemma 41.18.1. *Let S be a scheme. Let X be an algebraic space over S. The rule $U \mapsto \Gamma(U, \mathcal{O}_U)$ defines a sheaf of rings on $X_{étale}$.*

Proof. Immediate from the definition of a covering and Descent, Lemma 31.6.1. □

Definition 41.18.2. Let S be a scheme. Let X be an algebraic space over S. The *structure sheaf* of X is the sheaf of rings $\mathcal{O}_X$ on the small étale site $X_{étale}$ described in Lemma 41.18.1.

According to Lemma 41.15.12 the sheaf $\mathcal{O}_X$ corresponds to a system of étale sheaves $(\mathcal{O}_X)_U$ for U ranging through the objects of $X_{étale}$. It is clear from the proof of that lemma and our definition that we have simply $(\mathcal{O}_X)_U = \mathcal{O}_U$ where $\mathcal{O}_U$ is the structure sheaf of $U_{étale}$ as introduced in Descent, Definition 31.6.2. In particular, if X is a scheme we recover the sheaf $\mathcal{O}_X$ on the small étale site of X.

Via the equivalence $Sh(X_{étale}) = Sh(X_{spaces,étale})$ of Lemma 41.15.3 we may also think of $\mathcal{O}_X$ as a sheaf of rings on $X_{spaces,étale}$. It is explained in Remark 41.15.4 how to compute $\mathcal{O}_X(Y)$, and in particular $\mathcal{O}_X(X)$, when $Y \to X$ is an object of $X_{spaces,étale}$.

Lemma 41.18.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Then there is a canonical map $f^\sharp : f_{small}^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ such that*

$$(f_{small}, f^\sharp) : (X_{étale}, \mathcal{O}_X) \longrightarrow (Y_{étale}, \mathcal{O}_Y)$$

is a morphism of ringed topoi. Furthermore,
(1) *The construction $f \mapsto (f_{small}, f^\sharp)$ is compatible with compositions.*
(2) *If f is a morphism of schemes, then $f^\sharp$ is the map described in Descent, Remark 31.6.4.*

Proof. By Lemma 41.15.9 it suffices to give an f-map from $\mathcal{O}_Y$ to $\mathcal{O}_X$. In other words, for every commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, "g"'] & X \arrow[d, "f"] \\ V \arrow[r] & Y \end{tikzcd}$$

where $U \in X_{étale}$, $V \in Y_{étale}$ we have to give a map of rings $(f^\sharp)_{(U,V,g)} : \Gamma(V, \mathcal{O}_V) \to \Gamma(U, \mathcal{O}_U)$. Of course we just take $(f^\sharp)_{(U,V,g)} = g^\sharp$. It is clear that this is compatible with restriction mappings and hence indeed gives an f-map. We omit checking compatibility with compositions and agreement with the construction in Descent, Remark 31.6.4. $\square$

41.19. Stalks of the structure sheaf

This section is the analogue of Étale Cohomology, Section 41.19.

Lemma 41.19.1. *Let S be a scheme. Let X be an algebraic space over S. Let $\overline{x}$ be a geometric point of X. Let $(U, \overline{u})$ be an étale neighbourhood of $\overline{x}$ where U is a scheme. Then we have*

$$\mathcal{O}_{X,\overline{x}} = \mathcal{O}_{U,\overline{u}} = \mathcal{O}_{U,u}^{sh}$$

where the left hand side is the stalk of the structure sheaf of X, and the right hand side is the strict henselization of the local ring of U at the point u at which $\overline{u}$ is centered.

Proof. We know that the structure sheaf $\mathcal{O}_U$ on $U_{étale}$ is the restriction of the structure sheaf of X. Hence the first equality follows from Lemma 41.16.9 part (4). The second equality is explained in Étale Cohomology, Lemma 38.33.1. $\square$

Definition 41.19.2. Let S be a scheme. Let X be an algebraic space over S. Let $\overline{x}$ be a geometric point of X lying over the point $x \in |X|$.

(1) The *étale local ring of X at $\overline{x}$* is the stalk of the structure sheaf $\mathcal{O}_X$ on $X_{étale}$ at $\overline{x}$. Notation: $\mathcal{O}_{X,\overline{x}}$.
(2) The *strict henselization of X at $\overline{x}$* is the scheme $Spec(\mathcal{O}_{X,\overline{x}})$.

The isomorphism type of the strict henselization of X at $\overline{x}$ (as a scheme over X) depends only on the point $x \in |X|$ and not on the choice of the geometric point lying over x, see Remark 41.16.11.

Lemma 41.19.3. *Let S be a scheme. Let X be an algebraic space over S. The small étale site $X_{étale}$ endowed with its structure sheaf $\mathcal{O}_X$ is a locally ringed site, see Modules on Sites, Definition 16.34.4.*

Proof. This follows because the stalks $\mathcal{O}_{X,\overline{x}}$ are local, and because $S_{étale}$ has enough points, see Lemmas 41.19.1 and Theorem 41.16.12. See Modules on Sites, Lemma 16.34.2 and 16.34.3 for the fact that this implies the small étale site is locally ringed. $\square$

41.20. Dimension of local rings

It turns out the dimension of the local ring of an algebraic space is a well defined concept.

Lemma 41.20.1. *Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$ be a point. Let $d \in \{0, 1, 2, \ldots, \infty\}$. The following are equivalent*

(1) *for some scheme U and étale morphism $a : U \to X$ and point $u \in U$ with $a(u) = x$ we have $\dim(\mathcal{O}_{U,u}) = d$,*
(2) *for any scheme U, any étale morphism $a : U \to X$, and any point $u \in U$ with $a(u) = x$ we have $\dim(\mathcal{O}_{U,u}) = d$,*
(3) *$\dim(\mathcal{O}_{X,\overline{x}}) = d$ for some geometric point $\overline{x}$ lying over x, and*
(4) *$\dim(\mathcal{O}_{X,\overline{x}}) = d$ for any geometric point $\overline{x}$ lying over x.*

Proof. The equivalence of (1) and (2) follows from a combination of Lemma 41.7.4 and Descent, Lemma 31.17.3. The equivalence of (3) and (4) follows from the fact that the isomorphism type of $\mathcal{O}_{X,\overline{x}}$ only depends on $x \in |X|$, see Remark 41.16.11.

Using Lemma 41.19.1 the equivalence of (1)+(2) and (3)+(4) comes down to the following statement: Given any local ring R we have $\dim(R) = \dim(R^{sh})$. This is Étale Cohomology, Lemma 38.32.12. □

Definition 41.20.2. Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$ be a point. The *dimension of the local ring of X at x* is the element $d \in \{0, 1, 2, \ldots, \infty\}$ satisfying the equivalent conditions of Lemma 41.20.1.

41.21. Local irreducibility

A point on an algebraic space has a well defined étale local ring, which corresponds to the strict henselization of the local ring in the case of a scheme. In general it is impossible to read of from the étale local ring the irreducible components of the algebraic stack passing through the given point. Here is something we can do.

Lemma 41.21.1. *Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$ be a point. The following are equivalent*

1. *for any scheme U and étale morphism $a : U \to X$ and $u \in U$ with $a(u) = x$ the local ring $\mathcal{O}_{U,u}$ has a unique minimal prime,*
2. *for any scheme U and étale morphism $a : U \to X$ and $u \in U$ with $a(u) = x$ there is a unique irreducible component of U through u, and*
3. *$\mathcal{O}_{X,\overline{x}}$ has a unique minimal prime for any geometric point $\overline{x}$ lying over x.*

Proof. The equivalence of (1) and (2) follows from the fact that irreducible components of U passing through u are in $1-1$ correspondence with minimal primes of the local ring of U at u. Let $a : U \to X$ and $u \in U$ be as in (1). Then $\mathcal{O}_{U,u} \to \mathcal{O}_{X,\overline{x}}$ is flat in particular injective. Hence if $f, g \in \mathcal{O}_{U,u}$ are non-nilpotent elements such that $fg = 0$, then the same is true in $\mathcal{O}_{X,\overline{x}}$. Conversely, suppose that $f, g \in \mathcal{O}_{X,\overline{x}}$ are non-nilpotent such that $fg = 0$. Since $\mathcal{O}_{X,\overline{x}}$ is the filtered colimit of the rings $\mathcal{O}_{U,u}$ we see that f, g are the images of elements of $\mathcal{O}_{U,u}$ for some choice of $a : U \to X$. Hence we see that $\mathcal{O}_{U,u}$ doesn't have a unique minimal prime. In this way we see the equivalence of (1) and (3). □

Definition 41.21.2. Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. We say that X is *geometrically unibranch at x* if the equivalent conditions of Lemma 41.21.1 hold. We say that X is *geometrically unibranch* if X is geometrically unibranch at every $x \in |X|$.

To prove this is consistent with the definition of [**DG67**] for schemes we offer the following lemma (see [**Art66**, Lemma 2.2]).

Lemma 41.21.3. *Let A be a local ring. Let A^{sh} be a strict henselization of A. The following are equivalent*

1. *A^{sh} has a unique minimal prime, and*
2. *A has a unique minimal prime $\mathfrak{p}$ and the integral closure A' of $A/\mathfrak{p}$ in its fraction field is a local ring whose residue field is purely inseparable over the residue field of A.*

Proof. Denote $\mathfrak{m}$ the maximal ideal of the ring A. Denote κ, κ^{sh} the residue field of A, A^{sh}.

Assume (1). Let $\mathfrak{p}^{sh}$ be the unique minimal prime of A^{sh}. The flatness of $A \to A^{sh}$ implies that $\mathfrak{p} = A \cap \mathfrak{p}^{sh}$ is the unique minimal prime of A (by going down, see Algebra, Lemma 7.35.17). Also, since $A^{sh}/\mathfrak{p}A^{sh} = (A/\mathfrak{p})^{sh}$ (see Algebra, Lemma 7.139.22) is reduced by

Étale Cohomology, Lemma 38.32.9 we see that $\mathfrak{p}^{sh} = \mathfrak{p}A^{sh}$. Since $A \to A'$ is integral, every maximal ideal of A' lies over $\mathfrak{m}$ (by going up for integral ring maps, see Algebra, Lemma 7.32.20). If A' is not local, then we can find distinct maximal ideals $\mathfrak{m}_1$, $\mathfrak{m}_2$. Choosing elements $f_1, f_2 \in A'$ with $f_i \in \mathfrak{m}_i$, $f_i \not\in \mathfrak{m}_{3-i}$ we find a finite subalgebra $B = A[f_1, f_2] \subset A'$ with distinct maximal ideals $B \cap \mathfrak{m}_i$, $i = 1, 2$. If A' is local with maximal ideal $\mathfrak{m}'$, but $A/\mathfrak{m} \subset A'/\mathfrak{m}'$ is not purely inseparable, then we can find a $f \in A'$ whose image in $A'/\mathfrak{m}'$ generates finite, not purely inseparable extension of $A/\mathfrak{m}$ and we find a finite local subalgebra $B = A[f] \subset A'$ whose residue field is not a purely inseparable extension of $A/\mathfrak{m}$. Note that the inclusions

$$A/\mathfrak{p} \subset B \subset \kappa(\mathfrak{p})$$

give, on tensoring with the flat ring map $A \to A^{sh}$ the inclusions

$$A^{sh}/\mathfrak{p}^{sh} \subset B \otimes_A A^{sh} \subset \kappa(\mathfrak{p}) \otimes_A A^{sh} \subset \kappa(\mathfrak{p}^{sh})$$

the last inclusion because $\kappa(\mathfrak{p}) \otimes_A A^{sh} = \kappa(\mathfrak{p}) \otimes_{A/\mathfrak{p}} A^{sh}/\mathfrak{p}^{sh}$ is a localization of the domain $A^{sh}/\mathfrak{p}^{sh}$. Note that $B \otimes_A \kappa^{sh}$ has at least two maximal ideals because $B/\mathfrak{m}B$ either has two maximal ideals or one whose residue field is not purely inseparable over κ, and because κ^{sh} is separably algebraically closed. Hence, as A^{sh} is strictly henselian we see that $B \otimes_A A^{sh}$ is a product of ≥ 2 local rings, see Algebra, Lemma 7.139.7. But we've just seen that $B \otimes_A A^{sh}$ is a subring of a domain and we get a contradiction.

Assume (2). Let $A \to B$ be a local map of local rings which is a localization of an étale A-algebra. In particular $\mathfrak{m}_B$ is the unique prime containing $\mathfrak{m}_A B$. Then $B' = A' \otimes_A B$ is integral over B and the assumption that $A \to A'$ is local with purely inseparable residue field extension implies that B' is local. On the other hand, $A' \to B'$ is the localization of an étale ring map, hence B' is normal, see Algebra, Lemma 7.145.7. Thus B' is a (local) normal domain. Finally, we have

$$B/\mathfrak{p}B \subset B \otimes_A \kappa(\mathfrak{p}) = B' \otimes_{A'} f.f.(A') \subset f.f.(B')$$

Hence $B/\mathfrak{p}B$ is a domain, which implies that B has a unique minimal prime (since by flatness of $A \to B$ these all have to lie over $\mathfrak{p}$). Hence, by Lemma 41.21.1 we see that A^{sh} has a unique minimal prime. □

41.22. Regular algebraic spaces

We have already defined regular algebraic spaces in Section 41.7.

Lemma 41.22.1. *Let S be a scheme. Let X be a locally Noetherian algebraic space over S. The following are equivalent*

(1) *X is regular, and*
(2) *every étale local ring $\mathcal{O}_{X,\overline{x}}$ is regular.*

Proof. Let U be a scheme and let $U \to X$ be a surjective étale morphism. By assumption U is locally Noetherian. Moreover, every étale local ring $\mathcal{O}_{X,\overline{x}}$ is the strict henselization of a local ring on U and conversely, see Lemma 41.19.1. Thus by Étale Cohomology, Lemma 38.32.15 we see that (2) is equivalent to every local ring of U being regular, i.e., U being a regular scheme (see Properties, Lemma 23.9.2). This equivalent to (1) by Definition 41.7.2. □

41.23. Sheaves of modules on algebraic spaces

If X is an algebraic space, then a sheaf of modules on X is a sheaf of $\mathcal{O}_X$-modules on the small étale site of X where $\mathcal{O}_X$ is the structure sheaf of X. The category of sheaves of modules is denoted $Mod(\mathcal{O}_X)$.

Given a morphism $f : X \to Y$ of algebraic spaces, by Lemma 41.18.3 we get a morphism of ringed topoi and hence by Modules on Sites, Definition 16.13.1 we get well defined pullback and direct image functors

$$f^* : Mod(\mathcal{O}_Y) \longrightarrow Mod(\mathcal{O}_X), \quad f_* : Mod(\mathcal{O}_X) \longrightarrow Mod(\mathcal{O}_Y) \tag{41.23.0.1}$$

which are adjoint in the usual way. If $g : Y \to Z$ is another morphism of algebraic spaces over S, then we have $(g \circ f)^* = f^* \circ g^*$ and $(g \circ f)_* = g_* \circ f_*$ simply because the morphisms of ringed topoi compose in the corresponding way (by the lemma).

Lemma 41.23.1. *Let S be a scheme. Let $f : X \to Y$ be an étale morphism of algebraic spaces over S. Then $f^{-1}\mathcal{O}_Y = \mathcal{O}_X$, and $f^*\mathcal{G} = f_{small}^{-1}\mathcal{G}$ for any sheaf of $\mathcal{O}_Y$-modules $\mathcal{G}$. In particular, $f^* : Mod(\mathcal{O}_X) \to Mod(\mathcal{O}_Y)$ is exact.*

Proof. By the description of inverse image in Lemma 41.15.10 and the definition of the structure sheaves it is clear that $f_{small}^{-1}\mathcal{O}_Y = \mathcal{O}_X$. Since the pullback

$$f^*\mathcal{G} = f_{small}^{-1}\mathcal{G} \otimes_{f_{small}^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

by definition we conclude that $f^*\mathcal{G} = f_{small}^{-1}\mathcal{G}$. The exactness is clear because f_{small}^{-1} is exact, as f_{small} is a morphism of topoi. $\square$

We continue our abuse of notation introduced in Equation (41.15.10.1) by writing

$$\mathcal{G}|_{X_{\acute{e}tale}} = f^*\mathcal{G} = f_{small}^{-1}\mathcal{G} \tag{41.23.1.1}$$

in the situation of the lemma above. We will discuss this in a more technical fashion in Section 41.24.

Lemma 41.23.2. *Let S be a scheme. Let*

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

be a cartesian square of algebraic spaces over S. Let $\mathcal{F} \in Mod(\mathcal{O}_X)$. If g is étale, then $f'_*(\mathcal{F}|_{X'}) = (f_*\mathcal{F})|_{Y'}$ and $R^i f'_*(\mathcal{F}|_{X'}) = (R^i f_*\mathcal{F})|_{Y'}$ in $Mod(\mathcal{O}_{Y'})$.

Proof. This is a reformulation of Lemma 41.15.11 in the case of modules. $\square$

Lemma 41.23.3. *Let S be a scheme. Let X be an algebraic space over S. A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is given by the following data:*

1. *for every $U \in Ob(X_{\acute{e}tale})$ a sheaf $\mathcal{F}_U$ of $\mathcal{O}_U$-modules on $U_{\acute{e}tale}$,*
2. *for every $f : U' \to U$ in $X_{\acute{e}tale}$ an isomorphism $c_f : f_{small}^*\mathcal{F}_U \to \mathcal{F}_{U'}$.*

These data are subject to the condition that given any $f : U' \to U$ and $g : U'' \to U'$ in $X_{\acute{e}tale}$ the composition $g_{small}^{-1}c_f \circ c_g$ is equal to $c_{f \circ g}$.

Proof. Combine Lemmas 41.23.1 and 41.15.12, and use the fact that any morphism between objects of $X_{\acute{e}tale}$ is an étale morphism of schemes. $\square$

41.24. Étale localization

Reading this section should be avoided at all cost.

Let $X \to Y$ be an étale morphism of algebraic spaces. Then X is an object of $Y_{spaces,étale}$ and it is immediate from the definitions, see also the proof of Lemma 41.15.10, that

$$X_{spaces,étale} = Y_{spaces,étale}/X \tag{41.24.0.1}$$

where the right hand side is the localization of the site $Y_{spaces,étale}$ at the object X, see Sites, Definition 9.21.1. Moreover, this identification is compatible with the structure sheaves by Lemma 41.23.1. Hence the ringed site $(X_{spaces,étale}, \mathcal{O}_X)$ is identified with the localization of the ringed site $(Y_{spaces,étale}, \mathcal{O}_Y)$ at the object X:

$$(X_{spaces,étale}, \mathcal{O}_X) = (Y_{spaces,étale}/X, \mathcal{O}_Y|_{Y_{spaces,étale}/X}) \tag{41.24.0.2}$$

The localization of a ringed site used on the right hand side is defined in Modules on Sites, Definition 16.19.1.

Assume now $X \to Y$ is an étale morphism of algebraic spaces and X is a scheme. Then X is an object of $Y_{étale}$ and it follows that

$$X_{étale} = Y_{étale}/X \tag{41.24.0.3}$$

and

$$(X_{étale}, \mathcal{O}_X) = (Y_{étale}/X, \mathcal{O}_Y|_{Y_{étale}/X}) \tag{41.24.0.4}$$

as above.

Finally, if $X \to Y$ is an étale morphism of algebraic spaces and X is an affine scheme, then X is an object of $Y_{affine,étale}$ and

$$X_{affine,étale} = Y_{affine,étale}/X \tag{41.24.0.5}$$

and

$$(X_{affine,étale}, \mathcal{O}_X) = (Y_{affine,étale}/X, \mathcal{O}_Y|_{Y_{affine,étale}/X}) \tag{41.24.0.6}$$

as above.

Next, we show that these localizations are compatible with morphisms.

Lemma 41.24.1. *Let S be a scheme. Let*

$$\begin{tikzcd} U \arrow[r, "g"] \arrow[d, "p"'] & V \arrow[d, "q"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

be a commutative diagram of algebraic spaces over S with p and q étale. Via the identifications (41.24.0.2) for $U \to X$ and $V \to Y$ the morphism of ringed topoi

$$(g_{spaces,étale}, g^\sharp) : (Sh(U_{spaces,étale}), \mathcal{O}_U) \longrightarrow (Sh(V_{spaces,étale}), \mathcal{O}_V)$$

is 2-isomorphic to the morphism $(f_{spaces,étale,c}, f_c^\sharp)$ constructed in Modules on Sites, Lemma 16.20.2 starting with the morphism of ringed sites $(f_{spaces,étale}, f^\sharp)$ and the map $c : U \to V \times_Y X$ corresponding to g.

Proof. The morphism $(f_{spaces,étale,c}, f_c^\sharp)$ is defined as a composition $f' \circ j$ of a localization and a base change map. Similarly g is a composition $U \to V \times_Y X \to V$. Hence it suffices to prove the lemma in the following two cases: (1) $f = \text{id}$, and (2) $U = X \times_Y V$. In case (1) the morphism $g : U \to V$ is étale, see Lemma 41.13.6. Hence $(g_{spaces,étale}, g^\sharp)$ is a localization morphism by the discussion surrounding Equations (41.24.0.1) and (41.24.0.2) which is exactly the content of the lemma in this case. In case (2) the morphism $g_{spaces,étale}$ comes from the morphism of ringed sites given by the functor $V_{spaces,étale} \to U_{spaces,étale}$, $V'/V \mapsto V' \times_V U/U$ which is also what the morphism f' is defined by, see Sites, Lemma 9.24.1. We omit the verification that $(f')^\sharp = g^\sharp$ in this case (both are the restriction of $f^\sharp$ to $U_{spaces,étale}$). $\square$

Lemma 41.24.2. *Same notation and assumptions as in Lemma 41.24.1 except that we also assume U and V are schemes. Via the identifications (41.24.0.4) for $U \to X$ and $V \to Y$ the morphism of ringed topoi*

$$(g_{small}, g^\sharp) : (Sh(U_{étale}), \mathcal{O}_U) \longrightarrow (Sh(V_{étale}), \mathcal{O}_V)$$

is 2-isomorphic to the morphism $(f_{small,s}, f_s^\sharp)$ constructed in Modules on Sites, Lemma 16.22.3 starting with $(f_{small}, f^\sharp)$ and the map $s : h_U \to f_{small}^{-1} h_V$ corresponding to g.

Proof. Note that $(g_{small}, g^\sharp)$ is 2-isomorphic as a morphism of ringed topoi to the morphism of ringed topoi associated to the morphism of ringed sites $(g_{spaces,étale}, g^\sharp)$. Hence we conclude by Lemma 41.24.1 and Modules on Sites, Lemma 16.22.4. $\square$

41.25. Recovering morphisms

In this section we prove that the rule which associates to an algebraic space its locally ringed small étale topos is fully faithful in a suitable sense, see Theorem 41.25.4.

Lemma 41.25.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The morphism of ringed topoi $(f_{small}, f^\sharp)$ associated to f is a morphism of locally ringed topoi, see Modules on Sites, Definition 16.34.8.*

Proof. Note that the assertion makes sense since we have seen that $(X_{étale}, \mathcal{O}_{X_{étale}})$ and $(Y_{étale}, \mathcal{O}_{Y_{étale}})$ are locally ringed sites, see Lemma 41.19.3. Moreover, we know that $X_{étale}$ has enough points, see Theorem 41.16.12. Hence it suffices to prove that $(f_{small}, f^\sharp)$ satisfies condition (3) of Modules on Sites, Lemma 16.34.7. To see this take a point p of $X_{étale}$. By Lemma 41.16.13 p corresponds to a geometric point $\overline{x}$ of X. By Lemma 41.16.9 the point $q = f_{small} \circ p$ corresponds to the geometric point $\overline{y} = f \circ \overline{x}$ of Y. Hence the assertion we have to prove is that the induced map of étale local rings

$$\mathcal{O}_{Y,\overline{y}} \longrightarrow \mathcal{O}_{X,\overline{x}}$$

is a local ring map. You can prove this directly, but instead we deduce it from the corresponding result for schemes. To do this choose a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\psi"'] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U and V are schemes, and the vertical arrows are surjective étale (see Spaces, Lemma 40.11.4). Choose a lift $\overline{u} : \overline{x} \to U$ (possible by Lemma 41.16.5). Set $\overline{v} = \psi \circ \overline{u}$. We obtain a commutative diagram of étale local rings

$$\begin{tikzcd} \mathcal{O}_{U,\bar{u}} & \mathcal{O}_{V,\bar{v}} \arrow[l] \\ \mathcal{O}_{X,\bar{x}} \arrow[u] & \mathcal{O}_{Y,\bar{y}}. \arrow[l] \arrow[u] \end{tikzcd}$$

By Étale Cohomology, Lemma 38.40.1 the top horizontal arrow is a local ring map. Finally by Lemma 41.19.1 the vertical arrows are isomorphisms. Hence we win. $\square$

Lemma 41.25.2. *Let S be a scheme. Let X, Y be algebraic spaces over S. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let t be a 2-morphism from $(f_{small}, f^\sharp)$ to itself, see Modules on Sites, Definition 16.8.1. Then $t = id$.*

Proof. Let X', resp. Y' be X viewed as an algebraic space over $Spec(\mathbf{Z})$, see Spaces, Definition 40.16.2. It is clear from the construction that $(X_{small}, \mathcal{O})$ is equal to $(X'_{small}, \mathcal{O})$ and similarly for Y. Hence we may work with X' and Y'. In other words we may assume that $S = Spec(\mathbf{Z})$.

Assume $S = Spec(\mathbf{Z})$, $f : X \to Y$ and t are as in the lemma. This means that $t : f_{small}^{-1} \to f_{small}^{-1}$ is a transformation of functors such that the diagram

$$\begin{tikzcd} f_{small}^{-1}\mathcal{O}_Y \arrow[dr, "f^\sharp"'] & & f_{small}^{-1}\mathcal{O}_Y \arrow[ll, "t"'] \arrow[dl, "f^\sharp"] \\ & \mathcal{O}_X & \end{tikzcd}$$

is commutative. Suppose $V \to Y$ is étale with V affine. Write $V = Spec(B)$. Choose generators $b_j \in B$, $j \in J$ for B as a $\mathbf{Z}$-algebra. Set $T = Spec(\mathbf{Z}[\{x_j\}_{j\in J}])$. In the following we will use that $Mor_{Sch}(U, T) = \prod_{j\in J} \Gamma(U, \mathcal{O}_U)$ for any scheme U without further mention. The surjective ring map $\mathbf{Z}[x_j] \to B$, $x_j \mapsto b_j$ corresponds to a closed immersion $V \to T$. We obtain a monomorphism

$$i : V \longrightarrow T_Y = T \times Y$$

of algebraic spaces over Y. In terms of sheaves on $Y_{étale}$ the morphism i induces an injection $h_i : h_V \to \prod_{j\in J} \mathcal{O}_Y$ of sheaves. The base change $i' : X \times_Y V \to T_X$ of i to X is a monomorphism too (Spaces, Lemma 40.5.5). Hence $i' : X \times_Y V \to T_X$ is a monomorphism, which in turn means that $h_{i'} : h_{X\times_Y V} \to \prod_{j\in J} \mathcal{O}_X$ is an injection of sheaves. Via the identification $f_{small}^{-1} h_V = h_{X\times_Y V}$ of Lemma 41.16.9 the map $h_{i'}$ is equal to

$$f_{small}^{-1} h_V \xrightarrow{f^{-1}h_i} \prod\nolimits_{j\in J} f_{small}^{-1}\mathcal{O}_Y \xrightarrow{\prod f^\sharp} \prod\nolimits_{j\in J} \mathcal{O}_X$$

(verification omitted). This means that the map $t : f_{small}^{-1} h_V \to f_{small}^{-1} h_V$ fits into the commutative diagram

$$\begin{tikzcd} f_{small}^{-1} h_V \arrow[r, "f^{-1}h_i"] \arrow[d, "t"] & \prod_{j\in J} f_{small}^{-1}\mathcal{O}_Y \arrow[r, "\prod f^\sharp"] \arrow[d, "\prod t"] & \prod_{j\in J} \mathcal{O}_X \arrow[d, "\mathrm{id}"] \\ f_{small}^{-1} h_V \arrow[r, "f^{-1}h_i"] & \prod_{j\in J} f_{small}^{-1}\mathcal{O}_Y \arrow[r, "\prod f^\sharp"] & \prod_{j\in J} \mathcal{O}_X \end{tikzcd}$$

The commutativity of the right square holds by our assumption on t explained above. Since the composition of the horizontal arrows is injective by the discussion above we conclude that the left vertical arrow is the identity map as well. Any sheaf of sets on $Y_{\acute{e}tale}$ admits a surjection from a (huge) coproduct of sheaves of the form h_V with V affine (combine Lemma 41.15.5 with Sites, Lemma 9.12.4). Thus we conclude that $t : f_{small}^{-1} \to f_{small}^{-1}$ is the identity transformation as desired. $\square$

Lemma 41.25.3. *Let S be a scheme. Let X, Y be algebraic spaces over S. Any two morphisms $a, b : X \to Y$ of algebraic spaces over S for which there exists a 2-isomorphism $(a_{small}, a^\sharp) \cong (b_{small}, b^\sharp)$ in the 2-category of ringed topoi are equal.*

Proof. Let $t : a_{small}^{-1} \to b_{small}^{-1}$ be the 2-isomorphism. We may equivalently think of t as a transformation $t : a_{spaces,\acute{e}tale}^{-1} \to b_{spaces,\acute{e}tale}^{-1}$ since there is not difference between sheaves on $X_{\acute{e}tale}$ and sheaves on $X_{spaces,\acute{e}tale}$. Choose a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\alpha"'] \arrow[d, "p"'] & V \arrow[d, "q"] \\ X \arrow[r, "a"] & Y \end{tikzcd}$$

where U and V are schemes, and p and q are surjective étale. Consider the diagram

$$\begin{tikzcd} h_U \arrow[r, "\alpha"'] \arrow[d, equal] & a_{spaces,\acute{e}tale}^{-1} h_V \arrow[d, "t"] \\ h_U \arrow[r, dotted] & b_{spaces,\acute{e}tale}^{-1} h_V \end{tikzcd}$$

Since the sheaf $b_{spaces,\acute{e}tale}^{-1} h_V$ is isomorphic to $h_{V \times_{Y,b} X}$ we see that the dotted arrow comes from a morphism of schemes $\beta : U \to V$ fitting into a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\beta"'] \arrow[d, "p"'] & V \arrow[d, "q"] \\ X \arrow[r, "b"] & Y \end{tikzcd}$$

We claim that there exists a sequence of 2-isomorphisms

$$\begin{aligned} (\alpha_{small}, \alpha^\sharp) & \cong (\alpha_{spaces,\acute{e}tale}, \alpha^\sharp) \\ & \cong (a_{spaces,\acute{e}tale,c}, a_c^\sharp) \\ & \cong (b_{spaces,\acute{e}tale,d}, b_d^\sharp) \\ & \cong (\beta_{spaces,\acute{e}tale}, \beta^\sharp) \\ & \cong (\beta_{small}, \beta^\sharp) \end{aligned}$$

The first and the last 2-isomorphisms come from the identifications between sheaves on $U_{spaces,\acute{e}tale}$ and sheaves on $U_{\acute{e}tale}$ and similarly for V. The second and fourth 2-isomorphisms are those of Lemma 41.24.1 with $c : U \to X \times_{a,Y} V$ induced by α and $d : U \to X \times_{b,Y} V$ induced by β. The middle 2-isomorphism comes from the transformation t. Namely, the functor $a_{spaces,\acute{e}tale,c}^{-1}$ corresponds to the functor

$$(\mathcal{H} \to h_V) \longmapsto (a_{spaces,\acute{e}tale}^{-1}\mathcal{H} \times_{a_{spaces,\acute{e}tale}^{-1} h_V, \alpha} h_U \to h_U)$$

and similarly for $b^{-1}_{spaces,étale,d}$, see Sites, Lemma 9.24.3. This uses the identification of sheaves on $Y_{spaces,étale}/V$ as arrows $(\mathcal{H} \to h_V)$ in $Sh(Y_{spaces,étale})$ and similarly for U/X, see Sites, Lemma 9.21.4. Via this identification the structure sheaf $\mathcal{O}_V$ corresponds to the pair $(\mathcal{O}_Y \times h_V \to h_V)$ and similarly for $\mathcal{O}_U$, see Modules on Sites, Lemma 16.21.3. Since t switches α and β we see that t induces an isomorphism

$$t : a^{-1}_{spaces,étale}\mathcal{H} \times_{a^{-1}_{spaces,étale}h_V,\alpha} h_U \longrightarrow b^{-1}_{spaces,étale}\mathcal{H} \times_{b^{-1}_{spaces,étale}h_V,\beta} h_U$$

over h_U functorially in $(\mathcal{H} \to h_V)$. Also, t is compatible with $a^\sharp_c$ and $b^\sharp_d$ as t is compatible with $a^\sharp$ and $b^\sharp$ by our description of the structure sheaves $\mathcal{O}_U$ and $\mathcal{O}_V$ above. Hence, the morphisms of ringed topoi $(\alpha_{small}, \alpha^\sharp)$ and $(\beta_{small}, \beta^\sharp)$ are 2-isomorphic. By Étale Cohomology, Lemma 38.40.3 we conclude $\alpha = \beta$! Since $p : U \to X$ is a surjection of sheaves it follows that $a = b$. □

Here is the main result of this section.

Theorem 41.25.4. *Let X, Y be algebraic spaces over $Spec(\mathbf{Z})$. Let*

$$(g, g^\sharp) : (Sh(X_{étale}), \mathcal{O}_X) \longrightarrow (Sh(Y_{étale}), \mathcal{O}_Y)$$

be a morphism of locally ringed topoi. Then there exists a unique morphism of algebraic spaces $f : X \to Y$ such that $(g, g^\sharp)$ is isomorphic to $(f_{small}, f^\sharp)$. In other words, the construction

$$Spaces/Spec(\mathbf{Z}) \longrightarrow Locally\ ringed\ topoi, \quad X \longrightarrow (X_{étale}, \mathcal{O}_X)$$

is fully faithful (morphisms up to 2-isomorphisms on the right hand side).

Proof. The uniqueness we have seen in Lemma 41.25.3. Thus it suffices to prove existence. In this proof we will freely use the identifications of Equation (41.24.0.4) as well as the result of Lemma 41.24.2.

Let $U \in Ob(X_{étale})$, let $V \in Ob(Y_{étale})$ and let $s \in g^{-1}h_V(U)$ be a section. We may think of s as a map of sheaves $s : h_U \to g^{-1}h_V$. By Modules on Sites, Lemma 16.22.3 we obtain a commutative diagram of morphisms of ringed topoi

$$\begin{tikzcd}
(Sh(X_{étale}/U), \mathcal{O}_U) \arrow[r, "{(j,j^\sharp)}"'] \arrow[d, "{(g_s,g_s^\sharp)}"'] & (Sh(X_{étale}), \mathcal{O}_X) \arrow[d, "{(g,g^\sharp)}"] \\
(Sh(V_{étale}), \mathcal{O}_V) \arrow[r] & (Sh(Y_{étale}), \mathcal{O}_Y).
\end{tikzcd}$$

By Étale Cohomology, Theorem 38.40.5 we obtain a unique morphism of schemes $f_s : U \to V$ such that $(g_s, g_s^\sharp)$ is 2-isomorphic to $(f_{s,small}, f_s^\sharp)$. The construction $(U, V, s) \rightsquigarrow f_s$ just explained satisfies the following functoriality property: Suppose given morphisms $a : U' \to U$ in $X_{étale}$ and $b : V' \to V$ in $Y_{étale}$ and a map $s' : h_{U'} \to g^{-1}h_{V'}$ such that the diagram

$$\begin{tikzcd}
h_{U'} \arrow[r, "s'"'] \arrow[d, "a"'] & g^{-1}h_{V'} \arrow[d, "g^{-1}b"] \\
h_U \arrow[r, "s"] & g^{-1}h_V
\end{tikzcd}$$

commutes. Then the diagram

$$\begin{tikzcd} U' \arrow[r, "f_{s'}"'] \arrow[d, "a"'] & u(V') \arrow[d, "u(b)"] \\ U \arrow[r, "f_s"] & u(V) \end{tikzcd}$$

of schemes commutes. The reason this is true is that the same condition holds for the morphisms $(g_s, g_s^\sharp)$ constructed in Modules on Sites, Lemma 16.22.3 and the uniqueness in Étale Cohomology, Theorem 38.40.5.

The problem is to glue the morphisms f_s to a morphism of algebraic spaces. To do this first choose a scheme V and a surjective étale morphism $V \to Y$. This means that $h_V \to *$ is surjective and hence $g^{-1}h_V \to *$ is surjective too. This means there exists a scheme U and a surjective étale morphism $U \to X$ and a morphism $s : h_U \to g^{-1}h_V$. Next, set $R = V \times_Y V$ and $R' = U \times_X U$. Then we get $g^{-1}h_R = g^{-1}h_V \times g^{-1}h_V$ as g^{-1} is exact. Thus s induces a morphism $s \times s : h_{R'} \to g^{-1}h_R$. Applying the constructions above we see that we get a commutative diagram of morphisms of schemes

$$\begin{tikzcd} R' \arrow[r, "f_{s\times s}"'] \arrow[d, shift right] \arrow[d, shift left] & R \arrow[d, shift right] \arrow[d, shift left] \\ U \arrow[r, "f_s"] & V \end{tikzcd}$$

Since we have $X = U/R'$ and $Y = V/R$ (see Spaces, Lemma 40.9.1) we conclude that this diagram defines a morphism of algebraic spaces $f : X \to Y$ fitting into an obvious commutative diagram. Now we still have to show that $(f_{small}, f^\sharp)$ is 2-isomorphic to $(g, g^\sharp)$. Let $t_V : f_{s,small}^{-1} \to g_s^{-1}$ and $t_R : f_{s\times s,small}^{-1} \to g_{s\times s}^{-1}$ be the 2-isomorphisms which are given to us by the construction above. Let $\mathcal{G}$ be a sheaf on $Y_{étale}$. Then we see that t_V defines an isomorphism

$$f_{small}^{-1}\mathcal{G}|_{U_{étale}} = f_{s,small}^{-1}\mathcal{G}|_{V_{étale}} \xrightarrow{t_V} g_s^{-1}\mathcal{G}|_{V_{étale}} = g^{-1}\mathcal{G}|_{U_{étale}}.$$

Moreover, this isomorphism pulled back to R' via either projection $R' \to U$ is the isomorphism

$$f_{small}^{-1}\mathcal{G}|_{R'_{étale}} = f_{s\times s,small}^{-1}\mathcal{G}|_{R_{étale}} \xrightarrow{t_R} g_{s\times s}^{-1}\mathcal{G}|_{R_{étale}} = g^{-1}\mathcal{G}|_{R'_{étale}}.$$

Since $\{U \to X\}$ is a covering in the site $X_{spaces,étale}$ this means the first displayed isomorphism descends to an isomorphism $t : f_{small}^{-1}\mathcal{G} \to g^{-1}\mathcal{G}$ of sheaves (small detail omitted). The isomorphism is functorial in $\mathcal{G}$ since t_V and t_R are transformations of functors. Finally, t is compatible with $f^\sharp$ and $g^\sharp$ as t_V and t_R are (some details omitted). This finishes the proof of the theorem. $\square$

Lemma 41.25.5. *Let X, Y be algebraic spaces over* $\mathbf{Z}$. *If*

$$(g, g^\sharp) : (Sh(X_{étale}), \mathcal{O}_X) \longrightarrow (Sh(Y_{étale}), \mathcal{O}_Y)$$

is an isomorphism of ringed topoi, then there exists a unique morphism $f : X \to Y$ of algebraic spaces such that $(g, g^\sharp)$ is isomorphic to $(f_{small}, f^\sharp)$ and moreover f is an isomorphism of algebraic spaces.

Proof. By Theorem 41.25.4 it suffices to show that $(g, g^\sharp)$ is a morphism of locally ringed topoi. By Modules on Sites, Lemma 16.34.7 (and since the site $X_{étale}$ has enough points) it suffices to check that the map $\mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$ induced by $g^\sharp$ is a local ring map where $q = f \circ p$ and p is any point of $X_{\acute{e}tale}$. As it is an isomorphism this is clear. $\square$

41.26. Quasi-coherent sheaves on algebraic spaces

In Descent, Section 31.6 we have seen that for a scheme U, there is no difference between a quasi-coherent $\mathcal{O}_U$-module on U, or a quasi-coherent $\mathcal{O}$-module on the small étale site of U. Hence the following definition is compatible with our original notion of a quasi-coherent sheaf on a scheme (Schemes, Section 21.24), when applied to a representable algebraic space.

Definition 41.26.1. Let S be a scheme. Let X be an algebraic space over S. A *quasi-coherent $\mathcal{O}_X$-module* is a quasi-coherent module on the ringed site $(X_{\acute{e}tale}, \mathcal{O}_X)$ in the sense of Modules on Sites, Definition 16.23.1. The category of quasi-coherent sheaves on X is denoted $QCoh(\mathcal{O}_X)$ or $QCoh(X)$.

Note that as being quasi-coherent is an intrinsic notion (see Modules on Sites, Lemma 16.23.2) this is equivalent to saying that the corresponding $\mathcal{O}_X$-module on $X_{spaces,\acute{e}tale}$ is quasi-coherent.

As usual, quasi-coherent sheaves behave well with respect to pullback.

Lemma 41.26.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The pullback functor $f^* : Mod(\mathcal{O}_Y) \to Mod(\mathcal{O}_X)$ preserves quasi-coherent sheaves.*

Proof. This is a general fact, see Modules on Sites, Lemma 16.23.4. $\square$

Note that this pullback functor agrees with the usual pullback functor between quasi-coherent sheaves of modules if X and Y happen to be schemes, see Descent, Proposition 31.6.14. Here is the obligatory lemma comparing this with quasi-coherent sheaves on the objects of the small étale site of X.

Lemma 41.26.3. *Let S be a scheme. Let X be an algebraic space over S. A quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is given by the following data:*

1. *for every $U \in Ob(X_{\acute{e}tale})$ a quasi-coherent $\mathcal{O}_U$-module $\mathcal{F}_U$ on $U_{\acute{e}tale}$,*
2. *for every $f : U' \to U$ in $X_{\acute{e}tale}$ an isomorphism $c_f : f^*_{small}\mathcal{F}_U \to \mathcal{F}_{U'}$.*

These data are subject to the condition that given any $f : U' \to U$ and $g : U'' \to U'$ in $X_{\acute{e}tale}$ the composition $g^{-1}_{small}c_f \circ c_g$ is equal to $c_{f\circ g}$.

Proof. Combine Lemmas 41.26.2 and 41.23.3. $\square$

Lemma 41.26.4. *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x \in |X|$ be a point and let $\overline{x}$ be a geometric point lying over x. Finally, let $\varphi : (U, \overline{u}) \to (X, \overline{x})$ be an étale neighbourhood where U is a scheme. Then*

$$(\varphi^*\mathcal{F})_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{X,\overline{x}} = \mathcal{F}_{\overline{x}}$$

where $u \in U$ is the image of $\overline{u}$.

Proof. Note that $\mathcal{O}_{X,\overline{x}} = \mathcal{O}^{sh}_{U,u}$ by Lemma 41.19.1 hence the tensor product makes sense. Moreover, from Definition 41.16.6 it is clear that

$$\mathcal{F}_{\overline{u}} = colim(\varphi^*\mathcal{F})_u$$

where the colimit is over $\varphi : (U, \overline{u}) \to (X, \overline{x})$ as in the lemma. Hence there is a canonical map from left to right in the statement of the lemma. We have a similar colimit description for $\mathcal{O}_{X,\overline{x}}$ and by Lemma 41.26.3 we have

$$((\varphi')^*\mathcal{F})_{u'} = (\varphi^*\mathcal{F})_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{U',u'}$$

whenever $(U', \overline{u}') \to (U, \overline{u})$ is a morphism of étale neighbourhoods. To complete the proof we use that $\otimes$ commutes with colimits. □

Lemma 41.26.5. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module. Let $\overline{x}$ be a geometric point of X and let $\overline{y} = f \circ \overline{x}$ be the image in Y. Then there is a canonical isomorphism*

$$(f^*\mathcal{G})_{\overline{x}} = \mathcal{G}_{\overline{y}} \otimes_{\mathcal{O}_{Y,\overline{y}}} \mathcal{O}_{X,\overline{x}}$$

of the stalk of the pullback with the tensor product of the stalk with the local ring of X at $\overline{x}$.

Proof. Since $f^*\mathcal{G} = f_{small}^{-1}\mathcal{G} \otimes_{f_{small}^{-1}\mathcal{O}_Y} \mathcal{O}_X$ this follows from the description of stalks of pullbacks in Lemma 41.16.9 and the fact that taking stalks commutes with tensor products. A more direct way to see this is as follows. Choose a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\alpha"] \arrow[d, "p"] & V \arrow[d, "q"] \\ X \arrow[r, "a"] & Y \end{tikzcd}$$

where U and V are schemes, and p and q are surjective étale. By Lemma 41.16.4 we can choose a geometric point $\overline{u}$ of U such that $\overline{x} = p \circ \overline{u}$. Set $\overline{v} = \alpha \circ \overline{u}$. Then we see that

$$\begin{aligned} (f^*\mathcal{G})_{\overline{x}} &= (p^*f^*\mathcal{G})_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{X,\overline{x}} \\ &= (\alpha^*q^*\mathcal{G})_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{X,\overline{x}} \\ &= (q^*\mathcal{G})_v \otimes_{\mathcal{O}_{V,v}} \mathcal{O}_{U,u} \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{X,\overline{x}} \\ &= (q^*\mathcal{G})_v \otimes_{\mathcal{O}_{V,v}} \mathcal{O}_{X,\overline{x}} \\ &= (q^*\mathcal{G})_v \otimes_{\mathcal{O}_{V,v}} \mathcal{O}_{Y,\overline{y}} \otimes_{\mathcal{O}_{Y,\overline{y}}} \mathcal{O}_{X,\overline{x}} \\ &= \mathcal{G}_{\overline{y}} \otimes_{\mathcal{O}_{Y,\overline{y}}} \mathcal{O}_{X,\overline{x}} \end{aligned}$$

Here we have used Lemma 41.26.4 (twice) and the corresponding result for pullbacks of quasi-coherent sheaves on schemes, see Sheaves, Lemma 6.26.4. □

Lemma 41.26.6. *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. The following are equivalent*

1. *$\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module,*
2. *there exists an étale morphism $f : Y \to X$ of algebraic spaces over S with $|f| : |Y| \to |X|$ surjective such that $f^*\mathcal{F}$ is quasi-coherent on Y,*
3. *there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that $\varphi^*\mathcal{F}$ is a quasi-coherent $\mathcal{O}_U$-module, and*
4. *for every affine scheme U and étale morphism $\varphi : U \to X$ the restriction $\varphi^*\mathcal{F}$ is a quasi-coherent $\mathcal{O}_U$-module.*

Proof. It is clear that (1) implies (2) by considering id_X. Assume $f : Y \to X$ is as in (2), and let $V \to Y$ be a surjective étale morphism from a scheme towards Y. Then the composition $V \to X$ is surjective étale as well and by Lemma 41.26.2 the pullback of $\mathcal{F}$ to V is quasi-coherent as well. Hence we see that (2) implies (3).

Let $U \to X$ be as in (3). Let us use the abuse of notation introduced in Equation (41.23.1.1). As $\mathcal{F}|_{U_{étale}}$ is quasi-coherent there exists an étale covering $\{U_i \to U\}$ such that $\mathcal{F}|_{U_{i,étale}}$ has a global presentation, see Modules on Sites, Definition 16.17.1 and Lemma 16.23.3. Let $V \to X$ be an object of $X_{étale}$. Since $U \to X$ is surjective and étale, the family of maps $\{U_i \times_X V \to V\}$ is an étale covering of V. Via the morphisms $U_i \times_X V \to U_i$ we can restrict the global presentations of $\mathcal{F}|_{U_{i,étale}}$ to get a global presentation of $\mathcal{F}|_{(U_i \times_X V)_{étale}}$ Hence the sheaf $\mathcal{F}$ on $X_{étale}$ satisfies the condition of Modules on Sites, Definition 16.23.1 and hence is quasi-coherent.

The equivalence of (3) and (4) comes from the fact that any scheme has an affine open covering. □

Lemma 41.26.7. *Let S be a scheme. Let X be an algebraic space over S. The category $QCoh(\mathcal{O}_X)$ of quasi-coherent sheaves on X has the following properties:*

(1) *Any direct sum of quasi-coherent sheaves is quasi-coherent.*
(2) *Any colimit of quasi-coherent sheaves is quasi-coherent.*
(3) *The kernel and cokernel of a morphism of quasi-coherent sheaves is quasi-coherent.*
(4) *Given a short exact sequence of $\mathcal{O}_X$-modules $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ if two out of three are quasi-coherent so is the third.*
(5) *Given two quasi-coherent $\mathcal{O}_X$-modules the tensor product is quasi-coherent.*
(6) *Given two quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}$, $\mathcal{G}$ such that $\mathcal{F}$ is of finite presentation (see Section 41.27), then the internal hom $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is quasi-coherent.*

Proof. Note that we have the corresponding result for quasi-coherent modules on schemes, see Schemes, Section 21.24. We will reduce the lemma to this case by étale localization. Choose a scheme U and a surjective étale morphism $\varphi : U \to X$. In order to formulate this proof correctly, we temporarily go back to making the (pedantic) distinction between a quasi-coherent sheaf $\mathcal{G}$ on the scheme U and the associated quasi-coherent sheaf $\mathcal{G}^a$ (see Descent, Definition 31.6.2) on $U_{étale}$ We have a commutative diagram

$$\begin{tikzcd} QCoh(\mathcal{O}_X) \arrow[r] \arrow[d] & QCoh(\mathcal{O}_U) \arrow[d] \\ Mod(\mathcal{O}_X) \arrow[r] & Mod(\mathcal{O}_U) \end{tikzcd}$$

The bottom horizontal arrow is the restriction functor (41.23.1.1) $\mathcal{G} \mapsto \mathcal{G}|_{U_{étale}}$. This functor has both a left adjoint and a right adjoint, see Modules on Sites, Section 16.19, hence commutes with all limits and colimits. Moreover, we know that an object of $Mod(\mathcal{O}_X)$ is in $QCoh(\mathcal{O}_X)$ if and only if its restriction to U is in $QCoh(\mathcal{O}_U)$, see Lemma 41.26.6. Let $\mathcal{F}_i$ be a family of quasi-coherent $\mathcal{O}_X$-modules. Then $\bigoplus \mathcal{F}_i$ is an $\mathcal{O}_X$-module whose restriction to U is the direct sum of the restrictions. Let $\mathcal{G}_i$ be a quasi-coherent sheaf on U with $\mathcal{F}_i|_{U_{étale}} = \mathcal{G}_i^a$. Combining the above with Descent, Lemma 31.6.13 we see that

$$\left(\bigoplus \mathcal{F}_i\right)|_{U_{étale}} = \bigoplus \mathcal{F}_i|_{U_{étale}} = \bigoplus \mathcal{G}_i^a = \left(\bigoplus \mathcal{G}_i\right)^a$$

hence $\bigoplus \mathcal{F}_i$ is quasi-coherent and (1) follows. The other statements are proved just so (using the same references). □

It is in general not the case that the pushforward of a quasi-coherent sheaf along a morphism of algebraic spaces is quasi-coherent. We will return to this issue in Morphisms of Spaces, Section 42.15.

41.27. Properties of modules

In Modules on Sites, Sections 16.17, 16.23, and Definition 16.26.1 we have defined a number of intrinsic properties of modules of $\mathcal{O}$-module on any ringed topos. If X is an algebraic space, we will apply these notions freely to modules on the ringed site $(X_{étale}, \mathcal{O}_X)$, or equivalently on the ringed site $(X_{spaces,étale}, \mathcal{O}_X)$.

Global properties $\mathcal{P}$:

(1) *free*,
(2) *finite free*,
(3) *generated by global sections*,
(4) *generated by finitely many global sections*,
(5) having a *global presentation*, and
(6) having a *global finite presentation*.

Local properties $\mathcal{P}$:

(1) *locally free*,
(2) *finite locally free*,
(3) *locally generated by sections*,
(4) *finite type*,
(5) *quasi-coherent* (see Section 41.26),
(6) *of finite presentation*,
(7) *coherent*, and
(8) *flat*.

In each case, except for $\mathcal{P}$ =``coherent'', the property is preserved under pullback, see Modules on Sites, Lemma 16.17.2, Modules on Sites, Lemma 16.23.4, and Modules on Sites, Lemma 16.33.3. In particular, if $\mathcal{F}$ is an $\mathcal{O}_X$-module on $X_{étale}$ satisfying one of the properties $\mathcal{P}$ above and $\varphi : U \to X$ is a surjective étale morphism with U a scheme, then the pullback $\varphi^*\mathcal{F}$ has property $\mathcal{P}$ as a sheaf of modules on $U_{étale}$. Moreover, for each of the local properties $\mathcal{P}$, the fact that $\varphi^*\mathcal{G}$ has $\mathcal{P}$ implies that $\mathcal{G}$ has $\mathcal{P}$. This follows as $\{U \to X\}$ is a covering in $X_{spaces,étale}$ and Modules on Sites, Lemma 16.23.3. Finally, if $\mathcal{G}$ is assumed quasi-coherent and for any $\mathcal{P}$ except $\mathcal{P}$ =``coherent'' or ``locally free'', then $\mathcal{P}$ for $\varphi^*\mathcal{G}$ on $U_{étale}$ is equivalent to the corresponding property for $\varphi^*\mathcal{G}|_{U_{Zar}}$, i.e., it corresponds to $\mathcal{P}$ for $\varphi^*\mathcal{G}$ when we think of it as a quasi-coherent sheaf on the scheme U. See Descent, Lemma 31.6.12.

41.28. Locally projective modules

Recall that in Properties, Section 23.19 we defined the notion of a locally projective quasi-coherent module.

Lemma 41.28.1. *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The following are equivalent*

(1) *for some scheme U and surjective étale morphism $U \to X$ the restriction $\mathcal{F}|_U$ is locally projective on U, and*
(2) *for any scheme U and any étale morphism $U \to X$ the restriction $\mathcal{F}|_U$ is locally projective on U.*

Proof. Let $U \to X$ be as in (1) and let $V \to X$ be étale where V is a scheme. Then $\{U \times_X V \to V\}$ is an fppf covering of schemes. Hence if $\mathcal{F}|_U$ is locally projective, then $\mathcal{F}|_{U \times_X V}$ is locally projective (see Properties, Lemma 23.19.3) and hence $\mathcal{F}|_V$ is locally projective, see Descent, Lemma 31.5.5. $\square$

Definition 41.28.2. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. We say $\mathcal{F}$ is *locally projective* if the equivalent conditions of Lemma 41.28.1 are satisfied.

Lemma 41.28.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module. If $\mathcal{G}$ is locally projective on Y, then $f^*\mathcal{G}$ is locally projective on X.*

Proof. Choose a surjective étale morphism $V \to Y$ with V a scheme. Choose a surjective étale morphism $U \to V \times_Y X$ with U a scheme. Denote $\psi : U \to V$ the induced morphism. Then

$$f^*\mathcal{G}|_U = \psi^*(\mathcal{G}|_V)$$

Hence the lemma follows from the definition and the result in the case of schemes, see Properties, Lemma 23.19.3. □

41.29. Quasi-coherent sheaves and presentations

Let S be a scheme. Let X be an algebraic space over S. Let $X = U/R$ be a presentation of X coming from any surjective étale morphism $\varphi : U \to X$, see Spaces, Definition 40.9.3. In particular, we obtain a groupoid (U, R, s, t, c), such that $j = (t, s) : R \to U \times_S U$, see Groupoids, Lemma 35.11.3. In Groupoids, Definition 35.12.1 we have the defined the notion of a quasi-coherent sheaf on an arbitrary groupoid. With these notions in place we have the following observation.

Proposition 41.29.1. *With S, $\varphi : U \to X$, and (U, R, s, t, c) as above. For any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the sheaf $\varphi^*\mathcal{F}$ comes equipped with a canonical isomorphism*

$$\alpha : t^*\varphi^*\mathcal{F} \longrightarrow s^*\varphi^*\mathcal{F}$$

which satisfies the conditions of Groupoids, Definition 35.12.1 and therefore defines a quasi-coherent sheaf on (U, R, s, t, c). The functor $\mathcal{F} \mapsto (\varphi^*\mathcal{F}, \alpha)$ defines an equivalence of categories

$$\begin{matrix} \textit{Quasi-coherent} \\ \mathcal{O}_X\textit{-modules} \end{matrix} \longleftrightarrow \begin{matrix} \textit{Quasi-coherent modules} \\ \textit{on } (U, R, s, t, c) \end{matrix}$$

Proof. In the statement of the proposition, and in this proof we think of a quasi-coherent sheaf on a scheme as a quasi-coherent sheaf on the small étale site of that scheme. This is permissible by the results of Descent, Section 31.6.

The existence of α comes from the fact that $\varphi \circ t = \varphi \circ s$ and that pullback is functorial in the morphism, see discussion surrounding Equation (41.23.0.1). In exacty the same way, i.e., by functoriality of pullback, we see that the isomorphism α satisfies condition (1) of Groupoids, Definition 35.12.1. To see condition (2) of the definition it suffices to see that α is an isomorphism which is clear. The construction $\mathcal{F} \mapsto (\varphi^*\mathcal{F}, \alpha)$ is clearly functorial in the quasi-coherent sheaf $\mathcal{F}$. Hence we obtain the functor from left to right in the displayed formula of the lemma.

Conversely, suppose that $(\mathcal{F}, \alpha)$ is a quasi-coherent sheaf on (U, R, s, t, c). Let $V \to X$ be an object of $X_{étale}$. In this case the morphism $V' = U \times_X V \to V$ is a surjective étale morphism of schemes, and hence $\{V' \to V\}$ is an étale covering of V. Moreover, the quasi-coherent sheaf $\mathcal{F}$ pulls back to a quasi-coherent sheaf $\mathcal{F}'$ on V'. Since $R = U \times_X U$ with $t = \text{pr}_0$ and $s = \text{pr}_0$ we see that $V' \times_V V' = R \times_X V$ with projection maps $V' \times_V V' \to V'$ equal to the pullbacks of t and s. Hence α pulls back to an isomorphism $\alpha' : \text{pr}_0^*\mathcal{F}' \to \text{pr}_1^*\mathcal{F}'$, and the pair $(\mathcal{F}', \alpha')$ is a descend datum for quasi-coherent sheaves with respect to $\{V' \to V\}$. By

Descent, Proposition 31.4.2 this descent datum is effective, and we obtain a quasi-coherent $\mathcal{O}_V$-module $\mathcal{F}_V$ on $V_{étale}$. To see that this gives a quasi-coherent sheaf on $X_{étale}$ we have to show (by Lemma 41.26.3) that for any morphism $f : V_1 \to V_2$ in $X_{étale}$ there is a canonical isomorphism $c_f : \mathcal{F}_{V_1} \to \mathcal{F}_{V_2}$ compatible with compositions of morphisms. We omit the verification. We also omit the verification that this defines a functor from the category on the right to the category on the left which is inverse to the functor described above. □

Proposition 41.29.2. *Let S be a scheme Let X be an algebraic space over S. The inclusion functor* $QCoh(\mathcal{O}_X) \to Mod(\mathcal{O}_X)$ *has a right adjoint*

$$Q^7 : Mod(\mathcal{O}_X) \longrightarrow QCoh(\mathcal{O}_X)$$

such that for every quasi-coherent sheaf $\mathcal{F}$ *the adjunction mapping* $Q(\mathcal{F}) \to \mathcal{F}$ *is an isomorphism. Moreover, the category* $QCoh(\mathcal{O}_X)$ *has limits and enough injectives.*

Proof. This proof is a repeat of the proof in the case of schemes, see Properties, Proposition 23.21.4. We urge the reader to read that proof first.

The two assertions about $Q(\mathcal{F}) \to \mathcal{F}$ and limits in $QCoh(\mathcal{O}_X)$ are formal consequences of the existence of Q, the fact that the inclusion is fully faithful, and the fact that $Mod(\mathcal{O}_X)$ has limits (see Modules on Sites, Lemma 16.14.2). The existence of injectives follows from the existence of injectives in $Mod(\mathcal{O}_X)$ (see Injectives, Theorem 17.12.4) and Homology, Lemma 10.22.3. Thus it suffices to construct Q.

Choose a presentation $X = U/R$ so that (U, R, s, t, c) is an étale groupoid scheme and in particular s and t are flat morphisms of schemes. Pick a cardinal κ as in Groupoids, Lemma 35.12.7. Pick a collection $(\mathcal{E}_t, \alpha_t)_{t\in T}$ of κ-generated quasi-coherent modules on (U, R, s, t, c) as in Groupoids, Lemma 35.12.6. Let $\mathcal{F}_t$ be the quasi-coherent module on X which corresponds to the quasi-coherent module $(\mathcal{E}_t, \alpha_t)$ via the equivalence of categories of Proposition 41.29.1. Then we see that every quasi-coherent module $\mathcal{H}$ is the directed colimit of its quasi-coherent submodules which are isomorphic to one of the $\mathcal{F}_t$.

Given an object $\mathcal{G}$ of $QCoh(\mathcal{O}_X)$ we set

$$Q(\mathcal{G}) = colim_{(t,\psi)} \mathcal{F}_t$$

The colimit is over the category of pairs (t, ψ) where $t \in T$ and $\psi : \mathcal{F}_t \to \mathcal{G}$ is a morphism of $\mathcal{O}_X$-modules. A morphism $(t, \psi) \to (t', \psi')$ is given by a morphism $\beta : \mathcal{F}_t \to \mathcal{F}_{t'}$ such that $\psi' \circ \beta = \psi$. By Lemma 41.26.7 the colimit is quasi-coherent. Note that there is a canonical map $Q(\mathcal{G}) \to \mathcal{G}$ by definition of the colimit. The formula

$$Hom(\mathcal{H}, Q(\mathcal{G})) = Hom(\mathcal{H}, \mathcal{G})$$

holds for $\mathcal{H} = \mathcal{F}_t$ by construction. It follows formally from this and the fact that every $\mathcal{H}$ is a directed colimit of $\mathcal{O}_X$-modules isomorphic to $\mathcal{F}_t$ that this equality holds for any quasi-coherent module $\mathcal{H}$ on X. This finishes the proof. □

41.30. Morphisms towards schemes

Here is the analogue of Schemes, Lemma 21.6.4.

Lemma 41.30.1. *Let X be an algebraic space over* $\mathbf{Z}$. *Let T be an affine scheme. The map*

$$Mor(X, T) \longrightarrow Hom(\Gamma(T, \mathcal{O}_T), \Gamma(X, \mathcal{O}_X))$$

which maps f *to* $f^\sharp$ *(on global sections) is bijective.*

[7] This functor is sometimes called the *coherator*.

Proof. We construct the inverse of the map. Let $\varphi : \Gamma(T, \mathcal{O}_T) \to \Gamma(X, \mathcal{O}_X)$ be a ring map. Choose a presentation $X = U/R$, see Spaces, Definition 40.9.3. By Schemes, Lemma 21.6.4 the composition

$$\Gamma(T, \mathcal{O}_T) \to \Gamma(X, \mathcal{O}_X) \to \Gamma(U, \mathcal{O}_U)$$

corresponds to a unique morphism of schemes $g : U \to T$. By the same lemma the two compositions $R \to U \to T$ are equal. Hence we obtain a morphism $f : X = U/R \to T$ such that $U \to X \to T$ equals g. By construction the diagram

$$\begin{tikzcd} \Gamma(U, \mathcal{O}_U) & \Gamma(X, \mathcal{O}_X) \arrow[l, "f^\sharp"'] \\ & \Gamma(T, \mathcal{O}_T) \arrow[ul, "g^\sharp"] \arrow[u, "\varphi"'] \end{tikzcd}$$

commutes. Hence $f^\sharp$ equals φ because $U \to X$ is an étale covering and $\mathcal{O}_X$ is a sheaf on $X_{étale}$. The uniqueness of f follows from the uniqueness of g. $\square$

41.31. Quotients by free actions

Let S be a scheme. Let X be an algebraic space over S. Let G be an abstract group. Let $a : G \to \text{Aut}(X)$ be a homomorphism, i.e., a is an *action* of G on X. We will say the action is *free* if for every scheme T over S the map

$$G \times X(T) \longrightarrow X(T)$$

is free. (We cannot use a criterion as in Spaces, Lemma 40.14.3 because points may not have well defined residue fields.) In case the action is free we're going to construct the quotient X/G as an algebraic space. This is a special case of the general Bootstrap, Lemma 54.11.5 that we will prove later.

Lemma 41.31.1. *Let S be a scheme. Let X be an algebraic space over S. Let G be an abstract group with a free action on X. Then the quotient sheaf X/G is an algebraic space.*

Proof. The statement means that the sheaf F associated to the presheaf

$$T \longmapsto X(T)/G$$

is an algebraic space. To see this we will construct a presentation. Namely, choose a scheme U and a surjective étale morphism $\varphi : U \to X$. Set $V = \coprod_{g \in G} U$ and set $\psi : V \to X$ equal to $a(g) \circ \varphi$ on the component corresponding to $g \in G$. Let G act on V by permuting the components, i.e., $g_0 \in G$ maps the component corresponding to g to the component corresponding to $g_0 g$ via the identity morphism of U. Then ψ is a G-equivariant morphism, i.e., we reduce to the case dealt with in the next paragraph.

Assume that there exists a G-action on U and that $U \to X$ is surjective, étale and G-equivariant. In this case there is an induced action of G on $R = U \times_X U$ compatible with the projection mappings $t, s : R \to U$. Now we claim that

$$X/G = U/\coprod\nolimits_{g \in G} R$$

where the map

$$j : \coprod\nolimits_{g \in G} R \longrightarrow U \times_S U$$

is given by $(r, g) \mapsto (t(r), g(s(r)))$. Note that j is a monomorphism: If $(t(r), g(s(r))) = (t(r'), g'(s(r')))$, then $t(r) = t(r')$, hence r and r' have the same image in X under both s and t, hence $g = g'$ (as G acts freely on X), hence $s(r) = s(r')$, hence $r = r'$ (as R is an equivalence relation on U). Moreover j is an equivalence relation (details omitted). Both projections $\coprod_{g \in G} R \to U$ are étale, as s and t are étale. Thus j is an étale equivalence relation and $U/\coprod_{g \in G} R$ is an algebaic space by Spaces, Theorem 40.10.5. There is a map

$$U/\coprod\nolimits_{g \in G} R \longrightarrow X/G$$

induced by the map $U \to X$. We omit the proof that it is an isomorphism of sheaves. $\square$

41.32. Other chapters

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(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
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(7) Commutative Algebra
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CHAPTER 42

Morphisms of Algebraic Spaces

42.1. Introduction

In this chapter we introduce some types of morphisms of algebraic spaces. A reference is [**Knu71a**].

The goal is to extend the definition of each of the types of morphisms of schemes defined in the chapters on schemes, and on morphisms of schemes to the category of algebraic spaces. Each case is slightly different and it seems best to treat them all separately.

42.2. Conventions

The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $Spec(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

42.3. Properties of representable morphisms

Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces. In Spaces, Section 40.5 we defined what it means for f to have property $\mathcal{P}$ in case $\mathcal{P}$ is a property of morphisms of schemes which

(1) is preserved under any base change, see Schemes, Definition 21.18.3, and
(2) is fppf local on the base, see Descent, Definition 31.18.1.

Namely, in this case we say f has property $\mathcal{P}$ if and only if for every scheme U and any morphism $U \to Y$ the morphism of schemes $X \times_Y U \to U$ has property $\mathcal{P}$.

According to the lists in Spaces, Section 40.4 this applies to the following properties: (1)(a) closed immersions, (1)(b) open immersions, (1)(c) quasi-compact immersions, (2) quasi-compact, (3) universally-closed, (4) (quasi-)separated, (5) monomorphism, (6) surjective, (7) universally injective, (8) affine, (9) quasi-affine, (10) (locally) of finite type, (11) (locally) quasi-finite, (12) (locally) of finite presentation, (13) locally of finite type of relative dimension d, (14) universally open, (15) flat, (16) syntomic, (17) smooth, (18) unramified (resp. G-unramified), (19) étale, (20) proper, (21) finite or integral, (22) finite locally free, and (23) immersion.

In this chapter we will redefine these notions for not necessarily representable morphisms of algebraic spaces. Whenever we do this we will make sure that the new definition agrees with the old one, in order to avoid ambiguity.

Note that the definition above applies whenever X is a scheme, since a morphism from a scheme to an algebraic space is representable. And in particular it applies when both X

and Y are schemes. In Spaces, Lemma 40.5.3 we have seen that in this case the definitions match, and no ambiguity arise.

Furthermore, in Spaces, Lemma 40.5.5 we have seen that the property of representable morphisms of algebraic spaces so defined is stable under arbitrary base change by a morphism of algebraic spaces. And finally, in Spaces, Lemmas 40.5.4 and 40.5.7 we have seen that if $\mathcal{P}$ is stable under compositions, which holds for the properties (1)(a), (1)(b), (1)(c), (2) -- (23), except (13) above, then taking products of representable morphisms preserves property $\mathcal{P}$ and compositions of representable morphisms preserves property $\mathcal{P}$.

We will use these facts below, and whenever we do we will simply refer to this section as a reference.

42.4. Immersions

Open, closed and locally closed immersions of algebraic spaces were defined in Spaces, Section 40.12. Namely, a morphism of algebraic spaces is a *closed immersion* (resp. *open immersion*, resp. *immersion*) if it is representable and a closed immersion (resp. open immersion, resp. immersion) in the sense of Section 42.3.

In particular these types of morphisms are stable under base change and compositions of morphisms in the category of algebraic spaces over S, see Spaces, Lemmas 40.12.2 and 40.12.3.

Lemma 42.4.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

1. *f is a closed immersion (resp. open immersion, resp. immersion),*
2. *for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is a closed immersion (resp. open immersion, resp. immersion),*
3. *for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is a closed immersion (resp. open immersion, resp. immersion),*
4. *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is a closed immersion (resp. open immersion, resp. immersion), and*
5. *there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is a closed immersion (resp. open immersion, resp. immersion).*

Proof. Using that a base change of a closed immersion (resp. open immersion, resp. immersion) is another one it is clear that (1) implies (2) and (2) implies (3). Also (3) implies (4) since we can take V to be a disjoint union of affines, see Properties of Spaces, Lemma 41.6.1.

Assume $V \to Y$ is as in (4). Let $\mathcal{P}$ be the property closed immersion (resp. open immersion, resp. immersion) of morphisms of schemes. Note that property $\mathcal{P}$ is preserved under any base change and fppf local on the base (see Section 42.3). Moreover, morphisms of type $\mathcal{P}$ are separated and locally quasi-finite (in each of the three cases, see Schemes, Lemma 21.23.7, and Morphisms, Lemma 24.19.14). Hence by More on Morphisms, Lemma 33.35.1 the morphisms of type $\mathcal{P}$ satisfy descent for fppf covering. Thus Spaces, Lemma 40.11.3 applies and we see that $X \to Y$ is representable and has property $\mathcal{P}$, in other words (1) holds.

The equivalence of (1) and (5) follows from the fact that $\mathcal{P}$ is Zariski local on the target (since we saw above that $\mathcal{P}$ is in fact fppf local on the target). $\square$

Lemma 42.4.2. *Let S be a scheme. Let $i : Z \to X$ be an immersion of algebraic spaces over S. Then $|i|(|Z|) \subset |X|$ is a locally closed subset, and i is a closed immersion if and only if $|i|(|Z|) \subset |X|$ is a closed subset.*

Proof. Let U be a scheme and let $U \to X$ be a surjective étale morphism. By assumption $T = U \times_X Z$ is a scheme and the morphism $j : T \to U$ is an immersion of schemes. Moreover, note that $|j|(|T|) \subset |U|$ is the inverse image of $|i|(|Z|) \subset |X|$, see Properties of Spaces, Lemma 41.4.3. Recall that $|U| \to |X|$ is surjective and open, see Properties of Spaces, Lemma 41.4.6. Hence since $|T|$ is locally closed in $|U|$ it follows that $|i|(|Z|)$ is locally closed in $|X|$, see Topology, Lemma 5.15.2. And in the same way we see that $|i|(|Z|) \subset |X|$ is closed if and only if $T \subset U$ is closed, Thus we can combine Lemma 42.4.1 above and Schemes, Lemma 21.10.4 to finish the proof. □

Remark 42.4.3. Let S be a scheme. Let $i : Z \to X$ be an immersion of algebraic spaces over S. Since i is a monomorphism we may think of $|Z|$ as a subset of $|X|$; in the rest of this remark we do so. Let $\partial|Z|$ be the boundary of $|Z|$ in the topological space $|X|$. In a formula

$$\partial|Z| = \overline{|Z|} \setminus |Z|.$$

Let ∂Z be the reduced closed subspace of X with $|\partial Z| = \partial|Z|$ obtained by taking the reduced induced closed subspace structure, see Properties of Spaces, Definition 41.9.3. By construction we see that $|Z|$ is closed in $|X| \setminus |\partial Z| = |X \setminus \partial Z|$. Hence it is true that any immersion of algebraic spaces can be factored as a closed immmersion followed by an open immersion (but not the other way in general, see Morphisms, Example 24.2.10).

Remark 42.4.4. Let S be a scheme. Let X be an algebraic space over S. Let $T \subset |X|$ be a locally closed subset. Let ∂T be the boundary of T in the topological space $|X|$. In a formula

$$\partial T = \overline{T} \setminus T.$$

Let $U \subset X$ be the open subspace of X with $|U| = |X| \setminus \partial T$, see Properties of Spaces, Lemma 41.4.8. Let Z be the reduced closed subspace of U with $|Z| = T$ obtained by taking the reduced induced closed subspace structure, see Properties of Spaces, Definition 41.9.3. By construction $Z \to U$ is a closed immersion of algebraic spaces and $U \to X$ is an open immersion, hence $Z \to X$ is an immersion of algebraic spaces over S (see Spaces, Lemma 40.12.2). Note that Z is a reduced algebraic space and that $|Z| = T$ as subsets of $|X|$. We sometimes say Z is the *reduced induced subspace structure* on T.

42.5. Separation axioms

It makes sense to list some a priori properties of the diagonal of a morphism of algebraic spaces.

Lemma 42.5.1. *Let S be a scheme contained in Sch_{fppf}. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\Delta_{X/Y} : X \to X \times_Y X$ be the diagonal morphism. Then*

(1) *$\Delta_{X/Y}$ is representable,*
(2) *$\Delta_{X/Y}$ is locally of finite type,*
(3) *$\Delta_{X/Y}$ is a monomorphism,*
(4) *$\Delta_{X/Y}$ is separated, and*
(5) *$\Delta_{X/Y}$ is locally quasi-finite.*

Proof. We are going to use the fact that $\Delta_{X/S}$ is representable (by definition of an algebraic space) and that it satisfies properties (2) -- (5), see Spaces, Lemma 40.13.1. Note that we have a factorization

$$X \longrightarrow X \times_Y X \longrightarrow X \times_S X$$

of the diagonal $\Delta_{X/S} : X \to X \times_S X$. Since $X \times_Y X \to X \times_S X$ is a monomorphism, and since $\Delta_{X/S}$ is representable, it follows formally that $\Delta_{X/Y}$ is representable. In particular, the rest of the statements now make sense, see Section 42.3.

Choose a surjective étale morphism $U \to X$, with U a scheme. Consider the diagram

$$\begin{tikzcd} R = U \times_X U \arrow[r] \arrow[d] & U \times_Y U \arrow[r] \arrow[d] & U \times_S U \arrow[d] \\ X \arrow[r] & X \times_Y X \arrow[r] & X \times_S X \end{tikzcd}$$

Both squares are cartesian, hence so is the outer rectangle. The top row consists of schemes, and the vertical arrows are surjective étale morphisms. By Spaces, Lemma 40.11.2 the properties (2) -- (5) for $\Delta_{X/Y}$ are equivalent to those of $R \to U \times_Y U$. In the proof of Spaces, Lemma 40.13.1 we have seen that $R \to U \times_S U$ has properties (2) -- (5). The morphism $U \times_Y U \to U \times_S U$ is a monomorphism of schemes. These facts imply that $R \to U \times_S U$ have properties (2) -- (5).

Namely: For (3), note that $R \to U \times_Y U$ is a monomorphism as the composition $R \to U \times_S U$ is a monomorphism. For (2), note that $R \to U \times_Y U$ is locally of finite type, as the composition $R \to U \times_S U$ is locally of finite type (Morphisms, Lemma 24.14.8). A monomorphism which is locally of finite type is locally quasi-finite because it has finite fibres (Morphisms, Lemma 24.19.7), hence (5). A monomorphism is separated (Schemes, Lemma 21.23.3), hence (4). □

Definition 42.5.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\Delta_{X/Y} : X \to X \times_Y X$ be the diagonal morphism.

(1) We say f is *separated* if $\Delta_{X/Y}$ is a closed immersion.
(2) We say f is *locally separated*[1] if $\Delta_{X/Y}$ is an immersion.
(3) We say f is *quasi-separated* if $\Delta_{X/Y}$ is quasi-compact.

This definition makes sense since $\Delta_{X/Y}$ is representable, and hence we know what it means for it to have one of the properties described in the definition. We will see below (Lemma 42.5.13) that this definition matches the ones we already have for morphisms of schemes and representable morphisms.

Lemma 42.5.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is separated, then f is locally separated and f is quasi-separated.*

Proof. This is true, via the general principle Spaces, Lemma 40.5.8, because a closed immersion of schemes is an immersion and is quasi-compact. □

Lemma 42.5.4. *All of the separation axioms listed in Definition 42.5.2 are stable under base change.*

[1]In the literature this term often refers to quasi-separated and locally separated morphisms.

Proof. Let $f : X \to Y$ and $Y' \to Y$ be morphisms of algebraic spaces. Let $f' : X' \to Y'$ be the base change of f by $Y' \to Y$. Then $\Delta_{X'/Y'}$ is the base change of $\Delta_{X/Y}$ by the morphism $X' \times_{Y'} X' \to X \times_Y X$. By the results of Section 42.3 each of the properties of the diagonal used in Definition 42.5.2 is stable under base change. Hence the lemma is true. $\square$

Lemma 42.5.5. *Let S be a scheme. Let $f : X \to Z$, $g : Y \to Z$ and $Z \to T$ be morphisms of algebraic spaces over S. Consider the induced morphism $i : X \times_Z Y \to X \times_T Y$. Then*

1. *i is representable, locally of finite type, locally quasi-finite, separated and a monomorphism,*
2. *if $Z \to T$ is locally separated, then i is an immersion,*
3. *if $Z \to T$ is separated, then i is a closed immersion, and*
4. *if $Z \to T$ is quasi-separated, then i is quasi-compact.*

Proof. By general category theory the following diagram

$$\begin{tikzcd} X \times_Z Y \arrow[r, "i"'] \arrow[d] & X \times_T Y \arrow[d] \\ Z \arrow[r, "\Delta_{Z/T}"] & Z \times_T Z \end{tikzcd}$$

is a fibre product diagram. Hence i is the base change of the diagonal morphism $\Delta_{Z/T}$. Thus the lemma follows from Lemma 42.5.1, and the material in Section 42.3. $\square$

Lemma 42.5.6. *Let S be a scheme. Let T be an algebraic space over S. Let $g : X \to Y$ be a morphism of algebraic spaces over T. Consider the graph $i : X \to X \times_T Y$ of g. Then*

1. *i is representable, locally of finite type, locally quasi-finite, separated and a monomorphism,*
2. *if $Y \to T$ is locally separated, then i is an immersion,*
3. *if $Y \to T$ is separated, then i is a closed immersion, and*
4. *if $Y \to T$ is quasi-separated, then i is quasi-compact.*

Proof. This is a special case of Lemma 42.5.5 applied to the morphism $X = X \times_Y Y \to X \times_T Y$. $\square$

Lemma 42.5.7. *Let S be a schemes. Let $f : X \to T$ be a morphism of algebraic spaces over S. Let $s : T \to X$ be a section of f (in a formula $f \circ s = id_T$). Then*

1. *s is representable, locally of finite type, locally quasi-finite, separated and a monomorphism,*
2. *if f is locally separated, then s is an immersion,*
3. *if f is separated, then s is a closed immersion, and*
4. *if f is quasi-separated, then s is quasi-compact.*

Proof. This is a special case of Lemma 42.5.6 applied to $g = s$ so the morphism $i = s : T \to T \times_T X$. $\square$

Lemma 42.5.8. *All of the separation axioms listed in Definition 42.5.2 are stable under composition of morphisms.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of algebraic spaces to which the axiom in question applies. The diagonal $\Delta_{X/Z}$ is the composition

$$X \longrightarrow X \times_Y X \longrightarrow X \times_Z X.$$

Our separation axiom is defined by requiring the diagonal to have some property $\mathcal{P}$. By Lemma 42.5.5 above we see that the second arrow also has this property. Hence the lemma follows since the composition of (representable) morphisms with property $\mathcal{P}$also is a morphism with property $\mathcal{P}$, see Section 42.3. □

Lemma 42.5.9. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.*

(1) *If Y is separated and f is separated, then X is separated.*
(2) *If Y is quasi-separated and f is quasi-separated, then X is quasi-separated.*
(3) *If Y is locally separated and f is locally separated, then X is locally separated.*
(4) *If Y is separated over S and f is separated, then X is separated over S.*
(5) *If Y is quasi-separated over S and f is quasi-separated, then X is quasi-separated over S.*
(6) *If Y is locally separated over S and f is locally separated, then X is locally separated over S.*

Proof. Parts (4), (5), and (6) follow immediately from Lemma 42.5.8 and Spaces, Definition 40.13.2. Parts (1), (2), and (3) reduce to parts (4), (5), and (6) by thinking of X and Y as algebraic spaces over $Spec(\mathbf{Z})$, see Properties of Spaces, Definition 41.3.1. □

Lemma 42.5.10. *Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of algebraic spaces over S.*

(1) *If $g \circ f$ is separated then so is f.*
(2) *If $g \circ f$ is locally separated then so is f.*
(3) *If $g \circ f$ is quasi-separated then so is f.*

Proof. Consider the factorization

$$X \to X \times_Y X \to X \times_Z X$$

of the diagonal morphism of $g \circ f$. In any case the last morphism is a monomorphism. Hence for any scheme T and morphism $T \to X \times_Y X$ we have the equality

$$X \times_{(X \times_Y X)} T = X \times_{(X \times_Z X)} T.$$

Hence the result is clear. □

Lemma 42.5.11. *Let S be a scheme. Let X be an algebraic space over S.*

(1) *If X is separated then X is separated over S.*
(2) *If X is locally separated then X is locally separated over S.*
(3) *If X is quasi-separated then X is quasi-separated over S.*

Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(4) *If X is separated over S then f is separated.*
(5) *If X is locally separated over S then f is locally separated.*
(6) *If X is quasi-separated over S then f is quasi-separated.*

Proof. Parts (4), (5), and (6) follow immediately from Lemma 42.5.10 and Spaces, Definition 40.13.2. Parts (1), (2), and (3) follow from parts (4), (5), and (6) by thinking of X and Y as algebraic spaces over $Spec(\mathbf{Z})$, see Properties of Spaces, Definition 41.3.1. □

Lemma 42.5.12. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{P}$ be any of the separation axioms of Definition 42.5.2. The following are equivalent*

(1) *f is $\mathcal{P}$,*

(2) *for every scheme Z and morphism $Z \to Y$ the base change $Z \times_Y X \to Z$ of f is $\mathcal{P}$,*
(3) *for every affine scheme Z and every morphism $Z \to Y$ the base change $Z \times_Y X \to Z$ of f is $\mathcal{P}$,*
(4) *for every affine scheme Z and every morphism $Z \to Y$ the algebraic space $Z \times_Y X$ is $\mathcal{P}$ (see Properties of Spaces, Definition 41.3.1),*
(5) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that the base change $V \times_Y X \to V$ has $\mathcal{P}$, and*
(6) *there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ has $\mathcal{P}$.*

Proof. We will repeatedly use Lemma 42.5.4 without further mention. In particular, it is clear that (1) implies (2) and (2) implies (3).

Let us prove that (3) and (4) are equivalent. Note that if Z is an affine scheme, then the morphism $Z \to Spec(\mathbf{Z})$ is a separated morphism as a morphism of algebraic spaces over $Spec(\mathbf{Z})$. If $Z \times_Y X \to Z$ is $\mathcal{P}$, then $Z \times_Y X \to Spec(\mathbf{Z})$ is $\mathcal{P}$ as a composition (see Lemma 42.5.8). Hence the algebraic space $Z \times_Y X$ is $\mathcal{P}$. Conversely, if the algebraic space $Z \times_Y X$ is $\mathcal{P}$, then $Z \times_Y X \to Spec(\mathbf{Z})$ is $\mathcal{P}$, and hence by Lemma 42.5.10 we see that $Z \times_Y X \to Z$ is $\mathcal{P}$.

Let us prove that (3) implies (5). Assume (3). Let V be a scheme and let $V \to Y$ be étale surjective. We have to show that $V \times_Y X \to V$ has property $\mathcal{P}$. In other words, we have to show that the morphism

$$V \times_Y X \longrightarrow (V \times_Y X) \times_V (V \times_Y X) = V \times_Y X \times_Y X$$

has the corresponding property (i.e., is a closed immersion, immersion, or quasi-compact). Let $V = \bigcup V_j$ be an affine open covering of V. By assumption we know that each of the morphisms

$$V_j \times_Y X \longrightarrow V_j \times_Y X \times_Y X$$

does have the corresponding property. Since being a closed immersion, immersion, quasi-compact immersion, or quasi-compact is Zariski local on the target, and since the V_j cover V we get the desired conclusion.

Let us prove that (5) implies (1). Let $V \to Y$ be as in (5). Then we have the fibre product diagram

$$\begin{tikzcd}
V \times_Y X \arrow[r] \arrow[d] & X \arrow[d] \\
V \times_Y X \times_Y X \arrow[r] & X \times_Y X
\end{tikzcd}$$

By assumption the left vertical arrow is a closed immersion, immersion, quasi-compact immersion, or quasi-compact. It follows from Spaces, Lemma 40.5.6 that also the right vertical arrow is a closed immersion, immersion, quasi-compact immersion, or quasi-compact.

It is clear that (1) implies (6) by taking the covering $Y = Y$. Assume $Y = \bigcup Y_i$ is as in (6). Choose schemes V_i and surjective étale morphisms $V_i \to Y_i$. Note that the morphisms $V_i \times_Y X \to V_i$ have $\mathcal{P}$ as they are base changes of the morphisms $f^{-1}(Y_i) \to Y_i$. Set $V = \coprod V_i$. Then $V \to Y$ is a morphism as in (5) (details omitted). Hence (6) implies (5) and we are done. $\square$

Lemma 42.5.13. *Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S.*

1. *The morphism f is locally separated.*
2. *The morphism f is (quasi-)separated in the sense of Definition 42.5.2 above if and only if f is (quasi-)separated in the sense of Section 42.3.*

In particular, if $f : X \to Y$ is a morphism of schemes over S, then f is (quasi-)separated in the sense of Definition 42.5.2 if and only if f is (quasi-)separated as a morphism of schemes.

Proof. This is the equivalence of (1) and (2) of Lemma 42.5.12 combined with the fact that any morphism of schemes is locally separated, see Schemes, Lemma 21.21.2. ☐

42.6. Surjective morphisms

We have already defined in Section 42.3 what it means for a representable morphism of algebraic spaces to be surjective.

Lemma 42.6.1. *Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. Then f is surjective if and only if $|f| : |X| \to |Y|$ is surjective.*

Proof. Namely, if $f : X \to Y$ is representable, then it is surjective if and only if for every scheme T and every morphism $T \to Y$ the base change $f_T : T \times_Y X \to T$ of f is a surjective morphism of schemes, in other words, if and only if $|f_T|$ is surjective. By Properties of Spaces, Lemma 41.4.3 the map $|T \times_Y X| \to |T| \times_{|Y|} |X|$ is always surjective. Hence $|f_T| : |T \times_Y X| \to |T|$ is surjective if $|f| : |X| \to |Y|$ is surjective. Conversely, if $|f_T|$ is surjective for every $T \to Y$ as above, then by taking T to be the spectrum of a field we conclude that $|X| \to |Y|$ is surjective. ☐

This clears the way for the following definition.

Definition 42.6.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say f is *surjective* if the map $|f| : |X| \to |Y|$ of associated topological spaces is surjective.

Lemma 42.6.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

1. *f is surjective,*
2. *for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is surjective,*
3. *for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is surjective,*
4. *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is a surjective morphism,*
5. *there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that the composition $f \circ \varphi$ is surjective,*
6. *there exists a commutative diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U, V are schemes and the vertical arrows are surjective étale such that the top horizontal arrow is surjective, and

(7) *there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is surjective.*

Proof. Omitted. $\square$

Lemma 42.6.4. *The composition of surjective morphisms is surjective.*

Proof. This is immediate from the definition. $\square$

Lemma 42.6.5. *The base change of a surjective morphism is surjective.*

Proof. Follows immediately from Properties of Spaces, Lemma 41.4.3. $\square$

42.7. Open morphisms

For a representable morphism of algebraic spaces we have already defined (in Section 42.3) what it means to be universally open. Hence before we give the natural definition we check that it agrees with this in the representable case.

Lemma 42.7.1. *Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. The following are equivalent*

(1) *f is universally open, and*
(2) *for every morphism of algebraic spaces $Z \to Y$ the morphism of topological spaces $|Z \times_Y X| \to |Z|$ is open.*

Proof. Assume (1), and let $Z \to Y$ be as in (2). Choose a scheme V and a surjective étale morphism $V \to Y$. By assumption the morphism of schemes $V \times_Y X \to V$ is universally open. By Properties of Spaces, Section 41.4 in the commutative diagram

$$\begin{tikzcd} {|V \times_Y X|} \arrow[r] \arrow[d] & {|Z \times_Y X|} \arrow[d] \\ {|V|} \arrow[r] & {|Z|} \end{tikzcd}$$

the horizontal arrows are open and surjective, and moreover

$$|V \times_Y X| \longrightarrow |V| \times_{|Z|} |Z \times_Y X|$$

is surjective. Hence as the left vertical arrow is open it follows that the right vertical arrow is open. This proves (2). The implication (2) $\Rightarrow$ (1) is immediate from the definitions. $\square$

Thus we may use the following natural definition.

Definition 42.7.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is *open* if the map of topological spaces $|f| : |X| \to |Y|$ is open.
(2) We say f is *universally open* if for every morphism of algebraic spaces $Z \to Y$ the morphism of topological spaces

$$|Z \times_Y X| \to |Z|$$

is open, i.e., the base change $Z \times_Y X \to Z$ is open.

Note that an étale morphism of algebraic spaces is universally open, see Properties of Spaces, Definition 41.13.2 and Lemmas 41.13.7 and 41.13.5.

Lemma 42.7.3. *The base change of a universally open morphism of algebraic spaces by any morphism of algebraic spaces is universally open.*

Proof. This is immediate from the definition. □

Lemma 42.7.4. *The composition of a pair of (universally) open morphisms of algebraic spaces is (universally) open.*

Proof. Omitted. □

Lemma 42.7.5. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent*

(1) *f is universally open,*
(2) *for every scheme Z and every morphism $Z \to Y$ the projection $|Z \times_Y X| \to |Z|$ is open,*
(3) *for every affine scheme Z and every morphism $Z \to Y$ the projection $|Z \times_Y X| \to |Z|$ is open, and*
(4) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is a universally open morpism of algebraic spaces, and*
(5) *there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is universally open.*

Proof. We omit the proof that (1) implies (2), and that (2) implies (3).

Assume (3). Choose a surjective étale morphism $V \to Y$. We are going to show that $V \times_Y X \to V$ is a universally open morphism of algebraic spaces. Let $Z \to V$ be a morphism from an algebraic space to V. Let $W \to Z$ be a surjective étale morphism where $W = \coprod W_i$ is a disjoint union of affine schemes, see Properties of Spaces, Lemma 41.6.1. Then we have the following commutative diagram

$$\begin{tikzcd}
\coprod_i |W_i \times_Y X| \arrow[r, equal] \arrow[d] & |W \times_Y X| \arrow[r] \arrow[d] & |Z \times_Y X| \arrow[r, equal] \arrow[d] & |Z \times_V (V \times_Y X)| \arrow[dl] \\
\coprod |W_i| \arrow[r, equal] & |W| \arrow[r] & |Z| &
\end{tikzcd}$$

We have to show the south-east arrow is open. The middle horizontal arrows are surjective and open (Properties of Spaces, Lemma 41.13.7). By assumption (3), and the fact that W_i is affine we see that the left vertical arrows are open. Hence it follows that the right vertical arrow is open.

Assume $V \to Y$ is as in (4). We will show that f is universally open. Let $Z \to Y$ be a morphism of algebraic spaces. Consider the diagram

$$\begin{tikzcd}
|(V \times_Y Z) \times_V (V \times_Y X)| \arrow[r, equal] \arrow[dr] & |V \times_Y X| \arrow[r] \arrow[d] & |Z \times_Y X| \arrow[d] \\
& |V \times_Y Z| \arrow[r] & |Z|
\end{tikzcd}$$

The south-west arrow is open by assumption. The horizontal arrows are surjective and open because the corresponding morphisms of algebraic spaces are étale (see Properties of Spaces, Lemma 41.13.7). It follows that the right vertical arrow is open.

Of course (1) implies (5) by taking the covering $Y = Y$. Assume $Y = \bigcup Y_i$ is as in (5). Then for any $Z \to Y$ we get a corresponding Zariski covering $Z = \bigcup Z_i$ such that the base change of f to Z_i is open. By a simple topological argument this implies that $Z \times_Y X \to Z$ is open. Hence (1) holds. □

Lemma 42.7.6. *Let S be a scheme. Let $p : X \to Spec(k)$ be a morphism of algebraic spaces over S where k is a field. Then $p : X \to Spec(k)$ is universally open.*

Proof. Choose a scheme U and a surjective étale morphism $U \to X$. The composition $U \to Spec(k)$ is universally open (as a morphism of schemes) by Morphisms, Lemma 24.22.4. Let $Z \to Spec(k)$ be a morphism of schemes. Then $U \times_{Spec(k)} Z \to X \times_{Spec(k)} Z$ is surjective, see Lemma 42.6.5. Hence the first of the maps

$$|U \times_{Spec(k)} Z| \to |X \times_{Spec(k)} Z| \to |Z|$$

is surjective. Since the composition is open by the above we conclude that the second map is open as well. Whence p is universally open by Lemma 42.7.5. □

42.8. Submersive morphisms

Definition 42.8.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is *submersive*[2] if the continuous map $|X| \to |Y|$ is submersive, see Topology, Definition 5.15.1.
(2) We say f is *universally submersive* if for every morphism of algebraic spaces $Y' \to Y$ the base change $Y' \times_Y X \to Y'$ is submersive.

We note that a submersive morphism is in particular surjective.

42.9. Quasi-compact morphisms

By Section 42.3 we know what it means for a representable morphism of algebraic spaces to be quasi-compact. In order to formulate the definition for a general morphism of algebraic spaces we make the following observation.

Lemma 42.9.1. *Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is quasi-compact, and*
(2) *for every quasi-compact algebraic space Z and any morphism $Z \to Y$ the algebraic space $Z \times_Y X$ is quasi-compact.*

Proof. Assume (1), and let $Z \to Y$ be a morphism of algebraic spaces with Z quasi-compact. By Properties of Spaces, Definition 41.5.1 there exists a quasi-compact scheme U and a surjective étale morphism $U \to Z$. Since f is representable and quasi-compact we see by definition that $U \times_Y X$ is a scheme, and that $U \times_Y X \to U$ is quasi-compact. Hence $U \times_Y X$ is a quasi-compact scheme. The morphism $U \times_Y X \to Z \times_Y X$ is étale and surjective (as the base change of the representable étale and surjective morphism $U \to Z$, see Section 42.3). Hence by definition $Z \times_Y X$ is quasi-compact.

Assume (2). Let $Z \to Y$ be a morphism, where Z is a scheme. We have to show that $p : Z \times_Y X \to Z$ is quasi-compact. Let $U \subset Z$ be affine open. Then $p^{-1}(U) = U \times_Y Z$ and the scheme $U \times_Y Z$ is quasi-compact by assumption (2). Hence p is quasi-compact, see Schemes, Section 21.19. □

This motivates the following definition.

Definition 42.9.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say f is *quasi-compact* if for every quasi-compact algebraic space Z and morphism $Z \to Y$ the fibre product $Z \times_Y X$ is quasi-compact.

[2] This is very different from the notion of a submersion of differential manifolds.

By Lemma 42.9.1 above this agrees with the already existing notion for representable morphisms of algebraic spaces.

Lemma 42.9.3. *The base change of a quasi-compact morphism of algebraic spaces by any morphism of algebraic spaces is quasi-compact.*

Proof. Omitted. □

Lemma 42.9.4. *The composition of a pair of quasi-compact morphisms of algebraic spaces is quasi-compact.*

Proof. Omitted. □

Lemma 42.9.5. *Let S be a scheme.*

(1) *If $X \to Y$ is a surjective morphism of algebraic spaces over S, and X is quasi-compact then Y is quasi-compact.*
(2) *If*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

is a commutative diagram of morphisms of algebraic spaces over S and f is surjective and p is quasi-compact, then q is quasi-compact.

Proof. Assume X is quasi-compact and $X \to Y$ is surjective. By Definition 42.6.2 the map $|X| \to |Y|$ is surjective, hence we see Y is quasi-compact by Properties of Spaces, Lemma 41.5.2 and the topological fact that the image of a quasi-compact space under a continuous map is quasi-compact, see Topology, Lemma 5.9.5. Let f, p, q be as in (2). Let $T \to Z$ be a morphism whose source is a quasi-compact algebraic space. By assumption $T \times_Z X$ is quasi-compact. By Lemma 42.6.5 the morphism $T \times_Z X \to T \times_Z Y$ is surjective. Hence by part (1) we see $T \times_Z Y$ is quasi-compact too. Thus q is quasi-compact. □

Lemma 42.9.6. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $g : Y' \to Y$ be a universally open and surjective morphism of algebraic spaces such that the base change $f' : X' \to Y'$ is quasi-compact. Then f is quasi-compact.*

Proof. Let $Z \to Y$ be a morphism of algebraic spaces with Z quasi-compact. As g is universally open and surjective, we see that $Y' \times_Y Z \to Z$ is open and surjective. As every point of $|Y' \times_Y Z|$ has a fundamental system of quasi-compact open neighbourhoods (see Properties of Spaces, Lemma 41.5.5) we can find a quasi-compact open $W \subset |Y' \times_Y Z|$ which surjects onto Z. Denote $f'' : W \times_Y X \to W$ the base change of f' by $W \to Y'$. By assumption $W \times_Y X$ is quasi-compact. As $W \to Z$ is surjective we see that $W \times_Y X \to Z \times_Y X$ is surjective. Hence $Z \times_Y X$ is quasi-compact by Lemma 42.9.5. Thus f is quasi-compact. □

Lemma 42.9.7. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is quasi-compact,*
(2) *for every scheme Z and any morphism $Z \to Y$ the morphism of algebraic spaces $Z \times_Y X \to Z$ is quasi-compact,*
(3) *for every affine scheme Z and any morphism $Z \to Y$ the algebraic space $Z \times_Y X$ is quasi-compact,*

(4) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is a quasi-compact morphism of algebraic spaces, and*
(5) *there exists a surjective étale morphism $Y' \to Y$ of algebraic spaces such that $Y' \times_Y X \to Y'$ is a quasi-compact morphism of algebraic spaces, and*
(6) *there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is quasi-compact.*

Proof. We will use Lemma 42.9.3 without further mention. It is clear that (1) implies (2) and that (2) implies (3). Assume (3). Let Z be a quasi-compact algebraic space over S, and let $Z \to Y$ be a morphism. By Properties of Spaces, Lemma 41.6.3 there exists an affine scheme U and a surjective étale morphism $U \to Z$. Then $U \times_Y X \to Z \times_Y X$ is a surjective morphism of algebraic spaces, see Lemma 42.6.5. By assumption $|U \times_Y X|$ is quasi-compact. It surjects onto $|Z \times_Y X|$, hence we conclude that $|Z \times_Y X|$ is quasi-compact, see Topology, Lemma 5.9.5. This proves that (3) implies (1).

The implications (1) ⇒ (4), (4) ⇒ (5) are clear. The implication (5) ⇒ (1) follows from Lemma 42.9.6 and the fact that an étale morphism of algebraic spaces is universally open (see discussion following Definition 42.7.2).

Of course (1) implies (6) by taking the covering $Y = Y$. Assume $Y = \bigcup Y_i$ is as in (6). Let Z be affine and let $Z \to Y$ be a morphism. Then there exists a finite standard affine covering $Z = Z_1 \cup \ldots \cup Z_n$ such that each $Z_j \to Y$ factors through Y_{i_j} for some i_j. Hence the algebraic space

$$Z_j \times_Y X = Z_j \times_{Y_{i_j}} f^{-1}(Y_{i_j})$$

is quasi-compact. Since $Z \times_Y X = \bigcup_{j=1,\ldots,n} Z_j \times_Y X$ is a Zariski covering we see that $|Z \times_Y X| = \bigcup_{j=1,\ldots,n} |Z_j \times_Y X|$ (see Properties of Spaces, Lemma 41.4.8) is a finite union of quasi-compact spaces, hence quasi-compact. Thus we see that (6) implies (3). □

The following (and the next) lemma guarantees in particular that a morphism $X \to Spec(A)$ is quasi-compact as soon as X is a quasi-compact algebraic space

Lemma 42.9.8. *Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of algebraic spaces over S. If $g \circ f$ is quasi-compact and g is quasi-separated then f is quasi-compact.*

Proof. This is true because f equals the composition $(1, f) : X \to X \times_Z Y \to Y$. The first map is quasi-compact by Lemma 42.5.7 because it is a section of the quasi-separated morphism $X \times_Z Y \to X$ (a base change of g, see Lemma 42.5.4). The second map is quasi-compact as it is the base change of f, see Lemma 42.9.3. And compositions of quasi-compact morphisms are quasi-compact, see Lemma 42.9.4. □

Lemma 42.9.9. *Let $f : X \to Y$ be a morphism of algebraic spaces over a scheme S.*
(1) *If X is quasi-compact and Y is quasi-separated, then f is quasi-compact.*
(2) *If X is quasi-compact and quasi-separated and Y is quasi-separated, then f is quasi-compact and quasi-separated.*
(3) *A fibre product of quasi-compact and quasi-separated algebraic spaces is quasi-compact and quasi-separated.*

Proof. Part (1) follows from Lemma 42.9.8 with $Z = S = Spec(\mathbf{Z})$. Part (2) follows from (1) and Lemma 42.5.10. For (3) let $X \to Y$ and $Z \to Y$ be morphisms of quasi-compact and quasi-separated algebraic spaces. Then $X \times_Y Z \to Z$ is quasi-compact and quasi-separated as a base change of $X \to Y$ using (2) and Lemmas 42.9.3 and 42.5.4. Hence $X \times_Y Z$ is quasi-compact and quasi-separated as an algebraic space quasi-compact and quasi-separated over Z, see Lemmas 42.5.9 and 42.9.4. ☐

42.10. Universally closed morphisms

For a representable morphism of algebraic spaces we have already defined (in Section 42.3) what it means to be universally closed. Hence before we give the natural definition we check that it agrees with this in the representable case.

Lemma 42.10.1. *Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. The following are equivalent*

(1) *f is universally closed, and*
(2) *for every morphism of algebraic spaces $Z \to Y$ the morphism of topological spaces $|Z \times_Y X| \to |Z|$ is closed.*

Proof. Assume (1), and let $Z \to Y$ be as in (2). Choose a scheme V and a surjective étale morphism $V \to Y$. By assumption the morphism of schemes $V \times_Y X \to V$ is universally closed. By Properties of Spaces, Section 41.4 in the commutative diagram

$$\begin{tikzcd} {|V \times_Y X|} \arrow[r] \arrow[d] & {|Z \times_Y X|} \arrow[d] \\ {|V|} \arrow[r] & {|Z|} \end{tikzcd}$$

the horizontal arrows are open and surjective, and moreover

$$|V \times_Y X| \longrightarrow |V| \times_{|Z|} |Z \times_Y X|$$

is surjective. Hence as the left vertical arrow is closed it follows that the right vertical arrow is closed. This proves (2). The implication (2) $\Rightarrow$ (1) is immediate from the definitions. ☐

Thus we may use the following natural definition.

Definition 42.10.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is *closed* if the map of topological spaces $|X| \to |Y|$ is closed.
(2) We say f is *universally closed* if for every morphism of algebraic spaces $Z \to Y$ the morphism of topological spaces
$$|Z \times_Y X| \to |Z|$$
is closed, i.e., the base change $Z \times_Y X \to Z$ is closed.

Lemma 42.10.3. *The base change of a universally closed morphism of algebraic spaces by any morphism of algebraic spaces is universally closed.*

Proof. This is immediate from the definition. ☐

Lemma 42.10.4. *The composition of a pair of (universally) closed morphisms of algebraic spaces is (universally) closed.*

Proof. Omitted. ☐

Lemma 42.10.5. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent*

(1) *f is universally closed,*

(2) *for every scheme* Z *and every morphism* $Z \to Y$ *the projection* $|Z \times_Y X| \to |Z|$ *is closed,*
(3) *for every affine scheme* Z *and every morphism* $Z \to Y$ *the projection* $|Z \times_Y X| \to |Z|$ *is closed,*
(4) *there exists a scheme* V *and a surjective étale morphism* $V \to Y$ *such that* $V \times_Y X \to V$ *is a universally closed morpism of algebraic spaces, and*
(5) *there exists a Zariski covering* $Y = \bigcup Y_i$ *such that each of the morphisms* $f^{-1}(Y_i) \to Y_i$ *is universally closed.*

Proof. We omit the proof that (1) implies (2), and that (2) implies (3).

Assume (3). Choose a surjective étale morphism $V \to Y$. We are going to show that $V \times_Y X \to V$ is a universally closed morpism of algebraic spaces. Let $Z \to V$ be a morphism from an algebraic space to V. Let $W \to Z$ be a surjective étale morphism where $W = \coprod W_i$ is a disjoint union of affine schemes, see Properties of Spaces, Lemma 41.6.1. Then we have the following commutative diagram

$$\begin{tikzcd}
\coprod_i |W_i \times_Y X| \arrow[r, equal] \arrow[d] & |W \times_Y X| \arrow[r] \arrow[d] & |Z \times_Y X| \arrow[r, equal] \arrow[d] & |Z \times_V (V \times_Y X)| \arrow[dl] \\
\coprod |W_i| \arrow[r, equal] & |W| \arrow[r] & |Z| &
\end{tikzcd}$$

We have to show the south-east arrow is closed. The middle horizontal arrows are surjective and open (Properties of Spaces, Lemma 41.13.7). By assumption (3), and the fact that W_i is affine we see that the left vertical arrows are closed. Hence it follows that the right vertical arrow is closed.

Assume (4). We will show that f is universally closed. Let $Z \to Y$ be a morphism of algebraic spaces. Consider the diagram

$$\begin{tikzcd}
|(V \times_Y Z) \times_V (V \times_Y X)| \arrow[r, equal] \arrow[dr] & |V \times_Y X| \arrow[r] \arrow[d] & |Z \times_Y X| \arrow[d] \\
& |V \times_Y Z| \arrow[r] & |Z|
\end{tikzcd}$$

The south-west arrow is closed by assumption. The horizontal arrows are surjective and open because the corresponding morphisms of algebraic spaces are étale (see Properties of Spaces, Lemma 41.13.7). It follows that the right vertical arrow is closed.

Of course (1) implies (5) by taking the covering $Y = Y$. Assume $Y = \bigcup Y_i$ is as in (5). Then for any $Z \to Y$ we get a corresponding Zariski covering $Z = \bigcup Z_i$ such that the base change of f to Z_i is closed. By a simple topological argument this implies that $Z \times_Y X \to Z$ is closed. Hence (1) holds. ☐

Example 42.10.6. Strange example of a universally closed morphism. Let $\mathbf{Q} \subset k$ be a field of characteristic zero. Let $X = \mathbf{A}^1_k/\mathbf{Z}$ as in Spaces, Example 40.14.8. We claim the structure morphism $p : X \to Spec(k)$ is universally closed. Namely, if Z/k is a scheme, and $T \subset |X \times_k Z|$ is closed, then T corresponds to a $\mathbf{Z}$-invariant closed subset of $T' \subset |\mathbf{A}^1 \times Z|$. It is easy to see that this implies that T' is the inverse image of a subset T'' of Z. By Morphisms, Lemma 24.24.10 we have that $T'' \subset Z$ is closed. Of course T'' is the image of T. Hence p is universally closed by Lemma 42.10.5.

Lemma 42.10.7. *Let S be a scheme. A universally closed morphism of algebraic spaces over S is quasi-compact.*

Proof. This proof is a repeat of the proof in the case of schemes, see Morphisms, Lemma 24.40.9. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume that f is not quasi-compact. Our goal is to show that f is not universally closed. By Lemma 42.9.7 there exists an affine scheme Z and a morphism $Z \to Y$ such that $Z \times_Y X \to Z$ is not quasi-compact. To achieve our goal it suffices to show that $Z \times_Y X \to Z$ is not universally closed, hence we may assume that $Y = Spec(B)$ for some ring B.

Write $X = \bigcup_{i \in I} X_i$ where the X_i are quasi-compact open subspaces of X. For example, choose a surjective étale morphism $U \to X$ where U is a scheme, choose an affine open covering $U = \bigcup U_i$ and let $X_i \subset X$ be the image of U_i. We will use later that the morphisms $X_i \to Y$ are quasi-compact, see Lemma 42.9.8. Let $T = Spec(B[a_i; i \in I])$. Let $T_i = D(a_i) \subset T$. Let $Z \subset T \times_Y X$ be the reduced closed subspace whose underlying closed set of points is $|T \times_Y Z| \setminus \bigcup_{i \in I} |T_i \times_Y X_i|$, see Properties of Spaces, Lemma 41.9.1. (Note that by the results of Section 42.4 the algebraic spaces $T_i \times_Y X_i$ are open subspaces of $T \times_Y X$.) Here is a diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[rd] & T \times_Y X \arrow[r, "q"'] \arrow[d, "f_T"] & X \arrow[d, "f"] \\ & T \arrow[r, "p"] & Y \end{tikzcd}$$

It suffices to prove that the image $f_T(|Z|)$ is not closed in $|T|$.

We claim there exists a point $y \in Y$ such that there is no affine open neighborhood V of y in Y such that X_V is quasi-compact. If not then we can cover Y with finitely many such V and for each V the morphism $Y_V \to V$ is quasi-compact by Lemma 42.9.8 and then Lemma 42.9.7 implies f quasi-compact, a contradiction. Fix a $y \in Y$ as in the claim.

Let $t \in T$ be the point lying over y with $\kappa(t) = \kappa(y)$ such that $a_i = 1$ in $\kappa(t)$ for all i. Suppose $z \in |Z|$ with $f_T(z) = t$. Then $q(t) \in X_i$ for some i. Hence $f_T(z) \notin T_i$ by construction of Z, which contradicts the fact that $t \in T_i$ by construction. Hence we see that $t \in |T| \setminus f_T(|Z|)$.

Assume $f_T(|Z|)$ is closed in $|T|$. Then there exists an element $g \in B[a_i; i \in I]$ with $f_T(|Z|) \subset V(g)$ but $t \notin V(g)$. Hence the image of g in $\kappa(t)$ is nonzero. In particular some coefficient of g has nonzero image in $\kappa(y)$. Hence this coefficient is invertible on some affine open neighborhood V of y. Let J be the finite set of $j \in I$ such that the variable a_j appears in g. Since X_V is not quasi-compact and each $X_{i,V}$ is quasi-compact, we may choose a point $x \in |X_V| \setminus \bigcup_{j \in J} |X_{j,V}|$. In other words, $x \in |X| \setminus \bigcup_{j \in J} |X_j|$ and x lies above some $v \in V$. Since g has a coefficient that is invertible on V, we can find a point $t' \in T$ lying above v such that $t' \notin V(g)$ and $t' \in V(a_i)$ for all $i \notin J$. This is true because $V(a_i; i \in I \setminus J) = Spec(B[a_j; j \in J])$ and the set of points of this scheme lying over v is bijective with $Spec(\kappa(v)[a_j; j \in J])$ and g restricts to a nonzero element of this polynomial ring by construction. In other words $t' \notin T_i$ for each $i \notin J$. By Properties of Spaces, Lemma 41.4.3 we can find a point z of $X \times_Y T$ mapping to $x \in X$ and to $t' \in T$. Since $x \notin |X_j|$ for $j \in J$ and $t' \notin T_i$ for $i \in I \setminus J$ we see that $z \in |Z|$. On the other hand $f_T(z) = t' \notin V(g)$ which contradicts $f_T(Z) \subset V(g)$. Thus the assumption "$f_T(|Z|)$ closed" is wrong and we conclude indeed that f_T is not closed as desired. □

The target of a separated algebraic space under a surjective universally closed morphism is separated.

Lemma 42.10.8. *Let S be a scheme. Let B be an algebraic space over S. Let $f : X \to Y$ be a surjective universally closed morphism of algebraic spaces over B.*

(1) *If X is quasi-separated, then Y is quasi-separated.*
(2) *If X is separated, then Y is separated.*
(3) *If X is quasi-separated over B, then Y is quasi-separated over B.*
(4) *If X is separated over B, then Y is separated over B.*

Proof. Parts (1) and (2) are a consequence of (3) and (4) for $S = B = Spec(\mathbf{Z})$ (see Properties of Spaces, Definition 41.3.1). Consider the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\Delta_{X/B}"'] \arrow[d] & X \times_B X \arrow[d] \\ Y \arrow[r, "\Delta_{Y/B}"] & Y \times_B Y \end{tikzcd}$$

The left vertical arrow is surjective (i.e., universally surjective). The right vertical arrow is universally closed as a composition of the universally closed morphisms $X \times_B X \to X \times_B Y \to Y \times_B Y$. Hence it is also quasi-compact, see Lemma 42.10.7.

Assume X is quasi-separated over B, i.e., $\Delta_{X/B}$ is quasi-compact. Then if Z is quasi-compact and $Z \to Y \times_B Y$ is a morphism, then $Z \times_{Y \times_B Y} X \to Z \times_{Y \times_B Y} Y$ is surjective and $Z \times_{Y \times_B Y} X$ is quasi-compact by our remarks above. We conclude that $\Delta_{Y/B}$ is quasi-compact, i.e., Y is quasi-separated over B.

Assume X is quasi-separated over B, i.e., $\Delta_{X/B}$ is a closed immersion. Then if Z is affine, and $Z \to Y \times_B Y$ is a morphism, then $Z \times_{Y \times_B Y} X \to Z \times_{Y \times_B Y} Y$ is surjective and $Z \times_{Y \times_B Y} X \to Z$ is universally closed by our remarks above. We conclude that $\Delta_{Y/B}$ is universally closed. It follows that $\Delta_{Y/B}$ is representable, locally of finite type, a monomorphism (see Lemma 42.5.1) and universally closed, hence a closed immersion, see Étale Morphisms, Lemma 37.7.2 (and also the abstract principle Spaces, Lemma 40.5.8). Thus Y is separated over B. □

42.11. Valuative criteria

The formulation of the existence part of the valuative criterion is slightly different for morphisms of algebraic spaces, since it may be necessary to extend the fraction field of the valuation ring. See Example 42.11.4. Here is the definition.

Definition 42.11.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say *f satisfies the uniqueness part of the valuative criterion* if given any commutative solid diagram

$$\begin{tikzcd} Spec(K) \arrow[r] \arrow[d] & X \arrow[d] \\ Spec(A) \arrow[r] \arrow[ru, dashed] & Y \end{tikzcd}$$

where A is a valuation ring with field of fractions K, there exists at most one dotted arrow (without requiring existence). We say *f satisfies the existence part of the valuative criterion* if given any solid diagram as above there exists an extension $K \subset K'$ of fields, a valuation ring $A' \subset K'$ dominating A and a morphism $Spec(A') \to X$ such that the following diagram commutes

$$\begin{tikzcd} Spec(K') \arrow[r] \arrow[d] & Spec(K) \arrow[r] & X \arrow[d] \\ Spec(A') \arrow[r] \arrow[urr] & Spec(A) \arrow[r] & Y \end{tikzcd}$$

We say f *satisfies the valuative criterion* if f satisfies both the existence and uniqueness part.

It turns out that for algebraic spaces, it always sufffices to take a finite separable extension $K \subset K'$ above. See Lemma 42.11.3. Before we prove it we show that the criterion is identical to the criterion as formulated for morphisms of schemes in case the morphism of algebraic spaces is representable.

Lemma 42.11.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is representable. The following are equivalent*

(1) *f satisfies the existence part of the valuation criterion as in Definition 42.11.1 above, and*
(2) *given any commutative solid diagram*

$$\begin{tikzcd} Spec(K) \arrow[r] \arrow[d] & X \arrow[d] \\ Spec(A) \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where A is a valuation ring with field of fractions K, there exists a dotted arrow, i.e., f satisfies the existence part of the valuative criterion as in Schemes, Definition 21.20.3.

Proof. It suffices to show that given a commutative diagram of the form

$$\begin{tikzcd} Spec(K') \arrow[r] \arrow[d] & Spec(K) \arrow[r] & X \arrow[d] \\ Spec(A') \arrow[r] \arrow[urr, "\varphi"] & Spec(A) \arrow[r] & Y \end{tikzcd}$$

as in Definition 42.11.1, then we can find a morphism $Spec(A) \to X$ fitting into the diagram too. Set $X_A = Spec(A) \times_Y Y$. As f is representable we see that X_A is a scheme. The morphism φ gives a morphism $\varphi' : Spec(A') \to X_A$. Let $x \in X_A$ be the image of the closed point of $\varphi' : Spec(A') \to X_A$. Then we have the following commutative diagram of rings

$$\begin{tikzcd} K' & K \arrow[l] & \mathcal{O}_{X_A,x} \arrow[l] \arrow[dll] \\ A' \arrow[u] & A \arrow[l] & A \arrow[l] \arrow[u] \end{tikzcd}$$

Since A is a valuation ring, and since A' dominates A, we see that $K \cap A' = A$. Hence the ring map $\mathcal{O}_{X_A,x} \to K$ has image contained in A. Whence a morphism $Spec(A) \to X_A$ (see Schemes, Section 21.13) as desired. $\square$

Lemma 42.11.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent*

(1) *f satisfies the existence part of the valuation criterion as in Definition 42.11.1, and*
(2) *f satisfies the existence part of the valuation criterion as in Definition 42.11.1 modified by requiring the extension $K \subset K'$ to be finite separable.*

Proof. We have to show that (1) implies (2). Suppose given a diagram

$$\begin{tikzcd} Spec(K') \arrow[r] \arrow[d] & Spec(K) \arrow[r] & X \arrow[d] \\ Spec(A') \arrow[r] \arrow[urr] & Spec(A) \arrow[r] & Y \end{tikzcd}$$

as in Definition 42.11.1 with $K \subset K'$ arbitrary. Choose a scheme U and a surjective étale morphism $U \to X$. Then

$$Spec(A') \times_X U \longrightarrow Spec(A')$$

is surjective étale. Let p be a point of $Spec(A') \times_X U$ mapping to the closed point of $Spec(A')$. Let $p' \rightsquigarrow p$ be a generalization of p mapping to the generic point of $Spec(A')$. Such a generalization exists because generalizations lift along flat morphisms of schemes, see Morphisms, Lemma 24.24.8. Then p' corresponds to a point of the scheme $Spec(K') \times_X U$. Note that

$$Spec(K') \times_X U = Spec(K') \times_{Spec(K)} (Spec(K) \times_X U)$$

Hence p' maps to a point $q' \in Spec(K) \times_X U$ whose residue field is a finite separable extension of K. Finally, $p' \rightsquigarrow p$ maps to a specialization $u' \rightsquigarrow u$ on the scheme U. With all this notation we get the following diagram of rings

$$\begin{tikzcd} \kappa(p') & & \kappa(q') \arrow[ll] & \kappa(u') \arrow[l] \\ & \mathcal{O}_{Spec(A') \times_X U, p} \arrow[ul] & & \mathcal{O}_{U,u} \arrow[ll] \arrow[u] \\ K' \arrow[uu] & A' \arrow[l] \arrow[u] & A \arrow[l] \arrow[uu] & \end{tikzcd}$$

This means that the ring $B \subset \kappa(q')$ generated by the images of A and $\mathcal{O}_{U,u}$ maps to a subring of $\kappa(p')$ contained in the image B' of $\mathcal{O}_{Spec(A') \times_X U, p} \to \kappa(p')$. Note that B' is a local ring. Let $\mathfrak{m} \subset B$ be the maximal ideal. By construction $A \cap \mathfrak{m}$, (resp. $\mathcal{O}_{U,u} \cap \mathfrak{m}$, resp. $A' \cap \mathfrak{m}$) is the maximal ideal of A (resp. $\mathcal{O}_{U,u}$, resp. A'). Set $\mathfrak{q} = B \cap \mathfrak{m}$. This is a prime ideal such that $A \cap \mathfrak{q}$ is the maximal ideal of A. Hence $B_{\mathfrak{q}} \subset \kappa(q')$ is a local ring dominating A. By Algebra, Lemma 7.46.2 we can find a valuation ring $A_1 \subset \kappa(q')$ with field of fractions $\kappa(q')$ dominating $B_{\mathfrak{q}}$. The (local) ring map $\mathcal{O}_{U,u} \to A_1$ gives a morphism $Spec(A_1) \to U \to X$ such that the diagram

$$\begin{tikzcd} Spec(\kappa(q')) \arrow[r] \arrow[d] & Spec(K) \arrow[r] & X \arrow[d] \\ Spec(A_1) \arrow[r] \arrow[urr] & Spec(A) \arrow[r] & Y \end{tikzcd}$$

is commutative. Since $f.f.(A_1) = \kappa(q') \supset K$ is finite separable by construction the lemma is proved. $\square$

Example 42.11.4. Consider the algebraic space X constructed in Spaces, Example 40.14.2. Recall that it is the affine line with zero doubled in a Galois twisted relative to a degree two Galois extension $k \subset k'$. As such it comes with a morphism

$$\pi : X \longrightarrow S = \mathbf{A}^1_k$$

which is quasi-compact. We claim that π is universally closed. Namely, after base change by $Spec(k') \to Spec(k)$ the morphism π is identified with the morphism

$$\text{affine line with zero doubled} \longrightarrow \text{affine line}$$

which is universally closed (some details omitted). Since the morphism $Spec(k') \to Spec(k)$ is universally closed and surjective, a diagram chase shows that π is universally closed. On the other hand, consider the diagram

$$\begin{tikzcd}
Spec(k((x))) \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\
Spec(k[[x]]) \arrow[r] \arrow[ur, dotted] & \mathbf{A}^1_k
\end{tikzcd}$$

Since the unique point of X above $0 \in \mathbf{A}^1_k$ corresponds to a monomorphism $Spec(k') \to X$ it is clear there cannot exist a dotted arrow! This shows that a finite separable field extension is needed in general.

Lemma 42.11.5. *The base change of a morphism of algebraic spaces which satisfies the existence part of (resp. uniqueness part of) the valuative criterion by any morphism of algebraic spaces satisfies the existence part of (resp. uniqueness part of) the valuative criterion.*

Proof. Let $f : X \to Y$ be a morphism of algebraic spaces over the scheme S. Let $Z \to Y$ be any morphism of algebraic spaces over S. Consider a solid commutative diagram of the following shape

$$\begin{tikzcd}
Spec(K) \arrow[r] \arrow[d] & Z \times_Y X \arrow[r] \arrow[d] & X \arrow[d] \\
Spec(A) \arrow[r] \arrow[ur, dashed] \arrow[urr, dashed] & Z \arrow[r] & Y
\end{tikzcd}$$

Then the set of north-west dotted arrows making the diagram commute is in 1-1 correspondence with the set of west-north-west dotted arrows making the diagram commute. This proves the lemma in the case of ``uniqueness''. For the existence part, assume f satisfies the existence part of the valuative criterion. If we are given a solid commutative diagram as above, then by assumption there exists an extension $K \subset K'$ of fields and a valuation ring $A' \subset K'$ dominating A and a morphism $Spec(A') \to X$ fitting into the following commutative diagram

$$\begin{tikzcd}
Spec(K') \arrow[r] \arrow[d] & Spec(K) \arrow[r] & Z \times_Y X \arrow[r] & X \arrow[d] \\
Spec(A') \arrow[r] \arrow[urrr] & Spec(A) \arrow[r] & Z \arrow[r] & Y
\end{tikzcd}$$

And by the remarks above the skew arrow corresponds to an arrow $Spec(A') \to Z \times_Y X$ as desired. $\square$

Lemma 42.11.6. *The composition of two morphisms of algebraic spaces which satisfy the (existence part of, resp. uniqueness part of) the valuative criterion satisfies the (existence part of, resp. uniqueness part of) the valuative criterion.*

Proof. Let $f : X \to Y$, $g : Y \to Z$ be morphisms of algebraic spaces over the scheme S. Consider a solid commutative diagram of the following shape

$$\begin{tikzcd}
Spec(K) \arrow[r] \arrow[dd] & X \arrow[d, "f"] \\
 & Y \arrow[d, "g"] \\
Spec(A) \arrow[r] \arrow[ruu, dashed] \arrow[ru, dashed] & Z
\end{tikzcd}$$

If we have the uniqueness part for g, then there exists at most one north-west dotted arrow making the diagram commute. If we also have the uniqueness part for f, then we have at most one north-north-west dotted arrow making the diagram commute. The proof in the existence case comes from contemplating the following diagram

$$\begin{tikzcd}
Spec(K'') \arrow[r] \arrow[dd] & Spec(K') \arrow[r] & Spec(K) \arrow[r] & X \arrow[d, "f"] \\
 & & & Y \arrow[d, "g"] \\
Spec(A'') \arrow[r] \arrow[rrruu] & Spec(A') \arrow[r] \arrow[rru] & Spec(A) \arrow[r] & Z
\end{tikzcd}$$

Namely, the existence part for g gives us the extension K', the valuation ring A' and the arrow $Spec(A') \to Y$, whereupon the existence part for f gives us the extension K'', the valuation ring A'' and the arrow $Spec(A'') \to X$. $\square$

42.12. Valuative criterion for universal closedness

This is a little more involved than in the case of schemes, especially since the most optimistic guess is wrong. See discussion below.

Lemma 42.12.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume*

(1) *f is quasi-compact, and*
(2) *f satisfies the existence part of the valuative criterion.*

Then f is universally closed.

Proof. By Lemmas 42.9.3 and 42.11.5 properties (1) and (2) are preserved under any base change. By Lemma 42.10.5 we only have to show that $T \times_Y X \to T$ is universally closed, whenever T is an affine scheme over S mapping into Y. Hence it suffices to prove: If Y is an affine scheme, $f : X \to Y$ is quasi-compact and satisfies the existence part of the valuative criterion, then $f : |X| \to |Y|$ is closed. In this situation X is a quasi-compact algebraic space. By Properties of Spaces, Lemma 41.6.3 there exists an affine scheme U and a surjective étale morphism $\varphi : U \to X$. Let $T \subset |X|$ closed. The inverse image $\varphi^{-1}(T) \subset U$ is closed, and hence is the set of points of an affine closed subscheme $Z \subset U$.

Thus, by Algebra, Lemma 7.36.5 we see that $f(T) = f(\varphi(|Z|)) \subset |Y|$ is closed if it is closed under specialization.

Let $y' \rightsquigarrow y$ be a specialization in Y with $y' \in f(T)$. Choose a point $x' \in T \subset |X|$ mapping to y' under f. We may represent x' by a morphism $Spec(K) \to X$ for some field K. Thus we have the following diagram

$$\begin{tikzcd} Spec(K) \arrow[r, "x'"'] \arrow[d] & X \arrow[d, "f"] \\ Spec(\mathcal{O}_{Y,y}) \arrow[r] & Y, \end{tikzcd}$$

see Schemes, Section 21.13 for the existence of the left vertical map. Choose a valuation ring $A \subset K$ dominating the image of the ring map $\mathcal{O}_{Y,y} \to K$ (this is possible since the image is a local ring and not a field as $y' \neq y$, see Algebra, Lemma 7.46.2). By assumption there exists a field extension $K \subset K'$ and a valuation ring $A' \subset K'$ dominating A, and a morphism $Spec(A') \to X$ fitting into the commutative diagram. Since A' dominates A, and A dominates $\mathcal{O}_{Y,y}$ we see that the closed point of $Spec(A')$ maps to a point $x \in X$ with $f(x) = y$ which is a specialization of x'. Hence $x \in T$ as T is closed, and hence $y \in f(T)$ as desired. □

We also want to prove the converse of Lemma 42.12.1. Namely, we would like to show, under additional conditions, that a quasi-compact morphism is universallly closed if and only if the existence part of the valuative criterion holds. Example 42.10.6 shows that $\mathbf{A}^1_k/\mathbf{Z} \to Spec(k)$ is universally closed, but it is easy to see that the existence part of the valuative criterion fails. Hence some additional hypothesis is needed. We address this in Decent Spaces, Section 43.12 when the source of the morphism is a decent space. See also Decent Spaces, Lemma 43.13.8 for a slight weakening of the hypothesis.

42.13. Valuative criterion of separatedness

Lemma 42.13.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is separated, then f satisfies the uniqueness part of the valuative criterion.*

Proof. Let a diagram as in Definition 42.11.1 be given. Suppose there are two distinct morphisms $a, b : Spec(A) \to X$ fitting into the diagram. Let $Z \subset Spec(A)$ be the equalizer of a and b. Then $Z = Spec(A) \times_{(a,b), X \times_Y X, \Delta} X$. If f is separated, then Δ is a closed immersion, and this is a closed subscheme of $Spec(A)$. By assumption it contains the generic point of $Spec(A)$. Since A is a domain this implies $Z = Spec(A)$. Hence $a = b$ as desired. □

Lemma 42.13.2. *(Valuative criterion separatedness.) Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume*

(1) *the morphism f is quasi-separated, and*
(2) *the morphism f satisfies the uniqueness part of the valuative criterion.*

Then f is separated.

Proof. Assumption (1) means $\Delta_{X/Y}$ is quasi-compact. We claim the morphism $\Delta_{X/Y} : X \to X \times_Y X$ satisfies the existence part of the valuative criterion. Let a solid commutative

diagram

$$\begin{tikzcd} Spec(K) \arrow[r] \arrow[d] & X \arrow[d] \\ Spec(A) \arrow[r] \arrow[ru, dashed] & X \times_Y X \end{tikzcd}$$

be given. The lower right arrow corresponds to a pair of morphisms $a, b : Spec(A) \to X$ over Y. By assumption (2) we see that $a = b$. Hence using a as the dotted arrow works. Hence Lemma 42.12.1 applies, and we see that $\Delta_{X/Y}$ is universally closed. Since always $\Delta_{X/Y}$ is locally of finite type and separated, we conclude from More on Morphisms, Lemma 33.29.5 that $\Delta_{X/Y}$ is a finite morphism (also, use the general principle of Spaces, Lemma 40.5.8). At this point $\Delta_{X/Y}$ is a representable, finite monomorphism, hence a closed immersion by Morphisms, Lemma 24.42.13. □

42.14. Monomorphisms

A representable morphism $X \to Y$ of algebraic spaces is a monomorphism according to Section 42.3 if for every scheme Z and morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is representable by a monomorphism of schemes. This means exactly that $Z \times_Y X \to Z$ is an injective map of sheaves on $(Sch/S)_{fppf}$. Since this is supposed to hold for all Z and all maps $Z \to Y$ this is in turn equivalent to the map $X \to Y$ being an injective map of sheaves on $(Sch/S)_{fppf}$. Thus we may define a monomorphism of a (possibly nonrepresentable[3]) morphism of algebraic spaces as follows.

Definition 42.14.1. Let S be a scheme. A morphism of algebraic spaces over S is called a *monomorphism* if it is an injective map of sheaves, i.e., a monomorphism in the category of sheaves on $(Sch/S)_{fppf}$.

The following lemma shows that this also means that it is a monomorphism in the category of algebraic spaces over S.

Lemma 42.14.2. *Let S be a scheme. Let $j : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *j is a monomorphism (as in Definition 42.14.1),*
(2) *j is a monomorphism in the category of algebraic spaces over S, and*
(3) *the diagonal morphism $\Delta_{X/Y} : X \to X \times_Y X$ is an isomorphism.*

Proof. Note that $X \times_Y X$ is both the fibre product in the category of sheaves on $(Sch/S)_{fppf}$ and the fibre product in the category of algebraic spaces over S, see Spaces, Lemma 40.7.3. The equivalence of (1) and (3) is a general characterization of injective maps of sheaves on any site. The equivalence of (2) and (3) is a characterization of monomorphisms in any category with fibre products. □

Lemma 42.14.3. *A monomorphism of algebraic spaces is separated.*

Proof. This is true because an isomorphism is a closed immersion, and Lemma 42.14.2 above. □

Lemma 42.14.4. *A composition of monomorphisms is a monomorphism.*

Proof. True because a composition of injective sheaf maps is injective. □

[3]We do not know whether or not every monomorphism of algebraic spaces is representable. If you do, please email stacks.project@gmail.com.

Lemma 42.14.5. *The base change of a monomorphism is a monomorphism.*

Proof. This is a general fact about fibre products in a category of sheaves. □

Lemma 42.14.6. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent*

(1) *f is a monomorphism,*
(2) *for every scheme Z and morphism $Z \to Y$ the base change $Z \times_Y X \to Z$ of f is a monomorphism,*
(3) *for every affine scheme Z and every morphism $Z \to Y$ the base change $Z \times_Y X \to Z$ of f is a monomorphism,*
(4) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that the base change $V \times_Y X \to V$ is a monomorphism, and*
(5) *there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is a monomorphism.*

Proof. We will use without further mention that a base change of a monomorphism is a monomorphism, see Lemma 42.14.5. In particular it is clear that (1) ⇒ (2) ⇒ (3) ⇒ (4) (by taking V to be a disjoint union of affine schemes étale over Y, see Properties of Spaces, Lemma 41.6.1). Let V be a scheme, and let $V \to Y$ be a surjective étale morphism. If $V \times_Y X \to V$ is a monomorphism, then it follows that $X \to Y$ is a monomorphism. Namely, given any cartesian diagram of sheaves

$$\begin{tikzcd} \mathcal{F} \arrow[r, "a"] \arrow[d, "b"] & \mathcal{G} \arrow[d, "c"] \\ \mathcal{H} \arrow[r, "d"] & \mathcal{I} \end{tikzcd} \qquad \mathcal{F} = \mathcal{H} \times_{\mathcal{I}} \mathcal{G}$$

if c is a surjection of sheaves, and a is injective, then also d is injective. Thus (4) implies (1). Proof of the equivalence of (5) and (1) is omitted. □

Lemma 42.14.7. *An immersion of algebraic spaces is a monomorphism. In particular, any immersion is separated.*

Proof. Let $f : X \to Y$ be an immersion of algebraic spaces. For any morphism $Z \to Y$ with Z representable the base change $Z \times_Y X \to Z$ is an immersion of schemes, hence a monomorphism, see Schemes, Lemma 21.23.7. Hence f is representable, and a monomorphism. □

We will improve on the following lemma in Decent Spaces, Lemma 43.14.1.

Lemma 42.14.8. *Let S be a scheme. Let k be a field and let $Z \to Spec(k)$ be a monomorphism of algebraic spaces over S. Then either $Z = \emptyset$ or $Z = Spec(k)$.*

Proof. By Lemmas 42.14.3 and 42.5.9 we see that Z is a separated algebraic space. Hence there exists an open dense subspace $Z' \subset Z$ which is a scheme, see Properties of Spaces, Proposition 41.10.3. By Schemes, Lemma 21.23.10 we see that either $Z' = \emptyset$ or $Z' \cong Spec(k)$. In the first case we conclude that $Z = \emptyset$ and in the second case we conclude that $Z' = Z = Spec(k)$ as $Z \to Spec(k)$ is a monomorphism which is an isomorphism over Z'. □

Lemma 42.14.9. *Let S be a scheme. If $X \to Y$ is a monomorphism of algebraic spaces over S, then $|X| \to |Y|$ is injective.*

Proof. Immediate from the definitions. □

42.15. Pushforward of quasi-coherent sheaves

We first prove a simple lemma that relates pushforward of sheaves of modules for a morphism of algebraic spaces to pushforward of sheaves of modules for a morphism of schemes.

Lemma 42.15.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $U \to X$ be a surjective étale morphism from a scheme to X. Set $R = U \times_X U$ and denote $t, s : R \to U$ the projection morphisms as usual. Denote $a : U \to Y$ and $b : R \to Y$ the induced morphisms. For any object $\mathcal{F}$ of $Mod(\mathcal{O}_X)$ there exists an exact sequence*

$$0 \to f_*\mathcal{F} \to a_*(\mathcal{F}|_U) \to b_*(\mathcal{F}|_R)$$

where the second arrow is the difference $t^* - s^*$.

Proof. We denote $\mathcal{F}$ also its extension to a sheaf of modules on $X_{spaces,étale}$, see Properties of Spaces, Remark 41.15.4. Let $V \to Y$ be an object of $Y_{étale}$. Then $V \times_Y X$ is an object of $X_{spaces,étale}$, and by definition $f_*\mathcal{F}(V) = \mathcal{F}(V \times_Y X)$. Since $U \to X$ is surjective étale, we see that $\{V \times_Y U \to V \times_Y X\}$ is a covering. Also, we have $(V \times_Y U) \times_X (V \times_Y U) = V \times_Y R$. Hence, by the sheaf condition of $\mathcal{F}$ on $X_{spaces,étale}$ we have a short exact sequence

$$0 \to \mathcal{F}(V \times_Y X) \to \mathcal{F}(V \times_Y U) \to \mathcal{F}(V \times_Y R)$$

where the second arrow is the difference of restricting via t or s. This exact sequence is functorial in V and hence we obtain the lemma. □

Let S be a scheme. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of representable algebraic spacdes X and Y over S. By Descent, Proposition 31.6.14 the functor $f_* : QCoh(X) \to QCoh(Y)$ agrees with the usual functor if we think of X and Y as schemes.

More generally, suppose $f : X \to Y$ is a representable, quasi-compact, and quasi-separated morphism of algebraic spaces over S. Let V be a scheme and let $V \to Y$ be an étale surjective morphism. Let $U = V \times_Y X$ and let $f' : U \to V$ be the base change of f. Then for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have

$$f'_*(\mathcal{F}|_U) = (f_*\mathcal{F})|_V, \tag{42.15.1.1}$$

see Properties of Spaces, Lemma 41.23.2. And because $f' : U \to V$ is a quasi-compact and quasi-separated morphism of schemes, by the remark of the preceding paragraph we may compute $f'_*(\mathcal{F}|_U)$ by thinking of $\mathcal{F}|_U$ as a quasi-coherent sheaf on the scheme U, and f' as a morphism of schemes. We will frequently use this without further mention.

The next level of generality is to consider an arbitrary quasi-compact and quasi-separated morphism of algebraic spaces.

Lemma 42.15.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is quasi-compact and quasi-separated, then f_* transforms quasi-coherent $\mathcal{O}_X$-modules into quasi-coherent $\mathcal{O}_Y$-modules.*

Proof. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. We have to show that $f_*\mathcal{F}$ is a quasi-coherent sheaf on Y. For this it suffices to show that for any affine scheme V and étale morphism $V \to Y$ the restriction of $f_*\mathcal{F}$ to V is quasi-coherent, see Properties of Spaces, Lemma 41.26.6. Let $f' : V \times_Y X \to V$ be the base change of f by $V \to Y$. Note that f' is also quasi-compact and quasi-separated, see Lemmas 42.9.3 and 42.5.4. By (42.15.1.1) we know that the restriction of $f_*\mathcal{F}$ to V is f'_* of the restriction of $\mathcal{F}$ to $V \times_Y X$. Hence we may replace f by f', and assume that Y is an affine scheme.

Assume Y is an affine scheme. Since f is quasi-compact we see that X is quasi-compact. Thus we may choose an affine scheme U and a surjective étale morphism $U \to X$, see Properties of Spaces, Lemma 41.6.3. By Lemma 42.15.1 we get an exact sequence

$$0 \to f_*\mathcal{F} \to a_*(\mathcal{F}|_U) \to b_*(\mathcal{F}|_R).$$

where $R = U \times_X U$. As $X \to Y$ is quasi-separated we see that $R \to U \times_Y U$ is a quasi-compact monomorphism. This implies that R is a quasi-compact separated scheme (as U and Y are affine at this point). Hence $a : U \to Y$ and $b : R \to Y$ are quasi-compact and quasi-separated morphisms of schemes. Thus by Descent, Proposition 31.6.14 the sheaves $a_*(\mathcal{F}|_U)$ and $b_*(\mathcal{F}|_R)$ are quasi-coherent (see also the discussion preceding this lemma). This implies that $f_*\mathcal{F}$ is a kernel of quasi-coherent modules, and hence itself quasi-coherent, see Properties of Spaces, Lemma 41.26.7. □

Higher direct images are discussed in Cohomology of Spaces, Section 49.4.

42.16. Closed immersions

In this section we elucidate some of the results obtained previously on immersions of algebraic spaces; it should parallel Morphisms, Section 24.2.

Lemma 42.16.1. *Let S be a scheme. Let X be an algebraic space over S. For every closed immersion $i : Z \to X$ the sheaf $i_*\mathcal{O}_Z$ is a quasi-coherent $\mathcal{O}_X$-module, the map $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective and its kernel is a quasi-coherent sheaf of ideals. The rule $Z \mapsto Ker(\mathcal{O}_X \to i_*\mathcal{O}_Z)$ defines an inclusion reversing bijection*

$$\begin{matrix} \textit{closed subschemes} \\ Z \subset X \end{matrix} \longrightarrow \begin{matrix} \textit{quasi-coherent sheaves} \\ \textit{of ideals } \mathcal{I} \subset \mathcal{O}_X \end{matrix}$$

Moreover, given a closed subscheme Z corresponding to the quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ a morphism of algebraic spaces $h : Y \to X$ factors through Z if and only if the map $h^*\mathcal{I} \to h^*\mathcal{O}_X = \mathcal{O}_Y$ is zero.

Proof. Let $U \to X$ be a surjective étale morphism whose source is a scheme. Consider the diagram

$$\begin{tikzcd} U \times_X Z \arrow[r] \arrow[d, "i'"] & Z \arrow[d, "i"] \\ U \arrow[r] & X \end{tikzcd}$$

By Lemma 42.4.1 we see that i is a closed immersion if and only if i' is a closed immersion. By Properties of Spaces, Lemma 41.23.2 we see that $i'_*\mathcal{O}_{U \times_X Z}$ is the restriction of $i_*\mathcal{O}_Z$ to U. Hence the assertions on $\mathcal{O}_X \to i_*\mathcal{O}_Z$ are equivalent to the corresponding assertions on $\mathcal{O}_U \to i'_*\mathcal{O}_{U \times_X Z}$. And since i' is a closed immersion of schemes, these results follow from Morphisms, Lemma 24.2.1.

Let us prove that given a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ the formula

$$Z(T) = \{h : T \to X \mid h^*\mathcal{I} \to \mathcal{O}_T \text{ is zero}\}$$

defines a closed subspace of X. It is clearly a subfunctor of X. To show that $Z \to X$ is representable by closed immersions, let $\varphi : U \to X$ be a morphism from a scheme towards X. Then $Z \times_X U$ is represented by the analogous subfunctor of U corresponding to the sheaf of ideals $\mathrm{Im}(\varphi^*\mathcal{I} \to \mathcal{O}_U)$. By Properties of Spaces, Lemma 41.26.2 the $\mathcal{O}_U$-module $\varphi^*\mathcal{I}$ is quasi-coherent on on U, and hence $\mathrm{Im}(\varphi^*\mathcal{I} \to \mathcal{O}_U)$ is a quasi-coherent sheaf of ideals on U. By Schemes, Lemma 21.4.6 we conclude that $Z \times_X U$ is represented by the closed subscheme of U associated to $\mathrm{Im}(\varphi^*\mathcal{I} \to \mathcal{O}_U)$. Thus Z is a closed subspace of X.

In the formula for Z above the inputs T are schemes since algebraic spaces are sheaves on $(Sch/S)_{fppf}$. We omit the verification that the same formula remains true if T is an algebraic space. □

Lemma 42.16.2. *A closed immersion of algebraic spaces is quasi-compact.*

Proof. This follows from Schemes, Lemma 21.19.5 by general principles, see Spaces, Lemma 40.5.8. □

Lemma 42.16.3. *A closed immersion of algebraic spaces is separated.*

Proof. This follows from Schemes, Lemma 21.23.7 by general principles, see Spaces, Lemma 40.5.8. □

Lemma 42.16.4. *Let S be a scheme. Let $i : Z \to X$ be a closed immersion of algebraic spaces over S.*

(1) *The functor*
$$i_{small,*} : Sh(Z_{\acute{e}tale}) \longrightarrow Sh(X_{\acute{e}tale})$$
is fully faithful and its essential image is those sheaves of sets $\mathcal{F}$ on $X_{\acute{e}tale}$ whose restriction to $X \setminus Z$ is isomorphic to $*$, and

(2) *the functor*
$$i_{small,*} : Ab(Z_{\acute{e}tale}) \longrightarrow Ab(X_{\acute{e}tale})$$
is fully faithful and its essential image is those abelian sheaves on $X_{\acute{e}tale}$ whose support is contained in Z.

In both cases i_{small}^{-1} is a left inverse to the functor $i_{small,*}$.

Proof. Let U be a scheme and let $U \to X$ be surjective étale. Set $V = Z \times_X U$. Then V is a scheme and $i' : V \to U$ is a closed immersion of schemes. By Properties of Spaces, Lemma 41.15.11 for any sheaf $\mathcal{G}$ on Z we have
$$(i_{small}^{-1} i_{small,*}\mathcal{G})|_V = (i')^{-1}_{small} i'_{small,*}(\mathcal{G}|_V)$$
By Étale Cohomology, Proposition 38.46.4 the map $(i')^{-1}_{small} i'_{small,*}(\mathcal{G}|_V) \to \mathcal{G}|_V$ is an isomorphism. Since $V \to Z$ is surjective and étale this implies that $i_{small}^{-1} i_{small,*}\mathcal{G} \to \mathcal{G}$ is an isomorphism. This clearly implies that $i_{small,*}$ is fully faithful, see Sites, Lemma 9.36.1. To prove the statement on the essential image, consider a sheaf of sets $\mathcal{F}$ on $X_{\acute{e}tale}$ whose restriction to $X \setminus Z$ is isomorphic to $*$. As in the proof of Étale Cohomology, Proposition 38.46.4 we consider the adjunction mapping
$$\mathcal{F} \longrightarrow i_{small,*} i_{small}^{-1} \mathcal{F}.$$
As in the first part we see that the restriction of this map to U is an isomorphism by the corresponding result for the case of schemes. Since U is an étale covering of X we conclude it is an isomorphism. □

The following lemma holds more generally in the setting of a closed immersion of topoi (insert future reference here).

Lemma 42.16.5. *Let S be a scheme. Let $i : Z \to X$ be a closed immersion of algebraic spaces over S. Let $\mathcal{A}$ be a sheaf of rings on $X_{étale}$. Let $\mathcal{B}$ be a sheaf of rings on $Z_{étale}$. Let $\varphi : \mathcal{A} \to i_{small,*}\mathcal{B}$ (or what is the same thing $\varphi : i_{small}^{-1}\mathcal{A} \to \mathcal{B}$) be a homomorphism of sheaves of rings. Then for any sheaf of $\mathcal{A}$-modules $\mathcal{F}$ the adjunction mapping $\mathcal{F} \to i_{small,*}i_{small}^{-1}\mathcal{F}$ induces an isomorphism*

$$\mathcal{F} \otimes_{\mathcal{A}} i_{small,*}\mathcal{B} \longrightarrow i_{small,*}(i_{small}^{-1}\mathcal{F} \otimes_{i_{small}^{-1}\mathcal{A}} \mathcal{B}).$$

Proof. During this proof we drop the subscript $_{small}$ from the notation. There is a map $i^{-1}\mathcal{F} \to i^{-1}\mathcal{F} \otimes_{i^{-1}\mathcal{A}} \mathcal{B}$ to which we can apply i_* and compose with the adjunction map:

$$\mathcal{F} \longrightarrow i_*(i^{-1}\mathcal{F}) \longrightarrow i_*(i^{-1}\mathcal{F} \otimes_{i^{-1}\mathcal{A}} \mathcal{B}).$$

The composition is $\mathcal{A}$-linear where $\mathcal{A}$ acts on the target via φ. Note that this target $i_*(i^{-1}\mathcal{F} \otimes_{i^{-1}\mathcal{A}} \mathcal{B})$ has a canonical $i_*\mathcal{B}$-module structure. Hence by the universal property of tensor product we obtain a map as in the lemma.

Let $\mathcal{G}$ be a sheaf of $i_*\mathcal{B}$-modules on $X_{étale}$. Since the support of the sheaf of rings $i_*\mathcal{B}$ is contained in Z we see that the support of $\mathcal{G}$ is contained in Z. Hence by Lemma 42.16.4 we conclude that there exists a unique sheaf of $\mathcal{B}$-modules $\mathcal{H}$ and an isomorphism $i_*\mathcal{H} = \mathcal{G}$ as $i_*\mathcal{B}$-modules. To show that the map of the lemma is an isomorphism we show that the right hand side of the arrow satisfies the universal property enjoyed by the tensor product on the left (i.e., we will use Yoneda's lemma, see Categories, Lemma 4.3.5). To see this we have to show that maps into $\mathcal{G}$ agree. This can be seen using the following sequence of canonical isomorphisms

$$\begin{aligned}
Hom_{i_*\mathcal{B}}(\mathcal{F} \otimes_{\mathcal{A}} i_*\mathcal{B}, \mathcal{G}) &= Hom_{\mathcal{A}}(\mathcal{F}, \mathcal{G}) \\
&= Hom_{\mathcal{A}}(\mathcal{F}, i_*(\mathcal{H})) \\
&= Hom_{i^{-1}\mathcal{A}}(i^{-1}\mathcal{F}, \mathcal{H}) \\
&= Hom_{\mathcal{B}}(i^{-1}\mathcal{F} \otimes_{i^{-1}\mathcal{A}} \mathcal{B}, \mathcal{H}) \\
&= Hom_{i_*\mathcal{B}}(i_*(i^{-1}\mathcal{F} \otimes_{i^{-1}\mathcal{A}} \mathcal{B}), i_*\mathcal{H}) \\
&= Hom_{i_*\mathcal{B}}(i_*(i^{-1}\mathcal{F} \otimes_{i^{-1}\mathcal{A}} \mathcal{B}), \mathcal{G})
\end{aligned}$$

The fifth equality holds because of the equivalence of categories in Lemma 42.16.4. □

42.17. Closed immersions and quasi-coherent sheaves

This section is the analogue of Morphisms, Section 24.3.

Lemma 42.17.1. *Let S be a scheme. Let $i : Z \to X$ be a closed immersion of algebraic spaces over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaf of ideals cutting out Z.*

1. *For any $\mathcal{O}_X$-module $\mathcal{F}$ the adjunction map $\mathcal{F} \to i_*i^*\mathcal{F}$ induces an isomorphism $\mathcal{F}/\mathcal{I}\mathcal{F} \cong i_*i^*\mathcal{F}$.*
2. *The functor i^* is a left inverse to i_*, i.e., for any $\mathcal{O}_Z$-module $\mathcal{G}$ the adjunction map $i^*i_*\mathcal{G} \to \mathcal{G}$ is an isomorphism.*
3. *The functor*
$$i_* : QCoh(\mathcal{O}_Z) \longrightarrow QCoh(\mathcal{O}_X)$$
is exact, fully faithful, with essential image those quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}$ such that $\mathcal{I}\mathcal{F} = 0$.

Proof. During this proof we work exclusively with sheaves on the small étale sites, and we use $i_*, i^{-1}, \ldots$ to denote pushforward and pullback of sheaves of abelian groups instead of $i_{small,*}, i_{small}^{-1}$.

Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. By Lemma 42.16.5 we see that $i_*i^*\mathcal{F} = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_Z$. By Lemma 42.16.1 we see that we have a short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_X \to i_*\mathcal{O}_Z \to 0$$

It follows from properties of the tensor product that $\mathcal{F} \otimes_{\mathcal{O}_X} i_*\mathcal{O}_Z = \mathcal{F}/\mathcal{I}\mathcal{F}$. This proves (1) (except that we omit the verification that the map is induced by the adjunction mapping).

Let $\mathcal{G}$ be any $\mathcal{O}_Z$-module. By Lemma 42.16.4 we see that $i^{-1}i_*\mathcal{G} = \mathcal{G}$. Hence to prove (2) we have to show that the canonical map $\mathcal{G} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{O}_Z \to \mathcal{G}$ is an isomorphism. This follows from general properties of tensor products if we can show that $i^{-1}\mathcal{O}_X \to \mathcal{O}_Z$ is surjective. By Lemma 42.16.4 it suffices to prove that $i_*i^{-1}\mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective. Since the surjective map $\mathcal{O}_X \to i_*\mathcal{O}_Z$ factors through this map we see that (2) holds.

Finally we prove the most interesting part of the lemma, namely part (3). A closed immersion is quasi-compact and separated, see Lemmas 42.16.2 and 42.16.3. Hence Lemma 42.15.2 applies and the pushforward of a quasi-coherent sheaf on Z is indeed a quasi-coherent sheaf on X. Thus we obtain our functor $i_*^{QCoh} : QCoh(\mathcal{O}_Z) \to QCoh(\mathcal{O}_X)$. It is clear from part (2) that i_*^{QCoh} is fully faithful since it has a left inverse, namely i^*.

Now we turn to the description of the essential image of the functor i_*. It is clear that $\mathcal{I}(i_*\mathcal{G}) = 0$ for any $\mathcal{O}_Z$-module, since $\mathcal{I}$ is the kernel of the map $\mathcal{O}_X \to i_*\mathcal{O}_Z$ which is the map we use to put an $\mathcal{O}_X$-module structure on $i_*\mathcal{G}$. Next, suppose that $\mathcal{F}$ is any quasi-coherent $\mathcal{O}_X$-module such that $\mathcal{I}\mathcal{F} = 0$. Then we see that $\mathcal{F}$ is an $i_*\mathcal{O}_Z$-module because $i_*\mathcal{O}_Z = \mathcal{O}_X/\mathcal{I}$. Hence in particular its support is contained in Z. We apply Lemma 42.16.4 to see that $\mathcal{F} \cong i_*\mathcal{G}$ for some $\mathcal{O}_Z$-module $\mathcal{G}$. The only small detail left over is to see why $\mathcal{G}$ is quasi-coherent. This is true because $\mathcal{G} \cong i^*\mathcal{F}$ by part (2) and Properties of Spaces, Lemma 41.26.2. $\square$

Let $i : Z \to X$ be a closed immersion of algebraic spaces. Because of the lemma above we often, by abuse of notation, denote $\mathcal{F}$ the sheaf $i_*\mathcal{F}$ on X.

Lemma 42.17.2. *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{G} \subset \mathcal{F}$ be a $\mathcal{O}_X$-submodule. There exists a unique quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{G}' \subset \mathcal{G}$ with the following property: For every quasi-coherent $\mathcal{O}_X$-module $\mathcal{H}$ the map*

$$Hom_{\mathcal{O}_X}(\mathcal{H}, \mathcal{G}') \longrightarrow Hom_{\mathcal{O}_X}(\mathcal{H}, \mathcal{G})$$

is bijective. In particular $\mathcal{G}'$ is the largest quasi-coherent $\mathcal{O}_X$-submodule of $\mathcal{F}$ contained in $\mathcal{G}$.

Proof. Let $\mathcal{G}_a$, $a \in A$ be the set of quasi-coherent $\mathcal{O}_X$-submodules contained in $\mathcal{G}$. Then the image $\mathcal{G}'$ of

$$\bigoplus\nolimits_{a\in A} \mathcal{G}_a \longrightarrow \mathcal{F}$$

is quasi-coherent as the image of a map of quasi-coherent sheaves on X is quasi-coherent and since a direct sum of quasi-coherent sheaves is quasi-coherent, see Properties of Spaces, Lemma 41.26.7. The module $\mathcal{G}'$ is contained in $\mathcal{G}$. Hence this is the largest quasi-coherent $\mathcal{O}_X$-module contained in $\mathcal{G}$.

To prove the formula, let $\mathcal{H}$ be a quasi-coherent $\mathcal{O}_X$-module and let $\alpha : \mathcal{H} \to \mathcal{G}$ be an $\mathcal{O}_X$-module map. The image of the composition $\mathcal{H} \to \mathcal{G} \to \mathcal{F}$ is quasi-coherent as the image of a map of quasi-coherent sheaves. Hence it is contained in $\mathcal{G}'$. Hence α factors through $\mathcal{G}'$ as desired. $\square$

Lemma 42.17.3. *Let S be a scheme. Let $i : Z \to X$ be a closed immersion of algebraic spaces over S. There is a functor[4] $i^! : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Z)$ which is a right adjoint to i_*. (Compare Modules, Lemma 15.6.3.)*

Proof. Given quasi-coherent $\mathcal{O}_X$-module $\mathcal{G}$ we consider the subsheaf $\mathcal{H}_Z(\mathcal{G})$ of $\mathcal{G}$ of local sections annihilated by $\mathcal{I}$. By Lemma 42.17.2 there is a canonical largest quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{H}_Z(\mathcal{G})'$. By construction we have

$$Hom_{\mathcal{O}_X}(i_*\mathcal{F}, \mathcal{H}_Z(\mathcal{G})') = Hom_{\mathcal{O}_X}(i_*\mathcal{F}, \mathcal{G})$$

for any quasi-coherent $\mathcal{O}_Z$-module $\mathcal{F}$. Hence we can set $i^!\mathcal{G} = i^*(\mathcal{H}_Z(\mathcal{G})')$. Details omitted. $\square$

42.18. Universally injective morphisms

We have already defined in Section 42.3 what it means for a representable morphism of algebraic spaces to be universally injective. For a field K over S (recall this means that we are given a structure morphism $Spec(K) \to S$) and an algebraic space X over S we write $X(K) = Mor_S(Spec(K), X)$. We first translate the condition for representable morphisms into a condition on the functor of points.

Lemma 42.18.1. *Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. Then f is universally injective if and only if for all fields K the map $X(K) \to Y(K)$ is injective.*

Proof. We are going to use Morphisms, Lemma 24.10.2 without further mention. Suppose that f is universally injective. Then for any field K and any morphism $Spec(K) \to Y$ the morphism of schemes $Spec(K) \times_Y X \to Spec(K)$ is universally injective. Hence there exists at most one section of the morphism $Spec(K) \times_Y X \to Spec(K)$. Hence the map $X(K) \to Y(K)$ is injective. Conversely, suppose that for every field K the map $X(K) \to Y(K)$ is injective. Let $T \to Y$ be a morphism from a scheme into Y, and consider the base change $f_T : T \times_Y X \to T$. For any field K we have

$$(T \times_Y X)(K) = T(K) \times_{Y(K)} X(K)$$

by definition of the fibre product, and hence the injectivity of $X(K) \to Y(K)$ garantees the injectivity of $(T \times_Y X)(K) \to T(K)$ which means that f_T is universally injective as desired. $\square$

Next, we translate the property that the transformation between field valued points is injective into something more geometric.

Lemma 42.18.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

1. *the map $X(K) \to Y(K)$ is injective for every field K over S*
2. *for every morphism $Y' \to Y$ of algebraic spaces over S the induced map $|Y' \times_Y X| \to |Y'|$ is injective, and*
3. *the diagonal morphism $X \to X \times_Y X$ is surjective.*

[4] This is likely nonstandard notation.

Proof. Assume (1). Let $g : Y' \to Y$ be a morphism of algebraic spaces, and denote $f' : Y' \times_Y X \to Y'$ the base change of f. Let K_i, $i = 1, 2$ be fields and let $\varphi_i : Spec(K_i) \to Y' \times_Y X$ be morphisms such that $f' \circ \varphi_1$ and $f' \circ \varphi_2$ define the same element of $|Y'|$. By definition this means there exists a field Ω and embeddings $\alpha_i : K_i \subset \Omega$ such that the two morphisms $f' \circ \varphi_i \circ \alpha_i : Spec(\Omega) \to Y'$ are equal. Here is the corresponding commutative diagram

$$\begin{tikzcd}
Spec(\Omega) \arrow[r, "\alpha_2"'] \arrow[rd, "\alpha_1"] \arrow[rrd, bend right, to=3-3] & Spec(K_2) \arrow[rd, "\varphi_2"] & & \\
& Spec(K_1) \arrow[r, "\varphi_1"] & Y' \times_Y X \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
& & Y' \arrow[r, "g"] & Y.
\end{tikzcd}$$

In particular the compositions $g \circ f' \circ \varphi_i \circ \alpha_i$ are equal. By assumption (1) this implies that the morphism $g' \circ \varphi_i \circ \alpha_i$ are equal, where $g' : Y' \times_Y X \to X$ is the projection. By the universal property of the fibre product we conclude that the morphisms $\varphi_i \circ \alpha_i : Spec(\Omega) \to Y' \times_Y X$ are equal. In other words φ_1 and φ_2 define the same point of $Y' \times_Y X$. We conclude that (2) holds.

Assume (2). Let K be a field over S, and let $a, b : Spec(K) \to X$ be two morphisms such that $f \circ a = f \circ b$. Denote $c : Spec(K) \to Y$ the common value. By assumption $|Spec(K) \times_{c,Y} X| \to |Spec(K)|$ is injective. This means there exists a field Ω and embeddings $\alpha_i : K \to \Omega$ such that

$$\begin{tikzcd}
Spec(\Omega) \arrow[r, "\alpha_1"'] \arrow[d, "\alpha_2"'] & Spec(K) \arrow[d, "a"] \\
Spec(K) \arrow[r, "b"] & Spec(K) \times_{c,Y} X
\end{tikzcd}$$

is commutative. Composing with the projection to $Spec(K)$ we see that $\alpha_1 = \alpha_2$. Denote the common value α. Then we see that $\{\alpha : Spec(\Omega) \to Spec(K)\}$ is a fpqc covering of $Spec(K)$ such that the two morphisms a, b become equal on the members of the covering. By Properties of Spaces, Lemma 41.14.1 we conclude that $a = b$. We conclude that (1) holds.

Assume (3). Let $x, x' \in |X|$ be a pair of points such that $f(x) = f(x')$ in $|Y|$. By Properties of Spaces, Lemma 41.4.3 we see there exists a $x'' \in |X \times_Y X|$ whose projections are x and x'. By assumption and Properties of Spaces, Lemma 41.4.4 there exists a $x''' \in |X|$ with $\Delta_{X/Y}(x''') = x''$. Thus $x = x'$. In other words f is injective. Since condition (3) is stable under base change we see that f satsifies (2).

Assume (2). Then in particular $|X \times_Y X| \to |X|$ is injective which implies immediately that $|\Delta_{X/Y}| : |X| \to |X \times_Y X|$ is surjective, which implies that $\Delta_{X/Y}$ is surjective by Properties of Spaces, Lemma 41.4.4. □

By the two lemmas above the following definition does not conflict with the already defined notion of a universally injective representable morphism of algebraic spaces.

Definition 42.18.3. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say f is *universally injective* if for every morphism $Y' \to Y$ the induced map $|Y' \times_Y X| \to |Y'|$ is injective.

To be sure this means that any or all of the equivalent conditions of Lemma 42.18.2 hold.

Remark 42.18.4. A universally injective morphism of schemes is separated, see Morphisms, Lemma 24.10.3. This is not the case for morphisms of algebraic spaces. Namely, the algebraic space $X = \mathbf{A}^1_k/\{x \sim -x \mid x \neq 0\}$ constructed in Spaces, Example 40.14.1 comes equipped with a morphsm $X \to \mathbf{A}^1_k$ which maps the point with coordinate x to the point with coordinate x^2. This is an isomorphism away from 0, and there is a unique point of X lying above 0. As X isn't separated this is a universally injective morphism of algebraic spaces which is not separated.

Lemma 42.18.5. *The base change of a universally injective morphism is universally injective.*

Proof. Omitted. Hint: This is formal. $\square$

Lemma 42.18.6. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

1. *f is universally injective,*
2. *for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is universally injective,*
3. *for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is universally injective,*
4. *there exists a scheme Z and a surjective morphism $Z \to Y$ such that $Z \times_Y X \to Z$ is universally injective, and*
5. *there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is universally injective.*

Proof. We will use that being universally injective is preserved under base change (Lemma 42.18.5) without further mention in this proof. It is clear that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4).

Assume $g : Z \to Y$ as in (4). Let $y : Spec(K) \to Y$ be a morphism from the spectrum of a field into Y. By assumption we can find an extension field $\alpha : K \subset K'$ and a morphism $z : Spec(K') \to Z$ such that $y \circ \alpha = g \circ z$ (with obvious abuse of notation). By assumption the morphism $Z \times_Y X \to Z$ is universally injective, hence there is at most one lift of $g \circ z : Spec(K') \to Y$ to a morphism into X. Since $\{\alpha : Spec(K') \to Spec(K)\}$ is a fpqc covering this implies there is at most one lift of $y : Spec(K) \to Y$ to a morphism into X, see Properties of Spaces, Lemma 41.14.1. Thus we see that (1) holds.

We omit the verification that (5) is equivalent to (1). $\square$

Lemma 42.18.7. *A composition of universally injective morphisms is universally injective.*

Proof. Omitted. $\square$

42.19. Affine morphisms

We have already defined in Section 42.3 what it means for a representable morphism of algebraic spaces to be affine.

Lemma 42.19.1. *Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. Then f is affine if and only if for all affine schemes Z and morphisms $Z \to Y$ the scheme $X \times_Y Z$ is affine.*

Proof. This follows directly from the definition of an affine morphism of schemes (Morphisms, Definition 24.11.1). $\square$

This clears the way for the following definition.

Definition 42.19.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say f is *affine* if for every affine scheme Z and morphism $Z \to Y$ the algebraic space $X \times_Y Z$ is representable by an affine scheme.

Lemma 42.19.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is representable and affine,*
(2) *f is affine,*
(3) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is affine, and*
(4) *there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is affine.*

Proof. It is clear that (1) implies (2) and that (2) implies (3) by taking V to be a disjoint union of affines étale over Y, see Properties of Spaces, Lemma 41.6.1. Assume $V \to Y$ is as in (3). Then for every affine open W of V we see that $W \times_Y X$ is an affine open of $V \times_Y X$. Hence by Properties of Spaces, Lemma 41.10.1 we conclude that $V \times_Y X$ is a scheme. Moreover the morphism $V \times_Y X \to V$ is affine. This means we can apply Spaces, Lemma 40.11.3 because the class of affine morphisms satisfies all the required properties (see Morphisms, Lemmas 24.11.8 and Descent, Lemmas 31.19.16 and 31.33.1). The conclusion of applying this lemma is that f is representable and affine, i.e., (1) holds.

The equivalence of (1) and (4) follows from the fact that being affine is Zariski local on the target (the reference above shows that being affine is in fact fpqc local on the target). □

Lemma 42.19.4. *The composition of affine morphisms is affine.*

Proof. Omitted. □

Lemma 42.19.5. *The base change of an affine morphism is affine.*

Proof. Omitted. □

42.20. Quasi-affine morphisms

We have already defined in Section 42.3 what it means for a representable morphism of algebraic spaces to be quasi-affine.

Lemma 42.20.1. *Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. Then f is quasi-affine if and only if for all affine schemes Z and morphisms $Z \to Y$ the scheme $X \times_Y Z$ is quasi-affine.*

Proof. This follows directly from the definition of a quasi-affine morphism of schemes (Morphisms, Definition 24.12.1). □

This clears the way for the following definition.

Definition 42.20.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say f is *quasi-affine* if for every affine scheme Z and morphism $Z \to Y$ the algebraic space $X \times_Y Z$ is representable by a quasi-affine scheme.

Lemma 42.20.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is representable and quasi-affine,*
(2) *f is quasi-affine,*

(3) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is quasi-affine, and*
(4) *there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is quasi-affine.*

Proof. It is clear that (1) implies (2) and that (2) implies (3) by taking V to be a disjoint union of affines étale over Y, see Properties of Spaces, Lemma 41.6.1. Assume $V \to Y$ is as in (3). Then for every affine open W of V we see that $W \times_Y X$ is a quasi-affine open of $V \times_Y X$. Hence by Properties of Spaces, Lemma 41.10.1 we conclude that $V \times_Y X$ is a scheme. Moreover the morphism $V \times_Y X \to V$ is quasi-affine. This means we can apply Spaces, Lemma 40.11.3 because the class of quasi-affine morphisms satisfies all the required properties (see Morphisms, Lemmas 24.12.5 and Descent, Lemmas 31.19.18 and 31.34.1). The conclusion of applying this lemma is that f is representable and quasi-affine, i.e., (1) holds.

The equivalence of (1) and (4) follows from the fact that being quasi-affine is Zariski local on the target (the reference above shows that being quasi-affine is in fact fpqc local on the target). □

Lemma 42.20.4. *The composition of quasi-affine morphisms is quasi-affine.*

Proof. Omitted. □

Lemma 42.20.5. *The base change of a quasi-affine morphism is quasi-affine.*

Proof. Omitted. □

42.21. Types of morphisms étale local on source-and-target

Given a property of morphisms of schemes which is *étale local on the source-and-target*, see Descent, Definition 31.28.3 we may use it to define a corresponding property of morphisms of algebraic spaces, namely by imposing either of the equivalent conditions of the lemma below.

Lemma 42.21.1. *Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on the source-and-target. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Consider commutative diagrams*

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "a"'] & V \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where U and V are schemes and the vertical arrows are étale. The following are equivalent

(1) *for any diagram as above the morphism h has property $\mathcal{P}$, and*
(2) *for some diagram as above with $a : U \to X$ surjective the morphism h has property $\mathcal{P}$.*

If X and Y are representable, then this is also equivalent to f (as a morphism of schemes) having property $\mathcal{P}$. If $\mathcal{P}$ is also preserved under any base change, and fppf local on the base, then for representable morphisms f this is also equivalent to f having property $\mathcal{P}$ in the sense of Section 42.3.

Proof. Let us prove the equivalence of (1) and (2). The implication (1) ⇒ (2) is immediate (taking into account Spaces, Lemma 40.11.4). Assume

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd} \qquad \begin{tikzcd} U' \arrow[r, "h'"] \arrow[d] & V' \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

are two diagrams as in the lemma. Assume $U \to X$ is surjective and h has propery $\mathcal{P}$. To show that (2) implies (1) we have to prove that h' has $\mathcal{P}$. To do this consider the diagram

$$\begin{tikzcd} U \arrow[d, "h"] & U \times_X U' \arrow[l] \arrow[r] \arrow[d, "(h,h')"] & U' \arrow[d, "h'"] \\ V & V \times_Y V' \arrow[l] \arrow[r] & V' \end{tikzcd}$$

By Descent, Lemma 31.28.5 we see that h has $\mathcal{P}$implies (h, h') has $\mathcal{P}$and since $U \times_X U' \to U'$ is surjective this implies (by the same lemma) that h' has $\mathcal{P}$.

If X and Y are representable, then Descent, Lemma 31.28.5 applies which shows that (1) and (2) are equivalent to f having $\mathcal{P}$.

Finally, suppose f is representable, and U, V, a, b, h are as in part (2) of the lemma, and that $\mathcal{P}$is preserved under arbitrary base change. We have to show that for any scheme Z and morphism $Z \to X$ the base change $Z \times_Y X \to Z$ has property $\mathcal{P}$. Consider the diagram

$$\begin{tikzcd} Z \times_Y U \arrow[r] \arrow[d] & Z \times_Y V \arrow[d] \\ Z \times_Y X \arrow[r] & Z \end{tikzcd}$$

Note that the top horizontal arrow is a base change of h and hence has property $\mathcal{P}$. The left vertical arrow is étale and surjective and the right vertical arrow is étale. Thus Descent, Lemma 31.28.5 once again kicks in and shows that $Z \times_Y X \to Z$ has property $\mathcal{P}$. □

Definition 42.21.2. Let S be a scheme. Let $\mathcal{P}$be a property of morphisms of schemes which is étale local on the source-and-target. We say a morphism $f : X \to Y$ of algebraic spaces over S *has property* $\mathcal{P}$if the equivalent conditions of Lemma 42.21.1 hold.

Given a property of morphisms of germs of schemes which is *étale local on the source-and-target*, see Descent, Definition 31.29.1 we may use it to define a corresponding property of morphisms of algebraic spaces at a point, namely by imposing either of the equivalent conditions of the lemma below.

Lemma 42.21.3. *Let $\mathcal{Q}$ be a property of morphisms of germs which is étale local on the source-and-target. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $x \in |X|$ be a point of X. Consider the diagrams*

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "a"'] & V \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \qquad \begin{tikzcd} u \arrow[r] \arrow[d] & v \arrow[d] \\ x \arrow[r] & y \end{tikzcd}$$

where U and V are schemes, a, b are étale, and u, v, x, y are points of the corresponding spaces. The following are equivalent

(1) *for any diagram as above we have* $\mathcal{Q}((U,u) \to (V,v))$*, and*
(2) *for some diagram as above we have* $\mathcal{Q}((U,u) \to (V,v))$*.*

If X *and* Y *are representable, then this is also equivalent to* $\mathcal{Q}((X,x) \to (Y,y))$*.*

Proof. Omitted. Hint: Very similar to the proof of Lemma 42.21.1. □

Definition 42.21.4. Let $\mathcal{Q}$ be a property of morphisms of germs of schemes which is étale local on the source-and-target. Let S be a scheme. Given a morphism $f : X \to Y$ of algebraic spaces over S and a point $x \in |X|$ we say that *f has property $\mathcal{Q}$ at x* if the equivalent conditions of Lemma 42.21.3 hold.

The following lemma should not be used blindly to go from a property of morphisms to a property of morphisms at a point. For example if $\mathcal{P}$ is the property of being flat, then the property Q in the following lemma means ``$f$ is flat in an open neighbourhood of $x$'' which is not the same as ``f is flat at x''.

Lemma 42.21.5. *Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on the source-and-target. Consider the property $\mathcal{Q}$ of morphisms of germs associated to $\mathcal{P}$ in Descent, Lemma 31.29.2. Then*

(1) *$\mathcal{Q}$ is étale local on the source-and-target.*
(2) *given a morphism of algebraic spaces $f : X \to Y$ and $x \in |X|$ the following are equivalent*
    (a) *f has $\mathcal{Q}$ at x, and*
    (b) *there is an open neighbourhood $X' \subset X$ of x such that $X' \to Y$ has $\mathcal{P}$.*
(3) *given a morphism of algebraic spaces $f : X \to Y$ the following are equivalent:*
    (a) *f has $\mathcal{P}$,*
    (b) *for every $x \in |X|$ the morphism f has $\mathcal{Q}$ at x.*

Proof. See Descent, Lemma 31.29.2 for (1). The implication (1)(a) ⇒ (2)(b) follows on letting $X' = a(U) \subset X$ given a diagram as in Lemma 42.21.3. The implication (2)(b) ⇒ (1)(a) is clear. The equivalence of (3)(a) and (3)(b) follows from the corresponding result for morphisms of schemes, see Descent, Lemma 31.29.3. □

Remark 42.21.6. We will apply Lemma 42.21.5 above to all cases listed in Descent, Remark 31.28.7 except ``flat''. In each case we will do this by defining f to have property $\mathcal{P}$ at x if f has $\mathcal{P}$ in a neighbourhood of x.

42.22. Morphisms of finite type

The property ``locally of finite type'' of morphisms of schemes is étale local on the source-and-target, see Descent, Remark 31.28.7. It is also stable under base change and fpqc local on the target, see Morphisms, Lemma 24.14.4, and Descent, Lemmas 31.19.8. Hence, by Lemma 42.21.1 above, we may define what it means for a morphism of algebraic spaces to be locally of finite type as follows and it agrees with the already existing notion defined in Section 42.3 when the morphism is representable.

Definition 42.22.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say *f locally of finite type* if the equivalent conditions of Lemma 42.21.1 hold with $\mathcal{P}$= locally of finite type.
(2) Let $x \in |X|$. We say f is of *finite type at x* if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is locally of finite type.

(3) We say f is *of finite type* if it is locally of finite type and quasi-compact.

Consider the algebraic space $\mathbf{A}^1_k/\mathbf{Z}$ of Spaces, Example 40.14.8. The morphism $\mathbf{A}^1_k/\mathbf{Z} \to Spec(k)$ is of finite type.

Lemma 42.22.2. *The composition of finite type morphisms is of finite type. The same holds for locally of finite type.*

Proof. Omitted. □

Lemma 42.22.3. *A base change of a finite type morphism is finite type. The same holds for locally of finite type.*

Proof. Omitted. □

Lemma 42.22.4. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is locally of finite type,*
(2) *for every $x \in |X|$ the morphism f is of finite type at x,*
(3) *for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is locally of finite type,*
(4) *for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is locally of finite type,*
(5) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is locally of finite type,*
(6) *there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that the composition $f \circ \varphi$ is locally of finite type,*
(7) *for every commutative diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U, V are schemes and the vertical arrows are étale the top horizontal arrow is locally of finite type,
(8) *there exists a commutative diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U, V are schemes, the vertical arrows are étale, $U \to X$ is surjective, and the top horizontal arrow is locally of finite type, and
(9) *there exist Zariski coverings $Y = \bigcup_{i \in I} Y_i$, and $f^{-1}(Y_i) = \bigcup X_{ij}$ such that each morphism $X_{ij} \to Y_i$ is locally of finite type.*

Proof. Omitted. □

Lemma 42.22.5. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is locally of finite type and Y is locally Noetherian, then X is locally Noetherian.*

Proof. Let

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

be a commutative diagram where U, V are schemes and the vertical arrows are surjective étale. If f is locally of finite type, then $U \to V$ is locally of finite type. If Y is locally Noetherian, then V is locally Noetherian. By Morphisms, Lemma 24.14.6 we see that U is locally Noetherian, which means that X is locally Noetherian. □

Lemma 42.22.6. *Let S be a scheme. Let $f : X \to Y$, $g : Y \to Z$ be morphisms of algebraic spaces over S. If $g \circ f : X \to Z$ is locally of finite type, then $f : X \to Y$ is locally of finite type.*

Proof. We can find a diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & W \arrow[d] \\ X \arrow[r] & Y \arrow[r] & Z \end{tikzcd}$$

where U, V, W are schemes, the vertical arrows are étale and surjective, see Spaces, Lemma 40.11.4. At this point we can use Lemma 42.22.4 and Morphisms, Lemma 24.14.8 to conclude. □

Lemma 42.22.7. *An immersion is locally of finite type.*

Proof. Follows from the general principle Spaces, Lemma 40.5.8 and Morphisms, Lemmas 24.14.5. □

42.23. Points and geometric points

In this section we make some remarks on points and geometric points (see Properties of Spaces, Definition 41.16.1). One way to think about a geometric point of X is to consider a geometric point $\overline{s} : Spec(k) \to S$ of S and a lift of $\overline{s}$ to a morphism $\overline{x}$ into X. Here is a diagram

$$\begin{tikzcd} Spec(k) \arrow[r, "\overline{x}"'] \arrow[rd, "\overline{s}"'] & X \arrow[d] \\ & S. \end{tikzcd}$$

We often say ``let k be an algebraically closed field over S'' to indicate that $Spec(k)$ comes equipped with a morphism $Spec(k) \to S$. In this situation we write

$$X(k) = Mor_S(Spec(k), X) = \{\overline{x} \in X \text{ lying over } \overline{s}\}$$

for the set of k-valued points of X. In this case the map $X(k) \to |X|$ maps into the subset $|X_s| \subset |X|$. Here $X_s = Spec(\kappa(s)) \times_S X$, where $s \in S$ is the point corresponding to $\overline{s}$. As $Spec(\kappa(s)) \to S$ is a monomorphism, also the base change $X_s \to X$ is a monomorphism, and $|X_s|$ is indeed a subset of $|X|$.

Lemma 42.23.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is locally of finite type. The following are equivalent:*

(1) *f is surjective, and*

(2) *for every algebraically closed field k over S the induced map $X(k) \to Y(k)$ is surjective.*

Proof. Choose a diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

with U, V schemes over S and vertical arrows surjective and étale, see Spaces, Lemma 40.11.4. Since f is locally of finite type we see that $U \to V$ is locally of finite type.

Assume (1) and let $\overline{y} \in Y(k)$. Then $U \to Y$ is surjective and locally of finite type by Lemmas 42.6.4 and 42.22.2. Let $Z = U \times_{Y,\overline{y}} Spec(k)$. This is a scheme. The projection $Z \to Spec(k)$ is surjective and locally of finite type by Lemmas 42.6.5 and 42.22.3. It follows from Varieties, Lemma 28.12.1 that Z has a k valued point $\overline{z}$. The image $\overline{x} \in X(k)$ of $\overline{z}$ maps to $\overline{y}$ as desired.

Assume (2). By Properties of Spaces, Lemma 41.4.4 it suffices to show that $|X| \to |Y|$ is surjective. Let $y \in |Y|$. Choose a $u \in U$ mapping to y. Let $k \supset \kappa(u)$ be an algebraic closure. Denote $\overline{u} \in U(k)$ the corresponding point and $\overline{y} \in Y(k)$ its image. By assumption there exists a $\overline{x} \in X(k)$ mapping to $\overline{y}$. Then it is clear that the image $x \in |X|$ of $\overline{x}$ maps to y. $\square$

In order to state the next lemma we introduce the following notation. Given a scheme T we denote

$$\lambda(T) = \sup\{\aleph_0, |\kappa(t)|; t \in T\}.$$

In words $\lambda(T)$ is the smallest infinite cardinal bounding all the cardinalities of residue fields ot T. Note that if R is a ring then the cardinality of any residue field $\kappa(\mathfrak{p})$ of R is bounded by the cardinality of R (details omitted). This implies that $\lambda(T) \leq \text{size}(T)$ where $\text{size}(T)$ is the size of the scheme T as introduced in Sets, Section 3.9. If $K \subset L$ is a finitely generated field extension then $|K| \leq |L| \leq \max\{\aleph_0, |K|\}$. It follows that if $T' \to T$ is a morphism of schemes which is locally of finite type then $\lambda(T') \leq \lambda(T)$, and if $T' \to T$ is also surjective then equality holds. Next, suppose that S is a scheme and that X is an algebraic space over S. In this case we define

$$\lambda(X) := \lambda(U)$$

where U is any scheme over S which has a surjective étale morphism towards X. The reason that this is independent of the choice of U is that given a pair of such schemes U and U' the fibre product $U \times_X U'$ is a scheme which admits a surjective étale morphism to both U and U', whence $\lambda(U) = \lambda(U \times_X U') = \lambda(U')$ by the discussion above.

Lemma 42.23.2. *Let S be a scheme. Let X, Y be algebraic spaces over S.*

(1) *As k ranges over all algebraically closed fields over S the collection of geometric points $\overline{y} \in Y(k)$ cover all of $|Y|$.*
(2) *As k ranges over all algebraically closed fields over S with $|k| \geq \lambda(Y)$ and $|k| > \lambda(X)$ the geometric points $\overline{y} \in Y(k)$ cover all of $|Y|$.*
(3) *For any geometric point $\overline{s} : Spec(k) \to S$ where k has cardinality $> \lambda(X)$ the map*

$$X(k) \longrightarrow |X_s|$$

is surjective.

(4) *Let $X \to Y$ be a morphism of algebraic spaces over S. For any geometric point $\overline{s} : Spec(k) \to S$ where k has cardinality $> \lambda(X)$ the map*

$$X(k) \longrightarrow |X| \times_{|Y|} Y(k)$$

is surjective.

(5) *Let $X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*
   (a) *the map $X \to Y$ is surjective,*
   (b) *for all algebraically closed fields k over S with $|k| > \lambda(X)$, and $|k| \geq \lambda(Y)$ the map $X(k) \to Y(k)$ is surjective.*

Proof. To prove part (1) choose a surjective étale morphism $V \to Y$ where V is a scheme. For each $v \in V$ choose an algebraic closure $\kappa(v) \subset k_v$. Consider the morphisms $\overline{x} : Spec(k_v) \to V \to Y$. By construction of $|Y|$ these cover $|Y|$.

To prove part (2) we will use the following two facts whose proofs we omit: (i) If K is a field and $\overline{K}$ is algebraic closure then $|\overline{K}| \leq \max\{\aleph_0, |K|\}$. (ii) For any algebraically closed field k and any cardinal $\aleph$, $\aleph \geq |k|$ there exists an extension of algebraically closed fields $k \subset k'$ with $|k'| = \aleph$. Now we set $\aleph = \max\{\lambda(X), \lambda(Y)\}^+$. Here $\lambda^+ > \lambda$ indicates the next bigger cardinal, see Sets, Section 3.6. Now (i) implies that the fields k_u constructed in the first paragraph of the proof all have cardinality bounded by $\lambda(X)$. Hence by (ii) we can find extensions $k_u \subset k'_u$ such that $|k'_u| = \aleph$. The morphisms $\overline{x}' : Spec(k'_u) \to X$ cover $|X|$ as desired. To really finish the proof of (2) we need to show that the schemes $Spec(k'_u)$ are (isomorphic to) objects of Sch_{fppf} because our conventions are that all schemes are objects of Sch_{fppf}; the rest of this paragraph should be skipped by anyone who is not interested in set theoretical considerations. By construction there exists an object T of Sch_{fppf} such that $\lambda(X)$ and $\lambda(Y)$ are bounded by size(T). By our construction of the category Sch_{fppf} in Topologies, Definitions 30.7.6 as the category Sch_α constructed in Sets, Lemma 3.9.2 we see that any scheme whose size is $\leq$ size$(T)^+$ is isomorphic to an object of Sch_{fppf}. See the expression for the function *Bound* in Sets, Equation (3.9.1.1). Since $\aleph \leq$ size$(T)^+$ we conclude.

The notation X_s in part (3) means the fibre product $Spec(\kappa(s)) \times_S X$, where $s \in S$ is the point corresponding to $\overline{s}$. Hence part (2) follows from (4) with $Y = Spec(\kappa(s))$.

Let us prove (4). Let $X \to Y$ be a morphism of algebraic spaces over S. Let k be an algebraically closed field over S of cardinality $> \lambda(X)$. Let $\overline{y} \in Y(k)$ and $x \in |X|$ which map to the same element y of $|Y|$. We have to find $\overline{x} \in X(k)$ mapping to x and $\overline{y}$. Choose a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

with U, V schemes over S and vertical arrows surjective and étale, see Spaces, Lemma 40.11.4. Choose a $u \in |U|$ which maps to x, and denote $v \in |V|$ the image. We will think of $u = Spec(\kappa(u))$ and $v = Spec(\kappa(v))$ as schemes. Note that $V \times_Y Spec(k)$ is a scheme étale over k. Hence it is a disjoint union of spectra of finite separable extensions of k, see Morphisms, Lemma 24.35.7. As v maps to y we see that $v \times_Y Spec(k)$ is a nonempty scheme. As $v \to V$ is a monomorphism, we see that $v \times_Y Spec(k) \to V \times_Y Spec(k)$ is a monomorphism. Hence $v \times_Y Spec(k)$ is a disjoint union of spectra of finite separable extensions of k, by Schemes, Lemma 21.23.10. We conclude that the morphism $v \times_Y Spec(k) \to Spec(k)$ has a section, i.e., we can find a morphism $\overline{v} : Spec(k) \to V$ lying over v and over $\overline{y}$. Finally we consider the scheme

$$u \times_{V,\overline{v}} Spec(k) = Spec(\kappa(u) \otimes_{\kappa(v)} k)$$

where $\kappa(v) \to k$ is the field map defining the morphism $\overline{v}$. Since the cardinality of k is larger than the cardinality of $\kappa(u)$ by assumption we may apply Algebra, Lemma 7.31.12 to see that any maximal ideal $\mathfrak{m} \subset \kappa(u) \otimes_{\kappa(v)} k$ has a residue field which is algebraic over k and hence equal to k. Such a maximal ideal will hence produce a morphism $\overline{u} : Spec(k) \to U$ lying over u and mapping to $\overline{v}$. The composition $Spec(k) \to U \to X$ will be the desired geometric point $\overline{x} \in X(k)$. This concludes the proof of part (4).

Part (5) is a formal consequence of parts (2) and (4) and Properties of Spaces, Lemma 41.4.4. □

42.24. Points of finite type

Let S be a scheme. Let X be an algebraic space over S. A finite type point $x \in |X|$ is a point which can be represented by a morphism $Spec(k) \to X$ which is locally of finite type. Finite type points are a suitable replacement of closed points for algebraic spaces and algebraic stacks. There are always ``enough of them'' for example.

Lemma 42.24.1. *Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. The following are equivalent:*

(1) *There exists a morphism $Spec(k) \to X$ which is locally of finite type and represents x.*
(2) *There exists a scheme U, a closed point $u \in U$, and an étale morphism $\varphi : U \to X$ such that $\varphi(u) = x$.*

Proof. Let $u \in U$ and $U \to X$ be as in (2). Then $Spec(\kappa(u)) \to U$ is of finite type, and $U \to X$ is representable and locally of finite type (by the general principle Spaces, Lemma 40.5.8 and Morphisms, Lemmas 24.35.11 and 24.20.8). Hence we see (1) holds by Lemma 42.22.2.

Conversely, assume $Spec(k) \to X$ is locally of finite type and represents x. Let $U \to X$ be a surjective étale morphism where U is a scheme. By assumption $U \times_X Spec(k) \to U$ is locally of finite type. Pick a finite type point v of $U \times_X Spec(k)$ (there exists at least one, see Morphisms, Lemma 24.15.4). By Morphisms, Lemma 24.15.5 the image $u \in U$ of v is a finite type point of U. Hence by Morphisms, Lemma 24.15.4 after shrinking U we may assume that u is a closed point of U, i.e., (2) holds. □

Definition 42.24.2. Let S be a scheme. Let X be an algebraic space over S. We say a point $x \in |X|$ is a *finite type point*[5] if the equivalent conditions of Lemma 42.24.1 are satisfied. We denote $X_{\text{ft-pts}}$ the set of finite type points of X.

We can describe the set of finite type points as follows.

Lemma 42.24.3. *Let S be a scheme. Let X be an algebraic space over S. We have*

$$X_{\textit{ft-pts}} = \bigcup\nolimits_{\varphi : U \to X \textit{ étale}} |\varphi|(U_0)$$

where U_0 is the set of closed points of U. Here we may let U range over all schemes étale over X or over all affine schemes étale over X.

[5]This is a slight abuse of language as it would perhaps be more correct to say ``locally finite type point''.

Proof. Immediate from Lemma 42.24.1. □

Lemma 42.24.4. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is locally of finite type, then $f(X_{ft\text{-}pts}) \subset Y_{ft\text{-}pts}$.*

Proof. Take $x \in X_{\text{ft-pts}}$. Represent x by a locally finite type morphism $x : Spec(k) \to X$. Then $f \circ x$ is locally of finite type by Lemma 42.22.2. Hence $f(x) \in Y_{\text{ft-pts}}$. □

Lemma 42.24.5. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is locally of finite type and surjective, then $f(X_{ft\text{-}pts}) = Y_{ft\text{-}pts}$.*

Proof. We have $f(X_{\text{ft-pts}}) \subset Y_{\text{ft-pts}}$ by Lemma 42.24.4. Let $y \in |Y|$ be a finite type point. Represent y by a morphism $Spec(k) \to Y$ which is locally of finite type. As f is surjective the algebraic space $X_k = Spec(k) \times_Y X$ is nonempty, therefore has a finite type point $x \in |X_k|$ by Lemma 42.24.3. Now $X_k \to X$ is a morphism which is locally of finite type as a base change of $Spec(k) \to Y$ (Lemma 42.22.3). Hence the image of x in X is a finite type point by Lemma 42.24.4 which maps to y by construction. □

Lemma 42.24.6. *Let S be a scheme. Let X be an algebraic space over S. For any locally closed subset $T \subset |X|$ we have*

$$T \neq \emptyset \Rightarrow T \cap X_{ft\text{-}pts} \neq \emptyset.$$

In particular, for any closed subset $T \subset |X|$ we see that $T \cap X_{ft\text{-}pts}$ is dense in T.

Proof. Let $i : Z \to X$ be the reduced induce subspace structure on T, see Remark 42.4.4. Any immersion is locally of finite type, see Lemma 42.22.7. Hence by Lemma 42.24.4 we see $Z_{\text{ft-pts}} \subset X_{\text{ft-pts}} \cap T$. Finally, any nonempty affine scheme U with an étale morphism towards Z has at least one closed point. Hence Z has at least one finite type point by Lemma 42.24.3. The lemma follows. □

Here is another, more technical, characterization of a finite type point on an algebraic space.

Lemma 42.24.7. *Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. The following are equivalent:*

(1) *x is a finite type point,*
(2) *there exists an algebraic space Z whose underlying topological space $|Z|$ is a singleton, and a morphism $f : Z \to X$ which is locally of finite type such that $\{x\} = |f|(|Z|)$, and*
(3) *there exists an algebraic space Z and a morphism $f : Z \to X$ with the following properties:*
    (a) *there is a surjective étale morphism $z : Spec(k) \to Z$ where k is a field,*
    (b) *f is locally of finite type,*
    (c) *f is a monomorphism, and*
    (d) *$x = f(z)$.*

Proof. Assume x is a finite type point. Choose an affine scheme U, a closed point $u \in U$, and an étale morphism $\varphi : U \to X$ with $\varphi(u) = x$, see Lemma 42.24.3. Set $u = Spec(\kappa(u))$ as usual. The projection morphisms $u \times_X u \to u$ are the compositions

$$u \times_X u \to u \times_X U \to u \times_X X = u$$

where the first arrow is a closed immersion (a base change of $u \to U$) and the second arrow is étale (a base change of the étale morphism $U \to X$). Hence $u \times_X U$ is a disjoint union of spectra of finite separable extensions of k (see Morphisms, Lemma 24.35.7) and therefore the closed subscheme $u \times_X u$ is a disjoint union of finite separable extension of k, i.e., $u \times_X u \to u$ is étale. By Spaces, Theorem 40.10.5 we see that $Z = u/u \times_X u$ is an algebraic space. By construction the diagram

$$\begin{tikzcd} u \arrow[r] \arrow[d] & U \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

is commutative with étale vertical arrows. Hence $Z \to X$ is locally of finite type (see Lemma 42.22.4). By construction the morphism $Z \to X$ is a monomorphism and the image of z is x. Thus (3) holds.

It is clear that (3) implies (2). If (2) holds then x is a finite type point of X by Lemma 42.24.4 (and Lemma 42.24.6 to see that $Z_{\text{ft-pts}}$ is nonempty, i.e., the unique point of Z is a finite type point of Z). □

42.25. Quasi-finite morphisms

The property ``locally quasi-finite" of morphisms of schemes is étale local on the source-and-target, see Descent, Remark 31.28.7. It is also stable under base change and fpqc local on the target, see Morphisms, Lemma 24.19.13, and Descent, Lemma 31.19.22. Hence, by Lemma 42.21.1 above, we may define what it means for a morphism of algebraic spaces to be locally quasi-finite as follows and it agrees with the already existing notion defined in Section 42.3 when the morphism is representable.

Definition 42.25.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is *locally quasi-finite* if the equivalent conditions of Lemma 42.21.1 hold with $\mathcal{P}$= locally quasi-finite.
(2) Let $x \in |X|$. We say f is *quasi-finite at* x if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is locally quasi-finite.
(3) A morphism of algebraic spaces $f : X \to Y$ is *quasi-finite* if it is locally quasi-finite and quasi-compact.

The last part is compatible with the notion of quasi-finiteness for morphisms of schemes by Morphisms, Lemma 24.19.9.

Lemma 42.25.2. *The composition of quasi-finite morphisms is quasi-finite. The same holds for locally quasi-finite.*

Proof. Omitted. □

Lemma 42.25.3. *A base change of a quasi-finite morphism is quasi-finite. The same holds for locally quasi-finite.*

Proof. Omitted. □

The following lemma characterizes locally quasi-finite morphisms as those morphisms which are locally of finite type and have ``discrete fibres". However, it isn't enough to assume that $|X| \to |Y|$ has discrete fibres as the discussion in Examples, Section 64.30 shows.

Lemma 42.25.4. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces. Assume f is locally of finite type. The following are equivalent*

(1) *f is locally quasi-finite,*
(2) *for every morphism $Spec(k) \to Y$ where k is a field the space $|X_k|$ is discrete. Here $X_k = Spec(k) \times_Y X$.*

Proof. Assume f is locally quasi-finite. Let $Spec(k) \to Y$ be as in (2). Choose a surjective étale morphism $U \to X$ where U is a scheme. Then $U_k = Spec(k) \times_Y U \to X_k$ is an étale morphism of algebraic spaces by Properties of Spaces, Lemma 41.13.5. By Lemma 42.25.3 we see that $X_k \to Spec(k)$ is locally quasi-finite. By definition this means that $U_k \to Spec(k)$ is locally quasi-finite. Hence $|U_k|$ is discrete by Morphisms, Lemma 24.19.8. Since $|U_k| \to |X_k|$ is surjective and open we conclude that $|X_k|$ is discrete.

Conversely, assume (2). Choose a surjective étale morphism $V \to Y$ where V is a scheme. Choose a surjective étale morphism $U \to V \times_Y X$ where U is a scheme. Note that $U \to V$ is locally of finite type as f is locally of finite type. Picture

$$\begin{tikzcd} U \arrow[r] \arrow[dr] & X \times_Y V \arrow[r] \arrow[d] & V \arrow[d] \\ & X \arrow[r] & Y \end{tikzcd}$$

If f is not locally quasi-finite then $U \to V$ is not locally quasi-finite. Hence there exists a specialization $u \rightsquigarrow u'$ for some $u, u' \in U$ lying over the same point $v \in V$, see Morphisms, Lemma 24.19.6. We claim that u, u' do not have the same image in $X_v = Spec(\kappa(v)) \times_Y X$ which will contradict the assumption that $|X_v|$ is discrete as desired. Let $d = \text{trdeg}_{\kappa(v)}(\kappa(u))$ and $d' = \text{trdeg}_{\kappa(v)}(\kappa(u'))$. Then we see that $d > d'$ by Morphisms, Lemma 24.27.6. Note that U_v (the fibre of $U \to V$ over v) is the fibre product of U and X_v over $X \times_Y V$, hence $U_v \to X_v$ is étale (as a base change of the étale morphism $U \to X \times_Y V$). If $u, u' \in U_v$ map to the same element of $|X_v|$ then there exists a point $r \in R_v = U_v \times_{X_v} U_v$ with $t(r) = u$ and $s(r) = u'$, see Properties of Spaces, Lemma 41.4.3. Note that $s, t : R_v \to U_v$ are étale morphisms of schemes over $\kappa(v)$, hence $\kappa(u) \subset \kappa(r) \supset \kappa(u')$ are finite separable extensions of fields over $\kappa(v)$ (see Morphisms, Lemma 24.35.7). We conclude that the transcendence degrees are equal. This contradiction finishes the proof. $\square$

Lemma 42.25.5. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is locally quasi-finite,*
(2) *for every $x \in |X|$ the morphism f is quasi-finite at x,*
(3) *for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is locally quasi-finite,*
(4) *for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is locally quasi-finite,*
(5) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is locally quasi-finite,*
(6) *there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that the composition $f \circ \varphi$ is locally quasi-finite,*
(7) *for every commutative diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U, V are schemes and the vertical arrows are étale the top horizontal arrow is locally quasi-finite,

(8) *there exists a commutative diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U, V are schemes, the vertical arrows are étale, and $U \to X$ is surjective such that the top horizontal arrow is locally quasi-finite, and

(9) *there exist Zariski coverings $Y = \bigcup_{i \in I} Y_i$, and $f^{-1}(Y_i) = \bigcup X_{ij}$ such that each morphism $X_{ij} \to Y_i$ is locally quasi-finite.*

Proof. Omitted. $\square$

Lemma 42.25.6. *An immersion is locally quasi-finite.*

Proof. Omitted. $\square$

Lemma 42.25.7. *Let S be a scheme. Let $X \to Y \to Z$ be morphisms of algebraic spaces over S. If $X \to Z$ is locally quasi-finite, then $X \to Y$ is locally quasi-finite.*

Proof. Choose a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & W \arrow[d] \\ X \arrow[r] & Y \arrow[r] & Z \end{tikzcd}$$

with vertical arrows étale and surjective. (See Spaces, Lemma 40.11.4.) Apply Morphisms, Lemma 24.19.15 to the top row. $\square$

Lemma 42.25.8. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is locally of finite type and a monomorphism, then f is separated and locally quasi-finite.*

Proof. A monomorphism is separated, see Lemma 42.14.3. By Lemma 42.25.5 it suffices to prove the lemma after performing a base change by $Z \to Y$ with Z affine. Hence we may assume that Y is an affine scheme. Choose an affine scheme U and an étale morphism $U \to X$. Since $X \to Y$ is locally of finite type the morphism of affine schemes $U \to Y$ is of finite type. Since $X \to Y$ is a monomorphism we have $U \times_X U = U \times_Y U$. In particular the maps $U \times_Y U \to U$ are étale. Let $y \in Y$. Then either U_y is empty, or $Spec(\kappa(u)) \times_{Spec(\kappa(y))} U_y$ is isomorphic to the fibre of $U \times_Y U \to U$ over u for some $u \in U$ lying over y. This implies that the fibres of $U \to Y$ are finite discrete sets (as $U \times_Y U \to U$ is an étale morphism of affine schemes, see Morphisms, Lemma 24.35.7). Hence $U \to Y$ is quasi-finite, see Morphisms, Lemma 24.19.6. As $U \to X$ was an arbitrary étale morphism with U affine this implies that $X \to Y$ is locally quasi-finite. $\square$

42.26. Morphisms of finite presentation

The property ``locally of finite presentation'' of morphisms of schemes is étale local on the source-and-target, see Descent, Remark 31.28.7. It is also stable under base change and fpqc local on the target, see Morphisms, Lemma 24.20.4, and Descent, Lemma 31.19.9. Hence, by Lemma 42.21.1 above, we may define what it means for a morphism of algebraic spaces to be locally of finite presentation as follows and it agrees with the already existing notion defined in Section 42.3 when the morphism is representable.

Definition 42.26.1. Let S be a scheme. Let $X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is *locally of finite presentation* if the equivalent conditions of Lemma 42.21.1 hold with $\mathcal{P}$=``locally of finite presentation".
(2) Let $x \in |X|$. We say f is of *finite presentation at* x if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is locally of finite presentation[6].
(3) A morphism of algebraic spaces $f : X \to Y$ is *of finite presentation* if it is locally of finite presentation, quasi-compact and quasi-separated.

Note that a morphism of finite presentation is **not** just a quasi-compact morphism which is locally of finite presentation.

Lemma 42.26.2. *The composition of morphisms of finite presentation is of finite presentation. The same holds for locally of finite presentation.*

Proof. Omitted. $\square$

Lemma 42.26.3. *A base change of a morphism of finite presentation is of finite presentation The same holds for locally of finite presentation.*

Proof. Omitted. $\square$

Lemma 42.26.4. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is locally of finite presentation,*
(2) *for every $x \in |X|$ the morphism f is of finite presentation at x,*
(3) *for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is locally of finite presentation,*
(4) *for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is locally of finite presentation,*
(5) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is locally of finite presentation,*
(6) *there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that the composition $f \circ \varphi$ is locally of finite presentation,*
(7) *for every commutative diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U, V are schemes and the vertical arrows are étale the top horizontal arrow is locally of finite presentation,

[6] It seems awkward to use ``locally of finite presentation at $x$", but the current terminology may be misleading in the sense that ``of finite presentation at x" does **not** mean that there is an open neighbourhood $X' \subset X$ such that $f|_{X'}$ is of finite presentation.

(8) *there exists a commutative diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U, V are schemes, the vertical arrows are étale, and $U \to X$ is surjective such that the top horizontal arrow is locally of finite presentation, and

(9) *there exist Zariski coverings $Y = \bigcup_{i \in I} Y_i$, and $f^{-1}(Y_i) = \bigcup X_{ij}$ such that each morphism $X_{ij} \to Y_i$ is locally of finite presentation.*

Proof. Omitted. □

Lemma 42.26.5. *A morphism which is locally of finite presentation is locally of finite type. A morphism of finite presentation is of finite type.*

Proof. Let $f : X \to Y$ be a morphism of algebraic spaces which is locally of finite presentation. This means there exists a diagram as in Lemma 42.21.1 with h locally of finite presentation and surjective vertical arrow a. By Morphisms, Lemma 24.20.8 h is locally of finite type. Hence $X \to Y$ is locally of finite type by definition. If f is of finite presentation then it is quasi-compact and it follows that f is of finite type. □

Lemma 42.26.6. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is of finite presentation and Y is Noetherian, then X is Noetherian.*

Proof. Assume f is of finite presentation and Y Noetherian. By Lemmas 42.26.5 and 42.22.5 we see that X is locally Noetherian. As f is quasi-compact and Y is quasi-compact we see that X is quasi-compact. As f is of finite presentation it is quasi-separated (see Definition 42.26.1) and as Y is Noetherian it is quasi-separated (see Properties of Spaces, Definition 41.12.1). Hence X is quasi-separated by Lemma 42.5.9. Hence we have checked all three conditions of Properties of Spaces, Definition 41.12.1 and we win. □

Lemma 42.26.7. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.*

(1) *If Y is locally Noetherian and f locally of finite type then f is locally of finite presentation.*
(2) *If Y is locally Noetherian and f of finite type and quasi-separated then f is of finite presentation.*

Proof. Assume $f : X \to Y$ locally of finite type and Y locally Noetherian. This means there exists a diagram as in Lemma 42.21.1 with h locally of finite type and surjective vertical arrow a. By Morphisms, Lemma 24.20.9 h is locally of finite presentation. Hence $X \to Y$ is locally of finite presentation by definition. This proves (1). If f is of finite type and quasi-separated then it is also quasi-compact and quasi-separated and (2) follows immediately. □

Lemma 42.26.8. *Let S be a scheme. Let Y be an algebraic space over S which is quasi-compact and quasi-separated. If X is of finite presentation over Y, then X is quasi-compact and quasi-separated.*

Proof. Omitted. □

Lemma 42.26.9. *Let S be a scheme. Let $f : X \to Y$ and $Y \to Z$ be morphisms of algebraic spaces over S. If X is locally of finite presentation over Z, and Y is locally of finite type over Z, then f is locally of finite presentation.*

Proof. Choose a scheme W and a surjective étale morphism $W \to Z$. Then choose a scheme V and a surjective étale morphism $V \to W \times_Z Y$. Finally choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. By definition U is locally of finite presentation over W and V is locally of finite type over W. By Morphisms, Lemma 24.20.11 the morphism $U \to V$ is locally of finite presentation. Hence f is locally of finite presentation. □

Lemma 42.26.10. *An open immersion of algebraic spaces is locally of finite presentation.*

Proof. An open immersion is by definition representable, hence we can use the general principle Spaces, Lemma 40.5.8 and Morphisms, Lemma 24.20.5. □

42.27. Flat morphisms

The property ``flat'' of morphisms of schemes is étale local on the source-and-target, see Descent, Remark 31.28.7. It is also stable under base change and fpqc local on the target, see Morphisms, Lemma 24.24.7 and Descent, Lemma 31.19.13. Hence, by Lemma 42.21.1 above, we may define the notion of a flat morphism of algebraic spaces as follows and it agrees with the already existing notion defined in Section 42.3 when the morphism is representable.

Definition 42.27.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is *flat* if the equivalent conditions of Lemma 42.21.1 with $\mathcal{P}$=``flat''.
(2) Let $x \in |X|$. We say f is *flat at x* if the equivalent conditions of Lemma 42.21.3 holds with $\mathcal{Q}$ =``induced map local rings is flat''.

Note that the second part makes sense by Descent, Lemma 31.29.4.

Lemma 42.27.2. *The composition of flat morphisms is flat.*

Proof. Omitted. □

Lemma 42.27.3. *The base change of a flat morphism is flat.*

Proof. Omitted. □

Lemma 42.27.4. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is flat,*
(2) *for every $x \in |X|$ the morphism f is flat at x,*
(3) *for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is flat,*
(4) *for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is flat,*
(5) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is flat,*
(6) *there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that the composition $f \circ \varphi$ is flat,*

(7) *for every commutative diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U, V are schemes and the vertical arrows are étale the top horizontal arrow is flat,

(8) *there exists a commutative diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U, V are schemes, the vertical arrows are étale, and $U \to X$ is surjective such that the top horizontal arrow is flat, and

(9) *there exists a Zariski coverings $Y = \bigcup Y_i$ and $f^{-1}(Y_i) = \bigcup X_{ij}$ such that each morphism $X_{ij} \to Y_i$ is flat.*

Proof. Omitted. □

Lemma 42.27.5. *A flat morphism locally of finite presentation is universally open.*

Proof. Let $f : X \to Y$ be a flat morphism locally of finite presentation of algebraic spaces over S. Choose a diagram

$$\begin{tikzcd} U \arrow[r, "\alpha"'] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U and V are schemes and the vertical arrows are surjective and étale, see Spaces, Lemma 40.11.4. By Lemmas 42.27.4 and 42.26.4 the morphism α is flat and locally of finite presentation. Hence by Morphisms, Lemma 24.24.9 we see that α is universally open. Hence $X \to Y$ is universally open according to Lemma 42.7.5. □

Lemma 42.27.6. *Let S be a scheme. Let $f : X \to Y$ be a flat, quasi-compact, surjective morphism of algebraic spaces over S. A subset $T \subset |Y|$ is open (resp. closed) if and only $f^{-1}(|T|)$ is open (resp. closed) in $|X|$. In other words f is submersive, and in fact universally submersive.*

Proof. Choose affine schemes V_i and étale morphisms $V_i \to Y$ such that $V = \coprod V_i \to Y$ is surjective, see Properties of Spaces, Lemma 41.6.1. For each i the algebraic space $V_i \times_Y X$ is quasi-compact. Hence we can find an affine scheme U_i and a surjective étale morphism $U_i \to V_i \times_Y X$, see Properties of Spaces, Lemma 41.6.3. Then the composition $U_i \to V_i \times_Y X \to V_i$ is a surjective, flat morphism of affines. Of course then $U = \coprod U_i \to X$ is surjective and étale and $U = V \times_Y X$. Moreover, the morphism $U \to V$ is the disjoint union of the morphisms $U_i \to V_i$. Hence $U \to V$ is surjective, quasi-compact and flat. Consider the diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X \arrow[d] \\ V \arrow[r] & Y \end{tikzcd}$$

By definition of the topology on $|Y|$ the set T is closed (resp. open) if and only if $g^{-1}(T) \subset |V|$ is closed (resp. open). The same holds for $f^{-1}(T)$ and its inverse image in $|U|$. Since $U \to V$ is quasi-compact, surjective, and flat we win by Morphisms, Lemma 24.24.10. $\square$

Lemma 42.27.7. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\overline{x}$ be a geometric point of X lying over the point $x \in |X|$. Let $\overline{y} = f \circ \overline{x}$. The following are equivalent*

(1) *f is flat at x, and*
(2) *the map on étale local rings $\mathcal{O}_{Y,\overline{y}} \to \mathcal{O}_{X,\overline{x}}$ is flat.*

Proof. Choose a commutative diagram

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "a"'] & V \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where U and V are schemes, a, b are étale, and $u \in U$ mapping to x. We can find a geometric point $\overline{u} : Spec(k) \to U$ lying over u with $\overline{x} = a \circ \overline{u}$, see Properties of Spaces, Lemma 41.16.4. Set $\overline{v} = h \circ \overline{u}$ with image $v \in V$. We know that

$$\mathcal{O}_{X,\overline{x}} = \mathcal{O}^{sh}_{U,u} \quad \text{and} \quad \mathcal{O}_{Y,\overline{y}} = \mathcal{O}^{sh}_{V,v}$$

see Properties of Spaces, Lemma 41.19.1. We obtain a commutative diagram

$$\begin{tikzcd} \mathcal{O}_{U,u} \arrow[r] & \mathcal{O}_{X,\overline{x}} \\ \mathcal{O}_{V,v} \arrow[r] \arrow[u] & \mathcal{O}_{Y,\overline{y}} \arrow[u] \end{tikzcd}$$

of local rings with flat horizontal arrows. We have to show that the left vertical arrow is flat if and only if the right vertical arrow is. Algebra, Lemma 7.35.8 tells us $\mathcal{O}_{U,u}$ is flat over $\mathcal{O}_{V,v}$ if and only if $\mathcal{O}_{X,\overline{x}}$ is flat over $\mathcal{O}_{V,v}$. Hence the result follows from More on Flatness, Lemma 34.3.5. $\square$

Lemma 42.27.8. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Then f is flat if and only if the morphism of sites $(f_{small}, f^\sharp) : (X_{étale}, \mathcal{O}_X) \to (Y_{étale}, \mathcal{O}_Y)$ associated to f is flat.*

Proof. Flatness of $(f_{small}, f^\sharp)$ is defined in terms of flatness of $\mathcal{O}_X$ as a $f_{small}^{-1}\mathcal{O}_Y$-module. This can be checked at stalks, see Modules on Sites, Lemma 16.33.2 and Properties of Spaces, Theorem 41.16.12. But we've already seen that flatness of f can be checked on stalks, see Lemma 42.27.7. $\square$

42.28. Flat modules

In this section we define what it means for a module to be flat at a point. To do this we will use the notion of the stalk of a sheaf on the small étale site $X_{étale}$ of an algebraic space, see Properties of Spaces, Definition 41.16.6.

Lemma 42.28.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $x \in |X|$. The following are equivalent*

(1) *for some commutative diagram*

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "a"] & V \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where U and V are schemes, a, b are étale, and $u \in U$ mapping to x the module $a^*\mathcal{F}$ is flat at u over V,

(2) *the stalk $\mathcal{F}_{\overline{x}}$ is flat over the étale local ring $\mathcal{O}_{Y,\overline{y}}$ where $\overline{x}$ is any geometric point lying over x and $\overline{y} = f \circ \overline{x}$.*

Proof. During this proof we fix a geometric proof $\overline{x} : Spec(k) \to X$ over x and we denote $\overline{y} = f \circ \overline{x}$ its image in Y. Given a diagram as in (1) we can find a geometric point $\overline{u} : Spec(k) \to U$ lying over u with $\overline{x} = a \circ \overline{u}$, see Properties of Spaces, Lemma 41.16.4. Set $\overline{v} = h \circ \overline{u}$ with image $v \in V$. We know that

$$\mathcal{O}_{X,\overline{x}} = \mathcal{O}_{U,u}^{sh} \quad \text{and} \quad \mathcal{O}_{Y,\overline{y}} = \mathcal{O}_{V,v}^{sh}$$

see Properties of Spaces, Lemma 41.19.1. We obtain a commutative diagram

$$\begin{tikzcd} \mathcal{O}_{U,u} \arrow[r] & \mathcal{O}_{X,\overline{x}} \\ \mathcal{O}_{V,v} \arrow[r] \arrow[u] & \mathcal{O}_{Y,\overline{y}} \arrow[u] \end{tikzcd}$$

of local rings. Finally, we have

$$\mathcal{F}_{\overline{x}} = (\varphi^*\mathcal{F})_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{X,\overline{x}}$$

by Properties of Spaces, Lemma 41.26.4. Thus Algebra, Lemma 7.35.8 tells us $(\varphi^*\mathcal{F})_u$ is flat over $\mathcal{O}_{V,v}$ if and only if $\mathcal{F}_{\overline{x}}$ is flat over $\mathcal{O}_{V,v}$. Hence the result follows from More on Flatness, Lemma 34.3.5. ☐

Definition 42.28.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf on X.

(1) Let $x \in |X|$. We say $\mathcal{F}$ is *flat at x over Y* if the equivalent conditions of Lemma 42.28.1 hold.
(2) We say $\mathcal{F}$ is *flat over Y* if $\mathcal{F}$ is flat over Y at all $x \in |X|$.

Having defined this we have the obligatory base change lemma. This lemma implies that formation of the flat locus of a quasi-coherent sheaf commutes with flat base change.

Lemma 42.28.3. *Let S be a scheme. Let*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

be a cartesian diagram of algebraic spaces over S. Let $x' \in |X'|$ with image $x \in |X|$. Let $\mathcal{F}$ be a quasi-coherent sheaf on X and denote $\mathcal{F}' = (g')^*\mathcal{F}$.

(1) *If $\mathcal{F}$ is flat at x over Y then $\mathcal{F}'$ is flat at x' over Y'.*
(2) *If g is flat at $f'(x')$ and $\mathcal{F}'$ is flat at x' over Y', then $\mathcal{F}$ is flat at x over Y.*

In particular, if $\mathcal{F}$ is flat over Y, then $\mathcal{F}'$ is flat over Y'.

Proof. Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. Choose a scheme V' and a surjective étale morphism $V' \to V \times_Y Y'$. Then $U' = V' \times_V U$ is a scheme endowed with a surjective étale morphism $U' = V' \times_V U \to Y' \times_Y X = X'$. Pick $u' \in U'$ mapping to $x' \in |X'|$. Then we can check flatness of $\mathcal{F}'$ at x' over Y' in terms of flatness of $\mathcal{F}'|_{U'}$ at u' over V'. Hence the lemma follows from More on Morphisms, Lemma 33.11.2. □

The following lemma discusses ``composition'' of flat morphisms in terms of modules. It also shows that flatness satisfies a kind of top down descent.

Lemma 42.28.4. *Let S be a scheme. Let $X \to Y \to Z$ be morphisms of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $x \in |X|$ with image $y \in |Y|$.*

1. *If $\mathcal{F}$ is flat at x over Y and Y is flat at y over Z, then $\mathcal{F}$ is flat at x over Z.*
2. *Let $x : Spec(K) \to X$ be a representative of x. If*
   (a) *$\mathcal{F}$ is flat at x over Y,*
   (b) *$x^*\mathcal{F} \neq 0$, and*
   (c) *$\mathcal{F}$ is flat at x over Z,*
   then Y is flat at y over Z.
3. *Let $\overline{x}$ be a geometric point of X lying over x with image $\overline{y}$ in Y. If $\mathcal{F}_{\overline{x}}$ is a faithfully flat $\mathcal{O}_{Y,\overline{y}}$-module and $\mathcal{F}$ is flat at x over Z, then Y is flat at y over Z.*

Proof. Pick $\overline{x}$ and $\overline{y}$ as in part (3) and denote $\overline{z}$ the induced geometric point of Z. Via the characterization of flatness in Lemmas 42.28.1 and 42.27.7 the lemma reduces to a purely algebraic question on the local ring map $\mathcal{O}_{Z,\overline{z}} \to \mathcal{O}_{Y,\overline{y}}$ and the module $\mathcal{F}_{\overline{x}}$. Part (1) follows from Algebra, Lemma 7.35.3. We remark that condition (2)(b) guarantees that $\mathcal{F}_{\overline{x}}/\mathfrak{m}_{\overline{y}}\mathcal{F}_{\overline{x}}$ is nonzero. Hence (2)(a) + (2)(b) imply that $\mathcal{F}_{\overline{x}}$ is a faithfully flat $\mathcal{O}_{Y,\overline{y}}$-module, see Algebra, Lemma 7.35.14. Thus (2) is a special case of (3). Finally, (3) follows from Algebra, Lemma 7.35.9. □

Sometimes the base change happens ``up on top''. Here is a precise statement.

Lemma 42.28.5. *Let S be a scheme. Let $f : X \to Y$, $g : Y \to Z$ be morphisms of algebraic spaces over S. Let $\mathcal{G}$ be a quasi-coherent sheaf on Y. Let $x \in |X|$ with image $y \in |Y|$. If f is flat at x, then*

$$\mathcal{G} \text{ flat over } Z \text{ at } y \Leftrightarrow f^*\mathcal{G} \text{ flat over } Z \text{ at } x.$$

In particular: If f is surjective and flat, then $\mathcal{G}$ is flat over Z, if and only if $f^*\mathcal{G}$ is flat over Z.

Proof. Pick a geometric point $\overline{x}$ of X and denote $\overline{y}$ the image in Y and $\overline{z}$ the image in Z. Via the characterization of flatness in Lemmas 42.28.1 and 42.27.7 and the description of the stalk of $f^*\mathcal{G}$ at $\overline{x}$ of Properties of Spaces, Lemma 41.26.5 the lemma reduces to a purely algebraic question on the local ring maps $\mathcal{O}_{Z,\overline{z}} \to \mathcal{O}_{Y,\overline{y}} \to \mathcal{O}_{X,\overline{x}}$ and the module $\mathcal{G}_{\overline{y}}$. This algebraic statement is Algebra, Lemma 7.35.8. □

42.29. Generic flatness

This section is the analogue of Morphisms, Section 24.26.

Proposition 42.29.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. Assume*

1. *Y is reduced,*
2. *f is of finite type, and*

(3) *$\mathcal{F}$ is a finite type $\mathcal{O}_X$-module.*

Then there exists an open dense subspace $W \subset Y$ such that the base change $X_W \to W$ of f is flat, locally of finite presentation, and quasi-compact and such that $\mathcal{F}|_{X_W}$ is flat over W and of finite presentation over $\mathcal{O}_{X_W}$.

Proof. Let V be a scheme and let $V \to Y$ be a surjective étale morphism. Let $X_V = V \times_Y X$ and let $\mathcal{F}_V$ be the restriction of $\mathcal{F}$ to X_V. Suppose that the result holds for the morphism $X_V \to V$ and the sheaf $\mathcal{F}_V$. Then there exists an open subscheme $V' \subset V$ such that $X_{V'} \to V'$ is flat and of finite presentation and $\mathcal{F}_{V'}$ is an $\mathcal{O}_{X_{V'}}$-module of finite presentation flat over V'. Let $W \subset Y$ be the image of the étale morphism $V' \to Y$, see Properties of Spaces, Lemma 41.4.10. Then $V' \to W$ is a surjective étale morphism, hence we see that $X_W \to W$ is flat, locally of finite presentation, and quasi-compact by Lemmas 42.26.4, 42.27.4, and 42.9.7. By the discussion in Properties of Spaces, Section 41.27 we see that $\mathcal{F}_W$ is of finite presentation as a $\mathcal{O}_{X_W}$-module and by Lemma 42.28.3 we see that $\mathcal{F}_W$ is flat over W. This argument reduces the proposition to the case where Y is a scheme.

Suppose we can prove the proposition when Y is an affine scheme. Let $f : X \to Y$ be a finite type morphism of algebraic spaces over S with Y a scheme, and let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_X$-module. Choose an affine open covering $Y = \bigcup V_j$. By assumption we can find dense open $W_j \subset V_j$ such that $X_{W_j} \to W_j$ is flat, locally of finite presentation, and quasi-compact and such that $\mathcal{F}|_{X_{W_j}}$ is flat over W_j and of finite presentation as an $\mathcal{O}_{X_{W_j}}$-module. In this situation we simply take $W = \bigcup W_j$ and we win. Hence we reduce the proposition to the case where Y is an affine scheme.

Let Y be an affine scheme over S, let $f : X \to Y$ be a finite type morphism of algebraic spaces over S, and let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_X$-module. Since f is of finite type it is quasi-compact, hence X is quasi-compact. Thus we can find an affine scheme U and a surjective étale morphism $U \to X$, see Properties of Spaces, Lemma 41.6.3. Note that $U \to Y$ is of finite type (this is what it means for f to be of finite type in this case). Hence we can apply Morphisms, Proposition 24.26.2 to see that there exists a dense open $W \subset Y$ such that $U_W \to W$ is flat and of finite presentation and such that $\mathcal{F}|_{U_W}$ is flat over W and of finite presentation as an $\mathcal{O}_{U_W}$-module. According to our definitions this means that the base change $X_W \to W$ of f is flat, locally of finite presentation, and quasi-compact and $\mathcal{F}|_{X_W}$ is flat over W and of finite presentation over $\mathcal{O}_{X_W}$. □

We cannot improve the result of the lemma above to requiring $X_W \to W$ to be of finite presentation as $\mathbf{A}^1_{\mathbf{Q}}/\mathbf{Z} \to Spec(\mathbf{Q})$ gives a counter example. The problem is that the diagonal morphism $\Delta_{X/Y}$ may not be quasi-compact, i.e., f may not be quasi-separated. Clearly, this is also the only problem.

Proposition 42.29.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. Assume*

(1) *Y is reduced,*
(2) *f is quasi-separated,*
(3) *f is of finite type, and*
(4) *$\mathcal{F}$ is a finite type $\mathcal{O}_X$-module.*

Then there exists an open dense subspace $W \subset Y$ such that the base change $X_W \to W$ of f is flat and of finite presentation and such that $\mathcal{F}|_{X_W}$ is flat over W and of finite presentation over $\mathcal{O}_{X_W}$.

Proof. This follows immediately from Proposition 42.29.1 and the fact that ``of finite presentation" = ``locally of finite presentation" + ``quasi-compact" + ``quasi-separated". $\square$

42.30. Relative dimension

In this section we define the relative dimension of a morphism of algebraic spaces at a point, and some closely related properties.

Definition 42.30.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $x \in |X|$. Let $d, r \in \{0, 1, 2, \ldots, \infty\}$.

(1) We say the *dimension of the local ring of the fibre of f at x* is d if the equivalent conditions of Lemma 42.21.3 hold for the property $\mathcal{P}_d$ described in Descent, Lemma 31.29.6.
(2) We say the *transcendence degree of $x/f(x)$* is r if the equivalent conditions of Lemma 42.21.3 hold for the property $\mathcal{P}_r$ described in Descent, Lemma 31.29.7.
(3) We say the *f has relative dimension d at x* if the equivalent conditions of Lemma 42.21.3 hold for the property $\mathcal{P}_d$ described in Descent, Lemma 31.29.8.

Let us spell out what this means. Namely, choose some diagrams

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "a"'] & V \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \qquad \begin{tikzcd} u \arrow[r] \arrow[d] & v \arrow[d] \\ x \arrow[r] & y \end{tikzcd}$$

as in Lemma 42.21.3. Then we have

$$\begin{array}{ccc} \text{relative dimension of } f \text{ at } x & = & \dim_u(U_v) \\ \text{dimension of local ring of the fibre of } f \text{ at } x & = & \dim(\mathcal{O}_{U_v,u}) \\ \text{transcendence degree of } x/f(x) & = & \text{trdeg}_{\kappa(v)}(\kappa(u)) \end{array}$$

Note that if $Y = Spec(k)$ is the spectrum of a field, then the relative dimension of X/Y at x is the same as $\dim_x(X)$, the transcendence degree of $x/f(x)$ is the transcendence degree over k, and the dimension of the local ring of the fibre of f at x is just the dimension of the local ring at x, i.e., the relative notions become absolute notions in that case.

Definition 42.30.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $d \in \{0, 1, 2, \ldots\}$.

(1) We say f has *relative dimension $\leq d$* if f has relative dimension $\leq d$ at all $x \in |X|$.
(2) We say f has *relative dimension d* if f has relative dimension d at all $x \in |X|$.

Having relative dimension *equal* to d means roughly speaking that all nonempty fibres are equidimensional of dimension d.

Lemma 42.30.3. *Let S be a scheme. Let $X \to Y \to Z$ be morphisms of algebraic spaces over S. Let $x \in |X|$ and let $y \in |Y|$, $z \in |Z|$ be the images. Assume $X \to Y$ is locally quasi-finite and $Y \to Z$ locally of finite type. Then the transcendence degree of x/z is equal to the transcendence degree of y/z.*

Proof. We can choose commutative diagrams

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & W \arrow[d] \\ X \arrow[r] & Y \arrow[r] & Z \end{tikzcd} \qquad \begin{tikzcd} u \arrow[r] \arrow[d] & v \arrow[r] \arrow[d] & w \arrow[d] \\ x \arrow[r] & y \arrow[r] & z \end{tikzcd}$$

where U, V, W are schemes and the vertical arrows are étale. By definition the morphism $U \to V$ is locally quasi-finite which implies that $\kappa(v) \subset \kappa(u)$ is finite, see Morphisms, Lemma 24.19.5. Hence the result is clear. $\square$

42.31. Morphisms and dimensions of fibres

This section is the analogue of Morphisms, Section 24.27. The formulations in this section are a bit awkward since we do not have local rings of algebraic spaces at points.

Lemma 42.31.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $x \in |X|$. Assume f is locally of finite type. Then we have*

$$\begin{gathered}\text{relative dimension of } f \text{ at } x \\ = \\ \text{dimension of local ring of the fibre of } f \text{ at } x \\ + \\ \text{transcendence degree of } x/f(x)\end{gathered}$$

where the notation is as in Definition 42.30.1.

Proof. This follows immediately from Morphisms, Lemma 24.27.1 applied to $h : U \to V$ and $u \in U$ as in Lemma 42.21.3. $\square$

Lemma 42.31.2. *Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of algebraic spaces over S. Let $x \in |X|$ and set $y = f(x)$. Assume f and g locally of finite type. Then*

$$\begin{gathered}\text{relative dimension of } g \circ f \text{ at } x \\ \leq \\ \text{relative dimension of } f \text{ at } x \\ + \\ \text{relative dimension of } g \text{ at } y\end{gathered}$$

Moreover, equality holds if for some morphism $Spec(k) \to Z$ from the spectrum of a field in the class of $g(f(x)) = g(y)$ The morphism $X_k \to Y_k$ is flat at x. This holds for example if f is flat at x.

Proof. Choose a diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & W \arrow[d] \\ X \arrow[r] & Y \arrow[r] & Z \end{tikzcd}$$

with U, V, W schemes and vertical arrows étale and surjective. (See Spaces, Lemma 40.11.4.) Choose $u \in U$ mapping to x. Set v, w equal to the images of u in V, W. Apply Morphisms, Lemma 24.27.2 to the top row and the points u, v, w. Details omitted. $\square$

Lemma 42.31.3. *Let S be a scheme. Let*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

be a fibre product diagram of algebraic spaces over S. Let $x' \in |X'|$. Set $x = g'(x')$. Assume f locally of finite type. Then we have

$$\text{relative dimension of } f \text{ at } x = \text{relative dimension of } f' \text{ at } x'$$

Proof. Choose a surjective étale morphism $V \to Y$ with V a scheme. By Spaces, Lemma 40.11.4 we may choose morphisms of schemes $V' \to V$ lifting the morphism g and $U \to V$ lifting the morphism f such that $V' \to Y'$ and $U \to X$ are also surjective and étale. Set $U' = V' \times_V U$. Then the induced morphism $U' \to X'$ is also surjective and étale (argument omitted). Hence we can choose a $u' \in U'$ mapping to x'. At this point the result follows by applying Morphisms, Lemma 24.27.3 to the diagram of schemes involving U', U, V', V and the point u'. $\square$

Lemma 42.31.4. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $n \geq 0$. Assume f is locally of finite type. The set*

$$W_n = \{x \in |X| \text{ such that the relative dimension of } f \text{ at } x \leq n\}$$

is open in $|X|$.

Proof. Choose a diagram

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "a"] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U and V are schemes and the vertical arrows are surjective and étale, see Spaces, Lemma 40.11.4. By Morphisms, Lemma 24.27.4 the set U_n of points where h has relative dimension $\leq n$ is open in U. By our definition of relative dimension for morphisms of algebraic spaces at points we see that $U_n = a^{-1}(W_n)$. The lemma follows by definition of the toplogy on $|X|$. $\square$

Lemma 42.31.5. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S Let $n \geq 0$. Assume f is locally of finite presentation. The open*

$$W_n = \{x \in |X| \text{ such that the relative dimension of } f \text{ at } x \leq n\}$$

of Lemma 42.31.4 is retrocompact in $|X|$. (See Topology, Definition 5.9.1.)

Proof. Choose a diagram

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "a"] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U and V are schemes and the vertical arrows are surjective and étale, see Spaces, Lemma 40.11.4. In the proof of Lemma 42.31.4 we have seen that $a^{-1}(W_n) = U_n$ is the corresponding set for the morphism h. By Morphisms, Lemma 24.27.5 we see that U_n is retrocompact in U. The lemma follows by definition of the toplogy on $|X|$, compare with Properties of Spaces, Lemma 41.5.5 and its proof. $\square$

Lemma 42.31.6. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is locally of finite type. Then f is locally quasi-finite if and only if f has relative dimension 0 at each $x \in |X|$.*

Proof. Choose a diagram

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "a"'] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U and V are schemes and the vertical arrows are surjective and étale, see Spaces, Lemma 40.11.4. The definitions imply that h is locally quasi-finite if and only if f is locally quasi-finite, and that f has relative dimension 0 at all $x \in |X|$ if and only if h has relative dimension 0 at all $u \in U$. Hence the result follows from the result for h which is Morphisms, Lemma 24.28.5. □

Lemma 42.31.7. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is locally of finite type. Then there exists a canonical open subspace $X' \subset X$ such that $f|_{X'} : X' \to Y$ is locally quasi-finite, and such that the relative dimension of f at any $x \in |X|$, $x \notin |X'|$ is ≥ 1. Formation of X' commutes with arbitrary base change.*

Proof. Combine Lemmas 42.31.4, 42.31.6, and 42.31.3. □

Lemma 42.31.8. *Let S be a scheme. Consider a cartesian diagram*

$$\begin{tikzcd} X \arrow[d] & F \arrow[l, "p"'] \arrow[d] \\ Y & Spec(k) \arrow[l] \end{tikzcd}$$

where $X \to Y$ is a morphism of algebraic spaces over S which is locally of finite type and where k is a field over S. Let $z \in |F|$ be such that $\dim_z(F) = 0$. Then, after replacing X by an open subspace containing $p(z)$, the morphism

$$X \longrightarrow Y$$

is locally quasi-finite.

Proof. Let $X' \subset X$ be the open subspace over which f is locally quasi-finite found in Lemma 42.31.7. Since the formation of X' commutes with arbotrary base change we see that $z \in X' \times_Y Spec(k)$. Hence the lemma is clear. □

42.32. Syntomic morphisms

The property ``syntomic'' of morphisms of schemes is étale local on the source-and-target, see Descent, Remark 31.28.7. It is also stable under base change and fpqc local on the target, see Morphisms, Lemma 24.30.4 and Descent, Lemma 31.19.24. Hence, by Lemma 42.21.1 above, we may define the notion of a syntomic morphism of algebraic spaces as follows and it agrees with the already existing notion defined in Section 42.3 when the morphism is representable.

Definition 42.32.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is *syntomic* if the equivalent conditions of Lemma 42.21.1 hold with $\mathcal{P}$=``syntomic''.
(2) Let $x \in |X|$. We say f is *syntomic at x* if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is syntomic.

Lemma 42.32.2. *The composition of syntomic morphisms is syntomic.*

Proof. Omitted. ☐

Lemma 42.32.3. *The base change of a syntomic morphism is syntomic.*

Proof. Omitted. ☐

Lemma 42.32.4. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is syntomic,*
(2) *for every $x \in |X|$ the morphism f is syntomic at x,*
(3) *for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is syntomic,*
(4) *for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is syntomic,*
(5) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is a syntomic morphism,*
(6) *there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that the composition $f \circ \varphi$ is syntomic,*
(7) *for every commutative diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U, V are schemes and the vertical arrows are étale the top horizontal arrow is syntomic,
(8) *there exists a commutative diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U, V are schemes, the vertical arrows are étale, and $U \to X$ is surjective such that the top horizontal arrow is syntomic, and
(9) *there exist Zariski coverings $Y = \bigcup_{i \in I} Y_i$, and $f^{-1}(Y_i) = \bigcup X_{ij}$ such that each morphism $X_{ij} \to Y_i$ is syntomic.*

Proof. Omitted. ☐

42.33. Smooth morphisms

The property ``syntomic'' of morphisms of schemes is étale local on the source-and-target, see Descent, Remark 31.28.7. It is also stable under base change and fpqc local on the target, see Morphisms, Lemma 24.33.5 and Descent, Lemma 31.19.25. Hence, by Lemma 42.21.1 above, we may define the notion of a smooth morphism of algebraic spaces as follows and it agrees with the already existing notion defined in Section 42.3 when the morphism is representable.

Definition 42.33.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is *smooth* if the equivalent conditions of Lemma 42.21.1 hold with $\mathcal{P}$=``smooth''.

(2) Let $x \in |X|$. We say f is *smooth at x* if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is smooth.

Lemma 42.33.2. *The composition of smooth morphisms is smooth.*

Proof. Omitted. □

Lemma 42.33.3. *The base change of a smooth morphism is smooth.*

Proof. Omitted. □

Lemma 42.33.4. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is smooth,*
(2) *for every $x \in |X|$ the morphism f is smooth at x,*
(3) *for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is smooth,*
(4) *for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is smooth,*
(5) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is a smooth morphism,*
(6) *there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that the composition $f \circ \varphi$ is smooth,*
(7) *for every commutative diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U, V are schemes and the vertical arrows are étale the top horizontal arrow is smooth,
(8) *there exists a commutative diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U, V are schemes, the vertical arrows are étale, and $U \to X$ is surjective such that the top horizontal arrow is smooth, and
(9) *there exist Zariski coverings $Y = \bigcup_{i \in I} Y_i$, and $f^{-1}(Y_i) = \bigcup X_{ij}$ such that each morphism $X_{ij} \to Y_i$ is smooth.*

Proof. Omitted. □

Lemma 42.33.5. *A smooth morphism of algebraic spaces is locally of finite presentation.*

Proof. Let $X \to Y$ be a smooth morphism of algebraic spaces. By definition this means there exists a diagram as in Lemma 42.21.1 with h smooth and surjective vertical arrow a. By Morphisms, Lemma 24.33.8 h is locally of finite presentation. Hence $X \to Y$ is locally of finite presentation by definition. □

Lemma 42.33.6. *A smooth morphism of algebraic spaces is locally of finite type.*

Proof. Combine Lemmas 42.33.5 and 42.26.5. □

Lemma 42.33.7. *A smooth morphism of algebraic spaces is flat.*

Proof. Let $X \to Y$ be a smooth morphism of algebraic spaces. By definition this means there exists a diagram as in Lemma 42.21.1 with h smooth and surjective vertical arrow a. By Morphisms, Lemma 24.33.8 h is flat. Hence $X \to Y$ is flat by definition. $\square$

Lemma 42.33.8. *A smooth morphism of algebraic spaces is syntomic.*

Proof. Let $X \to Y$ be a smooth morphism of algebraic spaces. By definition this means there exists a diagram as in Lemma 42.21.1 with h smooth and surjective vertical arrow a. By Morphisms, Lemma 24.33.7 h is syntomic. Hence $X \to Y$ is syntomic by definition. $\square$

42.34. Unramified morphisms

The property ``unramified'' (resp. ``G-unramified'') of morphisms of schemes is étale local on the source-and-target, see Descent, Remark 31.28.7. It is also stable under base change and fpqc local on the target, see Morphisms, Lemma 24.34.5 and Descent, Lemma 31.19.26. Hence, by Lemma 42.21.1 above, we may define the notion of an unramified morphism (resp. G-unramified morphism) of algebraic spaces as follows and it agrees with the already existing notion defined in Section 42.3 when the morphism is representable.

Definition 42.34.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is *unramified* if the equivalent conditions of Lemma 42.21.1 hold with $\mathcal{P}=$ unramified.
(2) Let $x \in |X|$. We say f is *unramified at x* if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is unramified.
(3) We say f is *G-unramified* if the equivalent conditions of Lemma 42.21.1 hold with $\mathcal{P}=$ G-unramified.
(4) Let $x \in |X|$. We say f is *G-unramified at x* if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is G-unramified.

Because of the following lemma, from here on we will only develop theory for unramified morphisms, and whenever we want to use a G-unramified morphism we will simply say ``an unramified morphism locally of finite presentation''.

Lemma 42.34.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Then f is G-unramified if and only if f is unramified and locally of finite presentation.*

Proof. Consider any diagram as in Lemma 42.21.1. Then all we are saying is that the morphism h is G-unramified if and only if it is unramified and locally of finite presentation. This is clear from Morphisms, Definition 24.34.1. $\square$

Lemma 42.34.3. *The composition of unramified morphisms is unramified.*

Proof. Omitted. $\square$

Lemma 42.34.4. *The base change of an unramified morphism is unramified.*

Proof. Omitted. $\square$

Lemma 42.34.5. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is unramified,*

(2) *for every $x \in |X|$ the morphism f is unramified at x,*
(3) *for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is unramified,*
(4) *for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is unramified,*
(5) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is an unramified morphism,*
(6) *there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that the composition $f \circ \varphi$ is unramified,*
(7) *for every commutative diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U, V are schemes and the vertical arrows are étale the top horizontal arrow is unramified,
(8) *there exists a commutative diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U, V are schemes, the vertical arrows are étale, and $U \to X$ is surjective such that the top horizontal arrow is unramified, and
(9) *there exist Zariski coverings $Y = \bigcup_{i \in I} Y_i$, and $f^{-1}(Y_i) = \bigcup X_{ij}$ such that each morphism $X_{ij} \to Y_i$ is unramified.*

Proof. Omitted. □

Lemma 42.34.6. *An unramified morphism of algebraic spaces is locally of finite type.*

Proof. Via a diagram as in Lemma 42.21.1 this translates into Morphisms, Lemma 24.34.9. □

Lemma 42.34.7. *If f is unramified at x then f is quasi-finite at x. In particular, an unramified morphism is locally quasi-finite.*

Proof. Via a diagram as in Lemma 42.21.1 this translates into Morphisms, Lemma 24.34.10. □

Lemma 42.34.8. *An immersion of algebraic spaces is unramified.*

Proof. Let $i : X \to Y$ be an immersion of algebraic spaces. Choose a scheme V and a surjective étale morphism $V \to Y$. Then $V \times_Y X \to V$ is an immersion of schemes, hence unramified (see Morphisms, Lemmas 24.34.7 and 24.34.8). Thus by definition i is unramified. □

Lemma 42.34.9. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.*

(1) *If f is unramified, then the diagonal morphism $\Delta_{X/Y} : X \to X \times_Y X$ is an open immersion.*
(2) *If f is locally of finite type and $\Delta_{X/Y}$ is an open immersion, then f is unramified.*

Proof. We know in any case that $\Delta_{X/Y}$ is a representable monomorphism, see Lemma 42.5.1. Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme U and a surjective étale morphism $U \to X \times_Y V$. Consider the commutative diagram

$$\begin{tikzcd} U \arrow[r, "\Delta_{U/V}"'] \arrow[d] & U \times_V U \arrow[r] \arrow[d] & V \arrow[d, "\Delta_{V/Y}"] \\ X \arrow[r, "\Delta_{X/Y}"] & X \times_Y X \arrow[r] & V \times_Y V \end{tikzcd}$$

with cartesian right square. The left vertical arrow is surjective étale. The right vertical arrow is étale as a morphism between schemes étale over Y, see Properties of Spaces, Lemma 41.13.6. Hence the middle vertical arrow is étale too (but it need not be surjective).

Assume f is unramified. Then $U \to V$ is unramified, hence $\Delta_{U/V}$ is an open immersion by Morphisms, Lemma 24.34.13. Looking at the left square of the diagram above we conclude that $\Delta_{X/Y}$ is an étale morphism, see Properties of Spaces, Lemma 41.13.3. Hence $\Delta_{X/Y}$ is a representable étale monomorphism, which implies that it is an open immersion by Étale Morphisms, Theorem 37.14.1. (See also Spaces, Lemma 40.5.8 for the translation from schemes language into the language of functors.)

Assume that f is locally of finite type and that $\Delta_{X/Y}$ is an open immersion. This implies that $U \to V$ is locally of finite type too (by definition of a morphism of algebraic spaces which is locally of finite type). Looking at the displayed diagram above we conclude that $\Delta_{U/V}$ is étale as a morphism between schemes étale over $X \times_Y X$, see Properties of Spaces, Lemma 41.13.6. But since $\Delta_{U/V}$ is the diagonal of a morphism between schemes we see that it is in any case an immersion, see Schemes, Lemma 21.21.2. Hence it is an open immersion, and we conclude that $U \to V$ is unramified by Morphisms, Lemma 24.34.13. This in turn means that f is unramified by definition. □

Lemma 42.34.10. *Let S be a scheme. Consider a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "f"'] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

of algebraic spaces over S. Assume that $X \to Z$ is locally of finite type. Then there exists an open subspace $U(f) \subset X$ such that $|U(f)| \subset |X|$ is the set of points where f is unramified. Moreover, for any morphism of algebraic spaces $Z' \to Z$, if $f' : X' \to Y'$ is the base change of f by $Z' \to Z$, then $U(f')$ is the inverse image of $U(f)$ under the projection $X' \to X$.

Proof. This lemma is the analogue of Morphisms, Lemma 24.34.15 and in fact we will deduce the lemma from it. By Definition 42.34.1 the set $\{x \in |X| : f \text{ is unramified at } x\}$ is open in X. Hence we only need to prove the final statement. By Lemma 42.22.6 the morphism $X \to Y$ is locally of finite type. By Lemma 42.22.3 the morphism $X' \to Y'$ is locally of finite type.

Choose a scheme W and a surjective étale morphism $W \to Z$. Choose a scheme V and a surjective étale morphism $V \to W \times_Z Y$. Choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. Finally, choose a scheme W' and a surjective étale morphism $W' \to W \times_Z Z'$. Set $V' = W' \times_W V$ and $U' = W' \times_W U$, so that we obtain surjective étale morphisms $V' \to Y'$ and $U' \to X'$. We will use without further mention an étale morphism of algebraic spaces induces an open map of associated topological spaces (see Properties of Spaces, Lemma 41.13.7). This combined with Lemma 42.34.5 implies that $U(f)$ is the image in $|X|$ of the set T of points in U where the morphism $U \to V$ is unramified. Similarly, $U(f')$ is the image in $|X'|$ of the set T' of points in U' where the morphism $U' \to V'$ is unramified. Now, by construction the diagram

$$\begin{tikzcd} U' \arrow[r] \arrow[d] & U \arrow[d] \\ V' \arrow[r] & V \end{tikzcd}$$

is cartesian (in the category of schemes). Hence the aforementioned Morphisms, Lemma 24.34.15 applies to show that T' is the inverse image of T. Since $|U'| \to |X'|$ is surjective this implies the lemma. $\square$

Lemma 42.34.11. *Let S be a scheme. Let $X \to Y \to Z$ be morphisms of algebraic spaces over S. If $X \to Z$ is unramified, then $X \to Y$ is unramified.*

Proof. Choose a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & W \arrow[d] \\ X \arrow[r] & Y \arrow[r] & Z \end{tikzcd}$$

with vertical arrows étale and surjective. (See Spaces, Lemma 40.11.4.) Apply Morphisms, Lemma 24.34.16 to the top row. $\square$

42.35. Étale morphisms

The notion of an étale morphism of algebraic spaces was defined in Properties of Spaces, Definition 41.13.2. Here is the what it means for a morphism to be étale at a point.

Definition 42.35.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $x \in |X|$. We say f is *étale at x* if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is étale

Lemma 42.35.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

1. *f is étale,*
2. *for every $x \in |X|$ the morphism f is étale at x,*
3. *for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is étale,*
4. *for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is étale,*
5. *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is an étale morphism,*
6. *there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that the composition $f \circ \varphi$ is étale,*
7. *for every commutative diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U, V are schemes and the vertical arrows are étale the top horizontal arrow is étale,

(8) *there exists a commutative diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U, V are schemes, the vertical arrows are étale, and $U \to X$ surjective such that the top horizontal arrow is étale, and

(9) *there exist Zariski coverings $Y = \bigcup Y_i$ and $f^{-1}(Y_i) = \bigcup X_{ij}$ such that each morphism $X_{ij} \to Y_i$ is étale.*

Proof. Combine Properties of Spaces, Lemmas 41.13.3, 41.13.5 and 41.13.4. Some details omitted. ☐

Lemma 42.35.3. *The composition of two étale morphisms of algebraic spaces is étale.*

Proof. This is a copy of Properties of Spaces, Lemma 41.13.4. ☐

Lemma 42.35.4. *The base change of an étale morphism of algebraic spaces by any morphism of algebraic spaces is étale.*

Proof. This is a copy of Properties of Spaces, Lemma 41.13.5. ☐

Lemma 42.35.5. *An étale morphism of algebraic spaces is locally quasi-finite.*

Proof. Let $X \to Y$ be an étale morphism of algebraic spaces, see Properties of Spaces, Definition 41.13.2. By Properties of Spaces, Lemma 41.13.3 we see this means there exists a diagram as in Lemma 42.21.1 with h étale and surjective vertical arrow a. By Morphisms, Lemma 24.35.6 h is locally quasi-finite. Hence $X \to Y$ is locally quasi-finite by definition. ☐

Lemma 42.35.6. *An étale morphism of algebraic spaces is smooth.*

Proof. The proof is identical to the proof of Lemma 42.35.5. It uses the fact that an étale morphism of schemes is smooth (by definition of an étale morphism of schemes). ☐

Lemma 42.35.7. *An étale morphism of algebraic spaces is flat.*

Proof. The proof is identical to the proof of Lemma 42.35.5. It uses Morphisms, Lemma 24.35.12. ☐

Lemma 42.35.8. *An étale morphism of algebraic spaces is locally of finite presentation.*

Proof. The proof is identical to the proof of Lemma 42.35.5. It uses Morphisms, Lemma 24.35.11. ☐

Lemma 42.35.9. *An étale morphism of algebraic spaces is locally of finite type.*

Proof. An étale morphism is locally of finite presentation and a morphism locally of finite presentation is locally of finite type, see Lemmas 42.35.8 and 42.26.5. ☐

Lemma 42.35.10. *An étale morphism of algebraic spaces is unramified.*

Proof. The proof is identical to the proof of Lemma 42.35.5. It uses Morphisms, Lemma 24.35.5. ☐

Lemma 42.35.11. *Let S be a scheme. Let X, Y be algebraic spaces étale over an algebraic space Z. Any morphism $X \to Y$ over Z is étale.*

Proof. This is a copy of Properties of Spaces, Lemma 41.13.6. □

Lemma 42.35.12. *A locally finitely presented, flat, unramified morphism of algebraic spaces is étale.*

Proof. Let $X \to Y$ be a locally finitely presented, flat, unramified morphism of algebraic spaces. By Properties of Spaces, Lemma 41.13.3 we see this means there exists a diagram as in Lemma 42.21.1 with h locally finitely presented, flat, unramified and surjective vertical arrow a. By Morphisms, Lemma 24.35.16 h is étale. Hence $X \to Y$ is étale by definition. □

42.36. Proper morphisms

The notion of a proper morphism plays an important role in algebraic geometry. Here is the definition of a proper morphism of algebraic spaces.

Definition 42.36.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say f is *proper* if f is separated, finite type, and universally closed.

Some of the results in this section are results on universally closed morphisms.

Lemma 42.36.2. *A base change of a proper morphism is proper.*

Proof. See Lemmas 42.5.4, 42.22.3, and 42.10.3. □

Lemma 42.36.3. *A composition of proper morphisms is proper.*

Proof. See Lemmas 42.5.8, 42.22.2, and 42.10.4. □

Lemma 42.36.4. *A closed immersion of algebraic spaces is a proper morphism of algebraic spaces.*

Proof. As a closed immersion is by definition representable this follows from Spaces, Lemma 40.5.8 and the corresponding result for morphisms of schemes, see Morphisms, Lemma 24.40.6. □

Lemma 42.36.5. *Let S be a scheme. Consider a commutative diagram of algebraic spaces*

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & B & \end{tikzcd}$$

over S.

(1) *If $X \to B$ is universally closed and $Y \to B$ is separated, then the morphism $X \to Y$ is universally closed. In particular, the image of $|X|$ in $|Y|$ is closed.*
(2) *If $X \to B$ is proper and $Y \to B$ is separated, then the morphism $X \to Y$ is proper.*

Proof. Assume $X \to B$ is universally closed and $Y \to B$ is separated. We factor the morphism as $X \to X \times_B Y \to Y$. The first morphism is a closed immersion, see Lemma 42.5.6 hence universally closed. The projection $X \times_B Y \to Y$ is the base change of a unviversally closed morphism and hence universally closed, see Lemma 42.10.3. Thus $X \to Y$ is universally closed as the composition of universally closed morphisms, see Lemma 42.10.4. This proves (1). To deduce (2) combine (1) with Lemmas 42.5.10, 42.9.8, and 42.22.6. □

Lemma 42.36.6. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is separated,*
(2) *$\Delta_{X/Y} : X \to X \times_Y X$ is universally closed, and*
(3) *$\Delta_{X/Y} : X \to X \times_Y X$ is proper.*

Proof. The implication (1) $\Rightarrow$ (3) follows from Lemma 42.36.4. We will use Spaces, Lemma 40.5.8 without further mention in the rest of the proof. Recall that $\Delta_{X/Y}$ is a representable monomorphism which is locally of finite type, see Lemma 42.5.1. Since proper $\Rightarrow$ universally closed for morphisms of schemes we conclude that (3) implies (2). If $\Delta_{X/Y}$ is universally closed then Étale Morphisms, Lemma 37.7.2 implies that it is a closed immersion. Thus (2) $\Rightarrow$ (1) and we win. □

42.37. Integral and finite morphisms

We have already defined in Section 42.3 what it means for a representable morphism of algebraic spaces to be integral (resp. finite).

Lemma 42.37.1. *Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. Then f is integral (resp. finite) if and only if for all affine schemes Z and morphisms $Z \to Y$ the scheme $X \times_Y Z$ is affine and integral (resp. finite) over Z.*

Proof. This follows directly from the definition of an integral (resp. finite) morphism of schemes (Morphisms, Definition 24.42.1). □

This clears the way for the following definition.

Definition 42.37.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say that f is *integral* if for every affine scheme Z and morphisms $Z \to Y$ the algebraic space $X \times_Y Z$ is representable by an affine scheme integral over Z.
(2) We say that f is *finite* if for every affine scheme Z and morphisms $Z \to Y$ the algebraic space $X \times_Y Z$ is representable by an affine scheme finite over Z.

Lemma 42.37.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is representable and integral (resp. finite),*
(2) *f is integral (resp. finite),*
(3) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is integral (resp. finite), and*
(4) *there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is integral (resp. finite).*

Proof. It is clear that (1) implies (2) and that (2) implies (3) by taking V to be a disjoint union of affines étale over Y, see Properties of Spaces, Lemma 41.6.1. Assume $V \to Y$ is as in (3). Then for every affine open W of V we see that $W \times_Y X$ is an affine open of $V \times_Y X$. Hence by Properties of Spaces, Lemma 41.10.1 we conclude that $V \times_Y X$ is a scheme. Moreover the morphism $V \times_Y X \to V$ is affine. This means we can apply Spaces, Lemma 40.11.3 because the class of integral (resp. finite) morphisms satisfies all the required properties (see Morphisms, Lemmas 24.42.6 and Descent, Lemmas 31.19.20, 31.19.21, and 31.33.1). The conclusion of applying this lemma is that f is representable and integral (resp. finite), i.e., (1) holds.

The equivalence of (1) and (4) follows from the fact that being integral (resp. finite) is Zariski local on the target (the reference above shows that being integral or finite is in fact fpqc local on the target). □

Lemma 42.37.4. *The composition of integral (resp. finite) morphisms is integral (resp. finite).*

Proof. Omitted. □

Lemma 42.37.5. *The base change of an integral (resp. finite) morphism is integral (resp. finite).*

Proof. Omitted. □

Lemma 42.37.6. *A finite morphism of algebraic spaces is integral. An integral morphism of algebraic spaces which is locally of finite type is finite.*

Proof. In both cases the morphism is representable, and you can check the condition after a base change by an affine scheme mapping into Y, see Lemmas 42.37.3. Hence this lemma follows from the same lemma for the case of schemes, see Morphisms, Lemma 24.42.4. □

Lemma 42.37.7. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent*

(1) *f is integral, and*
(2) *f is affine and universally closed.*

Proof. In both cases the morphism is representable, and you can check the condition after a base change by an affine scheme mapping into Y, see Lemmas 42.37.3, 42.19.3, and 42.10.5. Hence the result follows from Morphisms, Lemma 24.42.7. □

Lemma 42.37.8. *A finite morphism of algebraic spaces is quasi-finite.*

Proof. Let $f : X \to Y$ be a morphism of algebraic spaces. By Definition 42.37.2 and Lemmas 42.9.7 and 42.25.5 both properties may be checked after base change to an affine over Y, i.e., we may assume Y affine. If f is finite then X is a scheme. Hence the result follows from the corresponding result for schemes, see Morphisms, Lemma 24.42.9. □

Lemma 42.37.9. *A finite morphism of algebraic spaces is proper.*

Proof. Let $f : X \to Y$ be a morphism of algebraic spaces. We think of proper as synonymous to ``finite type, separated, and universally closed''. By Definition 42.37.2 and Lemmas 42.22.4, 42.5.12, and 42.10.5 both properties may be checked after base change to an affine over Y, i.e., we may assume Y affine. If f is finite then X is a scheme. Hence the result follows from the corresponding result for schemes, see Morphisms, Lemma 24.42.10. □

42.38. Finite locally free morphisms

We have already defined in Section 42.3 what it means for a representable morphism of algebraic spaces to be finite locally free.

Lemma 42.38.1. *Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. Then f is finite locally free if and only if f is affine and the sheaf $f_*\mathcal{O}_X$ is a finite locally free $\mathcal{O}_Y$-module.*

Proof. Assume f is finite locally free (as defined in Section 42.3). This means that for every morphism $V \to Y$ whose source is a scheme the base change $f' : V \times_Y X \to V$ is a finite locally free morphism of schemes. This in turn means (by the definition of a finite locally free morphism of schemes) that $f'_*\mathcal{O}_{V\times_Y X}$ is a finite locally free $\mathcal{O}_V$-module. We may choose $V \to Y$ to be surjective and étale. By Properties of Spaces, Lemma 41.23.2 we conclude the restriction of $f_*\mathcal{O}_X$ to V is finite locally free. Hence by Modules on Sites, Lemma 16.23.3 applied to the sheaf $f_*\mathcal{O}_X$ on $Y_{spaces,étale}$ we conclude that $f_*\mathcal{O}_X$ is finite locally free.

Conversely, assume f is affine and that $f_*\mathcal{O}_X$ is a finite locally free $\mathcal{O}_Y$-module. Let V be a scheme, and let $V \to Y$ be a surjective étale morphism. Again by Properties of Spaces, Lemma 41.23.2 we see that $f'_*\mathcal{O}_{V\times_Y X}$ is finite locally free. Hence $f' : V \times_Y X \to V$ is finite locally free (as it is also affine). By Spaces, Lemma 40.11.3 we conclude that f is finite locally free (use Morphisms, Lemma 24.44.4 Descent, Lemmas 31.19.28 and 31.33.1). Thus we win. □

This clears the way for the following definition.

Definition 42.38.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say that f is *finite locally free* if f is affine and $f_*\mathcal{O}_X$ is a finite locally free $\mathcal{O}_Y$-module. In this case we say f is has *rank* or *degree* d if the sheaf $f_*\mathcal{O}_X$ is finite locally free of rank d.

Lemma 42.38.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is representable and finite locally free,*
(2) *f is finite locally free,*
(3) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is finite locally free, and*
(4) *there exists a Zariski covering $Y = \bigcup Y_i$ such that each morphism $f^{-1}(Y_i) \to Y_i$ is finite locally free.*

Proof. It is clear that (1) implies (2) and that (2) implies (3) by taking V to be a disjoint union of affines étale over Y, see Properties of Spaces, Lemma 41.6.1. Assume $V \to Y$ is as in (3). Then for every affine open W of V we see that $W \times_Y X$ is an affine open of $V \times_Y X$. Hence by Properties of Spaces, Lemma 41.10.1 we conclude that $V \times_Y X$ is a scheme. Moreover the morphism $V \times_Y X \to V$ is affine. This means we can apply Spaces, Lemma 40.11.3 because the class of finite locally free morphisms satisfies all the required properties (see Morphisms, Lemma 24.44.4 Descent, Lemmas 31.19.28 and 31.33.1). The conclusion of applying this lemma is that f is representable and finite locally free, i.e., (1) holds.

The equivalence of (1) and (4) follows from the fact that being finite locally free is Zariski local on the target (the reference above shows that being finite locally free is in fact fpqc local on the target). □

Lemma 42.38.4. *The composition of finite locally free morphisms is finite locally free.*

Proof. Omitted. □

Lemma 42.38.5. *The base change of a finite locally free morphism is finite locally free.*

Proof. Omitted. □

Lemma 42.38.6. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

1. *f is finite locally free,*
2. *f is finite, flat, and locally of finite presentation.*

If Y is locally Noetherian these are also equivalent to

3. *f is finite and flat.*

Proof. In each of the three cases the morphism is representable and you can check the property after base change by a surjective étale morphism $V \to Y$, see Lemmas 42.37.3, 42.38.3, 42.27.4, and 42.26.4. If Y is locally Noetherian, then V is locally Noetherian. Hence the result follows from the corresponding result in the schemes case, see Morphisms, Lemma 24.44.2. □

42.39. Separated, locally quasi-finite morphisms

We prove a result that is so interesting it deserves its own section. The result is that an algebraic space which is locally quasi-finite and separated over a scheme is a scheme. It implies that a separated and locally quasi-finite morphism is representable. But first... a lemma (which will be obsoleted by Proposition 42.39.2).

Lemma 42.39.1. *Let S be a scheme. Consider a commutative diagram*

$$\begin{tikzcd} V' \arrow[r] \arrow[rd] & T' \times_T X \arrow[r] \arrow[d] & X \arrow[d] \\ & T' \arrow[r] & T \end{tikzcd}$$

Assume

1. *$T' \to T$ is an étale morphism of affine schemes,*
2. *X is an algebraic space,*
3. *$X \to T$ is a separated, locally quasi-finite morphism,*
4. *V' is an open subspace of $T' \times_T X$, and*
5. *$V' \to T'$ is quasi-affine.*

In this situation the image U of V' in X is a quasi-compact open subspace of X which is representable.

Proof. We first make some trivial observations. Note that V' is a representable by Lemma 42.20.3. It is also quasi-compact (as a quasi-affine scheme over an affine scheme, see Morphisms, Lemma 24.12.2). Since $T' \times_T X \to X$ is étale (Properties of Spaces, Lemma 41.13.5) the map $|T' \times_T X| \to |X|$ is open, see Properties of Spaces, Lemma 41.13.7. Let $U \subset X$ be the open subspace corresponding to the image of $|V'|$, see Properties of Spaces, Lemma 41.4.8. As $|V'|$ is quasi-compact we see that $|U|$ is quasi-compact, hence U is a quasi-compact algebraic spaces, by Properties of Spaces, Lemma 41.5.2.

By Morphisms, Lemma 24.48.8 the morphism $T' \to T$ is universally bounded. Hence we can do induction on the integer n bounding the degree of the fibres of $T' \to T$, see Morphisms, Lemma 24.48.7 for a description of this integer in the case of an étale morphism. If $n = 1$, then $T' \to T$ is an open immersion (see Étale Morphisms, Theorem 37.14.1), and the result is clear. Assume $n > 1$.

Consider the affine scheme $T'' = T' \times_T T'$. As $T' \to T$ is étale we have a decomposition (into open and closed affine subschemes) $T'' = \Delta(T') \amalg T^*$. Namely $\Delta = \Delta_{T'/T}$ is open by

Morphisms, Lemma 24.34.13 and closed because $T' \to T$ is separated as a morphism of affines. As a base change the degrees of the fibres of the second projection $\text{pr}_1 : T' \times_T T' \to T'$ are bounded by n, see Morphisms, Lemma 24.48.4. On the other hand, $\text{pr}_1|_{\Delta(T')} : \Delta(T') \to T'$ is an isomorphism and every fibre has exactly one point. Thus, on applying Morphisms, Lemma 24.48.7 we conclude the degrees of the fibres of the restriction $\text{pr}_1|_{T^*} : T^* \to T'$ are bounded by $n - 1$. Hence the induction hypothesis applied to the diagram

$$\begin{tikzcd} p_0^{-1}(V') \cap X^* \arrow[r] \arrow[dr] & X^* \arrow[r, "p_1|_{X^*}"'] \arrow[d] & X' \arrow[d] \\ & T^* \arrow[r, "\text{pr}_1|_{T^*}"] & T' \end{tikzcd}$$

gives that $p_1(p_0^{-1}(V') \cap X^*)$ is a quasi-compact scheme. Here we set $X'' = T'' \times_T X$, $X^* = T^* \times_T X$, and $X' = T' \times_T X$, and $p_0, p_1 : X'' \to X'$ are the base changes of pr_0, pr_1. Most of the hypotheses of the lemma imply by base change the corresponding hypothesis for the diagram above. For example $p_0^{-1}(V') = T'' \times_{T'} V'$ is a scheme quasi-affine over T'' as a base change. Some verifications omitted.

By Properties of Spaces, Lemma 41.10.1 we conclude that

$$p_1(p_0^{-1}(V')) = V' \cup p_1(p_0^{-1}(V') \cap X^*)$$

is a quasi-compact scheme. Moreover, it is clear that $p_1(p_0^{-1}(V'))$ is the inverse image of the quasi-compact open subspace $U \subset X$ discussed in the first paragraph of the proof. In other words, $T' \times_T U$ is a scheme! Note that $T' \times_T U$ is quasi-compact and separated and locally quasi-finite over T', as $T' \times_T X \to T'$ is locally quasi-finite and separated being a base change of the original morphism $X \to T$ (see Lemmas 42.5.4 and 42.25.3). This implies by More on Morphisms, Lemma 33.29.3 that $T' \times_T U \to T'$ is quasi-affine.

By Descent, Lemma 31.35.1 this gives a descent datum on $T' \times_T U/T'$ relative to the étale covering $\{T' \to W\}$, where $W \subset T$ is the image of the morphism $T' \to T$. Because U' is quasi-affine over T' we see from Descent, Lemma 31.34.1 that this datum is effective, and by the last part of Descent, Lemma 31.35.1 this implies that U is a scheme as desired. Some minor details omitted. □

Proposition 42.39.2. *Let S be a scheme. Let $f : X \to T$ be a morphism of algebraic spaces. Assume*

(1) *T is representable,*
(2) *f is locally quasi-finite, and*
(3) *f is separated.*

Then X is representable.

Proof. Let $T = \bigcup T_i$ be an affine open covering of the scheme T. If we can show that the open subspaces $X_i = f^{-1}(T_i)$ are representable, then X is representable, see Properties of Spaces, Lemma 41.10.1. Note that $X_i = T_i \times_T X$ and that locally quasi-finite and separated are both stable under base change, see Lemmas 42.5.4 and 42.25.3. Hence we may assume T is an affine scheme.

By Properties of Spaces, Lemma 41.6.2 there exists a Zariski covering $X = \bigcup X_i$ such that each X_i has a surjective étale covering by an affine scheme. By Properties of Spaces, Lemma 41.10.1 again it suffices to prove the propostion for each X_i. Hence we may assume there exists an affine scheme U and a surjective étale morphism $U \to X$. This reduces us to the situation in the next paragraph.

Assume we have

$$U \longrightarrow X \longrightarrow T$$

where U and T are affine schemes, $U \to X$ is étale surjective, and $X \to T$ is separated and locally quasi-finite. By Lemmas 42.35.5 and 42.25.2 the morphism $U \to T$ is locally quasi-finite. Since U and T are affine it is quasi-finite. Set $R = U \times_X U$. Then $X = U/R$, see Spaces, Lemma 40.9.1. As $X \to T$ is separated the morphism $R \to U \times_T U$ is a closed immersion, see Lemma 42.5.5. In particular R is an affine scheme also. As $U \to X$ is étale the projection morphisms $t, s : R \to U$ are étale as well. In particular s and t are quasi-finite, flat and of finite presentation (see Morphisms, Lemmas 24.35.6, 24.35.12 and 24.35.11).

Let (U, R, s, t, c) be the groupoid associated to the étale equivalence relation R on U. Let $u \in U$ be a point, and denote $p \in T$ its image. We are going to use More on Groupoids, Lemma 36.12.2 for the groupoid (U, R, s, t, c) over the scheme T with points p and u as above. By the discussion in the previous paragraph all the assumptions (1) -- (7) of that lemma are satisfied. Hence we get an étale neighbourhood $(T', p') \to (T, p)$ and disjoint union decompositions

$$U_{T'} = U' \amalg W, \quad R_{T'} = R' \amalg W'$$

and $u' \in U'$ satisfying conclusions (a), (b), (c), (d), (e), (f), (g), and (h) of the aforementioned More on Groupoids, Lemma 36.12.2. We may and do assume that T' is affine (after possibly shrinking T'). Conclusion (h) implies that $R' = U' \times_{X_{T'}} U'$ with projection mappings identified with the restrictions of s' and t'. Thus $(U', R', s'|_{R'}, t'|_{R'}, c'|_{R' \times_{t', U', s'} R'})$ of conclusion (g) is an étale equivalence relation. By Spaces, Lemma 40.10.2 we conclude that U'/R' is an open subspace of $X_{T'}$. By conclusion (d) the schemes U', R' are affine and the morphisms $s'|_{R'}, t'|_{R'}$ are finite étale. Hence Groupoids, Proposition 35.19.8 kicks in and we see that U'/R' is an affine scheme.

We conclude that for every pair of points (u, p) as above we can find an étale neighbourhood $(T', p') \to (T, p)$ with $\kappa(p) = \kappa(p')$ and a point $u' \in U_{T'}$ mapping to u such that the image x' of u' in $|X_{T'}|$ has an open neighbourhood V' in $X_{T'}$ which is an affine scheme. We apply Lemma 42.39.1 to obtain an open subspace $W \subset X$ which is a scheme, and which contains x (the image of u in $|X|$). Since this works for every x we see that X is a scheme by Properties of Spaces, Lemma 41.10.1. This ends the proof. □

42.40. Applications

Here is another formulation of the result above.

Lemma 42.40.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is locally quasi-finite and separated, then f is representable.*

Proof. This is immediate from Proposition 42.39.2 and the fact that being locally quasi-finite and separated is preserved under any base change, see Lemmas 42.25.3 and 42.5.4. □

Lemma 42.40.2. *Let S be a scheme. Let $f : X \to Y$ be an étale and universally injective morphism of algebraic spaces over S. Then f is an open immersion.*

Proof. Let $T \to Y$ be a morphism from a scheme into Y. If we can show that $X \times_Y T \to T$ is an open immersion, then we are done. Since being étale and being universally injective are properties of morphisms stable under base change (see Lemmas 42.35.4 and 42.18.5) we may assume that Y is a scheme. Note that the diagonal $\Delta_{X/Y} : X \to X \times_Y X$ is étale, a monomorphism, and surjective by Lemma 42.18.2. Hence we see that $\Delta_{X/Y}$ is an isomorphism (see Spaces, Lemma 40.5.9), in particular we see that X is separated over Y. It follows that X is a scheme too, by Proposition 42.39.2. Finally, $X \to Y$ is an open immersion by the fundamental theorem for étale morphisms of schemes, see Étale Morphisms, Theorem 37.14.1. □

42.41. Universal homeomorphisms

In Morphisms, Section 24.43 we have shown that a morphism of schemes is a universal homeomorphism if and only if it is integral, universally injective and surjective. In particular the class of universal homeomorphisms of schemes is closed under composition and arbtriary base change and is fppf local on the base (as this is true for integral, universally injective, and surjective morphisms). Thus, if we apply the discussion in Section 42.3 to this notion we see that we know what it means for a representable morphism of algebraic spaces to be a universal homeomorphism.

Lemma 42.41.1. *Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. Then f is a universal homeomorphism (as discussed above) if and only if for every morphism of algebraic spaces $Z \to Y$ the base change map $Z \times_Y X \to Z$ induces a homeomorphism $|Z \times_Y X| \to |Z|$.*

Proof. If for every morphism of algebraic spaces $Z \to Y$ the base change map $Z \times_Y X \to Z$ induces a homeomorphism $|Z \times_Y X| \to |Z|$, then the same is true whenever Z is a scheme, which formally implies that f is a universal homeomorphism in the sense of Section 42.3. Conversely, if f is a universal homeomorphism in the sense of Section 42.3 then $X \to Y$ is integral, universally injective and surjective (see discussion above). Hence f is universally closed, see Lemma 42.37.7 and universally injective and (universally) surjective, i.e., f is a universal homeomorphism. □

Definition 42.41.2. Let S be a scheme. A morphisms $f : X \to Y$ of algebraic spaces over S is called a *universal homeomorphism* if and only if for every morphism of algebraic spaces $Z \to Y$ the base change $Z \times_Y X \to Z$ induces a homeomorphism $|Z \times_Y X| \to |Z|$.

This definition does not clash with the pre-existing definition for representable morphisms of algebraic spaces by our Lemma 42.41.1. For morphisms of algebraic spaces it is not the case that universal homeomorphisms are always integral.

Example 42.41.3. This is a continuation of Remark 42.18.4. Consider the algebraic space $X = \mathbf{A}^1_k/\{x \sim -x \mid x \neq 0\}$. There are morphisms

$$\mathbf{A}^1_k \longrightarrow X \longrightarrow \mathbf{A}^1_k$$

such that the first arrow is étale surjective, the second arrow is universally injective, and the composition is the map $x \mapsto x^2$. Hence the composition is universally closed. Thus it follows that the map $X \to \mathbf{A}^1_k$ is a universal homeomorphism, but $X \to \mathbf{A}^1_k$ is not separated.

Let S be a scheme. Let $f : X \to Y$ be a universal homeomorphism of algebraic spaces over S. Then f is universally closed, hence is quasi-compact, see Lemma 42.10.7. But f need not be separated (see example above), and not even quasi-separated: an example is to take infinite dimensional affine space $\mathbf{A}^\infty = \mathit{Spec}(k[x_1, x_2, \ldots])$ modulo the equivalence relation given by flipping finitely many signs of nonzero coordinates (details omitted).

42.42. Other chapters

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CHAPTER 43

Decent Algebraic Spaces

43.1. Introduction

In this chapter we talk study ``local" properties of general algebraic spaces, i.e., those algebraic spaces which aren't quasi-separated. Quasi-separated algebraic spaces are studied in [**Kol96**]. It turns out that essentially new phenomena happen, especially regarding points and specializations of points, on more general algebraic spaces. On the other hand, for most basic results on algebraic spaces, one needn't worry about these phenomena, which is why we have decided to have this material in a separate chapter following the standard development of the theory.

43.2. Conventions

The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $Spec(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

43.3. Universally bounded fibres

We briefly discuss what it means for a morphism from a scheme to an algebraic space to have universally bounded fibres. Please refer to Morphisms, Section 24.48 for similar definitions and results on morphisms of schemes.

Definition 43.3.1. Let S be a scheme. Let X be an algebraic space over S, and let U be a scheme over S. Let $f : U \to X$ be a morphism over S. We say the *fibres of f are universally bounded*[1] if there exists an integer n such that for all fields k and all morphisms $Spec(k) \to X$ the fibre product $Spec(k) \times_X U$ is a finite scheme over k whose degree over k is $\leq n$.

This definition makes sense because the fibre product $Spec(k) \times_Y X$ is a scheme. Moreover, if Y is a scheme we recover the notion of Morphisms, Definition 24.48.1 by virtue of Morphisms, Lemma 24.48.2.

Lemma 43.3.2. *Let S be a scheme. Let X be an algebraic space over S. Let $V \to U$ be a morphism of schemes over S, and let $U \to X$ be a morphism from U to X. If the fibres of $V \to U$ and $U \to X$ are universally bounded, then so are the fibres of $V \to X$.*

[1] This is probably nonstandard notation.

Proof. Let n be an integer which works for $V \to U$, and let m be an integer which works for $U \to X$ in Definition 43.3.1. Let $Spec(k) \to X$ be a morphism, where k is a field. Consider the morphisms

$$Spec(k) \times_X V \longrightarrow Spec(k) \times_X U \longrightarrow Spec(k).$$

By assumption the scheme $Spec(k) \times_X U$ is finite of degree at most m over k, and n is an integer which bounds the degree of the fibres of the first morphism. Hence by Morphisms, Lemma 24.48.3 we conclude that $Spec(k) \times_X V$ is finite over k of degree at most nm. $\square$

Lemma 43.3.3. *Let S be a scheme. Let $Y \to X$ be a representable morphism of algebraic spaces over S. Let $U \to X$ be a morphism from a scheme to X. If the fibres of $U \to X$ are universally bounded, then the fibres of $U \times_X Y \to Y$ are universally bounded.*

Proof. This is clear from the definition, and properties of fibre products. (Note that $U \times_X Y$ is a scheme as we assumed $Y \to X$ representable, so the definition applies.) $\square$

Lemma 43.3.4. *Let S be a scheme. Let $g : Y \to X$ be a representable morphism of algebraic spaces over S. Let $f : U \to X$ be a morphism from a scheme towards X. Let $f' : U \times_X Y \to Y$ be the base change of f. If*

$$Im(|f| : |U| \to |X|) \subset Im(|g| : |Y| \to |X|)$$

and f' has universally bounded fibres, then f has universally bounded fibres.

Proof. Let $n \geq 0$ be an integer bounding the degrees of the fibre products $Spec(k) \times_Y (U \times_X Y)$ as in Definition 43.3.1 for the morphism f'. We claim that n works for f also. Namely, suppose that $x : Spec(k) \to X$ is a morphism from the spectrum of a field. Then either $Spec(k) \times_X U$ is empty (and there is nothing to prove), or x is in the image of $|f|$. By Properties of Spaces, Lemma 41.4.3 and the assumption of the lemma we see that this means there exists a field extension $k \subset k'$ and a commutative diagram

$$\begin{tikzcd} Spec(k') \arrow[r] \arrow[d] & Y \arrow[d] \\ Spec(k) \arrow[r] & X \end{tikzcd}$$

Hence we see that

$$Spec(k') \times_Y (U \times_X Y) = Spec(k') \times_{Spec(k)} (Spec(k) \times_X U)$$

Since the scheme $Spec(k') \times_Y (U \times_X Y)$ is assumed finite of degree $\leq n$ over k' it follows that also $Spec(k) \times_X U$ is finite of degree $\leq n$ over k as desired. (Some details omitted.) $\square$

Lemma 43.3.5. *Let S be a scheme. Let X be an algebraic space over S. Consider a commutative diagram*

$$\begin{tikzcd} U \arrow[rr, "f"] \arrow[rd, "g"'] & & V \arrow[ld, "h"] \\ & X & \end{tikzcd}$$

where U and V are schemes. If g has universally bounded fibres, and f is surjective and flat, then also h has universally bounded fibres.

Proof. Assume g has universally bounded fibres, and f is surjective and flat. Say $n \geq 0$ is an integer which bounds the degrees of the schemes $Spec(k) \times_X U$ as in Definition 43.3.1. We claim n also works for h. Let $Spec(k) \to X$ be a morphism from the spectrum of a field to X. Consider the morphism of schemes

$$Spec(k) \times_X V \longrightarrow Spec(k) \times_X U$$

It is flat and surjective. By assumption the scheme on the left is finite of degree $\leq n$ over $Spec(k)$. It follows from Morphisms, Lemma 24.48.9 that the degree of the scheme on the right is also bounded by n as desired. □

Lemma 43.3.6. *Let S be a scheme. Let X be an algebraic space over S, and let U be a scheme over S. Let $\varphi : U \to X$ be a morphism over S. If the fibres of φ are universally bounded, then there exists an integer n such that each fibre of $|U| \to |X|$ has at most n elements.*

Proof. The integer n of Definition 43.3.1 works. Namely, pick $x \in |X|$. Represent x by a morphism $x : Spec(k) \to X$. Then we get a commutative diagram

$$\begin{tikzcd} Spec(k) \times_X U \arrow[r] \arrow[d] & U \arrow[d] \\ Spec(k) \arrow[r, "x"] & X \end{tikzcd}$$

which shows (via Properties of Spaces, Lemma 41.4.3) that the inverse image of x in $|U|$ is the image of the top horizontal arrow. Since $Spec(k) \times_X U$ is finite of degree $\leq n$ over k it has at most n points. □

43.4. Finiteness conditions and points

In this section we elaborate on the question of when points can be represented by monomorphisms from spectra of fields into the space.

Remark 43.4.1. Before we give the proof of the next lemma let us recall some facts about étale morphisms of schemes:

(1) An étale morphism is flat and hence generalizations lift along an étale morphism (Morphisms, Lemmas 24.35.12 and 24.24.8).
(2) An étale morphism is unramified, an unramified morphism is locally quasi-finite, hence fibres are discrete (Morphisms, Lemmas 24.35.16, 24.34.10, and 24.19.6).
(3) A quasi-compact étale morphism is quasi-finite and in particular has finite fibres (Morphisms, Lemmas 24.19.9 and 24.19.10).
(4) An étale scheme over a field k is a disjoint union of spectra of finite separable field extension of k (Morphisms, Lemma 24.35.7).

For a general discussion of étale morphisms, please see Étale Morphisms, Section 37.11.

Lemma 43.4.2. *Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. The following are equivalent:*

(1) *there exists a family of schemes U_i and étale morphisms $\varphi_i : U_i \to X$ such that $\coprod \varphi_i : \coprod U_i \to X$ is surjective, and such that for each i the fibre of $|U_i| \to |X|$ over x is finite, and*
(2) *for every affine scheme U and étale morphism $\varphi : U \to X$ the fibre of $|U| \to |X|$ over x is finite.*

Proof. The implication (2) $\Rightarrow$ (1) is trivial. Let $\varphi_i : U_i \to X$ be a family of étale morphisms as in (1). Let $\varphi : U \to X$ be an étale morphism from an affine scheme towards X. Consider the fibre product diagrams

$$\begin{tikzcd} U \times_X U_i \arrow[r, "p_i"] \arrow[d, "q_i"] & U_i \arrow[d, "\varphi_i"] \\ U \arrow[r, "\varphi"] & X \end{tikzcd} \qquad \begin{tikzcd} \coprod U \times_X U_i \arrow[r, "\coprod p_i"] \arrow[d, "\coprod q_i"] & \coprod U_i \arrow[d, "\coprod \varphi_i"] \\ U \arrow[r, "\varphi"] & X \end{tikzcd}$$

Since q_i is étale it is open (see Remark 43.4.1). Moreover, the morphism $\coprod q_i$ is surjective. Hence there exist finitely many indices $i_1, \ldots, i_n$ and a quasi-compact opens $W_{i_j} \subset U \times_X U_{i_j}$ which surject onto U. The morphism p_i is étale, hence locally quasi-finite (see remark on étale morphisms above). Thus we may apply Morphisms, Lemma 24.48.8 to see the fibres of $p_{i_j}|_{W_{i_j}} : W_{i_j} \to U_i$ are finite. Hence by Properties of Spaces, Lemma 41.4.3 and the assumption on φ_i we conclude that the fibre of φ over x is finite. In other words (2) holds. $\square$

Lemma 43.4.3. *Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. The following are equivalent:*

(1) *there exists a scheme U, an étale morphism $\varphi : U \to X$, and points $u, u' \in U$ mapping to x such that setting $R = U \times_X U$ the fibre of*
$$|R| \to |U| \times_{|X|} |U|$$
over (u, u') is finite,
(2) *for every scheme U, étale morphism $\varphi : U \to X$ and any points $u, u' \in U$ mapping to x setting $R = U \times_X U$ the fibre of*
$$|R| \to |U| \times_{|X|} |U|$$
over (u, u') is finite,
(3) *there exists a morphism $Spec(k) \to X$ with k a field in the equivalence class of x such that the projections $Spec(k) \times_X Spec(k) \to Spec(k)$ are étale and quasi-compact, and*
(4) *there exists a monomorphism $Spec(k) \to X$ with k a field in the equivalence class of x.*

Proof. Assume (1), i.e., let $\varphi : U \to X$ be an étale morphism from a scheme towards X, and let u, u' be points of U lying over x such that the fibre of $|R| \to |U| \times_{|X|} |U|$ over (u, u') is a finite set. In this proof we think of a point $u = Spec(\kappa(u))$ as a scheme. Note that $u \to U$, $u' \to U$ are monomorphisms (see Schemes, Lemma 21.23.6), hence $u \times_X u' \to R = U \times_X U$ is a monomorphism. In this language the assumption really means that $u \times_X u'$ is a scheme whose underlying topological space has finitely many points. Let $\psi : W \to X$ be an étale morphism from a scheme towards X. Let $w, w' \in W$ be points of W mapping to x. We have to show that $w \times_X w'$ is a scheme whose underlying topological space has finitely many points. Consider the fibre product diagram

$$\begin{tikzcd} W \times_X U \arrow[r, "p"] \arrow[d, "q"] & U \arrow[d, "\varphi"] \\ W \arrow[r, "\psi"] & X \end{tikzcd}$$

As x is the image of u and u' we may pick points $\tilde{w}, \tilde{w}'$ in $W \times_X U$ with $q(\tilde{w}) = w$, $q(\tilde{w}') = w'$, $u = p(\tilde{w})$ and $u' = p(\tilde{w}')$, see Properties of Spaces, Lemma 41.4.3. As p, q

are étale the field extensions $\kappa(w) \subset \kappa(\tilde{w}) \supset \kappa(u)$ and $\kappa(w') \subset \kappa(\tilde{w}') \supset \kappa(u')$ are finite separable, see Remark 43.4.1. Then we get a commutative diagram

$$\begin{tikzcd} w \times_X w' \arrow[d] & \tilde{w} \times_X \tilde{w}' \arrow[l] \arrow[r] \arrow[d] & u \times_X u' \arrow[d] \\ w \times_X w' & \tilde{w} \times_S \tilde{w}' \arrow[l] \arrow[r] & u \times_S u' \end{tikzcd}$$

where the squares are fibre product squares. The lower horizontal morpisms are étale and quasi-compact, as any scheme of the form $Spec(k) \times_S Spec(k')$ is affine, and by our observations about the field extensions above. Thus we see that the top horizontal arrows are étale and quasi-compact and hence have finite fibres. We have seen above that $|u \times_X u'|$ is finite, so we conclude that $|w \times_X w'|$ is finite. In other words, (2) holds.

Assume (2). Let $U \to X$ be an étale morphism from a scheme U such that x is in the image of $|U| \to |X|$. Let $u \in U$ be a point mapping to x. Then we have seen in the previous paragraph that $u = Spec(\kappa(u)) \to X$ has the property that $u \times_X u$ has a finite underlying topological space. On the other hand, the projection maps $u \times_X u \to u$ are the composition

$$u \times_X u \longrightarrow u \times_X U \longrightarrow u \times_X X = u,$$

i.e., the composition of a monomorphism (the base change of the monomorphism $u \to U$) by an étale morphism (the base change of the étale morphism $U \to X$). Hence $u \times_X U$ is a disjoint union of spectra of fields finite separable over $\kappa(u)$ (see Remark 43.4.1). Since $u \times_X u$ is finite the image of it in $u \times_X U$ is a finite disjoint union of spectra of fields finite separable over $\kappa(u)$. By Schemes, Lemma 21.23.10 we conclude that $u \times_X u$ is a finite disjoint union of spectra of fields finite separable over $\kappa(u)$. In other words, we see that $u \times_X u \to u$ is quasi-compact and étale. This means that (3) holds.

Let us prove that (3) implies (4). Let $Spec(k) \to X$ be a morphism from the spectrum of a field into X, in the equivalence class of x such that the two projections $t, s : R = Spec(k) \times_X Spec(k) \to Spec(k)$ are quasi-compact and étale. This means in particular that R is an étale equivalence relation on $Spec(k)$. By Spaces, Theorem 40.10.5 we know that the quotient sheaf $X' = Spec(k)/R$ is an algebraic space. By Groupoids, Lemma 35.17.6 the map $X' \to X$ is a monomorphism. Since s, t are quasi-compact, we see that R is quasi-compact and hence Properties of Spaces, Lemma 41.11.1 applies to X', and we see that $X' = Spec(k')$ for some field k'. Hence we get a factorization

$$Spec(k) \longrightarrow Spec(k') \longrightarrow X$$

which shows that $Spec(k') \to X$ is a monomorphism mapping to $x \in |X|$. In other words (4) holds.

Finally, we prove that (4) implies (1). Let $Spec(k) \to X$ be a monomorphism with k a field in the equivalence class of x. Let $U \to X$ be a surjectve étale morphism from a scheme U to X. Let $u \in U$ be a point over x. Since $Spec(k) \times_X u$ is nonempty, and since $Spec(k) \times_X u \to u$ is a monomorphism we conclude that $Spec(k) \times_X u = u$ (see Schemes, Lemma 21.23.10). Hence $u \to U \to X$ factors through $Spec(k) \to X$, here is a picture

$$\begin{tikzcd} u \arrow[r] \arrow[d] & U \arrow[d] \\ Spec(k) \arrow[r] & X \end{tikzcd}$$

Since the right vertical arrow is étale this implies that $k \subset \kappa(u)$ is a finite separable extension. Hence we conclude that

$$u \times_X u = u \times_{Spec(k)} u$$

is a finite scheme, and we win by the discussion of the meaning of property (1) in the first paragraph of this proof. □

Lemma 43.4.4. *Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. Let U be a scheme and let $\varphi : U \to X$ be an étale morphism. The following are equivalent:*

(1) *x is in the image of $|U| \to |X|$, and setting $R = U \times_X U$ the fibres of both*

$$|U| \longrightarrow |X| \quad \text{and} \quad |R| \longrightarrow |X|$$

over x are finite,

(2) *there exists a monomorphism $Spec(k) \to X$ with k a field in the equivalence class of x, and the fibre product $Spec(k) \times_X U$ is a finite nonempty scheme over k.*

Proof. Assume (1). This clearly implies the first condition of Lemma 43.4.3 and hence we obtain a monomorphism $Spec(k) \to X$ in the class of x. Taking the fibre product we see that $Spec(k) \times_X U \to Spec(k)$ is a scheme étale over $Spec(k)$ with finitely many points, hence a finite nonempty scheme over k, i.e., (2) holds.

Assume (2). By assumption x is in the image of $|U| \to |X|$. The finiteness of the fibre of $|U| \to |X|$ over x is clear since this fibre is equal to $|Spec(k) \times_X U|$ by Properties of Spaces, Lemma 41.4.3. The finiteness of the fibre of $|R| \to |X|$ above x is also clear since it is equal to the set underlying the scheme

$$(Spec(k) \times_X U) \times_{Spec(k)} (Spec(k) \times_X U)$$

which is finite over k. Thus (1) holds. □

Lemma 43.4.5. *Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. The following are equivalent:*

(1) *for every affine scheme U, any étale morphism $\varphi : U \to X$ setting $R = U \times_X U$ the fibres of both*

$$|U| \longrightarrow |X| \quad \text{and} \quad |R| \longrightarrow |X|$$

over x are finite,

(2) *there exist schemes U_i and étale morphisms $U_i \to X$ such that $\coprod U_i \to X$ is surjective and for each i, setting $R_i = U_i \times_X U_i$ the fibres of both*

$$|U_i| \longrightarrow |X| \quad \text{and} \quad |R_i| \longrightarrow |X|$$

over x are finite,

(3) *there exists a monomorphism $Spec(k) \to X$ with k a field in the equivalence class of x, and for any affine scheme U and étale morphism $U \to X$ the fibre product $Spec(k) \times_X U$ is a finite scheme over k, and*

(4) *there exists a quasi-compact monomorphism $Spec(k) \to X$ with k a field in the equivalence class of x.*

Proof. The equivalence of (1) and (3) follows on applying Lemma 43.4.4 to every étale morphism $U \to X$ with U affine. It is clear that (3) implies (2). Assume $U_i \to X$ and R_i are as in (2). We conclude from Lemma 43.4.2 that for any affine scheme U and étale morphism $U \to X$ the fibre of $|U| \to |X|$ over x is finite. Say this fibre is $\{u_1, \ldots, u_n\}$. Then, as Lemma 43.4.3 (1) applies to $U_i \to X$ for some i such that x is in the image of

$|U_i| \to |X|$, we see that the fibre of $|R = U \times_X U| \to |U| \times_{|X|} |U|$ is finite over (u_a, u_b), $a, b \in \{1, \ldots, n\}$. Hence the fibre of $|R| \to |X|$ over x is finite. In this way we see that (1) holds. At this point we know that (1), (2), and (3) are equivalent.

If (4) holds, then for any affine scheme U and étale morphism $U \to X$ the scheme $Spec(k) \times_X U$ is on the one hand étale over k (hence a disjoint union of spectra of finite separable extensions of k by Remark 43.4.1) and on the other hand quasi-compact over U (hence quasi-compact). Thus we see that (3) holds. Conversely, if $U_i \to X$ is as in (2) and $Spec(k) \to X$ is a monomorphism as in (3), then

$$\coprod Spec(k) \times_X U_i \longrightarrow \coprod U_i$$

is quasi-compact (because over each U_i we see that $Spec(k) \times_X U_i$ is a finite disjoint union spectra of fields). Thus $Spec(k) \to X$ is quasi-compact by Morphisms of Spaces, Lemma 42.9.7. $\square$

Lemma 43.4.6. *Let S be a scheme. Let X be an algebraic space over S. The following are equivalent:*

1. *there exist schemes U_i and étale morphisms $U_i \to X$ such that $\coprod U_i \to X$ is surjective and each $U_i \to X$ has universally bounded fibres, and*
2. *for every affine scheme U and étale morphism $\varphi : U \to X$ the fibres of $U \to X$ are universally bounded.*

Proof. The implication (2) $\Rightarrow$ (1) is trivial. Assume (1). Let $(\varphi_i : U_i \to X)_{i \in I}$ be a collection of étale morphisms from schemes towards X, covering X, such that each φ_i has universally bounded fibres. Let $\psi : U \to X$ be an étale morphism from an affine scheme towards X. For each i consider the fibre product diagram

$$\begin{tikzcd} U \times_X U_i \arrow[r, "p_i"'] \arrow[d, "q_i"'] & U_i \arrow[d, "\varphi_i"] \\ U \arrow[r, "\psi"] & X \end{tikzcd}$$

Since q_i is étale it is open (see Remark 43.4.1). Moreover, we have $U = \bigcup \mathrm{Im}(q_i)$, since the family $(\varphi_i)_{i \in I}$ is surjective. Since U is affine, hence quasi-compact we can finite finitely many $i_1, \ldots, i_n \in I$ and quasi-compact opens $W_j \subset U \times_X U_{i_j}$ such that $U = \bigcup p_{i_j}(W_j)$. The morphism p_{i_j} is étale, hence locally quasi-finite (see remark on étale morphisms above). Thus we may apply Morphisms, Lemma 24.48.8 to see the fibres of $p_{i_j}|_{W_j} : W_j \to U_{i_j}$ are universally bounded. Hence by Lemma 43.3.2 we see that the fibres of $W_j \to X$ are universally bounded. Thus also $\coprod_{j=1,\ldots,n} W_j \to X$ has universally bounded fibres. Since $\coprod_{j=1,\ldots,n} W_j \to X$ factors through the surjective étale map $\coprod q_{i_j}|_{W_j} : \coprod_{j=1,\ldots,n} W_j \to U$ we see that the fibres of $U \to X$ are universally bounded by Lemma 43.3.5. In other words (2) holds. $\square$

Lemma 43.4.7. *Let S be a scheme. Let X be an algebraic space over S. The following are equivalent:*

1. *there exists a Zariski covering $X = \bigcup X_i$ and for each i a scheme U_i and a quasi-compact surjective étale morphism $U_i \to X_i$, and*
2. *there exist schemes U_i and étale morphisms $U_i \to X$ such that the projections $U_i \times_X U_i \to U_i$ are quasi-compact and $\coprod U_i \to X$ is surjective.*

Proof. If (1) holds then the morphisms $U_i \to X_i \to X$ are étale (combine Morphisms, Lemma 24.35.3 and Spaces, Lemmas 40.5.4 and 40.5.3). Moreover, as $U_i \times_X U_i = U_i \times_{X_i} U_i$, both projections $U_i \times_X U_i \to U_i$ are quasi-compact.

If (2) holds then let $X_i \subset X$ be the open subspace corresponding to the image of the open map $|U_i| \to |X|$, see Properties of Spaces, Lemma 41.4.10. The morphisms $U_i \to X_i$ are surjective. Hence $U_i \to X_i$ is surjective étale, and the projections $U_i \times_{X_i} U_i \to U_i$ are quasi-compact, because $U_i \times_{X_i} U_i = U_i \times_X U_i$. Thus by Spaces, Lemma 40.11.2 the morphisms $U_i \to X_i$ are quasi-compact. □

43.5. Conditions on algebraic spaces

In this section we discuss the relationship between various natural conditions on algebraic spaces we have seen above. Please read Section 43.6 to get a feeling for the meaning of these conditions.

Lemma 43.5.1. *Let S be a scheme. Let X be an algebraic space over S. Consider the following conditions on X:*

(α) *For every $x \in |X|$, the equivalent conditions of Lemma 43.4.2 hold.*
(β) *For every $x \in |X|$, the equivalent conditions of Lemma 43.4.3 hold.*
(γ) *For every $x \in |X|$, the equivalent conditions of Lemma 43.4.5 hold.*
(δ) *The equivalent conditions of Lemma 43.4.6 hold.*
(ϵ) *The equivalent conditions of Lemma 43.4.7 hold.*
(ζ) *The space X is Zariski locally quasi-separated.*
(η) *The space X is quasi-separated*
(θ) *The space X is representable, i.e., X is a scheme.*
(ι) *The space X is a quasi-separated scheme.*

We have

$$\begin{tikzcd} & (\theta) \arrow[dr, Rightarrow] & & & & & \\ (\iota) \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & (\zeta) \arrow[r, Rightarrow] & (\epsilon) \arrow[r, Rightarrow] & (\delta) \arrow[r, Rightarrow] & (\gamma) \arrow[r, Leftrightarrow] & (\alpha)+(\beta) \\ & (\eta) \arrow[ur, Rightarrow] & & & & & \end{tikzcd}$$

Proof. The implication $(\gamma) \Leftrightarrow (\alpha) + (\beta)$ is immediate. The implications in the diamond on the left are clear from the definitions.

Assume (ζ), i.e., that X is Zariski locally quasi-separated. Then (ϵ) holds by Properties of Spaces, Lemma 41.6.5.

Assume (ϵ). By Lemma 43.4.7 there exists a Zariski open covering $X = \bigcup X_i$ such that for each i there exists a scheme U_i and a quasi-compact surjective étale morphism $U_i \to X_i$. Choose an i and an affine open subscheme $W \subset U_i$. It suffices to show that $W \to X$ has universally bounded fibres, since then the family of all these morphisms $W \to X$ covers X. To do this we consider the diagram

$$\begin{tikzcd} W \times_X U_i \arrow[r, "p"] \arrow[d, "q"'] & U_i \arrow[d] \\ W \arrow[r] & X \end{tikzcd}$$

Since $W \to X$ factors through X_i we see that $W \times_X U_i = W \times_{X_i} U_i$, and hence q is quasi-compact. Since W is affine this implies that the scheme $W \times_X U_i$ is quasi-compact. Thus we may apply Morphisms, Lemma 24.48.8 and we conclude that p has universally bounded fibres. From Lemma 43.3.4 we conclude that $W \to X$ has universally bounded fibres as well.

Assume (δ). Let U be an affine scheme, and let $U \to X$ be an étale morphism. By assumption the fibres of the morphism $U \to X$ are universally bounded. Thus also the fibres of both projections $R = U \times_X U \to U$ are universally bounded, see Lemma 43.3.3. And by Lemma 43.3.2 also the fibres of $R \to X$ are universally bounded. Hence for any $x \in X$ the fibres of $|U| \to |X|$ and $|R| \to |X|$ over x are finite, see Lemma 43.3.6. In other words, the equivalent conditions of Lemma 43.4.5 hold. This proves that $(\delta) \Rightarrow (\gamma)$. $\square$

Lemma 43.5.2. *Let S be a scheme. Let $\mathcal{P}$ be one of the properties (α), (β), (γ), (δ), (ϵ), (ζ), or (θ) of algebraic spaces listed in Lemma 43.5.1. Then if X is an algebraic space over S, and $X = \bigcup X_i$ is a Zariski open covering such that each X_i has $\mathcal{P}$, then X has $\mathcal{P}$.*

Proof. Let X be an algebraic space over S, and let $X = \bigcup X_i$ is a Zariski open covering such that each X_i has $\mathcal{P}$.

The case $\mathcal{P} = (\alpha)$. The condition (α) for X_i means that for every $x \in |X_i|$ and every affine scheme U, and étale morphism $\varphi : U \to X_i$ the fibre of $\varphi : |U| \to |X_i|$ over x is finite. Consider $x \in X$, an affine scheme U and an étale morphism $U \to X$. Since $X = \bigcup X_i$ is a Zariski open covering there exits a finite affine open covering $U = U_1 \cup \ldots \cup U_n$ such that each $U_j \to X$ factors through some X_{i_j}. By assumption the fibres of $|U_j| \to |X_{i_j}|$ over x are finite for $j = 1, \ldots, n$. Clearly this means that the fibre of $|U| \to |X|$ over x is finite. This proves the result for (α).

The case $\mathcal{P} = (\beta)$. The condition (β) for X_i means that every $x \in |X_i|$ is represented by a monomorphism from the spectrum of a field towards X_i. Hence the same follows for X as $X_i \to X$ is a monomorphism and $X = \bigcup X_i$.

The case $\mathcal{P} = (\gamma)$. Note that $(\gamma) = (\alpha) + (\beta)$ by Lemma 43.5.1 hence the lemma for (γ) follows from the cases treated above.

The case $\mathcal{P} = (\delta)$. The condition (δ) for X_i means there exist schemes U_{ij} and étale morphisms $U_{ij} \to X_i$ with universally bounded fibres which cover X_i. These schemes also give an étale surjective morphism $\coprod U_{ij} \to X$ and $U_{ij} \to X$ still has universally bounded fibres.

The case $\mathcal{P} = (\epsilon)$. The condition (ϵ) for X_i means we can find a set J_i and morphisms $\varphi_{ij} : U_{ij} \to X_i$ such that each φ_{ij} is étale, both projections $U_{ij} \times_{X_i} U_{ij} \to U_{ij}$ are quasi-compact, and $\coprod_{j \in J_i} U_{ij} \to X_i$ is surjective. In this case the compositions $U_{ij} \to X_i \to X$ are étale (combine Morphisms, Lemmas 24.35.3 and 24.35.9 and Spaces, Lemmas 40.5.4 and 40.5.3). Since $X_i \subset X$ is a subspace we see that $U_{ij} \times_{X_i} U_{ij} = U_{ij} \times_X U_{ij}$, and hence the condition on fibre products is preserved. And clearly $\coprod_{i,j} U_{ij} \to X$ is surjective. Hence X satisfies (ϵ).

The case $\mathcal{P} = (\zeta)$. The condition (ζ) for X_i means that X_i is Zariski locally quasi-separated. It is immediately clear that this means X is Zariski locally quasi-separated.

For (θ), see Properties of Spaces, Lemma 41.10.1. $\square$

Lemma 43.5.3. *Let S be a scheme. Let $\mathcal{P}$ be one of the properties (β), (γ), (δ), (ϵ), or (θ) of algebraic spaces listed in Lemma 43.5.1. Let X, Y be algebraic spaces over S. Let $X \to Y$ be a representable morphism. If Y has property $\mathcal{P}$, so does X.*

Proof. Assume $f : X \to Y$ is a representable morphism of algebraic spaces, and assume that Y has $\mathcal{P}$. Let $x \in |X|$, and set $y = f(x) \in |Y|$.

The case $\mathcal{P} = (\beta)$. Condition (β) for Y means there exists a monomorphism $Spec(k) \to Y$ representing y. The fibre product $X_y = Spec(k) \times_Y X$ is a scheme, and x corresponds to a point of X_y, i.e., to a monomorphism $Spec(k') \to X_y$. As $X_y \to X$ is a monomorphism also we see that x is represented by the monomorphism $Spec(k') \to X_y \to X$. In other words (β) holds for X.

The case $\mathcal{P} = (\gamma)$. Since $(\gamma) \Rightarrow (\beta)$ we have seen in the preceding paragraph that y and x can be represented by monomorphisms as in the following diagram

$$\begin{tikzcd} Spec(k') \arrow[r, "x"'] \arrow[d] & X \arrow[d] \\ Spec(k) \arrow[r, "y"] & Y \end{tikzcd}$$

Also, by definition of property (γ) via Lemma 43.4.5 (2) there exist schemes V_i and étale morphisms $V_i \to Y$ such that $\coprod V_i \to Y$ is surjective and for each i, setting $R_i = V_i \times_Y V_i$ the fibres of both

$$|V_i| \longrightarrow |Y| \quad \text{and} \quad |R_i| \longrightarrow |Y|$$

over y are finite. This means that the schemes $(V_i)_y$ and $(R_i)_y$ are finite schemes over $y = Spec(k)$. As $X \to Y$ is representable, the fibre products $U_i = V_i \times_Y X$ are schemes. The morphisms $U_i \to X$ are étale, and $\coprod U_i \to X$ is surjective. Finally, for each i we have

$$(U_i)_x = (V_i \times_Y X)_x = (V_i)_y \times_{Spec(k)} Spec(k')$$

and

$$(U_i \times_X U_i)_x = \big((V_i \times_Y X) \times_X (V_i \times_Y X)\big)_x = (R_i)_y \times_{Spec(k)} Spec(k')$$

hence these are finite over k' as base changes of the finite schemes $(V_i)_y$ and $(R_i)_y$. This implies that (γ) holds for X, again via the second condition of Lemma 43.4.5.

The case $\mathcal{P} = (\delta)$. Let $V \to Y$ be an étale morphism with V an affine scheme. Since Y has property (δ) this morphism has universally bounded fibres. By Lemma 43.3.3 the base change $V \times_Y X \to X$ also has universally bounded fibres. Hence the first part of Lemma 43.4.6 applies and we see that Y also has property (δ).

The case $\mathcal{P} = (\epsilon)$. We will repeatedly use Spaces, Lemma 40.5.5. Let $V_i \to Y$ be as in Lemma 43.4.7 (2). Set $U_i = X \times_Y V_i$. The morphisms $U_i \to X$ are étale, and $\coprod U_i \to X$ is surjective. Because $U_i \times_X U_i = X \times_Y (V_i \times_Y V_i)$ we see that the projections $U_i \times_Y U_i \to U_i$ are base changes of the projections $V_i \times_Y V_i \to V_i$, and so quasi-compact as well. Hence X satisfies Lemma 43.4.7 (2).

The case $\mathcal{P} = (\theta)$. In this case the result is Categories, Lemma 4.8.3. $\square$

43.6. Reasonable and decent algebraic spaces

In Lemma 43.5.1 we have seen a number of conditions on algebraic spaces related to the behaviour of étale morphisms from affine schemes into X and related to the existence of special étale coverings of X by schemes. We tabulate the different types of conditions here:

(α)	fi
(β)	points come from monomorphisms of spectra of fields
(γ)	points come from quasi-compact monomorphisms of spectra of fields
(δ)	fi
(ϵ)	cover by etale morphisms from schemes quasi-compact onto their image

The conditions in the following definition are not exactly conditions on the diagonal of X, but they are in some sense separation conditions on X.

Definition 43.6.1. Let S be a scheme. Let X be an algebraic space over S.

(1) We say X is *decent* if for every point $x \in X$ the equivalent conditions of Lemma 43.4.5 hold, in other words property (γ) of Lemma 43.5.1 holds.
(2) We say X is *reasonable* if the equivalent conditions of Lemma 43.4.6 hold, in other words property (δ) of Lemma 43.5.1 holds.
(3) We say X is *very reasonable* if the equivalent conditions of Lemma 43.4.7 hold, i.e., property (ϵ) of Lemma 43.5.1 holds.

In particular we have

$$\text{very reasonable} \Rightarrow \text{reasonable} \Rightarrow \text{decent}.$$

The notion of a very reasonable algebraic space was introduced because the assumption was sufficient to prove some of the results below, especially Proposition 43.8.1 and Proposition 43.9.6. We hope (in the future) to strengthen these results to the case where the space X is reasonable or even just decent. If there exists a scheme U and a surjective, étale, quasi-compact morphism $U \to X$, then X is very reasonable, see Lemma 43.4.7.

Lemma 43.6.2. *A scheme is very reasonable.*

Proof. This is true because the identity map is a quasi-compact, surjective étale morphism. □

Lemma 43.6.3. *Let S be a scheme. Let X be an algebraic space over S. If there exists a Zariski open covering $X = \bigcup X_i$ such that each X_i is very reasonable, then X is very reasonable.*

Proof. This is case (ϵ) of Lemma 43.5.2. □

Lemma 43.6.4. *An algebraic space which is Zariski locally quasi-separated is very reasonable. In particular any quasi-separated algebraic space is very reasonable.*

Proof. This is one of the implications of Lemma 43.5.1. □

Lemma 43.6.5. *Let S be a scheme. Let X, Y be algebraic spaces over S. Let $Y \to X$ be a representable morphism. If X is very reasonable, so is Y.*

Proof. This is case (ϵ) of Lemma 43.5.3. □

Remark 43.6.6. Very reasonable algebraic spaces form a stricly larger collection than Zariski locally quasi-separated algebraic spaces. Consider an algebraic space of the form $X = [U/G]$ (see Spaces, Definition 40.14.4) where G is a finite group acting without fixed points on a non-quasi-separated scheme U. Namely, in this case $U \times_X U = U \times G$ and clearly both projections to U are quasi-compact, hence X is very reasonable. On the other hand, the diagonal $U \times_X U \to U \times U$ is not quasi-compact, hence this algebraic space is not quasi-separated. Now, take U the infinite affine space over a field k of characteristic $\neq 2$ with zero doubled, see Schemes, Example 21.21.4. Let $0_1, 0_2$ be the two zeros of U. Let $G = \{+1, -1\}$, and let -1 act by -1 on all coordinates, and by switching 0_1 and 0_2. Then $[U/G]$ is very reasonable but not Zariski locally quasi-separated (details omitted).

Example 43.6.7. The algebraic space $\mathbf{A}^1_{\mathbf{Q}}/\mathbf{Z}$ constructed in Spaces, Example 40.14.8 is not decent as its ``generic point'' cannot be represented by a monomorphism from the spectrum of a point.

Remark 43.6.8. Reasonable algebraic spaces are technically easier to work with than very reasonable algebraic spaces. For example, if $X \to Y$ is a quasi-compact étale surjective morphism of algebraic spaces and X is reasonable, then so is Y, see Lemma 43.13.5 but we don't know if this is true for the property ``very reasonble''. On the other hand, we do not know whether a reasonable algebraic space has an open dense subspace which is a scheme, and we also do not know whether its underlying topological space is sober, whereas we do know that very reasonable spaces have those properties (see Proposition 43.8.1 and Proposition 43.9.6). Below we give another technical property enjoyed by reasonable algebraic spaces.

Lemma 43.6.9. *Let S be a scheme. Let X be a quasi-compact reasonable algebraic space. Then there exists a directed system of quasi-compact and quasi-separated algebraic spaces X_i such that $X = colim_i X_i$ (colimit in the category of sheaves).*

Proof. We sketch the proof. By Properties of Spaces, Lemma 41.6.3 we have $X = U/R$ with U affine. In this case, reasonable means $U \to X$ is universally bounded. Hence there exists an integer N such that the ``fibres'' of $U \to X$ have degree at most N, see Definition 43.3.1. Denote $s, t : R \to U$ and $c : R \times_{s,U,t} R \to R$ the groupoid structural maps.

Claim: for every quasi-compact open $A \subset R$ there exists an open $R' \subset R$ such that

(1) $A \subset R'$,
(2) R' is quasi-compact, and
(3) $(U, R', s|_{R'}, t|_{R'}, c|_{R' \times_{s,U,t} R'})$ is a groupoid scheme.

Note that $e : U \to R$ is open as it is a section of the étale morphism $s : R \to U$, see Étale Morphisms, Proposition 37.6.1. Moreover U is affine hence quasi-compact. Hence we may replace A by $A \cup e(U) \subset R$, and assume that A contains $e(U)$. Next, we define inductively $A^1 = A$, and

$$A^n = c(A^{n-1} \times_{s,U,t} A) \subset R$$

for $n \geq 2$. Arguing inductively, we see that A^n is quasi-compact for all $n \geq 2$, as the image of the quasi-compact fibre product $A^{n-1} \times_{s,U,t} A$. If k is an algebraically closed field over S, and we consider k-points then

$$A^n(k) = \left\{ (u, u') \in U(k) : \begin{matrix} \text{there exist } u = u_1, u_2, \ldots, u_n \in U(k) \text{ with} \\ (u_i, u_{i+1}) \in A \text{ for all } i = 1, \ldots, n-1. \end{matrix} \right\}$$

But as the fibres of $U(k) \to X(k)$ have size at most N we see that if $n > N$ then we get a repeat in the sequence above, and we can shorten it proving $A^N = A^n$ for all $n \geq N$. This implies that $R' = A^N$ gives a groupoid scheme $(U, R', s|_{R'}, t|_{R'}, c|_{R' \times_{s,U,t} R'})$, proving the claim above.

Consider the map of sheaves on $(Sch/S)_{fppf}$

$$colim_{R' \subset R} U/R' \longrightarrow U/R$$

where $R' \subset R$ runs over the quasi-compact open subschemes of R which give étale equivalence relations as above. Each of the quotients U/R' is an algebraic space (see Spaces, Theorem 40.10.5). Since R' is quasi-compact, and U affine the morphism $R' \to U \times_{Spec(\mathbf{Z})} U$ is quasi-compact, and hence U/R' is quasi-separated. Finally, if T is a quasi-compact scheme, then

$$colim_{R' \subset R} U(T)/R'(T) \longrightarrow U(T)/R(T)$$

is a bijection, since every morphism from T into R ends up in one of the open subrelations R' by the claim above. This clearly implies that the colimit of the sheaves U/R' is U/R. In other words the algebraic space $X = U/R$ is the colimit of the quasi-separated algebraic spaces U/R'. □

Warning: The following lemma should be used with caution, as the schemes U_i in it are not necessarily separated or even quasi-separated.

Lemma 43.6.10. *Let S be a scheme. Let X be a very reasonable algebraic space over S. There exists a set of schemes U_i and morphisms $U_i \to X$ such that*

(1) *each U_i is a quasi-compact scheme,*
(2) *each $U_i \to X$ is étale,*
(3) *both projections $U_i \times_X U_i \to U_i$ are quasi-compact, and*
(4) *the morphism $\coprod U_i \to X$ is surjective (and étale).*

Proof. Definition 43.6.1 says that there exist $U_i \to X$ such that (2), (3) and (4) hold. Fix i, and set $R_i = U_i \times_X U_i$, and denote $s, t : R_i \to U_i$ the projections. For any affine open $W \subset U_i$ the open $W' = t(s^{-1}(W)) \subset U_i$ is a quasi-compact R_i-invariant open (see Groupoids, Lemma 35.16.2). Hence W' is a quasi-compact scheme, $W' \to X$ is étale, and $W' \times_X W' = s^{-1}(W') = t^{-1}(W')$ so both projections $W' \times_X W' \to W'$ are quasi-compact. This means the family of $W' \to X$, where $W \subset U_i$ runs through the members of affine open coverings of the U_i gives what we want. □

43.7. Points and specializations

There exists an étale morphism of algebraic spaces $f : X \to Y$ and a nontrivial specializations between points in a fibre of $|f| : |X| \to |Y|$, see Examples, Lemma 64.30.1. If the source of the morphism is a scheme we can avoid this by imposing condition (α) on Y.

Lemma 43.7.1. *Let S be a scheme. Let X be an algebraic space over S. Let $U \to X$ be an étale morphism from a scheme to X. Assume $u, u' \in |U|$ map to the same point x of $|X|$, and $u' \rightsquigarrow u$. If the pair (X, x) satisfies the equivalent conditions of Lemma 43.4.2 then $u = u'$.*

Proof. Assume the pair (X, x) satisfies the equivalent conditions for Lemma 43.4.2. Let U be a scheme, $U \to X$ étale, and let $u, u' \in |U|$ map to x of $|X|$, and $u' \rightsquigarrow u$. We may and do replace U by an affine neighbourhood of u. Let $t, s : R = U \times_X U \to U$ be the étale projection maps.

Pick a point $r \in R$ with $t(r) = u$ and $s(r) = u'$. This is possible by Properties of Spaces, Lemma 41.4.5. Because generalizations lift along the étale morphism t (Remark 43.4.1) we can find a specialization $r' \rightsquigarrow r$ with $t(r') = u'$. Set $u'' = s(r')$. Then $u'' \rightsquigarrow u'$. Thus we may repeat and find $r'' \rightsquigarrow r'$ with $t(r'') = u''$. Set $u''' = s(r'')$, and so on. Here is a picture:

$$\begin{tikzcd} & r'' \arrow[dl, "t"'] \arrow[d, squiggly] \arrow[dr, "s"] & \\ u'' \arrow[d, squiggly] & r' \arrow[dl, "t"'] \arrow[d, squiggly] \arrow[dr, "s"] & u''' \arrow[d, squiggly] \\ u' \arrow[d, squiggly] & r \arrow[dl, "t"'] \arrow[dr, "s"] & u'' \arrow[d, squiggly] \\ u & & u' \end{tikzcd}$$

In Remark 43.4.1 we have seen that there are no specializations among points in the fibres of the étale morphism s. Hence if $u^{(n+1)} = u^{(n)}$ for some n, then also $r^{(n)} = r^{(n-1)}$ and hence also (by taking t) $u^{(n)} = u^{(n-1)}$. This then forces the whole tower to collapse, in particular $u = u'$. Thus we see that if $u \neq u'$, then all the specializations are strict and $\{u, u', u'', \ldots\}$ is an infinite set of points in U which map to the point x in $|X|$. As we chose U affine this contradicts the second part of Lemma 43.4.2, as desired. $\square$

Lemma 43.7.2. *Let S be an algebraic space. Let X be an algebraic space over S. Let $x, x' \in |X|$ and assume $x' \rightsquigarrow x$, i.e., x is a specialization of x'. Assume the pair (X, x') satisfies the equivalent conditions of Lemma 43.4.5. Then for every étale morphism $\varphi : U \to X$ from a scheme U and any $u \in U$ with $\varphi(u) = x$, exists a point $u' \in U$, $u' \rightsquigarrow u$ with $\varphi(u') = x'$.*

Proof. We may replace U by an affine open neighbourhood of u. Hence we may assume that U is affine. As x is in the image of the open map $|U| \to |X|$, so is x'. Thus we may replace X by the Zariski open subspace corresponding to the image of $|U| \to |X|$, see Properties of Spaces, Lemma 41.4.10. In other words we may assume that $U \to X$ is surjective and étale. Let $s, t : R = U \times_X U \to U$ be the projections. By our assumption that (X, x') satisfies the equivalent conditions of Lemma 43.4.5 we see that the fibres of $|U| \to |X|$ and $|R| \to |X|$ over x' are finite. Say $\{u'_1, \ldots, u'_n\} \subset U$ and $\{r'_1, \ldots, r'_m\} \subset R$ form the complete inverse image of $\{x'\}$. Consider the closed sets

$$T = \overline{\{u'_1\}} \cup \ldots \cup \overline{\{u'_n\}} \subset |U|, \quad T' = \overline{\{r'_1\}} \cup \ldots \cup \overline{\{r'_m\}} \subset |R|.$$

Trivially we have $s(T') \subset T$. Because R is an equivalence relation we also have $t(T') = s(T')$ as the set $\{r'_j\}$ is invariant under the inverse of R by construction. Let $w \in T$ be any point. Then $u'_i \rightsquigarrow w$ for some i. Choose $r \in R$ with $s(r) = w$. Since generalizations lift along $s : R \to U$, see Remark 43.4.1, we can find $r' \rightsquigarrow r$ with $s(r') = u'_i$. Then $r' = r'_j$ for some j and we conclude that $w \in s(T')$. Hence $T = s(T') = t(T')$ is an $|R|$-invariant closed set in $|U|$. This means T is the inverse image of a closed (!) subset $T'' = \varphi(T)$ of $|X|$, see Properties of Spaces, Lemmas 41.4.5 and 41.4.6. Hence $T'' = \overline{\{x'\}}$. Thus T contains some point u_1 mapping to x as $x \in T''$. I.e., we see that for some i there exists a specialization $u'_i \rightsquigarrow u_1$ which maps to the given specialization $x' \rightsquigarrow x$.

To finish the proof, choose a point $r \in R$ such that $s(r) = u$ and $t(r) = u_1$ (using Properties of Spaces, Lemma 41.4.3). As generalizations lift along t, and $u'_i \rightsquigarrow u_1$ we can find a specialization $r' \rightsquigarrow r$ such that $t(r') = u'_i$. Set $u' = s(r')$. Then $u' \rightsquigarrow u$ and $\varphi(u') = x'$ as desired. $\square$

Lemma 43.7.3. *Let S be a scheme. Let X be an algebraic space over S. Assume that X is decent. Then $|X|$ is Kolmogorov (see Topology, Definition 5.5.4).*

Proof. Let $x_1, x_2 \in |X|$ with $x_1 \rightsquigarrow x_2$ and $x_2 \rightsquigarrow x_1$. We have to show that $x_1 = x_2$. Pick a scheme U and an étale morphism $U \to X$ such that x_1, x_2 are both in the image of $|U| \to |X|$. By Lemma 43.7.2 we can find a specialization $u_1 \rightsquigarrow u_2$ in U mapping to $x_1 \rightsquigarrow x_2$. By Lemma 43.7.2 we can find $u_2' \rightsquigarrow u_1$ mapping to $x_2 \rightsquigarrow x_1$. This means that $u_2' \rightsquigarrow u_2$ is a specialization between points of U mapping to the same point of X, namely x_2. This is not possible, unless $u_2' = u_2$, see Lemma 43.7.1. Hence also $u_1 = u_2$ as desired. □

43.8. Schematic locus

Proposition 43.8.1. *Let S be a scheme. Let X be an algebraic space over S. If X is very reasonable, then there exists a dense open subspace of X which is a scheme.*

Proof. By Properties of Spaces, Lemma 41.10.1 and Lemma 43.4.7 we may assume that there exists a scheme U and a surjective quasi-compact, étale morphism $U \to X$. Set $R = U \times_X U$, and denote $s, t : R \to U$ the projections as usual. Note that s, t are surjective, quasi-compact and étale, hence also quasi-finite (see Étale Morphisms, Section 37.11). By More on Morphisms, Lemma 33.29.7 there exists a dense open subscheme $W \subset U$ such that $s^{-1}(W) \to W$ is finite. By Descent, Lemma 31.19.21 being finite is fpqc (and in particular étale) local on the target. Hence we may apply More on Groupoids, Lemma 36.5.4 which says that the largest open $W \subset U$ over which s is finite is R-invariant. It is still dense of course. The restriction R_W of R to W equals $R_W = s^{-1}(W) = t^{-1}(W)$ (see Groupoids, Definition 35.16.1 and discussion following it). By construction $s_W, t_W : R_W \to W$ are finite étale. If we can show the open subspace $W/R_W \subset X$ (see Spaces, Lemma 40.10.2) contains a dense open subspace which is a scheme, then the proposition follows for X. This reduces us to Properties of Spaces, Lemma 41.10.2. □

43.9. Points on very reasonable spaces

In this section we prove some properties of points on very reasonable algebraic spaces.

Lemma 43.9.1. *Let S be a scheme. Let X be an algebraic space over S. Consider the map*

$$\{Spec(k) \to X \text{ monomorphism}\} \longrightarrow |X|$$

This map is always injective. If X is very reasonable then this map is a bijection.

Proof. We have seen in Properties of Spaces, Lemma 41.4.11 that the map is an injection in general. By Lemma 43.5.1 it is surjective when X is very reasonable. □

The following lemma is a tiny bit stronger than Properties of Spaces, Lemma 41.11.1. We will improve this lemma in Lemma 43.11.1.

Lemma 43.9.2. *Let S be a scheme. Let k be a field. Let X be an algebraic space over S and assume that there exists a surjective étale morphism $Spec(k) \to X$. If X is very reasonable, then $X \cong Spec(k')$ where $k' \subset k$ is a finite separable extension.*

Proof. This can be proved directly by adding a few words to the proof of Properties of Spaces, Lemma 41.11.1, but we think it is fun to deduce it from the results obtained so far. By Lemma 43.9.1 we see that $Spec(k) \to X$ factors as $Spec(k) \to Spec(k') \to X$ where $Spec(k') \to X$ is a monomorphism. But since $Spec(k) \to X$ is a surjection of sheaves on $(Sch/S)_{fppf}$, we see that also $Spec(k') \to X$ is surjective (as a map of sheaves). But a map of sheaves which is both injective and surjective is an isomorphism. Finally, the fact that

$Spec(k) \to X$ is étale means that $k \otimes_{k'} k$ is étale over k, which implies easily that $k' \subset k$ is a finite separable extension. □

The following lemma shows that specialization of points behaves in a reasonable manner on very reasonable algebraic spaces. Spaces, Example 40.14.9 shows that this is **not** true in general.

Lemma 43.9.3. *Let S be a scheme. Let X be a very reasonable algebraic space over S. Let $U \to X$ be an étale morphism from a scheme to X. If $u, u' \in |U|$ map to the same point of $|X|$, and $u' \rightsquigarrow u$, then $u = u'$.*

Proof. Combine Lemmas 43.5.1 and 43.7.1. □

Lemma 43.9.4. *Let S be an algebraic space. Let X be an algebraic space over S. Let $x, x' \in |X|$ and assume $x' \rightsquigarrow x$, i.e., x is a specialization of x'. Assume X is very reasonable. Then for every étale morphism $\varphi : U \to X$ from a scheme U and any $u \in U$ with $\varphi(u) = x$, exists a point $u' \in U$, $u' \rightsquigarrow u$ with $\varphi(u') = x'$.*

Proof. Combine Lemmas 43.5.1 and 43.7.2. □

Lemma 43.9.5. *Let S be a scheme. Let X be a very reasonable algebraic space over S. Then $|X|$ is Kolmogorov (see Topology, Definition 5.5.4).*

Proof. Combine Lemmas 43.5.1 and 43.7.3. □

Proposition 43.9.6. *Let S be a scheme. Let X be a very reasonable algebraic space over S. Then the topological space $|X|$ is sober (see Topology, Definition 5.5.4).*

Proof. We have seen in Lemma 43.9.5 that $|X|$ is Kolmogorov. Hence it remains to show that every irreducible closed subset $T \subset |X|$ has a generic point. By Properties of Spaces, Lemma 41.9.1 there exists a closed subspace $Z \subset X$ with $|Z| = |T|$. By definition this means that $Z \to X$ is a representable morphism of algebraic spaces. Hence Z is a very reasonable algebraic space by Lemma 43.6.5. By Proposition 43.8.1 we see that there exists an open dense subspace $Z' \subset Z$ which is a scheme. This means that $|Z'| \subset T$ is open dense. Hence the topological space $|Z'|$ is irreducible, which means that Z' is an irreducible scheme. By Schemes, Lemma 21.11.1 we conclude that $|Z'|$ is the closure of a single point $\eta \in T$ and hence also $T = \overline{\{\eta\}}$, and we win. □

43.10. Reduced singleton spaces

A *singleton* space is an algebraic space X such that $|X|$ is a singleton. It turns out that these can be more interesting than just being the spectrum of a field, see Spaces, Example 40.14.7. We develop a tiny bit of machinery to be able to talk about these.

Lemma 43.10.1. *Let S be a scheme. Let Z be an algebraic space over S. Let k be a field and let $Spec(k) \to Z$ be surjective and flat. Then any morphism $Spec(k') \to Z$ where k' is a field is surjective and flat.*

Proof. Consider the fibre square

$$\begin{tikzcd} T \arrow[r] \arrow[d] & Spec(k) \arrow[d] \\ Spec(k') \arrow[r] & Z \end{tikzcd}$$

Note that $T \to Spec(k')$ is flat and surjective hence T is not empty. On the other hand $T \to Spec(k)$ is flat as k is a field. Hence $T \to Z$ is flat and surjective. It follows from Morphisms of Spaces, Lemma 42.28.5 that $Spec(k') \to Z$ is flat. It is surjective as by assumption $|Z|$ is a singleton. $\square$

Lemma 43.10.2. *Let S be a scheme. Let Z be an algebraic space over S. The following are equivalent*

(1) *Z is reduced and $|Z|$ is a singleton,*
(2) *there exists a surjective flat morphism $Spec(k) \to Z$ where k is a field, and*
(3) *there exists a locally of finite type, surjective, flat morphism $Spec(k) \to Z$ where k is a field.*

Proof. Assume (1). Let W be a scheme and let $W \to Z$ be a surjective étale morphism. Then W is a reduced scheme. Let $\eta \in W$ be a generic point of an irreducible component of W. Since W is reduced we have $\mathcal{O}_{W,\eta} = \kappa(\eta)$. It follows that the canonical morphism $\eta = Spec(\kappa(\eta)) \to W$ is flat. We see that the composition $\eta \to Z$ is flat (see Morphisms of Spaces, Lemma 42.27.2). It is also surjective as $|Z|$ is a singleton. In other words (2) holds.

Assume (2). Let W be a scheme and let $W \to Z$ be a surjective étale morphism. Choose a field k and a surjective flat morphism $Spec(k) \to Z$. Then $W \times_Z Spec(k)$ is a scheme étale over k. Hence $W \times_Z Spec(k)$ is a disjoint union of spectra of fields (see Remark 43.4.1), in particular reduced. Since $W \times_Z Spec(k) \to W$ is surjective and flat we conclude that W is reduced (Descent, Lemma 31.15.1). In other words (1) holds.

It is clear that (3) implies (2). Finally, assume (2). Pick a nonempty affine scheme W and an étale morphism $W \to Z$. Pick a closed point $w \in W$ and set $k = \kappa(w)$. The composition

$$Spec(k) \xrightarrow{w} W \longrightarrow Z$$

is locally of finite type by Morphisms of Spaces, Lemmas 42.22.2 and 42.35.9. It is also flat and surjective by Lemma 43.10.1. Hence (3) holds. $\square$

The following lemma singles out a slightly better class of singleton algebraic spaces than the preceding lemma.

Lemma 43.10.3. *Let S be a scheme. Let Z be an algebraic space over S. The following are equivalent*

(1) *Z is reduced, locally Noetherian, and $|Z|$ is a singleton, and*
(2) *there exists a locally finitely presented, surjective, flat morphism $Spec(k) \to Z$ where k is a field.*

Proof. Assume (2) holds. By Lemma 43.10.2 we see that Z is reduced and $|Z|$ is a singleton. Let W be a scheme and let $W \to Z$ be a surjective étale morphism. Choose a field k and a locally finitely presented, surjective, flat morphism $Spec(k) \to Z$. Then $W \times_Z Spec(k)$ is a scheme étale over k, hence a disjoint union of spectra of fields (see Remark 43.4.1), hence locally Noetherian. Since $W \times_Z Spec(k) \to W$ is flat, surjective, and locally of finite presentation, we see that $\{W \times_Z Spec(k) \to W\}$ is an fppf covering and we conclude that W is locally Noetherian (Descent, Lemma 31.12.1). In other words (1) holds.

Assume (1). Pick a nonempty affine scheme W and an étale morphism $W \to Z$. Pick a closed point $w \in W$ and set $k = \kappa(w)$. Because W is locally Noetherian the morphism

$w : Spec(k) \to W$ is of finite presentation, see Morphisms, Lemma 24.20.7. Hence the composition

$$Spec(k) \xrightarrow{w} W \longrightarrow Z$$

is locally of finite presentation by Morphisms of Spaces, Lemmas 42.26.2 and 42.35.8. It is also flat and surjective by Lemma 43.10.1. Hence (2) holds. □

Lemma 43.10.4. *Let S be a scheme. Let $Z' \to Z$ be a monomorphism of algebraic spaces over S. Assume there exists a field k and a locally finitely presented, surjective, flat morphism $Spec(k) \to Z$. Then either Z' is empty or $Z' = Z$.*

Proof. We may assume that Z' is nonempty. In this case the fibre product $T = Z' \times_Z Spec(k)$ is nonempty, see Properties of Spaces, Lemma 41.4.3. Now T is an algebraic space and the projection $T \to Spec(k)$ is a monomorphism. Hence $T = Spec(k)$, see Morphisms of Spaces, Lemma 42.14.8. We conclude that $Spec(k) \to Z$ factors through Z'. But as $Spec(k) \to Z$ is surjective, flat and locally of finite presentation, we see that $Spec(k) \to Z$ is surjective as a map of sheaves on $(Sch/S)_{fppf}$ (see Spaces, Remark 40.5.2) and we conclude that $Z' = Z$. □

The following lemma says that to each point of an algebraic space we can associate a canonical reduced, locally Noetherian singleton algebraic space.

Lemma 43.10.5. *Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. Then there exists a unique monomorphism $Z \to X$ of algebraic spaces over S such that Z is an algebraic space which satisfies the equivalent conditions of Lemma 43.10.3 and such that the image of $|Z| \to |X|$ is $\{x\}$.*

Proof. Choose a scheme U and a surjective étale morphism $U \to X$. Set $R = U \times_X U$ so that $X = U/R$ is a presentation (see Spaces, Section 40.9). Set

$$U' = \coprod\nolimits_{u \in U \text{ lying over } x} Spec(\kappa(u)).$$

The canonical morphism $U' \to U$ is a monomorphism. Let

$$R' = U' \times_X U' = R \times_{(U \times_S U)} (U' \times_S U').$$

Because $U' \to U$ is a monomorphism we see that the projections $s', t' : R' \to U'$ factor as a monomorphism followed by an étale morphism. Hence, as U' is a disjoint union of spectra of fields, using Remark 43.4.1, and using Schemes, Lemma 21.23.10 we conclude that R' is a disjoint union of spectra of fields and that the morphisms $s', t' : R' \to U'$ are étale. Hence $Z = U'/R'$ is an algebraic space by Spaces, Theorem 40.10.5. As R' is the restriction of R by $U' \to U$ we see $Z \to X$ is a monomorphism by Groupoids, Lemma 35.17.6. Since $Z \to X$ is a monomorphism we see that $|Z| \to |X|$ is injective, see Morphisms of Spaces, Lemma 42.14.9. By Properties of Spaces, Lemma 41.4.3 we see that

$$|U'| = |Z \times_X U'| \to |Z| \times_{|X|} |U'|$$

is surjective which implies (by our choice of U') that $|Z| \to |X|$ has image $\{x\}$. We conclude that $|Z|$ is a singleton. Finally, by construction U' is locally Noetherian and reduced, i.e., we see that Z satisfies the equivalent conditions of Lemma 43.10.3.

Let us prove uniqueness of $Z \to X$. Suppose that $Z' \to X$ is a second such monomorphism of algebraic spaces. Then the projections

$$Z' \longleftarrow Z' \times_X Z \longrightarrow Z$$

are monomorphisms. The algebraic space in the middle is nonempty by Properties of Spaces, Lemma 41.4.3. Hence the two projections are isomorphisms by Lemma 43.10.4 and we win. □

We introduce the following terminology which foreshadows the residual gerbes we will introduce later, see Properties of Stacks, Definition 60.11.8.

Definition 43.10.6. Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. The *residual space of X at x*[2] is the monomorphism $Z_x \to X$ constructed in Lemma 43.10.5.

In particular we know that Z_x is a locally Noetherian, reduced, singleton algebraic space and that there exists a field and a surjective, flat, locally finitely presented morphism

$$Spec(k) \longrightarrow Z_x.$$

It turns out that Z_x is a regular algebraic space as follows from the following lemma.

Lemma 43.10.7. *A reduced, locally Noetherian singleton algebraic space Z is regular.*

Proof. Let Z be a reduced, locally Noetherian singleton algebraic space over a scheme S. Let $W \to Z$ be a surjective étale morphism where W is a scheme. Let k be a field and let $Spec(k) \to Z$ be surjective, flat, and locally of finite presentation (see Lemma 43.10.3). The scheme $T = W \times_Z Spec(k)$ is étale over k in particular regular, see Remark 43.4.1. Since $T \to W$ is locally of finite presentation, flat, and surjective it follows that W is regular, see Descent, Lemma 31.15.2. By definition this means that Z is regular. □

43.11. Decent spaces

In this section we collect some useful facts on decent spaces.

Lemma 43.11.1. *Let S be a scheme. Let X be a decent reduced algebraic space over S. Assume that $|X|$ is a singleton. Then $X \cong Spec(k)$ for some field k.*

Proof. As $|X|$ is a singleton X is quasi-compact, see Properties of Spaces, Lemma 41.5.2. Let $U \to X$ be surjective étale with U an affine scheme, see Properties of Spaces, Lemma 41.6.3. Since X is reduced we see that U is reduced, see Properties of Spaces, Section 41.7. As X is decent there exists a monomorphism $Spec(k) \to X$ and $V = Spec(k) \times_X U$ is a scheme finite étale over $Spec(k)$. Namely, this follows from the definition of decent, see Definition 43.6.1, which says that the equivalent conditions of Lemma 43.4.5 hold at the unique point of X. Hence V is a finite disjoint union of spectra of finite separable field extensions of k, see Morphisms, Lemma 24.35.7. On the other hand $V \to U$ is a monomorphism (as $Spec(k) \to X$ is a monomorphism) and surjective (as $Spec(k) \to X$ is surjective by Properties of Spaces, Lemma 41.4.4). In particular U has finitely many points. By Lemma 43.7.1 there are no specializations among the points of U (note that decent implies the condition of that lemma are satisfied in view of Lemma 43.5.1). It follows that U is a finite discrete topological space. As U is also reduced it follows that U is a disjoint union of spectra of fields. By Schemes, Lemma 21.23.10 we conclude that $V \to U$ is an isomorphism. Hence we see that $U \to X$ factors through $Spec(k)$ which implies that $Spec(k) \to X$ is also a surjection of sheaves, whence an isomorphism as desired. □

Remark 43.11.2. We will see later (insert future reference here) that an algebraic space whose reduction is a scheme is a scheme. Hence it follows from Lemma 43.11.1 that a decent algebraic space with one point is a scheme.

[2] This is nonstandard notation.

43.12. Valuative criterion

For a quasi-compact morphism from a decent space the valuative criterion is necessary in order for the morphism to be universally closed.

Proposition 43.12.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume*

(1) *f is quasi-compact, and*
(2) *X is decent.*

Then f is universally closed if and only if the existence part of the valuative criterion holds.

Proof. In Morphisms of Spaces, Lemma 42.12.1 we have seen one of the implications. To prove the other, assume that f is universally closed. Let

$$\begin{tikzcd} Spec(K) \arrow[r] \arrow[d] & X \arrow[d] \\ Spec(A) \arrow[r] & Y \end{tikzcd}$$

be a diagram as in Morphisms of Spaces, Definition 42.11.1. Let $X_A = Spec(A) \times_Y X$, so that we have

$$\begin{tikzcd} Spec(K) \arrow[r] \arrow[rd] & X_A \arrow[d] \\ & Spec(A) \end{tikzcd}$$

By Morphisms of Spaces, Lemma 42.9.3 we see that $X_A \to Spec(A)$ is quasi-compact. Since $X_A \to X$ is representable, we see that X_A has is decent also, see Lemma 43.5.3. Moreover, as f is universally closed, we see that $X_A \to Spec(A)$ is universally closed. Hence we may and do replace X by X_A and Y by $Spec(A)$.

Let $x' \in |X|$ be the equivalence class of $Spec(K) \to X$. Let $y \in |Y| = |Spec(A)|$ be the closed point. Set $y' = f(x')$; it is the generic point of $Spec(A)$. Since f is universally closed we see that $f(\overline{\{x'\}})$ contains $\overline{\{y'\}}$, and hence contains y. Let $x \in \overline{\{x'\}}$ be a point such that $f(x) = y$. Let U be a scheme, and $\varphi : U \to X$ an étale morphism such that there exists a $u \in U$ with $\varphi(u) = x$. By Lemma 43.7.2 and our assumption that X is decent there exists a specialization $u' \rightsquigarrow u$ on U with $\varphi(u') = x'$. This means that there exists a common field extension $K \subset K' \supset \kappa(u')$ such that

$$\begin{tikzcd} Spec(K') \arrow[r] \arrow[d] & U \arrow[d] \\ Spec(K) \arrow[r] \arrow[rd] & X \arrow[d] \\ & Spec(A) \end{tikzcd}$$

is commutative. This gives the following commutative diagram of rings

$$\begin{tikzcd} K' & \mathcal{O}_{U,u} \arrow[l] \\ K \arrow[u] & \\ & A \arrow[ul] \arrow[uu] \end{tikzcd}$$

By Algebra, Lemma 7.46.2 we can find a valuation ring $A' \subset K'$ dominating the image of $\mathcal{O}_{U,u}$ in K'. Since by construction $\mathcal{O}_{U,u}$ dominates A we see that A' dominates A also. Hence we obtain a diagram resembling the second diagram of Morphisms of Spaces, Definition 42.11.1 and the proposition is proved. $\square$

43.13. Relative conditions

This is a (yet another) technical section dealing with conditions on algebraic spaces having to do with points. It is probably a good idea to skip this section.

Definition 43.13.1. Let S be a scheme. We say an algebraic space X over S *has property* (β) if X has the corresponding property of Lemma 43.5.1. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f *has property* (β) if for any scheme T and morphism $T \to Y$ the fibre product $T \times_Y X$ has property (β).
(2) We say f is *decent* if for any scheme T and morphism $T \to Y$ the fibre product $T \times_Y X$ is a decent algebraic space.
(3) We say f is *reasonable* if for any scheme T and morphism $T \to Y$ the fibre product $T \times_Y X$ is a reasonable algebraic space.
(4) We say f is *very reasonable* if for any scheme T and morphism $T \to Y$ the fibre product $T \times_Y X$ is a very reasonable algebraic space.

We refer to Remark 43.13.7 for an informal discussion. It will turn out that the class of very reasonable morphisms is not so useful, but that the classes of decent and reasonable morphisms are useful.

Lemma 43.13.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We have the following implications among the conditions on f:*

$$\begin{tikzcd} \textit{representable} \arrow[dr, Rightarrow] & & & \\ & \textit{very reasonable} \arrow[r, Rightarrow] & \textit{reasonable} \arrow[r, Rightarrow] & \textit{decent} \arrow[r, Rightarrow] & (\beta) \\ \textit{quasi-separated} \arrow[ur, Rightarrow] & & & \end{tikzcd}$$

Proof. This is clear from the definitions, Lemma 43.5.1 and Morphisms of Spaces, Lemma 42.5.12. $\square$

Lemma 43.13.3. *Having property (β), (γ), (δ), or (ϵ) is preserved under arbitrary base change.*

Proof. Omitted. □

Lemma 43.13.4. *Having property (β), being decent, or being reasonable is preserved under compositions.*

Proof. Let $\omega \in \{\beta, decent, reasonable\}$. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of algebraic spaces over the scheme S. Assume f and g both have property (ω). Then we have to show that for any scheme T and morphism $T \to Z$ the space $T \times_Z X$ has (ω). By Lemma 43.13.3 this reduces us to the following claim: Suppose that Y is an algebraic space having property (ω), and that $f : X \to Y$ is a morphism with (ω). Then X has (ω).

Let us prove the claim in case $\omega = \beta$. In this case we have to show that any $x \in |X|$ is represented by a monomorphism from the spectrum of a field into X. Let $y = f(x) \in |Y|$. By assumption there exists a field k and a monomorphism $Spec(k) \to Y$ representing y. Then x corresponds to a point x' of $Spec(k) \times_Y X$. By assumption x' is represented by a monomorphism $Spec(k') \to Spec(k) \times_Y X$. Clearly the composition $Spec(k') \to X$ is a monomorphism representing x.

Let us prove the claim in case $\omega = decent$. Let $x \in |X|$ and $y = f(x) \in |Y|$. By the result of the preceding paragraph we can choose a diagram

$$\begin{tikzcd} Spec(k') \arrow[r, "x"'] \arrow[d] & X \arrow[d, "f"] \\ Spec(k) \arrow[r, "y"] & Y \end{tikzcd}$$

whose horizontal arrows monomorphisms. As Y is decent the morphism y is quasi-compact. As f is decent the algebraic space $Spec(k) \times_Y X$ is decent. Hence the monomorphism $Spec(k') \to Spec(k) \times_Y X$ is quasi-compact. Then the monomorphism $x : Spec(k') \to X$ is quasi-compact as a composition of quasi-compact morphisms (use Morphisms of Spaces, Lemmas 42.9.3 and 42.9.4). As the point x was arbitrary this implies X is decent.

Let us prove the claim in case $\omega = reasonable$. Choose $V \to Y$ étale with V an affine scheme. Choose $U \to V \times_Y X$ étale with U an affine scheme. By assumption $V \to Y$ has universally bounded fibres. By Lemma 43.3.3 the morphism $V \times_Y X \to X$ has universally bounded fibres. By assumption on f we see that $U \to V \times_Y X$ has universally bounded fibres. By Lemma 43.3.2 the composition $U \to X$ has universally bounded fibres. Hence there exists sufficiently many étale morphisms $U \to X$ from schemes with universally bounded fibres, and we conclude that X is reasonable. □

Lemma 43.13.5. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{P} \in \{(\beta), decent, reasonable\}$. Assume*

(1) *f is quasi-compact,*
(2) *f is étale,*
(3) *$|f| : |X| \to |Y|$ is surjective, and*
(4) *the algebraic space X has property $\mathcal{P}$.*

Then Y has property $\mathcal{P}$.

Proof. Let us prove this in case $\mathcal{P} = (\beta)$. Let $y \in |Y|$ be a point. We have to show that y can be represented by a monomorphism from a field. Choose a point $x \in |X|$ with $f(x) = y$. By assumption we may represent x by a monomorphism $Spec(k) \to X$, with k a field. By

Lemma 43.4.3 it suffices to show that the projections $Spec(k) \times_Y Spec(k) \to Spec(k)$ are étale and quasi-compact. We can factor the first projection as

$$Spec(k) \times_Y Spec(k) \longrightarrow Spec(k) \times_Y X \longrightarrow Spec(k)$$

The first morphism is a monomorphism, and the second is étale and quasi-compact. By Properties of Spaces, Lemma 41.13.8 we see that $Spec(k) \times_Y X$ is a scheme. Hence it is a finite disjoint union of spectra of finite separable field extensions of k. By Schemes, Lemma 21.23.10 we see that the first arrow identifies $Spec(k) \times_Y Spec(k)$ with a finite disjoint union of spectra of finite separable field extensions of k. Hence the projection morphism is étale and quasi-compact.

Let us prove this in case $\mathcal{P} = decent$. We have already seen in the first paragraph of the proof that this implies that every $y \in |Y|$ can be represented by a monomorphism $y : Spec(k) \to Y$. Pick such a y. Pick an affine scheme U and an étale morphism $U \to X$ such that the image of $|U| \to |Y|$ contains y. By Lemma 43.4.5 it suffices to show that U_y is a finite scheme over k. The fibre product $X_y = Spec(k) \times_Y X$ is a quasi-compact étale algebraic space over k. Hence by Properties of Spaces, Lemma 41.13.8 it is a scheme. So it is a finite disjoint union of spectra of finite separable extensions of k. Say $X_y = \{x_1, \ldots, x_n\}$ so x_i is given by $x_i : Spec(k_i) \to X$ with $[k_i : k] < \infty$. By assumption X is decent, so the schemes $U_{x_i} = Spec(k_i) \times_X U$ are finite over k_i. Finally, we note that $U_y = \coprod U_{x_i}$ as a scheme and we conclude that U_y is finite over k as desired.

Let us prove this in case $\mathcal{P} = reasonable$. Pick an affine scheme V and an étale morphism $V \to Y$. We have the show the fibres of $V \to Y$ are universally bounded. The algebraic space $V \times_Y X$ is quasi-compact. Thus we can find an affine scheme W and a surjective étale morphism $W \to V \times_Y X$, see Properties of Spaces, Lemma 41.6.3. Here is a picture (solid diagram)

$$\begin{tikzcd} W \arrow[r] \arrow[dr] & V \times_Y X \arrow[r] \arrow[d] & X \arrow[d, "f"] & Spec(k) \arrow[l, dotted, "x"'] \arrow[dl, dotted, "y"] \\ & V \arrow[r] & Y & \end{tikzcd}$$

The morphism $W \to X$ is universally bounded by our assumption that the space X is reasonable. Let n be an integer bounding the degrees of the fibres of $W \to X$. We claim that the same integer works for bounding the fibres of $V \to Y$. Namely, suppose $y \in |Y|$ is a point. Then there exists a $x \in |X|$ with $f(x) = y$ (see above). This means we can find a field k and morphisms x, y given as dotted arrows in the diagram above. In particular we get a surjective étale morphism

$$Spec(k) \times_{x,X} W \to Spec(k) \times_{x,X} (V \times_Y X) = Spec(k) \times_{y,Y} V$$

which shows that the degree of $Spec(k) \times_{y,Y} V$ over k is less than or equal to the degree of $Spec(k) \times_{x,X} W$ over k, i.e., $\leq n$, and we win. (This last part of the argument is the same as the argument in the proof of Lemma 43.3.4. Unfortunately that lemma is not general enough because it only applies to representable morphisms.) □

Lemma 43.13.6. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{P} \in \{(\beta), decent, reasonable, very\ reasonable\}$. The following are equivalent*

(1) *f is $\mathcal{P}$,*
(2) *for every affine scheme Z and every morphism $Z \to Y$ the base change $Z \times_Y X \to Z$ of f is $\mathcal{P}$,*

(3) *for every affine scheme Z and every morphism $Z \to Y$ the algebraic space $Z \times_Y X$ is $\mathcal{P}$; and*
(4) *there exists a Zariski covering $Y = \bigcup Y_i$ such that each morphism $f^{-1}(Y_i) \to Y_i$ has $\mathcal{P}$.*

If $\mathcal{P} \in \{(\beta), decent, reasonable\}$, then this is also equivalent to

(5) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that the base change $V \times_Y X \to V$ has $\mathcal{P}$.*

Proof. The implications (1) ⇒ (2) ⇒ (3) ⇒ (4) are trivial. The implication (3) ⇒ (1) can be seen as follows. Let $Z \to Y$ be a morphism whose source is a scheme over S. Consider the algebraic space $Z \times_Y X$. If we assume (3), then for any affine open $W \subset Z$, the open subspace $W \times_Y X$ of $Z \times_Y X$ has property $\mathcal{P}$. Hence by Lemma 43.5.2 the space $Z \times_Y X$ has property $\mathcal{P}$, i.e., (1) holds. A similar argument (omitted) shows that (4) implies (1).

The implication (1) ⇒ (5) is trivial. Let $V \to Y$ be an étale morphism from a scheme as in (5). Let Z be an affine scheme, and let $Z \to Y$ be a morphism. Consider the diagram

$$\begin{tikzcd} Z \times_Y V \arrow[r, "q"'] \arrow[d, "p"'] & V \arrow[d] \\ Z \arrow[r] & Y \end{tikzcd}$$

Since p is étale, and hence open, we can choose finitely many affine open subschemes $W_i \subset Z \times_Y V$ such that $Z = \bigcup p(W_i)$. Consider the commutative diagram

$$\begin{tikzcd} V \times_Y X \arrow[d] & (\coprod W_i) \times_Y X \arrow[l] \arrow[r] \arrow[d] & Z \times_Y X \arrow[d] \\ V & \coprod W_i \arrow[l] \arrow[r] & Z \end{tikzcd}$$

We know $V \times_Y X$ has property $\mathcal{P}$. By Lemma 43.5.3 we see that $(\coprod W_i) \times_Y X$ has property $\mathcal{P}$. Note that the morphism $(\coprod W_i) \times_Y X \to Z \times_Y X$ is étale and quasi-compact as the base change of $\coprod W_i \to Z$. Hence by Lemma 43.13.5 we conclude that $Z \times_Y X$ has property $\mathcal{P}$. □

Remark 43.13.7. An informal description of the properties (β), decent, reasonable, very reasonable was given in Section 43.6. A morphism has one of these properties if (very) loosely speaking the fibres of the morphism have the corresponding properties. Being decent is useful to prove things about specializations of points on $|X|$. Being reasonable is a bit stronger, and technically quite easy to work with. Very reasonable is a good condition in the sense that it implies that X has a dense open subspace which is a scheme, and that $|X|$ is a sober topological space. This is not clear for reasonable spaces and probably not true; although Lemma 43.6.9 shows reasonable spaces are very close to being quasi-separated. On the other hand, we do not know whether the class of very reasonable morphisms is closed under composition, and we do not know whether very reasonable spaces satisfy a descent property as the one in Lemma 43.13.5 (even with f assumed representable).

Here is a lemma we promised earlier which uses decent morphisms.

Lemma 43.13.8. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is quasi-compact and decent. (For example if f is representable, or quasi-separated, see Lemma 43.13.2.) Then f is universally closed if and only if the existence part of the valuative criterion holds.*

Proof. In Morphisms of Spaces, Lemma 42.12.1 we proved that any quasi-compact morphism which satsifies the existence part of the valuative criterion is universally closed. To prove the other, assume that f is universally closed. In the proof of Proposition 43.12.1 we have seen that it suffices to show, for any valuation ring A, and any morphism $Spec(A) \to Y$, that the base change $f_A : X_A \to Spec(A)$ satisfies the existence part of the valuative criterion. By definition the algebraic space X_A has property (γ) and hence Proposition 43.12.1 applies to the morphism f_A and we win. □

43.14. Monomorphisms

Here is another case where monomorphisms are representable.

Lemma 43.14.1. *Let S be a scheme. Let Y be a disjoint union of spectra of zero dimensional local rings over S. Let $f : X \to Y$ be a monomorphism of algebraic spaces over S. Then f is representable, i.e., X is a scheme.*

Proof. This immediately reduces to the case $Y = Spec(A)$ where A is a zero dimensional local ring, i.e., $Spec(A) = \{\mathfrak{m}_A\}$ is a singleton. If $X = \emptyset$, then there is nothing to prove. If not, choose a nonempty affine scheme $U = Spec(B)$ and an étale morphism $U \to X$. As $|X|$ is a singleton (as a subset of $|Y|$, see Morphisms of Spaces, Lemma 42.14.9) we see that $U \to X$ is surjective. Note that $U \times_X U = U \times_Y U = Spec(B \otimes_A B)$. Thus we see that the ring maps $B \to B \otimes_A B$ are étale. Since

$$(B \otimes_A B)/\mathfrak{m}_A(B \otimes_A B) = (B/\mathfrak{m}_A B) \otimes_{A/\mathfrak{m}_A} (B/\mathfrak{m}_A B)$$

we see that $B/\mathfrak{m}_A B \to (B \otimes_A B)/\mathfrak{m}_A(B \otimes_A B)$ is flat and in fact free of rank equal to the dimension of $B/\mathfrak{m}_A B$ as a $A/\mathfrak{m}_A$-vector space. Since $B \to B \otimes_A B$ is étale, this can only happen if this dimension is finite (see for example Morphisms, Lemmas 24.48.7 and 24.48.8). Every prime of B lies over $\mathfrak{m}_A$ (the unique prime of A). Hence $Spec(B) = Spec(B/\mathfrak{m}_A)$ as a topological space, and this space is a finite discrete set as $B/\mathfrak{m}_A B$ is an Artinian ring, see Algebra, Lemmas 7.49.2 and 7.49.8. Hence all prime ideals of B are maximal and $B = B_1 \times \ldots \times B_n$ is a product of finitely many local rings of dimension zero, see Algebra, Lemma 7.49.7. Thus $B \to B \otimes_A B$ is finite étale as all the local rings B_i are henselian by Algebra, Lemma 7.139.11. Thus X is an affine scheme by Groupoids, Proposition 35.19.8. □

43.15. Other chapters

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CHAPTER 44

Topologies on Algebraic Spaces

44.1. Introduction

In this chapter we introduce some topologies on the category of algebraic spaces. Compare with the material in [**Gro71**], [**BLR90**], [**LMB00a**] and [**Knu71a**]. Before doing so we would like to point out that there are many different choices of sites (as defined in Sites, Definition 9.6.2) which give rise to the same notion of sheaf on the underlying category. Hence our choices may be slightly different from those in the references but ultimately lead to the same cohomology groups, etc.

44.2. The general procedure

In this section we explain a general procedure for producing the sites we will be working with. This discussion will make little or no sense unless the reader has read Topologies, Section 30.2.

Let S be a base scheme. Take any category Sch_α constructed as in Sets, Lemma 3.9.2 starting with S and any set of schemes over S you want to be included. Choose any set of coverings Cov_{fppf} on Sch_α as in Sets, Lemma 3.11.1 starting with the category Sch_α and the class of fppf coverings. Let Sch_{fppf} denote the big fppf site so obtained, and let $(Sch/S)_{fppf}$ denote the corresponding big fppf site of S. (The above is entirely as prescribed in Topologies, Section 30.7.)

Given choices as above the category of algebraic spaces over S has a set of isomorphism classes. One way to see this is to use the fact that any algebraic space over S is of the form U/R for some étale equivalence relation $j : R \to U \times_S U$ with $U, R \in Ob((Sch/S)_{fppf})$, see Spaces, Lemma 40.9.1. Hence we can find a full subcategory $Spaces/S$ of the category of algebraic spaces over S which has a set of objects such that each algebraic space is isomorphic to an object of $Spaces/S$. We fix a choice of such a category.

In the sections below, given a topology τ, the big site $(Spaces/S)_\tau$ (resp. the big site $(Spaces/X)_\tau$ of an algebraic space X over S) has as underlying category the category $Spaces/S$ (resp. the subcategory $Spaces/X$ of $Spaces/S$, see Categories, Example 4.2.13). The procedure for turning this into a site is as usual by defining a class of τ-coverings and using Sets, Lemma 3.11.1 to choose a sufficiently large set of coverings which defines the topology.

We point out that the *small étale site $X_{étale}$ of an algebraic space X* has already been defined in Properties of Spaces, Definition 41.15.1. Its objects are schemes étale over X, of which there are plenty by definition of an algebraic spaces. However, a more natural site, from the perspective of this chapter (compare Topologies, Definition 30.4.8) is the site $X_{spaces,étale}$ of Properties of Spaces, Definition 41.15.2. These two sites define the same topos, see Properties of Spaces, Lemma 41.15.3. We will not redefine these in this chapter; instead we will simply use them.

Finally, we intend not to define the Zariski sites, since these do not seem particularly useful (allthough the Zariski topology is occasionally useful).

44.3. Fpqc topology

We briefly discuss the notion of an fpqc covering of algebraic spaces. Please compare with Topologies, Section 30.8. We will show in Descent on Spaces, Proposition 45.4.1 that quasi-coherent sheaves descent along these.

Definition 44.3.1. Let S be a scheme, and let X be an algebraic space over S. An *fpqc covering of* X is a family of morphisms $\{f_i : X_i \to X\}_{i\in I}$ of algebraic spaces such that each f_i is flat and such that for every affine scheme Z and morphism $h : Z \to X$ there exists a standard fpqc covering $\{g_j : Z_j \to Z\}_{j=1,\ldots,n}$ which refines the family $\{X_i \times_X Z \to Z\}_{i\in I}$.

In other words, there exists indices $i_1, \ldots, i_n \in I$ and morphisms $h_j : U_j \to X_{i_j}$ such that $f_{i_j} \circ h_j = h \circ g_j$. Note that if X and all X_i are representable, this is the same as a fpqc covering of schemes by Topologies, Lemma 30.8.11.

Lemma 44.3.2. *Let S be a scheme. Let X be an algebraic space over S.*

1. *If $X' \to X$ is an isomorphism then $\{X' \to X\}$ is an fpqc covering of X.*
2. *If $\{X_i \to X\}_{i\in I}$ is an fpqc covering and for each i we have an fpqc covering $\{X_{ij} \to X_i\}_{j\in J_i}$, then $\{X_{ij} \to X\}_{i\in I, j\in J_i}$ is an fpqc covering.*
3. *If $\{X_i \to X\}_{i\in I}$ is an fpqc covering and $X' \to X$ is a morphism of algebraic spaces then $\{X' \times_X X_i \to X'\}_{i\in I}$ is an fpqc covering.*

Proof. Part (1) is clear. Consider $g : X' \to X$ and $\{X_i \to X\}_{i\in I}$ an fpqc covering as in (3). By Morphisms of Spaces, Lemma 42.27.3 the morphisms $X' \times_X X_i \to X'$ are flat. If $h' : Z \to X'$ is a morphism from an affine scheme towards X', then set $h = g \circ h' : Z \to X$. The assumption on $\{X_i \to X\}_{i\in I}$ means there exists a standard fpqc covering $\{Z_j \to Z\}_{j=1,\ldots,n}$ and morphisms $Z_j \to X_{i(j)}$ covering h for certain $i(j) \in I$. By the universal property of the fibre product we obtain morphisms $Z_j \to X' \times_X X_{i(j)}$ over h' also. Hence $\{X' \times_X X_i \to X'\}_{i\in I}$ is an fpqc covering. This proves (3).

Let $\{X_i \to X\}_{i\in I}$ and $\{X_{ij} \to X_i\}_{j\in J_i}$ be as in (2). Let $h : Z \to X$ be a morphism from an affine scheme towards X. By assumption there exists a standard fpqc covering $\{Z_j \to Z\}_{j=1,\ldots,n}$ and morphisms $h_j : Z_j \to X_{i(j)}$ covering h for some indices $i(j) \in I$. By assumption there exist standard fpqc coverings $\{Z_{j,l} \to Z_j\}_{l=1,\ldots,n(j)}$ and morphisms $Z_{j,l} \to X_{i(j)j(l)}$ covering h_j for some indices $j(l) \in J_{i(j)}$. By Topologies, Lemma 30.8.10 the family $\{Z_{j,l} \to Z\}$ is a standard fpqc covering. Hence we conclude that $\{X_{ij} \to X\}_{i\in I, j\in J_i}$ is an fpqc covering. □

Lemma 44.3.3. *Let S be a scheme, and let X be an algebraic space over S. Suppose that $\{f_i : X_i \to X\}_{i\in I}$ is a family of morphisms of algebraic spaces with target X. Let $U \to X$ be a surjective étale morphism from a scheme towards X. Then $\{f_i : X_i \to X\}_{i\in I}$ is an fpqc covering of X if and only if $\{U \times_X X_i \to U\}_{i\in I}$ is an fpqc covering of U.*

Proof. If $\{X_i \to X\}_{i\in I}$ is an fpqc covering, then so is $\{U \times_X X_i \to U\}_{i\in I}$ by Lemma 44.3.2. Assume that $\{U \times_X X_i \to U\}_{i\in I}$ is an fpqc covering. Let $h : Z \to X$ be a morphism from an affine scheme towards X. Then we see that $U \times_X Z \to Z$ is a surjective étale morphism of schemes, in particular open. Hence we can find finitely many affine opens $W_1, \ldots, W_t$ of $U \times_X Z$ whose images cover Z. For each j we may apply the condition that $\{U \times_X X_i \to U\}_{i\in I}$ is an fpqc covering to the morphism $W_j \to U$, and obtain a standard fpqc covering $\{W_{jl} \to W_j\}$ which refines $\{W_j \times_X X_i \to W_j\}_{i \in I}$. Hence $\{W_{jl} \to Z\}$ is a standard fpqc covering of Z (see Topologies, Lemma 30.8.10) which refines $\{Z \times_X X_i \to X\}$ and we win. □

Lemma 44.3.4. *Let S be a scheme, and let X be an algebraic space over S. Suppose that $\mathcal{U} = \{f_i : X_i \to X\}_{i \in I}$ is an fpqc covering of X. Then there exists a refinement $\mathcal{V} = \{g_i : T_i \to X\}$ of $\mathcal{U}$ which is an fpqc covering such that each T_i is a scheme.*

Proof. Omitted. Hint: For each i choose a scheme T_i and a surjective étale morphism $T_i \to X_i$. Then check that $\{T_i \to X\}$ is an fpqc covering. □

To be continued...

44.4. Fppf topology

In this section we discuss the notion of an fppf covering of algebraic spaces, and we define the big fppf site of an algebraic space. Please compare with Topologies, Section 30.7.

Definition 44.4.1. Let S be a scheme, and let X be an algebraic space over S. An *fppf covering of X* is a family of morphisms $\{f_i : X_i \to X\}_{i \in I}$ of algebraic spaces such that each f_i is flat and locally of finite presentation and such that $\bigcup_{i \in I} f_i(X_i) = X$.

This is exactly the same as Topologies, Definition 30.7.1. In particular, if X and all the X_i are schemes, then we recover the usual notion of an fppf covering of schemes.

Lemma 44.4.2. *Let S be a scheme. Let X be an algebraic space over S.*

(1) *If $X' \to X$ is an isomorphism then $\{X' \to X\}$ is an fppf covering of X.*
(2) *If $\{X_i \to X\}_{i \in I}$ is an fppf covering and for each i we have an fppf covering $\{X_{ij} \to X_i\}_{j \in J_i}$, then $\{X_{ij} \to X\}_{i \in I, j \in J_i}$ is an fppf covering.*
(3) *If $\{X_i \to X\}_{i \in I}$ is an fppf covering and $X' \to X$ is a morphism of algebraic spaces then $\{X' \times_X X_i \to X'\}_{i \in I}$ is an fppf covering.*

Proof. Omitted. □

Lemma 44.4.3. *Let S be a scheme, and let X be an algebraic space over S. Suppose that $\mathcal{U} = \{f_i : X_i \to X\}_{i \in I}$ is an fppf covering of X. Then there exists a refinement $\mathcal{V} = \{g_i : T_i \to X\}$ of $\mathcal{U}$ which is an fppf covering such that each T_i is a scheme.*

Proof. Omitted. Hint: For each i choose a scheme T_i and a surjective étale morphism $T_i \to X_i$. Then check that $\{T_i \to X\}$ is an fppf covering. □

Lemma 44.4.4. *Let S be a scheme. Let $\{f_i : X_i \to X\}_{i \in I}$ be an fppf covering of algebraic spaces over S. Then the map of sheaves*

$$\coprod X_i \longrightarrow X$$

is surjective.

Proof. Omitted. See Spaces, Remark 40.5.2 if you are confused about the meaning of this simple lemma. □

To be continued...

44.5. Syntomic topology

In this section we discuss the notion of a syntomic covering of algebraic spaces, and we define the big syntomic site of an algebraic space. Please compare with Topologies, Section 30.6.

Definition 44.5.1. Let S be a scheme, and let X be an algebraic space over S. A *syntomic covering of* X is a family of morphisms $\{f_i : X_i \to X\}_{i\in I}$ of algebraic spaces such that each f_i is syntomic and such that $\bigcup_{i\in I} f_i(X_i) = X$.

This is exactly the same as Topologies, Definition 30.6.1. In particular, if X and all the X_i are schemes, then we recover the usual notion of a syntomic covering of schemes.

Lemma 44.5.2. *Let S be a scheme. Let X be an algebraic space over S.*

(1) *If $X' \to X$ is an isomorphism then $\{X' \to X\}$ is a syntomic covering of X.*
(2) *If $\{X_i \to X\}_{i\in I}$ is a syntomic covering and for each i we have a syntomic covering $\{X_{ij} \to X_i\}_{j\in J_i}$, then $\{X_{ij} \to X\}_{i\in I, j\in J_i}$ is a syntomic covering.*
(3) *If $\{X_i \to X\}_{i\in I}$ is a syntomic covering and $X' \to X$ is a morphism of algebraic spaces then $\{X' \times_X X_i \to X'\}_{i\in I}$ is a syntomic covering.*

Proof. Omitted. □

To be continued...

44.6. Smooth topology

In this section we discuss the notion of a smooth covering of algebraic spaces, and we define the big smooth site of an algebraic space. Please compare with Topologies, Section 30.5.

Definition 44.6.1. Let S be a scheme, and let X be an algebraic space over S. A *smooth covering of* X is a family of morphisms $\{f_i : X_i \to X\}_{i\in I}$ of algebraic spaces such that each f_i is smooth and such that $\bigcup_{i\in I} f_i(X_i) = X$.

This is exactly the same as Topologies, Definition 30.5.1. In particular, if X and all the X_i are schemes, then we recover the usual notion of a smooth covering of schemes.

Lemma 44.6.2. *Let S be a scheme. Let X be an algebraic space over S.*

(1) *If $X' \to X$ is an isomorphism then $\{X' \to X\}$ is a smooth covering of X.*
(2) *If $\{X_i \to X\}_{i\in I}$ is a smooth covering and for each i we have a smooth covering $\{X_{ij} \to X_i\}_{j\in J_i}$, then $\{X_{ij} \to X\}_{i\in I, j\in J_i}$ is a smooth covering.*
(3) *If $\{X_i \to X\}_{i\in I}$ is a smooth covering and $X' \to X$ is a morphism of algebraic spaces then $\{X' \times_X X_i \to X'\}_{i\in I}$ is a smooth covering.*

Proof. Omitted. □

To be continued...

44.7. Étale topology

In this section we discuss the notion of a étale covering of algebraic spaces, and we define the big étale site of an algebraic space. Please compare with Topologies, Section 30.4.

Definition 44.7.1. Let S be a scheme, and let X be an algebraic space over S. A *étale covering of* X is a family of morphisms $\{f_i : X_i \to X\}_{i\in I}$ of algebraic spaces such that each f_i is étale and such that $\bigcup_{i\in I} f_i(X_i) = X$.

This is exactly the same as Topologies, Definition 30.4.1. In particular, if X and all the X_i are schemes, then we recover the usual notion of a étale covering of schemes.

Lemma 44.7.2. *Let S be a scheme. Let X be an algebraic space over S.*

1. *If $X' \to X$ is an isomorphism then $\{X' \to X\}$ is a étale covering of X.*
2. *If $\{X_i \to X\}_{i\in I}$ is a étale covering and for each i we have a étale covering $\{X_{ij} \to X_i\}_{j\in J_i}$, then $\{X_{ij} \to X\}_{i\in I, j\in J_i}$ is a étale covering.*
3. *If $\{X_i \to X\}_{i\in I}$ is a étale covering and $X' \to X$ is a morphism of algebraic spaces then $\{X' \times_X X_i \to X'\}_{i\in I}$ is a étale covering.*

Proof. Omitted. □

To be continued...

44.8. Zariski topology

In Spaces, Section 40.12 we introduced the notion of a Zariski covering of an algebraic space by open subspaces. Here is the corresponding notion with open subspaces replaces by open immersions.

Definition 44.8.1. Let S be a scheme, and let X be an algebraic space over S. A *Zariski covering of X* is a family of morphisms $\{f_i : X_i \to X\}_{i\in I}$ of algebraic spaces such that each f_i is an open immersion and such that $\bigcup_{i\in I} f_i(X_i) = X$.

Allthough Zariski coverings are occasionally useful the corresponding topology on the category of algebraic spaces is really too coarse, and not particularly useful. Still, it does define a site.

Lemma 44.8.2. *Let S be a scheme. Let X be an algebraic space over S.*

1. *If $X' \to X$ is an isomorphism then $\{X' \to X\}$ is a Zariski covering of X.*
2. *If $\{X_i \to X\}_{i\in I}$ is a Zariski covering and for each i we have a Zariski covering $\{X_{ij} \to X_i\}_{j\in J_i}$, then $\{X_{ij} \to X\}_{i\in I, j\in J_i}$ is a Zariski covering.*
3. *If $\{X_i \to X\}_{i\in I}$ is a Zariski covering and $X' \to X$ is a morphism of algebraic spaces then $\{X' \times_X X_i \to X'\}_{i\in I}$ is a Zariski covering.*

Proof. Omitted. □

44.9. Other chapters

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CHAPTER 45

Descent and Algebraic Spaces

45.1. Introduction

In the chapter on topologies on algebraic spaces (see Topologies on Spaces, Section 44.1) we introduced étale, fppf, smooth, syntomic and fpqc coverings of algebraic spaces. In this chapter we discuss what kind of structures over algebraic spaces can be descended through such coverings. See for example [**Gro95a**], [**Gro95b**], [**Gro95e**], [**Gro95f**], [**Gro95c**], and [**Gro95d**].

45.2. Conventions

The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $Spec(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

45.3. Descent data for quasi-coherent sheaves

This section is the analogue of Descent, Section 31.2 for algebraic spaces. It makes sense to read that section first.

Definition 45.3.1. Let S be a scheme. Let $\{f_i : X_i \to X\}_{i \in I}$ be a family of morphisms of algebraic spaces over S with fixed target X.

(1) A *descent datum* $(\mathcal{F}_i, \varphi_{ij})$ *for quasi-coherent sheaves* with respect to the given family is given by a quasi-coherent sheaf $\mathcal{F}_i$ on X_i for each $i \in I$, an isomorphism of quasi-coherent $\mathcal{O}_{X_i \times_X X_j}$-modules $\varphi_{ij} : \mathrm{pr}_0^*\mathcal{F}_i \to \mathrm{pr}_1^*\mathcal{F}_j$ for each pair $(i, j) \in I^2$ such that for every triple of indices $(i, j, k) \in I^3$ the diagram

$$\begin{tikzcd} \mathrm{pr}_0^*\mathcal{F}_i \arrow[rr, "\mathrm{pr}_{02}^*\varphi_{ik}"'] \arrow[dr, "\mathrm{pr}_{01}^*\varphi_{ij}"'] & & \mathrm{pr}_2^*\mathcal{F}_k \\ & \mathrm{pr}_1^*\mathcal{F}_j \arrow[ur, "\mathrm{pr}_{12}^*\varphi_{jk}"'] & \end{tikzcd}$$

of $\mathcal{O}_{X_i \times_X X_j \times_X X_k}$-modules commutes. This is called the *cocycle condition*.

(2) A *morphism* $\psi : (\mathcal{F}_i, \varphi_{ij}) \to (\mathcal{F}'_i, \varphi'_{ij})$ *of descent data* is given by a family $\psi = (\psi_i)_{i\in I}$ of morphisms of $\mathcal{O}_{X_i}$-modules $\psi_i : \mathcal{F}_i \to \mathcal{F}'_i$ such that all the diagrams

$$\begin{tikzcd} \mathrm{pr}_0^*\mathcal{F}_i \arrow[r, "\varphi_{ij}"'] \arrow[d, "\mathrm{pr}_0^*\psi_i"'] & \mathrm{pr}_1^*\mathcal{F}_j \arrow[d, "\mathrm{pr}_1^*\psi_j"] \\ \mathrm{pr}_0^*\mathcal{F}'_i \arrow[r, "\varphi'_{ij}"] & \mathrm{pr}_1^*\mathcal{F}'_j \end{tikzcd}$$

commute.

Lemma 45.3.2. *Let S be a scheme. Let $\mathcal{U} = \{U_i \to U\}_{i\in I}$ and $\mathcal{V} = \{V_j \to V\}_{j\in J}$ be families of morphisms of algebraic spaces over S with fixed targets. Let $(g, \alpha : I \to J, (g_i)) : \mathcal{U} \to \mathcal{V}$ be a morphism of families of maps with fixed target, see Sites, Definition 9.8.1. Let $(\mathcal{F}_j, \varphi_{jj'})$ be a descent datum for quasi-coherent sheaves with respect to the family $\{V_j \to V\}_{j\in J}$. Then*

(1) *The system*
$$\left(g_i^*\mathcal{F}_{\alpha(i)}, (g_i \times g_{i'})^*\varphi_{\alpha(i)\alpha(i')}\right)$$
is a descent datum with respect to the family $\{U_i \to U\}_{i\in I}$.

(2) *This construction is functorial in the descent datum $(\mathcal{F}_j, \varphi_{jj'})$.*

(3) *Given a second morphism $(g', \alpha' : I \to J, (g'_i))$ of families of maps with fixed target with $g = g'$ there exists a functorial isomorphism of descent data*
$$(g_i^*\mathcal{F}_{\alpha(i)}, (g_i \times g_{i'})^*\varphi_{\alpha(i)\alpha(i')}) \cong ((g'_i)^*\mathcal{F}_{\alpha'(i)}, (g'_i \times g'_{i'})^*\varphi_{\alpha'(i)\alpha'(i')}).$$

Proof. Omitted. Hint: The maps $g_i^*\mathcal{F}_{\alpha(i)} \to (g'_i)^*\mathcal{F}_{\alpha'(i)}$ which give the ismorphism of descent data in part (3) are the pullbacks of the maps $\varphi_{\alpha(i)\alpha'(i)}$ by the morphisms $(g_i, g'_i) : U_i \to V_{\alpha(i)} \times_V V_{\alpha'(i)}$. □

Let $g : U \to V$ be a morphism of algebraic spaces. The lemma above tells us that there is a well defined pullback functor between the categories of descent data relative to families of maps with target V and U provided there is a morphism between those families of maps which ``lives over g''.

Definition 45.3.3. Let S be a scheme. Let $\{U_i \to U\}_{i\in I}$ be a family of morphisms of algebraic spaces over S with fixed target.

(1) Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_U$-module. We call the unique descent on $\mathcal{F}$ datum with respect to the covering $\{U \to U\}$ the *trivial descent datum*.
(2) The pullback of the trival descent datum to $\{U_i \to U\}$ is called the *canonical descent datum*. Notation: $(\mathcal{F}|_{U_i}, can)$.
(3) A descent datum $(\mathcal{F}_i, \varphi_{ij})$ for quasi-coherent sheaves with respect to the given family is said to be *effective* if there exists a quasi-coherent sheaf $\mathcal{F}$ on U such that $(\mathcal{F}_i, \varphi_{ij})$ is isomorphic to $(\mathcal{F}|_{U_i}, can)$.

Lemma 45.3.4. *Let S be a scheme. Let U be an algebraic space over S. Let $\{U_i \to U\}$ be a Zariski covering of U, see Topologies on Spaces, Definition 44.8.1. Any descent datum on quasi-coherent sheaves for the family $\mathcal{U} = \{U_i \to U\}$ is effective. Moreover, the functor from the category of quasi-coherent $\mathcal{O}_U$-modules to the category of descent data with respect to $\{U_i \to U\}$ is fully faithful.*

Proof. Omitted. □

45.4. Fpqc descent of quasi-coherent sheaves

The main application of flat descent for modules is the corresponding descent statement for quasi-coherent sheaves with respect to fpqc-coverings.

Proposition 45.4.1. *Let S be a scheme. Let $\{X_i \to X\}$ be an fpqc covering of algebraic spaces over S, see Topologies on Spaces, Definition 44.3.1. Any descent datum on quasi-coherent sheaves for $\{X_i \to X\}$ is effective. Moreover, the functor from the category of quasi-coherent $\mathcal{O}_X$-modules to the category of descent data with respect to $\{X_i \to X\}$ is fully faithful.*

Proof. This is more or less a formal consequence of the corresponding result for schemes, see Descent, Proposition 31.4.2. Here is a strategy for a proof:

(1) The fact that $\{X_i \to X\}$ is a refinement of the trivial covering $\{X \to X\}$ gives, via Lemma 45.3.2, a functor $QCoh(X) \to DD(\{X_i \to X\})$ from the category of quasi-coherent $\mathcal{O}_X$-modules to the category of descent data for the given family.
(2) In order to prove the proposition we will construct a quasi-inverse functor $back : DD(\{X_i \to X\}) \to QCoh(X)$.
(3) Applying again Lemma 45.3.2 we see that there is a functor $DD(\{X_i \to X\}) \to DD(\{T_j \to X\})$ if $\{T_j \to X\}$ is a refinement of the given family. Hence in order to construct the functor *back* we may assume that each X_i is a scheme, see Topologies on Spaces, Lemma 44.3.4. This reduces us to the case where all the X_i are schemes.
(4) A quasi-coherent sheaf on X is by definition a quasi-coherent $\mathcal{O}_X$-module on $X_{étale}$. Now for any $U \in Ob(X_{étale})$ we get an fppf covering $\{U_i \times_X X_i \to U\}$ by schemes and a morphism $g : \{U_i \times_X X_i \to U\} \to \{X_i \to X\}$ of coverings lying over $U \to X$. Given a descent datum $\xi = (\mathcal{F}_i, \varphi_{ij})$ we obtain a quasi-coherent $\mathcal{O}_U$-module $\mathcal{F}_{\xi,U}$ corresponding to the pullback $g^*\xi$ of Lemma 45.3.2 to the covering of U and using effectivity for fppf covering of schemes, see Descent, Proposition 31.4.2.
(5) Check that $\xi \mapsto \mathcal{F}_{\xi,U}$ is functorial in ξ. Omitted.
(6) Check that $\xi \mapsto \mathcal{F}_{\xi,U}$ is compatible with morphisms $U \to U'$ of the site $X_{étale}$, so that the system of sheaves $\mathcal{F}_{\xi,U}$ corresponds to a quasi-coherent $\mathcal{F}_\xi$ on $X_{étale}$, see Properties of Spaces, Lemma 41.26.3. Details omitted.
(7) Check that $back : \xi \mapsto \mathcal{F}_\xi$ is quasi-inverse to the functor constructed in (1). Omitted.

This finishes the proof. □

45.5. Descent of finiteness properties of modules

This section is the analogue for the case of algebraic spaces of Descent, Section 31.5. The goal is to show that one can check a quasi-coherent module has a certain finiteness conditions by checking on the members of a covering. We will repeatedly use the following proof scheme. Suppose that X is an algebraic space, and that $\{X_i \to X\}$ is a fppf (resp. fpqc) covering. Let $U \to X$ be a surjective étale morphism such that U is a scheme. Then there exists an fppf (resp. fpqc) covering $\{Y_j \to X\}$ such that

(1) $\{Y_j \to X\}$ is a refinement of $\{X_i \to X\}$,
(2) each Y_j is a scheme, and
(3) each morphism $Y_j \to X$ factors though U, and
(4) $\{Y_j \to U\}$ is an fppf (resp. fpqc) covering of U.

Namely, first refine $\{X_i \to X\}$ by an fppf (resp. fpqc) covering such that each X_i is a scheme, see Topologies on Spaces, Lemma 44.4.3, resp. Lemma 44.3.4. Then set $Y_i = U \times_X X_i$. A quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is of finite type, of finite presentation, etc if and only if the quasi-coherent $\mathcal{O}_U$-module $\mathcal{F}|_U$ is of finite type, of finite presentation, etc. Hence we can use the existence of the refinement $\{Y_j \to X\}$ to reduce the proof of the following lemmas to the case of schemes. We will indicate this by saying that ``*the result follows from the case of schemes by étale localization*".

Lemma 45.5.1. *Let X be an algebraic space over a scheme S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i\in I}$ be an fpqc covering such that each $f_i^*\mathcal{F}$ is a finite type $\mathcal{O}_{X_i}$-module. Then $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module.*

Proof. This follows from the case of schemes, see Descent, Lemma 31.5.1, by étale localization. □

Lemma 45.5.2. *Let X be an algebraic space over a scheme S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i\in I}$ be an fpqc covering such that each $f_i^*\mathcal{F}$ is an $\mathcal{O}_{X_i}$-module of finite presentation. Then $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation.*

Proof. This follows from the case of schemes, see Descent, Lemma 31.5.2, by étale localization. □

Lemma 45.5.3. *Let X be an algebraic space over a scheme S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i\in I}$ be an fpqc covering such that each $f_i^*\mathcal{F}$ is a flat $\mathcal{O}_{X_i}$-module. Then $\mathcal{F}$ is a flat $\mathcal{O}_X$-module.*

Proof. This follows from the case of schemes, see Descent, Lemma 31.5.3, by étale localization. □

Lemma 45.5.4. *Let X be an algebraic space over a scheme S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i\in I}$ be an fpqc covering such that each $f_i^*\mathcal{F}$ is a finite locally free $\mathcal{O}_{X_i}$-module. Then $\mathcal{F}$ is a finite locally free $\mathcal{O}_X$-module.*

Proof. This follows from the case of schemes, see Descent, Lemma 31.5.4, by étale localization. □

The definition of a locally projective quasi-coherent sheaf can be found in Properties of Spaces, Section 41.28. It is also proved there that this notion is preserved under pullback.

Lemma 45.5.5. *Let X be an algebraic space over a scheme S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i\in I}$ be an fpqc covering such that each $f_i^*\mathcal{F}$ is a locally projective $\mathcal{O}_{X_i}$-module. Then $\mathcal{F}$ is a locally projective $\mathcal{O}_X$-module.*

Proof. This follows from the case of schemes, see Descent, Lemma 31.5.5, by étale localization. □

We also add here two results which are related to the results above, but are of a slightly different nature.

Lemma 45.5.6. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume f is a finite morphism. Then $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite type if and only if $f_*\mathcal{F}$ is an $\mathcal{O}_Y$-module of finite type.*

Proof. As f is finite it is representable. Choose a scheme V and a surjective étale morphism $V \to Y$. Then $U = V \times_Y X$ is a scheme with a surjective étale morphism towards X and a finite morphism $\psi : U \to V$ (the base change of f). Since $\psi_*(\mathcal{F}|_U) = f_*\mathcal{F}|_V$ the result of the lemma follows immediately from the schemes version which is Descent, Lemma 31.5.7. $\square$

Lemma 45.5.7. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume f is finite and of finite presentation. Then $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation if and only if $f_*\mathcal{F}$ is an $\mathcal{O}_Y$-module of finite presentation.*

Proof. As f is finite it is representable. Choose a scheme V and a surjective étale morphism $V \to Y$. Then $U = V \times_Y X$ is a scheme with a surjective étale morphism towards X and a finite morphism $\psi : U \to V$ (the base change of f). Since $\psi_*(\mathcal{F}|_U) = f_*\mathcal{F}|_V$ the result of the lemma follows immediately from the schemes version which is Descent, Lemma 31.5.8. $\square$

45.6. Fpqc coverings

This section is the analogue of Descent, Section 31.9. At the moment we do not know if all of the material for fpqc coverings of schemes holds also for algebraic spaces.

Lemma 45.6.1. *Let S be a scheme. Let $\{f_i : T_i \to T\}_{i\in I}$ be an fpqc covering of algebraic spaces over S. Suppose that for each i we have an open subspace $W_i \subset T_i$ such that for all $i, j \in I$ we have $pr_0^{-1}(W_i) = pr_1^{-1}(W_j)$ as open subspaces of $T_i \times_T T_j$. Then there exists a unique open subspace $W \subset T$ such that $W_i = f_i^{-1}(W)$ for each i.*

Proof. By Topologies on Spaces, Lemma 44.3.4 we may assume each T_i is a scheme. Choose a scheme U and a surjective étale morphism $U \to T$. Then $\{T_i \times_T U \to U\}$ is an fpqc covering of U and $T_i \times_T U$ is a scheme for each i. Hence we see that the collection of opens $W_i \times_T U$ comes from a unique open subscheme $W' \subset U$ by Descent, Lemma 31.9.2. As $U \to X$ is open we can define $W \subset X$ the Zariski open which is the image of W', see Properties of Spaces, Section 41.4. We omit the verification that this works, i.e., that W_i is the inverse image of W for each i. $\square$

We do not know whether the following lemma holds for fpqc instead of fppf, see Properties of Spaces, Remark 41.14.4.

Lemma 45.6.2. *Let S be a scheme. Let $\{T_i \to T\}$ be an fppf covering of algebraic spaces over S, see Topologies on Spaces, Definition 44.4.1. Then given an algebraic space B over S the sequence*

$$Mor_S(T, B) \longrightarrow \prod_i Mor_S(T_i, B) \rightrightarrows \prod_{i,j} Mor_S(T_i \times_T T_j, B)$$

is an equalizer diagram. In other words, every representable functor on the category of algebraic spaces over S satisfies the sheaf condition for the fppf topology.

Proof. We have seen in Topologies on Spaces, Lemma 44.4.4 that $\coprod T_i \to T$ is surjective as a map of sheaves. Also note that $T_i \times_T T_j$ is the fibre product as sheaves. Since we have

$$Mor_S(T, B) = Mor_{Sh((Sch/S)_{fppf})}(T, B)$$

by definition the lemma follows formally. $\square$

45.7. Descent of finiteness properties of morphisms

The following type of lemma is occasionally useful.

Lemma 45.7.1. *Let S be a scheme. Let $X \to Y \to Z$ be morphism of algebraic spaces. Let P be one of the following properties of morphisms of algebraic spaces over S: flat, locally finite type, locally finite presentation. Assume that $X \to Z$ has P and that $X \to Y$ is a surjection of sheaves on $(Sch/S)_{fppf}$. Then $Y \to Z$ is P.*

Proof. Choose a scheme W and a surjective étale morphism $W \to Z$. Choose a scheme V and a surjective étale morphism $V \to W \times_Z Y$. Choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. By assumption we can find an fppf covering $\{V_i \to V\}$ and lifts $V_i \to X$ of the morphism $V_i \to Y$. Since $U \to X$ is surjective étale we see that over the members of the fppf covering $\{V_i \times_X U \to V\}$ we have lifts into U. Hence $U \to V$ induces a surjection of sheaves on $(Sch/S)_{fppf}$. By our definition of what it means to have property P for a morphism of algebraic spaces (see Morphisms of Spaces, Definition 42.27.1, Definition 42.22.1, and Definition 42.26.1) we see that $U \to W$ has P and we have to show $V \to W$ has P. Thus we reduce the question to the case of morphisms of schemes which is treated in Descent, Lemma 31.10.8. □

45.8. Descending properties of spaces

In this section we put some results of the following kind.

Lemma 45.8.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $x \in |X|$. If f is flat at x and X is geometrically unibranch at x, then Y is geometrically unibranch at $f(x)$.*

Proof. Consider the map of étale local rings $\mathcal{O}_{Y,f(\overline{x})} \to \mathcal{O}_{X,\overline{x}}$. By Morphisms of Spaces, Lemma 42.27.7 this is flat. Hence if $\mathcal{O}_{X,\overline{x}}$ has a unique minimal prime, so does $\mathcal{O}_{Y,f(\overline{x})}$ (by going down, see Algebra, Lemma 7.35.17). □

Lemma 45.8.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is flat and surjective and X is reduced, then Y is reduced.*

Proof. Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme U and a surjective étale morphism $U \to X \times_Y V$. As f is surjective and flat, the morphism of schemes $U \to V$ is surjective and flat. In this way we reduce the problem to the case of schemes (as reducedness of X and Y is defined in terms of reducedness of U and V, see Properties of Spaces, Section 41.7). The case of schemes is Descent, Lemma 31.15.1. □

Lemma 45.8.3. *Let $f : X \to Y$ be a morphism of algebraic spaces. If f is locally of finite presentation, flat, and surjective and X is locally Noetherian, then Y is locally Noetherian.*

Proof. Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme U and a surjective étale morphism $U \to X \times_Y V$. As f is surjective, flat, and locally of finite presentation the morphism of schemes $U \to V$ is surjective, flat, and locally of finite presentation. In this way we reduce the problem to the case of schemes (as being locally Noetherian for X and Y is defined in terms of being locally Noetherian of U and V, see Properties of Spaces, Section 41.7). In the case of schemes the result follows from Descent, Lemma 31.12.1. □

Lemma 45.8.4. *Let $f : X \to Y$ be a morphism of algebraic spaces. If f is locally of finite presentation, flat, and surjective and X is regular, then Y is regular.*

Proof. By Lemma 45.8.3 we know that Y is locally Noetherian. Choose a scheme V and a surjective étale morphism $V \to Y$. It suffices to prove that the local rings of V are all regular local rings, see Properties, Lemma 23.9.2. Choose a scheme U and a surjective étale morphism $U \to X \times_Y V$. As f is surjective and flat the morphism of schemes $U \to V$ is surjective and flat. By assumption U is a regular scheme in particular all of its local rings are regular (by the lemma above). Hence the lemma follows from Algebra, Lemma 7.102.8. □

45.9. Descending properties of morphisms

In this section we introduce the notion of when a property of morphisms of algebraic spaces is local on the target in a topology. Please compare with Descent, Section 31.18.

Definition 45.9.1. Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S. Let $\tau \in \{fpqc, fppf, syntomic, smooth, étale\}$. We say $\mathcal{P}$ is *τ local on the base*, or *τ local on the target*, or *local on the base for the τ-topology* if for any τ-covering $\{Y_i \to Y\}_{i\in I}$ of algebraic spaces and any morphism of algebraic spaces $f : X \to Y$ we have

$$f \text{ has } \mathcal{P} \Leftrightarrow \text{each } Y_i \times_Y X \to Y_i \text{ has } \mathcal{P}.$$

To be sure, since isomorphisms are always coverings we see (or require) that property $\mathcal{P}$ holds for $X \to Y$ if and only if it holds for any arrow $X' \to Y'$ isomorphic to $X \to Y$. If a property is τ-local on the target then it is preserved by base changes by morphisms which occur in τ-coverings. Here is a formal statement.

Lemma 45.9.2. *Let S be a scheme. Let $\tau \in \{fpqc, fppf, syntomic, smooth, étale\}$. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S which is τ local on the target. Let $f : X \to Y$ have property $\mathcal{P}$. For any morphism $Y' \to Y$ which is flat, resp. flat and locally of finite presentation, resp. syntomic, resp. étale, the base change $f' : Y' \times_Y X \to Y'$ of f has property $\mathcal{P}$.*

Proof. This is true because we can fit $Y' \to Y$ into a family of morphisms which forms a τ-covering. □

A simple often used consequence of the above is that if $f : X \to Y$ has property $\mathcal{P}$ which is τ-local on the target and $f(X) \subset V$ for some open subspace $V \subset Y$, then also the induced morphism $X \to V$ has $\mathcal{P}$. Proof: The base change f by $V \to Y$ gives $X \to V$.

Lemma 45.9.3. *Let S be a scheme. Let $\tau \in \{fppf, syntomic, smooth, étale\}$. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S which is τ local on the target. For any morphism of algebraic spaces $f : X \to Y$ over S there exists a largest open subspace $W(f) \subset Y$ such that the restriction $X_{W(f)} \to W(f)$ has $\mathcal{P}$. Moreover,*

1. *if $g : Y' \to Y$ is a morphism of algebraic spaces which is flat and locally of finite presentation, syntomic, smooth, or étale and the base change $f' : X_{Y'} \to Y'$ has $\mathcal{P}$, then g factors through $W(f)$,*
2. *if $g : Y' \to Y$ is flat and locally of finite presentation, syntomic, smooth, or étale, then $W(f') = g^{-1}(W(f))$, and*
3. *if $\{g_i : Y_i \to Y\}$ is a τ-covering, then $g_i^{-1}(W(f)) = W(f_i)$, where f_i is the base change of f by $Y_i \to Y$.*

Proof. Consider the union $W_{set} \subset |Y|$ of the images $g(|Y'|) \subset |Y|$ of morphisms $g : Y' \to Y$ with the properties:

1. g is flat and locally of finite presentation, syntomic, smooth, or étale, and

(2) the base change $Y' \times_{g,Y} X \to Y'$ has property $\mathcal{P}$.

Since such a morphism g is open (see Morphisms of Spaces, Lemma 42.27.5) we see that W_{set} is an open subset of $|Y|$. Denote $W \subset Y$ the open subspace whose underlying set of points is W_{set}, see Properties of Spaces, Lemma 41.4.8. Since $\mathcal{P}$ is local in the τ topology the restriction $X_W \to W$ has property $\mathcal{P}$ because we are given a covering $\{Y' \to W\}$ of W such that the pullbacks have $\mathcal{P}$. This proves the existence and proves that $W(f)$ has property (1). To see property (2) note that $W(f') \supset g^{-1}(W(f))$ because $\mathcal{P}$ is stable under base change by flat and locally of finite presentation, syntomic, smooth, or étale morphisms, see Lemma 45.9.2. On the other hand, if $Y'' \subset Y'$ is an open such that $X_{Y''} \to Y''$ has property $\mathcal{P}$, then $Y'' \to Y$ factors through W by construction, i.e., $Y'' \subset g^{-1}(W(f))$. This proves (2). Assertion (3) follows from (2) because each morphism $Y_i \to Y$ is flat and locally of finite presentation, syntomic, smooth, or étale by our definition of a τ-covering. ☐

Lemma 45.9.4. *Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S. Assume*

(1) *if $X_i \to Y_i$, $i = 1, 2$ have property $\mathcal{P}$ so does $X_1 \amalg X_2 \to Y_1 \amalg Y_2$,*
(2) *a morphism of algebraic spaces $f : X \to Y$ has property $\mathcal{P}$ if and only if for every affine scheme Z and morphism $Z \to Y$ the base change $Z \times_Y X \to Z$ of f has property $\mathcal{P}$, and*
(3) *for any surjective flat morphism of affine schemes $Z' \to Z$ over S and a morphism $f : X \to Z$ from an algebraic space to Z we have*

$$f' : Z' \times_Z X \to Z' \text{ has } \mathcal{P} \Rightarrow f \text{ has } \mathcal{P}.$$

Then $\mathcal{P}$ is fpqc local on the base.

Proof. If $\mathcal{P}$ has property (2), then it is automatically stable under any base change. Hence the direct implication in Definition 45.9.1.

Let $\{Y_i \to Y\}_{i\in I}$ be an fpqc covering of algebraic spaces over S. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume each base change $f_i : Y_i \times_Y X \to Y_i$ has property $\mathcal{P}$. Our goal is to show that f has $\mathcal{P}$. Let Z be an affine scheme, and let $Z \to Y$ be a morphism. By (2) it suffices to show that the morphism of algebraic spaces $Z \times_Y X \to Z$ has $\mathcal{P}$. Since $\{Y_i \to Y\}_{i\in I}$ is an fpqc covering we know there exists a standard fpqc covering $\{Z_j \to Z\}_{j=1,\ldots,n}$ and morphisms $Z_j \to Y_{i_j}$ over Y for suitable indices $i_j \in I$. Since f_{i_j} has $\mathcal{P}$ we see that

$$Z_j \times_Y X = Z_j \times_{Y_{i_j}} (Y_{i_j} \times_Y X) \longrightarrow Z_j$$

has $\mathcal{P}$ as a base change of f_{i_j} (see first remark of the proof). Set $Z' = \coprod_{j=1,\ldots,n} Z_j$, so that $Z' \to Z$ is a flat and surjective morphism of affine schemes over S. By (1) we conclude that $Z' \times_Y X \to Z'$ has property $\mathcal{P}$. Since this is the base change of the morphism $Z \times_Y X \to Z$ by the morphism $Z' \to Z$ we conclude that $Z \times_Y X \to Z$ has property $\mathcal{P}$ as desired. ☐

45.10. Descending properties of morphisms in the fpqc topology

In this section we find a large number of properties of morphisms of algebraic spaces which are local on the base in the fpqc topology. Please compare with Descent, Section 31.19 for the case of morphisms of schemes.

Lemma 45.10.1. *Let S be a scheme. The property $\mathcal{P}(f) =$``f is quasi-compact'' is fpqc local on the base on algebraic spaces over S.*

Proof. We will use Lemma 45.9.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 42.9.7. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is quasi-compact. We have to show that f is quasi-compact. To see this, using Morphisms of Spaces, Lemma 42.9.7 again, it is enough to show that for every affine scheme Y and morphism $Y \to Z$ the fibre product $Y \times_Z X$ is quasi-compact. Here is a picture:

(45.10.1.1)
$$\begin{tikzcd}
Y \times_Z Z' \times_Z X \arrow[rr] \arrow[dd] \arrow[dr] & & Z' \times_Z X \arrow[dd, "f'"] \arrow[dr] & \\
& Y \times_Z X \arrow[rr] \arrow[dd] & & X \arrow[dd, "f"] \\
Y \times_Z Z' \arrow[rr] \arrow[dr] & & Z' \arrow[dr] & \\
& Y \arrow[rr] & & Z
\end{tikzcd}$$

Note that all squares are cartesian and the bottom square consists of affine schemes. The assumption that f' is quasi-compact combined with the fact that $Y \times_Z Z'$ is affine implies that $Y \times_Z Z' \times_Z X$ is quasi-compact. Since

$$Y \times_Z Z' \times_Z X \longrightarrow Y \times_Z X$$

is surjective as a base change of $Z' \to Z$ we conclude that $Y \times_Z X$ is quasi-compact, see Morphisms of Spaces, Lemma 42.9.5. This finishes the proof. □

Lemma 45.10.2. *Let S be a scheme. The property $\mathcal{P}(f) =$``f is quasi-separated'' is fpqc local on the base on algebraic spaces over S.*

Proof. A base change of a quasi-separated morphism is quasi-separated, see Morphisms of Spaces, Lemma 42.5.4. Hence the direct implication in Definition 45.9.1.

Let $\{Y_i \to Y\}_{i \in I}$ be an fpqc covering of algebraic spaces over S. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume each base change $X_i := Y_i \times_Y X \to Y_i$ is quasi-separated. This means that each of the morphisms

$$\Delta_i : X_i \longrightarrow X_i \times_{Y_i} X_i = Y_i \times_Y (X \times_Y X)$$

is quasi-compact. The base change of a fpqc covering is an fpqc covering, see Topologies on Spaces, Lemma 44.3.2 hence $\{Y_i \times_Y (X \times_Y X) \to X \times_Y X\}$ is an fpqc covering of algebraic spaces. Moreover, each Δ_i is the base change of the morphism $\Delta : X \to X \times_Y X$. Hence it follows from Lemma 45.10.1 that Δ is quasi-compact, i.e., f is quasi-separated. □

Lemma 45.10.3. *Let S be a scheme. The property $\mathcal{P}(f) =$``f is universally closed'' is fpqc local on the base on algebraic spaces over S.*

Proof. We will use Lemma 45.9.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 42.10.5. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is universally closed. We have to show that f is universally closed. To see this, using Morphisms of Spaces, Lemma 42.10.5 again, it is enough to show that for every affine scheme Y and morphism $Y \to Z$ the map

$|Y \times_Z X| \to |Y|$ is closed. Consider the cube (45.10.1.1). The assumption that f' is universally closed implies that $|Y \times_Z Z' \times_Z X| \to |Y \times_Z Z'|$ is closed. As $Y \times_Z Z' \to Y$ is surjective and flat as a base change of $Z' \to Z$ we see the map $|Y \times_Z Z'| \to |Y|$ is submersive, see Morphisms, Lemma 24.24.10. Moreover the map

$$|Y \times_Z Z' \times_Z X| \longrightarrow |Y \times_Z Z'| \times_{|Y|} |Y \times_Z X|$$

is surjective, see Properties of Spaces, Lemma 41.4.3. It follows by elementary topology that $|Y \times_Z X| \to |Y|$ is closed. □

Lemma 45.10.4. *Let S be a scheme. The property $\mathcal{P}(f)$ =``f is universally open'' is fpqc local on the base on algebraic spaces over S.*

Proof. The proof is the same as the proof of Lemma 45.10.3. □

Lemma 45.10.5. *The property $\mathcal{P}(f)$ =``f is surjective'' is fpqc local on the base.*

Proof. Omitted. (Hint: Use Properties of Spaces, Lemma 41.4.3.) □

Lemma 45.10.6. *The property $\mathcal{P}(f)$ =``f is universally injective'' is fpqc local on the base.*

Proof. We will use Lemma 45.9.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 42.10.5. Let $Z' \to Z$ be a flat surjective morphism of affine schemes over S and let $f : X \to Z$ be a morphism from an algebraic space to Z. Assume that the base change $f' : X' \to Z'$ is universally injective. Let K be a field, and let $a, b : Spec(K) \to X$ be two morphisms such that $f \circ a = f \circ b$. As $Z' \to Z$ is surjective there exists a field extension $K \subset K'$ and a morphism $Spec(K') \to Z'$ such that the following solid diagram commutes

$$\begin{tikzcd}
Spec(K') \arrow[dd] \arrow[dr, dashed, "{a',b'}"'] \arrow[drr] & & \\
& X' \arrow[r] \arrow[d] & Z' \arrow[d] \\
Spec(K) \arrow[r, "{a,b}"] & X \arrow[r] & Z
\end{tikzcd}$$

As the square is cartesian we get the two dotted arrows a', b' making the diagram commute. Since $X' \to Z'$ is universally injective we get $a' = b'$. This forces $a = b$ as $\{Spec(K') \to Spec(K)\}$ is an fpqc covering, see Properties of Spaces, Lemma 41.14.1. Hence f is universally injective as desired. □

Lemma 45.10.7. *The property $\mathcal{P}(f)$ =``f is locally of finite type'' is fpqc local on the base.*

Proof. We will use Lemma 45.9.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 42.22.4. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is locally of finite type. We have to show that f is locally of finite type. Let U be a scheme and let $U \to X$ be surjective and étale. By Morphisms of Spaces, Lemma 42.22.4 again, it is enough to show that $U \to Z$ is locally of finite type. Since f' is locally of finite type, and since $Z' \times_Z U$ is a scheme étale over $Z' \times_Z X$ we conclude (by the same lemma again) that $Z' \times_Z U \to Z'$ is locally of finite type. As $\{Z' \to Z\}$ is an fpqc covering we conclude that $U \to Z$ is locally of finite type by Descent, Lemma 31.19.8 as desired. □

Lemma 45.10.8. *The property $\mathcal{P}(f)$ =``f is locally of finite presentation'' is fpqc local on the base.*

Proof. We will use Lemma 45.9.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 42.26.4. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is locally of finite presentation. We have to show that f is locally of finite presentation. Let U be a scheme and let $U \to X$ be surjective and étale. By Morphisms of Spaces, Lemma 42.26.4 again, it is enough to show that $U \to Z$ is locally of finite presentation. Since f' is locally of finite presentation, and since $Z' \times_Z U$ is a scheme étale over $Z' \times_Z X$ we conclude (by the same lemma again) that $Z' \times_Z U \to Z'$ is locally of finite presentation. As $\{Z' \to Z\}$ is an fpqc covering we conclude that $U \to Z$ is locally of finite presentation by Descent, Lemma 31.19.9 as desired. □

Lemma 45.10.9. *The property $\mathcal{P}(f)$ =``f is of finite type'' is fpqc local on the base.*

Proof. Combine Lemmas 45.10.1 and 45.10.7. □

Lemma 45.10.10. *The property $\mathcal{P}(f)$ =``f is of finite presentation'' is fpqc local on the base.*

Proof. Combine Lemmas 45.10.1, 45.10.2 and 45.10.8. □

Lemma 45.10.11. *The property $\mathcal{P}(f)$ =``f is flat'' is fpqc local on the base.*

Proof. We will use Lemma 45.9.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 42.27.4. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is flat. We have to show that f is flat. Let U be a scheme and let $U \to X$ be surjective and étale. By Morphisms of Spaces, Lemma 42.27.4 again, it is enough to show that $U \to Z$ is flat. Since f' is flat, and since $Z' \times_Z U$ is a scheme étale over $Z' \times_Z X$ we conclude (by the same lemma again) that $Z' \times_Z U \to Z'$ is flat. As $\{Z' \to Z\}$ is an fpqc covering we conclude that $U \to Z$ is flat by Descent, Lemma 31.19.13 as desired. □

Lemma 45.10.12. *The property $\mathcal{P}(f)$ =``f is an open immersion'' is fpqc local on the base.*

Proof. We will use Lemma 45.9.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 42.4.1. Consider a cartesian diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ Z' \arrow[r] & Z \end{tikzcd}$$

of algebraic spaces over S where $Z' \to Z$ is a surjective flat morphism of affine schemes, and $X' \to Z'$ is an open immersion. We have to show that $X \to Z$ is an open immersion. Note that $|X'| \subset |Z'|$ corresponds to an open subscheme $U' \subset Z'$ (isomorphic to X') with the property that $\mathrm{pr}_0^{-1}(U') = \mathrm{pr}_1^{-1}(U')$ as open subschemes of $Z' \times_Z Z'$. Hence there exists an open subscheme $U \subset Z$ such that $X' = (Z' \to Z)^{-1}(U)$, see Descent, Lemma 31.9.2. As $X' \to Z'$ is quasi-separated also $X \to Z$ is quasi-separated, by Lemma 45.10.2. Hence X is quasi-separated over S (since Z is affine, hence separated, hence $Z \to S$ is separated, hence $X \to Z \to S$ is quasi-separated by Morphisms of Spaces, Lemma 42.5.8).

Thus by Properties of Spaces, Lemma 41.14.3 we see that X satisfies the sheaf condition for the fpqc topology. Now we have the fpqc covering $\mathcal{U} = \{U' \to U\}$ and the element $U' \to X' \to X \in \check{H}^0(\mathcal{U}, X)$. By the sheaf condition we obtain a morphism $U \to X$ such that

$$\begin{tikzcd} U' \arrow[r] \arrow[d, "\cong"] \arrow[dd, bend right] & U \arrow[d] \arrow[dd, bend left] \\ X' \arrow[r] \arrow[d] & X \arrow[d] \\ Z' \arrow[r] & Z \end{tikzcd}$$

is commutative. On the other hand, we know that for any scheme T pver S and T-valued point $T \to X$ the composition $T \to X \to Z$ is a morphism such that $Z' \times_Z T \to Z'$ factors through U'. Clearly this means that $T \to Z$ factors through U. In other words the map of sheaves $U \to X$ is bijective and we win. □

Lemma 45.10.13. *The property $\mathcal{P}(f)$ =``f is an isomorphism" is fpqc local on the base.*

Proof. Combine Lemmas 45.10.5 and 45.10.12. □

Lemma 45.10.14. *The property $\mathcal{P}(f)$ =``f is affine" is fpqc local on the base.*

Proof. We will use Lemma 45.9.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 42.19.3. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is affine. This implies that $Z' \times_Z X$ is representable, and that $Z' \times_Z X \to Z'$ is quasi-separated. Hence we conclude that $X \to Z$ is quasi-separated by Lemma 45.10.2. In particular X is quasi-separated over S, and satisfies the sheaf condition for the fpqc topology, see Properties of Spaces, Lemma 41.14.3.

Let X' be a scheme representing $Z' \times_Z X$. We obtain a canonical isomorphism

$$\varphi : X' \times_Z Z' \longrightarrow Z' \times_Z X'$$

since both schemes represent the algebraic space $Z' \times_Z Z' \times_Z X$. This is a descent datum for $X'/Z'/Z$, see Descent, Definition 31.30.1 (verification omitted, compare with Descent, Lemma 31.35.1). Since $X' \to Z'$ is affine this descent datum is effective, see Descent, Lemma 31.33.1. Thus there exists a scheme $Y \to Z$ over Z and an isomorphism $\psi : Z' \times_Z Y \to X'$ compatible with descent data. Of course $Y \to Z$ is affine (by construction or by Descent, Lemma 31.19.16). Note that $\mathcal{Y} = \{Z' \times_Z Y \to Y\}$ is a fpqc covering, and interpreting ψ as an element of $X(Z' \times_Z Y)$ we see that $\psi \in \check{H}^0(\mathcal{Y}, X)$. By the sheaf condition for X (see above) we obtain a morphism $Y \to X$. By construction the base change of this to Z' is an isomorphism, hence an isomorphism by Lemma 45.10.13. This proves that X is representable by an affine scheme and we win. □

Lemma 45.10.15. *The property $\mathcal{P}(f)$ =``f is a closed immersion" is fpqc local on the base.*

Proof. We will use Lemma 45.9.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 42.4.1. Consider a cartesian diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ Z' \arrow[r] & Z \end{tikzcd}$$

of algebraic spaces over S where $Z' \to Z$ is a surjective flat morphism of affine schemes, and $X' \to Z'$ is a closed immersion. We have to show that $X \to Z$ is a closed immersion. The morphism $X' \to Z'$ is affine. Hence by Lemma 45.10.14 we see that X is a scheme and $X \to Z$ is affine. It follows from Descent, Lemma 31.19.17 that $X \to Z$ is a closed immersion as desired. $\square$

Lemma 45.10.16. *The property $\mathcal{P}(f) =$ "f is separated" is fpqc local on the base.*

Proof. A base change of a separated morphism is separated, see Morphisms of Spaces, Lemma 42.5.4. Hence the direct implication in Definition 45.9.1.

Let $\{Y_i \to Y\}_{i \in I}$ be an fpqc covering of algebraic spaces over S. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume each base change $X_i := Y_i \times_Y X \to Y_i$ is separated. This means that each of the morphisms

$$\Delta_i : X_i \longrightarrow X_i \times_{Y_i} X_i = Y_i \times_Y (X \times_Y X)$$

is a closed immersion. The base change of a fpqc covering is an fpqc covering, see Topologies on Spaces, Lemma 44.3.2 hence $\{Y_i \times_Y (X \times_Y X) \to X \times_Y X\}$ is an fpqc covering of algebraic spaces. Moreover, each Δ_i is the base change of the morphism $\Delta : X \to X \times_Y X$. Hence it follows from Lemma 45.10.15 that Δ is a closed immersion, i.e., f is separated. $\square$

Lemma 45.10.17. *The property $\mathcal{P}(f) =$ "f is proper" is fpqc local on the base.*

Proof. The lemma follows by combining Lemmas 45.10.3, 45.10.16 and 45.10.9. $\square$

Lemma 45.10.18. *The property $\mathcal{P}(f) =$ "f is quasi-affine" is fpqc local on the base.*

Proof. We will use Lemma 45.9.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 42.20.3. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is quasi-affine. This implies that $Z' \times_Z X$ is representable, and that $Z' \times_Z X \to Z'$ is quasi-separated. Hence we conclude that $X \to Z$ is quasi-separated by Lemma 45.10.2. In particular X is quasi-separated over S, and satisfies the sheaf condition for the fpqc topology, see Properties of Spaces, Lemma 41.14.3.

Let X' be a scheme representing $Z' \times_Z X$. We obtain a canonical isomorphism

$$\varphi : X' \times_Z Z' \longrightarrow Z' \times_Z X'$$

since both schemes represent the algebraic space $Z' \times_Z Z' \times_Z X$. This is a descent datum for $X'/Z'/Z$, see Descent, Definition 31.30.1 (verification omitted, compare with Descent, Lemma 31.35.1). Since $X' \to Z'$ is quasi-affine this descent datum is effective, see Descent, Lemma 31.34.1. Thus there exists a scheme $Y \to Z$ over Z and an isomorphism $\psi : Z' \times_Z Y \to X'$ compatible with descent data. Of course $Y \to Z$ is quasi-affine (by construction or by Descent, Lemma 31.19.18). Note that $\mathcal{Y} = \{Z' \times_Z Y \to Y\}$ is a fpqc covering, and interpreting ψ as an element of $X(Z' \times_Z Y)$ we see that $\psi \in \check{H}^0(\mathcal{Y}, X)$. By the sheaf condition for X (see above) we obtain a morphism $Y \to X$. By construction the base change of this to Z' is an isomorphism, hence an isomorphism by Lemma 45.10.13. This proves that X is representable by a quasi-affine scheme and we win. $\square$

Lemma 45.10.19. *The property $\mathcal{P}(f) =$ "f is a quasi-compact immersion" is fpqc local on the base.*

Proof. We will use Lemma 45.9.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemmas 42.4.1 and 42.9.7. Consider a cartesian diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ Z' \arrow[r] & Z \end{tikzcd}$$

of algebraic spaces over S where $Z' \to Z$ is a surjective flat morphism of affine schemes, and $X' \to Z'$ is a quasi-compact immersion. We have to show that $X \to Z$ is a closed immersion. The morphism $X' \to Z'$ is quasi-affine. Hence by Lemma 45.10.18 we see that X is a scheme and $X \to Z$ is quasi-affine. It follows from Descent, Lemma 31.19.19 that $X \to Z$ is a quasi-compact immersion as desired. $\square$

Lemma 45.10.20. *The property $\mathcal{P}(f) =$``f is integral'' is fpqc local on the base.*

Proof. An integral morphism is the same thing as an affine, universally closed morphism. See Morphisms of Spaces, Lemma 42.37.7. Hence the lemma follows on combining Lemmas 45.10.3 and 45.10.14. $\square$

Lemma 45.10.21. *The property $\mathcal{P}(f) =$``f is finite'' is fpqc local on the base.*

Proof. An finite morphism is the same thing as an integral, morphism which is locally of finite type. See Morphisms of Spaces, Lemma 42.37.6. Hence the lemma follows on combining Lemmas 45.10.7 and 45.10.20. $\square$

Lemma 45.10.22. *The properties $\mathcal{P}(f) =$``$f$ is locally quasi-finite'' and $\mathcal{P}(f) =$``f is quasi-finite'' are fpqc local on the base.*

Proof. We have already seen that ``quasi-compact'' is fpqc local on the base, see Lemma 45.10.1. Hence it is enough to prove the lemma for ``locally quasi-finite''. We will use Lemma 45.9.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 42.25.5. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is locally quasi-finite. We have to show that f is locally quasi-finite. Let U be a scheme and let $U \to X$ be surjective and étale. By Morphisms of Spaces, Lemma 42.25.5 again, it is enough to show that $U \to Z$ is locally quasi-finite. Since f' is locally quasi-finite, and since $Z' \times_Z U$ is a scheme étale over $Z' \times_Z X$ we conclude (by the same lemma again) that $Z' \times_Z U \to Z'$ is locally quasi-finite. As $\{Z' \to Z\}$ is an fpqc covering we conclude that $U \to Z$ is locally quasi-finite by Descent, Lemma 31.19.22 as desired. $\square$

Lemma 45.10.23. *The property $\mathcal{P}(f) =$``f is syntomic'' is fpqc local on the base.*

Proof. We will use Lemma 45.9.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 42.32.4. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is syntomic. We have to show that f is syntomic. Let U be a scheme and let $U \to X$ be surjective and étale. By Morphisms of Spaces, Lemma 42.32.4 again, it is enough to show that $U \to Z$ is syntomic. Since f' is syntomic, and since $Z' \times_Z U$ is a scheme étale over $Z' \times_Z X$ we conclude (by the same lemma again) that $Z' \times_Z U \to Z'$ is syntomic. As $\{Z' \to Z\}$ is an fpqc covering we conclude that $U \to Z$ is syntomic by Descent, Lemma 31.19.24 as desired. $\square$

Lemma 45.10.24. *The property $\mathcal{P}(f)$ =``f is smooth'' is fpqc local on the base.*

Proof. We will use Lemma 45.9.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 42.33.4. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is smooth. We have to show that f is smooth. Let U be a scheme and let $U \to X$ be surjective and étale. By Morphisms of Spaces, Lemma 42.33.4 again, it is enough to show that $U \to Z$ is smooth. Since f' is smooth, and since $Z' \times_Z U$ is a scheme étale over $Z' \times_Z X$ we conclude (by the same lemma again) that $Z' \times_Z U \to Z'$ is smooth. As $\{Z' \to Z\}$ is an fpqc covering we conclude that $U \to Z$ is smooth by Descent, Lemma 31.19.25 as desired. □

Lemma 45.10.25. *The property $\mathcal{P}(f)$ =``f is unramified'' is fpqc local on the base.*

Proof. We will use Lemma 45.9.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 42.34.5. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is unramified. We have to show that f is unramified. Let U be a scheme and let $U \to X$ be surjective and étale. By Morphisms of Spaces, Lemma 42.34.5 again, it is enough to show that $U \to Z$ is unramified. Since f' is unramified, and since $Z' \times_Z U$ is a scheme étale over $Z' \times_Z X$ we conclude (by the same lemma again) that $Z' \times_Z U \to Z'$ is unramified. As $\{Z' \to Z\}$ is an fpqc covering we conclude that $U \to Z$ is unramified by Descent, Lemma 31.19.26 as desired. □

Lemma 45.10.26. *The property $\mathcal{P}(f)$ =``f is étale'' is fpqc local on the base.*

Proof. We will use Lemma 45.9.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 42.35.2. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is étale. We have to show that f is étale. Let U be a scheme and let $U \to X$ be surjective and étale. By Morphisms of Spaces, Lemma 42.35.2 again, it is enough to show that $U \to Z$ is étale. Since f' is étale, and since $Z' \times_Z U$ is a scheme étale over $Z' \times_Z X$ we conclude (by the same lemma again) that $Z' \times_Z U \to Z'$ is étale. As $\{Z' \to Z\}$ is an fpqc covering we conclude that $U \to Z$ is étale by Descent, Lemma 31.19.27 as desired. □

Lemma 45.10.27. *The property $\mathcal{P}(f)$ =``f is finite locally free'' is fpqc local on the base.*

Proof. Being finite locally free is equivalent to being finite, flat and locally of finite presentation (Morphisms of Spaces, Lemma 42.38.6). Hence this follows from Lemmas 45.10.21, 45.10.11, and 45.10.8. □

Lemma 45.10.28. *The property $\mathcal{P}(f)$ =``f is a monomorphism'' is fpqc local on the base.*

Proof. Let $f : X \to Y$ be a morphism of algebraic spaces. Let $\{Y_i \to Y\}$ be an fpqc covering, and assume each of the base changes $f_i : X_i \to Y_i$ of f is a monomorphism. We have to show that f is a monomorphism.

First proof. Note that f is a monomorphism if and only if $\Delta : X \to X \times_Y X$ is an isomorphism. By applying this to f_i we see that each of the morphisms

$$\Delta_i : X_i \longrightarrow X_i \times_{Y_i} X_i = Y_i \times_Y (X \times_Y X)$$

is an isomorphism. The base change of an fpqc covering is an fpqc covering, see Topologies on Spaces, Lemma 44.3.2 hence $\{Y_i \times_Y (X \times_Y X) \to X \times_Y X\}$ is an fpqc covering of algebraic spaces. Moreover, each Δ_i is the base change of the morphism $\Delta : X \to X \times_Y X$. Hence it follows from Lemma 45.10.13 that Δ is an isomorphism, i.e., f is a monomorphism.

Second proof. Let V be a scheme, and let $V \to Y$ be a surjective étale morphism. If we can show that $V \times_Y X \to V$ is a monomorphism, then it follows that $X \to Y$ is a monomorphism. Namely, given any cartesian diagram of sheaves

$$\begin{tikzcd} \mathcal{F} \arrow[r, "a"'] \arrow[d, "b"'] & \mathcal{G} \arrow[d, "c"] \\ \mathcal{H} \arrow[r, "d"] & \mathcal{I} \end{tikzcd} \qquad \mathcal{F} = \mathcal{H} \times_{\mathcal{I}} \mathcal{G}$$

if c is a surjection of sheaves, and a is injective, then also d is injective. This reduces the problem to the case where Y is a scheme. Moreover, in this case we may assume that the algebraic spaces Y_i are schemes also, since we can always refine the covering to place ourselves in this situation, see Topologies on Spaces, Lemma 44.3.4.

Assume $\{Y_i \to Y\}$ is an fpqc covering of schemes. Let $a, b : T \to X$ be two morphisms such that $f \circ a = f \circ b$. We have to show that $a = b$. Since f_i is a monomorphism we see that $a_i = b_i$, where $a_i, b_i : Y_i \times_Y T \to X_i$ are the base changes. In particular the compositions $Y_i \times_Y T \to T \to X$ are equal. Since $\{Y_i \times_Y T \to T\}$ is an fpqc covering we deduce that $a = b$ from Properties of Spaces, Lemma 41.14.1. $\square$

45.11. Descending properties of morphisms in the fppf topology

In this section we find some properties of morphisms of algebraic spaces for which we could not (yet) show they are local on the base in the fpqc topology which, however, are local on the base in the fppf topology.

Lemma 45.11.1. *The property $\mathcal{P}(f)$ = "f is an immersion" is fppf local on the base.*

Proof. Let $f : X \to Y$ be a morphism of algebraic spaces. Let $\{Y_i \to Y\}_{i \in I}$ be an fppf covering of Y. Let $f_i : X_i \to Y_i$ be the base change of f.

If f is an immersion, then each f_i is an immersion by Spaces, Lemma 40.12.3. This proves the direct implication in Definition 45.9.1.

Conversely, assume each f_i is an immersion. By Morphisms of Spaces, Lemma 42.14.7 this implies each f_i is separated. By Morphisms of Spaces, Lemma 42.25.6 this implies each f_i is locally quasi-finite. Hence we see that f is locally quasi-finite and separated, by applying Lemmas 45.10.16 and 45.10.22. By Morphisms of Spaces, Lemma 42.40.1 this implies that f is representable!

By Morphisms of Spaces, Lemma 42.4.1 it suffices to show that for every scheme Z and morphism $Z \to Y$ the base change $Z \times_Y X \to Z$ is an immersion. By Topologies on Spaces, Lemma 44.4.3 we can find an fppf covering $\{Z_i \to Z\}$ by schemes which refines the pull back of the covering $\{Y_i \to Y\}$ to Z. Hence we see that $Z \times_Y X \to Z$ (which is a morphism of schemes according to the result of the preceding paragraph) becomes an immersion after pulling back to the members of an fppf (by schemes) of Z. Hence $Z \times_Y X \to Z$ is an immersion by the result for schemes, see Descent, Lemma 31.20.1. $\square$

Lemma 45.11.2. *The property $\mathcal{P}(f)$ = "f is locally separated" is fppf local on the base.*

Proof. A base change of a locally separated morphism is locally separated, see Morphisms of Spaces, Lemma 42.5.4. Hence the direct implication in Definition 45.9.1.

Let $\{Y_i \to Y\}_{i\in I}$ be an fppf covering of algebraic spaces over S. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume each base change $X_i := Y_i \times_Y X \to Y_i$ is locally separated. This means that each of the morphisms

$$\Delta_i : X_i \longrightarrow X_i \times_{Y_i} X_i = Y_i \times_Y (X \times_Y X)$$

is an immersion. The base change of a fppf covering is an fppf covering, see Topologies on Spaces, Lemma 44.4.2 hence $\{Y_i \times_Y (X \times_Y X) \to X \times_Y X\}$ is an fppf covering of algebraic spaces. Moreover, each Δ_i is the base change of the morphism $\Delta : X \to X \times_Y X$. Hence it follows from Lemma 45.11.1 that Δ is a immersion, i.e., f is locally separated. $\square$

45.12. Properties of morphisms local on the source

In this section we define what it means for a property of morphisms of algebraic spaces to be local on the source. Please compare with Descent, Section 31.22.

Definition 45.12.1. Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S. Let $\tau \in \{fpqc, fppf, syntomic, smooth, étale\}$. We say $\mathcal{P}$ is *τ local on the source*, or *local on the source for the τ-topology* if for any morphism $f : X \to Y$ of algebraic spaces over S, and any τ-covering $\{X_i \to X\}_{i\in I}$ of algebraic spaces we have

$$f \text{ has } \mathcal{P} \Leftrightarrow \text{each } X_i \to Y \text{ has } \mathcal{P}.$$

To be sure, since isomorphisms are always coverings we see (or require) that property $\mathcal{P}$ holds for $X \to Y$ if and only if it holds for any arrow $X' \to Y'$ isomorphic to $X \to Y$. If a property is τ-local on the source then it is preserved by precomposing with morphisms which occur in τ-coverings. Here is a formal statement.

Lemma 45.12.2. *Let S be a scheme. Let $\tau \in \{fpqc, fppf, syntomic, smooth, étale\}$. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S which is τ local on the source. Let $f : X \to Y$ have property $\mathcal{P}$. For any morphism $a : X' \to X$ which is flat, resp. flat and locally of finite presentation, resp. syntomic, resp. smooth, resp. étale, the composition $f \circ a : X' \to Y$ has property $\mathcal{P}$.*

Proof. This is true because we can fit $X' \to X$ into a family of morphisms which forms a τ-covering. $\square$

Lemma 45.12.3. *Let S be a scheme. Let $\tau \in \{fpqc, fppf, syntomic, smooth, étale\}$. Suppose that $\mathcal{P}$ is a property of morphisms of schemes over S which is étale local on the source-and-target. Denote $\mathcal{P}_{spaces}$ the corresponding property of morphisms of algebraic spaces over S, see Morphisms of Spaces, Definition 42.21.2. If $\mathcal{P}$ is local on the source for the τ-topology, then $\mathcal{P}_{spaces}$ is local on the source for the τ-topology.*

Proof. Let $f : X \to Y$ be a morphism of of algebraic spaces over S. Let $\{X_i \to X\}_{i\in I}$ be a τ-covering of algebraic spaces. Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme U and a surjective étale morphism $U \to X \times_Y V$. For each i choose a scheme U_i and a surjective étale morphism $U_i \to X_i \times_X U$.

Note that $\{X_i \times_X U \to U\}_{i\in I}$ is a τ-covering. Note that each $\{U_i \to X_i \times_X U\}$ is an étale covering, hence a τ-covering. Hence $\{U_i \to U\}_{i\in I}$ is a τ-covering of algebraic spaces over S. But since U and each U_i is a scheme we see that $\{U_i \to U\}_{i\in I}$ is a τ-covering of schemes over S.

Now we have

$$\begin{aligned} f \text{ has } \mathcal{P}_{spaces} &\Leftrightarrow U \to V \text{ has } \mathcal{P} \\ &\Leftrightarrow \text{each } U_i \to V \text{ has } \mathcal{P} \\ &\Leftrightarrow \text{each } X_i \to Y \text{ has } \mathcal{P}_{spaces}. \end{aligned}$$

the first and last equivalence by the definition of $\mathcal{P}_{spaces}$ the middle equivalence because we assumed $\mathcal{P}$is local on the source in the τ-topology. ☐

45.13. Properties of morphisms local in the fpqc topology on the source

Here are some properties of morphisms that are fpqc local on the source.

Lemma 45.13.1. *The property $\mathcal{P}(f) =$``f is flat'' is fpqc local on the source.*

Proof. Follows from Lemma 45.12.3 using Morphisms of Spaces, Definition 42.27.1 and Descent, Lemma 31.23.1. ☐

45.14. Properties of morphisms local in the fppf topology on the source

Here are some properties of morphisms that are fppf local on the source.

Lemma 45.14.1. *The property $\mathcal{P}(f) =$``f is locally of finite presentation'' is fppf local on the source.*

Proof. Follows from Lemma 45.12.3 using Morphisms of Spaces, Definition 42.26.1 and Descent, Lemma 31.24.1. ☐

Lemma 45.14.2. *The property $\mathcal{P}(f) =$``f is locally of finite type'' is fppf local on the source.*

Proof. Follows from Lemma 45.12.3 using Morphisms of Spaces, Definition 42.22.1 and Descent, Lemma 31.24.2. ☐

Lemma 45.14.3. *The property $\mathcal{P}(f) =$``f is open'' is fppf local on the source.*

Proof. Follows from Lemma 45.12.3 using Morphisms of Spaces, Definition 42.7.2 and Descent, Lemma 31.24.3. ☐

Lemma 45.14.4. *The property $\mathcal{P}(f) =$``f is universally open'' is fppf local on the source.*

Proof. Follows from Lemma 45.12.3 using Morphisms of Spaces, Definition 42.7.2 and Descent, Lemma 31.24.4. ☐

45.15. Properties of morphisms local in the syntomic toplogy on the source

Here are some properties of morphisms that are syntomic local on the source.

Lemma 45.15.1. *The property $\mathcal{P}(f) =$``f is syntomic'' is syntomic local on the source.*

Proof. Follows from Lemma 45.12.3 using Morphisms of Spaces, Definition 42.32.1 and Descent, Lemma 31.25.1. ☐

45.16. Properties of morphisms local in the smooth topology on the source

Here are some properties of morphisms that are smooth local on the source.

Lemma 45.16.1. *The property $\mathcal{P}(f) =$``f is smooth'' is smooth local on the source.*

Proof. Follows from Lemma 45.12.3 using Morphisms of Spaces, Definition 42.33.1 and Descent, Lemma 31.26.1. ☐

45.17. Properties of morphisms local in the étale topology on the source

Here are some properties of morphisms that are étale local on the source.

Lemma 45.17.1. *The property $\mathcal{P}(f)$ = "f is étale" is étale local on the source.*

Proof. Follows from Lemma 45.12.3 using Morphisms of Spaces, Definition 42.35.1 and Descent, Lemma 31.27.1. □

Lemma 45.17.2. *The property $\mathcal{P}(f)$ = "f is locally quasi-finite" is étale local on the source.*

Proof. Follows from Lemma 45.12.3 using Morphisms of Spaces, Definition 42.25.1 and Descent, Lemma 31.27.2. □

Lemma 45.17.3. *The property $\mathcal{P}(f)$ = "f is unramified" is étale local on the source.*

Proof. Follows from Lemma 45.12.3 using Morphisms of Spaces, Definition 42.34.1 and Descent, Lemma 31.27.3. □

45.18. Properties of morphisms smooth local on source-and-target

Let $\mathcal{P}$ be a property of morphisms of algebraic spaces. There is an intuitive meaning to the phrase "$\mathcal{P}$ is smooth local on the source and target". However, it turns out that this notion is not the same as asking $\mathcal{P}$ to be both smooth local on the source and smooth local on the target. We have discussed a similar phenomenon (for the étale topology and the category of schemes) in great detail in Descent, Section 31.28 (for a quick overview take a look at Descent, Remark 31.28.8). However, there is an important difference between the case of the smooth and the étale topology. To see this difference we encourage the reader to ponder the difference beween Descent, Lemma 31.28.4 and Lemma 45.18.2 as well as the difference between Descent, Lemma 31.28.5 and Lemma 45.18.3. Namely, in the étale setting the choice of the étale "covering" of the target is immaterial, whereas in the smooth setting it is not.

Definition 45.18.1. Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S. We say $\mathcal{P}$ is *smooth local on source-and-target* if

1. (stable under precomposing with smooth maps) if $f : X \to Y$ is smooth and $g : Y \to Z$ has $\mathcal{P}$, then $g \circ f$ has $\mathcal{P}$,
2. (stable under smooth base change) if $f : X \to Y$ has $\mathcal{P}$ and $Y' \to Y$ is smooth, then the base change $f' : Y' \times_Y X \to Y'$ has $\mathcal{P}$, and
3. (locality) given a morphism $f : X \to Y$ the following are equivalent
   (a) f has $\mathcal{P}$,
   (b) for every $x \in |X|$ there exists a commutative diagram
$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "a"] & V \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$
   with smooth vertical arrows and $u \in |U|$ with $a(u) = x$ such that h has $\mathcal{P}$.

The above serves as our definition. In the lemmas below we will show that this is equivalent to $\mathcal{P}$ being local on the target, local on the source, and stable under post-composing by smooth morphisms.

Lemma 45.18.2. *Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S which is smooth local on source-and-target. Then*

(1) *$\mathcal{P}$ is smooth local on the source,*
(2) *$\mathcal{P}$ is smooth local on the target,*
(3) *$\mathcal{P}$ is stable under postcomposing with smooth morphisms: if $f : X \to Y$ has $\mathcal{P}$ and $g : Y \to Z$ is smooth, then $g \circ f$ has $\mathcal{P}$, and*

Proof. We write everything out completely.

Proof of (1). Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\{X_i \to X\}_{i \in I}$ be a smooth covering of X. If each composition $h_i : X_i \to Y$ has $\mathcal{P}$, then for each $|x| \in X$ we can find an $i \in I$ and a point $x_i \in |X_i|$ mapping to x. Then $(X_i, x_i) \to (X, x)$ is a smooth morphism of pairs, and $\mathrm{id}_Y : Y \to Y$ is a smooth morphism, and h_i is as in part (3) of Definition 45.18.1. Thus we see that f has $\mathcal{P}$. Conversely, if f has $\mathcal{P}$ then each $X_i \to Y$ has $\mathcal{P}$ by Definition 45.18.1 part (1).

Proof of (2). Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\{Y_i \to Y\}_{i \in I}$ be a smooth covering of Y. Write $X_i = Y_i \times_Y X$ and $h_i : X_i \to Y_i$ for the base change of f. If each $h_i : X_i \to Y_i$ has $\mathcal{P}$, then for each $x \in |X|$ we pick an $i \in I$ and a point $x_i \in |X_i|$ mapping to x. Then $(X_i, x_i) \to (X, x)$ is a smooth morphism of pairs, $Y_i \to Y$ is smooth, and h_i is as in part (3) of Definition 45.18.1. Thus we see that f has $\mathcal{P}$. Conversely, if f has $\mathcal{P}$, then each $X_i \to Y_i$ has $\mathcal{P}$ by Definition 45.18.1 part (2).

Proof of (3). Assume $f : X \to Y$ has $\mathcal{P}$ and $g : Y \to Z$ is smooth. For every $x \in |X|$ we can think of $(X, x) \to (X, x)$ as a smooth morphism of pairs, $Y \to Z$ is a smooth morphism, and $h = f$ is as in part (3) of Definition 45.18.1. Thus we see that $g \circ f$ has $\mathcal{P}$. □

The following lemma is the analogue of Morphisms, Lemma 24.13.4.

Lemma 45.18.3. *Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S which is smooth local on source-and-target. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(a) *f has property $\mathcal{P}$,*
(b) *for every $x \in |X|$ there exists a smooth morphism of pairs $a : (U, u) \to (X, x)$, a smooth morphism $b : V \to Y$, and a morphism $h : U \to V$ such that $f \circ a = b \circ h$ and h has $\mathcal{P}$,*
(c) *for some commutative diagram*

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "a"'] & V \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with a, b smooth and a surjective the morphism h has $\mathcal{P}$,
(d) *for any commutative diagram*

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "a"'] & V \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with b smooth and $U \to X \times_Y V$ smooth the morphism h has $\mathcal{P}$,
(e) *there exists a smooth covering $\{Y_i \to Y\}_{i \in I}$ such that each base change $Y_i \times_Y X \to Y_i$ has $\mathcal{P}$,*
(f) *there exists a smooth covering $\{X_i \to X\}_{i \in I}$ such that each composition $X_i \to Y$ has $\mathcal{P}$,*

(g) *there exists a smooth covering* $\{Y_i \to Y\}_{i \in I}$ *and for each* $i \in I$ *a smooth covering* $\{X_{ij} \to Y_i \times_Y X\}_{j \in J_i}$ *such that each morphism* $X_{ij} \to Y_i$ *has* $\mathcal{P}$.

Proof. The equivalence of (a) and (b) is part of Definition 45.18.1. The equivalence of (a) and (e) is Lemma 45.18.2 part (2). The equivalence of (a) and (f) is Lemma 45.18.2 part (1). As (a) is now equivalent to (e) and (f) it follows that (a) equivalent to (g).

It is clear that (c) implies (b). If (b) holds, then for any $x \in |X|$ we can choose a smooth morphism of pairs $a_x : (U_x, u_x) \to (X, x)$, a smooth morphism $b_x : V_x \to Y$, and a morphism $h_x : U_x \to V_x$ such that $f \circ a_x = b_x \circ h_x$ and h_x has $\mathcal{P}$. Then $h = \coprod h_x : \coprod U_x \to \coprod V_x$ with $a = \coprod a_x$ and $b = \coprod b_x$ is a diagram as in (c). (Note that h has property $\mathcal{P}$ as $\{V_x \to \coprod V_x\}$ is a smooth covering and $\mathcal{P}$ is smooth local on the target.) Thus (b) is equivalent to (c).

Now we know that (a), (b), (c), (e), (f), and (g) are equivalent. Suppose (a) holds. Let U, V, a, b, h be as in (d). Then $X \times_Y V \to V$ has $\mathcal{P}$ as $\mathcal{P}$ is stable under smooth base change, whence $U \to V$ has $\mathcal{P}$ as $\mathcal{P}$ is stable under precomposing with smooth morphisms. Conversely, if (d) holds, then setting $U = X$ and $V = Y$ we see that f has $\mathcal{P}$. □

Lemma 45.18.4. *Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S. Assume*

(1) *$\mathcal{P}$ is smooth local on the source,*
(2) *$\mathcal{P}$ is smooth local on the target, and*
(3) *$\mathcal{P}$ is stable under postcomposing with smooth morphisms: if $f : X \to Y$ has $\mathcal{P}$ and $Y \subset Z$ is a smooth morphism then $X \to Z$ has $\mathcal{P}$.*

Then $\mathcal{P}$ is smooth local on the source-and-target.

Proof. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which satisfies conditions (1), (2) and (3) of the lemma. By Lemma 45.12.2 we see that $\mathcal{P}$ is stable under precomposing with smooth morphisms. By Lemma 45.9.2 we see that $\mathcal{P}$ is stable under smooth base change. Hence it suffices to prove part (3) of Definition 45.18.1 holds.

More precisely, suppose that $f : X \to Y$ is a morphism of algebraic spaces over S which satisfies Definition 45.18.1 part (3)(b). In other words, for every $x \in X$ there exists a smooth morphism $a_x : U_x \to X$, a point $u_x \in |U_x|$ mapping to x, a smooth morphism $b_x : V_x \to Y$, and a morphism $h_x : U_x \to V_x$ such that $f \circ a_x = b_x \circ h_x$ and h_x has $\mathcal{P}$. The proof of the lemma is complete once we show that f has $\mathcal{P}$. Set $U = \coprod U_x$, $a = \coprod a_x$, $V = \coprod V_x$, $b = \coprod b_x$, and $h = \coprod h_x$. We obtain a commutative diagram

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "a"] & V \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with a, b smooth, a surjective. Note that h has $\mathcal{P}$ as each h_x does and $\mathcal{P}$ is smooth local on the target. Because a is surjective and $\mathcal{P}$ is smooth local on the source, it suffices to prove that $b \circ h$ has $\mathcal{P}$. This follows as we assumed that $\mathcal{P}$ is stable under postcomposing with a smooth morphism and as b is smooth. □

Remark 45.18.5. Using Lemma 45.18.4 and the work done in the earlier sections of this chapter it is easy to make a list of types of morphisms which are smooth local on the source-and-target. In each case we list the lemma which implies the property is smooth local on the source and the lemma which implies the property is smooth local on the target. In each case the third assumption of Lemma 45.18.4 is trivial to check, and we omit it. Here is the list:

(1) flat, see Lemmas 45.13.1 and 45.10.11,
(2) locally of finite presentation, see Lemmas 45.14.1 and 45.10.8,
(3) locally finite type, see Lemmas 45.14.2 and 45.10.7,
(4) universally open, see Lemmas 45.14.4 and 45.10.4,
(5) syntomic, see Lemmas 45.15.1 and 45.10.23,
(6) smooth, see Lemmas 45.16.1 and 45.10.24,
(7) add more here as needed.

45.19. Other chapters

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Commutative Algebra
(8) Brauer Groups
(9) Sites and Sheaves
(10) Homological Algebra
(11) Derived Categories
(12) More on Algebra
(13) Smoothing Ring Maps
(14) Simplicial Methods
(15) Sheaves of Modules
(16) Modules on Sites
(17) Injectives
(18) Cohomology of Sheaves
(19) Cohomology on Sites
(20) Hypercoverings
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(22) Constructions of Schemes
(23) Properties of Schemes
(24) Morphisms of Schemes
(25) Coherent Cohomology
(26) Divisors
(27) Limits of Schemes
(28) Varieties
(29) Chow Homology
(30) Topologies on Schemes
(31) Descent
(32) Adequate Modules
(33) More on Morphisms
(34) More on Flatness
(35) Groupoid Schemes
(36) More on Groupoid Schemes
(37) Étale Morphisms of Schemes
(38) Étale Cohomology
(39) Crystalline Cohomology
(40) Algebraic Spaces
(41) Properties of Algebraic Spaces
(42) Morphisms of Algebraic Spaces
(43) Decent Algebraic Spaces
(44) Topologies on Algebraic Spaces
(45) Descent and Algebraic Spaces
(46) More on Morphisms of Spaces
(47) Quot and Hilbert Spaces
(48) Spaces over Fields
(49) Cohomology of Algebraic Spaces
(50) Stacks
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CHAPTER 46

More on Morphisms of Spaces

46.1. Introduction

In this chapter we continue our study of properties of morphisms of algebraic spaces. A fundamental reference is [**Knu71a**].

46.2. Conventions

The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $Spec(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

46.3. Radicial morphisms

It turns out that a radicial morphism is not the same thing as a universally injective morphism, contrary to what happens with morphisms of schemes. In fact it is a bit stronger.

Definition 46.3.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say f is *radicial* if for any morphism $Spec(K) \to Y$ where K is a field the reduction $(Spec(K) \times_Y X)_{red}$ is either empty or representable by the spectrum of a purely inseparable field extension of K.

Lemma 46.3.2. *A radicial morphism of algebraic spaces is universally injective.*

Proof. Let S be a scheme. Let $f : X \to Y$ be a radicial morphism of algebraic spaces over S. It is clear from the definition that given a morphism $Spec(K) \to Y$ there is at most one lift of this morphism to a morphism into X. Hence we conclude that f is universally injective by Morphisms of Spaces, Lemma 42.18.2. □

Example 46.3.3. It is no longer true that universally injective is equivalent to radicial. For example the morphism

$$X = [Spec(\overline{\mathbf{Q}})/\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})] \longrightarrow S = Spec(\mathbf{Q})$$

of Spaces, Example 40.14.7 is universally injective, but is not radicial in the sense above.

Nonetheless it is often the case that the reverse implication holds.

Lemma 46.3.4. *Let S be a scheme. Let $f : X \to Y$ be a universally injective morphism of algebraic spaces over S.*

(1) *If f is decent then f is radicial.*
(2) *If f is quasi-separated then f is radicial.*
(3) *If f is locally separated then f is radicial.*

Proof. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is stable under base change and composition and holds for closed immersions. Assume $f : X \to Y$ has $\mathcal{P}$ and is universally injective. Then, in the situation of Definition 46.3.1 the morphism $(Spec(K) \times_Y X)_{red} \to Spec(K)$ is universally injective and has $\mathcal{P}$. This reduces the problem of proving

$$\mathcal{P} + \text{universally injective} \Rightarrow \text{radicial}$$

to the problem of proving that any nonempty reduced algebraic space X over field whose structure morphism $X \to Spec(K)$ is universally injective and $\mathcal{P}$ is representable by the spectrum of a field. Namely, then $X \to Spec(K)$ will be a morphism of schemes and we conclude by the equivalence of radicial and universally injective for morphisms of schemes, see Morphisms, Lemma 24.10.2.

Let us prove (1). Assume f is decent and universally injective. By Decent Spaces, Lemmas 43.13.3, 43.13.4, and 43.13.2 (to see that an immersion is decent) we see that the discussion in the first pararagraph applies. Let X be a nonempty decent reduced algebraic space universally injective over a field K. In particular we see that $|X|$ is a singleton. By Decent Spaces, Lemma 43.11.1 we conclude that $X \cong Spec(L)$ for some extension $K \subset L$ as desired.

A quasi-separated morphism is decent, see Decent Spaces, Lemma 43.13.2. Hence (1) implies (2).

Let us prove (3). Recall that the separation axioms are stable under base change and composition and that closed immersions are separated, see Morphisms of Spaces, Lemmas 42.5.4, 42.5.8, and 42.14.7. Thus the discussion in the first paragraph of the proof applies. Let X be a reduced algebraic space universally injective and locally separated over a field K. In particular $|X|$ is a singleton hence X is quasi-compact, see Properties of Spaces, Lemma 41.5.2. We can find a surjective étale morphism $U \to X$ with U affine, see Properties of Spaces, Lemma 41.6.3. Consider the morphism of schemes

$$j : U \times_X U \longrightarrow U \times_{Spec(K)} U$$

As $X \to Spec(K)$ is universally injective j is surjective, and as $X \to Spec(K)$ is locally separated j is an immersion. A surjective immersion is a closed immersion, see Schemes, Lemma 21.10.4. Hence $R = U \times_X U$ is affine as a closed subscheme of an affine scheme. In particular R is quasi-compact. It follows that $X = U/R$ is quasi-separated, and the result follows from (2). □

Remark 46.3.5. Let $X \to Y$ be a morphism of algebraic spaces. For some applications (of radicial morphisms) it is enough to require that for every $Spec(K) \to Y$ where K is a field

(1) the space $|Spec(K) \times_Y X|$ is a singleton,
(2) there exists a monomorphism $Spec(L) \to Spec(K) \times_Y X$, and
(3) $K \subset L$ is purely inseparable.

If needed later we will may call such a morphism *weakly radicial*. For example if $X \to Y$ is a surjective weakly radicial morphism then $X(k) \to Y(k)$ is surjective for every algebraically closed field k. Note that the base change $X_{\overline{\mathbf{Q}}} \to Spec(\overline{\mathbf{Q}})$ of the morphism in Example 46.3.3 is weakly radicial, but not radicial. The analogue of Lemma 46.3.4 is that if $X \to Y$ has property (β) and is universally injective, then it is weakly radicial (proof omitted).

46.4. Morphisms of finite presentation

In this section we generalize Limits, Proposition 27.4.1 to morphisms of algebraic spaces. The motivation for the following definition comes from the proposition just cited.

Definition 46.4.1. Let S be a scheme.

(1) A functor $F : (Sch/S)_{fppf}^{opp} \to Sets$ is said to be *locally of finite presentation* or *limit preserving* if for every affine scheme T over S which is a limit $T = lim\, T_i$ of a directed inverse system of affine schemes T_i over S, we have
$$F(T) = colim\, F(T_i).$$
We sometimes say that F is *locally of finite presentation over S*.

(2) Let $F, G : (Sch/S)_{fppf}^{opp} \to Sets$. A transformation of functors $a : F \to G$ is *locally of finite presentation* if for every scheme T over S and every $y \in G(T)$ the functor
$$F_y : (Sch/T)_{fppf}^{opp} \longrightarrow Sets, \quad T'/T \longmapsto \{x \in F(T') \mid a(x) = y|_{T'}\}$$
is locally of finite presentation over T[1]. We sometimes say that F is *relatively limit preserving* over G.

The functor F_y is in some sense the fiber of $a : F \to G$ over y, except that it is a presheaf on the big fppf site of T. A formula for this functor is:

$$F_y = F|_{(Sch/T)_{fppf}} \times_{G|_{(Sch/T)_{fppf}}} * \tag{46.4.1.1}$$

Here $*$ is the final object in the category of (pre)sheaves on $(Sch/T)_{fppf}$ (see Sites, Example 9.10.2) and the map $* \to G|_{(Sch/T)_{fppf}}$ is given by y. Note that if $j : (Sch/T)_{fppf} \to (Sch/S)_{fppf}$ is the localization functor, then the formula above becomes $F_y = j^{-1}F \times_{j^{-1}G} *$ and $j_!F_y$ is just the fiber product $F \times_{G,y} T$. (See Sites, Section 9.21, for information on localization, and especially Sites, Remark 9.21.9 for information on $j_!$ for presheaves.)

At this point we temporarily have two definitions of what it means for a morphism $X \to Y$ of algebraic spaces over S to be locally of finite presentation. Namely, one by Morphisms of Spaces, Definition 42.26.1 and one using that $X \to Y$ is a transformation of functors so that Definition 46.4.1 applies. We will show in Proposition 46.4.9 that these two definitions agree.

Lemma 46.4.2. *Let S be a scheme. Let $a : F \to G$ be a transformation of functors $(Sch/S)_{fppf}^{opp} \to Sets$. The following are equivalent*

(1) *F is relatively limit preserving over G, and*

(2) *for every every affine scheme T over S which is a limit $T = lim\, T_i$ of a directed inverse system of affine schemes T_i over S the diagram of sets*
$$\begin{tikzcd} colim_i\, F(T_i) \arrow[r] \arrow[d, "a"] & F(T) \arrow[d, "a"] \\ colim_i\, G(T_i) \arrow[r] & G(T) \end{tikzcd}$$
is a fibre product diagram.

[1] The characterization (2) in Lemma 46.4.2 may be easier to parse.

Proof. Assume (1). Consider $T = lim_{i\in I} T_i$ as in (2). Let (y, x_T) be an element of the fibre product $colim_i\, G(T_i) \times_{G(T)} F(T)$. Then y comes from $y_i \in G(T_i)$ for some i. Consider the functor F_{y_i} on $(Sch/T_i)_{fppf}$ as in Definition 46.4.1. We see that $x_T \in F_{y_i}(T)$. Moreover $T = lim_{i'\geq i} T_{i'}$ is a directed system of affine schemes over T_i. Hence (1) implies that x_T the image of a unique element x of $colim_{i'\geq i} F_{y_i}(T_{i'})$. Thus x is the unique element of $colim\, F(T_i)$ which maps to the pair (y, x_T). This proves that (2) holds.

Assume (2). Let T be a scheme and $y_T \in G(T)$. We have to show that F_{y_T} is limit preserving. Let $T' = lim_{i\in I} T'_i$ be an affine scheme over T which is the directed limit of affine scheme T'_i over T. Let $x_{T'} \in F_{y_T}$. Pick $i \in I$ which is possible as I is a directed partially ordered set. Denote $y_i \in F(T'_i)$ the image of $y_{T'}$. Then we see that $(y_i, x_{T'})$ is an element of the fibre product $colim_i\, G(T'_i) \times_{G(T')} F(T')$. Hence by (2) we get a unique element x of $colim_i\, F(T'_i)$ mapping to $(y_i, x_{T'})$. It is clear that x defines an element of $colim_i\, F_y(T'_i)$ mapping to $x_{T'}$ and we win. □

Lemma 46.4.3. *Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)^{opp}_{fppf} \to Sets$. Let $a : F \to G$, $b : G \to H$ be transformations of functors. If a and b are locally of finite presentation, then*

$$b \circ a : F \longrightarrow H$$

is locally of finite presentation.

Proof. Let $T = lim_{i\in I} T_i$ as in characterization (2) of Lemma 46.4.2. Consider the diagram

$$\begin{tikzcd} colim_i\, F(T_i) \arrow[r] \arrow[d, "a"] & F(T) \arrow[d, "a"] \\ colim_i\, G(T_i) \arrow[r] \arrow[d, "b"] & G(T) \arrow[d, "b"] \\ colim_i\, H(T_i) \arrow[r] & H(T) \end{tikzcd}$$

By assumption the two squares are fibre product squares. Hence the outer rectangle is a fibre product diagram too which proves the lemma. □

Lemma 46.4.4. *Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)^{opp}_{fppf} \to Sets$. Let $a : F \to G$, $b : H \to G$ be transformations of functors. Consider the fibre product diagram*

$$\begin{tikzcd} H \times_{b,G,a} F \arrow[r, "b'"'] \arrow[d, "a'"] & F \arrow[d, "a"] \\ H \arrow[r, "b"] & G \end{tikzcd}$$

If a is locally of finite presentation, then the base change a' is locally of finite presentation.

Proof. Omitted. Hint: This is formal. □

Lemma 46.4.5. *Let T be an affine scheme which is written as a limit $T = lim_{i\in I} T_i$ of a directed inverse system of affine schemes.*

(1) *Let $\mathcal{V} = \{V_j \to T\}_{j=1,\ldots,m}$ be a standard fppf covering of T, see Topologies, Definition 30.7.5. Then there exists an index i and a standard fppf covering $\mathcal{V}_i = \{V_{i,j} \to T_i\}_{j=1,\ldots,m}$ whose base change $T \times_{T_i} \mathcal{V}_i$ to T is isomorphic to $\mathcal{V}$.*

(2) *Let $\mathcal{V}_i$, $\mathcal{V}'_i$ be a pair of standard fppf coverings of T_i. If $f : T \times_{T_i} \mathcal{V} \to T \times_{T_i} \mathcal{V}'_i$ is a morphism of coverings of T, then there exists an index $i' \geq i$ and a morphism $f_{i'} : T_{i'} \times_{T_i} \mathcal{V} \to T_{i'} \times_{T_i} \mathcal{V}'_i$ whose base change to T is f.*

(3) *If $f, g : \mathcal{V} \to \mathcal{V}'_i$ are morphisms of standard fppf coverings of T_i whose base changes f_T, g_T to T are equal then there exists an index $i' \geq i$ such that $f_{T_{i'}} = g_{T_{i'}}$.*

In other words, the category of standard fppf coverings of T is the colimit over I of the categories of standard fppf coverings of T_i

Proof. By Limits, Lemma 27.6.1 the category of schemes of finite presentation over T is the colimit over I of the categories of finite presentation over T_i. By Limits, Lemmas 27.6.2 and 27.6.3 the same is true for category of schemes which are affine, flat and of finite presentation over T. To finish the proof of the lemma it suffices to show that if $\{V_{j,i} \to T_i\}_{j=1,\ldots,m}$ is a finite family of flat finitely presented morphisms with $V_{j,i}$ affine, and the base change $\coprod_j T \times_{T_i} V_{j,i} \to T$ is surjective, then for some $i' \geq i$ the morphism $\coprod T_{i'} \times_{T_i} V_{j,i} \to T_{i'}$ is surjective. Denote $W_{i'} \subset T_{i'}$, resp. $W \subset T$ the image. Of course $W = T$ by assumption. Since the morphisms are flat and of finite presentation we see that W_i is a quasi-compact open of T_i, see Morphisms, Lemma 24.24.9. Moreover, $W = T \times_{T_i} W_i$ (formation of image commutes with base change). Hence by Limits, Lemma 27.3.5 we conclude that $W_{i'} = T_{i'}$ for some large enough i' and we win. □

Lemma 46.4.6. *Let S be a scheme contained in Sch_{fppf}. Let $F : (Sch/S)_{fppf}^{opp} \to Sets$ be a functor. If F is locally of finite presentation over S then its sheafification $F^\#$ is locally of finite presentation over S.*

Proof. Assume F is locally of finite presentation. It suffices to show that F^+ is locally of finite presentation, since $F^\# = (F^+)^+$, see Sites, Theorem 9.10.10. Let T be an affine scheme over S, and let $T = lim T_i$ be written as the directed limit of an inverse system of affine S schemes. Recall that $F^+(T)$ is the colimit of $\check{H}^0(\mathcal{V}, F)$ where the limit is over all coverings of T in $(Sch/S)_{fppf}$. Any fppf covering of an affine scheme can be refined by a standard fppf covering, see Topologies, Lemma 30.7.4. Hence we can write

$$F^+(T) = colim_{\mathcal{V} \text{ standard covering } T} \check{H}^0(\mathcal{V}, F).$$

By Lemma 46.4.5 we may rewrite this as

$$colim_{i \in I}\, colim_{\mathcal{V}_i \text{ standard covering } T_i} \check{H}^0(T \times_{T_i} \mathcal{V}_i, F).$$

(The order of the colimits is irrelevant by Categories, Lemma 4.13.9.) Given a standard fppf covering $\mathcal{V}_i = \{V_j \to T_i\}_{j=1,\ldots,m}$ of T_i we see that

$$T \times_{T_i} V_j = lim_{i' \geq i} T_{i'} \times_T V_j$$

by Limits, Lemma 27.2.4, and similarly

$$T \times_{T_i} (V_j \times_{T_i} V_{j'}) = lim_{i' \geq i} T_{i'} \times_T (V_j \times_{T_i} V_{j'}).$$

As the presheaf F is locally of finite presentation this means that

$$\check{H}^0(T \times_{T_i} \mathcal{V}_i, F) = colim_{i' \geq i} \check{H}^0(T_{i'} \times_{T_i} \mathcal{V}_i, F).$$

Hence the colimit expression for $F^+(T)$ above collapses to

$$colim_{i \in I}\, colim_{\mathcal{V} \text{ standard covering } T_i} \check{H}^0(\mathcal{V}, F). = colim_{i \in I} F^+(T_i).$$

In other words $F^+(T) = colim_i F^+(T_i)$ and hence the lemma holds. □

Lemma 46.4.7. *Let S be a scheme. Let $F : (Sch/S)_{fppf}^{opp} \to Sets$ be a functor. Assume that*

(1) *F is a sheaf, and*
(2) *there exists an fppf covering $\{U_j \to S\}_{j\in J}$ such that $F|_{(Sch/U_j)_{fppf}}$ is locally of finite presentation.*

Then F is locally of finite presentation.

Proof. Let T be an affine scheme over S. Let I be a directed partially ordered set, and let T_i be an inverse system of affine schemes over S such that $T = lim\, T_i$. We have to show that the canonical map $colim\, F(T_i) \to F(T)$ is bijective.

Choose some $0 \in I$ and choose a standard fppf covering $\{V_{0,k} \to T_0\}_{k=1,\ldots,m}$ which refines the pullback $\{U_j \times_S T_0 \to T_0\}$ of the given fppf covering of S. For each $i \geq 0$ we set $V_{i,k} = T_i \times_{T_0} V_{0,k}$, and we set $V_k = T \times_{T_0} V_{0,k}$. Note that $V_k = lim_{i\geq 0} V_{i,k}$, see Limits, Lemma 27.2.4.

Suppose that $x, x' \in colim\, F(T_i)$ map to the same element of $F(T)$. Say x, x' are given by elements $x_i, x_i' \in F(T_i)$ for some $i \in I$ (we may choose the same i for both as I is directed). By assumption (2) and the fact that x_i, x_i' map to the same element of $F(T)$ this implies that

$$x_i|_{V_{i',k}} = x_i'|_{V_{i',k}}$$

for some suitably large $i' \in I$. We can choose the same i' for each k as $k \in \{1, \ldots, m\}$ ranges over a finite set. Since $\{V_{i',k} \to T_{i'}\}$ is an fppf covering and F is a sheaf this implies that $x_i|_{T_{i'}} = x_i'|_{T_{i'}}$ as desired. This proves that the map $colim\, F(T_i) \to F(T)$ is injective.

To show surjectivity we argue in a similar fashion. Let $x \in F(T)$. By assumption (2) for each k we can choose a i such that $x|_{V_k}$ comes from an element $x_{i,k} \in F(V_{i,k})$. As before we may choose a single i which works for all k. By the injectivity proved above we see that

$$x_{i,k}|_{V_{i',k}\times_{T_{i'}} V_{i',l}} = x_{i,l}|_{V_{i',k}\times_{T_{i'}} V_{i',l}}$$

for some large enough i'. Hence by the sheaf condition of F the elements $x_{i,k}|_{V_{i',k}}$ glue to an element $x_{i'} \in F(T_{i'})$ as desired. □

Lemma 46.4.8. *Let S be a scheme contained in Sch_{fppf}. Let $F, G : (Sch/S)_{fppf}^{opp} \to Sets$ be functors. If $a : F \to G$ is a transformation which is locally of finite presentation, then the induced transformation of sheaves $F^\# \to G^\#$ is of finite presentation.*

Proof. Suppose that T is a scheme and $y \in G^\#(T)$. We have to show the functor $F^\#_y : (Sch/T)_{fppf}^{opp} \to Sets$ constructed from $F^\# \to G^\#$ and y as in Definition 46.4.1 is locally of finite presentation. By Equation (46.4.1.1) we see that $F^\#_y$ is a sheaf. Choose an fppf covering $\{V_j \to T\}_{j\in J}$ such that $y|_{V_j}$ comes from an element $y_j \in F(V_j)$. Note that the restriction of $F^\#$ to $(Sch/V_j)_{fppf}$ is just $F^\#_{y_j}$. If we can show that $F^\#_{y_j}$ is locally of finite presentation then Lemma 46.4.7 garantees that $F^\#_y$ is locally of finite presentation and we win. This reduces us to the case $y \in G(T)$.

Let $y \in G(T)$. In this case we claim that $F^\#_y = (F_y)^\#$. This follows from Equation (46.4.1.1). Thus this case follows from Lemma 46.4.6. □

Proposition 46.4.9. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *The morphism f is a morphism of algebraic spaces which is locally of finite presentation, see Morphisms of Spaces, Definition 42.26.1.*

(2) *The morphism $f : X \to Y$ is locally of finite presentation as a transformation of functors, see Definition 46.4.1.*

Proof. Assume (1). Let T be a scheme and let $y \in Y(T)$. We have to show that $T \times_X Y$ is locally of finite presentation over T in the sense of Definition 46.4.1. Hence we are reduced to proving that if X is an algebraic space which is locally of finite presentation over S as an algebraic space, then it is locally of finite presentation as a functor $X : (Sch/S)_{fppf}^{opp} \to Sets$. To see this choose a presentation $X = U/R$, see Spaces, Definition 40.9.3. It follows from Morphisms of Spaces, Definition 42.26.1 that both U and R are schemes which are locally of finite presentation over S. Hence by Limits, Proposition 27.4.1 we have

$$U(T) = colim\, U(T_i), \quad R(T) = colim\, R(T_i)$$

whenever $T = lim_i T_i$ in $(Sch/S)_{fppf}$. It follows that the presheaf

$$(Sch/S)_{fppf}^{opp} \longrightarrow Sets, \quad W \longmapsto U(W)/R(W)$$

is locally of finite presentation. Hence by Lemma 46.4.6 its sheafification $X = U/R$ is locally of finite presentation too.

Assume (2). Choose a scheme V and a surjective étale morphism $V \to Y$. Next, choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. By Lemma 46.4.4 the transformation of functors $V \times_Y X \to V$ is locally of finite presentation. By Morphisms of Spaces, Lemma 42.35.8 the morphism of algebraic spaces $U \to V \times_Y X$ is locally of finite presentation, hence locally of finite presentation as a transformation of functors by the first part of the proof. By Lemma 46.4.3 the composition $U \to V \times_Y X \to V$ is locally of finite presentation as a transformation of functors. Hence the morphism of schemes $U \to V$ is locally of finite presentation by Limits, Proposition 27.4.1 (modulo a set theoretic remark, see last paragraph of the proof). This means, by definition, that (1) holds.

Set theoretic remark. Let $U \to V$ be a morphism of $(Sch/S)_{fppf}$. In the statement of Limits, Proposition 27.4.1 we characterize $U \to V$ as being locally of finite presentation if for *all* directed inverse systems $(T_i, f_{ii'})$ of affine schemes over V we have $U(T) = colim\, V(T_i)$, but in the current setting we may only consider affine schemes T_i over V which are (isomorphic to) an object of $(Sch/S)_{fppf}$. So we have to make sure that there are enough affines in $(Sch/S)_{fppf}$ to make the proof work. Inspecting the proof of (2) ⇒ (1) of Limits, Proposition 27.4.1 we see that the question reduces to the case that U and V are affine. Say $U = Spec(A)$ and $V = Spec(B)$. By construction of $(Sch/S)_{fppf}$ the spectrum of any ring of cardinality $\leq |B|$ is isomorphic to an object of $(Sch/S)_{fppf}$. Hence it suffices to observe that in the "only if" part of the proof of Algebra, Lemma 7.118.2 only A-algebras of cardinality $\leq |B|$ are used. □

Remark 46.4.10. Here is an important special case of Proposition 46.4.9. Let S be a scheme. Let X be an algebraic space over S. Then X is locally of finite presentation over S if and only if X, as a functor $(Sch/S)^{opp} \to Sets$, is limit preserving. Compare with Limits, Remark 27.4.2.

46.5. Conormal sheaf of an immersion

Let S be a scheme. Let $i : Z \to X$ be a closed immersion of algebraic spaces over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be the corresponding quasi-coherent sheaf of ideals, see Morphisms of Spaces, Lemma 42.16.1. Consider the short exact sequence

$$0 \to \mathcal{I}^2 \to \mathcal{I} \to \mathcal{I}/\mathcal{I}^2 \to 0$$

of quasi-coherent sheaves on X. Since the sheaf $\mathcal{I}/\mathcal{I}^2$ is annihilated by $\mathcal{I}$ it corresponds to a sheaf on Z by Morphisms of Spaces, Lemma 42.17.1. This quasi-coherent $\mathcal{O}_Z$-module is the *conormal sheaf of Z in X* and is often denoted $\mathcal{I}/\mathcal{I}^2$ by the abuse of notation mentioned in Morphisms of Spaces, Section 42.17.

In case $i : Z \to X$ is a (locally closed) immersion we define the conormal sheaf of i as the conormal sheaf of the closed immersion $i : Z \to X \setminus \partial Z$, see Morphisms of Spaces, Remark 42.4.3. It is often denoted $\mathcal{I}/\mathcal{I}^2$ where $\mathcal{I}$ is the ideal sheaf of the closed immersion $i : Z \to X \setminus \partial Z$.

Definition 46.5.1. Let $i : Z \to X$ be an immersion. The *conormal sheaf $\mathcal{C}_{Z/X}$ of Z in X* or the *conormal sheaf of i* is the quasi-coherent $\mathcal{O}_Z$-module $\mathcal{I}/\mathcal{I}^2$ described above.

In [**DG67**, IV Definition 16.1.2] this sheaf is denoted $\mathcal{N}_{Z/X}$. We will not follow this convention since we would like to reserve the notation $\mathcal{N}_{Z/X}$ for the *normal sheaf of the immersion*. It is defined as

$$\mathcal{N}_{Z/X} = \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{C}_{Z/X}, \mathcal{O}_Z) = \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Z)$$

provided the conormal sheaf is of finite presentation (otherwise the normal sheaf may not even be quasi-coherent). We will come back to the normal sheaf later (insert future reference here).

Lemma 46.5.2. *Let S be a scheme. Let $i : Z \to X$ be an immersion. Let $\varphi : U \to X$ be an étale morphism where U is a scheme. Set $Z_U = U \times_X Z$ which is a locally closed subscheme of U. Then*

$$\mathcal{C}_{Z/X}|_{Z_U} = \mathcal{C}_{Z_U/U}$$

canonically and functorially in U.

Proof. Let $T \subset X$ be a closed subspace such that i defines a closed immersion into $X \setminus T$. Let $\mathcal{I}$ be the quasi-coherent sheaf of ideals on $X \setminus T$ defining Z. Then the lemma just states that $\mathcal{I}|_{U \setminus \varphi^{-1}(T)}$ is the sheaf of ideals of the immersion $Z' \to U \setminus \varphi^{-1}(T)$. This is clear from the construction of $\mathcal{I}$ in Morphisms of Spaces, Lemma 42.16.1. $\square$

Lemma 46.5.3. *Let S be a scheme. Let*

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ Z' \arrow[r, "i'"] & X' \end{tikzcd}$$

be a commutative diagram of algebraic spaces over S. Assume i, i' immersions. There is a canonical map of $\mathcal{O}_Z$-modules

$$f^*\mathcal{C}_{Z'/X'} \longrightarrow \mathcal{C}_{Z/X}$$

Proof. First find open subspaces $U' \subset X'$ and $U \subset X$ such that $g(U) \subset U'$ and such that $i(Z) \subset U$ and $i(Z') \subset U'$ are closed (proof existence omitted). Replacing X by U and X' by U' we may assume that i and i' are closed immersions. Let $\mathcal{I}' \subset \mathcal{O}_{X'}$ and $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaves of ideals associated to i' and i, see Morphisms of Spaces, Lemma 42.16.1. Consider the composition

$$g^{-1}\mathcal{I}' \to g^{-1}\mathcal{O}_{X'} \xrightarrow{g^\sharp} \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I} = i_*\mathcal{O}_Z$$

Since $g(i(Z)) \subset Z'$ we conclude this composition is zero (see statement on factorizations in Morphisms of Spaces, Lemma 42.16.1). Thus we obtain a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{I} \arrow[r] & \mathcal{O}_X \arrow[r] & i_*\mathcal{O}_Z \arrow[r] & 0 \\ 0 \arrow[r] & g^{-1}\mathcal{I}' \arrow[r] \arrow[u] & g^{-1}\mathcal{O}_{X'} \arrow[r] \arrow[u] & g^{-1}i'_*\mathcal{O}_{Z'} \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

The lower row is exact since g^{-1} is an exact functor. By exactness we also see that $(g^{-1}\mathcal{I}')^2 = g^{-1}((\mathcal{I}')^2)$. Hence the diagram induces a map $g^{-1}(\mathcal{I}'/(\mathcal{I}')^2) \to \mathcal{I}/\mathcal{I}^2$. Pulling back (using i^{-1} for example) to Z we obtain $i^{-1}g^{-1}(\mathcal{I}'/(\mathcal{I}')^2) \to \mathcal{C}_{Z/X}$. Since $i^{-1}g^{-1} = f^{-1}(i')^{-1}$ this gives a map $f^{-1}\mathcal{C}_{Z'/X'} \to \mathcal{C}_{Z/X}$, which induces the desired map. $\square$

Lemma 46.5.4. *Let S be a scheme. The conormal sheaf of Definition 46.5.1, and its functoriality of Lemma 46.5.3 satisfy the following properties:*

1. *If $Z \to X$ is an immersion of schemes over S, then the conormal sheaf agrees with the one from Morphisms, Definition 24.31.1.*
2. *If in Lemma 46.5.3 all the spaces are schemes, then the map $f^*\mathcal{C}_{Z'/X'} \to \mathcal{C}_{Z/X}$ is the same as the one constructed in Morphisms, Lemma 24.31.3.*
3. *Given a commutative diagram*

$$\begin{tikzcd} Z \arrow[r, "i"'] \arrow[d, "f"'] & X \arrow[d, "g"] \\ Z' \arrow[r, "i'"] \arrow[d, "f'"'] & X' \arrow[d, "g'"] \\ Z'' \arrow[r, "i''"] & X'' \end{tikzcd}$$

then the map $(f' \circ f)^*\mathcal{C}_{Z''/X''} \to \mathcal{C}_{Z/X}$ is the same as the composition of $f^*\mathcal{C}_{Z'/X'} \to \mathcal{C}_{Z/X}$ with the pullback by f of $(f')^*\mathcal{C}_{Z''/X''} \to \mathcal{C}_{Z'/X'}$

Proof. Omitted. Note that Part (1) is a special case of Lemma 46.5.2. $\square$

Lemma 46.5.5. *Let S be a scheme. Let*

$$\begin{tikzcd} Z \arrow[r, "i"'] \arrow[d, "f"'] & X \arrow[d, "g"] \\ Z' \arrow[r, "i'"] & X' \end{tikzcd}$$

be a fibre product diagram of algebraic spaces over S. Assume i, i' immersions. Then the canonical map $f^*\mathcal{C}_{Z'/X'} \to \mathcal{C}_{Z/X}$ of Lemma 46.5.3 is surjective. If g is flat, then it is an isomorphism.

Proof. Choose a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X \arrow[d] \\ U' \arrow[r] & X' \end{tikzcd}$$

where U, U' are schemes and the horizontal arrows are surjective and étale, see Spaces, Lemma 40.11.4. Then using Lemmas 46.5.2 and 46.5.4 we see that the question reduces to the case of a morphism of schemes. In the schemes case this is Morphisms, Lemma 24.31.4. ☐

Lemma 46.5.6. *Let S be a scheme. Let $Z \to Y \to X$ be immersions of algebraic spaces. Then there is a canonical exact sequence*

$$i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

where the maps come from Lemma 46.5.3 and $i : Z \to Y$ is the first morphism.

Proof. Let U be a scheme and let $U \to X$ be a surjective étale morphism. Via Lemmas 46.5.2 and 46.5.4 the exactness of the sequence translates immediately into the exactness of the corresponding sequence for the immersions of schemes $Z \times_X U \to Y \times_X U \to U$. Hence the lemma follows from Morphisms, Lemma 24.31.5. ☐

46.6. Sheaf of differentials of a morphism

We suggest the reader take a look at the corresponding section in the chapter on commutative algebra (Algebra, Section 7.122), the corresponding section in the chapter on morphism of schemes (Morphisms, Section 24.32) as well as Modules on Sites, Section 16.29. We first show that the notion of sheaf of differentials for a morphism of schemes agrees with the corresponding morphism of small étale (ringed) sites.

To clearly state the following lemma we temporarily go back to denoting $\mathcal{F}^a$ the sheaf of $\mathcal{O}_{X_{étale}}$-modules associated to a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ on the scheme X, see Descent, Definition 31.6.2.

Lemma 46.6.1. *Let $f : X \to Y$ be a morphism of schemes. Let $f_{small} : X_{étale} \to Y_{étale}$ be the associated morphism of small étale sites, see Descent, Remark 31.6.4. Then there is a canonical isomorphism*

$$(\Omega_{X/Y})^a = \Omega_{X_{étale}/Y_{étale}}$$

compatible with universal derivations. Here the first module is the sheaf on $X_{étale}$ associated to the quasi-coherent $\mathcal{O}_X$-module $\Omega_{X/Y}$, see Morphisms, Definition 24.32.4, and the second module is the one from Modules on Sites, Definition 16.29.3.

Proof. Let $h : U \to X$ be an étale morphism. In this case the natural map $h^*\Omega_{X/Y} \to \Omega_{U/Y}$ is an isomorphism, see More on Morphisms, Lemma 33.7.7. This means that there is a natural $\mathcal{O}_{Y_{étale}}$-derivation

$$\mathrm{d}^a : \mathcal{O}_{X_{étale}} \longrightarrow (\Omega_{X/Y})^a$$

since we have just seen that the value of $(\Omega_{X/Y})^a$ on any object U of $X_{étale}$ is canonically identified with $\Gamma(U, \Omega_{U/Y})$. By the universal property of $\mathrm{d}_{X/Y} : \mathcal{O}_{X_{étale}} \to \Omega_{X_{étale}/Y_{étale}}$ there is a unique $\mathcal{O}_{X_{étale}}$-linear map $c : \Omega_{X_{étale}/Y_{étale}} \to (\Omega_{X/Y})^a$ such that $\mathrm{d}^a = c \circ \mathrm{d}_{X/Y}$.

Conversely, suppose that $\mathcal{F}$ is an $\mathcal{O}_{X_{étale}}$-module and $D : \mathcal{O}_{X_{étale}} \to \mathcal{F}$ is a $\mathcal{O}_{Y_{étale}}$-derivation. Then we can simply restrict D to the small Zariski site X_{Zar} of X. Since sheaves on X_{Zar} agree with sheaves on X, see Descent, Remark 31.6.3, we see that $D|_{X_{Zar}} : \mathcal{O}_X \to \mathcal{F}|_{X_{Zar}}$ is just a ``usual" Y-derivation. Hence we obtain a map $\psi : \Omega_{X/Y} \longrightarrow \mathcal{F}|_{X_{Zar}}$ such that $D|_{X_{Zar}} = \psi \circ \mathrm{d}$. In particular, if we apply this with $\mathcal{F} = \Omega_{X_{étale}/Y_{étale}}$ we obtain a map

$$c' : \Omega_{X/Y} \longrightarrow \Omega_{X_{étale}/Y_{étale}}|_{X_{Zar}}$$

Denote $\mathrm{id}_{étale,Zar} : X_{étale} \to X_{Zar}$ the morphism of ringed sites discussed in Descent, Remark 31.6.4 and Lemma 31.6.5. Since the restriction functor $\mathcal{F} \mapsto \mathcal{F}|_{X_{Zar}}$ is equal to

$\mathrm{id}_{\acute{e}tale,Zar,*}$, since $\mathrm{id}_{\acute{e}tale,Zar}^*$ is left adjoint to $\mathrm{id}_{\acute{e}tale,Zar,*}$ and since $(\Omega_{X/Y})^a = \mathrm{id}_{\acute{e}tale,Zar}^*\Omega_{X/Y}$ we see that c' is adjoint to a map

$$c'' : (\Omega_{X/Y})^a \longrightarrow \Omega_{X_{\acute{e}tale}/Y_{\acute{e}tale}}.$$

We claim that c'' and c' are mutually inverse. This claim finishes the proof of the lemma. To see this it is enough to show that $c''(\mathrm{d}(f)) = \mathrm{d}_{X/Y}(f)$ and $c(\mathrm{d}_{X/Y}(f)) = \mathrm{d}(f)$ if f is a local section of $\mathcal{O}_X$ over an open of X. We omit the verification. □

This clears the way for the following definition. For an alternative, see Remark 46.6.5.

Definition 46.6.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The *sheaf of differentials* $\Omega_{X/Y}$ *of* X *over* Y is sheaf of differentials (Modules on Sites, Definition 16.29.6) for the morphism of ringed topoi

$$(f_{small}, f^\sharp) : (X_{\acute{e}tale}, \mathcal{O}_X) \to (Y_{\acute{e}tale}, \mathcal{O}_Y)$$

of Properties of Spaces, Lemma 41.18.3. The *universal Y-derivation* will be denoted $\mathrm{d}_{X/Y} : \mathcal{O}_X \to \Omega_{X/Y}$.

By Lemma 46.6.1 this does not conflict with the already existing notion in case X and Y are representable. From now on, if X and Y are representable, we no longer distinguish between the sheaf of differentials defined above and the one defined in Morphisms, Definition 24.32.4. We want to relate this to the usual modules of differentials for morphisms of schemes. Here is the key lemma.

Lemma 46.6.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Consider any commutative diagram*

$$\begin{tikzcd} U \arrow[r, "\psi"'] \arrow[d, "a"'] & V \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where the vertical arrows are étale morphisms of algebraic spaces. Then

$$\Omega_{X/Y}|_{U_{\acute{e}tale}} = \Omega_{U/V}$$

In particular, if U, V are schemes, then this is equal to the usual sheaf of differentials of the morphism of schemes $U \to V$.

Proof. By Properties of Spaces, Lemma 41.15.10 and Equation (41.15.10.1) we may think of the restriction of a sheaf on $X_{\acute{e}tale}$ to $U_{\acute{e}tale}$ as the pullback by a_{small}. Similarly for b. By Modules on Sites, Lemma 16.29.4 we have

$$\Omega_{X/Y}|_{U_{\acute{e}tale}} = \Omega_{\mathcal{O}_{U_{\acute{e}tale}}/a_{small}^{-1}f_{small}^{-1}\mathcal{O}_{Y_{\acute{e}tale}}}$$

Since $a_{small}^{-1}f_{small}^{-1}\mathcal{O}_{Y_{\acute{e}tale}} = \psi_{small}^{-1}b_{small}^{-1}\mathcal{O}_{Y_{\acute{e}tale}} = \psi_{small}^{-1}\mathcal{O}_{V_{\acute{e}tale}}$ we see that the lemma holds. □

Lemma 46.6.4. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Then $\Omega_{X/Y}$ is a quasi-coherent $\mathcal{O}_X$-module.*

Proof. Choose a diagram as in Lemma 46.6.3 with a and b surjective and U and V schemes. Then we see that $\Omega_{X/Y}|_U = \Omega_{U/V}$ which is quasi-coherent by Morphisms, Definition 24.32.4. Hence we conclude that $\Omega_{X/Y}$ is quasi-coherent by Properties of Spaces, Lemma 41.26.6. □

Remark 46.6.5. Now that we know that $\Omega_{X/Y}$ is quasi-coherent we can attempt to construct it in another manner. For example we can use the result of Properties of Spaces, Section 41.29 to construct the sheaf of differentials by glueing. For example if Y is a scheme and if $U \to X$ is a surjective étale morphism from a scheme towards X, then we see that $\Omega_{U/Y}$ is a quasi-coherent $\mathcal{O}_U$-module, and since $s, t : R \to U$ are étale we get an isomorphism

$$\alpha : s^*\Omega_{U/Y} \to \Omega_{R/Y} \to t^*\Omega_{U/Y}$$

by using Morphisms, Lemma 24.33.16. You check that this satisfies the cocycle condition and you're done. If Y is not a scheme, then you define $\Omega_{U/Y}$ as the cokernel of the map $(U \to Y)^*\Omega_{Y/S} \to \Omega_{U/S}$, and proceed as before. This two step process is a little bit ugly. Another possibility is to glue the sheaves $\Omega_{U/V}$ for any diagram as in Lemma 46.6.3 but this is not very elegant either. Both approaches will work however, and will give a slightly more elementary construction of the sheaf of differentials.

Lemma 46.6.6. *Let S be a scheme. Let*

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

be a commutative diagram of algebraic spaces. The map $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_{X'}$ composed with the map $f_*d_{X'/Y'} : f_*\mathcal{O}_{X'} \to f_*\Omega_{X'/Y'}$ is a Y-derivation. Hence we obtain a canonical map of $\mathcal{O}_X$-modules $\Omega_{X/Y} \to f_*\Omega_{X'/Y'}$, and by adjointness of f_* and f^* a canonical $\mathcal{O}_{X'}$-module homomorphism

$$c_f : f^*\Omega_{X/Y} \longrightarrow \Omega_{X'/Y'}.$$

It is uniquely characterized by the property that $f^*d_{X/Y}(t)$ mapsto $d_{X'/Y'}(f^*t)$ for any local section t of $\mathcal{O}_X$.

Proof. This is a special case of Modules on Sites, Lemma 16.29.7. □

Lemma 46.6.7. *Let S be a scheme. Let*

$$\begin{tikzcd} X'' \arrow[r, "g"] \arrow[d] & X' \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ Y'' \arrow[r] & Y' \arrow[r] & Y \end{tikzcd}$$

be a commutative diagram of algebraic spaces over S. Then we have

$$c_{f\circ g} = c_g \circ g^*c_f$$

as maps $(f \circ g)^*\Omega_{X/Y} \to \Omega_{X''/Y''}$.

Proof. Omitted. Hint: Use the characterization of $c_f, c_g, c_{f\circ g}$ in terms of the effect these maps have on local sections. □

Lemma 46.6.8. *Let S be a scheme. Let $f : X \to Y$, $g : Y \to B$ be morphisms of algebraic spaces over S. Then there is a canonical exact sequence*

$$f^*\Omega_{Y/B} \to \Omega_{X/B} \to \Omega_{X/Y} \to 0$$

where the maps come from applications of Lemma 46.6.6.

Proof. Follows from the schemes version, see Morphisms, Lemma 24.32.11, of this result via étale localization, see Lemma 46.6.3. □

Lemma 46.6.9. *Let S be a scheme. If $X \to Y$ is an immersion of algebraic spaces over S then $\Omega_{X/S}$ is zero.*

Proof. Follows from the schemes version, see Morphisms, Lemma 24.32.16, of this result via étale localization, see Lemma 46.6.3. □

Lemma 46.6.10. *Let S be a scheme. Let B be an algebraic space over S. Let $i : Z \to X$ be an immersion of algebraic spaces over B. There is a canonical exact sequence*

$$\mathcal{C}_{Z/X} \to i^*\Omega_{X/B} \to \Omega_{Z/B} \to 0$$

where the first arrow is induced by $\mathrm{d}_{X/B}$ and the second arrow comes from Lemma 46.6.6.

Proof. This is the algebraic spaces version of Morphisms, Lemma 24.32.17 and will be a consequence of that lemma by étale localization, see Lemmas 46.6.3 and 46.5.2. However, we should make sure we can define the first arrow globally. Hence we explain the meaning of ``induced by $\mathrm{d}_{X/B}$'' here. Namely, we may assume that i is a closed immersion after replacing X by an open subspace. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaf of ideals corresponding to $Z \subset X$. Then $\mathrm{d}_{X/S} : \mathcal{I} \to \Omega_{X/S}$ maps the subsheaf $\mathcal{I}^2 \subset \mathcal{I}$ to $\mathcal{I}\Omega_{X/S}$. Hence it induces a map $\mathcal{I}/\mathcal{I}^2 \to \Omega_{X/S}/\mathcal{I}\Omega_{X/S}$ which is $\mathcal{O}_X/\mathcal{I}$-linear. By Morphisms of Spaces, Lemma 42.17.1 this corresponds to a map $\mathcal{C}_{Z/X} \to i^*\Omega_{X/S}$ as desired. □

Lemma 46.6.11. *Let S be a scheme. Let B be an algebraic space over S. Let $i : Z \to X$ be an immersion of schemes over B, and assume i (étale locally) has a left inverse. Then the canonical sequence*

$$0 \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/B} \to \Omega_{Z/B} \to 0$$

of Lemma 46.6.10 is (étale locally) split exact.

Proof. Clarification: we claim that if $g : X \to Z$ is a left inverse of i, then i^*c_g is a right inverse of the map $i^*\Omega_{X/B} \to \Omega_{Z/B}$. Having said this, the result follows from the corresponding result for morphisms of schemes by étale localization, see Lemmas 46.6.3 and 46.5.2. □

Lemma 46.6.12. *Let S be a scheme. Let $X \to Y$ be a morphism of algebraic spaces over S. Let $g : Y' \to Y$ be a morphism of algebraic spaces over S. Let $X' = X_{Y'}$ be the base change of X. Denote $g' : X' \to X$ the projection. Then the map*

$$(g')^*\Omega_{X/Y} \to \Omega_{X'/Y'}$$

of Lemma 46.6.6 is an isomorphism.

Proof. Follows from the schemes version, see Morphisms, Lemma 24.32.12 and étale localization, see Lemma 46.6.3. □

Lemma 46.6.13. *Let S be a scheme. Let $f : X \to B$ and $g : Y \to B$ be morphisms of algebraic spaces over S with the same target. Let $p : X \times_B Y \to X$ and $q : X \times_B Y \to Y$ be the projection morphisms. The maps from Lemma 46.6.6*

$$p^*\Omega_{X/S} \oplus q^*\Omega_{Y/S} \longrightarrow \Omega_{X \times_S Y/S}$$

give an isomorphism.

Proof. Follows from the schemes version, see Morphisms, Lemma 24.32.13 and étale localization, see Lemma 46.6.3. □

Lemma 46.6.14. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is locally of finite type, then $\Omega_{X/Y}$ is a finite type $\mathcal{O}_X$-module.*

Proof. Follows from the schemes version, see Morphisms, Lemma 24.32.14 and étale localization, see Lemma 46.6.3. ☐

Lemma 46.6.15. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is locally of finite type, then $\Omega_{X/Y}$ is an $\mathcal{O}_X$-module of finite presentation.*

Proof. Follows from the schemes version, see Morphisms, Lemma 24.32.15 and étale localization, see Lemma 46.6.3. ☐

46.7. Topological invariance of an étale site

We show that the site $X_{spaces,étale}$ is a ``topological invariant''. We will prove later that actually also $X_{étale}$, which consists of the representable objects in $X_{spaces,étale}$, is a topological invariant too (insert future reference here).

Theorem 46.7.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is integral, universally injective and surjective. The functor*

$$V \longmapsto V_X = X \times_Y V$$

defines an equivalence of categories $Y_{spaces,étale} \to X_{spaces,étale}$.

Proof. The morphism f is representable and a universal homeomorphism, see Morphisms of Spaces, Section 42.41.

We first prove that the functor is faithful. Suppose that V', V are objects of $Y_{spaces,étale}$ and that $a, b : V' \to V$ are distinct morphisms over Y. Since V', V are étale over Y the equalizer

$$E = V' \times_{(a,b), V\times_Y V, \Delta_{V/Y}} V$$

of a, b is étale over Y also. Hence $E \to V'$ is an étale monomorphism (i.e., an open immersion) which is an isomorphism if and only if it is surjective. Since $X \to Y$ is a universal homeomorphism we see that this is the case if and only if $E_X = V'_X$, i.e., if and only if $a_X = b_X$.

Next, we prove that the functor is fully faithful. Suppose that V', V are objects of $Y_{spaces,étale}$ and that $c : V'_X \to V_X$ is a morphism over X. We want to construct a morphism $a : V' \to V$ over Y such that $a_X = c$. Let $a' : V'' \to V'$ be a surjective étale morphism such that V'' is a separated algebraic space. If we can construct a morphism $a'' : V'' \to V$ such that $a''_X = c \circ a'_X$, then the two compositions

$$V'' \times_{V'} V'' \xrightarrow{\text{pr}_i} V'' \xrightarrow{a''} V$$

will be equal by the faithfulness of the functor proved in the first paragraph. Hence a'' will factor through a unique morphism $a : V' \to V$ as V' is (as a sheaf) the quotient of V'' by the equivalence relation $V'' \times_{V'} V''$. Hence we may assume that V' is separated. In this case the graph

$$\Gamma_c \subset (V' \times_Y V)_X$$

is open and closed (details omitted). Since $X \to Y$ is a universal homeomorphism, there exists an open and closed subspace $\Gamma \subset V' \times_Y V$ such that $\Gamma_X = \Gamma_c$. The projection $\Gamma \to V'$ is an étale morphism whose base change to X is an isomorphism. Hence $\Gamma \to V'$ is étale, universally injective, and surjective, so an isomorphism by Morphisms of Spaces, Lemma 42.40.2. Thus Γ is the graph of a morphism $a : V' \to V$ as desired.

Finally, we prove that the functor is essentially surjective. Suppose that U is an object of $X_{spaces,étale}$. We have to find an object V of $Y_{spaces,étale}$ such that $V_X \cong U$. Let $U' \to U$ be a surjective étale morphism such that $U' \cong V'_X$ and $U' \times_U U' \cong V''_X$ for some objects V'', V' of $Y_{spaces,étale}$. Then by fully faithfulness of the functor we obtain morphisms $s, t : V'' \to V'$ with $t_X = \text{pr}_0$ and $s_X = \text{pr}_1$ as morphisms $U' \times_U U' \to U'$. Using that $(\text{pr}_0, \text{pr}_1) : U' \times_U U' \to U' \times_S U'$ is an étale equivalence relation, and that $U' \to V'$ and $U' \times_U U' \to V''$ are universally injective and surjective we deduce that $(t, s) : V'' \to V' \times_S V'$ is an étale equivalence relation. Then the quotient $V = V'/V''$ (see Spaces, Theorem 40.10.5) is an algebraic space V over Y. There is a morphism $V' \to V$ such that $V'' = V' \times_V V'$. Thus we obtain a morphism $V \to Y$ (see Descent on Spaces, Lemma 45.6.2). On base change to X we see that we have a morphism $U' \to V_X$ and a compatibe isomorphism $U' \times_{V_X} U' = U' \times_U U'$, which implies that $V_X \cong U$ (by the lemma just cited once more).

Pick a scheme W and a surjective étale morphism $W \to Y$. Pick a scheme U' and a surjective étale morphism $U' \to U \times_X W_X$. Note that U' and $U' \times_U U'$ are schemes étale over X whose structure morphism to X factors through the scheme W_X. Hence by Étale Cohomology, Theorem 38.45.1 there exist schemes V', V'' étale over W whose base change to W_X is isomorphic to respectively U' and $U' \times_U U'$. This finishes the proof. □

Remark 46.7.2. A universal homeomorphism of algebraic spaces need not be representable, see Morphisms of Spaces, Example 42.41.3. The argument in the proof of Theorem 46.7.1 above cannot be used in this case. In fact we do not know whether given a universal homeomorphism of algebraic spaces $f : X \to Y$ the categories $X_{spaces,étale}$ and $Y_{spaces,étale}$ are equivalent. If you do, please email stacks.project@gmail.com.

46.8. Thickenings

The following terminology may not be completely standard, but it is convenient.

Definition 46.8.1. Thickenings. Let S be a scheme.

(1) We say an algebraic space X' is a *thickening* of an algebraic space X if X is a closed subspace of X' and the associated topological spaces are equal.
(2) We say X' is a *first order thickening* of X if X is a closed subspace of X' and the quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_{X'}$ defining X has square zero.
(3) Given two thickenings $X \subset X'$ and $Y \subset Y'$ a *morphism of thickenings* is a morphism $f' : X' \to Y'$ such that $f(X) \subset Y$, i.e., such that $f'|_X$ factors through the closed subspace Y. In this situation we set $f = f'|_X : X \to Y$ and we say that $(f, f') : (X \subset X') \to (Y \subset Y')$ is a morphism of thickenings.
(4) Let B be an algebraic space. We similarly define *thickenings over* B, and *morphisms of thickenings over* B. This means that the spaces X, X', Y, Y' above are algebraic spaces endowed with a structure morphism to B, and that the morphisms $X \to X'$, $Y \to Y'$ and $f' : X' \to Y'$ are morphisms over B.

The fundamental equivalence. Note that if $X \subset X'$ is a thickening, then $X \to X'$ is integral and universally bijective. This implies that

$$X_{spaces,étale} = X'_{spaces,étale} \tag{46.8.1.1}$$

via the pullback functor, see Theorem 46.7.1. Hence we may think of $\mathcal{O}_{X'}$ as a sheaf on $X_{spaces,étale}$. Thus a canonical equivalence of locally ringed topoi

$$(Sh(X'_{spaces,étale}), \mathcal{O}_{X'}) \cong (Sh(X_{spaces,étale}), \mathcal{O}_{X'}) \tag{46.8.1.2}$$

Below we will frequently combine this with the fully faithfulness result of Properties of Spaces, Theorem 41.25.4. For example the closed immersion $i_X : X \to X'$ corresponds to the surjective map $i_X^\sharp : \mathcal{O}_{X'} \to \mathcal{O}_X$.

Let S be a scheme, and let B be an algebraic space over S. Let $(f, f') : (X \subset X') \to (Y \subset Y')$ be a morphism of thickenings over B. Note that the diagram of continuous functors

$$
\begin{tikzcd}
X_{spaces,étale} & Y_{spaces,étale} \arrow[l] \\
X'_{spaces,étale} \arrow[u] & Y'_{spaces,étale} \arrow[l] \arrow[u]
\end{tikzcd}
$$

is commutative and the vertical arrows are equivalences. Hence $f_{spaces,étale}$, f_{small}, $f'_{spaces,étale}$, and f'_{small} all define the same morphism of topoi. Thus we may think of

$$(f')^\sharp : f_{spaces,étale}^{-1}\mathcal{O}_{Y'} \longrightarrow \mathcal{O}_{X'}$$

as a map of sheaves of $\mathcal{O}_B$-algebras fitting into the commutative diagram

$$
\begin{tikzcd}
f_{spaces,étale}^{-1}\mathcal{O}_Y \arrow[r, "f^\sharp"'] & \mathcal{O}_X \\
f_{spaces,étale}^{-1}\mathcal{O}_{Y'} \arrow[u, "i_Y^\sharp"] \arrow[r, "(f')^\sharp"] & \mathcal{O}_{X'} \arrow[u, "i_X^\sharp"']
\end{tikzcd}
$$

Here $i_X : X \to X'$ and $i_Y : Y \to Y'$ are the names of the given closed immersions.

Lemma 46.8.2. *Let S be a scheme. Let B be an algebraic space over S. Let $X \subset X'$ and $Y \subset Y'$ be thickenings of algebraic spaces over B. Let $f : X \to Y$ be a morphism of algebraic spaces over B. Given any map of $\mathcal{O}_B$-algebras*

$$\alpha : f_{spaces,étale}^{-1}\mathcal{O}_{Y'} \to \mathcal{O}_{X'}$$

such that

$$
\begin{tikzcd}
f_{spaces,étale}^{-1}\mathcal{O}_Y \arrow[r, "f^\sharp"'] & \mathcal{O}_X \\
f_{spaces,étale}^{-1}\mathcal{O}_{Y'} \arrow[u, "i_Y^\sharp"] \arrow[r, "\alpha"] & \mathcal{O}_{X'} \arrow[u, "i_X^\sharp"']
\end{tikzcd}
$$

commutes, there exists a unique morphism of (f, f') of thickenings over B such that $\alpha = (f')^\sharp$.

Proof. To find f', by Properties of Spaces, Theorem 41.25.4, all we have to do is show that the morphism of ringed topoi

$$(f_{spaces,étale}, \alpha) : (Sh(X_{spaces,étale}), \mathcal{O}_{X'}) \longrightarrow (Sh(Y_{spaces,étale}), \mathcal{O}_{Y'})$$

is a morphism of locally ringed topoi. This follows directly from the definition of morphisms of locally ringed topoi (Modules on Sites, Definition 16.34.8), the fact that $(f, f^\sharp)$ is a morphism of locally ringed topoi (Properties of Spaces, Lemma 41.25.1), that α fits into the given commutative diagram, and the fact that the kernels of $i_X^\sharp$ and $i_Y^\sharp$ are locally nilpotent. Finally, the fact that $f' \circ i_X = i_Y \circ f$ follows from the commutativity of the diagram and another application of Properties of Spaces, Theorem 41.25.4. We omit the verification that f' is a morphism over B. $\square$

Lemma 46.8.3. *Let S be a scheme. Let $X \subset X'$ be a thickening of algebraic spaces over S. For any open subspace $U \subset X$ there exists a unique open subspace $U' \subset X'$ such that $U = X \times_{X'} U'$.*

Proof. Let $U' \to X'$ be the object of $X'_{spaces,étale}$ corresponding to the object $U \to X$ of $X_{spaces,étale}$ via (46.8.1.1). The morphism $U' \to X'$ is étale and universally injective, hence an open immersion, see Morphisms of Spaces, Lemma 42.40.2. □

Finite order thickenings. Let $i_X : X \to X'$ be a thickening of algebraic spaces. Any local section of the kernel $\mathcal{I} = \text{Ker}(i_X^\sharp) \subset \mathcal{O}_{X'}$ is locally nilpotent. Let us say that $X \subset X'$ is a *finite order thickening* if the ideal sheaf $\mathcal{I}$ is ``globally" nilpotent, i.e., if there exists an $n \geq 0$ such that $\mathcal{I}^{n+1} = 0$. Technically the class of finite order thickenings $X \subset X'$ is much easier to handle than the general case. Namely, in this case we have a filtration

$$0 \subset \mathcal{I}^n \subset \mathcal{I}^{n-1} \subset \ldots \subset \mathcal{I} \subset \mathcal{O}_{X'}$$

and we see that X' is filtered by closed subspaces

$$X = X_0 \subset X_1 \subset \ldots \subset X_{n-1} \subset X_{n+1} = X'$$

such that each pair $X_i \subset X_{i+1}$ is a first order thickening over B. Using simple induction arguments many results proved for first order thickenings can be rephrased as results on finite order thickenings.

Lemma 46.8.4. *Let S be a scheme. Let $X \subset X'$ be a finite order thickening of algebraic spaces over S. Let U be an affine object of $X_{spaces,étale}$. Then*

$$\Gamma(U, \mathcal{O}_{X'}) \to \Gamma(U, \mathcal{O}_X)$$

is surjective where we think of $\mathcal{O}_{X'}$ as a sheaf on $X_{spaces,étale}$ via (46.8.1.2).

Proof. We may assume that $X \subset X'$ is a first order thickening by the principle explained above. Denote $\mathcal{I}$ the kernel of the surjection $\mathcal{O}_{X'} \to \mathcal{O}_X$. As $\mathcal{I}$ is a quasi-coherent $\mathcal{O}_{X'}$-module and since $\mathcal{I}^2 = 0$ by the definition of a first order thickening we may apply Morphisms of Spaces, Lemma 42.17.1 to see that $\mathcal{I}$ is a quasi-coherent $\mathcal{O}_X$-module. Hence the lemma follows from the long exact cohomology sequence associated to the short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$$

and the fact that $H^1_{étale}(U, \mathcal{I}) = 0$ as $\mathcal{I}$ is quasi-coherent, see Descent, Proposition 31.6.10 and Coherent, Lemma 25.2.2. □

Lemma 46.8.5. *Let S be a scheme. Let $X \subset X'$ be a finite order thickening of algebraic spaces over S. If X is (representable by) a scheme, then so is X'.*

Proof. It suffices to prove this when X' is a first order thickening of X. By Properties of Spaces, Lemma 41.10.1 there is a largest open subspace of X' which is a scheme. Thus we have to show that every point x of $|X'| = |X|$ is contained in an open subspace of X' which is a scheme. Using Lemma 46.8.3 we may replace $X \subset X'$ by $U \subset U'$ with $x \in U$ and U an affine scheme. Hence we may assume that X is affine. Thus we reduce to the case discussed in the next paragraph.

Assume $X \subset X'$ is a first order thickening where X is an affine scheme. Set $A = \Gamma(X, \mathcal{O}_X)$ and $A' = \Gamma(X', \mathcal{O}_{X'})$. By Lemma 46.8.4 the map $A \to A'$ is surjective. The kernel I is an ideal of square zero. By Properties of Spaces, Lemma 41.30.1 we obtain a canonical morphism $f : X' \to Spec(A')$ which fits into the following commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, equal] & X' \arrow[d, "f"] \\ Spec(A) \arrow[r] & Spec(A') \end{tikzcd}$$

Because the horizontal arrows are thickenings it is clear that f is universally injective and surjective. Hence it suffices to show that f is étale, since then Morphisms of Spaces, Lemma 42.40.2 will imply that f is an isomorphism.

To prove that f is étale choose an affine scheme U' and an étale morphism $U' \to X'$. It suffices to show that $U' \to X' \to Spec(A')$ is étale, see Properties of Spaces, Definition 41.13.2. Write $U' = Spec(B')$. Set $U = X \times_{X'} U'$. Since U is a closed subspace of U', it is a closed subscheme, hence $U = Spec(B)$ with $B' \to B$ surjective. Denote $J = \mathrm{Ker}(B' \to B)$ and note that $J = \Gamma(U, \mathcal{I})$ where $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_{X'} \to \mathcal{O}_X)$ on $X_{spaces,étale}$ as in the proof of Lemma 46.8.4. The morphism $U' \to X' \to Spec(A')$ induces a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r] & B' \arrow[r] & B \arrow[r] & 0 \\ 0 \arrow[r] & I \arrow[r] \arrow[u] & A' \arrow[r] \arrow[u] & A \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

Now, since $\mathcal{I}$ is a quasi-coherent $\mathcal{O}_X$-module we have $\mathcal{I} = (\tilde{I})^a$, see Descent, Definition 31.6.2 for notation and Descent, Proposition 31.6.11 for why this is true. Hence we see that $J = I \otimes_A B$. Finally, note that $A \to B$ is étale as $U \to X$ is étale as the base change of the étale morphism $U' \to X'$. We conclude that $A' \to B'$ is étale by Algebra, Lemma 7.132.11. □

The following lemma will be superseded by the more general (insert future reference here).

Lemma 46.8.6. *Let S be a scheme. Let $X \subset X'$ be a first order thickening of algebraic spaces over S. The functor*

$$V' \longmapsto V = X \times_{X'} V'$$

defines an equivalence of categories $X'_{étale} \to X_{étale}$.

Proof. The functor $V' \mapsto V$ defines an equivalence of categories $X'_{spaces,étale} \to X_{spaces,étale}$, see Theorem 46.7.1. Thus it suffices to show that V is a scheme if and only if V' is a scheme. This is the content of Lemma 46.8.5. □

First order thickening are described as follows.

Lemma 46.8.7. *Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Consider a short exact sequence*

$$0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{O}_X \to 0$$

of sheaves on $X_{étale}$ where $\mathcal{A}$ is a sheaf of $f^{-1}\mathcal{O}_B$-algebras, $\mathcal{A} \to \mathcal{O}_X$ is a surjection of sheaves of $f^{-1}\mathcal{O}_B$-algebras, and $\mathcal{I}$ is its kernel. If

(1) *$\mathcal{I}$ is an ideal of square zero in $\mathcal{A}$, and*
(2) *$\mathcal{I}$ is quasi-coherent as an $\mathcal{O}_X$-module*

then there exists a first order thickening $X \subset X'$ over B and an isomorphism $\mathcal{O}_{X'} \to \mathcal{A}$ of $f^{-1}\mathcal{O}_B$-algebras compatible with the surjections to $\mathcal{O}_X$.

Proof. In this proof we redo some of the arguments used in the proofs of Lemmas 46.8.4 and 46.8.5. We first handle the case $B = S = Spec(\mathbf{Z})$. Let U be an affine scheme, and let $U \to X$ be étale. Then

$$0 \to \mathcal{I}(U) \to \mathcal{A}(U) \to \mathcal{O}_X(U) \to 0$$

is exact as $H^1(U_{étale}, \mathcal{I}) = 0$ as $\mathcal{I}$ is quasi-coherent, see Descent, Proposition 31.6.10 and Coherent, Lemma 25.2.2. If $V \to U$ is a morphism of affine objects of $X_{spaces,étale}$ then

$$\mathcal{I}(V) = \mathcal{I}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V)$$

since $\mathcal{I}$ is a quasi-coherent $\mathcal{O}_X$-module, see Descent, Proposition 31.6.11. Hence $\mathcal{A}(U) \to \mathcal{A}(V)$ is an étale ring map, see Algebra, Lemma 7.132.11. Hence we see that

$$U \longmapsto U' = Spec(\mathcal{A}(U))$$

is a functor from $X_{affine,étale}$ to the category of affine schemes and étale morphisms. In fact, we claim that this functor can be extended to a functor $U \mapsto U'$ on all of $X_{étale}$. To see this, if U is an object of $X_{étale}$, note that

$$0 \to \mathcal{I}|_{U_{Zar}} \to \mathcal{A}|_{U_{Zar}} \to \mathcal{O}_X|_{U_{Zar}} \to 0$$

and $\mathcal{I}|_{U_{Zar}}$ is a quasi-coherent sheaf on U, see Descent, Proposition 31.6.14. Hence by More on Morphisms, Lemma 33.2.2 we obtain a first order thickening $U \subset U'$ of schemes such that $\mathcal{O}_{U'}$ is isomorphic to $\mathcal{A}|_{U_{Zar}}$. It is clear that this construction is compatible with the construction for affines above.

Choose a presentation $X = U/R$, see Spaces, Definition 40.9.3 so that $s, t : R \to U$ define an étale equivalence relation. Applying the functor above we obtain an étale equivalence relation $s', t' : R' \to U'$ in schemes. Consider the algebraic space $X' = U'/R'$ (see Spaces, Theorem 40.10.5). The morphism $X = U/R \to U'/R' = X'$ is a first order thickening. Consider $\mathcal{O}_{X'}$ viewed as a sheaf on $X_{étale}$. By construction we have an isomorphism

$$\gamma : \mathcal{O}_{X'}|_{U_{étale}} \longrightarrow \mathcal{A}|_{U_{étale}}$$

such that $s^{-1}\gamma$ agrees with $t^{-1}\gamma$ on $R_{étale}$. Hence by Properties of Spaces, Lemma 41.15.13 this implies that γ comes from a unique isomorphism $\mathcal{O}_{X'} \to \mathcal{A}$ as desired.

To handle the case of a general base algebraic space B, we first construct X' as an algebraic space over $\mathbf{Z}$ as above. Then we use the isomorphism $\mathcal{O}_{X'} \to \mathcal{A}$ to define $f^{-1}\mathcal{O}_B \to \mathcal{O}_{X'}$. According to Lemma 46.8.2 this defines a morphism $X' \to B$ compatible with the given morphism $X \to B$ and we are done. □

46.9. First order infinitesimal neighbourhood

A natural construction of first order thickenings is the following. Suppose that $i : Z \to X$ be an immersion of algebraic spaces. Choose an open subspace $U \subset X$ such that i identifies Z with a closed subspace $Z \subset U$ (see Morphisms of Spaces, Remark 42.4.3). Let $\mathcal{I} \subset \mathcal{O}_U$ be the quasi-coherent sheaf of ideals defining Z in U, see Morphisms of Spaces, Lemma 42.16.1. Then we can consider the closed subspace $Z' \subset U$ defined by the quasi-coherent sheaf of ideals $\mathcal{I}^2$.

Definition 46.9.1. Let $i : Z \to X$ be an immersion of algebraic spaces. The *first order infinitesimal neighbourhood* of Z in X is the first order thickening $Z \subset Z'$ over X described above.

This thickening has the following universal property (which will assuage any fears that the construction above depends on the choice of the open U).

Lemma 46.9.2. *Let $i : Z \to X$ be an immersion of algebraic spaces. The first order infinitesimal neighbourhood Z' of Z in X has the following universal property: Given any commutative diagram*

$$\begin{tikzcd} Z \arrow[d, "i"'] & T \arrow[l, "a"'] \arrow[d] \\ X & T' \arrow[l, "b"] \end{tikzcd}$$

where $T \subset T'$ is a first order thickening over X, there exists a unique morphism $(a', a) : (T \subset T') \to (Z \subset Z')$ of thickenings over X.

Proof. Let $U \subset X$ be the open subspace used in the construction of Z', i.e., an open such that Z is identified with a closed subspace of U cut out by the quasi-coherent sheaf of ideals $\mathcal{I}$. Since $|T| = |T'|$ we see that $|b|(|T'|) \subset |U|$. Hence we can think of b as a morphism into U, see Properties of Spaces, Lemma 41.4.9. Let $\mathcal{J} \subset \mathcal{O}_{T'}$ be the square zero quasi-coherent sheaf of ideals cutting out T. By the commutativity of the diagram we have $b|_T = i \circ a$ where $i : Z \to U$ is the closed immersion. We conclude that $b^\sharp(b^{-1}\mathcal{I}) \subset \mathcal{J}$ by Morphisms of Spaces, Lemma 42.16.1. As T' is a first order thickening of T we see that $\mathcal{J}^2 = 0$ hence $b^\sharp(b^{-1}(\mathcal{I}^2)) = 0$. By Morphisms of Spaces, Lemma 42.16.1 this implies that b factors through Z'. Letting $a' : T' \to Z'$ be this factorization we win. $\square$

Lemma 46.9.3. *Let $i : Z \to X$ be an immersion of algebraic spaces. Let $Z \subset Z'$ be the first order infinitesimal neighbourhood of Z in X. Then the diagram*

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & Z' \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

induces a map of conormal sheaves $\mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Z'}$ by Lemma 46.5.3. This map is an isomorphism.

Proof. This is clear from the construction of Z' above. $\square$

46.10. Formally smooth, étale, unramified transformations

Recall that a ring map $R \to A$ is called *formally smooth*, resp. *formally étale*, resp. *formally unramified* (see Algebra, Definition 7.127.1, resp. Definition 7.137.1, resp. Definition 7.135.1) if for every commutative solid diagram

$$\begin{tikzcd} A \arrow[r] \arrow[rd, dashed] & B/I \\ R \arrow[u] \arrow[r] & B \arrow[u] \end{tikzcd}$$

where $I \subset B$ is an ideal of square zero, there exists a, resp. exists a unique, resp. exists at most one dotted arrow which makes the diagram commute. This motivates the following analogue for morphisms of algebraic spaces, and more generally functors.

Definition 46.10.1. Let S be a scheme. Let $a : F \to G$ be a transformation of functors $F, G : (Sch/S)_{fppf}^{opp} \to Sets$. Consider commutative solid diagrams of the form

$$\begin{tikzcd} F & T \arrow[l] \arrow[d, "i"] \\ G \arrow[from=u, "a"'] & T' \arrow[l] \arrow[ul, dashed] \end{tikzcd}$$

where T and T' are affine schemes and i is a closed immersion defined by an ideal of square zero.

(1) We say a is *formally smooth* if given any solid diagram as above there exists a dotted arrow making the diagram commute[2].
(2) We say a is *formally étale* if given any solid diagram as above there exists exactly one dotted arrow making the diagram commute.
(3) We say a is *formally unramified* if given any solid diagram as above there exists at most one dotted arrow making the diagram commute.

Lemma 46.10.2. *Let S be a scheme. Let $a : F \to G$ be a transformation of functors $F, G : (Sch/S)_{fppf}^{opp} \to Sets$. Then a is formally étale if and only if a is both formally smooth and formally unramified.*

Proof. Formal from the definition. □

Lemma 46.10.3. *Composition.*

(1) *A composition of formally smooth transformations of functors is formally smooth.*
(2) *A composition of formally étale transformations of functors is formally étale.*
(3) *A composition of formally unramified transformations of functors is formally unramified.*

Proof. This is formal. □

Lemma 46.10.4. *Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)_{fppf}^{opp} \to Sets$. Let $a : F \to G$, $b : H \to G$ be transformations of functors. Consider the fibre product diagram*

$$\begin{tikzcd} H \times_{b,G,a} F \arrow[r, "b'"'] \arrow[d, "a'"'] & F \arrow[d, "a"] \\ H \arrow[r, "b"] & G \end{tikzcd}$$

(1) *If a is formally smooth, then the base change a' is formally smooth.*
(2) *If a is formally étale, then the base change a' is formally étale.*
(3) *If a is formally unramified, then the base change a' is formally unramified.*

Proof. This is formal. □

Lemma 46.10.5. *Let S be a scheme. Let $F, G : (Sch/S)_{fppf}^{opp} \to Sets$. Let $a : F \to G$ be a representable tranformation of functors.*

(1) *If a is smooth then a is formally smooth.*
(2) *If a is étale, then a is formally étale.*
(3) *If a is unramified, then a is formally unramified.*

[2]This is just one possible definition that one can make here. Another slightly weaker condition would be to require that the dotted arrow exists fppf locally on T'. This weaker notion has in some sense better formal properties.

Proof. Consider a solid commutative diagram

$$\begin{tikzcd} F \arrow[d, "a"'] & T \arrow[l] \arrow[d, "i"] \\ G & T' \arrow[l] \arrow[ul, dashed] \end{tikzcd}$$

as in Definition 46.10.1. Then $F \times_G T'$ is a scheme smooth (resp. étale, resp. unramified) over T'. Hence by More on Morphisms, Lemma 33.9.7 (resp. Lemma 33.6.9, resp. Lemma 33.4.8) we can fill in (resp. uniquely fill in, resp. fill in in at most one way) the dotted arrow in the diagram

$$\begin{tikzcd} F \times_G T' \arrow[d] & T \arrow[l] \arrow[d, "i"] \\ T' & T' \arrow[l] \arrow[ul, dashed] \end{tikzcd}$$

an hence we also obtain the corresponding assertion in the first diagram. □

Lemma 46.10.6. *Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)_{fppf}^{opp} \to Sets$. Let $a : F \to G$, $b : G \to H$ be transformations of functors. Assume that a is representable, surjective, and étale.*

(1) *If b is formally smooth, then $b \circ a$ is formally smooth.*
(2) *If b is formally étale, then $b \circ a$ is formally étale.*
(3) *If b is formally unramified, then $b \circ a$ is formally unramified.*

Conversely, consider a solid commutative diagram

$$\begin{tikzcd} G \arrow[d, "b"'] & T \arrow[l] \arrow[d, "i"] \\ H & T' \arrow[l] \arrow[ul, dashed] \end{tikzcd}$$

with T' an affine scheme over S and $i : T \to T'$ a closed immersion defined by an ideal of square zero.

(4) *If $b \circ a$ is formally smooth, then for every $t \in T$ there exists an étale morphism of affines $U' \to T'$ and a morphism $U' \to G$ such that*

$$\begin{tikzcd} G \arrow[d, "b"'] & T \arrow[l] & T \times_{T'} U' \arrow[l] \arrow[d] \\ H & T' \arrow[l] & U' \arrow[l] \arrow[ull] \end{tikzcd}$$

commutes and t is in the image of $U' \to T'$.

(5) *If $b \circ a$ is formally unramified, then there exists at most one dotted arrow in the diagram above, i.e., b is formally unramified.*
(6) *If $b \circ a$ is formally étale, then there exists exactly one dotted arrow in the diagram above, i.e., b is formally étale.*

Proof. Assume b is formally smooth (resp. formally étale, resp. formally unramified). Since an étale morphism is both smooth and unramified we see that a is representable and smooth (resp. étale, resp. unramified). Hence parts (1), (2) and (3) follow from a combination of Lemma 46.10.5 and Lemma 46.10.3.

Assume that $b \circ a$ is formally smooth. Consider a diagram as in the statement of the lemma. Let $W = F \times_G T$. By assumption W is a scheme surjective étale over T. By Étale Morphisms, Theorem 37.15.2 there exists a scheme W' étale over T' such that $W = T \times_{T'} W'$. Choose an affine open subscheme $U' \subset W'$ such that t is in the image of $U' \to T'$. Because $b \circ a$ is formally smooth we see that the exist morphisms $U' \to F$ such that

$$\begin{tikzcd} F \arrow[d, "b \circ a"'] & W \arrow[l] & T \times_{T'} U' \arrow[l] \arrow[d] \\ H & T' \arrow[l] & U' \arrow[l] \arrow[ull] \end{tikzcd}$$

commutes. Taking the composition $U' \to F \to G$ gives a map as in part (5) of the lemma.

Assume that $f, g : T' \to G$ are two dotted arrows fitting into the diagram of the lemma. Let $W = F \times_G T$. By assumption W is a scheme surjective étale over T. By Étale Morphisms, Theorem 37.15.2 there exists a scheme W' étale over T' such that $W = T \times_{T'} W'$. Since a is formally étale the compositions

$$W' \to T' \xrightarrow{f} G \quad \text{and} \quad W' \to T' \xrightarrow{g} G$$

lift to morphisms $f', g' : W' \to F$ (lift on affine opens and glue by uniqueness). Now if $b \circ a : F \to H$ is formally unramified, then $f' = g'$ and hence $f = g$ as $W' \to T'$ is an étale covering. This proves part (6) of the lemma.

Assume that $b \circ a$ is formally étale. Then by part (4) we can étale locally on T' find a dotted arrow fitting into the diagram and by part (5) this dotted arrow is unique. Hence we may glue the local solutions to get assertion (6). Some details omitted. □

Remark 46.10.7. It is tempting to think that in the situation of Lemma 46.10.6 we have ``$b$ formally smooth'' $\Leftrightarrow$ ``$b \circ a$ formally smooth''. However, this is likely not true in general.

Lemma 46.10.8. *Let S be a scheme. Let $F, G, H : (Sch/S)_{fppf}^{opp} \to Sets$. Let $a : F \to G$, $b : G \to H$ be transformations of functors. Assume b is formally unramified.*

(1) *If $b \circ a$ is formally unramified then a is formally unramified.*
(2) *If $b \circ a$ is formally étale then a is formally étale.*
(3) *If $b \circ a$ is formally smooth then a is formally smooth.*

Proof. Let $T \subset T'$ be a closed immersion of affine schemes defined by an ideal of square zero. Let $g' : T' \to G$ and $f : T \to F$ be given such that $g'|_T = a \circ f$. Because b is formally unramified, there is a one to one correspondence between

$$\{f' : T' \to F \mid f = f'|_T \text{ and } a \circ f' = g'\}$$

and

$$\{f' : T' \to F \mid f = f'|_T \text{ and } b \circ a \circ f' = b \circ g'\}.$$

From this the lemma follows formally. □

46.11. Formally unramified morphisms

In this section we work out what it means that a morphism of algebraic spaces is formally unramified.

Definition 46.11.1. Let S be a scheme. A morphism $f : X \to Y$ of algebraic spaces over S is said to be *formally unramified* if it is formally unramified as a transformation of functors as in Definition 46.10.1.

We will not restate the results proved in the more general setting of formally unramified transformations of functors in Section 46.10. It turns out we can characterize this property in terms of vanishing of the module of relative differentials, see Lemma 46.11.6.

Lemma 46.11.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is formally unramified,*
(2) *for every diagram*

$$\begin{tikzcd} U \arrow[r, "\psi"'] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where U and V are schemes and the vertical arrows are étale the morphism of schemes ψ is formally unramified (as in More on Morphisms, Definition 33.4.1), and

(3) *for one such diagram with surjective vertical arrows the morphism ψ is formally unramified.*

Proof. Assume f is formally unramified. By Lemma 46.10.5 the morphisms $U \to X$ and $V \to Y$ are formally unramified. Thus by Lemma 46.10.3 the composition $U \to Y$ is formally unramified. Then it follows from Lemma 46.10.8 that $U \to V$ is formally unramified. Thus (1) implies (2). And (2) implies (3) trivially

Assume given a diagram as in (3). By Lemma 46.10.5 the morphism $V \to Y$ is formally unramified. Thus by Lemma 46.10.3 the composition $U \to Y$ is formally unramified. Then it follows from Lemma 46.10.6 that $X \to Y$ is formally unramified, i.e., (1) holds. □

Lemma 46.11.3. *Let S be a scheme. If $f : X \to Y$ is a formally unramified morphism of algebraic spaces over S, then given any solid commutative diagram*

$$\begin{tikzcd} X \arrow[d, "f"'] & T \arrow[l] \arrow[d, "i"] \\ S & T' \arrow[l] \arrow[ul, dashed] \end{tikzcd}$$

where $T \subset T'$ is a first order thickening of algebraic spaces over S there exists at most one dotted arrow making the diagram commute. In other words, in Definition 46.11.1 the condition that T be an affine scheme may be dropped.

Proof. This is true because there exists a surjective étale morphism $U' \to T'$ where U' is a disjoint union of affine schemes (see Properties of Spaces, Lemma 41.6.1) and a morphism $T' \to X$ is determined by its restriction to U'. □

Lemma 46.11.4. *A composition of formally unramified morphisms is formally unramified.*

Proof. This is formal. □

Lemma 46.11.5. *A base change of a formally unramified morphism is formally unramified.*

Proof. This is formal. □

Lemma 46.11.6. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is formally unramified, and*

(2) $\Omega_{X/Y} = 0$.

Proof. This is a combination of Lemma 46.11.2, More on Morphisms, Lemma 33.4.7, and Lemma 46.6.3. □

Lemma 46.11.7. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *The morphism f is unramified,*
(2) *the morphism f is locally of finite type and $\Omega_{X/Y} = 0$, and*
(3) *the morphism f is locally of finite type and formally unramified.*

Proof. Choose a diagram

$$\begin{tikzcd} U \arrow[r, "\psi"] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where U and V are schemes and the vertical arrows are étale and surjective. Then we see

$$\begin{aligned} f \text{ unramified} &\Leftrightarrow \psi \text{ unramified} \\ &\Leftrightarrow \psi \text{ locally finite type and } \Omega_{U/V} = 0 \\ &\Leftrightarrow f \text{ locally finite type and } \Omega_{X/Y} = 0 \\ &\Leftrightarrow f \text{ locally finite type and formally unramified} \end{aligned}$$

Here we have used Morphisms, Lemma 24.34.2 and Lemma 46.11.6. □

Lemma 46.11.8. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is unramified and a monomorphism,*
(2) *f is unramified and universally injective,*
(3) *f is locally of finite type and a monomorphism,*
(4) *f is universally injective, locally of finite type, and formally unramified.*

Moreover, in this case f is also representable, separated, and locally quasi-finite.

Proof. We have seen in Lemma 46.11.7 that being formally unramified and locally of finite type is the same thing as being unramified. Hence (4) is equivalent to (2). A monomorphism is certainly formally unramified hence (3) implies (4). It is clear that (1) implies (3). Finally, if (2) holds, then $\Delta : X \to X \times_Y X$ is both an open immersion (Morphisms of Spaces, Lemma 42.34.9) and surjective (Morphisms of Spaces, Lemma 42.18.2) hence an isomorphism, i.e., f is a monomorphism. In this way we see that (2) implies (1). Finally, we see that f is representable, separated, and locally quasi-finite by Morphisms of Spaces, Lemmas 42.25.8 and 42.40.1. □

46.12. Universal first order thickenings

Let S be a scheme. Let $h : Z \to X$ be a morphism of algebraic spaces over S. A *universal first order thickening* of Z over X is a first order thickening $Z \subset Z'$ over X such that given any first order thickening $T \subset T'$ over X and a solid commutative diagram

(46.12.0.1)
$$\begin{tikzcd} & Z \arrow[ld] & & T \arrow[ll, "a"'] \arrow[rd] & \\ Z' \arrow[rrd] & & & & T' \arrow[llll, dotted, "a'"'] \arrow[lld, "b"] \\ & & X & & \end{tikzcd}$$

there exists a unique dotted arrow making the diagram commute. Note that in this situation $(a, a') : (T \subset T') \to (Z \subset Z')$ is a morphism of thickenings over X. Thus if a universal first order thickening exists, then it is unique up to unique isomorphism. In general a universal first order thickening does not exist, but if h is formally unramified then it does. Before we prove this, let us show that a universal first order thickening in the category of schemes is a universal first order thickening in the category of algebraic spaces.

Lemma 46.12.1. *Let S be a scheme. Let $h : Z \to X$ be a morphism of algebraic spaces over S. Let $Z \subset Z'$ be a first order thickening over X. The following are equivalent*

(1) *$Z \subset Z'$ is a universal first order thickening,*
(2) *for any diagram (46.12.0.1) with T' a scheme a unique dotted arrow exists making the diagram commute, and*
(3) *for any diagram (46.12.0.1) with T' an affine scheme a unique dotted arrow exists making the diagram commute.*

Proof. The implications (1) ⇒ (2) ⇒ (3) are formal. Assume (3) a assume given an arbitrary diagram (46.12.0.1). Choose a presentation $T' = U'/R'$, see Spaces, Definition 40.9.3. We may assume that $U' = \coprod U'_i$ is a disjoint union of affines, so $R' = U' \times_{T'} U' = \coprod_{i,j} U'_i \times'_T U'_j$. For each pair (i, j) choose an affine open covering $U'_i \times'_T U'_j = \bigcup_k R'_{ijk}$. Denote U_i, R_{ijk} the fibre products with T over T'. Then each $U_i \subset U'_i$ and $R_{ijk} \subset R'_{ijk}$ is a first order thickening of affine schemes. Denote $a_i : U_i \to Z$, resp. $a_{ijk} : R_{ijk} \to Z$ the composition of $a : T \to Z$ with the morphism $U_i \to T$, resp. $R_{ijk} \to T$. By (3) applied to $a_i : U_i \to Z$ we obtain unique morphisms $a'_i : U'_i \to Z'$. By (3) applied to a_{ijk} we see that the two compositions $R'_{ijk} \to R'_i \to Z'$ and $R'_{ijk} \to R'_j \to Z'$ are equal. Hence $a' = \coprod a'_i : U' = \coprod U'_i \to Z'$ descends to the quotient sheaf $T' = U'/R'$ and we win. □

Lemma 46.12.2. *Let S be a scheme. Let $Z \to Y \to X$ be morphisms of algebraic spaces over S. If $Z \subset Z'$ is a universal first order thickening of Z over Y and $Y \to X$ is formally étale, then $Z \subset Z'$ is a universal first order thickening of Z over X.*

Proof. This is formal. Namely, by Lemma 46.12.1 it suffices to consider solid commutative diagrams (46.12.0.1) with T' an affine scheme. The composition $T \to Z \to Y$ lifts uniquely to $T' \to Y$ as $Y \to X$ is assumed formally étale. Hence the fact that $Z \subset Z'$ is a universal first order thickening over Y produces the desired morphism $a' : T' \to Z'$. □

Lemma 46.12.3. *Let S be a scheme. Let $Z \to Y \to X$ be morphisms of algebraic spaces over S. Assume $Z \to Y$ is étale.*

(1) *If $Y \subset Y'$ is a universal first order thickening of Y over X, then the unique étale morphism $Z' \to Y'$ such that $Z = Y \times_{Y'} Z'$ (see Theorem 46.7.1) is a universal first order thickening of Z over X.*

(2) *If $Z \to Y$ is surjective and $(Z \subset Z') \to (Y \subset Y')$ is an étale morphism of first order thickenings over X and Z' is a universal first order thickening of Z over X, then Y' is a universal first order thickening of Y over X.*

Proof. Proof of (1). By Lemma 46.12.1 it suffices to consider solid commutative diagrams (46.12.0.1) with T' an affine scheme. The composition $T \to Z \to Y$ lifts uniquely to $T' \to Y'$ as Y' is the universal first order thickening. Then the fact that $Z' \to Y'$ is étale implies (see Lemma 46.10.5) that $T' \to Y'$ lifts to the desired morphism $a' : T' \to Z'$.

Proof of (2). Let $T \subset T'$ be a first order thickening over X and let $a : T \to Y$ be a morphism. Set $W = T \times_Y Z$ and denote $c : W \to Z$ the projection Let $W' \to T'$ be the unique étale morphism such that $W = T \times_{T'} W'$, see Theorem 46.7.1. Note that $W' \to T'$ is surjective as $Z \to Y$ is surjective. By assumption we obtain a unique morphism $c' : W' \to Z'$ over X restricting to c on W. By uniqueness the two restrictions of c' to $W' \times_{T'} W'$ are equal (as the two restrictions of c to $W \times_T W$ are equal). Hence c' descends to a unique morphism $a' : T' \to Y'$ and we win. □

Lemma 46.12.4. *Let S be a scheme. Let $h : Z \to X$ be a formally unramified morphism of algebraic spaces over S. There exists a universal first order thickening $Z \subset Z'$ of Z over X.*

Proof. Choose any commutative diagram

$$\begin{tikzcd} V \arrow[r] \arrow[d] & U \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

where V and U are schemes and the vertical arrows are étale. Note that $V \to U$ is a formally unramified morphism of schemes, see Lemma 46.11.2. Combining Lemma 46.12.1 and More on Morphisms, Lemma 33.5.1 we see that a universal first order thickening $V \subset V'$ of V over U exists. By Lemma 46.12.2 part (1) V' is a universal first order thickening of V over X.

Fix a scheme U and a surjective étale morphism $U \to X$. The argument above shows that for any $V \to Z$ étale with V a scheme such that $V \to Z \to X$ factors through U a universal first order thickening $V \subset V'$ of V over X exists (but does not depend on the chosen factorization of $V \to X$ through U). Now we may choose V such that $V \to Z$ is surjective étale (see Spaces, Lemma 40.11.4). Then $R = V \times_Z V$ a scheme étale over Z such that $R \to X$ factors through U also. Hence we obtain universal first order thickenings $V \subset V'$ and $R \subset R'$ over X. As $V \subset V'$ is a universal first order thickening, the two projections $s, t : R \to V$ lift to morphisms $s', t' : R' \to V'$. By Lemma 46.12.3 as R' is the universal first order thickening of R over X these morphisms are étale. Then $(t', s') : R' \to V'$ is an étale equivalence relation and we can set $Z' = V'/R'$. Since $V' \to Z'$ is surjective étale and v' is the universal first order thickening of V over X we conclude from Lemma 46.12.2 part (2) that Z' is a universal first order thickening of Z over X. □

Definition 46.12.5. Let S be a scheme. Let $h : Z \to X$ be a formally unramified morphism of algebraic spaces over S.

(1) The *universal first order thickening* of Z over X is the thickening $Z \subset Z'$ constructed in Lemma 46.12.4.
(2) The *conormal sheaf of Z over X* is the conormal sheaf of Z in its universal first order thickening Z' over X.

We often denote the conormal sheaf $\mathcal{C}_{Z/X}$ in this situation.

Thus we see that there is a short exact sequence of sheaves

$$0 \to \mathcal{C}_{Z/X} \to \mathcal{O}_{Z'} \to \mathcal{O}_Z \to 0$$

on $Z_{étale}$ and $\mathcal{C}_{Z/X}$ is a quasi-coherent $\mathcal{O}_Z$-module. The following lemma proves that there is no conflict between this definition and the definition in case $Z \to X$ is an immersion.

Lemma 46.12.6. *Let S be a scheme. Let $i : Z \to X$ be an immersion of algebraic spaces over S. Then*

(1) *i is formally unramified,*
(2) *the universal first order thickening of Z over X is the first order infinitesimal neighbourhood of Z in X of Definition 46.9.1,*
(3) *the conormal sheaf of i in the sense of Definition 46.5.1 agrees with the conormal sheaf of i in the sense of Definition 46.12.5.*

Proof. An immersion of algebraic spaces is by definition a representable morphism. Hence by Morphisms, Lemmas 24.34.7 and 24.34.8 an immersion is unramified (via the abstract principle of Spaces, Lemma 40.5.8). Hence it is formally unramified by Lemma 46.11.7. The other assertions follow by combining Lemmas 46.9.2 and 46.9.3 and the definitions. □

Lemma 46.12.7. *Let S be a scheme. Let $Z \to X$ be a formally unramified morphism of algebraic spaces over S. Then the universal first order thickening Z' is formally unramified over X.*

Proof. Let $T \subset T'$ be a first order thickening of affine schemes over X. Let

$$\begin{tikzcd} Z' \arrow[d] & T \arrow[l, "c"'] \arrow[d] \\ X & T' \arrow[l] \arrow[ul, "a,b"'] \end{tikzcd}$$

be a commutative diagram. Set $T_0 = c^{-1}(Z) \subset T$ and $T'_a = a^{-1}(Z)$ (scheme theoretically). Since Z' is a first order thickening of Z, we see that T' is a first order thickening of T'_a. Moreover, since $c = a|_T$ we see that $T_0 = T \cap T'_a$ (scheme theoretically). As T' is a first order thickening of T it follows that T'_a is a first order thickening of T_0. Now $a|_{T'_a}$ and $b|_{T'_a}$ are morphisms of T'_a into Z' over X which agree on T_0 as morphisms into Z. Hence by the universal property of Z' we conclude that $a|_{T'_a} = b|_{T'_a}$. Thus a and b are morphism from the first order thickening T' of T'_a whose restrictions to T'_a agree as morphisms into Z. Thus using the universal property of Z' once more we conclude that $a = b$. In other words, the defining property of a formally unramified morphism holds for $Z' \to X$ as desired. □

Lemma 46.12.8. *Let S be a scheme Consider a commutative diagram of algebraic spaces over S*

$$\begin{tikzcd} Z \arrow[r, "h"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ W \arrow[r, "h'"] & Y \end{tikzcd}$$

with h and h' formally unramified. Let $Z \subset Z'$ be the universal first order thickening of Z over X. Let $W \subset W'$ be the universal first order thickening of W over Y. There exists a *canonical morphism $(f, f') : (Z, Z') \to (W, W')$ of thickenings over Y which fits into the following commutative diagram*

$$
\begin{tikzcd}
 & & & Z' \arrow[dl] \arrow[d, "f'"] \\
Z \arrow[urrr] \arrow[rr] \arrow[d, "f"] & & X \arrow[d] & W' \arrow[dl] \\
W \arrow[urrr] \arrow[rr] & & Y &
\end{tikzcd}
$$

In particular the morphism (f, f') of thickenings induces a morphism of conormal sheaves $f^*\mathcal{C}_{W/Y} \to \mathcal{C}_{Z/X}$.

Proof. The first assertion is clear from the universal property of W'. The induced map on conormal sheaves is the map of Lemma 46.5.3 applied to $(Z \subset Z') \to (W \subset W')$. □

Lemma 46.12.9. *Let S be a scheme. Let*

$$
\begin{tikzcd}
Z \arrow[r, "h"'] \arrow[d, "f"'] & X \arrow[d, "g"] \\
W \arrow[r, "h'"] & Y
\end{tikzcd}
$$

be a fibre product diagram of algebraic spaces over S with h' formally unramified. Then h is formally unramified and if $W \subset W'$ is the universal first order thickening of W over Y, then $Z = X \times_Y W \subset X \times_Y W'$ is the universal first order thickening of Z over X. In particular the canonical map $f^*\mathcal{C}_{W/Y} \to \mathcal{C}_{Z/X}$ of Lemma 46.12.8 is surjective.

Proof. The morphism h is formally unramified by Lemma 46.11.5. It is clear that $X \times_Y W'$ is a first order thickening. It is straightforward to check that it has the universal property because W' has the universal property (by mapping properties of fibre products). See Lemma 46.5.5 for why this implies that the map of conormal sheaves is surjective. □

Lemma 46.12.10. *Let S be a scheme. Let*

$$
\begin{tikzcd}
Z \arrow[r, "h"'] \arrow[d, "f"'] & X \arrow[d, "g"] \\
W \arrow[r, "h'"] & Y
\end{tikzcd}
$$

be a fibre product diagram of algebraic spaces over S with h' formally unramified and g flat. In this case the corresponding map $Z' \to W'$ of universal first order thickenings is flat, and $f^*\mathcal{C}_{W/Y} \to \mathcal{C}_{Z/X}$ is an isomorphism.

Proof. Flatness is preserved under base change, see Morphisms of Spaces, Lemma 42.27.3. Hence the first statement follows from the description of W' in Lemma 46.12.9. It is clear that $X \times_Y W'$ is a first order thickening. It is straightforward to check that it has the universal property because W' has the universal property (by mapping properties of fibre products). See Lemma 46.5.5 for why this implies that the map of conormal sheaves is an isomorphism. □

Lemma 46.12.11. *Taking the universal first order thickenings commutes with étale localization. More precisely, let $h : Z \to X$ be a formally unramified morphism of algebraic* *spaces over a base scheme S. Let*

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

be a commutative diagram with étale vertical arrows. Let Z' be the universal first order thickening of Z over X. Then $V \to U$ is formally unramified and the universal first order thickening V' of V over U is étale over Z'. In particular, $\mathcal{C}_{Z/X}|_V = \mathcal{C}_{V/U}$.

Proof. The first statement is Lemma 46.11.2. The compatibility of universal first order thickenings is a consequence of Lemmas 46.12.2 and 46.12.3. □

Lemma 46.12.12. *Let S be a scheme. Let B be an algebraic space over S. Let $h : Z \to X$ be a formally unramified morphism of algebraic spaces over B. Let $Z \subset Z'$ be the universal first order thickening of Z over X with structure morphism $h' : Z' \to X$. The canonical map*

$$dh' : (h')^*\Omega_{X/B} \to \Omega_{Z'/B}$$

induces an isomorphism $h^*\Omega_{X/B} \to \Omega_{Z'/B} \otimes \mathcal{O}_Z$.

Proof. The map $c_{h'}$ is the map defined in Lemma 46.6.6. If $i : Z \to Z'$ is the given closed immersion, then $i^*c_{h'}$ is a map $h^*\Omega_{X/S} \to \Omega_{Z'/S} \otimes \mathcal{O}_Z$. Checking that it is an isomorphism reduces to the case of schemes by étale localization, see Lemma 46.12.11 and Lemma 46.6.3. In this case the result is More on Morphisms, Lemma 33.5.9. □

Lemma 46.12.13. *Let S be a scheme. Let B be an algebraic space over S. Let $h : Z \to X$ be a formally unramified morphism of algebraic spaces over B. There is a canonical exact sequence*

$$\mathcal{C}_{Z/X} \to h^*\Omega_{X/B} \to \Omega_{Z/B} \to 0.$$

The first arrow is induced by $d_{Z'/B}$ where Z' is the universal first order neighbourhood of Z over X.

Proof. We know that there is a canonical exact sequence

$$\mathcal{C}_{Z/Z'} \to \Omega_{Z'/S} \otimes \mathcal{O}_Z \to \Omega_{Z/S} \to 0.$$

see Lemma 46.6.10. Hence the result follows on applying Lemma 46.12.12. □

Lemma 46.12.14. *Let S be a scheme. Let*

$$\begin{array}{ccc} Z & \xrightarrow{i} & X \\ & \searrow^{j} & \downarrow \\ & & Y \end{array}$$

be a commutative diagram of algebraic spaces over S where i and j are formally unramified. Then there is a canonical exact sequence

$$\mathcal{C}_{Z/Y} \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/Y} \to 0$$

where the first arrow comes from Lemma 46.12.8 and the second from Lemma 46.12.13.

Proof. Since the maps have been defined, checking the sequence is exact reduces to the case of schemes by étale localization, see Lemma 46.12.11 and Lemma 46.6.3. In this case the result is More on Morphisms, Lemma 33.5.11. □

Lemma 46.12.15. *Let S be a scheme. Let $Z \to Y \to X$ be formally unramified morphisms of algebraic spaces over S.*

1. *If $Z \subset Z'$ is the universal first order thickening of Z over X and $Y \subset Y'$ is the universal first order thickening of Y over X, then there is a morphism $Z' \to Y'$ and $Y \times_{Y'} Z'$ is the universal first order thickening of Z over Y.*
2. *There is a canonical exact sequence*
$$i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$
where the maps come from Lemma 46.12.8 and $i : Z \to Y$ is the first morphism.

Proof. The map $h : Z' \to Y'$ in (1) comes from Lemma 46.12.8. The assertion that $Y \times_{Y'} Z'$ is the universal first order thickening of Z over Y is clear from the universal properties of Z' and Y'. By Lemma 46.5.6 we have an exact sequence
$$(i')^*\mathcal{C}_{Y\times_{Y'} Z'/Z'} \to \mathcal{C}_{Z/Z'} \to \mathcal{C}_{Z/Y\times_{Y'} Z'} \to 0$$
where $i' : Z \to Y\times_{Y'} Z'$ is the given morphism. By Lemma 46.5.5 there exists a surjection $h^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{Y\times_{Y'} Z'/Z'}$. Combined with the equalities $\mathcal{C}_{Y/Y'} = \mathcal{C}_{Y/X}$, $\mathcal{C}_{Z/Z'} = \mathcal{C}_{Z/X}$, and $\mathcal{C}_{Z/Y\times_{Y'} Z'} = \mathcal{C}_{Z/Y}$ this proves the lemma. □

46.13. Formally étale morphisms

In this section we work out what it means that a morphism of algebraic spaces is formally étale.

Definition 46.13.1. Let S be a scheme. A morphism $f : X \to Y$ of algebraic spaces over S is said to be *formally étale* if it is formally étale as a transformation of functors as in Definition 46.10.1.

We will not restate the results proved in the more general setting of formally étale transformations of functors in Section 46.10.

Lemma 46.13.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

1. *f is formally étale,*
2. *for every diagram*
$$\begin{tikzcd} U \arrow[r, "\psi"'] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$
where U and V are schemes and the vertical arrows are étale the morphism of schemes ψ is formally étale (as in More on Morphisms, Definition 33.6.1), and
3. *for one such diagram with surjective vertical arrows the morphism ψ is formally étale.*

Proof. Assume f is formally étale. By Lemma 46.10.5 the morphisms $U \to X$ and $V \to Y$ are formally étale. Thus by Lemma 46.10.3 the composition $U \to Y$ is formally étale. Then it follows from Lemma 46.10.8 that $U \to V$ is formally étale. Thus (1) implies (2). And (2) implies (3) trivially

Assume given a diagram as in (3). By Lemma 46.10.5 the morphism $V \to Y$ is formally étale. Thus by Lemma 46.10.3 the composition $U \to Y$ is formally étale. Then it follows from Lemma 46.10.6 that $X \to Y$ is formally étale, i.e., (1) holds. □

Lemma 46.13.3. *Let S be a scheme. Let $f : X \to Y$ be a formally étale morphism of algebraic spaces over S. Then given any solid commutative diagram*

$$\begin{tikzcd} X \arrow[d, "f"'] & T \arrow[l, "a"'] \arrow[d, "i"] \\ Y & T' \arrow[l] \arrow[lu, dashed] \end{tikzcd}$$

where $T \subset T'$ is a first order thickening of algebraic spaces over Y there exists exactly one dotted arrow making the diagram commute. In other words, in Definition 46.13.1 the condition that T be affine may be dropped.

Proof. Let $U' \to T'$ be a surjective étale morphism where $U' = \coprod U'_i$ is a disjoint union of affine schemes. Let $U_i = T \times_{T'} U'_i$. Then we get morphisms $a'_i : U'_i \to X$ such that $a'_i|_{U_i}$ equals the composition $U_i \to T \to X$. By uniqueness (see Lemma 46.11.3) we see that a'_i and a'_j agree on the fibre product $U'_i \times_{T'} U'_j$. Hence $\coprod a'_i : U' \to X$ descends to give a unique morphism $a' : T' \to X$. $\square$

Lemma 46.13.4. *A composition of formally étale morphisms is formally étale.*

Proof. This is formal. $\square$

Lemma 46.13.5. *A base change of a formally étale morphism is formally étale.*

Proof. This is formal. $\square$

Lemma 46.13.6. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S The following are equivalent:*

(1) *f is formally étale,*
(2) *f is formally unramified and the universal first order thickening of X over Y is equal to X,*
(3) *f is formally unramified and $\mathcal{C}_{X/Y} = 0$, and*
(4) *$\Omega_{X/Y} = 0$ and $\mathcal{C}_{X/Y} = 0$.*

Proof. Actually, the last assertion only make sense because $\Omega_{X/Y} = 0$ implies that $\mathcal{C}_{X/Y}$ is defined via Lemma 46.11.6 and Definition 46.12.5. This also makes it clear that (3) and (4) are equivalent.

Either of the assumptions (1), (2), and (3) imply that f is formally unramified. Hence we may assume f is formally unramified. The equivalence of (1), (2), and (3) follow from the universal property of the universal first order thickening X' of X over S and the fact that $X = X' \Leftrightarrow \mathcal{C}_{X/Y} = 0$ since after all by definition $\mathcal{C}_{X/Y} = \mathcal{C}_{X/X'}$ is the ideal sheaf of X in X'. $\square$

Lemma 46.13.7. *An unramified flat morphism is formally étale.*

Proof. Follows from the case of schemes, see More on Morphisms, Lemma 33.6.7 and étale localization, see Lemmas 46.11.2 and 46.13.2 and Morphisms of Spaces, Lemma 42.27.4. $\square$

Lemma 46.13.8. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *The morphism f is étale, and*
(2) *the morphism f is locally of finite presentation and formally étale.*

Proof. Follows from the case of schemes, see More on Morphisms, Lemma 33.6.9 and étale localization, see Lemma 46.13.2 and Morphisms of Spaces, Lemmas 42.26.4 and 42.35.2. □

46.14. Infinitesimal deformations of maps

In this section we explain how a derivation can be used to infinitesimally move a map. Throughout this section we use that a sheaf on a thickening X' of X can be seen as a sheaf on X, see Equations (46.8.1.1) and (46.8.1.2).

Lemma 46.14.1. *Let S be a scheme. Let B be an algebraic space over S. Let $X \subset X'$ and $Y \subset Y'$ be two first order thickenings of algebraic spaces over B. Let $(a, a'), (b, b') : (X \subset X') \to (Y \subset Y')$ be two morphisms of thickenings over B. Assume that*

(1) *$a = b$, and*
(2) *the two maps $a^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{X/X'}$ (Lemma 46.5.3) are equal.*

Then the map $(a')^\sharp - (b')^\sharp$ factors as

$$\mathcal{O}_{Y'} \to \mathcal{O}_Y \xrightarrow{D} a_*\mathcal{C}_{X/X'} \to a_*\mathcal{O}_{X'}$$

where D is an $\mathcal{O}_B$-derivation.

Proof. Instead of working on Y we work on X. The advantage is that the pullback functor a^{-1} is exact. Using (1) and (2) we obtain a commutive diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{C}_{X/X'} \arrow[r] & \mathcal{O}_{X'} \arrow[r] & \mathcal{O}_X \arrow[r] & 0 \\ 0 \arrow[r] & a^{-1}\mathcal{C}_{Y/Y'} \arrow[r] \arrow[u] & a^{-1}\mathcal{O}_{Y'} \arrow[r] \arrow[u, shift left, "(a')^\sharp"'] \arrow[u, shift right, "(b')^\sharp"'] & a^{-1}\mathcal{O}_Y \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

Now it is a general fact that in such a situation the difference of the $\mathcal{O}_B$-algebra maps $(a')^\sharp$ and $(b')^\sharp$ is an $\mathcal{O}_B$-derivation from $a^{-1}\mathcal{O}_Y$ to $\mathcal{C}_{X/X'}$. By adjointness of the functors a^{-1} and a_* this is the same thing as an $\mathcal{O}_B$-derivation from $\mathcal{O}_Y$ into $a_*\mathcal{C}_{X/X'}$. Some details omitted. □

Note that in the situation of the lemma above we may write D as

$$D = \mathrm{d}_{Y/B} \circ \theta \tag{46.14.1.1}$$

where θ is an $\mathcal{O}_Y$-linear map $\theta : \Omega_{Y/B} \to a_*\mathcal{C}_{X/X'}$. Of course, then by adjunction again we may view θ as an $\mathcal{O}_X$-linear map $\theta : a^*\Omega_{Y/B} \to \mathcal{C}_{X/X'}$.

Lemma 46.14.2. *Let S be a scheme. Let B be an algebraic space over S. Let $(a, a') : (X \subset X') \to (Y \subset Y')$ be a morphism of first order thickenings over B. Let*

$$\theta : a^*\Omega_{Y/B} \to \mathcal{C}_{X/X'}$$

be an $\mathcal{O}_X$-linear map. Then there exists a unique morphism of pairs $(b, b') : (X \subset X') \to (Y \subset Y')$ such that (1) and (2) of Lemma 46.14.1 hold and the derivation D and θ are related by Equation (46.14.1.1).

Proof. Consider the map

$$\alpha = (a')^\sharp + D : a^{-1}\mathcal{O}_{Y'} \to \mathcal{O}_{X'}$$

where D is as in Equation (46.14.1.1). As D is an $\mathcal{O}_B$-derivation it follows that α is a map of sheaves of $\mathcal{O}_B$-algebras. By construction we have $i_X^\sharp \circ \alpha = a^\sharp \circ i_Y^\sharp$ where $i_X : X \to X'$

and $i_Y : Y \to Y'$ are the given closed immersions. By Lemma 46.8.2 we obtain a unique morphism $(a, b') : (X \subset X') \to (Y \subset Y')$ of thickenings over B such that $\alpha = (b')^\sharp$. Setting $b = a$ we win. $\square$

Lemma 46.14.3. *Let S be a scheme. Let B be an algebraic space over S. Let $X \subset X'$ and $Y \subset Y'$ be first order thickenings over B. Assume given a morphism $a : X \to Y$ and a map $A : a^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{X/X'}$ of $\mathcal{O}_X$-modules. For an object U' of $(X')_{spaces,étale}$ with $U = X \times_{X'} U'$ consider morphisms $a' : U' \to Y'$ such that*

1. *a' is a morphism over B,*
2. *$a'|_U = a|_U$, and*
3. *the induced map $a^*\mathcal{C}_{Y/Y'}|_U \to \mathcal{C}_{X/X'}|_U$ is the restriction of A to U.*

Then the rule

$$U' \mapsto \{a' : U' \to Y' \text{ such that (1), (2), (3) hold.}\} \quad (46.14.3.1)$$

defines a sheaf of sets on $(X')_{spaces,étale}$.

Proof. Denote $\mathcal{F}$ the rule of the lemma. The restriction mapping $\mathcal{F}(U') \to \mathcal{F}(V')$ for $V' \subset U' \subset X'$ of $\mathcal{F}$ is really the restriction map $a' \mapsto a'|_{V'}$. With this definition in place it is clear that $\mathcal{F}$ is a sheaf since morphisms of algebraic spaces satisfy étale descent, see Descent on Spaces, Lemma 45.6.2. $\square$

Lemma 46.14.4. *Same notation and assumptions as in Lemma 46.14.3. We identify sheaves on X and X' via (46.8.1.1). There is an action of the sheaf*

$$\mathcal{H}om_{\mathcal{O}_X}(a^*\Omega_{Y/B}, \mathcal{C}_{X/X'})$$

on the sheaf (46.14.3.1). Moreover, the action is simply transitive for any object U' of $(X')_{spaces,étale}$ over which the sheaf (46.14.3.1) has a section.

Proof. This is a combination of Lemmas 46.14.1, 46.14.2, and 46.14.3. $\square$

Remark 46.14.5. A special case of Lemmas 46.14.1, 46.14.2, 46.14.3, and 46.14.4 is where $Y = Y'$. In this case the map A is always zero. The sheaf of Lemma 46.14.3 is just given by the rule

$$U' \mapsto \{a' : U' \to Y \text{ over } S \text{ with } a'|_U = a|_U\}$$

and we act on this by the sheaf $\mathcal{H}om_{\mathcal{O}_X}(a^*\Omega_{Y/B}, \mathcal{C}_{X/X'})$. The action of a local section θ on a' is sometimes indicated by $\theta \cdot a'$. Note that this means nothing else than the fact that $(a')^\sharp$ and $(\theta \cdot a')^\sharp$ differ by a derivation D which is related to θ by Equation (46.14.1.1).

46.15. Infinitesimal deformations of algebraic spaces

The following simple lemma is often a convenient tool to check whether an infinitesimal deformation of a map is flat.

Lemma 46.15.1. *Let S be a scheme. Let $(f, f') : (X \subset X') \to (Y \subset Y')$ be a morphism of first order thickenings of algebraic spaces over S. Assume that f is flat. Then the following are equivalent*

1. *f' is flat and $X = Y \times_{Y'} X'$, and*
2. *the canonical map $f^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{X/X'}$ is an isomorphism.*

Proof. Choose a scheme V' and a surjective étale morphism $V' \to Y'$. Choose a scheme U' and a surjective étale morphism $U' \to X' \times_{Y'} V'$. Set $U = X \times_{X'} U'$ and $V = Y \times_{Y'} V'$. According to our definition of a flat morphism of algebraic spaces we see that the induced map $g : U \to V$ is a flat morphism of schemes and that f' is flat if and only if the corresponding morphism $g' : U' \to V'$ is flat. Also, $X = Y \times_{Y'} X'$ if and only if $U = V \times_{V'} V'$. Finally, the map $f^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{X/X'}$ is an isomorphism if and only if $g^*\mathcal{C}_{V/V'} \to \mathcal{C}_{U/U'}$ is an isomorphism. Hence the lemma follows from its analogue for morphisms of schemes, see More on Morphisms, Lemma 33.8.1. □

46.16. Formally smooth morphisms

In this section we introduce the notion of a formally smooth morphism $X \to Y$ of algebraic spaces. Such a morphism is characterized by the property that T-valued points of X lift to inifinitesimal thickenings of T provided T is affine. The main result is that a morphism which is formally smooth and locally of finite presentation is smooth, see Lemma 46.16.6. It turns out that this criterion is often easier to use than the Jacobian criterion.

Definition 46.16.1. Let S be a scheme. A morphism $f : X \to Y$ of algebraic spaces over S is said to be *formally smooth* if it is formally smooth as a transformation of functors as in Definition 46.10.1.

In the cases of formally unramified and formally étale morphisms the condition that T' be affine could be dropped, see Lemmas 46.11.3 and 46.13.3. This is no longer true in the case of formally smooth morphisms. In fact, a slightly more natural condition would be that we should be able to fill in the dotted arrow étale locally on T'. In fact, analyzing the proof of Lemma 46.16.6 shows that this would be equivalent to the definition as it currently stands. It is also true that requiring the existence of the dotted arrow fppf locally on T' would be sufficient, but that is slightly more difficult to prove.

We will not restate the results proved in the more general setting of formally smooth transformations of functors in Section 46.10.

Lemma 46.16.2. *A composition of formally smooth morphisms is formally smooth.*

Proof. Omitted. □

Lemma 46.16.3. *A base change of a formally smooth morphism is formally smooth.*

Proof. Omitted, but see Algebra, Lemma 7.127.2 for the algebraic version. □

Lemma 46.16.4. *Let $f : X \to S$ be a morphism of schemes. Then f is formally étale if and only if f is formally smooth and formally unramified.*

Proof. Omitted. □

Here is a helper lemma which will be superseded by Lemma 46.16.9.

Lemma 46.16.5. *Let S be a scheme. Let*

$$\begin{tikzcd} U \arrow[r, "\psi"'] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

be a commutative diagram of morphisms of algebraic spaces over S. If the vertical arrows are étale and f is formally smooth, then ψ is formally smooth.

Proof. By Lemma 46.10.5 the morphisms $U \to X$ and $V \to Y$ are formally étale. By Lemma 46.10.3 the composition $U \to Y$ is formally smooth. By Lemma 46.10.8 we see $\psi : U \to V$ is formally smooth. $\square$

The following lemma is the main result of this section. It implies, combined with Proposition 46.4.9, that we can recognize whether a morphism of algebraic spaces $f : X \to Y$ is smooth in terms of ``simple'' properties of the transformation of functors $X \to Y$.

Lemma 46.16.6. *(Infinitesimal lifting criterion) Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

1. *The morphism f is smooth.*
2. *The morphism f is locally of finite presentation, and formally smooth.*

Proof. Assume $f : X \to S$ is locally of finite presentation and formally smooth. Consider a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\psi"] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where U and V are schemes and the vertical arrows are étale and surjective. By Lemma 46.16.5 we see $\psi : U \to V$ is formally smooth. By Morphisms of Spaces, Lemma 42.26.4 the morphism ψ is locally of finite presentation. Hence by the case of schemes the morphism ψ is smooth, see More on Morphisms, Lemma 33.9.7. Hence f is smooth, see Morphisms of Spaces, Lemma 42.33.4.

Conversely, assume that $f : X \to Y$ is smooth. Consider a solid commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & T \arrow[l, "a"'] \arrow[d, "i"] \\ Y & T' \arrow[l] \arrow[ul, dashed] \end{tikzcd}$$

as in Definition 46.16.1. We will show the dotted arrow exists thereby proving that f is formally smooth. Let $\mathcal{F}$ be the sheaf of sets on $(T')_{spaces,étale}$ of Lemma 46.14.3, see also Remark 46.14.5. Let

$$\mathcal{H} = \mathcal{H}om_{\mathcal{O}_T}(a^*\Omega_{X/Y}, \mathcal{C}_{T/T'})$$

be the sheaf of $\mathcal{O}_T$-modules on $T_{étale}$ introduced in Lemma 46.14.4. The action $\mathcal{H} \times \mathcal{F} \to \mathcal{F}$ turns $\mathcal{F}$ into a pseudo $\mathcal{H}$-torsor, see Cohomology on Sites, Definition 19.5.1. Our goal is to show that $\mathcal{F}$ is a trivial $\mathcal{H}$-torsor. There are two steps: (I) To show that $\mathcal{F}$ is a torsor we have to show that $\mathcal{F}$ has étale locally a section. (II) To show that $\mathcal{F}$ is the trivial torsor it suffices to show that $H^1(T_{étale}, \mathcal{H}) = 0$, see Cohomology on Sites, Lemma 19.5.3.

First we prove (I). To see this choose a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\psi"] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where U and V are schemes and the vertical arrows are étale and surjective. As f is assumed smooth we see that ψ is smooth and hence formally smooth by Lemma 46.10.5. By the same lemma the morphism $V \to Y$ is formally étale. Thus by Lemma 46.10.3 the composition $U \to Y$ is formally smooth. Then (I) follows from Lemma 46.10.6 part (4).

Finally we prove (II). By Lemma 46.6.15 we see that $\Omega_{X/S}$ is of finite presentation. Hence $a^*\Omega_{X/S}$ is of finite presentation (see Properties of Spaces, Section 41.27). Hence the sheaf $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_T}(a^*\Omega_{X/Y}, \mathcal{C}_{T/T'})$ is quasi-coherent by Properties of Spaces, Lemma 41.26.7. Thus by Descent, Proposition 31.6.10 and Coherent, Lemma 25.2.2 we have

$$H^1(T_{spaces,\acute{e}tale}, \mathcal{H}) = H^1(T_{\acute{e}tale}, \mathcal{H}) = H^1(T, \mathcal{H}) = 0$$

as desired. □

We do a bit more work to show that being formally smooth is étale local on the source. To begin we show that a formally smooth morphism has a nice sheaf of differentials. The notion of a locally projective quasi-coherent module is defined in Properties of Spaces, Section 41.28.

Lemma 46.16.7. *Let S be a scheme. Let $f : X \to Y$ be a formally smooth morphism of algebraic spaces over S. Then $\Omega_{X/Y}$ is locally projective on X.*

Proof. Choose a diagram

$$\begin{tikzcd} U \arrow[r, "\psi"'] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where U and V are affine(!) schemes and the vertical arrows are étale. By Lemma 46.16.5 we see $\psi : U \to V$ is formally smooth. Hence $\Gamma(V, \mathcal{O}_V) \to \Gamma(U, \mathcal{O}_U)$ is a formally smooth ring map, see More on Morphisms, Lemma 33.9.6. Hence by Algebra, Lemma 7.127.7 the $\Gamma(U, \mathcal{O}_U)$-module $\Omega_{\Gamma(U,\mathcal{O}_U)/\Gamma(V,\mathcal{O}_V)}$ is projective. Hence $\Omega_{U/V}$ is locally projective, see Properties, Section 23.19. Since $\Omega_{X/Y}|_U = \Omega_{U/V}$ we see that $\Omega_{X/Y}$ is locally projective too. (Because we can find an étale covering of X by the affine U's fitting into diagrams as above -- details omitted.) □

Lemma 46.16.8. *Let T be an affine scheme. Let $\mathcal{F}$, $\mathcal{G}$ be quasi-coherent $\mathcal{O}_T$-modules on $T_{\acute{e}tale}$. Consider the internal hom sheaf $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_T}(\mathcal{F}, \mathcal{G})$ on $T_{\acute{e}tale}$. If $\mathcal{F}$ is locally projective, then $H^1(T_{\acute{e}tale}, \mathcal{H}) = 0$.*

Proof. By the definition of a locally projective sheaf on an algebraic space (see Properties of Spaces, Definition 41.28.2) we see that $\mathcal{F}_{Zar} = \mathcal{F}|_{T_{Zar}}$ is a locally projective sheaf on the scheme T. Thus $\mathcal{F}_{Zar}$ is a direct summand of a free $\mathcal{O}_{T_{Zar}}$-module. Whereupon we conclude (as $\mathcal{F} = (\mathcal{F}_{Zar})^a$, see Descent, Proposition 31.6.11) that $\mathcal{F}$ is a direct summand of a free $\mathcal{O}_T$-module on $T_{\acute{e}tale}$. Hence we may assume that $\mathcal{F} = \bigoplus_{i \in I} \mathcal{O}_T$ is a free module. In this case $\mathcal{H} = \prod_{i \in I} \mathcal{G}$ is a product of quasi-coherent modules. By Cohomology on Sites, Lemma 19.12.5 we conclude that $H^1 = 0$ because the cohomology of a quasi-coherent sheaf on an affine scheme is zero, see Descent, Proposition 31.6.10 and Coherent, Lemma 25.2.2. □

Lemma 46.16.9. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is formally smooth,*
(2) *for every diagram*

$$\begin{tikzcd} U \arrow[r, "\psi"'] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where U and V are schemes and the vertical arrows are étale the morphism of schemes ψ is formally smooth (as in More on Morphisms, Definition 33.4.1), and

(3) *for one such diagram with surjective vertical arrows the morphism ψ is formally smooth.*

Proof. We have seen that (1) implies (2) and (3) in Lemma 46.16.5. Assume (3). The proof that f is formally smooth is entirely similar to the proof of (1) $\Rightarrow$ (2) of Lemma 46.16.6.

Consider a solid commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & T \arrow[l, "a"'] \arrow[d, "i"] \\ Y & T' \arrow[l] \arrow[ul, dashed] \end{tikzcd}$$

as in Definition 46.16.1. We will show the dotted arrow exists thereby proving that f is formally smooth. Let $\mathcal{F}$ be the sheaf of sets on $(T')_{spaces,étale}$ of Lemma 46.14.3, see also Remark 46.14.5. Let

$$\mathcal{H} = \mathcal{H}om_{\mathcal{O}_T}(a^*\Omega_{X/Y}, \mathcal{C}_{T/T'})$$

be the sheaf of $\mathcal{O}_T$-modules on $T_{étale}$ introduced in Lemma 46.14.4. The action $\mathcal{H} \times \mathcal{F} \to \mathcal{F}$ turns $\mathcal{F}$ into a pseudo $\mathcal{H}$-torsor, see Cohomology on Sites, Definition 19.5.1. Our goal is to show that $\mathcal{F}$ is a trivial $\mathcal{H}$-torsor. There are two steps: (I) To show that $\mathcal{F}$ is a torsor we have to show that $\mathcal{F}$ has étale locally a section. (II) To show that $\mathcal{F}$ is the trivial torsor it suffices to show that $H^1(T_{étale}, \mathcal{H}) = 0$, see Cohomology on Sites, Lemma 19.5.3.

First we prove (I). To see this consider a diagram (which exists because we are assuming (3))

$$\begin{tikzcd} U \arrow[r, "\psi"] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where U and V are schemes, the vertical arrows are étale and surjective, and ψ is formally smooth. By Lemma 46.10.5 the morphism $V \to Y$ is formally étale. Thus by Lemma 46.10.3 the composition $U \to Y$ is formally smooth. Then (I) follows from Lemma 46.10.6 part (4).

Finally we prove (II). By Lemma 46.16.7 we see that $\Omega_{U/V}$ locally projective. Hence $\Omega_{X/Y}$ is locally projective, see Descent on Spaces, Lemma 45.5.5. Hence $a^*\Omega_{X/Y}$ is locally projective, see Properties of Spaces, Lemma 41.28.3. Hence

$$H^1(T_{étale}, \mathcal{H}) = H^1(T_{étale}, \mathcal{H}om_{\mathcal{O}_T}(a^*\Omega_{X/Y}, \mathcal{C}_{T/T'}) = 0$$

by Lemma 46.16.8 as desired. $\square$

Lemma 46.16.10. *The property $\mathcal{P}(f)=$"f is formally smooth" is fpqc local on the base.*

Proof. Let $f : X \to Y$ be a morphism of algebraic spaces over a scheme S. Choose an index set I and diagrams

$$\begin{tikzcd} U_i \arrow[r, "\psi_i"] \arrow[d] & V_i \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with étale vertical arrows and U_i, V_i affine schemes. Moreover, assume that $\coprod U_i \to X$ and $\coprod V_i \to Y$ are surjective, see Properties of Spaces, Lemma 41.6.1. By Lemma 46.16.9 we see that f is formally smooth if and only if each of the morphisms ψ_i are formally smooth. Hence we reduce to the case of a morphism of affine schemes. In this case the result follows from Algebra, Lemma 7.127.15. Some details omitted. □

Lemma 46.16.11. *Let S be a scheme. Let $f : X \to Y$, $g : Y \to Z$ be morphisms of algebraic spaces over S. Assume f is formally smooth. Then*

$$0 \to f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Z} \to 0$$

Lemma 46.6.8 is short exact.

Proof. Follows from the case of schemes, see More on Morphisms, Lemma 33.9.9, by étale localization, see Lemmas 46.16.9 and 46.6.3. □

Lemma 46.16.12. *Let S be a scheme. Let B be an algebraic space over S. Let $h : Z \to X$ be a formally unramified morphism of algebraic spaces over B. Assume that Z is formally smooth over B. Then the canonical exact sequence*

$$0 \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/B} \to \Omega_{Z/B} \to 0$$

of Lemma 46.12.13 is short exact.

Proof. Let $Z \to Z'$ be the universal first order thickening of Z over X. From the proof of Lemma 46.12.13 we see that our sequence is identified with the sequence

$$\mathcal{C}_{Z/Z'} \to \Omega_{Z'/B} \otimes \mathcal{O}_Z \to \Omega_{Z/B} \to 0.$$

Since $Z \to S$ is formally smooth we can étale locally on Z' find a left inverse $Z' \to Z$ over B to the inclusion map $Z \to Z'$. Thus the sequence is étale locally split, see Lemma 46.6.11. □

Lemma 46.16.13. *Let S be a scheme. Let*

$$\begin{tikzcd} Z \arrow[r, "i"'] \arrow[rd, "j"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

be a commutative diagram of algebraic spaces over S where i and j are formally unramified and f is formally smooth. Then the canonical exact sequence

$$0 \to \mathcal{C}_{Z/Y} \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/Y} \to 0$$

of Lemma 46.12.14 is exact and locally split.

Proof. Denote $Z \to Z'$ the universal first order thickening of Z over X. Denote $Z \to Z''$ the universal first order thickening of Z over Y. By Lemma 46.12.13 here is a canonical morpism $Z' \to Z''$ so that we have a commutative diagram

$$\begin{tikzcd} Z \arrow[r, "i'"'] \arrow[rd, "j'"'] & Z' \arrow[r, "a"'] \arrow[d, "k"] & X \arrow[d, "f"] \\ & Z'' \arrow[r, "b"] & Y \end{tikzcd}$$

The sequence above is identified with the sequence

$$\mathcal{C}_{Z/Z''} \to \mathcal{C}_{Z/Z'} \to (i')^*\Omega_{Z'/Z''} \to 0$$

via our definitions concering conormal sheaves of formally unramified morphisms. Let $U'' \to Z''$ be an étale morphism with U'' affine. Denote $U \to Z$ and $U' \to Z'$ the corresponding affine schemes étale over Z and Z'. As f is formally smooth there exists a morphism $h : U'' \to X$ which agrees with i on U and such that $f \circ h$ equals $b|_{U''}$. Since Z' is the universal first order thickening we obtain a unique morphism $g : U'' \to Z'$ such that $g = a \circ h$. The universal property of Z'' implies that $k \circ g$ is the inclusion map $U'' \to Z''$. Hence g is a left inverse to k. Picture

$$\begin{tikzcd} U \arrow[r] \arrow[d] & Z' \arrow[d, "k"] \\ U'' \arrow[r] \arrow[ur, "g"] & Z'' \end{tikzcd}$$

Thus g induces a map $\mathcal{C}_{Z/Z'}|_U \to \mathcal{C}_{Z/Z''}|_U$ which is a left inverse to the map $\mathcal{C}_{Z/Z''} \to \mathcal{C}_{Z/Z'}$ over U. ☐

46.17. Openness of the flat locus

This section is analogue of More on Morphisms, Section 33.11. Note that we have defined the notion of flatness for quasi-coherent modules on algebraic spaces in Morphisms of Spaces, Section 42.28.

Theorem 46.17.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Assume f is locally of finite presentation and that $\mathcal{F}$ is an $\mathcal{O}_X$-module which is locally of finite presentation. Then*

$$\{x \in |X| : \mathcal{F} \text{ is flat over } Y \text{ at } x\}$$

is open in $|X|$.

Proof. Choose a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\alpha"'] \arrow[d, "p"'] & V \arrow[d, "q"] \\ X \arrow[r, "a"] & Y \end{tikzcd}$$

with U, V schemes and p, q surjective and étale as in Spaces, Lemma 40.11.4. By More on Morphisms, Theorem 33.11.1 the set $U' = \{u \in |U| : p^*\mathcal{F} \text{ is flat over } V \text{ at } u\}$ is open in U. By Morphisms of Spaces, Definition 42.28.2 the image of U' in $|X|$ is the set of the theorem. Hence we are done because the map $|U| \to |X|$ is open, see Properties of Spaces, Lemma 41.4.6. ☐

Lemma 46.17.2. *Let S be a scheme. Let*

$$\begin{tikzcd} X' \arrow[r, "g'"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

be a cartesian diagram of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume g is flat, f is locally of finite presentation, and $\mathcal{F}$ is locally of finite presentation. Then

$$\{x' \in |X'| : (g')^*\mathcal{F} \text{ is flat over } Y' \text{ at } x'\}$$

is the inverse image of the open subset of Theorem 46.17.1 under the continuous map $|g'| : |X'| \to |X|$.

Proof. This follows from Morphisms of Spaces, Lemma 42.28.3. ☐

46.18. Critère de platitude par fibres

Let S be a scheme. Consider a commutative diagram of algebraic spaces over S

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[rd, "g"'] & & Y \arrow[ld, "h"] \\ & Z & \end{tikzcd}$$

and a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$. Given a point $x \in |X|$ we consider the question as to whether $\mathcal{F}$ is flat over Y at x. If $\mathcal{F}$ is flat over Z at x, then the theorem below states this question is intimately related to the question of whether the restriction of $\mathcal{F}$ to the fibre of $X \to Z$ over $g(x)$ is flat over the fibre of $Y \to Z$ over $g(x)$. To make sense out of this we offer the following preliminary lemma.

Lemma 46.18.1. *In the situation above the following are equivalent*

(1) *Pick a geometric point $\overline{x}$ of X lying over x. Set $\overline{y} = f \circ \overline{x}$ and $\overline{z} = g \circ \overline{x}$. Then the module $\mathcal{F}_{\overline{x}}/\mathfrak{m}_{\overline{z}}\mathcal{F}_{\overline{x}}$ is flat over $\mathcal{O}_{Y,\overline{y}}/\mathfrak{m}_{\overline{z}}\mathcal{O}_{Y,\overline{y}}$.*

(2) *Pick a morphism $x : Spec(K) \to X$ in the equivalence class of x. Set $z = g \circ x$, $X_z = Spec(K) \times_{z,Z} X$, $Y_z = Spec(K) \times_{z,Z} Y$, and $\mathcal{F}_z$ the pullback of $\mathcal{F}$ to X_z. Then $\mathcal{F}_z$ is flat at x over Y_z (as defined in Morphisms of Spaces, Definition 42.28.2).*

(3) *Pick a commutative diagram*

$$\begin{tikzcd} & & U \arrow[r] \arrow[lld, "a"'] \arrow[rd] & V \arrow[lld, "b"] \arrow[d] \\ X \arrow[r, "f"] \arrow[rd, "g"'] & Y \arrow[d, "h"] & & W \arrow[llld, "c"] \\ & Z & & \end{tikzcd}$$

where U, V, W are schemes, and a, b, c are étale, and a point $u \in U$ mapping to x. Let $w \in W$ be the image of u. Let $\mathcal{F}_w$ be the pullback of $\mathcal{F}$ to the fibre U_w of $U \to W$ at w. Then $\mathcal{F}_w$ is flat over V_w at u.

Proof. Note that in (2) the morphism $x : Spec(K) \to X$ defines a K-rational point of X_z, hence the statement makes sense. Moreover, note that we can always choose a diagram as in (3) by: first choosing a scheme W and a surjective étale morphism $W \to Z$, then choosing a scheme V and a surjective étale morphism $V \to W \times_Z Y$, and finally choosing a scheme U and a surjective étale morphism $U \to V \times_Y X$. Having made these choices we set $U \to W$ equal to the composition $U \to V \to W$ and we can pick a point $u \in U$ mapping to x because the morphism $U \to X$ is surjective.

Suppose given both a diagram as in (3) and a geometric point $\overline{x} : Spec(k) \to X$ as in (1). By Properties of Spaces, Lemma 41.16.4 we can choose a geometric point $\overline{u} : Spec(k) \to U$ lying over u such that $\overline{x} = a \circ \overline{u}$. Denote $\overline{v} : Spec(k) \to V$ and $\overline{w} : Spec(k) \to W$ the induced geometric points of V and W. In this setting we know that $\mathcal{O}_{X,\overline{x}} = \mathcal{O}_{U,u}^{sh}$ and similarly for Y and Z, see Properties of Spaces, Lemma 41.19.1. In the same vein we have

$$\mathcal{F}_{\overline{x}} = (a^*\mathcal{F})_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{U,u}^{sh}$$

see Properties of Spaces, Lemma 41.26.4. Note that the stalk of $\mathcal{F}_w$ at u is given by

$$(\mathcal{F}_w)_u = (a^*\mathcal{F})_u/\mathfrak{m}_w(a^*\mathcal{F})_u$$

and the local ring of V_w at v is given by

$$\mathcal{O}_{V_w,v} = \mathcal{O}_{V,v}/\mathfrak{m}_w\mathcal{O}_{V,v}.$$

Since $\mathfrak{m}_{\overline{z}} = \mathfrak{m}_w\mathcal{O}_{Z,\overline{z}} = \mathfrak{m}_w\mathcal{O}^{sh}_{W,w}$ we see that

$$\begin{aligned}
\mathcal{F}_{\overline{x}}/\mathfrak{m}_{\overline{z}}\mathcal{F}_{\overline{x}} &= (a^*\mathcal{F})_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{X,\overline{x}}/\mathfrak{m}_{\overline{z}}\mathcal{O}_{X,\overline{x}} \\
&= (\mathcal{F}_w)_u \otimes_{\mathcal{O}_{U_w,u}} \mathcal{O}^{sh}_{U,u}/\mathfrak{m}_w\mathcal{O}^{sh}_{U,u} \\
&= (\mathcal{F}_w)_u \otimes_{\mathcal{O}_{U_w,u}} \mathcal{O}^{sh}_{U_w,\overline{u}} \\
&= (\mathcal{F}_w)_{\overline{u}}
\end{aligned}$$

the penultimate equality by Algebra, Lemma 7.139.22 and the last equality by Properties of Spaces, Lemma 41.26.4. The same arguments applied to the structure sheaves of V and Y show that

$$\mathcal{O}^{sh}_{V_w,\overline{v}} = \mathcal{O}^{sh}_{V,v}/\mathfrak{m}_w\mathcal{O}^{sh}_{V,v} = \mathcal{O}_{Y,\overline{y}}/\mathfrak{m}_{\overline{z}}\mathcal{O}_{Y,\overline{y}}.$$

OK, and now we can use Morphisms of Spaces, Lemma 42.28.1 to see that (1) is equivalent to (3).

Finally we prove the equivalence of (2) and (3). To do this we pick a field extension $\tilde{K}$ of K and and a morphism $\tilde{x} : Spec(\tilde{K}) \to U$ which lies over u (this is possible because $u \times_{X,x} Spec(K)$ is a nonempty scheme). Set $\tilde{z} : Spec(\tilde{K}) \to U \to W$ be the composition. We obtain a commutative diagram

$$\begin{tikzcd}
 & & & U_w \times_w \tilde{z} \arrow[rr] \arrow[dlll, "a"'] \arrow[dr] & & V_w \times_w \tilde{z} \arrow[dlll, "b"'] \arrow[dl] \\
X_z \arrow[rr, "f"'] \arrow[dr, "g"'] & & Y_z \arrow[dl, "h"] & & \tilde{z} \arrow[dlll, "c"'] & \\
 & z & & & &
\end{tikzcd}$$

where $z = Spec(K)$ and $w = Spec(\kappa(w))$. Now it is clear that $\mathcal{F}_w$ and $\mathcal{F}_z$ pull back to the same module on $U_w \times_w \tilde{z}$. This leads to a commutative diagram

$$\begin{tikzcd}
X_z \arrow[d] & U_w \times_w \tilde{z} \arrow[l] \arrow[r] \arrow[d] & U_w \arrow[d] \\
Y_z & V_w \times_w \tilde{z} \arrow[l] \arrow[r] & V_w
\end{tikzcd}$$

both of whose squares are cartesian and whose bottom horizontal arrows are flat: the lower left horizontal arrow is the composition of the morphism $Y \times_Z \tilde{z} \to Y \times_Z z = Y_z$ (base change of a flat morphism), the étale morphism $V \times_Z \tilde{z} \to Y \times_Z \tilde{z}$, and the étale morphism $V \times_W \tilde{z} \to V \times_Z \tilde{z}$. Thus it follows from Morphisms of Spaces, Lemma 42.28.3 that

$$\mathcal{F}_z \text{ flat at } x \text{ over } Y_z \Leftrightarrow \mathcal{F}|_{U_w \times_w \tilde{z}} \text{ flat at } \tilde{x} \text{ over } V_w \times_w \tilde{z} \Leftrightarrow \mathcal{F}_w \text{ flat at } u \text{ over } V_w$$

and we win. $\square$

Definition 46.18.2. Let S be a scheme. Let $X \to Y \to Z$ be morphisms of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x \in |X|$ be a point and denote $z \in |Z|$ its image.

(1) We say *the restriction of $\mathcal{F}$ to its fibre over z is flat at x over the fibre of Y over z* if the equivalent conditions of Lemma 46.18.1 are satisfied.

(2) We say *the fibre of X over z is flat at x over the fibre of Y over z* if the quivalent conditions of Lemma 46.18.1 holds with $\mathcal{F} = \mathcal{O}_X$.
(3) We say *the fibre of X over z is flat over the fibre of Y over z* if for all $x \in |X|$ lying over z the fibre of X over z is flat at x over the fibre of Y over z

With this definition in hand we can state the criterion as follows. (We leave the Noetherian version for later; insert future reference here.)

Theorem 46.18.3. *Let S be a scheme. Let $f : X \to Y$ and $Y \to Z$ be a morphisms of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume*

(1) *X is locally of finite presentation over Z,*
(2) *$\mathcal{F}$ an $\mathcal{O}_X$-module of finite presentation, and*
(3) *Y is locally of finite type over Z.*

Let $x \in |X|$ and let $y \in |Y|$ and $z \in |Z|$ be the images of x. If $\mathcal{F}_{\overline{x}} \neq 0$, then the following are equivalent:

(1) *$\mathcal{F}$ is flat over Z at x and the restriction of $\mathcal{F}$ to its fibre over z is flat at x over the fibre of Y over z, and*
(2) *Y is flat over Z at y and $\mathcal{F}$ is flat over Y at x.*

Moreover, the set of points x where (1) and (2) hold is open in $Supp(\mathcal{F})$.

Proof. Choose a diagram as in Lemma 46.18.1 part (3). It follows from the definitions that this reduces to the corresponding theorem for the morphisms of schemes $U \to V \to W$, the quasi-coherent sheaf $a^*\mathcal{F}$, and the point $u \in U$. Thus the theorem follows from the corresponding result for schemes which is More on Morphisms, Theorem 33.12.2. □

Lemma 46.18.4. *Let S be a scheme. Let $f : X \to Y$ and $Y \to Z$ be a morphism of algebraic spaces over S. Assume*

(1) *X is locally of finite presentation over Z,*
(2) *X is flat over Z,*
(3) *for every $z \in |Z|$ the fibre of X over z is flat over the fibre of Y over z, and*
(4) *Y is locally of finite type over Z.*

Then f is flat. If f is also surjective, then Y is flat over Z.

Proof. This is a special case of Theorem 46.18.3. □

Lemma 46.18.5. *Let S be a scheme. Let $f : X \to Y$ and $Y \to Z$ be morphisms of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume*

(1) *X is locally of finite presentation over Z,*
(2) *$\mathcal{F}$ an $\mathcal{O}_X$-module of finite presentation,*
(3) *$\mathcal{F}$ is flat over Z, and*
(4) *Y is locally of finite type over Z.*

Then the set

$$A = \{x \in |X| : \mathcal{F} \textit{ flat at } x \textit{ over } Y\}.$$

is open in $|X|$ and its formation commutes with arbitrary base change: If $Z' \to Z$ is a morphism of algebraic spaces, and A' is the set of points of $X' = X \times_Z Z'$ where $\mathcal{F}' = \mathcal{F} \times_Z Z'$ is flat over $Y' = Y \times_Z Z'$, then A' is the inverse image of A under the continuous map $|X'| \to |X|$.

Proof. One way to prove this is to translate the proof as given in More on Morphisms, Lemma 33.12.4 into the category of algebraic spaces. Instead we will prove this by reducing to the case of schemes instead. Namely, choose a diagram as in Lemma 46.18.1 part (3) such that a, b, and c are surjective. It follows from the definitions that this reduces to the corresponding theorem for the morphisms of schemes $U \to V \to W$, the quasi-coherent sheaf $a^*\mathcal{F}$, and the point $u \in U$. The only minor point to make is that given a morphism of algebraic spaces $Z' \to Z$ we choose a scheme W' and a surjective étale morphism $W' \to W \times_Z Z'$. Then we set $U' = W' \times_W U$ and $V' = W' \times_W V$. We write a', b', c' for the morphisms from U', V', W' to X', Y', Z'. In this case A, resp. A' are images of the open subsets of U, resp. U' associated to $a^*\mathcal{F}$, resp. $(a')^*\mathcal{F}'$. This indeed does reduce the lemma to More on Morphisms, Lemma 33.12.4. □

Lemma 46.18.6. *Let S be a scheme. Let $f : X \to Y$ and $Y \to Z$ be a morphism of algebraic spaces over S. Assume*

(1) *X is locally of finite presentation over Z,*
(2) *X is flat over Z, and*
(3) *Y is locally of finite type over Z.*

Then the set

$$\{x \in |X| : X \text{ flat at } x \text{ over } Y\}.$$

is open in $|X|$ and its formation commutes with arbitrary base change $Z' \to Z$.

Proof. This is a special case of Lemma 46.18.5. □

46.19. Slicing Cohen-Macaulay morphisms

Let S be a scheme. Let X be an algebraic space over S. Let $f_1, \ldots, f_r \in \Gamma(X, \mathcal{O}_X)$. In this case we denote $V(f_1, \ldots, f_r)$ the *closed subspace of X cut out by $f_1, \ldots, f_r$*. More precisely, we can define $V(f_1, \ldots, f_r)$ as the closed subspace of X corresponding to the quasi-coherent sheaf of ideals generated by $f_1, \ldots, f_r$, see Morphisms of Spaces, Lemma 42.16.1. Alternatively, we can choose a presentation $X = U/R$ and consider the closed subscheme $Z \subset U$ cut out by $f_1|U, \ldots, f_r|_U$. It is clear that Z is an R-invariant (see Groupoids, Definition 35.16.1) closed subscheme and we may set $V(f_1, \ldots, f_r) = Z/R_Z$.

Lemma 46.19.1. *Let S be a scheme. Consider a cartesian diagram*

$$\begin{tikzcd} X \arrow[d] & F \arrow[l, "p"'] \arrow[d] \\ Y & Spec(k) \arrow[l] \end{tikzcd}$$

where $X \to Y$ is a morphism of algebraic spaces over S which is flat and locally of finite presentation, and where k is a field over S. Let $f_1, \ldots, f_r \in \Gamma(X, \mathcal{O}_X)$ and $z \in |F|$ such that $f_1, \ldots, f_r$ map to a regular sequence in the local ring $\mathcal{O}_{F,\bar{z}}$. Then, after replacing X by an open subspace containing $p(z)$, the morphism

$$V(f_1, \ldots, f_r) \longrightarrow Y$$

is flat and locally of finite presentation.

Proof. Set $Z = V(f_1, \ldots, f_r)$. It is clear that $Z \to X$ is locally of finite presentation, hence the composition $Z \to Y$ is locally of finite presentation, see Morphisms of Spaces, Lemma 42.26.2. Hence it suffices to show that $Z \to Y$ is flat in a neighbourhood of $p(z)$. Let $k \subset k'$ be an extension field. Then $F' = F \times_{Spec(k)} Spec(k')$ is surjective and flat over F, hence we can find a point $z' \in |F'|$ mapping to z and the local ring map $\mathcal{O}_{F,\bar{z}} \to \mathcal{O}_{F',\bar{z}'}$ is flat, see Morphisms of Spaces, Lemma 42.27.7. Hence the image of $f_1, \ldots, f_r$ in $\mathcal{O}_{F',\bar{z}'}$ is a regular sequence too, see Algebra, Lemma 7.65.7. Thus, during the proof we may replace

k by an extension field. In particular, we may assume that $z \in |F|$ comes from a section $z : Spec(k) \to F$ of the structure morphism $F \to Spec(k)$.

Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme U and a surjective étale morphism $U \to X \times_Y V$. After possibly enlarging k once more we may assume that $Spec(k) \to F \to X$ factors through U (as $U \to X$ is surjective). Let $u : Spec(k) \to U$ be such a factorization and denote $v \in V$ the image of u. Note that the morphisms

$$U_v \times_{Spec(\kappa(v))} Spec(k) = U \times_V Spec(k) \to U \times_Y Spec(k) \to F$$

are étale (the first as the base change of $V \to V \times_Y V$ and the second as the base change of $U \to X$). Moreover, by construction the point $u : Spec(k) \to U$ gives a point of the left most space which maps to z on the right. Hence the elements $f_1, \ldots, f_r$ map to a regular sequence in the local ring on the right of the following map

$$\mathcal{O}_{U_v,u} \longrightarrow \mathcal{O}_{U_v \times_{Spec(\kappa(v))} Spec(k), \bar{u}} = \mathcal{O}_{U \times_V Spec(k), \bar{u}}.$$

But since the displayed arrow is flat (combine More on Flatness, Lemma 34.3.5 and Morphisms of Spaces, Lemma 42.27.7) we see from Algebra, Lemma 7.65.7 that $f_1, \ldots, f_r$ maps to a regular sequence in $\mathcal{O}_{U_v,u}$. By More on Morphisms, Lemma 33.16.2 we conclude that the morphism of schemes

$$V(f_1, \ldots, f_r) \times_X U = V(f_1|_U, \ldots, f_r|_U) \to V$$

is flat in an open neighbourhood U' of u. Let $X' \subset X$ be the open subspace corresponding to the image of $|U'| \to |X|$ (see Properties of Spaces, Lemmas 41.4.6 and 41.4.8). We conclude that $V(f_1, \ldots, f_r) \cap X' \to Y$ is flat (see Morphisms of Spaces, Definition 42.27.1) as we have the commutative diagram

$$\begin{tikzcd} V(f_1, \ldots, f_r) \times_X U' \arrow[r] \arrow[d, "a"'] & V \arrow[d, "b"] \\ V(f_1, \ldots, f_r) \cap X' \arrow[r] & Y \end{tikzcd}$$

with a, b étale and a surjective. □

46.20. The structure of quasi-finite morphisms

Lemma 46.20.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

1. *f is a closed immersion,*
2. *f is universally closed, unramified, and a monomorphism,*
3. *f is universally closed, unramified, and universally injective,*
4. *f is universally closed, locally of finite type, and a monomorphism,*
5. *f is universally closed, universally injective, locally of finite type, and formally unramified.*

Proof. The equivalence of (2) -- (5) follows immediately from Lemma 46.11.8. Moreover, if (2) -- (5) are satisfied then f is representable. Similarly, if (1) is satified then f is representable. Hence the result follows from the case of schemes, see Étale Morphisms, Lemma 37.7.2. □

46.21. Regular immersions

This section is the analogue of Divisors, Section 26.13 for morphisms of schemes. The reader is encouraged to read up on regular immersions of schemes in that section first.

In Divisors, Section 26.13 we defined four types of regular immersions for morphisms of schemes. Of these only three are (as far as we know) local on the target for the étale topology; as usual plain old regular immersions aren't. This is why for morphisms of algebraic spaces we cannot actually define regular immersions. (These kinds of annoyances prompted Grothendieck and his school to replace original notion of a regular immersion by a Koszul-regular immersions, see [**BGI71**, Exposee VII, Definition 1.4].) But we can define Koszul-regular, H_1-regular, and quasi-regular immersions. Another remark is that since Koszul-regular immersions are not preserved by arbitrary base change, we cannot use the strategy of Morphisms of Spaces, Section 42.3 to define them. Similarly, as Koszul-regular immersions are not étale local on the source, we cannot use Morphisms of Spaces, Lemma 42.21.1 to define them either. We replace this lemma instead by the following.

Lemma 46.21.1. *Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on the target. Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. Consider commutative diagrams*

$$\begin{tikzcd} X \times_Y V \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where V is a scheme and $V \to Y$ is étale. The following are equivalent

1. *for any diagram as above the projection $X \times_Y V \to V$ has property $\mathcal{P}$, and*
2. *for some diagram as above with $V \to Y$ surjective the projection $X \times_Y V \to V$ has property $\mathcal{P}$.*

If X and Y are representable, then this is also equivalent to f (as a morphism of schemes) having property $\mathcal{P}$.

Proof. Let us prove the equivalence of (1) and (2). The implication (1) $\Rightarrow$ (2) is immediate. Assume

$$\begin{tikzcd} X \times_Y V \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd} \qquad \begin{tikzcd} X \times_Y V' \arrow[r] \arrow[d] & V' \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

are two diagrams as in the lemma. Assume $V \to Y$ is surjective and $X \times_Y V \to V$ has propery $\mathcal{P}$. To show that (2) implies (1) we have to prove that $X \times_Y V' \to V'$ has $\mathcal{P}$. To do this consider the diagram

$$\begin{tikzcd} X \times_Y V \arrow[d] & (X \times_Y V) \times_X (X \times_Y V') \arrow[l] \arrow[r] \arrow[d] & X \times_Y V' \arrow[d] \\ V & V \times_Y V' \arrow[l] \arrow[r] & V' \end{tikzcd}$$

By our assumption that $\mathcal{P}$ is étale local on the source, we see that $\mathcal{P}$ is preserved under étale base change, see Descent, Lemma 31.18.2. Hence if the left vertical arrow has $\mathcal{P}$ the so does the middle vertical arrow. Since $U \times_X U' \to U'$ is surjective and étale (hence defines an étale covering of U') this implies (as $\mathcal{P}$ is assumed local for the étale topology on the target) that the left vertical arrow has $\mathcal{P}$.

If X and Y are representable, then we can take $\text{id}_Y : Y \to Y$ as our étale covering to see the final statement of the lemma is true. □

Note that ``being a Koszul-regular (resp. H_1-regular, resp. quasi-regular) immersion" is a property of morphisms of schemes which is fpqc local on the target, see Descent, Lemma 31.19.30. Hence the following definition now makes sense.

Definition 46.21.2. Let S be a scheme. Let $i : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say i is a *Koszul-regular immersion* if i is representable and the equivalent conditions of Lemma 46.21.1 hold with $\mathcal{P}(f)$ =``f is a Koszul-regular immersion".
(2) We say i is an H_1*-regular immersion* if i is representable and the equivalent conditions of Lemma 46.21.1 hold with $\mathcal{P}(f)$ =``f is an H_1-regular immersion".
(3) We say i is a *quasi-regular immersion* if i is representable and the equivalent conditions of Lemma 46.21.1 hold with $\mathcal{P}(f)$ =``f is a quasi-regular immersion".

Lemma 46.21.3. *Let S be a scheme. Let $i : Z \to X$ be an immersion of algebraic spaces over S. We have the following implications: i is Koszul-regular $\Rightarrow$ i is H_1-regular $\Rightarrow$ i is quasi-regular.*

Proof. Via the definition this lemma immediately reduces to Divisors, Lemma 26.13.2. □

To be continued...

46.22. Pseudo-coherent morphisms

This section is the analogue of More on Morphisms, Section 33.36 for morphisms of schemes. The reader is encouraged to read up on pseudo-coherent morphisms of schemes in that section first.

The property ``pseudo-coherent" of morphisms of schemes is étale local on the source-and-target. To see this use More on Morphisms, Lemmas 33.36.9 and 33.36.12 and Descent, Lemma 31.28.6. By Morphisms of Spaces, Lemma 42.21.1 we may define the notion of a pseudo-coherent morphism of algebraic spaces as follows and it agrees with the already existing notion defined in More on Morphisms, Section 33.36 when the algebraic spaces in question are representable.

Definition 46.22.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is *pseudo-coherent* if the equivalent conditions of Morphisms of Spaces, Lemma 42.21.1 hold with $\mathcal{P}$=``pseudo-coherent".
(2) Let $x \in |X|$. We say f is *pseudo-coherent at x* if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is pseudo-coherent.

Beware that a base change of a pseudo-coherent morphism is not pseudo-coherent in general.

Lemma 46.22.2. *A flat base change of a pseudo-coherent morphism is pseudo-coherent.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 33.36.3. ☐

Lemma 46.22.3. *A composition of pseudo-coherent morphisms of schemes is pseudo-coherent.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 33.36.4. ☐

Lemma 46.22.4. *A pseudo-coherent morphism is locally of finite presentation.*

Proof. Immediate from the definitions. ☐

Lemma 46.22.5. *A flat morphism which is locally of finite presentation is pseudo-coherent.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 33.36.6. ☐

Lemma 46.22.6. *Let $f : X \to Y$ be a morphism of algebraic spaces pseudo-coherent over a base algebraic space B. Then f is pseudo-coherent.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 33.36.7. ☐

Lemma 46.22.7. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If Y is locally Noetherian, then f is pseudo-coherent if and only if f is locally of finite type.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 33.36.8. ☐

46.23. Perfect morphisms

This section is the analogue of More on Morphisms, Section 33.37 for morphisms of schemes. The reader is encouraged to read up on perfect morphisms of schemes in that section first.

The property ``perfect'' of morphisms of schemes is étale local on the source-and-target. To see this use More on Morphisms, Lemmas 33.37.10 and 33.37.12 and Descent, Lemma 31.28.6. By Morphisms of Spaces, Lemma 42.21.1 we may define the notion of a perfect morphism of algebraic spaces as follows and it agrees with the already existing notion defined in More on Morphisms, Section 33.37 when the algebraic spaces in question are representable.

Definition 46.23.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is *perfect* if the equivalent conditions of Morphisms of Spaces, Lemma 42.21.1 hold with $\mathcal{P}$=``perfect''.
(2) Let $x \in |X|$. We say f is *perfect at x* if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is perfect.

Note that a perfect morphism is pseudo-coherent, hence locally of finite presentation. Beware that a base change of a perfect morphism is not perfect in general.

Lemma 46.23.2. *A flat base change of a perfect morphism is perfect.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 33.37.3. ☐

Lemma 46.23.3. *A composition of perfect morphisms of schemes is perfect.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 33.37.4. □

Lemma 46.23.4. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent*

(1) *f is flat and perfect, and*
(2) *f is flat and locally of finite presentation.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 33.37.5. □

46.24. Local complete intersection morphisms

This section is the analogue of More on Morphisms, Section 33.38 for morphisms of schemes. The reader is encouraged to read up on local complete intersection morphisms of schemes in that section first.

The property ``being a local complete intersection morphism" of morphisms of schemes is étale local on the source-and-target. To see this use More on Morphisms, Lemmas 33.38.11 and 33.38.12 and Descent, Lemma 31.28.6. By Morphisms of Spaces, Lemma 42.21.1 we may define the notion of a local complete intersection morphism of algebraic spaces as follows and it agrees with the already existing notion defined in More on Morphisms, Section 33.38 when the algebraic spaces in question are representable.

Definition 46.24.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is a *Koszul morphism*, or that f is a *local complete intersection morphism* if the equivalent conditions of Morphisms of Spaces, Lemma 42.21.1 hold with $\mathcal{P}(f) =$``f is a local complete intersection morphism".
(2) Let $x \in |X|$. We say f is *Koszul at x* if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is a local complete intersection morphism.

In some sense the defining property of a local complete intersection morphism is the result of the following lemma.

Lemma 46.24.2. *Let S be a scheme. Let $f : X \to Y$ be a local complete intersection morphism of algebraic spaces over S. Let P be an algebraic space smooth over Y. Let $U \to X$ be an étale morphism of algebraic spaces and let $i : U \to P$ an immersion of algebraic spaces over Y. Picture:*

$$\begin{tikzcd} X \arrow[dr] & U \arrow[l] \arrow[r, "i"'] \arrow[d] & P \arrow[dl] \\ & Y & \end{tikzcd}$$

Then i is a Koszul-regular immersion of algebraic spaces.

Proof. Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme W and a surjective étale morphism $W \to P \times_Y V$. Set $U' = U \times_P W$, which is a scheme étale over U. We have to show that $U' \to W$ is a Koszul-regular immersion of schemes, see Definition 46.21.2. By Definition 46.24.1 above the morphism of schemes $U' \to V$ is a local complete intersection morphism. Hence the result follows from More on Morphisms, Lemma 33.38.3. □

It seems like a good idea to collect here some properties in common with all Koszul morphisms.

Lemma 46.24.3. *Let S be a scheme. Let $f : X \to Y$ be a local complete intersection morphism of algebraic spaces over S. Then*

(1) *f is locally of finite presentation,*
(2) *f is pseudo-coherent, and*
(3) *f is perfect.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 33.38.4. $\square$

Beware that a base change of a Koszul morphism is not Koszul in general.

Lemma 46.24.4. *A flat base change of a local complete intersection morphism is a local complete intersection morphism.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 33.38.6. $\square$

Lemma 46.24.5. *A composition of local complete intersection morphisms of schemes is a local complete intersection morphism.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 33.38.7. $\square$

Lemma 46.24.6. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent*

(1) *f is flat and a local complete intersection morphism, and*
(2) *f is syntomic.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 33.38.8. $\square$

Lemma 46.24.7. *Let S be a scheme. Consider a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

of algebraic spaces over S. Assume that both p and q are flat and locally of finite presentation. Then there exists an open subspace $U(f) \subset X$ such that $|U(f)| \subset |X|$ is the set of points where f is Koszul. Moreover, for any morphism of algebraic spaces $Z' \to Z$, if $f' : X' \to Y'$ is the base change of f by $Z' \to Z$, then $U(f')$ is the inverse image of $U(f)$ under the projection $X' \to X$.

Proof. This lemma is the analogue of More on Morphisms, Lemma 33.38.13 and in fact we will deduce the lemma from it. By Definition 46.24.1 the set $\{x \in |X| : f \text{ is Koszul at } x\}$ is open in $|X|$ hence by Properties of Spaces, Lemma 41.4.8 it corresponds to an open subspace $U(f)$ of X. Hence we only need to prove the final statement.

Choose a scheme W and a surjective étale morphism $W \to Z$. Choose a scheme V and a surjective étale morphism $V \to W \times_Z Y$. Choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. Finally, choose a scheme W' and a surjective étale morphism $W' \to W \times_Z Z'$. Set $V' = W' \times_W V$ and $U' = W' \times_W U$, so that we obtain surjective étale morphisms $V' \to Y'$ and $U' \to X'$. We will use without further mention an étale morphism of algebraic spaces induces an open map of associated topological spaces (see Properties of Spaces, Lemma 41.13.7). Note that by definition $U(f)$ is the image in $|X|$ of the set T of points in U where the morphism of schemes $U \to V$ is Koszul. Similarly, $U(f')$ is the image in $|X'|$ of the set T' of points in U' where the morphism of schemes $U' \to V'$ is Koszul. Now, by construction the diagram

$$\begin{tikzcd} U' \arrow[r] \arrow[d] & U \arrow[d] \\ V' \arrow[r] & V \end{tikzcd}$$

is cartesian (in the category of schemes). Hence the aforementioned More on Morphisms, Lemma 33.38.13 applies to show that T' is the inverse image of T. Since $|U'| \to |X'|$ is surjective this implies the lemma. □

Lemma 46.24.8. *Let S be a scheme. Let $f : X \to Y$ be a local complete intersection morphism of algebraic spaces over S. Then f is unramified if and only if f is formally unramified and in this case the conormal sheaf $\mathcal{C}_{X/Y}$ is finite locally free on X.*

Proof. This follows from the corresponding result for morphisms of schemes, see More on Morphisms, Lemma 33.38.14, by étale localization, see Lemma 46.12.11. (Note that in the situation of this lemma the morphism $V \to U$ is unramified and a local complete intersection morphism by definition.) □

Lemma 46.24.9. *Let S be a scheme. Let $Z \to Y \to X$ be formally unramified morphisms of algebraic spaces over S. Assume that $Z \to Y$ is a local complete intersection morphism. The exact sequence*

$$0 \to i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

of Lemma 46.5.6 is short exact.

Proof. Choose a scheme U and a surjective étale morphism $U \to X$. Choose a scheme V and a surjective étale morphism $V \to U \times_X Y$. Choose a scheme W and a surjective étale morphism $W \to V \times_Y Z$. By Lemma 46.12.11 the morphisms $W \to V$ and $V \to U$ are formally unramified. Moreover the sequence $i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$ restricts to the corresponding sequence $i^*\mathcal{C}_{V/U} \to \mathcal{C}_{W/U} \to \mathcal{C}_{W/V} \to 0$ for $W \to V \to U$. Hence the result follows from the result for schemes (More on Morphisms, Lemma 33.38.15) as by definition the morphism $W \to V$ is a local complete intersection morphism. □

46.25. Exact sequences of differentials and conormal sheaves

In this section we collect some results on exact sequences of conormal sheaves and sheaves of differentials. In some sense these are all realizations of the triangle of cotangent complexes associated to composable morphisms of algebraic spaces.

In the sequences below each of the maps are as constructed in either Lemma 46.6.6 or Lemma 46.12.8. Let S be a scheme. Let $g : Z \to Y$ and $f : Y \to X$ be morphisms of algebraic spaces over S.

(1) There is a canonical exact sequence

$$g^*\Omega_{Y/X} \to \Omega_{Z/X} \to \Omega_{Z/Y} \to 0,$$

see Lemma 46.6.8. If $g : Z \to Y$ is formally smooth, then this sequence is a short exact sequence, see Lemma 46.16.11.

(2) If g is formally unramified, then there is a canonical exact sequence
$$\mathcal{C}_{Z/Y} \to g^*\Omega_{Y/X} \to \Omega_{Z/X} \to 0,$$
see Lemma 46.12.13. If $f \circ g : Z \to X$ is formally smooth, then this sequence is a short exact sequence, see Lemma 46.16.12.

(3) if g and $f \circ g$ are formally unramified, then there is a canonical exact sequence
$$\mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to g^*\Omega_{Y/X} \to 0,$$
see Lemma 46.12.14. If $f : Y \to X$ is formally smooth, then this sequence is a short exact sequence, see Lemma 46.16.13.

(4) if g and f are formally unramified, then there is a canonical exact sequence
$$g^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0.$$
see Lemma 46.12.15. If $g : Z \to Y$ is a local complete intersection morphism, then this sequence is a short exact sequence, see Lemma 46.24.9.

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CHAPTER 47

Quot and Hilbert Spaces

47.1. Introduction

The purpose of this chapter is to write about Quot and Hilbert functors and to prove that these are algebraic spaces provided certain technical conditions are satisfied. In this chapter we will discuss this in the setting of algebraic space. A reference is Grothendieck's lectures, see [**Gro95a**], [**Gro95b**], [**Gro95e**], [**Gro95f**], [**Gro95c**], and [**Gro95d**]. Another reference is the paper [**OS03a**], allthough this paper discusses the more general case of Quot and Hilbert spaces associated to a morphism of algebraic stacks which we will discuss in another chapter, see (insert future reference here).

In the case of Hilbert spaces there is a more general notion of ``Hilbert stacks'' which we will discuss in a separate chapter, see (insert future reference here).

47.2. Conventions

The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $Spec(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

47.3. When is a morphism an isomorphism?

More generally we can ask: ``When does a morphism have property $\mathcal{P}$?'' A more precise question is the following. Suppose given a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

of algebraic spaces. Does there exist a monomorphism of algebraic spaces $W \to Z$ with the following two properties:

(1) the base change $f_W : X_W \to Y_W$ has property $\mathcal{P}$, and
(2) any morphism $Z' \to Z$ of algebraic spaces factors through W if and only if the base change $f_{Z'} : X_{Z'} \to Y_{Z'}$ has property $\mathcal{P}$.

In many cases, if $W \to Z$ exists, then it is an immersion, open immersion, or closed immersion.

The answer to this question may depend on auxiliary properties of the morphisms f, p, and q. An example is $\mathcal{P}(f)$ =``$f$ is flat'' which we have discussed for morphisms of schemes in the case $Y = S$ in great detail in the chapter ``More on Flatness", starting with More on Flatness, Section 34.20.

Lemma 47.3.1. *Consider a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

of algebraic spaces. Assume that p is locally of finite type and closed. Then there exists an open subspace $W \subset Z$ such that a morphism $Z' \to Z$ factors through W if and only if the base change $f_{Z'} : X_{Z'} \to Y_{Z'}$ is unramified.

Proof. By Morphisms of Spaces, Lemma 42.34.10 there exists an open subspace $U(f) \subset X$ which is the set of points where f is unramified. Moreover, formation of $U(f)$ commutes with arbitrary base change. Let $W \subset Z$ be the open subspace (see Properties of Spaces, Lemma 41.4.8) with underlying set of points

$$|W| = |Z| \setminus |p|\left(|X| \setminus |U(f)|\right)$$

i.e., $z \in |Z|$ is a point of W if and only if f is unramified at every point of X above z. Note that this is open because we assumed that p is closed. Since the formation of $U(f)$ commutes with arbitrary base change we immediately see (using Properties of Spaces, Lemma 41.4.9) that W has the desired universal property. $\square$

Lemma 47.3.2. *Consider a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

of algebraic spaces. Assume that

1. *p is locally of finite type,*
2. *p is closed, and*
3. *$p_2 : X \times_Y X \to Z$ is closed.*

Then there exists an open subspace $W \subset Z$ such that a morphism $Z' \to Z$ factors through W if and only if the base change $f_{Z'} : X_{Z'} \to Y_{Z'}$ is unramified and universally injective.

Proof. After replacing Z by the open subspace found in Lemma 47.3.1 we may assume that f is already unramified; note that this does not destroy assumption (2) or (3). By Morphisms of Spaces, Lemma 42.34.9 we see that $\Delta_{X/Y} : X \to X \times_Y X$ is an open immersion. This remains true after any base change. Hence by Morphisms of Spaces, Lemma 42.18.2 we see that $f_{Z'}$ is universally injective if and only if the base change of the diagonal $X_{Z'} \to (X \times_Y X)_{Z'}$ is an isomorphism. Let $W \subset Z$ be the open subspace (see Properties of Spaces, Lemma 41.4.8) with underlying set of points

$$|W| = |Z| \setminus |p_2|\left(|X \times_Y X| \setminus \mathrm{Im}(|\Delta_{X/Y}|)\right)$$

i.e., $z \in |Z|$ is a point of W if and only if the fibre of $|X \times_Y X| \to |Z|$ over z is in the image of $|X| \to |X \times_Y X|$. Then it is clear from the discussion above that the restriction $p^{-1}(W) \to q^{-1}(W)$ of f is unramified and universally injective.

Conversely, suppose that $f_{Z'}$ is unramified and universally injective. In order to show that $Z' \to Z$ factors through W it suffices to show that $|Z'| \to |Z|$ has image contained in $|W|$, see Properties of Spaces, Lemma 41.4.9. Hence it suffices to prove the result when Z' is the spectrum of a field. Denote $z \in |Z|$ the image of $|Z'| \to |Z|$. The discussion above shows that

$$|X_{Z'}| \longrightarrow |(X \times_Y X)_{Z'}|$$

is surjective. By Properties of Spaces, Lemma 41.4.3 in the commutative diagram

$$\begin{tikzcd} |X_{Z'}| \arrow[r] \arrow[d] & |(X \times_Y X)_{Z'}| \arrow[d] \\ |p|^{-1}(\{z\}) \arrow[r] & |p_2|^{-1}(\{z\}) \end{tikzcd}$$

the vertical arrows are surjective. It follows that $z \in |W|$ as desired. $\square$

Lemma 47.3.3. *Consider a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

of algebraic spaces. Assume that

(1) *p is locally of finite type,*
(2) *p is universally closed, and*
(3) *$q : Y \to Z$ is separated.*

Then there exists an open subspace $W \subset Z$ such that a morphism $Z' \to Z$ factors through W if and only if the base change $f_{Z'} : X_{Z'} \to Y_{Z'}$ is a closed immersion.

Proof. We will use the characterization of closed immersions as universally closed, unramified, and universally injective morphisms, see More on Morphisms of Spaces, Lemma 46.20.1. First, note that since p is universally closed and q is separated, we see that f is universally closed, see Morphisms of Spaces, Lemma 42.36.5. It follows that any base change of f is universally closed, see Morphisms of Spaces, Lemma 42.10.3. Thus to finish the proof of the lemma it suffices to prove that the assumptions of Lemma 47.3.2 are satisfied. The projection $\text{pr}_0 : X \times_Y X \to X$ is universally closed as a base change of f, see Morphisms of Spaces, Lemma 42.10.3. Hence $X \times_Y X \to Z$ is universally closed as a composition of universally closed morphisms (see Morphisms of Spaces, Lemma 42.10.4). This finishes the proof of the lemma. $\square$

Lemma 47.3.4. *Consider a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

of algebraic spaces. Assume that

(1) *p is locally of finite presentation,*
(2) *p is flat,*
(3) *p is closed, and*
(4) *q is locally of finite type.*

Then there exists an open subspace $W \subset Z$ such that a morphism $Z' \to Z$ factors through W if and only if the base change $f_{Z'} : X_{Z'} \to Y_{Z'}$ is flat.

Proof. By More on Morphisms of Spaces, Lemma 46.18.6 the set

$$A = \{x \in |X| : X \text{ flat at } x \text{ over } Y\}.$$

is open in $|X|$ and its formation commutes with arbitrary base change. Let $W \subset Z$ be the open subspace (see Properties of Spaces, Lemma 41.4.8) with underlying set of points

$$|W| = |Z| \setminus |p|(|X| \setminus A)$$

i.e., $z \in |Z|$ is a point of W if and only if the whole fibre of $|X| \to |Z|$ over z is contained in A. This is open because p is closed. Since the formation of A commutes with arbitrary base change it follows that W works. $\square$

Lemma 47.3.5. *Consider a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

of algebraic spaces. Assume that

(1) *p is locally of finite presentation,*
(2) *p is flat,*
(3) *p is closed,*
(4) *q is locally of finite type, and*
(5) *q is closed.*

Then there exists an open subspace $W \subset Z$ such that a morphism $Z' \to Z$ factors through W if and only if the base change $f_{Z'} : X_{Z'} \to Y_{Z'}$ is surjective and flat.

Proof. By Lemma 47.3.4 we may assume that f is flat. Note that f is locally of finite presentation by Morphisms of Spaces, Lemma 42.26.9. Hence f is open, see Morphisms of Spaces, Lemma 42.27.5. Let $W \subset Z$ be the open subspace (see Properties of Spaces, Lemma 41.4.8) with underlying set of points

$$|W| = |Z| \setminus |q|\left(|Y| \setminus |f|(|X|)\right).$$

in other words for $z \in |Z|$ we have $z \in |W|$ if and only if the whole fibre of $|Y| \to |Z|$ over z is in the image of $|X| \to |Y|$. Since q is closed this set is open in $|Z|$. The morphism $X_W \to Y_W$ is surjective by construction. Finally, suppose that $X_{Z'} \to Y_{Z'}$ is surjective. In order to show that $Z' \to Z$ factors through W it suffices to show that $|Z'| \to |Z|$ has image contained in $|W|$, see Properties of Spaces, Lemma 41.4.9. Hence it suffices to prove the result when Z' is the spectrum of a field. Denote $z \in |Z|$ the image of $|Z'| \to |Z|$. By Properties of Spaces, Lemma 41.4.3 in the commutative diagram

$$\begin{tikzcd} {|X_{Z'}|} \arrow[r] \arrow[d] & {|Y_{Z'}|} \arrow[d] \\ {|p|^{-1}(\{z\})} \arrow[r] & {|q|^{-1}(\{z\})} \end{tikzcd}$$

the vertical arrows are surjective. It follows that $z \in |W|$ as desired. $\square$

Lemma 47.3.6. *Consider a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

of algebraic spaces. Assume that

(1) *p is locally of finite presentation,*
(2) *p is flat,*
(3) *p is universally closed,*
(4) *q is locally of finite type,*
(5) *q is closed, and*
(6) *q is separated.*

Then there exists an open subspace $W \subset Z$ such that a morphism $Z' \to Z$ factors through W if and only if the base change $f_{Z'} : X_{Z'} \to Y_{Z'}$ is an isomorphism.

Proof. By Lemma 47.3.5 there exists an open subspace $W_1 \subset Z$ such that $f_{Z'}$ is surjective and flat if and only if $Z' \to Z$ factors through W_1. By Lemma 47.3.3 there exists an open subspace $W_2 \subset Z$ such that $f_{Z'}$ is a closed immersion if and only if $Z' \to Z$ factors through W_2. We claim that $W = W_1 \cap W_2$ works. Certainly, if $f_{Z'}$ is an isomorphism, then $Z' \to Z$ factors through W. Hence it suffices to show that f_W is an isomorphism. By construction f_W is a surjective flat closed immersion. In particular f_W is representable. Since a surjective flat closed immersion of schemes is an isomorphism (see Morphisms, Lemma 24.25.1) we win. (Note that actually f_W is locally of finite presentation, whence open, so you can avoid the use of this lemma if you like.) $\square$

Lemma 47.3.7. *Consider a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

of algebraic spaces. Assume that

(1) *p is flat and locally of finite presentation,*
(2) *p is closed, and*
(3) *q is flat and locally of finite presentation,*

Then there exists an open subspace $W \subset Z$ such that a morphism $Z' \to Z$ factors through W if and only if the base change $f_{Z'} : X_{Z'} \to Y_{Z'}$ is a local complete intersection morphism.

Proof. By More on Morphisms of Spaces, Lemma 46.24.7 there exists an open subspace $U(f) \subset X$ which is the set of points where f is Koszul. Moreover, formation of $U(f)$ commutes with arbitrary base change. Let $W \subset Z$ be the open subspace (see Properties of Spaces, Lemma 41.4.8) with underlying set of points

$$|W| = |Z| \setminus |p| (|X| \setminus |U(f)|)$$

i.e., $z \in |Z|$ is a point of W if and only if f is Koszul at every point of X above z. Note that this is open because we assumed that p is closed. Since the formation of $U(f)$ commutes with arbitrary base change we immediately see (using Properties of Spaces, Lemma 41.4.9) that W has the desired universal property. $\square$

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CHAPTER 48

Algebraic Spaces over Fields

48.1. Introduction

This chapter is the analogue of the chapter on varieties in the setting of algebraic spaces. A reference for algebraic spaces is [**Kol96**].

48.2. Conventions

The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $Spec(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

48.3. Geometric components

Lemma 48.3.1. *Let k be an algebraically closed field. Let A, B be strictly henselian local k-algebras with residue field equal to k. Let C be the strict henselization of $A \otimes_k B$ at the maximal ideal $\mathfrak m_A \otimes_k B + A \otimes_k \mathfrak m_B$. Then the minimal primes of C correspond 1-to-1 to pairs of minimal primes of A and B.*

Proof. First note that a minimal prime $\mathfrak r$ of C maps to a minimal prime $\mathfrak p$ in A and to a minimal prime $\mathfrak q$ of B because the ring maps $A \to C$ and $B \to C$ are flat (by going down for flat ring map Algebra, Lemma 7.35.17). Hence it suffices to show that the strict henselization of $(A/\mathfrak p \otimes_k B/\mathfrak q)_{\mathfrak m_A \otimes_k B + A \otimes_k \mathfrak m_B}$ has a unique minimal prime ideal. By Algebra, Lemma 7.139.22 the rings $A/\mathfrak p$, $B/\mathfrak q$ are strictly henselian. Hence we may assume that A and B are strictly henselian local domains and our goal is to show that C has a unique minimal prime. By Properties of Spaces, Lemma 41.21.3. we see that the integral closure A' of A in its fraction field is a normal local domain with residue field k and similarly for the integral closure B' of B into its fraction field. By Algebra, Lemma 7.147.4 we see that $A' \otimes_k B'$ is a normal ring. Hence its localization

$$R = (A' \otimes_k B')_{\mathfrak m_{A'} \otimes_k B' + A' \otimes_k \mathfrak m_{B'}}$$

is a normal local domain. Note that $A \otimes_k B \to A' \otimes_k B'$ is integral (hence gong up holds -- Algebra, Lemma 7.32.20) and that $\mathfrak m_{A'} \otimes_k B' + A' \otimes_k \mathfrak m_{B'}$ is the unique maximal ideal of $A' \otimes_k B'$ lying over $\mathfrak m_A \otimes_k B + A \otimes_k \mathfrak m_B$. Hence we see that

$$R = (A' \otimes_k B')_{\mathfrak m_A \otimes_k B + A \otimes_k \mathfrak m_B}$$

by Algebra, Lemma 7.36.11. It follows that

$$(A \otimes_k B)_{\mathfrak m_A \otimes_k B + A \otimes_k \mathfrak m_B} \longrightarrow R$$

is integral. We conclude that R is the integral closure of $(A \otimes_k B)_{\mathfrak{m}_A \otimes_k B + A \otimes_k \mathfrak{m}_B}$ in its fraction field, and by Properties of Spaces, Lemma 41.21.3 once again we conclude that C has a unique prime ideal. $\square$

48.4. Schematic locus

Lemma 48.4.1. *Let k be a field. Let X be an algebraic space over $Spec(k)$. If X is locally of finite type over k and has dimension 0, then X is a scheme.*

Proof. Let U be an affine scheme and let $U \to X$ be an étale morphism. Set $R = U \times_X U$. Note that the two projection morphisms $s, t : R \to U$ are étale morphisms of schemes. By Properties of Spaces, Definition 41.8.2 we see that $\dim(U) = 0$ and similarly $\dim(R) = 0$. On the other hand, the morphism $U \to Spec(k)$ is locally of finite type as the composition of the étale morphism $U \to X$ and $X \to Spec(k)$, see Morphisms of Spaces, Lemmas 42.22.2 and 42.35.9. Similarly, $R \to Spec(k)$ is locally of finite type. Hence by Varieties, Lemma 28.13.2 we see that U and R are disjoint unions of spectra of local Artinian k-algebras A finite over k. In particular, as

$$R = U \times_X U \longrightarrow U \times_{Spec(k)} U$$

is a monomorphism, we see that R is a finite union of spectra of finite k-algebras. It follows that R is affine, see Schemes, Lemma 21.6.8. Applying Varieties, Lemma 28.13.2 once more we see that R is finite over k. Hence s, t are finite, see Morphisms, Lemma 24.42.12. Thus Groupoids, Proposition 35.19.8 shows that the open subspace U/R of X is an affine scheme. Since the schematic locus of X is an open subspace (see Properties of Spaces, Lemma 41.10.1), and since $U \to X$ was an arbtrary étale morphisms from an affine scheme we conclude that X is a scheme. $\square$

Lemma 48.4.2. *Let k be a field. Let X be an algebraic space over k. The following are equivalent*

(1) *X is locally quasi-finite over k,*
(2) *X is locally of finite type over k and has dimension 0,*
(3) *X is a scheme and is locally quasi-finite over k,*
(4) *X is a scheme and is locally of finite type over k and has dimension 0, and*
(5) *X is a disjoint union of spectra of Artinian local k-algebras A over k with $\dim_k(A) < \infty$.*

Proof. Because we are over a field relative dimension of X/k is the same as the dimension of X. Hence by Morphisms of Spaces, Lemma 42.31.6 we see that (1) and (2) are equivalent. Hence it follows from Lemma 48.4.1 (and trivial implications) that (1) -- (4) are equivalent. Finally, Varieties, Lemma 28.13.2 shows that (1) -- (4) are equivalent with (5). $\square$

Lemma 48.4.3. *Let k be a field. Let $f : X \to Y$ be a monomorphism of algebraic spaces over k. If Y is locally quasi-finite over k so is X.*

Proof. Assume Y is locally quasi-finite over k. By Lemma 48.4.2 we see that $Y = \coprod Spec(A_i)$ where each A_i is an Artinian local ring finite over k. By Decent Spaces, Lemma 43.14.1 we see that X is a scheme. Consider $X_i = f^{-1}(Spec(A_i))$. Then X_i has either one or zero points. If X_i has zero points there is nothing to prove. If X_i has one point, then $X_i = Spec(B_i)$ with B_i a zero dimensional local ring and $A_i \to B_i$ is an epimorphism of rings. In particular $A_i/\mathfrak{m}_{A_i} = B_i/\mathfrak{m}_{A_i}B_i$ and we see that $A_i \to B_i$ is surjective by Nakayama's lemma, Algebra, Lemma 7.14.5 (because $\mathfrak{m}_{A_i}$ is a nilpotent ideal!). Thus B_i

is a finite local k-algebra, and we conclude by Lemma 48.4.2 that $X \to Spec(k)$ is locally quasi-finite. ☐

48.5. Spaces smooth over fields

Lemma 48.5.1. *Let k be a field. Let X be an algebraic space smooth over k. Then X is a regular algebraic space.*

Proof. Choose a scheme U and a surjective étale morphism $U \to X$. The morphism $U \to Spec(k)$ is smooth as a composition of an étale (hence smooth) morphism and a smooth morphism (see Morphisms of Spaces, Lemmas 42.35.6 and 42.33.2). Hence U is regular by Varieties, Lemma 28.15.3. By Properties of Spaces, Definition 41.7.2 this means that X is regular. ☐

48.6. Other chapters

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CHAPTER 49

Cohomology of Algebraic Spaces

49.1. Introduction

In this chapter we write about cohomology of algebraic spaces. This mean in particular cohomology of quasi-coherent sheaves, i.e., we prove analogues of the results in the chapter entitled ``Coherent Cohomology''. Some of the results in this chapter can be found in [**Knu71b**].

49.2. Conventions

The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $Spec(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

49.3. Derived category of quasi-coherent modules

Let S be a scheme. In Descent, Lemma 31.8.1 we proved that the category $D_{QCoh}(\mathcal{O}_S)$ can be defined in terms of complexes of $\mathcal{O}_S$-modules on the scheme S or by complexes of $\mathcal{O}$-modules on the small étale site of S. Hence the following definition is compatible with the definition in the case of schemes.

Definition 49.3.1. Let S be a scheme. Let X be an algebraic space over S. The *derived category of $\mathcal{O}_X$-modules with quasi-coherent cohomology sheaves* is denoted $D_{QCoh}(\mathcal{O}_X)$.

This makes sense by Properties of Spaces, Lemma 41.26.7 and Derived Categories, Section 11.12.

49.4. Higher direct images

Let S be a scheme. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of representable algebraic spaces X and Y over S. Let $\mathcal{F}$ be a quasi-coherent module on X. By Descent, Lemma 31.6.15 the sheaf $R^i f_*\mathcal{F}$ agrees with the usual higher direct image (computed for the Zariski topologies) if we think of X and Y as schemes.

More generally, suppose $f : X \to Y$ is a representable, quasi-compact, and quasi-separated morphism of algebraic spaces over S. Let V be a scheme and let $V \to Y$ be an étale surjective morphism. Let $U = V \times_Y X$ and let $f' : U \to V$ be the base change of f. Then for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have

$$R^i f'_*(\mathcal{F}|_U) = (R^i f_*\mathcal{F})|_V, \tag{49.4.0.1}$$

see Properties of Spaces, Lemma 41.23.2. And because $f' : U \to V$ is a quasi-compact and quasi-separated morphism of schemes, by the remark of the preceding paragraph we may compute $R^i f'_*(\mathcal{F}|_U)$ by thinking of $\mathcal{F}|_U$ as a quasi-coherent sheaf on the scheme U, and f' as a morphism of schemes. We will frequently use this without further mention.

Next, we prove that higher direct images of quasi-coherent sheaves are quasi-coherent for any quasi-compact and quasi-separated morphism of algebraic spaces. In the proof we use a trick; a ``better'' proof would use a relative Cech complex, as discussed in Sheaves on Stacks, Sections 58.17 and 58.18 ff.

Lemma 49.4.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is quasi-compact and quasi-separated, then $R^i f_*$ transforms quasi-coherent $\mathcal{O}_X$-modules into quasi-coherent $\mathcal{O}_Y$-modules and induces a functor $Rf_* : D^+_{QCoh}(\mathcal{O}_X) \to D^+_{QCoh}(\mathcal{O}_Y)$.*

Proof. Let $V \to Y$ be an étale morphism where V is an affine scheme. Set $U = V \times_Y X$ and denote $f' : U \to V$ the induced morphism. Let $\mathcal{I}^\bullet$ be a bounded above complex of injective $\mathcal{O}_X$-modules. By Properties of Spaces, Lemma 41.23.2 we have

$$f'_*(\mathcal{I}^\bullet|_U) = (f_*\mathcal{I}^\bullet)|_V.$$

The complex $\mathcal{I}^\bullet|_U$ is a bounded below complex of injective $\mathcal{O}_U$-modules, see Cohomology on Sites, Lemma 19.8.1. Since the property of being a quasi-coherent module is local in the étale topology on Y (see Properties of Spaces, Lemma 41.26.6) we may replace Y by V, i.e., we may assume Y is an affine scheme.

Assume Y is affine. Since f is quasi-compact we see that X is quasi-compact. Thus we may choose an affine scheme U and a surjective étale morphism $g : U \to X$, see Properties of Spaces, Lemma 41.6.3. Note that the morphism $g : U \to X$ is representable, separated and quasi-compact because X is quasi-separated. Hence the lemma holds for g (either by the discussion above the lemma or by applying the reduction in the first paragraph of this proof). It also holds for $f \circ g : U \to Y$ (as this is a morphism of affine schemes). Moreover, for an injective $\mathcal{O}_U$-module $\mathcal{I}$ the module $g_*\mathcal{I}$ is injective (see Homology, Lemma 10.22.1) whence $Rf_* \circ Rg_* = R(g \circ f)_*$, see Derived Categories, Lemma 11.21.1.

In the situation described in the previous paragraph we will show by induction on n that IH_n: for any quasi-coherent sheaf $\mathcal{F}$ on X the sheaves $R^i f\mathcal{F}$ are quasi-coherent for $i \leq n$. The case $n = 0$ follows from Morphisms of Spaces, Lemma 42.15.2. Assume IH_n. In the rest of the proof we show that IH_{n+1} holds.

The hypothesis IH_n implies, via the spectral sequence of Derived Categories, Lemma 11.20.3, that $R^i f_*\mathcal{G}^\bullet$ is quasi-coherent for $i \leq n$ if $\mathcal{G}^\bullet$ is a complex of $\mathcal{O}_X$-modules with $H^j(\mathcal{G}^\bullet) = 0$ for $j < 0$ and $H^j(\mathcal{G}^\bullet)$ is quasi-coherent for all j. Suppose $\mathcal{H}$ is a quasi-coherent $\mathcal{O}_U$-module. Consider the distinguished triangle

$$g_*\mathcal{H} \to Rg_*\mathcal{H} \to \tau_{\geq 1} Rg_*\mathcal{H} \to g_*\mathcal{H}[1].$$

Note that $Rg_*\mathcal{H}$ and $Rf_*Rg_*\mathcal{H} = R(f \circ g)_*\mathcal{H}$ have quasi-coherent cohomology sheaves (see above). Combined with the remark above we conclude that IH_n implies that $R^i f_*g_*\mathcal{H}$ is quasi-coherent for $i \leq n+1$.

Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Consider the exact sequence

$$0 \to \mathcal{F} \to g_*g^*\mathcal{F} \to \mathcal{G} \to 0$$

where $\mathcal{G}$ is the cokernel of the first map. Applying the long exact cohomology sequence we obtain

$$R^n f_*g_*g^*\mathcal{F} \to R^n f_*\mathcal{G} \to R^{n+1} f_*\mathcal{F} \to R^{n+1} f_*g_*g^*\mathcal{F} \to R^{n+1} f_*\mathcal{G}$$

By the above we see that $R^{n+1}f_*g_*g^*\mathcal{F}$ is quasi-coherent. Thus $R^{n+1}f_*\mathcal{F}$ has a 2-step filtration where the first step is quasi-coherent and the second a subsheaf of a quasi-coherent sheaf. Applying this to $R^{n+1}f_*\mathcal{G}$ we find an exact sequence $0 \to \mathcal{A} \to R^{n+1}f_*\mathcal{G} \to \mathcal{B}$ wit $\mathcal{A}$, $\mathcal{B}$ quasi-coherent $\mathcal{O}_Y$-modules. Then the kernel $\mathcal{K}$ of $R^{n+1}f_*g_*g^*\mathcal{F} \to R^{n+1}f_*\mathcal{G} \to \mathcal{B}$ is quasi-coherent, whereupon we obtain a map $\mathcal{K} \to \mathcal{A}$ whose kernel $\mathcal{K}'$ is quasi-coherent too. Hence $R^{n+1}f_*\mathcal{F}$ sits in an exact sequence

$$R^n f_*g_*g^*\mathcal{F} \to R^n f_*\mathcal{G} \to R^{n+1}f_*\mathcal{F} \to \mathcal{K}' \to 0$$

and we win. □

49.5. Colimits and cohomology

The following lemma in particular applies to diagrams of quasi-coherent sheaves.

Lemma 49.5.1. *Let S be a scheme. Let X be an algebraic space over S. If X is quasi-compact and quasi-separated, then*

$$colim_i\, H^p(X, \mathcal{F}_i) \longrightarrow H^p(X, colim_i\, \mathcal{F}_i)$$

for every filtered diagram of abelian sheaves on $X_{étale}$.

Proof. This follows from Cohomology on Sites, Lemma 19.16.2. Namely, let $\mathcal{B} \subset Ob(X_{spaces,étale})$ be the set of quasi-compact and quasi-separated spaces étale over X. Note that if $U \in \mathcal{B}$ then, because U is quasi-compact, the collection of finite coverings $\{U_i \to U\}$ with $U_i \in \mathcal{B}$ is cofinal in the set of coverings of U in $X_{étale}$. By Morphisms of Spaces, Lemma 42.9.9 the set $\mathcal{B}$ satisfies all the assumptions of Cohomology on Sites, Lemma 19.16.2. Since $X \in \mathcal{B}$ we win. □

49.6. The alternating Čech complex

Let S be a scheme. Let $f : U \to X$ be an étale morphism of algebraic spaces over S. The functor

$$j : U_{spaces,étale} \longrightarrow X_{spaces,étale}, \quad V/U \longmapsto V/X$$

induces an equivalence of $U_{spaces,étale}$ with the localization $X_{spaces,étale}/U$, see Properties of Spaces, Section 41.24. Hence there exist functors

$$f_! : Ab(U_{étale}) \longrightarrow Ab(X_{étale}), \quad f_! : Mod(\mathcal{O}_U) \longrightarrow Mod(\mathcal{O}_X),$$

which are left adjoint to

$$f^{-1} : Ab(X_{étale}) \longrightarrow Ab(U_{étale}), \quad f^* : Mod(\mathcal{O}_X) \longrightarrow Mod(\mathcal{O}_U)$$

see Modules on Sites, Section 16.19. Warning: This functor, a priori, has nothing to do with cohomology with compact supports! We dubbed this functor ``extension by zero" in the reference above. Note that the two versions of $f_!$ agree as $f^* = f^{-1}$ for sheaves of $\mathcal{O}_X$-modules.

As we are going to use this construction below let us recall some of its properties. Given an abelian sheaf $\mathcal{G}$ on $U_{étale}$ the sheaf $f_!$ is the sheafification of the presheaf

$$V/X \longmapsto f_!\mathcal{G}(V) = \bigoplus\nolimits_{\varphi \in Mor_X(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U),$$

see Modules on Sites, Lemma 16.19.2. Moreover, if $\mathcal{G}$ is an $\mathcal{O}_U$-module, then $f_!\mathcal{G}$ is the sheafification of the exact same presheaf of abelian groups which is endowed with an

$\mathcal{O}_X$-module structure in an obvious way (see loc. cit.). Let $\overline{x} : Spec(k) \to X$ be a geometric point. Then there is a canonical identification

$$(f_!\mathcal{G})_{\overline{x}} = \bigoplus\nolimits_{\overline{u}} \mathcal{G}_{\overline{u}}$$

where the sum is over all $\overline{u} : Spec(k) \to U$ such that $f \circ \overline{u} = \overline{x}$, see Modules on Sites, Lemma 16.32.1. In the following we are going to study the sheaf $f_!\underline{\mathbf{Z}}$. Here $\underline{\mathbf{Z}}$ denotes the constant sheaf on $X_{\acute{e}tale}$ or $U_{\acute{e}tale}$.

Lemma 49.6.1. *Let S be a scheme. Let $f_i : U_i \to X$ be étale morphisms of algebraic spaces over S. Then there are isomorphisms*

$$f_{1,!}\underline{\mathbf{Z}} \otimes_{\mathbf{Z}} f_{2,!}\underline{\mathbf{Z}} \longrightarrow f_{12,!}\underline{\mathbf{Z}}$$

where $f_{12} : U_1 \times_X U_2 \to X$ is the structure morphism and

$$(f_1 \amalg f_2)_!\underline{\mathbf{Z}} \longrightarrow f_{1,!}\underline{\mathbf{Z}} \oplus f_{2,!}\underline{\mathbf{Z}}$$

Proof. Once we have defined the map it will be an isomorphism by our description of stalks above. To define the map it suffices to work on the level of presheaves. Thus we have to define a map

$$\left(\bigoplus\nolimits_{\varphi_1 \in Mor_X(V,U_1)} \mathbf{Z}\right) \otimes_{\mathbf{Z}} \left(\bigoplus\nolimits_{\varphi_2 \in Mor_X(V,U_2)} \mathbf{Z}\right) \longrightarrow \bigoplus\nolimits_{\varphi \in Mor_X(V,U_1 \times_X U_2)} \mathbf{Z}$$

We map the element $1_{\varphi_1} \otimes 1_{\varphi_2}$ to the element $1_{\varphi_1 \times \varphi_2}$ with obvious notation. We omit the proof of the second equality. $\square$

Another important feature is the trace map

$$\mathrm{Tr}_f : f_!\underline{\mathbf{Z}} \longrightarrow \underline{\mathbf{Z}}.$$

The trace map is adjoint to the map $\mathbf{Z} \to f^{-1}\underline{\mathbf{Z}}$ (which is an isomorphism). If $\overline{x}$ is above, then Tr_f on stalks at $\overline{x}$ is the map

$$(\mathrm{Tr}_f)_{\overline{x}} : (f_!\underline{\mathbf{Z}})_{\overline{x}} = \bigoplus\nolimits_{\overline{u}} \mathbf{Z} \longrightarrow \mathbf{Z} = \underline{\mathbf{Z}}_{\overline{x}}$$

which sums the given integers. This is true because it is adjoint to the map $1 : \mathbf{Z} \to f^{-1}\underline{\mathbf{Z}}$. In particular, if f is surjective as well as étale then Tr_f is surjective.

Assume that $f : U \to X$ is a surjective étale morphism of algebraic spaces. Consider the *Koszul complex* associated to the trace map we discussed above

$$\ldots \to \wedge^3 f_!\underline{\mathbf{Z}} \to \wedge^2 f_!\underline{\mathbf{Z}} \to f_!\underline{\mathbf{Z}} \to \underline{\mathbf{Z}} \to 0$$

Here the exterior powers are over the sheaf of rings $\underline{\mathbf{Z}}$. The maps are defined by the rule

$$e_1 \wedge \ldots \wedge e_n \longmapsto \sum\nolimits_{i=1,\ldots,n} (-1)^{i+1} \mathrm{Tr}_f(e_i) e_1 \wedge \ldots \wedge \widehat{e_i} \wedge \ldots \wedge e_n$$

where $e_1, \ldots, e_n$ are local sections of $f_!\underline{\mathbf{Z}}$. Let $\overline{x}$ be a geometric point of X and set $M_{\overline{x}} = (f_!\underline{\mathbf{Z}})_{\overline{x}} = \bigoplus_{\overline{u}} \mathbf{Z}$. Then the stalk of the complex above at $\overline{x}$ is the complex

$$\ldots \to \wedge^3 M_{\overline{x}} \to \wedge^2 M_{\overline{x}} \to M_{\overline{x}} \to \mathbf{Z} \to 0$$

which is exact because $M_{\overline{x}} \to \mathbf{Z}$ is surjective, see More on Algebra, Lemma 12.21.5. Hence if we let $K^\bullet = K^\bullet(f)$ be the complex with $K^i = \wedge^{i+1} f_!\underline{\mathbf{Z}}$, then we obtain a quasi-isomorphism

$$K^\bullet \longrightarrow \underline{\mathbf{Z}}[0] \tag{49.6.1.1}$$

We use the complex $K^\bullet$ to define what we call the alternating Čech complex associated to $f : U \to X$.

Definition 49.6.2. Let S be a scheme. Let $f : U \to X$ be a surjective étale morphism of algebraic spaces over S. Let $\mathcal{F}$ be an object of $Ab(X_{étale})$. The *alternating Čech complex*[1] $\check{\mathcal{C}}^\bullet_{alt}(f, \mathcal{F})$ associated to $\mathcal{F}$ and f is the complex

$$Hom(K^0, \mathcal{F}) \to Hom(K^1, \mathcal{F}) \to Hom(K^2, \mathcal{F}) \to \ldots$$

with Hom groups computed in $Ab(X_{étale})$.

The reader may verify that if $U = \coprod U_i$ and $f|_{U_i} : U_i \to X$ is the open immersion of a subspace, then $\check{\mathcal{C}}^\bullet_{alt}(f, \mathcal{F})$ agrees with the complex introduced in Cohomology, Section 18.17 for the Zariski covering $X = \bigcup U_i$ and the restriction of $\mathcal{F}$ to the Zariski site of X. What is more important however, is to relate the cohomology of the alternating Čech complex to the cohomology.

Lemma 49.6.3. *Let S be a scheme. Let $f : U \to X$ be a surjective étale morphism of algebraic spaces over S. Let $\mathcal{F}$ be an object of $Ab(X_{étale})$. There exists a canonical map*

$$\check{\mathcal{C}}^\bullet_{alt}(f, \mathcal{F}) \longrightarrow R\Gamma(X, \mathcal{F})$$

in $D(Ab)$. Moreover, there is a spectral sequence with E_1-page

$$E_1^{p,q} = Ext^q_{Ab(X_{étale})}(K^p, \mathcal{F})$$

converging to $H^{p+q}(X, \mathcal{F})$ where $K^p = \wedge^{p+1} f_!\underline{\mathbf{Z}}$.

Proof. Recall that we have the quasi-isomorphism $K^\bullet \to \underline{\mathbf{Z}}[0]$, see (49.6.1.1). Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in $Ab(X_{étale})$. Consider the double complex $A^{\bullet,\bullet}$ with terms

$$A^{p,q} = Hom(K^p, \mathcal{I}^q)$$

where the differential $d_1^{p,q} : A^{p,q} \to A^{p+1,q}$ is the one coming from the differential $K^{p+1} \to K^p$ and the differential $d_2^{p,q} : A^{p,q} \to A^{p,q+1}$ is the one coming from the differential $\mathcal{I}^q \to \mathcal{I}^{q+1}$. Denote $sA^\bullet$ the total complex associated to the double complex $A^{\bullet,\bullet}$. We will use the two spectral sequences $({}'E_r, {}'d_r)$ and $({}''E_r, {}''d_r)$ associated to this double complex, see Homology, Section 10.19.

Because $K^\bullet$ is a resolution of $\underline{\mathbf{Z}}$ we see that the complexes

$$A^{\bullet,q} : Hom(K^0, \mathcal{I}^q) \to Hom(K^1, \mathcal{I}^q) \to Hom(K^2, \mathcal{I}^q) \to \ldots$$

are acyclic in positive degrees and have H^0 equal to $\Gamma(X, \mathcal{I}^q)$. Hence by Homology, Lemma 10.19.6 and its proof the spectral sequence $({}''E_r, {}''d_r)$ degenerates, and the natural map

$$\mathcal{I}^\bullet(X) \longrightarrow sA^\bullet$$

is a quasi-isomorphism of complexes of abelian groups. In particular we conclude that $H^n(sA^\bullet) = H^n(X, \mathcal{F})$.

The map $\check{\mathcal{C}}^\bullet_{alt}(f, \mathcal{F}) \to R\Gamma(X, \mathcal{F})$ of the lemma is the composition of $\check{\mathcal{C}}^\bullet_{alt}(f, \mathcal{F}) \to SA^\bullet$ with the inverse of the displayed quasi-isomorphism.

Finally, consider the spectral sequence $({}'E_r, {}'d_r)$. We have

$$E_1^{p,q} = q\text{th cohomology of } Hom(K^p, \mathcal{I}^0) \to Hom(K^p, \mathcal{I}^1) \to Hom(K^p, \mathcal{I}^2) \to \ldots$$

This proves the lemma. □

It follows from the lemma that it is important to understand the ext groups $\mathrm{Ext}_{Ab(X_{étale})}(K^p, \mathcal{F})$, i.e., the right derived functors of $\mathcal{F} \mapsto Hom(K^p, \mathcal{F})$.

[1] This may be nonstandard notation

Lemma 49.6.4. *Let S be a scheme. Let $f : U \to X$ be a surjective, étale, and separated morphism of algebraic spaces over S. For $p \geq 0$ set*

$$W_p = U \times_X \ldots \times_X U \setminus \textit{all diagonals}$$

where the fibre product has $p+1$ factors. There is a free action of S_{p+1} on W_p over X and

$$Hom(K^p, \mathcal{F}) = S_{p+1}\textit{-anti-invariant elements of } \mathcal{F}(W_p)$$

functorially in $\mathcal{F}$ where $K^p = \wedge^{p+1} f_! \underline{\mathbf{Z}}$.

Proof. Because $U \to X$ is separated the diagonal $U \to U \times_X U$ is a closed immersion. Since $U \to X$ is étale the diagonal $U \to U \times_X U$ is an open immersion, see Morphisms of Spaces, Lemmas 42.35.10 and 42.34.9. Hence W_p is an open and closed subspace of $U^{p+1} = U \times_X \ldots \times_X U$. The action of S_{p+1} on W_p is free as we've thrown out the fixed points of the action. By Lemma 49.6.1 we see that

$$(f_! \underline{\mathbf{Z}})^{\otimes p+1} = f^{p+1}_! \underline{\mathbf{Z}} = (W_p \to X)_! \underline{\mathbf{Z}} \oplus Rest$$

where $f^{p+1} : U^{p+1} \to X$ is the structure morphism. Looking at stalks over a geometric point $\overline{x}$ of X we see that

$$\left(\bigoplus\nolimits_{\overline{u} \mapsto \overline{x}} \mathbf{Z}\right)^{\otimes p+1} \longrightarrow (W_p \to X)_! \underline{\mathbf{Z}}_{\overline{x}}$$

is the quotient whose kernel is generated by all tensors $1_{\overline{u}_0} \otimes \ldots \otimes 1_{\overline{u}_p}$ where $\overline{u}_i = \overline{u}_j$ for some $i \neq j$. Thus the quotient map

$$(f_! \underline{\mathbf{Z}})^{\otimes p+1} \longrightarrow \wedge^{p+1} f_! \underline{\mathbf{Z}}$$

factors through $(W_p \to X)_! \underline{\mathbf{Z}}$, i.e., we get

$$(f_! \underline{\mathbf{Z}})^{\otimes p+1} \longrightarrow (W_p \to X)_! \underline{\mathbf{Z}} \longrightarrow \wedge^{p+1} f_! \underline{\mathbf{Z}}$$

This already proves that $Hom(K^p, \mathcal{F})$ is (functorially) a subgroup of

$$Hom((W_p \to X)_! \underline{\mathbf{Z}}, \mathcal{F}) = \mathcal{F}(W_p)$$

To identify it with the S_{p+1}-anti-invariants we have to prove that the surjection $(W_p \to X)_! \underline{\mathbf{Z}} \to \wedge^{p+1} f_! \underline{\mathbf{Z}}$ is the maximal S_{p+1}-anti-invariant quotient. In other words, we have to show that $\wedge^{p+1} f_! \underline{\mathbf{Z}}$ is the quotient of $(W_p \to X)_! \underline{\mathbf{Z}}$ by the subsheaf generated by the local sections $s - \text{sign}(\sigma)\sigma(s)$ where s is a local section of $(W_p \to X)_! \underline{\mathbf{Z}}$. This can be checked on the stacks, where it is clear. □

Lemma 49.6.5. *Let S be a scheme. Let W be an algebraic space over S. Let G be a finite group acting freely on W. Let $U = W/G$, see Properties of Spaces, Lemma 41.31.1. Let $\chi : G \to \{+1, -1\}$ be a character. Then there exists a rank 1 locally free sheaf of $\mathbf{Z}$-modules $\underline{\mathbf{Z}}(\chi)$ on $U_{étale}$ such that for every abelian sheaf $\mathcal{F}$ on $U_{étale}$ we have*

$$H^0(W, \mathcal{F}|_W)^\chi = H^0(U, \mathcal{F} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi))$$

Proof. The quotient morphism $q : W \to U$ is a G-torsor, i.e., there exists a surjective étale morphism $U' \to U$ such that $W \times_U U' = \coprod_{g \in G} U'$ as spaces with G-action over U'. (Namely, $U' = W$ works.) Hence $q_* \underline{\mathbf{Z}}$ is a finite locally free $\mathbf{Z}$-module with an action of G. For any geometric point $\overline{u}$ of U, then we get G-equivariant isomorphisms

$$(q_* \underline{\mathbf{Z}})_{\overline{u}} = \bigoplus\nolimits_{\overline{w} \mapsto \overline{u}} \mathbf{Z} = \bigoplus\nolimits_{g \in G} \mathbf{Z} = \mathbf{Z}[G]$$

where the second = uses a geometric point $\overline{w}_0$ lying over $\overline{u}$ and maps the summand corresponding to $g \in G$ to the summand corresponding to $g(\overline{w}_0)$. We have

$$H^0(W, \mathcal{F}|_W) = H^0(U, \mathcal{F} \otimes_{\mathbf{Z}} q_*\underline{\mathbf{Z}})$$

because $q_*\mathcal{F}|_W = \mathcal{F} \otimes_{\mathbf{Z}} q_*\underline{\mathbf{Z}}$ as one can check by restricting to U'. Let

$$\underline{\mathbf{Z}}(\chi) = (q_*\underline{\mathbf{Z}})^\chi \subset q_*\underline{\mathbf{Z}}$$

be the subsheaf of sections that transform according to χ. For any geometric point $\overline{u}$ of U we have

$$\underline{\mathbf{Z}}(\chi)_{\overline{u}} = \mathbf{Z} \cdot \sum\nolimits_g \chi(g)g \subset \mathbf{Z}[G] = (q_*\underline{\mathbf{Z}})_{\overline{u}}$$

It follows that $\underline{\mathbf{Z}}(\chi)$ is locally free of rank 1 (more precisely, this should be checked after restricting to U'). Note that for any $\mathbf{Z}$-module M the χ-semi-invariants of $M[G]$ are the elements of the form $m \cdot \sum_g \chi(g)g$. Thus we see that for any abelian sheaf $\mathcal{F}$ on U we have

$$\left(\mathcal{F} \otimes_{\mathbf{Z}} q_*\underline{\mathbf{Z}}\right)^\chi = \mathcal{F} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi)$$

because we have equality at all stalks. The result of the lemma follows by taking global sections. □

Now we can put everything together and obtain the following pleasing result.

Lemma 49.6.6. *Let S be a scheme. Let $f : U \to X$ be a surjective, étale, and separated morphism of algebraic spaces over S. For $p \geq 0$ set*

$$W_p = U \times_X \ldots \times_X U \setminus \textit{all diagonals}$$

(with $p + 1$ factors) as in Lemma 49.6.4. Let $\chi_p : S_{p+1} \to \{+1, -1\}$ be the sign character. Let $U_p = W_p/S_{p+1}$ and $\underline{\mathbf{Z}}(\chi_p)$ be as in Lemma 49.6.5. Then the spectral sequence of Lemma 49.6.3 has E_1-page

$$E_1^{p,q} = H^q(U_p, \mathcal{F}|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p))$$

and converges to $H^{p+q}(X, \mathcal{F})$.

Proof. Note that since the action of S_{p+1} on W_p is over X we do obtain a morphism $U_p \to X$. Since $W_p \to X$ is étale and since $W_p \to U_p$ is surjective étale, it follows that also $U_p \to X$ is étale, see Descent on Spaces, Lemma 45.17.1. Therefore an injective object of $Ab(X_{étale})$ restricts to an injective object of $Ab(U_{p,étale})$, see Cohomology on Sites, Lemma 19.8.1. Moreover, the functor $\mathcal{G} \mapsto \mathcal{G} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p))$ is an auto-equivalence of $Ab(U_p)$, whence transforms injective objects into injective objects and is exact (because $\underline{\mathbf{Z}}(\chi_p)$ is an invertible $\underline{\mathbf{Z}}$-module). Thus given an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in $Ab(X_{étale})$ the complex

$$\Gamma(U_p, \mathcal{I}^0|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p)) \to \Gamma(U_p, \mathcal{I}^1|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p)) \to \Gamma(U_p, \mathcal{I}^2|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p)) \to \ldots$$

computes $H^*(U_p, \mathcal{F}|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p))$. On the other hand, by Lemma 49.6.5 it is equal to the complex of S_{p+1}-anti-invariants in

$$\Gamma(W_p, \mathcal{I}^0) \to \Gamma(W_p, \mathcal{I}^1) \to \Gamma(W_p, \mathcal{I}^2) \to \ldots$$

which by Lemma 49.6.4 is equal to the complex

$$Hom(K^p, \mathcal{I}^0) \to Hom(K^p, \mathcal{I}^1) \to Hom(K^p, \mathcal{I}^2) \to \ldots$$

which computes $\mathrm{Ext}^*_{Ab(X_{étale})}(K^p, \mathcal{F})$. Putting everything together we win. □

49.7. Higher vanishing for quasi-coherent sheaves

In this section we show that given a quasi-compact and quasi-separated algebraic space X there exists an integer $n = n(X)$ such that the cohomology of any quasi-coherent sheaf on X vanishes beyond degree n.

Lemma 49.7.1. *With S, W, G, U, χ as in Lemma 49.6.5. If $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_U$-module, then so is $\mathcal{F} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi)$.*

Proof. The $\mathcal{O}_U$-module structure is clear. To check that $\mathcal{F} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi)$ is quasi-coherent it suffices to check étale locally. Hence the lemma follows as $\underline{\mathbf{Z}}(\chi)$ is finite locally free as a $\underline{\mathbf{Z}}$-module. □

The following proposition is interesting even if X is a scheme. It is the natural generalization of Coherent, Lemma 25.5.1. Before we state it, observe that given an étale morphism $f : U \to X$ from an affine scheme towards a quasi-separated algebraic space X the fibres of f are universally bounded, in particular there exists an integer d such that the fibres of $|U| \to |X|$ all have size at most d; this is the implication $(\eta) \Rightarrow (\delta)$ of Decent Spaces, Lemma 43.5.1.

Proposition 49.7.2. *Let S be a scheme. Let X be an algebraic space over S. Assume X is quasi-compact and separated. Let U be an affine scheme, and let $f : U \to X$ be a surjective étale morphism. Let d be an upper bound for the size of the fibres of $|U| \to |X|$. Then for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have $H^q(X, \mathcal{F}) = 0$ for $q \geq d$.*

Proof. We will use the spectral sequence of Lemma 49.6.6. The lemma applies since f is separated as U is separated, see Morphisms of Spaces, Lemma 42.5.10. Since X is separated the scheme $U \times_X \ldots \times_X U$ is a closed subscheme of $U \times_{Spec(\mathbf{Z})} \ldots \times_{Spec(\mathbf{Z})} U$ hence is affine. Thus W_p is affine. Hence $U_p = W_p/S_{p+1}$ is an affine scheme by Groupoids, Proposition 35.19.8. The discussion in Section 49.4 shows that cohomology of quasi-coherent sheaves on W_p (as an algebraic space) agrees with the cohomology of the corresponding quasi-coherent sheaf on the underlying affine scheme, hence vanishes in positive degrees by Coherent, Lemma 25.2.2. By Lemma 49.7.1 the sheaves $\mathcal{F}|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p)$ are quasi-coherent. Hence $H^q(W_p, \mathcal{F}|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p))$ is zero when $q > 0$. By our definition of the integer d we see that $W_p = \emptyset$ for $p \geq d$. Hence also $H^0(W_p, \mathcal{F}|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p))$ is zero when $p \geq d$. This proves the proposition. □

In the following lemma we esthablish that a quasi-compact and quasi-separated algebraic space has finite cohomological dimension for quasi-coherent modules. We are explicit about the bound only because we will use it later to prove a similar result for higher direct images.

Lemma 49.7.3. *Let S be a scheme. Let X be an algebraic space over S. Assume X is quasi-compact and quasi-separated. Then we can choose*

1. *an affine scheme U,*
2. *a surjective étale morphism $f : U \to X$,*
3. *an integer d bounding the degrees of the fibres of $U \to X$,*
4. *for every $p = 0, 1, \ldots, d$ a surjective étale morphism $V_p \to U_p$ from an affine scheme V_p where U_p is as in Lemma 49.6.6, and*
5. *an integer d_p bounding the degree of the fibres of $V_p \to U_p$.*

Then for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have $H^q(X, \mathcal{F}) = 0$ for $q \geq \max(d_p + p)$.

Proof. Since X is quasi-compact we can find a surjective étale morphism $U \to X$ with U affine, see Properties of Spaces, Lemma 41.6.3. By Decent Spaces, Lemma 43.5.1 the fibres of f are universally bounded, hence we can find d. We have $U_p = W_p/S_{p+1}$ and $W_p \subset U \times_X \ldots \times_X U$ is open and closed. Since X is quasi-separated the schemes W_p are quasi-compact, hence U_p is quasi-compact. Since U is separated, the schemes W_p are separated, hence U_p is separated by (the absolute version of) Spaces, Lemma 40.14.5. By Properties of Spaces, Lemma 41.6.3 we can find the morphisms $V_p \to W_p$. By Decent Spaces, Lemma 43.5.1 we can find the integers d_p.

At this point the proof uses the spectral sequence

$$E_1^{p,q} = H^q(U_p, \mathcal{F}|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p)) \Rightarrow H^{p+q}(X, \mathcal{F})$$

see Lemma 49.6.6. By definition of the integer d we see that $U_p = 0$ for $p \geq d$. By Proposition 49.7.2 and Lemma 49.7.1 we see that $H^q(U_p, \mathcal{F}|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p))$ is zero for $q \geq d_p$ for $p = 0, \ldots, d$. Whence the lemma. □

49.8. Vanishing for higher direct images

We apply the results of Section 49.7 to obtain vanishing of higher direct images of quasi-coherent sheaves for quasi-compact and quasi-separated morphisms. This is useful because it allows one to argue by descending induction on the cohomological degree in certain situations.

Lemma 49.8.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume that*

1. *f is quasi-compact and quasi-separated, and*
2. *Y is quasi-compact.*

Then there exists an integer $n(X \to Y)$ such that for any algebraic space Y', any morphism $Y' \to Y$ and any quasi-coherent sheaf $\mathcal{F}'$ on $X' = Y' \times_Y X$ the higher direct images $R^i f'_*\mathcal{F}'$ are zero for $i \geq n(X \to Y)$.

Proof. Let $V \to Y$ be a surjective étale morphism where V is an affine scheme, see Properties of Spaces, Lemma 41.6.3. Suppose we prove the result for the base change $f_V : V \times_Y X \to V$. Then the result holds for f with $n(X \to Y) = n(X_V \to V)$. Namely, if $Y' \to Y$ and $\mathcal{F}'$ are as in the lemma, then $R^i f'_*\mathcal{F}'|_{V \times_Y Y'}$ is equal to $R^i f'_{V,*}\mathcal{F}'|_{X'_V}$ where $f'_V : X'_V = V \times_Y Y' \times_Y X \to V \times_Y Y' = Y'_V$, see Properties of Spaces, Lemma 41.23.2. Thus we may assume that Y is an affine scheme.

Moreover, to prove the vanishing for all $Y' \to Y$ and $\mathcal{F}'$ it suffices to do so when Y' is an affine scheme. In this case, $R^i f'_*\mathcal{F}'$ is quasi-coherent by Lemma 49.4.1. Hence it suffices to prove that $H^i(X', \mathcal{F}') = 0$, because $H^i(X', \mathcal{F}') = H^0(Y', R^i f'_*\mathcal{F}')$ by Cohomology on Sites, Lemma 19.14.5 and the vanishing of higher cohomology of quasi-coherent sheaves on affine algebraic spaces (Proposition 49.7.2).

Choose $U \to X$, d, $V_p \to U_p$ and d_p as in Lemma 49.7.3. For any affine scheme Y' and morphism $Y' \to Y$ denote $X' = Y' \times_Y X$, $U' = Y' \times_Y U$, $V'_p = Y' \times_Y V_p$. Then $U' \to X'$, $d' = d$, $V'_p \to U'_p$ and $d'_p = d$ is a collection of choices as in Lemma 49.7.3 for the algebraic space X' (details omitted). Hence we see that $H^i(X', \mathcal{F}') = 0$ for $i \geq \max(p + d_p)$ and we win. □

Lemma 49.8.2. *Let S be a scheme. Let $f : X \to Y$ be an affine morphism of algebraic spaces over S. Then $R^i f_*\mathcal{F} = 0$ for $i > 0$ and any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$.*

Proof. Recall that an affine morphism of algebraic spaces is representable. Hence this follows from (49.4.0.1) and Coherent, Lemma 25.2.3. □

49.9. Cohomology and base change, I

Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Suppose further that $g : Y' \to Y$ is a morphism of algebraic spaces over S. Denote $X' = X_{Y'} = Y' \times_Y X$ the base change of X and denote $f' : X' \to Y'$ the base change of f. Also write $g' : X' \to X$ the projection, and set $\mathcal{F}' = (g')^*\mathcal{F}$. Here is a diagram representing the situation:

(49.9.0.1)
$$\begin{tikzcd} \mathcal{F}' = (g')^*\mathcal{F} & X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] & \mathcal{F} \\ Rf'_*\mathcal{F}' & Y' \arrow[r, "g"] & Y & Rf_*\mathcal{F} \end{tikzcd}$$

Here is the basic result for a flat base change.

Lemma 49.9.1. *In the situation above, assume that g is flat and that f is quasi-compact and quasi-separated. Then the base change map for any $i \geq 0$ we have*

$$R^p f'_*\mathcal{F}' = g^* R^p f_*\mathcal{F}$$

with notation as in (49.9.0.1).

Proof. The morphism g' is flat by Morphisms of Spaces, Lemma 42.27.3. Note that flatness of g and g' is equivalent to flatness of the morphisms of small étale ringed sites, see Morphisms of Spaces, Lemma 42.27.8. Hence we can apply Cohomology on Sites, Lemma 19.15.1 to obtain a base change map

$$g^* R^p f_*\mathcal{F} \longrightarrow R^p f'_*\mathcal{F}'$$

To prove this map is an isomorphism we can work locally in the étale topology on Y'. Thus we may assume that Y and Y' are affine schemes. Say $Y = \text{Spec}(A)$ and $Y' = \text{Spec}(B)$. In this case we are really trying to show that the map

$$H^p(X, \mathcal{F}) \otimes_A B \longrightarrow H^p(X_B, \mathcal{F}_B)$$

is an isomorphism where $X_B = \text{Spec}(B) \times_{\text{Spec}(A)} X$ and $\mathcal{F}_B$ is the pullback of $\mathcal{F}$ to X_B.

Fix $A \to B$ a flat ring map and let X be a quasi-compact and quasi-separated algebraic space over A. Note that $g' : X_B \to X$ is affine as a base change of $Spec(B) \to Spec(A)$. Hence the higher direct images $R^i(g')_*\mathcal{F}_B$ are zero by Lemma 49.8.2. Thus $H^p(X_B, \mathcal{F}_B) = H^p(X, g'_*\mathcal{F}_B)$, see Cohomology on Sites, Lemma 19.14.5. Moreover, we have

$$g'_*\mathcal{F}_B = \mathcal{F} \otimes_{\underline{A}} \underline{B}$$

where $\underline{A}$, $\underline{B}$ denotes the constant sheaf of rings with value A, B. Namely, it is clear that there is a map from right to left. For any affine scheme U étale over X we have

$$\begin{aligned} g'_*\mathcal{F}_B(U) &= \mathcal{F}_B(Spec(B) \times_{Spec(A)} U) \\ &= \Gamma(Spec(B) \times_{Spec(A)} U, (Spec(B) \times_{Spec(A)} U \to U)^*\mathcal{F}|_U) \\ &= B \otimes_A \mathcal{F}(U) \end{aligned}$$

hence the map is an isomorphism. Write $B = colim\, M_i$ as a filtered colimit of finite free A-modules M_i using Lazard's theorem, see Algebra, Theorem 7.75.4. We deduce that

$$\begin{aligned} H^p(X, g'_*\mathcal{F}_B) &= H^p(X, \mathcal{F} \otimes_{\underline{A}} \underline{B}) \\ &= H^p(X, colim_i\, \mathcal{F} \otimes_{\underline{A}} \underline{M_i}) \\ &= colim_i\, H^p(X, \mathcal{F} \otimes_{\underline{A}} \underline{M_i}) \\ &= colim_i\, H^p(X, \mathcal{F}) \otimes_A M_i \\ &= H^p(X, \mathcal{F}) \otimes_A colim_i\, M_i \\ &= H^p(X, \mathcal{F}) \otimes_A B \end{aligned}$$

The first equality because $g'_*\mathcal{F}_B = \mathcal{F} \otimes_{\underline{A}} \underline{B}$ as seen above. The second because $\otimes$ commutes with colimits. The third equality because cohomology on X commutes with colimits (see Lemma 49.5.1). The fourth equality because M_i is finite free (i.e., because cohomology commutes with finite direct sums). The fith because $\otimes$ commutes with colimits. The sixth by choice of our system. □

49.10. Other chapters

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CHAPTER 50

Stacks

50.1. Introduction

In this very short chapter we introduce stacks, and stacks in groupoids. See [**DM69a**], and [**Vis**].

50.2. Presheaves of morphisms associated to fibred categories

Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category, see Categories, Section 4.30. Suppose that $x, y \in Ob(\mathcal{S}_U)$ are objects in the fibre category over U. We are going to define a functor

$$Mor(x, y) : (\mathcal{C}/U)^{opp} \longrightarrow Sets.$$

In other words this will be a presheaf on $\mathcal{C}/U$, see Sites, Definition 9.2.2. Make a choice of pullbacks as in Categories, Definition 4.30.5. Then, for $f : V \to U$ we set

$$Mor(x, y)(f : V \to U) = Mor_{\mathcal{S}_V}(f^*x, f^*y).$$

Let $f' : V' \to U$ be a second object of $\mathcal{C}/U$. We also have to define the restriction map corresponding to a morphism $g : V'/U \to V/U$ in $\mathcal{C}/U$, in other words $g : V' \to V$ and $f' = f \circ g$. This will be a map

$$Mor_{\mathcal{S}_V}(f^*x, f^*y) \longrightarrow Mor_{\mathcal{S}_{V'}}(f'^*x, f'^*y), \quad \phi \longmapsto \phi|_{V'}$$

This map will basically be g^*, except that this transforms an element ϕ of the left hand side into an element $g^*\phi$ of $Mor_{\mathcal{S}_{V'}}(g^*f^*x, g^*f^*y)$. At this point we use the transformation $\alpha_{g,f}$ of Categories, Lemma 4.30.6. In a formula, the restriction map is described by

$$\phi|_{V'} = (\alpha_{g,f})_y^{-1} \circ g^*\phi \circ (\alpha_{g,f})_x.$$

Of course, nobody thinks of this restriction map in this way. We will only do this once in order to verify the following lemma.

Lemma 50.2.1. *This actually does give a presheaf.*

Proof. Let $g : V'/U \to V/U$ be as above and similarly $g' : V''/U \to V'/U$ be morphisms in $\mathcal{C}/U$. So $f' = f \circ g$ and $f'' = f' \circ g' = f \circ g \circ g'$. Let $\phi \in Mor_{\mathcal{S}_V}(f^*x, f^*y)$. Then we have

$$\begin{aligned}
& (\alpha_{g\circ g',f})_y^{-1} \circ (g \circ g')^*\phi \circ (\alpha_{g\circ g',f})_x \\
= \quad & (\alpha_{g\circ g',f})_y^{-1} \circ (\alpha_{g',g})_{f^*y}^{-1} \circ (g')^*g^*\phi \circ (\alpha_{g',g})_{f^*x} \circ (\alpha_{g\circ g',f})_x \\
= \quad & (\alpha_{g',f'})_y^{-1} \circ (g')^*(\alpha_{g,f})_y^{-1} \circ (g')^*g^*\phi \circ (g')^*(\alpha_{g,f})_x \circ (\alpha_{g',f'})_x \\
= \quad & (\alpha_{g',f'})_y^{-1} \circ (g')^*\Big((\alpha_{g,f})_y^{-1} \circ g^*\phi \circ (\alpha_{g,f})_x\Big) \circ (\alpha_{g',f'})_x
\end{aligned}$$

which is what we want, namely $\phi|_{V''} = (\phi|_{V'})|_{V''}$. The first equality holds because $\alpha_{g',g}$ is a transformation of functors, and hence

$$\begin{tikzcd} (g \circ g')^* f^* x \arrow[r, "(g \circ g')^* \phi"'] \arrow[d, "(\alpha_{g',g})_{f^* x}"'] & (g \circ g')^* f^* y \arrow[d, "(\alpha_{g',g})_{f^* y}"] \\ (g')^* g^* f^* x \arrow[r, "(g')^* g^* \phi"] & (g')^* g^* f^* y \end{tikzcd}$$

commutes. The second equality holds because of property (d) of a pseudo functor since $f' = f \circ g$ (see Categories, Definition 4.26.5). The last equality follows from the fact that $(g')^*$ is a functor. $\square$

From now on we often omit mentioning the transformations $\alpha_{g,f}$ and we simply identify the functors $g^* \circ f^*$ and $(f \circ g)^*$. In particular, given $g : V'/U \to V/U$ the restriction mappings for the presheaf $Mor(x, y)$ will sometimes be denoted $\phi \mapsto g^*\phi$. We formalize the construction in a definition.

Definition 50.2.2. Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category, see Categories, Section 4.30. Given an object U of $\mathcal{C}$ and objects x, y of the fibre category, the *presheaf of morphisms from x to y* is the presheaf

$$(f : V \to U) \longmapsto Mor_{\mathcal{S}_V}(f^*x, f^*y)$$

described above. It is denoted $Mor(x, y)$. The subpresheaf $Isom(x, y)$ whose values over V is the set of isomorphisms $f^*x \to f^*y$ in the fibre category $\mathcal{S}_V$ is called the *presheaf of isomorphisms from x to y*.

If $\mathcal{S}$ is fibred in groupoids then of course $Isom(x, y) = Mor(x, y)$, and it is customary to use the $Isom$ notation.

Lemma 50.2.3. *Let $F : \mathcal{S}_1 \to \mathcal{S}_2$ be a 1-morphism of fibred categories over the category $\mathcal{C}$. Let $U \in Ob(\mathcal{C})$ and $x, y \in Ob(\mathcal{S}_U)$. Then F defines a canonical morphism of presheaves*

$$Mor_{\mathcal{S}_1}(x, y) \longrightarrow Mor_{\mathcal{S}_2}(F(x), F(y))$$

on $\mathcal{C}/U$.

Proof. By Categories, Definition 4.30.8 the functor F maps strongly cartesian morphisms to strongly cartesian morphisms. Hence if $f : V \to U$ is a morphism in $\mathcal{C}$, then there are canonical isomorphisms $\alpha_V : f^*F(x) \to F(f^*x)$, $\beta_V : f^*F(y) \to F(f^*y)$ such that $f^*F(x) \to F(f^*x) \to F(x)$ is the canonical morphism $f^*F(x) \to F(x)$, and similarly for β_V. Thus we may define

$$\begin{tikzcd} Mor_{\mathcal{S}_1}(x, y)(f : V \to U) \arrow[r, equal] & Mor_{\mathcal{S}_{1,V}}(f^*x, f^*y) \arrow[d] \\ Mor_{\mathcal{S}_2}(F(x), F(y))(f : V \to U) \arrow[r, equal] & Mor_{\mathcal{S}_{2,V}}(f^*F(x), f^*F(y)) \end{tikzcd}$$

by $\phi \mapsto \beta_V^{-1} \circ F(\phi) \circ \alpha_V$. We omit the verification that this is compatible with the restriction mappings. $\square$

Remark 50.2.4. Suppose that $p : \mathcal{S} \to \mathcal{C}$ is fibred in groupoids. In this case we can prove Lemma 50.2.1 using Categories, Lemma 4.33.4 which says that $\mathcal{S} \to \mathcal{C}$ is equivalent to the category associated to a contravariant functor $F : \mathcal{C} \to$ Groupoids. In the case of the fibred category associated to F we have $g^* \circ f^* = (f \circ g)^*$ on the nose and there is no need to use the maps $\alpha_{g,f}$. In this case the lemma is (even more) trivial. Of course then one uses that the $Mor(x, y)$ presheaf is unchanged when passing to an equivalent fibred category which follows from Lemma 50.2.3.

Lemma 50.2.5. *Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category, see Categories, Section 4.30. Let $U \in Ob(\mathcal{C})$ and let $x, y \in Ob(\mathcal{S}_U)$. Denote $x, y : \mathcal{C}/U \to \mathcal{S}$ also the corresponding 1-morphisms, see Categories, Lemma 4.38.1. Then*

(1) *the 2-fibre product $\mathcal{S} \times_{\mathcal{S}\times\mathcal{S},(x,y)} \mathcal{C}/U$ is fibred in setoids over $\mathcal{C}/U$, and*
(2) *$Isom(x, y)$ is the presheaf of sets corresponding to this category fibred in setoids, see Categories, Lemma 4.36.6.*

Proof. Omitted. Hint: Objects of the 2-fibre product are $(a : V \to U, z, a : V \to U, (\alpha, \beta))$ where $\alpha : z \to a^*x$ and $\beta : z \to a^*y$ are isomorphisms in $\mathcal{S}_V$. Thus the relationship with $Isom(x, y)$ comes by assigning to such an object the isomorphism $\beta \circ \alpha^{-1}$. □

50.3. Descent data in fibred categories

In this section we define the notion of a descent datum in the abstract setting of a fibred category. Before we do so we point out that this is completely analogous to descent data for quasi-coherent sheaves (Descent, Section 31.2) and descent data for schemes over schemes (Descent, Section 31.30).

We will use the convention where the projection maps $\mathrm{pr}_i : X \times \ldots \times X \to X$ are labeled starting with $i = 0$. Hence we have $\mathrm{pr}_0, \mathrm{pr}_1 : X \times X \to X$, $\mathrm{pr}_0, \mathrm{pr}_1, \mathrm{pr}_2 : X \times X \times X \to X$, etc.

Definition 50.3.1. Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category. Make a choice of pullbacks as in Categories, Definition 4.30.5. Let $\mathcal{U} = \{f_i : U_i \to U\}_{i\in I}$ be a family of morphisms of $\mathcal{C}$. Assume all the fibre products $U_i \times_U U_j$, and $U_i \times_U U_j \times_U U_k$ exist.

(1) A *descent datum (X_i, φ_{ij}) in $\mathcal{S}$ relative to the family $\{f_i : U_i \to U\}$* is given by an object X_i of $\mathcal{S}_{U_i}$ for each $i \in I$, an isomorphism $\varphi_{ij} : \mathrm{pr}_0^*X_i \to \mathrm{pr}_1^*X_j$ in $\mathcal{S}_{U_i\times_U U_j}$ for each pair $(i, j) \in I^2$ such that for every triple of indices $(i, j, k) \in I^3$ the diagram
$$\begin{tikzcd} \mathrm{pr}_0^*X_i \arrow[rr, "\mathrm{pr}_{02}^*\varphi_{ik}"'] \arrow[dr, "\mathrm{pr}_{01}^*\varphi_{ij}"'] & & \mathrm{pr}_2^*X_k \\ & \mathrm{pr}_1^*X_j \arrow[ur, "\mathrm{pr}_{12}^*\varphi_{jk}"'] & \end{tikzcd}$$
in the category $\mathcal{S}_{U_i\times_U U_j\times_U U_k}$ commutes. This is called the *cocycle condition*.
(2) A *morphism $\psi : (X_i, \varphi_{ij}) \to (X'_i, \varphi'_{ij})$ of descent data* is given by a family $\psi = (\psi_i)_{i\in I}$ of morphisms $\psi_i : X_i \to X'_i$ in $\mathcal{S}_{U_i}$ such that all the diagrams
$$\begin{tikzcd} \mathrm{pr}_0^*X_i \arrow[r, "\varphi_{ij}"'] \arrow[d, "\mathrm{pr}_0^*\psi_i"'] & \mathrm{pr}_1^*X_j \arrow[d, "\mathrm{pr}_1^*\psi_j"] \\ \mathrm{pr}_0^*X'_i \arrow[r, "\varphi'_{ij}"] & \mathrm{pr}_1^*X'_j \end{tikzcd}$$
in the categories $\mathcal{S}_{U_i\times_U U_j}$ commute.
(3) The category of descent data relative to $\mathcal{U}$ is denoted $DD(\mathcal{U})$.

The fibre products $U_i \times_U U_j$ and $U_i \times_U U_j \times_U U_k$ will exist if each of the morphisms $f_i : U_i \to U$ is *representable*, see Categories, Definition 4.6.3. Recall that in a site one of the conditions for a covering $\{U_i \to U\}$ is that each of the morphisms is representable, see Sites, Definition 9.6.2 part (3). In fact the main interest in the definition above is where $\mathcal{C}$ is a site and $\{U_i \to U\}$ is a covering of $\mathcal{C}$. However, a descent datum is just an abstract gadget that can be defined as above. This is useful: for example, given a fibred category over $\mathcal{C}$ one can look at the collection of families with respect to which descent data are effective, and try to use these as the family of coverings for a site.

Remarks 50.3.2. Two remarks on Definition 50.3.1 are in order. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category. Let $\{f_i : U_i \to U\}_{i\in I}$, and (X_i, φ_{ij}) be as in Definition 50.3.1.

(1) There is a diagonal morphism $\Delta : U_i \to U_i \times_U U_i$. We can pull back φ_{ii} via this morphism to get an automorphism $\Delta^*\varphi_{ii} \in \mathrm{Aut}_{U_i}(x_i)$. On pulling back the cocycle condition for the triple (i, i, i) by $\Delta_{123} : U_i \to U_i \times_U U_i \times_U U_i$ we deduce that $\Delta^*\varphi_{ii} \circ \Delta^*\varphi_{ii} = \Delta^*\varphi_{ii}$; thus $\Delta^*\varphi_{ii} = \mathrm{id}_{x_i}$.
(2) There is a morphism $\Delta_{13} : U_i \times_U U_j \to U_i \times_U U_j \times_U U_i$ and we can pull back the cocycle condition for the triple (i, j, i) to get the identity $(\sigma^*\varphi_{ji}) \circ \varphi_{ij} = \mathrm{id}_{\mathrm{pr}_0^* x_i}$, where $\sigma : U_i \times_U U_j \to U_j \times_U U_i$ is the switching morphism.

Lemma 50.3.3. *(Pullback of descent data.) Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category. Make a choice pullbacks as in Categories, Definition 4.30.5. Let $\mathcal{U} = \{f_i : U_i \to U\}_{i\in I}$, and $\mathcal{V} = \{V_j \to V\}_{j\in J}$ be a families of morphisms of $\mathcal{C}$ with fixed target. Assume all the fibre products $U_i \times_U U_{i'}$, $U_i \times_U U_{i'} \times_U U_{i''}$, $V_j \times_V V_{j'}$, and $V_j \times_V V_{j'} \times_V V_{j''}$ exist. Let $\alpha : I \to J$, $h : U \to V$ and $g_i : U_i \to V_{\alpha(i)}$ be a morphism of families of maps with fixed target, see Sites, Definition 9.8.1.*

(1) *Let $(Y_j, \varphi_{jj'})$ be a descent datum relative to the family $\{V_j \to V\}$. The system*
$$\left(g_i^* Y_{\alpha(i)}, (g_i \times g_{i'})^* \varphi_{\alpha(i)\alpha(i')}\right)$$
is a descent datum relative to $\mathcal{U}$.
(2) *This construction defines a functor between descent data relative to $\mathcal{V}$ and descent data relative to $\mathcal{U}$.*
(3) *Given a second $\alpha' : I \to J$, $h' : U \to V$ and $g'_i : U_i \to V_{\alpha'(i)}$ morphism of families of maps with fixed target, then if $h = h'$ the two resulting functors between descent data are canonically isomorphic.*

Proof. Omitted. $\square$

Definition 50.3.4. With $\mathcal{U} = \{U_i \to U\}_{i\in I}$, $\mathcal{V} = \{V_j \to V\}_{j\in J}$, $\alpha : I \to J$, $h : U \to V$, and $g_i : U_i \to V_{\alpha(i)}$ as in Lemma 50.3.3 the functor
$$(Y_j, \varphi_{jj'}) \longmapsto (g_i^* Y_{\alpha(i)}, (g_i \times g_{i'})^* \varphi_{\alpha(i)\alpha(i')})$$
constructed in that lemma is called the *pullback functor* on descent data.

Given $h : U \to V$, if there exists a morphism $\tilde{h} : \mathcal{U} \to \mathcal{V}$ covering h then $\tilde{h}^*$ is independent of the choice of $\tilde{h}$ as we saw in Lemma 50.3.3. Hence we will sometimes simply write h^* to indicate the pullback functor.

Definition 50.3.5. Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category. Make a choice of pullbacks as in Categories, Definition 4.30.5. Let $\mathcal{U} = \{f_i : U_i \to U\}_{i\in I}$ be a family of morphisms with target U. Assume all the fibre products $U_i \times_U U_j$ and $U_i \times_U U_j \times_U U_k$ exist.

(1) Given an object X of $\mathcal{S}_U$ the *trivial descent datum* is the descent datum (X, id_X) with respect to the family $\{\text{id}_U : U \to U\}$.
(2) Given an object X of $\mathcal{S}_U$ we have a *canonical descent datum* on the family of objects f_i^*X by pulling back the trivial descent datum (X, id_X) via the obvious map $\{f_i : U_i \to U\} \to \{\text{id}_U : U \to U\}$. We denote this descent datum (f_i^*X, can).
(3) A descent datum (X_i, φ_{ij}) relative to $\{f_i : U_i \to U\}$ is called *effective* if there exists an object X of $\mathcal{S}_U$ such that (X_i, φ_{ij}) is isomorphic to (f_i^*X, can).

Note that the rule that associates to $X \in \mathcal{S}_U$ its canonical descent datum relative to $\mathcal{U}$ defines a functor

$$\mathcal{S}_U \longrightarrow DD(\mathcal{U}).$$

A descent datum is effective if and only if it is in the essential image of this functor. Let us make explicit the canonical descent datum as follows.

Lemma 50.3.6. *In the situation of Definition 50.3.5 part (2) the maps $can_{ij} : pr_0^*f_i^*X \to pr_1^*f_j^*X$ are equal to $(\alpha_{pr_1, f_j})_X \circ (\alpha_{pr_0, f_i})_X^{-1}$ where $\alpha_{\cdot,\cdot}$ is as in Categories, Lemma 4.30.6 and where we use the equality $f_i \circ pr_0 = f_j \circ pr_1$ as maps $U_i \times_U U_j \to U$.*

Proof. Omitted. □

50.4. Stacks

Here is the definition of a stack. It mixes the notion of a fibred category with the notion of descent.

Definition 50.4.1. Let $\mathcal{C}$ be a site. A *stack* over $\mathcal{C}$ is a category $p : \mathcal{S} \to \mathcal{C}$ over $\mathcal{C}$ which satisfies the following conditions:

(1) $p : \mathcal{S} \to \mathcal{C}$ is a fibred category, see Categories, Definition 4.30.4,
(2) for any $U \in Ob(\mathcal{C})$ and any $x, y \in \mathcal{S}_U$ the presheaf $Mor(x, y)$ (see Definition 50.2.2) is a sheaf on the site $\mathcal{C}/U$, and
(3) for any covering $\mathcal{U} = \{f_i : U_i \to U\}_{i \in I}$ of the site $\mathcal{C}$, any descent datum in $\mathcal{S}$ relative to $\mathcal{U}$ is effective.

We find the formulation above the most convenient way to think about a stack. Namely, given a category over $\mathcal{C}$ in order to verify that it is a stack you proceed to check properties (1), (2) and (3) in that order. Certainly properties (2) and (3) do not make sense if the category isn't fibred. Without (2) we cannot prove that the descent in (3) is unique up to unique isomorphism and functorial.

The following lemma provides an alternative definition.

Lemma 50.4.2. *Let $\mathcal{C}$ be a site. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category over $\mathcal{C}$. The following are equivalent*

(1) *$\mathcal{S}$ is a stack over $\mathcal{C}$, and*
(2) *for any covering $\mathcal{U} = \{f_i : U_i \to U\}_{i \in I}$ of the site $\mathcal{C}$ the functor*

$$\mathcal{S}_U \longrightarrow DD(\mathcal{U})$$

which associates to an object its canonical descent datum is an equivalence.

Proof. Omitted. □

Lemma 50.4.3. *Let $p : \mathcal{S} \to \mathcal{C}$ be a stack over the site $\mathcal{C}$. Let $\mathcal{S}'$ be a subcategory of $\mathcal{S}$. Assume*

(1) *if* $\varphi : y \to x$ *is a strongly cartesian morphism of* $\mathcal{S}$ *and* x *is an object of* $\mathcal{S}'$*, then* y *is isomorphic to an object of* $\mathcal{S}'$*,*
(2) $\mathcal{S}'$ *is a full subcategory of* $\mathcal{S}$*, and*
(3) *if* $\{f_i : U_i \to U\}$ *is a covering of* $\mathcal{C}$*, and* x *an object of* $\mathcal{S}$ *over* U *such that* f_i^*x *is isomorphic to an object of* $\mathcal{S}'$ *for each* i*, then* x *is isomorphic to an object of* $\mathcal{S}'$*.*

Then $\mathcal{S}' \to \mathcal{C}$ *is a stack.*

Proof. Omitted. Hints: The first condition garantees that $\mathcal{S}'$ is a fibred category. The second condition garantees that the $Isom$-presheaves of $\mathcal{S}'$ are sheaves (as they are identical to their counter parts in $\mathcal{S}$). The third condition garantees that the descent condition holds in $\mathcal{S}'$ as we can first descend in $\mathcal{S}$ and then (3) implies the resulting object is isomorphic to an object of $\mathcal{S}'$. $\square$

Lemma 50.4.4. *Let* $\mathcal{C}$ *be a site. Let* $\mathcal{S}_1$*,* $\mathcal{S}_2$ *be categories over* $\mathcal{C}$*. Suppose that* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *are equivalent as categories over* $\mathcal{C}$*. Then* $\mathcal{S}_1$ *is a stack over* $\mathcal{C}$ *if and only if* $\mathcal{S}_2$ *is a stack over* $\mathcal{C}$*.*

Proof. Let $F : \mathcal{S}_1 \to \mathcal{S}_2$, $G : \mathcal{S}_2 \to \mathcal{S}_1$ be functors over $\mathcal{C}$, and let $i : F \circ G \to \mathrm{id}_{\mathcal{S}_2}$, $j : G \circ F \to \mathrm{id}_{\mathcal{S}_1}$ be isomorphisms of functors over $\mathcal{C}$. By Categories, Lemma 4.30.7 we see that $\mathcal{S}_1$ is fibred if and only if $\mathcal{S}_2$ is fibred over $\mathcal{C}$. Hence we may assume that both $\mathcal{S}_1$ and $\mathcal{S}_2$ are fibred. Moreover, the proof of Categories, Lemma 4.30.7 shows that F and G map strongly cartesian morphisms to strongly cartesian morphisms, i.e., F and G are 1-morphisms of fibred cateogires over $\mathcal{C}$. This means that given $U \in Ob(\mathcal{C})$, and $x, y \in \mathcal{S}_{1,U}$ then the presheaves

$$Mor_{\mathcal{S}_1}(x, y), Mor_{\mathcal{S}_1}(F(x), F(y)) : (\mathcal{C}/U)^{opp} \longrightarrow Sets.$$

are identified, see Lemma 50.2.3. Hence the first is a sheaf if and only if the second is a sheaf. Finally, we have to show that if every descent datum in $\mathcal{S}_1$ is effective, then so is every descent datum in $\mathcal{S}_2$. To do this, let $(X_i, \varphi_{ii'})$ be a descent datum in $\mathcal{S}_2$ relative the covering $\{U_i \to U\}$ of the site $\mathcal{C}$. Then $(G(X_i), G(\varphi_{ii'}))$ is a descent datum in $\mathcal{S}_1$ relative the covering $\{U_i \to U\}$. Let X be an object of $\mathcal{S}_{1,U}$ such that the descent datum (f_i^*X, can) is isomorphic to $(G(X_i), G(\varphi_{ii'}))$. Then $F(X)$ is an object of $\mathcal{S}_{2,U}$ such that the descent datum $(f_i^*F(X), can)$ is isomorphic to $(F(G(X_i)), F(G(\varphi_{ii'})))$ which in turn is isomorphic to the original descent datum $(X_i, \varphi_{ii'})$ using i. $\square$

The 2-category of stacks over $\mathcal{C}$ is defined as follows.

Definition 50.4.5. Let $\mathcal{C}$ be a site. The 2-*category of stacks over* $\mathcal{C}$ is the sub 2-category of the 2-category of fibred categories over $\mathcal{C}$ (see Categories, Definition 4.30.8) defined as follows:

(1) Its objects will be stacks $p : \mathcal{S} \to \mathcal{C}$.
(2) Its 1-morphisms $(\mathcal{S}, p) \to (\mathcal{S}', p')$ will be functors $G : \mathcal{S} \to \mathcal{S}'$ such that $p' \circ G = p$ and such that G maps strongly cartesian morphisms to strongly cartesian morphisms.
(3) Its 2-morphisms $t : G \to H$ for $G, H : (\mathcal{S}, p) \to (\mathcal{S}', p')$ will be morphisms of functors such that $p'(t_x) = \mathrm{id}_{p(x)}$ for all $x \in Ob(\mathcal{S})$.

Lemma 50.4.6. *Let* $\mathcal{C}$ *be a site. The* $(2,1)$*-category of stacks over* $\mathcal{C}$ *has 2-fibre products, and they are described as in Categories, Lemma 4.29.3.*

Proof. Let $f : \mathcal{X} \to \mathcal{S}$ and $g : \mathcal{Y} \to \mathcal{S}$ be 1-morphisms of stacks over $\mathcal{C}$ as defined above. The category $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ described in Categories, Lemma 4.29.3 is a fibred category according to Categories, Lemma 4.30.9. (This is where we use that f and g preserve strongly cartesian morphisms.) It remains to show that the morphism presheaves are sheaves and that descent relative to coverings of $\mathcal{C}$ is effective.

Recall that an object of $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ is given by a quadruple (U, x, y, ϕ). It lies over the object U of $\mathcal{C}$. Next, let (U, x', y', ϕ') be second object lying over U. Recall that $\phi : f(x) \to g(y)$, and $\phi' : f(x') \to g(y')$ are isomorphisms in the category $\mathcal{S}_U$. Let us use these isomorphisms to identify $z = f(x) = g(y)$ and $z' = f(x') = g(y')$. With this identifications it is clear that

$$Mor((U, x, y, \phi), (U, x', y', \phi')) = Mor(x, x') \times_{Mor(z,z')} Mor(y, y')$$

as presheaves. However, as the fibred product in the category of presheaves preserves sheaves (Sites, Lemma 9.10.1) we see that this is a sheaf.

Let $\mathcal{U} = \{f_i : U_i \to U\}_{i\in I}$ be a covering of the site $\mathcal{C}$. Let (X_i, χ_{ij}) be a descent datum in $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ relative to $\mathcal{U}$. Write $X_i = (U_i, x_i, y_i, \phi_i)$ as above. Write $\chi_{ij} = (\varphi_{ij}, \psi_{ij})$ as in the definition of the category $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ (see Categories, Lemma 4.29.3). It is clear that (x_i, φ_{ij}) is a descent datum in $\mathcal{X}$ and that (y_i, ψ_{ij}) is a descent datum in $\mathcal{Y}$. Since $\mathcal{X}$ and $\mathcal{Y}$ are stacks these descent data are effective. Thus we get $x \in Ob(\mathcal{X}_U)$, and $y \in Ob(\mathcal{Y}_U)$ with $x_i = x|_{U_i}$, and $y_i = y|_{U_i}$ compatibly with descent data. Set $z = f(x)$ and $z' = g(y)$ which are both objects of $\mathcal{S}_U$. The morphisms ϕ_i are elements of $Isom(z, z')(U_i)$ with the property that $\phi_i|_{U_i\times_U U_j} = \phi_j|_{U_i\times_U U_j}$. Hence by the sheaf property of $Isom(z, z')$ we obtain an isomorphism $\phi : z = f(x) \to z' = g(y)$. We omit the verification that the canonical descent datum associated to the object (U, x, y, ϕ) of $(\mathcal{X}\times_{\mathcal{S}}\mathcal{Y})_U$ is isomorphic to the descent datum we started with. □

Lemma 50.4.7. *Let $\mathcal{C}$ be a site. Let $\mathcal{S}_1$, $\mathcal{S}_2$ be stacks over $\mathcal{C}$. Let $F : \mathcal{S}_1 \to \mathcal{S}_2$ be a 1-morphism. Then the following are equivalent*

1. *F is fully faithful,*
2. *for every $U \in Ob(\mathcal{C})$ and for every $x, y \in Ob(\mathcal{S}_{1,U})$ the map*
$$F : Mor_{\mathcal{S}_1}(x, y) \longrightarrow Mor_{\mathcal{S}_2}(x, y)$$
is an isomorphism of sheaves on $\mathcal{C}/U$.

Proof. Omitted. □

Lemma 50.4.8. *Let $\mathcal{C}$ be a site. Let $\mathcal{S}_1$, $\mathcal{S}_2$ be stacks over $\mathcal{C}$. Let $F : \mathcal{S}_1 \to \mathcal{S}_2$ be a 1-morphism which is fully faithful. Then the following are equivalent*

1. *F is an equivalence,*
2. *for every $U \in Ob(\mathcal{C})$ and for every $x \in Ob(\mathcal{S}_{2,U})$ there exists a covering $\{f_i : U_i \to U\}$ such that f_i^*x is in the essential image of the functor $F : \mathcal{S}_{1,U_i} \to \mathcal{S}_{2,U_i}$.*

Proof. The implication (1) ⇒ (2) is immediate. To see that (2) implies (1) we have to show that every x as in (2) is in the essential image of the functor F. To do this choose a covering as in (2), $x_i \in Ob(\mathcal{S}_{1,U_i})$, and isomorphisms $\varphi_i : F(x_i) \to f_i^*x$. Then we get a descent datum for $\mathcal{S}_1$ relative to $\{f_i : U_i \to U\}$ by taking

$$\varphi_{ij} : x_i|_{U_i\times_U U_j} \longrightarrow x_j|_{U_i\times_U U_j}$$

the arrow such that $F(\varphi_{ij}) = \varphi_j^{-1} \circ \varphi_i$. This descent datum is effective by the axioms of a stack, and hence we obtain an object x_1 of $\mathcal{S}_1$ over U. We omit the verification that $F(x_1)$ is isomorphic to x over U. □

Remark 50.4.9. (Cutting down a ``big'' stack to get a stack.) Let $\mathcal{C}$ be a site. Suppose that $p : \mathcal{S} \to \mathcal{C}$ is functor from a ``big'' category to $\mathcal{C}$, i.e., suppose that the collection of objects of $\mathcal{S}$ forms a proper class. Finally, suppose that $p : \mathcal{S} \to \mathcal{C}$ satisfies conditions (1), (2), (3) of Definition 50.4.1. In general there is no way to replace $p : \mathcal{S} \to \mathcal{C}$ by a equivalent category such that we obtain a stack. The reason is that it can happen that a fibre categories $\mathcal{S}_U$ may have a proper class of isomorphism classes of objects. On the other hand, suppose that

(4) for every $U \in Ob(\mathcal{C})$ there exists a set $S_U \subset Ob(\mathcal{S}_U)$ such that every object of $\mathcal{S}_U$ is isomorphic in $\mathcal{S}_U$ to an element of S_U.

In this case we can find a full subcategory $\mathcal{S}_{small}$ of $\mathcal{S}$ such that, setting $p_{small} = p|_{\mathcal{S}_{small}}$, we have

(a) the functor $p_{small} : \mathcal{S}_{small} \to \mathcal{C}$ defines a stack, and
(b) the inclusion $\mathcal{S}_{small} \to \mathcal{S}$ is fully faithful and essentially surjective.

(Hint: For every $U \in Ob(\mathcal{C})$ let $\alpha(U)$ denote the smallest ordinal such that $Ob(\mathcal{S}_U) \cap V_{\alpha(U)}$ surjects onto the set of isomorphism classes of $\mathcal{S}_U$, and set $\alpha = \sup_{U \in Ob(\mathcal{C})} \alpha(U)$. Then take $Ob(\mathcal{S}_{small}) = Ob(\mathcal{S}) \cap V_\alpha$. For notation used see Sets, Section 3.5.)

50.5. Stacks in groupoids

Among stacks those which are fibred in groupoids are somewhat easier to comprehend. We redefine them as follows.

Definition 50.5.1. A *stack in groupoids* over a site $\mathcal{C}$ is a category $p : \mathcal{S} \to \mathcal{C}$ over $\mathcal{C}$ such that

(1) $p : \mathcal{S} \to \mathcal{C}$ is fibred in groupoids over $\mathcal{C}$ (see Categories, Definition 4.32.1),
(2) for all $U \in Ob(\mathcal{C})$, for all $x, y \in Ob(\mathcal{S}_U)$ the presheaf $Isom(x, y)$ is a sheaf on the site $\mathcal{C}/U$, and
(3) for all coverings $\mathcal{U} = \{U_i \to U\}$ in $\mathcal{C}$, all descent data (x_i, ϕ_{ij}) for $\mathcal{U}$ are effective.

Usually the hardest part to check is the third condition. Here is the lemma comparing this with the notion of a stack.

Lemma 50.5.2. *Let $\mathcal{C}$ be a site. Let $p : \mathcal{S} \to \mathcal{C}$ be a category over $\mathcal{C}$. The following are equivalent*

(1) *$\mathcal{S}$ is a stack in groupoids over $\mathcal{C}$,*
(2) *$\mathcal{S}$ is a stack over $\mathcal{C}$ and all fibre categories are groupoids, and*
(3) *$\mathcal{S}$ is fibred in groupoids over $\mathcal{C}$ and is a stack over $\mathcal{C}$.*

Proof. Omitted, but see Categories, Lemma 4.32.2. ☐

Lemma 50.5.3. *Let $\mathcal{C}$ be a site. Let $p : \mathcal{S} \to \mathcal{C}$ be a stack. Let $p' : \mathcal{S}' \to \mathcal{C}$ be the category fibred in groupoids associated to $\mathcal{S}$ constructed in Categories, Lemma 4.32.3. Then $p' : \mathcal{S}' \to \mathcal{C}$ is a stack in groupoids.*

Proof. Recall that the morphisms in $\mathcal{S}'$ are exactly the strongly cartesian morphisms of $\mathcal{S}$, and that any isomorphism of $\mathcal{S}$ is such a morphism. Hence descent data in $\mathcal{S}'$ are exactly the same thing as descent data in $\mathcal{S}$. Now apply Lemma 50.4.2. Some details omitted. ☐

Lemma 50.5.4. *Let $\mathcal{C}$ be a site. Let $\mathcal{S}_1$, $\mathcal{S}_2$ be categories over $\mathcal{C}$. Suppose that $\mathcal{S}_1$ and $\mathcal{S}_2$ are equivalent as categories over $\mathcal{C}$. Then $\mathcal{S}_1$ is a stack in groupoids over $\mathcal{C}$ if and only if $\mathcal{S}_2$ is a stack in groupoids over $\mathcal{C}$.*

Proof. Follows by combining Lemmas 50.5.2 and 50.4.4. $\square$

The 2-category of stacks in groupoids over $\mathcal{C}$ is defined as follows.

Definition 50.5.5. Let $\mathcal{C}$ be a site. The 2-*category of stacks in groupoids over* $\mathcal{C}$ is the sub 2-category of the 2-category of stacks over $\mathcal{C}$ (see Definition 50.4.5) defined as follows:

(1) Its objects will be stacks in groupoids $p : \mathcal{S} \to \mathcal{C}$.
(2) Its 1-morphisms $(\mathcal{S}, p) \to (\mathcal{S}', p')$ will be functors $G : \mathcal{S} \to \mathcal{S}'$ such that $p' \circ G = p$. (Since every morphism is strongly cartesian every functor preserves them.)
(3) Its 2-morphisms $t : G \to H$ for $G, H : (\mathcal{S}, p) \to (\mathcal{S}', p')$ will be morphisms of functors such that $p'(t_x) = \mathrm{id}_{p(x)}$ for all $x \in Ob(\mathcal{S})$.

Note that any 2-morphism is automatically an isomorphism, so that in fact the 2-category of stacks in groupoids over $\mathcal{C}$ is a (strict) $(2, 1)$-category.

Lemma 50.5.6. *Let $\mathcal{C}$ be a category. The* 2*-category of stacks in groupoids over $\mathcal{C}$ has 2-fibre products, and they are described as in Categories, Lemma 4.29.3.*

Proof. This is clear from Categories, Lemma 4.32.7 and Lemmas 50.5.2 and 50.4.6. $\square$

50.6. Stacks in setoids

This is just a brief section saying that a stack in sets is the same thing as a sheaf of sets. Please consult Categories, Section 4.36 for notation.

Definition 50.6.1. Let $\mathcal{C}$ be a site.

(1) A *stack in setoids* over $\mathcal{C}$ is a stack over $\mathcal{C}$ all of whose fibre categories are setoids.
(2) A *stack in sets*, or a *stack in discrete categories* is a stack over $\mathcal{C}$ all of whose fibre categories are discrete.

From the discussion in Section 50.5 this is the same thing as a stack in groupoids whose fibre categories are setoids (resp. discrete). Moreover, it is also the same thing as a category fibred in setoids (resp. sets) which is a stack.

Lemma 50.6.2. *Let $\mathcal{C}$ be a site. Under the equivalence*

$$\left\{\begin{matrix} \textit{the category of presheaves} \\ \textit{of sets over } \mathcal{C} \end{matrix}\right\} \leftrightarrow \left\{\begin{matrix} \textit{the category of categories} \\ \textit{fi}\mathcal{C} \end{matrix}\right\}$$

of Categories, Lemma 4.35.6 the stacks in sets correspond precisely to the sheaves.

Proof. Omitted. Hint: Show that effectivity of descent corresponds exactly to the sheaf condition. $\square$

Lemma 50.6.3. *Let $\mathcal{C}$ be a site. Let $\mathcal{S}$ be a category fibred in setoids over $\mathcal{C}$. Then $\mathcal{S}$ is a stack in setoids if and only if the unique equivalent category $\mathcal{S}'$ fibred in sets (see Categories, Lemma 4.36.5) is a stack in sets. In other words, if and only if the presheaf*

$$U \longmapsto Ob(\mathcal{S}_U)/\cong$$

is a sheaf.

Proof. Omitted. $\square$

Lemma 50.6.4. *Let $\mathcal{C}$ be a site. Let $\mathcal{S}_1$, $\mathcal{S}_2$ be categories over $\mathcal{C}$. Suppose that $\mathcal{S}_1$ and $\mathcal{S}_2$ are equivalent as categories over $\mathcal{C}$. Then $\mathcal{S}_1$ is a stack in setoids over $\mathcal{C}$ if and only if $\mathcal{S}_2$ is a stack in setoids over $\mathcal{C}$.*

Proof. By Categories, Lemma 4.36.5 we see that a category $\mathcal{S}$ over $\mathcal{C}$ is fibred in setoids over $\mathcal{C}$ if and only if it is equivalent over $\mathcal{C}$ to a category fibred in sets. Hence we see that $\mathcal{S}_1$ is fibred in setoids over $\mathcal{C}$ if and only if $\mathcal{S}_2$ is fibred in setoids over $\mathcal{C}$. Hence now the lemma follows from Lemma 50.6.3. ☐

The 2-category of stacks in setoids over $\mathcal{C}$ is defined as follows.

Definition 50.6.5. Let $\mathcal{C}$ be a site. The 2-*category of stacks in setoids over* $\mathcal{C}$ is the sub 2-category of the 2-category of stacks over $\mathcal{C}$ (see Definition 50.4.5) defined as follows:

(1) Its objects will be stacks in setoids $p : \mathcal{S} \to \mathcal{C}$.
(2) Its 1-morphisms $(\mathcal{S}, p) \to (\mathcal{S}', p')$ will be functors $G : \mathcal{S} \to \mathcal{S}'$ such that $p' \circ G = p$. (Since every morphism is strongly cartesian every functor preserves them.)
(3) Its 2-morphisms $t : G \to H$ for $G, H : (\mathcal{S}, p) \to (\mathcal{S}', p')$ will be morphisms of functors such that $p'(t_x) = \mathrm{id}_{p(x)}$ for all $x \in Ob(\mathcal{S})$.

Note that any 2-morphism is automatically an isomorphism, so that in fact the 2-category of stacks in setoids over $\mathcal{C}$ is a (strict) (2, 1)-category.

Lemma 50.6.6. *Let $\mathcal{C}$ be a site. The 2-category of stacks in setoids over $\mathcal{C}$ has 2-fibre products, and they are described as in Categories, Lemma 4.29.3.*

Proof. This is clear from Categories, Lemmas 4.32.7 and 4.36.4 and Lemmas 50.5.2 and 50.4.6. ☐

Lemma 50.6.7. *Let $\mathcal{C}$ be a site. Let $\mathcal{S}, \mathcal{T}$ be stacks in groupoids over $\mathcal{C}$ and let $\mathcal{R}$ be a stack in setoids over $\mathcal{C}$. Let $f : \mathcal{T} \to \mathcal{S}$ and $g : \mathcal{R} \to \mathcal{S}$ be 1-morphisms. If f is faithful, then the 2-fibre product*

$$\mathcal{T} \times_{f,\mathcal{S},g} \mathcal{R}$$

is a stack in setoids over $\mathcal{C}$.

Proof. Immediate from the explict description of the 2-fibre product in Categories, Lemma 4.29.3. ☐

Lemma 50.6.8. *Let $\mathcal{C}$ be a site. Let $\mathcal{S}$ be a stack in groupoids over $\mathcal{C}$ and let $\mathcal{S}_i$, $i = 1, 2$ be stacks in setoids over $\mathcal{C}$. Let $f_i : \mathcal{S}_i \to \mathcal{S}$ be 1-morphisms. Then the 2-fibre product*

$$\mathcal{S}_1 \times_{f_1,\mathcal{S},f_2} \mathcal{S}_2$$

is a stack in setoids over $\mathcal{C}$.

Proof. This is a special case of Lemma 50.6.7 as f_2 is faithful. ☐

Lemma 50.6.9. *Let $\mathcal{C}$ be a site. Let*

$$\begin{tikzcd} \mathcal{T}_2 \arrow[r] \arrow[d, "G'"'] & \mathcal{T}_1 \arrow[d, "G"] \\ \mathcal{S}_2 \arrow[r, "F"] & \mathcal{S}_1 \end{tikzcd}$$

be a 2-cartesian diagram of stacks in groupoids over $\mathcal{C}$. Assume

(1) *for every $U \in Ob(\mathcal{C})$ and $x \in Ob((\mathcal{S}_1)_U)$ there exists a covering $\{U_i \to U\}$ such that $x|_{U_i}$ is in the essential image of $F : (\mathcal{S}_2)_{U_i} \to (\mathcal{S}_1)_{U_i}$, and*
(2) *G' is faithful,*

then G is faithful.

Proof. We may assume that $\mathcal{T}_2$ is the category $\mathcal{S}_2 \times_{\mathcal{S}_1} \mathcal{T}_1$ described in Categories, Lemma 4.29.3. By Categories, Lemma 4.32.8 the faithfulness of G, G' can be checked on fibre categories. Suppose that y, y' are objects of $\mathcal{T}_1$ over the object U of $\mathcal{C}$. Let $\alpha, \beta : y \to y'$ be morphisms of $(\mathcal{T}_1)_U$ such that $G(\alpha) = G(\beta)$. Our object is to show that $\alpha = \beta$. Considering instead $\gamma = \alpha^{-1} \circ \beta$ we see that $G(\gamma) = \mathrm{id}_{G(y)}$ and we have to show that $\gamma = \mathrm{id}_y$. By assumption we can find a covering $\{U_i \to U\}$ such that $G(y)|_{U_i}$ is in the essential image of $F : (\mathcal{S}_2)_{U_i} \to (\mathcal{S}_1)_{U_i}$. Since it suffices to show that $\gamma|_{U_i} = \mathrm{id}$ for each i, we may therefore assume that we have $f : F(x) \to G(y)$ for some object x of $\mathcal{S}_2$ over U and morphisms f of $(\mathcal{S}_1)_U$. In this case we get a morphism

$$(1, \gamma) : (U, x, y, f) \longrightarrow (U, x, y, f)$$

in the fibre category of $\mathcal{S}_2 \times_{\mathcal{S}_1} \mathcal{T}_1$ over U whose image under G' in $\mathcal{S}_1$ is id_x. As G' is faithful we conclude that $\gamma = \mathrm{id}_y$ and we win. $\square$

Lemma 50.6.10. *Let $\mathcal{C}$ be a site. Let*

$$\begin{tikzcd} \mathcal{T}_2 \arrow[r] \arrow[d] & \mathcal{T}_1 \arrow[d, "G"] \\ \mathcal{S}_2 \arrow[r, "F"] & \mathcal{S}_1 \end{tikzcd}$$

be a 2-cartesian diagram of stacks in groupoids over $\mathcal{C}$. If

(1) *$F : \mathcal{S}_2 \to \mathcal{S}_1$ is fully faithful,*
(2) *for every $U \in Ob(\mathcal{C})$ and $x \in Ob((\mathcal{S}_1)_U)$ there exists a covering $\{U_i \to U\}$ such that $x|_{U_i}$ is in the essential image of $F : (\mathcal{S}_2)_{U_i} \to (\mathcal{S}_1)_{U_i}$, and*
(3) *$\mathcal{T}_2$ is a stack in setoids.*

then $\mathcal{T}_1$ is a stack in setoids.

Proof. We may assume that $\mathcal{T}_2$ is the category $\mathcal{S}_2 \times_{\mathcal{S}_1} \mathcal{T}_1$ described in Categories, Lemma 4.29.3. Pick $U \in Ob(\mathcal{C})$ and $y \in Ob((\mathcal{T}_1)_U)$. We have to show that the sheaf $Aut(y)$ on $\mathcal{C}/U$ is trivial. To to this we may replace U by the members of a covering of U. Hence by assumption (2) we may assume that there exists an object $x \in Ob((\mathcal{S}_2)_U)$ and an isomorphism $f : F(x) \to G(y)$. Then $y' = (U, x, y, f)$ is an object of $\mathcal{T}_2$ over U which is mapped to y under the projection $\mathcal{T}_2 \to \mathcal{T}_1$. Because F is fully faithful by (1) the map $Aut(y') \to Aut(y)$ is surjective, use the explicit description of morphisms in $\mathcal{T}_2$ in Categories, Lemma 4.29.3. Since by (3) the sheaf $Aut(y')$ is trivial we get the the result of the lemma. $\square$

50.7. The inertia stack

Let $p : \mathcal{S} \to \mathcal{C}$ and $p' : \mathcal{S}' \to \mathcal{C}$ be fibred categories over the category $\mathcal{C}$. Let $F : \mathcal{S} \to \mathcal{S}'$ be a 1-morphism of fibred categories over $\mathcal{C}$. Recall that we have defined in Categories, Definition 4.31.2 an *relative inertia fibred category* $\mathcal{I}_{\mathcal{S}/\mathcal{S}'} \to \mathcal{C}$ as the category whose objects are pairs (x, α) where $x \in Ob(\mathcal{S})$ and $\alpha : x \to x$ with $F(\alpha) = \mathrm{id}_{F(x)}$. There is also an absolute version, namely the *inertia* $\mathcal{I}_{\mathcal{S}}$ of $\mathcal{S}$. These inertia categories are actually stacks over $\mathcal{C}$ provided that $\mathcal{S}$ and $\mathcal{S}'$ are stacks.

Lemma 50.7.1. *Let $\mathcal{C}$ be a site. Let $p : \mathcal{S} \to \mathcal{C}$ and $p' : \mathcal{S}' \to \mathcal{C}$ be stacks over the site $\mathcal{C}$. Let $F : \mathcal{S} \to \mathcal{S}'$ be a 1-morphism of stacks over $\mathcal{C}$.*

(1) *The inertia $\mathcal{I}_{\mathcal{S}/\mathcal{S}'}$ and $\mathcal{I}_{\mathcal{S}}$ are stacks over $\mathcal{C}$.*
(2) *If $\mathcal{S}, \mathcal{S}'$ are stacks in groupoids over $\mathcal{S}$, then so are $\mathcal{I}_{\mathcal{S}/\mathcal{S}'}$ and $\mathcal{I}_{\mathcal{S}}$.*
(3) *If $\mathcal{S}, \mathcal{S}'$ are stacks in setoids over $\mathcal{S}$, then so are $\mathcal{I}_{\mathcal{S}/\mathcal{S}'}$ and $\mathcal{I}_{\mathcal{S}}$.*

Proof. The first three assertions follow from Lemmas 50.4.6, 50.5.6, and 50.6.6 and the equivalence in Categories, Lemma 4.31.1 part (1). □

Lemma 50.7.2. *Let $\mathcal{C}$ be a site. If $\mathcal{S}$ is a stack in groupoids, then the canonical* 1*-morphism $\mathcal{I}_\mathcal{S} \to \mathcal{S}$ is an equivalence if and only if $\mathcal{S}$ is a stack in setoids.*

Proof. Follows directly from Categories, Lemma 4.36.7. □

50.8. Stackification of fibred categories

Here is the result.

Lemma 50.8.1. *Let $\mathcal{C}$ be a site. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category over $\mathcal{C}$. There exists a stack $p' : \mathcal{S}' \to \mathcal{C}$ and a* 1*-morphism $G : \mathcal{S} \to \mathcal{S}'$ of fibred categories over $\mathcal{C}$ (see Categories, Definition 4.30.8) such that*

1. *for every $U \in Ob(\mathcal{C})$, and any $x, y \in Ob(\mathcal{S}_U)$ the map*
$$Mor(x, y) \longrightarrow Mor(G(x), G(y))$$
induced by G identifies the right hand side with the sheafification of the left hand side, and
2. *for every $U \in Ob(\mathcal{C})$, and any $x' \in Ob(\mathcal{S}'_U)$ there exists a covering $\{U_i \to U\}_{i\in I}$ such that for every $i \in I$ the object $x'|_{U_i}$ is in the essential image of the functor $G : \mathcal{S}_U \to \mathcal{S}'_U$.*

Moreover the stack $\mathcal{S}'$ is determined up to unique 2*-isomorphism by these conditions.*

Proof by naive method. In this proof method we proceed in stages:

First, given x lying over U and any object y of $\mathcal{S}$, we say that two morphisms $a, b : x \to y$ of $\mathcal{S}$ lying over the same arrow of $\mathcal{C}$ are *locally equal* if there exists a covering $\{f_i : U_i \to U\}$ of $\mathcal{C}$ such that the compositions
$$f_i^*x \to x \xrightarrow{a} y, \quad f_i^*x \to x \xrightarrow{b} y$$
are equal. This gives an equivalence relation $\sim$ on arrows of $\mathcal{S}$. If $b \sim b'$ then $a\circ b\circ c \sim a\circ b'\circ c$ (verification omitted). Hence we can quotient out by this equivalence relation to obtain a new category $\mathcal{S}^1$ over $\mathcal{C}$ to gether with a morphism $G^1 : \mathcal{S} \to \mathcal{S}^1$.

One checks that G^1 preserves strongly cartesian morphisms and that $\mathcal{S}^1$ is a fibred category over $\mathcal{C}$. Checks omitted. Thus we reduce to the case where locally equal morphisms are equal.

Next, we add morphisms as follows. Given x lying over U and any object y of lying over V a *locally defined morphism from x to y* is given by

1. a morphism $f : U \to V$,
2. a covering $\{f_i : U_i \to U\}$ of U, and
3. morphisms $a_i : f_i^*x \to Y$ with $p(a_i) = h \circ f_i$

with the property that the compositions
$$(f_i \times f_j)^*x \to f_i^*x \xrightarrow{a_i} y, \quad (f_i \times f_j)^*x \to f_j^*x \xrightarrow{a_j} y$$
are equal. Note that a usual morphism $a : x \to y$ gives a locally defined morphism $(p(a) : U \to V, \{\mathrm{id}_U\}, a)$. We say two locally defined morphisms $(f, \{f_i : U_i \to U\}, a_i)$ and $(g, \{g_j : U_i \to U\}, b_j)$ are *equal* if $f = g$ and the compositions
$$(f_i \times g_j)^*x \to f_i^*x \xrightarrow{a_i} y, \quad (f_i \times g_j)^*x \to g_j^*x \xrightarrow{b_j} y$$

are equal (this is the right condition since we are in the situation where locally equal morphisms are equal). To compose locally defined morphisms $(f, \{f_i : U_i \to U\}, a_i)$ from x to y and $(g, \{g_j : V_j \to V\}, b_j)$ from y to z lying over W, just take $g \circ f : U \to W$, the covering $\{U_i \times_V V_j \to U\}$, and as maps the compositions

$$x|_{U_i \times_V V_j} \xrightarrow{\text{pr}_0^* a_i} y|_{V_j} \xrightarrow{b_j} z$$

We omit the verification that this is a locally defined morphism.

One checks that $\mathcal{S}^2$ with the same objects as $\mathcal{S}$ and with locally defined morphisms as morphisms is a category over $\mathcal{C}$, that there is a functor $G^2 : \mathcal{S} \to \mathcal{S}^2$ over $\mathcal{C}$, that this functor preserves strongly cartesian objects, and that $\mathcal{S}^2$ is a fibred category over $\mathcal{C}$. Checks omitted. This reduces one to the case where the morphism presheaves of $\mathcal{S}$ are all sheaves, by checking that the effect of using locally defined morphisms is to take the sheafification of the (separated) morphisms presheaves.

Finally, in the case where the morphism presheaves are all sheaves we have to add objects in order to make sure descent conditions are effective in the end result. The simplest way to do this is to consider the category $\mathcal{S}'$ whose objects are pairs $(\mathcal{U}, \xi)$ where $\mathcal{U} = \{U_i \to U\}$ is a covering of $\mathcal{C}$ and $\xi = (X_i, \varphi_{ii'})$ is a descent datum relative $\mathcal{U}$. Suppose given two such data $(\mathcal{U}, \xi) = (\{f_i : U_i \to U\}, x_i, \varphi_{ii'})$ and $(\mathcal{V}, \eta) = (\{g_j : V_j \to V\}, y_j, \psi_{jj'})$. We define

$$Mor_{\mathcal{S}'}((\mathcal{U}, \xi), (\mathcal{V}, \eta))$$

as the set of (f, a_{ij}), where $f : U \to V$ and

$$a_{ij} : x_i|_{U_i \times_V V_j} \longrightarrow y_j$$

are morphisms of $\mathcal{S}$ lying over $U_i \times_V V_j \to V_j$. These have to satisfy the following condition: for any $i, i' \in I$ and $j, j' \in J$ set $W = (U_i \times_U U_{i'}) \times_V (V_j \times_V V_{j'})$. Then

$$\begin{tikzcd} x_i|_W \arrow[r, "a_{ij}|_W"'] \arrow[d, "\varphi_{ii'}|_W"'] & y_j|_W \arrow[d, "\psi_{jj'}|_W"] \\ x_{i'}|_W \arrow[r, "a_{i'j'}|_W"] & y_{j'}|_W \end{tikzcd}$$

commutes. At this point you have to verify the following things:

(1) there is a well defined composition on morphisms as above,
(2) this turns $\mathcal{S}'$ into a category over $\mathcal{C}$,
(3) there is a functor $G : \mathcal{S} \to \mathcal{S}'$ over $\mathcal{C}$,
(4) for x, y objects of $\mathcal{S}$ we have $Mor_{\mathcal{S}}(x, y) = Mor_{\mathcal{S}'}(G(x), G(y))$,
(5) any object of $\mathcal{S}'$ locally comes from an object of $\mathcal{S}$, i.e., part (2) of the lemma holds,
(6) G preserves strongly cartesian morphisms,
(7) $\mathcal{S}'$ is a fibred category over $\mathcal{C}$, and
(8) $\mathcal{S}'$ is a stack over $\mathcal{C}$.

This is all not hard but there is a lot of it. Details omitted. □

Less naive proof. Here is a less naive proof. By Categories, Lemma 4.33.4 there exists an equivalence of fibred categories $\mathcal{S} \to \mathcal{S}'$ where $\mathcal{S}'$ is a split fibred category, i.e., one in which the pullback functors compose on the nose. Obviously the lemma for $\mathcal{S}'$ implies the lemma for $\mathcal{S}$. Hence we may think of $\mathcal{S}$ as a presheaf in categories.

Consider the 2-category *Cat* temporarily as a category by forgetting about 2-morphisms. Let us think of a category as a quintuple $(\text{Ob}, \text{Arrows}, s, t, \circ)$ as in Categories, Section 4.2. Consider the forgetful functor

$$forget : Cat \to Sets, \quad (\text{Ob}, \text{Arrows}, s, t, \circ) \mapsto \text{Ob} \coprod \text{Arrows}.$$

Then $forget$ is faithful, *Cat* has limits and $forget$ commutes with them, *Cat* has directed colimits and $forget$ commutes with them, and $forget$ reflects isomorphisms. Hence, according to the first part of Sites, Section 9.38 we can sheafify presheaves with values in *Cat*, and the result commutes with $forget$. Applying this to $\mathcal{S}$ we obtain a sheafification $\mathcal{S}^\#$ which has a sheaf of objects and a sheaf of morphisms both of which are the sheafifications of the corresponding presheaves for $\mathcal{S}$. In this case it is quite easy to see that the map $\mathcal{S} \to \mathcal{S}^\#$ has the properties (1) and (2) of the lemma.

However, the category $\mathcal{S}^\#$ may not yet be a stack since, allthough the presheaf of objects is a sheaf, the descent condition may not yet be satisfied. To remedy this we have to add more objects. But the argument above does reduce us to the case where $\mathcal{S} = \mathcal{S}_F$ for some sheaf(!) $F : \mathcal{C}^{opp} \to Cat$ of categories. In this case consider the functor $F' : \mathcal{C}^{opp} \to Cat$ defined by

1. The set $Ob(F'(U))$ is the set of pairs $(\mathcal{U}, \xi)$ where $\mathcal{U} = \{U_i \to U\}$ is a covering of U and $\xi = (x_i, \varphi_{ii'})$ is a descent datum relative to $\mathcal{U}$.
2. A morphism in $F'(U)$ from $(\mathcal{U}, \xi)$ to $(\mathcal{V}, \eta)$ is an element of
$$colim\ Mor_{DD(\mathcal{W})}(a^*\xi, b^*\eta)$$
where the colimit is over all common refinements $a : \mathcal{W} \to \mathcal{U}$, $b : \mathcal{W} \to \mathcal{V}$. This colimit is filtered (verification omitted). Hence composition of morphisms in $F(U)$ is defined by finding a common refinement and composing in $DD(\mathcal{W})$.
3. Given $h : V \to U$ and an object $(\mathcal{U}, \xi)$ of $F'(U)$ we set $F'(h)(\mathcal{U}, \xi)$ equal to $(V \times_U \mathcal{U}, \text{pr}_1^*\xi)$. More precisely, if $\mathcal{U} = \{U_i \to U\}$ and $\xi = (x_i, \varphi_{ii'})$, then $V \times_U \mathcal{U} = \{V \times_U U_i \to V\}$ which comes with a canonical morphism $\text{pr}_1 : V \times_U \mathcal{U} \to \mathcal{U}$ and $\text{pr}_1^*\xi$ is the pullback of ξ with respect to this morphism (see Definition 50.3.4).
4. Given $h : V \to U$, objects $(\mathcal{U}, \xi)$ and $(\mathcal{V}, \eta)$ and a morphism between them, represented by $a : \mathcal{W} \to \mathcal{U}$, $b : \mathcal{W} \to \mathcal{V}$, and $\alpha : a^*\xi \to b^*\eta$, then $F'(h)(\alpha)$ is represented by $a' : V \times_U \mathcal{W} \to V \times_U \mathcal{U}$, $b' : V \times_U \mathcal{W} \to V \times_U \mathcal{V}$, and the pullback α' of the morphism α via the map $V \times_U \mathcal{W} \to \mathcal{W}$. This works since pullbacks in $\mathcal{S}_F$ commute on the nose.

There is a map $F \to F'$ given by associating to an object x of $F(U)$ the object $(\{U \to U\}, (x, triv))$ of $F'(U)$. At this point you have to check that the corresponding functor $\mathcal{S}_F \to \mathcal{S}_{F'}$ has properties (1) and (2) of the lemma, and finally that $\mathcal{S}_{F'}$ is a stack. Details omitted. □

Lemma 50.8.2. *Let $\mathcal{C}$ be a site. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category over $\mathcal{C}$. Let $p' : \mathcal{S}' \to \mathcal{C}$ and $G : \mathcal{S} \to \mathcal{S}'$ the the stack and 1-morphism constructed in Lemma 50.8.1. This construction has the following universal property: Given a stack $q : \mathcal{X} \to \mathcal{C}$ and a 1-morphism $F : \mathcal{S} \to \mathcal{S}'$ of fibred categories over $\mathcal{C}$ there exists a 1-morphism $H : \mathcal{S}' \to \mathcal{X}$ such that the diagram*

$$\begin{tikzcd} \mathcal{S} \arrow[rr, "F"] \arrow[dr, "G"'] & & \mathcal{X} \\ & \mathcal{S}' \arrow[ur, "H"'] & \end{tikzcd}$$

is 2*-commutative.*

Proof. Omitted. Hint: Suppose that $x' \in Ob(\mathcal{S}'_U)$. By the result of Lemma 50.8.1 there exists a covering $\{U_i \to U\}_{i \in I}$ such that $x'|_{U_i} = G(x_i)$ for some $x_i \in Ob(\mathcal{S}_{U_i})$. Moreover, there exist coverings $\{U_{ijk} \to U_i \times_U U_j\}$ and isomorphisms $\alpha_{ijk} : x_i|_{U_{ijk}} \to x_j|_{U_{ijk}}$ with $G(\alpha_{ijk}) = \mathrm{id}_{x'|_{U_{ijk}}}$. Set $y_i = F(x_i)$. Then you can check that

$$F(\alpha_{ijk}) : y_i|_{U_{ijk}} \to y_j|_{U_{ijk}}$$

agree on overlaps and therefore (as $\mathcal{X}$ is a stack) define a morphism $\beta_{ij} : y_i|_{U_i \times_U U_j} \to y_j|_{U_i \times_U U_j}$. Next, you check that the β_{ij} define a descent datum. Since $\mathcal{X}$ is a stack these descent data are effective and we find an object y of $\mathcal{X}_U$ agreeing with $G(x_i)$ over U_i. The hint is to set $H(x') = y$. $\square$

Lemma 50.8.3. *Notation and assumptions as in Lemma 50.8.2. There is a canonical equivalence of categories*

$$Mor_{Fib/\mathcal{C}}(\mathcal{S}, \mathcal{X}) = Mor_{Stacks/\mathcal{C}}(\mathcal{S}', \mathcal{X})$$

given by the constructions in the proof of the aforementioned lemma.

Proof. Omitted. $\square$

Lemma 50.8.4. *Let $\mathcal{C}$ be a site. Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Z} \to \mathcal{Y}$ be morphisms of fibred categories over $\mathcal{C}$. In this case the stackification of the* 2*-fibre product is the* 2*-fibre product of the stackifications.*

Proof. Let us denote $\mathcal{X}', \mathcal{Y}', \mathcal{Z}'$ the stackifications and $\mathcal{W}$ the stackification of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$. By construction of 2-fibre products there is a canonical 1-morphism $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{X}' \times_{\mathcal{Y}'} \mathcal{Z}'$. As the second 2-fibre product is a stack (see Lemma 50.4.6) this 1-morphism induces a 1-morphism $h : \mathcal{W} \to \mathcal{X}' \times_{\mathcal{Y}'} \mathcal{Z}'$ by the universal property of stackification, see Lemma 50.8.2. Now h is a morphism of stacks, and we may check that it is an equivalence using Lemmas 50.4.7 and 50.4.8.

Thus we first prove that h induces isomorphisms of Mor-sheaves. Let ξ, ξ' be objects of $\mathcal{W}$ over $U \in Ob(\mathcal{C})$. We want to show that

$$h : Mor(\xi, \xi') \longrightarrow Mor(h(\xi), h(\xi'))$$

is an isomorphism. To do this we may work locally on U (see Sites, Section 9.22). Hence by construction of $\mathcal{W}$ (see Lemma 50.8.1) we may assume that ξ, ξ' actually come from objects (x, y, α) and (x', y', α') of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ over U. By the same lemma once more we see that in this case $Mor(\xi, \xi')$ is the sheafification of

$$V/U \longmapsto Mor_{\mathcal{X}_V}(x|_V, x'|_V) \times_{Mor_{\mathcal{Z}_V}(f(x)|_V, f(x')|_V)} Mor_{\mathcal{Y}_V}(y|_V, y'|_V)$$

and that $Mor(h(\xi), h(\xi'))$ is equal to the fibre product

$$Mor(i(x), i(x')) \times_{Mor(k(f(x)), k(f(x')))} Mor(j(x), j(x'))$$

where $i : \mathcal{X} \to \mathcal{X}'$, $j : \mathcal{Y} \to \mathcal{Y}'$, and $k : \mathcal{Z} \to \mathcal{Z}'$ are the canonical functors. Thus the first displayed map of this paragraph is an isomorphism as sheafification is exact (and hence the sheafification of a fibre product of presheaves is the fibre product of the sheafifications).

Finally, we have to check that any object of $\mathcal{X}' \times_{\mathcal{Y}'} \mathcal{Z}'$ over U is locally on U in the essential image of h. Write such an object as a triple (x', y', α). Then x' locally comes from an object of $\mathcal{X}$, y' locally comes from an object of $\mathcal{Y}$, and having made suitable replacements for x', y' the morphism α of $\mathcal{Z}'_U$ locally comes from a morphism of $\mathcal{Z}$. In other words, we have shown that any object of $\mathcal{X}' \times_{\mathcal{Y}'} \mathcal{Z}'$ over U is locally on U in the essential image of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{X}' \times_{\mathcal{Y}'} \mathcal{Z}'$, hence a fortiori it is locally in the essential image of h. $\square$

Lemma 50.8.5. *Let $\mathcal{C}$ be a site. Let $\mathcal{X}$ be a fibred category over $\mathcal{C}$. The stackification of the inertia fibred category $\mathcal{I}_{\mathcal{X}}$ is intertia of the stackification of $\mathcal{X}$.*

Proof. This follows from the fact that stackification is compatible with 2-fibre products by Lemma 50.8.4 and the fact that there is a formula for the inertia in terms of 2-fibre products of categories over $\mathcal{C}$, see Categories, Lemma 4.31.1. $\square$

50.9. Stackification of categories fibred in groupoids

Here is the result.

Lemma 50.9.1. *Let $\mathcal{C}$ be a site. Let $p : \mathcal{S} \to \mathcal{C}$ be a category fibred in groupoids over $\mathcal{C}$. There exists a stack in groupoids $p' : \mathcal{S}' \to \mathcal{C}$ and a 1-morphism $G : \mathcal{S} \to \mathcal{S}'$ of categories fibred in groupoids over $\mathcal{C}$ (see Categories, Definition 4.32.6) such that*

(1) *for every $U \in Ob(\mathcal{C})$, and any $x, y \in Ob(\mathcal{S}_U)$ the map*
$$Mor(x, y) \longrightarrow Mor(G(x), G(y))$$
induced by G identifies the right hand side with the sheafification of the left hand side, and

(2) *for every $U \in Ob(\mathcal{C})$, and any $x' \in Ob(\mathcal{S}'_U)$ there exists a covering $\{U_i \to U\}_{i\in I}$ such that for every $i \in I$ the object $x'|_{U_i}$ is in the essential image of the functor $G : \mathcal{S}_{U_i} \to \mathcal{S}'_{U_i}$.*

Moreover the stack in groupoids $\mathcal{S}'$ is determined up to unique 2-isomorphism by these conditions.

Proof. Apply Lemma 50.8.1. The result will be a stack in groupoids by applying Lemma 50.5.2. $\square$

Lemma 50.9.2. *Let $\mathcal{C}$ be a site. Let $p : \mathcal{S} \to \mathcal{C}$ be a category fibred in groupoids over $\mathcal{C}$. Let $p' : \mathcal{S}' \to \mathcal{C}$ and $G : \mathcal{S} \to \mathcal{S}'$ the the stack in groupoids and 1-morphism constructed in Lemma 50.9.1. This construction has the following universal property: Given a stack in groupoids $q : \mathcal{X} \to \mathcal{C}$ and a 1-morphism $F : \mathcal{S} \to \mathcal{X}$ of categories over $\mathcal{C}$ there exists a 1-morphism $H : \mathcal{S}' \to \mathcal{X}$ such that the diagram*

$$\begin{tikzcd} \mathcal{S} \arrow[rr, "F"] \arrow[dr, "G"'] & & \mathcal{X} \\ & \mathcal{S}' \arrow[ur, "H"'] & \end{tikzcd}$$

is 2-commutative.

Proof. This is a special case of Lemma 50.8.2. $\square$

Lemma 50.9.3. *Let $\mathcal{C}$ be a site. Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be morphisms categories fibred in groupoids over $\mathcal{C}$. In this case the stackification of the 2-fibre product is the 2-fibre product of the stackifications.*

Proof. This is a special case of Lemma 50.8.4. $\square$

50.10. Inherited topologies

It turns out that a fibred category over a site inherits a canonical topology from the underlying site.

Lemma 50.10.1. *Let $\mathcal{C}$ be a site. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category. Let $Cov(\mathcal{S})$ be the set of families $\{x_i \to x\}_{i \in I}$ of morphisms in $\mathcal{S}$ with fixed target such that (a) each $x_i \to x$ is strongly cartesian, and (b) $\{p(x_i) \to p(x)\}_{i\in I}$ is a covering of $\mathcal{C}$. Then $(\mathcal{S}, Cov(\mathcal{S}))$ is a site.*

Proof. We have to check the three conditions of Sites, Definition 9.6.2.

(1) If $x \to y$ is an isomorphism of $\mathcal{S}$, then it is strongly cartesian by Categories, Lemma 4.30.2 and $p(x) \to p(y)$ is an isomorphism of $\mathcal{C}$. Thus $\{p(x) \to p(y)\}$ is a covering of $\mathcal{C}$ whence $\{x \to y\} \in \text{Cov}(\mathcal{S})$.
(2) If $\{x_i \to x\}_{i\in I} \in \text{Cov}(\mathcal{S})$ and for each i we have $\{y_{ij} \to x_i\}_{j \in J_i} \in \text{Cov}(\mathcal{S})$, then each composition $p(y_{ij}) \to p(x)$ is strongly cartesian by Categories, Lemma 4.30.2 and $\{p(y_{ij}) \to p(x)\}_{i\in I, j\in J_i} \in \text{Cov}(\mathcal{C})$. Hence also $\{y_{ij} \to x\}_{i\in I, j \in J_i} \in \text{Cov}(\mathcal{S})$.
(3) Suppose $\{x_i \to x\}_{i\in I} \in \text{Cov}(\mathcal{S})$ and $y \to x$ is a morphism of $\mathcal{S}$. As $\{p(x_i) \to p(x)\}$ is a covering of $\mathcal{C}$ we see that $p(x_i) \times_{p(x)} p(y)$ exists. Hence Categories, Lemma 4.30.11 implies that $x_i \times_x y$ exists, that $p(x_i \times_x y) = p(x_i) \times_{p(x)} p(y)$, and that $x_i \times_x y \to y$ is strongly cartesian. Since also $\{p(x_i) \times_{p(x)} p(y) \to p(y)\}_{i \in I} \in \text{Cov}(\mathcal{C})$ we conclude that $\{x_i \times_x y \to y\}_{i\in I} \in \text{Cov}(\mathcal{S})$

This finishes the proof. □

Note that if $p : \mathcal{S} \to \mathcal{C}$ is fibred in groupoids, then the coverings of the site $\mathcal{S}$ in Lemma 50.10.1 are characterized by

$$\{x_i \to x\} \in \text{Cov}(\mathcal{S}) \Leftrightarrow \{p(x_i) \to p(x)\} \in \text{Cov}(\mathcal{C})$$

because every morphism of $\mathcal{S}$ is strongly cartesian.

Definition 50.10.2. Let $\mathcal{C}$ be a site. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category. We say $(\mathcal{S}, \text{Cov}(\mathcal{S}))$ as in Lemma 50.10.1 is the *structure of site on $\mathcal{S}$ inherited from $\mathcal{C}$*. We sometimes indicate this by saying that *$\mathcal{S}$ is endowed with the topology inherited from $\mathcal{C}$*.

In particular we obtain a topos of sheaves $Sh(\mathcal{S})$ in this situation. It turns out that this topos is functorial with respect to 1-morphisms of fibred categories.

Lemma 50.10.3. *Let $\mathcal{C}$ be a site. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of fibred categories over $\mathcal{C}$. Then F is a continuous and cocontinuous functor between the structure of sites inherited from $\mathcal{C}$. Hence F induces a morphism of topoi $f : Sh(\mathcal{X}) \to Sh(\mathcal{Y})$ with $f_* = {}_sF = {}_pF$ and $f^{-1} = F^s = F^p$. In particular $f^{-1}(\mathcal{G})(x) = \mathcal{G}(F(x))$ for a sheaf $\mathcal{G}$ on $\mathcal{Y}$ and object x of $\mathcal{X}$.*

Proof. We first prove that F is continuous. Let $\{x_i \to x\}_{i\in I}$ be a covering of $\mathcal{X}$. By Categories, Definition 4.30.8 the functor F transforms strongly cartesian morphisms into strongly cartesian morphisms, hence $\{F(x_i) \to F(x)\}_{i\in I}$ is a covering of $\mathcal{Y}$. This proves part (1) of Sites, Definition 9.13.1. Moreover, let $x' \to x$ be a morphism of $\mathcal{X}$. By Categories, Lemma 4.30.11 the fibre product $x_i \times_x x'$ exists and $x_i \times_x x' \to x'$ is strongly cartesian. Hence $F(x_i \times_x x') \to F(x')$ is strongly cartesian. By Categories, Lemma 4.30.11 applied to $\mathcal{Y}$ this means that $F(x_i \times_x x') = F(x_i) \times_{F(x)} F(x')$. This proves part (2) of Sites, Definition 9.13.1 and we conclude that F is continuous.

Next we prove that F is cocontinuous. Let $x \in Ob(\mathcal{X})$ and let $\{y_i \to F(x)\}_{i \in I}$ be a covering in $\mathcal{Y}$. Denote $\{U_i \to U\}_{i \in I}$ the corresponding covering of $\mathcal{C}$. For each i choose a strongly cartesian morphism $x_i \to x$ in $\mathcal{X}$ lying over $U_i \to U$. Then $F(x_i) \to F(x)$ and $y_i \to F(x)$ are both a strongly cartesian morphisms in $\mathcal{Y}$ lying over $U_i \to U$. Hence there exists a unique isomorphism $F(x_i) \to y_i$ in $\mathcal{Y}_{U_i}$ compatible with the maps to $F(x)$. Thus $\{x_i \to x\}_{i \in I}$ is a covering of $\mathcal{X}$ such that $\{F(x_i) \to F(x)\}_{i \in I}$ is isomorphic to $\{y_i \to F(x)\}_{i \in I}$. Hence F is cocontinuous, see Sites, Definition 9.18.1.

The final assertion follows from the first two, see Sites, Lemmas 9.19.1, 9.18.2, and 9.19.5. □

Lemma 50.10.4. *Let $\mathcal{C}$ be a site. Let $p : \mathcal{X} \to \mathcal{C}$ and $q : \mathcal{Y} \to \mathcal{C}$ be stacks in groupoids. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories over $\mathcal{C}$. If F turns $\mathcal{X}$ into a category fibred in groupoids over $\mathcal{Y}$, then $\mathcal{X}$ is a stack in groupoids over $\mathcal{Y}$ (with topology inherited from $\mathcal{C}$).*

Proof. Let us prove descent for objects. Let $\{y_i \to y\}$ be a covering of $\mathcal{Y}$. Let (x_i, φ_{ij}) be a descent datum in $\mathcal{X}$ with respect to this covering. Then (x_i, φ_{ij}) is also a descent datum with respect to the covering $\{q(y_i) \to q(y)\}$ of $\mathcal{C}$. As $\mathcal{X}$ is a stack in groupoids we obtain an object x over $q(y)$ and isomorphisms $\psi_i : x|_{q(y_i)} \to x_i$ over $q(y_i)$ compatible with the φ_{ij}, i.e., such that

$$\varphi_{ij} = \psi_j|_{q(y_i)\times_{q(y)}q(y_j)} \circ \psi_i^{-1}|_{q(y_i)\times_{q(y)}q(y_j)}.$$

Consider the sheaf $I = Isom_{\mathcal{Y}}(F(x), y)$ on $\mathcal{C}/p(x)$. Note that $s_i = F(\psi_i) \in I(q(x_i))$ because $F(x_i) = y_i$. Because $F(\varphi_{ij}) = \text{id}$ (as we started with a descent datum over $\{y_i \to y\}$) the displayed formula shows that $s_i|_{q(y_i)\times_{q(y)}q(y_j)} = s_j|_{q(y_i)\times_{q(y)}q(y_j)}$. Hence the local sections s_i glue to $s : F(x) \to y$. As F is fibred in groupoids we see that x is isomorphic to an object x' with $F(x') = y$. We omit the verification that x' in the fibre category of $\mathcal{X}$ over y is a solution to the problem of descent posed by the descent datum (x_i, φ_{ij}). We also omit the proof of the sheaf property of the $Isom$-presheaves of $\mathcal{X}/\mathcal{Y}$. □

50.11. Gerbes

Gerbes are a special kind of stacks in groupoids.

Definition 50.11.1. A *gerbe* over a site $\mathcal{C}$ is a category $p : \mathcal{S} \to \mathcal{C}$ over $\mathcal{C}$ such that

1. $p : \mathcal{S} \to \mathcal{C}$ is a stack in groupoids over $\mathcal{C}$ (see Definition 50.5.1),
2. for $U \in Ob(\mathcal{C})$ there exists a covering $\{U_i \to U\}$ in $\mathcal{C}$ such that $\mathcal{S}_{U_i}$ is nonempty, and
3. for $U \in Ob(\mathcal{C})$ and $x, y \in Ob(\mathcal{S}_U)$ there exists a covering $\{U_i \to U\}$ in $\mathcal{C}$ such that $x|_{U_i} \cong y|_{U_i}$ in $\mathcal{S}_{U_i}$.

In other words, a gerbe is a stack in groupoids such that any two objects are locally isomorphic and such that objects exist locally.

Lemma 50.11.2. *Let $\mathcal{C}$ be a site. Let $\mathcal{S}_1$, $\mathcal{S}_2$ be categories over $\mathcal{C}$. Suppose that $\mathcal{S}_1$ and $\mathcal{S}_2$ are equivalent as categories over $\mathcal{C}$. Then $\mathcal{S}_1$ is a gerbe over $\mathcal{C}$ if and only if $\mathcal{S}_2$ is a gerbe over $\mathcal{C}$.*

Proof. Assume $\mathcal{S}_1$ is a gerbe over $\mathcal{C}$. By Lemma 50.5.4 we see $\mathcal{S}_2$ is a stack in groupoids over $\mathcal{C}$. Let $F : \mathcal{S}_1 \to \mathcal{S}_2$, $G : \mathcal{S}_2 \to \mathcal{S}_1$ be equivalences of categories over $\mathcal{C}$. Given $U \in Ob(\mathcal{C})$ we see that there exists a covering $\{U_i \to U\}$ such that $(\mathcal{S}_1)_{U_i}$ is nonempty. Applying F we see that $(\mathcal{S}_2)_{U_i}$ is nonempty. Given $U \in Ob(\mathcal{C})$ and $x, y \in Ob((\mathcal{S}_2)_U)$ there exists a covering $\{U_i \to U\}$ in $\mathcal{C}$ such that $G(x)|_{U_i} \cong G(y)|_{U_i}$ in $(\mathcal{S}_1)_{U_i}$. By Categories, Lemma 4.32.8 this implies $x|_{U_i} \cong y|_{U_i}$ in $(\mathcal{S}_2)_{U_i}$. $\square$

We want to generalize the definition of gerbes a bit. Namely, let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over a site $\mathcal{C}$. We want to say what it means for $\mathcal{X}$ to be a gerbe over $\mathcal{Y}$. By Section 50.10 the category $\mathcal{Y}$ inherits the structure of a site from $\mathcal{C}$. A naive guess is: Just require that $\mathcal{X} \to \mathcal{Y}$ *is* a gerbe in the sense above. Except the notion so obtained is not invariants under replacing $\mathcal{X}$ by an equivalent stack in groupoids over $\mathcal{C}$; this is even the case for the property of being fibred in groupoids over $\mathcal{Y}$. However, it turns out that we can replace $\mathcal{X}$ by an equivalent stack in groupoids over $\mathcal{Y}$ which is fibred in groupoids over $\mathcal{Y}$, and then the property of being a gerbe over $\mathcal{Y}$ is independent of this choice. Here is the precise formulation.

Lemma 50.11.3. *Let $\mathcal{C}$ be a site. Let $p : \mathcal{X} \to \mathcal{C}$ and $q : \mathcal{Y} \to \mathcal{C}$ be stacks in groupoids. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories over $\mathcal{C}$. The following are equivalent*

(1) *For some (equivalently any) factorization $F = F' \circ a$ where $a : \mathcal{X} \to \mathcal{X}'$ is an equivalence of categories over $\mathcal{C}$ and F' is fibred in groupoids, the map $F' : \mathcal{X}' \to \mathcal{Y}$ is a gerbe (with the topology on $\mathcal{Y}$ inherited from $\mathcal{C}$).*
(2) *The following two conditions are satisfied*
    (a) *for $y \in Ob(\mathcal{Y})$ lying over $U \in Ob(\mathcal{C})$ there exists a covering $\{U_i \to U\}$ in $\mathcal{C}$ and objects x_i of $\mathcal{X}$ over U_i such that $F(x_i) \cong y|_{U_i}$ in $\mathcal{Y}_{U_i}$, and*
    (b) *for $U \in Ob(\mathcal{C})$, $x, x' \in Ob(\mathcal{X}_U)$, and $b : F(x) \to F(x')$ in $\mathcal{Y}_U$ there exists a covering $\{U_i \to U\}$ in $\mathcal{C}$ and morphisms $a_i : x|_{U_i} \to x'|_{U_i}$ in $\mathcal{X}_{U_i}$ with $F(a_i) = b|_{U_i}$.*

Proof. By Categories, Lemma 4.32.14 there exists a factorization $F = F' \circ a$ where $a : \mathcal{X} \to \mathcal{X}'$ is an equivalence of categories over $\mathcal{C}$ and F' is fibred in groupoids. By Categories, Lemma 4.32.15 given any two such factorizations $F = F' \circ a = F'' \circ b$ we have that $\mathcal{X}'$ is equivalent to $\mathcal{X}''$ as categories over $\mathcal{Y}$. Hence Lemma 50.11.2 guarantees that the condition (1) is independent of the choice of the factorization. Moreover, this means that we may assume $\mathcal{X}' = \mathcal{X} \times_{F,\mathcal{Y},\text{id}} \mathcal{Y}$ as in the proof of Categories, Lemma 4.32.14

Let us prove that (a) and (b) imply that $\mathcal{X}' \to \mathcal{Y}$ is a gerbe. First of all, by Lemma 50.10.4 we see that $\mathcal{X}' \to \mathcal{Y}$ is a stack in groupoids. Next, let y be an object of $\mathcal{Y}$ lying over $U \in Ob(\mathcal{C})$. By (a) we can find a covering $\{U_i \to U\}$ in $\mathcal{C}$ and objects x_i of $\mathcal{X}$ over U_i and isomorphisms $f_i : F(x_i) \to y|_{U_i}$ in $\mathcal{Y}_{U_i}$. Then $(U_i, x_i, y|_{U_i}, f_i)$ are objects of $\mathcal{X}'_{U_i}$, i.e., the second condition of Definition 50.11.1 holds. Finally, let (U, x, y, f) and (U, x', y, f') be objects of $\mathcal{X}'$ lying over the same object $y \in Ob(\mathcal{Y})$. Set $b = (f')^{-1} \circ f$. By condition (b) we can find a convering $\{U_i \to U\}$ and isomorphisms $a_i : x|_{U_i} \to x'|_{U_i}$ in $\mathcal{X}_{U_i}$ with $F(a_i) = b|_{U_i}$. Then

$$(a_i, \text{id}) : (U, x, y, f)|_{U_i} \to (U, x', y, f')|_{U_i}$$

is a morphism in $\mathcal{X}'_{U_i}$ as desired. This proves that (2) implies (1).

To prove that (1) implies (2) one reads the arguments in the preceding paragraph backwards. Details omitted. $\square$

Definition 50.11.4. Let $\mathcal{C}$ be a site. Let $\mathcal{X}$ and $\mathcal{Y}$ be stacks in groupoids over $\mathcal{C}$. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories over $\mathcal{C}$. We say $\mathcal{X}$ is a *gerbe over* $\mathcal{Y}$ if the equivalent conditions of Lemma 50.11.3 are satisfied.

This definition does not conflict with Definition 50.11.1 when $\mathcal{Y} = \mathcal{C}$ because in this case we may take $\mathcal{X}' = \mathcal{X}$ in part (1) of Lemma 50.11.3. Note that conditions (2)(a) and (2)(b) of Lemma 50.11.3 are quite close in spirit to conditions (2) and (3) of Definition 50.11.1. Namely, (2)(a) says that the map of presheaves of isomorphism classes of objects becomes a surjection after sheafification. Moreover, (2)(b) says that

$$Isom_{\mathcal{X}}(x, x') \longrightarrow Isom_{\mathcal{Y}}(F(x), F(x'))$$

is a surjection of sheaves on $\mathcal{C}/U$ for any U and $x, x' \in Ob(\mathcal{X}_U)$.

Lemma 50.11.5. *Let $\mathcal{C}$ be a site. Let*

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "G'"'] \arrow[d, "F'"'] & \mathcal{X} \arrow[d, "F"] \\ \mathcal{Y}' \arrow[r, "G"] & \mathcal{Y} \end{tikzcd}$$

be a 2-fibre product of stacks in groupoids over $\mathcal{C}$. If $\mathcal{X}$ is a gerbe over $\mathcal{Y}$, then $\mathcal{X}'$ is a gerbe over $\mathcal{Y}'$.

Proof. By the uniqueness property of a 2-fibre product may assume that $\mathcal{X}' = \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}$ as in Categories, Lemma 4.29.3. Let us prove properties (2)(a) and (2)(b) of Lemma 50.11.3 for $\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Y}'$.

Let y' be an object of $\mathcal{Y}'$ lying over the object U of $\mathcal{C}$. By assumption there exists a covering $\{U_i \to U\}$ of U and objects $x_i \in \mathcal{X}_{U_i}$ with isomorphisms $\alpha_i : G(y')|_{U_i} \to F(x_i)$. Then $(U_i, y'|_{U_i}, x_i, \alpha_i)$ is an object of $\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}$ over U_i whose image in $\mathcal{Y}'$ is $y'|_{U_i}$. Thus (2)(a) holds.

Let $U \in Ob(\mathcal{C})$, let x'_1, x'_2 be objects of $\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}$ over U, and let $b' : F'(x'_1) \to F'(x'_2)$ be a morphism in $\mathcal{Y}'_U$. Write $x'_i = (U, y'_i, x_i, \alpha_i)$. Note that $F'(x'_i) = x_i$ and $G'(x'_i) = y'_i$. By assumption there exists a covering $\{U_i \to U\}$ in $\mathcal{C}$ and morphisms $a_i : x_1|_{U_i} \to x_2|_{U_i}$ in $\mathcal{X}_{U_i}$ with $F(a_i) = G(b')|_{U_i}$. Then $(b'|_{U_i}, a_i)$ is a morphism $x'_1|_{U_i} \to x'_2|_{U_i}$ as required in (2)(b). □

Lemma 50.11.6. *Let $\mathcal{C}$ be a site. Let $F : \mathcal{X} \to \mathcal{Y}$ and $G : \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms of stacks in groupoids over $\mathcal{C}$. If $\mathcal{X}$ is a gerbe over $\mathcal{Y}$ and $\mathcal{Y}$ is a gerbe over $\mathcal{Z}$, then $\mathcal{X}$ is a gerbe over $\mathcal{Z}$.*

Proof. Let us prove properties (2)(a) and (2)(b) of Lemma 50.11.3 for $\mathcal{X} \to \mathcal{Z}$.

Let z be an object of $\mathcal{Z}$ lying over the object U of $\mathcal{C}$. By assumption on G there exists a covering $\{U_i \to U\}$ of U and objects $y_i \in \mathcal{Y}_{U_i}$ such that $G(y_i) \cong z|_{U_i}$. By assumption on F there exist coverings $\{U_{ij} \to U_i\}$ and objects $x_{ij} \in \mathcal{X}_{U_{ij}}$ such that $F(x_{ij}) \cong y_i|_{U_{ij}}$. Then $\{U_{ij} \to U\}$ is a covering of $\mathcal{C}$ and $(G \circ F)(x_{ij}) \cong z|_{U_{ij}}$. Thus (2)(a) holds.

Let $U \in Ob(\mathcal{C})$, let x_1, x_2 be objects of $\mathcal{X}$ over U, and let $c : (G \circ F)(x_1) \to (G \circ F)(x_2)$ be a morphism in $\mathcal{Z}_U$. By assumption on G there exists a covering $\{U_i \to U\}$ of U and morphisms $b_i : F(x_1)|_{U_i} \to F(x_2)|_{U_i}$ in $\mathcal{Y}_{U_i}$ such that $G(b_i) = c|_{U_i}$. By assumption on F there exist coverings $\{U_{ij} \to U_i\}$ and morphisms $a_{ij} : x_1|_{U_{ij}} \to x_2|_{U_{ij}}$ in $\mathcal{X}_{U_{ij}}$ such that $F(a_{ij}) = b_i|_{U_{ij}}$. Then $\{U_{ij} \to U\}$ is a covering of $\mathcal{C}$ and $(G \circ F)(a_{ij}) = c|_{U_{ij}}$ as required in (2)(b). □

Lemma 50.11.7. *Let $\mathcal{C}$ be a site. Let*

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "G'"] \arrow[d, "F'"'] & \mathcal{X} \arrow[d, "F"] \\ \mathcal{Y}' \arrow[r, "G"] & \mathcal{Y} \end{tikzcd}$$

be a 2-cartesian diagram of stacks in groupoids over $\mathcal{C}$. If for every $U \in Ob(\mathcal{C})$ and $x \in Ob(\mathcal{Y}_U)$ there exists a covering $\{U_i \to U\}$ such that $x|_{U_i}$ is in the essential image of $G : \mathcal{Y}'_{U_i} \to \mathcal{Y}_{U_i}$ and $\mathcal{X}'$ is a gerbe over $\mathcal{Y}'$, then $\mathcal{X}$ is a gerbe over $\mathcal{Y}$.

Proof. By the uniqueness property of a 2-fibre product may assume that $\mathcal{X}' = \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}$ as in Categories, Lemma 4.29.3. Let us prove properties (2)(a) and (2)(b) of Lemma 50.11.3 for $\mathcal{X} \to \mathcal{Y}$.

Let y be an object of $\mathcal{Y}$ lying over the object U of $\mathcal{C}$. By assumption there exists a covering $\{U_i \to U\}$ of U and objects $y'_i \in \mathcal{Y}'_{U_i}$ with $G(y'_i) \cong y|_{U_i}$. By (2)(a) for $\mathcal{X}' \to \mathcal{Y}'$ there exist coverings $\{U_{ij} \to U_i\}$ and objects x'_{ij} of $\mathcal{X}'$ over U_{ij} with $F'(x'_{ij})$ isomorphic to the restriction of y'_i to U_{ij}. Then $\{U_{ij} \to U\}$ is a covering of $\mathcal{C}$ and $G'(x'_{ij})$ are objects of $\mathcal{X}$ over U_{ij} whose images in $\mathcal{Y}$ are isomorphic to the restrictions $y|_{U_{ij}}$. This proves (2)(a) for $\mathcal{X} \to \mathcal{Y}$.

Let $U \in Ob(\mathcal{C})$, let x_1, x_2 be objects of $\mathcal{X}$ over U, and let $b : F(x_1) \to F(x_2)$ be a morphism in $\mathcal{Y}_U$. By assumption we may choose a covering $\{U_i \to U\}$ and objects y'_{1i}, y'_{2i} of $\mathcal{Y}'$ over U_i such that there exist isomorphisms $\alpha_{1i} : G(y'_{1i}) \cong F(x_1)|_{U_i}$ and $\alpha_{2i} : G(y'_{2i}) \cong F(x_2)|_{U_i}$. Then we get objects $x'_{1i} = (U_i, y'_{1i}, x_1|_{U_i}, \alpha_{1i})$ and $x'_{2i} = (U_i, y'_{1i}, x_1|_{U_i}, \alpha_{2i})$ of $\mathcal{X}'$ over U_i. The restriction $b|_{U_i}$ is a morphism $F'(x'_{1i}) \to F'(x'_{2i})$. By (2)(b) for $\mathcal{X}' \to \mathcal{Y}'$ there exist coverings $\{U_{ij} \to U_i\}$ and morphisms $a'_{ij} : x'_{1i}|_{U_{ij}} \to x'_{2i}|_{U_{ij}}$ such that $F'(a'_{ij}) = b|_{U_{ij}}$. Unwinding the definition of morphisms in $\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}$ we see that $G'(a'_{ij}) : x_1|_{U_{ij}} \to x_2|_{U_{ij}}$ are the morphism we're looking for, i.e., (2)(b) holds for $\mathcal{X} \to \mathcal{Y}$. $\square$

50.12. Functoriality for stacks

In this section we study what happens if we want to change the base site of a stack. This section can be skipped on a first reading.

Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between categories. Let $p : \mathcal{S} \to \mathcal{D}$ be a category over $\mathcal{D}$. In this situation we denote $u^p\mathcal{S}$ the category over $\mathcal{C}$ defined as follows

1. An object of $u^p\mathcal{S}$ is a pair (U, y) consisting of an object U of $\mathcal{C}$ and an object y of $\mathcal{S}_{u(U)}$.
2. A morphism $(a, \beta) : (U, y) \to (U', y')$ is given by a morphism $a : U \to U'$ of $\mathcal{C}$ and a morphism $\beta : y \to y'$ of $\mathcal{S}$ such that $p(\beta) = u(a)$.

Note that with these definitions the fibre category of $u^p\mathcal{S}$ over U is equal to the fibre category of $\mathcal{S}$ over $u(U)$.

Lemma 50.12.1. *In the situation above, if $\mathcal{S}$ is a fibred category over $\mathcal{D}$ then $u^p\mathcal{S}$ is a fibred category over $\mathcal{C}$.*

Proof. Please take a look at the discussion surrounding Categories, Definitions 4.30.1 and 4.30.4 before reading this proof. Let $(a, \beta) : (U, y) \to (U', y')$ be a morphism of $u^p\mathcal{S}$. We claim that (a, β) is strongly cartesian if and only if β is strongly cartesian. First, assume β

is strongly cartesian. Consider any second morphism $(a_1, \beta_1) : (U_1, y_1) \to (U', y')$ of $u^p\mathcal{S}$. Then

$$
\begin{aligned}
& Mor_{u^p\mathcal{S}}((U_1, y_1), (U, y)) \\
& = Mor_{\mathcal{C}}(U_1, U) \times_{Mor_{\mathcal{D}}(u(U_1), u(U))} Mor_{\mathcal{S}}(y_1, y) \\
& = Mor_{\mathcal{C}}(U_1, U) \times_{Mor_{\mathcal{D}}(u(U_1), u(U))} Mor_{\mathcal{S}}(y_1, y') \times_{Mor_{\mathcal{D}}(u(U_1), u(U'))} Mor_{\mathcal{D}}(u(U_1), u(U)) \\
& = Mor_{\mathcal{S}}(y_1, y') \times_{Mor_{\mathcal{D}}(u(U_1), u(U'))} Mor_{\mathcal{C}}(U_1, U) \\
& = Mor_{u^p\mathcal{S}}((U_1, y_1), (U', y')) \times_{Mor_{\mathcal{C}}(U_1, U')} Mor_{\mathcal{C}}(U_1, U)
\end{aligned}
$$

the second equality as β is strongly cartesian. Hence we see that indeed (a, β) is strongly cartesian. Conversely, suppose that (a, β) is strongly cartesian. Choose a strongly cartesian morphism $\beta' : y'' \to y'$ in $\mathcal{S}$ with $p(\beta') = u(a)$. Then bot $(a, \beta) : (U, y) \to (U, y')$ and $(a, \beta') : (U, y'') \to (U, y)$ are strongly cartesian and lift a. Hence, by the uniqueness of strongly cartesian morphisms (see discussion in Categories, Section 4.30) there exists an isomorphism $\iota : y \to y''$ in $\mathcal{S}_{u(U)}$ such that $\beta = \beta' \circ \iota$, which implies that β is strongly cartesian in $\mathcal{S}$ by Categories, Lemma 4.30.2.

Finally, we have to show that given (U', y') and $U \to U'$ we can find a strongly cartesian morphism $(U, y) \to (U', y')$ in $u^p\mathcal{S}$ lifting the morphism $U \to U'$. This follows from the above as by assumption we can find a strongly cartesian morphism $y \to y'$ lifting the morphism $u(U) \to u(U')$. □

Lemma 50.12.2. *Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous functor of sites. Let $p : \mathcal{S} \to \mathcal{D}$ be a stack over $\mathcal{D}$. Then $u^p\mathcal{S}$ is a stack over $\mathcal{S}$.*

Proof. We have seen in Lemma 50.12.1 that $u^p\mathcal{S}$ is a fibred category over $\mathcal{S}$. Moreover, in the proof of that lemma we have seen that a morphism (a, β) of $u^p\mathcal{S}$ is strongly cartesian if and only β is strongly cartesian in $\mathcal{S}$. Hence, given a morphism $a : U \to U'$ of $\mathcal{C}$, not only do we have the equalities $(u^p\mathcal{S})_U = \mathcal{S}_U$ and $(u^p\mathcal{S})_{U'} = \mathcal{S}_{U'}$, but via these equalities the pullback functors agree; in a formula $a^*(U', y') = (U, u(a)^*y')$.

Having said this, let $\mathcal{U} = \{U_i \to U\}$ be a covering of $\mathcal{C}$. As u is continuous we see that $\mathcal{V} = \{u(U_i) \to u(U)\}$ is a covering of $\mathcal{D}$, and that $u(U_i \times_U U_j) = u(U_i) \times_{u(U)} u(U_j)$ and similarly for the triple fibre products $U_i \times_U U_j \times_U U_k$. As we have the identifications of fibre categories and pullbacks we see that descend data relative to $\mathcal{U}$ are identical to descend data relative to $\mathcal{V}$. Since by assumption we have effective descent in $\mathcal{S}$ we conclude the same holds for $u^p\mathcal{S}$. □

Lemma 50.12.3. *Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous functor of sites. Let $p : \mathcal{S} \to \mathcal{D}$ be a stack in groupoids over $\mathcal{D}$. Then $u^p\mathcal{S}$ is a stack in groupoids over $\mathcal{S}$.*

Proof. This follows immediately from Lemma 50.12.2 and the fact that all fibre categories are groupoids. □

Definition 50.12.4. Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites given by the continuous functor $u : \mathcal{C} \to \mathcal{D}$. Let $\mathcal{S}$ be a fibred category over $\mathcal{D}$. In this setting we write $f_*\mathcal{S}$ for the fibred category $u^p\mathcal{S}$ defined above. We say that $f_*\mathcal{S}$ is the *pushforward of $\mathcal{S}$ along f*.

By the results above we know that $f_*\mathcal{S}$ is a stack (in groupoids) if $\mathcal{S}$ is a stack (in groupoids). It is harder to define the pullback of a stack (and we'll need additional assumptions for our particular construction -- feel free to write up and submit a more general construction). We do this in several steps.

Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between categories. Let $p : \mathcal{S} \to \mathcal{C}$ be a category over $\mathcal{C}$. In this setting we define a category $u_{pp}\mathcal{S}$ as follows:

(1) An object of $u_{pp}\mathcal{S}$ is a triple $(U, \phi : V \to u(U), x)$ where $U \in Ob(\mathcal{C})$, the map $\phi : V \to u(U)$ is a morphism in $\mathcal{D}$, and $x \in Ob(\mathcal{S}_U)$.
(2) A morphism

$$(U_1, \phi_1 : V_1 \to u(U_1), x_1) \longrightarrow (U_2, \phi_2 : V_2 \to u(U_2), x_2)$$

of $u_{pp}\mathcal{S}$ is given by a (a, b, α) where $a : U_1 \to U_2$ is a morphism of $\mathcal{C}$, $b : V_1 \to V_2$ is a morphism of $\mathcal{D}$, and $\alpha : x_1 \to x_2$ is morphism of $\mathcal{S}$, such that $p(\alpha) = a$ and the diagram

$$\begin{tikzcd} V_1 \arrow[r, "b"'] \arrow[d, "\phi_1"'] & V_2 \arrow[d, "\phi_2"] \\ u(U_1) \arrow[r, "u(a)"] & u(U_2) \end{tikzcd}$$

commutes in $\mathcal{D}$.

We think of $u_{pp}\mathcal{S}$ as a category over $\mathcal{D}$ via

$$p_{pp} : u_{pp}\mathcal{S} \longrightarrow \mathcal{D}, \quad (U, \phi : V \to u(U), x) \longmapsto V.$$

The fibre category of $u_{pp}\mathcal{S}$ over an object V of $\mathcal{D}$ does not have a simple description. Moreover, it is in general not the case that $u_{pp}\mathcal{S}$ is a fibred category over $\mathcal{D}$ if $\mathcal{S}$ is a fibred category over $\mathcal{C}$.

Lemma 50.12.5. *In the situation above assume*

(1) $p : \mathcal{S} \to \mathcal{C}$ *is a fibred category,*
(2) $\mathcal{C}$ *has nonempty finite limits, and*
(3) $u : \mathcal{C} \to \mathcal{D}$ *commutes with nonempty finite limits.*

Consider the set $R \subset Arrows(u_{pp}\mathcal{S})$ *of morphisms of the form*

$$(a, id_V, \alpha) : (U', \phi' : V \to u(U'), x') \longrightarrow (U, \phi : V \to u(U), x)$$

with α *strongly cartesian. Then* R *is a right multiplicative system.*

Proof. According to Categories, Definition 4.24.1 we have to check RMS1, RMS2, RMS3. Condition RMS1 holds as a composition of strongly cartesian morphisms is strongly cartesian, see Categories, Lemma 4.30.2.

To check RMS2 suppose we have a morphism

$$(a, b, \alpha) : (U_1, \phi_1 : V_1 \to u(U_1), x_1) \longrightarrow (U, \phi : V \to u(U), x)$$

of $u_{pp}\mathcal{S}$ and a morphism

$$(c, \mathrm{id}_V, \gamma) : (U', \phi' : V \to u(U'), x') \longrightarrow (U, \phi : V \to u(U), x)$$

with γ strongly cartesian from R. In this situation set $U_1' = U_1 \times_U U'$, and denote $a' : U_1' \to U'$ and $c' : U_1' \to U_1$ the projections. As $u(U_1') = u(U_1) \times_{u(U)} u(U')$ we see that $\phi_1' = (\phi_1, \phi') : V_1 \to u(U_1')$ is a morphism in $\mathcal{D}$. Let $\gamma_1 : x_1' \to x_1$ be a strongly cartesian morphism of $\mathcal{S}$ with $p(\gamma_1) = \phi_1'$ (which exists because $\mathcal{S}$ is a fibred category over $\mathcal{C}$). Then as $\gamma : x' \to x$ is strongly cartesian there exists a unique morphism $\alpha' : x_1' \to x'$ with $p(\alpha') = a'$. At this point we see that

$$(a', b, \alpha') : (U_1, \phi_1 : V_1 \to u(U_1'), x_1') \longrightarrow (U, \phi : V \to u(U'), x')$$

is a morphism and that

$$(c', \mathrm{id}_{V_1}, \gamma_1) : (U_1', \phi_1' : V_1 \to u(U_1'), x_1') \longrightarrow (U_1, \phi : V_1 \to u(U_1), x_1)$$

is an element of R which form a solution of the existence problem posed by RMS2.

Finally, suppose that

$$(a, b, \alpha), (a', b', \alpha') : (U_1, \phi_1 : V_1 \to u(U_1), x_1) \longrightarrow (U, \phi : V \to u(U), x)$$

are two morphisms of $u_{pp}\mathcal{S}$ and suppose that

$$(c, \mathrm{id}_V, \gamma) : (U, \phi : V \to u(U), x) \longrightarrow (U', \phi : V \to u(U'), x')$$

is an element of R which equalizes the morphisms (a, b, α) and (a', b', α'). This implies in particular that $b = b'$. Let $d : U_2 \to U_1$ be the equalizer of a, a' which exists (see Categories, Lemma 4.16.3). Moreover, $u(d) : u(U_2) \to u(U_1)$ is the equalizer of $u(a), u(a')$ hence (as $b = b'$) there is a morphism $\phi_2 : V_1 \to u(U_2)$ such that $\phi_1 = u(d) \circ \phi_1$. Let $\delta : x_2 \to x_1$ be a strongly cartesian morphism of $\mathcal{S}$ with $p(\delta) = u(d)$. Now we claim that $\alpha \circ \delta = \alpha' \circ \delta$. This is true because γ is strongly cartesian, $\gamma \circ \alpha \circ \delta = \gamma \circ \alpha' \circ \delta$, and $p(\alpha \circ \delta) = p(\alpha' \circ \delta)$. Hence the arrow

$$(d, \mathrm{id}_{V_1}, \delta) : (U_2, \phi_2 : V_1 \to u(U_2), x_2) \longrightarrow (U_1, \phi_1 : V_1 \to u(U_1), x_1)$$

is an element of R and equalizes (a, b, α) and (a', b', α'). Hence R satisfies RMS3 as well. □

Lemma 50.12.6. *With notation and assumptions as in Lemma 50.12.5. Set $u_p\mathcal{S} = R^{-1}u_{pp}\mathcal{S}$, see Categories, Section 4.24. Then $u_p\mathcal{S}$ is a fibred category over $\mathcal{D}$.*

Proof. We use the description of $u_p\mathcal{S}$ given just above Categories, Lemma 4.24.9. Note that the functor $p_{pp} : u_{pp}\mathcal{S} \to \mathcal{D}$ transforms every element of R to an identity morphism. Hence by Categories, Lemma 4.24.13 we obtain a canonical functor $p_p : u_p\mathcal{S} \to \mathcal{D}$ extending the given functor. This is how we think of $u_p\mathcal{S}$ as a category over $\mathcal{D}$.

First we want to characterize the $\mathcal{D}$-strongly cartesian morphisms in $u_p\mathcal{S}$. A morphism $f : X \to Y$ of $u_p\mathcal{S}$ is the equivalence class of a pair $(f' : X' \to Y, r : X' \to X)$ with $r \in R$. In fact, in $u_p\mathcal{S}$ we have $f = (f', 1) \circ (r, 1)^{-1}$ with obvious notation. Note that an isomorphism is always strongly cartesian, as are compositions of strongly cartesian morphisms, see Categories, Lemma 4.30.2. Hence f is strongly cartesian if and only if $(f', 1)$ is so. Thus the following claim completely characterizes strongly cartesian morphisms. Claim: A morphism

$$(a, b, \alpha) : X_1 = (U_1, \phi_1 : V_1 \to u(U_1), x_1) \longrightarrow (U_2, \phi_2 : V_2 \to u(U_2), x_2) = X_2$$

of $u_{pp}\mathcal{S}$ has image $f = ((a, b, \alpha), 1)$ strongly cartesian in $u_p\mathcal{S}$ if and only if α is a strongly cartesian morphism of $\mathcal{S}$.

Assume α strongly cartesian. Let $X = (U, \phi : V \to u(U), x)$ be another object, and let $f_2 : X \to X_2$ be a morphism of $u_p\mathcal{S}$ such that $p_p(f_2) = b \circ b_1$ for some $b_1 : U \to U_1$. To show that f is strongly cartesian we have to show that there exists a unique morphism $f_1 : X \to X_1$ in $u_p\mathcal{S}$ such that $p_p(f_1) = b_1$ and $f_2 = f \circ f_1$ in $u_p\mathcal{S}$. Write $f_2 = (f_2' : X' \to X_2, r : X' \to X)$. Again we can write $f_2 = (f_2', 1) \circ (r, 1)^{-1}$ in $u_p\mathcal{S}$. Since $(r, 1)$ is an isomorphism whose image in $\mathcal{D}$ is an identity we see that finding a morphism $f_1 : X \to X_1$ with the required properties is the same thing as finding a morphism $f_1' : X' \to X_1$ in $u_p\mathcal{S}$ with $p(f_1') = b_1$

and $f_2' = f \circ f_1'$. Hence we may assume that f_2 is of the form $f_2 = ((a_2, b_2, \alpha_2), 1)$ with $b_2 = b \circ b_1$. Here is a picture

$$\begin{tikzcd} & (U_1, V_1 \to u(U_1), x_1) \arrow[d, "{(a,b,\alpha)}"] \\ (U, V \to u(U), x) \arrow[r, "{(a_2,b_2,\alpha_2)}"] & (U_2, V_2 \to u(U_2), x_2) \end{tikzcd}$$

Now it is clear how to construct the morphism f_1. Namely, set $U' = U \times_{U_2} U_1$ with projections $c : U' \to U$ and $a_1 : U' \to U_1$. Pick a strongly cartesian morphism $\gamma : x' \to x$ lifting the morphism c. Since $b_2 = b \circ b_1$, and since $u(U') = u(U) \times_{u(U_2)} u(U_1)$ we see that $\phi' = (\phi, \phi_1 \circ b_1) : V \to u(U')$. Since α is strongly cartesian, and $a \circ a_1 = a_2 \circ c = p(\alpha_2 \circ \gamma)$ there exists a morphism $\alpha_1 : x' \to x_1$ lifting a_1 such that $\alpha \circ \alpha_1 = \alpha_2 \circ \gamma$. Set $X' = (U', \phi' : V \to u(U'), x')$. Thus we see that

$$f_1 = ((a_1, b_1, \alpha_1) : X' \to X_1, (c, \mathrm{id}_V, \gamma) : X' \to X) : X \longrightarrow X_1$$

works, in fact the diagram

$$\begin{tikzcd} (U', \phi' : V \to u(U'), x') \arrow[r, "{(a_1,b_1,\alpha_1)}"'] \arrow[d, "{(c,\mathrm{id}_V,\gamma)}"'] & (U_1, V_1 \to u(U_1), x_1) \arrow[d, "{(a,b,\alpha)}"] \\ (U, V \to u(U), x) \arrow[r, "{(a_2,b_2,\alpha_2)}"] & (U_2, V_2 \to u(U_2), x_2) \end{tikzcd}$$

is commutative by construction. This proves existence.

Next we prove uniqueness, still in the special case $f = ((a, b, \alpha), 1)$ and $f_2 = ((a_2, b_2, \alpha_2), 1)$. We strongly advise the reader to skip this part. Suppose that $g_1, g_1' : X \to X_1$ are two morphisms of $u_p\mathcal{S}$ such that $p_p(g_1) = p_p(g_1') = b_1$ and $f_2 = f \circ g_1 = f \circ g_1'$. Our goal is to show that $g_1 = g_1'$. By Categories, Lemma 4.24.10 we may represent g_1 and g_1' as the equivalence classes of $(f_1 : X' \to X_1, r : X' \to X)$ and $(f_1' : X' \to X_1, r : X' \to X)$ for some $r \in R$. By Categories, Lemma 4.24.11 we see that $f_2 = f \circ g_1 = f \circ g_1'$ means that there exists a morphism $r' : X'' \to X'$ in $u_{pp}\mathcal{S}$ such that $r' \circ r \in R$ and

$$(a, b, \alpha) \circ f_1 \circ r' = (a, b, \alpha) \circ f_1' \circ r' = (a_2, b_2, \alpha_2) \circ r'$$

in $u_{pp}\mathcal{S}$. Note that now g_1 is represented by $(f_1 \circ r', r \circ r')$ and similarly for g_1'. Hence we may assume that

$$(a, b, \alpha) \circ f_1 = (a, b, \alpha) \circ f_1' = (a_2, b_2, \alpha_2).$$

Write $r = (c, \mathrm{id}_V, \gamma) : (U', \phi' : V \to u(U'), x')$, $f_1 = (a_1, b_1, \alpha_1)$, and $f_1' = (a_1', b_1, \alpha_1')$. Here we have used the condition that $p_p(g_1) = p_p(g_1')$. The equalities above are now equivalent to $a \circ a_1 = a \circ a_1' = a_2 \circ c$ and $\alpha \circ \alpha_1 = \alpha \circ \alpha_1' = \alpha_2 \circ \gamma$. It need not be the case that $a_1 = a_1'$ in this situation. Thus we have to precompose by one more morphism from R. Namely, let $U'' = \mathrm{Eq}(a_1, a_1')$ be the equalizer of a_1 and a_1' which is a subobject of U'. Denote $c' : U'' \to U'$ the canonical monomorphism. Because of the relations among the morphisms above we see that $V \to u(U')$ maps into $u(U'') = u(\mathrm{Eq}(a_1, a_1')) = \mathrm{Eq}(u(a_1), u(a_1'))$. Hence we get a new object $(U'', \phi'' : V \to u(U''), x'')$, where $\gamma' : x'' \to x'$ is a strongly cartesian morphism lifting γ. Then we see that we may precompose f_1 and f_1' with the element $(c', \mathrm{id}_V, \gamma')$ of R. After doing this, i.e., replacing $(U', \phi' : V \to u(U'), x')$ with $(U'', \phi'' : V \to u(U''), x'')$, we get back to the previous situation where in addition we now have that $a_1 = a_1'$. In this case it follows formally from the fact that α is strongly cartesian (!) that $\alpha_1 = \alpha_1'$. This shows that $g_1 = g_1'$ as desired.

We omit the proof of the fact that for any strongly cartesian morphism of $u_p\mathcal{S}$ of the form $((a, b, \alpha), 1)$ the morphism α is strongly cartesian in $\mathcal{S}$. (We do not need the characterization of strongly cartesian morphisms in the rest of the proof, although we do use it later in this section.)

Let $(U, \phi : V \to u(U), x)$ be an object of $u_p\mathcal{S}$. Let $b : V' \to V$ be a morphism of $\mathcal{D}$. Then the morphism

$$(\text{id}_U, b, \text{id}_x) : (U, \phi \circ b : V' \to u(U), x) \longrightarrow (U, \phi : V \to u(U), x)$$

is strongly cartesian by the result of the preceding paragraphs and we win. □

Lemma 50.12.7. *With notation and assumptions as in Lemma 50.12.6. If $\mathcal{S}$ is fibred in groupoids, then $u_p\mathcal{S}$ is fibred in groupoids.*

Proof. By Lemma 50.12.6 we know that $u_p\mathcal{S}$ is a fibred category. Let $f : X \to Y$ be a morphism of $u_p\mathcal{S}$ with $p_p(f) = \text{id}_V$. We are done if we can show that f is invertible, see Categories, Lemma 4.32.2. Write f as the equivalence class of a pair $((a, b, \alpha), r)$ with $r \in R$. Then $p_p(r) = \text{id}_V$, hence $p_{pp}((a, b, \alpha)) = \text{id}_V$. Hence $b = \text{id}_V$. But any morphism of $\mathcal{S}$ is strongly cartesian, see Categories, Lemma 4.32.2 hence we see that $(a, b, \alpha) \in R$ is invertible in $u_p\mathcal{S}$ as desired. □

Lemma 50.12.8. *Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Let $p : \mathcal{S} \to \mathcal{C}$ and $q : \mathcal{T} \to \mathcal{D}$ be categories over $\mathcal{C}$ and $\mathcal{D}$. Assume that*

(1) *$p : \mathcal{S} \to \mathcal{C}$ is a fibred category,*
(2) *$q : \mathcal{T} \to \mathcal{D}$ is a fibred category,*
(3) *$\mathcal{C}$ has nonempty finite limits, and*
(4) *$u : \mathcal{C} \to \mathcal{D}$ commutes with nonempty finite limits.*

Then we have a canonical equivalence of categories

$$Mor_{Fib/\mathcal{C}}(\mathcal{S}, u^p\mathcal{T}) = Mor_{Fib/\mathcal{D}}(u_p\mathcal{S}, \mathcal{T})$$

of morphism categories.

Proof. In this proof we use the notation x/U to denote an object x of $\mathcal{S}$ which lies over U in $\mathcal{C}$. Similarly y/V denotes an object y of $\mathcal{T}$ which lies over V in $\mathcal{D}$. In the same vein $\alpha/a : x/U \to x'/U'$ denotes the morphism $\alpha : x \to x'$ with image $a : U \to U'$ in $\mathcal{C}$.

Let $G : u_p\mathcal{S} \to \mathcal{T}$ be a 1-morphism of fibred categories over $\mathcal{D}$. Denote $G' : u_{pp}\mathcal{S} \to \mathcal{T}$ the composition of G with the canonical (localization) functor $u_{pp}\mathcal{S} \to u_p\mathcal{S}$. Then consider the functor $H : \mathcal{S} \to u^p\mathcal{T}$ given by

$$H(x/U) = (U, G'(U, \text{id}_{u(U)} : u(U) \to u(U), x))$$

on objects and by

$$H((\alpha, a) : x/U \to x'/U') = G'(a, u(a), \alpha)$$

on morphisms. Since G transforms strongly cartesian morphisms into strongly cartesian morphisms, we see that if α is strongly cartesian, then $H(\alpha)$ is strongly cartesian. Namely, we've seen in the proof of Lemma 50.12.6 that in this case the map $(a, u(a), \alpha)$ becomes strongly cartesian in $u_p\mathcal{S}$. Clearly this construction is functorial in G and we obtain a functor

$$A : Mor_{Fib/\mathcal{D}}(u_p\mathcal{S}, \mathcal{T}) \longrightarrow Mor_{Fib/\mathcal{C}}(\mathcal{S}, u^p\mathcal{T})$$

Conversely, let $H : \mathcal{S} \to u^p\mathcal{T}$ be a 1-morphism of fibred categories. Recall that an object of $u^p\mathcal{T}$ is a pair (U, y) with $y \in Ob(\mathcal{T}_{u(U)})$. We denote $\text{pr} : u^p\mathcal{T} \to \mathcal{T}$ the functor $(U, y) \mapsto y$. In this case we define a functor $G' : u_{pp}\mathcal{S} \to \mathcal{T}$ by the rules

$$G'(U, \phi : V \to u(U), x) = \phi^*\text{pr}(H(x))$$

on objects and we let

$$G'((a, b, \alpha) : (U, \phi : V \to u(U), x) \to (U', \phi' : V' \to u(U'), x')) = \beta$$

be the unique morphism $\beta : \phi^*\text{pr}(H(x)) \to (\phi')^*\text{pr}(H(x'))$ such that $q(\beta) = b$ and the diagram

$$\begin{tikzcd} \phi^*\text{pr}(H(x)) \arrow[r, "\beta"'] \arrow[d] & (\phi')^*\text{pr}(H(x')) \arrow[d] \\ \text{pr}(H(x)) \arrow[r, "\text{pr}(H(a,\alpha))"] & \text{pr}(H(x')) \end{tikzcd}$$

Such a morphism exists and is unique because $\mathcal{T}$ is a fibred category.

We check that $G'(r)$ is an isomorphism if $r \in R$. Namely, if

$$(a, \text{id}_V, \alpha) : (U', \phi' : V \to u(U'), x') \longrightarrow (U, \phi : V \to u(U), x)$$

with α strongly cartesian is an element of the right multiplicative system R of Lemma 50.12.5 then $H(\alpha)$ is strongly cartesian, and $\text{pr}(H(\alpha))$ is strongly cartesian, see proof of Lemma 50.12.1. Hence in this case the morphism β has $q(\beta) = \text{id}_V$ and is strongly cartesian. Hence β is an isomorphism by Categories, Lemma 4.30.2. Thus by Categories, Lemma 4.24.13 we obtain a canonical extension $G : u_p\mathcal{S} \to \mathcal{T}$.

Next, let us prove that G transforms strongly cartesian morphisms into strongly cartesian morphisms. Suppose that $f : X \to Y$ is a strongly cartesian. By the characterization of strongly cartesian morphisms in $u_p\mathcal{S}$ we can write f as $((a, b, \alpha) : X' \to Y, r : X' \to Y)$ where $r \in R$ and α strongly cartesian in $\mathcal{S}$. By the above it suffices to show that $G'(a, b\alpha)$ is strongly cartesian. As before the condition that α is strongly cartesian implies that $\text{pr}(H(a, \alpha)) : \text{pr}(H(x)) \to \text{pr}(H(x'))$ is strongly cartesian in $\mathcal{T}$. Since in the commutative square above now all arrows except possibly β is strongly cartesian it follows that also β is strongly cartesian as desired. Clearly the construction $H \mapsto G$ is functorial in H and we obtain a functor

$$B : Mor_{Fib/\mathcal{C}}(\mathcal{S}, u^p\mathcal{T}) \longrightarrow Mor_{Fib/\mathcal{D}}(u_p\mathcal{S}, \mathcal{T})$$

To finish the proof of the lemma we have to show that the functors A and B are mutually quasi-inverse. We omit the verifications. □

Definition 50.12.9. Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites given by a continuous functor $u : \mathcal{C} \to \mathcal{D}$ satisyfing the hypotheses and conclusions of Sites, Proposition 9.14.6. Let $\mathcal{S}$ be a stack over $\mathcal{C}$. In this setting we write $f^{-1}\mathcal{S}$ for the stackification of the fibred category $u_p\mathcal{S}$ over $\mathcal{D}$ constructed above. We say that $f^{-1}\mathcal{S}$ is the *pullback of $\mathcal{S}$ along f*.

Of course, if $\mathcal{S}$ is a stack in groupoids, then $f^{-1}\mathcal{S}$ is a stack in groupoids by Lemmas 50.9.1 and 50.12.7.

Lemma 50.12.10. *Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites given by a continuous functor $u : \mathcal{C} \to \mathcal{D}$ satisyfing the hypotheses and conclusions of Sites, Proposition 9.14.6. Let $p : \mathcal{S} \to \mathcal{C}$ and $q : \mathcal{T} \to \mathcal{D}$ be stacks. Then we have a canonical equivalence of categories*

$$Mor_{Stacks/\mathcal{C}}(\mathcal{S}, f_*\mathcal{T}) = Mor_{Stacks/\mathcal{D}}(f^{-1}\mathcal{S}, \mathcal{T})$$

of morphism categories.

Proof. For $i = 1, 2$ an i-morphism of stacks is the same thing as a i-morphism of fibred categories, see Definition 50.4.5. By Lemma 50.12.8 we have already

$$Mor_{Fib/\mathcal{C}}(\mathcal{S}, u^p\mathcal{T}) = Mor_{Fib/\mathcal{D}}(u_p\mathcal{S}, \mathcal{T})$$

Hence the result follows from Lemma 50.8.3 as $u^p\mathcal{T} = f_*\mathcal{T}$ and $f^{-1}\mathcal{S}$ is the stackification of $u_p\mathcal{S}$. □

Lemma 50.12.11. *Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites given by a continuous functor $u : \mathcal{C} \to \mathcal{D}$ satisyfing the hypotheses and conclusions of Sites, Proposition 9.14.6. Let $\mathcal{S} \to \mathcal{C}$ be a fibred category, and let $\mathcal{S} \to \mathcal{S}'$ be the stackification of $\mathcal{S}$. Then $f^{-1}\mathcal{S}'$ is the stackification of $u_p\mathcal{S}$.*

Proof. Omitted. Hint: This is the analogue of Sites, Lemma 9.13.4. □

The following lemma tells us that the 2-category of stacks over Sch_{fppf} is a ``full 2-sub category'' of the 2-category of stacks over Sch'_{fppf} provided that Sch'_{fppf} contains Sch_{fppf} (see Topologies, Section 30.10).

Lemma 50.12.12. *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor satisfying the assumptions of Sites, Lemma 9.19.8. Let $f : \mathcal{D} \to \mathcal{C}$ be the corresponding morphism of sites. Then*

1. *for every stack $p : \mathcal{S} \to \mathcal{C}$ the canonical functor $\mathcal{S} \to f_*f^{-1}\mathcal{S}$ is an equivalence of stacks,*
2. *given stacks $\mathcal{S}, \mathcal{S}'$ over $\mathcal{C}$ the construction f^{-1} induces an equivalence*
$$Mor_{Stacks/\mathcal{C}}(\mathcal{S}, \mathcal{S}') \longrightarrow Mor_{Stacks/\mathcal{D}}(f^{-1}\mathcal{S}, f^{-1}\mathcal{S}')$$
of morphism categories.

Proof. Note that by Lemma 50.12.10 we have an equivalence of categories

$$Mor_{Stacks/\mathcal{D}}(f^{-1}\mathcal{S}, f^{-1}\mathcal{S}') = Mor_{Stacks/\mathcal{C}}(\mathcal{S}, f_*f^{-1}\mathcal{S}')$$

Hence (2) follows from (1).

To prove (1) we are going to use Lemma 50.4.8. This lemma tells us that we have to show that $can : \mathcal{S} \to f_*f^{-1}\mathcal{S}$ is fully faithful and that all objects of $f_*f^{-1}\mathcal{S}$ are locally in the essential image.

We quickly describe the functor can, see proof of Lemma 50.12.8. To do this we introduce the functor $c'' : \mathcal{S} \to u_{pp}\mathcal{S}$ defined by $c''(x/U) = (U, \text{id} : u(U) \to u(U), x)$, and $c''(\alpha/a) = (a, u(a), \alpha)$. We set $c' : \mathcal{S} \to u_p\mathcal{S}$ equal to the composition of c'' and the canonical functor $u_{pp}\mathcal{S} \to u_p\mathcal{S}$. We set $c : \mathcal{S} \to f^{-1}\mathcal{S}$ equal to the composition of c' and the canonical functor $u_p\mathcal{S} \to f^{-1}\mathcal{S}$. Then $can : \mathcal{S} \to f_*f^{-1}\mathcal{S}$ is the functor which to x/U associates the pair $(U, c(x))$ and to α/a the morphism $(a, c(\alpha))$.

Fully faithfulness. To prove this we are going to use Lemma 50.4.7. Let $U \in Ob(\mathcal{C})$. Let $x, y \in \mathcal{S}_U$. First off, as u is fully faithful, we have

$$Mor_{(f_*f^{-1}\mathcal{S})_U}(can(x), can(y)) = Mor_{(f^{-1}\mathcal{S})_{u(U)}}(c(x), c(y))$$

directly from the definition of f_*. Similarly holds after pull back to any U'/U. Because $f^{-1}\mathcal{S}$ is the stackification of $u_p\mathcal{S}$, and since u is continuous and cocontinuous the presheaf

$$U'/U \longmapsto Mor_{(f^{-1}\mathcal{S})_{u(U')}}(c(x|_{U'}), c(y|_{U'}))$$

is the sheafification of the presheaf

$$U'/U \longmapsto Mor_{(u_p\mathcal{S})_{u(U')}}(c'(x|_{U'}), c'(y|_{U'}))$$

Hence to finish the proof of fully faithfulness it suffices to show that for any U and x, y the map

$$Mor_{\mathcal{S}_U}(x, y) \longrightarrow Mor_{(u_p\mathcal{S})_U}(c'(x), c'(y))$$

is bijective. A morphism $f : x \to y$ in $u_p\mathcal{S}$ over $u(U)$ is given by an equivalence class of diagrams

$$\begin{tikzcd} (U', \phi : u(U) \to u(U'), x') \arrow[r, "(a,b,\alpha)"'] \arrow[d, "(c, \mathrm{id}_{u(U)}, \gamma)"'] & (U, \mathrm{id} : u(U) \to u(U), y) \\ (U, \mathrm{id} : u(U) \to u(U), x) \end{tikzcd}$$

with γ strongly cartesian and $b = \mathrm{id}_{u(U)}$. But since u is fully faithful we can write $\phi = u(c')$ for some morphism $c' : U \to U'$ and then we see that $a \circ c' = \mathrm{id}_U$ and $c \circ c' = \mathrm{id}_{U'}$. Because γ is strongly cartesian we can find a morphism $\gamma' : x \to x'$ lifting c' such that $\gamma \circ \gamma' = \mathrm{id}_x$. By definition of the equivalence classes defining morphisms in $u_p\mathcal{S}$ it follows that the morphism

$$(U, \mathrm{id} : u(U) \to u(U), x) \xrightarrow[(\mathrm{id}, \mathrm{id}, \alpha \circ \gamma')]{} (U, \mathrm{id} : u(U) \to u(U), y)$$

of $u_{pp}\mathcal{S}$ induces the morphism f in $u_p\mathcal{S}$. This proves that the map is surjective. We omit the proof that it is injective.

Finally, we have to show that any object of $f_*f^{-1}\mathcal{S}$ locally comes from an object of $\mathcal{S}$. This is clear from the constructions (details omitted). □

50.13. Stacks and localization

Let $\mathcal{C}$ be a site. Let U be an object of $\mathcal{C}$. We want to understand stacks over $\mathcal{C}/U$ as stacks over $\mathcal{C}$ together with a morphism towards U. The following lemma is the reason why this is easier to do when the presheaf h_U is a sheaf.

Lemma 50.13.1. *Let $\mathcal{C}$ be a site. Let $U \in Ob(\mathcal{C})$. Then $j_U : \mathcal{C}/U \to \mathcal{C}$ is a stack over $\mathcal{C}$ if and only if h_U is a sheaf.*

Proof. Combine Lemma 50.6.3 with Categories, Example 4.35.7. □

Assume that $\mathcal{C}$ is a site, and U is an object of $\mathcal{C}$ whose associated representable presheaf is a sheaf. We denote $j : \mathcal{C}/U \to \mathcal{C}$ the localization functor.

Construction A. Let $p : \mathcal{S} \to \mathcal{C}/U$ be a stack over the site $\mathcal{C}/U$. We define a stack $j_!p : j_!\mathcal{S} \to \mathcal{C}$ as follows:

(1) As a category $j_!\mathcal{S} = \mathcal{S}$, and
(2) the functor $j_!p : j_!\mathcal{S} \to \mathcal{C}$ is just the composition $j \circ p$.

We omit the verification that this is a stack (hint: Use that h_U is a sheaf to glue morphisms to U). There is a canonical functor

$$j_!\mathcal{S} \longrightarrow \mathcal{C}/U$$

namely the functor p which is a 1-morphism of stacks over $\mathcal{C}$.

Construction B. Let $q : \mathcal{T} \to \mathcal{C}$ be a stack over $\mathcal{C}$ which is endowed with a morphism of stacks $p : \mathcal{T} \to \mathcal{C}/U$ over $\mathcal{C}$. In this case it is automically the case that $p : \mathcal{T} \to \mathcal{C}/U$ is a stack over $\mathcal{C}/U$.

Lemma 50.13.2. *Assume that $\mathcal{C}$ is a site, and U is an object of $\mathcal{C}$ whose associated representable presheaf is a sheaf. Constructions A and B above define mutually inverse (!) functors of 2-categories*

$$\left\{\begin{matrix} \text{2-category of} \\ \text{stacks over } \mathcal{C}/U \end{matrix}\right\} \leftrightarrow \left\{\begin{matrix} \text{2-category of pairs } (\mathcal{X}, p) \text{ consisting} \\ \text{of a stack } \mathcal{X} \text{ over } \mathcal{C} \text{ and a morphism} \\ p : \mathcal{X} \to \mathcal{C}/U \text{ of stacks over } \mathcal{C} \end{matrix}\right\}$$

Proof. This is clear. □

50.14. Other chapters

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(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Commutative Algebra
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(12) More on Algebra
(13) Smoothing Ring Maps
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(15) Sheaves of Modules
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(36) More on Groupoid Schemes
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CHAPTER 51

Formal Deformation Theory

51.1. Introduction

This chapter develops formal deformation theory in a form applicable later in the stacks project, closely following Rim [**GRR72**, Exposee VI] and Schlessinger [**Sch68**]. We strongly encourage the reader new to this topic to read the paper by Schlessinger first, as it is sufficiently general for most applications, and Schlessinger's results are indeed used in most papers that use this kind of formal deformation theory.

Let Λ be a complete Noetherian local ring with residue field k, and let $\mathcal{C}_\Lambda$ denote the category of Artinian local Λ-algebras with residue field k. Given a functor $F : \mathcal{C}_\Lambda \to Sets$ such that $F(k)$ is a one element set, Schlessinger's paper introduced conditions (H1)-(H4) such that:

(1) F has a ``hull'' if and only if (H1)-(H3) hold.
(2) F is prorepresentable if and only (H1)-(H4) hold.

The purpose of this chapter is to generalize these results in two ways exactly as is done in Rim's paper:

(A) The functor F is replaced by a category $\mathcal{F}$ cofibered in groupoids over $\mathcal{C}_\Lambda$, see Section 51.3.
(B) We let Λ be a Noetherian ring and $\Lambda \to k$ a finite ring map to a field. The category $\mathcal{C}_\Lambda$ is the category of Artinian local Λ-algebras A endowed with a given identification $A/\mathfrak{m}_A = k$.

The analogue of the condition that $F(k)$ is a one element set is that $\mathcal{F}(k)$ is the trivial groupoid. If $\mathcal{F}$ satisfies this condition then we say it is a *predeformation category*, but in general we do not make this assumption. Rim's paper [**GRR72**, Exposee VI] is the original source for the results in this document. We also mention the useful paper [**TV**], which discusses deformation theory with groupoids but in less generality than we do here.

An important role is played by the ``completion'' $\widehat{\mathcal{C}}_\Lambda$ of the category $\mathcal{C}_\Lambda$. An object of $\widehat{\mathcal{C}}_\Lambda$ is a Noetherian complete local Λ-algebra R whose residue field is identified with k, see Section 51.4. On the one hand $\mathcal{C}_\Lambda \subset \widehat{\mathcal{C}}_\Lambda$ is a strictly full subcategory and on the other hand $\widehat{\mathcal{C}}_\Lambda$ is a full subcategory of the category of pro-objects of $\mathcal{C}_\Lambda$. A functor $\mathcal{C}_\Lambda \to Sets$ is *prorepresentable* if it is isomorphic to the restriction of a representable functor $\underline{R} = Mor_{\widehat{\mathcal{C}}_\Lambda}(R, -)$ to $\mathcal{C}_\Lambda$ where $R \in Ob(\widehat{\mathcal{C}}_\Lambda)$.

Categories cofibred in groupoids are dual to categories fibred in groupoids; we introduced them in Section 51.5. A *smooth* morphism of categories cofibred in groupoids over $\mathcal{C}_\Lambda$ is one that satisfies the infinitesimal lifting criterion for objects, see Section 51.8. This is analogous to the definition of a formally smooth ring map, see Algebra, Definition 7.127.1 and is exactly dual to the notion in Criteria for Representability, Section 59.6. This is an important notion as we eventually want to prove that certain kinds of categories cofibred in groupoids have a smooth prorepresentable presentation, much like the characterization of algebraic stacks in Algebraic Stacks, Sections 57.16 and 57.17. A *versal formal object* of a category $\mathcal{F}$ cofibred in groupoids over $\mathcal{C}_\Lambda$ is an object $\xi \in \widehat{\mathcal{F}}(R)$ of the completion such that the associated morphism $\underline{\xi} : \underline{R} \to \mathcal{F}$ is smooth.

In Section 51.9, we define conditions (S1) and (S2) on $\mathcal{F}$ generalizing Schlessinger's (H1) and (H2). The analogue of Schlessinger's (H3)---the condition that $\mathcal{F}$ has finite dimensional tangent space---is not given a name. A key step in the development of the theory is the existence of versal formal objects for predeformation categories satisfying (S1), (S2) and (H3), see Lemma 51.12.4. Schlessinger's notion of a *hull* for a functor $F : \mathcal{C}_\Lambda \to Sets$ is, in our terminology, a versal formal object $\xi \in \widehat{F}(R)$ such that the induced map of tangent spaces $d\underline{\xi} : T\underline{R} \to TF$ is an isomorphism. In the literature a hull is often called a ``miniversal'' object. We do not do so, and here is why. It can happen that a functor has a versal formal object without having a hull. Moreover, we show in Section 51.13 that if a predeformation category has a versal formal object, then it always has a *minimal* one (as defined in Definition 51.13.4) which is unique up to isomorphism, see Lemma 51.13.5. But it can happen that the minimal versal formal object does not induce an isomorphism on tangent spaces! (See Examples 51.14.3 and 51.14.8.)

Keeping in mind the differences pointed out above, Theorem 51.14.5 is the direct generalization of (1) above: it recovers Schlessinger's result in the case that $\mathcal{F}$ is a functor and it characterizes minimal versal formal objects, in the presence of conditions (S1) and (S2), in terms of the map $d\underline{\xi} : T\underline{R} \to TF$ on tangent spaces.

In Section 51.15, we define Rim's condition (RS) on $\mathcal{F}$ generalizing Schlessinger's (H4). A *deformation category* is defined as a predeformation category satisfying (RS). The analogue to prorepresentable functors are the categories cofibred in groupoids over $\mathcal{C}_\Lambda$ which have a *presentation by a smooth prorepresentable groupoid in functors* on $\mathcal{C}_\Lambda$, see Definitions 51.19.1, 51.20.1, and 51.21.1. This notion of a presentation takes into account the groupoid structure of the fibers of $\mathcal{F}$. In Theorem 51.24.5 we prove that $\mathcal{F}$ has a presentation by a smooth prorepresentable groupoid in functors if and only if $\mathcal{F}$ has a finite dimensional tangent space and finite dimensonsal infinitesimal automorphism space. This is the generalization of (2) above: it reduces to Schlessinger's result in the case that $\mathcal{F}$ is a functor. There is a final Section 51.25 where we discuss how to use minimal versal formal objects to produce a (unique up to isomorphism) minimal presentation by a smooth prorepresentable groupoid in functors.

We also find the following conceptual explanation for Schlessinger's conditions. If a predeformation category $\mathcal{F}$ satisfies (RS), then the associated functor of isomorphism classes $\overline{\mathcal{F}} : \mathcal{C}_\Lambda \to Sets$ satisfies (H1) and (H2) (Lemmas 51.15.6 and 51.9.5). Conversely, if a functor $F : \mathcal{C}_\Lambda \to Sets$ arises naturally as the functor of isomorphism classes of a category $\mathcal{F}$ cofibered in groupoids, then it seems to happen in practice that an argument showing F satisfies (H1) and (H2) will also show $\mathcal{F}$ satisfies (RS) (see Section 51.27 for examples). Moreover, if $\mathcal{F}$ satisfies (RS), then condition (H4) for $\overline{\mathcal{F}}$ has a simple interpretation in terms of extending automorphisms of objects of $\mathcal{F}$ (Lemma 51.15.7). These observations suggest that (RS) should be regarded as the fundamental deformation theoretic glueing condition.

51.2. Notation and Conventions

A ring is commutative with 1. The maximal ideal of a local ring A is denoted by $\mathfrak{m}_A$. The set of positive integers is denoted by $\mathbf{N} = \{1, 2, 3, \ldots\}$. If U is an object of a category $\mathcal{C}$, we denote by $\underline{U}$ the functor $Mor_{\mathcal{C}}(U, -) : \mathcal{C} \to Sets$, see Remarks 51.5.2 (12). Warning: this may conflict with the notation in other chapters where sometimes use $\underline{U}$ to denote $h_U(-) = Mor_{\mathcal{C}}(-, U)$.

Throughout this chapter Λ is a Noetherian ring and $\Lambda \to k$ is a finite ring map from Λ to a field. The kernel of this map is denoted $\mathfrak{m}_\Lambda$ and the image $k' \subset k$. It turns out that $\mathfrak{m}_\Lambda$ is a maximal ideal, $k' = \Lambda/\mathfrak{m}_\Lambda$ is a field, and the extension $k' \subset k$ is finite. See discussion surrounding (51.3.3.1).

51.3. The category $\mathcal{C}_\Lambda$

Motivation. An important application of formal deformation theory is to criteria for representability by algebraic spaces. Suppose given a locally Noetherian base change S and a functor $F : (Sch/S)_{fppf}^{opp} \to Sets$. Let k be a finite type field over S, i.e., we are given a finite type morphism $Spec(k) \to S$. One of Artin's criteria is that for any element $x \in F(Spec(k))$ the predeformation functor associated to the triple (S, k, x) should be prorepresentable. By Morphisms, Lemma 24.15.1 the condition that k is of finite type over S means that there exists an affine open $Spec(\Lambda) \subset S$ such that k is a finite Λ-algebra. This motivates why we work throughout this chapter with a base category as follows.

Definition 51.3.1. Let Λ be a Noetherian ring and let $\Lambda \to k$ be a finite ring map where k is a field. We define $\mathcal{C}_\Lambda$ to be the category with

(1) objects are pairs (A, φ) where A is an Artinian local Λ-algebra and where $\varphi : A/\mathfrak{m}_A \to k$ is a Λ-algebra isomorphism, and
(2) morphisms $f : (B, \psi) \to (A, \varphi)$ are local Λ-algebra homomorphisms such that $\varphi \circ (f \bmod \mathfrak{m}) = \psi$.

We say we are in the *classical case* if Λ is a Noetherian complete local ring and k is its residue field.

Note that if $\Lambda \to k$ is surjective and if A is an Artinian local Λ-algebra, then the identification φ, if it exists, is unique. Moreover, in this case any Λ-algebra map $A \to B$ is going to be compatible with the identifications. Hence in this case $\mathcal{C}_\Lambda$ is just the category of local Artinian Λ-algebras whose residue field ``is'' k. By abuse of notation we also denote objects of $\mathcal{C}_\Lambda$ simply A in the general case. Moreover, we will often write $A/\mathfrak{m} = k$, i.e., we will pretend all rings in $\mathcal{C}_\Lambda$ have residue field k (since all ring maps in $\mathcal{C}_\Lambda$ are compatible with the given identifications this should never cause any problems). Throughout the rest of this chapter the base ring Λ and the field k are fixed. The category $\mathcal{C}_\Lambda$ will be the base category for the cofibered categories considered below.

Definition 51.3.2. Let $f : B \to A$ be a ring map in $\mathcal{C}_\Lambda$. We say f is a *small extension* if it is surjective and $\ker(f)$ is a nonzero principal ideal which is annihilated by $\mathfrak{m}_B$.

By the following lemma we can often reduce arguments involving surjective ring maps in $\mathcal{C}_\Lambda$ to the case of small extensions.

Lemma 51.3.3. *Let $f : B \to A$ be a surjective ring map in $\mathcal{C}_\Lambda$. Then f can be factored as a composition of small extensions.*

Proof. Let I be the kernel of f. The maximal ideal $\mathfrak{m}_B$ is nilpotent since B is Artinian, say $\mathfrak{m}_B^n = 0$. Hence we get a factorization

$$B = B/I\mathfrak{m}_B^{n-1} \to B/I\mathfrak{m}_B^{n-2} \to \cdots \to B/I \cong A$$

of f into a composition of surjective maps whose kernels are annihilated by the maximal ideal. Thus it suffices to prove the lemma when f itself is such a map, i.e. when I is annihilated by $\mathfrak{m}_B$. In this case I is a k-vector space, which has finite dimension, see Algebra, Lemma 7.49.8. Take a basis $x_1, \ldots, x_n$ of I as a k-vector space to get a factorization

$$B \to B/(x_1) \to \cdots \to B/(x_1, \ldots, x_n) \cong A$$

of f into a composition of small extensions. □

The next lemma says that we can compute the length of a module over a local Λ-algebra with residue field k in terms of the length over Λ. To explain the notation in the statement, let $k' \subset k$ be the image of our fixed finite ring map $\Lambda \to k$. Note that k/k' is a finite extension of rings. Hence k' is a field and k'/k is a finite extension, see Algebra, Lemma 7.32.16. Moreover, as $\Lambda \to k'$ is surjective we see that its kernel is a maximal ideal $\mathfrak{m}_\Lambda$. Thus

$$[k : k'] = [k : \Lambda/\mathfrak{m}_\Lambda] < \infty \tag{51.3.3.1}$$

and in the classical case we have $k = k'$. The notation $k' = \Lambda/\mathfrak{m}_\Lambda$ will be fixed throughout this chapter.

Lemma 51.3.4. *Let A be a local Λ-algebra with residue field k. Let M be an A-module. Then $[k : k']length_A(M) = length_\Lambda(M)$. In the classical case we have $length_A(M) = length_\Lambda(M)$.*

Proof. If M is a simple A-module then $M \cong k$ as an A-module, see Algebra, Lemma 7.48.10. In this case $\text{length}_A(M) = 1$ and $\text{length}_\Lambda(M) = [k' : k]$, see Algebra, Lemma 7.48.6. If $\text{length}_A(M)$ is finite, then the result follows on choosing a filtration of M by A-submodules with simple quotients using additivity, see Algebra, Lemma 7.48.3. If $\text{length}_A(M)$ is infinite, the result follows from the obvious inequality $\text{length}_A(M) \le \text{length}_\Lambda(M)$. □

Lemma 51.3.5. *Let $A \to B$ be a ring map in $\mathcal{C}_\Lambda$. The following are equivalent*

(1) *f is surjective,*
(2) *$\mathfrak{m}_A/\mathfrak{m}_A^2 \to \mathfrak{m}_B/\mathfrak{m}_B^2$ is surjective, and*
(3) *$\mathfrak{m}_A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2) \to \mathfrak{m}_B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2)$ is surjective.*

Proof. For any ring map $f : A \to B$ in $\mathcal{C}_\Lambda$ we have $f(\mathfrak{m}_A) \subset \mathfrak{m}_B$ for example because $\mathfrak{m}_A$, $\mathfrak{m}_B$ is the set of nilpotent elements of A, B. Suppose f is surjective. Let $y \in \mathfrak{m}_B$. Choose $x \in A$ with $f(x) = y$. Since f induces an isomorphism $A/\mathfrak{m}_A \to B/\mathfrak{m}_B$ we see that $x \in \mathfrak{m}_A$. Hence the induced map $\mathfrak{m}_A/\mathfrak{m}_A^2 \to \mathfrak{m}_B/\mathfrak{m}_B^2$ is surjective. In this way we see that (1) implies (2).

It is clear that (2) implies (3). The map $A \to B$ gives rise to a canonical commutative diagram

$$\begin{tikzcd}
\mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2 \otimes_{k'} k \arrow[r] \arrow[d] & \mathfrak{m}_A/\mathfrak{m}_A^2 \arrow[r] \arrow[d] & \mathfrak{m}_A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2) \arrow[r] \arrow[d] & 0 \\
\mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2 \otimes_{k'} k \arrow[r] & \mathfrak{m}_B/\mathfrak{m}_B^2 \arrow[r] & \mathfrak{m}_B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Hence if (3) holds, then so does (2).

Assume (2). To show that $A \to B$ is surjective it suffices by Nakayama's lemma (Algebra, Lemma 7.14.5) to show that $A/\mathfrak{m}_A \to B/\mathfrak{m}_A B$ is surjective. (Note that $\mathfrak{m}_A$ is a nilpotent ideal.) As $k = A/\mathfrak{m}_A = B/\mathfrak{m}_B$ it suffices to show that $\mathfrak{m}_A B \to \mathfrak{m}_B$ is surjective. Applying

Nakayama's lemma once more we see that it suffces to see that $\mathfrak{m}_A B/\mathfrak{m}_A\mathfrak{m}_B \to \mathfrak{m}_B/\mathfrak{m}_B^2$ is surjective which is what we assumed. □

If $A \to B$ is a ring map in $\mathcal{C}_\Lambda$, then the map $\mathfrak{m}_A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2) \to \mathfrak{m}_B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2)$ is the map on relative cotangent spaces. Here is a formal definition.

Definition 51.3.6. Let $R \to S$ be a local homomorphism of local rings. The *relative cotangent space*[1] of R over S is the $S/\mathfrak{m}_S$-vector space $\mathfrak{m}_S/(\mathfrak{m}_R S + \mathfrak{m}_S^2)$.

If $f_1 : A_1 \to A$ and $f_2 : A_2 \to A$ are two ring maps, then the fiber product $A_1 \times_A A_2$ is the subring of $A_1 \times A_2$ consisting of elements whose two projections to A are equal. Throughout this chapter we will be considering conditions involving such a fiber product when f_1 and f_2 are in $\mathcal{C}_\Lambda$. It isn't always the case that the fibre product is an object of $\mathcal{C}_\Lambda$.

Example 51.3.7. Let p be a prime number and let $n \in \mathbf{N}$. Let $\Lambda = \mathbf{F}_p(t_1, t_2, \ldots, t_n)$ and let $k = \mathbf{F}_p(x_1, \ldots, x_n)$ with map $\Lambda \to k$ given by $t_i \mapsto x_i^p$. Let $A = k[\epsilon] = k[x]/(x^2)$. Then A is an object of $\mathcal{C}_\Lambda$. Suppose that $D : k \to k$ is a derivation of k over Λ, for example $D = \partial/\partial x_i$. Then the map

$$f_D : k \longrightarrow k[\epsilon], \quad a \mapsto a + D(a)\epsilon$$

is a morphism of $\mathcal{C}_\Lambda$. Set $A_1 = A_2 = k$ and set $f_1 = f_{\partial/\partial x_1}$ and $f_2(a) = a$. Then $A_1 \times_A A_2 = \{a \in k \mid \partial/\partial x_1(a) = 0\}$ which does not surject onto k. Hence the fibre product isn't an object of $\mathcal{C}_\Lambda$.

It turns out that this problem can only occur if the residue field extension $k' \subset k$ (51.3.3.1) is inseparable and neither f_1 nor f_2 is surjective.

Lemma 51.3.8. *Let $f_1 : A_1 \to A$ and $f_2 : A_2 \to A$ be ring maps in $\mathcal{C}_\Lambda$. Then:*

(1) *If f_1 or f_2 is surjective, then $A_1 \times_A A_2$ is in $\mathcal{C}_\Lambda$.*
(2) *If f_2 is a small extension, then so is $A_1 \times_A A_2 \to A_1$.*
(3) *If the field extension $k' \subset k$ is separable, then $A_1 \times_A A_2$ is in $\mathcal{C}_\Lambda$.*

Proof. The ring $A_1 \times_A A_2$ is a Λ-algebra via the map $\Lambda \to A_1 \times_A A_2$ induced by the maps $\Lambda \to A_1$ and $\Lambda \to A_2$. It is a local ring with unique maximal ideal

$$\mathfrak{m}_{A_1} \times_{\mathfrak{m}_A} \mathfrak{m}_{A_2} = \mathrm{Ker}(A_1 \times_A A_2 \longrightarrow k)$$

A ring is Artinian if and only if it has finite length as a module over itself, see Algebra, Lemma 7.49.8. Since A_1 and A_2 are Artinian, Lemma 51.3.4 implies $\mathrm{length}_\Lambda(A_1)$ and $\mathrm{length}_\Lambda(A_2)$, and hence $\mathrm{length}_\Lambda(A_1 \times A_2)$, are all finite. As $A_1 \times_A A_2 \subset A_1 \times A_2$ is a Λ-submodule, this implies $\mathrm{length}_{A_1\times_A A_2}(A_1 \times_A A_2) \leq \mathrm{length}_\Lambda(A_1 \times_A A_2)$ is finite. So $A_1 \times_A A_2$ is Artinian. Thus the only thing that is keeping $A_1 \times_A A_2$ from being an object of $\mathcal{C}_\Lambda$ is the possibility that its residue field maps to a proper subfield of k via the map $A_1 \times_A A_2 \to A \to A/\mathfrak{m}_A = k$ above.

Proof of (1). If f_2 is surjective, then the projection $A_1 \times_A A_2 \to A_1$ is surjective. Hence the composition $A_1 \times_A A_2 \to A_1 \to A_1/\mathfrak{m}_{A_1} = k$ is surjective and we conclude that $A_1 \times_A A_2$ is an object of $\mathcal{C}_\Lambda$.

Proof of (2). If f_2 is a small extension then $A_2 \to A$ and $A_1 \times_A A_2 \to A_1$ are both surjective with the same kernel. Hence the kernel of $A_1 \times_A A_2 \to A_1$ is a 1-dimensional k-vector space and we see that $A_1 \times_A A_2 \to A_1$ is a small extension.

[1] Caution: We will see later that in our general setting the tangent space of an object $A \in \mathcal{C}_\Lambda$ over Λ should not be defined simply as the k-linear dual of the relative cotangent space. In fact, the correct definition of the relative cotangent space is $\Omega_{S/R} \otimes_S S/\mathfrak{m}_S$.

Proof of (3). Choose $\overline{x} \in k$ such that $k = k'(\overline{x})$ (see Algebra, Lemma 7.38.5). Let $P'(T) \in k'[T]$ be the minimal polynomial of $\overline{x}$ over k'. Since k/k' is separable we see that $\mathrm{d}P/\mathrm{d}T(\overline{x}) \neq 0$. Choose a monic $P \in \Lambda[T]$ which maps to P' under the surjective map $\Lambda[T] \to k'[T]$. Because A, A_1, A_2 are henselian, see Algebra, Lemma 7.139.11, we can find $x, x_1, x_2 \in A, A_1, A_2$ with $P(x) = 0, P(x_1) = 0, P(x_2) = 0$ and such that the image of x, x_1, x_2 in k is $\overline{x}$. Then $(x_1, x_2) \in A_1 \times_A A_2$ because x_1, x_2 map to $x \in A$ by uniqueness, see Algebra, Lemma 7.139.2. Hence the residue field of $A_1 \times_A A_2$ contains a generator of k over k' and we win. □

Next we define essential surjections in $\mathcal{C}_\Lambda$. A necessary and sufficient condition for a surjection in $\mathcal{C}_\Lambda$ to be essential is given in Lemma 51.3.12.

Definition 51.3.9. Let $f : B \to A$ be a ring map in $\mathcal{C}_\Lambda$. We say f is an *essential surjection* if it has the following properties:

(1) f is surjective.
(2) If $g : C \to B$ is a ring map in $\mathcal{C}_\Lambda$ such that $f \circ g$ is surjective, then g is surjective.

Using Lemma 51.3.5, we can characterize essential surjections in $\mathcal{C}_\Lambda$ as follows.

Lemma 51.3.10. *Let $f : B \to A$ be a ring map in $\mathcal{C}_\Lambda$. The following are equivalent*

(1) *f is an essential surjection,*
(2) *the map $B/\mathfrak{m}_B^2 \to A/\mathfrak{m}_A^2$ is an essential surjection, and*
(3) *the map $B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2) \to A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2)$ is an essential surjection.*

Proof. Assume (3). Let $C \to B$ be a ring map in $\mathcal{C}_\Lambda$ such that $C \to A$ is surjective. Then $C \to A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2)$ is surjective too. We conclude that $C \to B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2)$ is surjective by our assumption. Hence $C \to B$ is surjective by applying Lemma 51.3.5 (2 times).

Assume (1). Let $C \to B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2)$ be a morphism of $\mathcal{C}_\Lambda$ such that $C \to A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2)$ is surjective. Set $C' = C \times_{B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2)} B$ which is an object of $\mathcal{C}_\Lambda$ by Lemma 51.3.8. Note that $C' \to A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2)$ is still surjective, hence $C' \to A$ is surjective by Lemma 51.3.5. Thus $C' \to B$ is surjective by our assumption. This implies that $C' \to B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2)$ is surjective, which implies by the construction of C' that $C \to B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2)$ is surjective.

In the first paragraph we proved (3) ⇒ (1) and in the second paragraph we proved (1) ⇒ (3). The equivalence of (2) and (3) is a special case of the equivalence of (1) and (3), hence we are done. □

To analyze essential surjections in $\mathcal{C}_\Lambda$ a bit more we introduce some notation. Suppose that A is an object of $\mathcal{C}_\Lambda$. There is a canonical exact sequence

$$\mathfrak{m}_A/\mathfrak{m}_A^2 \xrightarrow{\mathrm{d}_A} \Omega_{A/\Lambda} \otimes_A k \to \Omega_{k/\Lambda} \to 0 \tag{51.3.10.1}$$

see Algebra, Lemma 7.122.9. Note that $\Omega_{k/\Lambda} = \Omega_{k/k'}$ with k' as in (51.3.3.1). Let $H_1(L_{k/\Lambda})$ be the first homology module of the naive cotangent complex of k over Λ, see Algebra, Definition 7.123.1. Then we can extend (51.3.10.1) to the exact sequence

$$H_1(L_{k/\Lambda}) \to \mathfrak{m}_A/\mathfrak{m}_A^2 \xrightarrow{\mathrm{d}_A} \Omega_{A/\Lambda} \otimes_A k \to \Omega_{k/\Lambda} \to 0, \tag{51.3.10.2}$$

see Algebra, Lemma 7.123.3. If $B \to A$ is a ring map in $\mathcal{C}_\Lambda$ then we obtain a commutative diagram

$$\begin{tikzcd} H_1(L_{k/\Lambda}) \arrow[r] \arrow[d, equal] & \mathfrak{m}_B/\mathfrak{m}_B^2 \arrow[r, "\mathrm{d}_B"'] \arrow[d] & \Omega_{B/\Lambda} \otimes_B k \arrow[r] \arrow[d] & \Omega_{k/\Lambda} \arrow[r] \arrow[d, equal] & 0 \\ H_1(L_{k/\Lambda}) \arrow[r] & \mathfrak{m}_A/\mathfrak{m}_A^2 \arrow[r, "\mathrm{d}_A"] & \Omega_{A/\Lambda} \otimes_A k \arrow[r] & \Omega_{k/\Lambda} \arrow[r] & 0 \end{tikzcd} \tag{51.3.10.3}$$

with exact rows.

Lemma 51.3.11. *There is a canonical map*

$$\mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2 \longrightarrow H_1(L_{k/\Lambda}).$$

If $k' \subset k$ is separable (for example if the characteristic of k is zero), then this map induces an isomorphism $\mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2 \otimes_{k'} k = H_1(L_{k/\Lambda})$. If $k = k'$ (for example in the classical case), then $\mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2 = H_1(L_{k/\Lambda})$. The composition

$$\mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2 \longrightarrow H_1(L_{k/\Lambda}) \longrightarrow \mathfrak{m}_A/\mathfrak{m}_A^2$$

comes from the canonical map $\mathfrak{m}_\Lambda \to \mathfrak{m}_A$.

Proof. Note that $H_1(L_{k'/\Lambda}) = \mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2$ as $\Lambda \to k'$ is surjective with kernel $\mathfrak{m}_\Lambda$. The map arises from functoriality of the naive cotangent complex. If $k' \subset k$ is separable, then $k' \to k$ is an étale ring map, see Algebra, Lemma 7.132.4. Thus its naive cotangent complex has trivial homology groups, see Algebra, Definition 7.132.1. Then Algebra, Lemma 7.123.3 applied to the ring maps $\Lambda \to k' \to k$ implies that $\mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2 \otimes_{k'} k = H_1(L_{k/\Lambda})$. We omit the proof of the final statement. □

Lemma 51.3.12. *Let $f : B \to A$ be a ring map in $\mathcal{C}_\Lambda$. Notation as in (51.3.10.3).*

(1) *The equivalent conditions of Lemma 51.3.10 characterizing when f is surjective are also equivalent to*
   (a) *$Im(d_B) \to Im(d_A)$ is surjective, and*
   (b) *the map $\Omega_{B/\Lambda} \otimes_B k \to \Omega_{A/\Lambda} \otimes_A k$ is surjective.*
(2) *The following are equivalent*
   (a) *f is an essential surjection,*
   (b) *the map $Im(d_B) \to Im(d_A)$ is an isomorphism, and*
   (c) *the map $\Omega_{B/\Lambda} \otimes_B k \to \Omega_{A/\Lambda} \otimes_A k$ is an isomorphism.*
(3) *If k/k' is separable, then f is an essential surjection if and only if the map $\mathfrak{m}_B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2) \to \mathfrak{m}_A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2)$ is an isomorphism.*
(4) *If f is a small extension, then f is not essential if and only if f has a section $s : A \to B$ in $\mathcal{C}_\Lambda$ with $f \circ s = id_A$.*

Proof. Proof of (1). It follows from (51.3.10.3) that (1)(a) and (1)(b) are equivalent. Also, if $A \to B$ is surjective, then (1)(a) and (1)(b) hold. Assume (1)(a). Since the kernel of d_A is the image of $H_1(L_{k/\Lambda})$ which also maps to $\mathfrak{m}_B/\mathfrak{m}_B^2$ we conclude that $\mathfrak{m}_B/\mathfrak{m}_B^2 \to \mathfrak{m}_A/\mathfrak{m}_A^2$ is surjective. Hence $B \to A$ is surjective by Lemma 51.3.5. This finishes the proof of (1).

Proof of (2). The equivalence of (2)(b) and (2)(c) is immediate from (51.3.10.3).

Assume (2)(b). Let $g : C \to B$ be a ring map in $\mathcal{C}_\Lambda$ such that $f \circ g$ is surjective. We conclude that $\mathfrak{m}_C/\mathfrak{m}_C^2 \to \mathfrak{m}_A/\mathfrak{m}_A^2$ is surjective by Lemma 51.3.5. Hence $\mathrm{Im}(\mathrm{d}_C) \to \mathrm{Im}(\mathrm{d}_A)$ is surjective and by the assumption we see that $\mathrm{Im}(\mathrm{d}_C) \to \mathrm{Im}(\mathrm{d}_B)$ is surjective. It follows that $C \to B$ is surjective by (1).

Assume (2)(a). Then f is surjective and we see that $\Omega_{B/\Lambda} \otimes_B k \to \Omega_{A/\Lambda} \otimes_A k$ is surjective. Let K be the kernel. Note that $K = \mathrm{d}_B(\mathrm{Ker}(\mathfrak{m}_B/\mathfrak{m}_B^2 \to \mathfrak{m}_A/\mathfrak{m}_A^2))$ by (51.3.10.3). Choose a splitting

$$\Omega_{B/\Lambda} \otimes_B k = \Omega_{A/\Lambda} \otimes_A k \oplus K$$

of k-vector space. The map $\mathrm{d} : B \to \Omega_{B/\Lambda}$ induces via the projection onto K a map $D : B \to K$. Set $C = \{b \in B \mid D(b) = 0\}$. The Leibniz rule shows that this is a Λ-subalgebra of B. Let $\overline{x} \in k$. Choose $x \in B$ mapping to $\overline{x}$. If $D(x) \neq 0$, then we can find an element $y \in \mathfrak{m}_B$ such that $D(y) = D(x)$. Hence $x - y \in C$ is an element which maps to $\overline{x}$. Thus $C \to k$ is surjective and C is an object of $\mathcal{C}_\Lambda$. Similarly, pick $\omega \in \mathrm{Im}(\mathrm{d}_A)$. We can find $x \in \mathfrak{m}_B$ such that $\mathrm{d}_B(x)$ maps to ω by (1). If $D(x) \neq 0$, then we can find an element $y \in \mathfrak{m}_B$ which maps to zero in $\mathfrak{m}_A/\mathfrak{m}_A^2$ such that $D(y) = D(x)$. Hence $z = x - y$ is an element of $\mathfrak{m}_C$ whose image $\mathrm{d}_C(z) \in \Omega_{C/k} \otimes_C k$ maps to ω. Hence $\mathrm{Im}(\mathrm{d}_C) \to \mathrm{Im}(\mathrm{d}_A)$ is surjective. We conclude that $C \to A$ is surjective by (1). Hence $C \to B$ is surjective by assumption. Hence $D = 0$, i.e., $K = 0$, i.e., (2)(c) holds. This finishes the proof of (2).

Proof of (3). If k'/k is separable, then $H_1(L_{k/\Lambda}) = \mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2 \otimes_{k'} k$, see Lemma 51.3.11. Hence $\mathrm{Im}(\mathrm{d}_A) = \mathfrak{m}_A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2)$ and similarly for B. Thus (3) follows from (2).

Proof of (4). A section s of f is not surjective (by definition a small extension has nontrivial kernel), hence f is not essentially surjective. Conversely, assume f is a small surjection but not an essential surjection. Choose a ring map $C \to B$ in $\mathcal{C}_\Lambda$ which is not surjective, such that $C \to A$ is surjective. Let $C' \subset B$ be the image of $C \to B$. Then $C' \neq B$ but C' surjects onto A. Since $f : B \to A$ is a small extension, $\mathrm{length}_C(B) = \mathrm{length}_C(A) + 1$. Thus $\mathrm{length}_C(C') \leq \mathrm{length}_C(A)$ since C' is a proper subring of B. But $C' \to A$ is surjective, so in fact we must have $\mathrm{length}_C(C') = \mathrm{length}_C(A)$ and $C' \to A$ is an isomorphism which gives us our section. $\square$

Example 51.3.13. Let $\Lambda = k[[x]]$ be the power series ring in 1 variable over k. Set $A = k$ and $B = \Lambda/(x^2)$. Then $B \to A$ is an essential surjection by Lemma 51.3.12 because it is a small extension and the map $B \to A$ does not have a right inverse (in the category $\mathcal{C}_\Lambda$). But the map

$$k \cong \mathfrak{m}_B/\mathfrak{m}_B^2 \longrightarrow \mathfrak{m}_A/\mathfrak{m}_A^2 = 0$$

is not an isomorphism. Thus in Lemma 51.3.12 (3) it is necessary to consider the map of relative cotangent spaces $\mathfrak{m}_B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2) \to \mathfrak{m}_A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2)$.

51.4. The category $\widehat{\mathcal{C}}_\Lambda$

The following ``completion'' of the category $\mathcal{C}_\Lambda$ will serve as the base category of the completion of a category cofibered in groupoids over $\mathcal{C}_\Lambda$ (Section 51.7).

Definition 51.4.1. Let Λ be a Noetherian ring and let $\Lambda \to k$ be a finite ring map where k is a field. We define $\widehat{\mathcal{C}}_\Lambda$ to be the category with

(1) objects are pairs (R, φ) where R is a Noetherian complete local Λ-algebra and where $\varphi : R/\mathfrak{m}_R \to k$ is a Λ-algebra isomorphism, and
(2) morphisms $f : (S, \psi) \to (R, \varphi)$ are local Λ-algebra homomorphisms such that $\varphi \circ (f \bmod \mathfrak{m}) = \psi$.

As in the discussion following Definition 51.3.1 we will usually denote an object of $\widehat{\mathcal{C}}_\Lambda$ simply R, with the identification $R/\mathfrak{m}_R = k$ understood. In this section we discuss some basic properties of objects and morphisms of the category $\widehat{\mathcal{C}}_\Lambda$ paralleling our discussion of the category $\mathcal{C}_\Lambda$ in the previous section.

Our first observation is that any object $A \in \mathcal{C}_\Lambda$ is an object of $\widehat{\mathcal{C}}_\Lambda$ as an Artinian local ring is always Noetherian and complete with respect to its maximal ideal (which is after all a nilpotent ideal). Moreover, it is clear from the definitions that $\mathcal{C}_\Lambda \subset \widehat{\mathcal{C}}_\Lambda$ is the strictly full subcategory consisting of all Artinian rings. As it turns out, conversely every object of $\widehat{\mathcal{C}}_\Lambda$ is a limit of objects of $\mathcal{C}_\Lambda$.

Suppose that R is an object of $\widehat{\mathcal{C}}_\Lambda$. Consider the rings $R_n = R/\mathfrak{m}_R^n$ for $n \in \mathbf{N}$. These are Noetherian local rings with a unique nilpotent prime ideal, hence Artinian, see Algebra, Proposition 7.57.6. The ring maps

$$\ldots \to R_{n+1} \to R_n \to \ldots \to R_2 \to R_1 = k$$

are all surjective. Completeness of R by definition means that $R = lim\, R_n$. If $f : R \to S$ is a ring map in $\widehat{\mathcal{C}}_\Lambda$ then we obtain a system of ring maps $f_n : R_n \to S_n$ whose limit is the given map.

Lemma 51.4.2. *Let $f : R \to S$ be a ring map in $\widehat{\mathcal{C}}_\Lambda$. The following are equivalent*

(1) *f is surjective,*
(2) *the map $\mathfrak{m}_R/\mathfrak{m}_R^2 \to \mathfrak{m}_S/\mathfrak{m}_S^2$ is surjective, and*
(3) *the map $\mathfrak{m}_R/(\mathfrak{m}_\Lambda R + \mathfrak{m}_R^2) \to \mathfrak{m}_S/(\mathfrak{m}_\Lambda S + \mathfrak{m}_S^2)$ is surjective.*

Proof. Note that for $n \geq 2$ we have the equality of relative cotangent spaces

$$\mathfrak{m}_R/(\mathfrak{m}_\Lambda R + \mathfrak{m}_R^2) = \mathfrak{m}_{R_n}/(\mathfrak{m}_\Lambda R_n + \mathfrak{m}_{R_n}^2)$$

and similarly for S. Hence by Lemma 51.3.5 we see that $R_n \to S_n$ is surjective for all n. Now let K_n be the kernel of $R_n \to S_n$. Then the sequences

$$0 \to K_n \to R_n \to S_n \to 0$$

form an exact sequence of directed inverse systems. The system (K_n) is Mittag-Leffler since each K_n is Artinian. Hence by Algebra, Lemma 7.80.4 taking limits preserves exactness. So $lim\, R_n \to lim\, S_n$ is surjective, i.e., f is surjective. $\square$

Lemma 51.4.3. *The category $\widehat{\mathcal{C}}_\Lambda$ admits pushouts.*

Proof. Let $R \to S_1$ and $R \to S_2$ be morphisms of $\widehat{\mathcal{C}}_\Lambda$. Consider the ring $C = S_1 \otimes_R S_2$. This ring has a finitely generated maximal ideal $\mathfrak{m} = \mathfrak{m}_{S_1} \otimes S_2 + S_1 \otimes \mathfrak{m}_{S_2}$ with residue field k. Set $C^\wedge$ equal to the completion of C with respect to $\mathfrak{m}$. Then $C^\wedge$ is a Noetherian ring complete with respect to the maximal ideal $\mathfrak{m}^\wedge = \mathfrak{m}C^\wedge$ whose residue field is identified with k, see Algebra, Lemma 7.90.9. Hence $C^\wedge$ is an object of $\widehat{\mathcal{C}}_\Lambda$. Then $S_1 \to C^\wedge$ and $S_2 \to C^\wedge$ turn $C^\wedge$ into a pushout over R in $\widehat{\mathcal{C}}_\Lambda$ (details omitted). $\square$

We will not need the following lemma.

Lemma 51.4.4. *The category $\widehat{\mathcal{C}}_\Lambda$ admits coproducts of pairs of objects.*

Proof. Let R and S be objects of $\widehat{\mathcal{C}}_\Lambda$. Consider the ring $C = R \otimes_\Lambda S$. There is a canonical surjective map $C \to R \otimes_\Lambda S \to k \otimes_\Lambda k \to k$ where the last map is the multiplication map. The kernel of $C \to k$ is a maximal ideal $\mathfrak{m}$. Note that $\mathfrak{m}$ is generated by $\mathfrak{m}_R C$, $\mathfrak{m}_S C$ and finitely many elements of C which map to generators of the kernel of $k \otimes_\Lambda k \to k$. Hence $\mathfrak{m}$ is a finitely generated ideal. Set $C^\wedge$ equal to the completion of C with respect to $\mathfrak{m}$. Then $C^\wedge$ is a Noetherian ring complete with respect to the maximal ideal $\mathfrak{m}^\wedge = \mathfrak{m}C^\wedge$ with residue field k, see Algebra, Lemma 7.90.9. Hence $C^\wedge$ is an object of $\widehat{\mathcal{C}}_\Lambda$. Then $R \to C^\wedge$ and $S \to C^\wedge$ turn $C^\wedge$ into a coproduct in $\widehat{\mathcal{C}}_\Lambda$ (details omitted). $\square$

An empty coproduct in a category is an initial object of the category. In the classical case $\widehat{\mathcal{C}}_\Lambda$ has an initial object, namely Λ itself. More generally, if $k' = k$, then the completion $\Lambda^\wedge$ of Λ with respect to $\mathfrak{m}_\Lambda$ is an initial object. More generally still, if $k' \subset k$ is separable, then $\widehat{\mathcal{C}}_\Lambda$ has an initial object too. Namely, choose a monic polynomial $P \in \Lambda[T]$ such that $k \cong k'[T]/(P')$ where $p' \in k'[T]$ is the image of P. Then $R = \Lambda^\wedge[T]/(P)$ is an initial object, see proof of Lemma 51.3.8.

If R is an initial object as above, then we have $\mathcal{C}_\Lambda = \mathcal{C}_R$ and $\widehat{\mathcal{C}}_\Lambda = \widehat{\mathcal{C}}_R$ which effectively brings the whole discussion in this chapter back to the classical case. But, if $k' \subset k$ is inseparable, then an initial object does not exist.

Lemma 51.4.5. *Let S be an object of $\widehat{\mathcal{C}}_\Lambda$. Then* $\dim_k Der_\Lambda(S, k) < \infty$.

Proof. Let $x_1, \ldots, x_n \in \mathfrak{m}_S$ map to a k-basis for the relative cotangent space $\mathfrak{m}_S/(\mathfrak{m}_\Lambda S + \mathfrak{m}_S^2)$. Choose $y_1, \ldots, y_m \in S$ whose images in k generate k over k'. We claim that $\dim_k \mathrm{Der}_\Lambda(S, k) \leq n + m$. To see this it suffices to prove that if $D(x_i) = 0$ and $D(y_j) = 0$, then $D = 0$. Let $a \in S$. We can find a polynomial $P = \sum \lambda_J y^J$ with $\lambda_J \in \Lambda$ whose image in k is the same as the image of a in k. Then we see that $D(a-P) = D(a) - D(P) = D(a)$ by our assumption that $D(y_j) = 0$ for all j. Thus we may assume $a \in \mathfrak{m}_S$. Write $a = \sum a_i x_i$ with $a_i \in S$. By the Leibniz rule

$$D(a) = \sum x_i D(a_i) + \sum a_i D(x_i) = \sum x_i D(a_i)$$

as we assumed $D(x_i) = 0$. We have $\sum x_i D(a_i) = 0$ as multiplication by x_i is zero on k. $\square$

Lemma 51.4.6. *Let $f : R \to S$ be a morphism of $\widehat{\mathcal{C}}_\Lambda$. If $Der_\Lambda(S, k) \to Der_\Lambda(R, k)$ is injective, then f is surjective.*

Proof. If f is not surjective, then $\mathfrak{m}_S/(\mathfrak{m}_R S + \mathfrak{m}_S^2)$ is nonzero by Lemma 51.4.2. Then also $Q = S/(f(R) + \mathfrak{m}_R S + \mathfrak{m}_S^2)$ is nonzero. Note that Q is a $k = R/\mathfrak{m}_R$-vector space via f. We turn Q into an S-module via $S \to k$. The quotient map $D : S \to Q$ is an R-derivation: if $a_1, a_2 \in S$, we can write $a_1 = f(b_1) + a_1'$ and $a_2 = f(b_2) + a_2'$ for some $b_1, b_2 \in R$ and $a_1', a_2' \in \mathfrak{m}_S$. Then b_i and a_i have the same image in k for $i = 1, 2$ and

$$\begin{aligned} a_1 a_2 &= (f(b_1) + a_1')(f(b_2) + a_2') \\ &= f(b_1)a_2' + f(b_2)a_1' \\ &= f(b_1)(f(b_2) + a_2') + f(b_2)(f(b_1) + a_1') \\ &= f(b_1)a_2 + f(b_2)a_1 \end{aligned}$$

in Q which proves the Leibnize rule. Hence $D : S \to Q$ is a Λ-derivation which is zero on composing with $R \to S$. Since $Q \neq 0$ there also exist derivations $D : S \to k$ which are zero on composing with $R \to S$, i.e., $\mathrm{Der}_\Lambda(S, k) \to \mathrm{Der}_\Lambda(R, k)$ is not injective. $\square$

Lemma 51.4.7. *Let R be an object of $\widehat{\mathcal{C}}_\Lambda$. Let (J_n) be a decreasing sequence of ideals such that $\mathfrak{m}_R^n \subset J_n$. Set $J = \bigcap J_n$. Then the sequence (J_n/J) defines the $\mathfrak{m}_{R/J}$-adic topology on R/J.*

Proof. It is clear that $\mathfrak{m}_{R/J}^n \subset J_n/J$. Thus it suffices to show that for every n there exists an N such that $J_N/J \subset \mathfrak{m}_{R/J}^n$. This is equivalent to $J_N \subset \mathfrak{m}_R^n + J$. For each n the ring $R/\mathfrak{m}_R^n$ is Artinian, hence there exists a N_n such that

$$J_{N_n} + \mathfrak{m}_R^n = J_{N_n+1} + \mathfrak{m}_R^n = \ldots$$

Set $E_n = (J_{N_n} + \mathfrak{m}_R^n)/\mathfrak{m}_R^n$. Set $E = \lim E_n \subset \lim R/\mathfrak{m}_R^n = R$. Note that $E \subset J$ as for any $f \in E$ and any m we have $f \in J_m + \mathfrak{m}_R^n$ for all $n \gg 0$, so $f \in J_m$ by Artin-Rees, see Algebra, Lemma 7.47.6. Since the transition maps $E_n \to E_{n-1}$ are all surjective, we see that J surjects onto E_n. Hence for $N = N_n$ works. ☐

Lemma 51.4.8. *Let $\ldots \to A_3 \to A_2 \to A_1$ be a sequence of surjective ring maps in $\mathcal{C}_\Lambda$. If $\dim_k(\mathfrak{m}_{A_n}/\mathfrak{m}_{A_n}^2)$ is bounded, then $S = \lim A_n$ is an object in $\widehat{\mathcal{C}}_\Lambda$ and the ideals $I_n = Ker(S \to A_n)$ define the $\mathfrak{m}_S$-adic topology on S.*

Proof. We will use freely that the maps $S \to A_n$ are surjective for all n. Note that the maps $\mathfrak{m}_{A_{n+1}}/\mathfrak{m}_{A_{n+1}}^2 \to \mathfrak{m}_{A_n}/\mathfrak{m}_{A_n}^2$ are surjective, see Lemma 51.4.2. Hence for n sufficiently large the dimension $\dim_k(\mathfrak{m}_{A_n}/\mathfrak{m}_{A_n}^2)$ stabilizes to an integer, say r. Thus we can find $x_1, \ldots, x_r \in \mathfrak{m}_S$ whose images in A_n generate $\mathfrak{m}_{A_n}$. Moreover, pick $y_1, \ldots, y_t \in S$ whose images in k generate k over Λ. Then we get a ring map $P = \Lambda[z_1, \ldots, z_{r+t}] \to S$, $z_i \mapsto x_i$ and $z_{r+j} \mapsto y_j$ such that the composition $P \to S \to A_n$ is surjective for all n. Let $\mathfrak{m} \subset P$ be the kernel of $P \to k$. Let $R = P^\wedge$ be the $\mathfrak{m}$-adic completion of P; this is an object of $\widehat{\mathcal{C}}_\Lambda$. Since we still have the compatible system of (surjective) maps $R \to A_n$ we get a map $R \to S$. Set $J_n = \mathrm{Ker}(R \to A_n)$. Set $J = \bigcap J_n$. By Lemma 51.4.7 we see that $R/J = \lim R/J_n = \lim A_n = S$ and that the ideals $J_n/J = I_n$ define the $\mathfrak{m}$-adic topology. (Note that for each n we have $\mathfrak{m}_R^{N_n} \subset J_n$ for some N_n and not necessarily $N_n = n$, so a renumbering of the ideals J_n may be necessary before applying the lemma.) ☐

Lemma 51.4.9. *Let $R', R \in Ob(\widehat{\mathcal{C}}_\Lambda)$. Suppose that $R = R' \oplus I$ for some ideal I of R. Let $x_1, \ldots, x_r \in I$ map to a basis of $I/\mathfrak{m}_R I$. Set $S = R'[[X_1, \ldots, X_r]]$ and consider the R'-algebra map $S \to R$ mapping X_i to x_i. Assume that for every $n \gg 0$ the map $S/\mathfrak{m}_S^n \to R/\mathfrak{m}_R^n$ has a left inverse in $\mathcal{C}_\Lambda$. Then $S \to R$ is an isomorphism.*

Proof. As $R = R' \oplus I$ we have

$$\mathfrak{m}_R/\mathfrak{m}_R^2 = \mathfrak{m}_{R'}/\mathfrak{m}_{R'}^2 \oplus I/\mathfrak{m}_R I$$

and similarly

$$\mathfrak{m}_R/\mathfrak{m}_R^2 = \mathfrak{m}_{R'}/\mathfrak{m}_{R'}^2 \oplus \bigoplus kX_i$$

Hence for $n > 1$ the map $S/\mathfrak{m}_S^n \to R/\mathfrak{m}_R^n$ induces an isomorphism on cotangent spaces. Thus a left inverse $h_n : R/\mathfrak{m}_R^n \to S/\mathfrak{m}_S^n$ is surjective by Lemma 51.4.2. Since h_n is injective as a left inverse it is an isomorphism. Thus the canonical surjections $S/\mathfrak{m}_S^n \to R/\mathfrak{m}_R^n$ are all isomorphisms and we win. ☐

51.5. Categories cofibered in groupoids

In developing the theory we work with categories *cofibered* in groupoids. We assume as known the definition and basic properties of categories *fibered* in groupoids, see Categories, Section 4.32.

Definition 51.5.1. Let $\mathcal{C}$ be a category. A *category cofibered in groupoids over $\mathcal{C}$* is a category $\mathcal{F}$ equipped with a functor $p : \mathcal{F} \to \mathcal{C}$ such that $\mathcal{F}^{opp}$ is a category fibered in groupoids over $\mathcal{C}^{opp}$ via $p^{opp} : \mathcal{F}^{opp} \to \mathcal{C}^{opp}$.

Explicitly, $p : \mathcal{F} \to \mathcal{C}$ is cofibered in groupoids if the following two conditions hold:

(1) For every morphism $f : U \to V$ in $\mathcal{C}$ and every object x lying over U, there is a morphism $x \to y$ of $\mathcal{F}$ lying over f.

(2) For every pair of morphisms $a : x \to y$ and $b : x \to z$ of $\mathcal{F}$ and any morphism $f : p(y) \to p(z)$ such that $p(b) = f \circ p(a)$, there exists a unique morphism $c : y \to z$ of $\mathcal{F}$ lying over f such that $b = c \circ a$.

Remarks 51.5.2. Everything about categories fibered in groupoids translates directly to the cofibered setting. The following remarks are meant to fix notation. Let $\mathcal{C}$ be a category.

(1) We often omit the functor $p : \mathcal{F} \to \mathcal{C}$ from the notation.
(2) The fiber category over an object U in $\mathcal{C}$ is denoted by $\mathcal{F}(U)$. Its objects are those of $\mathcal{F}$ lying over U and its morphisms are those of $\mathcal{F}$ lying over id_U. If x, y are objects of $\mathcal{F}(U)$, we sometimes write $Mor_U(x, y)$ for $Mor_{\mathcal{F}(U)}(x, y)$.
(3) The fibre categories $\mathcal{F}(U)$ are groupoids, see Categories, Lemma 4.32.2. Hence the morphisms in $\mathcal{F}(U)$ are all isomorphisms. We sometimes write $\mathrm{Aut}_U(x)$ for $Mor_{\mathcal{F}(U)}(x, x)$.
(4) Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}$, let $f : U \to V$ be a morphism in $\mathcal{C}$, and let $x \in Ob(\mathcal{F}(U))$. A *pushforward* of x along f is a morphism $x \to y$ of $\mathcal{F}$ lying over f. A pushforward is unique up to unique isomorphism (see the discussion following Categories, Definition 4.30.1). We sometimes write $x \to f_*x$ for ``the'' pushforward of x along f.
(5) A *choice of pushforwards for* $\mathcal{F}$ is the choice of a pushforward of x along f for every pair (x, f) as above. We can make such a choice of pushforwards for $\mathcal{F}$ by the axiom of choice.
(6) Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}$. Given a choice of pushforwards for $\mathcal{F}$, there is an associated pseudo-functor $\mathcal{C} \to$ *Groupoids*. We will never use this construction so we give no details.
(7) A morphism of categories cofibered in groupoids over $\mathcal{C}$ is a functor commuting with the projections to $\mathcal{C}$. If $\mathcal{F}$ and $\mathcal{F}'$ are categories cofibered in groupoids over $\mathcal{C}$, we denote the morphisms from $\mathcal{F}$ to $\mathcal{F}'$ by $Mor_{\mathcal{C}}(\mathcal{F}, \mathcal{F}')$.
(8) Categories cofibered in groupoids form a (2, 1)-category $\mathrm{Cof}(\mathcal{C})$. Its 1-morphisms are the morphisms described in (7). If $p : \mathcal{F} \to C$ and $p' : \mathcal{F}' \to \mathcal{C}$ are categories cofibered in groupoids and $\varphi, \psi : \mathcal{F} \to \mathcal{F}'$ are 1-morphisms, then a 2-morphism $t : \varphi \to \psi$ is a morphism of functors such that $p'(t_x) = \mathrm{id}_{p(x)}$ for all $x \in Ob(\mathcal{F})$.
(9) Let $F : \mathcal{C} \to$ *Groupoids* be a functor. There is a category cofibered in groupoids $\mathcal{F} \to \mathcal{C}$ associated to F as follows. An object of $\mathcal{F}$ is a pair (U, x) where $U \in Ob(\mathcal{C})$ and $x \in Ob(F(U))$. A morphism $(U, x) \to (V, y)$ is a pair (f, a) where $f \in Mor_{\mathcal{C}}(U, V)$ and $a \in Mor_{F(V)}(F(f)(x), y)$. The functor $\mathcal{F} \to \mathcal{C}$ sends (U, x) to U. See Categories, Section 4.34.
(10) Let $\mathcal{F}$ be cofibered in groupoids over $\mathcal{C}$. For $U \in Ob(\mathcal{C})$ set $\overline{F}(U)$ equal to the set of isomorphisms classes of the category $\mathcal{F}(U)$. If $f : U \to V$ is a morphism of $\mathcal{C}$, then we obtain a map of sets $\overline{F}(U) \to \overline{F}(V)$ by mapping the isomorphism class of x to the isomorphism class of a pushforward f_*x of x see (4). Then $\overline{\mathcal{F}} : \mathcal{C} \to$ *Sets* is a functor. Similarly, if $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of cofibered categories, we denote by $\overline{\varphi} : \overline{\mathcal{F}} \to \overline{\mathcal{G}}$ the associated morphism of functors.
(11) Let $F : \mathcal{C} \to$ *Sets* be a functor. We can think of a set as a discrete category, i.e., as a groupoid with only identity morphisms. Then the construction (9) associates to F a category cofibered in sets. This defines a fully faithful embedding of the category of functors $\mathcal{C} \to$ *Sets* to the category of categories cofibered in groupoids over $\mathcal{C}$. We identify the category of functors with its image under this

embedding. Hence if $F : \mathcal{C} \to Sets$ is a functor, we denote the associated category cofibered in sets also by F; and if $\varphi : F \to G$ is a morphism of functors, we denote still by φ the corresponding morphism of categories cofibered in sets, and vice-versa. See Categories, Section 4.35.

(12) Let U be an object of $\mathcal{C}$. We write $\underline{U}$ for the functor $Mor_{\mathcal{C}}(U, -) : \mathcal{C} \to Sets$. This defines a fully faithful embedding of $\mathcal{C}^{opp}$ into the category of functors $\mathcal{C} \to Sets$. Hence, if $f : U \to V$ is a morphism, we are justified in denoting still by f the induced morphism $\underline{V} \to \underline{U}$, and vice-versa.

(13) Fiber products of categories cofibered in groupoids: If $\mathcal{F} \to \mathcal{H}$ and $\mathcal{G} \to \mathcal{H}$ are morphisms of categories cofibered in groupoids over $\mathcal{C}_\Lambda$, then a construction of their 2-fiber product is given by the construction for their 2-fiber product as categories over $\mathcal{C}_\Lambda$, as described in Categories, Lemma 4.29.3.

(14) Restricting the base category: Let $p : \mathcal{F} \to \mathcal{C}$ be a category cofibered in groupoids, and let $\mathcal{C}'$ be a full subcategory of $\mathcal{C}$. The restriction $\mathcal{F}|_{\mathcal{C}'}$ is the full subcategory of $\mathcal{F}$ whose objects lie over objects of $\mathcal{C}'$. It is a category cofibered in groupoids via the functor $p|_{\mathcal{C}'} : \mathcal{F}|_{\mathcal{C}'} \to \mathcal{C}'$.

51.6. Prorepresentable functors and predeformation categories

Our basic goal is to understand categories cofibered in groupoids over $\mathcal{C}_\Lambda$ and $\widehat{\mathcal{C}}_\Lambda$. Since $\mathcal{C}_\Lambda$ is a full subcategory of $\widehat{\mathcal{C}}_\Lambda$ we can restrict categories cofibred in groupoids over $\widehat{\mathcal{C}}_\Lambda$ to $\mathcal{C}_\Lambda$, see Remarks 51.5.2 (14). In particular we can do this with functors, in particular with representable functors. The functors on $\mathcal{C}_\Lambda$ one obtains in this way are called prorepresentable functors.

Definition 51.6.1. Let $F : \mathcal{C}_\Lambda \to Sets$ be a functor. We say F is *prorepresentable* if there exists an isomorphism $F \cong \underline{R}|_{\mathcal{C}_\Lambda}$ of functors for some $R \in Ob(\widehat{\mathcal{C}}_\Lambda)$.

Note that if $F : \mathcal{C}_\Lambda \to Sets$ is prorepresentable by $R \in Ob(\widehat{\mathcal{C}}_\Lambda)$, then

$$F(k) = Mor_{\widehat{\mathcal{C}}_\Lambda}(R, k) = \{*\}$$

is a singleton. The categories cofibered in groupoids over $\mathcal{C}_\Lambda$ that are arise in deformation theory will often satisfy an analogous condition.

Definition 51.6.2. A *predeformation category* $\mathcal{F}$ is a category cofibered in groupoids over $\mathcal{C}_\Lambda$ such that $\mathcal{F}(k)$ is equivalent to a category with a single object and a single morphism, i.e., $\mathcal{F}(k)$ contains at least one object and there is a unique morphism between any two objects. A *morphism of predeformation categories* is a morphism of categories cofibered in groupoids over $\mathcal{C}_\Lambda$.

A feature of a predeformation category is the following. Let $x_0 \in Ob(\mathcal{F}(k))$. Then every object of $\mathcal{F}$ comes equipped with a unique morphism to x_0. Namely, if x is an object of $\mathcal{F}$ over A, then we can choose a pushforward $x \to q_*x$ where $q : A \to k$ is the quotient map. There is a unique isomorphism $q_*x \to x_0$ and the composition $x \to q_*x \to x_0$ is the desired morphism.

Remark 51.6.3. We say that a functor $F : \mathcal{C}_\Lambda \to Sets$ is a *predeformation functor* if the associated cofibered set is a predeformation category, i.e. if $F(k)$ is a one element set. Thus if $\mathcal{F}$ is a predeformation category, then $\overline{\mathcal{F}}$ is a predeformation functor.

Remark 51.6.4. Let $p : \mathcal{F} \to \mathcal{C}_\Lambda$ be a category cofibered in groupoids, and let $x \in Ob(\mathcal{F}(k))$. We denote by $\mathcal{F}_x$ the category of objects over x. An object of $\mathcal{F}_x$ is an arrow $y \to x$. A morphism $(y \to x) \to (z \to x)$ in $\mathcal{F}_x$ is a commutative diagram

$$\begin{tikzcd} y \arrow[rr] \arrow[dr] & & z \arrow[dl] \\ & x & \end{tikzcd}$$

There is a forgetful functor $\mathcal{F}_x \to \mathcal{F}$. We define the functor $p_x : \mathcal{F}_x \to \mathcal{C}_\Lambda$ as the composition $\mathcal{F}_x \to \mathcal{F} \xrightarrow{p} \mathcal{C}_\Lambda$. Then $p_x : \mathcal{F}_x \to \mathcal{C}_\Lambda$ is a predeformation category (proof omitted). In this way we can pass from an arbitrary category cofibered in groupoids over $\mathcal{C}_\Lambda$ to a predeformation category at any $x \in Ob(\mathcal{F}(k))$.

51.7. Formal objects and completion categories

In this section we discuss how to go between categories cofibred in groupoids over $\mathcal{C}_\Lambda$ to categories cofibred in groupoids over $\widehat{\mathcal{C}}_\Lambda$ and vice versa.

Definition 51.7.1. Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. The *category $\widehat{\mathcal{F}}$ of formal objects of $\mathcal{F}$* is the category with the following objects and morphisms.

(1) A *formal object $\xi = (R, \xi_n, f_n)$ of $\mathcal{F}$* consists of an object R of $\widehat{\mathcal{C}}_\Lambda$, and a collection indexed by $n \in \mathbf{N}$ of objects ξ_n of $\mathcal{F}(R/\mathfrak{m}_R^n)$ and morphisms $f_n : \xi_{n+1} \to \xi_n$ lying over the projection $R/\mathfrak{m}_R^{n+1} \to R/\mathfrak{m}_R^n$.
(2) Let $\xi = (R, \xi_n, f_n)$ and $\eta = (S, \eta_n, g_n)$ be formal objects of $\mathcal{F}$. A *morphism $a : \xi \to \eta$ of formal objects* consists of a map $a_0 : R \to S$ in $\widehat{\mathcal{C}}_\Lambda$ and a collection $a_n : \xi_n \to \eta_n$ of morphisms of $\mathcal{F}$ lying over $R/\mathfrak{m}_R^n \to S/\mathfrak{m}_S^n$, such that for every n the diagram

$$\begin{tikzcd} \xi_{n+1} \arrow[r, "f_n"] \arrow[d, "a_{n+1}"'] & \xi_n \arrow[d, "a_n"] \\ \eta_{n+1} \arrow[r, "g_n"] & \eta_n \end{tikzcd}$$

commutes.

The category of formal objects comes with a functor $\hat{p} : \widehat{\mathcal{F}} \to \widehat{\mathcal{C}}_\Lambda$ which sends an object (R, ξ_n, f_n) to R and a morphism $(R, \xi_n, f_n) \to (S, \eta_n, g_n)$ to the map $R \to S$.

Lemma 51.7.2. *Let $p : \mathcal{F} \to \mathcal{C}_\Lambda$ be a category cofibered in groupoids. Then $\hat{p} : \widehat{\mathcal{F}} \to \widehat{\mathcal{C}}_\Lambda$ is a category cofibered in groupoids.*

Proof. Let $R \to S$ be a ring map in $\widehat{\mathcal{C}}_\Lambda$. Let (R, ξ_n, f_n) be an object of $\widehat{\mathcal{F}}$. For each n choose a pushforward $\xi_n \to \eta_n$ of ξ_n along $R/\mathfrak{m}_R^n \to S/\mathfrak{m}_S^n$. For each n there exists a unique morphism $g_n : \eta_{n+1} \to \eta_n$ in $\mathcal{F}$ lying over $S/\mathfrak{m}_S^{n+1} \to S/\mathfrak{m}_S^n$ such that

$$\begin{tikzcd} \xi_{n+1} \arrow[r, "f_n"'] \arrow[d] & \xi_n \arrow[d] \\ \eta_{n+1} \arrow[r, "g_n"] & \eta_n \end{tikzcd}$$

commutes (by the first axiom of a category cofibred in groupoids). Hence we obtain a morphism $(R, \xi_n, f_n) \to (S, \eta_n, g_n)$ lying over $R \to S$, i.e., the first axiom of a category cofibred in groupoids holds for $\widehat{\mathcal{F}}$. To see the second axiom suppose that we have morphisms $a : (R, \xi_n, f_n) \to (S, \eta_n, g_n)$ and $b : (R, \xi_n, f_n) \to (T, \theta_n, h_n)$ in $\widehat{\mathcal{F}}$ and a morphism $c_0 : S \to T$ in $\widehat{\mathcal{C}}_\Lambda$ such that $c_0 \circ a_0 = b_0$. By the second axiom of a category cofibred in groupoids for $\mathcal{F}$ we obtain unique maps $c_n : \eta_n \to \theta_n$ lying over $S/\mathfrak{m}_S^n \to T/\mathfrak{m}_T^n$ such that $c_n \circ a_n = b_n$. Setting $c = (c_n)_{n \geq 0}$ gives the desired morphism $c : (S, \eta_n, g_n) \to (T, \theta_n, h_n)$ in $\widehat{\mathcal{F}}$ (we omit the verification that $h_n \circ c_{n+1} = c_n \circ g_n$). □

Definition 51.7.3. Let $p : \mathcal{F} \to \mathcal{C}_\Lambda$ be a category cofibered in groupoids. The category cofibered in groupoids $\hat{p} : \widehat{\mathcal{F}} \to \widehat{\mathcal{C}}_\Lambda$ is called the *completion of* $\mathcal{F}$.

If $\mathcal{F}$ is a category cofibered in groupoids over $\mathcal{C}_\Lambda$, we have defined $\widehat{\mathcal{F}}(R)$ for $R \in Ob(\widehat{\mathcal{C}}_\Lambda)$ in terms of the filtration of R by powers of its maximal ideal. But suppose $\mathcal{I} = (I_n)$ is a filtration of R by ideals inducing the $\mathfrak{m}_R$-adic topology. We define $\widehat{\mathcal{F}}_\mathcal{I}(R)$ to be the category with the following objects and morphisms:

(1) An object is a collection $(\xi_n, f_n)_{n \in \mathbf{N}}$ of objects ξ_n of $\mathcal{F}(R/I_n)$ and morphisms $f_n : \xi_{n+1} \to \xi_n$ lying over the projections $R/I_{n+1} \to R/I_n$.
(2) A morphism $a : (\xi_n, f_n) \to (\eta_n, g_n)$ consists of a collection $a_n : \xi_n \to \eta_n$ of morphisms in $\mathcal{F}(R/I_n)$, such that for every n the diagram

$$\begin{tikzcd} \xi_{n+1} \arrow[r, "f_n"] \arrow[d, "a_{n+1}"'] & \xi_n \arrow[d, "a_n"] \\ \eta_{n+1} \arrow[r, "g_n"] & \eta_n \end{tikzcd}$$

commutes.

Lemma 51.7.4. *In the situation above, $\widehat{\mathcal{F}}_\mathcal{I}(R)$ is equivalent to the category $\widehat{\mathcal{F}}(R)$.*

Proof. An equivalence $\widehat{\mathcal{F}}_\mathcal{I}(R) \to \widehat{\mathcal{F}}(R)$ can be defined as follows. For each n, let $m(n)$ be the least m that $I_m \subset \mathfrak{m}_R^n$. Given an object (ξ_n, f_n) of $\widehat{\mathcal{F}}_\mathcal{I}(R)$, let η_n be the pushforward of $\xi_{m(n)}$ along $R/I_{m(n)} \to R/\mathfrak{m}_R^n$. Let $g_n : \eta_{n+1} \to \eta_n$ be the unique morphism of $\mathcal{F}$ lying over $R/\mathfrak{m}_R^{n+1} \to R/\mathfrak{m}_R^n$ such that

$$\begin{tikzcd} \xi_{m(n+1)} \arrow[r, "f_{m(n)} \circ \ldots \circ f_{m(n+1)-1}"'] \arrow[d] & \xi_{m(n)} \arrow[d] \\ \eta_{n+1} \arrow[r, "g_n"] & \eta_n \end{tikzcd}$$

commutes (existence and uniqueness is garanteed by the axioms of a cofibred category). The functor $\widehat{\mathcal{F}}_\mathcal{I}(R) \to \widehat{\mathcal{F}}(R)$ sends (ξ_n, f_n) to (R, η_n, g_n). We omit the verification that this is indeed an equivalence of categories. □

Remark 51.7.5. Let $p : \mathcal{F} \to \mathcal{C}_\Lambda$ be a category cofibered in groupoids. Suppose that for each $R \in Ob(\widehat{\mathcal{C}}_\Lambda)$ we are given a filtration $\mathcal{I}_R$ of R by ideals. If $\mathcal{I}_R$ induces the $\mathfrak{m}_R$-adic topology on R for all R, then one can define a category $\widehat{\mathcal{F}}_\mathcal{I}$ by mimicking the definition of $\widehat{\mathcal{F}}$. This category comes equipped with a morphism $\hat{p}_\mathcal{I} : \widehat{\mathcal{F}}_\mathcal{I} \to \widehat{\mathcal{C}}_\Lambda$ making it into a category cofibered in groupoids such that $\widehat{\mathcal{F}}_\mathcal{I}(R)$ is isomorphic to $\widehat{\mathcal{F}}_{\mathcal{I}_R}(R)$ as defined above. The categories cofibered in groupoids $\widehat{\mathcal{F}}_\mathcal{I}$ and $\widehat{\mathcal{F}}$ are equivalent, by using over an object $R \in Ob(\widehat{\mathcal{C}}_\Lambda)$ the equivalence of Lemma 51.7.4.

Remark 51.7.6. Let $F : \mathcal{C}_\Lambda \to Sets$ be a functor. Identifying functors with cofibered sets, the completion of F is the functor $\widehat{F} : \widehat{\mathcal{C}}_\Lambda \to Sets$ given by $\widehat{F}(S) = \lim F(S/\mathfrak{m}_S^n)$. This agrees with the definition in Schlessinger's paper [**Sch68**].

Remark 51.7.7. Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$. We claim that there is a canonical equivalence

$$can : \widehat{\mathcal{F}}|_{\mathcal{C}_\Lambda} \longrightarrow \mathcal{F}.$$

Namely, let $A \in Ob(\mathcal{C}_\Lambda)$ and let (A, ξ_n, f_n) be an object of $\widehat{\mathcal{F}}|_{\mathcal{C}_\Lambda}(A)$. Since A is Artinian there is a minimal $m \in \mathbf{N}$ such that $\mathfrak{m}_A^m = 0$. Then can sends (A, ξ_n, f_n) to ξ_m. This functor is an equivalence of categories cofibered in groupoids by Categories, Lemma 4.32.8 because it is an equivalence on all fibre categories by Lemma 51.7.4 and the fact that the $\mathfrak{m}_A$-adic topology on a local Artinian ring A comes from the zero ideal. We will frequently identify $\mathcal{F}$ with a full subcategory of $\widehat{\mathcal{F}}$ via a quasi-inverse to the functor can.

Remark 51.7.8. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of categories cofibered in groupoids over $\mathcal{C}_\Lambda$. Then there is an induced morphism $\widehat{\varphi} : \widehat{\mathcal{F}} \to \widehat{\mathcal{G}}$ of categories cofibered in groupoids over $\widehat{\mathcal{C}}_\Lambda$. It sends an object $\xi = (R, \xi_n, f_n)$ of $\widehat{\mathcal{F}}$ to $(R, \varphi(\xi_n), \varphi(f_n))$, and it sends a morphism $(a_0 : R \to S, a_n : \xi_n \to \eta_n)$ between objects ξ and η of $\widehat{\mathcal{F}}$ to $(a_0 : R \to S, \varphi(a_n) : \varphi(\xi_n) \to \varphi(\eta_n))$. Finally, if $t : \varphi \to \varphi'$ is a 2-morphism between 1-morphisms $\varphi, \varphi' : \mathcal{F} \to \mathcal{G}$ of categories cofibred in groupoids, then we obtain a 2-morphism $\hat{t} : \widehat{\varphi} \to \widehat{\varphi}'$. Namely, for $\xi = (R, \xi_n, f_n)$ as above we set $\hat{t}_\xi = (t_{\varphi(\xi_n)})$. Hence completion defines a functor between 2-categories

$$\widehat{\ } : \text{Cof}(\mathcal{C}_\Lambda) \longrightarrow \text{Cof}(\widehat{\mathcal{C}}_\Lambda)$$

from the 2-category of categories cofibred in groupoids over $\mathcal{C}_\Lambda$ to the 2-category of categories cofibred in groupoids over $\widehat{\mathcal{C}}_\Lambda$.

Remark 51.7.9. We claim the completion functor of Remark 51.7.8 and the restriction functor $|_{\mathcal{C}_\Lambda} : \text{Cof}(\widehat{\mathcal{C}}_\Lambda) \to \text{Cof}(\mathcal{C}_\Lambda)$ of Remarks 51.5.2 (14) are ``2-adjoint'' in the following precise sense. Let $\mathcal{F} \in Ob(\text{Cof}(\mathcal{C}_\Lambda))$ and let $\mathcal{G} \in Ob(\text{Cof}(\widehat{\mathcal{C}}_\Lambda))$. Then there is an equivalence of categories

$$\Phi : Mor_{\mathcal{C}_\Lambda}(\mathcal{G}|_{\mathcal{C}_\Lambda}, \mathcal{F}) \longrightarrow Mor_{\widehat{\mathcal{C}}_\Lambda}(\mathcal{G}, \widehat{\mathcal{F}})$$

To describe this equivalence, we define canonical morphisms $\mathcal{G} \to \widehat{\mathcal{G}|_{\mathcal{C}_\Lambda}}$ and $\widehat{\mathcal{F}}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ as follows

(1) Let $R \in Ob(\widehat{\mathcal{C}}_\Lambda)$ and let ξ be an object of the fiber category $\mathcal{G}(R)$. Choose a pushforward $\xi \to \xi_n$ of ξ to $R/\mathfrak{m}_R^n$ for each $n \in \mathbf{N}$, and let $f_n : \xi_{n+1} \to \xi_n$ be the induced morphism. Then $\mathcal{G} \to \widehat{\mathcal{G}|_{\mathcal{C}_\Lambda}}$ sends ξ to (R, ξ_n, f_n).
(2) This is the equivalence $can : \widehat{\mathcal{F}}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ of Remark 51.7.7.

Having said this, the equivalence $\Phi : Mor_{\mathcal{C}_\Lambda}(\mathcal{G}|_{\mathcal{C}_\Lambda}, \mathcal{F}) \to Mor_{\widehat{\mathcal{C}}_\Lambda}(\mathcal{G}, \widehat{\mathcal{F}})$ sends a morphism $\varphi : \mathcal{G}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ to

$$\mathcal{G} \to \widehat{\mathcal{G}|_{\mathcal{C}_\Lambda}} \xrightarrow{\widehat{\varphi}} \widehat{\mathcal{F}}$$

There is a quasi-inverse $\Psi : Mor_{\widehat{\mathcal{C}}_\Lambda}(\mathcal{G}, \widehat{\mathcal{F}}) \to Mor_{\mathcal{C}_\Lambda}(\mathcal{G}|_{\mathcal{C}_\Lambda}, \mathcal{F})$ to Φ which sends $\psi : \mathcal{G} \to \widehat{\mathcal{F}}$ to

$$\mathcal{G}|_{\mathcal{C}_\Lambda} \xrightarrow{\psi|_{\mathcal{C}_\Lambda}} \widehat{\mathcal{F}}|_{\mathcal{C}_\Lambda} \to \mathcal{F}.$$

We omit the verification that Φ and Ψ are quasi-inverse. We also do not address functoriality of Φ (because it would lead into 3-category territory which we want to avoid at all cost).

Remark 51.7.10. For a category $\mathcal{C}$ we denote by CofSet($\mathcal{C}$) the category of cofibered sets over $\mathcal{C}$. It is a 1-category isomorphic the category of functors $\mathcal{C} \to \textit{Sets}$. See Remarks 51.5.2 (11). The completion and restriction functors restrict to functors$\hat{}$: CofSet($\mathcal{C}_\Lambda$) $\to$ CofSet($\widehat{\mathcal{C}}_\Lambda$) and $|_{\mathcal{C}_\Lambda}$: CofSet($\widehat{\mathcal{C}}_\Lambda$) $\to$ CofSet($\mathcal{C}_\Lambda$) which we denote by the same symbols. As functors on the categories of cofibered sets, completion and restriction are adjoints in the usual 1-categorical sense: the same construction as in Remark 51.7.9 defines a functorial bijection

$$Mor_{\mathcal{C}_\Lambda}(G|_{\mathcal{C}_\Lambda}, F) \longrightarrow Mor_{\widehat{\mathcal{C}}_\Lambda}(G, \widehat{F})$$

for $F \in Ob(\text{CofSet}(\mathcal{C}_\Lambda))$ and $G \in Ob(\text{CofSet}(\widehat{\mathcal{C}}_\Lambda))$. Again the map $\widehat{F}|_{\mathcal{C}_\Lambda} \to F$ is an isomorphism.

Remark 51.7.11. Let $G : \widehat{\mathcal{C}}_\Lambda \to \textit{Sets}$ be a functor that commutes with limits. Then the map $G \to \widehat{G|_{\mathcal{C}_\Lambda}}$ described in Remark 51.7.9 is an isomorphism. Indeed, if S is an object of $\widehat{\mathcal{C}}_\Lambda$, then we have canonical bijections

$$\widehat{G|_{\mathcal{C}_\Lambda}}(S) = lim_n\, G(S/\mathfrak{m}_S^n) = G(lim_n\, S/\mathfrak{m}_S^n) = G(S).$$

In particular, if R is an object of $\widehat{\mathcal{C}}_\Lambda$ then $\underline{R} = \widehat{\underline{R}|_{\mathcal{C}_\Lambda}}$ because the representable functor $\underline{R}$ commutes with limits by definition of limits.

Remark 51.7.12. Let R be an object of $\widehat{\mathcal{C}}_\Lambda$. It defines a functor $\underline{R} : \widehat{\mathcal{C}}_\Lambda \to \textit{Sets}$ as described in Remarks 51.5.2 (12). As usual we identify this functor with the associated cofibered set. If $\mathcal{F}$ is a cofibered category over $\mathcal{C}_\Lambda$, then there is an equivalence of categories

$$Mor_{\mathcal{C}_\Lambda}(\underline{R}|_{\mathcal{C}_\Lambda}, \mathcal{F}) \longrightarrow \widehat{\mathcal{F}}(R). \tag{51.7.12.1}$$

It is given by the composition

$$Mor_{\mathcal{C}_\Lambda}(\underline{R}|_{\mathcal{C}_\Lambda}, \mathcal{F}) \xrightarrow{\Phi} Mor_{\widehat{\mathcal{C}}_\Lambda}(\underline{R}, \widehat{\mathcal{F}}) \xrightarrow{\sim} \widehat{\mathcal{F}}(R)$$

where Φ is as in Remark 51.7.9 and the second equivalence comes from the 2-Yoneda lemma (the cofibered analogue of Categories, Lemma 4.38.1). Explicitly, the equivalence sends a morphism $\varphi : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ to the formal object $(R, \varphi(R \to R/\mathfrak{m}_R^n), \varphi(f_n))$ in $\widehat{\mathcal{F}}(R)$, where $f_n : R/\mathfrak{m}_R^{n+1} \to R/\mathfrak{m}_R^n$ is the projection.

Assume a choice of pushforwards for $\mathcal{F}$ has been made. Given any $\xi \in Ob(\widehat{\mathcal{F}}(R))$ we construct an explicit $\underline{\xi} : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ which maps to ξ under (51.7.12.1). Namely, say $\xi = (R, \xi_n, f_n)$. An object α in $\underline{R}|_{\mathcal{C}_\Lambda}$ is the same thing as a morphism $\alpha : R \to A$ of $\widehat{\mathcal{C}}_\Lambda$ with A Artinian. Let $m \in \mathbf{N}$ be minimal such that $\mathfrak{m}_A^m = 0$. Then α factors through a unique $\alpha_m : R/\mathfrak{m}_R^m \to A$ and we can set $\underline{\xi}(\alpha) = \alpha_{m,*}\xi_m$. We omit the description of $\underline{\xi}$ on morphisms and we omit the proof that $\underline{\xi}$ maps to ξ via (51.7.12.1).

Assume a choice of pushforwards for $\widehat{\mathcal{F}}$ has been made. In this case the proof of Categories, Lemma 4.38.1 gives an explicit quasi-inverse

$$\iota : \widehat{\mathcal{F}}(R) \longrightarrow Mor_{\widehat{\mathcal{C}}_\Lambda}(\underline{R}, \widehat{\mathcal{F}})$$

to the 2-Yoneda equivalence which takes ξ to the the morphism $\iota(\xi) : \underline{R} \to \widehat{\mathcal{F}}$ sending $f \in \underline{R}(S) = Mor_{\mathcal{C}_\Lambda}(R, S)$ to $f_*\xi$. A quasi-inverse to (51.7.12.1) is then

$$\widehat{\mathcal{F}}(R) \xrightarrow{\iota} Mor_{\widehat{\mathcal{C}}_\Lambda}(\underline{R}, \widehat{\mathcal{F}}) \xrightarrow{\Psi} Mor_{\mathcal{C}_\Lambda}(\underline{R}|_{\mathcal{C}_\Lambda}, \mathcal{F})$$

where Ψ is as in Remark 51.7.9. Given $\xi \in Ob(\widehat{\mathcal{F}}(R))$ we have $\Psi(\iota(\xi)) \cong \underline{\xi}$ where $\underline{\xi}$ is as in the previous paragraph, because both are mapped to ξ under the equivalence of categories (51.7.12.1). Using $\underline{R} = \widehat{\underline{R}|_{\mathcal{C}_\Lambda}}$ (see Remark 51.7.11) and unwinding the definitions of Φ and Ψ we conclude that $\iota(\xi)$ is isomorphic to the completion of $\underline{\xi}$.

Remark 51.7.13. Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$. Let $\xi = (R, \xi_i, f_n)$ and $\eta = (S, \eta_n, g_n)$ be formal objects of $\mathcal{F}$. Let $a = (a_n) : \xi \to \eta$ be a morphism of formal objects, i.e., a morphism of $\widehat{\mathcal{F}}$. Let $f = \hat{p}(a) = a_0 : R \to S$ be the projection of a in $\widehat{\mathcal{C}}_\Lambda$. Then we obtain a 2-commutative diagram

$$\begin{tikzcd} \underline{R}|_{\mathcal{C}_\Lambda} \arrow[dr, "\underline{\xi}"'] & & \underline{S}|_{\mathcal{C}_\Lambda} \arrow[ll, "f"'] \arrow[dl, "\underline{\eta}"] \\ & \mathcal{F} & \end{tikzcd}$$

where $\underline{\xi}$ and $\underline{\eta}$ are the morphisms constructed in Remark 51.7.12. To see this let $\alpha : S \to A$ be an object of $\underline{S}|_{\mathcal{C}_\Lambda}$ (see loc. cit.). Let $m \in \mathbf{N}$ be minimal such that $\mathfrak{m}_A^m = 0$. We get a commutative diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d, "f"] & R/\mathfrak{m}_R^m \arrow[d, "f_m"'] \arrow[dr, "\beta_m"] & \\ S \arrow[r] & S/\mathfrak{m}_S^m \arrow[r, "\alpha_m"] & A \end{tikzcd}$$

such that the bottom arrows compose to give α. Then $\underline{\eta}(\alpha) = \alpha_{m,*}\eta_m$ and $\underline{\xi}(\alpha \circ f) = \beta_{m,*}\xi_m$. The morphism $a_m : \xi_m \to \eta_m$ lies over f_m hence we obtain a canonical morphism

$$\underline{\xi}(\alpha \circ f) = \beta_{m,*}\xi_m \longrightarrow \underline{\eta}(\alpha) = \alpha_{m,*}\eta_m$$

lying over id_A such that

$$\begin{tikzcd} \xi_m \arrow[r] \arrow[d, "a_m"] & \beta_{m,*}\xi_m \arrow[d] \\ \eta_m \arrow[r] & \alpha_{m,*}\eta_m \end{tikzcd}$$

commutes by the axioms of a category cofibred in groupoids. This defines a transformation of functors $\underline{\xi} \circ f \to \underline{\eta}$ which witnesses the 2-commutativity of the first diagram of this remark.

Remark 51.7.14. According to Remark 51.7.12, giving a formal object ξ of $\mathcal{F}$ is equivalent to giving a prorepresentable functor $U : \mathcal{C}_\Lambda \to Sets$ and a morphism $U \to \mathcal{F}$.

51.8. Smooth morphisms

In this section we discuss smooth morphisms of categories cofibered in groupoids over $\mathcal{C}_\Lambda$.

Definition 51.8.1. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of categories cofibered in groupoids over $\mathcal{C}_\Lambda$. We say φ is *smooth* if it satisfies the following condition: Let $B \to A$ be a surjective ring map in $\mathcal{C}_\Lambda$. Let $y \in Ob(\mathcal{G}(B)), x \in Ob(\mathcal{F}(A))$, and $y \to \varphi(x)$ be a morphism lying over $B \to A$. Then there exists $x' \in Ob(\mathcal{F}(B))$, a morphism $x' \to x$ lying over $B \to A$, and a morphism $\varphi(x') \to y$ lying over $\mathrm{id} : B \to B$, such that the diagram

$$\begin{tikzcd} \varphi(x') \arrow[r] \arrow[dr] & y \arrow[d] \\ & \varphi(x) \end{tikzcd}$$

commutes.

Lemma 51.8.2. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of categories cofibered in groupoids over $\mathcal{C}_\Lambda$. Then φ is smooth if the condition in Definition 51.8.1 is assumed to hold only for small extensions $B \to A$.*

Proof. Let $B \to A$ be a surjective ring map in $\mathcal{C}_\Lambda$. Let $y \in Ob(\mathcal{G}(B))$, $x \in Ob(\mathcal{F}(A))$, and $y \to \varphi(x)$ be a morphism lying over $B \to A$. By Lemma 51.3.3 we can factor $B \to A$ into small extensions $B = B_n \to B_{n-1} \to \ldots \to B_0 = A$. We argue by induction on n. If $n = 1$ the result is true by assumption. If $n > 1$, then denote $f : B = B_n \to B_{n-1}$ and denote $g : B_{n-1} \to B_0 = A$. Choose a pushforward $y \to f_*y$ of y along f, so that the morphism $y \to \varphi(x)$ factors as $y \to f_*y \to \varphi(x)$. By the induction hypothesis we can find $x_{n-1} \to x$ lying over $g : B_{n-1} \to A$ and $a : \varphi(x_{n-1}) \to f_*y$ lying over $\mathrm{id} : B_{n-1} \to B_{n-1}$ such that

$$\begin{tikzcd} \varphi(x_{n-1}) \arrow[r, "a"'] \arrow[dr] & f_*y \arrow[d] \\ & \varphi(x) \end{tikzcd}$$

commutes. We can apply the assumption to the composition $y \to \varphi(x_{n-1})$ of $y \to f_*y$ with $a^{-1} : f_*y \to \varphi(x_{n-1})$. We obtain $x_n \to x_{n-1}$ lying over $B_n \to B_{n-1}$ and $\varphi(x_n) \to y$ lying over $\mathrm{id} : B_n \to B_n$ so that the diagram

$$\begin{tikzcd} \varphi(x_n) \arrow[r] \arrow[d] & y \arrow[d] \\ \varphi(x_{n-1}) \arrow[r, "a"] \arrow[dr] & f_*y \arrow[d] \\ & \varphi(x) \end{tikzcd}$$

commutes. Then the composition $x_n \to x_{n-1} \to x$ and $\varphi(x_n) \to y$ are the morphisms required by the definition of smoothness. $\square$

Remark 51.8.3. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of categories cofibered in groupoids over $\mathcal{C}_\Lambda$. Let $B \to A$ be a ring map in $\mathcal{C}_\Lambda$. Choices of pushforwards along $B \to A$ for objects in the fiber categories $\mathcal{F}(B)$ and $\mathcal{G}(B)$ determine functors $\mathcal{F}(B) \to \mathcal{F}(A)$ and $\mathcal{G}(B) \to \mathcal{G}(A)$ fitting into a 2-commutative diagram

$$\begin{tikzcd} \mathcal{F}(B) \arrow[r, "\varphi"] \arrow[d] & \mathcal{G}(B) \arrow[d] \\ \mathcal{F}(A) \arrow[r, "\varphi"] & \mathcal{G}(A). \end{tikzcd}$$

Hence there is an induced functor $\mathcal{F}(B) \to \mathcal{F}(A) \times_{\mathcal{G}(A)} \mathcal{G}(B)$. Unwinding the definitions shows that $\varphi : \mathcal{F} \to \mathcal{G}$ is smooth if and only if this induced functor is essentially surjective whenever $B \to A$ is surjective (or equivalently, by Lemma 51.8.2, whenever $B \to A$ is a small extension).

Remark 51.8.4. The characterization of smooth morphisms in Remark 51.8.3 is analogous to Schlessinger's notion of a smooth morphism of functors, cf. [**Sch68**, Definition 2.2.]. In fact, when $\mathcal{F}$ and $\mathcal{G}$ are cofibered in sets then our notion is equivalent to Schlessinger's. Namely, in this case let $F, G : \mathcal{C}_\Lambda \to Sets$ be the corresponding functors, see Remarks 51.5.2 (11). Then $F \to G$ is smooth if and only if for every surjection of rings $B \to A$ in $\mathcal{C}_\Lambda$ the map $F(B) \to F(A) \times_{G(A)} G(B)$ is surjective.

Remark 51.8.5. Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Then the morphism $\mathcal{F} \to \overline{\mathcal{F}}$ is smooth.

If $R \to S$ is a ring map $\widehat{\mathcal{C}}_\Lambda$, then there is an induced morphism $\underline{S} \to \underline{R}$ between the functors $\underline{S}, \underline{R} : \widehat{\mathcal{C}}_\Lambda \to Sets$. In this situation, smoothness of the restriction $\underline{S}|_{\mathcal{C}_\Lambda} \to \underline{R}|_{\mathcal{C}_\Lambda}$ is a familiar notion:

Lemma 51.8.6. *Let $R \to S$ be a ring map in $\widehat{\mathcal{C}}_\Lambda$. Then the induced morphism $\underline{S}|_{\mathcal{C}_\Lambda} \to \underline{R}|_{\mathcal{C}_\Lambda}$ is smooth if and only if S is a power series ring over R.*

Proof. Assume S is a power series ring over R. Say $S = R[[x_1, \ldots, x_n]]$. Smoothness of $\underline{S}|_{\mathcal{C}_\Lambda} \to \underline{R}|_{\mathcal{C}_\Lambda}$ means the following (see Remark 51.8.4): Given a surjective ring map $B \to A$ in $\mathcal{C}_\Lambda$, a ring map $R \to B$, a ring map $S \to A$ such that the solid diagram

$$\begin{tikzcd} S \arrow[r] \arrow[dr, dotted] & A \\ R \arrow[u] \arrow[r] & B \arrow[u] \end{tikzcd}$$

is commutative then a dotted arrow exists making the diagram commute. (Note the similarity with Algebra, Definition 7.127.1.) To construct the dotted arrow choose elements $b_i \in B$ whose images in A are equal to the images of x_i in A. Note that $b_i \in \mathfrak{m}_B$ as x_i maps to an element of $\mathfrak{m}_A$. Hence there is a unique R-algebra map $R[[x_1, \ldots, x_n]] \to B$ which maps x_i to b_i and which can serve as our dotted arrow.

Conversely, assume $\underline{S}|_{\mathcal{C}_\Lambda} \to \underline{R}|_{\mathcal{C}_\Lambda}$ is smooth. Let $x_1, \ldots, x_n \in S$ be elements whose images form a basis in the relative cotangent space $\mathfrak{m}_S/(\mathfrak{m}_R S + \mathfrak{m}_S^2)$ of S over R. Set $T = R[[X_1, \ldots, X_n]]$. Note that both

$$S/(\mathfrak{m}_R S + \mathfrak{m}_S^2) \cong R/\mathfrak{m}_R[x_1, \ldots, x_n]/(x_i x_j)$$

and

$$T/(\mathfrak{m}_R T + \mathfrak{m}_T^2) \cong R/\mathfrak{m}_R[X_1, \ldots, X_n]/(X_i X_j).$$

Let $S/(\mathfrak{m}_R S + \mathfrak{m}_S^2) \to T/(\mathfrak{m}_R T + \mathfrak{m}_T^2)$ be the local R-algebra isomorphism given by mapping the class of x_i to the class of X_i. Let $f_1 : S \to T/(\mathfrak{m}_R T + \mathfrak{m}_T^2)$ be the composition $S \to S/(\mathfrak{m}_R S + \mathfrak{m}_S^2) \to T/(\mathfrak{m}_R T + \mathfrak{m}_T^2)$. The assumption that $\underline{S}|_{\mathcal{C}_\Lambda} \to \underline{R}|_{\mathcal{C}_\Lambda}$ is smooth means we can lift f_1 to a map $f_2 : S \to T/\mathfrak{m}_T^2$, then to a map $f_3 : S \to T/\mathfrak{m}_T^3$, and so on, for all $n \geq 1$. Thus we get an induced map $f : S \to T = \lim T/\mathfrak{m}_T^n$ of local R-algebras. By our choice of f_1, the map f induces an isomorphism $\mathfrak{m}_S/(\mathfrak{m}_R S + \mathfrak{m}_S^2) \to \mathfrak{m}_T/(\mathfrak{m}_R T + \mathfrak{m}_T^2)$ of relative cotangent spaces. Hence f is surjective by Lemma 51.4.2 (where we think of f as a map in $\widehat{\mathcal{C}}_R$). Choose preimages $y_i \in S$ of $X_i \in T$ under f. As T is a power series ring over R there exists a local R-algebra homomorphism $s : T \to S$ mapping X_i to y_i. By construction $f \circ s = \mathrm{id}$. Then s is injective. But s induces an isomorphism on relative cotangent spaces since f does, so it is also surjective by Lemma 51.4.2 again. Hence s and f are isomorphisms. □

Smooth morphisms satisfy the following functorial properties.

Lemma 51.8.7. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ and $\psi : \mathcal{G} \to \mathcal{H}$ be morphisms of categories cofibered in groupoids over $\mathcal{C}_\Lambda$.*

1. *If φ and ψ are smooth, then $\psi \circ \varphi$ is smooth.*
2. *If φ is essentially surjective and $\psi \circ \varphi$ is smooth, then ψ is smooth.*
3. *If $\mathcal{G}' \to \mathcal{G}$ is a morphism of categories cofibered in groupoids and φ is smooth, then $\mathcal{F} \times_{\mathcal{G}} \mathcal{G}' \to \mathcal{G}'$ is smooth.*

Proof. Statements (1) and (2) follow immediately from the definitions. Proof of (3) omitted. Hints: use the formulation of smoothness given in Remark 51.8.3 and use that $\mathcal{F} \times_{\mathcal{G}} \mathcal{G}'$ is the 2-fibre product, see Remarks 51.5.2 (13). □

Lemma 51.8.8. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a smooth morphism of categories cofibered in groupoids over $\mathcal{C}_\Lambda$. Assume $\varphi : \mathcal{F}(k) \to \mathcal{G}(k)$ is essentially surjective. Then $\varphi : \mathcal{F} \to \mathcal{G}$ and $\widehat{\varphi} : \widehat{\mathcal{F}} \to \widehat{\mathcal{G}}$ are essentially surjective.*

Proof. Let y be an object of $\mathcal{G}$ lying over $A \in Ob(\mathcal{C}_\Lambda)$. Let $y \to y_0$ be a pushforward of y along $A \to k$. By the assumption on essential surjectivity of $\varphi : \mathcal{F}(k) \to \mathcal{G}(k)$ there exist an object x_0 of $\mathcal{F}$ lying over k and an isomorphism $y_0 \to \varphi(x_0)$. Smoothness of φ implies there exists an object x of $\mathcal{F}$ over A whose image $\varphi(x)$ is isomorphic to y. Thus $\varphi : \mathcal{F} \to \mathcal{G}$ is essentially surjective.

Let $\eta = (R, \eta_n, g_n)$ be an object of $\widehat{\mathcal{G}}$. We construct an object ξ of $\widehat{\mathcal{F}}$ with an isomorphism $\eta \to \varphi(\xi)$. By the assumption on essential surjectivity of $\varphi : \mathcal{F}(k) \to \mathcal{G}(k)$, there exists a morphism $\eta_1 \to \varphi(\xi_1)$ in $\mathcal{G}(k)$ for some $\xi_1 \in Ob(\mathcal{F}(k))$. The morphism $\eta_2 \xrightarrow{g_1} \eta_1 \to \varphi(\xi_1)$ lies over the surjective ring map $R/\mathfrak{m}_R^2 \to k$, hence by smoothness of φ there exists $\xi_2 \in Ob(\mathcal{F}(R/\mathfrak{m}_R^2))$, a morphism $f_1 : \xi_2 \to \xi_1$ lying over $R/\mathfrak{m}_R^2 \to k$, and a morphism $\eta_2 \to \varphi(\xi_2)$ such that

$$
\begin{tikzcd}
\varphi(\xi_2) \arrow[r, "\varphi(f_1)"] & \varphi(\xi_1) \\
\eta_2 \arrow[u] \arrow[r, "g_1"] & \eta_1 \arrow[u]
\end{tikzcd}
$$

commutes. Continuing in this way we construct an object $\xi = (R, \xi_n, f_n)$ of $\widehat{\mathcal{F}}$ and a morphism $\eta \to \varphi(\xi) = (R, \varphi(\xi_n), \varphi(f_n))$ in $\widehat{\mathcal{G}}(R)$. □

Remark 51.8.9. Let $p : \mathcal{F} \to \mathcal{C}_\Lambda$ be a category cofibered in groupoids. We can consider $\mathcal{C}_\Lambda$ as the trivial category cofibered in groupoids over $\mathcal{C}_\Lambda$, and then p is a morphism of categories cofibered in groupoids over $\mathcal{C}_\Lambda$. We say $\mathcal{F}$ is *smooth* if its structure morphism p is smooth. This is the ``absolute" notion of smoothness for a category cofibered in groupoids over $\mathcal{C}_\Lambda$.

Example 51.8.10. Let $R \in Ob(\widehat{\mathcal{C}}_\Lambda)$. When is $\underline{R}|_{\mathcal{C}_\Lambda}$ smooth? In the classical case this means that R is a power series ring over Λ, see Lemma 51.8.6. (Strictly speaking this uses that $\underline{\Lambda}|_{\mathcal{C}_\Lambda} = \mathcal{C}_\Lambda$ because Λ is an initial object of $\widehat{\mathcal{C}}_\Lambda$ in the classical case.) In the general case we can construct examples as follows. Pick an integer $n \geq 0$ and a maximal ideal $\mathfrak{m} \subset \Lambda[x_1, \ldots, x_n]$ lying over $\mathfrak{m}_\Lambda$ so that

$$k' = \Lambda/\mathfrak{m}_\Lambda \longrightarrow \Lambda[x_1, \ldots, x_n]/\mathfrak{m}$$

is isomorphic to $k' \to k$. Fix such an identification $k = \Lambda[x_1, \ldots, x_n]/\mathfrak{m}$. Set $R = \Lambda[x_1, \ldots, x_n]^\wedge$ equal to the $\mathfrak{m}$-adic completion of $\Lambda[x_1, \ldots, x_n]$. Then R is an object of $\widehat{\mathcal{C}}_\Lambda$. Namely, it is a complete local Noetherian ring (see Algebra, Lemma 7.90.10) and its residue field is identified with k. We claim that $\underline{R}|_{\mathcal{C}_\Lambda}$ is smooth. To see this we have to show: Given a surjection $B \to A$ in $\mathcal{C}_\Lambda$ and a map $R \to A$ there exists a lift of this map to B. This is clear as we can first lift the composition $\Lambda[x_1, \ldots, x_n] \to R \to A$ to a map $\Lambda[x_1, \ldots, x_n] \to B$ and then observe that this latter map factors through the completion R as B is complete (being Artinian). In fact, it turns out that whenever $\underline{R}|_{\mathcal{C}_\Lambda}$ is smooth, then R is isomorphic to a completion of a smooth algebra over Λ, but we won't use this.

Example 51.8.11. Here is a more explicit example of an R as in Example 51.8.10. Let p be a prime number and let $n \in \mathbf{N}$. Let $\Lambda = \mathbf{F}_p(t_1, t_2, \ldots, t_n)$ and let $k = \mathbf{F}_p(x_1, \ldots, x_n)$ with map $\Lambda \to k$ given by $t_i \mapsto x_i^p$. Then we can take

$$R = \Lambda[x_1, \ldots, x_n]^\wedge_{(x_1^p - t_1, \ldots, x_n^p - t_n)}$$

We cannot do ``better'' in this example, i.e., we cannot approximate $\mathcal{C}_\Lambda$ by a smaller smooth object of $\widehat{\mathcal{C}}_\Lambda$ (one can argue that the dimension of R has to be at least n since the map $\Omega_{R/\Lambda} \otimes_R k \to \Omega_{k/\Lambda}$ is surjective). We will discuss this phenomenon later in more detail.

Remark 51.8.12. Suppose $\mathcal{F}$ is a predeformation category admitting a smooth morphism $\varphi : \mathcal{U} \to \mathcal{F}$ from a predeformation category $\mathcal{U}$. Then by Lemma 51.8.8 φ is essentially surjective, so by Lemma 51.8.7 $p : \mathcal{F} \to \mathcal{C}_\Lambda$ is smooth if and only if the composition $\mathcal{U} \xrightarrow{\varphi} \mathcal{F} \xrightarrow{p} \mathcal{C}_\Lambda$ is smooth, i.e. $\mathcal{F}$ is smooth if and only if $\mathcal{U}$ is smooth.

Later we are interested in producing smooth morphisms from prorepresentable functors to predeformation categories $\mathcal{F}$. By the discussion in Remark 51.7.12 these morphisms correspond to certain formal objects of $\mathcal{F}$ More precisely, these are the so-called versal formal objects of $\mathcal{F}$.

Definition 51.8.13. Let $\mathcal{F}$ be a category cofibered in groupoids. Let ξ be a formal object of $\mathcal{F}$ lying over $R \in Ob(\widehat{\mathcal{C}}_\Lambda)$. We say ξ is *versal* if the corresponding morphism $\underline{\xi} : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ of Remark 51.7.12 is smooth.

Remark 51.8.14. Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$, and let ξ be a formal object of $\mathcal{F}$. It follows from the definition of smoothness that versality of ξ is equivalent to the following condition: If

$$\begin{tikzcd} & y \arrow[d] \\ \xi \arrow[r] & x \end{tikzcd}$$

is a diagram in $\widehat{\mathcal{F}}$ such that $y \to x$ lies over a surjective map $B \to A$ of Artinian rings (we may assume it is a small extension), then there exists a morphism $\xi \to y$ such that

$$\begin{tikzcd} & y \arrow[d] \\ \xi \arrow[r] \arrow[ur] & x \end{tikzcd}$$

commutes. In particular, the condition that ξ be versal does not depend on the choices of pushforwards made in the construction of $\underline{\xi} : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ in Remark 51.7.12.

Lemma 51.8.15. *Let $\mathcal{F}$ be a predeformation category. Let ξ be a versal formal object of $\mathcal{F}$. For any formal object η of $\widehat{\mathcal{F}}$, there exists a morphism $\xi \to \eta$.*

Proof. By assumption the morphism $\underline{\xi} : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ is smooth. Then $\iota(\xi) : \underline{R} \to \widehat{\mathcal{F}}$ is the completion of $\underline{\xi}$, see Remark 51.7.12. By Lemma 51.8.8 there exists an object f of $\underline{R}$ such that $\iota(\xi)(f) = \eta$. Then f is a ring map $f : R \to S$ in $\widehat{\mathcal{C}}_\Lambda$. And $\iota(\xi)(f) = \eta$ means that $f_*\xi \cong \eta$ which means exactly that there is a morphism $\xi \to \eta$ lying over f. $\square$

51.9. Schlessinger's conditions

In the following we often consider fibre products $A_1 \times_A A_2$ of rings in the category $\mathcal{C}_\Lambda$. We have seen in Example 51.3.7 that such a fibre product may not always be an object of $\mathcal{C}_\Lambda$. However, in virtually all cases below one of the two maps $A_i \to A$ is surjective and $A_1 \times_A A_2$ will be an object of $\mathcal{C}_\Lambda$ by Lemma 51.3.8. We will use this result without further mention.

We denote by $k[\epsilon]$ the ring of dual numbers over k. More generally, for a k-vector space V, we denote by $k[V]$ the k-algebra whose underlying vector space is $k \oplus V$ and whose multiplication is given by $(a, v) \cdot (a', v') = (aa', av' + a'v)$. When $V = k$, $k[V]$ is the ring of dual numbers over k. For any finite dimensional k-vector space V the ring $k[V]$ is in $\mathcal{C}_\Lambda$.

Definition 51.9.1. Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. We define *conditions (S1) and (S2)* on $\mathcal{F}$ as follows:

(S1) Every diagram in $\mathcal{F}$

$$\begin{tikzcd} & x_2 \arrow[d] \\ x_1 \arrow[r] & x \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} & A_2 \arrow[d] \\ A_1 \arrow[r] & A \end{tikzcd}$$

in $\mathcal{C}_\Lambda$ with $A_2 \to A$ surjective can be completed to a commutative diagram

$$\begin{tikzcd} y \arrow[r] \arrow[d] & x_2 \arrow[d] \\ x_1 \arrow[r] & x \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} A_1 \times_A A_2 \arrow[r] \arrow[d] & A_2 \arrow[d] \\ A_1 \arrow[r] & A. \end{tikzcd}$$

(S2) The condition of (S1) holds for diagrams in $\mathcal{F}$ lying over a diagram in $\mathcal{C}_\Lambda$ of the form

$$\begin{tikzcd} & k[\epsilon] \arrow[d] \\ A \arrow[r] & k. \end{tikzcd}$$

Moreover, if we have two commutative diagrams in $\mathcal{F}$

$$\begin{tikzcd} y \arrow[r, "c"] \arrow[d, "a"'] & x_\epsilon \arrow[d, "e"] \\ x \arrow[r, "d"] & x_0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} y' \arrow[r, "c'"] \arrow[d, "a'"'] & x_\epsilon \arrow[d, "e"] \\ x \arrow[r, "d"] & x_0 \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} A \times_k k[\epsilon] \arrow[r] \arrow[d] & k[\epsilon] \arrow[d] \\ A \arrow[r] & k \end{tikzcd}$$

then there exists a morphism $b : y \to y'$ in $\mathcal{F}(A \times_k k[\epsilon])$ such that $a = a' \circ b$.

We can partly explain the meaning of conditions (S1) and (S2) in terms of fibre categories. Suppose that $f_1 : A_1 \to A$ and $f_2 : A_2 \to A$ are ring maps in $\mathcal{C}_\Lambda$ with f_2 surjective. Denote $p_i : A_1 \times_A A_2 \to A_i$ the projection maps. Assume a choice of pushforwards for $\mathcal{F}$ has been made. Then the commutative diagram of rings tranlates into a 2-commutative diagram

$$\begin{tikzcd} \mathcal{F}(A_1 \times_A A_2) \arrow[r, "p_{2,*}"'] \arrow[d, "p_{1,*}"'] & \mathcal{F}(A_2) \arrow[d, "f_{2,*}"] \\ \mathcal{F}(A_1) \arrow[r, "f_{1,*}"] & \mathcal{F}(A) \end{tikzcd}$$

of fibre categories whence a functor

$$\mathcal{F}(A_1 \times_A A_2) \to \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2) \tag{51.9.1.1}$$

into the 2-fibre product of categories. Condition (S1) requires that this functor be essentially surjective. The first part of condition (S2) requires that this functor be a essentially surjective if f_2 equals the map $k[\epsilon] \to k$. Moreover in this case, the second part of (S2) implies that two objects which become isomorphic in the target are isomorphic in the source (but it is *not* equivalent to this statement). The advantage of stating the conditions as in the definition is that no choices have to be made.

Lemma 51.9.2. *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Then $\mathcal{F}$ satisfies (S1) if the condition of (S1) is assumed to hold only when $A_2 \to A$ is a small extension.*

Proof. Proof omitted. Hints: apply Lemma 51.3.3 and use induction similar to the proof of Lemma 51.8.2. ☐

Remark 51.9.3. When $\mathcal{F}$ is cofibered in sets, conditions (S1) and (S2) are exactly conditions (H1) and (H2) from Schlessinger's paper [**Sch68**]. Namely, for a functor $F : \mathcal{C}_\Lambda \to$ *Sets*, conditions (S1) and (S2) state:

(S1) If $A_1 \to A$ and $A_2 \to A$ are maps in $\mathcal{C}_\Lambda$ with $A_2 \to A$ surjective, then the induced map $F(A_1 \times_A A_2) \to F(A_1) \times_{F(A)} F(A_2)$ is surjective.

(S2) If $A \to k$ is a map in $\mathcal{C}_\Lambda$, then the induced map $F(A \times_k k[\epsilon]) \to F(A) \times_{F(k)} F(k[\epsilon])$ is bijective.

The injectivity of the map $F(A \times_k k[\epsilon]) \to F(A) \times_{F(k)} F(k[\epsilon])$ comes from the second part of condition (S2) and the fact that morphisms are identities.

Lemma 51.9.4. *Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$. If $\mathcal{F}$ satisfies (S2), then the condition of (S2) also holds when $k[\epsilon]$ is replaced by $k[V]$ for any finite dimensional k-vector space V.*

Proof. In the case that $\mathcal{F}$ is cofibred in sets, i.e., corresponds to a functor $F : \mathcal{C}_\Lambda \to$ *Sets* this follows from the description of (S2) for F in Remark 51.9.3 and the fact that $k[V] \cong k[\epsilon] \times_k \ldots \times_k k[\epsilon]$ with $\dim_k V$ factors. The case of functors is what we will use in the rest of this chapter.

We prove the general case by induction on $\dim(V)$. If $\dim(V) = 1$, then $k[V] \cong k[\epsilon]$ and the result holds by assumption. If $\dim(V) > 1$ we write $V = V' \oplus k\epsilon$. Pick a diagram

$$\begin{tikzcd} & x_V \arrow[d] \\ x \arrow[r] & x_0 \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} & k[V] \arrow[d] \\ A \arrow[r] & k \end{tikzcd}$$

Choose a morphism $x_V \to x_{V'}$ lying over $k[V] \to k[V']$ and a morphism $x_V \to x_\epsilon$ lying over $k[V] \to k[\epsilon]$. Note that the morphism $x_V \to x_0$ factors as $x_V \to x_{V'} \to x_0$ and as $x_V \to x_\epsilon \to x_0$. By induction hypothesis we can find a diagram

$$\begin{tikzcd} y' \arrow[r] \arrow[d] & x_{V'} \arrow[d] \\ x \arrow[r] & x_0 \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} A \times_k k[V'] \arrow[r] \arrow[d] & k[V'] \arrow[d] \\ A \arrow[r] & k \end{tikzcd}$$

This gives us a commutative diagram

$$\begin{tikzcd} & x_\epsilon \arrow[d] \\ y' \arrow[r] & x_0 \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} & k[\epsilon] \arrow[d] \\ A \times_k k[V'] \arrow[r] & k \end{tikzcd}$$

Hence by (S2) we get a commutative diagram

$$\begin{tikzcd} y \arrow[r] \arrow[d] & x_\epsilon \arrow[d] \\ y' \arrow[r] & x_0 \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} (A \times_k k[V']) \times_k k[\epsilon] \arrow[r] \arrow[d] & k[\epsilon] \arrow[d] \\ A \times_k k[V'] \arrow[r] & k \end{tikzcd}$$

Note that $(A \times_k k[V']) \times_k k[\epsilon] = A \times_k k[V' \oplus k\epsilon] = A \times_k k[V]$. We claim that y fits into the correct commutative diagram. To see this we let $y \to y_V$ be a morphism lying over $A \times_k k[V] \to k[V]$. We can factor the morphisms $y \to y' \to x_{V'}$ and $y \to x_\epsilon$ through the morphism $y \to y_V$ (by the axioms of categories cofibred in groupoids). Hence we see that both y_V and x_V fit into commutative diagrams

$$\begin{tikzcd} y_V \arrow[r] \arrow[d] & x_\epsilon \arrow[d] \\ x_{V'} \arrow[r] & x_0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} x_V \arrow[r] \arrow[d] & x_\epsilon \arrow[d] \\ x_{V'} \arrow[r] & x_0 \end{tikzcd}$$

and hence by the second part of (S2) there exists an isomorphism $y_V \to x_V$ compatible with $y_V \to x_{V'}$ and $x_V \to x_{V'}$ and in particular compatible with the maps to x_0. The composition $y \to y_V \to x_V$ then fits into the required commutative diagram

$$\begin{tikzcd} y \arrow[r] \arrow[d] & x_V \arrow[d] \\ x \arrow[r] & x_0 \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} A \times_k k[V] \arrow[r] \arrow[d] & k[V] \arrow[d] \\ A \arrow[r] & k \end{tikzcd}$$

In this way we see that the first part of $(S2)$ holds with $k[\epsilon]$ replaced by $k[V]$.

To prove the second part suppose given two commutative diagrams

$$\begin{tikzcd} y \arrow[r] \arrow[d] & x_V \arrow[d] \\ x \arrow[r] & x_0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} y' \arrow[r] \arrow[d] & x_V \arrow[d] \\ x \arrow[r] & x_0 \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} A \times_k k[V] \arrow[r] \arrow[d] & k[V] \arrow[d] \\ A \arrow[r] & k \end{tikzcd}$$

We will use the morphisms $x_V \to x_{V'} \to x_0$ and $x_V \to x_\epsilon \to x_0$ introduced in the first paragraph of the proof. Choose morphisms $y \to y_{V'}$ and $y' \to y'_{V'}$ lying over $A \times_k k[V] \to A \times_k k[V']$. The axioms of a cofibred category imply we can find commutative diagrams

$$\begin{tikzcd} y_{V'} \arrow[r] \arrow[d] & x_{V'} \arrow[d] \\ x \arrow[r] & x_0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} y'_{V'} \arrow[r] \arrow[d] & x_{V'} \arrow[d] \\ x \arrow[r] & x_0 \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} A \times_k k[V'] \arrow[r] \arrow[d] & k[V'] \arrow[d] \\ A \arrow[r] & k \end{tikzcd}$$

By induction hypothesis we obtain an isomorphism $b : y_{V'} \to y'_{V'}$ compatible with the morphisms $y_{V'} \to x$ and $y'_{V'} \to x$, in particular compatible with the morphisms to x_0. Then we have commutative diagrams

$$\begin{tikzcd} y \arrow[r] \arrow[d] & x_\epsilon \arrow[d] \\ y'_{V'} \arrow[r] & x_0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} y' \arrow[r] \arrow[d] & x_\epsilon \arrow[d] \\ y'_{V'} \arrow[r] & x_0 \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} A \times_k k[\epsilon] \arrow[r] \arrow[d] & k[\epsilon] \arrow[d] \\ A \arrow[r] & k \end{tikzcd}$$

where the morphism $y \to y'_{V'}$ is the composition $y \to y_{V'} \xrightarrow{b} y'_{V'}$ and where the morphisms $y \to x_\epsilon$ and $y' \to x_\epsilon$ are the compositions of the maps $y \to x_V$ and $y' \to x_V$ with the morphism $x_V \to x_\epsilon$. Then the second part of (S2) garantees the existence of an isomorphism $y \to y'$ compatible with the maps to $y'_{V'}$, in particular compatible with the maps to x (because b was compatible with the maps to x). ☐

Lemma 51.9.5. *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$.*

(1) *If $\mathcal{F}$ satisfies (S1), then so does $\overline{\mathcal{F}}$.*
(2) *If $\mathcal{F}$ satisfies (S2), then so does $\overline{\mathcal{F}}$ provided at least one of the following conditions is satisfied*
   (a) *$\mathcal{F}$ is a predeformation category,*
   (b) *the category $\mathcal{F}(k)$ is a set or a setoid, or*
   (c) *for any morphism $x_\epsilon \to x_0$ of $\mathcal{F}$ lying over $k[\epsilon] \to k$ the pushforward map $Aut_{k[\epsilon]}(x_\epsilon) \to Aut_k(x_0)$ is surjective.*

Proof. Assume $\mathcal{F}$ has (S1). Suppose we have ring maps $f_i : A_i \to A$ in $\mathcal{C}_\Lambda$ with f_2 surjective. Let $x_i \in \mathcal{F}(A_i)$ such that the pushforwards $f_{1,*}(x_1)$ and $f_{2,*}(x_2)$ are isomorphic. Then we can denote x an object of $\mathcal{F}$ over A isomorphic to both of these and we obtain a diagram as in (S1). Hence we find an object y of $\mathcal{F}$ over $A_1 \times_A A_2$ whose pushforward to A_1, resp. A_2 is isomorphic to x_1, resp. x_2. In this way we see that (S1) holds for $\overline{\mathcal{F}}$.

Assume $\mathcal{F}$ has (S2). The first part of (S2) for $\overline{\mathcal{F}}$ follows as in the argument above. The second part of (S2) for $\overline{\mathcal{F}}$ signifies that the map

$$\overline{\mathcal{F}}(A \times_k k[\epsilon]) \to \overline{\mathcal{F}}(A) \times_{\overline{\mathcal{F}}(k)} \overline{\mathcal{F}}(k[\epsilon])$$

is injective for any ring A in $\mathcal{C}_\Lambda$. Suppose that $y, y' \in \mathcal{F}(A \times_k k[\epsilon])$. Using the axioms of cofibred categories we can choose commutative diagrams

$$\begin{tikzcd} y \arrow[r, "c"] \arrow[d, "a"] & x_\epsilon \arrow[d, "e"] \\ x \arrow[r, "d"] & x_0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} y' \arrow[r, "c'"] \arrow[d, "a'"] & x'_\epsilon \arrow[d, "e'"] \\ x' \arrow[r, "d'"] & x'_0 \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} A \times_k k[\epsilon] \arrow[r] \arrow[d] & k[\epsilon] \arrow[d] \\ A \arrow[r] & k \end{tikzcd}$$

Assume that there exist isomorphisms $\alpha : x \to x'$ in $\mathcal{F}(A)$ and $\beta : x_\epsilon \to x'_\epsilon$ in $\mathcal{F}(k[\epsilon])$. This also means there exists an isomorphism $\gamma : x_0 \to x'_0$ compatible with α. To prove (S2) for $\overline{\mathcal{F}}$ we have to show that there exists an isomorphism $y \to y'$ in $\mathcal{F}(A \times_k k[\epsilon])$. By (S2) for $\mathcal{F}$ such a morphism will exist if we can choose the isomorphisms α and β and γ such that

$$\begin{tikzcd}
x \arrow[r] \arrow[d, "\alpha"] & x_0 \arrow[d, "\gamma"] & x_\epsilon \arrow[l, "e"'] \arrow[d, "\beta"] \\
x' \arrow[r] & x'_0 & x'_\epsilon \arrow[l, "e'"]
\end{tikzcd}$$

is commutative (because then we can replace x by x' and x_ϵ by x'_ϵ in the previous displayed diagram). The left hand square commutes by our choice of γ. We can factor $e' \circ \beta$ as $\gamma' \circ e$ for some second map $\gamma' : x_0 \to x'_0$. Now the question is whether we can arrange it so that $\gamma = \gamma'$? This is clear if $\mathcal{F}(k)$ is a set, or a setoid. Moreover, if $\text{Aut}_{k[\epsilon]}(x_\epsilon) \to \text{Aut}_k(x_0)$ is surjective, then we can adjust the choice of β by precomposing with an automorphism of x_ϵ whose image is $\gamma^{-1} \circ \gamma'$ to make things work. $\square$

Lemma 51.9.6. *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Let $x_0 \in Ob(\mathcal{F}(k))$. Let $\mathcal{F}_{x_0}$ be the category cofibred in groupoids over $\mathcal{C}_\Lambda$ constructed in Remark 51.6.4.*

(1) *If $\mathcal{F}$ satisfies (S1), then so does $\mathcal{F}_{x_0}$.*
(2) *If $\mathcal{F}$ satisfies (S2), then so does $\mathcal{F}_{x_0}$.*

Proof. Any diagram as in Definition 51.9.1 in $\mathcal{F}_{x_0}$ gives rise to a diagram in $\mathcal{F}$ and the output of condition (S1) or (S2) for this diagram in $\mathcal{F}$ can be viewed as an output for $\mathcal{F}_{x_0}$ as well. $\square$

Lemma 51.9.7. *Let $p : \mathcal{F} \to \mathcal{C}_\Lambda$ be a category cofibered in groupoids. Consider a diagram of $\mathcal{F}$*

$$\begin{tikzcd}
y \arrow[r] \arrow[d, "a"'] & x_\epsilon \arrow[d, "e"] \\
x \arrow[r, "d"] & x_0
\end{tikzcd} \quad \textit{lying over} \quad \begin{tikzcd}
A \times_k k[\epsilon] \arrow[r] \arrow[d] & k[\epsilon] \arrow[d] \\
A \arrow[r] & k.
\end{tikzcd}$$

in $\mathcal{C}_\Lambda$. Assume $\mathcal{F}$ satisfies (S2). Then there exists a morphism $s : x \to y$ with $a \circ s = id_x$ if and only if there exists a morphism $s_\epsilon : x \to x_\epsilon$ with $e \circ s_\epsilon = d$.

Proof. The ``only if" direction is clear. Conversely, assume there exists a morphism $s_\epsilon : x \to x_\epsilon$ with $e \circ s_\epsilon = d$. Note that $p(s_\epsilon) : A \to k[\epsilon]$ is a ring map compatible with the map $A \to k$. Hence we obtain

$$\sigma = (\text{id}_A, p(s_\epsilon)) : A \to A \times_k k[\epsilon].$$

Choose a pushforward $x \to \sigma_* x$. By construction we can factor s_ϵ as $x \to \sigma_* x \to x_\epsilon$. Moreover, as σ is a section of $A \times_k k[\epsilon] \to A$, we get a morphism $\sigma_* x \to x$ such that $x \to \sigma_* x \to x$ is id_x. Because $e \circ s_\epsilon = d$ we find that the diagram

$$\begin{tikzcd}
\sigma_* x \arrow[r] \arrow[d] & x_\epsilon \arrow[d, "e"] \\
x \arrow[r, "d"] & x_0
\end{tikzcd}$$

is commutative. Hence by (S2) we obtain a morphism $\sigma_*x \to y$ such that $\sigma_*x \to y \to x$ is the given map $\sigma_*x \to x$. The solution to the problem is now to take $a : x \to y$ equal to the composition $x \to \sigma_*x \to y$. $\square$

Lemma 51.9.8. *Consider a commutative diagram in a predeformation category $\mathcal{F}$*

$$\begin{tikzcd} y \arrow[r] \arrow[d] & x_2 \arrow[d, "a_2"] \\ x_1 \arrow[r, "a_1"] & x \end{tikzcd} \quad \textit{lying over} \quad \begin{tikzcd} A_1 \times_A A_2 \arrow[r] \arrow[d] & A_2 \arrow[d, "f_2"] \\ A_1 \arrow[r, "f_1"] & A \end{tikzcd}$$

in $\mathcal{C}_\Lambda$ where $f_2 : A_2 \to A$ is a small extension. Assume there is a map $h : A_1 \to A_2$ such that $f_2 = f_1 \circ h$. Let $I = Ker(f_2)$. Consider the ring map

$$g : A_1 \times_A A_2 \longrightarrow k[I] = k \oplus I, \quad (u, v) \longmapsto \overline{u} \oplus (v - h(u))$$

Choose a pushforward $y \to g_*y$. Assume $\mathcal{F}$ satisfies (S2). If there exists a morphism $x_1 \to g_*y$, then there exists a morphism $b : x_1 \to x_2$ such that $a_1 = a_2 \circ b$.

Proof. Note that $\mathrm{id}_{A_1} \times g : A_1 \times_A A_2 \to A_1 \times_k k[I]$ is an isomorphism and that $k[I] \cong k[\epsilon]$. Hence we have a diagram

$$\begin{tikzcd} y \arrow[r] \arrow[d] & g_*y \arrow[d] \\ x_1 \arrow[r] & x_0 \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} A_1 \times_k k[\epsilon] \arrow[r] \arrow[d] & k[\epsilon] \arrow[d] \\ A_1 \arrow[r] & k. \end{tikzcd}$$

where x_0 is an object of $\mathcal{F}$ lying over k (every object of $\mathcal{F}$ has a unique morphism to x_0, see discussion following Definition 51.6.2). If we have a morphism $x_1 \to g_*y$ then Lemma 51.9.7 provides us with a section $s : x_1 \to y$ of the map $y \to x_1$. Composing this with the map $y \to x_2$ we obtain $b : x_1 \to x_2$ which has the property that $a_1 = a_2 \circ b$ because the diagram of the lemma commutes and because s is a section. $\square$

51.10. Tangent spaces of functors

Let R be a ring. We write $\mathrm{Mod}(R)$ for the category of R-modules and $\mathrm{Mod}_{fg}(R)$ for the category of finitely generated R-modules.

Definition 51.10.1. Let $L : \mathrm{Mod}_{fg}(R) \to \mathrm{Mod}(R)$ be a functor. We say that L is *R-linear* if for every $M, N \in Ob(\mathrm{Mod}_{fg}(R))$ the map

$$L : Hom_R(M, N) \longrightarrow Hom_R(L(M), L(N))$$

is a map of R-modules.

Remark 51.10.2. One can define the notion of an R-linearity for any functor between categories enriched over $\mathrm{Mod}(R)$. We made the definition specifically for a functor $L : \mathrm{Mod}_{fg}(R) \to \mathrm{Mod}(R)$ because this is the case that occurs below.

Remark 51.10.3. If $L : \mathrm{Mod}_{fg}(R) \to \mathrm{Mod}(R)$ is an R-linear functor, then L preserves finite products and sends the zero module to the zero module, see Homology, Lemma 10.3.7. On the other hand, if a functor $\mathrm{Mod}_{fg}(R) \to \mathit{Sets}$ preserves finite products and sends the zero module to a one element set, then it has a unique lift to a R-linear functor, see Lemma 51.10.4.

Lemma 51.10.4. *Let $L : Mod_{fg}(R) \to Sets$ be a functor. Suppose $L(0)$ is a one element set and L preserves finite products. Then there exists a unique R-linear functor $\widetilde{L} : Mod_{fg}(R) \to Mod(R)$ such that the diagram*

$$\begin{tikzcd} & Mod(R) \arrow[dr, "forget"] & \\ Mod_{fg}(R) \arrow[ur, "\widetilde{L}"] \arrow[rr, "L"] & & Sets \end{tikzcd}$$

commutes.

Proof. Let M be a finitely generated R-module. We define $\widetilde{L}(M)$ to be the set $L(M)$ with the following R-module structure.

Multiplication: If $r \in R$, multiplication by r on $L(M)$ is defined to be the map $L(M) \to L(M)$ induced by the multiplication map $r\cdot : M \to M$.

Addition: The sum map $M \times M \to M : (m_1, m_2) \mapsto m_1 + m_2$ induces a map $L(M \times M) \to L(M)$. By assumption $L(M \times M)$ is canonically isomorphic to $L(M) \times L(M)$. Addition on $L(M)$ is defined by the map $L(M) \times L(M) \cong L(M \times M) \to L(M)$.

Zero: There is a unique map $0 \to M$. The zero element of $L(M)$ is the image of $L(0) \to L(M)$.

We omit the verification that this defines an R-module $\widetilde{L}(M)$, the unique such that is R-linearly functorial in M. □

Lemma 51.10.5. *Let $L_1, L_2 : Mod_{fg}(R) \to Sets$ be functors that take 0 to a one element set and preserve finite products. Let $t : L_1 \to L_2$ be a morphism of functors. Then t induces a morphism $\tilde{t} : \widetilde{L}_1 \to \widetilde{L}_2$ between the functors guaranteed by Lemma 51.10.4, which is given simply by $\tilde{t}_M = t_M : \widetilde{L}_1(M) \to \widetilde{L}_2(M)$ for each $M \in Ob(Mod_{fg}(R))$. In other words, $t_M : \widetilde{L}_1(M) \to \widetilde{L}_2(M)$ is a map of R-modules.*

Proof. Omitted. □

In the case $R = K$ is a field, a K-linear functor $L : \text{Mod}_{fg}(K) \to \text{Mod}(K)$ is determined by its value $L(K)$.

Lemma 51.10.6. *Let K be a field. Let $L : Mod_{fg}(K) \to Mod(K)$ be a K-linear functor. Then L is isomorphic to the functor $L(K) \otimes_K - : Mod_{fg}(K) \to Mod(K)$.*

Proof. For $V \in Ob(\text{Mod}_{fg}(K))$, the isomorphism $L(K) \otimes_K V \to L(V)$ is given on pure tensors by $x \otimes v \mapsto L(f_v)(x)$, where $f_v : K \to V$ is the K-linear map sending $1 \mapsto v$. When $V = K$, this is the isomorphism $L(K) \otimes_K K \to L(K)$ given by multiplication by K. For general V, it is an isomorphism by the case $V = K$ and the fact that L commutes with finite products (Remark 51.10.3). □

For a ring R and an R-module M, let $R[M]$ be the R-algebra whose underlying R-module is $R \oplus M$ and whose multiplication is given by $(r, m) \cdot (r', m') = (rr', rm' + r'm)$. When $M = R$ this is the ring of dual numbers over R, which we denote by $R[\epsilon]$.

Now let S be a ring and assume R is an S-algebra. Then the assignment $M \mapsto R[M]$ determines a functor $\text{Mod}(R) \to S\text{-Alg}/R$, where $S\text{-Alg}/R$ denotes the category of S-algebras over R. Note that $S\text{-Alg}/R$ admits finite products: if $A_1 \to R$ and $A_2 \to R$ are two objects, then $A_1 \times_R A_2$ is a product.

Lemma 51.10.7. *Let R be an S-algebra. Then the functor $Mod(R) \to S\text{-}Alg/R$ described above preserves finite products.*

Proof. This is merely the statement that if M and N are R-modules, then the map $R[M \times N] \to R[M] \times_R R[N]$ is an isomorphism in S-Alg$/R$. □

Lemma 51.10.8. *Let R be an S-algebra, and let $\mathcal{C}$ be a strictly full subcategory of $S\text{-}Alg/R$ containing $R[M]$ for all $M \in Ob(Mod_{fg}(R))$. Let $F : \mathcal{C} \to Sets$ be a functor. Suppose that $F(R)$ is a one element set and that for any $M, N \in Ob(Mod_{fg}(R))$, the induced map*

$$F(R[M] \times_R R[N]) \to F(R[M]) \times F(R[N])$$

is a bijection. Then $F(R[M])$ has a natural R-module structure for any $M \in Ob(Mod_{fg}(R))$.

Proof. Note that $R \cong R[0]$ and $R[M] \times_R R[N] \cong R[M \times N]$ hence R and $R[M] \times_R R[N]$ are objects of $\mathcal{C}$ by our assumptions on $\mathcal{C}$. Thus the conditions on F make sense. The functor $\text{Mod}(R) \to S\text{-Alg}/R$ of Lemma 51.10.7 restricts to a functor $\text{Mod}_{fg}(R) \to \mathcal{C}$ by the assumption on $\mathcal{C}$. Let L be the composition $\text{Mod}_{fg}(R) \to \mathcal{C} \to Sets$, i.e., $L(M) = F(R[M])$. Then L preserves finite products by Lemma 51.10.7 and the assumption on F. Hence Lemma 51.10.4 shows that $L(M) = F(R[M])$ has a natural R-module structure for any $M \in Ob(\text{Mod}_{fg}(R))$. □

Definition 51.10.9. Let $\mathcal{C}$ be a category as in Lemma 51.10.8. Let $F : \mathcal{C} \to Sets$ be a functor such that $F(R)$ is a one element set. The *tangent space TF of F* is $F(R[\epsilon])$.

When $F : \mathcal{C} \to Sets$ satisfies the hypotheses of Lemma 51.10.8, the tangent space TF has a natural R-module structure.

Example 51.10.10. Since $\mathcal{C}_\Lambda$ contains all $k[V]$ for finite dimensional vector spaces V we see that Definition 51.10.9 applies with $S = \Lambda$, $R = k$, $\mathcal{C} = \mathcal{C}_\Lambda$, and $F : \mathcal{C}_\Lambda \to Sets$ a predeformation functor. The tangent space is $TF = F(k[\epsilon])$.

Example 51.10.11. Let us work out the tangent space of Example 51.10.10 when $F : \mathcal{C}_\Lambda \to Sets$ is a prorepresentable functor, say $F = \underline{S}|_{\mathcal{C}_\Lambda}$ for $S \in Ob(\widehat{\mathcal{C}}_\Lambda)$. Then F commutes with arbitrary limits and thus satisfies the hypotheses of Lemma 51.10.8. We compute

$$TF = F(k[\epsilon]) = Mor_{\mathcal{C}_\Lambda}(S, k[\epsilon]) = \text{Der}_\Lambda(S, k)$$

and more generally for a finite dimensional k-vector space V we have

$$F(k[V]) = Mor_{\mathcal{C}_\Lambda}(S, k[V]) = \text{Der}_\Lambda(S, V).$$

Explicitly, a Λ-algebra map $f : S \to k[V]$ compatible with the augmentations $q : S \to k$ and $k[V] \to k$ corresponds to the derivation D defined by $s \mapsto f(s) - q(s)$. Conversely, a Λ-derivation $D : S \to V$ corresponds to $f : S \to k[V]$ in $\mathcal{C}_\Lambda$ defined by the rule $f(s) = q(s) + D(s)$. Since these identifications are functorial we see that the k-vector spaces structures on TF and $\text{Der}_\Lambda(S, k)$ correspond (see Lemma 51.10.5). It follows that $\dim_k TF$ is finite by Lemma 51.4.5.

Example 51.10.12. The computation of Example 51.10.11 simplifies in the classical case. Namely, in this case the tangent space of the functor $F = \underline{S}|_{\mathcal{C}_\Lambda}$ is simply the relative cotangent space of S over Λ, in a formula $TF = T_{S/\Lambda}$. In fact, this works more generally when the field extension $k' \subset k$ is separable. See Exercises, Exercise 65.28.2.

Lemma 51.10.13. *Let $F, G : \mathcal{C} \to Sets$ be functors satisfying the hypotheses of Lemma 51.10.8. Let $t : F \to G$ be a morphism of functors. For any $M \in Ob(Mod_{fg}(R))$, the map $t_{R[M]} : F(R[M]) \to G(R[M])$ is a map of R-modules, where $F(R[M])$ and $G(R[M])$ are* *given the R-module structure from Lemma 51.10.8. In particular, $t_{R[\epsilon]} : TF \to TG$ is a map of R-modules.*

Proof. Follows from Lemma 51.10.5. □

Example 51.10.14. Suppose that $f : R \to S$ is a ring map in $\widehat{\mathcal{C}}_\Lambda$. Set $F = \underline{R}|_{\mathcal{C}_\Lambda}$ and $G = \underline{S}|_{\mathcal{C}_\Lambda}$. The ring map f induces a transformation of functors $G \to F$. By Lemma 51.10.13 we get a k-linear map $TG \to TF$. This is the map

$$TG = \mathrm{Der}_\Lambda(S, k) \longrightarrow \mathrm{Der}_\Lambda(R, k) = TF$$

as follows from the canonical identifications $F(k[V]) = \mathrm{Der}_\Lambda(R, V)$ and $G(k[V]) = \mathrm{Der}_\Lambda(S, V)$ of Example 51.10.11 and the rule for computing the map on tangent spaces.

Lemma 51.10.15. *Let $F : \mathcal{C} \to Sets$ be a functor satisfying the hypotheses of Lemma 51.10.8. Assume $R = K$ is a field. Then $F(K[V]) \cong TF \otimes_K V$ for any finite dimensional K-vector space V.*

Proof. Follows from Lemma 51.10.6. □

51.11. Tangent spaces of predeformation categories

We will define tangent spaces of predeformation functors using the general Definition 51.10.9. We have spelled this out in Example 51.10.10. It applies to predeformation categories by looking at the associated functor of isomorphism classes.

Definition 51.11.1. Let $\mathcal{F}$ be a predeformation category. The *tangent space $T\mathcal{F}$ of $\mathcal{F}$* is the set $\overline{\mathcal{F}}(k[\epsilon])$ of isomorphism classes of objects in the fiber category $\mathcal{F}(k[\epsilon])$.

Thus $T\mathcal{F}$ is nothing but the tangent space of the associated functor $\overline{\mathcal{F}} : \mathcal{C}_\Lambda \to Sets$. It has a natural vector space structure when $\mathcal{F}$ satisfies (S2), or, in fact, as long as $\overline{\mathcal{F}}$ does.

Lemma 51.11.2. *Let $\mathcal{F}$ be a predeformation category such that $\overline{\mathcal{F}}$ satisfies (S2). Then $T\mathcal{F}$ has a natural k-vector space structure. For any finite dimensional vector space V we have $\overline{\mathcal{F}}(k[V]) = T\mathcal{F} \otimes_k V$ functorially in V.*

Proof. Let us write $F = \overline{\mathcal{F}} : \mathcal{C}_\Lambda \to Sets$. This is a predeformation functor and F satisfies (S2). By Lemma 51.9.4 (and the translation of Remark 51.9.3) we see that

$$F(A \times_k k[V]) \longrightarrow F(A) \times F(k[V])$$

is a bijection for every finite dimensional vector space V and every $A \in Ob(\mathcal{C}_\Lambda)$. In particular, if $A = k[W]$ then we see that $F(k[W] \times_k k[V]) = F(k[W]) \times F(k[V])$. In other words, the hypotheses of Lemma 51.10.8 hold and we see that $TF = T\mathcal{F}$ has a natural k-vector space structure. The final assertion follows from Lemma 51.10.15. □

A morphism of predeformation categories induces a map on tangent spaces.

Definition 51.11.3. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism predeformation categories. The *differential $d\varphi : T\mathcal{F} \to T\mathcal{G}$ of φ* is the map obtained by evaluating the morphism of functors $\overline{\varphi} : \overline{\mathcal{F}} \to \overline{\mathcal{G}}$ at $A = k[\epsilon]$.

Lemma 51.11.4. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism predeformation categories. Assume $\overline{\mathcal{F}}$ and $\overline{\mathcal{G}}$ both satisfy (S2). Then $d\varphi : T\mathcal{F} \to T\mathcal{G}$ is k-linear.*

Proof. In the proof of Lemma 51.11.2 we have seen that $\overline{\mathcal{F}}$ and $\overline{\mathcal{G}}$ satisfy the hypotheses of Lemma 51.10.8. Hence the lemma follows from Lemma 51.10.13. □

Remark 51.11.5. We can globalize the notions of tangent space and differential to arbitrary categories cofibered in groupoids as follows. Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$, and let $x \in Ob(\mathcal{F}(k))$. As in Remark 51.6.4, we get a predeformation category $\mathcal{F}_x$. We define the *tangent space $T_x\mathcal{F}$ of $\mathcal{F}$ at x* to be the tangent space $T\mathcal{F}_x$ of $\mathcal{F}_x$. Similarly, if $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of categories cofibered in groupoids over $\mathcal{C}_\Lambda$ and $x \in Ob(\mathcal{F}(k))$, then there is an induced morphism $\varphi_x : \mathcal{F}_x \to \mathcal{G}_{\varphi(x)}$. We define the *differential $d_x\varphi : T_x\mathcal{F} \to T_{\varphi(x)}\mathcal{G}$ of φ at x* to be the map $d\varphi_x : T\mathcal{F}_x \to T\mathcal{G}_{\varphi(x)}$. If both $\mathcal{F}$ and $\mathcal{G}$ satisfy (S2) then all of these tangent spaces have a natural k-vector space structure and all the differentials $d_x\varphi : T_x\mathcal{F} \to T_{\varphi(x)}\mathcal{G}$ are k-linear (use Lemmas 51.9.6 and 51.11.4).

The following observations are uninteresting in the classical case or when $k' \subset k$ is a separable field extension, because then $\mathrm{Der}_\Lambda(k, k)$ and $\mathrm{Der}_\Lambda(V, k)$ are zero. There is a canonical identification

$$Mor_{\mathcal{C}_\Lambda}(k, k[\epsilon]) = \mathrm{Der}_\Lambda(k, k).$$

Namely, for $D \in \mathrm{Der}_\Lambda(k, k)$ let $f_D : k \to k[\epsilon]$ be the map $a \mapsto a + D(a)\epsilon$. More generally, given a finite dimensional vector space V over k we have

$$Mor_{\mathcal{C}_\Lambda}(k, k[V]) = \mathrm{Der}_\Lambda(k, V)$$

and we will use the same notation f_D for the map associated to the derivation D. We also have

$$Mor_{\mathcal{C}_\Lambda}(k[W], k[V]) = Hom_k(V, W) \oplus \mathrm{Der}_\Lambda(k, V)$$

where (φ, D) corresponds to the map $f_{\varphi,D} : a + w \mapsto a + \varphi(w) + D(a)$. We will sometimes write $f_{1,D} : a+v \to a+v+D(a)$ for the automorphism of $k[V]$ determined by the derivation $D : k \to V$. Note that $f_{1,D} \circ f_{1,D'} = f_{1,D+D'}$.

Let $\mathcal{F}$ be a predeformation category over $\mathcal{C}_\Lambda$. Let $x_0 \in Ob(\mathcal{F}(k))$. By the above there is a canonical map

$$\gamma_V : \mathrm{Der}_\Lambda(k, V) \longrightarrow \overline{\mathcal{F}}(k[V])$$

defined by $D \mapsto f_{D,*}(x_0)$. Moreover, there is an action

$$a_V : \mathrm{Der}_\Lambda(k, V) \times \overline{\mathcal{F}}(k[V]) \longrightarrow \overline{\mathcal{F}}(k[V])$$

defined by $(D, x) \mapsto f_{1,D,*}(x)$. These two maps are compatible, i.e., $f_{1,D,*}f_{D',*}x_0 = f_{D+D',*}x_0$ as follows from a computation of the compositions of these maps. Note that the maps γ_V and a_V are independent of the choice of x_0 as there is a unique x_0 up to isomorphism.

Lemma 51.11.6. *Let $\mathcal{F}$ be a predeformation category over $\mathcal{C}_\Lambda$. If $\overline{\mathcal{F}}$ has (S2) then the maps γ_V are k-linear and we have $a_V(D, x) = x + \gamma_V(D)$.*

Proof. In the proof of Lemma 51.11.2 we have seen that the functor $V \mapsto \overline{\mathcal{F}}(k[V])$ transforms 0 to a singleton and products to products. The same is true of the functor $V \mapsto \mathrm{Der}_\Lambda(k, V)$. Hence γ_V is linear by Lemma 51.10.5. Let $D : k \to V$ be a Λ-derivation. Set $D_1 : k \to V^{\oplus 2}$ equal to $a \mapsto (D(a), 0)$. Then

$$\begin{tikzcd} k[V \times V] \arrow[r, "+"] \arrow[d, "f_{1,D_1}"] & k[V] \arrow[d, "f_{1,D}"] \\ k[V \times V] \arrow[r, "+"] & k[V] \end{tikzcd}$$

commutes. Unwinding the definitions and using that $\overline{F}(V \times V) = \overline{F}(V) \times \overline{F}(V)$ this means that $a_D(x_1) + x_2 = a_D(x_1 + x_2)$ for all $x_1, x_2 \in \overline{F}(V)$. Thus it suffices to show that $a_V(D, 0) =$

$0 + \gamma_V(D)$ where $0 \in \overline{F}(V)$ is the zero vector. By definition this is the element $f_{0,*}(x_0)$. Since $f_D = f_{1,D} \circ f_0$ the desired result follows. □

A special case of the constructions above are the map

$$\gamma : \mathrm{Der}_\Lambda(k, k) \longrightarrow T\mathcal{F} \tag{51.11.6.1}$$

and the action

$$a : \mathrm{Der}_\Lambda(k, k) \times T\mathcal{F} \longrightarrow T\mathcal{F} \tag{51.11.6.2}$$

defined for any predeformation category $\mathcal{F}$. Note that if $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of predeformation categories, then we get commutative diagrams

$$\begin{tikzcd} \mathrm{Der}_\Lambda(k,k) \arrow[r, "\gamma"'] \arrow[dr, "\gamma"'] & T\mathcal{F} \arrow[d, "d\varphi"'] \\ & T\mathcal{G} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathrm{Der}_\Lambda(k,k) \times T\mathcal{F} \arrow[r, "a"'] \arrow[d, "1\times d\varphi"'] & T\mathcal{F} \arrow[d, "d\varphi"] \\ \mathrm{Der}_\Lambda(k,k) \times T\mathcal{G} \arrow[r, "a"] & T\mathcal{G} \end{tikzcd}$$

51.12. Versal formal objects

The existence of a versal formal object forces $\mathcal{F}$ to have property (S1).

Lemma 51.12.1. *Let $\mathcal{F}$ be a predeformation category. Assume $\mathcal{F}$ has a versal formal object. Then $\mathcal{F}$ satisfies (S1).*

Proof. Let ξ be a versal formal object of $\mathcal{F}$. Let

$$\begin{tikzcd} & x_2 \arrow[d] \\ x_1 \arrow[r] & x \end{tikzcd}$$

be a diagram in $\mathcal{F}$ such that $x_2 \to x$ lies over a surjective ring map. Since the natural morphism $\widehat{\mathcal{F}}|_{\mathcal{C}_\Lambda} \xrightarrow{\sim} \mathcal{F}$ is an equivalence (see Remark 51.7.7), we can consider this diagram also as a diagram in $\widehat{\mathcal{F}}$. By Lemma 51.8.15 there exists a morphism $\xi \to x_1$, so by Remark 51.8.14 we also get a morphism $\xi \to x_2$ making the diagram

$$\begin{tikzcd} \xi \arrow[r] \arrow[d] & x_2 \arrow[d] \\ x_1 \arrow[r] & x \end{tikzcd}$$

commute. If $x_1 \to x$ and $x_2 \to x$ lie above ring maps $A_1 \to A$ and $A_2 \to A$ then taking the pushforward of ξ to $A_1 \times_A A_2$ gives an object y as required by (S1). □

In the case that our cofibred category satisfies (S1) and (S2) we can characterize the versal formal objects as follows.

Lemma 51.12.2. *Let $\mathcal{F}$ be a predeformation category satisfying (S1) and (S2). Let ξ be a formal object of $\mathcal{F}$ corresponding to $\underline{\xi} : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$, see Remark 51.7.12. Then ξ is versal if and only if the following two conditions hold:*

1. *the map $d\underline{\xi} : T\underline{R}|_{\mathcal{C}_\Lambda} \to T\mathcal{F}$ on tangent spaces is surjective, and*

(2) *given a diagram in* $\widehat{\mathcal{F}}$

$$\begin{tikzcd} & y \arrow[d] \\ \xi \arrow[r] & x \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} & B \arrow[d, "f"] \\ R \arrow[r] & A \end{tikzcd}$$

in $\widehat{\mathcal{C}}_\Lambda$ *with* $B \to A$ *a small extension of Artinian rings, then there exists a ring map* $R \to B$ *such that*

$$\begin{tikzcd} & B \arrow[d, "f"] \\ R \arrow[r] \arrow[ur] & A \end{tikzcd}$$

commutes.

Proof. If ξ is versal then (1) holds by Lemma 51.8.8 and (2) holds by Remark 51.8.14. Assume (1) and (2) hold. By Remark 51.8.14 we must show that given a diagram in $\widehat{\mathcal{F}}$ as in (2), there exists $\xi \to y$ such that

$$\begin{tikzcd} & y \arrow[d] \\ \xi \arrow[r] \arrow[ur] & x \end{tikzcd}$$

commutes. Let $b : R \to B$ be the map guaranteed by (2). Denote $y' = b_*\xi$ and choose a factorization $\xi \to y' \to x$ lying over $R \to B \to A$ of the given morphism $\xi \to x$. By (S1) we obtain a commutative diagram

$$\begin{tikzcd} z \arrow[r] \arrow[d] & y \arrow[d] \\ y' \arrow[r] & x \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} B \times_A B \arrow[r] \arrow[d] & B \arrow[d, "f"] \\ B \arrow[r, "f"] & A. \end{tikzcd}$$

Set $I = \text{Ker}(k)$. Let $\overline{g} : B \times_A B \to k[I]$ be the ring map $(u, v) \mapsto \overline{u} \oplus (v - u)$, cf. Lemma 51.9.8. By (1) there exists a morphism $\xi \to \overline{g}_* z$ which lies over a ring map $i : R \to k[\epsilon]$. Choose an Artinian quotient $b_1 : R \to B_1$ such that both $b : R \to B$ and $i : R \to k[\epsilon]$ factor through $R \to B_1$, i.e., giving $h : B_1 \to B$ and $i' : B_1 \to k[\epsilon]$. Choose a pushforward $y_1 = b_{1,*}\xi$, a factorization $\xi \to y_1 \to y'$ lying over $R \to B_1 \to B$ of $\xi \to y'$, and a factorization $\xi \to y_1 \to \overline{g}_* z$ lying over $R \to B_1 \to k[\epsilon]$ of $\xi \to \overline{g}_* z$. Applying (S1) once more we obtain

$$\begin{tikzcd} z_1 \arrow[r] \arrow[d] & z \arrow[r] \arrow[d] & y \arrow[d] \\ y_1 \arrow[r] & y' \arrow[r] & x \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} B_1 \times_A B \arrow[r] \arrow[d] & B \times_A B \arrow[r] \arrow[d] & B \arrow[d, "f"] \\ B_1 \arrow[r] & B \arrow[r] & A. \end{tikzcd}$$

Note that the map $g : B_1 \times_A B \to k[I]$ of Lemma 51.9.8 (defined using h) is the composition of $B_1 \times_A B \to B \times_A B$ and the map $\overline{g}$ above. By construction there exists a morphism $y_1 \to g_* z_1 \cong \overline{g}_* z$! Hence Lemma 51.9.8 applies (to the outer rectangles in the diagrams above) to give a morphism $y_1 \to y$ and precomposing with $\xi \to y_1$ gives the desired morphism $\xi \to y$. $\square$

If $\mathcal{F}$ has property (S1) then the ``largest quotient where a lift exists'' exists. Here is a precise statement.

Lemma 51.12.3. *Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$ which has (S1). Let $B \to A$ be a surjection in $\mathcal{C}_\Lambda$ with kernel I annihilated by $\mathfrak{m}_B$. Let $x \in \mathcal{F}(A)$. The set of ideals*

$$\mathcal{J} = \{J \subset I \mid \text{there exists an } y \to x \text{ lying over } B/J \to A\}$$

has a smallest element.

Proof. Note that $\mathcal{J}$ is nonempty as $I \in \mathcal{J}$. Also, if $J \in \mathcal{J}$ and $J \subset J' \subset I$ then $J' \in \mathcal{J}$ because we can pushforward the object y to an object y' over B/J'. Let J and K be elements of the displayed set. We claim that $J \cap K \in \mathcal{J}$ which will prove the lemma. Since I is a k-vector space we can find an ideal $J \subset J' \subset I$ such that $J \cap K = J' \cap K$ and such that $J' + K = I$. By the above we may replace J by J' and assume that $J + K = I$. In this case

$$A/(J \cap K) = A/J \times_{A/I} A/K.$$

Hence the existence of an element $z \in \mathcal{F}(A/(J \cap K))$ mapping to x follows, via (S1), from the existence of the elements we have assumed exist over A/J and A/K. □

We will improve on the following result later.

Lemma 51.12.4. *Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$. Assume the following conditions hold:*

(1) *$\mathcal{F}$ is a predeformation category.*
(2) *$\mathcal{F}$ satisfies (S1).*
(3) *$\mathcal{F}$ satisfies (S2).*
(4) *$\dim_k T\mathcal{F}$ is finite.*

Then $\mathcal{F}$ has a versal formal object.

Proof. Assume (1), (2), (3), and (4) hold. Choose an object $R \in Ob(\widehat{\mathcal{C}}_\Lambda)$ such that $\underline{R}|_{\mathcal{C}_\Lambda}$ is smooth, see Example 51.8.10. Let $r = \dim_k T\mathcal{F}$ and put $S = R[[X_1, \ldots, X_r]]$.

We are going to inductively construct for $n \geq 2$ pairs $(J_n, f_{n-1} : \xi_n \to \xi_{n-1})$ where $J_n \subset S$ is an decreasing sequence of ideals and $f_{n-1} : \xi_n \to \xi_{n-1}$ is a morphism of $\mathcal{F}$ lying over the projection $S/J_n \to S/J_{n-1}$.

Step 1. Let $J_1 = \mathfrak{m}_S$. Let ξ_1 be the unique (up to unique isomorphism) object of $\mathcal{F}$ over $k = S/J_1 = S/\mathfrak{m}_S$

Step 2. Let $J_2 = \mathfrak{m}_S^2 + \mathfrak{m}_R S$. Then $S/J_2 = k[V]$ with $V = kX_1 \oplus \ldots \oplus kX_r$. By (S2) for $\overline{\mathcal{F}}$ we get a bijection

$$\overline{\mathcal{F}}(S/J_2) \longrightarrow T\mathcal{F} \otimes_k V,$$

see Lemmas 51.9.5 and 51.11.2. Choose a basis $\theta_1, \ldots, \theta_r$ for $T\mathcal{F}$ and set $\xi_2 = \sum \theta_i \otimes X_i \in Ob(\mathcal{F}(S/J_2))$. The point of this choice is that

$$d\xi_2 : Mor_{\mathcal{C}_\Lambda}(S/J_2, k[\epsilon]) \longrightarrow T\mathcal{F}$$

is surjective. Let $f_1 : \xi_2 \to \xi_1$ be the unique morphism.

Induction step. Assume $(J_n, f_{n-1} : \xi_n \to \xi_{n-1})$ has been constructed for some $n \geq 2$. There is a minimal element J_{n+1} of the set of ideals $J \subset S$ satisfying: (a) $\mathfrak{m}_S J_n \subset J \subset J_n$ and (b) there exists a morphism $\xi_{n+1} \to \xi_n$ lying over $S/J \to S/J_n$, see Lemma 51.12.3. Let $f_n : \xi_{n+1} \to \xi_n$ be any morphism of $\mathcal{F}$ lying over $S/J_{n+1} \to S/J_n$.

Set $J = \bigcap J_n$. Set $\overline{S} = S/J$. Set $\overline{J}_n = J_n/J$. By Lemma 51.4.7 the sequence of ideals $(\overline{J}_n)$ induces the $\mathfrak{m}_{\overline{S}}$-adic topology on $\overline{S}$. Since (ξ_n, f_n) is an object of $\widehat{\mathcal{F}}_{\mathcal{J}}(\overline{S})$, where $\mathcal{J}$ is the filtration $(\overline{J}_n)$ of $\overline{S}$, we see that (ξ_n, f_n) induces an object ξ of $\widehat{\mathcal{F}}(\overline{S})$. see Lemma 51.7.4.

We prove ξ is versal. For versality it suffices to check conditions (1) and (2) of Lemma 51.12.2. Condition (1) follows from our choice of ξ_2 in Step 2 above. Suppose given a diagram in $\widehat{\mathcal{F}}$

$$\begin{tikzcd} & y \arrow[d] \\ \eta \arrow[r] & x \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} & B \arrow[d, "f"] \\ \overline{S} \arrow[r] & A \end{tikzcd}$$

in $\widehat{\mathcal{C}}_\Lambda$ with $f : B \to A$ a small extension of Artinian rings. We have to show there is a map $\overline{S} \to B$ fitting into the diagram on the right. Choose n such that $\overline{S} \to A$ factors through $\overline{S} \to S/J_n$. This is possible as the sequence $(\overline{J}_n)$ induces the $\mathfrak{m}_{\overline{S}}$-adic topology as we saw above. The pushforward of ξ along $\overline{S} \to S/J_n$ is ξ_n. We may factor $\xi \to x$ as $\xi \to \xi_n \to x$ hence we get a diagram in $\mathcal{F}$

$$\begin{tikzcd} & y \arrow[d] \\ \xi_n \arrow[r] & x \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} & B \arrow[d, "f"] \\ S/J_n \arrow[r] & A. \end{tikzcd}$$

To check condition (2) of Lemma 51.12.2 it suffices to complete the diagram

$$\begin{tikzcd} S/J_{n+1} \arrow[r, dashed] \arrow[d] & B \arrow[d, "f"] \\ S/J_n \arrow[r] & A \end{tikzcd}$$

or equivalently, to complete the diagram

$$\begin{tikzcd} & S/J_n \times_A B \arrow[d, "p_1"] \\ S/J_{n+1} \arrow[r] \arrow[ru, dashed] & S/J_n. \end{tikzcd}$$

If p_1 has a section we are done. If not, by Lemma 51.3.8 (2) p_1 is a small extension, so by Lemma 51.3.12 (4) p_1 is an essential surjection. Recall that $S = R[[X_1, \ldots, X_r]]$ and that we chose R such that $\underline{R}|_{\mathcal{C}_\Lambda}$ is smooth. Hence there exists a map $h : R \to B$ lifting the map $R \to S \to S/J_n \to A$. By the universal property of a power series ring there is an R-algebra map $h : S = R[[X_1, \ldots, X_2]] \to B$ lifting the given map $S \to S/J_n \to A$. This induces a map $g : S \to S/J_n \times_A B$ making the solid square in the diagram

$$\begin{tikzcd} S \arrow[r, "g"'] \arrow[d] & S/J_n \times_A B \arrow[d, "p_1"] \\ S/J_{n+1} \arrow[r] \arrow[ru, dashed] & S/J_n \end{tikzcd}$$

commute. Then g is a surjection since p_1 is an essential surjection. We claim the ideal $K = \mathrm{Ker}(g)$ of S satisfies conditions (a) and (b) of the construction of J_{n+1} in the induction step above. Namely, $K \subset J_n$ is clear and $\mathfrak{m}_S J_n \subset K$ as p_1 is a small extension; this proves (a). By (S1) applied to

$$\begin{tikzcd} & y \arrow[d] \\ \xi_n \arrow[r] & x, \end{tikzcd}$$

there exists a lifting of ξ_n to $S/K \cong S/J_n \times_A B$, so (b) holds. Since J_{n+1} was the minimal ideal with properties (a) and (b) this implies $J_{n+1} \subset K$. Thus the desired map $S/J_{n+1} \to S/K \cong S/J_n \times_A B$ exists. □

51.13. Minimal versal formal objects

We do a little bit of work to try and understand (non)uniqueness of versal formal objects. It turns out that if a predeformation category has a versal formal object, then it has a minimal versal formal object and any two such are isomorphic. Moreover, all versal formal objects are ``more or less'' the same up to replacing the base ring by a power series extension.

Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$. For every object x of $\mathcal{F}$ lying over $A \in Ob(\mathcal{C}_\Lambda)$ consider the category $\mathcal{S}_x$ with objects

$$Ob(\mathcal{S}_x) = \{x' \to x \mid x' \to x \text{ lies over } A' \subset A\}$$

and morphisms are morphisms over x. For every $y \to x$ in $\mathcal{F}$ lying over $f : B \to A$ in $\mathcal{C}_\Lambda$ there is a functor $f_* : \mathcal{S}_y \to \mathcal{S}_x$ defined as follows: Given $y' \to y$ lying over $B' \subset B$ set $A' = f(B')$ and let $y' \to x'$ be over $B' \to f(B')$ be the pushforward of y'. By the axioms of a category cofibred in groupoids we obtain a unique morphism $x' \to x$ lying over $f(B') \to A$ such that

$$\begin{tikzcd} y' \arrow[r] \arrow[d] & x' \arrow[d] \\ y \arrow[r] & x \end{tikzcd}$$

commutes. Then $x' \to x$ is an object of $\mathcal{S}_x$. We say an object $x' \to x$ of $\mathcal{S}_x$ is *minimal* if any morphism $(x'_1 \to x) \to (x' \to x)$ in $\mathcal{S}_x$ is an isomorphism, i.e., x' and x'_1 are defined over the same subring of A. Since A has finite length as a Λ-module we see that minimal objects always exist.

Lemma 51.13.1. *Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$ which has (S1).*

1. *For $y \to x$ in $\mathcal{F}$ a minimal object in $\mathcal{S}_y$ maps to a minimal object of $\mathcal{S}_x$.*
2. *For $y \to x$ in $\mathcal{F}$ lying over a surjection $f : B \to A$ in $\mathcal{C}_\Lambda$ every minimal object of $\mathcal{S}_x$ is the image of a minimal object of $\mathcal{S}_y$.*

Proof. Proof of (1). Say $y \to x$ lies over $f : B \to A$. Let $y' \to y$ lying over $B' \subset B$ be a minimal object of $\mathcal{S}_y$. Let

$$\begin{tikzcd} y' \arrow[r] \arrow[d] & x' \arrow[d] \\ y \arrow[r] & x \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} B' \arrow[r] \arrow[d] & f(B') \arrow[d] \\ B \arrow[r] & A \end{tikzcd}$$

be as in the construction of f_* above. Suppose that $(x'' \to x) \to (x' \to x)$ is a morphism of $\mathcal{S}_x$ with $x'' \to x'$ lying over $A'' \subset f(B')$. By (S1) there exists $y'' \to y'$ lying over $B' \times_{f(B')} A'' \to B'$. Since $y' \to y$ is minimal we conclude that $B' \times_{f(B')} A'' \to B'$ is an isomorphism, which implies that $A'' = f(B')$, i.e., $x' \to x$ is minimal.

Proof of (2). Suppose $f : B \to A$ is surjective and $y \to x$ lies over f. Let $x' \to x$ be a minimal object of $\mathcal{S}_x$ lying over $A' \subset A$. By (S1) there exists $y' \to y$ lying over $B' = f^{-1}(A') = B \times_A A' \to B$ whose image in $\mathcal{S}_x$ is $x' \to x$. So $f_*(y' \to y) = x' \to x$. Choose a morphism $(y'' \to y) \to (y' \to y)$ in $\mathcal{S}_y$ with $y'' \to y$ a minimal object (this is possible by the remark on lengths above the lemma). Then $f_*(y'' \to y)$ is an object of $\mathcal{S}_x$ which maps to $x' \to x$ (by functoriality of f_*) hence is isomorphic to $x' \to x$ by minimality of $x' \to x$. $\square$

Lemma 51.13.2. *Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$ which has (S1). Let ξ be a versal formal object of $\mathcal{F}$ lying over R. There exists a morphism $\xi' \to \xi$ lying over $R' \subset R$ with the following minimality properties*

(1) *for every $f : R \to A$ with $A \in Ob(\mathcal{C}_\Lambda)$ the pushforwards*

$$\begin{tikzcd} \xi' \arrow[r] \arrow[d] & x' \arrow[d] & \text{lying over} & R' \arrow[r] \arrow[d] & f(R') \arrow[d] \\ \xi \arrow[r] & x & & R \arrow[r] & A \end{tikzcd}$$

produce a minimal object $x' \to x$ of $\mathcal{S}_x$, and

(2) *for any morphism of formal objects $\xi'' \to \xi'$ the corresponding morphism $R'' \to R'$ is surjective.*

Proof. Write $\xi = (R, \xi_n, f_n)$. Set $R'_1 = k$ and $\xi'_1 = \xi_1$. Suppose that we have constructed minimal objects $\xi'_m \to \xi_m$ of $\mathcal{S}_{\xi_m}$ lying over $R'_m \subset R/\mathfrak{m}_R^m$ for $m \leq n$ and morphisms $f'_m : \xi'_{m+1} \to \xi'_m$ compatible with f_m for $m \leq n-1$. By Lemma 51.13.1 (2) there exists a minimal object $\xi'_{n+1} \to \xi_{n+1}$ lying over $R'_{n+1} \subset R/\mathfrak{m}_R^{n+1}$ whose image is $\xi'_n \to \xi_n$ over $R'_n \subset R/\mathfrak{m}_R^n$. This produces the commutative diagram

$$\begin{tikzcd} \xi'_{n+1} \arrow[r, "f'_n"'] \arrow[d] & \xi'_n \arrow[d] \\ \xi_{n+1} \arrow[r, "f_n"] & \xi_n \end{tikzcd}$$

by construction. Moreover the ring map $R'_{n+1} \to R'_n$ is surjective. Set $R' = lim_n R'_n$. Then $R' \to R$ is injective.

However, it isn't a priori clear that R' is Noetherian. To prove this we use that ξ is versal. Namely, versality implies that there exists a morphism $\xi \to \xi'_n$ in $\widehat{\mathcal{F}}$, see Lemma 51.8.15. The corresponding map $R \to R'_n$ has to be surjective (as $\xi'_n \to \xi_n$ is minimal in $\mathcal{S}_{\xi_n}$). Thus the dimensions of the cotangent spaces are bounded and Lemma 51.4.8 implies R' is Noetherian, i.e., an object of $\widehat{\mathcal{C}}_\Lambda$. By Lemma 51.7.4 (plus the result on filtrations of Lemma 51.4.8) the sequence of elements ξ'_n defines a formal object ξ' over R' and we have a map $\xi' \to \xi$.

By construction (1) holds for $R \to R/\mathfrak{m}_R^n$ for each n. Since each $R \to A$ as in (1) factors through $R \to R/\mathfrak{m}_R^n \to A$ we see that (1) for $x' \to x$ over $f(R) \subset A$ follows from the minimality of $\xi'_n \to \xi_n$ over $R'_n \to R/\mathfrak{m}_R^n$ by Lemma 51.13.1 (1).

If $R'' \to R'$ as in (2) is not surjective, then $R'' \to R' \to R'_n$ would not be surjective for some n and $\xi'_n \to \xi_n$ wouldn't be minimal, a contradiction. This contradiction proves (2). $\square$

Lemma 51.13.3. *Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$ which has (S1). Let ξ be a versal formal object of $\mathcal{F}$ lying over R. Let $\xi' \to \xi$ be a morphism of formal objects lying over $R' \subset R$ as constructed in Lemma 51.13.2. Then*

$$R \cong R'[[x_1, \ldots, x_r]]$$

is a power series ring over R'. Moreover, ξ' is a versal formal object too.

Proof. By Lemma 51.8.15 there exists a morphism $\xi \to \xi'$. By Lemma 51.13.2 the corresponding map $f : R \to R'$ induces a surjection $f|_{R'} : R' \to R'$. This is an isomorphism by Algebra, Lemma 7.28.8. Hence $I = \text{Ker}(f)$ is an ideal of R such that $R = R' \oplus I$. Let $x_1, \ldots, x_n \in I$ be elements which form a basis for $I/\mathfrak{m}_R I$. Consider the map $S = R'[[X_1, \ldots, X_r]] \to R$ mapping X_i to x_i. For every $n \geq 1$ we get a surjection of Artinian R'-algebras $B = S/\mathfrak{m}_S^n \to R/\mathfrak{m}_R^n = A$. Denote $y \in Ob(\mathcal{F}(B)$, resp. $x \in Ob(\mathcal{F}(A))$ the pushforward of ξ' along $R' \to S \to B$, resp. $R' \to S \to A$. Note that x is also the pushforward of ξ along $R \to A$ as ξ is the pushforward of ξ' along $R' \to R$. Thus we have a solid diagram

$$\begin{tikzcd} & y \arrow[d] \\ \xi \arrow[r] \arrow[ur, dotted] & x \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} & S/\mathfrak{m}_S^n \arrow[d] \\ R \arrow[r] \arrow[ur, dotted] & R/\mathfrak{m}_R^n \end{tikzcd}$$

Because ξ is versal, using Remark 51.8.14 we obtain the dotted arrows fitting into these diagrams. In particular, the maps $S/\mathfrak{m}_S^n \to R/\mathfrak{m}_R^n$ have sections $h_n : R/\mathfrak{m}_R^n \to S/\mathfrak{m}_S^n$. It follows from Lemma 51.4.9 that $S \to R$ is an isomorphism.

As ξ is a pushforward of ξ' along $R' \to R$ we obtain from Remark 51.7.13 a commutative diagram

$$\begin{tikzcd} \underline{R}|_{\mathcal{C}_\Lambda} \arrow[rr] \arrow[dr, "\underline{\xi}"'] & & \underline{R'}|_{\mathcal{C}_\Lambda} \arrow[dl, "\underline{\xi'}"] \\ & \mathcal{F} & \end{tikzcd}$$

Since $R' \to R$ has a left inverse (namely $R \to R/I = R'$) we see that $\underline{R}|_{\mathcal{C}_\Lambda} \to \underline{R'}|_{\mathcal{C}_\Lambda}$ is essentially surjective. Hence by Lemma 51.8.7 we see that $\underline{\xi'}$ is smooth, i.e., ξ' is a versal formal object. □

Motivated by the preceding lemmas we make the following definition.

Definition 51.13.4. Let $\mathcal{F}$ be a predeformation category. We say a versal formal object ξ of $\mathcal{F}$ is *minimal*[2] if for any morphism of formal objects $\xi' \to \xi$ the underlying map on rings is surjective. Sometimes a minimal versal formal object is called *miniversal*.

The work in this section shows this definition is reasonable. First of all, the existence of a versal formal object implies that $\mathcal{F}$ has (S1). Then the preceding lemmas show there exists a minimal versal formal object. Finally, any two minimal versal formal objects are isomorphic. Here is a summary of our results (with detailed proofs).

Lemma 51.13.5. *Let $\mathcal{F}$ be a predeformation category which has a versal formal object. Then*

[2]This may be nonstandard terminology. Many authors tie this notion in with properties of tangent spaces. We will make the link in Section 51.14.

(1) *$\mathcal{F}$ has a minimal versal formal object,*
(2) *minimal versal objects are unique up to isomorphism, and*
(3) *any versal object is the pushforward of a minimal versal object along a power series ring extension.*

Proof. Suppse $\mathcal{F}$ has a versal formal object ξ over R. Then it satisfies (S1), see Lemma 51.12.1. Let $\xi' \to \xi$ over $R' \subset R$ be any of the morphisms constructed in Lemma 51.13.2. By Lemma 51.13.3 we see that ξ' is versal, hence it is a minimal versal formal object (by construction). This proves (1). Also, $R \cong R'[[x_1, \ldots, x_n]]$ which proves (3).

Suppose that ξ_i/R_i are two minimal versal formal objects. By Lemma 51.8.15 there exist morphisms $\xi_1 \to \xi_2$ and $\xi_2 \to \xi_1$. The corresponding ring maps $f : R_1 \to R_2$ and $g : R_2 \to R_1$ are surjective by minimality. Hence the compositions $g \circ f : R_1 \to R_1$ and $f \circ g : R_2 \to R_2$ are isomorphisms by Algebra, Lemma 7.28.8. Thus f and g are isomorphisms whence the maps $\xi_1 \to \xi_2$ and $\xi_2 \to \xi_1$ are isomorphisms (because $\widehat{\mathcal{F}}$ is cofibred in groupoids by Lemma 51.7.2). This proves (2) and finishes the proof of the lemma. □

51.14. Miniversal formal objects and tangent spaces

The general notion of minimality introduced in Definition 51.13.4 can sometimes be deduced from the behaviour on tangent spaces. Let ξ be a formal object of the predeformation category $\mathcal{F}$ and let $\underline{\xi} : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ be the corresponding morphism. Then we can consider the following two conditions

(51.14.0.1) $$d\underline{\xi} : \mathrm{Der}_\Lambda(R, k) \to T\mathcal{F} \text{ is bijective}$$

(51.14.0.2) $$d\underline{\xi} : \mathrm{Der}_\Lambda(R, k) \to T\mathcal{F} \text{ is bijective on } \mathrm{Der}_\Lambda(k, k)\text{-orbits.}$$

Here we are using the identification $T\underline{R}|_{\mathcal{C}_\Lambda} = \mathrm{Der}_\Lambda(R, k)$ of Example 51.10.11 and the action (51.11.6.2) of derivations on the tangent spaces. If $k' \subset k$ is separable, then $\mathrm{Der}_\Lambda(k, k) = 0$ and the two conditions are equivalent. It turns out that, in the presence of condition (S2) a versal formal object is minimal if and only if $\underline{\xi}$ satisfies (51.14.0.2). Moreover, if $\underline{\xi}$ satisfies (51.14.0.1), then $\mathcal{F}$ satisfies (S2).

Lemma 51.14.1. *Let $\mathcal{F}$ be a predeformation category. Let ξ be a versal formal object of $\mathcal{F}$ such that (51.14.0.2) holds. Then ξ is a minimal versal formal object. In particular, such ξ are unique up to isomorphism.*

Proof. If ξ is not minimal, then there exists a morphism $\xi' \to \xi$ lying over $R' \to R$ such that $R = R'[[x_1, \ldots, x_n]]$ with $n > 0$, see Lemma 51.13.5. Thus $d\underline{\xi}$ factors as

$$\mathrm{Der}_\Lambda(R, k) \to \mathrm{Der}_\Lambda(R', k) \to T\mathcal{F}$$

and we see that (51.14.0.2) cannot hold because $D : f \mapsto \partial/\partial x_1(f) \bmod \mathfrak{m}_R$ is an element of the kernel of the first arrow which is not in the image of $\mathrm{Der}_\Lambda(k, k) \to \mathrm{Der}_\Lambda(R, k)$. □

Lemma 51.14.2. *Let $\mathcal{F}$ be a predeformation category. Let ξ be a versal formal object of $\mathcal{F}$ such that (51.14.0.1) holds. Then*

(1) *$\mathcal{F}$ satisfies (S1).*
(2) *$\mathcal{F}$ satisfies (S2).*
(3) *$\dim_k T\mathcal{F}$ is finite.*

Proof. Condition (S1) holds by Lemma 51.12.1. The first part of (S2) holds since (S1) holds. Let

$$\begin{tikzcd} y \arrow[r, "c"] \arrow[d, "a"] & x_\epsilon \arrow[d, "e"] \\ x \arrow[r, "d"] & x_0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} y' \arrow[r, "c'"] \arrow[d, "a'"] & x_\epsilon \arrow[d, "e"] \\ x \arrow[r, "d"] & x_0 \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} A \times_k k[\epsilon] \arrow[r] \arrow[d] & k[\epsilon] \arrow[d] \\ A \arrow[r] & k \end{tikzcd}$$

be diagrams as in the second part of (S2). As above we can find morphisms $b : \xi \to y$ and $b' : \xi \to y'$ such that

$$\begin{tikzcd} \xi \arrow[r, "b'"] \arrow[d, "b"] & y' \arrow[d, "a'"] \\ y \arrow[r, "a"] & x \end{tikzcd}$$

commutes. Let $p : \mathscr{F} \to \mathscr{C}_\Lambda$ denote the structure morphism. Say $\hat{p}(\xi) = R$, i.e., ξ lies over $R \in Ob(\widehat{\mathscr{C}}_\Lambda)$. We see that the pushforward of ξ via $p(c) \circ p(b)$ is x_ϵ and that the pushforward of ξ via $p(c') \circ p(b')$ is x_ϵ. Since ξ satisfies (51.14.0.1), we see that $p(c) \circ p(b) = p(c') \circ p(b')$ as maps $R \to k[\epsilon]$. Hence $p(b) = p(b')$ as maps from $R \to A \times_k k[\epsilon]$. Thus we see that y and y' are isomorphic to the pushforward of ξ along this map and we get a unique morphism $y \to y'$ over $A \times_k k[\epsilon]$ compatible with b and b' as desired.

Finally, by Example 51.10.11 we see $\dim_k T\mathscr{F} = \dim_k T\underline{R}|_{\mathscr{C}_\Lambda}$ is finite. □

Example 51.14.3. There exist predeformation categories which have a versal formal object satisfying (51.14.0.2) but which do not satisfy (S2). A quick example is to take $F = \underline{k[\epsilon]}/G$ where $G \subset \mathrm{Aut}_{\mathscr{C}_\Lambda}(k[\epsilon])$ is a finite nontrivial subgroup. Namely, the map $\underline{k[\epsilon]} \to F$ is smooth, but the tangent space of F does not have a natural k-vector space structure (as it is a quotient of a k-vector space by a finite group).

Lemma 51.14.4. *Let $\mathscr{F}$ be a predeformation category satisfying (S2) which has a versal formal object. Then its minimal versal formal object satisfies (51.14.0.2).*

Proof. Let ξ be a minimal versal formal object for $\mathscr{F}$, see Lemma 51.13.5. Say ξ lies over $R \in Ob(\widehat{\mathscr{C}}_\Lambda)$. In order to parse (51.14.0.2) we point out that $T\mathscr{F}$ has a natural k-vector space structure (see Lemma 51.11.2), that $d\underline{\xi} : \mathrm{Der}_\Lambda(R, k) \to T\mathscr{F}$ is linear (see Lemma 51.11.4), and that the action of $\mathrm{Der}_\Lambda(k, k)$ is given by addition (see Lemma 51.11.6). Consider the diagram

$$\begin{tikzcd} & Hom_k(\mathfrak{m}_R/\mathfrak{m}_R^2, k) & \\ K \arrow[r] & \mathrm{Der}_\Lambda(R, k) \arrow[u] \arrow[r, "d\underline{\xi}"] & T\mathscr{F} \\ & \mathrm{Der}_\Lambda(k, k) \arrow[u] \arrow[ur] & \end{tikzcd}$$

The vector space K is the kernel of $d\underline{\xi}$. Note that the middle column is exact in the middle as it is dual to the sequence (51.3.10.1). If (51.14.0.2) fails, then we can find a nonzero element $D \in K$ which does not map to zero in $Hom_k(\mathfrak{m}_R/\mathfrak{m}_R^2, k)$. This means there exists an $t \in \mathfrak{m}_R$ such that $D(t) = 1$. Set $R' = \{a \in R \mid D(a) = 0\}$. As D is a derivation this is a subring of R. Since $D(t) = 1$ we see that $R' \to k$ is surjective (compare with the proof of Lemma 51.3.12). Note that $\mathfrak{m}_{R'} = \text{Ker}(D : \mathfrak{m}_R \to k)$ is an ideal of R and $\mathfrak{m}_R^2 \subset \mathfrak{m}_{R'}$. Hence

$$\mathfrak{m}_R/\mathfrak{m}_R^2 = \mathfrak{m}_{R'}/\mathfrak{m}_R^2 + k\bar{t}$$

which implies that the map

$$R'/\mathfrak{m}_R^2 \times_k k[\epsilon] \to R/\mathfrak{m}_R^2$$

sending ϵ to $\bar{t}$ is an isomorphism. In particular there is a map $R/\mathfrak{m}_R^2 \to R'/\mathfrak{m}_R^2$.

Let $\xi \to y$ be a morphism lying over $R \to R/\mathfrak{m}_R^2$. Let $y \to x$ be a morphism lying over $R/\mathfrak{m}_R^2 \to R'/\mathfrak{m}_R^2$. Let $y \to x_\epsilon$ be a morphism lying over $R/\mathfrak{m}_R^2 \to k[\epsilon]$. Let x_0 be the unique (up to unique isomorphism) object of $\mathcal{F}$ over k. By the axioms of a category cofibred in groupoids we obtain a commutative diagram

$$\begin{tikzcd} y \arrow[r] \arrow[d] & x_\epsilon \arrow[d] \\ x \arrow[r] & x_0 \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} R'/\mathfrak{m}_R^2 \times_k k[\epsilon] \arrow[r] \arrow[d] & k[\epsilon] \arrow[d] \\ R'/\mathfrak{m}_R^2 \arrow[r] & k. \end{tikzcd}$$

Because $D \in K$ we see that x_ϵ is isomorphic to $0 \in \mathcal{F}(k[\epsilon])$, i.e., x_ϵ is the pushforward of x_0 via $k \to k[\epsilon]$, $a \mapsto a$. Hence by Lemma 51.9.7 we see that there exists a morphism $x \to y$. Since $\text{length}_\Lambda(R'/\mathfrak{m}_R^2) < \text{length}_\Lambda(R/\mathfrak{m}_R^2)$ the corresponding ring map $R'/\mathfrak{m}_R^2 \to R/\mathfrak{m}_R^2$ is not surjective. This contradicts the minimality of ξ/R, see part (1) of Lemma 51.13.2. This contradiction shows that such a D cannot exist, hence we win. □

Theorem 51.14.5. *Let $\mathcal{F}$ be a predeformation category. Consider the following conditions*

(1) *$\mathcal{F}$ has a minimal versal formal object satisfying (51.14.0.1),*
(2) *$\mathcal{F}$ has a minimal versal formal object satisfying (51.14.0.2),*
(3) *the following conditions hold:*
    (a) *$\mathcal{F}$ satisfies (S1).*
    (b) *$\mathcal{F}$ satisfies (S2).*
    (c) *$\dim_k T\mathcal{F}$ is finite.*

We always have

$$(1) \Rightarrow (3) \Rightarrow (2).$$

If $k' \subset k$ is separable, then all three are equivalent.

Proof. Lemma 51.14.2 shows that (1) ⇒ (3). Lemmas 51.12.4 and 51.14.4 show that (3) ⇒ (2). If $k' \subset k$ is separable then $\text{Der}_\Lambda(k, k) = 0$ and we see that (51.14.0.1) = (51.14.0.2), i.e., (1) is the same as (2).

An alternative proof of (3) ⇒ (1) in the classical case is to add a few words to the proof of Lemma 51.12.4 to see that one can right away construct a versal object which satisfies (51.14.0.1) in this case. This avoids the use of Lemma 51.12.4 in the classical case. Details omitted. □

Remark 51.14.6. When $\mathcal{F}$ is a predeformation functor, the condition $\dim_k T\mathcal{F} < \infty$ is precisely condition (H3) from Schlessinger's paper. In the classical case (or the case where $k' \subset k$ is separable), Theorem 51.14.5 recovers Schlessinger's theorem on the existence of ``hulls''. In our terminology a *hull* is a versal formal object ξ for a predeformation functor such that $d\xi$ is an isomorphism.

Remark 51.14.7. Let $\mathcal{F}$ be a predeformation category satisfying (S1), (S2), and $\dim_k T\mathcal{F} < \infty$. Then $\overline{\mathcal{F}}$ also satisfies (S1), (S2), and $\dim_k T\overline{\mathcal{F}} < \infty$, see Lemma 51.9.5. Thus, if $k' \subset k$ is separable, then $\overline{\mathcal{F}}$ has a hull (see Remark 51.14.6). In fact, if ξ is a minimal versal object for $\mathcal{F}$ lying over R, then the composition

$$\underline{R}|_{\mathcal{C}_\Lambda} \longrightarrow \mathcal{F} \longrightarrow \overline{\mathcal{F}}$$

is smooth and identifies tangent spaces, i.e., the image $\overline{\xi}$ of ξ in $\overline{\mathcal{F}}$ is a hull. This follows from the fact that $\mathcal{F} \to \overline{\mathcal{F}}$ identifies tangent spaces.

Example 51.14.8. In Example 51.8.10 we constructed objects $R \in \widehat{\mathcal{C}}_\Lambda$ such that $\underline{R}|_{\mathcal{C}_\Lambda}$ is smooth. We can reformulate this as follows. Let $\mathcal{F} = \mathcal{C}_\Lambda$ considered as cofibred in groupoids via the identity functor. In other words, $\mathcal{F}$ is the category cofibred in sets corresponding to the functor $F : A \mapsto \{*\}$ (this is the final object in the category of functors $\mathcal{C}_\Lambda \to \textit{Sets}$). The condition that $\underline{R}|_{\mathcal{C}_\Lambda}$ is smooth means exactly that $\underline{R}|_{\mathcal{C}_\Lambda} \to F$ is smooth, i.e., that $\xi = *$ is a formal versal object of $\mathcal{F}$ over R. Hence $\mathcal{F}$ has a versal formal object. In fact, it is easy to see that $\mathcal{F}$ satisfies condition (3) of Theorem 51.14.5. The theorem implies that (2) holds. This means we can find a minimal versal formal object $* \in \widehat{\mathcal{F}}(S)$ over some $S \in \widehat{\mathcal{C}}_\Lambda$ such that $d* : \mathrm{Der}_\Lambda(S, k) \to 0$ is bijective on $\mathrm{Der}_\Lambda(k, k)$-orbits. Clearly this means that the injection $\mathrm{Der}_\Lambda(k, k) \to \mathrm{Der}_\Lambda(S, k)$ is also surjective. In other words, the exact sequence (51.3.10.2) turns into a pair of isomorphisms

$$H_1(L_{k/\Lambda}) = \mathfrak{m}_S/\mathfrak{m}_S^2 \quad \text{and} \quad \Omega_{S/\Lambda} \otimes_S k = \Omega_{k/\Lambda}.$$

(The first arrow is injective because of the formal smoothness of S over Λ; details omitted.) Of course the existence of such a ring S can be proved directly by judiciously slicing the ring R constructed in Example 51.8.10.

51.15. Rim-Schlessinger conditions and deformation categories

There is a very natural property of categories fibred in groupoids over $\mathcal{C}_\Lambda$ which is easy to check in practice and which implies Schlessinger's properties (S1) and (S2) we have introduced earlier.

Definition 51.15.1. Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. We say that $\mathcal{F}$ satisfies *condition (RS)* if for every diagram in $\mathcal{F}$

$$\begin{tikzcd} & x_2 \arrow[d] \\ x_1 \arrow[r] & x \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} & A_2 \arrow[d] \\ A_1 \arrow[r] & A \end{tikzcd}$$

in $\mathcal{C}_\Lambda$ with $A_2 \to A$ surjective, there exits a fiber product $x_1 \times_x x_2$ in $\mathcal{F}$ such that the diagram

$$\begin{tikzcd} x_1 \times_x x_2 \arrow[r] \arrow[d] & x_2 \arrow[d] \\ x_1 \arrow[r] & x \end{tikzcd} \quad \text{lies over} \quad \begin{tikzcd} A_1 \times_A A_2 \arrow[r] \arrow[d] & A_2 \arrow[d] \\ A_1 \arrow[r] & A. \end{tikzcd}$$

Lemma 51.15.2. *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$ satisfying (RS). Given a commutative diagram in $\mathcal{F}$*

$$\begin{tikzcd} y \arrow[r] \arrow[d] & x_2 \arrow[d] \\ x_1 \arrow[r] & x \end{tikzcd} \quad \textit{lying over} \quad \begin{tikzcd} A_1 \times_A A_2 \arrow[r] \arrow[d] & A_2 \arrow[d] \\ A_1 \arrow[r] & A. \end{tikzcd}$$

with $A_2 \to A$ surjective, then it is a fiber square.

Proof. Since $\mathcal{F}$ satisfies (RS), there exists a fiber product diagram

$$\begin{tikzcd} x_1 \times_x x_2 \arrow[r] \arrow[d] & x_2 \arrow[d] \\ x_1 \arrow[r] & x \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} A_1 \times_A A_2 \arrow[r] \arrow[d] & A_2 \arrow[d] \\ A_1 \arrow[r] & A. \end{tikzcd}$$

The induced map $y \to x_1 \times_x x_2$ lies over $\text{id} : A_1 \times_A A_1 \to A_1 \times_A A_1$, hence it is an isomorphism. □

Lemma 51.15.3. *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Then $\mathcal{F}$ satisfies (RS) if the condition in Definition 51.15.1 is assumed to hold only when $A_2 \to A$ is a small extension.*

Proof. Apply Lemma 51.3.3. The proof is similar to that of Lemma 51.8.2. □

Lemma 51.15.4. *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. The following are equivalent*

(1) *$\mathcal{F}$ satisfies (RS),*
(2) *the functor $\mathcal{F}(A_1 \times_A A_2) \to \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$ see (51.9.1.1) is an equivalence of categories whenever $A_2 \to A$ is surjective, and*
(3) *same as in (2) whenever $A_2 \to A$ is a small extension.*

Proof. Assume (1). By Lemma 51.15.2 we see that every object of $\mathcal{F}(A_1 \times_A A_2)$ is of the form $x_1 \times_x x_2$. Moreover

$$Mor_{A_1 \times_A A_2}(x_1 \times_x x_2, y_1 \times_y y_2) = Mor_{A_1}(x_1, y_1) \times_{Mor_A(x,y)} Mor_{A_2}(x_2, y_2).$$

Hence we see that $\mathcal{F}(A_1 \times_A A_2)$ is a 2-fibre product of $\mathcal{F}(A_1)$ with $\mathcal{F}(A_2)$ over $\mathcal{F}(A)$ by Categories, Remark 4.28.5. In other words, we see that (2) holds.

The implication (2) ⇒ (3) is immediate.

Assume (3). Let $q_1 : A_1 \to A$ and $q_2 : A_2 \to A$ be given with q_2 a small extension. We will use the description of the 2-fibre product $\mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$ from Categories, Remark 4.28.5. Hence let $y \in \mathcal{F}(A_1 \times_A A_2)$ correspond to $(x_1, x_2, x, a_1 : x_1 \to x, a_2 : x_2 \to x)$. Let z be an object of $\mathcal{F}$ lying over C. Then

$$\begin{aligned} Mor_{\mathcal{F}}(z, y) &= \{(f, \alpha) \mid f : C \to A_1 \times_A A_2, \alpha : f_* z \to y\} \\ &= \{(f_1, f_2, \alpha_1, \alpha_2) \mid f_i : C \to A_i,\ \alpha_i : f_{i,*} z \to x_i, \\ &\qquad q_1 \circ f_1 = q_2 \circ f_2,\ q_{1,*}\alpha_1 = q_{2,*}\alpha_2\} \\ &= Mor_{\mathcal{F}}(z, x_1) \times_{Mor_{\mathcal{F}}(z,x)} Mor_{\mathcal{F}}(z, x_2) \end{aligned}$$

whence y is a fibre product of x_1 and x_2 over x. Thus we see that $\mathcal{F}$ satisfies (RS) in case $A_2 \to A$ is a small extension. Hence (RS) holds by Lemma 51.15.3. □

Remark 51.15.5. When $\mathcal{F}$ is cofibered in sets, condition (RS) is exactly condition (H4) from Schlessinger's paper [**Sch68**, Theorem 2.11]. Namely, for a functor $F : \mathcal{C}_\Lambda \to \textit{Sets}$, condition (RS) states: If $A_1 \to A$ and $A_2 \to A$ are maps in $\mathcal{C}_\Lambda$ with $A_2 \to A$ surjective, then the induced map $F(A_1 \times_A A_2) \to F(A_1) \times_{F(A)} F(A_2)$ is bijective.

Lemma 51.15.6. *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. The condition (RS) for $\mathcal{F}$ implies both (S1) and (S2) for $\mathcal{F}$.*

Proof. Using the reformulation of Lemma 51.15.4 and the explanation of (S1) following Definition 51.9.1 it is immediate that (RS) implies (S1). This proves the first part of (S2). The second part of (S2) follows because Lemma 51.15.2 tells us that $y = x_1 \times_{d,x_0,e} x_2 = y'$ if y, y' are as in the second part of the definition of (S2) in Definition 51.9.1. (In fact the morphism $y \to y'$ is compatible with both a, a' and c, c'!) $\square$

The following lemma is the analogue of Lemma 51.9.5. Recall that if $\mathcal{F}$ is a category cofibred in groupoids over $\mathcal{C}_\Lambda$ and x is an object of $\mathcal{F}$ lying over A, then we denote $\text{Aut}_A(x) = \mathit{Mor}_A(x, x) = \mathit{Mor}_{\mathcal{F}(A)}(x, x)$. If $x' \to x$ is a morphism of $\mathcal{F}$ lying over $A' \to A$ then there is a well defined map of groups $\text{Aut}_{A'}(x') \to \text{Aut}_A(x)$.

Lemma 51.15.7. *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$ satisfying (RS). The following conditions are equivalent:*

(1) *$\overline{\mathcal{F}}$ satisfies (RS).*
(2) *Let $f_1 : A_1 \to A$ and $f_2 : A_2 \to A$ be ring maps in $\mathcal{C}_\Lambda$ with f_2 surjective. The induced map of sets of isomorphism classes*
$$\overline{\mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)} \to \overline{\mathcal{F}}(A_1) \times_{\overline{\mathcal{F}}(A)} \overline{\mathcal{F}}(A_2)$$
is injective.
(3) *For every morphism $x' \to x$ in $\mathcal{F}$ lying over a surjective ring map $A' \to A$, the map $Aut_{A'}(x') \to Aut_A(x)$ is surjective.*
(4) *For every morphism $x' \to x$ in $\mathcal{F}$ lying over a small extension $A' \to A$, the map $Aut_{A'}(x') \to Aut_A(x)$ is surjective.*

Proof. We prove that (1) is equivalent to (2) and (2) is equivalent to (3). The equivalence of (3) and (4) follows from Lemma 51.3.3.

Let $f_1 : A_1 \to A$ and $f_2 : A_2 \to A$ be ring maps in $\mathcal{C}_\Lambda$ with f_2 surjective. By Remark 51.15.5 we see $\overline{\mathcal{F}}$ satisfies (RS) if and only if the map
$$\overline{\mathcal{F}}(A_1 \times_A A_2) \to \overline{\mathcal{F}}(A_1) \times_{\overline{\mathcal{F}}(A)} \overline{\mathcal{F}}(A_2)$$
is bijective for any such f_1, f_2. This map is at least surjective since that is the condition of (S1) and $\overline{\mathcal{F}}$ satisfies (S1) by Lemmas 51.15.6 and 51.9.5. Moreover, this map factors as
$$\overline{\mathcal{F}}(A_1 \times_A A_2) \longrightarrow \overline{\mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)} \longrightarrow \overline{\mathcal{F}}(A_1) \times_{\overline{\mathcal{F}}(A)} \overline{\mathcal{F}}(A_2),$$
where the first map is a bijection since
$$\mathcal{F}(A_1 \times_A A_2) \longrightarrow \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$$
is an equivalence by (RS) for $\mathcal{F}$. Hence (1) is equivalent to (2).

Assume (2) holds. Let $x' \to x$ be a morphism in $\mathcal{F}$ lying over a surjective ring map $f : A' \to A$. Let $a \in \text{Aut}_A(x)$. The objects
$$(x', x', a : x \to x),\ (x', x', \text{id} : x \to x)$$

of $\mathcal{F}(A') \times_{\mathcal{F}(A)} \mathcal{F}(A')$ have the same image in $\overline{\mathcal{F}}(A') \times_{\overline{\mathcal{F}}(A)} \overline{\mathcal{F}}(A')$. By (2) there exists maps $b_1, b_2 : x' \to x'$ such that

$$\begin{tikzcd} x \arrow[r, "a"] \arrow[d, "f_*b_1"'] & x \arrow[d, "f_*b_2"] \\ x \arrow[r, "\mathrm{id}"] & x \end{tikzcd}$$

commutes. Hence $b_2^{-1} \circ b_1 \in \mathrm{Aut}_{A'}(x')$ has image $a \in \mathrm{Aut}_A(x)$. Hence (3) holds.

Assume (3) holds. Suppose

$$(x_1, x_2, a : (f_1)_*x_1 \to (f_2)_*x_2),\ (x'_1, x'_2, a' : (f_1)_*x'_1 \to (f_2)_*x'_2)$$

are objects of $\mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$ with the same image in $\overline{\mathcal{F}}(A_1) \times_{\overline{\mathcal{F}}(A)} \overline{\mathcal{F}}(A_2)$. Then there are morphisms $b_1 : x_1 \to x'_1$ in $\mathcal{F}(A_1)$ and $b_2 : x_2 \to x'_2$ in $\mathcal{F}(A_2)$. By (3) we can modify b_2 by an automorphism of x_2 over A_2 so that the diagram

$$\begin{tikzcd} (f_1)_*x_1 \arrow[r, "a"] \arrow[d, "(f_1)_*b_1"'] & (f_2)_*x_2 \arrow[d, "(f_2)_*b_2"] \\ (f_1)_*x'_1 \arrow[r, "a'"] & (f_2)_*x'_2. \end{tikzcd}$$

commutes. This proves $(x_1, x_2, a) \cong (x'_1, x'_2, a')$ in $\overline{\mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)}$. Hence (2) holds. □

Finally we define the notion of a deformation category.

Definition 51.15.8. A *deformation category* is a predeformation category $\mathcal{F}$ satisfying (RS). A morphism of deformation categories is a morphism of categories over $\mathcal{C}_\Lambda$.

Remark 51.15.9. We say that a functor $F : \mathcal{C}_\Lambda \to Sets$ is a *deformation functor* if the associated cofibered set is a deformation category, i.e. if $F(k)$ is a one element set and F satisfies (RS). If $\mathcal{F}$ is a deformation category, then $\overline{\mathcal{F}}$ is a predeformation functor but not necessarily a deformation functor, as Lemma 51.15.7 shows.

Example 51.15.10. A prorepresentable functor F is a deformation functor. Namely, suppose $R \in Ob(\widehat{\mathcal{C}}_\Lambda)$ and $F(A) = Mor_{\widehat{\mathcal{C}}_\Lambda}(R, A)$. There is a unique morphism $R \to k$, so $F(k)$ is a one element set. Since

$$Hom_\Lambda(R, A_1 \times_A A_2) = Hom_\Lambda(R, A_1) \times_{Hom_\Lambda(R,A)} Hom_\Lambda(R, A_2)$$

the same is true for maps in $\widehat{\mathcal{C}}_\Lambda$ and we see that F has (RS).

The following is one of our typical remarks on passing from a category cofibered in groupoids to the predeformation category at a point over k: it says that this process preserves (RS).

Lemma 51.15.11. *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Let $x_0 \in Ob(\mathcal{F}(k))$. Let $\mathcal{F}_{x_0}$ be the category cofibred in groupoids over $\mathcal{C}_\Lambda$ constructed in Remark 51.6.4. If $\mathcal{F}$ satisfies (RS), then so does $\mathcal{F}_{x_0}$. In particular, $\mathcal{F}_{x_0}$ is a deformation category.*

Proof. Any diagram as in Definition 51.15.1 in $\mathcal{F}_{x_0}$ gives rise to a diagram in $\mathcal{F}$ and the output of (RS) for this diagram in $\mathcal{F}$ can be viewed as an output for $\mathcal{F}_{x_0}$ as well. □

The following lemma is the analogue of the fact that 2-fibre products of algebraic stacks are algebraic stacks.

Lemma 51.15.12. *Let*

$$\begin{tikzcd} \mathcal{H} \times_{\mathcal{F}} \mathcal{G} \arrow[r] \arrow[d] & \mathcal{G} \arrow[d, "g"] \\ \mathcal{H} \arrow[r, "f"] & \mathcal{F} \end{tikzcd}$$

be 2*-fibre product of categories cofibered in groupoids over* $\mathcal{C}_\Lambda$*. If* $\mathcal{F}, \mathcal{G}, \mathcal{H}$ *all satisfy (RS), then* $\mathcal{H} \times_{\mathcal{F}} \mathcal{G}$ *satisfies (RS).*

Proof. If A is an object of $\mathcal{C}_\Lambda$, then an object of the fiber category of $\mathcal{H} \times_{\mathcal{F}} \mathcal{G}$ over A is a triple (u, v, a) where $u \in \mathcal{H}(A)$, $v \in \mathcal{G}(A)$, and $a : f(u) \to g(v)$ is a morphism in $\mathcal{F}(A)$. Consider a diagram in $\mathcal{H} \times_{\mathcal{F}} \mathcal{G}$

$$\begin{tikzcd} & (u_2, v_2, a_2) \arrow[d] & & & A_2 \arrow[d] \\ (u_1, v_1, a_1) \arrow[r] & (u, v, a) & \text{lying over} & A_1 \arrow[r] & A \end{tikzcd}$$

in $\mathcal{C}_\Lambda$ with $A_2 \to A$ surjective. Since $\mathcal{H}$ and $\mathcal{G}$ satisfy (RS), there are fiber products $u_1 \times_u u_2$ and $v_1 \times_v v_2$ lying over $A_1 \times_A A_2$. Since $\mathcal{F}$ satisfies (RS), Lemma 51.15.2 shows

$$\begin{tikzcd} f(u_1 \times_u u_2) \arrow[r] \arrow[d] & f(u_2) \arrow[d] & \text{and} & g(v_1 \times_v v_2) \arrow[r] \arrow[d] & g(v_2) \arrow[d] \\ f(u_1) \arrow[r] & f(u) & & g(v_1) \arrow[r] & g(v) \end{tikzcd}$$

are both fiber squares in $\mathcal{F}$. Thus we can view $a_1 \times_a a_2$ as a morphism from $f(u_1 \times_u u_2)$ to $g(v_1 \times_v v_2)$ over $A_1 \times_A A_2$. It follows that

$$\begin{tikzcd} (u_1 \times_u u_2, v_1 \times_v v_2, a_1 \times_a a_2) \arrow[r] \arrow[d] & (u_2, v_2, a_2) \arrow[d] \\ (u_1, v_1, a_1) \arrow[r] & (u, v, a) \end{tikzcd}$$

is a fiber square in $\mathcal{H} \times_{\mathcal{F}} \mathcal{G}$ as desired. $\square$

51.16. Lifts of objects

The content of this section is that the tangent space has a principal homogeneous action on the set of lifts along a small surjection in the case of a deformation category.

Definition 51.16.1. Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Let $f : A' \to A$ be a map in $\mathcal{C}_\Lambda$. Let $x \in \mathcal{F}(A)$. The category $\mathit{Lift}(x, f)$ of lifts of x along f is the category with the following objects and morphisms.

(1) Objects: A *lift of x along f* is a morphism $x' \to x$ lying over f.
(2) Morphisms: A *morphism of lifts* from $a_1 : x'_1 \to x$ to $a_2 : x'_2 \to x$ is a morphism $b : x'_1 \to x'_2$ in $\mathcal{F}(A')$ such that $a_2 = a_1 \circ b$.

The set $\mathrm{Lift}(x, f)$ of lifts of x along f is the set of isomorphism classes of $\mathit{Lift}(x, f)$.

Remark 51.16.2. When the map $f : A' \to A$ is clear from the context, we may write $\mathit{Lift}(x, A')$ and $\mathrm{Lift}(x, A')$ in place of $\mathit{Lift}(x, f)$ and $\mathrm{Lift}(x, f)$.

Remark 51.16.3. Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$. Let $x_0 \in Ob(\mathcal{F}(k))$. Let V be a finite dimesional vector space. Then $\text{Lift}(x_0, k[V])$ is the set of isomorphism classes of $\mathcal{F}_{x_0}(k[V])$ where $\mathcal{F}_{x_0}$ is the predeformation category of objects in $\mathcal{F}$ lying over x_0, see Remark 51.6.4. Hence if $\mathcal{F}$ satisfies (S2), then so does $\mathcal{F}_{x_0}$ (see Lemma 51.9.6) and by Lemma 51.11.2 we see that

$$\text{Lift}(x_0, k[V]) = T\mathcal{F}_{x_0} \otimes_k V$$

as k-vector spaces.

Remark 51.16.4. Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$ satisfying (RS). Let

$$\begin{tikzcd} A_1 \times_A A_2 \arrow[r] \arrow[d] & A_2 \arrow[d] \\ A_1 \arrow[r] & A \end{tikzcd}$$

be a fibre square in $\mathcal{C}_\Lambda$ such that either $A_1 \to A$ or $A_2 \to A$ is surjective. Let $x \in Ob(\mathcal{F}(A))$. Given lifts $x_1 \to x$ and $x_2 \to x$ of x to A_1 and A_2, we get by (RS) a lift $x_1 \times_x x_2 \to x$ of x to $A_1 \times_A A_2$. Conversely, by Lemma 51.15.2 any lift of x to $A_1 \times_A A_2$ is of this form. Hence a bijection

$$\text{Lift}(x, A_1) \times \text{Lift}(x, A_2) \longrightarrow \text{Lift}(x, A_1 \times_A A_2).$$

Similarly, if $x_1 \to x$ is a fixed lifting of x to A_1, then there is a bijection

$$\text{Lift}(x_1, A_1 \times_A A_2) \longrightarrow \text{Lift}(x, A_2).$$

Now let

$$\begin{tikzcd} A_1' \times_A A_2 \arrow[r] \arrow[d] & A_1 \times_A A_2 \arrow[r] \arrow[d] & A_2 \arrow[d] \\ A_1' \arrow[r] & A_1 \arrow[r] & A \end{tikzcd}$$

be a composition of fibre squares in $\mathcal{C}_\Lambda$ with both $A_1' \to A_1$ and $A_1 \to A$ surjective. Let $x_1 \to x$ be a morphism lying over $A_1 \to A$. Then by the above we have bijections

$$\begin{aligned} \text{Lift}(x_1, A_1' \times_A A_2) &= \text{Lift}(x_1, A_1') \times \text{Lift}(x_1, A_1 \times_A A_2) \\ &= \text{Lift}(x_1, A_1') \times \text{Lift}(x, A_2). \end{aligned}$$

Lemma 51.16.5. *Let $\mathcal{F}$ be a deformation category. Let $A' \to A$ be a surjective ring map in $\mathcal{C}_\Lambda$ whose kernel I is annihilated by $\mathfrak{m}_{A'}$. Let $x \in Ob(\mathcal{F}(A))$. If $\text{Lift}(x, A')$ is nonempty, then there is a free and transitive action of $T\mathcal{F} \otimes_k I$ on $\text{Lift}(x, A')$.*

Proof. Consider the ring map $g : A' \times_A A' \to k[I]$ defined by the rule $g(a_1, a_2) = \overline{a_1} \oplus a_2 - a_1$ (compare with Lemma 51.9.8). There is an isomorphism

$$A' \times_A A' \xrightarrow{\sim} A' \times_k k[I]$$

given by $(a_1, a_2) \mapsto (a_1, g(a_1, a_2))$. This isomorphism commutes with the projections to A' on the first factor, and hence with the projections of $A' \times_A A'$ and $A' \times_k k[I]$ to A. Thus there is a bijection

$$\text{Lift}(x, A' \times_A A') \longrightarrow \text{Lift}(x, A' \times_k k[I]) \tag{51.16.5.1}$$

By Remark 51.16.4 there is a bijection

$$\text{Lift}(x, A') \times \text{Lift}(x, A') \longrightarrow \text{Lift}(x, A' \times_A A') \tag{51.16.5.2}$$

There is a commutative diagram

$$\begin{tikzcd} A' \times_k k[I] \arrow[r] \arrow[d] & A \times_k k[I] \arrow[r] \arrow[d] & k[I] \arrow[d] \\ A' \arrow[r] & A \arrow[r] & k. \end{tikzcd}$$

Thus if we choose a pushforward $x \to x_0$ of x along $A \to k$, we obtain by the end of Remark 51.16.4 a bijection

$$\mathrm{Lift}(x, A' \times_k k[I]) \longrightarrow \mathrm{Lift}(x, A') \times \mathrm{Lift}(x_0, k[I]) \tag{51.16.5.3}$$

Composing (51.16.5.2), (51.16.5.1), and (51.16.5.3) we get a bijection

$$\Phi : \mathrm{Lift}(x, A') \times \mathrm{Lift}(x, A') \longrightarrow \mathrm{Lift}(x, A') \times \mathrm{Lift}(x_0, k[I]).$$

This bijection commutes with the projections on the first factors. By Remark 51.16.3 we see that $\mathrm{Lift}(x_0, k[I]) = T\mathcal{F} \otimes_k I$. If pr_2 is the second projection of $\mathrm{Lift}(x, A') \times \mathrm{Lift}(x, A')$, then we get a map

$$a = \mathrm{pr}_2 \circ \Phi^{-1} : \mathrm{Lift}(x, A') \times (T\mathcal{F} \otimes_k I) \longrightarrow \mathrm{Lift}(x, A').$$

Unwinding all the above we see that $a(x' \to x, \theta)$ is the unique lift $x'' \to x$ such that $g_*(x', x'') = \theta$ in $\mathrm{Lift}(x_0, k[I]) = T\mathcal{F} \otimes_k I$. To see this is an action of $T\mathcal{F} \otimes_k I$ on $\mathrm{Lift}(x, A')$ we have to show the following: if x', x'', x''' are lifts of x and $g_*(x', x'') = \theta$, $g_*(x'', x''') = \theta'$, then $g_*(x', x''') = \theta + \theta'$. This follows from the commutative diagram

$$\begin{tikzcd} A' \times_A A' \times_A A' \arrow[r, "(a_1,a_2,a_3)\mapsto(g(a_1,a_2),g(a_2,a_3))"'] \arrow[dr, "(a_1,a_2,a_3)\mapsto g(a_1,a_3)"'] & k[I] \times_k k[I] = k[I \times I] \arrow[d, "+"] \\ & k[I] \end{tikzcd}$$

The action is free and transitive because Φ is bijective. $\square$

Remark 51.16.6. The action of Lemma 51.16.5 is functorial. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of deformation categories. Let $A' \to A$ be a surjective ring map whose kernel I is annihilated by $\mathfrak{m}_{A'}$. Let $x \in Ob(\mathcal{F}(A))$. In this situation φ induces the vertical arrows in the following commutative diagram

$$\begin{tikzcd} \mathrm{Lift}(x, A') \times (T\mathcal{F} \otimes_k I) \arrow[r] \arrow[d, "(\varphi, d\varphi \otimes \mathrm{id}_I)"'] & \mathrm{Lift}(x, A') \arrow[d, "\varphi"] \\ \mathrm{Lift}(\varphi(x), A') \times (T\mathcal{G} \otimes_k I) \arrow[r] & \mathrm{Lift}(\varphi(x), A') \end{tikzcd}$$

The commutativity follows as each of the maps (51.16.5.2), (51.16.5.1), and (51.16.5.3) of the proof of Lemma 51.16.5 gives rise to a similar commutative diagram.

51.17. Schlessinger's theorem on prorepresentable functors

We deduce Schlessinger's theorem characterizing prorepresentable functors on $\mathcal{C}_\Lambda$.

Lemma 51.17.1. *Let $F, G : \mathcal{C}_\Lambda \to Sets$ be deformation functors. Let $\varphi : F \to G$ be a smooth morphism which induces an isomorphism $d\varphi : TF \to TG$ of tangent spaces. Then φ is an isomorphism.*

Proof. We prove $F(A) \to G(A)$ is a bijection for all $A \in Ob(\mathcal{C}_\Lambda)$ by induction on $\text{length}_A(A)$. For $A = k$ the statement follows from the assumption that F and G are deformation functors. Suppose that the statement holds for rings of length less than n and let A' be a ring of length n. Choose a small extension $f : A' \to A$. We have a commutative diagram

$$\begin{tikzcd} F(A') \arrow[r] \arrow[d, "F(f)"'] & G(A') \arrow[d, "G(f)"] \\ F(A) \arrow[r, "\sim"] & G(A) \end{tikzcd}$$

where the map $F(A) \to G(A)$ is a bijection. By smoothness of $F \to G$, $F(A') \to G(A')$ is surjective (Lemma 51.8.8). Thus we can check bijectivity by checking it on fibers $F(f)^{-1}(x) \to G(f)^{-1}(\varphi(x))$ for $x \in F(A)$ such that $F(f)^{-1}(x)$ is nonempty. These fibers are precisely $\text{Lift}(x, A')$ and $\text{Lift}(\varphi(x), A')$ and by assumption we have an isomorphism $d\varphi \otimes \text{id} : TF \otimes_k \text{Ker}(f) \to TG \otimes_k \text{Ker}(f)$. Thus, by Lemma 51.16.5 and Remark 51.16.6, for $x \in F(A)$ such that $F(f)^{-1}(x)$ is nonempty the map $F(f)^{-1}(x) \to G(f)^{-1}(\varphi(x))$ is a map of sets commuting with free transitive actions by $TF \otimes_k \text{Ker}(f)$. Hence it is bijective. □

Note that in case $k' \subset k$ is separable condition (c) in the theorem below is empty.

Theorem 51.17.2. *Let $F : \mathcal{C}_\Lambda \to Sets$ be a functor. Then F is prorepresentable if and only if (a) F is a deformation functor, (b) $\dim_k TF$ is finite, and (c) $\gamma : Der_\Lambda(k, k) \to TF$ is injective.*

Proof. Assume F is prorepresentable by $R \in \widehat{\mathcal{C}}_\Lambda$. We see F is a deformation functor by Example 51.15.10. We see $\dim_k TF$ is finite by Example 51.10.11. Finally, $\text{Der}_\Lambda(k, k) \to TF$ is identified with $\text{Der}_\Lambda(k, k) \to \text{Der}_\Lambda(R, k)$ by Example 51.10.14 which is injective because $R \to k$ is surjective.

Conversely, assume (a), (b), and (c) hold. By Lemma 51.15.6 we see that (S1) and (S2) hold. Hence by Theorem 51.14.5 there exists a minimal versal formal object ξ of F such that (51.14.0.2) holds. Say ξ lies over R. The map

$$d\underline{\xi} : \text{Der}_\Lambda(R, k) \to T\mathcal{F}$$

is bijective on $\text{Der}_\Lambda(k, k)$-orbits. Since the action of $\text{Der}_\Lambda(k, k)$ on the left hand side is free by (c) and Lemma 51.11.6 we see that the map is bijective. Thus we see that $\underline{\xi}$ is an isomorphism by Lemma 51.17.1. □

51.18. Infinitesimal automorphisms

Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Given a morphism $x' \to x$ in $\mathcal{F}$ lying over $A' \to A$, there is an induced homomorphism

$$\text{Aut}_{A'}(x') \to \text{Aut}_A(x).$$

Lemma 51.15.7 says that the cokernel of this homomorphism determines whether condition (RS) on $\mathcal{F}$ passes to $\overline{\mathcal{F}}$. In this section we study the kernel of this homomorphism. We will see that it also gives a measure of how far $\mathcal{F}$ is from $\overline{\mathcal{F}}$.

Definition 51.18.1. Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Let $x' \to x$ be a morphism in $\mathcal{F}$ lying over $A' \to A$. The *group of infinitesimal automorphisms of x' over x* is the kernel of $\text{Aut}_{A'}(x') \to \text{Aut}_A(x)$. Notation $\text{Inf}(x'/x) = \text{Ker}(\text{Aut}_{A'}(x') \to \text{Aut}_A(x))$.

Definition 51.18.2. Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Let $x_0 \in Ob(\mathcal{F}(k))$. Assume a choice of pushforward $x_0 \to x_0'$ of x_0 along the map $k \to k[\epsilon]$, $a \mapsto a$ has been made. Then there is a unique map $x_0' \to x_0$ such that $x_0 \to x_0' \to x_0$ is the identity on x_0. The *group of infinitesimal automorphisms of* x_0 is $\mathrm{Inf}_{x_0}(\mathcal{F}) := \mathrm{Inf}(x_0'/x_0)$.

Remark 51.18.3. Up to isomorphism, $\mathrm{Inf}_{x_0}(\mathcal{F})$ does not depend on the choice of pushforward $x_0 \to x_0'$. Moreover, if $y_0 \in \mathcal{F}(k)$ and $x_0 \cong y_0$ in $\mathcal{F}(k)$, then $\mathrm{Inf}_{x_0}(\mathcal{F}) \cong \mathrm{Inf}_{y_0}(\mathcal{F})$.

Remark 51.18.4. When $\mathcal{F}$ is a predeformation category, $\mathrm{Aut}_k(x_0)$ is trivial and hence $\mathrm{Inf}_{x_0}(\mathcal{F}) = \mathrm{Aut}_{k[\epsilon]}(x_0')$.

We will see that $\mathrm{Inf}_{x_0}(\mathcal{F})$ has a natural k-vector space structure when $\mathcal{F}$ satisfies (RS). At the same time, we will see that if $\mathcal{F}$ satisfies (RS), then the infinitesimal automorphisms $\mathrm{Inf}(x'/x)$ of a morphism $x' \to x$ lying over a small extension are governed by $\mathrm{Inf}_{x_0}(\mathcal{F})$, where x_0 is a pushforward of x to $\mathcal{F}(k)$. In order to do this, we introduce the automorphism functor for any object $x \in Ob(\mathcal{F})$ as follows.

Definition 51.18.5. Let $p : \mathcal{F} \to \mathcal{C}$ be a category cofibered in groupoids over an arbitrary base category $\mathcal{C}$. Assume a choice of pushforwards has been made. Let $x \in Ob(\mathcal{F})$ and let $U = p(x)$. Let $U/\mathcal{C}$ denote the category of objects under U. The *automorphism functor of* x is the functor $Aut(x) : U/\mathcal{C} \to Sets$ sending an object $f : U \to V$ to $\mathrm{Aut}_V(f_*x)$ and sending a morphism

$$\begin{tikzcd} V' \arrow[rr] & & V \\ & U \arrow[ul, "f'"] \arrow[ur, "f"'] & \end{tikzcd}$$

to the homomorphism $\mathrm{Aut}_{V'}(f'_*x) \to \mathrm{Aut}_V(f_*x)$ coming from the unique morphism $f'_*x \to f_*x$ lying over $V' \to V$ and compatible with $x \to f'_*x$ and $x \to f_*x$.

We will be concerned with the automorphism functors of objects in a category cofibered in groupoids $\mathcal{F}$ over $\mathcal{C}_\Lambda$. If $A \in Ob(\mathcal{C}_\Lambda)$, then the category $A/\mathcal{C}_\Lambda$ is nothing but the category $\mathcal{C}_A$, i.e. the category defined in Section 51.3 where we take $\Lambda = A$ and $k = A/\mathfrak{m}_A$. Hence the automorphism functor of an object $x \in Ob(\mathcal{F}(A))$ is a functor $Aut(x) : \mathcal{C}_A \to Sets$.

The following lemma could be deduced from Lemma 51.15.12 by thinking about the ``inertia'' of a category cofibred in groupoids, see for example Stacks, Section 50.7 and Categories, Section 4.31. However, it is easier to see it directly.

Lemma 51.18.6. *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$ satisfying (RS). Let $x \in Ob(\mathcal{F}(A))$. Then $Aut(x) : \mathcal{C}_A \to Sets$ satisfies (RS).*

Proof. It follows that $Aut(x)$ satisfies (RS) from the fully faithfulness of the functor $\mathcal{F}(A_1 \times_A A_2) \to \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$ in Lemma 51.15.4. □

Lemma 51.18.7. *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$ satisfying (RS). Let $x \in Ob(\mathcal{F}(A))$. Let x_0 be a pushforward of x to $\mathcal{F}(k)$.*

(1) *$T_{id_{x_0}} Aut(x)$ has a natural k-vector space structure such that addition agrees with composition in $T_{id_{x_0}} Aut(x)$. In particular, composition in $T_{id_{x_0}} Aut(x)$ is commutative.*
(2) *There is a canonical isomorphism $T_{id_{x_0}} Aut(x) \to T_{id_{x_0}} Aut(x_0)$ of k-vector spaces.*

Proof. We apply Remark 51.6.4 to the functor $Aut(x) : \mathcal{C}_A \to Sets$ and the element $\mathrm{id}_{x_0} \in Aut(x)(k)$ to get a predeformation functor $F = Aut(x)_{\mathrm{id}_{x_0}}$. By Lemmas 51.18.6 and 51.15.11 F is a deformation functor. By definition $T_{\mathrm{id}_{x_0}} Aut(x) = TF = F(k[\epsilon])$ which has a natural k-vector space structure specified by Lemma 51.10.8.

Addition is defined as the composition

$$F(k[\epsilon]) \times F(k[\epsilon]) \longrightarrow F(k[\epsilon] \times_k k[\epsilon]) \longrightarrow F(k[\epsilon])$$

where the first map is the inverse of the bijection guaranteed by (RS) and the second is induced by the k-algebra map $k[\epsilon] \times_k k[\epsilon] \to k[\epsilon]$ which maps $(\epsilon, 0)$ and $(0, \epsilon)$ to ϵ. If $A \to B$ is a ring map in $\mathcal{C}_\Lambda$, then $F(A) \to F(B)$ is a homomorphism where $F(A) = Aut(x)_{\mathrm{id}_{x_0}}(A)$ and $F(B) = Aut(x)_{\mathrm{id}_{x_0}}(B)$ are groups under composition. We conclude that $+ : F(k[\epsilon]) \times F(k[\epsilon]) \to F(k[\epsilon])$ is a homomorphism where $F(k[\epsilon])$ is regarded as a group under composition. With $\mathrm{id} \in F(k[\epsilon])$ the unit element we see that $+(v, \mathrm{id}) = +(\mathrm{id}, v) = v$ for any $v \in F(k[\epsilon])$ because (id, v) is the pushforward of v along the ring map $k[\epsilon] \to k[\epsilon] \times_k k[\epsilon]$ with $\epsilon \mapsto (\epsilon, 0)$. In general, given a group G with multiplication $\circ$ and $+ : G \times G \to G$ is a homomorphism such that $+(g, 1) = +(1, g) = g$, where 1 is the identity of G, then $+ = \circ$. This shows addition in the k-vector space structure on $F(k[\epsilon])$ agrees with composition.

Finally, (2) is a matter of unwinding the definitions. Namely $T_{\mathrm{id}_{x_0}} Aut(x)$ is the set of automorphisms α of the pushforward of x along $A \to k \to k[\epsilon]$ which are trivial modulo ϵ. On the other hand $T_{\mathrm{id}_{x_0}} Aut(x_0)$ is the set of automorphisms of the pushforward of x_0 along $k \to k[\epsilon]$ which are trivial modulo ϵ. Since x_0 is the pushforward of x along $A \to k$ the result is clear. □

Remark 51.18.8. We point out some basic relationships between infinitesimal automorphism groups, liftings, and tangent spaces to automorphism functors. Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Let $x' \to x$ be a morphism lying over a ring map $A' \to A$. Let x_0 be a pushforward of x to $\mathcal{F}(k)$. Then from the definitions we have an equality

$$\mathrm{Inf}(x'/x) = \mathrm{Lift}(\mathrm{id}_x, A')$$

where the liftings are of id_x as an object of $Aut(x')$. If $x_0 \in Ob(\mathcal{F}(k))$ and x'_0 is the pushforward to $\mathcal{F}(k[\epsilon])$, then applying this to $x'_0 \to x_0$ we get

$$\mathrm{Inf}_{x_0}(\mathcal{F}) = \mathrm{Lift}(\mathrm{id}_{x_0}, k[\epsilon]) = T_{\mathrm{id}_{x_0}} Aut(x_0),$$

the last equality following directly from the definitions.

Lemma 51.18.9. *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$ satisfying (RS). Let $x_0 \in Ob(\mathcal{F}(k))$. Then $Inf_{x_0}(\mathcal{F})$ is equal as a set to $T_{id_{x_0}} Aut(x_0)$, and so has a natural k-vector space structure such that addition agrees with composition of automorphisms.*

Proof. The equality of sets is as in the end of Remark 51.18.8 and the statement about the vector space structure follows from Lemma 51.18.7. □

Lemma 51.18.10. *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$ satisfying (RS). Let $x' \to x$ be a morphism lying over a surjective ring map $A' \to A$ with kernel I annihilated by $\mathfrak{m}_{A'}$. Let x_0 be a pushforward of x to $\mathcal{F}(k)$. Then $Inf(x'/x)$ has a free and transitive action by $T_{id_{x_0}} Aut(x') \otimes_k I = Inf_{x_0}(\mathcal{F}) \otimes_k I$.*

Proof. This is just the analogue of Lemma 51.16.5 in the setting of automorphism sheaves. To be precise, we apply Remark 51.6.4 to the functor $Aut(x') : \mathcal{C}_{A'} \to Sets$ and the element

$\mathrm{id}_{x_0} \in Aut(x)(k)$ to get a predeformation functor $F = Aut(x')_{\mathrm{id}_{x_0}}$. By Lemmas 51.18.6 and 51.15.11 F is a deformation functor. Hence Lemma 51.16.5 gives a free and transitive action of $TF \otimes_k I$ on $\mathrm{Lift}(\mathrm{id}_x, A')$, because as $\mathrm{Lift}(\mathrm{id}_x, A')$ is a group it is always nonempty. Note that we have equalities of vector spaces

$$TF = T_{\mathrm{id}_{x_0}} Aut(x') \otimes_k I = \mathrm{Inf}_{x_0}(\mathcal{F}) \otimes_k I$$

by Lemma 51.18.7. The equality $\mathrm{Inf}(x'/x) = \mathrm{Lift}(\mathrm{id}_x, A')$ of Remark 51.18.8 finishes the proof. □

Lemma 51.18.11. *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$ satisfying (RS). Let $x' \to x$ be a morphism in $\mathcal{F}$ lying over a surjective ring map. Let x_0 be a pushforward of x to $\mathcal{F}(k)$. If $Inf_{x_0}(\mathcal{F}) = 0$ then $Inf(x'/x) = 0$.*

Proof. Follows from Lemmas 51.3.3 and 51.18.10. □

Lemma 51.18.12. *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$ satisfying (RS). Let $x_0 \in Ob(\mathcal{F}(k))$. Then $Inf_{x_0}(\mathcal{F}) = 0$ if and only if the natural morphism $\mathcal{F}_{x_0} \to \overline{\mathcal{F}_{x_0}}$ of categories cofibered in groupoids is an equivalence.*

Proof. The morphism $\mathcal{F}_{x_0} \to \overline{\mathcal{F}_{x_0}}$ is an equivalence if and only if $\mathcal{F}_{x_0}$ is fibered in setoids, cf. Categories, Section 4.36 (a setoid is by definition a groupoid in which the only automorphism of any object is the identity). We prove that $\mathrm{Inf}_{x_0}(\mathcal{F}) = 0$ if and only if this condition holds for $\mathcal{F}_{x_0}$. Obviously if $\mathcal{F}_{x_0}$ is fibered in setoids then $\mathrm{Inf}_{x_0}(\mathcal{F}) = 0$. Conversely assume $\mathrm{Inf}_{x_0}(\mathcal{F}) = 0$. Let A be an object of $\mathcal{C}_\Lambda$. Then by Lemma 51.18.11, $\mathrm{Inf}(x/x_0) = 0$ for any object $x \to x_0$ of $\mathcal{F}_{x_0}(A)$. Since by definition $\mathrm{Inf}(x/x_0)$ equals the group of automorphisms of $x \to x_0$ in $\mathcal{F}_{x_0}(A)$, this proves $\mathcal{F}_{x_0}(A)$ is a setoid. □

51.19. Groupoids in functors on an arbitrary category

We begin with generalities on groupoids in functors on an arbitrary category. In the next section we will pass to the category $\mathcal{C}_\Lambda$. For clarity we shall sometimes refer to an ordinary groupoid, i.e., a category whose morphisms are all isomorphisms, as a groupoid category.

Definition 51.19.1. Let $\mathcal{C}$ be a category. The *category of groupoids in functors on $\mathcal{C}$* is the category with the following objects and morphisms.

(1) Objects: A *groupoid in functors on $\mathcal{C}$* is a quintuple (U, R, s, t, c) where $U, R : \mathcal{C} \to Sets$ are functors and $s, t : R \to U$ and $c : R \times_{s,U,t} R \to R$ are morphisms with the following property: For any object T of $\mathcal{C}$, the quintuple

$$(U(T), R(T), s, t, c)$$

is a groupoid category.

(2) Morphisms: A *morphism* $(U, R, s, t, c) \to (U', R', s', t', c')$ *of groupoids in functors on $\mathcal{C}$* consists of morphisms $U \to U'$ and $R \to R'$ with the following property: For any object T of $\mathcal{C}$, the induced maps $U(T) \to U'(T)$ and $R(T) \to R'(T)$ define a functor between groupoid categories

$$(U(T), R(T), s, t, c) \to (U'(T), R'(T), s', t', c').$$

Remark 51.19.2. A groupoid in functors on $\mathcal{C}$ amounts to the data of a functor $\mathcal{C} \to Groupoids$, and a morphism of groupoids in functors on $\mathcal{C}$ amounts to a morphism of the corresponding functors $\mathcal{C} \to Groupoids$ (where $Groupoids$ is regarded as a 1-category). However, for our purposes it is more convenient to use the terminology of groupoids in functors. In fact, thinking of a groupoid in functors as the corresponding functor $\mathcal{C} \to$

Groupoids, or equivalently as the category cofibered in groupoids associated to that functor, can lead to confusion (Remark 51.21.2).

Remark 51.19.3. Let (U, R, s, t, c) be a groupoid in functors on a category $\mathcal{C}$. There are unique morphisms $e : U \to R$ and $i : R \to R$ such that for every object T of $\mathcal{C}$, $e : U(T) \to R(T)$ sends $x \in U(T)$ to the identity morphism on x and $i : R(T) \to R(T)$ sends $a \in U(T)$ to the inverse of a in the groupoid category $(U(T), R(T), s, t, c)$. We will sometimes refer to s, t, c, e, and i as ``source'', ``target'', ``composition'', ``identity'', and ``inverse''.

Definition 51.19.4. Let $\mathcal{C}$ be a category. A groupoid in functors on $\mathcal{C}$ is *representable* if it is isomorphic to one of the form $(\underline{U}, \underline{R}, s, t, c)$ where U and R are objects of $\mathcal{C}$ and the pushout $R \amalg_{s,U,t} R$ exists.

Remark 51.19.5. Hence a representable groupoid in functors on $\mathcal{C}$ is given by objects U and R of $\mathcal{C}$ and morphisms $s, t : U \to R$ and $c : R \to R \amalg_{s,U,t} R$ such that $(\underline{U}, \underline{R}, s, t, c)$ satisfies the condition of Definition 51.19.1. The reason for requiring the existence of the pushout $R \coprod_{s,U,t} R$ is so that the composition morphism c is defined at the level of morphisms in $\mathcal{C}$. This requirement will always be satisfied below when we consider representable groupoids in functors on $\widehat{\mathcal{C}}_\Lambda$, since by Lemma 51.4.3 the category $\widehat{\mathcal{C}}_\Lambda$ admits pushouts.

Remark 51.19.6. We will say ``*let* $(\underline{U}, \underline{R}, s, t, c)$ *be a groupoid in functors on* $\mathcal{C}$'' to mean that we have a representable groupoid in functors. Thus this means that U and R are objects of $\mathcal{C}$, there are morphisms $s, t : U \to R$, the pushout $R \amalg_{s,U,t} R$ exists, there is a morphism $c : R \to R \amalg_{s,U,t} R$, and $(\underline{U}, \underline{R}, s, t, c)$ is a groupoid in functors on $\mathcal{C}$.

We introduce notation for restriction of groupoids in functors. This will be relevant below in situations where we restrict from $\widehat{\mathcal{C}}_\Lambda$ to $\mathcal{C}_\Lambda$.

Definition 51.19.7. Let (U, R, s, t, c) be a groupoid in functors on a category $\mathcal{C}$. Let $\mathcal{C}'$ be a subcategory of $\mathcal{C}$. The *restriction* $(U, R, s, t, c)|_{\mathcal{C}'}$ *of* (U, R, s, t, c) *to* $\mathcal{C}'$ is the groupoid in functors on $\mathcal{C}'$ given by $(U|_{\mathcal{C}'}, R|_{\mathcal{C}'}, s|_{\mathcal{C}'}, t|_{\mathcal{C}'}, c|_{\mathcal{C}'})$.

Remark 51.19.8. In the situation of Definition 51.19.7, we often denote $s|_{\mathcal{C}'}, t|_{\mathcal{C}'}, c|_{\mathcal{C}'}$ simply by s, t, c.

Definition 51.19.9. Let (U, R, s, t, c) be a groupoid in functors on a category $\mathcal{C}$.

(1) The assignment $T \mapsto (U(T), R(T), s, t, c)$ determines a functor $\mathcal{C} \to$ *Groupoids*. The *quotient category cofibered in groupoids* $[U/R] \to \mathcal{C}$ is the category cofibered in groupoids over $\mathcal{C}$ associated to this functor (as in Remarks 51.5.2 (9)).
(2) The *quotient morphism* $U \to [U/R]$ is the morphism of categories cofibered in groupoids over $\mathcal{C}$ defined by the rules
  (a) $x \in U(T)$ maps to the object $(T, x) \in Ob([U/R](T))$, and
  (b) $x \in U(T)$ and $f : T \to T'$ give rise to the morphism $(f, \mathrm{id}_{U(f)(x)}) : (T, x) \to (T, U(f)(x))$ lying over $f : T \to T'$.

51.20. Groupoids in functors on $\mathcal{C}_\Lambda$

In this section we discuss groupoids in functors on $\mathcal{C}_\Lambda$. Our eventual goal is to show that prorepresentable groupoids in functors on $\mathcal{C}_\Lambda$ serve as ``presentations'' for well-behaved deformation categories in the same way that smooth groupoids in algebraic spaces serve as presentations for algebraic stacks, cf. Algebraic Stacks, Section 57.16.

Definition 51.20.1. A groupoid in functors on $\mathcal{C}_\Lambda$ is *prorepresentable* if it is isomorphic to $(\underline{R_0}, \underline{R_1}, s, t, c)|_{\mathcal{C}_\Lambda}$ for some representable groupoid in functors $(\underline{R_0}, \underline{R_1}, s, t, c)$ on the category $\widehat{\mathcal{C}}_\Lambda$.

Let (U, R, s, t, c) be a groupoid in functors on $\mathcal{C}_\Lambda$. Taking completions, we get a quintuple $(\widehat{U}, \widehat{R}, \hat{s}, \hat{t}, \hat{c})$. By Remark 51.7.10 completion as a functor on CofSet$(\mathcal{C}_\Lambda)$ is a right adjoint, so it commutes with limits. In particular, there is a canonical isomorphism

$$\widehat{R \times_{s,U,t} R} \longrightarrow \widehat{R} \times_{\hat{s},\widehat{U},\hat{t}} \widehat{R},$$

so $\hat{c}$ can be regarded as a functor $\widehat{R} \times_{\hat{s},\widehat{U},\hat{t}} \widehat{R} \to \widehat{R}$. Then $(\widehat{U}, \widehat{R}, \hat{s}, \hat{t}, \hat{c})$ is a groupoid in functors on $\widehat{\mathcal{C}}_\Lambda$, with identity and inverse morphisms being the completions of those of (U, R, s, t, c).

Definition 51.20.2. Let (U, R, s, t, c) be a groupoid in functors on $\mathcal{C}_\Lambda$. The *completion* $(U, R, s, t, c)^\wedge$ *of* (U, R, s, t, c) is the groupoid in functors $(\widehat{U}, \widehat{R}, \hat{s}, \hat{t}, \hat{c})$ on $\widehat{\mathcal{C}}_\Lambda$ described above.

Remark 51.20.3. Let (U, R, s, t, c) be a groupoid in functors on $\mathcal{C}_\Lambda$. Then there is a canonical isomorphism $(U, R, s, t, c)^\wedge|_{\mathcal{C}_\Lambda} \cong (U, R, s, t, c)$, see Remark 51.7.7. On the other hand, let (U, R, s, t, c) be a groupoid in functors on $\widehat{\mathcal{C}}_\Lambda$ such that $U, R : \widehat{\mathcal{C}}_\Lambda \to$ *Sets* both commute with limits, e.g. if U, R are representable. Then there is a canonical isomorphism $((U, R, s, t, c)|_{\mathcal{C}_\Lambda})^\wedge \cong (U, R, s, t, c)$. This follows from Remark 51.7.11.

Lemma 51.20.4. *Let (U, R, s, t, c) be a groupoid in functors on $\mathcal{C}_\Lambda$.*

(1) *(U, R, s, t, c) is prorepresentable if and only if its completion is representable as a groupoid in functors on $\widehat{\mathcal{C}}_\Lambda$.*
(2) *(U, R, s, t, c) is prorepresentable if and only if U and R are prorepresentable.*

Proof. Part (1) follows from Remark 51.20.3. For (2), the ``only if'' direction is clear from the definition of a prorepresentable groupoid in functors. Conversely, assume U and R are prorepresentable, say $U \cong \underline{R_0}|_{\mathcal{C}_\Lambda}$ and $R \cong \underline{R_1}|_{\mathcal{C}_\Lambda}$ for objects R_0 and R_1 of $\widehat{\mathcal{C}}_\Lambda$. Since $\underline{R_0} \cong \widehat{\underline{R_0}|_{\mathcal{C}_\Lambda}}$ and $\underline{R_1} \cong \widehat{\underline{R_1}|_{\mathcal{C}_\Lambda}}$ by Remark 51.7.11 we see that the completion $(U, R, s, t, c)^\wedge$ is a groupoid in functors of the form $(\underline{R_0}, \underline{R_1}, \hat{s}, \hat{t}, \hat{c})$. By Lemma 51.4.3 the pushout $\underline{R_1} \times_{\hat{s},\underline{R_1},\hat{t}} \underline{R_1}$ exists. Hence $(\underline{R_0}, \underline{R_1}, \hat{s}, \hat{t}, \hat{c})$ is a representable groupoid in functors on $\widehat{\mathcal{C}}_\Lambda$. Finally, the restriction $(\underline{R_0}, \underline{R_1}, s, t, c)|_{\mathcal{C}_\Lambda}$ gives back (U, R, s, t, c) by Remark 51.20.3 hence (U, R, s, t, c) is prorepresentable by definition. □

51.21. Smooth groupoids in functors on $\mathcal{C}_\Lambda$

The notion of smoothness for groupoids in functors on $\mathcal{C}_\Lambda$ is defined as follows.

Definition 51.21.1. Let (U, R, s, t, c) be a groupoid in functors on $\mathcal{C}_\Lambda$. We say (U, R, s, t, c) is *smooth* if $s, t : R \to U$ are smooth.

Remark 51.21.2. We note that this terminology is potentially confusing: if (U, R, s, t, c) is a smooth groupoid in functors, then the quotient $[U/R]$ need not be a smooth category cofibred in groupoids as defined in Remark 51.8.9. However smoothness of (U, R, s, t, c) does imply (in fact is equivalent to) smoothness of the quotient morphism $U \to [U/R]$ as we shall see in Lemma 51.21.4.

Remark 51.21.3. Let $(\underline{R_0}, \underline{R_1}, s, t, c)|_{\mathcal{C}_\Lambda}$ be a prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$. Then $(\underline{R_0}, \underline{R_1}, s, t, c)|_{\mathcal{C}_\Lambda}$ is smooth if and only if R_1 is a power series over R_0 via both s and t. This follows from Lemma 51.8.6.

Lemma 51.21.4. *Let (U, R, s, t, c) be a groupoid in functors on $\mathcal{C}_\Lambda$. The following are equivalent:*

(1) *The groupoid in functors (U, R, s, t, c) is smooth.*
(2) *The morphism $s : R \to U$ is smooth.*
(3) *The morphism $t : R \to U$ is smooth.*
(4) *The quotient morphism $U \to [U/R]$ is smooth.*

Proof. Statement (2) is equivalent to (3) since the inverse $i : R \to R$ of (U, R, s, t, c) is an isomorphism and $t = s \circ i$. By definition (1) is equivalent to (2) and (3) together, hence it is equivalent to either of them individually.

Finally we prove (2) is equivalent to (4). Unwinding the definitions:

(2) Smoothness of $s : R \to U$ amounts to the following condition: If $f : B \to A$ is a surjective ring map in $\mathcal{C}_\Lambda$, $a \in R(A)$, and $y \in U(B)$ such that $s(a) = U(f)(y)$, then there exists $a' \in R(B)$ such that $R(f)(a') = a$ and $s(a') = y$.
(4) Smoothness of $U \to [U/R]$ amounts to the following condition: If $f : B \to A$ is a surjective ring map in $\mathcal{C}_\Lambda$ and $(f, a) : (B, y) \to (A, x)$ is a morphism of $[U/R]$, then there exists $x' \in U(B)$ and $b \in R(B)$ with $s(b) = x'$, $t(b) = y$ such that $c(a, R(f)(b)) = e(x)$. Here $e : U \to R$ denotes the identity and the notation (f, a) is as in Remarks 51.5.2 (9); in particular $a \in R(A)$ with $s(a) = U(f)(y)$ and $t(a) = x$.

If (4) holds and f, a, y as in (2) are given, let $x = t(a)$ so that we have a morphism $(f, a) : (B, y) \to (A, x)$. Then (4) produces x' and b, and $a' = i(b)$ satisfies the requirements of (2). Conversely, assume (2) holds and let $(f, a) : (B, y) \to (A, x)$ as in (4) be given. Then (2) produces $a' \in R(B)$, and $x' = t(a')$ and $b = i(a')$ satisfy the requirements of (4). ☐

51.22. Deformation categories as quotients of groupoids in functors

We discuss conditions on a groupoid in functors on $\mathcal{C}_\Lambda$ which guarantee that the quotient is a deformation category, and we calculate the tangent and infinitesimal automorphism spaces of such a quotient.

Lemma 51.22.1. *Let (U, R, s, t, c) be a smooth groupoid in functors on $\mathcal{C}_\Lambda$. Assume U and R satisfy (RS). Then $[U/R]$ satisfies (RS).*

Proof. Let

$$\begin{tikzcd} & (A_2, x_2) \arrow[d, "(f_2, a_2)"] \\ (A_1, x_1) \arrow[r, "(f_1, a_1)"] & (A, x) \end{tikzcd}$$

be a diagram in $[U/R]$ such that $f_2 : A_2 \to A$ is surjective. The notation is as in Remarks 51.5.2 (9). Hence $f_1 : A_1 \to A$, $f_2 : A_2 \to A$ are maps in $\mathcal{C}_\Lambda$, $x \in U(A)$, $x_1 \in U(A_1)$, $x_2 \in U(A_2)$, and $a_1, a_2 \in R(A)$ with $s(a_1) = U(f_1)(x_1)$, $t(a_1) = x$ and $s(a_2) = U(f_2)(x_2)$, $t(a_2) = x$. We construct a fiber product lying over $A_1 \times_A A_2$ for this diagram in $[U/R]$ as follows.

Let $a = c(i(a_1), a_2)$, where $i : R \to R$ is the inverse morphism. Then $a \in R(A)$, $x_2 \in U(A_2)$ and $s(a) = U(f_2)(x_2)$. Hence an element $(a, x_2) \in R(A) \times_{s,U(A),U(f_2)} U(A_2)$. By smoothness of $s : R \to U$ there is an element $\tilde{a} \in R(A_2)$ with $R(f_2)(\tilde{a}) = a$ and $s(\tilde{a}) = x_2$. In particular $U(f_2)(t(\tilde{a})) = t(a) = U(f_1)(x_1)$. Thus x_1 and $t(\tilde{a})$ define an element

$$(x_1, t(\tilde{a})) \in U(A_1) \times_{U(A)} U(A_2).$$

By the assumption that U satisfies (RS), we have an identification $U(A_1) \times_{U(A)} U(A_2) = U(A_1 \times_A A_2)$. Let us denote $x_1 \times t(\tilde{a}) \in U(A_1 \times_A A_2)$ the element corresponding to $(x_1, t(\tilde{a})) \in U(A_1) \times_{U(A)} U(A_2)$. Let p_1, p_2 be the projections of $A_1 \times_A A_2$. We claim

$$\begin{tikzcd} (A_1 \times_A A_2, x_1 \times t(\tilde{a})) \arrow[r, "(p_2, i(\tilde{a}))"'] \arrow[d, "(p_1, e(x_1))"'] & (A_2, x_2) \arrow[d, "(f_2, a_2)"] \\ (A_1, x_1) \arrow[r, "(f_1, a_1)"] & (A, x) \end{tikzcd}$$

is a fiber square in $[U/R]$. (Note $e : U \to R$ denotes the identity.)

The diagram is commutative because $c(a_2, R(f_2)(i(\tilde{a}))) = c(a_2, i(a)) = a_1$. To check it is a fiber square, let

$$\begin{tikzcd} (B, z) \arrow[r, "(g_2, b_2)"'] \arrow[d, "(g_1, b_1)"'] & (A_2, x_2) \arrow[d, "(f_2, a_2)"] \\ (A_1, x_1) \arrow[r, "(f_1, a_1)"] & (A, x) \end{tikzcd}$$

be a commutative diagram in $[U/R]$. We will show there is a unique morphism $(g, b) : (B, z) \to (A_1 \times_A A_2, x_1 \times t(\tilde{a}))$ compatible with the morphisms to (A_1, x_1) and (A_2, x_2). We must take $g = (g_1, g_2) : B \to A_1 \times_A A_2$. Since by assumption R satisfies (RS), we have an identification $R(A_1 \times_A A_2) = R(A_1) \times_{R(A)} R(A_2)$. Hence we can write $b = (b_1', b_2')$ for some $b_1' \in R(A_1)$, $b_2' \in R(A_2)$ which agree in $R(A)$. Then $((g_1, g_2), (b_1', b_2')) : (B, z) \to (A_1 \times_A A_2, x_1 \times t(\tilde{a}))$ will commute with the projections if and only if $b_1' = b_1$ and $b_2' = c(\tilde{a}, b_2)$ proving unicity and existence. □

Lemma 51.22.2. *Let (U, R, s, t, c) be a smooth groupoid in functors on $\mathcal{C}_\Lambda$. Assume U and R are deformation functors. Then:*

1. *The quotient $[U/R]$ is a deformation category.*
2. *The tangent space of $[U/R]$ is*

$$T[U/R] = Coker(ds - dt : TR \to TU).$$

3. *Let x_0 be the unique object of $[U/R](k)$. The space of infinitesimal automorphisms of $[U/R]$ is*

$$Inf_{x_0}([U/R]) = Ker(ds \oplus dt : TR \to TU \oplus TU).$$

Proof. Since U and R are deformation functors $[U/R]$ is a predeformation category. Since (RS) holds for deformation functors by definition we see that (RS) holds for [U/R] by Lemma 51.22.1. Hence $[U/R]$ is a deformation category. Statements (2) and (3) follow directly from the definitions. □

51.23. Presentations of categories cofibered in groupoids

A presentation is defined as follows.

Definition 51.23.1. Let $\mathcal{F}$ be a category cofibered in groupoids over a category $\mathcal{C}$. Let (U, R, s, t, c) be a groupoid in functors on $\mathcal{C}$. A *presentation of $\mathcal{F}$ by (U, R, s, t, c)* is an equivalence $\varphi : [U/R] \to \mathcal{F}$ of categories cofibered in groupoids over $\mathcal{C}$.

The following two general lemmas will be used to get presentations.

Lemma 51.23.2. *Let $\mathcal{F}$ be category cofibered in groupoids over a category $\mathcal{C}$. Let $U : \mathcal{C} \to Sets$ be a functor. Let $f : U \to \mathcal{F}$ be a morphism of categories cofibered in groupoids over $\mathcal{C}$. Define R, s, t, c as follows:*

1. *$R : \mathcal{C} \to Sets$ is the functor $U \times_{f,\mathcal{F},f} U$.*
2. *$t, s : R \to U$ are the first and second projections, respectively.*
3. *$c : R \times_{s,U,t} R \to R$ is the morphism given by projection onto the first and last factors of $U \times_{f,\mathcal{F},f} U \times_{f,\mathcal{F},f} U$ under the canonical isomorphism $R \times_{s,U,t} R \to U \times_{f,\mathcal{F},f} U \times_{f,\mathcal{F},f} U$.*

Then (U, R, s, t, c) is a groupoid in functors on $\mathcal{C}$.

Proof. Omitted. □

Lemma 51.23.3. *Let $\mathcal{F}$ be category cofibered in groupoids over a category $\mathcal{C}$. Let $U : \mathcal{C} \to Sets$ be a functor. Let $f : U \to \mathcal{F}$ be a morphism of categories cofibered in groupoids over $\mathcal{C}$. Let (U, R, s, t, c) be the groupoid in functors on $\mathcal{C}$ constructed from $f : U \to \mathcal{F}$ in Lemma 51.23.2. Then there is a natural morphism $[f] : [U/R] \to \mathcal{F}$ such that:*

1. *$[f] : [U/R] \to \mathcal{F}$ is fully faithful.*
2. *$[f] : [U/R] \to \mathcal{F}$ is an equivalence if and only if $f : U \to \mathcal{F}$ is essentially surjective.*

Proof. Omitted. □

51.24. Presentations of deformation categories

According to the next lemma, a smooth morphism from a predeformation functor to a predeformation category $\mathcal{F}$ gives rise to a presentation of $\mathcal{F}$ by a smooth groupoid in functors.

Lemma 51.24.1. *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Let $U : \mathcal{C}_\Lambda \to Sets$ be a functor. Let $f : U \to \mathcal{F}$ be a smooth morphism of categories cofibered in groupoids. Then:*

1. *If (U, R, s, t, c) is the groupoid in functors on $\mathcal{C}_\Lambda$ constructed from $f : U \to \mathcal{F}$ in Lemma 51.23.2, then (U, R, s, t, c) is smooth.*
2. *If $f : U(k) \to \mathcal{F}(k)$ is essentially surjective, then the morphism $[f] : [U/R] \to \mathcal{F}$ of Lemma 51.23.3 is an equivalence.*

Proof. From the construction of Lemma 51.23.2 we have a commutative diagram

$$\begin{tikzcd} R = U \times_{f,\mathcal{F},f} U \arrow[r, "s"'] \arrow[d, "t"'] & U \arrow[d, "f"] \\ U \arrow[r, "f"] & \mathcal{F} \end{tikzcd}$$

where t, s are the first and second projections. So t, s are smooth by Lemma 51.8.7. Hence (1) holds.

If the assumption of (2) holds, then by Lemma 51.8.8 the morphism $f : U \to \mathcal{F}$ is essentially surjective. Hence by Lemma 51.23.3 the morphism $[f] : [U/R] \to \mathcal{F}$ is an equivalence. □

Lemma 51.24.2. *Let $\mathcal{F}$ be a deformation category. Let $U, V : \mathcal{C}_\Lambda \to Sets$ be deformation functors. Let $f : U \to \mathcal{F}$ and $g : V \to \mathcal{F}$ be morphisms of categories cofibered in groupoids. Then:*

(1) *$U \times_{f,\mathcal{F},g} V$ is a deformation functor.*
(2) *Let u_0 be the unique element of $U(k)$ and u_0' its pushforward to $U(k[\epsilon])$; define v_0 and v_0' similarly. There is an exact sequence of k-vector spaces*
$$0 \to K \to T(U \times_{f,\mathcal{F},g} V) \to TU \oplus TV$$
where K is the subspace of $T(U \times_{f,\mathcal{F},g} V)$ consisting of elements of the form $(u_0', v_0', a : f(u_0') \to g(v_0'))$, $a \in Hom_{k[\epsilon]}(f(u_0'), g(v_0'))$.

Proof. Part (1) follows from Lemma 51.15.12 and the fact that $(U \times_{f,\mathcal{F},g} V)(k)$ is a singleton as $U(k)$, $V(k)$ are singletons and $\mathcal{F}(k)$ is a setoid with exactly one isomorphism class.

Taking the differentials of the projections of $U \times_{f,\mathcal{F},g} V$ to U and V gives k-linear maps $T(U \times_{f,\mathcal{F},g} V) \to TU$ and $T(U \times_{f,\mathcal{F},g} V) \to TV$ by Lemma 51.11.4. Hence a k-linear map $T(U \times_{f,\mathcal{F},g} V) \to TU \oplus TV$. Explicitly, this map sends an element $(u, v, a : f(u) \to g(v))$ of $T(U \times_{f,\mathcal{F},g} V)$ to (u, v). So the kernel is exactly K (this proves that K is a subspace). Hence (2) holds. □

Lemma 51.24.3. *Let $\mathcal{F}$ be a deformation category. Let $U : \mathcal{C}_\Lambda \to Sets$ be a deformation functor. Let $f : U \to \mathcal{F}$ be a morphism of categories cofibered in groupoids. Let u_0 be the unique element of $U(k)$. Then $U \times_{f,\mathcal{F},f} U$ is a deformation functor with tangent space fitting into an exact sequence of k-vector spaces*
$$0 \to Inf_{f(u_0)}(\mathcal{F}) \to T(U \times_{f,\mathcal{F},g} U) \to TU \oplus TU$$

Proof. Follows from Lemma 51.24.2 and Definition 51.18.2. □

Lemma 51.24.4. *Let $\mathcal{F}$ be a deformation category. Let $U : \mathcal{C}_\Lambda \to Sets$ be a prorepresentable functor. Let $f : U \to \mathcal{F}$ be a morphism of categories cofibered in groupoids. Let (U, R, s, t, c) be the groupoid in functors on $\mathcal{C}_\Lambda$ constructed from $f : U \to \mathcal{F}$ in Lemma 51.23.2. Assume $\dim_k Inf_{x_0}(\mathcal{F})$ is finite for $x_0 \in Ob(\mathcal{F}(k))$. Then (U, R, s, t, c) is prorepresentable.*

Proof. Note that U is a deformation functor by Example 51.15.10. By Lemma 51.24.3 we see that $R = U \times_{f,\mathcal{F},f} U$ is a deformation functor whose tangent space $TR = T(U \times_{f,\mathcal{F},f} U)$ sits in an exact sequence $0 \to \mathrm{Inf}_{x_0}(\mathcal{F}) \to TR \to TU \oplus TU$. Since we have assumed the first space has finite dimension and since TU has finite dimension by Example 51.10.11 we see that $\dim TR < \infty$. The map $\gamma : \mathrm{Der}_\Lambda(k, k) \to TR$ see (51.11.6.1) is injective because its composition with $TR \to TU$ is injective by Theorem 51.17.2 for the prorepresentable functor U. Thus R is prorepresentable by Theorem 51.17.2. It follows from Lemma 51.20.4 that (U, R, s, t, c) is prorepresentable. □

Theorem 51.24.5. *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Then $\mathcal{F}$ admits a presentation by a smooth prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$ if and only if the following conditions hold:*

(1) *$\mathcal{F}$ is a deformation category.*
(2) *$\dim_k T\mathcal{F}$ is finite.*

(3) $\dim_k Inf_{x_0}(\mathcal{F})$ *is finite for some* $x_0 \in Ob(\mathcal{F}(k))$.

Proof. Recall that a prorepresentable functor is a deformation functor, see Example 51.15.10. Thus if $\mathcal{F}$ is equivalent to a smooth prorepresentable groupoid in functors, then conditions (1), (2), and (3) follow from Lemma 51.22.2 (1), (2), and (3).

Conversely, assume conditions (1), (2), and (3) hold. Condition (1) implies that (S1) and (S2) are satisfied, see Lemma 51.15.6. By Lemma 51.12.4 there exists a versal formal object ξ. Setting $U = \underline{R}|_{\mathcal{C}_\Lambda}$ the associated map $\underline{\xi} : U \to \mathcal{F}$ is smooth (this is the definition of a versal formal object). Let (U, R, s, t, c) be the groupoid in functors constructed in Lemma 51.23.2 from the map $\underline{\xi}$. By Lemma 51.24.1 we see that (U, R, s, t, c) is a smooth groupoid in functors and that $[U/R] \to \mathcal{F}$ is an equivalence. By Lemma 51.24.4 we see that (U, R, s, t, c) is prorepresentable. Hence $[U/R] \to \mathcal{F}$ is the desired presentation of $\mathcal{F}$. $\square$

51.25. Remarks regarding minimality

The main theorem of this chapter is Theorem 51.24.5 above. It describes completely those categories cofibred in groupoids over $\mathcal{C}_\Lambda$ which have a presentation by a smooth prorepresentable groupoid in functors. In this section we briefly discuss how the minimality discussed in Sections 51.13 and 51.14 can be used to obtain a ``minimal'' smooth prorepresentable presentation.

Definition 51.25.1. Let (U, R, s, t, c) be a smooth prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$.

(1) We say (U, R, s, t, c) is *normalized* if the groupoid $(U(k[\epsilon]), R(k[\epsilon]), s, t, c)$ is totally disconnected, i.e., there are no morphisms between distinct objects.
(2) We say (U, R, s, t, c) is *minimal* if the $U \to [U/R]$ is given by a minimal versal formal object of $[U/R]$.

The difference between the two notions is related to the difference between conditions (51.14.0.1) and (51.14.0.2) and disappears when $k' \subset k$ is separable. Also a normalized smooth prorepresentable groupoid in functors is minimal as the following lemma shows. Here is a precise statement.

Lemma 51.25.2. *Let* (U, R, s, t, c) *be a smooth prorepresentable groupoid in functors on* $\mathcal{C}_\Lambda$.

(1) (U, R, s, t, c) *is normalized if and only if the morphism* $U \to [U/R]$ *induces an isomorphism on tangent spaces, and*
(2) (U, R, s, t, c) *is minimal if and only if the kernel of* $TU \to T[U/R]$ *is contained in the image of* $Der_\Lambda(k, k) \to TU$.

Proof. Part (1) follows immediately from the definitions. To see part (2) set $\mathcal{F} = [U/R]$. Since $\mathcal{F}$ has a presentation it is a deformation category, see Theorem 51.24.5. In particular it satisfies (RS), (S1), and (S2), see Lemma 51.15.6. Recall that minimal versal formal objects are unique up to isomorphism, see Lemma 51.13.5. By Theorem 51.14.5 a minimal versal object induces a map $\underline{\xi} : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ satisfying (51.14.0.2). Since $U \cong \underline{R}|_{\mathcal{C}_\Lambda}$ over $\mathcal{F}$ we see that $TU \to T\mathcal{F} = T[U/R]$ satisfies the property as stated in the lemma. $\square$

The quotient of a minimal prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$ does not admit autoequivalences which are not automorphisms. To prove this, we first note the following lemma.

Lemma 51.25.3. *Let $U : \mathcal{C}_\Lambda \to Sets$ be a prorepresentable functor. Let $\varphi : U \to U$ be a morphism such that $d\varphi : TU \to TU$ is an isomorphism. Then φ is an isomorphism.*

Proof. If $U \cong \underline{R}|_{\mathcal{C}_\Lambda}$ for some $R \in Ob(\widehat{\mathcal{C}}_\Lambda)$, then completing φ gives a morphism $\underline{R} \to \underline{R}$. If $f : R \to R$ is the corresponding morphism in $\widehat{\mathcal{C}}_\Lambda$, then f induces an isomorphism $\mathrm{Der}_\Lambda(R, k) \to \mathrm{Der}_\Lambda(R, k)$, see Example 51.10.14. In particular f is a surjection by Lemma 51.4.6. As a surjective endomorphism of a Noetherian ring is an isomorphism (see Algebra, Lemma 7.28.8) we conclude f, hence $\underline{R} \to \underline{R}$, hence $\varphi : U \to U$ is an isomorphism. □

Lemma 51.25.4. *Let (U, R, s, t, c) be a minimal smooth prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$. If $\varphi : [U/R] \to [U/R]$ is an equivalence of categories cofibered in groupoids, then φ is an isomorphism.*

Proof. A morphism $\varphi : [U/R] \to [U/R]$ is the same thing as a morphism $\varphi : (U, R, s, t, c) \to (U, R, s, t, c)$ of groupoids in functors over $\mathcal{C}_\Lambda$ as defined in Definition 51.19.1. Denote $\phi : U \to U$ and $\psi : R \to R$ the corresponding morphisms. Because the diagram

$$\begin{tikzcd} & \mathrm{Der}_\Lambda(k,k) \arrow[dl, "\gamma"'] \arrow[dr, "\gamma"] & \\ TU \arrow[rr, "d\phi"] \arrow[d] & & TU \arrow[d] \\ T[U/R] \arrow[rr, "d\varphi"] & & T[U/R] \end{tikzcd}$$

is commutative, since $d\varphi$ is bijective, and since we have the characterization of minimality in Lemma 51.25.2 we conclude that $d\phi$ is injective (hence bijective by dimension reasons). Thus $\phi : U \to U$ is an isomorphism by Lemma 51.25.3. We can use a similar argument, using the exact sequence

$$0 \to \mathrm{Inf}_{x_0}([U/R]) \to TR \to TU \oplus TU$$

of Lemma 51.24.3 to prove that $\psi : R \to R$ is an isomorphism. But is also a consequence of the fact that $R = U \times_{[U/R]} U$ and that φ and ϕ are isomorphisms. □

Lemma 51.25.5. *Let (U, R, s, t, c) and (U', R', s', t', c') be minimal smooth prorepresentable groupoids in functors on $\mathcal{C}_\Lambda$. If $\varphi : [U/R] \to [U'/R']$ is an equivalence of categories cofibered in groupoids, then φ is an isomorphism.*

Proof. Let $\psi : [U'/R'] \to [U/R]$ be a quasi-inverse to φ. Then $\psi \circ \varphi$ and $\varphi \circ \psi$ are isomorphisms by Lemma 51.25.4, hence φ and ψ are isomorphisms. □

The following lemma summarizes some of the things we have seen earlier in this chapter.

Lemma 51.25.6. *Let $\mathcal{F}$ be a deformation category such that $\dim_k T\mathcal{F} < \infty$ and $\dim_k Inf_{x_0}(\mathcal{F}) < \infty$ for some $x_0 \in Ob(\mathcal{F}(k))$. Then there exists a minimal versal formal object ξ of $\mathcal{F}$. Say ξ lies over $R \in Ob(\widehat{\mathcal{C}}_\Lambda)$. Let $U = \underline{R}|_{\mathcal{C}_\Lambda}$. Let $f = \underline{\xi} : U \to \mathcal{F}$ be the associated morphism. Let (U, R, s, t, c) be the groupoid in functors on $\mathcal{C}_\Lambda$ constructed from $f : U \to \mathcal{F}$ in Lemma 51.23.2. Then (U, R, s, t, c) is a minimal smooth prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$ and there is an equivalence $[U/R] \to \mathcal{F}$.*

Proof. As $\mathcal{F}$ is a deformation category it satisfies (S1) and (S2), see Lemma 51.15.6. By Lemma 51.12.4 there exists a versal formal object. By Lemma 51.13.5 there exists a minimal versal formal object ξ/R as in the statement of the lemma. Setting $U = \underline{R}|_{\mathcal{C}_\Lambda}$ the associated map $\underline{\xi} : U \to \mathcal{F}$ is smooth (this is the definition of a versal formal object). Let (U, R, s, t, c) be the groupoid in functors constructed in Lemma 51.23.2 from the map $\underline{\xi}$. By Lemma 51.24.1 we see that (U, R, s, t, c) is a smooth groupoid in functors and that $[U/R] \to \mathcal{F}$ is an equivalence. By Lemma 51.24.4 we see that (U, R, s, t, c) is prorepresentable. Finally, (U, R, s, t, c) is minimal because $U \to [U/R] = \mathcal{F}$ corresponds to the minimal versal formal object ξ. $\square$

Presentations by minimal prorepresentable groupoids in functors satisfy the following uniqueness property.

Lemma 51.25.7. *Let $\mathcal{F}$ be category cofibered in groupoids over $\mathcal{C}_\Lambda$. Assume there exist presentations of $\mathcal{F}$ by minimal smooth prorepresentable groupoids in functors (U, R, s, t, c) and (U', R', s', t', c'). Then (U, R, s, t, c) and (U', R', s', t', c') are isomorphic.*

Proof. Follows from Lemma 51.25.5 and the observation that a morphism $[U/R] \to [U'/R']$ is the same thing as a morphism of groupoids in functors (by our explicit construction of $[U/R]$ in Definition 51.19.9). $\square$

In summary we have proved the following theorem.

Theorem 51.25.8. *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Consider the following conditions*

1. *$\mathcal{F}$ admits a presentation by a normalized smooth prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$,*
2. *$\mathcal{F}$ admits a presentation by a smooth prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$,*
3. *$\mathcal{F}$ admits a presentation by a minimal smooth prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$, and*
4. *$\mathcal{F}$ satisfies the following conditions*
   (a) *$\mathcal{F}$ is a deformation category.*
   (b) *$\dim_k T\mathcal{F}$ is finite.*
   (c) *$\dim_k Inf_{x_0}(\mathcal{F})$ is finite for some $x_0 \in Ob(\mathcal{F}(k))$.*

Then (2), (3), (4) are equivalent and are implied by (1). If $k' \subset k$ is separable, then (1), (2), (3), (4) are all equivalent. Furthermore, the minimal smooth prorepresentable groupoids in functors which provide a presentation of $\mathcal{F}$ are unique up to isomorphism.

Proof. We see that (1) implies (3) and is equivalent to (3) if $k' \subset k$ is separable from Lemma 51.25.2. It is clear that (3) implies (2). We see that (2) implies (4) by Theorem 51.24.5. We see that (4) implies (3) by Lemma 51.25.6. This proves all the implications. The final uniqueness statement follows from Lemma 51.25.7. $\square$

51.26. The Deformation Category of a Point of an Algebraic Stack

To do: Show that an algebraic stack of finite type over a locally Noetherian base satisfies (RS) at any finite type point (this may have to go in a later chapter). This will provide some motivation for Artin's criteria later. A perhaps more roundabout way of showing this (which does give more information, though) is to show that a groupoid presentation of the stack gives rise to a presentation of the deformation category at any point.

51.27. Examples

List of things that should go here:

(1) Describe the general outline of an example.
(2) Deformations of schemes:
    (a) The Rim-Schlessinger condition.
    (b) Computing the tangent space.
    (c) Computing the infinitesimal deformations.
    (d) The deformation category of an affine hypersurface.
(3) Deformations of representations of abstract groups.
(4) Deformations of representations of topological groups (e.g., profinite ones).
(5) Deformations of sheaves (for example fix X/S, a finite type point s of S, and a quasi-coherent sheaf $\mathcal{F}_s$ over X_s).
(6) Deformations of algebraic spaces (very similar to deformations of schemes; maybe even easier?).
(7) Deformations of maps (eg morphisms between schemes; you can fix both or one of the target and/or source).
(8) Add more here.

51.28. Other chapters

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Commutative Algebra
(8) Brauer Groups
(9) Sites and Sheaves
(10) Homological Algebra
(11) Derived Categories
(12) More on Algebra
(13) Smoothing Ring Maps
(14) Simplicial Methods
(15) Sheaves of Modules
(16) Modules on Sites
(17) Injectives
(18) Cohomology of Sheaves
(19) Cohomology on Sites
(20) Hypercoverings
(21) Schemes
(22) Constructions of Schemes
(23) Properties of Schemes
(24) Morphisms of Schemes
(25) Coherent Cohomology
(26) Divisors
(27) Limits of Schemes
(28) Varieties
(29) Chow Homology
(30) Topologies on Schemes
(31) Descent
(32) Adequate Modules
(33) More on Morphisms
(34) More on Flatness
(35) Groupoid Schemes
(36) More on Groupoid Schemes
(37) Étale Morphisms of Schemes
(38) Étale Cohomology
(39) Crystalline Cohomology
(40) Algebraic Spaces
(41) Properties of Algebraic Spaces
(42) Morphisms of Algebraic Spaces
(43) Decent Algebraic Spaces
(44) Topologies on Algebraic Spaces
(45) Descent and Algebraic Spaces
(46) More on Morphisms of Spaces
(47) Quot and Hilbert Spaces
(48) Spaces over Fields
(49) Cohomology of Algebraic Spaces
(50) Stacks
(51) Formal Deformation Theory
(52) Groupoids in Algebraic Spaces
(53) More on Groupoids in Spaces
(54) Bootstrap
(55) Examples of Stacks
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(57) Algebraic Stacks
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CHAPTER 52

Groupoids in Algebraic Spaces

52.1. Introduction

This chapter is devoted to generalities concering groupoids in algebraic spaces. We recommend reading the beautiful paper [**KM97a**] by Keel and Mori.

A lot of what we say here is a repeat of what we said in the chapter on groupoid schemes, see Groupoids, Section 35.1. The discussion of quotient stacks is new here.

52.2. Conventions

The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $Spec(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

We continue our convention to label projection maps starting with index 0, so we have $\text{pr}_0 : X \times_S Y \to X$ and $\text{pr}_1 : X \times_S Y \to Y$.

52.3. Notation

Let S be a scheme; this will be our base scheme and all algebraic spaces will be over S. Let B be an algebraic space over S; this will be our base algebraic space, and often other algebraic spaces, and schemes will be over B. If we say that X is an algebraic space over B, then we mean that X is an algebraic space over S which comes equipped with structure morphism $X \to B$. Moreover, we try to reserve the letter T to denote a ``test" scheme over B. In other words T is a scheme which comes equipped with a structure morphism $T \to B$. In this situation we denote $X(T)$ for the set of T-valued points of X *over* B. In a formula:

$$X(T) = Mor_B(T, X).$$

Similarly, given a second algebraic space Y over B we set

$$X(Y) = Mor_B(Y, X).$$

Suppose we are given algebraic spaces X, Y over B as above and a morphism $f : X \to Y$ over B. For any scheme T over B we get an induced map of sets

$$f : X(T) \longrightarrow Y(T)$$

which is functorial in the scheme T over B. As f is a map of sheaves on $(Sch/S)_{fppf}$ over the sheaf B it is clear that f determines and is determined by this rule. More generally, we use the same notation for maps between fibre products. For example, if X, Y, Z are algebraic spaces over B, and if $m : X \times_B Y \to Z \times_B Z$ is a morphism of algebraic spaces over B, then we think of m as corresponding to a collection of maps between T-valued points

$$X(T) \times Y(T) \longrightarrow Z(T) \times Z(T).$$

And so on and so forth.

Finally, given two maps $f, g : X \to Y$ of algebraic spaces over B, if the induced maps $f, g : X(T) \to Y(T)$ are equal for every scheme T over B, then $f = g$, and hence also $f, g : X(Z) \to Y(Z)$ are equal for every third algebraic space Z over B. Hence, for example, to check the axioms for an group algebraic space G over B, it suffices to check commutativity of diagram on T-valued points where T is a scheme over B as we do in Definition 52.5.1 below.

52.4. Equivalence relations

Please refer to Groupoids, Section 35.3 for notation.

Definition 52.4.1. Let $B \to S$ as in Section 52.3. Let U be an algebraic space over B.

1. A *pre-relation* on U over B is any morphism $j : R \to U \times_B U$ of algebraic spaces over B. In this case we set $t = \text{pr}_0 \circ j$ and $s = \text{pr}_1 \circ j$, so that $j = (t, s)$.
2. A *relation* on U over B is a monomorphism $j : R \to U \times_B U$ of algebraic spaces over B.
3. A *pre-equivalence relation* is a pre-relation $j : R \to U \times_B U$ such that the image of $j : R(T) \to U(T) \times U(T)$ is an equivalence relation for all schemes T over B.
4. We say a morphism $R \to U \times_B U$ of algebraic spaces over B is an *equivalence relation on U over B* if and only if for every T over B the T-valued points of R define an equivalence relation on the set of T-valued points of U.

In other words, an equivalence relation is a pre-equivalence relation such that j is a relation.

Lemma 52.4.2. *Let $B \to S$ as in Section 52.3. Let U be an algebraic space over B. Let $j : R \to U \times_B U$ be a pre-relation. Let $g : U' \to U$ be a morphism of algebraic spaces over B. Finally, set*

$$R' = (U' \times_B U') \times_{U \times_B U} R \xrightarrow{j'} U' \times_B U'$$

Then j' is a pre-relation on U' over B. If j is a relation, then j' is a relation. If j is a pre-equivalence relation, then j' is a pre-equivalence relation. If j is an equivalence relation, then j' is an equivalence relation.

Proof. Omitted. □

Definition 52.4.3. Let $B \to S$ as in Section 52.3. Let U be an algebraic space over B. Let $j : R \to U \times_B U$ be a pre-relation. Let $g : U' \to U$ be a morphism of algebraic spaces over B. The pre-relation $j' : R' \to U' \times_B U'$ is called the *restriction*, or *pullback* of the pre-relation j to U'. In this situation we sometimes write $R' = R|_{U'}$.

Lemma 52.4.4. *Let $B \to S$ as in Section 52.3. Let $j : R \to U \times_B U$ be a pre-relation of algebraic spaces over B. Consider the relation on $|U|$ defined by the rule*

$$x \sim y \Leftrightarrow \exists\, r \in |R| : t(r) = x, s(r) = y.$$

If j is a pre-equivalence relation then this is an equivalence relation.

Proof. Suppose that $x \sim y$ and $y \sim z$. Pick $r \in |R|$ with $t(r) = x$, $s(r) = y$ and pick $r' \in |R|$ with $t(r') = y$, $s(r') = z$. We may pick a field K such that r and r' can be represented by morphisms $r, r' : Spec(K) \to R$ with $s \circ r = t \circ r'$. Denote $x = t \circ r$, $y = s \circ r = t \circ r'$, and $z = s \circ r'$, so $x, y, z : Spec(K) \to U$. By construction $(x, y) \in j(R(K))$ and $(y, z) \in j(R(K))$. Since j is a pre-equivalence relation we see that also $(x, z) \in j(R(K))$. This clearly implies that $x \sim z$.

The proof that $\sim$ is reflexive and symmetric is omitted. □

52.5. Group algebraic spaces

Please refer to Groupoids, Section 35.4 for notation.

Definition 52.5.1. Let $B \to S$ as in Section 52.3.

(1) A *group algebraic space over B* is a pair (G, m), where G is an algebraic space over B and $m : G \times_B G \to G$ is a morphism of algebraic spaces over B with the following property: For every scheme T over B the pair $(G(T), m)$ is a group.
(2) A *morphism $\psi : (G, m) \to (G', m')$ of group algebraic spaces over B* is a morphism $\psi : G \to G'$ of algebraic spaces over B such that for every T/B the induced map $\psi : G(T) \to G'(T)$ is a homomorphism of groups.

Let (G, m) be a group algebraic space over the algebraic space B. By the discussion in Groupoids, Section 35.4 we obtain morphisms of algebraic spaces over B (identity) $e : B \to G$ and (inverse) $i : B \to B$ such that for every T the quadruple $(G(T), m, e, i)$ satisfies the axioms of a group.

Let (G, m), (G', m') be group algebraic spaces over B. Let $f : G \to G'$ be a morphism of algebraic spaces over B. It follows from the definition that f is a morphism of group algebraic spaces over B if and only if the following diagram is commutative:

$$\begin{tikzcd} G \times_B G \arrow[r, "f\times f"'] \arrow[d, "m"'] & G' \times_B G' \arrow[d, "m"] \\ G \arrow[r, "f"] & G' \end{tikzcd}$$

Lemma 52.5.2. *Let $B \to S$ as in Section 52.3. Let (G, m) be a group algebraic space over B. Let $B' \to B$ be a morphism of algebraic spaces. The pullback $(G_{B'}, m_{B'})$ is a group algebraic space over B'.*

Proof. Omitted. □

52.6. Properties of group algebraic spaces

In this section we collect some simple properties of group algebraic spaces which hold over any base.

Lemma 52.6.1. *Let S be a scheme. Let B be an algebraic space over S. Let G be a group algebraic space over B. Then $G \to B$ is separated (resp. quasi-separated, resp. locally separated) if and only if the identity morphism $e : B \to G$ is a closed immersion (resp. quasi-compact, resp. an immersion).*

Proof. We recall that by Morphisms of Spaces, Lemma 42.5.7 we have that e is a closed immersion (resp. quasi-compact, resp. an immersion) if $G \to B$ is separated (resp. quasi-separated, resp. locally separated). For the converse, consider the diagram

$$\begin{tikzcd} G \arrow[r, "\Delta_{G/B}"'] \arrow[d] & G \times_B G \arrow[d, "{(g,g') \mapsto m(i(g),g')}"] \\ B \arrow[r, "e"] & G \end{tikzcd}$$

It is an exercise in the functorial point of view in algebraic geometry to show that this diagram is cartesian. In other words, we see that $\Delta_{G/B}$ is a base change of e. Hence if e is a closed immersion (resp. quasi-compact, resp. an immersion) so is $\Delta_{G/B}$, see Spaces, Lemma 40.12.3 (resp. Morphisms of Spaces, Lemma 42.9.3, resp. Spaces, Lemma 40.12.3). $\square$

52.7. Examples of group algebraic spaces

If $G \to S$ is a group scheme over the base scheme S, then the base change G_B to any algebraic space B over S is an group algebraic space over B by Lemma 52.5.2. We will frequently use this in the examples below.

Example 52.7.1. (Multiplicative group algebraic space.) Let $B \to S$ as in Section 52.3. Consider the functor which associates to any scheme T over B the group $\Gamma(T, \mathcal{O}_T^*)$ of units in the global sections of the structure sheaf. This is representable by the group algebraic space

$$\mathbf{G}_{m,B} = B \times_S \mathbf{G}_{m,S}$$

over B. Here $\mathbf{G}_{m,S}$ is the multiplicative group scheme over S, see Groupoids, Example 35.5.1.

Example 52.7.2. (Roots of unity as a group algebraic space.) Let $B \to S$ as in Section 52.3. Let $n \in \mathbf{N}$. Consider the functor which associates to any scheme T over B the subgroup of $\Gamma(T, \mathcal{O}_T^*)$ consisting of nth roots of unity. This is representable by the group algebraic space

$$\mu_{n,B} = B \times_S \mu_{n,S}$$

over B. Here $\mu_{n,S}$ is the group scheme of nth roots of unity over S, see Groupoids, Example 35.5.2.

Example 52.7.3. (Additive group algebraic space.) Let $B \to S$ as in Section 52.3. Consider the functor which associates to any scheme T over B the group $\Gamma(T, \mathcal{O}_T)$ of global sections of the structure sheaf. This is representable by the group algebraic space

$$\mathbf{G}_{a,B} = B \times_S \mathbf{G}_{a,S}$$

over B. Here $\mathbf{G}_{a,S}$ is the additive group scheme over S, see Groupoids, Example 35.5.3.

Example 52.7.4. (General linear group algebraic space.) Let $B \to S$ as in Section 52.3. Let $n \geq 1$. Consider the functor which associates to any scheme T over B the group

$$\text{GL}_n(\Gamma(T, \mathcal{O}_T))$$

of invertible $n \times n$ matrices over the global sections of the structure sheaf. This is representable by the group algebraic space

$$\text{GL}_{n,B} = B \times_S \text{GL}_{n,S}$$

over B. Here $\mathbf{G}_{m,S}$ is the general linear group scheme over S, see Groupoids, Example 35.5.4.

Example 52.7.5. Let $B \to S$ as in Section 52.3. Let $n \geq 1$. The determinant defines a morphisms of group algebraic spaces

$$\det : \mathrm{GL}_{n,B} \longrightarrow \mathbf{G}_{m,B}$$

over B. It is the base change of the determinant morphism over S from Groupoids, Example 35.5.5.

Example 52.7.6. Let $B \to S$ as in Section 52.3. (Constant group algebraic space.) Let G be an abstract group. Consider the functor which associates to any scheme T over B the group of locally constant maps $T \to G$ (where T has the Zariski topology and G the discrete topology). This is representable by the group algebraic space

$$G_B = B \times_S G_S$$

over B. Here G_S is the constant group scheme introduced in Groupoids, Example 35.5.6.

52.8. Actions of group algebraic spaces

Please refer to Groupoids, Section 35.8 for notation.

Definition 52.8.1. Let $B \to S$ as in Section 52.3. Let (G, m) be a group algebraic space over B. Let X be an algebraic space over B.

(1) An *action of G on the algebraic space X/B* is a morphism $a : G \times_B X \to X$ over B such that for every scheme T over B the map $a : G(T) \times X(T) \to X(T)$ defines the structure of a $G(T)$-set on $X(T)$.
(2) Suppose that X, Y are algebraic spaces over B each endowed with an action of G. An *equivariant* or more precisely a *G-equivariant* morphism $\psi : X \to Y$ is a morphism of algebraic spaces over B such that for every T over B the map $\psi : X(T) \to Y(T)$ is a morphism of $G(T)$-sets.

In situation (1) this means that the diagrams

$$\begin{tikzcd} G \times_B G \times_B X \arrow[r, "1_G \times a"] \arrow[d, "m \times 1_X"'] & G \times_B X \arrow[d, "a"] \\ G \times_B X \arrow[r, "a"] & X \end{tikzcd} \qquad \begin{tikzcd} G \times_B X \arrow[r, "a"] & X \\ X \arrow[u, "e \times 1_X"] \arrow[ur, "1_X"'] & \end{tikzcd} \tag{52.8.1.1}$$

are commutative. In situation (2) this just means that the diagram

$$\begin{tikzcd} G \times_B X \arrow[r, "\mathrm{id} \times f"] \arrow[d, "a"'] & G \times_B Y \arrow[d, "a"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

commutes.

Definition 52.8.2. Let $B \to S$, $G \to B$, and $X \to B$ as in Definition 52.8.1. Let $a : G \times_B X \to X$ be an action of G on X/B. We say the action is *free* if for every scheme T over B the action $a : G(T) \times X(T) \to X(T)$ is a free action of the group $G(T)$ on the set $X(T)$.

Lemma 52.8.3. *Situation as in Definition 52.8.2, The action a is free if and only if*

$$G \times_B X \to X \times_B X, \quad (g, x) \mapsto (a(g, x), x)$$

is a monomorphism of algebraic spaces.

Proof. Immediate from the definitions. $\square$

52.9. Principal homogeneous spaces

This section is the analogue of Groupoids, Section 35.9. We suggest reading that section first.

Definition 52.9.1. Let S be a scheme. Let B be an algebraic space over S. Let (G, m) be a group algebraic space over B. Let X be an algebraic space over B, and let $a : G \times_B X \to X$ be an action of G on X.

(1) We say X is a *pseudo G-torsor* or that X is *formally principally homogeneous under G* if the induced morphism $G \times_B X \to X \times_B X$, $(g, x) \mapsto (a(g, x), x)$ is an isomorphism.
(2) A pseudo G-torsor X is called *trivial* if there exists an G-equivariant isomorphism $G \to X$ over B where G acts on G by left multiplication.

It is clear that if $B' \to B$ is a morphism of algebraic spaces then the pullback $X_{B'}$ of a pseudo G-torsor over B is a pseudo $G_{B'}$-torsor over B'.

Lemma 52.9.2. *In the situation of Definition 52.9.1.*

(1) *The algebraic space X is a pseudo G-torsor if and only if for every scheme T over B the set $X(T)$ is either empty or the action of the group $G(T)$ on $X(T)$ is simply transitive.*
(2) *A pseudo G-torsor X is trivial if and only if the morphism $X \to B$ has a section.*

Proof. Omitted. $\square$

Definition 52.9.3. Let S be a scheme. Let B be an algebraic space over S. Let (G, m) be a group algebraic space over B. Let X be a pseudo G-torsor over B.

(1) We say X is a *principal homogeneous space*, or more precisely a *principal homogeneous G-space over B* if there exists a fpqc covering[1] $\{B_i \to B\}_{i \in I}$ such that each $X_{B_i} \to B_i$ has a section (i.e., is a trivial pseudo G_{B_i}-torsor).
(2) Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. We say X is a *G-torsor in the τ topology*, or a *τ G-torsor*, or simply a *τ torsor* if there exists a τ covering $\{B_i \to B\}_{i \in I}$ such that each $X_{B_i} \to B_i$ has a section.
(3) If X is a G-torsor, then we say that it is *quasi-isotrivial* if it is a torsor for the étale topology.
(4) If X is a G-torsor, then we say that it is *locally trivial* if it is a torsor for the Zariski topology.

We sometimes say ``let X be a G-principal homogeneous space over B" to indicate that X is an algebraic space over B equippend with an action of G which turns it into a principal homogeneous space over B. Next we show that this agrees with the notation introduced earlier when both apply.

Lemma 52.9.4. *Let S be a scheme. Let (G, m) be a group algebraic space over S. Let X be an algebraic space over S, and let $a : G \times_S X \to X$ be an action of G on X. Then X is a G-torsor in the $fppf$-topology in the sense of Definition 52.9.3 if and only if X is a G-torsor on $(Sch/S)_{fppf}$ in the sense of Cohomology on Sites, Definition 19.5.1.*

Proof. Omitted. $\square$

[1] The default type of torsor in Groupoids, Definition 35.9.3 is a pseudo torsor which is trivial on an fpqc covering. Since G, as an algebraic space, can be seen a sheaf of groups there already is a notion of a G-torsor which corresponds to fppf-torsor, see Lemma 52.9.4. Hence we use ``principal homogeneous space" for a pseudo torsor which is fpqc locally trivial, and we try to avoid using the word torsor in this situation.

52.10. Equivariant quasi-coherent sheaves

Please compare with Groupoids, Section 35.10.

Definition 52.10.1. Let $B \to S$ as in Section 52.3. Let (G, m) be a group algebraic space over B, and let $a : G \times_B X \to X$ be an action of G on the algebraic space X over B. An *G-equivariant quasi-coherent $\mathcal{O}_X$-module*, or simply a *equivariant quasi-coherent $\mathcal{O}_X$-module*, is a pair $(\mathcal{F}, \alpha)$, where $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module, and α is a $\mathcal{O}_{G\times_B X}$-module map

$$\alpha : a^*\mathcal{F} \longrightarrow \mathrm{pr}_1^*\mathcal{F}$$

where $\mathrm{pr}_1 : G \times_B X \to X$ is the projection such that

(1) the diagram

$$\begin{tikzcd} (1_G \times a)^*\mathrm{pr}_2^*\mathcal{F} \arrow[r, "\mathrm{pr}_{12}^*\alpha"'] & \mathrm{pr}_2^*\mathcal{F} \\ (1_G \times a)^*a^*\mathcal{F} \arrow[u, "(1_G\times a)^*\alpha"] \arrow[r, equal] & (m \times 1_X)^*a^*\mathcal{F} \arrow[u, "(m\times 1_X)^*\alpha"'] \end{tikzcd}$$

is a commutative in the category of $\mathcal{O}_{G\times_B G\times_B X}$-modules, and

(2) the pullback

$$(e \times 1_X)^*\alpha : \mathcal{F} \longrightarrow \mathcal{F}$$

is the identity map.

For explanation compare with the relevant diagrams of Equation (52.8.1.1).

Note that the commutativity of the first diagram garantees that $(e \times 1_X)^*\alpha$ is an idempotent operator on $\mathcal{F}$, and hence condition (2) is just the condition that it is an isomorphism.

Lemma 52.10.2. *Let $B \to S$ as in Section 52.3. Let G be a group algebraic space over B. Let $f : X \to Y$ be a G-equivariant morphism between algebraic spaces over B endowed with G-actions. Then pullback f^* given by $(\mathcal{F}, \alpha) \mapsto (f^*\mathcal{F}, (1_G \times f)^*\alpha)$ defines a functor from the category of G-equivariant sheaves on X to the category of quasi-coherent G-equivariant sheaves on Y.*

Proof. Omitted. □

52.11. Groupoids in algebraic spaces

Please refer to Groupoids, Section 35.11 for notation.

Definition 52.11.1. Let $B \to S$ as in Section 52.3.

(1) A *groupoid in algebraic spaces over B* is a quintuple (U, R, s, t, c) where U and R are algebraic spaces over B, and $s, t : R \to U$ and $c : R \times_{s,U,t} R \to R$ are morphisms of algebraic spaces over B with the following property: For any scheme T over B the quintuple

$$(U(T), R(T), s, t, c)$$

is a groupoid category.

(2) A *morphism $f : (U, R, s, t, c) \to (U', R', s', t', c')$ of groupoids in algebraic spaces over B* is given by morphisms of algebraic spaces $f : U \to U'$ and $f : R \to R'$ over B with the following property: For any scheme T over B the maps f define a functor from the groupoid category $(U(T), R(T), s, t, c)$ to the groupoid category $(U'(T), R'(T), s', t', c')$.

Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Note that there are unique morphisms of algebraic spaces $e : U \to R$ and $i : R \to R$ over B such that for every scheme T over B the induced map $e : U(T) \to R(T)$ is the identity, and $i : R(T) \to R(T)$ is the inverse of the groupoid category. The septuple (U, R, s, t, c, e, i) satisfies commutative diagrams corresponding to each of the axioms (1), (2)(a), (2)(b), (3)(a) and (3)(b) of Groupoids, Section 35.11. Conversely given a septuple with this property the quintuple (U, R, s, t, c) is a groupoid in algebraic spaces over B. Note that i is an isomorphism, and e is a section of both s and t. Moreover, given a groupoid in algebraic spaces over B we denote

$$j = (t, s) : R \longrightarrow U \times_B U$$

which is compatible with our conventions in Section 52.4 above. We sometimes say ``let (U, R, s, t, c, e, i) be a groupoid in algebraic spaces over B'' to stress the existence of identity and inverse.

Lemma 52.11.2. *Let $B \to S$ as in Section 52.3. Given a groupoid in algebraic spaces (U, R, s, t, c) over B the morphism $j : R \to U \times_B U$ is a pre-equivalence relation.*

Proof. Omitted. This is a nice exercise in the definitions. ☐

Lemma 52.11.3. *Let $B \to S$ as in Section 52.3. Given an equivalence relation $j : R \to U$ over B there is a unique way to extend it to a groupoid in algebraic spaces (U, R, s, t, c) over B.*

Proof. Omitted. This is a nice exercise in the definitions. ☐

Lemma 52.11.4. *Let $B \to S$ as in Section 52.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. In the commutative diagram*

$$\begin{tikzcd}
 & U & \\
R \arrow[ur, "t"] \arrow[d, "s"'] & R \times_{s,U,t} R \arrow[l, "pr_0"'] \arrow[r, "c"] \arrow[d, "pr_1"] & R \arrow[ul, "t"'] \arrow[d, "s"] \\
U & R \arrow[l, "t"] \arrow[r, "s"'] & U
\end{tikzcd}$$

the two lower squares are fibre product squares. Moreover, the triangle on top (which is really a square) is also cartesian.

Proof. Omitted. Exercise in the definitions and the functorial point of view in algebraic geometry. ☐

Lemma 52.11.5. *Let $B \to S$ be as in Section 52.3. Let (U, R, s, t, c, e, i) be a groupoid in algebraic spaces over B. The diagram*

(52.11.5.1)
$$\begin{tikzcd}
R \times_{t,U,t} R \arrow[r, shift left, "pr_1"] \arrow[r, shift right, "pr_0"'] \arrow[d, "pr_0 \times c \circ (i,1)"'] & R \arrow[r, "t"] \arrow[d, "id_R"] & U \arrow[d, "id_U"] \\
R \times_{s,U,t} R \arrow[r, shift left, "c"] \arrow[r, shift right, "pr_0"'] \arrow[d, "pr_1"'] & R \arrow[r, "t"] \arrow[d, "s"] & U \\
R \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & U &
\end{tikzcd}$$

is commutative. The two top rows are isomorphic via the vertical maps given. The two lower left squares are cartesian.

Proof. The commutativity of the diagram follows from the axioms of a groupoid. Note that, in terms of groupoids, the top left vertical arrow assigns to a pair of morphisms (α, β) with the same target, the pair of morphisms $(\alpha, \alpha^{-1} \circ \beta)$. In any groupoid this defines a bijection between Arrows $\times_{t,\mathrm{Ob},t}$ Arrows and Arrows $\times_{s,\mathrm{Ob},t}$ Arrows. Hence the second assertion of the lemma. The last assertion follows from Lemma 52.11.4. □

52.12. Quasi-coherent sheaves on groupoids

Please compare with Groupoids, Section 35.12.

Definition 52.12.1. Let $B \to S$ as in Section 52.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. A *quasi-coherent module on* (U, R, s, t, c) is a pair $(\mathcal{F}, \alpha)$, where $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_U$-module, and α is a $\mathcal{O}_R$-module map

$$\alpha : t^*\mathcal{F} \longrightarrow s^*\mathcal{F}$$

such that

(1) the diagram

$$\begin{tikzcd}
& \mathrm{pr}_1^*t^*\mathcal{F} \arrow[r, "\mathrm{pr}_1^*\alpha"'] \arrow[ld, equal] & \mathrm{pr}_1^*s^*\mathcal{F} \arrow[rd, equal] & \\
\mathrm{pr}_0^*s^*\mathcal{F} & & & c^*s^*\mathcal{F} \\
& \mathrm{pr}_0^*t^*\mathcal{F} \arrow[lu, "\mathrm{pr}_0^*\alpha"] \arrow[r, equal] & c^*t^*\mathcal{F} \arrow[ru, "c^*\alpha"'] &
\end{tikzcd}$$

is a commutative in the category of $\mathcal{O}_{R\times_{s,U,t}R}$-modules, and

(2) the pullback

$$e^*\alpha : \mathcal{F} \longrightarrow \mathcal{F}$$

is the identity map.

Compare with the commutative diagrams of Lemma 52.11.4.

The commutativity of the first diagram forces the operator $e^*\alpha$ to be idempotent. Hence the second condition can be reformulated as saying that $e^*\alpha$ is an isomorphism. In fact, the condition implies that α is an isomorphism.

Lemma 52.12.2. *Let S be a scheme, let (U, R, s, t, c) be a groupoid scheme over S. If $(\mathcal{F}, \alpha)$ is a quasi-coherent module on (U, R, s, t, c) then α is an isomorphism.*

Proof. Pull back the commutative diagram of Definition 52.12.1 by the morphism $(i, 1) : R \to R \times_{s,U,t} R$. Then we see that $i^*\alpha \circ \alpha = s^*e^*\alpha$. Pulling back by the morphism $(1, i)$ we obtain the relation $\alpha \circ i^*\alpha = t^*e^*\alpha$. By the second assumption these morphisms are the identity. Hence $i^*\alpha$ is an inverse of α. □

Lemma 52.12.3. *Let $B \to S$ as in Section 52.3. Consider a morphism $f : (U, R, s, t, c) \to (U', R', s', t', c')$ of groupoid in algebraic spacess over B. Then pullback f^* given by*

$$(\mathcal{F}, \alpha) \mapsto (f^*\mathcal{F}, f^*\alpha)$$

defines a functor from the category of quasi-coherent sheaves on (U', R', s', t', c') to the category of quasi-coherent sheaves on (U, R, s, t, c).

Proof. Omitted. □

Lemma 52.12.4. *Let $B \to S$ be as in Section 52.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. The category of quasi-coherent modules on (U, R, s, t, c) has colimits.*

Proof. Let $i \mapsto (\mathcal{F}_i, \alpha_i)$ be a diagram over the index category $\mathcal{I}$. We can form the colimit $\mathcal{F} = colim \mathcal{F}_i$ which is a quasi-coherent sheaf on U, see Properties of Spaces, Lemma 41.26.7. Since colimits commute with pullback we see that $s^*\mathcal{F} = colim\, s^*\mathcal{F}_i$ and similarly $t^*\mathcal{F} = colim\, t^*\mathcal{F}_i$. Hence we can set $\alpha = colim\, \alpha_i$. We omit the proof that $(\mathcal{F}, \alpha)$ is the colimit of the diagram in the category of quasi-coherent modules on (U, R, s, t, c). □

Lemma 52.12.5. *Let $B \to S$ as in Section 52.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. If s, t are flat, then the category of quasi-coherent modules on (U, R, s, t, c) is abelian.*

Proof. Let $\varphi : (\mathcal{F}, \alpha) \to (\mathcal{G}, \beta)$ be a homomorphism of quasi-coherent modules on (U, R, s, t, c). Since s is flat we see that

$$0 \to s^*\mathrm{Ker}(\varphi) \to s^*\mathcal{F} \to s^*\mathcal{G} \to s^*\mathrm{Coker}(\varphi) \to 0$$

is exact and similarly for pullback by t. Hence α and β induce isomorphisms $\kappa : t^*\mathrm{Ker}(\varphi) \to s^*\mathrm{Ker}(\varphi)$ and $\lambda : t^*\mathrm{Coker}(\varphi) \to s^*\mathrm{Coker}(\varphi)$ which satisfy the cocycle condition. Then it is straightforward to verify that $(\mathrm{Ker}(\varphi), \kappa)$ and $(\mathrm{Coker}(\varphi), \lambda)$ are a kernel and cokernel in the category of quasi-coherent modules on (U, R, s, t, c). Moreover, the condition $\mathrm{Coim}(\varphi) = \mathrm{Im}(\varphi)$ follows because it holds over U. □

52.13. Crystals in quasi-coherent sheaves

Let (I, Φ, j) be a pair consisting of a set I and a pre-relation $j : \Phi \to I \times I$. Assume given for every $i \in I$ a scheme X_i and for every $\phi \in \Phi$ a morphisms of schemes $f_\phi : X_{i'} \to X_i$ where $j(\phi) = (i, i')$. Set $X = (\{X_i\}_{i \in I}, \{f_\phi\}_{\phi \in \Phi})$. Define a *crystal in quasi-coherent modules on X* as a rule which associates to every $i \in Ob(\mathcal{I})$ a quasi-coherent sheaf $\mathcal{F}_i$ on X_i and for every $\phi \in \Phi$ with $j(\phi) = (i, i')$ an isomorphism

$$\alpha_\phi : f_\phi^*\mathcal{F}_i \longrightarrow \mathcal{F}_{i'}$$

of quasi-coherent sheaves on $X_{i'}$. These crystals in quasi-coherent modules form an additive category $CQC(X)$[2]. This category has colimits (proof is the same as the proof of Lemma 52.12.4). If all the morphisms f_ϕ are flat, then $CQC(X)$ is abelian (proof is the same as the proof of Lemma 52.12.5). Let κ be a cardinal. We say that a crystal in quasi-coherent modules $\mathcal{F}$ on X is *κ-generated* if each $\mathcal{F}_i$ is κ-generated (see Properties, Definition 23.21.1).

Lemma 52.13.1. *In the situation above, if all the morphisms f_ϕ are flat, then there exists a cardinal κ such that every object $(\{\mathcal{F}_i\}_{i \in I}, \{\alpha_\phi\}_{\phi \in \Phi})$ of $CQC(X)$ is the directed colimit of its κ-generated submodules.*

[2]We could single out a set of triples $\phi, \phi', \phi'' \in \Phi$ with $j(\phi) = (i, i')$, $j(\phi') = (i', i'')$, and $j(\phi'') = (i, i'')$ such that $f_{\phi''} = f_\phi \circ f_{\phi'}$ and require that $\alpha_{\phi'} \circ f_{\phi'}^*\alpha_\phi = \alpha_{\phi''}$ for these triples. This would define an additive subcategory. For example the data (I, Φ) could be the set of objects and arrows of an index category and X could be a diagram of schemes over this index category. The result of Lemma 52.13.1 immediately gives the corresponding result in the subcategory.

Proof. In the the lemma and in this proof a *submodule* of $(\{\mathcal{F}_i\}_{i\in I}, \{\alpha_\phi\}_{\phi\in\Phi})$ means the data of a quasi-coherent submodule $\mathcal{G}_i \subset \mathcal{F}_i$ for all i such that $\alpha_\phi(f_\phi^*\mathcal{G}_i) = \mathcal{G}_{i'}$ as subsheaves of $\mathcal{F}_{i'}$ for all $\phi \in \Phi$. This makes sense because since f_ϕ is flat the pullback f_ϕ^* is exact, i.e., preserves subsheaves. The proof will be a variant to the proof of Properties, Lemma 23.21.3. We urge the reader to read that proof first.

We claim that it suffices to prove the lemma in case all the schemes X_i are affine. To see this let

$$J = \coprod\nolimits_{i\in I}\{U \subset X_i \text{ affine open}\}$$

and let

$$\begin{aligned}\Psi = &\coprod\nolimits_{\phi\in\Phi}\{(U,V) \mid U \subset X_i, V \subset X_{i'} \text{ affine open with } f_\phi(U) \subset V\}\\ &\amalg \coprod\nolimits_{i\in I}\{(U,U') \mid U,U' \subset X_i \text{ affine open with } U \subset U'\}\end{aligned}$$

endowed with the obvious map $\Psi \to J \times J$. Then our $(\mathcal{F}, \alpha)$ induces a crystal in quasi-coherent sheaves $(\{\mathcal{H}_j\}_{j\in J}, \{\beta_\psi\}_{\psi\in\Psi})$ on $Y = (J, \Psi)$ by setting $\mathcal{H}_{(i,U)} = \mathcal{F}_i|_U$ for $(i,U) \in J$ and setting β_ψ for $\psi \in \Psi$ equal to the restriction of α_ϕ to U if $\psi = (\phi, U, V)$ and equal to $\text{id} : (\mathcal{F}_i|_{U'})|_U \to \mathcal{F}_i|_U$ when $\psi = (i, U, U')$. Moreover, submodules of $(\{\mathcal{H}_j\}_{j\in J}, \{\beta_\psi\}_{\psi\in\Psi})$ correspond 1-to-1 with submodules of $(\{\mathcal{F}_i\}_{i\in I}, \{\alpha_\phi\}_{\phi\in\Phi})$. We omit the proof (hint: use Sheaves, Section 6.30). Moreover, it is clear that if κ works for Y, then the same κ works for X (by the definition of κ-generated modules). Hence it suffices to proof the lemma for crystals in quasi-coherent sheaves on Y.

Assume that all the schemes X_i are affine. Let κ be an infinite cardinal larger than the cardinality of I or Φ. Let $(\{\mathcal{F}_i\}_{i\in I}, \{\alpha_\phi\}_{\phi\in\Phi})$ be an object of $CQC(X)$. For each i write $X_i = Spec(A_i)$ and $M_i = \Gamma(X_i, \mathcal{F}_i)$. For every $\phi \in \Phi$ with $j(\phi) = (i,i')$ the map α_ϕ translates into an $A_{i'}$-module isomorphism

$$\alpha_\phi : M_i \otimes_{A_i} A_{i'} \longrightarrow M_{i'}$$

Using the axiom of choice choose a rule

$$(\phi, m) \longmapsto S(\phi, m')$$

where the source is the collection of pairs (ϕ, m') such that $\phi \in \Phi$ with $j(\phi) = (i,i')$ and $m' \in M_{i'}$ and where the output is a finite subset $S(\phi, m') \subset M_i$ so that

$$m' = \alpha_\phi\left(\sum\nolimits_{m\in S(\phi,m')} m \otimes a'_m\right)$$

for some $a'_m \in A_{i'}$.

Having made these choices we claim that any section of any $\mathcal{F}_i$ over any X_i is in a κ-generated submodule. To see this suppose that we are given a collection $\mathcal{S} = \{S_i\}_{i\in I}$ of subsets $S_i \subset M_i$ each with cardinality at most κ. Then we define a new collection $\mathcal{S}' = \{S'_i\}_{i\in I}$ with

$$S'_i = S_i \cup \bigcup\nolimits_{(\phi,m'),\ j(\phi)=(i,i'),\ m'\in S_{i'}} S(\phi, m')$$

Note that each S'_i still has cardinality at most κ. Set $\mathcal{S}^{(0)} = \mathcal{S}$, $\mathcal{S}^{(1)} = \mathcal{S}'$ and by induction $\mathcal{S}^{(n+1)} = (\mathcal{S}^{(n)})'$. Then set $S_i^{(\infty)} = \bigcup_{n\geq 0} S_i^{(n)}$ and $\mathcal{S}^{(\infty)} = \{S_i^{(\infty)}\}_{i\in I}$. By construction, for every $\phi \in \Phi$ with $j(\phi) = (i,i')$ and every $m' \in S_{i'}^{(\infty)}$ we can write m' as a finite linear combination of images $\alpha_\phi(m \otimes 1)$ with $m \in S_i^{(\infty)}$. Thus we see that setting N_i equal to the A_i-submodule of M_i generated by $S_i^{(\infty)}$ the corresponding quasi-coherent submodules $\widetilde{N_i} \subset \mathcal{F}_i$ form a κ-generated submodule. This finishes the proof. □

Lemma 52.13.2. *Let $B \to S$ as in Section 52.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. If s, t are flat, then there exists a set T and a family of objects $(\mathcal{F}_t, \alpha_t)_{t\in T}$ of $QCoh(U, R, s, t, c)$ such that every object $(\mathcal{F}, \alpha)$ is the directed colimit of its submodules isomorphic to one of the objects $(\mathcal{F}_t, \alpha_t)$.*

Proof. This lemma is a generalization of Groupoids, Lemma 35.12.7 which deals with the case of a groupoid in schemes. We can't quite use the same argument, so we use the material on ``crystals of quasi-coherent sheaves'' we developed above.

Choose a scheme W and a surjective étale morphism $W \to U$. Choose a scheme V and a surjective étale morphism $V \to W \times_{U,s} R$. Choose a scheme V' and a surjective étale morphism $V' \to R \times_{t,U} W$. Consider the collection of schemes

$$I = \{W, W \times_U W, V, V', V \times_R V'\}$$

and the set of morphisms of schemes

$$\Phi = \{\text{pr}_i : W \times_U W \to W, V \to W, V' \to W, V \times_R V' \to V, V \times_R V' \to V'\}$$

Set $X = (I, \Phi)$. Recall that we have defined a category $CQC(X)$ of crystals of quasi-coherent sheaves on X. There is a functor

$$QCoh(U, R, s, t, c) \longrightarrow CQC(X)$$

which assigns to $(\mathcal{F}, \alpha)$ the sheaf $\mathcal{F}|_W$ on W, the sheaf $\mathcal{F}|_{W\times_U W}$ on $W \times_U W$, the pull back of $\mathcal{F}$ via $V \to W \times_{U,s} R \to W \to U$ on V, the pull back of $\mathcal{F}$ via $V' \to R \times_{t,U} W \to W \to U$ on V', and finally the pull back of $\mathcal{F}$ via $V \times_R V' \to V \to W \times_{U,s} R \to W \to U$ on $V \times_R V'$. As comparison maps $\{\alpha_\phi\}_{\phi\in\Phi}$ we use the obvious ones (coming from associativity of pullbacks) except for the map $\phi = \text{pr}_{V'} : V \times_R V' \to V'$ we use the pullback of $\alpha : t^*\mathcal{F} \to s^*\mathcal{F}$ to $V \times_R V'$. This makes sense because of the following commutative diagram

$$\begin{tikzcd}
& V \times_R V' \arrow[dl] \arrow[dr] & \\
V \arrow[dr] \arrow[dd] & & V' \arrow[dl] \arrow[dd] \\
& R \arrow[dd, shift right, "s"'] \arrow[dd, shift left, "t"] & \\
W \arrow[dr] & & W \arrow[dl] \\
& U &
\end{tikzcd}$$

The functor displayed above isn't an equivalence of categories. However, since $W \to U$ is surjective étale it is faithful[3]. Since all the morphisms in the diagram above are flat we see that it is an exact functor of abelian categories. Moreover, we claim that given $(\mathcal{F}, \alpha)$ with image $(\{\mathcal{F}_i\}_{i\in I}, \{\alpha_\phi\}_{\phi\in\Phi})$ there is a 1-to-1 correspondence between quasi-coherent submodules of $(\mathcal{F}, \alpha)$ and $(\{\mathcal{F}_i\}_{i\in I}, \{\alpha_\phi\}_{\phi\in\Phi})$. Namely, given a submodule of $(\{\mathcal{F}_i\}_{i\in I}, \{\alpha_\phi\}_{\phi\in\Phi})$ compatibility of the submodule over W with the projection maps $W \times_U W \to W$ will guarantee the submodule comes from a quasi-coherent submodule of $\mathcal{F}$ (by

[3]In fact the functor is fully faithful, but we won't need this.

Properties of Spaces, Proposition 41.29.1) and compatibility with $\alpha_{\mathrm{pr}_{V'}}$ will insure this subsheaf is compatible with α (details omitted).

Choose a cardinal κ as in Lemma 52.13.1 for the system $X = (I, \Phi)$. It is clear from Properties, Lemma 23.21.2 that there is a set of isomorphism classes of κ-generated crystals in quasi-coherent sheaves on X. Hence the result is clear. □

52.14. Groupoids and group spaces

Please compare with Groupoids, Section 35.13.

Lemma 52.14.1. *Let $B \to S$ as in Section 52.3. Let (G, m) be a group algebraic space over B with identity e_G and inverse i_G. Let X be an algebraic space over B and let $a : G \times_B X \to X$ be an action of G on X over B. Then we get a groupoid in algebraic spaces (U, R, s, t, c, e, i) over B in the following manner:*

1. *We set $U = X$, and $R = G \times_B X$.*
2. *We set $s : R \to U$ equal to $(g, x) \mapsto x$.*
3. *We set $t : R \to U$ equal to $(g, x) \mapsto a(g, x)$.*
4. *We set $c : R \times_{s,U,t} R \to R$ equal to $((g, x), (g', x')) \mapsto (m(g, g'), x')$.*
5. *We set $e : U \to R$ equal to $x \mapsto (e_G(x), x)$.*
6. *We set $i : R \to R$ equal to $(g, x) \mapsto (i_G(g), a(g, x))$.*

Proof. Omitted. Hint: It is enough to show that this works on the set level. For this use the description above the lemma describing g as an arrow from v to $a(g, v)$. □

Lemma 52.14.2. *Let $B \to S$ as in Section 52.3. Let (G, m) be a group algebraic space over B. Let X be an algebraic space over B and let $a : G \times_B X \to X$ be an action of G on X over B. Let (U, R, s, t, c) be the groupoid in algebraic spaces constructed in Lemma 52.14.1. The rule $(\mathcal{F}, \alpha) \mapsto (\mathcal{F}, \alpha)$ defines an equivalence of categories between G-equivariant $\mathcal{O}_X$-modules and the category of quasi-coherent modules on (U, R, s, t, c).*

Proof. The assertion makes sense because $t = a$ and $s = \mathrm{pr}_1$ as morphisms $R = G \times_B X \to X$, see Definitions 52.10.1 and 52.12.1. Using the translation in Lemma 52.14.1 the commutativity requirements of the two definitions match up exactly. □

52.15. The stabilizer group algebraic space

Please compare with Groupoids, Section 35.14. Given a groupoid in algebraic spaces we get a group algebraic space as follows.

Lemma 52.15.1. *Let $B \to S$ as in Section 52.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. The algebraic space G defined by the cartesian square*

$$\begin{tikzcd} G \arrow[r] \arrow[d] & R \arrow[d, "j=(t,s)"] \\ U \arrow[r, "\Delta"] & U \times_B U \end{tikzcd}$$

is a group algebraic space over U with compostion law m induced by the composition law c.

Proof. This is true because in a groupoid category the set of self maps of any object forms a group. □

Since Δ is a monomorphism we see that $G = j^{-1}(\Delta_{U/B})$ is a subsheaf of R. Thinking of it in this way, the structure morphism $G = j^{-1}(\Delta_{U/B}) \to U$ is induced by either s or t (it is the same), and m is induced by c.

Definition 52.15.2. Let $B \to S$ as in Section 52.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. The group algebraic space $j^{-1}(\Delta_{U/B}) \to U$ is called the *stabilizer of the groupoid in algebraic spaces* (U, R, s, t, c).

In the literature the stabilizer group algebraic space is often denoted S (because the word stabilizer starts with an ``s" presumably); we cannot do this since we have already used S for the base scheme.

Lemma 52.15.3. *Let $B \to S$ as in Section 52.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B, and let G/U be its stabilizer. Denote R_t/U the algebraic space R seen as an algebraic space over U via the morphism $t : R \to U$. There is a canonical left action*

$$a : G \times_U R_t \longrightarrow R_t$$

induced by the composition law c.

Proof. In terms of points over T/B we define $a(g, r) = c(g, r)$. $\square$

52.16. Restricting groupoids

Please refer to Groupoids, Section 35.15 for notation.

Lemma 52.16.1. *Let $B \to S$ as in Section 52.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let $g : U' \to U$ be a morphism of algebraic spaces. Consider the following diagram*

$$\begin{tikzcd}
R' \arrow[r] \arrow[d] \arrow[rr, bend left, "s'"] \arrow[dd, bend right, "t'"'] & R \times_{s,U} U' \arrow[r] \arrow[d] & U' \arrow[d, "g"] \\
U' \times_{U,t} R \arrow[r] \arrow[d] & R \arrow[r, "s"] \arrow[d, "t"] & U \\
U' \arrow[r, "g"] & U &
\end{tikzcd}$$

where all the squares are fibre product squares. Then there is a canonical composition law $c' : R' \times_{s',U',t'} R' \to R'$ such that (U', R', s', t', c') is a groupoid in algebraic spaces over B and such that $U' \to U$, $R' \to R$ defines a morphism $(U', R', s', t', c') \to (U, R, s, t, c)$ of groupoids in algebraic spaces over B. Moreover, for any scheme T over B the functor of groupoids

$$(U'(T), R'(T), s', t', c') \to (U(T), R(T), s, t, c)$$

is the restriction (see Groupoids, Section 35.15) of $(U(T), R(T), s, t, c)$ via the map $U'(T) \to U(T)$.

Proof. Omitted. $\square$

Definition 52.16.2. Let $B \to S$ as in Section 52.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let $g : U' \to U$ be a morphism of algebraic spaces over B. The morphism of groupoids in algebraic spaces $(U', R', s', t', c') \to (U, R, s, t, c)$ constructed in Lemma 52.16.1 is called the *restriction of* (U, R, s, t, c) *to* U'. We sometime use the notation $R' = R|_{U'}$ in this case.

Lemma 52.16.3. *The notions of restricting groupoids and (pre-)equivalence relations defined in Definitions 52.16.2 and 52.4.3 agree via the constructions of Lemmas 52.11.2 and 52.11.3.*

Proof. What we are saying here is that R' of Lemma 52.16.1 is also equal to

$$R' = (U' \times_B U') \times_{U \times_B U} R \longrightarrow U' \times_B U'$$

In fact this might have been a clearer way to state that lemma. □

52.17. Invariant subspaces

In this section we discuss briefly the notion of an invariant subspace.

Definition 52.17.1. Let $B \to S$ as in Section 52.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over the base B.

(1) We say an open subspace $W \subset U$ is *R-invariant* if $t(s^{-1}(W)) \subset W$.
(2) A locally closed subspace $Z \subset U$ is called *R-invariant* if $t^{-1}(Z) = s^{-1}(Z)$ as locally closed subspaces of R.
(3) A monomorphism of algebraic spaces $T \to U$ is *R-invariant* if $T \times_{U,t} R = R \times_{s,U} T$ as algebraic spaces over R.

For an open subspace $W \subset U$ the R-invariance is also equivalent to requiring that $s^{-1}(W) = t^{-1}(W)$. If $W \subset U$ is R-invariant then the restriction of R to W is just $R_W = s^{-1}(W) = t^{-1}(W)$. Similarly, if $Z \subset U$ is an R-invariant locally closed subspace, then the restriction of R to Z is just $R_Z = s^{-1}(Z) = t^{-1}(Z)$.

Lemma 52.17.2. *Let $B \to S$ as in Section 52.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B.*

(1) *If s and t are open, then for every open $W \subset U$ the open $s(t^{-1}(W))$ is R-invariant.*
(2) *If s and t are open and quasi-compact, then U has an open covering consisting of R-invariant quasi-compact open subspaces.*

Proof. Assume s and t open and $W \subset U$ open. Since s is open we see that $W' = s(t^{-1}(W))$ is an open subspace of U. Now it is quite easy to using the functorial point of view that this is an R-invariant open subset of U, but we are going to argue this directly by some diagrams, since we think it is instructive. Note that $t^{-1}(W')$ is the image of the morphism

$$A := t^{-1}(W) \times_{s|_{t^{-1}(W)}, U, t} R \xrightarrow{\text{pr}_1} R$$

and that $s^{-1}(W')$ is the image of the morphism

$$B := R \times_{s, U, s|_{t^{-1}(W)}} t^{-1}(W) \xrightarrow{\text{pr}_0} R.$$

The algebraic spaces A, B on the left of the arrows above are open subspaces of $R \times_{s,U,t} R$ and $R \times_{s,U,s} R$ respectively. By Lemma 52.11.4 the diagram

$$\begin{tikzcd} R \times_{s,U,t} R \arrow[rr, "(\text{pr}_1, c)"'] \arrow[dr, "\text{pr}_1"'] & & R \times_{s,U,s} R \arrow[dl, "\text{pr}_0"] \\ & R & \end{tikzcd}$$

is commutative, and the horizontal arrow is an isomorphism. Moreover, it is clear that $(\text{pr}_1, c)(A) = B$. Hence we conclude $s^{-1}(W') = t^{-1}(W')$, and W' is R-invariant. This proves (1).

Assume now that s, t are both open and quasi-compact. Then, if $W \subset U$ is a quasi-compact open, then also $W' = s(t^{-1}(W))$ is a quasi-compact open, and invariant by the discussion above. Letting W range over images of affines étale over U we see (2). □

52.18. Quotient sheaves

Let S be a scheme, and let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation over B. For each scheme S' over S we can take the equivalence relation $\sim_{S'}$ generated by the image of $j(S') : R(S') \to U(S') \times U(S')$. Hence we get a presheaf

$$(52.18.0.1) \qquad \begin{array}{ccc} (Sch/S)_{fppf}^{opp} & \longrightarrow & Sets, \\ S' & \longmapsto & U(S')/\sim_{S'} \end{array}$$

Note that since j is a morphism of algebraic spaces over B and into $U \times_B U$ there is a canonical transformation of presheaves from the presheaf (52.18.0.1) to B.

Definition 52.18.1. Let $B \to S$ and the pre-relation $j : R \to U \times_B U$ be as above. In this setting the *quotient sheaf* U/R associated to j is the sheafification of the presheaf (52.18.0.1) on $(Sch/S)_{fppf}$. If $j : R \to U \times_B U$ comes from the action of a group algebraic space G over B on U as in Lemma 52.14.1 then we denote the quotient sheaf U/G.

This means exactly that the diagram

$$R \rightrightarrows U \longrightarrow U/R$$

is a coequalizer diagram in the category of sheaves of sets on $(Sch/S)_{fppf}$. Again there is a canonical map of sheaves $U/R \to B$ as j is a morphism of algebraic spaces over B into $U \times_B U$.

Remark 52.18.2. A variant of the construction above would have been to sheafify the functor

$$\begin{array}{ccc} (Spaces/B)_{fppf}^{opp} & \longrightarrow & Sets, \\ X & \longmapsto & U(X)/\sim_X \end{array}$$

where now $\sim_X \subset U(X) \times U(X)$ is the equivalence relation generated by the image of $j : R(X) \to U(X) \times U(X)$. Here of course $U(X) = Mor_B(X, U)$ and $R(X) = Mor_B(X, R)$. In fact, the result would have been the same, via the identifications of (insert future reference in Topologies of Spaces here).

Definition 52.18.3. In the situation of Definition 52.18.1. We say that the pre-relation j has a *quotient representable by an algebraic space* if the sheaf U/R is an algebraic space. We say that the pre-relation j has a *representable quotient* if the sheaf U/R is representable by a scheme. We will say a groupoid in algebraic spaces (U, R, s, t, c) over B has a *representable quotient* (resp. *quotient representable by an algebraic space* if the quotient U/R with $j = (t, s)$ is representable (resp. an algebraic space).

If the quotient U/R is representable by M (either a scheme or an algebraic space over S), then it comes equipped with a canonical structure morphism $M \to B$ as we've seen above.

The following lemma characterizes M representing the quotient. It applies for example if $U \to M$ is flat, of finite presentation and surjective, and $R \cong U \times_M U$.

Lemma 52.18.4. *In the situation of Definition 52.18.1. Assume there is an algebraic space M over S, and a morphism $U \to M$ such that*

(1) *the morphism $U \to M$ equalizes s, t,*
(2) *the map $U \to M$ is a surjection of sheaves, and*

(3) *the induced map* $(t, s) : R \to U \times_M U$ *is a surjection of sheaves.*

In this case M *represents the quotient sheaf* U/R.

Proof. Condition (1) says that $U \to M$ factors through U/R. Condition (2) says that $U/R \to M$ is surjective as a map of sheaves. Condition (3) says that $U/R \to M$ is injective as a map of sheaves. Hence the lemma follows. □

The following lemma is wrong if we do not require j to be a pre-equivalence relation (but just a pre-relation say).

Lemma 52.18.5. *Let* S *be a scheme. Let* B *be an algebraic space over* S. *Let* $j : R \to U \times_B U$ *be a pre-equivalence relation over* B. *For a scheme* S' *over* S *and* $a, b \in U(S')$ *the following are equivalent:*

(1) *a and b map to the same element of* $(U/R)(S')$, *and*
(2) *there exists an fppf covering* $\{f_i : S_i \to S'\}$ *of* S' *and morphisms* $r_i : S_i \to R$ *such that* $a \circ f_i = s \circ r_i$ *and* $b \circ f_i = t \circ r_i$.

In other words, in this case the map of sheaves

$$R \longrightarrow U \times_{U/R} U$$

is surjective.

Proof. Omitted. Hint: The reason this works is that the presheaf (52.18.0.1) in this case is really given by $T \mapsto U(T)/j(R(T))$ as $j(R(T)) \subset U(T) \times U(T)$ is an equivalence relation, see Definition 52.4.1. □

Lemma 52.18.6. *Let* S *be a scheme. Let* B *be an algebraic space over* S. *Let* $j : R \to U \times_B U$ *be a pre-relation over* B *and* $g : U' \to U$ *a morphism of algebraic spaces over* B. *Let* $j' : R' \to U' \times_B U'$ *be the restriction of* j *to* U'. *The map of quotient sheaves*

$$U'/R' \longrightarrow U/R$$

is injective. If $U' \to U$ *is surjective as a map of sheaves, for example if* $\{g : U' \to U\}$ *is an fppf covering (see Topologies on Spaces, Definition 44.4.1), then* $U'/R' \to U/R$ *is an isomorphism of sheaves.*

Proof. Suppose $\xi, \xi' \in (U'/R')(S')$ are sections which map to the same section of U/R. Then we can find an fppf covering $\mathcal{S} = \{S_i \to S'\}$ of S' such that $\xi|_{S_i}, \xi'|_{S_i}$ are given by $a_i, a_i' \in U'(S_i)$. By Lemma 52.18.5 and the axioms of a site we may after refining $\mathcal{T}$ assume there exist morphisms $r_i : S_i \to R$ such that $g \circ a_i = s \circ r_i$, $g \circ a_i' = t \circ r_i$. Since by construction $R' = R \times_{U \times_S U} (U' \times_S U')$ we see that $(r_i, (a_i, a_i')) \in R'(S_i)$ and this shows that a_i and a_i' define the same section of U'/R' over S_i. By the sheaf condition this implies $\xi = \xi'$.

If $U' \to U$ is a surjective map of sheaves, then $U'/R' \to U/R$ is surjective also. Finally, if $\{g : U' \to U\}$ is a fppf covering, then the map of sheaves $U' \to U$ is surjective, see Topologies on Spaces, Lemma 44.4.4. □

Lemma 52.18.7. *Let* S *be a scheme. Let* B *be an algebraic space over* S. *Let* (U, R, s, t, c) *be a groupoid in algebraic spaces over* B. *Let* $g : U' \to U$ *a morphism of algebraic spaces over* B. *Let* (U', R', s', t', c') *be the restriction of* (U, R, s, t, c) *to* U'. *The map of quotient sheaves*

$$U'/R' \longrightarrow U/R$$

is injective. If the composition

$$\begin{tikzcd} U' \times_{g,U,t} R \arrow[r, "pr_1"'] \arrow[rr, bend left, "h"] & R \arrow[r, "s"'] & U \end{tikzcd}$$

is a surjection of fppf sheaves then the map is bijective. This holds for example if $\{h : U' \times_{g,U,t} R \to U\}$ *is an* $fppf$*-covering, or if* $U' \to U$ *is a surjection of sheaves, or if* $\{g : U' \to U\}$ *is a covering in the fppf topology.*

Proof. Injectivity follows on combining Lemmas 52.11.2 and 52.18.6. To see surjectivity (see Sites, Section 9.11 for a characterization of surjective maps of sheaves) we argue as follows. Suppose that T is a scheme and $\sigma \in U/R(T)$. There exists a covering $\{T_i \to T\}$ such that $\sigma|_{T_i}$ is the image of some element $f_i \in U(T_i)$. Hence we may assume that σ if the image of $f \in U(T)$. By the assumption that h is a surjection of sheaves, we can find an fppf covering $\{\varphi_i : T_i \to T\}$ and morphisms $f_i : T_i \to U' \times_{g,U,t} R$ such that $f \circ \varphi_i = h \circ f_i$. Denote $f_i' = \mathrm{pr}_0 \circ f_i : T_i \to U'$. Then we see that $f_i' \in U'(T_i)$ maps to $g \circ f_i' \in U(T_i)$ and that $g \circ f_i' \sim_{T_i} h \circ f_i = f \circ \varphi_i$ notation as in (52.18.0.1). Namely, the element of $R(T_i)$ giving the relation is $\mathrm{pr}_1 \circ f_i$. This means that the restriction of σ to T_i is in the image of $U'/R'(T_i) \to U/R(T_i)$ as desired.

If $\{h\}$ is an fppf covering, then it induces a surjection of sheaves, see Topologies on Spaces, Lemma 44.4.4. If $U' \to U$ is surjective, then also h is surjective as s has a section (namely the neutral element e of the groupoid scheme). $\square$

52.19. Quotient stacks

In this section and the next few sections we describe a kind of generalization of Section 52.18 above and Groupoids, Section 35.17. It is different in the following way: We are going to take quotient stacks instead of quotient sheaves.

Let us assume we have a scheme S, and algebraic space B over S and a groupoid in algebraic spaces (U, R, s, t, c) over B. Given these data we consider the functor

(52.19.0.1)
$$\begin{matrix} (Sch/S)_{fppf}^{opp} & \longrightarrow & Groupoids \\ S' & \longmapsto & (U(S'), R(S'), s, t, c) \end{matrix}$$

By Categories, Example 4.34.1 this ``presheaf in groupoids'' corresponds to a category fibred in groupoids over $(Sch/S)_{fppf}$. In this chapter we will denote this

$$[U/_pR] \to (Sch/S)_{fppf}$$

where the subscript $_p$ is there to distinguish from the quotient stack.

Definition 52.19.1. Quotient stacks. Let $B \to S$ be as above.

(1) Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. The *quotient stack*
$$p : [U/R] \longrightarrow (Sch/S)_{fppf}$$
of (U, R, s, t, c) is the stackification (see Stacks, Lemma 50.9.1) of the category fibred in groupoids $[U/_pR]$ over $(Sch/S)_{fppf}$ associated to (52.19.0.1).

(2) Let (G, m) be a group algebraic space over B. Let $a : G \times_B X \to X$ be an action of G on an algebraic space over B. The *quotient stack*
$$p : [X/G] \longrightarrow (Sch/S)_{fppf}$$
is the quotient stack associated to the groupoid in algebraic spaces $(X, G \times_B X, s, t, c)$ over B of Lemma 52.14.1.

Thus $[U/R]$ and $[X/G]$ are stacks in groupoids over $(Sch/S)_{fppf}$. These stacks will be very important later on and hence it makes sense to give a detailed description. Recall that given an algebraic space X over S we use the notation $\mathcal{S}_X \to (Sch/S)_{fppf}$ to denote the stack in sets associated to the sheaf X, see Categories, Lemma 4.35.6 and Stacks, Lemma 50.6.2.

Lemma 52.19.2. *Assume $B \to S$ and (U, R, s, t, c) as in Definition 52.19.1 (1). There are canonical 1-morphisms $\pi : \mathcal{S}_U \to [U/R]$, and $[U/R] \to \mathcal{S}_B$ of stacks in groupoids over $(Sch/S)_{fppf}$. The composition $\mathcal{S}_U \to \mathcal{S}_B$ is the 1-morphism associated to the structure morphism $U \to B$.*

Proof. During this proof let us denote $[U/_pR]$ the category fibred in groupoids associated to the presheaf in groupoids (52.19.0.1). By construction of the stackification there is a 1-morphism $[U/_pR] \to [U/R]$. The 1-morphism $\mathcal{S}_U \to [U/R]$ is simply the composition $\mathcal{S}_U \to [U/_pR] \to [U/R]$, where the first arrow associates to the scheme S'/S and morphism $x : S' \to U$ over S the object $x \in U(S')$ of the fibre category of $[U/_pR]$ over S'.

To construct the 1-morphism $[U/R] \to \mathcal{S}_B$ it is enough to construct the 1-morphism $[U/_pR] \to \mathcal{S}_B$, see Stacks, Lemma 50.9.2. On objects over S'/S we just use the map

$$U(S') \longrightarrow B(S')$$

coming from the structure morphism $U \to B$. And clearly, if $a \in R(S')$ is an ``arrow'' with source $s(a) \in U(S')$ and target $t(a) \in U(S')$, then since s and t are morphisms *over* B these both map to the same element $\overline{a}$ of $B(S')$. Hence we can map an arrow $a \in R(S')$ to the identity morphism of $\overline{a}$. (This is good because the fibre category $(\mathcal{S}_B)_{S'}$ only contains identities.) We omit the verification that this rule is compatible with pullback on these split fibred categories, and hence defines a 1-morphism $[U/_pR] \to \mathcal{S}_B$ as desired.

We omit the verification of the last statement. □

Lemma 52.19.3. *Assumptions and notation as in Lemma 52.19.2. There exists a canonical 2-morphism $\alpha : \pi \circ s \to \pi \circ t$ making the diagram*

$$\begin{tikzcd} \mathcal{S}_R \arrow[r, "s"'] \arrow[d, "t"'] & \mathcal{S}_U \arrow[d, "\pi"] \\ \mathcal{S}_U \arrow[r, "\pi"] & {[U/R]} \end{tikzcd}$$

2-commutative.

Proof. Let S' be a scheme over S. Let $r : S' \to R$ be a morphism over S. Then $r \in R(S')$ is an isomorphism between the objects $s \circ r, t \circ r \in U(S')$. Moreover, this construction is compatible with pullbacks. This gives a canonical 2-morphism $\alpha_p : \pi_p \circ s \to \pi_p \circ t$ where $\pi_p : \mathcal{S}_U \to [U/_pR]$ is as in the proof of Lemma 52.19.2. Thus even the diagram

$$\begin{tikzcd} \mathcal{S}_R \arrow[r, "s"'] \arrow[d, "t"] & \mathcal{S}_U \arrow[d, "\pi_p"] \\ \mathcal{S}_U \arrow[r, "\pi_p"] & {[U/_pR]} \end{tikzcd}$$

is 2-commutative. Thus a fortiori the diagram of the lemma is 2-commutative. □

Remark 52.19.4. In future chapters we will use the ambiguous notation where instead of writing $\mathcal{S}_X$ for the stack in sets associated to X we simply write X. Using this notation the diagram of Lemma 52.19.3 becomes the familiar diagram

$$\begin{tikzcd} R \arrow[r, "s"'] \arrow[d, "t"'] & U \arrow[d, "\pi"] \\ U \arrow[r, "\pi"] & {[U/R]} \end{tikzcd}$$

In the following sections we will show that this diagram has many good properties. In particular we will show that it is a 2-fibre product (Section 52.21) and that it is close to being a 2-coequalizer of s and t (Section 52.22).

52.20. Functoriality of quotient stacks

A morphism of groupoids in algebraic spaces gives an associated morphism of quotient stacks.

Lemma 52.20.1. *Let S be a scheme. Let B be an algebraic space over S. Let $f : (U, R, s, t, c) \to (U', R', s', t', c')$ be a morphism of groupoids in algebraic spacess over B. Then f induces a canonical 1-morphism of quotient stacks*

$$[f] : [U/R] \longrightarrow [U'/R'].$$

Proof. Denote $[U/_pR]$ and $[U'/_pR']$ the categories fibred in groupoids over the base site $(Sch/S)_{fppf}$ associated to the functors (52.19.0.1). It is clear that f defines a 1-morphism $[U/_pR] \to [U'/_pR']$ which we can compose with the stackyfication functor for $[U'/R']$ to get $[U/_pR] \to [U'/R']$. Then, by the universal property of the stackyfication functor $[U/_pR] \to [U/R]$, see Stacks, Lemma 50.9.2 we get $[U/R] \to [U'/R']$. $\square$

Let $B \to S$ and $f : (U, R, s, t, c) \to (U', R', s', t', c')$ be as in Lemma 52.20.1. In this situation, we define a third groupoid in algebraic spaces over B as follows, using the language of T-valued points where T is a (varying) scheme over B:

(1) $U'' = U \times_{f,U',t'} R'$ so that a T-valued point is a pair (u, r') with $f(u) = t'(r')$,
(2) $R'' = R \times_{f\circ s,U',t'} R'$ so that a T-valued point is a pair (r, r') with $f(s(r)) = t'(r')$,
(3) $s'' : R'' \to U''$ is given by $s''(r, r') = (s(r), r')$,
(4) $t'' : R'' \to U''$ is given by $t''(r, r') = (t(r), c'(f(r), r'))$,
(5) $c'' : R'' \times_{s'',U'',t''} R'' \to R''$ is given by $c''((r_1, r_1'), (r_2, r_2')) = (c(r_1, r_2), r_2')$.

The formula for c'' makes sense as $s''(r_1, r_1') = t''(r_2, r_2')$. It is clear that c'' is associative. The identity e'' is given by $e''(u, r) = (e(u), r)$. The inverse of (r, r') is given by $(i(r), c'(f(r), r'))$. Thus we do indeed get a groupoid in algebraic spaces over B.

Clearly the maps $U'' \to U$ and $R'' \to R$ define a morphism $g : (U'', R'', s'', t'', c'') \to (U, R, s, t, c)$ of groupoids in algebraic spaces over B. Moreover, the maps $U'' \to U'$, $(u, r') \mapsto s'(r')$ and $R'' \to U'$, $(r, r') \mapsto s'(r')$ show that in fact $(U'', R'', s'', t'', c'')$ is a groupoid in algebraic spaces over U'.

Lemma 52.20.2. *Notation and assumption as in Lemma 52.20.1. Let $(U'', R'', s'', t'', c'')$ be the groupoid in algebraic spaces over B constructed above. There is a 2-commutative* *square*

$$\begin{tikzcd} {[U''/R'']} \arrow[r, "{[g]}"'] \arrow[d] & {[U/R]} \arrow[d, "{[f]}"] \\ \mathcal{S}_{U'} \arrow[r] & {[U'/R']} \end{tikzcd}$$

which identifies $[U''/R'']$ *with the* 2*-fibre product.*

Proof. The maps $[f]$ and $[g]$ come from an application of Lemma 52.20.1 and the other two maps come from Lemma 52.19.2 (and the fact that $(U'', R'', s'', t'', c'')$ lives over U'). To show the 2-fibre product property, it suffices to prove the lemma for the diagram

$$\begin{tikzcd} {[U''/_pR'']} \arrow[r, "{[g]}"'] \arrow[d] & {[U/_pR]} \arrow[d, "{[f]}"] \\ \mathcal{S}_{U'} \arrow[r] & {[U'/_pR']} \end{tikzcd}$$

of categories fibred in groupoids, see Stacks, Lemma 50.9.3. In other words, it suffices to show that an object of the 2-fibre product $\mathcal{S}_U \times_{[U'/_pR']} [U/_pR]$ over T corresponds to a T-valued point of U'' and similarly for morphisms. And of course this is exactly how we constructed U'' and R'' in the first place.

In detail, an object of $\mathcal{S}_U \times_{[U'/_pR']} [U/_pR]$ over T is a triple (u', u, r') where u' is a T-valued point of U', u is a T-valued point of U, and r' is a morphism from u' to $f(u)$ in $[U'/R']_T$, i.e., r' is a T-valued point of R with $s'(r') = u'$ and $t'(r') = f(u)$. Clearly we can forget about u' without losing information and we see that these objects are in one-to-one correspondence with T-valued points of R''.

Similarly for morphisms: Let (u'_1, u_1, r'_1) and (u'_2, u_2, r'_2) be two objects of the fibre product over T. Then a morphism from (u'_2, u_2, r'_2) to (u'_1, u_1, r'_1) is given by $(1, r)$ where $1 : u'_1 \to u'_2$ means simply $u'_1 = u'_2$ (this is so because $\mathcal{S}_U$ is fibred in sets), and r is a T-valued point of R with $s(r) = u_2$, $t(r) = u_1$ and moreover $c'(f(r), r'_2) = r'_1$. Hence the arrow

$$(1, r) : (u'_2, u_2, r'_2) \to (u'_1, u_1, r'_1)$$

is completely determined by knowing the pair (r, r'_2). Thus the functor of arrows is represented by R'', and moreover the morphisms s'', t'', and c'' clearly correspond to source, target and composition in the 2-fibre product $\mathcal{S}_U \times_{[U'/_pR']} [U/_pR]$. $\square$

52.21. The 2-cartesian square of a quotient stack

In this section we compute the $Isom$-sheaves for a quotient stack and we deduce that the defining diagram of a quotient stack is a 2-fibre product.

Lemma 52.21.1. *Assume $B \to S$, (U, R, s, t, c) and $\pi : \mathcal{S}_U \to [U/R]$ are as in Lemma 52.19.2. Let S' be a scheme over S. Let $x, y \in Ob([U/R]_{S'})$ be objects of the quotient stack over S'. If $x = \pi(x')$ and $y = \pi(y')$ for some morphisms $x', y' : S' \to U$, then*

$$Isom(x, y) = S' \times_{(y',x'), U \times_S U} R$$

as sheaves over S'.

Proof. Let $[U/_pR]$ be the category fibred in groupoids associated to the presheaf in groupoids (52.19.0.1) as in the proof of Lemma 52.19.2. By construction the sheaf $Isom(x, y)$ is the sheaf associated to the presheaf $Isom(x', y')$. On the other hand, by definition of morphisms in $[U/_pR]$ we have

$$Isom(x', y') = S' \times_{(y',x'),U\times_S U} R$$

and the right hand side is an algebraic space, therefore a sheaf. □

Lemma 52.21.2. *Assume $B \to S$, (U, R, s, t, c), and $\pi : \mathcal{S}_U \to [U/R]$ are as in Lemma 52.19.2. The 2-commutative square*

$$\begin{tikzcd} \mathcal{S}_R \arrow[r, "s"'] \arrow[d, "t"'] & \mathcal{S}_U \arrow[d, "\pi"] \\ \mathcal{S}_U \arrow[r, "\pi"] & {[U/R]} \end{tikzcd}$$

of Lemma 52.19.3 is a 2-fibre product of stacks in groupoids of $(Sch/S)_{fppf}$.

Proof. According to Stacks, Lemma 50.5.6 the lemma makes sense. It also tells us that we have to show that the functor

$$\mathcal{S}_R \longrightarrow \mathcal{S}_U \times_{[U/R]} \mathcal{S}_U$$

which maps $r : T \to R$ to $(T, t(r), s(r), \alpha(r))$ is an equivalence, where the right hand side is the 2-fibre product as described in Categories, Lemma 4.29.3. This is, after spelling out the definitions, exactly the content of Lemma 52.21.1. (Alternative proof: Work out the meaning of Lemma 52.20.2 in this situation will give you the result also.) □

Lemma 52.21.3. *Assume $B \to S$ and (U, R, s, t, c) are as in Definition 52.19.1 (1). For any scheme T over S and objects x, y of $[U/R]$ over T the sheaf $Isom(x, y)$ on $(Sch/T)_{fppf}$ has the following property: There exists a fppf covering $\{T_i \to T\}_{i\in I}$ such that $Isom(x, y)|_{(Sch/T_i)_{fppf}}$ is representable by an algebraic space.*

Proof. Follows immediately from Lemma 52.21.1 and the fact that both x and y locally in the fppf topology come from objects of $\mathcal{S}_U$ by construction of the quotient stack. □

52.22. The 2-coequalizer property of a quotient stack

On a groupoid we have the composition, which leads to a cocyle condition for the canonical 2-morphism of the lemma above. To give the precise formulation we will use the notation introduced in Categories, Sections 4.25 and 4.26.

Lemma 52.22.1. *Assumptions and notation as in Lemmas 52.19.2 and 52.19.3. The vertical composition of*

$$\begin{tikzcd}[column sep=huge] \mathcal{S}_{R\times_{s,U,t}R} \arrow[r, bend left=30, "\pi\circ s\circ pr_1=\pi\circ s\circ c"{name=U}] \arrow[r, "\pi\circ t\circ pr_1=\pi\circ s\circ pr_0"'{name=M}] \arrow[r, bend right=30, "\pi\circ t\circ pr_0=\pi\circ t\circ c"'{name=D}] & {[U/R]} \arrow[from=U, to=M, phantom, "\Downarrow\alpha\star id_{pr_1}"] \arrow[from=M, to=D, phantom, "\Downarrow\alpha\star id_{pr_0}"] \end{tikzcd}$$

is the 2-morphism $\alpha \star id_c$. In a formula $\alpha \star id_c = (\alpha \star id_{pr_0}) \circ (\alpha \star id_{pr_1})$.

Proof. We make two remarks:

(1) The formula $\alpha \star \mathrm{id}_c = (\alpha \star \mathrm{id}_{\mathrm{pr}_0}) \circ (\alpha \star \mathrm{id}_{\mathrm{pr}_1})$ only makes sense if you realize the *equalities* $\pi \circ s \circ \mathrm{pr}_1 = \pi \circ s \circ c$, $\pi \circ t \circ \mathrm{pr}_1 = \pi \circ s \circ \mathrm{pr}_0$, and $\pi \circ t \circ \mathrm{pr}_0 = \pi \circ t \circ c$. Namely, the second one implies the vertical composition $\circ$ makes sense, and the other two garantee the two sides of the formula are 2-morphisms with the same source and target.
(2) The reason the lemma holds is that composition in the category fibred in groupoids $[U/_pR]$ associated to the presheaf in groupoids (52.19.0.1) comes from the composition law $c : R \times_{s,U,t} R \to R$.

We omit the proof of the lemma. □

Note that, in the situation of the lemma, we actually have the equalities $s \circ \mathrm{pr}_1 = s \circ c$, $t \circ \mathrm{pr}_1 = s \circ \mathrm{pr}_0$, and $t \circ \mathrm{pr}_0 = t \circ c$ before composing with π. Hence the formula in the lemma below makes sense in exactly the same way that the formula in the lemma above makes sense.

Lemma 52.22.2. *Assumptions and notation as in Lemmas 52.19.2 and 52.19.3. The* 2*-commutative diagram of Lemma 52.19.3 is a* 2*-coequalizer in the following sense: Given*

(1) *a stack in groupoids $\mathcal{X}$ over $(Sch/S)_{fppf}$,*
(2) *a* 1*-morphism $f : \mathcal{S}_U \to \mathcal{X}$, and*
(3) *a* 2*-arrow $\beta : f \circ s \to f \circ t$*

such that

$$\beta \star id_c = (\beta \star id_{pr_0}) \circ (\beta \star id_{pr_1})$$

then there exists a 1*-morphism $[U/R] \to \mathcal{X}$ which makes the diagram*

$$\begin{tikzcd} \mathcal{S}_R \arrow[r, "s"'] \arrow[d, "t"] & \mathcal{S}_U \arrow[d] \arrow[ddr, "f"] & \\ \mathcal{S}_U \arrow[r] \arrow[drr, "f"'] & {[U/R]} \arrow[dr] & \\ & & \mathcal{X} \end{tikzcd}$$

2*-commute.*

Proof. Suppose given $\mathcal{X}$, f and β as in the lemma. By Stacks, Lemma 50.9.2 it suffices to construct a 1-morphism $g : [U/_pR] \to \mathcal{X}$. First we note that the 1-morphism $\mathcal{S}_U \to [U/_pR]$ is bijective on objects. Hence on objects we can set $g(x) = f(x)$ for $x \in Ob(\mathcal{S}_U) = Ob([U/_pR])$. A morphism $\varphi : x \to y$ of $[U/_pR]$ arises from a commutative diagram

$$\begin{tikzcd} S_2 \arrow[r, "x"'] \arrow[dr, "\varphi"'] \arrow[dd, "h"'] & U \\ & R \arrow[u, "s"'] \arrow[d, "t"] \\ S_1 \arrow[r, "y"] & U. \end{tikzcd}$$

Thus we can set $g(\varphi)$ equal to the composition

$$
\begin{tikzcd}
f(x) \arrow[r, equal] \arrow[rrrrrd] & f(s \circ \varphi) \arrow[r, equal] & (f \circ s)(\varphi) \arrow[r, "\beta"] & (f \circ t)(\varphi) \arrow[r, equal] & f(\varphi \circ t) \arrow[r, equal] & f(y \circ h) \arrow[d] \\
 & & & & & f(y).
\end{tikzcd}
$$

The vertical arrow is the result of applying the functor f to the canonical morphism $y \circ h \to y$ in $\mathcal{S}_U$ (namely, the strongly cartesian morphism lifting h with target y). Let us verify that f so defined is compatible with composition, at least on fibre categories. So let S' be a scheme over S, and let $a : S' \to R \times_{s,U,t} R$ be a morphism. In this situation we set $x = s \circ \mathrm{pr}_1 \circ a = s \circ c \circ a$, $y = t \circ \mathrm{pr}_1 \circ a = s \circ \mathrm{pr}_0 \circ a$, and $z = t \circ \mathrm{pr}_0 \circ a = t \circ \mathrm{pr}_0 \circ c$ to get a commutative diagram

$$
\begin{tikzcd}
x \arrow[rr, "c \circ a"] \arrow[rd, "\mathrm{pr}_1 \circ a"'] & & z \\
 & y \arrow[ru, "\mathrm{pr}_0 \circ a"'] &
\end{tikzcd}
$$

in the fibre category $[U/_pR]_{S'}$. Moreover, any commutative triangle in this fibre category has this form. Then we see by our definitions above that f maps this to a commutative diagram if and only if the diagram

$$
\begin{tikzcd}
 & (f \circ s)(c \circ a) \arrow[r, "\beta"] & (f \circ t)(c \circ a) \arrow[rd, equal] & \\
(f \circ s)(\mathrm{pr}_1 \circ a) \arrow[ru, equal] \arrow[rd, "\beta"] & & & (f \circ t)(\mathrm{pr}_0 \circ a) \\
 & (f \circ t)(\mathrm{pr}_1 \circ a) \arrow[r, equal] & (f \circ s)(\mathrm{pr}_0 \circ a) \arrow[ru, "\beta"] &
\end{tikzcd}
$$

is commutative which is exactly the condition expressed by the formula in the lemma. We omit the verification that f maps identities to idenities and is compatible with composition for arbitrary morphisms. □

52.23. Explicit description of quotient stacks

In order to formulate the result we need to introduce some notation. Assume $B \to S$ and (U, R, s, t, c) are as in Definition 52.19.1 (1). Let T be a scheme over S. Let $\mathcal{T} = \{T_i \to T\}_{i \in I}$ be an fppf covering. A $[U/R]$-*descent datum* relative to $\mathcal{T}$ is given by a system (u_i, r_{ij}) where

(1) for each i a morphism $u_i : T_i \to U$, and
(2) for each i, j a morphism $r_{ij} : T_i \times_T T_j \to R$

such that

(a) as morphisms $T_i \times_T T_j \to U$ we have
$$s \circ r_{ij} = u_i \circ \mathrm{pr}_0 \quad \text{and} \quad t \circ r_{ij} = u_j \circ \mathrm{pr}_1,$$
(b) as morphisms $T_i \times_T T_j \times_T T_k \to R$ we have
$$c \circ (r_{jk} \circ \mathrm{pr}_{12}, r_{ij} \circ \mathrm{pr}_{01}) = r_{ik} \circ \mathrm{pr}_{02}.$$

A *morphism* $(u_i, r_{ij}) \to (u'_i, r'_{ij})$ *between two* $[U/R]$-*descent data* over the same covering $\mathcal{T}$ is a collection $(r_i : T_i \to R)$ such that

(α) as morphisms $T_i \to U$ we have

$$u_i = s \circ r_i \quad \text{and} \quad u'_i = t \circ r_i$$

(β) as morphisms $T_i \times_T T_j \to R$ we have

$$c \circ (r'_{ij}, r_i \circ \text{pr}_0) = c \circ (r_j \circ \text{pr}_1, r_{ij}).$$

There is a natural composition law on morphisms of descent data relative to a fixed covering and we obtain a category of descent data. This category is a groupoid. Finally, if $\mathcal{T}' = \{T'_j \to T\}_{j \in J}$ is a second fppf covering which refines $\mathcal{T}$ then there is a notion of pullback of descent data. This is particularly easy to describe explicitly in this case. Namely, if $\alpha : J \to I$ and $\varphi_j : T'_j \to T_{\alpha(i)}$ is the morphism of coverings, then the pullback of the descent datum $(u_i, r_{ii'})$ is simply

$$(u_{\alpha(i)} \circ \varphi_j, r_{\alpha(j)\alpha(j')} \circ \varphi_j \times \varphi_{j'}).$$

Pullback defined in this manner defines a functor from the category of descent data over $\mathcal{T}$ to the category of descend data over $\mathcal{T}'$.

Lemma 52.23.1. *Assume $B \to S$ and (U, R, s, t, c) are as in Definition 52.19.1 (1). Let $\pi : \mathcal{S}_U \to [U/R]$ be as in Lemma 52.19.2. Let T be a scheme over S.*

1. *for every object x of the fibre category $[U/R]_T$ there exists an fppf covering $\{f_i : T_i \to T\}_{i \in I}$ such that $f_i^* x \cong \pi(u_i)$ for some $u_i \in U(T_i)$,*
2. *the composition of the isomorphisms*
$$\pi(u_i \circ pr_0) = pr_0^* \pi(u_i) \cong pr_0^* f_i^* x \cong pr_1^* f_j^* x \cong pr_1^* \pi(u_j) = \pi(u_j \circ pr_1)$$
are of the form $\pi(r_{ij})$ for certain morphisms $r_{ij} : T_i \times_T T_j \to R$,
3. *the system (u_i, r_{ij}) forms a $[U/R]$-descent datum as defined above,*
4. *any $[U/R]$-descent datum (u_i, r_{ij}) arises in this manner,*
5. *if x corresponds to (u_i, r_{ij}) as above, and $y \in Ob([U/R]_T)$ corresponds to (u'_i, r'_{ij}) then there is a canonical bijection*
$$Mor_{[U/R]_T}(x, y) \longleftrightarrow \left\{ \begin{matrix} \textit{morphisms } (u_i, r_{ij}) \to (u'_i, r'_{ij}) \\ \textit{of } [U/R]\textit{-descent data} \end{matrix} \right\}$$
6. *this correspondence is compatible with refinements of fppf coverings.*

Proof. Statement (1) is part of the construction of the stackyfication. Part (2) follows from Lemma 52.21.1. We omit the verification of (3). Part (4) is a translation of the fact that in a stack all descent data are effective. We omit the verifications of (5) and (6). $\square$

52.24. Restriction and quotient stacks

In this section we study what happens to the quotient stack when taking a restriction.

Lemma 52.24.1. *Notation and assumption as in Lemma 52.20.1. The morphism of quotient stacks*

$$[f] : [U/R] \longrightarrow [U'/R']$$

is fully faithful if and only if R is the restriction of R via the morphism $f : U \to U'$.

Proof. Let x, y be objects of $[U/R]$ over a scheme T/S. Let x', y' be the images of x, y in the category $[U'/' R]_T$. The functor $[f]$ is fully faithful if and only if the map of sheaves

$$Isom(x, y) \longrightarrow Isom(x', y')$$

is an isomorphism for every T, x, y. We may test this locally on T (in the fppf topology). Hence, by Lemma 52.23.1 we may assume that x, y come from $a, b \in U(T)$. In that case we see that x', y' correspond to $f \circ a, f \circ b$. By Lemma 52.21.1 the displayed map of sheaves in this case becomes

$$T \times_{(a,b), U \times_B U} R \longrightarrow T \times_{f \circ a, f \circ b, U' \times_B U'} R'.$$

This is an isomorphism if R is the restriction, because in that case $R = (U \times_B U) \times_{U' \times_B U'} R'$, see Lemma 52.16.3 and its proof. Conversely, if the last displayed map is an isomorphism for all T, a, b, then it follows that $R = (U \times_B U) \times_{U' \times_B U'} R'$, i.e., R is the restriction of R'. $\square$

Lemma 52.24.2. *Notation and assumption as in Lemma 52.20.1. The morphism of quotient stacks*

$$[f] : [U/R] \longrightarrow [U'/R']$$

is an equivalence if and only if

(1) *(U, R, s, t, c) is the restriction of (U', R', s', t', c') via $f : U \to U'$, and*
(2) *the map*

$$\begin{tikzcd} U \times_{f, U', t'} R' \arrow[r, "pr_1"'] \arrow[rr, bend left, "h"] & R' \arrow[r, "s'"'] & U' \end{tikzcd}$$

is a surjection of sheaves.

Part (2) holds for example if $\{h : U \times_{f, U', t'} R' \to U'\}$ is an fppf covering, or if $f : U \to U'$ is a surjection of sheaves, or if $\{f : U \to U'\}$ is an fppf covering.

Proof. We already know that part (1) is equivalent to fully faithfulness by Lemma 52.24.1. Hence we may assume that (1) holds and that $[f]$ is fully faithful. Our goal is to show, under these assumptions, that $[f]$ is an equivalence if and only if (2) holds. We may use Stacks, Lemma 50.4.8 which characterizes equivalences.

Assume (2). We will use Stacks, Lemma 50.4.8 to prove $[f]$ is an equivalence. Suppose that T is a scheme and $x' \in Ob([U'/R']_T)$. There exists a covering $\{g_i : T_i \to T\}$ such that $g_i^* x'$ is the image of some element $a_i' \in U'(T_i)$, see Lemma 52.23.1. Hence we may assume that x' is the image of $a' \in U'(T)$. By the assumption that h is a surjection of sheaves, we can find an fppf covering $\{\varphi_i : T_i \to T\}$ and morphisms $b_i : T_i \to U \times_{g, U', t'} R'$ such that $a' \circ \varphi_i = h \circ b_i$. Denote $a_i = \mathrm{pr}_0 \circ b_i : T_i \to U$. Then we see that $a_i \in U(T_i)$ maps to $f \circ a_i \in U'(T_i)$ and that $f \circ a_i \cong_{T_i} h \circ b_i = a' \circ \varphi_i$, where $\cong_{T_i}$ denotes isomorphism in the fibre category $[U'/R']_{T_i}$. Namely, the element of $R'(T_i)$ giving the isomorphism is $\mathrm{pr}_1 \circ b_i$. This means that the restriction of x to T_i is in the essential image of the functor $[U/R]_{T_i} \to [U'/R']_{T_i}$ as desired.

Assume $[f]$ is an equivalence. Let $\xi' \in [U'/R']_{U'}$ denote the object corresponding to the identity morphism of U'. Applying Stacks, Lemma 50.4.8 we see there exists an fppf covering $\mathcal{U}' = \{g_i' : U_i' \to U'\}$ such that $(g_i')^* \xi' \cong [f](\xi_i)$ for some ξ_i in $[U/R]_{U_i'}$. After refining the covering $\mathcal{U}'$ (using Lemma 52.23.1) we may assume ξ_i comes from a morphism $a_i : U_i' \to U$. The fact that $[f](\xi_i) \cong (g_i')^* \xi'$ means that, after possibly refining the covering $\mathcal{U}'$ once more, there exist morphisms $r_i' : U_i' \to R'$ with $t' \circ r_i' = f \circ a_i$ and $s' \circ r_i' = \mathrm{id}_{U'} \circ g_i'$. Picture

$$\begin{tikzcd} U \arrow[d, "f"] & & U_i' \arrow[ll, "a_i"'] \arrow[dl, "r_i'"'] \arrow[d, "g_i'"] \\ U' & R' \arrow[l, "t'"'] \arrow[r, "s'"] & U' \end{tikzcd}$$

Thus $(a_i, r'_i) : U'_i \to U \times_{g,U',t'} R'$ are morphisms such that $h \circ (a_i, r'_i) = g'_i$ and we conclude that $\{h : U \times_{g,U',t'} R' \to U'\}$ can be refined by the fppf covering $\mathcal{U}'$ which means that h induces a surjection of sheaves, see Topologies on Spaces, Lemma 44.4.4.

If $\{h\}$ is an fppf covering, then it induces a surjection of sheaves, see Topologies on Spaces, Lemma 44.4.4. If $U' \to U$ is surjective, then also h is surjective as s has a section (namely the neutral element e of the groupoid in algebraic spaces). $\square$

Lemma 52.24.3. *Notation and assumption as in Lemma 52.20.1. Assume that*

$$\begin{tikzcd} R \arrow[r, "f"'] \arrow[d, "s"'] & R' \arrow[d, "s'"] \\ U \arrow[r, "f"] & U' \end{tikzcd}$$

is cartesian. Then

$$\begin{tikzcd} \mathcal{S}_U \arrow[r] \arrow[d] & {[U/R]} \arrow[d, "{[f]}"] \\ \mathcal{S}_{U'} \arrow[r] & {[U'/R']} \end{tikzcd}$$

is a 2*-fibre product square.*

Proof. Applying the inverse isomorphisms $i : R \to R$ and $i' : R' \to R'$ to the (first) cartesian diagram of the statement of the lemma we see that

$$\begin{tikzcd} R \arrow[r, "f"'] \arrow[d, "t"'] & R' \arrow[d, "t'"] \\ U \arrow[r, "f"] & U' \end{tikzcd}$$

is cartesian as well. By Lemma 52.20.2 we have a 2-fibre square

$$\begin{tikzcd} {[U''/R'']} \arrow[r] \arrow[d] & {[U/R]} \arrow[d] \\ \mathcal{S}_{U'} \arrow[r] & {[U'/R']} \end{tikzcd}$$

where $U'' = U \times_{f,U',t'} R'$ and $R'' = R \times_{f \circ s,U',t'} R'$. By the above we see that $(t, f) : R \to U''$ is an isomorphism, and that

$$R'' = R \times_{f \circ s,U',t'} R' = R \times_{s,U} U \times_{f,U',t'} R' = R \times_{s,U,t} \times R.$$

Explicitly the isomorphism $R \times_{s,U,t} R \to R''$ is given by the rule $(r_0, r_1) \mapsto (r_0, f(r_1))$. Moreover, s'', t'', c'' translate into the maps

$$R \times_{s,U,t} R \to R, \quad s''(r_0, r_1) = r_1, \quad t''(r_0, r_1) = c(r_0, r_1)$$

and

$$\begin{array}{rccc} c'' : & (R \times_{s,U,t} R) \times_{s'',R,t''} (R \times_{s,U,t} R) & \longrightarrow & R \times_{s,U,t} R, \\ & ((r_0, r_1), (r_2, r_3)) & \longmapsto & (c(r_0, r_2), r_3). \end{array}$$

Precomposing with the isomorphism

$$R \times_{s,U,s} R \longrightarrow R \times_{s,U,t} R, \quad (r_0, r_1) \longmapsto (c(r_0, i(r_1)), r_1)$$

we see that t'' and s'' turn into pr_0 and pr_1 and that c'' turns into $\mathrm{pr}_{02} : R \times_{s,U,s} R \times_{s,U,s} R \to R \times_{s,U,s} R$. Hence we see that there is an isomorphism $[U''/R''] \cong [R/R \times_{s,U,s} R]$ where as a groupoid in algebraic spaces $(R, R \times_{s,U,s} R, s'', t'', c'')$ is the restriction of the trivial groupoid $(U, U, \mathrm{id}, \mathrm{id}, \mathrm{id})$ via $s : R \to U$. Since $s : R \to U$ is a surjection of fppf sheaves (as it has a right inverse) the morphism

$$[U''/R''] \cong [R/R \times_{s,U,s} R] \longrightarrow [U/U] = \mathcal{S}_U$$

is an equivalence by Lemma 52.24.2. This proves the lemma. □

52.25. Inertia and quotient stacks

The (relative) inertia stack of a stack in groupoids is defined in Stacks, Section 50.7. The actual construction, in the setting of fibred categories, and some of its properties is in Categories, Section 4.31.

Lemma 52.25.1. *Assume $B \to S$ and (U, R, s, t, c) as in Definition 52.19.1 (1). Let G/U be the stabilizer group algebraic space of the groupoid (U, R, s, t, c, e, i), see Definition 52.15.2. Set $R' = R \times_{s,U} G$ and set*

(1) $s' : R' \to G$, $(r, g) \mapsto g$,
(2) $t' : R' \to G$, $(r, g) \mapsto c(r, c(g, i(r)))$,
(3) $c' : R' \times_{s',G,t'} R' \to R'$, $((r_1, g_1), (r_2, g_2)) \mapsto (c(r_1, r_2), g_1)$.

Then (G, R', s', t', c') is a groupoid in algebraic spaces over B and

$$\mathcal{I}_{[U/R]} = [G/R'].$$

i.e., the associated quotient stack is the inertia stack of $[U/R]$.

Proof. By Stacks, Lemma 50.8.5 it suffices to prove that $\mathcal{I}_{[U/_pR]} = [G/_pR']$. Let T be a scheme over S. Recall that an object of the inertia fibred category of $[U/_pR]$ over T is given by a pair (x, g) where x is an object of $[U/_pR]$ over T and g is an automorphism of x in its fibre category over T. In other words, $x : T \to U$ and $g : T \to R$ such that $x = s \circ g = t \circ g$. This means exactly that $g : T \to G$. A morphism in the inertia fibred category from $(x, g) \to (y, h)$ over T is given by $r : T \to R$ such that $s(r) = x$, $t(r) = y$ and $c(r, g) = c(h, r)$, see the commutative diagram in Categories, Lemma 4.31.1. In a formula

$$h = c(r, c(g, i(r))) = c(c(r, g), i(r)).$$

The notation $s(r)$, etc is a short hand for $s \circ r$, etc. The composition of $r_1 : (x_2, g_2) \to (x_1, g_1)$ and $r_2 : (x_1, g_1) \to (x_2, g_2)$ is $c(r_1, r_2) : (x_1, g_1) \to (x_3, g_3)$.

Note that in the above we could have written g in stead of (x, g) for an object of $\mathcal{I}_{[U/_pR]}$ over T as x is the image of g under the structure morphism $G \to U$. Then the morphisms $g \to h$ in $\mathcal{I}_{[U/_pR]}$ over T correspond exactly to morphisms $r' : T \to R'$ with $s'(r') = g$ and $t'(r') = h$. Moreover, the composition corresponds to the rule explained in (3). Thus the lemma is proved. □

Lemma 52.25.2. *Assume $B \to S$ and (U, R, s, t, c) as in Definition 52.19.1 (1). Let G/U be the stabilizer group algebraic space of the groupoid (U, R, s, t, c, e, i), see Definition 52.15.2. There is a canonical 2-cartesian diagram*

$$\begin{tikzcd} \mathcal{S}_G \arrow[r] \arrow[d] & \mathcal{S}_U \arrow[d] \\ \mathcal{I}_{[U/R]} \arrow[r] & [U/R] \end{tikzcd}$$

of stacks in groupoids of $(Sch/S)_{fppf}$.

Proof. By Lemma 52.24.3 it suffices to prove that the morphism $s' : R' \to G$ of Lemma 52.25.1 isomorphic to the base change of s by the structure morphism $G \to U$. This base change property is clear from the construction of s'. □

52.26. Gerbes and quotient stacks

In this section we relate quotient stacks to the discussion Stacks, Section 50.11 and especially gerbes as defined in Stacks, Definition 50.11.4. The stacks in groupoids occuring in this section are generally speaking not algebraic stacks!

Lemma 52.26.1. *Notation and assumption as in Lemma 52.20.1. The morphism of quotient stacks*

$$[f] : [U/R] \longrightarrow [U'/R']$$

turns $[U/R]$ into a gerbe over $[U'/R']$ if $f : U \to U'$ and $R \to R'|_U$ are surjective maps of fppf sheaves. Here $R'|_U$ is the restriction of R' to U via $f : U \to U'$.

Proof. We will verify that Stacks, Lemma 50.11.3 properties (2) (a) and (2) (b) hold. Property (2)(a) holds because $U \to U'$ is a surjective map of sheaves (use Lemma 52.23.1 to see that objects in $[U'/R']$ locally come from U'). To prove (2)(b) let x, y be objects of $[U/R]$ over a scheme T/S. Let x', y' be the images of x, y in the category $[U'/'R]_T$. Condition (2)(b) requires us to check the map of sheaves

$$Isom(x, y) \longrightarrow Isom(x', y')$$

on $(Sch/T)_{fppf}$ is surjective. To see this we may work fppf locally on T and assume that come from $a, b \in U(T)$. In that case we see that x', y' correspond to $f \circ a, f \circ b$. By Lemma 52.21.1 the displayed map of sheaves in this case becomes

$$T \times_{(a,b),U\times_B U} R \longrightarrow T \times_{f\circ a, f\circ b, U'\times_B U'} R' = T \times_{(a,b),U\times_B U} R'|_U.$$

Hence the assumption that $R \to R'|_U$ is a surjective map of fppf sheaves on $(Sch/S)_{fppf}$ implies the desired surjectivity. □

Lemma 52.26.2. *Let S be a scheme. Let B be an algebraic space over S. Let G be a group algebraic space over B. Endow B with the trivial action of G. The morphism*

$$[B/G] \longrightarrow \mathcal{S}_B$$

(Lemma 52.19.2) turns $[B/G]$ into a gerbe over B.

Proof. Immediate from Lemma 52.26.1 as the morphisms $B \to B$ and $B \times_B G \to B$ are surjective as morphisms of sheaves. □

52.27. Quotient stacks and change of big site

We suggest skipping this section on a first reading. Pullbacks of stacks are defined in Stacks, Section 50.12.

Lemma 52.27.1. *Suppose given big sites Sch_{fppf} and Sch'_{fppf}. Assume that Sch_{fppf} is contained in Sch'_{fppf}, see Topologies, Section 30.10. Let $S \in Ob(Sch_{fppf})$. Let $B, U, R \in Sh((Sch/S)_{fppf})$ be algebraic spaces, and let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let $f : (Sch'/S)_{fppf} \to (Sch/S)_{fppf}$ the morphism of sites corresponding to the inclusion functor $u : Sch_{fppf} \to Sch'_{fppf}$. Then we have a canonical equivalence*

$$[f^{-1}U/f^{-1}R] \longrightarrow f^{-1}[U/R]$$

of stacks in groupoids over $(Sch'/S)_{fppf}$.

Proof. Note that $f^{-1}B, f^{-1}U, f^{-1}R \in Sh((Sch'/S)_{fppf})$ are algebraic spaces by Spaces, Lemma 40.15.1 and hence $(f^{-1}U, f^{-1}R, f^{-1}s, f^{-1}t, f^{-1}c)$ is a groupoid in algebraic spaces over $f^{-1}B$. Thus the statement makes sense.

The category $u_p[U/_pR]$ is the localization of the category $u_{pp}[U/_pR]$ at right multiplicative system I of morphisms. An object of $u_{pp}[U/_pR]$ is a triple

$$(T', \phi : T' \to T, x)$$

where $T' \in Ob((Sch'/S)_{fppf})$, $T \in Ob((Sch/S)_{fppf})$, ϕ is a morphism of schemes over S, and $x : T \to U$ is a morphism of sheaves on $(Sch/S)_{fppf}$. Note that the morphism of schemes $\phi : T' \to T$ is the same thing as a morphism $\phi : T' \to u(T)$, and since $u(T)$ represents $f^{-1}T$ it is the same thing as a morphism $T' \to f^{-1}T$. Moreover, as f^{-1} on algebraic spaces is fully faithful, see Spaces, Lemma 40.15.2, we may think of x as a morphism $x : f^{-1}T \to f^{-1}U$ as well. From now on we will make such identifications without further mention. A morphism

$$(a, a', \alpha) : (T'_1, \phi_1 : T'_1 \to T_1, x_1) \longrightarrow (T'_2, \phi_2 : T'_2 \to T_2, x_2)$$

of $u_{pp}[U/_pR]$ is a commutative diagram

$$\begin{tikzcd} & & U \\ T'_1 \arrow[r, "\phi_1"'] \arrow[d, "a'"'] & T_1 \arrow[r, "\alpha"'] \arrow[d, "a"'] \arrow[ru, "x_1"] & R \arrow[u, "s"'] \arrow[d, "t"] \\ T'_2 \arrow[r, "\phi_2"] & T_2 \arrow[r, "x_2"] & U \end{tikzcd}$$

and such a morphism is an element of I if and only if $T'_1 = T'_2$ and $a' = \text{id}$. We define a functor

$$u_{pp}[U/_pR] \longrightarrow [f^{-1}U/_pf^{-1}R]$$

by the rules

$$(T', \phi : T' \to T, x) \longmapsto (x \circ \phi : T' \to f^{-1}U)$$

on objects and

$$(a, a', \alpha) \longmapsto (\alpha \circ \phi_1 : T'_1 \to f^{-1}R)$$

on morphisms as above. It is clear that elements of I are transformed into isomorphisms as $(f^{-1}U, f^{-1}R, f^{-1}s, f^{-1}t, f^{-1}c)$ is a groupoid in algebraic spaces over $f^{-1}B$. Hence this functor factors in a canonical way through a functor

$$u_p[U/_pR] \longrightarrow [f^{-1}U/_pf^{-1}R]$$

Applying stackification we obtain a functor of stacks

$$f^{-1}[U/R] \longrightarrow [f^{-1}U/f^{-1}R]$$

over $(Sch'/S)_{fppf}$, as by Stacks, Lemma 50.12.11 the stack $f^{-1}[U/R]$ is the stackification of $u_p[U/_pR]$.

At this point we have a morphism of stacks, and to verify that it is an equivalence it suffices to show that it is fully faithful and that objects are locally in the essential image, see Stacks, Lemmas 50.4.7 and 50.4.8. The statement on objects holds as $f^{-1}R$ admits a surjective étale morphism $f^{-1}W \to f^{-1}R$ for some object W of $(Sch/S)_{fppf}$. To show that the functor is ``full'', it suffices to show that morphisms are locally in the image of the functor which holds as $f^{-1}U$ admits a surjective étale morphism $f^{-1}W \to f^{-1}U$ for some object W of $(Sch/S)_{fppf}$. We omit the proof that the functor is faithful. $\square$

52.28. Separation conditions

This really means conditions on the morphism $j : R \to U \times_B U$ when given a groupoid in algebraic spaces (U, R, s, t, c) over B. As in the previous section we first formulate the corresponding diagram.

Lemma 52.28.1. *Let $B \to S$ be as in Section 52.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let $G \to U$ be the stabilizer group algebraic space. The commutative diagram*

$$\begin{tikzcd} R \arrow[r, "f \mapsto (f, s(f))"'] \arrow[d, "\Delta_{R/U\times_B U}"] & R \times_{s,U} U \arrow[r] \arrow[d] & U \arrow[d] \\ R \times_{(U \times_B U)} R \arrow[r, "(f,g) \mapsto (f, f^{-1} \circ g)"] & R \times_{s,U} G \arrow[r] & G \end{tikzcd}$$

the two left horizontal arrows are isomorphisms and the right square is a fibre product square.

Proof. Omitted. Exercise in the definitions and the functorial point of view in algebraic geometry. $\square$

Lemma 52.28.2. *Let $B \to S$ be as in Section 52.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let $G \to U$ be the stabilizer group algebraic space.*

(1) *The following are equivalent*
   (a) *$j : R \to U \times_B U$ is separated,*
   (b) *$G \to U$ is separated, and*
   (c) *$e : U \to G$ is a closed immersion.*
(2) *The following are equivalent*
   (a) *$j : R \to U \times_B U$ is locally separated,*
   (b) *$G \to U$ is locally separated, and*
   (c) *$e : U \to G$ is an immersion.*
(3) *The following are equivalent*
   (a) *$j : R \to U \times_B U$ is quasi-separated,*
   (b) *$G \to U$ is quasi-separated, and*
   (c) *$e : U \to G$ is quasi-compact.*

Proof. The group algebraic space $G \to U$ is the base change of $R \to U \times_B U$ by the diagonal morphism $U \to U \times_B U$, see Lemma 52.15.1. Hence if j is separated (resp. locally separated, resp. quasi-separated), then $G \to U$ is separated (resp. locally separated, resp. quasi-separated). See Morphisms of Spaces, Lemma 42.5.4. Thus (a) $\Rightarrow$ (b) in (1), (2), and (3).

Conversely, if $G \to U$ is separated (resp. locally separated, resp. quasi-separated), then the morphism $e : U \to G$, as a section of the structure morphism $G \to U$ is a closed immersion (resp. an immersion, resp. quasi-compact), see Morphisms of Spaces, Lemma 42.5.7. Thus (b) $\Rightarrow$ (c) in (1), (2), and (3).

If e is a closed immersion (resp. an immersion, resp. quasi-compact) then by the result of Lemma 52.28.1 (and Spaces, Lemma 40.12.3, and Morphisms of Spaces, Lemma 42.9.3) we see that $\Delta_{R/U\times_B U}$ is a closed immersion (resp. an immerion, resp. quasi-compact). Thus (c) $\Rightarrow$ (a) in (1), (2), and (3). $\square$

52.29. Other chapters

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(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Commutative Algebra
(8) Brauer Groups
(9) Sites and Sheaves
(10) Homological Algebra
(11) Derived Categories
(12) More on Algebra
(13) Smoothing Ring Maps
(14) Simplicial Methods
(15) Sheaves of Modules
(16) Modules on Sites
(17) Injectives
(18) Cohomology of Sheaves
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(23) Properties of Schemes
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(36) More on Groupoid Schemes
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CHAPTER 53

More on Groupoids in Spaces

53.1. Introduction

This chapter is devoted to advanced topics on groupoids in algebraic spaces. Even though the results are stated in terms of groupoids in algebraic spaces, the reader should keep in mind the 2-cartesian diagram

(53.1.0.1)
$$\begin{tikzcd} R \arrow[r] \arrow[d] & U \arrow[d] \\ U \arrow[r] & {[U/R]} \end{tikzcd}$$

where $[U/R]$ is the quotient stack, see Groupoids in Spaces, Remark 52.19.4. Many of the results are motivated by thinking about this diagram. See for example the beautiful paper [**KM97a**] by Keel and Mori.

53.2. Notation

We continue to abide by the conventions and notation introduced in Groupoids in Spaces, Section 52.3.

53.3. Useful diagrams

We briefly restate the results of Groupoids in Spaces, Lemmas 52.11.4 and 52.11.5 for easy reference in this chapter. Let S be a scheme. Let B be an algebraic space over S. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. In the commutative diagram

(53.3.0.2)
$$\begin{tikzcd} & U & \\ R \arrow[ur, "t"] \arrow[d, "s"'] & R \times_{s,U,t} R \arrow[l, "\mathrm{pr}_0"'] \arrow[r, "c"] \arrow[d, "\mathrm{pr}_1"] & R \arrow[ul, "t"'] \arrow[d, "s"] \\ U & R \arrow[l, "t"'] \arrow[r, "s"] & U \end{tikzcd}$$

the two lower squares are fibre product squares. Moreover, the triangle on top (which is really a square) is also cartesian.

The diagram

$$
\begin{tikzcd}
R \times_{t,U,t} R \arrow[r, shift left, "\mathrm{pr}_1"] \arrow[r, shift right, "\mathrm{pr}_0"'] \arrow[d, "\mathrm{pr}_0 \times c \circ (i,1)"'] & R \arrow[r, "t"] \arrow[d, "\mathrm{id}_R"] & U \arrow[d, "\mathrm{id}_U"] \\
R \times_{s,U,t} R \arrow[r, shift left, "c"] \arrow[r, shift right, "\mathrm{pr}_0"'] \arrow[d, "\mathrm{pr}_1"'] & R \arrow[r, "t"] \arrow[d, "s"] & U \\
R \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & U &
\end{tikzcd}
\tag{53.3.0.3}
$$

is commutative. The two top rows are isomorphic via the vertical maps given. The two lower left squares are cartesian.

53.4. Properties of groupoids

This section is the analogue of More on Groupoids, Section 36.5. The reader is strongly encouraged to read that section first.

The following lemma is the analogue of More on Groupoids, Lemma 36.5.4.

Lemma 53.4.1. *Let $B \to S$ be as in Section 53.2. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let $\tau \in \{fppf, étale, smooth, syntomic\}$. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is τ-local on the target (Descent on Spaces, Definition 45.9.1). Assume $\{s : R \to U\}$ and $\{t : R \to U\}$ are coverings for the τ-topology. Let $W \subset U$ be the maximal open subspace such that $s^{-1}(W) \to W$ has property $\mathcal{P}$. Then W is R-invariant (Groupoids in Spaces, Definition 52.17.1).*

Proof. The existence and properties of the open $W \subset U$ are described in Descent on Spaces, Lemma 45.9.3. In Diagram (53.3.0.2) let $W_1 \subset R$ be the maximal open subscheme over which the morphism $\mathrm{pr}_1 : R \times_{s,U,t} R \to R$ has property $\mathcal{P}$. It follows from the aforementioned Descent on Spaces, Lemma 45.9.3 and the assumption that $\{s : R \to U\}$ and $\{t : R \to U\}$ are coverings for the τ-topology that $t^{-1}(W) = W_1 = s^{-1}(W)$ as desired. $\square$

Lemma 53.4.2. *Let $B \to S$ be as in Section 53.2. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let $G \to U$ be its stabilizer group algebraic space. Let $\tau \in \{fppf, étale, smooth, syntomic\}$. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is τ-local on the target. Assume $\{s : R \to U\}$ and $\{t : R \to U\}$ are coverings for the τ-topology. Let $W \subset U$ be the maximal open subspace such that $G_W \to W$ has property $\mathcal{P}$. Then W is R-invariant (see Groupoids in Spaces, Definition 52.17.1).*

Proof. The existence and properties of the open $W \subset U$ are described in Descent on Spaces, Lemma 45.9.3. The morphism

$$G \times_{U,t} R \longrightarrow R \times_{s,U} G, \quad (g, r) \longmapsto (r, r^{-1} \circ g \circ r)$$

is an isomorphism of algebraic spaces over R (where $\circ$ denotes composition in the groupoid). Hence $s^{-1}(W) = t^{-1}(W)$ by the properties of W proved in the aforementioned Descent on Spaces, Lemma 45.9.3. $\square$

53.5. Comparing fibres

This section is the analogue of More on Groupoids, Section 36.6. The reader is strongly encouraged to read that section first.

Lemma 53.5.1. *Let $B \to S$ be as in Section 53.2. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let K be a field and let $r, r' : Spec(K) \to R$ be morphisms such that $t \circ r = t \circ r' : Spec(K) \to U$. Set $u = s \circ r$, $u' = s \circ r'$ and denote $F_u = Spec(K) \times_{u,U,s} R$ and $F_{u'} = Spec(K) \times_{u',U,s} R$ the fibre products. Then $F_u \cong F_{u'}$ as algebraic spaces over K.*

Proof. We use the properties and the existence of Diagram (53.3.0.2). There exists a morphism $\xi : Spec(K) \to R \times_{s,U,t} R$ with $\mathrm{pr}_0 \circ \xi = r$ and $c \circ \xi = r'$. Let $\tilde{r} = \mathrm{pr}_1 \circ \xi : Spec(K) \to R$. Then looking at the bottom two squares of Diagram (53.3.0.2) we see that both F_u and $F_{u'}$ are identified with the algebraic space $Spec(K) \times_{\tilde{r},R,\mathrm{pr}_1} (R \times_{s,U,t} R)$. □

Actually, in the situation of the lemma the morphisms of pairs $s : (R, r) \to (U, u)$ and $s : (R, r') \to (U, u')$ are locally isomorphic in the τ-topology, provided $\{s : R \to U\}$ is a τ-covering. We will insert a precise statement here if needed.

53.6. Restricting groupoids

In this section we collect a bunch of lemmas on properties of groupoids which are inherited by restrictions. Most of these lemmas can be proved by contemplating the defining diagram

$$\tag{53.6.0.1}
\begin{tikzcd}
R' \arrow[r] \arrow[d] \arrow[rr, bend left, "s'"] \arrow[dd, bend right, "t'"'] & R \times_{s,U} U' \arrow[r] \arrow[d] & U' \arrow[d, "g"] \\
U' \times_{U,t} R \arrow[r] \arrow[d] & R \arrow[r, "s"] \arrow[d, "t"] & U \\
U' \arrow[r, "g"] & U &
\end{tikzcd}$$

of a restriction. See Groupoids in Spaces, Lemma 52.16.1.

Lemma 53.6.1. *Let S be a scheme. Let B be an algebraic space over S. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let $g : U' \to U$ be a morphism of algebraic spaces over B. Let (U', R', s', t', c') be the restriction of (U, R, s, t, c) via g.*

1. *If s, t are locally of finite type and g is locally of finite type, then s', t' are locally of finite type.*
2. *If s, t are locally of finite presentation and g is locally of finite presentation, then s', t' are locally of finite presentation.*
3. *If s, t are flat and g is flat, then s', t' are flat.*
4. *Add more here.*

Proof. The property of being locally of finite type is stable under composition and arbitrary base change, see Morphisms of Spaces, Lemmas 42.22.2 and 42.22.3. Hence (1) is clear from Diagram (53.6.0.1). For the other cases, see Morphisms of Spaces, Lemmas 42.26.2, 42.26.3, 42.27.2, and 42.27.3. □

53.7. Properties of groups over fields and groupoids on fields

The reader is advised to first look at the corresponding sections for groupoid schemes, see Groupoids, Section 35.7 and More on Groupoids, Section 36.9.

Situation 53.7.1. Here S is a scheme, k is a field over S, and (G, m) is a group algebraic spaces over $Spec(k)$.

Situation 53.7.2. Here S is a scheme, B is an algebraic space, and (U, R, s, t, c) is a groupoid in algebraic spaces over B with $U = Spec(k)$ for some field k.

Note that in Situation 53.7.1 we obtain a groupoid in algebraic spaces

$$(Spec(k), G, p, p, m) \tag{53.7.2.1}$$

where $p : G \to Spec(k)$ is the structure morphism of G, see Groupoids in Spaces, Lemma 52.14.1. This is a situation as in Situation 53.7.2. We will use this without further mention in the rest of this section.

Lemma 53.7.3. *In Situation 53.7.2 the composition morphism $c : R \times_{s,U,t} R \to R$ is flat and universally open. In Situation 53.7.1 the group law $m : G \times_k G \to G$ is flat and universally open.*

Proof. The composition is isomorphic to the projection map $\text{pr}_1 : R \times_{t,U,t} R \to R$ by Diagram (53.3.0.3). The projection is flat as a base change of the flat morphism t and open by Morphisms of Spaces, Lemma 42.7.6. The second assertion follows immediately from the first because m matches c in (53.7.2.1). $\square$

Lemma 53.7.4. *In Situation 53.7.2. Let $k \subset k'$ be a field extension, $U' = Spec(k')$ and let (U', R', s', t', c') be the restriction of (U, R, s, t, c) via $U' \to U$. In the defining diagram*

$$\begin{tikzcd}
R' \arrow[r] \arrow[d] \arrow[dr, dotted] \arrow[rr, bend left, "s'"] \arrow[dd, bend right=60, "t'"'] & R \times_{s,U} U' \arrow[r] \arrow[d] & U' \arrow[d] \\
U' \times_{U,t} R \arrow[r] \arrow[d] & R \arrow[r, "s"] \arrow[d, "t"] & U \\
U' \arrow[r] & U &
\end{tikzcd}$$

all the morphisms are surjective, flat, and universally open. The dotted arrow $R' \to R$ is in addition affine.

Proof. The morphism $U' \to U$ equals $Spec(k') \to Spec(k)$, hence is affine, surjective and flat. The morphisms $s, t : R \to U$ and the morphism $U' \to U$ are universally open by Morphisms, Lemma 24.22.4. Since R is not empty and U is the spectrum of a field the morphisms $s, t : R \to U$ are surjective and flat. Then you conclude by using Morphisms of Spaces, Lemmas 42.6.5, 42.6.4, 42.7.4, 42.19.5, 42.19.4, 42.27.3, and 42.27.2. $\square$

Lemma 53.7.5. *In Situation 53.7.2. For any point $r \in |R|$ there exist*

1. *a field extension $k \subset k'$ with k' algebraically closed,*
2. *a point $r' : Spec(k') \to R'$ where (U', R', s', t', c') is the restriction of (U, R, s, t, c) via $Spec(k') \to Spec(k)$*

such that

(1) *the point r' maps to r under the morphism $R' \to R$, and*
(2) *the maps $s' \circ r', t' \circ r' : Spec(k') \to Spec(k')$ are automorphisms.*

Proof. Let's represent r by a morphism $r : Spec(K) \to R$ for some field K. To prove the lemma we have to find an algebraically closed field k' and a commutative diagram

$$\begin{tikzcd} k' & k' \arrow[l, "1"'] & \\ k' \arrow[u, "\tau"] & K \arrow[ul, "\sigma"] & k \arrow[l, "s"] \arrow[ul, "i"'] \\ & k \arrow[ul, "i"] \arrow[u, "t"'] & \end{tikzcd}$$

where $s, t : k \to K$ are the field maps coming from $s \circ r$ and $t \circ r$. In the proof of More on Groupoids, Lemma 36.9.5 it is shown how to construct such a diagram. $\square$

Lemma 53.7.6. *In Situation 53.7.2. If $r : Spec(k) \to R$ is a morphism such that $s \circ r, t \circ r$ are automorphisms of $Spec(k)$, then the map*

$$R \longrightarrow R, \quad x \longmapsto c(r, x)$$

is an automorphism $R \to R$ which maps e to r.

Proof. Proof is identical to the proof of More on Groupoids, Lemma 36.9.6. $\square$

Lemma 53.7.7. *In Situation 53.7.2 the algebraic space R is geometrically unibranch. In Situation 53.7.1 the algebraic space G is geometrically unibranch.*

Proof. Let $r \in |R|$. We have to show that R is geometrically unibranch at r. Combining Lemma 53.7.4 with Descent on Spaces, Lemma 45.8.1 we see that it suffices to prove this in case k is algebraically closed and r comes from a morphism $r : Spec(k) \to R$ such that $s \circ r$ and $t \circ r$ are automorphisms of $Spec(k)$. By Lemma 53.7.6 we reduce to the case that $r = e$ is the identity of R and k is algebraically closed.

Assume $r = e$ and k is algebraically closed. Let $A = \mathcal{O}_{R,e}$ be the étale local ring of R at e and let $C = \mathcal{O}_{R \times_{s,U,t} R,(e,e)}$ be the étale local ring of $R \times_{s,U,t} R$ at (e, e). By Spaces over Fields, Lemma 48.3.1 the minimal prime ideals $\mathfrak{q}$ of C correspond 1-to-1 to pairs of minimal primes $\mathfrak{p}, \mathfrak{p}' \subset A$. On the other hand, the composition law induces a flat ring map

$$\begin{tikzcd} A \arrow[r, "c^\sharp"'] & C & & \mathfrak{q} \\ & A \otimes_{s^\sharp,k,t^\sharp} A \arrow[u] & & \mathfrak{p} \otimes A + A \otimes \mathfrak{p}' \arrow[u, mapsto] \end{tikzcd}$$

Note that $(c^\sharp)^{-1}(\mathfrak{q})$ contains both $\mathfrak{p}$ and $\mathfrak{p}'$ as the diagrams

$$\begin{tikzcd} A \arrow[r, "c^\sharp"'] & C & A \arrow[r, "c^\sharp"'] & C \\ A \otimes_{s^\sharp,k} k \arrow[u] & A \otimes_{s^\sharp,k,t^\sharp} A \arrow[l, "1\otimes e^\sharp"'] \arrow[u] & k \otimes_{k,t^\sharp} A \arrow[u] & A \otimes_{s^\sharp,k,t^\sharp} A \arrow[l, "e^\sharp\otimes 1"'] \arrow[u] \end{tikzcd}$$

commute by (53.3.0.2). Since $c^\sharp$ is flat (as c is a flat morphism by Lemma 53.7.3), we see that $(c^\sharp)^{-1}(\mathfrak{q})$ is a minimal prime of A. Hence $\mathfrak{p} = (c^\sharp)^{-1}(\mathfrak{q}) = \mathfrak{p}'$. $\square$

In the following lemma we use dimension of algebraic spaces (at a point) as defined in Properties of Spaces, Section 41.8. We also use the dimension of the local ring defined in Properties of Spaces, Section 41.20 and transcendence degree of points, see Morphisms of Spaces, Section 42.30.

Lemma 53.7.8. *In Situation 53.7.2 assume s, t are locally of finite type. For all $r \in |R|$*

1. $\dim(R) = \dim_r(R)$,
2. *the transcendence degree of r over $Spec(k)$ via s equals the transcendence degree of r over $Spec(k)$ via t, and*
3. *if the transcendence degree mentioned in (2) is* 0, *then* $\dim(R) = \dim(\mathcal{O}_{R,\overline{r}})$.

Proof. Let $r \in |R|$. Denote $\text{trdeg}(r/_s k)$ the transcendence degree of r over $Spec(k)$ via s. Choose an étale morphism $\varphi : V \to R$ where V is a scheme and $v \in V$ mapping to r. Using the definitions mentioned above the lemma we see that

$$\dim_r(R) = \dim_v(V) = \dim(\mathcal{O}_{V,v}) + \text{trdeg}_{s(k)}(\kappa(v)) = \dim(\mathcal{O}_{R,\overline{r}}) + \text{trdeg}(r/_s k)$$

and similarly for t (the second equality by Morphisms, Lemma 24.27.1). Hence we see that $\text{trdeg}(r/_s k) = \text{trdeg}(r/_t k)$, i.e., (2) holds.

Let $k \subset k'$ be a field extension. Note that the restriction R' of R to $Spec(k')$ (see Lemma 53.7.4) is obtained from R by two base changes by morphisms of fields. Thus Morphisms of Spaces, Lemma 42.31.3 shows the dimension of R at a point is unchanged by this operation. Hence in order to prove (1) we may assume, by Lemma 53.7.5, that r is represented by a morphism $r : Spec(k) \to R$ such that both $s \circ r$ and $t \circ r$ are automorphisms of $Spec(k)$. In this case there exists an automorphism $R \to R$ which maps r to e (Lemma 53.7.6). Hence we see that $\dim_r(R) = \dim_e(R)$ for any r. By definition this means that $\dim_r(R) = \dim(R)$.

Part (3) is a formal consequence of the results obtained in the discussion above. □

Lemma 53.7.9. *In Situation 53.7.1 assume G locally of finite type. For all $g \in |G|$*

1. $\dim(G) = \dim_g(G)$,
2. *if the transcendence degree of g over k is* 0, *then* $\dim(G) = \dim(\mathcal{O}_{G,\overline{g}})$.

Proof. Immediate from Lemma 53.7.8 via (53.7.2.1). □

Lemma 53.7.10. *In Situation 53.7.2 assume s, t are locally of finite type. Let $G = Spec(k) \times_{\Delta, Spec(k) \times_B Spec(k), t \times s} R$ be the stabilizer group algebraic space. Then we have* $\dim(R) = \dim(G)$.

Proof. Since G and R are equidimensional (see Lemmas 53.7.8 and 53.7.9) it suffices to prove that $\dim_e(R) = \dim_e(G)$. Let V be an affine scheme, $v \in V$, and let $\varphi : V \to R$ be an étale morphism of schemes such that $\varphi(v) = e$. Note that V is a Noetherian scheme as $s \circ \varphi$ is locally of finite type as a composition of morphisms locally of finite type and as V is quasi-compact (use Morphisms of Spaces, Lemmas 42.22.2, 42.35.8, and 42.26.5 and Morphisms, Lemma 24.14.6). Hence V is locally connected (see Properties, Lemma 23.5.5 and Topology, Lemma 5.6.6). Thus we may replace V by the connected component containing v (it is still affine as it is an open and closed subscheme of V). Set $T = V_{red}$ equal to the reduction of V. Consider the two morphisms $a, b : T \to Spec(k)$ given by $a = s \circ \varphi|_T$ and $b = t \circ \varphi|_T$. Note that a, b induce the same field map $k \to \kappa(v)$ because $\varphi(v) = e$! Let $k_a \subset \Gamma(T, \mathcal{O}_T)$ be the integral closure of $a^\sharp(k) \subset \Gamma(T, \mathcal{O}_T)$. Similarly, let $k_b \subset \Gamma(T, \mathcal{O}_T)$ be the integral closure of $b^\sharp(k) \subset \Gamma(T, \mathcal{O}_T)$. By Varieties, Proposition 28.18.1 we see that

$k_a = k_b$. Thus we obtain the following commutative diagram

$$\begin{tikzcd} k \arrow[dr, "a"] \arrow[drrr] & & & \\ & k_a = k_b \arrow[r] & \Gamma(T, \mathcal{O}_T) \arrow[r] & \kappa(v) \\ k \arrow[ur, "b"] \arrow[urrr] & & & \end{tikzcd}$$

As discussed above the long arrows are equal. Since $k_a = k_b \to \kappa(v)$ is injective we conclude that the two morphisms a and b agree. Hence $T \to R$ factors through G. It follows that $R_{red} = G_{red}$ in an open neighbourhood of e which certainly implies that $\dim_e(R) = \dim_e(G)$. □

53.8. The finite part of a morphism

Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. For an algebraic space or a scheme T over S consider pairs (a, Z) where

$$(53.8.0.1) \quad \begin{array}{l} a : T \to Y \text{ is a morphism over } S, \\ Z \subset T \times_Y X \text{ is an open subspace such that } \mathrm{pr}_0|_Z : Z \to T \text{ is finite.} \end{array}$$

Suppose $h : T' \to T$ is a morphism of algebraic spaces over S and (a, Z) is a pair over T. Set $a' = a \circ h$ and $Z' = (h \times \mathrm{id}_X)^{-1}(Z) = T' \times_T Z$. Then the pair (a', Z') satisfies (1), (2) over T'. This follows as finite morphisms are preserved under base change, see Morphisms of Spaces, Lemma 42.37.5. Thus we obtain a functor

$$(53.8.0.2) \quad \begin{array}{cccc} (X/Y)_{fin} : & (Sch/S)^{opp} & \longrightarrow & Sets \\ & T & \longmapsto & \{(a, Z) \text{ as above}\} \end{array}$$

For applications we are mainly interested in this functor $(X/Y)_{fin}$ when f is separated and locally of finite type. To get an idea of what this is all about, take a look at Remark 53.8.6.

Lemma 53.8.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Then we have*

1. *The presheaf $(X/Y)_{fin}$ satisfies the sheaf condition for the fppf topology.*
2. *If T is an algebraic space over S, then there is a canonical bijection*

$$Mor_{Sh((Sch/S)_{fppf})}(T, (X/Y)_{fin}) = \{(a, Z) \text{ satisfying 53.8.0.1}\}$$

Proof. Let T be an algebraic space over S. Let $\{T_i \to T\}$ be an fppf covering (by algebraic spaces). Let $s_i = (a_i, Z_i)$ be pairs over T_i satisfying 53.8.0.1 such that we have $s_i|_{T_i \times_T T_j} = s_j|_{T_i \times_T T_j}$. First, this implies in particular that a_i and a_j define the same morphism $T_i \times_T T_j \to Y$. By Descent on Spaces, Lemma 45.6.2 we deduce that there exists a unique morphism $a : T \to Y$ such that a_i equals the composition $T_i \to T \to Y$. Second, this implies that $Z_i \subset T_i \times_Y X$ are open subspaces whose inverse images in $(T_i \times_T T_j) \times_Y X$ are equal. Since $\{T_i \times_Y X \to T \times_Y X\}$ is an fppf covering we deduce that there exists a unique open subspace $Z \subset T \times_Y X$ which restricts back to Z_i over T_i, see Descent on Spaces, Lemma 45.6.1. We claim that the projection $Z \to T$ is finite. This follows as being finite is local for the fpqc topology, see Descent on Spaces, Lemma 45.10.21.

Note that the result of the preceding paragraph in particular implies (1).

Let T be an algebraic space over S. In order to prove (2) we will construct mutually inverse maps between the displayed sets. In the following when we say ``pair'' we mean a pair satisfying conditions 53.8.0.1.

Let $v : T \to (X/Y)_{fin}$ be a natural transformation. Choose a scheme U and a surjective étale morphism $p : U \to T$. Then $v(p) \in (X/Y)_{fin}(U)$ corresponds to a pair (a_U, Z_U) over U. Let $R = U \times_T U$ with projections $t, s : R \to U$. As v is a transformation of functors we see that the pullbacks of (a_U, Z_U) by s and t agree. Hence, since $\{U \to T\}$ is an fppf covering, we may apply the result of the first paragraph that deduce that there exists a unique pair (a, Z) over T.

Conversely, let (a, Z) be a pair over T. Let $U \to T$, $R = U \times_T U$, and $t, s : R \to U$ be as above. Then the restriction $(a, Z)|_U$ gives rise to a tranformation of functors $v : h_U \to (X/Y)_{fin}$ by the Yoneda lemma (Categories, Lemma 4.3.5). As the two pullbacks $s^*(a, Z)|_U$ and $t^*(a, Z)|_U$ are equal, we see that v coequalizes the two maps $h_t, h_s : h_R \to h_U$. Since $T = U/R$ is the fppf quotient sheaf by Spaces, Lemma 40.9.1 and since $(X/Y)_{fin}$ is an fppf sheaf by (1) we conclude that v factors through a map $T \to (X/Y)_{fin}$.

We omit the verification that the two constructions above are mutually inverse. ☐

Lemma 53.8.2. *Let S be a scheme. Consider a commutative diagram*

$$\begin{tikzcd} X' \arrow[rr, "j"] \arrow[dr] & & X \arrow[dl] \\ & Y & \end{tikzcd}$$

of algebraic spaces over S. If j is an open immersion, then there is a canonical injective map of sheaves $j : (X'/Y)_{fin} \to (X/Y)_{fin}$.

Proof. If (a, Z) is a pair over T for X'/Y, then $(a, j(Z))$ is a pair over T for X/Y. ☐

Lemma 53.8.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite type. Let $X' \subset X$ be the maximal open subspace over which f is locally quasi-finite, see Morphisms of Spaces, Lemma 42.31.7. Then $(X/Y)_{fin} = (X'/Y)_{fin}$.*

Proof. Lemma 53.8.2 gives us an injective map $(X'/Y)_{fin} \to (X/Y)_{fin}$. Morphisms of Spaces, Lemma 42.31.7 assures us that formation of X' commutes with base change. Hence everything comes down to proving that if $Z \subset X$ is a open subspace such that $f|_Z : Z \to Y$ is finite, then $Z \subset X'$. This is true because a finite morphism is locally quasi-finite, see Morphisms of Spaces, Lemma 42.37.8. ☐

Lemma 53.8.4. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let T be an algebraic space over S, and let (a, Z) be a pair as in 53.8.0.1. If f is separated, then Z is closed in $T \times_Y X$.*

Proof. A finite morphism of algebraic spaces is universally closed by Morphisms of Spaces, Lemma 42.37.9. Since f is separated so is the morphism $T \times_Y X \to T$, see Morphisms of Spaces, Lemma 42.5.4. Thus the closedness of Z follows from Morphisms of Spaces, Lemma 42.36.5. ☐

Remark 53.8.5. Let $f : X \to Y$ be a separated morphism of algebraic spaces. The sheaf $(X/Y)_{fin}$ comes with a natural map $(X/Y)_{fin} \to Y$ by mapping the pair $(a, Z) \in (X/Y)_{fin}(T)$ to the element $a \in Y(T)$. We can use Lemma 53.8.4 to define operations

$$\star_i : (X/Y)_{fin} \times_Y (X/Y)_{fin} \longrightarrow (X/Y)_{fin}$$

by the rules

$$\begin{aligned} \star_1 &: ((a, Z_1), (a, Z_2)) \longmapsto (a, Z_1 \cup Z_2) \\ \star_2 &: ((a, Z_1), (a, Z_2)) \longmapsto (a, Z_1 \cap Z_2) \\ \star_3 &: ((a, Z_1), (a, Z_2)) \longmapsto (a, Z_1 \setminus Z_2) \\ \star_4 &: ((a, Z_1), (a, Z_2)) \longmapsto (a, Z_2 \setminus Z_1). \end{aligned}$$

The reason this works is that $Z_1 \cap Z_2$ is both open and closed inside Z_1 and Z_2 (which also implies that $Z_1 \cup Z_2$ is the disjoint union of the other three pieces). Thus we can think of $(X/Y)_{fin}$ as an $\mathbf{F}_2$-algebras (without unit) over Y with multiplication given by $ss' = \star_2(s, s')$, and addition given by

$$s + s' = \star_1(\star_3(s, s'), \star_4(s, s'))$$

which boils down to taking the symmetric difference. Note that in this sheaf of algebras $0 = (1_Y, \emptyset)$ and that indeed $s + s = 0$ for any local section s. If $f : X \to Y$ is finite, then this algebra has a unit namely $1 = (1_Y, X)$ and $\star_3(s, s') = s(1 + s')$, and $\star_4(s, s') = (1 + s)s'$.

Remark 53.8.6. Let $f : X \to Y$ be a separated, locally quasi-finite morphism of schemes. In this case the sheaf $(X/Y)_{fin}$ is closely related to the sheaf $f_!\mathbf{F}_2$ (insert future reference here) on $Y_{étale}$. Namely, if $V \to Y$ is étale, and $s \in \Gamma(V, f_!\mathbf{F}_2)$, then $s \in \Gamma(V \times_Y X, \mathbf{F}_2)$ is a section with proper support $Z = \text{Supp}(s)$ over V. Since f is als locally quasi-finite we see that the projection $Z \to V$ is actually finite. Since the support of a section of a constant abelian sheaf is open we see that the pair $(V \to Y, \text{Supp}(s))$ satisfies 53.8.0.1. In fact, $f_!\mathbf{F}_2 \cong (X/Y)_{fin}|_{Y_{étale}}$ in this case which also explains the $\mathbf{F}_2$-algebra structure introduced in Remark 53.8.5.

Lemma 53.8.7. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The diagonal of $(X/Y)_{fin} \to Y$*

$$(X/Y)_{fin} \longrightarrow (X/Y)_{fin} \times_Y (X/Y)_{fin}$$

is representable (by schemes) and an open immersion and the ``absolute" diagonal

$$(X/Y)_{fin} \longrightarrow (X/Y)_{fin} \times (X/Y)_{fin}$$

is representable (by schemes).

Proof. The second statement follows from the first as the absolute diagonal is the composition of the relative diagonal and a base change of the diagonal of Y (which is representable by schemes), see Spaces, Section 40.3. To prove the first assertion we have to show the following: Given a scheme T and two pairs (a, Z_1) and (a, Z_2) over T with identical first component satisfying 53.8.0.1 there is an open subscheme $V \subset T$ with the following property: For any morphism of schemes $h : T' \to T$ we have

$$h(T') \subset V \Leftrightarrow \left(T' \times_T Z_1 = T' \times_T Z_2 \text{ as subspaces of } T' \times_Y X\right)$$

Let us construct V. Note that $Z_1 \cap Z_2$ is open in Z_1 and in Z_2. Since $\text{pr}_0|_{Z_i} : Z_i \to T$ is finite, hence proper (see Morphisms of Spaces, Lemma 42.37.9) we see that

$$E = \text{pr}_0|_{Z_1}\left(Z_1 \setminus Z_1 \cap Z_2\right) \cup \text{pr}_0|_{Z_2}\left(Z_2 \setminus Z_1 \cap Z_2\right)$$

is closed in T. Now it is clear that $V = T \setminus E$ works. □

Lemma 53.8.8. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Suppose that U is a scheme, $U \to Y$ is an étale morphism and $Z \subset U \times_Y X$ is an open subspace finite over U. Then the induced morphism $U \to (X/Y)_{fin}$ is étale.*

Proof. This is formal from the description of the diagonal in Lemma 53.8.7 but we write it out since it is an important step in the development of the theory. We have to check that for any scheme T over S and a morphism $T \to (X/Y)_{fin}$ the projection map

$$T \times_{(X/Y)_{fin}} U \longrightarrow T$$

is étale. Note that

$$T \times_{(X/Y)_{fin}} U = (X/Y)_{fin} \times_{((X/Y)_{fin} \times_Y (X/Y)_{fin})} (T \times_Y U)$$

Applying the result of Lemma 53.8.7 we see that $T \times_{(X/Y)_{fin}} U$ is represented by an open subscheme of $T \times_Y U$. As the projection $T \times_Y U \to T$ is étale by Morphisms of Spaces, Lemma 42.35.4 we conclude. $\square$

Lemma 53.8.9. *Let S be a scheme. Let*

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

be a fibre product square of algebraic spaces over S. Then

$$\begin{tikzcd} (X'/Y')_{fin} \arrow[r] \arrow[d] & (X/Y)_{fin} \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

is a fibre product square of sheaves on $(Sch/S)_{fppf}$.

Proof. It follows immediately from the definitions that the sheaf $(X'/Y')_{fin}$ is equal to the sheaf $Y' \times_Y (X/Y)_{fin}$. $\square$

Lemma 53.8.10. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is separated and locally quasi-finite, then there exists a scheme U étale over Y and a surjective étale morphism $U \to (X/Y)_{fin}$ over Y.*

Proof. Note that the assertion makes sense by the result of Lemma 53.8.7 on the diagonal of $(X/Y)_{fin}$, see Spaces, Lemma 40.5.10. Let V be a scheme and let $V \to Y$ be a surjective étale morphism. By Lemma 53.8.9 the morphism $(V \times_Y X/V)_{fin} \to (X/Y)_{fin}$ is a base change of the map $V \to Y$ and hence is surjective and étale, see Spaces, Lemma 40.5.5. Hence it suffices to prove the lemma for $(V \times_Y X/V)_{fin}$. (Here we implicitly use that the composition of representable, surjective, and étale transformations of functors is again representable, surjective, and étale, see Spaces, Lemmas 40.3.2 and 40.5.4, and Morphisms, Lemmas 24.9.2 and 24.35.3.) Note that the properties of being separated and locally quasi-finite are preserved under base change, see Morphisms of Spaces, Lemmas 42.5.4 and 42.25.3. Hence $V \times_Y X \to V$ is separated and locally quasi-finite as well, and by Morphisms of Spaces, Proposition 42.39.2 we see that $V \times_Y X$ is a scheme as well. Thus we may assume that $f : X \to Y$ is a separated and locally quasi-finite morphism of schemes.

Pick a point $y \in Y$. Pick $x_1, \ldots, x_n \in X$ points lying over y. Pick an étale neighbourhood $a : (U, u) \to (Y, y)$ and a decomposition

$$U \times_S X = W \coprod \coprod_{i=1,\ldots,n} \coprod_{j=1,\ldots,m_j} V_{i,j}$$

as in More on Morphisms, Lemma 33.28.5. Pick any subset

$$I \subset \{(i,j) \mid 1 \leq i \leq n,\ 1 \leq j \leq m_i\}.$$

Given these choices we obtain a pair (a, Z) with $Z = \bigcup_{(i,j)\in I} V_{i,j}$ which satisfies conditions 53.8.0.1. In other words we obtain a morphism $U \to (X/Y)_{fin}$. The construction of this morphism depends on all the things we picked above, so we should really write

$$U(y, n, x_1, \ldots, x_n, a, I) \longrightarrow (X/Y)_{fin}$$

This morphism is étale by Lemma 53.8.8.

Claim: The disjoint union of all of these is surjective onto $(X/Y)_{fin}$. It is clear that if the claim holds, then the lemma is true.

To show surjectivity we have to show the following (see Spaces, Remark 40.5.2): Given a scheme T over S, a point $t \in T$, and a map $T \to (X/Y)_{fin}$ we can find a datum $(y, n, x_1, \ldots, x_n, a, I)$ as above such that t is in the image of the projection map

$$U(y, n, x_1, \ldots, x_n, a, I) \times_{(X/Y)_{fin}} T \longrightarrow T.$$

To prove this we may clearly replace T by $Spec(\overline{\kappa(t)})$ and $T \to (X/Y)_{fin}$ by the composition $Spec(\overline{\kappa(t)}) \to T \to (X/Y)_{fin}$. In other words, we may assume that T is the spectrum of an algebraically closed field.

Let $T = Spec(k)$ be the spectrum of an algebraically closed field k. The morphism $T \to (X/Y)_{fin}$ is given by a pair $(T \to Y, Z)$ satisfying conditions 53.8.0.1. Here is a picture:

$$\begin{tikzcd} & Z \arrow[r] \arrow[d] & X \arrow[d] \\ Spec(k) \arrow[r, equal] & T \arrow[r] & Y \end{tikzcd}$$

Let $y \in Y$ be the image point of $T \to Y$. Since Z is finite over k it has finitely many points. Thus there exist finitely many points $x_1, \ldots, x_n \in X$ such that the image of Z in X is contained in $\{x_1, \ldots, x_n\}$. Choose $a : (U, u) \to (Y, y)$ adapted to y and $x_1, \ldots, x_n$ as above, which gives the diagram

$$\begin{tikzcd} W \amalg \coprod_{i=1,\ldots,n} \coprod_{j=1,\ldots,m_j} V_{i,j} \arrow[r] \arrow[d] & X \arrow[d] \\ U \arrow[r] & Y. \end{tikzcd}$$

Since k is algebraically closed and $\kappa(y) \subset \kappa(u)$ is finite separable we may factor the morphism $T = Spec(k) \to Y$ through the morphism $u = Spec(\kappa(u)) \to Spec(\kappa(y)) = y \subset Y$. With this choice we obtain the commutative diagram:

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & W \amalg \coprod_{i=1,\ldots,n} \coprod_{j=1,\ldots,m_j} V_{i,j} \arrow[r] \arrow[d] & X \arrow[d] \\ Spec(k) \arrow[r] & U \arrow[r] & Y \end{tikzcd}$$

We know that the image of the left upper arrow ends up in $\coprod V_{i,j}$. Recall also that Z is an open subscheme of $Spec(k) \times_Y X$ by definition of $(X/Y)_{fin}$ and that the right hand square is a fibre product square. Thus we see that

$$Z \subset \coprod\nolimits_{i=1,\ldots,n} \coprod\nolimits_{j=1,\ldots,m_j} Spec(k) \times_U V_{i,j}$$

is an open subscheme. By construction (see More on Morphisms, Lemma 33.28.5) each $V_{i,j}$ has a unique point $v_{i,j}$ lying over u with purely inseparable residue field extension $\kappa(u) \subset \kappa(v_{i,j})$. Hence each scheme $Spec(k) \times_U V_{i,j}$ has exactly one point. Thus we see that

$$Z = \coprod\nolimits_{(i,j)\in I} Spec(k) \times_U V_{i,j}$$

for a unique subset $I \subset \{(i,j) \mid 1 \leq i \leq n,\ 1 \leq j \leq m_i\}$. Unwinding the definitions this shows that

$$U(y, n, x_1, \ldots, x_n, a, I) \times_{(X/Y)_{fin}} T$$

with I as found above is nonempty as desired. □

Proposition 53.8.11. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is separated and locally of finite type. Then $(X/Y)_{fin}$ is an algebraic space. Moreover, the morphism $(X/Y)_{fin} \to Y$ is étale.*

Proof. By Lemma 53.8.3 we may replace X by the open subscheme which is locally quasi-finite over Y. Hence we may assume that f is separated and locally quasi-finite. We will check the three conditions of Spaces, Definition 40.6.1. Condition (1) follows from Lemma 53.8.1. Condition (2) follows from Lemma 53.8.7. Finally, condition (3) follows from Lemma 53.8.10. Thus $(X/Y)_{fin}$ is an algebraic space. Moreover, that lemma shows that there exists a commutative diagram

$$\begin{tikzcd} U \arrow[rr] \arrow[dr] & & (X/Y)_{fin} \arrow[dl] \\ & Y & \end{tikzcd}$$

with horizontal arrow surjective and étale and south-east arrow étale. By Properties of Spaces, Lemma 41.13.3 this implies that the south-west arrow is étale as well. □

Remark 53.8.12. The condition that f be separated cannot be dropped from Proposition 53.8.11. An example is to take X the affine line with zero doubled, see Schemes, Example 21.14.3, $Y = \mathbf{A}^1_k$ the affine line, and $X \to Y$ the obvious map. Recall that over $0 \in Y$ there are two points 0_1 and 0_2 in X. Thus $(X/Y)_{fin}$ has four points over 0, namely $\emptyset, \{0_1\}, \{0_2\}, \{0_1, 0_2\}$. Of these four points only three can be lifted to an open subscheme of $U \times_Y X$ finite over U for $U \to Y$ étale, namely $\emptyset, \{0_1\}, \{0_2\}$. This shows that $(X/Y)_{fin}$ if representable by an algebraic space is not étale over Y. Similar arguments show that $(X/Y)_{fin}$ is really not an algebraic space. Details omitted.

Remark 53.8.13. Let $Y = \mathbf{A}^1_{\mathbf{R}}$ be the affine line over the real numbers, and let $X = Spec(\mathbf{C})$ mapping to the $\mathbf{R}$-rational point 0 in Y. In this case the morphism $f : X \to Y$ is finite, but it is not the case that $(X/Y)_{fin}$ is a scheme. Namely, one can show that in this case the algebraic space $(X/Y)_{fin}$ is isomorphic to the algebraic space of Spaces, Example 40.14.2 associated to the extension $\mathbf{R} \subset \mathbf{C}$. Thus it is really necessary to leave the category of schemes in order to represent the sheaf $(X/Y)_{fin}$, even when f is a finite morphism.

Lemma 53.8.14. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is separated, flat, and locally of finite presentation. In this case*

(1) *$(X/Y)_{fin} \to Y$ is separated, representable, and étale, and*

(2) *if Y is a scheme, then $(X/Y)_{fin}$ is (representable by) a scheme.*

Proof. Since f is in particular separated and locally of finite type (see Morphisms of Spaces, Lemma 42.26.5) we see that $(X/Y)_{fin}$ is an algebraic space by Proposition 53.8.11. To prove that $(X/Y)_{fin} \to Y$ is separated we have to show the following: Given a scheme T and two pairs (a, Z_1) and (a, Z_2) over T with identical first component satisfying 53.8.0.1 there is a closed subscheme $V \subset T$ with the following property: For any morphism of schemes $h : T' \to T$ we have

$$h \text{ factors through } V \Leftrightarrow \left(T' \times_T Z_1 = T' \times_T Z_2 \text{ as subspaces of } T' \times_Y X\right)$$

In the proof of Lemma 53.8.7 we have seen that $V = T' \setminus E$ is an open subscheme of T' with closed complement

$$E = \mathrm{pr}_0|_{Z_1}\left(Z_1 \setminus Z_1 \cap Z_2)\right) \cup \mathrm{pr}_0|_{Z_2}\left(Z_2 \setminus Z_1 \cap Z_2)\right).$$

Thus everything comes down to showing that E is also open. By Lemma 53.8.4 we see that Z_1 and Z_2 are closed in $T' \times_Y X$. Hence $Z_1 \setminus Z_1 \cap Z_2$ is open in Z_1. As f is flat and locally of finite presentation, so is $\mathrm{pr}_0|_{Z_1}$. This is true as Z_1 is an open subspace of the base change $T' \times_Y X$, and Morphisms of Spaces, Lemmas 42.26.3 and Lemmas 42.27.3. Hence $\mathrm{pr}_0|_{Z_1}$ is open, see Morphisms of Spaces, Lemma 42.27.5. Thus $\mathrm{pr}_0|_{Z_1}\left(Z_1 \setminus Z_1 \cap Z_2)\right)$ is open and it follows that E is open as desired.

We have already seen that $(X/Y)_{fin} \to Y$ is étale, see Proposition 53.8.11. Hence now we know it is locally quasi-finite (see Morphisms of Spaces, Lemma 42.35.5) and separated, hence representable by Morphisms of Spaces, Lemma 42.40.1. The final assertion is clear (if you like you can use Morphisms of Spaces, Proposition 42.39.2). □

Variant: Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\sigma : Y \to X$ be a section of f. For an algebraic space or a scheme T over S consider pairs (a, Z) where

$$\begin{array}{l} a : T \to Y \text{ is a morphism over } S, \\ Z \subset T \times_Y X \text{ is an open subspace such that } \mathrm{pr}_0|_Z : Z \to T \text{ is finite, and} \\ (1_T, \sigma \circ a) : T \to T \times_Y X \text{ factors through } Z. \end{array} \tag{53.8.14.1}$$

We will denote $(X/Y, \sigma)_{fin}$ the subfunctor of $(X/Y)_{fin}$ parametrizing these pairs.

Lemma 53.8.15. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\sigma : Y \to X$ be a section of f. Consider the transformation of functors*

$$t : (X/Y, \sigma)_{fin} \longrightarrow (X/Y)_{fin}.$$

defined above. Then

(1) *t is representable by open immersions,*
(2) *if f is separated, then t is representable by open and closed immersions,*
(3) *if $(X/Y)_{fin}$ is an algebraic space, then $(X/Y, \sigma)_{fin}$ is an algebraic space and an open subspace of $(X/Y)_{fin}$, and*
(4) *if $(X/Y)_{fin}$ is a scheme, then $(X/Y, \sigma)_{fin}$ is an open subscheme of it.*

Proof. Omitted. Hint: Given a pair (a, Z) over T as in (53.8.0.1) the inverse image of Z by $(1_T, \sigma \circ a) : T \to T \times_Y X$ is the open subscheme of T we are looking for. □

53.9. Finite collections of arrows

Let $\mathcal{C}$ be a groupoid, see Categories, Definition 4.2.5. As discussed in Groupoids, Section 35.11 this corresponds to a septuple $(\text{Ob}, \text{Arrows}, s, t, c, e, i)$.

Using this data we can make another groupoid $\mathcal{C}_{fin}$ as follows:

(1) An object of $\mathcal{C}_{fin}$ consists of a finite subset $Z \subset \text{Arrows}$ with the following properties:
   (a) $s(Z) = \{u\}$ is a singleton, and
   (b) $e(u) \in Z$.
(2) A morphism of $\mathcal{C}_{fin}$ consists of a pair (Z, z), where Z is an object of $\mathcal{C}_{fin}$ and $z \in Z$.
(3) The source of (Z, z) is Z.
(4) The target of (Z, z) is $t(Z, z) = \{z' \circ z^{-1}; z' \in Z\}$.
(5) Given $(Z_1, z_1), (Z_2, z_2)$ such that $s(Z_1, z_1) = t(Z_2, z_2)$ the composition $(Z_1, z_1) \circ (Z_2, z_2)$ is $(Z_2, z_1 \circ z_2)$.

We omit the verification that this defines a groupoid. Pictorially an object of $\mathcal{C}_{fin}$ can be viewed as a diagram

$$\begin{tikzcd}
 & \bullet \\
\bullet \arrow[loop left, "e"] \arrow[ur] \arrow[r] \arrow[dr] & \bullet \\
 & \bullet
\end{tikzcd}$$

To make a morphism of $\mathcal{C}_{fin}$ you pick one of the arrows and you precompose the other arrows by its inverse. For example if we pick the middle horizontal arrow then the target is the picture

$$\begin{tikzcd}
 & \bullet & \\
\bullet & \bullet \arrow[l] \arrow[u] \arrow[d] \arrow[loop right, "e"] & \\
 & \bullet &
\end{tikzcd}$$

Note that the cardinalities of $s(Z, z)$ and $t(Z, z)$ are equal. So $\mathcal{C}_{fin}$ is really a countable disjoint union of groupoids.

53.10. The finite part of a groupoid

In this section we are going to use the idea explained in Section 53.9 to take the finite part of a groupoid in algebraic spaces.

Let S be a scheme. Let B be an algebraic space over S. Let (U, R, s, t, c, e, i) be a groupoid in algebraic spaces over B. Assumption: The morphisms s, t are separated and locally of finite type. This notation and assumption will we be fixed throughout this section.

Denote R_s the algebraic space R seen as an algebraic space over U via s. Let $U' = (R_s/U, e)_{fin}$. Since s is separated and locally of finite type, by Proposition 53.8.11 and Lemma 53.8.15, we see that U' is an algebraic space endowed with an étale morphism $g : U' \to U$. Moreover, by Lemma 53.8.1 there exists a universal open subspace $Z_{univ} \subset$

$R \times_{s,U,g} U'$ which is finite over U' and such that $(1_{U'}, e \circ g) : U' \to R \times_{s,U,g} U'$ factors through Z_{univ}. Moreover, by Lemma 53.8.4 the open subspace Z_{univ} is also closed in $R \times_{s,U',g} U$. Picture so far:

$$\begin{tikzcd}
Z_{univ} \arrow[d] \arrow[rd] & \\
R \times_{s,U,g} U' \arrow[r] \arrow[d] & U' \arrow[d, "g"] \\
R \arrow[r, "s"] & U
\end{tikzcd}$$

Let T be a scheme over B. We see that a T-valued point of Z_{univ} may be viewed as a triple (u, Z, z) where

(1) $u : T \to U$ is a T-valued point of U,
(2) $Z \subset R \times_{s,U,u} T$ is an open and closed subspace finite over T such that $(e \circ u, 1_T)$ factors through it, and
(3) $z : T \to R$ is a T-valued point of R with $s \circ z = u$ and such that $(z, 1_T)$ factors through Z.

Having said this, it is morally clear from the discussion in Section 53.9 that we can turn (Z_{univ}, U') into a groupoid in algebraic spaces over B. To make sure will define the morphisms s', t', c', e', i' one by one using the functorial point of view. (Please don't read this before reading and understanding the simple construction in Section 53.9.)

The morphism $s' : Z_{univ} \to U'$ corresponds to the rule

$$s' : (u, Z, z) \mapsto (u, Z).$$

The morphism $t' : Z_{univ} \to U'$ is given by the rule

$$t' : (u, Z, z) \mapsto (t \circ z, c(Z, i \circ z)).$$

The entry $c(Z, i \circ z)$ makes sense as the map $c(-, i \circ z) : R \times_{s,U,u} T \to R \times_{s,U,t\circ z} T$ is an isomorphism with inverse $c(-, z)$. The morphism $e' : U' \to Z_{univ}$ is given by the rule

$$e' : (u, Z) \mapsto (u, Z, (e \circ u, 1_T)).$$

Note that this makes sense by the requirement that $(e \circ u, 1_T)$ factors through Z. The morphism $i' : Z_{univ} \to Z_{univ}$ is given by the rule

$$i' : (u, Z, z) \mapsto (t \circ z, c(Z, i \circ z), i \circ z).$$

Finally, composition is defined by the rule

$$c' : ((u_1, Z_1, z_1), (u_2, Z_2, z_2)) \mapsto (u_2, Z_2, z_1 \circ z_2).$$

We omit the verification that the axioms of a groupoid in algebraic spaces hold for $(U', Z_{univ}, s', t', c', e', i')$.

A final piece of information is that there is a canonical morphism of groupoids

$$(U', Z_{univ}, s', t', c', e', i') \longrightarrow (U, R, s, t, c, e, i)$$

Namely, the morphism $U' \to U$ is the morphism $g : U' \to U$ which is defined by the rule $(u, Z) \mapsto u$. The morphism $Z_{univ} \to R$ is defined by the rule $(u, Z, z) \mapsto z$. This finishes the construction. Let us summarize our findings as follows.

Lemma 53.10.1. *Let S be a scheme. Let B be an algebraic space over S. Let (U, R, s, t, c, e, i) be a groupoid in algebraic spaces over B. Assume the morphisms s, t are separated and locally of finite type. There exists a canonical morphism*

$$(U', Z_{univ}, s', t', c', e', i') \longrightarrow (U, R, s, t, c, e, i)$$

of groupoids in algebraic spaces over B where

(1) *$g : U' \to U$ is identified with $(R_s/U, e)_{fin} \to U$, and*
(2) *$Z_{univ} \subset R \times_{s,U,g} U'$ is the universal open (and closed) subspace finite over U' which contains the base change of the unit e.*

Proof. See discussion above. □

53.11. Étale localization of groupoid schemes

In this section we prove results similar to [**KM97a**, Proposition 4.2]. We try to be a bit more general, and we try to avoid using Hilbert schemes by using the finite part of a morphism instead. The goal is to "split" a groupoid in algebraic spaces over a point after étale localization. Here is the definition (very similar to [**KM97a**, Definition 4.1]).

Definition 53.11.1. Let S be a scheme. Let B be an algebraic space over S Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let $u \in |U|$ be a point.

(1) We say R is *split over u* if there exists an open subspace $P \subset R$ such that
  (a) $(U, P, s|_P, t|_P, c|_{P \times_{s,U,t} P})$ is a groupoid in algebraic spaces over B,
  (b) $s|_P, t|_P$ are finite, and
  (c) $\{r \in |R| : s(r) = u, t(r) = u\} \subset P$.

  The choice of such a P will be called a *splitting of R over u*.
(2) We say R is *quasi-split over u* if there exists an open subspace $P \subset R$ such that
  (a) $(U, P, s|_P, t|_P, c|_{P \times_{s,U,t} P})$ is a groupoid in algebraic spaces over B,
  (b) $s|_P, t|_P$ are finite, and
  (c) $e(u) \in |P|$[1].

  The choice of such a P will be called a *quasi-splitting of R over u*.

Note the similarity of the conditions on P to the conditions on pairs in (53.8.0.1). In particular, if s, t are separated, then P is also closed in R (see Lemma 53.8.4).

Suppose we start with a groupoid in algebraic spaces (U, R, s, t, c) over B and a point $u \in |U|$. Since the goal is to split the groupoid after étale localization we may as well replace U by an affine scheme (what we mean is that this is harmless for any possible application). Moreover, the additional hypotheses we are going to have to impose will force R to be a scheme at least in a neighbourhood of $\{r \in |R| : s(r) = u, t(r) = u\}$ or $e(u)$. This is why we start with a groupoid scheme as described below. However, our technique of proof leads us outside of the category of schemes, which is why we have formulated a splitting for the case of groupoids in algebraic spaces above. On the other hand, we know of no applications but the case where the morphisms s, t are also flat and of finite presentation, in which case we end up back in the category of schemes.

Situation 53.11.2. (Assumptions for splitting.) Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $u \in U$ be a point. Assume that

(1) $s, t : R \to U$ are separated,
(2) s, t are locally of finite type,

[1] This condition is implied by (a).

(3) the set $\{r \in R : s(r) = u, t(r) = u\}$ is finite, and
(4) s is quasi-finite at each point of the set in (3).

Note that assumptions (3) and (4) are implied by the assumption that the fibre $s^{-1}(\{u\})$ is finite, see Morphisms, Lemma 24.19.7.

Situation 53.11.3. (Assumptions for quasi-splitting.) Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $u \in U$ be a point. Assume that

(1) $s, t : R \to U$ are separated,
(2) s, t are locally of finite type, and
(3) s is quasi-finite at $e(u)$.

It turns out that for applications to the existence theorems for algebraic spaces the case of quasi-splittings is sufficient. In fact, it is for us somehow a more natural case to consider, as in the stacks project there are no finiteness conditions on the diagonal of an algebaic space, hence the assumption that $\{r \in R : s(r) = u, t(r) = u\}$ is finite need not hold even for a presentation $X = U/R$ of an algebraic space X.

Lemma 53.11.4. *Assumptions and notation as in Situation 53.11.2. Then there exists an algebraic space U', an étale morphism $U' \to U$, and a point $u' : Spec(\kappa(u)) \to U'$ lying over $u : Spec(\kappa(u)) \to U$ such that the restriction $R' = R|_{U'}$ of R to U' splits over u'.*

Proof. Let $f : (U', Z_{univ}, s', t', c') \to (U, R, s, t, c)$ be as constructed in Lemma 53.10.1. Recall that $R' = R \times_{(U \times_S U)} (U' \times_S U')$. Thus we get a morphism $(f, t', s') : Z_{univ} \to R'$ of groupoids in algebraic spaces

$$(U', Z_{univ}, s', t', c') \to (U', R', s', t', c')$$

(by abuse of notation we indicate the morphisms in the two groupoids by the same symbols). Now, as $Z \subset R \times_{s,U,g} U'$ is open and $R' \to R \times_{s,U,g} U'$ is étale (as a base change of $U' \to U$) we see that $Z_{univ} \to R'$ is an open immersion. By construction the morphisms $s', t' : Z_{univ} \to U'$ are finite. It remains to find the point u' of U'.

We think of u as a morphism $Spec(\kappa(u)) \to U$ as in the statement of the lemma. Set $F_u = R \times_{s,U} Spec(\kappa(u))$. The set $\{r \in R : s(r) = u, t(r) = u\}$ is finite by assumption and $F_u \to Spec(\kappa(u))$ is quasi-finite at each of its elements. Hence we can find a decomposition into open and closed subschemes

$$F_u = Z_u \coprod Rest$$

for some scheme Z_u finite over $\kappa(u)$ whose support is $\{r \in R : s(r) = u, t(r) = u\}$. Note that $e(u) \in Z_u$. Hence by the construction of U' in Section 53.10 (u, Z_u) defines a $Spec(\kappa(u))$-valued point u' of U'.

We still have to show that the set $\{r' \in |R'| : s'(r') = u', t'(r') = u'\}$ is contained in $|Z_{univ}|$. Pick any point r' in this set and represent it by a morphism $r' : Spec(k) \to R'$. Denote $z : Spec(k) \to R$ the composition of r' with the map $R' \to R$. Since $\kappa(u) = \kappa(u')$, and since $s'(r') = u'$, $t(r') = u'$ no information is lost by considering the point z rather than the point r', i.e., we can recover r' from the point z. For example z is an element of the set $\{r \in R : s(r) = u, t(r) = u\}$ by our assumption on r'. The composition $s \circ z : Spec(k) \to U$ factors through u, so we may think of $s \circ z$ as a morphism $Spec(k) \to Spec(\kappa(u))$. Hence we can consider the triple

$$(s \circ z, Z_u \times_{Spec(\kappa(u)), s \circ z} Spec(k), z)$$

where Z_u is as above. This defines a $Spec(k)$-valued point of Z_{univ} above whose image under the map $Z_{univ} \to R'$ is the point r' by the relationship between z and r' mentioned above. This finishes the proof. □

Lemma 53.11.5. *Assumptions and notation as in Situation 53.11.3. Then there exists an algebraic space U', an étale morphism $U' \to U$, and a point $u' : Spec(\kappa(u)) \to U'$ lying over $u : Spec(\kappa(u)) \to U$ such that the restriction $R' = R|_{U'}$ of R to U' is quasi-split over u'.*

Proof. The proof is almost exactly the same as the proof of Lemma 53.11.4. Let $f : (U', Z_{univ}, s', t', c') \to (U, R, s, t, c)$ be as constructed in Lemma 53.10.1. Recall that $R' = R \times_{(U \times_S U)} (U' \times_S U')$. Thus we get a morphism $(f, t', s') : Z_{univ} \to R'$ of groupoids in algebraic spaces

$$(U', Z_{univ}, s', t', c') \to (U', R', s', t', c')$$

(by abuse of notation we indicate the morphisms in the two groupoids by the same symbols). Now, as $Z \subset R \times_{s,U,g} U'$ is open and $R' \to R \times_{s,U,g} U'$ is étale (as a base change of $U' \to U$) we see that $Z_{univ} \to R'$ is an open immersion. By construction the morphisms $s', t' : Z_{univ} \to U'$ are finite. It remains to find the point u' of U'.

We think of u as a morphism $Spec(\kappa(u)) \to U$ as in the statement of the lemma. Set $F_u = R \times_{s,U} Spec(\kappa(u))$. The morphism $F_u \to Spec(\kappa(u))$ is quasi-finite at $e(u)$ by assumption. Hence we can find a decomposition into open and closed subschemes

$$F_u = Z_u \coprod Rest$$

for some scheme Z_u finite over $\kappa(u)$ whose support is $e(u)$. Hence by the construction of U' in Section 53.10 (u, Z_u) defines a $Spec(\kappa(u))$-valued point u' of U'. To finish the proof we have to show that $e'(u') \in Z_{univ}$ which is clear. □

Finally, when we add additional assumptions we obtain schemes.

Lemma 53.11.6. *Assumptions and notation as in Situation 53.11.2. Assume in addition that s, t are flat and locally of finite presentation. Then there exists a scheme U', a separated étale morphism $U' \to U$, and a point $u' \in U'$ lying over u with $\kappa(u) = \kappa(u')$ such that the restriction $R' = R|_{U'}$ of R to U' splits over u'.*

Proof. This follows from the construction of U' in the proof of Lemma 53.11.4 because in this case $U' = (R_s/U, e)_{fin}$ is a scheme separated over U by Lemmas 53.8.14 and 53.8.15. □

Lemma 53.11.7. *Assumptions and notation as in Situation 53.11.3. Assume in addition that s, t are flat and locally of finite presentation. Then there exists a scheme U', a separated étale morphism $U' \to U$, and a point $u' \in U'$ lying over u with $\kappa(u) = \kappa(u')$ such that the restriction $R' = R|_{U'}$ of R to U' is quasi-split over u'.*

Proof. This follows from the construction of U' in the proof of Lemma 53.11.5 because in this case $U' = (R_s/U, e)_{fin}$ is a scheme separated over U by Lemmas 53.8.14 and 53.8.15. □

In fact we can obtain affine schemes by applying an earlier result on finite locally free groupoids.

Lemma 53.11.8. *Assumptions and notation as in Situation 53.11.2. Assume in addition that s, t are flat and locally of finite presentation and that U is affine. Then there exists an affine scheme U', an étale morphism $U' \to U$, and a point $u' \in U'$ lying over u with $\kappa(u) = \kappa(u')$ such that the restriction $R' = R|_{U'}$ of R to U' splits over u'.*

Proof. Let $U' \to U$ and $u' \in U'$ be the étale morphism of schemes we found in Lemma 53.11.6. Let $P \subset R'$ be the splitting of R' over u'. By More on Groupoids, Lemma 36.8.1 the morphisms $s', t' : R' \to U'$ are flat and locally of finite presentation. They are finite by assumption. Hence s', t' are finite locally free, see Morphisms, Lemma 24.44.2. In particular $t(s^{-1}(u'))$ is a finite set of points $\{u'_1, u'_2, \ldots, u'_n\}$ of U'. Choose a quasi-compact open $W \subset U'$ containing each u'_i. As U is affine the morphism $W \to U$ is quasi-compact (see Schemes, Lemma 21.19.2). The morphism $W \to U$ is also locally quasi-finite (see Morphisms, Lemma 24.35.6) and separated. Hence by More on Morphisms, Lemma 33.29.3 (a version of Zariski's Main Theorem) we conclude that W is quasi-affine. By Properties, Lemma 23.26.5 we see that $\{u'_1, \ldots, u'_n\}$ are contained in an affine open of U'. Thus we may apply Groupoids, Lemma 35.20.1 to conclude that there exists an affine P-invariant open $U'' \subset U'$ which contains u'.

To finish the proof denote $R'' = R|_{U''}$ the restriction of R to U''. This is the same as the restriction of R' to U''. As $P \subset R'$ is an open and closed subscheme, so is $P|_{U''} \subset R''$. By construction the open subscheme $U'' \subset U'$ is P-invariant which means that $P|_{U''} = (s'|_P)^{-1}(U'') = (t'|_P)^{-1}(U'')$ (see discussion in Groupoids, Section 35.16) so the restrictions of s'' and t'' to $P|_{U''}$ are still finite. The sub groupoid scheme $P|_{U''}$ is still a splitting of R'' over u''; above we verified (a), (b) and (c) holds as $\{r' \in R' : t'(r') = u', s'(r') = u'\} = \{r'' \in R'' : t''(r'') = u', s''(r'') = u'\}$ trivially. The lemma is proved. □

Lemma 53.11.9. *Assumptions and notation as in Situation 53.11.3. Assume in addition that s, t are flat and locally of finite presentation and that U is affine. Then there exists an affine scheme U', an étale morphism $U' \to U$, and a point $u' \in U'$ lying over u with $\kappa(u) = \kappa(u')$ such that the restriction $R' = R|_{U'}$ of R to U' is quasi-split over u'.*

Proof. The proof of this lemma is literally the same as the proof of Lemma 53.11.8 except that ``splitting'' needs to be replaced by ``quasi-splitting'' (2 times) and that the reference to Lemma 53.11.6. needs to be replaced by a reference to Lemma 53.11.7. □

53.12. Other chapters

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CHAPTER 54

Bootstrap

54.1. Introduction

In this chapter we use the material from the preceding sections to give criteria under which a presheaf of sets on the category of schemes is an algebraic space. Some of this material comes from the work of Artin, see [**Art69c**], [**Art70a**], [**Art73a**], [**Art71c**], [**Art71a**], [**Art69a**], [**Art69e**], and [**Art74a**]. However, our method will be to use as much as possible arguments similar to those of the paper by Keel and Mori, see [**KM97a**].

54.2. Conventions

The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $Spec(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

54.3. Morphisms representable by algebraic spaces

Here we define the notion of one presheaf being relatively representabe by algebraic spaces over another, and we prove some properties of this notion.

Definition 54.3.1. Let S be a scheme contained in Sch_{fppf}. Let F, G be presheaves on Sch_{fppf}/S. We say a morphism $a : F \to G$ is *representable by algebraic spaces* if for every $U \in Ob((Sch/S)_{fppf})$ and any $\xi : U \to G$ the fiber product $U \times_{\xi, G} F$ is an algebraic space.

Here is a sanity check.

Lemma 54.3.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Then f is representable by algebraic spaces.*

Proof. This is formal. It relies on the fact that the category of algebraic spaces over S has fibre products, see Spaces, Lemma 40.7.3. $\square$

Lemma 54.3.3. *Let S be a scheme. Let*

$$\begin{tikzcd} G' \times_G F \arrow[r] \arrow[d, "a'"] & F \arrow[d, "a"] \\ G' \arrow[r] & G \end{tikzcd}$$

be a fibre square of presheaves on $(Sch/S)_{fppf}$. If a is representable by algebraic spaces so is a'.

Proof. Omitted. Hint: This is formal. $\square$

Lemma 54.3.4. *Let S be a scheme contained in Sch_{fppf}. Let $F, G : (Sch/S)_{fppf}^{opp} \to Sets$. Let $a : F \to G$ be representable by algebraic spaces. If G is a sheaf, then so is F.*

Proof. (Same as the proof of Spaces, Lemma 40.3.5.) Let $\{\varphi_i : T_i \to T\}$ be a covering of the site $(Sch/S)_{fppf}$. Let $s_i \in F(T_i)$ which satisfy the sheaf condition. Then $\sigma_i = a(s_i) \in G(T_i)$ satisfy the sheaf condition also. Hence there exists a unique $\sigma \in G(T)$ such that $\sigma_i = \sigma|_{T_i}$. By assumption $F' = h_T \times_{\sigma, G, a} F$ is a sheaf. Note that $(\varphi_i, s_i) \in F'(T_i)$ satisfy the sheaf condition also, and hence come from some unique $(\mathrm{id}_T, s) \in F'(T)$. Clearly s is the section of F we are looking for. $\square$

Lemma 54.3.5. *Let S be a scheme contained in Sch_{fppf}. Let $F, G : (Sch/S)_{fppf}^{opp} \to Sets$. Let $a : F \to G$ be representable by algebraic spaces. Then $\Delta_{F/G} : F \to F \times_G F$ is representable by algebraic spaces.*

Proof. (Same as the proof of Spaces, Lemma 40.3.6.) Let U be a scheme. Let $\xi = (\xi_1, \xi_2) \in (F \times_G F)(U)$. Set $\xi' = a(\xi_1) = a(\xi_2) \in G(U)$. By assumption there exist an algebraic space V and a morphism $V \to U$ representing the fibre product $U \times_{\xi', G} F$. In particular, the elements ξ_1, ξ_2 give morphisms $f_1, f_2 : U \to V$ over U. Because V represents the fibre product $U \times_{\xi', G} F$ and because $\xi' = a \circ \xi_1 = a \circ \xi_2$ we see that if $g : U' \to U$ is a morphism then

$$g^*\xi_1 = g^*\xi_2 \Leftrightarrow f_1 \circ g = f_2 \circ g.$$

In other words, we see that $U \times_{\xi, F \times_G F} F$ is represented by $V \times_{\Delta, V \times V, (f_1, f_2)} U$ which is an algebraic space. $\square$

The proof of Lemma 54.3.6 below is actually slightly tricky. Namely, we cannot use the argument of the proof of Spaces, Lemma 40.11.1 because we do not yet know that a composition of transformations representable by algebraic spaces is representable by algebraic spaces. In fact, we will use this lemma to prove that statement.

Lemma 54.3.6. *Let S be a scheme contained in Sch_{fppf}. Let $F, G : (Sch/S)_{fppf}^{opp} \to Sets$. Let $a : F \to G$ be representable by algebraic spaces. If G is an algebraic space, then so is F.*

Proof. We have seen in Lemma 54.3.4 that F is a sheaf.

Let U be a scheme and let $U \to G$ be a surjective étale morphism. In this case $U \times_G F$ is an algebraic space. Let W be a scheme and let $W \to U \times_G F$ be a surjective étale morphism.

First we claim that $W \to F$ is representable. To see this let X be a scheme and let $X \to F$ be a morphism. Then

$$W \times_F X = W \times_{U \times_G F} U \times_G F \times_F X = W \times_{U \times_G F} (U \times_G X)$$

Since both $U \times_G F$ and G are algebraic spaces we see that this is a scheme.

Next, we claim that $W \to F$ is surjective and étale (this makes sense now that we know it is representable). This follows from the formula above since both $W \to U \times_G F$ and $U \to G$ are étale and surjective, hence $W \times_{U \times_G F} (U \times_G X) \to U \times_G X$ and $U \times_G X \to X$ are surjective and étale, and the composition of surjective étale morphisms is surjective and étale.

Set $R = W \times_F W$. By the above R is a scheme and the projections $t, s : R \to W$ are étale. It is clear that R is an equivalence relation, and $W \to F$ is a surjection of sheaves. Hence R is an étale equivalence relation and $F = W/R$. Hence F is an algebraic space by Spaces, Theorem 40.10.5. $\square$

Lemma 54.3.7. *Let S be a scheme. Let $a : F \to G$ be a map of presheaves on $(Sch/S)_{fppf}$. Suppose $a : F \to G$ is representable by algebraic spaces. If X is an algebraic space over S, and $X \to G$ is a map of presheaves then $X \times_G F$ is an algebraic space.*

Proof. By Lemma 54.3.3 the transformation $X \times_G F \to X$ is representable by algebraic spaces. Hence it is an algebraic space by Lemma 54.3.6. □

Lemma 54.3.8. *Let S be a scheme. Let*

$$F \xrightarrow{a} G \xrightarrow{b} H$$

be maps of presheaves on $(Sch/S)_{fppf}$. If a and b are representable by algebraic spaces, so is $b \circ a$.

Proof. Let T be a scheme over S, and let $T \to H$ be a morphism. By assumption $T \times_H G$ is an algebraic space. Hence by Lemma 54.3.7 we see that $T \times_H F = (T \times_H G) \times_G F$ is an algebraic space as well. □

Lemma 54.3.9. *Let S be a scheme. Let $F_i, G_i : (Sch/S)_{fppf}^{opp} \to Sets$, $i = 1, 2$. Let $a_i : F_i \to G_i$, $i = 1, 2$ be representable by algebraic spaces. Then*

$$a_1 \times a_2 : F_1 \times F_2 \longrightarrow G_1 \times G_2$$

is a representable by algebraic spaces.

Proof. Write $a_1 \times a_2$ as the composition $F_1 \times F_2 \to G_1 \times F_2 \to G_1 \times G_2$. The first arrow is the base change of a_1 by the map $G_1 \times F_2 \to G_1$, and the second arrow is the base change of a_2 by the map $G_1 \times G_2 \to G_2$. Hence this lemma is a formal consequence of Lemmas 54.3.8 and 54.3.3. □

54.4. Properties of maps of presheaves representable by algebraic spaces

Here is the definition that makes this work.

Definition 54.4.1. Let S be a scheme. Let $a : F \to G$ be a map of presheaves on $(Sch/S)_{fppf}$ which is representable by algebraic spaces. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which

(1) is preserved under any base change, and
(2) is fppf local on the base, see Descent on Spaces, Definition 45.9.1.

In this case we say that a has *property $\mathcal{P}$* if for every scheme U and $\xi : U \to G$ the resulting morphism of algebraic spaces $U \times_G F \to U$ has property $\mathcal{P}$.

It is important to note that we will only use this definition for properties of morphisms that are stable under base change, and local in the fppf topology on the base. This is not because the definition doesn't make sense otherwise; rather it is because we may want to give a different definition which is better suited to the property we have in mind.

The definition above applies for example to the properties of being ``surjective'', ``flat'', ``étale'', and ``locally of finite presentation''. In other words, a is *surjective* (resp. *étale*, *flat*, *locally of finite presentation*) if for every scheme T and map $\xi : T \to G$ the morphism of algebraic spaces $T \times_{\xi,G} F \to T$ is surjective (resp. étale, flat, locally of finite presentation).

By Lemma 54.3.2 any morphism between algebraic spaces over S is representable by algebraic spaces. And by Morphisms of Spaces, Lemma 42.6.3 (resp. 42.35.2, 42.27.4, 42.26.4) the definition of surjective (resp. étale, flat, locally of finite presentation) above agrees with the already existing definition of morphisms of algebraic spaces.

Some formal lemmas follow.

Lemma 54.4.2. *Let S be a scheme. Let $\mathcal{P}$ be a property as in Definition 54.4.1. Let*

$$\begin{tikzcd} G' \times_G F \arrow[r] \arrow[d, "a'"] & F \arrow[d, "a"] \\ G' \arrow[r] & G \end{tikzcd}$$

be a fibre square of presheaves on $(Sch/S)_{fppf}$. If a is representable by algebraic spaces and has $\mathcal{P}$ so does a'.

Proof. Omitted. Hint: This is formal. □

Lemma 54.4.3. *Let S be a scheme. Let $\mathcal{P}$ be a property as in Definition 54.4.1, and assume $\mathcal{P}$ is stable under composition. Let*

$$F \xrightarrow{a} G \xrightarrow{b} H$$

be maps of presheaves on $(Sch/S)_{fppf}$. If a, b are representable by algebraic spaces and has $\mathcal{P}$ so does $b \circ a$.

Proof. Omitted. Hint: See Lemma 54.3.8 and use stability under composition. □

Lemma 54.4.4. *Let S be a scheme. Let $F_i, G_i : (Sch/S)_{fppf}^{opp} \to Sets$, $i = 1, 2$. Let $a_i : F_i \to G_i$, $i = 1, 2$ be representable by algebraic spaces. Let $\mathcal{P}$ be a property as in Definition 54.4.1 which is stable under composition. If a_1 and a_2 have property $\mathcal{P}$ so does $a_1 \times a_2 : F_1 \times F_2 \longrightarrow G_1 \times G_2$.*

Proof. Note that the lemma makes sense by Lemma 54.3.9. Proof omitted. □

Lemma 54.4.5. *Let S be a scheme. Let $F, G : (Sch/S)_{fppf}^{opp} \to Sets$ be sheaves. Let $a : F \to G$ be representable by algebraic spaces, flat, locally of finite presentation, and surjective. Then $a : F \to G$ is surjective as a map of sheaves.*

Proof. Let T be a scheme over S and let $g : T \to G$ be a T-valued point of G. By assumption $T' = F \times_G T$ is an algebraic space and the morphism $T' \to T$ is a flat, locally of finite presentation, and surjective morphism of algebraic spaces. Let $U \to T'$ be a surjective étale morphism, where U is a scheme. Then by the definition of flat morphisms of algebraic spaces the morphism of schemes $U \to T$ is flat. Similarly for ``locally of finite presentation". The morphism $U \to T$ is surjective also, see Morphisms of Spaces, Lemma 42.6.3. Hence we see that $\{U \to T\}$ is an fppf covering such that $g|_U \in G(U)$ comes from an element of $F(U)$, namely the map $U \to T' \to F$. This proves the map is surjective as a map of sheaves, see Sites, Definition 9.11.1. □

54.5. Bootstrapping the diagonal

Lemma 54.5.1. *Let S be a scheme. If F is a presheaf on $(Sch/S)_{fppf}$. The following are equivalent:*

1. *$\Delta_F : F \to F \times F$ is representable by algebraic spaces,*
2. *for every scheme T any map $T \to F$ is representable by algebraic spaces, and*
3. *for every algebraic space X any map $X \to F$ is representable by algebraic spaces.*

Proof. Assume (1). Let $X \to F$ be as in (3). Let T be a scheme, and let $T \to F$ be a morphism. Then we have

$$T \times_F X = (T \times_S X) \times_{F \times F, \Delta} F$$

which is an algebraic space by Lemma 54.3.7 and (1). Hence $X \to F$ is representable, i.e., (3) holds. The implication (3) $\Rightarrow$ (2) is trivial. Assume (2). Let T be a scheme, and let $(a, b) : T \to F \times F$ be a morphism. Then

$$F \times_{\Delta_F, F \times F} T = T \times_{a, F, b} T$$

which is an algebraic space by assumption. Hence Δ_F is representable by algebraic spaces, i.e., (1) holds. □

In particular if F is a presheaf satisfying the equivalent conditions of the lemma, then for any morphism $X \to F$ where X is an algebraic space it makes sense to say that $X \to F$ is surjective (resp. étale, flat, locally of finite presentation) by using Definition 54.4.1.

Before we actually do the bootstrap we prove a fun lemma.

Lemma 54.5.2. *Let S be a scheme. Let*

$$\begin{tikzcd} E \arrow[r, "a"] \arrow[d, "f"'] & F \arrow[d, "g"] \\ H \arrow[r, "b"] & G \end{tikzcd}$$

be a cartesian diagram of sheaves on $(Sch/S)_{fppf}$, so $E = H \times_G F$. If

(1) *g is representable by algebraic spaces, surjective, flat, and locally of finite presentation, and*
(2) *a is representable by algebraic spaces, separated, and locally quasi-finite*

then b is representable (by schemes) as well as separated and locally quasi-finite.

Proof. Let T be a scheme, and let $T \to G$ be a morphism. We have to show that $T \times_G H$ is an algebraic space, and that the morphism $T \times_G H \to T$ is separated and locally quasi-finite. Thus we may base change the whole diagram to T and assume that G is a scheme. In this case F is an algebraic space. Let U be a scheme, and let $U \to F$ be a surjective étale morphism. Then $U \to F$ is representable, surjective, flat and locally of finite presentation by Morphisms of Spaces, Lemmas 42.35.7 and 42.35.8. By Lemma 54.3.8 $U \to G$ is surjective, flat and locally of finite presentation also. Note that the base change $E \times_F U \to U$ of a is still separated and locally quasi-finite (by Lemma 54.4.2). Hence we may replace the upper part of the diagram of the lemma by $E \times_F U \to U$. In other words, we may assume that $F \to G$ is a surjective, flat morphism of schemes which is locally of finite presentation. In particular, $\{F \to G\}$ is an fppf covering of schemes. By Morphisms of Spaces, Proposition 42.39.2 we conclude that E is a scheme also. By Descent, Lemma 31.35.1 the fact that $E = H \times_G F$ means that we get a descent datum on E relative to the fppf covering $\{F \to G\}$. By More on Morphisms, Lemma 33.35.1 this descent datum is effective. By Descent, Lemma 31.35.1 again this implies that H is a scheme. By Descent, Lemmas 31.19.5 and 31.19.22 it now follows that b is separated and locally quasi-finite. □

Here is the result that the section title refers to.

Lemma 54.5.3. *Let S be a scheme. Let $F : (Sch/S)_{fppf}^{opp} \to Sets$ be a functor. Assume that*

(1) *the presheaf F is a sheaf,*

(2) *there exists an algebraic space X and a map $X \to F$ which is representable by algebraic spaces, surjective, flat and locally of finite presentation.*

Then Δ_F is representable (by schemes).

Proof. Let $U \to X$ be a surjective étale morphism from a scheme towards X. Then $U \to X$ is representable, surjective, flat and locally of finite presentation by Morphisms of Spaces, Lemmas 42.35.7 and 42.35.8. By Lemma 54.4.3 the composition $U \to F$ is representable by algebraic spaces, surjective, flat and locally of finite presentation also. Thus we see that $R = U \times_F U$ is an algebraic space, see Lemma 54.3.7. The morphism of algebraic spaces $R \to U \times_S U$ is a monomorphism, hence separated (as the diagonal of a monomorphism is an isomorphism, see Morphisms of Spaces, Lemma 42.14.2). Since $U \to F$ is locally of finite presentation, both morphisms $R \to U$ are locally of finite presentation, see Lemma 54.4.2. Hence $R \to U \times_S U$ is locally of finite type (use Morphisms of Spaces, Lemmas 42.26.5 and 42.22.6). Altogether this means that $R \to U \times_S U$ is a monomorphism which is locally of finite type, hence a separated and locally quasi-finite morphism, see Morphisms of Spaces, Lemma 42.25.8.

Now we are ready to prove that Δ_F is representable. Let T be a scheme, and let $(a, b) : T \to F \times F$ be a morphism. Set

$$T' = (U \times_S U) \times_{F \times F} T.$$

Note that $U \times_S U \to F \times F$ is representable by algebraic spaces, surjective, flat and locally of finite presentation by Lemma 54.4.4. Hence T' is an algebraic space, and the projection morphism $T' \to T$ is surjective, flat, and locally of finite presentation. Consider $Z = T \times_{F \times F} F$ (this is a sheaf) and

$$Z' = T' \times_{U \times_S U} R = T' \times_T Z.$$

We see that Z' is an algebraic space, and $Z' \to T'$ is separated and locally quasi-finite by the discussion in the first paragraph of the proof which showed that R is an algebraic space and that the morphism $R \to U \times_S U$ has those properties. Hence we may apply Lemma 54.5.2 to the diagram

$$\begin{tikzcd} Z' \arrow[r] \arrow[d] & T' \arrow[d] \\ Z \arrow[r] & T \end{tikzcd}$$

and we conclude. □

54.6. Bootstrap

We warn the reader right away that the result of this section will be superseded by the stronger Theorem 54.10.1. On the other hand, the theorem in this section is quite a bit easier to prove and still provides quite a bit of insight into how things work, especially for those readers mainly interested in Deligne-Mumford stacks.

In Spaces, Section 40.6 we defined an algebraic space as a sheaf in the fppf topology whose diagonal is representable, and such that there exist a surjective étale morphism from a scheme towards it. In this section we show that a sheaf in the fppf topology whose diagonal is representable by algebraic spaces and which has an étale surjective covering by an algebraic space is also an algebraic space. In other words, the category of algebraic spaces is an enlargement of the category of schemes by those fppf sheaves F which have a representable diagonal and an étale covering by a scheme. The result of this section says that doing the same process again starting with the category of algebraic spaces, does not lead to yet another category.

Another motivation for the material in this section is that it will guarantee later that a Deligne-Mumford stack whose inertia stack is trivial is equivalent to an algebraic space, see Algebraic Stacks, Lemma 57.13.2.

Here is the main result of this section (as we mentioned above this will be superseded by the stronger Theorem 54.10.1).

Theorem 54.6.1. *Let S be a scheme. Let $F : (Sch/S)_{fppf}^{opp} \to Sets$ be a functor. Assume that*

(1) *the presheaf F is a sheaf,*
(2) *the diagonal morphism $F \to F \times F$ is representable by algebraic spaces, and*
(3) *there exists an algebraic space X and a map $X \to F$ which is surjective, and étale.*

Then F is an algebraic space.

Proof. We will use the remarks directly below Definition 54.4.1 without further mention. In the situation of the theorem, let $U \to X$ be a surjective étale morphism from a scheme towards X. By Lemma 54.3.8 $U \to F$ is surjective and étale also. Hence the theorem boils down to proving that Δ_F is representable. This follows immediately from Lemma 54.5.3. On the other hand we can circumvent this lemma and show directly F is an algebraic space as in the next paragraph.

Let U be a scheme, and let $U \to F$ be surjective and étale. Set $R = U \times_F U$, which is an algebraic space (see Lemma 54.5.1). The morphism of algebraic spaces $R \to U \times_S U$ is a monomorphism, hence separated (as the diagonal of a monomorphism is an isomorphism). Moreover, since $U \to F$ is étale, we see that $R \to U$ is étale, by Lemma 54.4.2. In particular, we see that $R \to U$ is locally quasi-finite, see Morphisms of Spaces, Lemma 42.35.5. We conclude that also $R \to U \times_S U$ is locally quasi-finite by Morphisms of Spaces, Lemma 42.25.7. Hence Morphisms of Spaces, Proposition 42.39.2 applies and R is a scheme. Hence $F = U/R$ is an algebraic space according to Spaces, Theorem 40.10.5. $\square$

54.7. Finding opens

First we prove a lemma which is a slight improvement and generalization of Spaces, Lemma 40.10.2 to quotient sheaves associated to groupoids.

Lemma 54.7.1. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $g : U' \to U$ be a morphism. Assume*

(1) *the composition*

$$\begin{tikzcd} U' \times_{g,U,t} R \arrow[r, "pr_1"'] \arrow[rr, bend left, "h"] & R \arrow[r, "s"'] & U \end{tikzcd}$$

has an open image $W \subset U$, and
(2) *the resulting map $h : U' \times_{g,U,t} R \to W$ defines a surjection of sheaves in the fppf topology.*

Let $R' = R|_{U'}$ be the restriction of R to U. Then the map of quotient sheaves

$$R'/U' \to R/U$$

in the fppf topology is representable, and is an open immersion.

Proof. Note that W is an R-invariant open subscheme of U. This is true because the set of points of W is the set of points of U which are equivalent in the sense of Groupoids, Lemma 35.3.4 to a point of $g(U') \subset U$ (the lemma applies as $j : R \to U \times_S U$ is a pre-equivalence relation by Groupoids, Lemma 35.11.2). Also $g : U' \to U$ factors through W. Let $R|_W$ be the restriction of R to W. Then it follows that R' is also the restriction of $R|_W$ to U'. Hence we can factor the map of sheaves of the lemma as

$$U'/R' \longrightarrow W/R|_W \longrightarrow U/R$$

By Groupoids, Lemma 35.17.6 we see that the first arrow is an isomorphism of sheaves. Hence it suffices to show the lemma in case g is the immersion of an R-invariant open into U.

Assume $U' \subset U$ is an R-invariant open and g is the inclusion morphism. Set $F = U/R$ and $F' = U'/R'$. By Groupoids, Lemma 35.17.5 or 35.17.6 the map $F' \to F$ is injective. Let $\xi \in F(T)$. We have to show that $T \times_{\xi, F} F'$ is representable by an open subscheme of T. There exists an fppf covering $\{f_i : T_i \to T\}$ such that $\xi|_{T_i}$ is the image via $U \to U/R$ of a morphism $a_i : T_i \to U$. Set $V_i = s_i^{-1}(U')$. We claim that $V_i \times_T T_j = T_i \times_T V_j$ as open subschemes of $T_i \times_T T_j$.

As $a_i \circ \text{pr}_0$ and $a_j \circ \text{pr}_1$ are morphisms $T_i \times_T T_j \to U$ which both map to the section $\xi|_{T_i \times_T T_j} \in F(T_i \times_T T_j)$ we can find an fppf covering $\{f_{ijk} : T_{ijk} \to T_i \times_T T_j\}$ and morphisms $r_{ijk} : T_{ijk} \to R$ such that

$$a_i \circ \text{pr}_0 \circ f_{ijk} = s \circ r_{ijk}, \quad a_j \circ \text{pr}_1 \circ f_{ijk} = t \circ r_{ijk},$$

see Groupoids, Lemma 35.17.4. Since U' is R-invariant we have $s^{-1}(U') = t^{-1}(U')$ and hence $f_{ijk}^{-1}(V_i \times_T T_j) = f_{ijk}^{-1}(T_i \times_T V_j)$. As $\{f_{ijk}\}$ is surjective this implies the claim above. Hence by Descent, Lemma 31.9.2 there exists an open subscheme $V \subset T$ such that $f_i^{-1}(V) = V_i$. We claim that V represents $T \times_{\xi, F} F'$.

As a first step, we will show that $\xi|_V$ lies in $F'(V) \subset F(V)$. Namely, the family of morphisms $\{V_i \to V\}$ is an fppf covering, and by construction we have $\xi|_{V_i} \in F'(V_i)$. Hence by the sheaf property of F' we get $\xi|_V \in F'(V)$. Finally, let $T' \to T$ be a morphism of schemes and that $\xi|_{T'} \in F'(T')$. To finish the proof we have to show that $T' \to T$ factors through V. We can find a fppf covering $\{T'_j \to T'\}_{j \in J}$ and morphisms $b_j : T'_j \to U'$ such that $\xi|_{T'_j}$ is the image via $U' \to U/R$ of b_j. Clearly, it is enough to show that the compositions $T'_j \to T$ factor through V. Hence we may assume that $\xi|_{T'}$ is the image of a morphism $b : T' \to U'$. Now, it is enough to show that $T' \times_T T_i \to T_i$ factors through V_i. Over the scheme $T' \times_T T_i$ the restriction of ξ is the image of two elements of $(U/R)(T' \times_T T_i)$, namely $a_i \circ \text{pr}_1$, and $b \circ \text{pr}_0$, the second of which factors through the R-invariant open U'. Hence by Groupoids, Lemma 35.17.4 there exists a covering $\{h_k : Z_k \to T' \times_T T_i\}$ and morphisms $r_k : Z_k \to R$ such that $a_i \circ \text{pr}_1 \circ h_k = s \circ r_k$ and $b \circ \text{pr}_0 \circ h_k = t \circ r_k$. As U' is an R-invariant open the fact that b has image in U' then implies that each $a_i \circ \text{pr}_1 \circ h_k$ has image in U'. It follows from this that $T' \times_T T_i \to T_i$ has image in V_i by definition of V_i which concludes the proof. □

54.8. Slicing equivalence relations

In this section we explain how to ``improve" a given equivalence relation by slicing. This is not a kind of ``étale slicing" that you may be used to but a much coarser kind of slicing.

Lemma 54.8.1. *Let S be a scheme. Let $j : R \to U \times_S U$ be an equivalence relation on schemes over S. Assume $s, t : R \to U$ are flat and locally of finite presentation. Then there exists an equivalence relation $j' : R' \to U' \times_S U'$ on schemes over S, and an isomorphism*

$$U'/R' \longrightarrow U/R$$

induced by a morphism $U' \to U$ which maps R' into R such that $s', t' : R \to U$ are flat, locally of finite presentation and locally quasi-finite.

Proof. We will prove this lemma in several steps. We will use without further mention that an equivalence relation gives rise to a groupoid scheme and that the restriction of an equivalence relation is an equivalence relation, see Groupoids, Lemmas 35.3.2, 35.11.3, and 35.15.3.

Step 1: We may assume that $s, t : R \to U$ are locally of finite presentation and Cohen-Macaulay morphisms. Namely, as in More on Groupoids, Lemma 36.7.1 let $g : U' \to U$ be the open subscheme such that $t^{-1}(U') \subset R$ is the maximal open over which $s : R \to U$ is Cohen-Macaulay, and denote R' the restriction of R to U'. By the lemma cited above we see that

$$\begin{tikzcd} t^{-1}(U') \arrow[r, equal] & U' \times_{g,U,t} R \arrow[r, "\mathrm{pr}_1"'] \arrow[rr, bend left, "h"] & R \arrow[r, "s"'] & U \end{tikzcd}$$

is surjective. Since h is flat and locally of finite presentation, we see that $\{h\}$ is a fppf covering. Hence by Groupoids, Lemma 35.17.6 we see that $U'/R' \to U/R$ is an isomorphism. By the construction of U' we see that s', t' are Cohen-Macaulay and locally of finite presentation.

Step 2. Assume s, t are Cohen-Macaulay and locally of finite presentation. Let $u \in U$ be a point of finite type. By More on Groupoids, Lemma 36.11.4 there exists an affine scheme U' and a morphism $g : U' \to U$ such that

(1) g is an immersion,
(2) $u \in U'$,
(3) g is locally of finite presentation,
(4) h is flat, locally of finite presentation and locally quasi-finite, and
(5) the morphisms $s', t' : R' \to U'$ are flat, locally of finite presentation and locally quasi-finite.

Here we have used the notation introduced in More on Groupoids, Situation 36.11.1.

Step 3. For each point $u \in U$ which is of finite type choose a $g_u : U'_u \to U$ as in Step 2 and denote R'_u the restriction of R to U'_u. Denote $h_u = s \circ \mathrm{pr}_1 : U'_u \times_{g_u,U,t} R \to U$. Set $U' = \coprod_{u \in U} U'_u$, and $g = \coprod g_u$. Let R' be the restriction of R to U as above. We claim that the pair (U', g) works[1]. Note that

$$\begin{aligned} R' &= \coprod\nolimits_{u_1,u_2 \in U} (U'_{u_1} \times_{g_{u_1},U,t} R) \times_R (R \times_{s,U,g_{u_2}} U'_{u_2}) \\ &= \coprod\nolimits_{u_1,u_2 \in U} (U'_{u_1} \times_{g_{u_1},U,t} R) \times_{h_{u_1},U,g_{u_2}} U'_{u_2} \end{aligned}$$

[1]Here we should check that U' is not too large, i.e., that it is isomorphic to an object of the category Sch_{fppf}, see Section 54.2. This is a purely set theoretical matter; let us use the notion of size of a scheme introduced in Sets, Section 3.9. Note that each U'_u has size at most the size of U and that the cardinality of the index set is at most the cardinality of $|U|$ which is bounded by the size of U. Hence U' is isomorphic to an object of Sch_{fppf} by Sets, Lemma 3.9.9 part (6).

Hence the projection $s' : R' \to U' = \coprod U'_{u_2}$ is flat, locally of finite presentation and locally quasi-finite as a base change of $\coprod h_{u_1}$. Finally, by construction the morphism $h : U' \times_{g,U,t} R \to U$ is equal to $\coprod h_u$ hence its image contains all points of finite type of U. Since each h_u is flat and locally of finite presentation we conclude that h is flat and locally of finite presentation. In particular, the image of h is open (see Morphisms, Lemma 24.24.9) and since the set of points of finite type is dense (see Morphisms, Lemma 24.15.7) we conclude that the image of h is U. This implies that $\{h\}$ is an fppf covering. By Groupoids, Lemma 35.17.6 this means that $U'/R' \to U/R$ is an isomorphism. This finishes the proof of the lemma. □

54.9. Quotient by a subgroupoid

We need one more lemma before we can do our final bootstrap. Let us discuss what is going on in terms of ``plain'' groupoids before embarking on the scheme theoretic version.

Let $\mathcal{C}$ be a groupoid, see Categories, Definition 4.2.5. As discussed in Groupoids, Section 35.11 this corresponds to a quintuple $(\text{Ob}, \text{Arrows}, s, t, c)$. Suppose we are given a subset $P \subset \text{Arrows}$ such that $(\text{Ob}, P, s|_P, t|_P, c|_P)$ is also a groupoid and such that there are no non-trivial automorphisms in P. Then we can construct the quotient groupoid $(\overline{\text{Ob}}, \overline{\text{Arrows}}, \overline{s}, \overline{t}, \overline{c})$ as follows:

(1) $\overline{\text{Ob}} = \text{Ob}/P$ is the set of P-isomorphism classes,
(2) $\overline{\text{Arrows}} = P\backslash\text{Arrows}/P$ is the set of arrows in $\mathcal{C}$ up to pre-composing and post-composing by arrows of P,
(3) the source and target maps $\overline{s}, \overline{t} : P\backslash\text{Arrows}/P \to \text{Ob}/P$ are induced by s, t,
(4) composition is defined by the rule $\overline{c}(\overline{a}, \overline{b}) = \overline{c(a, b)}$ which is well defined.

In fact, it turns out that the original groupoid $(\text{Ob}, \text{Arrows}, s, t, c)$ is canonically isomorphic to the restriction (see discussion in Groupoids, Section 35.15) of the groupoid $(\overline{\text{Ob}}, \overline{\text{Arrows}}, \overline{s}, \overline{t}, \overline{c})$ via the quotient map $g : \text{Ob} \to \overline{\text{Ob}}$. Recall that this means that

$$\text{Arrows} = \text{Ob} \times_{g, \overline{\text{Ob}}, \overline{t}} \overline{\text{Arrows}} \times_{\overline{s}, \overline{\text{Ob}}, g} \text{Ob}$$

which holds as P has no nontrivial automorphisms. We omit the details.

The following lemma holds in much greater generality, but this is the version we use in the proof of the final bootstrap (after which we can more easily prove the more general versions of this lemma).

Lemma 54.9.1. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $P \to R$ be monomorphism of schemes. Assume that*

(1) *$(U, P, s|_P, t|_P, c|_{P \times_{s,U,t} P})$ is a groupoid scheme,*
(2) *$s|_P, t|_P : P \to U$ are finite locally free,*
(3) *$j|_P : P \to U \times_S U$ is a monomorphism.*
(4) *U is affine, and*
(5) *$j : R \to U \times_S U$ is separated and locally quasi-finite,*

Then U/P is representable by an affine scheme $\overline{U}$, the quotient morphism $U \to \overline{U}$ is finite locally free, and $P = U \times_{\overline{U}} U$. Moreover, R is the restriction of a groupoid scheme $(\overline{U}, \overline{R}, \overline{s}, \overline{t}, \overline{c})$ on $\overline{U}$ via the quotient morphism $U \to \overline{U}$.

Proof. Conditions (1), (2), (3), and (4) and Groupoids, Proposition 35.19.8 imply the affine scheme $\overline{U}$ representing U/P exists, the morphism $U \to \overline{U}$ is finite locally free, and $P = U \times_{\overline{U}} U$. The identification $P = U \times_{\overline{U}} U$ is such that $t|_P = \text{pr}_0$ and $s|_P = \text{pr}_1$, and such that composition is equal to $\mathrm{pr}_{02} : U \times_{\overline{U}} U \times_{\overline{U}} U \to U \times_{\overline{U}} U$. A product of finite locally free morphisms is finite locally free (see Spaces, Lemma 40.5.7 and Morphisms, Lemmas 24.44.4 and 24.44.3). To get $\overline{R}$ we are going to descend the scheme R via the finite locally free morphism $U \times_S U \to \overline{U} \times_S \overline{U}$. Namely, note that

$$(U \times_S U) \times_{(\overline{U} \times_S \overline{U})} (U \times_S U) = P \times_S P$$

by the above. Thus giving a descent datum (see Descent, Definition 31.30.1) for $R/U \times_S U/\overline{U} \times_S \overline{U}$ consists of an isomorphism

$$\varphi : R \times_{(U \times_S U), t \times t} (P \times_S P) \longrightarrow (P \times_S P) \times_{s \times s, (U \times_S U)} R$$

over $P \times_S P$ satisfying a cocycle condition. We define φ on T-valued points by the rule

$$\varphi : (r, (p, p')) \longmapsto ((p, p'), p^{-1} \circ r \circ p')$$

where the composition is taken in the groupoid category $(U(T), R(T), s, t, c)$. This makes sense because for $(r, (p, p'))$ to be a T-valued point of the source of φ it needs to be the case that $t(r) = t(p)$ and $s(r) = t(p')$. Note that this map is an isomorphism with inverse given by $((p, p'), r') \mapsto (p \circ r' \circ (p')^{-1}, (p, p'))$. To check the cocycle condition we have to verify that $\varphi_{02} = \varphi_{12} \circ \varphi_{01}$ as maps over

$$(U \times_S U) \times_{(\overline{U} \times_S \overline{U})} (U \times_S U) \times_{(\overline{U} \times_S \overline{U})} (U \times_S U) = (P \times_S P) \times_{s \times s, (U \times_S U), t \times t} (P \times_S P)$$

By explicit calculation we see that

$$\begin{array}{lllc} \varphi_{02} & (r, (p_1, p_1'), (p_2, p_2')) & \mapsto & ((p_1, p_1'), (p_2, p_2'), (p_1 \circ p_2)^{-1} \circ r \circ (p_1' \circ p_2')) \\ \varphi_{01} & (r, (p_1, p_1'), (p_2, p_2')) & \mapsto & ((p_1, p_1'), p_1^{-1} \circ r \circ p_1', (p_2, p_2')) \\ \varphi_{12} & ((p_1, p_1'), r, (p_2, p_2')) & \mapsto & ((p_1, p_1'), (p_2, p_2'), p_2^{-1} \circ r \circ p_2') \end{array}$$

(with obvious notation) which implies what we want. As j is separated and locally quasi-finite by (5) we may apply More on Morphisms, Lemma 33.35.1 to get a scheme $\overline{R} \to \overline{U} \times_S \overline{U}$ and an isomorphism

$$R \to \overline{R} \times_{(\overline{U} \times_S \overline{U})} (U \times_S U)$$

which identifies the descent datum φ with the canonical descent datum on $\overline{R} \times_{(\overline{U} \times_S \overline{U})} (U \times_S U)$, see Descent, Definition 31.30.10.

Since $U \times_S U \to \overline{U} \times_S \overline{U}$ is finite locally free we conclude that $R \to \overline{R}$ is finite locally free as a base change. Hence $R \to \overline{R}$ is surjective as a map of sheaves on $(Sch/S)_{fppf}$. Our choice of φ implies that given T-valued points $r, r' \in R(T)$ these have the same image in $\overline{R}$ if and only if $p^{-1} \circ r \circ p'$ for some $p, p' \in P(T)$. Thus $\overline{R}$ represents the sheaf

$$T \longmapsto \overline{R(T)} = P(T) \backslash R(T) / P(T)$$

with notation as in the discussion preceding the lemma. Hence we can define the groupoid structure on $(\overline{U} = U/P, \overline{R} = P \backslash R/P)$ exactly as in the discussion of the ``plain'' groupoid case. It follows from this that (U, R, s, t, c) is the pullback of this groupoid structure via the morphism $U \to \overline{U}$. This concludes the proof. □

54.10. Final bootstrap

The results in this section go quite a bit beyond the earlier results.

Theorem 54.10.1. *Let S be a scheme. Let $F : (Sch/S)_{fppf}^{opp} \to Sets$ be a functor. Any one of the following conditions implies that F is an algebraic space:*

(1) *$F = U/R$ where (U, R, s, t, c) is a groupoid in algebraic spaces over S such that s, t are flat and locally of finite presentation, and $j = (t, s) : R \to U \times_S U$ is an equivalence relation,*
(2) *$F = U/R$ where (U, R, s, t, c) is a groupoid scheme over S such that s, t are flat and locally of finite presentation, and $j = (t, s) : R \to U \times_S U$ is an equivalence relation,*
(3) *F is a sheaf and there exists an algebraic space U and a morphism $U \to F$ which is which is representable by algebraic spaces, surjective, flat and locally of finite presentation,*
(4) *F is a sheaf and there exists a scheme U and a morphism $U \to F$ which is which is representable (by algebraic spaces or schemes), surjective, flat and locally of finite presentation,*
(5) *F is a sheaf, Δ_F is representable by algebraic spaces, and there exists an algebraic space U and a morphism $U \to F$ which is surjective, flat, and locally of finite presentation, or*
(6) *F is a sheaf, Δ_F is representable, and there exists a scheme U and a morphism $U \to F$ which is surjective, flat, and locally of finite presentation.*

Proof. Trivial observations: (6) is a special case of (5) and (4) is a special case of (3). We first prove that cases (5) and (3) reduce to case (1). Namely, by bootstrapping the diagonal Lemma 54.5.3 we see that (3) implies (5). In case (5) we set $R = U \times_F U$ which is an algebraic space by assumption. Moreover, by assumption both projections $s, t : R \to U$ are surjective, flat and locally of finite presentation. The map $j : R \to U \times_S U$ is clearly an equivalence relation. By Lemma 54.4.5 the map $U \to F$ is a surjection of sheaves. Thus $F = U/R$ which reduces us to case (1).

Next, we show that (1) reduces to (2). Namely, let (U, R, s, t, c) be a groupoid in algebraic spaces over S such that s, t are flat and locally of finite presentation, and $j = (t, s) : R \to U \times_S U$ is an equivalence relation. Choose a scheme U' and a surjective étale morphism $U' \to U$. Let $R' = R|_{U'}$ be the restriction of R to U'. By Groupoids in Spaces, Lemma 52.18.6 we see that $U/R = U'/R'$. Since $s', t' : R' \to U'$ are also flat and locally of finite presentation (see More on Groupoids in Spaces, Lemma 53.6.1) this reduces us to the case where U is a scheme. As j is an equivalence relation we see that j is a monomorphism. As $s : R \to U$ is locally of finite presentation we see that $j : R \to U \times_S U$ is locally of finite type, see Morphisms of Spaces, Lemma 42.22.6. By Morphisms of Spaces, Lemma 42.25.8 we see that j is locally quasi-finite and separated. Hence if U is a scheme, then R is a scheme by Morphisms of Spaces, Proposition 42.39.2. Thus we reduce to proving the theorem in case (2).

Assume $F = U/R$ where (U, R, s, t, c) is a groupoid scheme over S such that s, t are flat and locally of finite presentation, and $j = (t, s) : R \to U \times_S U$ is an equivalence relation. By Lemma 54.8.1 we reduce to that case where s, t are flat, locally of finite presentation, and locally quasi-finite. Let $U = \bigcup_{i \in I} U_i$ be an affine open covering (with index set I of cardinality $\leq$ than the size of U to avoid set theoretic problems later -- most readers can safely ignore this remark). Let $(U_i, R_i, s_i, t_i, c_i)$ be the restriction of R to U_i. It is clear that s_i, t_i are still flat, locally of finite presentation, and locally quasi-finite as R_i is the open subscheme $s^{-1}(U_i) \cap t^{-1}(U_i)$ of R and s_i, t_i are the restrictions of s, t to this open. By Lemma 54.7.1 (or the simpler Spaces, Lemma 40.10.2) the map $U_i/R_i \to U/R$ is representable by open immersions. Hence if we can show that $F_i = U_i/R_i$ is an algebraic space, then $\coprod_{i \in I} F_i$ is an algebraic space by Spaces, Lemma 40.8.3. As $U = \bigcup U_i$ is an open covering it is clear that $\coprod F_i \to F$ is surjective. Thus it follows that U/R is an algebraic space, by Spaces, Lemma 40.8.4. In this way we reduce to the case where U is affine and s, t are flat, locally of finite presentation, and locally quasi-finite and j is an equivalence.

Assume (U, R, s, t, c) is a groupoid scheme over S, with U affine, such that s, t are flat, locally of finite presentation, and locally quasi-finite, and j is an equivalence relation. Choose $u \in U$. We apply More on Groupoids in Spaces, Lemma 53.11.9 to $u \in U, R, s, t, c$. We obtain an affine scheme U', an étale morphism $g : U' \to U$, a point $u' \in U'$ with $\kappa(u) = \kappa(u')$ such that the restriction $R' = R|_{U'}$ is quasi-split over u'. Note that the image $g(U')$ is open as g is étale and contains u'. Hence, repeatedly applying the lemma, we can find finitely many points $u_i \in U$, $i = 1, \ldots, n$, affine schemes U_i', étale morphisms $g_i : U_i' \to U$, points $u_i' \in U_i'$ with $g(u_i') = u_i$ such that (a) each restriction R_i' is quasi-split over some point in U_i' and (b) $U = \bigcup_{i=1,\ldots,n} g_i(U_i')$. Now we rerun the last part of the argument in the preceding paragraph: Using Lemma 54.7.1 (or the simpler Spaces, Lemma 40.10.2) the map $U_i'/R_i' \to U/R$ is representable by open immersions. If we can show that $F_i = U_i'/R_i'$ is an algebraic space, then $\coprod_{i \in I} F_i$ is an algebraic space by Spaces, Lemma 40.8.3. As $\{g_i : U_i' \to U\}$ is an étale covering it is clear that $\coprod F_i \to F$ is surjective. Thus it follows that U/R is an algebraic space, by Spaces, Lemma 40.8.4. In this way we reduce to the case where U is affine and s, t are flat, locally of finite presentation, and locally quasi-finite, j is an equivalence, and R is quasi-split over u for some $u \in U$.

Assume (U, R, s, t, c) is a groupoid scheme over S, with U affine, $u \in U$ such that s, t are flat, locally of finite presentation, and locally quasi-finite and $j = (t, s) : R \to U \times_S U$ is an equivalence relation and R is quasi-split over u. Let $P \subset R$ be a quasi-splitting of R over u. By Lemma 54.9.1 we see that (U, R, s, t, c) is the restriction of a groupoid $(\overline{U}, \overline{R}, \overline{s}, \overline{t}, \overline{c})$ by a surjective finite locally free morphism $U \to \overline{U}$ such that $P = U \times_{\overline{U}} U$. Note that s, t are the base changes of the morphsms $\overline{s}, \overline{t}$ by $U \to \overline{U}$. As $\{U \to \overline{U}\}$ is an fppf covering we conclude $\overline{s}, \overline{t}$ are flat, locally of finite presentation, and locally quasi-finite, see Descent, Lemmas 31.19.13, 31.19.9, and 31.19.22. Consider the commutative diagram

$$
\begin{tikzcd}
U \times_{\overline{U}} U \arrow[r, equal] \arrow[dr] & P \arrow[r] \arrow[d] & R \arrow[d] \\
& \overline{U} \arrow[r, "\overline{e}"] & \overline{R}
\end{tikzcd}
$$

It is a general fact about restrictions that the outer four corners form a cartesian diagram. By the equality we see the inner square is cartesian. Since P is open in R (by definition of a quasi-splitting) we conclude that $\overline{e}$ is an open immersion by Descent, Lemma 31.19.14. An application of Groupoids, Lemma 35.17.5 shows that $U/R = \overline{U}/\overline{R}$. Hence we have reduced to the case where (U, R, s, t, c) is a groupoid scheme over S, with U affine, $u \in U$ such that s, t are flat, locally of finite presentation, and locally quasi-finite and $j = (t, s) : R \to U \times_S U$ is an equivalence relation and $e : U \to R$ is an open immersion!

But of course, if e is an open immersion and s, t are flat and locally of finite presentation then the morphisms t, s are étale. For example you can see this by applying More on Groupoids, Lemma 36.4.1 which shows that $\Omega_{R/U} = 0$ which in turn implies that $s, t : R \to U$ is G-unramified (see Morphisms, Lemma 24.34.2), which in turn implies that s, t are étale (see Morphisms, Lemma 24.35.16). And if s, t are étale then finally U/R is an algebraic space by Spaces, Theorem 40.10.5. $\square$

54.11. Applications

As a first application we obtain the following fundamental fact:

$$\boxed{\text{A sheaf which is fppf locally an algebraic space is an algebraic space.}}$$

This is the content of the following lemma. Note that assumption (2) is equivalent to the condition that $F|_{(Sch/S_i)_{fppf}}$ is an algebraic space, see Spaces, Lemma 40.16.4. Assumption (3) is a set theoretic condition which may be ignored by those not worried about set theoretic questions.

Lemma 54.11.1. *Let S be a scheme. Let $F : (Sch/S)_{fppf}^{opp} \to Sets$ be a functor. Let $\{S_i \to S\}_{i\in I}$ be a covering of $(Sch/S)_{fppf}$. Assume that*

(1) *F is a sheaf,*
(2) *each $F_i = h_{S_i} \times F$ is an algebraic space, and*
(3) *$\coprod_{i\in I} F_i$ is an algebraic space (see Spaces, Lemma 40.8.3).*

Then F is an algebraic space.

Proof. Consider the morphism $\coprod F_i \to F$. This is the base change of $\coprod S_i \to S$ via $F \to S$. Hence it is representable, locally of finite presentation, flat and surjective by our definition of an fppf covering and Lemma 54.4.2. Thus Theorem 54.10.1 applies to show that F is an algebraic space. $\square$

Here is a special case where we do not need to worry about set theoretical issues.

Lemma 54.11.2. *Let S be a scheme. Let $F : (Sch/S)_{fppf}^{opp} \to Sets$ be a functor. Let $\{S_i \to S\}_{i\in I}$ be a covering of $(Sch/S)_{fppf}$. Assume that*

(1) *F is a sheaf,*
(2) *each $F_i = h_{S_i} \times F$ is an algebraic space, and*
(3) *the morphisms $F_i \to S_i$ are of finite type.*

Then F is an algebraic space.

Proof. We will use Lemma 54.11.1 above. To do this we will show that the assumption that F_i is of finite type over S_i to prove that the set theoretic condition in the lemma is satisfied (after perhaps refining the given covering of S a bit). We suggest the reader skip the rest of the proof.

If $S_i' \to S_i$ is a morphism of schemes then

$$h_{S_i'} \times F = h_{S_i'} \times_{h_{S_i}} h_{S_i} \times F = h_{S_i'} \times_{h_{S_i}} F_i$$

is an algebraic space of finite type over S_i', see Spaces, Lemma 40.7.3 and Morphisms of Spaces, Lemma 42.22.3. Thus we may refine the given covering. After doing this we may assume: (a) each S_i is affine, and (b) the cardinality of I is at most the cardinality of the set of points of S. (Since to cover all of S it is enough that each point is in the image of $S_i \to S$ for some i.)

Since each S_i is affine and each F_i of finite type over S_i we conclude that F_i is quasi-compact. Hence by Properties of Spaces, Lemma 41.6.3 we can find an affine $U_i \in Ob((Sch/S)_{fppf})$ and a surjective étale morphism $U_i \to F_i$. The fact that $F_i \to S_i$ is locally of finite type then implies that $U_i \to S_i$ is locally of finite type, and in particular $U_i \to S$ is locally of finite type. By Sets, Lemma 3.9.7 we conclude that $\text{size}(U_i) \leq \text{size}(S)$. Since also $|I| \leq \text{size}(S)$ we conclude that $\coprod_{i\in I} U_i$ is isomorphic to an object of $(Sch/S)_{fppf}$ by Sets, Lemma 3.9.5

and the construction of *Sch*. This implies that $\coprod F_i$ is an algebraic space by Spaces, Lemma 40.8.3 and we win. □

Lemma 54.11.3. *Assume $B \to S$ and (U, R, s, t, c) are as in Groupoids in Spaces, Definition 52.19.1 (1). For any scheme T over S and objects x, y of $[U/R]$ over T the sheaf $Isom(x, y)$ on $(Sch/T)_{fppf}$ is an algebraic space.*

Proof. By Groupoids in Spaces, Lemma 52.21.3 there exists an fppf covering $\{T_i \to T\}_{i\in I}$ such that $Isom(x, y)|_{(Sch/T_i)_{fppf}}$ is an algebraic space for each i. By Spaces, Lemma 40.16.4 this means that each $F_i = h_{S_i} \times Isom(x, y)$ is an algebraic space. Thus to prove the lemma we only have to verify the set theoretic condition that $\coprod F_i$ is an algebraic space of Lemma 54.11.1 above to conclude. To do this we use Spaces, Lemma 40.8.3 which requires showing that I and the F_i are not ``too large". We suggest the reader skip the rest of the proof.

Choose $U' \in Ob(Sch/S)_{fppf}$ and a surjective étale morphism $U' \to U$. Let R' be the restriction of R to U'. Since $[U/R] = [U'/R']$ we may, after replacing U by U', assume that U is a scheme. (This step is here so that the fibre products below are over a scheme.)

Note that if we refine the covering $\{T_i \to T\}$ then it remains true that each F_i is an algebraic space. Hence we may assume that each T_i is affine. Since $T_i \to T$ is locally of finite presentation, this then implies that $\text{size}(T_i) \leq \text{size}(T)$, see Sets, Lemma 3.9.7. We may also assume that the cardinality of the index set I is at most the cardinality of the set of points of T since to get a covering it suffices to check that each point of T is in the image. Hence $|I| \leq \text{size}(T)$. Choose $W \in Ob((Sch/S)_{fppf})$ and a surjective étale morphism $W \to R$. Note that in the proof of Groupoids in Spaces, Lemma 52.21.3 we showed that F_i is representable by $T_i \times_{(y_i,x_i),U\times_B U} R$ for some $x_i, y_i : T_i \to U$. Hence now we see that $V_i = T_i \times_{(y_i,x_i),U\times_B U} W$ is a scheme which comes with an étale surjection $V_i \to F_i$. By Sets, Lemma 3.9.6 we see that

$$\text{size}(V_i) \leq \max\{\text{size}(T_i), \text{size}(W)\} \leq \max\{\text{size}(T), \text{size}(W)\}$$

Hence, by Sets, Lemma 3.9.5 we conclude that

$$\text{size}(\coprod\nolimits_{i\in I} V_i) \leq \max\{|I|, \text{size}(T), \text{size}(W)\}.$$

Hence we conclude by our construction of *Sch* that $\coprod_{i\in I} V_i$ is isomorphic to an object V of $(Sch/S)_{fppf}$. This verifies the hypothesis of Spaces, Lemma 40.8.3 and we win. □

Lemma 54.11.4. *Let S be a scheme. Consider an algebraic space F of the form $F = U/R$ where (U, R, s, t, c) is a groupoid in algebraic spaces over S such that s, t are flat and locally of finite presentation, and $j = (t, s) : R \to U \times_S U$ is an equivalence relation. Then $U \to F$ is surjective, flat, and locally of finite presentation.*

Proof. This is almost but not quite a triviality. Namely, by Groupoids in Spaces, Lemma 52.18.5 and the fact that j is a monomorphism we see that $R = U \times_F U$. Choose a scheme W and a surjective étale morphism $W \to F$. As $U \to F$ is a surjection of sheaves we can find an fppf covering $\{W_i \to W\}$ and maps $W_i \to U$ lifting the morphisms $W_i \to F$. Then we see that

$$W_i \times_F U = W_i \times_U U \times_F U = W_i \times_{U,t} R$$

and the projection $W_i \times_F U \to W_i$ is the base change of $t : R \to U$ hence flat and locally of finite presentation, see Morphisms of Spaces, Lemmas 42.27.3 and 42.26.3. Hence by Descent on Spaces, Lemmas 45.10.11 and 45.10.8 we see that $U \to F$ is flat and locally of finite presentation. It is surjective by Spaces, Remark 40.5.2. □

Lemma 54.11.5. *Let S be a scheme. Let $X \to B$ be a morphism of algebraic spaces over S. Let G be a group algebraic space over B and let $a : G \times_B X \to X$ be an action of G on X/B. If*

(1) *a is a free action, and*
(2) *$G \to B$ is flat and locally of finite presentation,*

then X/G (see Groupoids in Spaces, Definition 52.18.1) is an algebraic space and $X \to X/G$ is surjective, flat, and locally of finite presentation.

Proof. The fact that X/G is an algebraic space is immediate from Theorem 54.10.1 and the definitions. Namely, $X/G = X/R$ where $R = G \times_B X$. The morphisms $s, t : G \times_B X \to X$ are flat and locally of finite presentation (clear for s as a base change of $G \to B$ and by symmetry using the inverse it follows for t) and the morphism $j : G \times_B X \to X \times_B X$ is a monomorphism by Groupoids in Spaces, Lemma 52.8.3 as the action is free. The asssertions about the morphism $X \to X/G$ follow from Lemma 54.11.4. $\square$

Lemma 54.11.6. *Let $\{S_i \to S\}_{i \in I}$ be a covering of $(Sch/S)_{fppf}$. Let G be a group algebraic space over S, and denote $G_i = G_{S_i}$ the base changes. Suppose given*

(1) *for each $i \in I$ an fppf G_i-torsor X_i over S_i, and*
(2) *for each $i, j \in I$ a $G_{S_i \times_S S_j}$-equivariant isomorphism $\varphi_{ij} : X_i \times_S S_j \to S_i \times_S X_j$ satisfying the cocycle condition over every $S_i \times_S S_j \times_S S_j$.*

Then there exists an fppf G-torsor X over S whose base change to S_i is isomorphic to X_i such that we recover the descent datum φ_{ij}.

Proof. We may think of X_i as a sheaf on $(Sch/S_i)_{fppf}$, see Spaces, Section 40.16. By Sites, Section 9.22 the descent datum (X_i, φ_{ij}) is effective in the sense that there exists a unique sheaf X on $(Sch/S)_{fppf}$ which recovers the algebraic spaces X_i after restricting back to $(Sch/S_i)_{fppf}$. Hence we see that $X_i = h_{S_i} \times X$. By Lemma 54.11.1 we see that X is an algebraic space, modulo verifying that $\coprod X_i$ is an algebraic space which we do at the end of the proof. By the equivalence of categories in Sites, Lemma 9.22.3 the action maps $G_i \times_{S_i} X_i \to X_i$ glue to give a map $a : G \times_S X \to X$. Now we have to show that a is an action and that X is a pseudo-torsor, and fppf locally trivial (see Groupoids in Spaces, Definition 52.9.3). These may be checked fppf locally, and hence follow from the corresponding properties of the actions $G_i \times_{S_i} X_i \to X_i$. Hence the lemma is true.

We suggest the reader skip the rest of the proof, which is purely set theoretical. Pick coverings $\{S_{ij} \to S_j\}_{j \in J_i}$ of $(Sch/S)_{fppf}$ which trivialize the G_i torsors X_i (possible by assumption, and Topologies, Lemma 30.7.7 part (1)). Then $\{S_{ij} \to S\}_{i \in I, j \in J_i}$ is a covering of $(Sch/S)_{fppf}$ and hence we may assume that each X_i is the trivial torsor! Of course we may also refine the covering further, hence we may assume that each S_i is affine and that the index set I has cardinality bounded by the cardinality of the set of points of S. Choose $U \in Ob((Sch/S)_{fppf})$ and a surjective étale morphism $U \to G$. Then we see that $U_i = U \times_S S_i$ comes with an étale surjective morphism to $X_i \cong G_i$. By Sets, Lemma 3.9.6 we see $\text{size}(U_i) \leq \max\{\text{size}(U), \text{size}(S_i)\}$. By Sets, Lemma 3.9.7 we have $\text{size}(S_i) \leq \text{size}(S)$. Hence we see that $\text{size}(U_i) \leq \max\{\text{size}(U), \text{size}(S)\}$ for all $i \in I$. Together with the bound on $|I|$ we found above we conclude from Sets, Lemma 3.9.5 that $\text{size}(\coprod U_i) \leq \max\{\text{size}(U), \text{size}(S)\}$. Hence Spaces, Lemma 40.8.3 applies to show that $\coprod X_i$ is an algebraic space which is what we had to prove. $\square$

54.12. Algebraic spaces in the étale topology

Let S be a scheme. Instead of working with sheaves over the big fppf site $(Sch/S)_{fppf}$ we could work with sheaves over the big étale site $(Sch/S)_{étale}$. All of the material in Algebraic Spaces, Sections 40.3 and 40.5 makes sense for sheaves over $(Sch/S)_{étale}$. Thus we get a second notion of algebraic spaces by working in the étale topology. This notion is (a priori) weaker then the notion introduced in Algebraic Spaces, Definition 40.6.1 since a sheaf in the fppf topology is certainly a sheaf in the étale topology. However, the notions are equivalent as is shown by the following lemma.

Lemma 54.12.1. *Denote the common underlying category of Sch_{fppf} and $Sch_{étale}$ by Sch_α (see Topologies, Remark 30.9.1). Let S be an object of Sch_α.*

$$F : (Sch_\alpha/S)^{opp} \longrightarrow Sets$$

be a presheaf with the following properties:

(1) *F is a sheaf for the étale topology,*
(2) *the diagonal $\Delta : F \to F \times F$ is representable, and*
(3) *there exists $U \in Ob(Sch_\alpha/S)$ and $U \to F$ which is surjective and étale.*

Then F is an algebraic space in the sense of Algebraic Spaces, Definition 40.6.1.

Proof. Note that properties (2) and (3) of the lemma and the corresponding properties (2) and (3) of Algebraic Spaces, Definition 40.6.1 are independent of the topology. This is true because these properties involve only the notion of a fibre product of presheaves, maps of presheaves, the notion of a representable transformation of functors, and what it means for such a transformation to be surjective and étale. Thus all we have to prove is that an étale sheaf F with properties (2) and (3) is also an fppf sheaf.

To do this, let $R = U \times_F U$. By (2) the presheaf R is representable by a scheme and by (3) the projections $R \to U$ are étale. Thus $j : R \to U \times_S U$ is an étale equivalence relation. Moreover the map $U \to F$ identifies F as the quotient of U by R for the étale topology (follows exactly as in the proof of Algebraic Spaces, Lemma 40.9.1). Next, let U/R denote the quotient sheaf in the fppf topology which is an algebraic space by Spaces, Theorem 40.10.5. Thus we have morphisms (tranformations of functors)

$$U \to F \to U/R.$$

By the aforementioned Spaces, Theorem 40.10.5 the composition is representable, surjective, and étale. Hence for any scheme T and morphism $T \to U/R$ the fibre product $V = T \times_{U/R} U$ is a scheme surjective and étale over T. In other words, $\{V \to U\}$ is an étale covering. This proves that $U \to U/R$ is surjective as a map of sheaves in the étale topology. It follows that $F \to U/R$ is surjective as a map of sheaves in the étale topology. On the other hand, the map $F \to U/R$ is injective (as a map of presheaves) since $R = U \times_{U/R} U$ again by Spaces, Theorem 40.10.5. It follows that $F \to U/R$ is an isomorphism of étale sheaves, see Sites, Lemma 9.11.2 which concludes the proof. □

54.13. Other chapters

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(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Commutative Algebra
(8) Brauer Groups
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(10) Homological Algebra

(11) Derived Categories
(12) More on Algebra
(13) Smoothing Ring Maps
(14) Simplicial Methods
(15) Sheaves of Modules
(16) Modules on Sites
(17) Injectives
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(36) More on Groupoid Schemes
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CHAPTER 55

Examples of Stacks

55.1. Introduction

This is a discussion of examples of stacks in algebraic geometry. Some of them are algebraic stacks, some are not. We will discuss which are algebraic stacks in a later chapter. This means that in this chapter we mainly worry about the descent conditions. See [**Vis**] for example.

55.2. Notation

In this chapter we fix a suitable big fppf site Sch_{fppf} as in Topologies, Definition 30.7.6. So, if not explicitly stated otherwise all schemes will be objects of Sch_{fppf}. We will always work relative to a base S contained in Sch_{fppf}. And we will then work with the big fppf site $(Sch/S)_{fppf}$, see Topologies, Definition 30.7.8. The absolute case can be recovered by taking $S = Spec(\mathbf{Z})$.

55.3. Examples of stacks

We first give some important examples of stacks over $(Sch/S)_{fppf}$.

55.4. Quasi-coherent sheaves

We define a category $QCoh$ as follows:

(1) An object of $QCoh$ is a pair $(X, \mathcal{F})$, where X/S is an object of $(Sch/S)_{fppf}$, and $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module, and
(2) a morphism $(f, \varphi) : (Y, \mathcal{G}) \to (X, \mathcal{F})$ is a pair consisting of a morphism $f : Y \to X$ of schemes over S and an f-map (see Sheaves, Section 6.26) $\varphi : \mathcal{F} \to \mathcal{G}$.
(3) The composition of morphisms

$$(Z, \mathcal{H}) \xrightarrow{(g,\psi)} (Y, \mathcal{G}) \xrightarrow{(f,\phi)} (X, \mathcal{F})$$

is $(f \circ g, \psi \circ \phi)$ where $\psi \circ \phi$ is the composition of f-maps.

Thus $QCoh$ is a category and

$$p : QCoh \to (Sch/S)_{fppf}, \quad (X, \mathcal{F}) \mapsto X$$

is a functor. Note that the fibre category of $QCoh$ over a scheme X is just the category $QCoh(X)$ of quasi-coherent $\mathcal{O}_X$-modules. We remark for later use that given $(X, \mathcal{F}), (Y, \mathcal{G}) \in Ob(QCoh)$ we have

$$Mor_{QCoh}((Y, \mathcal{G}), (X, \mathcal{F})) = \coprod\nolimits_{f \in Mor_S(Y,X)} Mor_{QCoh(Y)}(f^*\mathcal{F}, \mathcal{G}) \tag{55.4.0.1}$$

See the discussion on f-maps of modules in Sheaves, Section 6.26.

The category $QCoh$ is not a stack over $(Sch/S)_{fppf}$ because its collection of objects is a proper class. On the other hand we will see that it does satisfy all the axioms of a stack. We will get around the set theoretical issue in Section 55.5.

Lemma 55.4.1. *A morphism $(f, \varphi) : (Y, \mathcal{G}) \to (X, \mathcal{F})$ of $QCoh$ is strongly cartesian if and only if the map φ induces an isomorphism $f^*\mathcal{F} \to \mathcal{G}$.*

Proof. Let $(X, \mathcal{F}) \in Ob(QCoh)$. Let $f : Y \to X$ be a morphism of $(Sch/S)_{fppf}$. Note that there is a canonical f-map $c : \mathcal{F} \to f^*\mathcal{F}$ and hence we get a morphism $(f, c) : (Y, f^*\mathcal{F}) \to (X, \mathcal{F})$. We claim that (f, c) is strongly cartesian. Namely, for any object $(Z, \mathcal{H})$ of $QCoh$ we have

$$\begin{aligned} Mor_{QCoh}((Z, \mathcal{H}), (Y, f^*\mathcal{F})) &= \coprod\nolimits_{g \in Mor_S(Z,Y)} Mor_{QCoh(Z)}(g^*f^*\mathcal{F}, \mathcal{H}) \\ &= \coprod\nolimits_{g \in Mor_S(Z,Y)} Mor_{QCoh(Z)}((f \circ g)^*\mathcal{F}, \mathcal{H}) \\ &= Mor_{QCoh}((Z, \mathcal{H}), (X, \mathcal{F})) \times_{Mor_S(Z,X)} Mor_S(Z, Y) \end{aligned}$$

where we have used Equation (55.4.0.1) twice. This proves that the condition of Categories, Definition 4.30.1 holds for (f, c), and hence our claim is true. Now by Categories, Lemma 4.30.2 we see that isomorphisms are strongly cartesian and compositions of strongly cartesian morphisms are strongly cartesian which proves the ``if" part of the lemma. For the converse, note that given $(X, \mathcal{F})$ and $f : Y \to X$, if there exists a strongly cartesian morphism lifting f with target $(X, \mathcal{F})$ then it has to be isomorphic to (f, c) (see discussion following Categories, Definition 4.30.1). Hence the "only if" part of the lemma holds. □

Lemma 55.4.2. *The functor $p : QCoh \to (Sch/S)_{fppf}$ satisfies conditions (1), (2) and (3) of Stacks, Definition 50.4.1.*

Proof. It is clear from Lemma 55.4.1 that $QCoh$ is a fibred category over $(Sch/S)_{fppf}$. Given covering $\mathcal{U} = \{X_i \to X\}_{i \in I}$ of $(Sch/S)_{fppf}$ the functor

$$QCoh(T) \longrightarrow DD(\mathcal{U})$$

is fully faithful and essentially surjective, see Descent, Proposition 31.4.2. Hence Stacks, Lemma 50.4.2 applies to show that $QCoh$ satisfies all the axioms of a stack. □

55.5. The stack of finitely generated quasi-coherent sheaves

It turns out that we can get a stack of quasi-coherent sheaves if we only consider finite type quasi-coherent modules. Let us denote

$$p_{fg} : QCoh_{fg} \to (Sch/S)_{fppf}$$

the full subcategory of $QCoh$ over $(Sch/S)_{fppf}$ consisting of pairs $(T, \mathcal{F})$ such that $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_T$-module of finite type.

Lemma 55.5.1. *The functor $p_{fg} : QCoh_{fg} \to (Sch/S)_{fppf}$ satisfies conditions (1), (2) and (3) of Stacks, Definition 50.4.1.*

Proof. We will verify assumptions (1), (2), (3) of Stacks, Lemma 50.4.3 to prove this. By Lemma 55.4.1 a morphism $(Y, \mathcal{G}) \to (X, \mathcal{F})$ is strongly cartesian if and only if it induces an isomorphism $f^*\mathcal{F} \to \mathcal{G}$. By Modules, Lemma 15.9.2 the pullback of a finite type $\mathcal{O}_X$-module is of finite type. Hence assumption (1) of Stacks, Lemma 50.4.3 holds. Assumption (2) holds trivially. Finally, to prove assumption (3) we have to show: If $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module and $\{f_i : X_i \to X\}$ is an fppf covering such that each $f_i^*\mathcal{F}$ is of finite type, then $\mathcal{F}$ is of finite type. Considering the restriction of $\mathcal{F}$ to an affine open of X

this reduces to the following algebra statement: Suppose that $R \to S$ is a finitely presented, faithfully flat ring map and M an R-module. If $M \otimes_R S$ is a finitely generated S-module, then M is a finitely generated R-module. A stronger form of the algebra fact can be found in Algebra, Lemma 7.77.2. □

Lemma 55.5.2. *Let $(X, \mathcal{O}_X)$ be a ringed space.*

1. *The category of finite type $\mathcal{O}_X$-modules has a set of isomorphism classes.*
2. *The category of finite type quasi-coherent $\mathcal{O}_X$-modules has a set of isomorphism classes.*

Proof. Part (2) follows from part (1) as the category in (2) is a full subcategory of the category in (1). Consider any open covering $\mathcal{U} : X = \bigcup_{i \in I} U_i$. Denote $j_i : U_i \to X$ the inclusion maps. Consider any map $r : I \to \mathbf{N}$. If $\mathcal{F}$ is an $\mathcal{O}_X$-module whose restriction to U_i is generated by at most $r(i)$ sections from $\mathcal{F}(U_i)$, then $\mathcal{F}$ is a quotient of the sheaf

$$\mathcal{H}_{\mathcal{U},r} = \bigoplus\nolimits_{i \in I} j_{i,!}\mathcal{O}_{U_i}^{\oplus r(i)}$$

By definition, if $\mathcal{F}$ is of finite type, then there exists some open covering with $\mathcal{U}$ whose index set is $I = X$ such that this condition is true. Hence it suffices to show that there is a set of possible choices for $\mathcal{U}$ (obvious), a set of possible choices for $r : I \to \mathbf{N}$ (obvious), and a set of possible quotient modules of $\mathcal{H}_{\mathcal{U},r}$ for each $\mathcal{U}$ and r. In other words, it suffices to show that given an $\mathcal{O}_X$-module $\mathcal{H}$ there is at most a set of isomorphism classes of quotients. This last assertion becomes obvious by thinking of the kernels of a quotient map $\mathcal{H} \to \mathcal{F}$ as being parametrized by a subset of the power set of $\prod_{U \subset X \text{ open}} \mathcal{H}(U)$. □

Lemma 55.5.3. *There exists a subcategory $QCoh_{fg,small} \subset QCoh_{fg}$ with the following properties:*

1. *the inclusion functor $QCoh_{fg,small} \to QCoh_{fg}$ is fully faithful and essentially surjective, and*
2. *the functor $p_{fg,small} : QCoh_{fg,small} \to (Sch/S)_{fppf}$ turns $QCoh_{fg,small}$ into a stack over $(Sch/S)_{fppf}$.*

Proof. We have seen in Lemmas 55.5.1 and 55.5.2 that $p_{fg} : QCoh_{fg} \to (Sch/S)_{fppf}$ satisfies (1), (2) and (3) of Stacks, Definition 50.4.1 as well as the additional condition (4) of Stacks, Remark 50.4.9. Hence we obtain $QCoh_{fg,small}$ from the discussion in that remark. □

We will often perform the replacement

$$QCoh_{fg} \leadsto QCoh_{fg,small}$$

without further remarking on it, and by abuse of notation we will simply denote $QCoh_{fg}$ this replacement.

Remark 55.5.4. Note that the whole discussion in this section works if we want to consider those quasi-coherent sheaves which are locally generated by at most κ sections, for some infinite cardinal κ, e.g., $\kappa = \aleph_0$.

55.6. Algebraic spaces

We define a category *Spaces* as follows:

1. An object of *Spaces* is a morphism $X \to U$ of algebraic spaces over S, where U is representable by an object of $(Sch/S)_{fppf}$, and

(2) a morphism $(f, g) : (X \to U) \to (Y \to V)$ is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"'] \arrow[d] & Y \arrow[d] \\ U \arrow[r, "g"] & V \end{tikzcd}$$

of morphisms of algebraic spaces over S.

Thus *Spaces* is a category and

$$p : Spaces \to (Sch/S)_{fppf}, \quad (X \to U) \mapsto U$$

is a functor. Note that the fibre category of *Spaces* over a scheme U is just the category $Spaces/U$ of algebraic spaces over U (see Topologies on Spaces, Section 44.2). Hence we sometimes think of an object of *Spaces* as a pair X/U consisting of a scheme U and an algebraic space X over U. We remark for later use that given $(X/U), (Y/V) \in Ob(Spaces)$ we have

$$Mor_{Spaces}(X/U, Y/V) = \coprod\nolimits_{g \in Mor_S(U,V)} Mor_{Spaces/U}(X, U \times_{g,V} Y) \tag{55.6.0.1}$$

The category *Spaces* is almost, but not quite a stack over $(Sch/S)_{fppf}$. The problem is a set theoretical issue as we will explain below.

Lemma 55.6.1. *A morphism $(f, g) : X/U \to Y/V$ of Spaces is strongly cartesian if and only if the map f induces an isomorphism $X \to U \times_{g,V} Y$.*

Proof. Let $Y/V \in Ob(Spaces)$. Let $g : U \to V$ be a morphism of $(Sch/S)_{fppf}$. Note that the projection $p : U \times_{g,V} Y \to Y$ gives rise a morphism $(p, g) : U \times_{g,V} Y/U \to Y/V$ of *Spaces*. We claim that (p, g) is strongly cartesian. Namely, for any object Z/W of *Spaces* we have

$$\begin{aligned} Mor_{Spaces}(Z/W, U \times_{g,V} Y/U) &= \coprod\nolimits_{h \in Mor_S(W,U)} Mor_{Spaces/W}(Z, W \times_{h,U} U \times_{g,V} Y) \\ &= \coprod\nolimits_{h \in Mor_S(W,U)} Mor_{Spaces/W}(Z, W \times_{g \circ h,V} Y) \\ &= Mor_{Spaces}(Z/W, Y/V) \times_{Mor_S(W,V)} Mor_S(W, U) \end{aligned}$$

where we have used Equation (55.6.0.1) twice. This proves that the condition of Categories, Definition 4.30.1 holds for (p, g), and hence our claim is true. Now by Categories, Lemma 4.30.2 we see that isomorphisms are strongly cartesian and compositions of strongly cartesian morphisms are strongly cartesian which proves the ``if'' part of the lemma. For the converse, note that given Y/V and $g : U \to V$, if there exists a strongly cartesian morphism lifting g with target Y/V then it has to be isomorphic to (p, g) (see discussion following Categories, Definition 4.30.1). Hence the "only if" part of the lemma holds. □

Lemma 55.6.2. *The functor $p : Spaces \to (Sch/S)_{fppf}$ satisfies conditions (1) and (2) of Stacks, Definition 50.4.1.*

Proof. It is follows from Lemma 55.6.1 that *Spaces* is a fibred category over $(Sch/S)_{fppf}$ which proves (1). Suppose that $\{U_i \to U\}_{i \in I}$ is a covering of $(Sch/S)_{fppf}$. Suppose that X, Y are algebraic spaces over U. Finally, suppose that $\varphi_i : X_{U_i} \to Y_{U_i}$ are morphisms of $Spaces/U_i$ such that φ_i and φ_j restrict to the same morphisms $X_{U_i \times_U U_j} \to Y_{U_i \times_U U_j}$ of algebraic spaces over $U_i \times_U U_j$. To prove (2) we have to show that there exists a unique morphism $\varphi : X \to Y$ over U whose base change to U_i is equal to φ_i. As a morphism from X to Y is the same thing as a map of sheaves this follows directly from Sites, Lemma 9.22.1. □

Remark 55.6.3. Ignoring set theoretical difficulties[1] *Spaces* also satisfies descent for objects and hence is a stack. Namely, we have to show that given

(1) an fppf covering $\{U_i \to U\}_{i \in I}$,
(2) for each $i \in I$ an algebraic space X_i/U_i, and
(3) for each $i, j \in I$ an isomorphism $\varphi_{ij} : X_i \times_U U_j \to U_i \times_U X_j$ of algebraic spaces over $U_i \times_U U_j$ satisfying the cocycle condition over $U_i \times_U U_j \times_U U_k$,

there exists an algebraic space X/U and isomorphisms $X_{U_i} \cong X_i$ over U_i recovering the isomorphisms φ_{ij}. First, note that by Sites, Lemma 9.22.2 there exists a sheaf X on $(Sch/U)_{fppf}$ recovering the X_i and the φ_{ij}. Then by Bootstrap, Lemma 54.11.1 we see that X is an algebraic space (if we ignore the set theoretic condition of that lemma). We will use this argument in the next section to show that if we consider only algebraic spaces of finite type, then we obtain a stack.

55.7. The stack of finite type algebraic spaces

It turns out that we can get a stack of spaces if we only consider spaces of finite type. Let us denote

$$p_{ft} : Spaces_{ft} \to (Sch/S)_{fppf}$$

the full subcategory of *Spaces* over $(Sch/S)_{fppf}$ consisting of pairs X/U such that $X \to U$ is a morphism of finite type.

Lemma 55.7.1. *The functor $p_{ft} : Spaces_{ft} \to (Sch/S)_{fppf}$ satisfies the conditions (1), (2) and (3) of Stacks, Definition 50.4.1.*

Proof. We are going to write this out in ridiculous detail (which may make it hard to see what is going on).

We have seen in Lemma 55.6.1 that a morphism $(f, g) : X/U \to Y/V$ of *Spaces* is strongly cartesian if the induced morphism $f : X \to U \times_V Y$ is an isomorphism. Note that if $Y \to V$ is of finite type then also $U \times_V Y \to U$ is of finite type, see Morphisms of Spaces, Lemma 42.22.3. So if $(f, g) : X/U \to Y/V$ of *Spaces* is strongly cartesian in *Spaces* and Y/V is an object of $Spaces_{ft}$ then automatically also X/U is an object of $Spaces_{ft}$, and of course (f, g) is also strongly cartesian in $Spaces_{ft}$. In this way we conclude that $Spaces_{ft}$ is a fibred category over $(Sch/S)_{fppf}$. This proves (1).

The argument above also shows that the inclusion functor $Spaces_{ft} \to Spaces$ transforms strongly cartesian morphisms into strongly cartesian morphisms. In other words $Spaces_{ft} \to Spaces$ is a 1-morphism of fibred categories over $(Sch/S)_{fppf}$.

Let $U \in Ob((Sch/S)_{fppf})$. Let X, Y be algebraic spaces of finite type over U. By Stacks, Lemma 50.2.3 we obtain a map of presheaves

$$Mor_{Spaces_{ft}}(X, Y) \longrightarrow Mor_{Spaces}(X, Y)$$

which is an isomorphism as $Spaces_{ft}$ is a full subcategory of *Spaces*. Hence the left hand side is a sheaf, because in Lemma 55.6.2 we showed the right hand side is a sheaf. This proves (2).

To prove condition (3) of Stacks, Definition 50.4.1 we have to show the following: Given

(1) a covering $\{U_i \to U\}_{i \in I}$ of $(Sch/S)_{fppf}$,

[1] The difficulty is not that *Spaces* is a proper class, since by our definition of an algebraic space over S there is only a set worth of isomorphism classes of algebraic spaces over S. It is rather that arbitrary disjoint unions of algebraic spaces may end up being too large, hence lie outside of our chosen ``partial universe'' of sets.

(2) for each $i \in I$ an algebraic space X_i of finite type over U_i, and
(3) for each $i, j \in I$ an isomorphism $\varphi_{ij} : X_i \times_U U_j \to U_i \times_U X_j$ of algebraic spaces over $U_i \times_U U_j$ satisfying the cocycle condition over $U_i \times_U U_j \times_U U_k$,

there exists an algebraic space X of finite type over U and isomorphisms $X_{U_i} \cong X_i$ over U_i recovering the isomorphisms φ_{ij}. By Sites, Lemma 9.22.2 there exists a sheaf X on $(Sch/U)_{fppf}$ recovering the X_i and the φ_{ij}. Then by Bootstrap, Lemma 54.11.2 we see that X is an algebraic space. By Descent on Spaces, Lemma 45.10.8 we see that $X \to U$ is of finite type which concludes the proof. □

Lemma 55.7.2. *There exists a subcategory $Spaces_{ft,small} \subset Spaces_{ft}$ with the following properties:*

(1) *the inclusion functor $Spaces_{ft,small} \to Spaces_{ft}$ is fully faithful and essentially surjective, and*
(2) *the functor $p_{ft,small} : Spaces_{ft,small} \to (Sch/S)_{fppf}$ turns $Spaces_{ft,small}$ into a stack over $(Sch/S)_{fppf}$.*

Proof. We have seen in Lemmas 55.7.1 that $p_{fg} : QCoh_{fg} \to (Sch/S)_{fppf}$ satisfies (1), (2) and (3) of Stacks, Definition 50.4.1. The additional condition (4) of Stacks, Remark 50.4.9 holds because every algebraic space X over S is of the form U/R for $U, R \in Ob((Sch/S)_{fppf})$, see Spaces, Lemma 40.9.1. Thus there is only a set worth of isomorphism classes of objects. Hence we obtain $Spaces_{ft,small}$ from the discussion in that remark. □

We will often perform the replacement

$$Spaces_{ft} \leadsto Spaces_{ft,small}$$

without further remarking on it, and by abuse of notation we will simply denote $Spaces_{ft}$ this replacement.

Remark 55.7.3. Note that the whole discussion in this section works if we want to consider those algebraic spaces X/U which are locally of finite type such that the inverse image in X of an affine open of U can be covered by countably many affines. If needed we can also introduce the notion of a morphism of κ-type (meaning some bound on the number of generators of ring extensions and some bound on the cardinality of the affines over a given affine in the base) where κ is a cardinal, and then we can produce a stack

$$Spaces_{\kappa} \longrightarrow (Sch/S)_{fppf}$$

in exactly the same manner as above (provided we make sure that Sch is large enough depending on κ).

55.8. Examples of stacks in groupoids

The examples above are examples of stacks which are not stacks in groupoids. In the rest of this chapter we give algebraic geometric examples of stacks in groupoids.

55.9. The stack associated to a sheaf

Let $F : (Sch/S)_{fppf}^{opp} \to Sets$ be a presheaf. We obtain a category fibred in sets

$$p_F : \mathcal{S}_F \to (Sch/S)_{fppf},$$

see Categories, Example 4.35.5. This is a stack in sets if and only if F is a sheaf, see Stacks, Lemma 50.6.3.

55.10. The stack in groupoids of finitely generated quasi-coherent sheaves

Let $p : QCoh_{fg} \to (Sch/S)_{fppf}$ be the stack introduced in Section 55.5 (using the abuse of notation introduced there). We can turn this into a stack in groupoids $p' : QCoh'_{fg} \to (Sch/S)_{fppf}$ by the procedure of Categories, Lemma 4.32.3, see Stacks, Lemma 50.5.3. In this particular case this simply means $QCoh'_{fg}$ has the same objects as $QCoh_{fg}$ but the morphsms are pairs $(f, g) : (U, \mathcal{F}) \to (U', \mathcal{F}')$ where g is an isomorphism $g : f^*\mathcal{F}' \to \mathcal{F}$.

55.11. The stack in groupoids of finite type algebraic spaces

Let $p : Spaces_{ft} \to (Sch/S)_{fppf}$ be the stack introduced in Section 55.7 (using the abuse of notation introduced there). We can turn this into a stack in groupoids $p' : Spaces'_{ft} \to (Sch/S)_{fppf}$ by the procedure of Categories, Lemma 4.32.3, see Stacks, Lemma 50.5.3. In this particular case this simply means $Spaces'_{ft}$ has the same objects as $Spaces_{ft}$, i.e., finite type morphisms $X \to U$ where X is an algebraic space over S and U is a scheme over S. But the morphsms $(f, g) : X/U \to Y/V$ are now commutative diagrams

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ U \arrow[r, "g"] & V \end{tikzcd}$$

which are cartesian.

55.12. Quotient stacks

Let (U, R, s, t, c) be a groupoid in algebraic spaces over S. In this case the quotient stack

$$[U/R] \longrightarrow (Sch/S)_{fppf}$$

is a stack in groupoids by construction, see Groupoids in Spaces, Definition 52.19.1. It is even the case that the $Isom$-sheaves are representable by algebraic spaces, see Bootstrap, Lemma 54.11.3. These quotient stacks are of fundamental importance to the theory of algebraic stacks.

A special case of the construction above is the quotient stack

$$[X/G] \longrightarrow (Sch/S)_{fppf}$$

associated to a datum $(B, G/B, m, X/B, a)$. Here

(1) B is an algebraic space over S,
(2) (G, m) is a group algebraic space over B,
(3) X is an algebraic space over B, and
(4) $a : G \times_B X \to X$ is an action of G on X over B.

Namely, by Groupoids in Spaces, Definition 52.19.1 the stack in groupoids $[X/G]$ is the quotient stack $[X/G \times_B X]$ given above. It behooves us to spell out what the category $[X/G]$ really looks like. We will do this in Section 55.14.

55.13. Classifying torsors

We want to carefully explain a number of variants of what it could mean to study the stack of torsors for a group algebraic space G or a sheaf of groups $\mathcal{G}$.

55.13.1. Torsors for a sheaf of groups. Let $\mathcal{G}$ be a sheaf of groups on $(Sch/S)_{fppf}$. For $U \in Ob((Sch/S)_{fppf})$ we denote $\mathcal{G}|_U$ the restriction of $\mathcal{G}$ to $(Sch/U)_{fppf}$. We define a category $\mathcal{G}$*-Torsors* as follows:

(1) An object of $\mathcal{G}$*-Torsors* is a pair $(U, \mathcal{F})$ where U is an object of $(Sch/S)_{fppf}$ and $\mathcal{F}$ is a $\mathcal{G}|_U$-torsor, see Cohomology on Sites, Definition 19.5.1.
(2) A morphism $(U, \mathcal{F}) \to (V, \mathcal{H})$ is given by a pair (f, α), where $f : U \to V$ is a morphism of schemes over S, and $\alpha : f^{-1}\mathcal{H} \to \mathcal{F}$ is an isomorphism of $\mathcal{G}|_U$-torsors.

Thus $\mathcal{G}$*-Torsors* is a category and

$$p : \mathcal{G}\text{-}Torsors \longrightarrow (Sch/S)_{fppf}, \quad (U, \mathcal{F}) \longmapsto U$$

is a functor. Note that the fibre category of $\mathcal{G}$*-Torsors* over U is the category of $\mathcal{G}|_U$-torsors which is a groupoid.

Lemma 55.13.2. *Up to a replacement as in Stacks, Remark 50.4.9 the functor*

$$p : \mathcal{G}\text{-}Torsors \longrightarrow (Sch/S)_{fppf}$$

defines a stack in groupoids over $(Sch/S)_{fppf}$.

Proof. The most difficult part of the proof is to show that we have descent for objects. Let $\{U_i \to U\}_{i \in I}$ be a covering of $(Sch/S)_{fppf}$. Suppose that for each i we are given a $\mathcal{G}|_{U_i}$-torsor $\mathcal{F}_i$, and for each $i, j \in I$ an isomorphism $\varphi_{ij} : \mathcal{F}_i|_{U_i \times_U U_j} \to \mathcal{F}_j|_{U_i \times_U U_j}$ of $\mathcal{G}|_{U_i \times_U U_j}$-torsors satisfying a suitable cocycle condition on $U_i \times_U U_j \times_U U_k$. Then by Sites, Section 9.22 we obtain a sheaf $\mathcal{F}$ on $(Sch/U)_{fppf}$ whose restriction to each U_i recovers $\mathcal{F}_i$ as well as recovering the descent data. By the equivalence of categories in Sites, Lemma 9.22.3 the action maps $\mathcal{G}|_{U_i} \times \mathcal{F}_i \to \mathcal{F}_i$ glue to give a map $a : \mathcal{G}|_U \times \mathcal{F} \to \mathcal{F}$. Now we have to show that a is an action and that $\mathcal{F}$ becomes a $\mathcal{G}|_U$-torsor. Both properties may be checked locally, and hence follow from the corresponding properties of the actions $\mathcal{G}|_{U_i} \times \mathcal{F}_i \to \mathcal{F}_i$. This proves that descent for objects holds in $\mathcal{G}$*-Torsors*. Some details omitted. □

55.13.3. Variant on torsors for a sheaf. The construction of Subsection 55.13.1 can be generalized slightly. Namely, let $\mathcal{G} \to \mathcal{B}$ be a map of sheaves on $(Sch/S)_{fppf}$ and let

$$m : \mathcal{G} \times_{\mathcal{B}} \mathcal{G} \longrightarrow \mathcal{G}$$

be a group law on $\mathcal{G}/\mathcal{B}$. In other words, the pair $(\mathcal{G}, m)$ is a group object of the topos $Sh((Sch/S)_{fppf})/\mathcal{B}$. See Sites, Section 9.26 for information regarding localizations of topoi. In this setting we can define a category $\mathcal{G}/\mathcal{B}$*-Torsors* as follows (where we use the Yoneda embedding to think of schemes as sheaves):

(1) An object of $\mathcal{G}/\mathcal{B}$*-Torsors* is a triple $(U, b, \mathcal{F})$ where
    (a) U is an object of $(Sch/S)_{fppf}$,
    (b) $b : U \to \mathcal{B}$ is a section of $\mathcal{B}$ over U, and
    (c) $\mathcal{F}$ is a $U \times_{b,\mathcal{B}} \mathcal{G}$-torsor over U.
(2) A morphism $(U, b, \mathcal{F}) \to (U', b', \mathcal{F}')$ is given by a pair (f, g), where $f : U \to U'$ is a morphism of schemes over S such that $b = b' \circ f$, and $g : f^{-1}\mathcal{F}' \to \mathcal{F}$ is an isomorphism of $U \times_{b,\mathcal{B}} \mathcal{G}$-torsors.

Thus $\mathcal{G}/\mathcal{B}$*-Torsors* is a category and

$$p : \mathcal{G}/\mathcal{B}\text{-}Torsors \longrightarrow (Sch/S)_{fppf}, \quad (U, b, \mathcal{F}) \longmapsto U$$

is a functor. Note that the fibre category of $\mathcal{G}/\mathcal{B}$*-Torsors* over U is the disjoint union over $b : U \to \mathcal{B}$ of the categories of $U \times_{b,\mathcal{B}} \mathcal{G}$-torsors, hence is a groupoid.

In the special case $\mathcal{B} = S$ we recover the category $\mathcal{G}$*-Torsors* introduced in Subsection 55.13.1.

Lemma 55.13.4. *Up to a replacement as in Stacks, Remark 50.4.9 the functor*

$$p : \mathcal{G}/\mathcal{B}\textit{-Torsors} \longrightarrow (Sch/S)_{fppf}$$

defines a stack in groupoids over $(Sch/S)_{fppf}$.

Proof. This proof is a repeat of the proof of Lemma 55.13.2. The reader is encouraged to read that proof first since the notation is less cumbersome. The most difficult part of the proof is to show that we have descent for objects. Let $\{U_i \to U\}_{i \in I}$ be a covering of $(Sch/S)_{fppf}$. Suppose that for each i we are given a pair $(b_i, \mathcal{F}_i)$ consisting of a morphism $b_i : U_i \to \mathcal{B}$ and a $U_i \times_{b_i, \mathcal{B}} \mathcal{G}$-torsor $\mathcal{F}_i$, and for each $i, j \in I$ we have $b_i|_{U_i \times_U U_j} = b_j|_{U_i \times_U U_j}$ and we are given an isomorphism $\varphi_{ij} : \mathcal{F}_i|_{U_i \times_U U_j} \to \mathcal{F}_j|_{U_i \times_U U_j}$ of $(U_i \times_U U_j) \times_{\mathcal{B}} \mathcal{G}$-torsors satisfying a suitable cocycle condition on $U_i \times_U U_j \times_U U_k$. Then by Sites, Section 9.22 we obtain a sheaf $\mathcal{F}$ on $(Sch/U)_{fppf}$ whose restriction to each U_i recovers $\mathcal{F}_i$ as well as recovering the descent data. By the sheaf axiom for $\mathcal{B}$ the morphisms b_i come from a unique morphism $b : U \to \mathcal{B}$. By the equivalence of categories in Sites, Lemma 9.22.3 the action maps $(U_i \times_{b_i, \mathcal{B}} \mathcal{G}) \times_{U_i} \mathcal{F}_i \to \mathcal{F}_i$ glue to give a map $(U \times_{b, \mathcal{B}} \mathcal{G}) \times \mathcal{F} \to \mathcal{F}$. Now we have to show that this is an action and that $\mathcal{F}$ becomes a $U \times_{b, \mathcal{B}} \mathcal{G}$-torsor. Both properties may be checked locally, and hence follow from the corresponding properties of the actions on the $\mathcal{F}_i$. This proves that descent for objects holds in $\mathcal{G}/\mathcal{B}$*-Torsors*. Some details omitted. □

55.13.5. Principal homogeneous spaces. Let B be an algebraic space over S. Let G be a group algebraic space over B. We define a category G*-Principal* as follows:

(1) An object of G*-Principal* is a triple (U, b, X) where
   (a) U is an object of $(Sch/S)_{fppf}$,
   (b) $b : U \to B$ is a morphism over S, and
   (c) X is a principal homogeneous G_U-space over U where $G_U = U \times_{b,B} G$.
   See Groupoids in Spaces, Definition 52.9.3.
(2) A morphism $(U, b, X) \to (U', b', X')$ is given by a pair (f, g), where $f : U \to U'$ is a morphism of schemes over B, and $g : X \to U \times_{f,U'} X'$ is an isomorphism of principal homogeneous G_U-spaces.

Thus G*-Principal* is a category and

$$p : G\textit{-Principal} \longrightarrow (Sch/S)_{fppf}, \quad (U, b, X) \longmapsto U$$

is a functor. Note that the fibre category of G*-Principal* over U is the disjoint union over $b : U \to B$ of the categories of principal homogeneous $U \times_{b,B} G$-spaces, hence is a groupoid.

In the special case $S = B$ the objects are simply pairs (U, X) where U is a scheme over S, and X is a principal homogeneous G_U-space over U. Moreover, morphisms are simply cartesian diagrams

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d] & X' \arrow[d] \\ U \arrow[r, "f"] & U' \end{tikzcd}$$

where g is G-equivariant.

Remark 55.13.6. We conjecture that up to a replacement as in Stacks, Remark 50.4.9 the functor

$$p : G\text{-}Principal \longrightarrow (Sch/S)_{fppf}$$

defines a stack in groupoids over $(Sch/S)_{fppf}$. This would follow if one could show that given

(1) a covering $\{U_i \to U\}_{i \in I}$ of $(Sch/S)_{fppf}$,
(2) an group algebraic space H over U,
(3) for every i a principal homogeneous H_{U_i}-space X_i over U_i, and
(4) H-equivariant isomorphisms $\varphi_{ij} : X_{i,U_i \times_U U_j} \to X_{j,U_i \times_U U_j}$ satisfying the cocycle condition,

there exists a principal homogeneous H-space X over U which recovers (X_i, φ_{ij}). The technique of the proof of Bootstrap, Lemma 54.11.6 reduces this to a set theoretical question, so the reader who ignores set theoretical questions will ``know" that the result is true. In `http://math.columbia.edu/~dejong/wordpress/?p=591` there is a suggestion as to how to approach this problem.

55.13.7. Variant on principal homogeneous spaces. Let S be a scheme. Let $B = S$. Let G be a group scheme over $B = S$. In this setting we can define a full subcategory $G\text{-}Principal\text{-}Schemes \subset G\text{-}Principal$ whose objects are pairs (U, X) where U is an object of $(Sch/S)_{fppf}$ and $X \to U$ is a principal homogeneous G-space over U which is representable, i.e., a scheme.

It is in general not the case that $G\text{-}Principal\text{-}Schemes$ is a stack in groupoids over $(Sch/S)_{fppf}$. The reason is that in general there really do exist principal homogeneous spaces which are not schemes, hence descent for objects will not be satisfied in general.

55.13.8. Torsors in fppf topology. Let B be an algebraic space over S. Let G be a group algebraic space over B. We define a category $G\text{-}Torsors$ as follows:

(1) An object of $G\text{-}Torsors$ is a triple (U, b, X) where
  (a) U is an object of $(Sch/S)_{fppf}$,
  (b) $b : U \to B$ is a morphism, and
  (c) X is an fppf G_U-torsor over U where $G_U = U \times_{b,B} G$.
  See Groupoids in Spaces, Definition 52.9.3.
(2) A morphism $(U, b, X) \to (U', b', X')$ is given by a pair (f, g), where $f : U \to U'$ is a morphism of schemes over B, and $g : X \to U \times_{f,U'} X'$ is an isomorphism of G_U-torsors.

Thus $G\text{-}Torsors$ is a category and

$$p : G\text{-}Torsors \longrightarrow (Sch/S)_{fppf}, \quad (U, a, X) \longmapsto U$$

is a functor. Note that the fibre category of $G\text{-}Torsors$ over U is the disjoint union over $b : U \to B$ of the categories of fppf $U \times_{b,B} G$-torsors, hence is a groupoid.

In the special case $S = B$ the objects are simply pairs (U, X) where U is a scheme over S, and X is an fppf G_U-torsor over U. Moreover, morphisms are simply cartesian diagrams

$$\begin{tikzcd} X \arrow[r, "g"'] \arrow[d] & X' \arrow[d] \\ U \arrow[r, "f"] & U' \end{tikzcd}$$

where g is G-equivariant.

Lemma 55.13.9. *Up to a replacement as in Stacks, Remark 50.4.9 the functor*

$$p : G\text{-}Torsors \longrightarrow (Sch/S)_{fppf}$$

defines a stack in groupoids over $(Sch/S)_{fppf}$.

Proof. The most difficult part of the proof is to show that we have descent for objects, which is Bootstrap, Lemma 54.11.6. Some details omitted. □

Lemma 55.13.10. *Let B be an algebraic space over S. Let G be a group algebraic space over B. Denote $\mathcal{G}$, resp. $\mathcal{B}$ the algebraic space G, resp. B seen as a sheaf on $(Sch/S)_{fppf}$. The functor*

$$G\text{-}Torsors \longrightarrow \mathcal{G}/\mathcal{B}\text{-}Torsors$$

which associates to a triple (U, b, X) the triple $(U, b, \mathcal{X})$ where $\mathcal{X}$ is X viewed as a sheaf is an equivalence of stacks in groupoids over $(Sch/S)_{fppf}$.

Proof. We will use the result of Stacks, Lemma 50.4.8 to prove this. The functor is fully faithful since the category of algebraic spaces over S is a full subcategory of the category of sheaves on $(Sch/S)_{fppf}$. Moreover, all objects (on both sides) are locally trivial torsors so condition (2) of the lemma referenced above holds. Hence the functor is an equivalence. □

55.13.11. Variant on torsors in fppf topology. Let S be a scheme. Let $B = S$. Let G be a group scheme over $B = S$. In this setting we can define a full subcategory G-*Torsors-Schemes* $\subset$ G-*Torsors* whose objects are pairs (U, X) where U is an object of $(Sch/S)_{fppf}$ and $X \to U$ is an fppf G-torsor over U which is representable, i.e., a scheme.

It is in general not the case that G-*Torsors-Schemes* is a stack in groupoids over $(Sch/S)_{fppf}$. The reason is that in general there really do exist fppf G-torsors which are not schemes, hence descent for objects will not be satisfied in general.

55.14. Quotients by group actions

At this point we have introduced enough notation that we can work out in more detail what the stacks $[X/G]$ of Section 55.12 look like.

Situation 55.14.1. Here

1. (1) S is a scheme contained in Sch_{fppf},
2. (2) B is an algebraic space over S,
3. (3) (G, m) is a group algebraic space over B,
4. (4) $\pi : X \to B$ is an algebraic space over B, and
5. (5) $a : G \times_B X \to X$ is an action of G on X over B.

In this situation we construct a category $[[X/G]]$[2] as follows:

1. (1) An object of $[[X/G]]$ consists of a quadruple $(U, b, P, \varphi : P \to X)$ where
   - (a) U is an object of $(Sch/S)_{fppf}$,
   - (b) $b : U \to B$ is a morphism over S,
   - (c) P is an fppf G_U-torsor over U where $G_U = U \times_{b,B} G$, and

[2] The notation $[[X/G]]$ with double brackets serves to distinguish this category from the stack $[X/G]$ introduced earlier. In Proposition 55.14.4 we show that the two are canonically equivalent. Afterwards we will use the notation $[X/G]$ to indicate either.

(d) $\varphi : P \to X$ is a G-equivariant morphism fitting into the commutative diagram

$$\begin{tikzcd} P \arrow[r, "\varphi"] \arrow[d] & X \arrow[d] \\ U \arrow[r, "b"] & B \end{tikzcd}$$

(2) A morphism of $[[X/G]]$ is a pair $(f, g) : (U, b, P, \varphi) \to (U', b', P', \varphi')$ where $f : U \to U'$ is a morphism of schemes over B and $g : P \to P'$ is a G-equivariant morphism over f which induces an isomorphism $P \cong U \times_{f,U'} P'$, and has the property that $\varphi = \varphi' \circ g$. In other words (f, g) fits into the following commutative diagram

$$\begin{tikzcd} P \arrow[r, "g"] \arrow[d] \arrow[rrd, "\varphi"] & P' \arrow[d] \arrow[rd, "\varphi'"] & \\ U \arrow[r, "f"] \arrow[rrd, "b"'] & U' \arrow[rd, "b'"] & X \arrow[d] \\ & & B \end{tikzcd}$$

Thus $[[X/G]]$ is a category and

$$p : [[X/G]] \longrightarrow (Sch/S)_{fppf}, \quad (U, b, P, \varphi) \longmapsto U$$

is a functor. Note that the fibre category of $[[X/G]]$ over U is the disjoint union over $b \in Mor_S(U, B)$ of $U \times_{b,B} G$-torsors P endowed with a G-equivariant morphism to X. Hence the fibre categories of $[[X/G]]$ are groupoids.

Note that the functor

$$[[X/G]] \longrightarrow G\text{-}Torsors, \quad (U, b, P, \varphi) \longmapsto (U, b, P)$$

is a 1-morphism of categories over $(Sch/S)_{fppf}$.

Lemma 55.14.2. *Up to a replacement as in Stacks, Remark 50.4.9 the functor*

$$p : [[X/G]] \longrightarrow (Sch/S)_{fppf}$$

defines a stack in groupoids over $(Sch/S)_{fppf}$.

Proof. The most difficult part of the proof is to show that we have descent for objects. Suppose that $\{U_i \to U\}_{i\in I}$ is a covering in $(Sch/S)_{fppf}$. Let $\xi_i = (U_i, b_i, P_i, \varphi_i)$ be objects of $[[X/G]]$ over U_i, and let $\varphi_{ij} : \mathrm{pr}_0^*\xi_i \to \mathrm{pr}_1^*\xi_j$ be a descent datum. This in particular implies that we get a descent datum on the triples (U_i, b_i, P_i) for the stack in groupoids G-*Torsors* by applying the functor $[[X/G]] \to G$-*Torsors* above. Hence we may assume that $b_i = b|_{U_i}$ for some morphism $b : U \to B$, and that $P_i = U_i \times_U P$ for some fppf $G_U = U \times_{b,B} G$-torsor P over U. Then finally the morphisms φ_i are compatible with the canonical descent datum on the restrictions $U_i \times_U P$ and hence define a morphism $\varphi : P \to X$. (For example you can use Sites, Lemma 9.22.3 or you can use Descent on Spaces, Lemma 45.6.2 to get φ.) This proves descent for objects. We omit the proof of the other two defining properties of a stack in groupoids. $\square$

Remark 55.14.3. Let S be a scheme. Let G be an abstract group. Let X be an algebraic space over S. Let $G \to \mathrm{Aut}_S(X)$ be a group homomorphism. In this setting we can define $[[X/G]]$ similarly to the above as follows:

(1) An object of $[[X/G]]$ consists of a triple $(U, P, \varphi : P \to X)$ where
   (a) U is an object of $(Sch/S)_{fppf}$,
   (b) P is a sheaf on $(Sch/U)_{fppf}$ which comes with an action of G that turns it into a torsor under the constant sheaf with value G, and
   (c) $\varphi : P \to X$ is a G-equivariant map of sheaves.
(2) A morphism $(f, g) : (U, P, \varphi) \to (U', P', \varphi')$ is given by a morphism of schemes $f : T \to T'$ and a G-equivariant isomorphism $g : P \to f^{-1}P'$ such that $\varphi = \varphi' \circ g$.

In exactly the same manner as above we obtain a functor

$$[[X/G]] \longrightarrow (Sch/S)_{fppf}$$

which turns $[[X/G]]$ into a stack in groupoids over $(Sch/S)_{fppf}$. The constant sheaf $\underline{G}$ is (provided the cardinality of G is not too large) representable by G_S on $(Sch/S)_{fppf}$ and this version of $[[X/G]]$ is equivalent to the stack $[[X/G_S]]$ introduced above.

Proposition 55.14.4. *In Situation 55.14.1 there exists a canonical equivalence*

$$[X/G] \longrightarrow [[X/G]]$$

of stacks in groupoids over $(Sch/S)_{fppf}$.

Proof. We write this out in detail, to make sure that all the definitions work out in exactly the correct manner. Recall that $[X/G]$ is the quotient stack associated to the groupoid in algebraic spaces $(X, G \times_B X, s, t, c)$, see Groupoids in Spaces, Definition 52.19.1. This means that $[X/G]$ is the stackification of the category fibred in groupoids $[X/_pG]$ associated to the functor

$$(Sch/S)_{fppf} \longrightarrow Groupoids, \quad U \longmapsto (X(U), G(U) \times_{B(U)} X(U), s, t, c)$$

where $s(g, x) = x$, $t(g, x) = a(g, x)$, and $c((g, x), (g', x')) = (m(g, g'), x')$. By the construction of Categories, Example 4.34.1 an object of $[X/_pG]$ is a pair (U, x) with $x \in X(U)$ and a morphism $(f, g) : (U, x) \to (U', x')$ of $[X/_pG]$ is given by a morphism of schemes $f : U \to U'$ and an element $g \in G(U)$ such that $a(g, x) = x' \circ f$. Hence we can define a 1-morphism of stacks in groupoids

$$F_p : [X/_pG] \longrightarrow [[X/G]]$$

by the following rules: On objects we set

$$F_p(U, x) = (U, \pi \circ x, G \times_{B, \pi\circ x} U, a \circ (\mathrm{id}_G \times x))$$

This makes sense because the diagram

$$\begin{tikzcd} G \times_{B,\pi\circ x} U \arrow[r, "\mathrm{id}_G \times x"'] \arrow[d] & G \times_{B,\pi} X \arrow[r, "a"'] & X \arrow[d, "\pi"] \\ U \arrow[rr, "\pi\circ x"] & & B \end{tikzcd}$$

commutes, and the two horizontal arrows are G-equivariant if we think of the fibre products as trivial G-torsors over U, resp. X. On morphisms $(f, g) : (U, x) \to (U', x')$ we set $F_p(f, g) = (f, R_g)$ where R_g denotes right translation by g. More precisely, the morphism of $F_p(f, g) : F_p(U, x) \to F_p(U', x')$ is given by the cartesian diagram

$$\begin{tikzcd} G \times_{B,\pi\circ x} U \arrow[r, "R_g"'] \arrow[d] & G \times_{B,\pi\circ x'} U' \arrow[d] \\ U \arrow[r, "f"] & U' \end{tikzcd}$$

where R_g on T-valued points is given by

$$R_g(g', u) = (m(g', g), f(u))$$

To see that this works we have to verify that

$$a \circ (\mathrm{id}_G \times x) = a \circ (\mathrm{id}_G \times x') \circ R_g$$

which is true because the right hand side applied to the T-valued point (g', u) gives

$$\begin{aligned} a((\mathrm{id}_G \times x')(m(g', g), f(u))) &= a(m(g', g), x'(f(u))) \\ &= a(g', a(g, x'(f(u)))) \\ &= a(g', x(u)) \end{aligned}$$

because $a(g, x) = x' \circ f$ as desired.

By the universal property of stackification from Stacks, Lemma 50.9.2 we obtain a canonical extension $F : [X/G] \to [[X/G]]$ of the 1-morphism F_p above. We first prove that F is fully faithful. To do this, since both source and target are stacks in groupoids, it suffices to prove that the $Isom$-sheaves are identified under F. Pick a scheme U and objects ξ, ξ' of $[X/G]$ over U. We want to show that

$$F : Isom_{[X/G]}(\xi, \xi') \longrightarrow Isom_{[[X/G]]}(F(\xi), F(\xi'))$$

is an isomorphism of sheaves. To do this it suffices to work locally on U, and hence we may assume that ξ, ξ' come from objects (U, x), (U, x') of $[X/_pG]$ over U; this follows directly from the construction of the stackification, and it is also worked out in detail in Groupoids in Spaces, Section 52.23. Either by directly using the description of morphisms in $[X/_pG]$ above, or using Groupoids in Spaces, Lemma 52.21.1 we see that in this case

$$Isom_{[X/G]}(\xi, \xi') = U \times_{(x,x'), X\times_S X, (s,t)} (G \times_B X)$$

A T-valued point of this fibre product corresponds to a pair (u, g) with $u \in U(T)$, and $g \in G(T)$ such that $a(g, x \circ u) = x' \circ u$. (Note that this implies $\pi \circ x \circ u = \pi \circ x' \circ u$.) On the other hand, a T-valued point of $Isom_{[[X/G]]}(F(\xi), F(\xi'))$ by definition corresponds to a morphism $u : T \to U$ such that $\pi \circ x \circ u = \pi \circ x' \circ u : T \to B$ and an isomorphism

$$R : G \times_{B,\pi\circ x\circ u} T \longrightarrow G \times_{B,\pi\circ x'\circ u} T$$

of trivial G_T-torsors compatible with the given maps to X. Since the torsors are trivial we see that $R = R_g$ (right multiplication) by some $g \in G(T)$. Compatibility with the maps $a\circ(1_G, x\circ u), a\circ(1_G, x'\circ u) : G\times_B T \to X$ is equivalent to the condition that $a(g, x\circ u) = x'\circ u$. Hence we obtain the desired equality of $Isom$-sheaves.

Now that we know that F is fully faithful we see that Stacks, Lemma 50.4.8 applies. Thus to show that F is an equivalence it suffices to show that objects of $[[X/G]]$ are fppf locally in the essential image of F. This is clear as fppf torsors are locally trivial, and hence we win. □

55.15. The Picard stack

Let S be a scheme. Let $\pi : X \to B$ be a morphism of algebraic spaces over S. We define a category $Pic_{X/B}$ as follows:

(1) An object is a triple $(U, b, \mathcal{L})$, where
  (a) U is an object of $(Sch/S)_{fppf}$,
  (b) $b : U \to B$ is a morphism over S, and
  (c) $\mathcal{L}$ is in invertible sheaf on the base change $X_U = U \times_{b,B} X$.

(2) A morphism $(f, g) : (U, b, \mathscr{L}) \to (U', b', \mathscr{L}')$ is given by a morphism of schemes $f : U \to U'$ over B and an isomorphism $g : f^*\mathscr{L}' \to \mathscr{L}$.

The composition of $(f, g) : (U, b, \mathscr{L}) \to (U', b', \mathscr{L}')$ with $(f', g') : (U', b', \mathscr{L}') \to (U'', b'', \mathscr{L}'')$ is given by $(f \circ f', g \circ f^*(g'))$. Thus we get a category $Pic_{X/B}$ and

$$p : Pic_{X/B} \longrightarrow (Sch/S)_{fppf}, \quad (U, b, \mathscr{L}) \longmapsto U$$

is a functor. Note that the fibre category of $Pic_{X/B}$ over U is the disjoint union over $b \in Mor_S(U, B)$ of the categories of invertible sheaves on $X_U = U \times_{b,B} X$. Hence the fibre categories are groupoids.

Lemma 55.15.1. *Up to a replacement as in Stacks, Remark 50.4.9 the functor*

$$Pic_{X/B} \longrightarrow (Sch/S)_{fppf}$$

defines a stack in groupoids over $(Sch/S)_{fppf}$.

Proof. As usual, the hardest part is to show descent for objects. To see this let $\{U_i \to U\}$ be a covering of $(Sch/S)_{fppf}$. Let $\xi_i = (U_i, b_i, \mathscr{L}_i)$ be an object of $Pic_{X/B}$ lying over U, and let $\varphi_{ij} : \mathrm{pr}_0^*\xi_i \to \mathrm{pr}_1^*\xi_j$ be a descent datum. This implies in particular that the morphisms b_i are the restrictions of a morphism $b : U \to B$. Write $X_U = U \times_{b,B} X$ and $X_i = U_i \times_{b_i,B} X = U_i \times_U U \times_{b,B} X = U_i \times_U X_U$. Observe that $\mathscr{L}_i$ is an invertible $\mathscr{O}_{X_i}$-module. Note that $\{X_i \to X_U\}$ forms an fppf covering as well. Moreover, the descent datum φ_{ij} translates into a descent datum on the invertible sheaves $\mathscr{L}_i$ relative to the fppf covering $\{X_i \to X_U\}$. Hence by Descent on Spaces, Proposition 45.4.1 we obtain a unique invertible sheaf $\mathscr{L}$ on X_U which recovers $\mathscr{L}_i$ and the descent data over X_i. The triple $(U, b, \mathscr{L})$ is therefore the object of $Pic_{X/B}$ over U we were looking for. Details omitted. $\square$

55.16. Examples of inertia stacks

Here are some examples of inertia stacks.

Example 55.16.1. Let S be a scheme. Let G be a commutative group. Let $X \to S$ be a scheme over S. Let $a : G \times X \to X$ be an action of G on X. For $g \in G$ we denote $g : X \to X$ the corresponding automorphism. In this case the inertia stack of $[X/G]$ (see Remark 55.14.3) is given by

$$I_{[X/G]} = \coprod\nolimits_{g \in G} [X^g/G],$$

where, given an element g of G, the symbol X^g denotes the scheme $X^g = \{x \in X \mid g(x) = x\}$. In a formula X^g is really the fibre product

$$X^g = X \times_{(1,1), X \times_S X, (g,1)} X.$$

Indeed, for any S-scheme T, a T-point on the inertia stack of $[X/G]$ consists of a triple $(P/T, \phi, \alpha)$ consisting of a G-torsor $P \to T$ together with a G-equivariant isomorphism $\phi : P \to X$, together with an automorphism α of $P \to T$ over T such that $\phi \circ \alpha = \phi$. Since G is a sheaf of *commutative* groups, α is, locally in the fppf topology over T, given by multiplication by some element g of G. The condition that $\phi \circ \alpha = \phi$ means that ϕ factors through the inclusion of X^g in X, i.e., ϕ is obtained by composing that inclusion with a morphism $P \to X^\gamma$. The above discussion allows us to define a morphism of fibred categories $I_{[X/G]} \to \coprod_{g \in G}[X^g/G]$ given on T-points by the discussion above. We omit showing that this is an equivalence.

Example 55.16.2. Let $X \to S$ be a morphism of schemes. Assume that for any $T \to S$ the base change $f_T : X_T \to T$ has the property that the map $\mathcal{O}_T \to f_{T,*}\mathcal{O}_{X_T}$ is an isomorphism. (This implies that f is *cohomologically flat in dimension* 0 (insert future reference here) but is stronger.) Consider the Picard stack $Pic_{X/S}$, see Section 55.15. The points of its inertia stack over an S-scheme T consist of pairs $(\mathcal{L}, \alpha)$ where $\mathcal{L}$ is a line bundle on X_T and α is an automorphism of that line bundle. I.e., we can think of α as an element of $H^0(X_T, \mathcal{O}_{X_T})^\times = H^0(T, \mathcal{O}_T^*)$ by our condition. Note that $H^0(T, \mathcal{O}_T^*) = \mathbf{G}_{m,S}(T)$, see Groupoids, Example 35.5.1. Hence the inertia stack of $Pic_{X/S}$ is

$$I_{Pic_{X/S}} = \mathbf{G}_{m,S} \times_S Pic_{X/S}.$$

as a stack over $(Sch/S)_{fppf}$.

55.17. Finite Hilbert stacks

We formulate this in somewhat greater generality than is perhaps strictly needed. Fix a 1-morphism

$$F : \mathcal{X} \longrightarrow \mathcal{Y}$$

of stacks in groupoids over $(Sch/S)_{fppf}$. For each integer $d \geq 1$ consider a category $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ defined as follows:

(1) An object (U, Z, y, x, α) where U, Z are objects of in $(Sch/S)_{fppf}$ and Z is a finite locally free of degree d over U, where $y \in Ob(\mathcal{Y}_U)$, $x \in Ob(\mathcal{X}_Z)$ and $\alpha : y|_Z \to F(x)$ is an isomorphism[3].

(2) A morphism $(U, Z, y, x, \alpha) \to (U', Z', y', x', \alpha')$ is given by a morphism of schemes $f : U \to U'$, a morphism of schemes $g : Z \to Z'$ which induces an isomorphism $Z \to Z' \times_U U'$, and isomorphisms $b : y \to f^*y'$, $a : x \to g^*x'$ inducing a commutative diagram

$$\begin{tikzcd} y|_Z \arrow[r, "\alpha"'] \arrow[d, "b|_Z"'] & F(x) \arrow[d, "F(a)"] \\ f^*y'|_Z \arrow[r, "\alpha'"] & F(g^*x') \end{tikzcd}$$

It is clear from the definitions that there is a canonical forgetful functor

$$p : \mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \longrightarrow (Sch/S)_{fppf}$$

which assignes to the quintuple (U, Z, y, x, α) the scheme U and to the morphism $(f, g, b, a) : (U, Z, y, x, \alpha) \to (U', Z', y', x', \alpha')$ the morphism $f : U \to U'$.

---

[3] This means the data gives rise, via the 2-Yoneda lemma (Categories, Lemma 4.38.1), to a 2-commutative diagram

$$\begin{tikzcd} (Sch/Z)_{fppf} \arrow[r, "x"'] \arrow[d] & \mathcal{X} \arrow[d, "F"] \\ (Sch/U)_{fppf} \arrow[r, "y"] & \mathcal{Y} \end{tikzcd}$$

of stacks in groupoids over $(Sch/S)_{fppf}$. Alternatively, we may picture α as a 2-morphism

$$\begin{tikzcd} (Sch/Z)_{fppf} \arrow[r, bend left, "y\circ(Z\to U)"] \arrow[r, bend right, "F\circ x"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{Y}. \end{tikzcd}$$

Lemma 55.17.1. *The category $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ endowed with the functor p above defines a stack in groupoids over $(Sch/S)_{fppf}$.*

Proof. As usual, the hardest part is to show descent for objects. To see this let $\{U_i \to U\}$ be a covering of $(Sch/S)_{fppf}$. Let $\xi_i = (U_i, Z_i, y_i, x_i, \alpha_i)$ be an object of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ lying over U_i, and let $\varphi_{ij} : \mathrm{pr}_0^*\xi_i \to \mathrm{pr}_1^*\xi_j$ be a descent datum. First, observe that φ_{ij} induces a descent datum $(Z_i/U_i, \varphi_{ij})$ which is effective by Descent, Lemma 31.33.1 This produces a scheme Z/U which is finite locally free of degree d by Descent, Lemma 31.19.28. From now on we identify Z_i with $Z \times_U U_i$. Next, the objects y_i in the fibre categories $\mathcal{Y}_{U_i}$ descend to an object y in $\mathcal{Y}_U$ because $\mathcal{Y}$ is a stack in groupoids. Similarly the objects x_i in the fibre categories $\mathcal{X}_{Z_i}$ descend to an object x in $\mathcal{X}_Z$ because $\mathcal{X}$ is a stack in groupoids. Finally, the given isomorpisms

$$\alpha_i : (y|_Z)_{Z_i} = y_i|_{Z_i} \longrightarrow F(x_i) = F(x|_{Z_i})$$

glue to a morphism $\alpha : y|_Z \to F(x)$ as the $\mathcal{Y}$ is a stack and hence $Isom_{\mathcal{Y}}(y|_Z, F(x))$ is a sheaf. Details omitted. $\square$

Definition 55.17.2. We will denote $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ the *degree d finite Hilbert stack of $\mathcal{X}$ over $\mathcal{Y}$* constructed above. If $\mathcal{Y} = S$ we write $\mathcal{H}_d(\mathcal{X}) = \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$. If $\mathcal{X} = \mathcal{Y} = S$ we denote it $\mathcal{H}_d$.

Note that given $F : \mathcal{X} \to \mathcal{Y}$ as above we have the following natural 1-morphisms of stacks in groupoids over $(Sch/S)_{fppf}$:

(55.17.2.1)
$$\begin{tikzcd} \mathcal{H}_d(\mathcal{X}) \arrow[dr] & \mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \arrow[l] \arrow[r] \arrow[d] & \mathcal{Y} \\ & \mathcal{H}_d & \end{tikzcd}$$

Each of the arrows is given by a "forgetful functor".

Lemma 55.17.3. *The 1-morphism $\mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \to \mathcal{H}_d(\mathcal{X})$ is faithful.*

Proof. To check that $\mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \to \mathcal{H}_d(\mathcal{X})$ is faithful it suffices to prove that it is faithful on fibre categories. Suppose that $\xi = (U, Z, y, x, \alpha)$ and $\xi' = (U, Z', y', x', \alpha')$ are two objects of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ over the scheme U. Let $(g, b, a), (g', b', a') : \xi \to \xi'$ be two morphisms in the fibre category of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ over U. The image of these morphisms in $\mathcal{H}_d(\mathcal{X})$ agree if and only if $g = g'$ and $a = a'$. Then the commutative diagram

$$\begin{tikzcd} y|_Z \arrow[rr, "\alpha"] \arrow[d, "b|_Z,\, b'|_Z"'] & & F(x) \arrow[d, "F(a)=F(a')"] \\ y'|_Z \arrow[r, "\alpha'"] & F(g^*x') = F((g')^*x') \end{tikzcd}$$

implies that $b|_Z = b'|_Z$. Since $Z \to U$ is finite locally free of degree d we see $\{Z \to U\}$ is an fppf covering, hence $b = b'$. $\square$

55.18. Other chapters

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CHAPTER 56

Quotients of Groupoids

56.1. Introduction

This chapter is devoted to generalities concering groupoids and their quotients (as far as they exist). There is a lot of literature on this subject, see for example [**MFK94**], [**Ses72**], [**Kol97**], [**KM97a**], [**Kol08**] and many more.

56.2. Conventions and notation

In this chapter the conventions and notation are those introduced in Groupoids in Spaces, Sections 52.2 and 52.3.

56.3. Invariant morphisms

Definition 56.3.1. Let S be a scheme, and let B be an algebraic space over S. Let $j = (t, s) : R \to U \times_B U$ be a pre-relation of algebraic spaces over B. We say a morphism $\phi : U \to X$ of algebraic spaces over B is *R-invariant* if the diagram

$$\begin{tikzcd} R \arrow[r, "s"] \arrow[d, "t"'] & U \arrow[d, "\phi"] \\ U \arrow[r, "\phi"] & X \end{tikzcd}$$

is commutative. If $j : R \to U \times_B U$ comes from the action of a group algebraic space G on U over B as in Groupoids in Spaces, Lemma 52.14.1, then we say that ϕ is *G-invariant*.

In other words, a morphism $U \to X$ is R-invariant if it equalizes s and t. We can reformulate this in terms of associated quotient sheaves as follows.

Lemma 56.3.2. *Let S be a scheme, and let B be an algebraic space over S. Let $j = (t, s) : R \to U \times_B U$ be a pre-relation of algebraic spaces over B. A morphism of algebraic spaces $\phi : U \to X$ is R-invariant if and only if it factors as $U \to U/R \to X$.*

Proof. This is clear from the definition of the quotient sheaf in Groupoids in Spaces, Section 52.18. □

Lemma 56.3.3. *Let S be a scheme, and let B be an algebraic space over S. Let $j = (t, s) : R \to U \times_B U$ be a pre-relation of algebraic spaces over B. Let $U \to X$ be an R-invariant morphism of algebraic spaces over B. Let $X' \to X$ be any morphism of algebraic spaces.*

(1) *Setting $U' = X' \times_X U$, $R' = X' \times_X R$ we obtain a pre-relation $j' : R' \to U' \times_B U'$.*
(2) *The pre-relation $j' : R' \to U' \times_B U'$ is the restriction of $j : R \to U \times_B U$ via $U' \to U$, see Groupoids in Spaces, Definition 52.4.3.*
(3) *If j is a relation, then j' is a relation.*
(4) *If j is a pre-equivalence relation, then j' is a pre-equivalence relation.*
(5) *If j is an equivalence relation, then j' is an equivalence relation.*

(6) *If j comes from a groupoid in algebraic spaces (U, R, s, t, c) over B, then j' comes from the restriction of this groupoid to U', see Groupoids in Spaces, Definition 52.16.2.*
(7) *If j comes from the action of a group algebraic space G/B on U as in Groupoids in Spaces, Lemma 52.14.1 then j' comes from the induced action of G on U'.*

Proof. Omitted. Hint: Functorial point of view combined with the picture:

$$\begin{tikzcd}
R' = X' \times_X R \arrow[rr] \arrow[dr] \arrow[dd] & & X' \times_X U = U' \arrow[dr] \arrow[dd] & \\
& R \arrow[rr] \arrow[dd] & & U \arrow[dd] \\
U' = X' \times_X U \arrow[rr] \arrow[dr] & & X' \arrow[dr] & \\
& U \arrow[rr] & & X
\end{tikzcd}$$

□

Definition 56.3.4. In the situation of Lemma 56.3.3 we call $j' : R' \to U' \times_B U'$ the *pullback* of the pre-relation j to X'. We say it is a *flat pullback* if $X' \to X$ is a flat morphism of algebraic spaces.

56.4. Categorical quotients

This is the most basic kind of quotient one can consider.

Definition 56.4.1. Let S be a scheme, and let B be an algebraic space over S. Let $j = (t, s) : R \to U \times_B U$ be pre-relation in algebraic spaces over B.

(1) We say a morphism $\phi : U \to X$ of algebraic spaces over B is a *categorical quotient* if it is R-invariant, and for every R-invariant morphism $\psi : U \to Y$ of algebraic spaces over B there exists a unique morphism $\chi : X \to Y$ such that $\psi = \phi \circ \chi$.
(2) Let $\mathcal{C}$ be a full subcategory of the category of algebraic spaces over B. Assume U, R are objects of $\mathcal{C}$. In this situation we say a morphism $\phi : U \to X$ of algebraic spaces over B is a *categorical quotient in* $\mathcal{C}$ if $X \in Ob(\mathcal{C})$, and ϕ is R-invariant, and for every R-invariant morphism $\psi : U \to Y$ with $Y \in Ob(\mathcal{C})$ there exists a unique morphism $\chi : X \to Y$ such that $\psi = \phi \circ \chi$.
(3) If $B = S$ and $\mathcal{C}$ is the category of schemes over S, then we say $U \to X$ is a *categorical quotient in the category of schemes*, or simply a *categorical quotient in schemes*.

We often single out a category $\mathcal{C}$ of algebraic spaces over B by some separation axiom, see Example 56.4.3 for some standard cases. Note that if $\phi : U \to X$ is a categorical quotient if and only if $U \to X$ is a coequalizer for the morphisms $t, s : R \to U$ in the category. Hence we immediately deduce the following lemma.

Lemma 56.4.2. *Let S be a scheme, and let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation in algebraic spaces over B. If a categorical quotient in the category of algebaic spaces over B exists, then it is unique up to unique isomorphism. Similarly for categorical quotients in full subcategories of Spaces/B.*

Proof. See Categories, Section 4.11. □

Example 56.4.3. Let S be a scheme, and let B be an algebraic space over S. Here are some standard examples of categories $\mathcal{C}$ that we often come up when applying Definition 56.4.1:

(1) $\mathcal{C}$ is the category of all algebraic spaces over B,
(2) B is separated and $\mathcal{C}$ is the category of all separated algebraic spaces over B,
(3) B is quasi-separated and $\mathcal{C}$ is the category of all quasi-separated algebraic spaces over B,
(4) B is locally separated and $\mathcal{C}$ is the category of all locally separated algebraic spaces over B,
(5) B is decent and $\mathcal{C}$ is the category of all decent algebraic spaces over B, and
(6) $S = B$ and $\mathcal{C}$ is the category of schemes over S.

In this case, if $\phi : U \to X$ is a categorical quotient then we say $U \to X$ is (1) a *categorical quotient*, (2) a *categorical quotient in separated algebraic spaces*, (3) a *categorical quotient in quasi-separated algebraic spaces*, (4) a *categorical quotient in locally separated algebraic spaces*, (5) a *categorical quotient in decent algebraic spaces*, (6) a *categorical quotient in schemes*.

Definition 56.4.4. Let S be a scheme, and let B be an algebraic space over S. Let $\mathcal{C}$ be a full subcategory of the category of algebraic spaces over B closed under fibre products. Let $j = (t, s) : R \to U \times_B U$ be pre-relation in $\mathcal{C}$, and let $U \to X$ be an R-invariant morphism with $X \in Ob(\mathcal{C})$.

(1) We say $U \to X$ is a *universal categorical quotient* in $\mathcal{C}$ if for every morphism $X' \to X$ in $\mathcal{C}$ the morphism $U' = X' \times_X U \to X'$ is the categorical quotient in $\mathcal{C}$ of the pullback $j' : R' \to U'$ of j.
(2) We say $U \to X$ is a *uniform categorical quotient* in $\mathcal{C}$ if for every flat morphism $X' \to X$ in $\mathcal{C}$ the morphism $U' = X' \times_X U \to X'$ is the categorical quotient in $\mathcal{C}$ of the pullback $j' : R' \to U'$ of j.

Lemma 56.4.5. *In the situation of Definition 56.4.1. If $\phi : U \to X$ is a categorical quotient and U is reduced, then X is reduced. The same holds for categorical quotients in a category of spaces $\mathcal{C}$ listed in Example 56.4.3.*

Proof. Let X_{red} be the reduction of the algebraic space X. Since U is reduced the morphism $\phi : U \to X$ factors through $i : X_{red} \to X$ (insert future reference here). Denote this morphism by $\phi_{red} : U \to X_{red}$. Since $\phi \circ s = \phi \circ t$ we see that also $\phi_{red} \circ s = \phi_{red} \circ t$ (as $i : X_{red} \to X$ is a monomorphism). Hence by the universal property of ϕ there exists a morphism $\chi : X \to X_{red}$ such that $\phi_{red} = \phi \circ \chi$. By uniqueness we see that $i \circ \chi = \mathrm{id}_X$ and $\chi \circ i = \mathrm{id}_{X_{red}}$. Hence i is an isomorphism and X is reduced.

To show that this argument works in a category $\mathcal{C}$ one just needs to show that the reduction of an object of $\mathcal{C}$ is an object of $\mathcal{C}$. We omit the verification that this holds for each of the standard examples. □

56.5. Quotients as orbit spaces

Let $j = (t, s) : R \to U \times_B U$ be a pre-relation. If j is a pre-equivalence relation, then loosely speaking the ``orbits'' of R on U are the subsets $t(s^{-1}(\{u\}))$ of U. However, if j is just a pre-relation, then we need to take the equivalence relation generated by R.

Definition 56.5.1. Let S be a scheme, and let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation over B. If $u \in |U|$, then the *orbit*, or more precisely the *R-orbit* of u is

$$O_u = \left\{ u' \in |U| \ : \ \begin{matrix} \exists n \geq 1,\ \exists u_0, \ldots, u_n \in |U| \text{ such that} \\ \text{for all } i \in \{0, \ldots, n-1\} \text{ either} \\ u_i = u_{i+1} \text{ or} \\ \exists r \in |R|,\ s(r) = u_i, t(r) = u_{i+1} \text{ or} \\ \exists r \in |R|,\ t(r) = u_i, s(r) = u_{i+1} \end{matrix} \right\}$$

It is clear that these are the equivalence classes of an equivalence relation, i.e., we have $u' \in O_u$ if and only if $u \in O_{u'}$. The following lemma is a reformulation of Groupoids in Spaces, Lemma 52.4.4.

Lemma 56.5.2. *Let $B \to S$ as in Section 56.2. Let $j : R \to U \times_B U$ be a pre-equivalence relation of algebraic spaces over B. Then*

$$O_u = \{u' \in |U| \text{ such that } \exists r \in |R|,\ s(r) = u,\ t(r) = u'\}.$$

Proof. By the aforemetioned Groupoids in Spaces, Lemma 52.4.4 we see that the orbits O_u as defined in the lemma give a disjoint union decomposition of $|U|$. Thus we see they are equal to the orbits as defined in Definition 56.5.1. □

Lemma 56.5.3. *In the situation of Definition 56.5.1. Let $\phi : U \to X$ be an R-invariant morphism of algebraic spaces over B. Then $|\phi| : |U| \to |X|$ is constant on the orbits.*

Proof. To see this we just have to show that $\phi(u) = \phi(u')$ for all $u, u' \in |U|$ such that there exists an $r \in |R|$ such that $s(r) = u$ and $t(r) = u'$. And this is clear since ϕ equalizes s and t. □

There are several problems with considering the orbits $O_u \subset |U|$ as a tool for singling out properties of quotient maps. One issue is the following. Suppose that $Spec(k) \to B$ is a geometric point of B. Consider the canonical map

$$U(k) \longrightarrow |U|.$$

Then it is usually not the case that the equivalence classes of the equivalence relation generated by $j(R(k)) \subset U(k) \times U(k)$ are the inverse images of the orbits $O_u \subset |U|$. A silly example is to take $S = B = Spec(\mathbf{Z})$, $U = R = Spec(k)$ with $s = t = \text{id}_k$. Then $|U| = |R|$ is a single point but $U(k)/R(k)$ is enormous. A more interesting example is to take $S = B = Spec(\mathbf{Q})$, choose some of number fields $K \subset L$, and set $U = Spec(L)$ and $R = Spec(L \otimes_K L)$ with obvious maps $s, t : R \to U$. In this case $|U|$ still has just one point, but the quotient

$$U(k)/R(k) = Hom(K, k)$$

consists of more than one element. We conclude from both examples that if $U \to X$ is an R-invariant map and if we want it to ``separate orbits'' we get a much stronger and interesting notion by considering the induced maps $U(k) \to X(k)$ and ask that those maps separate orbits.

There is an issue with this too. Namely, suppose that $S = B = Spec(\mathbf{R})$, $U = Spec(\mathbf{C})$, and $R = Spec(\mathbf{C}) \amalg Spec(K)$ for some field extension $\sigma : \mathbf{C} \to K$. Let the maps s, t be given by the identity on the component $Spec(\mathbf{C})$, but by $\sigma, \sigma \circ \tau$ on the second component where τ is complex conjugation. If K is a nontrivial extension of $\mathbf{C}$, then the two points $1, \tau \in U(\mathbf{C})$ are not equivalent under $j(R(\mathbf{C}))$. But after choosing an extension $\mathbf{C} \subset \Omega$ of sufficiently large cardinality (for example larger than the cardinality of K) then the images of $1, \tau \in U(\mathbf{C})$ in $U(\Omega)$ do become equivalent! It seems intuitively clear that this happens either because $s, t : R \to U$ are not locally of finite type or because the cardinality of the field k is not large enough.

Keeping this in mind we make the following definition.

Definition 56.5.4. Let S be a scheme, and let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation over B. Let $Spec(k) \to B$ be a geometric point of B.

(1) We say $\overline{u}, \overline{u}' \in U(k)$ are *weakly R-equivalent* if they are in the same equivalence class for the equivalence relation generated by the relation $j(R(k)) \subset U(k) \times U(k)$.
(2) We say $\overline{u}, \overline{u}' \in U(k)$ are *R-equivalent* if for some overfield $k \subset \Omega$ the images in $U(\Omega)$ are weakly R-equivalent.
(3) The *weak orbit*, or more precisely the *weak R-orbit* of $\overline{u} \in U(k)$ is set of all elements of $U(k)$ which are weakly R-equivalent to $\overline{u}$.
(4) The *orbit*, or more precisely the *R-orbit* of $\overline{u} \in U(k)$ is set of all elements of $U(k)$ which are R-equivalent to $\overline{u}$.

It turns out that in good cases orbits and weak orbits agree, see Lemma 56.5.7. The following lemma illustrates the difference in the special case of a pre-equivalence relation.

Lemma 56.5.5. *Let S be a scheme, and let B be an algebraic space over S. Let $Spec(k) \to B$ be a geometric point of B. Let $j : R \to U \times_B U$ be a pre-equivalence relation over B. In this case the weak orbit of $\overline{u} \in U(k)$ is simply*

$$\{\overline{u}' \in U(k) \text{ such that } \exists \overline{r} \in R(k),\ s(\overline{r}) = \overline{u},\ t(\overline{r}) = \overline{u}'\}$$

and the orbit of $\overline{u} \in U(k)$ is

$$\{\overline{u}' \in U(k) : \exists \text{ field extension } k \subset K,\ \exists\, r \in R(K),\ s(r) = \overline{u},\ t(r) = \overline{u}'\}$$

Proof. This is true because by definition of a pre-equivalence relation the image $j(R(k)) \subset U(k) \times U(k)$ is an equivalence relation. $\square$

Let us describe the recipe for turning any pre-relation into a pre-equivalence relation. We will use the morphisms

$$(56.5.5.1) \qquad \begin{array}{lcccccc} j_{diag} & : & U & \longrightarrow & U \times_B U, & u & \longmapsto & (u, u) \\ j_{flip} & : & R & \longrightarrow & U \times_B U, & r & \longmapsto & (s(r), t(r)) \\ j_{comp} & : & R \times_{s,U,t} R & \longrightarrow & U \times_B U, & (r, r') & \longmapsto & (t(r), s(r')) \end{array}$$

We define $j_1 = (t_1, s_1) : R_1 \to U \times_B U$ to be the morphism

$$j \amalg j_{diag} \amalg j_{flip} : R \amalg U \amalg R \longrightarrow U \times_B U$$

with notation as in Equation (56.5.5.1). For $n > 1$ we set

$$j_n = (t_n, s_n) : R_n = R_1 \times_{s_1, U, t_{n-1}} R_{n-1} \longrightarrow U \times_B U$$

where t_n comes from t_1 precomposed with projection onto R_1 and s_n comes from s_{n-1} precomposed with projection onto R_{n-1}. Finally, we denote

$$j_\infty = (t_\infty, s_\infty) : R_\infty = \coprod\nolimits_{n \geq 1} R_n \longrightarrow U \times_B U.$$

Lemma 56.5.6. *Let S be a scheme, and let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation over B. Then $j_\infty : R_\infty \to U \times_B U$ is a pre-equivalence relation over B. Moreover*

(1) *$\phi : U \to X$ is R-invariant if and only if it is R_∞-invariant,*

(2) *the canonical map of quotient sheaves $U/R \to U/R_\infty$ (see Groupoids in Spaces, Section 52.18) is an ismorphism,*
(3) *weak R-orbits agree with weak R_∞-orbits,*
(4) *R-orbits agree with R_∞-orbits,*
(5) *if s, t are locally of finite type, then s_∞, t_∞ are locally of finite type,*
(6) *add more here as needed.*

Proof. Omitted. Hint for (5): Any property of s, t which is stable under composition and stable under base change, and Zariski local on the source will be inherited by s_∞, t_∞. $\square$

Lemma 56.5.7. *Let S be a scheme, and let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation over B. Let $Spec(k) \to B$ be a geometric point of B.*

(1) *If $s, t : R \to U$ are locally of finite type then weak R-equivalence on $U(k)$ agrees with R-equivalence, and weak R-orbits agree with R-orbits on $U(k)$.*
(2) *If k has sufficiently large cardinality then weak R-equivalence on $U(k)$ agrees with R-equivalence, and weak R-orbits agree with R-orbits on $U(k)$.*

Proof. We first prove (1). Assume s, t locally of finite type. By Lemma 56.5.6 we may assume that R is a pre-equivalence relation. Let k be an algebraically closed field over B. Suppose $\overline{u}, \overline{u}' \in U(k)$ are R-equivalent. Then for some extension field $k \subset \Omega$ there exists a point $\overline{r} \in R(\Omega)$ mapping to $(\overline{u}, \overline{u}') \in (U \times_B U)(\Omega)$, see Lemma 56.5.5. Hence

$$Z = R \times_{j, U \times_B U, (\overline{u}, \overline{u}')} Spec(k)$$

is nonempty. As s is locally of finite type we see that also j is locally of finite type, see Morphisms of Spaces, Lemma 42.22.6. This implies Z is a nonempty algebraic space locally of finite type over the algebraically closed field k (use Morphisms of Spaces, Lemma 42.22.3). Thus Z has a k-valued point, see Morphisms of Spaces, Lemma 42.23.1. Hence we conclude there exists a $\overline{r} \in R(k)$ with $j(\overline{r}) = (\overline{u}, \overline{u}')$, and we conclude that $\overline{u}, \overline{u}'$ are R-equivalent as desired.

The proof of part (2) is the same, except that it uses Morphisms of Spaces, Lemma 42.23.2 instead of Morphisms of Spaces, Lemma 42.23.1. This shows that the assertion holds as soon as $|k| > \lambda(R)$ with $\lambda(R)$ as introduced just above Morphisms of Spaces, Lemma 42.23.1. $\square$

In the following definition we use the terminology ``k is a field over B'' to mean that $Spec(k)$ comes equipped with a morphism $Spec(k) \to B$.

Definition 56.5.8. Let S be a scheme, and let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation over B.

(1) We say $\phi : U \to X$ is *set-theoretically R-invariant* if and only if the map $U(k) \to X(k)$ equalizes the two maps $s, t : R(k) \to U(k)$ for every algebraically closed field k over B.
(2) We say $\phi : U \to X$ *separates orbits*, or *separates R-orbits* if it is set-theoretically R-invariant and $\phi(\overline{u}) = \phi(\overline{u}')$ in $X(k)$ implies that $\overline{u}, \overline{u}' \in U(k)$ are in the same orbit for every algebraically closed field k over B.

In Example 56.5.12 we show that being set-theoretically invariant is ``too weak'' a notion in the category of algebraic spaces. A more geometric reformulation of what it means to be set-theoretically invariant or to separate orbits is in Lemma 56.5.17.

Lemma 56.5.9. *In the situation of Definition 56.5.8. A morphism $\phi : U \to X$ is set-theoretically R-invariant if and only if for any algebraically closed field k over B the map $U(k) \to X(k)$ is constant on orbits.*

Proof. This is true because the condition is supposed to hold for all algebraically closed fields over B. □

Lemma 56.5.10. *In the situation of Definition 56.5.8. An invariant morphism is set-theoretically invariant.*

Proof. This is immediate from the definitions. □

Lemma 56.5.11. *In the situation of Definition 56.5.8. Let $\phi : U \to X$ be a morphism of algebraic spaces over B. Assume*

(1) *ϕ is set-theoretically R-invariant,*
(2) *R is reduced, and*
(3) *X is locally separated over B.*

Then ϕ is R-invariant.

Proof. Consider the equalizer

$$Z = R \times_{(\phi,\phi)\circ j, X\times_B X, \Delta_{X/B}} X$$

algebraic space. Then $Z \to R$ is an immersion by assumption (3). By assumption (1) $|Z| \to |R|$ is surjective. This implies that $Z \to R$ is a bijective closed immersion (use Schemes, Lemma 21.10.4) and by assumption (2) we conclude that $Z = R$. □

Example 56.5.12. There exist reduced quasi-separated algebraic spaces X, Y and a pair of morphisms $a, b : Y \to X$ which agree on all k-valued points but are not equal. To get an example take $Y = Spec(k[[x]])$ and

$$X = \mathbf{A}^1_k \Big/ \big(\Delta \amalg \{(x, -x) \mid x \neq 0\}\big)$$

the algebraic space of Spaces, Example 40.14.1. The two morphisms $a, b : Y \to X$ come from the two maps $x \mapsto x$ and $x \mapsto -x$ from Y to $\mathbf{A}^1_k = Spec(k[x])$. On the generic point the two maps are the same because on the open part $x \neq 0$ of the space X the functions x and $-x$ are equal. On the closed point the maps are obviously the same. It is also true that $a \neq b$. This implies that Lemma 56.5.11 does not hold with assumption (3) replaced by the assumption that X be quasi-separated. Namely, consider the diagram

$$\begin{tikzcd} Y \arrow[r, "1"'] \arrow[d, "-1"'] & Y \arrow[d, "a"] \\ Y \arrow[r, "a"] & X \end{tikzcd}$$

then the composition $a \circ (-1) = b$. Hence we can set $R = Y$, $U = Y$, $s = 1$, $t = -1$, $\phi = a$ to get an example of a set-theoretically invariant morphism which is not invariant.

The example above is instructive because the map $Y \to X$ even separates orbits. It shows that in the category of algebraic spaces there are simply too many set-theoretically invariant morphisms lying around. Next, let us define what it means for R to be a set-theoretic equivalence relation, while remembering that we need to allow for field extensions to make this work correctly.

Definition 56.5.13. Let S be a scheme, and let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation over B.

(1) We say j is a *set-theoretic pre-equivalence relation* if

$$\overline{u} \sim_R \overline{u}' \Leftrightarrow \begin{matrix} \exists \text{ field extension } k \subset K,\ \exists\, r \in R(K), \\ s(r) = \overline{u},\ t(r) = \overline{u}' \end{matrix}$$

defines an equivalence relation on $U(k)$ for all algebraically closed fields k over B.

(2) We say j is a *set-theoretic equivalence relation* if j is universally injective and a set-theoretic pre-equivalence relation.

Let us reformulate this in more geometric terms.

Lemma 56.5.14. *In the situation of Definition 56.5.13. The following are equivalent:*

(1) *The morphism j is a set-theoretic pre-equivalence relation.*
(2) *The subset $j(|R|) \subset |U \times_B U|$ contains the image of $|j'|$ for any of the morphisms j' as in Equation (56.5.5.1).*
(3) *For every algebraically closed field k over B of sufficiently large cardinality the subset $j(R(k)) \subset U(k) \times U(k)$ is an equivalence relation.*

If s, t are locally of finite type these are also equivalent to

(4) *For every algebraically closed field k over B the subset $j(R(k)) \subset U(k) \times U(k)$ is an equivalence relation.*

Proof. Assume (2). Let k be an algebraically closed field over B. We are going to show that $\sim_R$ is an equivalence relation. Suppose that $\overline{u}_i : Spec(k) \to U$, $i = 1, 2$ are k-valued points of U. Suppose that $(\overline{u}_1, \overline{u}_2)$ is the image of a K-valued point $r \in R(K)$. Consider the solid commutative diagram

$$\begin{tikzcd} Spec(K') \arrow[r, dotted] \arrow[d, dotted] & Spec(k) \arrow[d, "{(\overline{u}_2,\overline{u}_1)}"'] & Spec(K) \arrow[l] \arrow[d] \\ R \arrow[r, "j"] & U \times_B U & R \arrow[l, "j_{flip}"'] \end{tikzcd}$$

We also denote $r \in |R|$ the image of r. By assumption the image of $|j_{flip}|$ is contained in the image of $|j|$, in other words there exists a $r' \in |R|$ such that $|j|(r') = |j_{flip}|(r)$. But note that $(\overline{u}_2, \overline{u}_1)$ is in the equivalence class that defines $|j|(r')$ (by the commutativity of the solid part of the diagram). This means there exists a field extension $k \subset K'$ and a morphism $r' : Spec(K) \to R$ (abusively denoted r' as well) with $j \circ r' = (\overline{u}_2, \overline{u}_1) \circ i$ where $i : Spec(K') \to Spec(K)$ is the obvious map. In other words the dotted part of the diagram commutes. This proves that $\sim_R$ is a symmetric relation on $U(k)$. In the similar way, using that the image of $|j_{diag}|$ is contained in the image of $|j|$ we see that $\sim_R$ is reflexive (details omitted).

To show that $\sim_R$ is transitive assume given $\overline{u}_i : Spec(k) \to U$, $i = 1, 2, 3$ and field extensions $k \subset K_i$ and points $r_i : Spec(K_i) \to R$, $i = 1, 2$ such that $j(r_1) = (\overline{u}_1, \overline{u}_2)$ and $j(r_1) = (\overline{u}_2, \overline{u}_3)$. Then we may choose a commutative diagram of fields

$$\begin{tikzcd} K & K_2 \arrow[l] \\ K_1 \arrow[u] & k \arrow[l] \arrow[u] \end{tikzcd}$$

and we may think of $r_1, r_2 \in R(K)$. We consider the commutative solid diagram

$$\begin{tikzcd} Spec(K') \arrow[r, dotted] \arrow[d, dotted] & Spec(k) \arrow[d, "(\overline{u}_1,\overline{u}_3)"] & Spec(K) \arrow[l] \arrow[d, "(r_1,r_2)"] \\ R \arrow[r, "j"] & U \times_B U & R \times_{s,U,t} R \arrow[l, "j_{comp}"'] \end{tikzcd}$$

By exactly the same reasoning as in the first part of the proof, but this time using that $|j_{comp}|((r_1, r_2))$ is in the image of $|j|$, we conclude that a field K' and dotted arrows exist making the diagram commute. This proves that $\sim_R$ is transitive and concludes the proof that (2) implies (1).

Assume (1) and let k be an algebraically closed field over B whose cardinality is larger than $\lambda(R)$, see Morphisms of Spaces, Lemma 42.23.2. Suppose that $\overline{u} \sim_R \overline{u}'$ with $\overline{u}, \overline{u}' \in U(k)$. By assumption there exists a point in $|R|$ mapping to $(\overline{u}, \overline{u}') \in |U \times_B U|$. Hence by Morphisms of Spaces, Lemma 42.23.2 we conclude there exists an $\overline{r} \in R(k)$ with $j(\overline{r}) = (\overline{u}, \overline{u}')$. In this way we see that (1) implies (3).

Assume (3). Let us show that $\mathrm{Im}(|j_{comp}|) \subset \mathrm{Im}(|j|)$. Pick any point $c \in |R \times_{s,U,t} R|$. We may represent this by a morphism $\overline{c} : Spec(k) \to R \times_{s,U,t} R$, with k over B having sufficiently large cardinality. By assumption we see that $j_{comp}(\overline{c}) \in U(k) \times U(k) = (U \times_B U)(k)$ is also the image $j(\overline{r})$ for some $\overline{r} \in R(k)$. Hence $j_{comp}(c) = j(r)$ in $|U \times_B U|$ as desired (with $r \in |R|$ the equivalence class of $\overline{r}$). The same argument shows also that $\mathrm{Im}(|j_{diag}|) \subset \mathrm{Im}(|j|)$ and $\mathrm{Im}(|j_{flip}|) \subset \mathrm{Im}(|j|)$ (details omitted). In this way we see that (3) implies (2). At this point we have shown that (1), (2) and (3) are all equivalent.

It is clear that (4) implies (3) (without any assumptions on s, t). To finish the proof of the lemma we show that (1) implies (4) if s, t are locally of finite type. Namely, let k be an algebraically closed field over B. Suppose that $\overline{u} \sim_R \overline{u}'$ with $\overline{u}, \overline{u}' \in U(k)$. By assumption the algebraic space $Z = R \times_{j, U \times_B U, (\overline{u}, \overline{u}')} Spec(k)$ is nonempty. On the other hand, since $j = (t, s)$ is locally of finite type the morphism $Z \to Spec(k)$ is locally of finite type as well (use Morphisms of Spaces, Lemmas 42.22.6 and 42.22.3). Hence Z has a k point by Morphisms of Spaces, Lemma 42.23.1 and we conclude that $(\overline{u}, \overline{u}') \in j(R(k))$ as desired. This finishes the proof of the lemma. □

Lemma 56.5.15. *In the situation of Definition 56.5.13. The following are equivalent:*

(1) *The morphism j is a set-theoretic equivalence relation.*
(2) *The morphism j is universally injective and $j(|R|) \subset |U \times_B U|$ contains the image of $|j'|$ for any of the morphisms j' as in Equation (56.5.5.1).*
(3) *For every algebraically closed field k over B of sufficiently large cardinality the map $j : R(k) \to U(k) \times U(k)$ is injective and its image is an equivalence relation.*

If j is decent, or locally separated, or quasi-separated these are also equivalent to

(4) *For every algebraically closed field k over B the map $j : R(k) \to U(k) \times U(k)$ is injective and its image is an equivalence relation.*

Proof. The implications (1) ⇒ (2) and (2) ⇒ (3) follow from Lemma 56.5.14 and the definitions. The same lemma shows that (3) implies j is a set-theoretic pre-equivalence relation. But of course condition (3) also implies that j is universally injective, see Morphisms of Spaces, Lemma 42.18.2, so that j is indeed a set-theoretic equivalence relation. At this point we know that (1), (2), (3) are all equivalent.

Condition (4) implies (3) without any further hypotheses on j. Assume j is decent, or locally separated, or quasi-separated and the equivalent conditions (1), (2), (3) hold. By More on Morphisms of Spaces, Lemma 46.3.4 we see that j is radicial. Let k be any algebraically closed field over B. Let $\overline{u}, \overline{u}' \in U(k)$ with $\overline{u} \sim_R \overline{u}'$. We see that $R \times_{U \times_B U, (\overline{u}, \overline{u}')} Spec(k)$ is nonempty. Hence, as j is radicial, its reduction is the spectrum of a field purely inseparable over k. As $k = \overline{k}$ we see that it is the spectrum of k. Whence a point $\overline{r} \in R(k)$ with $t(\overline{r}) = \overline{u}$ and $s(\overline{r}) = \overline{u}'$ as desired. □

Lemma 56.5.16. *Let S be a scheme, and let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation over B.*

(1) *If j is a pre-equivalence relation, then j is a set-theoretic pre-equivalence relation. This holds in particular when j comes from a groupoid in algebraic spaces, or from an action of a group algebraic space on U.*
(2) *If j is an equivalence relation, then j is a set-theoretic equivalence relation.*

Proof. Omitted. □

Lemma 56.5.17. *Let $B \to S$ be as in Section 56.2. Let $j : R \to U \times_B U$ be a pre-relation. Let $\phi : U \to X$ be a morphism of algebraic spaces over B. Consider the diagram*

$$\begin{tikzcd} (U \times_X U) \times_{(U \times_B U)} R \arrow[r, "p"'] \arrow[d, "q"] & R \arrow[d, "j"] \\ U \times_X U \arrow[r, "c"] & U \times_B U \end{tikzcd}$$

Then we have:

(1) *The morphism ϕ is set-theoretically invariant if and only if p is surjective.*
(2) *If j is a set-theoretic pre-equivalence relation then ϕ separates orbits if and only if p and q are surjective.*
(3) *If p and q are surjective, then j is a set-theoretic pre-equivalence relation (and ϕ separates orbits).*
(4) *If ϕ is R-invariant and j is a set-theoretic pre-equivalence relation, then ϕ separates orbits if and only if the induced morphism $R \to U \times_X U$ is surjective.*

Proof. Assume ϕ is set-theoretically invariant. This means that for any algebraically closed field k over B and any $\overline{r} \in R(k)$ we have $\phi(s(\overline{r})) = \phi(t(\overline{r}))$. Hence $((\phi(t(\overline{r})), \phi(s(\overline{r}))), \overline{r})$ defines a point in the fibre product mapping to $\overline{r}$ via p. This shows that p is surjective. Conversely, assume p is surjective. Pick $\overline{r} \in R(k)$. As p is surjective, we can find a field extension $k \subset K$ and a K-valued point $\tilde{r}$ of the fibre product with $p(\tilde{r}) = \overline{r}$. Then $q(\tilde{r}) \in U \times_X U$ maps to $(t(\overline{r}), s(\overline{r}))$ in $U \times_B U$ and we conclude that $\phi(s(\overline{r})) = \phi(t(\overline{r}))$. This proves that ϕ is set-theoretically invariant.

The proofs of (2), (3), and (4) are omitted. Hint: Assume k is an algebraically closed field over B of large cardinality. Consider the associated diagram of sets

$$\begin{tikzcd} (U(k) \times_{X(k)} U(k)) \times_{U(k) \times U(k)} R(k) \arrow[r, "p"'] \arrow[d, "q"] & R(k) \arrow[d, "j"] \\ U(k) \times_{X(k)} U(k) \arrow[r, "c"] & U(k) \times U(k) \end{tikzcd}$$

By the lemmas above the equivalences posed in (2), (3), and (4) become set-theoretic questions related to the diagram we just displayed, using that surjectivity translates into surjectivity on k-valued points by Morphisms of Spaces, Lemma 42.23.2. □

Because we have seen above that the notion of a set-theoretically invariant morphism is a rather weak one in the category of algebraic spaces, we define an orbit space for a pre-relation as follows.

Definition 56.5.18. Let $B \to S$ as in Section 56.2. Let $j : R \to U \times_B U$ be a pre-relation. We say $\phi : U \to X$ is an *orbit space for R* if

(1) ϕ is R-invariant,
(2) ϕ separates R-orbits, and
(3) ϕ is surjective.

The definition of separating R-orbits involves a discussion of points with values in algebraically closed fields. But as we've seen in many cases this just corresponds to the surjectivity of certain canonically associated morphisms of algebraic spaces. We summarize some of the discussion above in the following characterization of orbit spaces.

Lemma 56.5.19. *Let $B \to S$ as in Section 56.2. Let $j : R \to U \times_B U$ be a set-theoretic pre-equivalence relation. A morphism $\phi : U \to X$ is an orbit space for R if and only if*

(1) *$\phi \circ s = \phi \circ t$, i.e., ϕ is invariant,*
(2) *the induced morphism $(t, s) : R \to U \times_X U$ is surjective, and*
(3) *the morphism $\phi : U \to X$ is surjective.*

This characterization applies for example if j is a pre-equivalence relation, or comes from a groupoid in algebraic spaces over B, or comes from the action of a group algebraic space over B on U.

Proof. Follows immediately from Lemma 56.5.17 part (4). □

In the following lemma it is (probably) not good enough to assume just that the morphisms s, t are locally of finite type. The reason is that it may happen that some map $\phi : U \to X$ is an orbit space, yet is not locally of finite type. In that case $U(k) \to X(k)$ may not be surjective for all algebraically closed fields k over B.

Lemma 56.5.20. *Let $B \to S$ as in Section 56.2. Let $j = (t, s) : R \to U \times_B U$ be a pre-relation. Assume R, U are locally of finite type over B. Let $\phi : U \to X$ be an R-invariant morphism of algebraic spaces over B. Then ϕ is an orbit space for R if and only if the natural map*

$$U(k)/\big(\textit{equivalence relation generated by } j(R(k))\big) \longrightarrow X(k)$$

is bijective for all algebraically closed fields k over B.

Proof. Note that since U, R are locally of finite type over B all of the morphisms s, t, j, ϕ are locally of finite type, see Morphisms of Spaces, Lemma 42.22.6. We will also use without further mention Morphisms of Spaces, Lemma 42.23.1. Assume ϕ is an orbit space. Let k be any algebraically closed field over B. Let $\overline{x} \in X(k)$. Consider $U \times_{\phi, X, \overline{x}} Spec(k)$. This is a nonempty algebraic space which is locally of finite type over k. Hence it has a k-valued point. This shows the displayed map of the lemma is surjective. Suppose that $\overline{u}, \overline{u}' \in U(k)$ map to the same element of $X(k)$. By Definition 56.5.8 this means that $\overline{u}, \overline{u}'$ are in the same R-orbit. By Lemma 56.5.7 this means that they are equivalent under the equivalence relation generated by $j(R(k))$. Thus the displayed morphism is injective.

Conversely, assume the displayed map is bijective for all algebraically closed fields k over B. This condition clearly implies that ϕ is surjective. We have already assumed that ϕ is R-invariant. Finally, the injectivity of all the displayed maps implies that ϕ separates orbits. Hence ϕ is an orbit space. □

56.6. Coarse quotients

We only add this here so that we can later say that coarse quotients correspond to coarse moduli spaces (or moduli schemes).

Definition 56.6.1. Let S be a scheme and B an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation. A morphism $\phi : U \to X$ of algebraic spaces over B is called a *coarse quotient* if

(1) ϕ is a categorical quotient, and
(2) ϕ is an orbit space.

If $S = B$, U, R are all schemes, then we say a morphism of schemes $\phi : U \to X$ is a *coarse quotient in schemes* if

(1) ϕ is a categorical quotient in schemes, and
(2) ϕ is an orbit space.

In many situations the algebraic spaces R and U are locally of finite type over B and the orbit space condition simply means that

$$U(k)/\big(\text{equivalence relation generated by } j(R(k))\big) \cong X(k)$$

for all algebraically closed fields k. See Lemma 56.5.20. If j is also a (set-theoretic) pre-equivalence relation, then the condition is simply equivalent to $U(k)/j(R(k)) \to X(k)$ being bijective for all algebraically closed fields k.

56.7. Topological properties

Let S be a scheme and B an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation. We say a subset $T \subset |U|$ is *R-invariant* if $s^{-1}(T) = t^{-1}(T)$ as subsets of $|R|$. Note that if T is closed, then it may not be the case that the corresponding reduced closed subspace of U is R-invariant (as in Groupoids in Spaces, Definition 52.17.1) because the pullbacks $s^{-1}(T)$, $t^{-1}(T)$ may not be reduced. Here are some conditions that we can consider for an invariant morphism $\phi : U \to X$.

Definition 56.7.1. Let S be a scheme and B an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation. Let $\phi : U \to X$ be an R-invariant morphism of algebraic spaces over B.

(56.7.1.1) The morphism ϕ is submersive.
(56.7.1.2) For any R-invariant closed subset $Z \subset |U|$ the image $\phi(Z)$ is closed in $|X|$.
(56.7.1.3) Condition (56.7.1.2) holds and for any pair of R-invariant closed subsets $Z_1, Z_2 \subset |U|$ we have
$$\phi(Z_1 \cap Z_2) = \phi(Z_1) \cap \phi(Z_2)$$
(56.7.1.4) The morphism $(t, s) : R \to U \times_X U$ is universally submersive.

For each of these properties we can also require them to hold after any flat pullback, or after any pullback, see Definition 56.3.4. In this case we say condition (56.7.1.1), (56.7.1.2), (56.7.1.3), or (56.7.1.4) holds *uniformly* or *universally*.

56.8. Invariant functions

In some cases it is convenient to pin down the structure sheaf of a quotient by requiring any invariant function to be a local section of the structure sheaf of the quotient.

Definition 56.8.1. Let S be a scheme and B an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation. Let $\phi : U \to X$ be an R-invariant morphism. Denote $\phi' = \phi \circ s = \phi \circ t : R \to X$.

(1) We denote $(\phi_* \mathcal{O}_U)^R$ the $\mathcal{O}_X$-sub-algebra of $\phi_* \mathcal{O}_U$ which is the equalizer of the two maps

$$\phi_* \mathcal{O}_U \underset{\phi_* t^\sharp}{\overset{\phi_* s^\sharp}{\rightrightarrows}} \phi'_* \mathcal{O}_R$$

on $X_{étale}$. We sometimes call this the *sheaf of R-invariant functions on X*.

(2) We say *the functions on X are the R-invariant functions on U* if the natural map $\mathcal{O}_X \to (\phi_* \mathcal{O}_U)^R$ is an isomorphism.

Of course we can require this property holds after any (flat or any) pullback, leading to a (uniform or) universal notion. This condition is often thrown in with other conditions in order to obtain a (more) unique quotient. And of course a good deal of motivation for the whole subject comes from the following special case: $U = Spec(A)$ is an affine scheme over a field $S = B = Spec(k)$ and where $R = G \times U$, with G an affine group scheme over k. In this case you have the option of taking for the quotient:

$$X = Spec(A^G)$$

so that at least the condition of the definition above is satisfied. Even though this is a nice thing you can do it is often not the right quotient; for example if $U = \mathrm{GL}_{n,k}$ and G is the group of upper triangular matrices, then the above gives $X = Spec(k)$, whereas a much better quotient (namely the flag variety) exists.

56.9. Good quotients

Especially when taking quotients by group actions the following definition is useful.

Definition 56.9.1. Let S be a scheme and B an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation. A morphism $\phi : U \to X$ of algebraic spaces over B is called a *good quotient* if

(1) ϕ is invariant,
(2) ϕ is affine,
(3) ϕ is surjective,
(4) condition (56.7.1.3) holds universally, and
(5) the functions on X are the R-invariant functions on U.

In [**Ses72**] Seshadri gives almost the same definition, except that instead of (4) he simply requires the condition (56.7.1.3) to hold -- he does not require it to hold universally.

56.10. Geometric quotients

This is Mumford's definition of a geometric quotient (at least the definition from the first edition of GIT; as far as we can tell later editions changed ``universally submersive'' to ``submersive'').

Definition 56.10.1. Let S be a scheme and B an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation. A morphism $\phi : U \to X$ of algebraic spaces over B is called a *geometric quotient* if

1. ϕ is an orbit space,
2. condition (56.7.1.1) holds universally, i.e., ϕ is universally submersive, and
3. the functions on X are the R-invariant functions on U.

56.11. Other chapters

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CHAPTER 57

Algebraic Stacks

57.1. Introduction

This is where we define algebraic stacks and make some very elementary observations. The general philosophy will be to have no separation conditions whatsoever and add those conditions necessary to make lemmas, propositions, theorems true/provable. Thus the notions discussed here differ slightly from those in other places in the literature, e.g., [**LMB00a**].

This chapter is not an introduction to algebraic stacks. For an informal discussion of algebraic stacks, please take a look at Introducing Algebraic Stacks, Section 63.1.

57.2. Conventions

The conventions we use in this chapter are the same as those in the chapter on algebraic spaces. For convenience we repeat them here.

We work in a suitable big fppf site Sch_{fppf} as in Topologies, Definition 30.7.6. So, if not explicitly stated otherwise all schemes will be objects of Sch_{fppf}. We discuss what changes if you change the big fppf site in Section 57.18.

We will always work relative to a base S contained in Sch_{fppf}. And we will then work with the big fppf site $(Sch/S)_{fppf}$, see Topologies, Definition 30.7.8. The absolute case can be recovered by taking $S = Spec(\mathbf{Z})$.

If U, T are schemes over S, then we denote $U(T)$ for the set of T-valued points *over* S. In a formula: $U(T) = Mor_S(T, U)$.

Note that any fpqc covering is a universal effective epimorphism, see Descent, Lemma 31.9.3. Hence the topology on Sch_{fppf} is weaker than the canonical topology and all representable presheaves are sheaves.

57.3. Notation

We use the letters S, T, U, V, X, Y to indicate schemes. We use the letters $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ to indicate categories (fibred, fibred in groupoids, stacks, ...) over $(Sch/S)_{fppf}$. We use small case letters f, g for functors such as $f : \mathcal{X} \to \mathcal{Y}$ over $(Sch/S)_{fppf}$. We use capital F, G, H for algebraic spaces over S, and more generally for presheaves of sets on $(Sch/S)_{fppf}$. (In future chapters we will revert to using also X, Y, etc for algebraic spaces.)

The reason for these choices is that we want to clearly distinguish between the different types of objects in this chapter, to build the foundations.

57.4. Representable categories fibred in groupoids

Let S be a scheme contained in Sch_{fppf}. The basic object of study in this chapter will be a category fibred in groupoids $p : \mathcal{X} \to (Sch/S)_{fppf}$, see Categories, Definition 4.32.1. We will often simply say ``let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$" to indicate this situation. A 1-morphism $\mathcal{X} \to \mathcal{Y}$ of categories in groupoids over $(Sch/S)_{fppf}$ will be a 1-morphism in the 2-category of categories fibred in groupoids over $(Sch/S)_{fppf}$, see Categories, Definition 4.32.6. It is simply a functor $\mathcal{X} \to \mathcal{Y}$ over $(Sch/S)_{fppf}$. We recall this is really a $(2,1)$-category and that all 2-fibre products exist.

Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Recall that $\mathcal{X}$ is said to be *representable* if there exists a scheme $U \in Ob((Sch/S)_{fppf})$ and an equivalence

$$j : \mathcal{X} \longrightarrow (Sch/U)_{fppf}$$

of categories over $(Sch/S)_{fppf}$, see Categories, Definition 4.37.1. We will sometimes say that $\mathcal{X}$ is *representable by a scheme* to distinguish from the case where $\mathcal{X}$ is representable by an algebraic space (see below).

If $\mathcal{X}, \mathcal{Y}$ are fibred in groupoids and representable by U, V, then we have

$$Mor_{Cat/(Sch/S)_{fppf}}(\mathcal{X}, \mathcal{Y}) \Big/ \text{2-isomorphism} = Mor_{Sch/S}(U, V) \tag{57.4.0.1}$$

see Categories, Lemma 4.37.3. More precisely, any 1-morphism $\mathcal{X} \to \mathcal{Y}$ gives rise to a morphism $U \to V$. Conversely, given a morphism of schemes $U \to V$ over S there exists a 1-morphism $\phi : \mathcal{X} \to \mathcal{Y}$ which gives rise to $U \to V$ and which is unique up to unique 2-isomorphism.

57.5. The 2-Yoneda lemma

Let $U \in Ob((Sch/S)_{fppf})$, and let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. We will frequently use the 2-Yoneda lemma, see Categories, Lemma 4.38.1. Technically it says that there is an equivalence of categories

$$Mor_{Cat/(Sch/S)_{fppf}}((Sch/U)_{fppf}, \mathcal{X}) \longrightarrow \mathcal{X}_U, \quad f \longmapsto f(U/U).$$

It says that 1-morphisms $(Sch/U)_{fppf} \to \mathcal{X}$ correspond to objects x of the fibre category $\mathcal{X}_U$. Namely, given a 1-morphism $f : (Sch/U)_{fppf} \to \mathcal{X}$ we obtain the object $x = f(U/U) \in Ob(\mathcal{X}_U)$. Conversely, given a choice of pullbacks for $\mathcal{X}$ as in Categories, Definition 4.30.5, and an object x of $\mathcal{X}_U$, we obtain a functor $(Sch/U)_{fppf} \to \mathcal{X}$ defined by the rule

$$(\varphi : V \to U) \longmapsto \varphi^* x$$

on objects. By abuse of notation we use $x : (Sch/U)_{fppf} \to \mathcal{X}$ to indicate this functor. It indeed has the property that $x(U/U) = x$ and moreover, given any other functor f with $f(U/U) = x$ there exists a unique 2-isomorphism $x \to f$. In other words the functor x is well determined by the object x up to unique 2-isomorphism.

We will use this without further mention in the following.

57.6. Representable morphisms of categories fibred in groupoids

Let $\mathcal{X}$, $\mathcal{Y}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $f : \mathcal{X} \to \mathcal{Y}$ be a *representable 1-morphism*, see Categories, Definition 4.38.5. This means that for every $U \in Ob((Sch/S)_{fppf})$ and any $y \in Ob(\mathcal{Y}_U)$ the 2-fibre product $(Sch/U)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$ is representable. Choose a representing object V_y and an equivalence

$$(Sch/V_y)_{fppf} \longrightarrow (Sch/U)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}.$$

The projection $(Sch/V_y)_{fppf} \to (Sch/U)_{fppf} \times_{\mathcal{Y}} \mathcal{Y} \to (Sch/U)_{fppf}$ comes from a morphism of schemes $f_y : V_y \to U$, see Section 57.4. We represent this by the diagram

(57.6.0.2)
$$\begin{tikzcd} V_y \arrow[r, squiggly] \arrow[d, "f_y"'] & (Sch/V_y)_{fppf} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "f"] \\ U \arrow[r, squiggly] & (Sch/U)_{fppf} \arrow[r, "y"] & \mathcal{Y} \end{tikzcd}$$

where the squiggly arrows represent the 2-Yoneda embedding. Here are some lemmas about this notion that work in great generality (namely, they work for categories fibred in groupoids over any base category which has fibre products).

Lemma 57.6.1. *Let S, X, Y be objects of Sch_{fppf}. Let $f : X \to Y$ be a morphism of schemes. Then the 1-morphism induced by f*

$$(Sch/X)_{fppf} \longrightarrow (Sch/Y)_{fppf}$$

is a representable 1-morphism.

Proof. This is formal and relies only on the fact that the category $(Sch/S)_{fppf}$ has fibre products. □

Lemma 57.6.2. *Let S be an object of Sch_{fppf}. Consider a 2-commutative diagram*

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d, "f'"'] & \mathcal{X} \arrow[d, "f"] \\ \mathcal{Y}' \arrow[r] & \mathcal{Y} \end{tikzcd}$$

of 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume the horizontal arrows are equivalences. Then f is representable if and only if f' is representable.

Proof. Omitted. □

Lemma 57.6.3. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$ Let $f : \mathcal{X} \to \mathcal{Y}$, $g : \mathcal{Y} \to \mathcal{Z}$ be representable 1-morphisms. Then*

$$g \circ f : \mathcal{X} \longrightarrow \mathcal{Z}$$

is a representable 1-morphism.

Proof. This is entirely formal and works in any category. □

Lemma 57.6.4. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$ Let $f : \mathcal{X} \to \mathcal{Y}$ be a representable 1-morphism. Let $g : \mathcal{Z} \to \mathcal{Y}$ be any 1-morphism. Consider the fibre product diagram*

$$\begin{tikzcd} \mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X} \arrow[r, "g'"'] \arrow[d, "f'"'] & \mathcal{X} \arrow[d, "f"] \\ \mathcal{Z} \arrow[r, "g"] & \mathcal{Y} \end{tikzcd}$$

Then the base change f' is a representable 1-morphism.

Proof. This is entirely formal and works in any category. □

Lemma 57.6.5. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}_i, \mathcal{Y}_i$ be categories fibred in groupoids over $(Sch/S)_{fppf}$, $i = 1, 2$. Let $f_i : \mathcal{X}_i \to \mathcal{Y}_i$, $i = 1, 2$ be representable 1-morphisms. Then*

$$f_1 \times f_2 : \mathcal{X}_1 \times \mathcal{X}_2 \longrightarrow \mathcal{Y}_1 \times \mathcal{Y}_2$$

is a representable 1-morphism.

Proof. Write $f_1 \times f_2$ as the composition $\mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{Y}_1 \times \mathcal{X}_2 \to \mathcal{Y}_1 \times \mathcal{Y}_2$. The first arrow is the base change of f_1 by the map $\mathcal{Y}_1 \times \mathcal{X}_2 \to \mathcal{Y}_1$, and the second arrow is the base change of f_2 by the map $\mathcal{Y}_1 \times \mathcal{Y}_2 \to \mathcal{Y}_2$. Hence this lemma is a formal consequence of Lemmas 57.6.3 and 57.6.4. □

57.7. Split categories fibred in groupoids

Let S be a scheme contained in Sch_{fppf}. Recall that given a ``presheaf of groupoids"

$$F : (Sch/S)_{fppf}^{opp} \longrightarrow Groupoids$$

we get a category fibred in groupoids $\mathcal{S}_F$ over $(Sch/S)_{fppf}$, see Categories, Example 4.34.1. Any category fibred in groupoids isomorphic (!) to one of these is called a *split category fibred in groupoids*. Any category fibred in groupoids is equivalent to a split one.

If F is a presheaf of sets then $\mathcal{S}_F$ is fibred in sets, see Categories, Definition 4.35.2, and Categories, Example 4.35.5. The rule $F \mapsto \mathcal{S}_F$ is in some sense fully faithful on presheaves, see Categories, Lemma 4.35.6. If F, G are presheaves, then

$$\mathcal{S}_{F \times G} = \mathcal{S}_F \times_{(Sch/S)_{fppf}} \mathcal{S}_G$$

and if $F \to H$ and $G \to H$ are maps of presheaves of sets, then

$$\mathcal{S}_{F \times_H G} = \mathcal{S}_F \times_{\mathcal{S}_H} \mathcal{S}_G$$

where the right hand sides are 2-fibre products. This is immediate from the definitions as the fibre categories of $\mathcal{S}_F, \mathcal{S}_G, \mathcal{S}_H$ have only identity morphisms.

An even more special case is where $F = h_X$ is a representable presheaf. In this case we have $\mathcal{S}_{h_X} = (Sch/X)_{fppf}$, see Categories, Example 4.35.7.

We will use the notation $\mathcal{S}_F$ without further mention in the following.

57.8. Categories fibred in groupoids representable by algebraic spaces

A slightly weaker notion than being representable is the notion of being representable by algebraic spaces which we discuss in this section. This discussion might have been avoided had we worked with some category $Spaces_{fppf}$ of algebraic spaces instead of the category Sch_{fppf}. However, it seems to us natural to consider the category of schemes as the natural collection of ``test objects" over which the fibre categories of an algebraic stack are defined.

In analogy with Categories, Definitions 4.37.1 we make the following definition.

Definition 57.8.1. Let S be a scheme contained in Sch_{fppf}. A category fibred in groupoids $p : \mathcal{X} \to (Sch/S)_{fppf}$ is called *representable by an algebraic space over S* if there exists an algebraic space F over S and an equivalence $j : \mathcal{X} \to \mathcal{S}_F$ of categories over $(Sch/S)_{fppf}$.

We continue our abuse of notation in surpressing the equivalence j whenever we encounter such a situation. It follows formally from the above that if $\mathcal{X}$ is representable (by a scheme), then it is representable by an algebraic space. Here is the analogue of Categories, Lemma 4.37.2.

Lemma 57.8.2. *Let S be a scheme contained in Sch_{fppf}. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Then $\mathcal{X}$ is representable by an algebraic space over S if and only if the following conditions are satisfied:*

(1) *$\mathcal{X}$ is fibred in setoids*[1]*, and*
(2) *the presheaf $U \mapsto Ob(\mathcal{X}_U)/\cong$ is an algebraic space.*

Proof. Omitted, but see Categories, Lemma 4.37.2. □

If $\mathcal{X}, \mathcal{Y}$ are fibred in groupoids and representable by algebraic spaces F, G over S, then we have

$$Mor_{Cat/(Sch/S)_{fppf}}(\mathcal{X}, \mathcal{Y})\Big/\text{2-isomorphism} = Mor_{Sch/S}(F, G) \tag{57.8.2.1}$$

see Categories, Lemma 4.36.6. More precisely, any 1-morphism $\mathcal{X} \to \mathcal{Y}$ gives rise to a morphism $F \to G$. Conversely, give a morphism of sheaves $F \to G$ over S there exists a 1-morphism $\phi : \mathcal{X} \to \mathcal{Y}$ which gives rise to $F \to G$ and which is unique up to unique 2-isomorphism.

57.9. Morphisms representable by algebraic spaces

In analogy with Categories, Definition 4.38.5 we make the following definition.

Definition 57.9.1. Let S be a scheme contained in Sch_{fppf}. A 1-morphism $f : \mathcal{X} \to \mathcal{Y}$ of categories fibred in groupoids over $(Sch/S)_{fppf}$ is called *representable by algebraic spaces* if for any $U \in Ob((Sch/S)_{fppf})$ and any $y : (Sch/U)_{fppf} \to \mathcal{Y}$ the category fibred in groupoids

$$(Sch/U)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$$

over $(Sch/U)_{fppf}$ is representable by an algebraic space over U.

Choose an algebraic space F_y over U which represents $(Sch/U)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$. We may think of F_y as an algebraic stack over S which comes equipped with a canonical morphism $f_y : F_y \to U$ over S, see Spaces, Section 40.16. Here is the diagram

$$\begin{tikzcd} F_y \arrow[d, "f_y"'] & (Sch/U)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X} \arrow[l, squiggly] \arrow[r, "\mathrm{pr}_1"'] \arrow[d, "\mathrm{pr}_0"'] & \mathcal{X} \arrow[d, "f"] \\ U & (Sch/U)_{fppf} \arrow[l, squiggly] \arrow[r, "y"] & \mathcal{Y} \end{tikzcd} \tag{57.9.1.1}$$

where the squiggly arrows represent the construction which associates to a stack fibred in setoids its associated sheaf of isomorphism classes of objects. The right square is 2-commutative, and is a 2-fibre product square.

Here is the analogue of Categories, Lemma 4.38.7.

Lemma 57.9.2. *Let S be a scheme contained in Sch_{fppf}. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. The following are necessary and sufficient conditions for f to be representable by algebraic spaces:*

(1) *for each scheme U/S the functor $f_U : \mathcal{X}_U \longrightarrow \mathcal{Y}_U$ between fibre categories is faithful, and*

[1] This means that it is fibred in groupoids and objects in the fibre categories have no nontrivial automorphisms, see Categories, Definition 4.35.2.

(2) *for each U and each $y \in Ob(\mathcal{Y}_U)$ the presheaf*
$$(h : V \to U) \longmapsto \{(x, \phi) \mid x \in Ob(\mathcal{X}_V), \phi : h^*y \to f(x)\}/\cong$$
is an algebraic space over U.

Here we have made a choice of pullbacks for $\mathcal{Y}$.

Proof. This follows from the description of fibre categories of the 2-fibre products $(Sch/U)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$ in Categories, Lemma 4.38.3 combined with Lemma 57.8.2. □

Here are some lemmas about this notion that work in great generality.

Lemma 57.9.3. *Let S be an object of Sch_{fppf}. Consider a 2-commutative diagram*
$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d, "f'"'] & \mathcal{X} \arrow[d, "f"] \\ \mathcal{Y}' \arrow[r] & \mathcal{Y} \end{tikzcd}$$
of 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume the horizontal arrows are equivalences. Then f is representable by algebraic spaces if and only if f' is representable by algebraic spaces.

Proof. Omitted. □

Lemma 57.9.4. *Let S be an object of Sch_{fppf}. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over S. If $\mathcal{X}$ and $\mathcal{Y}$ are representable by algebraic spaces over S, then the 1-morphism f is representable by algebraic spaces.*

Proof. Omitted. This relies only on the fact that the category of algebraic spaces over S has fibre products, see Spaces, Lemma 40.7.3. □

Lemma 57.9.5. *Let S be an object of Sch_{fppf}. Let $a : F \to G$ be a map of presheaves of sets on $(Sch/S)_{fppf}$. Denote $a' : \mathcal{S}_F \to \mathcal{S}_G$ the associated map of categories fibred in sets. Then a is representable by algebraic spaces (see Bootstrap, Definition 54.3.1) if and only if a' is representable by algebraic spaces.*

Proof. Omitted. □

Lemma 57.9.6. *Let S be an object of Sch_{fppf}. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in setoids over $(Sch/S)_{fppf}$. Let F, resp. G be the presheaf which to T associates the set of isomorphism classes of objects of $\mathcal{X}_T$, resp. $\mathcal{Y}_T$. Let $a : F \to G$ be the map of presheaves corresponding to f. Then a is representable by algebraic spaces (see Bootstrap, Definition 54.3.1) if and only if f is representable by algebraic spaces.*

Proof. Omitted. Hint: Combine Lemmas 57.9.3 and 57.9.5. □

Lemma 57.9.7. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism representable by algebraic spaces. Let $g : \mathcal{Z} \to \mathcal{Y}$ be any 1-morphism. Consider the fibre product diagram*
$$\begin{tikzcd} \mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X} \arrow[r, "g'"'] \arrow[d, "f'"'] & \mathcal{X} \arrow[d, "f"] \\ \mathcal{Z} \arrow[r, "g"] & \mathcal{Y} \end{tikzcd}$$
Then the base change f' is a 1-morphism representable by algebraic spaces.

Proof. This is formal. □

Lemma 57.9.8. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$ Let $f : \mathcal{X} \to \mathcal{Y}$, $g : \mathcal{Z} \to \mathcal{Y}$ be 1-morphisms. Assume*

(1) *f is representable by algebraic spaces, and*
(2) *$\mathcal{Z}$ is representable by an algebraic space over S.*

Then the 2-fibre product $\mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X}$ is representable by an algebraic space.

Proof. This is a reformulation of Bootstrap, Lemma 54.3.6. First note that $\mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X}$ is fibred in setoids over $(Sch/S)_{fppf}$. Hence it is equivalent to $\mathcal{S}_F$ for some presheaf F on $(Sch/S)_{fppf}$, see Categories, Lemma 4.36.5. Moreover, let G be an algebraic space which represents $\mathcal{Z}$. The 1-morphism $\mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X} \to \mathcal{Z}$ is representable by algebraic spaces by Lemma 57.9.7. And $\mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X} \to \mathcal{Z}$ corresponds to a morphism $F \to G$ by Categories, Lemma 4.36.6. Then $F \to G$ is representable by algebraic spaces by Lemma 57.9.6. Hence Bootstrap, Lemma 54.3.6 implies that F is an algebraic space as desired. □

Let S, $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$, f, g be as in Lemma 57.9.8. Let F and G be algebraic spaces over S such that F represents $\mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X}$ and G represents $\mathcal{Z}$. The 1-morphism $f' : \mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X} \to \mathcal{Z}$ corresponds to a morphism $f' : F \to G$ of algebraic spaces by (57.8.2.1). Thus we have the following diagram

(57.9.8.1)
$$\begin{tikzcd} F \arrow[d, "f'"'] & \mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X} \arrow[l, squiggly] \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "f"] \\ G & \mathcal{Z} \arrow[l, squiggly] \arrow[r, "g"] & \mathcal{Y} \end{tikzcd}$$

where the squiggly arrows represent the construction which associates to a stack fibred in setoids its associated sheaf of isomorphism classes of objects. The middle square is 2-commutative with equivalences as horizontal arrows.

Lemma 57.9.9. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$. If $f : \mathcal{X} \to \mathcal{Y}$, $g : \mathcal{Y} \to \mathcal{Z}$ are 1-morphisms representable by algebraic spaces, then*

$$g \circ f : \mathcal{X} \longrightarrow \mathcal{Z}$$

is a 1-morphism representable by algebraic spaces.

Proof. This follows from Lemma 57.9.8. Details omitted. □

Lemma 57.9.10. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}_i, \mathcal{Y}_i$ be categories fibred in groupoids over $(Sch/S)_{fppf}$, $i = 1, 2$. Let $f_i : \mathcal{X}_i \to \mathcal{Y}_i$, $i = 1, 2$ be 1-morphisms representable by algebraic spaces. Then*

$$f_1 \times f_2 : \mathcal{X}_1 \times \mathcal{X}_2 \longrightarrow \mathcal{Y}_1 \times \mathcal{Y}_2$$

is a 1-morphism representable by algebraic spaces.

Proof. Write $f_1 \times f_2$ as the composition $\mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{Y}_1 \times \mathcal{X}_2 \to \mathcal{Y}_1 \times \mathcal{Y}_2$. The first arrow is the base change of f_1 by the map $\mathcal{Y}_1 \times \mathcal{X}_2 \to \mathcal{Y}_1$, and the second arrow is the base change of f_2 by the map $\mathcal{Y}_1 \times \mathcal{Y}_2 \to \mathcal{Y}_2$. Hence this lemma is a formal consequence of Lemmas 57.9.9 and 57.9.7. □

57.10. Properties of morphisms representable by algebraic spaces

Here is the definition that makes this work.

Definition 57.10.1. Let S be a scheme contained in Sch_{fppf}. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume f is representable by algebraic spaces. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which

(1) is preserved under any base change, and
(2) is fppf local on the base, see Descent on Spaces, Definition 45.9.1.

In this case we say that f has *property* $\mathcal{P}$ if for every $U \in Ob((Sch/S)_{fppf})$ and any $y \in \mathcal{Y}_U$ the resulting morphism of algebraic spaces $f_y : F_y \to U$, see diagram (57.9.1.1), has property $\mathcal{P}$.

It is important to note that we will only use this definition for properties of morphisms that are stable under base change, and local in the fppf topology on the target. This is not because the definition doesn't make sense otherwise; rather it is because we may want to give a different definition which is better suited to the property we have in mind.

Lemma 57.10.2. *Let S be an object of Sch_{fppf}. Let $\mathcal{P}$ be as in Definition 57.10.1. Consider a 2-commutative diagram*

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d, "f'"'] & \mathcal{X} \arrow[d, "f"] \\ \mathcal{Y}' \arrow[r] & \mathcal{Y} \end{tikzcd}$$

of 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume the horizontal arrows are equivalences and f (or equivalently f') is representably by algebraic spaces. Then f has $\mathcal{P}$ if and only if f' has $\mathcal{P}$.

Proof. Note that this makes sense by Lemma 57.9.3. Proof omitted. □

Here is a sanity check.

Lemma 57.10.3. *Let S be a scheme contained in Sch_{fppf}. Let $a : F \to G$ be a map of presheaves on $(Sch/S)_{fppf}$. Let $\mathcal{P}$ be as in Definition 57.10.1. Assume a is representable by algebraic spaces. Then $a : F \to G$ has property $\mathcal{P}$ (see Bootstrap, Definition 54.4.1) if and only if the corresponding morphism $\mathcal{S}_F \to \mathcal{S}_G$ of categories fibred in groupoids has propery $\mathcal{P}$.*

Proof. Note that the lemma makes sense by Lemma 57.9.5. Proof omitted. □

Lemma 57.10.4. *Let S be an object of Sch_{fppf}. Let $\mathcal{P}$ be as in Definition 57.10.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in setoids over $(Sch/S)_{fppf}$. Let F, resp. G be the presheaf which to T associates the set of isomorphism classes of objects of $\mathcal{X}_T$, resp. $\mathcal{Y}_T$. Let $a : F \to G$ be the map of presheaves corresponding to f. Then a has $\mathcal{P}$ if and only if f has $\mathcal{P}$.*

Proof. The lemma makes sense by Lemma 57.9.6. The lemma follows on combining Lemmas 57.10.2 and 57.10.3. □

Lemma 57.10.5. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\mathcal{P}$ be a property as in Definition 57.10.1 which is stable under composition. Let $f : \mathcal{X} \to \mathcal{Y}$, $g : \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms which are representable by algebraic spaces. If f and g have property $\mathcal{P}$ so does $g \circ f : \mathcal{X} \to \mathcal{Z}$.*

Proof. Note that the lemma makes sense by Lemma 57.9.9. Proof omitted. $\square$

Lemma 57.10.6. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\mathcal{P}$ be a property as in Definition 57.10.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism representable by algebraic spaces. Let $g : \mathcal{Z} \to \mathcal{Y}$ be any 1-morphism. Consider the 2-fibre product diagram*

$$\begin{tikzcd} \mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X} \arrow[r, "g'"'] \arrow[d, "f'"'] & \mathcal{X} \arrow[d, "f"] \\ \mathcal{Z} \arrow[r, "g"] & \mathcal{Y} \end{tikzcd}$$

If f has $\mathcal{P}$, then the base change f' has $\mathcal{P}$.

Proof. The lemma makes sense by Lemma 57.9.7. Proof omitted. $\square$

Lemma 57.10.7. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\mathcal{P}$ be a property as in Definition 57.10.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism representable by algebraic spaces. Let $g : \mathcal{Z} \to \mathcal{Y}$ be any 1-morphism. Consider the fibre product diagram*

$$\begin{tikzcd} \mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X} \arrow[r, "g'"'] \arrow[d, "f'"'] & \mathcal{X} \arrow[d, "f"] \\ \mathcal{Z} \arrow[r, "g"] & \mathcal{Y} \end{tikzcd}$$

Assume that for every scheme U and object x of $\mathcal{Y}_U$, there exists an fppf covering $\{U_i \to U\}$ such that $x|_{U_i}$ is in the essential image of the functor $g : \mathcal{Z}_{U_i} \to \mathcal{Y}_{U_i}$. In this case, if f' has $\mathcal{P}$, then f has $\mathcal{P}$.

Proof. Proof omitted. Hint: Compare with the proof of Spaces, Lemma 40.5.6. $\square$

Lemma 57.10.8. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{P}$ be a property as in Definition 57.10.1 which is stable under composition. Let $\mathcal{X}_i, \mathcal{Y}_i$ be categories fibred in groupoids over $(Sch/S)_{fppf}$, $i = 1, 2$. Let $f_i : \mathcal{X}_i \to \mathcal{Y}_i$, $i = 1, 2$ be 1-morphisms representable by algebraic spaces. If f_1 and f_2 have property $\mathcal{P}$ so does $f_1 \times f_2 : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{Y}_1 \times \mathcal{Y}_2$.*

Proof. The lemma makes sense by Lemma 57.9.10. Proof omitted. $\square$

Lemma 57.10.9. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$, $\mathcal{Y}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism representable by algebraic spaces. Let $\mathcal{P}$, $\mathcal{P}'$ be properties as in Definition 57.10.1. Suppose that for any morphism of algebraic spaces $a : F \to G$ we have $\mathcal{P}(a) \Rightarrow \mathcal{P}'(a)$. If f has property $\mathcal{P}$ then f has property $\mathcal{P}'$.*

Proof. Formal. $\square$

Lemma 57.10.10. *Let S be a scheme contained in Sch_{fppf}. Let $j : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume j is representable by algebraic spaces and a monomorphism (see Definition 57.10.1 and Descent on Spaces, Lemma 45.10.28). Then j is fully faithful on fibre categories.*

Proof. We have seen in Lemma 57.9.2 that j is faithful on fibre categories. Consider a scheme U, two objects u, v of $\mathcal{X}_U$, and an isomorphism $t : j(u) \to j(v)$ in $\mathcal{Y}_U$. We have to construct an isomorphism in $\mathcal{X}_U$ between u and v. By the 2-Yoneda lemma (see Section

57.5) we think of u, v as 1-morphisms $u, v : (Sch/U)_{fppf} \to \mathcal{X}$ and we consider the 2-fibre product

$$(Sch/U)_{fppf} \times_{j\circ v, \mathcal{Y}} \mathcal{X}.$$

By assumption this is representable by an algebraic space $F_{j\circ v}$, over U and the morphism $F_{j\circ v} \to U$ is a monomorphism. But since $(1_U, v, 1_{j(v)})$ gives a 1-morphism of $(Sch/U)_{fppf}$ into the displayed 2-fibre product, we see that $F_{j\circ v} = U$ (here we use that if $V \to U$ is a monomorphism of algebraic spaces which has a section, then $V = U$). Therefore the 1-morphism projecting to the first coordinate

$$(Sch/U)_{fppf} \times_{j\circ v, \mathcal{Y}} \mathcal{X} \to (Sch/U)_{fppf}$$

is an equivalence of fibre categories. Since $(1_U, u, t)$ and $(1_U, v, 1_{j(v)})$ give two objects in $((Sch/U)_{fppf} \times_{j\circ v, \mathcal{Y}} \mathcal{X})_U$ which have the same first coordinate, there must be a 2-morphism between them in the 2-fibre product. This is by definition a morphism $\tilde{t} : u \to v$ such that $j(\tilde{t}) = t$. □

Here is a characterization of those categories fibred in groupoids for which the diagonal is representable by algebraic spaces.

Lemma 57.10.11. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. The following are equivalent:*

(1) *the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces,*
(2) *for every scheme U over S, and any $x, y \in Ob(\mathcal{X}_U)$ the sheaf $Isom(x, y)$ is representable by an algebraic space over U,*
(3) *for every scheme U over S, and any $x \in Ob(\mathcal{X}_U)$ the associated 1-morphism $x : (Sch/U)_{fppf} \to \mathcal{X}$ is representable by algebraic spaces,*
(4) *for every pair of schemes T_1, T_2 over S, and any $x_i \in Ob(\mathcal{X}_{T_i})$, $i = 1, 2$ the 2-fibre product $(Sch/T_1)_{fppf} \times_{x_1, \mathcal{X}, x_2} (Sch/T_2)_{fppf}$ is representable by an algebraic space,*
(5) *for every representable category fibred in groupoids $\mathcal{U}$ over $(Sch/S)_{fppf}$ every 1-morphism $\mathcal{U} \to \mathcal{X}$ is representable by algebraic spaces,*
(6) *for every pair $\mathcal{T}_1, \mathcal{T}_2$ of representable categories fibred in groupoids over $(Sch/S)_{fppf}$ and any 1-morphisms $x_i : \mathcal{T}_i \to \mathcal{X}$, $i = 1, 2$ the 2-fibre product $\mathcal{T}_1 \times_{x_1, \mathcal{X}, x_2} \mathcal{T}_2$ is representable by an algebraic space,*
(7) *for every category fibred in groupoids $\mathcal{U}$ over $(Sch/S)_{fppf}$ which is representable by an algebraic space every 1-morphism $\mathcal{U} \to \mathcal{X}$ is representable by algebraic spaces,*
(8) *for every pair $\mathcal{T}_1, \mathcal{T}_2$ of categories fibred in groupoids over $(Sch/S)_{fppf}$ which are representable by algebraic spaces, and any 1-morphisms $x_i : \mathcal{T}_i \to \mathcal{X}$ the 2-fibre product $\mathcal{T}_1 \times_{x_1, \mathcal{X}, x_2} \mathcal{T}_2$ is representable by an algebraic space.*

Proof. The equivalence of (1) and (2) follows from Stacks, Lemma 50.2.5 and the definitions. Let us prove the equivalence of (1) and (3). Write $\mathcal{C} = (Sch/S)_{fppf}$ for the base category. We will use some of the observations of the proof of the similar Categories, Lemma 4.38.8. We will use the symbol $\cong$ to mean ``equivalence of categories fibred in groupoids over $\mathcal{C} = (Sch/S)_{fppf}$". Assume (1). Suppose given U and x as in (3). For any scheme V and $y \in Ob(\mathcal{X}_V)$ we see (compare reference above) that

$$\mathcal{C}/U \times_{x, \mathcal{X}, y} \mathcal{C}/V \cong (\mathcal{C}/U \times_S V) \times_{(x,y), \mathcal{X} \times \mathcal{X}, \Delta} \mathcal{X}$$

which is representable by an algebraic space by assumption. Conversely, assume (3). Consider any scheme U over S and a pair (x, x') of objects of $\mathcal{X}$ over U. We have to show that

$\mathcal{X} \times_{\Delta, \mathcal{X} \times \mathcal{X}, (x, x')} U$ is representable by an algebraic space. This is clear because (compare reference above)

$$\mathcal{X} \times_{\Delta, \mathcal{X} \times \mathcal{X}, (x, x')} \mathcal{C}/U \cong (\mathcal{C}/U \times_{x, \mathcal{X}, x'} \mathcal{C}/U) \times_{\mathcal{C}/U \times_S U, \Delta} \mathcal{C}/U$$

and the right hand side is representable by an algebraic space by assumption and the fact that the category of algebraic spaces over S has fibre products and contains U and S.

The equivalences (3) $\Leftrightarrow$ (4), (5) $\Leftrightarrow$ (6), and (7) $\Leftrightarrow$ (8) are formal. The equivalences (3) $\Leftrightarrow$ (5) and (4) $\Leftrightarrow$ (6) follow from Lemma 57.9.3. Assume (3), and let $\mathcal{U} \to \mathcal{X}$ be as in (7). To prove (7) we have to show that for every scheme V and 1-morphism $y : (Sch/V)_{fppf} \to \mathcal{X}$ the 2-fibre product $(Sch/V)_{fppf} \times_{y, \mathcal{X}} \mathcal{U}$ is representable by an algebraic space. Property (3) tells us that y is representable by algebraic spaces hence Lemma 57.9.8 implies what we want. Finally, (7) directly implies (3). $\square$

In the situation of the lemma, for any 1-morphism $x : (Sch/U)_{fppf} \to \mathcal{X}$ as in the lemma, it makes sense to say that x has property $\mathcal{P}$, for any property as in Definition 57.10.1. In particular this holds for $\mathcal{P}$ = ``surjective'', $\mathcal{P}$ = ``smooth'', and $\mathcal{P}$ = ``étale'', see Descent on Spaces, Lemmas 45.10.5, 45.10.24, and 45.10.26. We will use these three cases in the definitions of algebraic stacks below.

57.11. Stacks in groupoids

Let S be a scheme contained in Sch_{fppf}. Recall that a category $p : \mathcal{X} \to (Sch/S)_{fppf}$ over $(Sch/S)_{fppf}$ is said to be a *stack in groupoids* (see Stacks, Definition 50.5.1) if and only if

1. $p : \mathcal{X} \to \mathcal{C}$ is fibred in groupoids over $(Sch/S)_{fppf}$,
2. for all $U \in Ob((Sch/S)_{fppf})$, for all $x, y \in Ob(\mathcal{X}_U)$ the presheaf $Isom(x, y)$ is a sheaf on the site $(Sch/U)_{fppf}$, and
3. for all coverings $\mathcal{U} = \{U_i \to U\}$ in $(Sch/S)_{fppf}$, all descent data (x_i, ϕ_{ij}) for $\mathcal{U}$ are effective.

For examples see Examples of Stacks, Section 55.8 ff.

57.12. Algebraic stacks

Here is the definition of an algebraic stack. We remark that condition (2) implies we can make sense out of the condition in part (3) that $(Sch/U)_{fppf} \to \mathcal{X}$ is smooth and surjective, see discussion following Lemma 57.10.11.

Definition 57.12.1. Let S be a base scheme contained in Sch_{fppf}. An *algebraic stack over S* is a category

$$p : \mathcal{X} \to (Sch/S)_{fppf}$$

over $(Sch/S)_{fppf}$ with the following properties:

1. The category $\mathcal{X}$ is a stack in groupoids over $(Sch/S)_{fppf}$.
2. The diagonal $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces.
3. There exists a scheme $U \in Ob((Sch/S)_{fppf})$ and a 1-morphism $(Sch/U)_{fppf} \to \mathcal{X}$ which is surjective and smooth[2].

[2]In future chapters we will denote this simply $U \to \mathcal{X}$ as is customary in the literature. Another good alterative would be to formulate this condition as the existence of a representable category fibred in groupoids $\mathcal{U}$ and a surjective smooth 1-morphism $\mathcal{U} \to \mathcal{X}$.

There are some differences with other definitions found in the literature.

The first is that we require $\mathcal{X}$ to be a stack in groupoids in the fppf topology, whereas in many references the étale topology is used. It somehow seems to us that the fppf topology is the natural topology to work with. In the end the resulting 2-category of algebraic stacks ends up being the same. This is explained in Criteria for Representability, Section 59.19.

The second is that we only require the diagonal map of $\mathcal{X}$ to be representable by algebraic spaces, whereas in most references some other conditions are imposed. Our point of view is to try to prove a certain number of the results that follow only assuming that the diagonal of $\mathcal{X}$ be representable by algebraic spaces, and simply add an additional hypothesis wherever this is necessary. It has the added benefit that any algebraic space (as defined in Spaces, Definition 40.6.1) gives rise to an algebraic stack.

The third is that in some papers it is required that there exists a scheme U and a surjective and étale morphism $U \to \mathcal{X}$. In the groundbreaking paper [**DM69a**] where algebraic stacks were first introduced Deligne and Mumford used this definition and showed that the moduli stack of stable genus $g > 1$ curves is an algebraic stack which has an étale covering by a scheme. Micheal Artin, see [**Art74b**], realized that many natural results on algebraic stacks generalize to the case where one only assume a smooth covering by a scheme. Hence our choice above. To distinguish the two cases one sees the terms ``Deligne-Mumford stack'' and ``Artin stack'' used in the literature. We will reserve the term ``Artin stack'' for later use (insert future reference here), and continue to use ``algebraic stack'', but we will use ``Deligne-Mumford stack'' to indicate those algebraic stacks which have an étale covering by a scheme.

Definition 57.12.2. Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$ be an algebraic stack over S. We say $\mathcal{X}$ is a *Deligne-Mumford stack* if there exists a scheme U and a surjective étale morphism $(Sch/U)_{fppf} \to \mathcal{X}$.

We will compare our notion of a Deligne-Mumford stack with the notion as defined in the paper by Deligne and Mumford later (see insert future reference here).

The category of algebraic stacks over S forms a 2-category. Here is the precise definition.

Definition 57.12.3. Let S be a scheme contained in Sch_{fppf}. The *2-category of algebraic stacks over S* is the sub 2-category of the 2-category of categories fibred in groupoids over $(Sch/S)_{fppf}$ (see Categories, Definition 4.32.6) defined as follows:

(1) Its objects are those categories fibred in groupoids over $(Sch/S)_{fppf}$ which are algebraic stacks over S.
(2) Its 1-morphisms $f : \mathcal{X} \to \mathcal{Y}$ are any functors of categories over $(Sch/S)_{fppf}$, as in Categories, Definition 4.29.1.
(3) Its 2-morphisms are transformations between functors over $(Sch/S)_{fppf}$, as in Categories, Definition 4.29.1.

In other words this 2-category is the full sub 2-category of $Cat/(Sch/S)_{fppf}$ whose objects are algebraic stacks. Note that every 2-morphism is automatically an isomorphism. Hence this is actually a (2, 1)-category and not just a 2-category.

We will see later (insert future reference here) that this 2-category has 2-fibre products.

Similar to the remark above the 2-category of algebraic stacks over S is a full sub 2-category of the 2-category of categories fibred in groupoids over $(Sch/S)_{fppf}$. It turns out that it is closed under equivalences. Here is the precise statement.

Lemma 57.12.4. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$, $\mathcal{Y}$ be categories over $(Sch/S)_{fppf}$. Assume $\mathcal{X}$, $\mathcal{Y}$ are equivalent as categories over $(Sch/S)_{fppf}$. Then $\mathcal{X}$ is an algebraic stack if and only if $\mathcal{Y}$ is an algebraic stack. Similarly, $\mathcal{X}$ is a Deligne-Mumford stack if and only if $\mathcal{Y}$ is a Deligne-Mumford stack.*

Proof. Assume $\mathcal{X}$ is an algebraic stack (resp. a Deligne-Mumford stack). By Stacks, Lemma 50.5.4 this implies that $\mathcal{Y}$ is a stack in groupoids over Sch_{fppf}. Choose an equivalence $f : \mathcal{X} \to \mathcal{Y}$ over Sch_{fppf}. This gives a 2-commutative diagram

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "f"'] \arrow[d, "\Delta_{\mathcal{X}}"'] & \mathcal{Y} \arrow[d, "\Delta_{\mathcal{Y}}"] \\
\mathcal{X} \times \mathcal{X} \arrow[r, "f \times f"] & \mathcal{Y} \times \mathcal{Y}
\end{tikzcd}$$

whose horizontal arrows are equivalences. This implies that $\Delta_{\mathcal{Y}}$ is representable by algebraic spaces according to Lemma 57.9.3. Finally, let U be a scheme over S, and let $x : (Sch/U)_{fppf} \to \mathcal{X}$ be a 1-morphism which is surjective and smooth (resp. étale). Considering the diagram

$$\begin{tikzcd}
(Sch/U)_{fppf} \arrow[r, "\mathrm{id}"'] \arrow[d, "x"'] & (Sch/U)_{fppf} \arrow[d, "f \circ x"] \\
\mathcal{X} \arrow[r, "f"] & \mathcal{Y}
\end{tikzcd}$$

and applying Lemma 57.10.2 we conclude that $x \circ f$ is surjective and smooth (resp. étale) as desired. $\square$

57.13. Algebraic stacks and algebraic spaces

In this section we discuss some simple criteria which imply that an algebraic stack is an algebraic space. The main result is that this happens exactly when objects of fibre categories have no nontrivial automorphisms. This is not a triviality! Before we come to this we first do a sanity check.

Lemma 57.13.1. *Let S be a scheme contained in Sch_{fppf}.*

1. *A category fibred in groupoids $p : \mathcal{X} \to (Sch/S)_{fppf}$ which is representable by an algebraic space is a Deligne-Mumford stack.*
2. *If F is an algebraic space over S, then the associated category fibred in groupoids $p : \mathcal{S}_F \to (Sch/S)_{fppf}$ is a Deligne-Mumford stack.*
3. *If $X \in Ob((Sch/S)_{fppf})$, then $(Sch/X)_{fppf} \to (Sch/S)_{fppf}$ is a Deligne-Mumford stack.*

Proof. It is clear that (2) implies (3). Parts (1) and (2) are equivalent by Lemma 57.12.4. Hence it suffices to prove (2). First, we note that $\mathcal{S}_F$ is stack in sets since F is a sheaf (see Examples of Stacks, Section 55.9). A fortiori it is a stack in groupoids. Second the diagonal morphism $\mathcal{S}_F \to \mathcal{S}_F \times \mathcal{S}_F$ is the same as the morphism $\mathcal{S}_F \to \mathcal{S}_{F \times F}$ which comes from the diagonal of F. Hence this is representable by algebraic spaces according to Lemma 57.9.4. Actually it is even representable (by schemes), as the diagonal of an algebraic space is representable, but we do not need this. Let U be a scheme and let $h_U \to F$ be a surjective étale morphism. We may think of this a surjective étale morphism of algebraic spaces. Hence by Lemma 57.10.3 the corresponding 1-morphism $(Sch/U)_{fppf} \to \mathcal{S}_F$ is surjective and étale. $\square$

The following result says that a Deligne-Mumford stack whose inertia is trivial ``is'' an algebraic space. This lemma will be obsoleted by the stronger Proposition 57.13.3 below which says that this holds more generally for algebraic stacks...

Lemma 57.13.2. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$ be an algebraic stack over S. The following are equivalent*

(1) *$\mathcal{X}$ is a Deligne-Mumford stack and is a stack in setoids,*
(2) *$\mathcal{X}$ is a Deligne-Mumford stack such that the canonical 1-morphism $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is an equivalence, and*
(3) *$\mathcal{X}$ is representable by an algebraic space.*

Proof. The equivalence of (1) and (2) follows from Stacks, Lemma 50.7.2. The implication (3) ⇒ (1) follows from Lemma 57.13.1. Finally, assume (1). By Stacks, Lemma 50.6.3 there exists a sheaf F on $(Sch/S)_{fppf}$ and an equivalence $j : \mathcal{X} \to \mathcal{S}_F$. By Lemma 57.9.5 the fact that $\Delta_\mathcal{X}$ is representable by algebraic spaces, means that $\Delta_F : F \to F \times F$ is representable by algebraic spaces. Let U be a scheme, and let $x : (Sch/U)_{fppf} \to \mathcal{X}$ be a surjective étale morphism. The composition $j \circ x : (Sch/U)_{fppf} \to \mathcal{S}_F$ corresponds to a morphism $h_U \to F$ of sheaves. By Bootstrap, Lemma 54.5.1 this morphism is representable by algebraic spaces. Hence by Lemma 57.10.4 we conclude that $h_U \to F$ is surjective and étale. Finally, we apply Bootstrap, Theorem 54.6.1 to see that F is an algebraic space. □

Proposition 57.13.3. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$ be an algebraic stack over S. The following are equivalent*

(1) *$\mathcal{X}$ is a stack in setoids,*
(2) *the canonical 1-morphism $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is an equivalence, and*
(3) *$\mathcal{X}$ is representable by an algebraic space.*

Proof. The equivalence of (1) and (2) follows from Stacks, Lemma 50.7.2. The implication (3) ⇒ (1) follows from Lemma 57.13.2. Finally, assume (1). By Stacks, Lemma 50.6.3 there exists an equivalence $j : \mathcal{X} \to \mathcal{S}_F$ where F is a sheaf on $(Sch/S)_{fppf}$. By Lemma 57.9.5 the fact that $\Delta_\mathcal{X}$ is representable by algebraic spaces, means that $\Delta_F : F \to F \times F$ is representable by algebraic spaces. Let U be a scheme and let $x : (Sch/U)_{fppf} \to \mathcal{X}$ be a surjective smooth morphism. The composition $j \circ x : (Sch/U)_{fppf} \to \mathcal{S}_F$ corresponds to a morphism $h_U \to F$ of sheaves. By Bootstrap, Lemma 54.5.1 this morphism is representable by algebraic spaces. Hence by Lemma 57.10.4 we conclude that $h_U \to F$ is surjective and smooth. In particular it is surjective, flat and locally of finite presentation (by Lemma 57.10.9 and the fact that a smooth morphism of algebraic spaces is flat and locally of finite presentation, see Morphisms of Spaces, Lemmas 42.33.5 and 42.33.7). Finally, we apply Bootstrap, Theorem 54.10.1 to see that F is an algebraic space. □

57.14. 2-Fibre products of algebraic stacks

The 2-category of algebraic stacks has products and 2-fibre products. The first lemma is really a special case of Lemma 57.14.3 but its proof is slightly easier.

Lemma 57.14.1. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$, $\mathcal{Y}$ be algebraic stacks over S. Then $\mathcal{X} \times_{(Sch/S)_{fppf}} \mathcal{Y}$ is an algebraic stack, and is a product in the 2-category of algebraic stacks over S.*

Proof. An object of $\mathcal{X} \times_{(Sch/S)_{fppf}} \mathcal{Y}$ over T is just a pair (x, y) where x is an object of $\mathcal{X}_T$ and y is an object of $\mathcal{Y}_T$. Hence it is immediate from the definitions that $\mathcal{X} \times_{(Sch/S)_{fppf}} \mathcal{Y}$ is a stack in groupoids. If (x, y) and (x', y') are two objects of $\mathcal{X} \times_{(Sch/S)_{fppf}} \mathcal{Y}$ over T, then

$$Isom((x, y), (x', y')) = Isom(x, x') \times Isom(y, y').$$

Hence it follows from the equivalences in Lemma 57.10.11 and the fact that the category of algebraic spaces has products that the diagonal of $\mathcal{X} \times_{(Sch/S)_{fppf}} \mathcal{Y}$ is representable by algebraic spaces. Finally, suppose that $U, V \in Ob((Sch/S)_{fppf})$, and let x, y be surjective smooth morphisms $x : (Sch/U)_{fppf} \to \mathcal{X}$, $y : (Sch/Y)_{fppf} \to \mathcal{Y}$. Note that

$$(Sch/U \times_S V)_{fppf} = (Sch/U)_{fppf} \times_{(Sch/S)_{fppf}} (Sch/V)_{fppf}.$$

The object $(\text{pr}_U^*x, \text{pr}_V^*y)$ of $\mathcal{X} \times_{(Sch/S)_{fppf}} \mathcal{Y}$ over $(Sch/U \times_S V)_{fppf}$ thus defines a 1-morphism

$$(Sch/U \times_S V)_{fppf} \longrightarrow \mathcal{X} \times_{(Sch/S)_{fppf}} \mathcal{Y}$$

which is the composition of base changes of x and y, hence is surjective and smooth, see Lemmas 57.10.6 and 57.10.5. We conclude that $\mathcal{X} \times_{(Sch/S)_{fppf}} \mathcal{Y}$ is indeed an algebraic stack. We omit the verification that it really is a product. □

Lemma 57.14.2. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{Z}$ be a stack in groupoids over $(Sch/S)_{fppf}$ whose diagonal is representable by algebraic spaces. Let $\mathcal{X}$, $\mathcal{Y}$ be algebraic stacks over S. Let $f : \mathcal{X} \to \mathcal{Z}$, $g : \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms of stacks in groupoids. Then the 2-fibre product $\mathcal{X} \times_{f,\mathcal{Z},g} \mathcal{Y}$ is an algebraic stack.*

Proof. We have to check conditions (1), (2), and (3) of Definition 57.12.1. The first condition follows from Stacks, Lemma 50.5.6.

The second condition we have to check is that the $Isom$-sheaves are representable by algebraic spaces. To do this, suppose that T is a scheme over S, and u, v are objects of $(\mathcal{X} \times_{f,\mathcal{Z},g} \mathcal{Y})_T$. By our construction of 2-fibre products (which goes all the way back to Categories, Lemma 4.29.3) we may write $u = (x, y, \alpha)$ and $v = (x', y', \alpha')$. Here $\alpha : f(x) \to g(y)$ and similarly for α'. Then it is clear that

$$\begin{tikzcd} Isom(u, v) \arrow[r] \arrow[d] & Isom(y, y') \arrow[d, "\phi \mapsto g(\phi) \circ \alpha"] \\ Isom(x, x') \arrow[r, "\psi \mapsto \alpha' \circ f(\psi)"] & Isom(f(x), g(y')) \end{tikzcd}$$

is a cartesian diagram of sheaves on $(Sch/T)_{fppf}$. Since by assumption the sheaves $Isom(y, y')$, $Isom(x, x')$, $Isom(f(x), g(y'))$ are algebraic spaces (see Lemma 57.10.11) we see that $Isom(u, v)$ is an algebraic space.

Let $U, V \in Ob((Sch/S)_{fppf})$, and let x, y be surjective smooth morphisms $x : (Sch/U)_{fppf} \to \mathcal{X}$, $y : (Sch/Y)_{fppf} \to \mathcal{Y}$. Consider the morphism

$$(Sch/U)_{fppf} \times_{f \circ x, \mathcal{Z}, g \circ y} (Sch/V)_{fppf} \longrightarrow \mathcal{X} \times_{f,\mathcal{Z},g} \mathcal{Y}.$$

As the diagonal of $\mathcal{Z}$ is representable by algebraic spaces the source of this arrow is representable by an algebraic space F, see Lemma 57.10.11. Moreover, the morphism is the composition of base changes of x and y, hence surjective and smooth, see Lemmas 57.10.6 and 57.10.5. Choosing a scheme W and a surjective étale morphism $W \to F$ we see that the composition of the displayed 1-morphism with the corresponding 1-morphism

$$(Sch/W)_{fppf} \longrightarrow (Sch/U)_{fppf} \times_{f \circ x, \mathcal{Z}, g \circ y} (Sch/V)_{fppf}$$

is surjective and smooth which proves the last condition. □

Lemma 57.14.3. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be algebraic stacks over S. Let $f : \mathcal{X} \to \mathcal{Z}$, $g : \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms of algebraic stacks. Then the 2-fibre product $\mathcal{X} \times_{f,\mathcal{Z},g} \mathcal{Y}$ is an algebraic stack. It is also the 2-fibre product in the 2-category of algebraic stacks over $(Sch/S)_{fppf}$.*

Proof. The fact that $\mathcal{X} \times_{f,\mathcal{Z},g} \mathcal{Y}$ is an algebraic stack follows from the stronger Lemma 57.14.2. The fact that $\mathcal{X} \times_{f,\mathcal{Z},g} \mathcal{Y}$ is a 2-fibre product in the 2-category of algebraic stacks over S follows formally from the fact that the 2-category of algebraic stacks over S is a full sub 2-category of the 2-category of stacks in groupoids over $(Sch/S)_{fppf}$. ☐

57.15. Algebraic stacks, overhauled

Some basic results on algebraic stacks.

Lemma 57.15.1. *Let S be a scheme contained in Sch_{fppf}. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of algebraic stacks over S. Let $V \in Ob((Sch/S)_{fppf})$. Let $y : (Sch/V)_{fppf} \to \mathcal{Y}$ be surjective and smooth. Then there exists an object $U \in Ob((Sch/S)_{fppf})$ and a 2-commutative diagram*

$$\begin{tikzcd} (Sch/U)_{fppf} \arrow[r, "a"'] \arrow[d, "x"'] & (Sch/V)_{fppf} \arrow[d, "y"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}$$

with x surjective and smooth.

Proof. First choose $W \in Ob((Sch/S)_{fppf})$ and a surjective smooth 1-morphism $z : (Sch/W)_{fppf} \to \mathcal{X}$. As $\mathcal{Y}$ is an algebraic stack we may choose an equivalence

$$j : \mathcal{S}_F \longrightarrow (Sch/W)_{fppf} \times_{f \circ z, \mathcal{Y}, y} (Sch/V)_{fppf}$$

where F is an algebraic space. By Lemma 57.10.6 the morphism $\mathcal{S}_F \to (Sch/W)_{fppf}$ is surjective and smooth as a base change of y. Hence by Lemma 57.10.5 we see that $\mathcal{S}_F \to \mathcal{X}$ is surjective and smooth. Choose an object $U \in Ob((Sch/S)_{fppf})$ and a surjective étale morphism $U \to F$. Then applying Lemma 57.10.5 once more we obtain the desired properties. ☐

This lemma is a generalization of Proposition 57.13.3.

Lemma 57.15.2. *Let S be a scheme contained in Sch_{fppf}. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of algebraic stacks over S. The following are equivalent:*

1. *for $U \in Ob((Sch/S)_{fppf})$ the functor $f : \mathcal{X}_U \to \mathcal{Y}_U$ is faithful,*
2. *the functor f is faithful, and*
3. *f is representable by algebraic spaces.*

Proof. Parts (1) and (2) are equivalent by general properties of 1-morphisms of categories fibred in groupoids, see Categories, Lemma 4.32.8. We see that (3) implies (2) by Lemma 57.9.2. Finally, assume (2). Let U be a scheme. Let $y \in Ob(\mathcal{Y}_U)$. We have to prove that

$$\mathcal{W} = (Sch/U)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$$

is representable by an algebraic space over U. Since $(Sch/U)_{fppf}$ is an algebraic stack we see from Lemma 57.14.3 that $\mathcal{W}$ is an algebraic stack. On the other hand the explicit description of objects of $\mathcal{W}$ as triples $(V, x, \alpha : y(V) \to f(x))$ and the fact that f is faithful, shows that the fibre categories of $\mathcal{W}$ are setoids. Hence Proposition 57.13.3 garantees that $\mathcal{W}$ is representable by an algebraic space. ☐

Lemma 57.15.3. *Let S be a scheme contained in Sch_{fppf}. Let $u : \mathcal{U} \to \mathcal{X}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. If*

(1) *$\mathcal{U}$ is representable by an algebraic space, and*
(2) *u is representable by algebraic spaces, surjective and smooth,*

then $\mathcal{X}$ is an algebraic stack over S.

Proof. We have to show that $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces, see Definition 57.12.1. Given two schemes T_1, T_2 over S denote $\mathcal{T}_i = (Sch/T_i)_{fppf}$ the associated representable fibre categories. Suppose given 1-morphisms $f_i : \mathcal{T}_i \to \mathcal{X}$. According to Lemma 57.10.11 it suffices to prove that the 2-fibered product $\mathcal{T}_1 \times_{\mathcal{X}} \mathcal{T}_2$ is representable by an algebraic space. By Stacks, Lemma 50.6.8 this is in any case a stack in setoids. Thus $\mathcal{T}_1 \times_{\mathcal{X}} \mathcal{T}_2$ corresponds to some sheaf F on $(Sch/S)_{fppf}$, see Stacks, Lemma 50.6.3. Let U be the algebraic space which represents $\mathcal{U}$. By assumption

$$\mathcal{T}_i' = \mathcal{U} \times_{u,\mathcal{X},f_i} \mathcal{T}_i$$

is representable by an algebraic space T_i' over S. Hence $\mathcal{T}_1' \times_{\mathcal{U}} \mathcal{T}_2'$ is representable by the algebraic space $T_1' \times_U T_2'$. Consider the commutative diagram

$$\begin{tikzcd}
 & \mathcal{T}_1 \times_{\mathcal{X}} \mathcal{T}_2 \arrow[rr] \arrow[dd] & & \mathcal{T}_1 \arrow[dd] \\
\mathcal{T}_1' \times_{\mathcal{U}} \mathcal{T}_2' \arrow[ur] \arrow[rr] \arrow[dd] & & \mathcal{T}_1' \arrow[ur] \arrow[dd] & \\
 & \mathcal{T}_2 \arrow[rr] & & \mathcal{X} \\
\mathcal{T}_2' \arrow[ur] \arrow[rr] & & \mathcal{U} \arrow[ur] &
\end{tikzcd}$$

In this diagram the bottom square, the right square, the back square, and the front square are 2-fibre products. A formal argument then shows that $\mathcal{T}_1' \times_{\mathcal{U}} \mathcal{T}_2' \to \mathcal{T}_1 \times_{\mathcal{X}} \mathcal{T}_2$ is the ``base change'' of $\mathcal{U} \to \mathcal{X}$, more precisely the diagram

$$\begin{tikzcd}
\mathcal{T}_1' \times_{\mathcal{U}} \mathcal{T}_2' \arrow[r] \arrow[d] & \mathcal{U} \arrow[d] \\
\mathcal{T}_1 \times_{\mathcal{X}} \mathcal{T}_2 \arrow[r] & \mathcal{X}
\end{tikzcd}$$

is a 2-fibre square. Hence $T_1' \times_U T_2' \to F$ is representable by algebraic spaces, smooth, and surjective, see Lemmas 57.9.6, 57.9.7, 57.10.4, and 57.10.6. Therefore F is an algebraic space by Bootstrap, Theorem 54.10.1 and we win. □

An application of Lemma 57.15.3 is that something which is an algebraic space over an algebraic stack is an algebraic stack. This is the analogue of Bootstrap, Lemma 54.3.6. Actually, it suffices to assume the morphism $\mathcal{X} \to \mathcal{Y}$ is ``algebraic'', as we will see in Criteria for Representability, Lemma 59.8.2.

Lemma 57.15.4. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of stacks in groupoids over $(Sch/S)_{fppf}$. Assume that*

(1) *$\mathcal{X} \to \mathcal{Y}$ is representable by algebraic spaces, and*

(2) *$\mathcal{Y}$ is an algebraic stack over S.*

Then $\mathcal{X}$ is an algebraic stack over S.

Proof. Let $\mathcal{V} \to \mathcal{Y}$ be a surjective smooth 1-morphism from a representable stack in groupoids to $\mathcal{Y}$. This exists by Definition 57.12.1. Then the 2-fibre product $\mathcal{U} = \mathcal{V} \times_{\mathcal{Y}} \mathcal{X}$ is representable by an algebraic space by Lemma 57.9.8. The 1-morphism $\mathcal{U} \to \mathcal{X}$ is representable by algebraic spaces, smooth, and surjective, see Lemmas 57.9.7 and 57.10.6. By Lemma 57.15.3 we conclude that $\mathcal{X}$ is an algebraic stack. $\square$

Lemma 57.15.5. *Let S be a scheme contained in Sch_{fppf}. Let $j : \mathcal{X} \to \mathcal{Y}$ be a 1 morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume j is representable by algebraic spaces and a monomorphism[3]. Then, if $\mathcal{Y}$ is a stack in groupoids (resp. an algebraic stack), so is $\mathcal{X}$.*

Proof. We prove that $\mathcal{X}$ is a stack. The case of algebraic stacks will then follow from Lemma 57.15.4. It suffices to check effectiveness of descent for $\mathcal{X}$. Fix a scheme T and an fppf covering $\{f_i : T_i \to T\}$. Suppose we have objects x_i of the fibre categories $\mathcal{X}_{T_i}$ together with a descent datum. Then since $\mathcal{Y}$ is a stack, there exists an object y in the fibre category $\mathcal{Y}_T$ such that $f_i^*(y) \simeq j(x_i)$ in $\mathcal{Y}_{T_i}$. By hypothesis, the 2-fibered product

$$\mathcal{X} \times_{j,\mathcal{Y},y} (Sch/T)_{fppf}$$

is representable by an algebraic space U such that the induced morphism $U \to T$ is a monomorphism of algebraic spaces. By the universal property of the 2-fibre product and the fact that $f_i^*(y) \cong j(x_i)$, we have that $f_i : T_i \to T$ factors through $U \to T$ for all i. Hence $U \to T$ is a monomorphism of fppf sheaves, but also surjective as $\{f_i : T_i \to T\}$ is a covering. We conclude that $U = T$. Thus y comes from some object x of the fibre category $\mathcal{X}_T$. We have $f_i^* x \cong x_i$ in the fibre category $\mathcal{X}_{T_i}$ as the functor j is fully faithful on fibre categories, see Lemma 57.10.10. $\square$

57.16. From an algebraic stack to a presentation

Given an algebraic stack over S we obtain a groupoid in algebraic spaces over S whose associated quotient stack is the algebraic stack.

Recall that if (U, R, s, t, c) is a groupoid in algebraic spaces over S then $[U/R]$ denotes the quotient stack associated to this datum, see Groupoids in Spaces, Definition 52.19.1. In general $[U/R]$ is **not** an algebraic stack. In particular the stack $[U/R]$ occuring in the following lemma is in general not algebraic.

Lemma 57.16.1. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$ be an algebraic stack over S. Let $\mathcal{U}$ be an algebraic stack over S which is representable by an algebraic space. Let $f : \mathcal{U} \to \mathcal{X}$ be a 1-morphism. Then*

(1) *the 2-fibre product $\mathcal{R} = \mathcal{U} \times_{f,\mathcal{X},f} \mathcal{U}$ is representable by an algebraic space,*
(2) *there is a canonical equivalence*

$$\mathcal{U} \times_{f,\mathcal{X},f} \mathcal{U} \times_{f,\mathcal{X},f} \mathcal{U} = \mathcal{R} \times_{pr_1,\mathcal{U},pr_0} \mathcal{R},$$

(3) *the projection pr_{02} induces via (2) a 1-morphism*

$$pr_{02} : \mathcal{R} \times_{pr_1,\mathcal{U},pr_0} \mathcal{R} \longrightarrow \mathcal{R}$$

[3] For example an open immersion.

(4) *let U, R be the algebraic spaces representing $\mathcal{U}, \mathcal{R}$ and $t, s : R \to U$ and $c : R \times_{s,U,t} R \to U$ are the morphisms corresponding to the 1-morphisms $pr_0, pr_1 : \mathcal{R} \to \mathcal{U}$ and $pr_{02} : \mathcal{R} \times_{pr_1,\mathcal{U},pr_0} \mathcal{R} \to \mathcal{R}$ above, then the quintuple (U, R, s, t, c) is a groupoid in algebraic spaces over S,*
(5) *the morphism f induces a canonical 1-morphism $f_{can} : [U/R] \to \mathcal{X}$ of stacks in groupoids over $(Sch/S)_{fppf}$, and*
(6) *the 1-morphism $f_{can} : [U/R] \to \mathcal{X}$ is fully faithful.*

Proof. Proof of (1). By definition $\Delta_{\mathcal{X}}$ is representable by algebraic spaces so Lemma 57.10.11 applies to show that $\mathcal{U} \to \mathcal{X}$ is representable by algebraic spaces. Hence the result follows from Lemma 57.9.8.

Let T be a scheme over S. By construction of the 2-fibre product (see Categories, Lemma 4.29.3) we see that the objects of the fibre category $\mathcal{R}_T$ are triples (a, b, α) where $a, b \in Ob(\mathcal{U}_T)$ and $\alpha : f(a) \to f(b)$ is a morphism in the fibre category $\mathcal{X}_T$.

Proof of (2). The equivalence comes from repeatedly applying Categories, Lemmas 4.28.8 and 4.28.10. Let us identify $\mathcal{U} \times_{\mathcal{X}} \mathcal{U} \times_{\mathcal{X}} \mathcal{U}$ with $(\mathcal{U} \times_{\mathcal{X}} \mathcal{U}) \times_{\mathcal{X}} \mathcal{U}$. If T is a scheme over S, then on fibre categories over T this equivalence maps the object $((a, b, \alpha), c, \beta)$ on the left hand side to the object $((a, b, \alpha), (b, c, \beta))$ of the right hand side.

Proof of (3). The 1-morphism pr_{02} is constructed in the proof of Categories, Lemma 4.28.9. In terms of the description of objects of the fibre category above we see that $((a, b, \alpha), (b, c, \beta))$ maps to $(a, c, \beta \circ \alpha)$.

Unfortunately, this is *not compatible* with our conventions on groupoids where we always have $j = (t, s) : R \to U$, and we ``think" of a T-valued point r of R as a morphism $r : s(r) \to t(r)$. However, this does not affect the proof of (4), since the opposite of a groupoid is a groupoid. But in the proof of (5) it is responsible for the inverses in the displayed formula below.

Proof of (4). Recall that the sheaf U is isomorphic to the sheaf $T \mapsto Ob(\mathcal{U}_T)/\cong$, and similarly for R, see Lemma 57.8.2. It follows from Categories, Lemma 4.36.8 that this description is compatible with 2-fibre products so we get a similar matching of $\mathcal{R} \times_{\text{pr}_1,\mathcal{U},\text{pr}_0} \mathcal{R}$ and $R \times_{s,U,t} R$. The morphisms $t, s : R \to U$ and $c : R \times_{s,U,t} R \to R$ we get from the general equality (57.8.2.1). Explicitly these maps are the transformations of functors that come from letting pr_0, pr_0, pr_{02} act on isomorphism classes of objects of fibre categories. Hence to show that we obtain a groupoid in algebraic spaces it suffices to show that for every scheme T over S the structure

$$(Ob(\mathcal{U}_T)/\cong, Ob(\mathcal{R}_T)/\cong, \text{pr}_1, \text{pr}_0, \text{pr}_{02})$$

is a groupoid which is clear from our description of objects of $\mathcal{R}_T$ above.

Proof of (5). We will eventually apply Groupoids in Spaces, Lemma 52.22.2 to obtain the functor $[U/R] \to \mathcal{X}$. Consider the 1-morphism $f : \mathcal{U} \to \mathcal{X}$. We have a 2-arrow $\tau : f \circ \text{pr}_1 \to f \circ \text{pr}_0$ by definition of $\mathcal{R}$ as the 2-fibre product. Namely, on an object (a, b, α) of $\mathcal{R}$ over T it is the map $\alpha^{-1} : b \to a$. We claim that

$$\tau \circ \text{id}_{\text{pr}_{02}} = (\tau \star \text{id}_{\text{pr}_0}) \circ (\tau \star \text{id}_{\text{pr}_1}).$$

This identity says that given an object $((a, b, \alpha), (b, c, \beta))$ of $\mathcal{R} \times_{\text{pr}_1,\mathcal{U},\text{pr}_0} \mathcal{R}$ over T, then the composition of

$$c \xrightarrow{\beta^{-1}} b \xrightarrow{\alpha^{-1}} a$$

is the same as the arrow $(\beta \circ \alpha)^{-1} : a \to c$. This is clearly true, hence the claim holds. In this way we see that all the assumption of Groupoids in Spaces, Lemma 52.22.2 are satisfied for the structure $(\mathcal{U}, \mathcal{R}, \text{pr}_0, \text{pr}_1, \text{pr}_{02})$ and the 1-morphism f and the 2-morphism τ. Except, to apply the lemma we need to prove this holds for the structure $(\mathcal{S}_U, \mathcal{S}_R, s, t, c)$ with suitable morphisms.

Now there should be some general abstract nonsense argument which transfer these data between the two, but it seems to be quite long. Instead, we use the following trick. Pick a quasi-inverse $j^{-1} : \mathcal{S}_U \to \mathcal{U}$ of the canonical equivalence $j : \mathcal{U} \to \mathcal{S}_U$ which comes from $U(T) = Ob(\mathcal{U}_T)/\cong$. This just means that for every scheme T/S and every object $a \in \mathcal{U}_T$ we have picked out a particular element of its isomorphism class, namely $j^{-1}(j(a))$. Using j^{-1} we may therefore see $\mathcal{S}_U$ as a subcategory of $\mathcal{U}$. Having chosen this subcategory we can consider those objects (a, b, α) of $\mathcal{R}_T$ such that a, b are objects of $(\mathcal{S}_U)_T$, i.e., such that $j^{-1}(j(a)) = a$ and $j^{-1}(j(b)) = b$. Then it is clear that this forms a subcategory of $\mathcal{R}$ which maps isomorphically to $\mathcal{S}_R$ via the canonical equivalence $\mathcal{R} \to \mathcal{S}_R$. Moreover, this is clearly compatible with forming the 2-fibre product $\mathcal{R} \times_{\text{pr}_1, \mathcal{U}, \text{pr}_0} \mathcal{R}$. Hence we see that we may simply restrict f to $\mathcal{S}_U$ and restrict τ to a transformation beween functors $\mathcal{S}_R \to \mathcal{X}$. Hence it is clear that the displayed equality of Groupoids in Spaces, Lemma 52.22.2 holds since it holds even as an equality of transformations of functors $\mathcal{R} \times_{\text{pr}_1, \mathcal{U}, \text{pr}_0} \mathcal{R} \to \mathcal{X}$ before restricting to the subcategory $\mathcal{S}_{R \times_{s,U,t} R}$.

This proves that Groupoids in Spaces, Lemma 52.22.2 applies and we get our desired morphism of stacks $f_{can} : [U/R] \to \mathcal{X}$. We briefly spell out how f_{can} is defined in this special case. On an object a of $\mathcal{S}_U$ over T we have $f_{can}(a) = f(a)$, where we think of $\mathcal{S}_U \subset \mathcal{U}$ by the chosen embedding above. If a, b are objects of $\mathcal{S}_U$ over T, then a morphism $\varphi : a \to b$ in $[U/R]$ is by definition an object of the form $\varphi = (b, a, \alpha)$ of $\mathcal{R}$ over T. (Note switch.) And the rule in the proof of Groupoids in Spaces, Lemma 52.22.2 is that

$$f_{can}(\varphi) = \left(f(a) \xrightarrow{\alpha^{-1}} f(b) \right). \tag{57.16.1.1}$$

Proof of (6). Both $[U/R]$ and $\mathcal{X}$ are stacks. Hence given a scheme T/S and objects a, b of $[U/R]$ over T we obtain a transformation of fppf sheaves

$$Isom(a, b) \longrightarrow Isom(f_{can}(a), f_{can}(b))$$

on $(Sch/T)_{fppf}$. We have to show that this is an isomorphism. We may work fppf locally on T, hence we may assume that a, b come from morphisms $a, b : T \to U$. By the embedding $\mathcal{S}_U \subset \mathcal{U}$ above we may also think of a, b as objects of $\mathcal{U}$ over T. In Groupoids in Spaces, Lemma 52.21.1 we have seen that the left hand sheaf is represented by the algebraic space

$$R \times_{(t,s), U \times_S U, (b,a)} T$$

over T. On the other hand, the right hand side is by Stacks, Lemma 50.2.5 equal to the sheaf associated to the following stack in setoids:

$$\mathcal{X} \times_{\mathcal{X} \times \mathcal{X}, (f \circ b, f \circ a)} T = \mathcal{X} \times_{\mathcal{X} \times \mathcal{X}, (f,f)} (\mathcal{U} \times \mathcal{U}) \times_{\mathcal{U} \times \mathcal{U}, (b,a)} T = \mathcal{R} \times_{(\text{pr}_0, \text{pr}_1), \mathcal{U} \times \mathcal{U}, (b,a)} T$$

which is representable by the fibre product desplayed above. At this point we have shown that the two $Isom$-sheaves are isomorphic. Our 1-morphism $f_{can} : [U/R] \to \mathcal{X}$ induces this isomorphism on $Isom$-sheaves by Equation (57.16.1.1). $\square$

We can use the previous very abstract lemma to produce presentations.

Lemma 57.16.2. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$ be an algebraic stack over S. Let U be an algebraic space over S. Let $f : \mathcal{S}_U \to \mathcal{X}$ be a surjective smooth morphism. Let (U, R, s, t, c) be the groupoid in algebraic spaces and $f_{can} : [U/R] \to \mathcal{X}$ be the result of applying Lemma 57.16.1 to U and f. Then*

(1) *the morphisms s, t are smooth, and*
(2) *the 1-morphism $f_{can} : [U/R] \to \mathcal{X}$ is an equivalence.*

Proof. The morphisms s, t are smooth by Lemmas 57.10.2 and 57.10.3. As the 1-morphism f is smooth and surjective it is clear that given any scheme T and any object $a \in Ob(\mathcal{X}_T)$ there exists a smooth and surjective morphism $T' \to T$ such that $a|'_T$ comes from an object of $[U/R]_{T'}$. Since $f_{can} : [U/R] \to \mathcal{X}$ is fully faithful, we deduce that $[U/R] \to \mathcal{X}$ is essentially surjective as descent data on objects are effective on both sides, see Stacks, Lemma 50.4.8. ☐

Remark 57.16.3. If the morphism $f : \mathcal{S}_U \to \mathcal{X}$ of Lemma 57.16.2 is only assumed surjective, flat and locally of finite presentation, then it will still be the case that $f_{can} : [U/R] \to \mathcal{X}$ is an equivalence. In this case the morphisms s, t will be flat and locally of finite presentation, but of course not smooth in general.

Lemma 57.16.2 suggests the following definitions.

Definition 57.16.4. Let S be a scheme. Let B be an algebraic space over S. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. We say (U, R, s, t, c) is a *smooth groupoid*[4] if $s, t : R \to U$ are smooth morphisms of algebraic spaces.

Definition 57.16.5. Let $\mathcal{X}$ be an algebraic stack over S. A *presentation* of $\mathcal{X}$ is given by a smooth groupoid (U, R, s, t, c) in algebraic spaces over S, and an equivalence $f : [U/R] \to \mathcal{X}$.

We have seen above that every algebraic stack has a presentation. Our next task is to show that every smooth groupoid in algebraic spaces over S gives rise to an algebraic stack.

57.17. The algebraic stack associated to a smooth groupoid

In this section we start with a smooth groupoid in algebraic spaces and we show that the associated quotient stack is an algebraic stack.

Lemma 57.17.1. *Let S be a scheme contained in Sch_{fppf}. Let (U, R, s, t, c) be a groupoid in algebraic spaces over S. Then the diagonal of $[U/R]$ is representable by algebraic spaces.*

Proof. It suffices to show that the $Isom$-sheaves are algebraic spaces, see Lemma 57.10.11. This follows from Bootstrap, Lemma 54.11.3. ☐

Lemma 57.17.2. *Let S be a scheme contained in Sch_{fppf}. Let (U, R, s, t, c) be a smooth groupoid in algebraic spaces over S. Then the morphism $\mathcal{S}_U \to [U/R]$ is smooth and surjective.*

Proof. Let T be a scheme and let $x : (Sch/T)_{fppf} \to [U/R]$ be a 1-morphism. We have to show that the projection

$$\mathcal{S}_U \times_{[U/R]} (Sch/T)_{fppf} \longrightarrow (Sch/T)_{fppf}$$

is surjective and smooth. We already know that the left hand side is representable by an algebraic space F, see Lemmas 57.17.1 and 57.10.11. Hence we have to show the corresponding morphism $F \to T$ of algebraic spaces is surjective and smooth. Since we are

[4]This terminology might be a bit confusing: it does not imply that $[U/R]$ is smooth over anything.

working with properties of morphisms of algebraic spaces which are local on the target in the fppf topology we may check this fppf locally on T. By construction, there exists an fppf covering $\{T_i \to T\}$ of T such that $x|_{(Sch/T_i)_{fppf}}$ comes from a morphism $x_i : T_i \to U$. (Note that $F \times_T T_i$ represents the 2-fibre product $\mathcal{S}_U \times_{[U/R]} (Sch/T_i)_{fppf}$ so everything is compatible with the base change via $T_i \to T$.) Hence we may assume that x comes from $x : T \to U$. In this case we see that

$$\mathcal{S}_U \times_{[U/R]} (Sch/T)_{fppf} = (\mathcal{S}_U \times_{[U/R]} \mathcal{S}_U) \times_{\mathcal{S}_U} (Sch/T)_{fppf} = \mathcal{S}_R \times_{\mathcal{S}_U} (Sch/T)_{fppf}$$

The first equality by Categories, Lemma 4.28.10 and the second equality by Groupoids in Spaces, Lemma 52.21.2. Clearly the last 2-fibre product is represented by the algebraic space $F = R \times_{s,U,x} T$ and the projection $R \times_{s,U,x} T \to T$ is smooth as the base change of the smooth morphism of algebraic spaces $s : R \to U$. It is also surjective as s has a section (namely the identity $e : U \to R$ of the groupoid). This proves the lemma. □

Here is the main result of this section.

Theorem 57.17.3. *Let S be a scheme contained in Sch_{fppf}. Let (U, R, s, t, c) be a smooth groupoid in algebraic spaces over S. Then the quotient stack $[U/R]$ is an algebraic stack over S.*

Proof. We check the three conditions of Definition 57.12.1. By construction we have that $[U/R]$ is a stack in groupoids which is the first condition.

The second condition follows from the stronger Lemma 57.17.1.

Finally, we have to show there exists a scheme W over S and a surjective smooth 1-morphism $(Sch/W)_{fppf} \longrightarrow \mathcal{X}$. First choose $W \in Ob((Sch/S)_{fppf})$ and a surjective étale morphism $W \to U$. Note that this gives a surjective étale morphism $\mathcal{S}_W \to \mathcal{S}_U$ of categories fibred in sets, see Lemma 57.10.3. Of course then $\mathcal{S}_W \to \mathcal{S}_U$ is also surjective and smooth, see Lemma 57.10.9. Hence $\mathcal{S}_W \to \mathcal{S}_U \to [U/R]$ is surjective and smooth by a combination of Lemmas 57.17.2 and 57.10.5. □

57.18. Change of big site

In this section we briefly discuss what happens when we change big sites. The upshot is that we can always enlarge the big site at will, hence we may assume any set of schemes we want to consider is contained in the big fppf site over which we consider our algebraic space. We encourage the reader to skip this section.

Pullbacks of stacks is defined in Stacks, Section 50.12.

Lemma 57.18.1. *Suppose given big sites Sch_{fppf} and Sch'_{fppf}. Assume that Sch_{fppf} is contained in Sch'_{fppf}, see Topologies, Section 30.10. Let S be an object of Sch_{fppf}. Let Let $f : (Sch'/S)_{fppf} \to (Sch/S)_{fppf}$ the morphism of sites corresponding to the inclusion functor $u : (Sch/S)_{fppf} \to (Sch'/S)_{fppf}$. Let $\mathcal{X}$ be a stack in groupoids over $(Sch/S)_{fppf}$.*

1. *if $\mathcal{X}$ is representable by some $X \in Ob((Sch/S)_{fppf})$, then $f^{-1}\mathcal{X}$ is representable too, in fact it is representable by the same scheme X, now viewed as an object of $(Sch'/S)_{fppf}$,*
2. *if $\mathcal{X}$ is representable by $F \in Sh((Sch/S)_{fppf})$ which is an algebraic space, then $f^{-1}\mathcal{X}$ is representable by the algebraic space $f^{-1}F$,*
3. *if $\mathcal{X}$ is an algebraic stack, then $f^{-1}\mathcal{X}$ is an algebraic stack, and*
4. *if $\mathcal{X}$ is a Deligne-Mumford stack, then $f^{-1}\mathcal{X}$ is a Deligne-Mumford stack too.*

Proof. Let us prove (3). By Lemma 57.16.2 we may write $\mathcal{X} = [U/R]$ for some smooth groupoid in algebraic spaces (U, R, s, t, c). By Groupoids in Spaces, Lemma 52.27.1 we see that $f^{-1}[U/R] = [f^{-1}U/f^{-1}R]$. Of course $(f^{-1}U, f^{-1}R, f^{-1}s, f^{-1}t, f^{-1}c)$ is a smooth groupoid in algebraic spaces too. Hence (3) is proved.

Now the other cases (1), (2), (4) each mean that $\mathcal{X}$ has a presentation $[U/R]$ of a particular kind, and hence translate into the same kind of presentation for $f^{-1}\mathcal{X} = [f^{-1}U/f^{-1}R]$. Whence the lemma is proved. □

It is not true (in general) that the restriction of an algebraic space over the bigger site is an algebraic space over the smaller site (simply by reasons of cardinality). Hence we can only ever use a simple lemma of this kind to enlarge the base category and never to shrink it.

Lemma 57.18.2. *Suppose Sch_{fppf} is contained in Sch'_{fppf}. Let S be an object of Sch_{fppf}. Denote Algebraic-Stacks/S the 2-category of algebraic spaces over S defined using Sch_{fppf}. Similarly, denote Algebraic-Stacks$'/S$ the 2-category of algebraic spaces over S defined using Sch'_{fppf}. The rule $\mathcal{X} \mapsto f^{-1}\mathcal{X}$ of Lemma 57.18.1 defines a functor of 2-categories*

$$\textit{Algebraic-Stacks}/S \longrightarrow \textit{Algebraic-Stacks}'/S$$

which defines equivalences of morphism categories

$$Mor_{\textit{Algebraic-Stacks}/S}(\mathcal{X}, \mathcal{Y}) \longrightarrow Mor_{\textit{Algebraic-Stacks}'/S}(f^{-1}\mathcal{X}, f^{-1}\mathcal{Y})$$

for every objects $\mathcal{X}, \mathcal{Y}$ of Algebraic-Stacks/S. An object $\mathcal{X}'$ of Algebraic-Stacks$'/S$ is equivalence to $f^{-1}\mathcal{X}$ for some $\mathcal{X}$ in Algebraic-Stacks/S if and only if it has a presentation $\mathcal{X} = [U'/R']$ with U', R' isomorphic to $f^{-1}U$, $f^{-1}R$ for some $U, R \in Spaces/S$.

Proof. The statement on morphism categories is a consequence of the more general Stacks, Lemma 50.12.12. The characterization of the ``essential image'' follows from the description of f^{-1} in the proof of Lemma 57.18.1. □

57.19. Change of base scheme

In this section we briefly discuss what happens when we change base schemes. The upshot is that given a morphism $S \to S'$ of base schemes, any algebraic stack over S can be viewed as an algebraic stack over S'.

Lemma 57.19.1. *Let Sch_{fppf} be a big fppf site. Let $S \to S'$ be a morphism of this site. The constructions A and B of Stacks, Section 50.13 above give ismorphisms of 2-categories*

$$\left\{ \begin{matrix} \textit{2-category of algebraic} \\ \textit{stacks } \mathcal{X} \textit{ over } S \end{matrix} \right\} \leftrightarrow \left\{ \begin{matrix} \textit{2-category of pairs } (\mathcal{X}', f) \textit{ consisting of an} \\ \textit{algebraic stack } \mathcal{X}' \textit{ over } S' \textit{ and a morphism} \\ f : \mathcal{X}' \to (Sch/S)_{fppf} \textit{ of algebraic stacks over } S' \end{matrix} \right\}$$

Proof. The statement makes sense as the functor $j : (Sch/S)_{fppf} \to (Sch/S')_{fppf}$ is the localization functor assocated to the object S/S' of $(Sch/S')_{fppf}$. By Stacks, Lemma 50.13.2 the only thing to show is that the constructions A and B preserve the subcategories of algebraic stacks. But for example, if $\mathcal{X} = [U/R]$ then we have construction A applied to $\mathcal{X}$ just produces $\mathcal{X}' = \mathcal{X}$. Conversely, if $\mathcal{X}' = [U'/R']$ the the morphism p induces morphisms of algebraic spaces $U' \to S$ and $R' \to S$, and then $\mathcal{X} = [U'/R']$ but now viewed as a stack over S. Hence the lemma is clear. □

Definition 57.19.2. Let Sch_{fppf} be a big fppf site. Let $S \to S'$ be a morphism of this site. If $p : \mathcal{X} \to (Sch/S)_{fppf}$ is an algebraic stack over S, then $\mathcal{X}$ *viewed as an algebraic stack over* S' is the algebraic stack

$$\mathcal{X} \longrightarrow (Sch/S')_{fppf}$$

gotten by applying construction A of Lemma 57.19.1 to $\mathcal{X}$.

Conversely, what if we start with an algebraic stack $\mathcal{X}'$ over S' and we want to get an algebraic stack over S? Well, then we consider the 2-fibre product

$$\mathcal{X}'_S = (Sch/S)_{fppf} \times_{(Sch/S')_{fppf}} \mathcal{X}'$$

which is an algebraic stack over S' according to Lemma 57.14.3. Moreover, it comes equipped with a natural 1-morphism $p : \mathcal{X}'_S \to (Sch/S)_{fppf}$ and hence by Lemma 57.19.1 it corresponds in a canonical way to an algebraic stack over S.

Definition 57.19.3. Let Sch_{fppf} be a big fppf site. Let $S \to S'$ be a morphism of this site. Let $\mathcal{X}'$ be an algebraic stack over S'. The *change of base of* $\mathcal{X}'$ is the algebraic space $\mathcal{X}'_S$ over S described above.

57.20. Other chapters

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CHAPTER 58

Sheaves on Algebraic Stacks

58.1. Introduction

There is a myriad of ways to think about sheaves on algebraic stacks. In this chapter we discuss one approach, which is particularly well adapted to our foundations for algebraic stacks. Whenever we introduce a type of sheaves we will indicate the precise relationship with similar notions in the literature. The goal of this chapter is to state those results that are either obviously true or straightforward to prove and leave more intricate constructions till later.

In fact, it turns out that to develop a fully fledged theory of constructible étale sheaves and/or an adequate discussion of derived categories of complexes $\mathcal{O}$-modules whose cohomology sheaves are quasi-coherent takes a significant amount of work, see [**Ols07b**]. We will return to these issues later (insert future reference here).

In the literature and in research papers on sheaves on algebraic stacks the lisse-étale site of an algebraic stack often plays a prominent role. However, it is a problematic beast, because it turns out that a morphism of algebraic stacks does not induce a morphism of lisse-étale topoi. We have therefore made the design decision to avoid any mention of the lisse-étale site as long as possible. Arguments that traditionally use the lisse-étale site will be replaced by an argument using a Čech covering in the site $\mathcal{X}_{smooth}$ defined below.

58.2. Conventions

The conventions we use in this chapter are the same as those in the chapter on algebraic stacks, see Algebraic Stacks, Section 57.2. For convenience we repeat them here.

We work in a suitable big fppf site Sch_{fppf} as in Topologies, Definition 30.7.6. So, if not explicitly stated otherwise all schemes will be objects of Sch_{fppf}. We record what changes if you change the big fppf site elsewhere (insert future reference here).

We will always work relative to a base S contained in Sch_{fppf}. And we will then work with the big fppf site $(Sch/S)_{fppf}$, see Topologies, Definition 30.7.8. The absolute case can be recovered by taking $S = Spec(\mathbf{Z})$.

58.3. Presheaves

In this section we define presheaves on categories fibred in groupoids over $(Sch/S)_{fppf}$, but most of the discussion works for categories over any base category. This section also serves to introduce the notation we will use later on.

Definition 58.3.1. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids.

(1) A *presheaf on $\mathcal{X}$* is a presheaf on the underlying category of $\mathcal{X}$.
(2) A *morphism of presheaves on $\mathcal{X}$* is a morphism of presheaves on the underlying category of $\mathcal{X}$.

We denote $PSh(\mathcal{X})$ the category of presheaves on $\mathcal{X}$.

This defines presheaves of sets. Of course we can also talk about presheaves of pointed sets, abelian groups, groups, monoids, rings, modules over a fixed ring, and lie algebras over a fixed field, etc. The category of *abelian presheaves*, i.e., presheaves of abelian groups, is denoted $PAb(\mathcal{X})$.

Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Recall that this means just that f is a functor over $(Sch/S)_{fppf}$. The material in Sites, Section 9.17 provides us with a pair of adjoint functors[1]

(58.3.1.1) $$f^p : PSh(\mathcal{Y}) \longrightarrow PSh(\mathcal{X}) \quad \text{and} \quad {}_pf : PSh(\mathcal{X}) \longrightarrow PSh(\mathcal{Y}).$$

The adjointness is

$$Mor_{PSh(\mathcal{X})}(f^p\mathcal{G}, \mathcal{F}) = Mor_{PSh(\mathcal{Y})}(\mathcal{G}, {}_pf\mathcal{F})$$

where $\mathcal{F} \in Ob(PSh(\mathcal{X}))$ and $\mathcal{G} \in Ob(PSh(\mathcal{Y}))$. We call $f^p\mathcal{G}$ the *pullback* of $\mathcal{G}$. It follows from the definitions that

$$f^p\mathcal{G}(x) = \mathcal{G}(f(x))$$

for any $x \in Ob(\mathcal{X})$. The presheaf ${}_pf\mathcal{F}$ is called the *pushforward* of $\mathcal{F}$. It is described by the formula

$$({}_pf\mathcal{F})(y) = lim_{f(x)\to y}\, \mathcal{F}(x).$$

The rest of this section should probably be moved to the chapter on sites and in any case should be skipped on a first reading.

Lemma 58.3.2. *Let $f\mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. Then $(g \circ f)^p = f^p \circ g^p$ and there is a canonical isomorphism ${}_p(g\circ f) \to {}_pg\circ {}_pf$ compatible with with adjointness of $(f^p, {}_pf)$, $(g^p, {}_pg)$, and $((g\circ f)^p, {}_p(g\circ f))$.*

Proof. Let $\mathcal{H}$ be a presheaf on $\mathcal{Z}$. Then $(g \circ f)^p\mathcal{H} = f^p(g^p\mathcal{H})$ is given by the equalities

$$(g \circ f)^p\mathcal{H}(x) = \mathcal{H}((g \circ f)(x)) = \mathcal{H}(g(f(x))) = f^p(g^p\mathcal{H})(x).$$

We omit the verification that this is compatible with restriction maps.

Next, we define the transformation ${}_p(g \circ f) \to {}_pg \circ {}_pf$. Let $\mathcal{F}$ be a presheaf on $\mathcal{X}$. If z is an object of $\mathcal{Z}$ then we get a category $\mathcal{J}$ of quadruples $(x, f(x) \to y, y, g(y) \to z)$ and a category $\mathcal{I}$ of pairs $(x, g(f(x)) \to z)$. There is a canonical functor $\mathcal{J} \to \mathcal{I}$ sending the object $(x, \alpha : f(x) \to y, y, \beta : g(y) \to z)$ to $(x, \beta \circ f(\alpha) : g(f(x)) \to z)$. This gives the arrow in

$$\begin{aligned} ({}_p(g \circ f)\mathcal{F})(z) &= lim_{g(f(x))\to z}\, \mathcal{F}(x) \\ &= lim_{\mathcal{I}}\, \mathcal{F} \\ &\to lim_{\mathcal{J}}\, \mathcal{F} \\ &= lim_{g(y)\to z} \left(lim_{f(x)\to y}\, \mathcal{F}(x) \right) \\ &= ({}_pg \circ {}_pf\mathcal{F})(x) \end{aligned}$$

by Categories, Lemma 4.13.8. We omit the verification that this is compatible with restriction maps. An alternative to this direct construction is to define ${}_p(g \circ f) \cong {}_pg \circ {}_pf$ as the unique map compatible with the adjointness properties. This also has the advantage that one does not need to prove the compatibility.

[1]These functors will be denoted f^{-1} and f_* after Lemma 58.4.4 has been proved.

Compatibility with adjointness of $(f^p, {}_pf)$, $(g^p, {}_pg)$, and $((g \circ f)^p, {}_p(g \circ f))$ means that given presheaves $\mathcal{H}$ and $\mathcal{F}$ as above we have a commutative diagram

$$\begin{tikzcd} Mor_{PSh(\mathcal{X})}(f^pg^p\mathcal{H}, \mathcal{F}) \arrow[r, equal] \arrow[d, equal] & Mor_{PSh(\mathcal{Y})}(g^p\mathcal{H}, {}_pf\mathcal{F}) \arrow[r, equal] & Mor_{PSh(\mathcal{Y})}(\mathcal{H}, {}_pg_pf\mathcal{F}) \\ Mor_{PSh(\mathcal{X})}((g \circ f)^p\mathcal{G}, \mathcal{F}) \arrow[rr, equal] & & Mor_{PSh(\mathcal{Y})}(\mathcal{G}, {}_p(g \circ f)\mathcal{F}) \arrow[u] \end{tikzcd}$$

Proof omitted. $\square$

Lemma 58.3.3. *Let $f, g : \mathcal{X} \to \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $t : f \to g$ be a 2-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assigned to t there are canonical isomorphisms of functors*

$$t^p : g^p \longrightarrow f^p \quad \text{and} \quad {}_pt : {}_pf \longrightarrow {}_pg$$

which compatible with adjointness of $(f^p, {}_pf)$ and $(g^p, {}_pg)$ and with vertical and horizontal composition of 2-morphisms.

Proof. Let $\mathcal{G}$ be a presheaf on $\mathcal{Y}$. Then $t^p : g^p\mathcal{G} \to f^p\mathcal{G}$ is given by the family of maps

$$g^p\mathcal{G}(x) = \mathcal{G}(g(x)) \xrightarrow{\mathcal{G}(t_x)} \mathcal{G}(f(x)) = f^p\mathcal{G}(x)$$

parametrized by $x \in Ob(\mathcal{X})$. This makes sense as $t_x : f(x) \to g(x)$ and $\mathcal{G}$ is a contravariant functor. We omit the verification that this is compatible with restriction mappings.

To define the transformation ${}_pt$ for $y \in Ob(\mathcal{Y})$ define ${}^f_y\mathcal{I}$, resp. ${}^g_y\mathcal{I}$ to be the category of pairs $(x, \psi : f(x) \to y)$, resp. $(x, \psi : g(x) \to y)$, see Sites, Section 9.17. Note that t defines a functor ${}_yt : {}^g_y\mathcal{I} \to {}^f_y\mathcal{I}$ given by the rule

$$(x, g(x) \to y) \longmapsto (x, f(x) \xrightarrow{t_x} g(x) \to y).$$

Note that for $\mathcal{F}$ a presheaf on $\mathcal{X}$ the composition of ${}_yt$ with $\mathcal{F} : {}^f_y\mathcal{I}^{opp} \to Sets$, $(x, f(x) \to y) \mapsto \mathcal{F}(x)$ is equal to $\mathcal{F} : {}^g_y\mathcal{I}^{opp} \to Sets$. Hence by Categories, Lemma 4.13.8 we get for every $y \in Ob(\mathcal{Y})$ a canonical map

$$({}_pf\mathcal{F})(y) = lim_{{}^f_y\mathcal{I}}\mathcal{F} \longrightarrow lim_{{}^g_y\mathcal{I}}\mathcal{F} = ({}_pg\mathcal{F})(y)$$

We omit the verification that this is compatible with restriction mappings. An alternative to this direct construction is to define ${}_pt$ as the unique map compatible with the adjointness properties of the pairs $(f^p, {}_pf)$ and $(g^p, {}_pg)$ (see below). This also has the advantage that one does not need to prove the compatibility.

Compatibility with adjointness of $(f^p, {}_pf)$ and $(g^p, {}_pg)$ means that given presheaves $\mathcal{G}$ and $\mathcal{F}$ as above we have a commutative diagram

$$\begin{tikzcd} Mor_{PSh(\mathcal{X})}(f^p\mathcal{G}, \mathcal{F}) \arrow[r, equal] \arrow[d, "- \circ t^p"'] & Mor_{PSh(\mathcal{Y})}(\mathcal{G}, {}_pf\mathcal{F}) \arrow[d, "{}_pt \circ -"] \\ Mor_{PSh(\mathcal{X})}(g^p\mathcal{G}, \mathcal{F}) \arrow[r, equal] & Mor_{PSh(\mathcal{Y})}(\mathcal{G}, {}_pg\mathcal{F}) \end{tikzcd}$$

Proof omitted. Hint: Work through the proof of Sites, Lemma 9.17.2 and observe the compatibility from the explicit description of the horizontal and vertical maps in the diagram.

We omit the verification that this is compatible with vertical and horizontal compositions. Hint: The proof of this for t^p is straightforward and one can conclude that this holds for the ${}_pt$ maps using compatibility with adjointness. □

58.4. Sheaves

We first make an observation that is important and trivial (especially for those readers who do not worry about set theoretical issues).

Consider a big fppf site Sch_{fppf} as in Topologies, Definition 30.7.6 and denote its underlying category Sch_α. Besides being the underlying category of a fppf site, the category Sch_α can also can serve as the underlying category for a big Zariski site, a big étale site, a big smooth site, and a big syntomic site, see Topologies, Remark 30.9.1. We denote these sites Sch_{Zar}, $Sch_{\acute{e}tale}$, Sch_{smooth}, and $Sch_{syntomic}$. In this situation, since we have defined the big Zariski site $(Sch/S)_{Zar}$ of S, the big étale site $(Sch/S)_{\acute{e}tale}$ of S, the big smooth site $(Sch/S)_{smooth}$ of S, the big syntomic site $(Sch/S)_{syntomic}$ of S, and the big fppf site $(Sch/S)_{fppf}$ of S as the localizations (see Sites, Section 9.21) Sch_{Zar}/S, $Sch_{\acute{e}tale}/S$, Sch_{smooth}/S, $Sch_{syntomic}/S$, and Sch_{fppf}/S of these (absolute) big sites we see that all of these have the same underlying category, namely Sch_α/S.

It follows that if we have a category $p : \mathcal{X} \to (Sch/S)_{fppf}$ fibred in groupoids, then $\mathcal{X}$ inherits a Zariski, étale, smooth, syntomic, and fppf topology, see Stacks, Definition 50.10.2.

Definition 58.4.1. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$.

(1) The *associated Zariski site*, denoted $\mathcal{X}_{Zar}$, is the structure of site on $\mathcal{X}$ inherited from $(Sch/S)_{Zar}$.
(2) The *associated étale site*, denoted $\mathcal{X}_{\acute{e}tale}$, is the structure of site on $\mathcal{X}$ inherited from $(Sch/S)_{\acute{e}tale}$.
(3) The *associated smooth site*, denoted $\mathcal{X}_{smooth}$, is the structure of site on $\mathcal{X}$ inherited from $(Sch/S)_{smooth}$.
(4) The *associated syntomic site*, denoted $\mathcal{X}_{syntomic}$, is the structure of site on $\mathcal{X}$ inherited from $(Sch/S)_{syntomic}$.
(5) The *associated fppf site*, denoted $\mathcal{X}_{fppf}$, is the structure of site on $\mathcal{X}$ inherited from $(Sch/S)_{fppf}$.

This definition makes sense by the discussion above. If $\mathcal{X}$ is an algebraic stack, the literature calls $\mathcal{X}_{fppf}$ (or a site equivalent to it) the *big fppf site* of $\mathcal{X}$ and similarly for the other ones. We may occasionally use this terminology to distinguish this construction from others.

Remark 58.4.2. We only use this notation when the symbol $\mathcal{X}$ refers to a category fibred in groupoids, and not a scheme, an algebraic space, etc. In this way we will avoid confusion with the small étale site of a scheme, or algebraic space which is denoted $X_{\acute{e}tale}$ (in which case we use a roman capital instead of a calligraphic one).

Now that we have these topologies defined we can say what it means to have a sheaf on $\mathcal{X}$, i.e., define the corresponding topoi.

Definition 58.4.3. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Let $\mathcal{F}$ be a presheaf on $\mathcal{X}$.

(1) We say $\mathcal{F}$ is a *Zariski sheaf*, or a *sheaf for the Zariski topology* if $\mathcal{F}$ is a sheaf on the associated Zariski site $\mathcal{X}_{Zar}$.
(2) We say $\mathcal{F}$ is an *étale sheaf*, or a *sheaf for the étale topology* if $\mathcal{F}$ is a sheaf on the associated étale site $\mathcal{X}_{\acute{e}tale}$.

(3) We say $\mathcal{F}$ is a *smooth sheaf*, or a *sheaf for the smooth topology* if $\mathcal{F}$ is a sheaf on the associated smooth site $\mathcal{X}_{smooth}$.
(4) We say $\mathcal{F}$ is a *syntomic sheaf*, or a *sheaf for the syntomic topology* if $\mathcal{F}$ is a sheaf on the associated syntomic site $\mathcal{X}_{syntomic}$.
(5) We say $\mathcal{F}$ is an *fppf sheaf*, or a *sheaf*, or a *sheaf for the fppf topology* if $\mathcal{F}$ is a sheaf on the associated fppf site $\mathcal{X}_{fppf}$.

A morphism of sheaves is just a morphism of presheaves. We denote these categories of sheaves $Sh(\mathcal{X}_{Zar})$, $Sh(\mathcal{X}_{étale})$, $Sh(\mathcal{X}_{smooth})$, $Sh(\mathcal{X}_{syntomic})$, and $Sh(\mathcal{X}_{fppf})$.

Of course we can also talk about sheaves of pointed sets, abelian groups, groups, monoids, rings, modules over a fixed ring, and lie algebras over a fixed field, etc. The category of *abelian sheaves*, i.e., sheaves of abelian groups, is denoted $Ab(\mathcal{X}_{fppf})$ and similarly for the other topologies. If $\mathcal{X}$ is an algebraic stack, then $Sh(\mathcal{X}_{fppf})$ is equivalent (modulo set theoretical problems) to what in the literature would be termed the *category of sheaves on the big fppf site of $\mathcal{X}$*. Similar for other topologies. We may occasionally use this terminology to distinguish this construction from others.

Since the topologies are listed in increasing order of strength we have the following strictly full inclusions

$$Sh(\mathcal{X}_{fppf}) \subset Sh(\mathcal{X}_{syntomic}) \subset Sh(\mathcal{X}_{smooth}) \subset Sh(\mathcal{X}_{étale}) \subset Sh(\mathcal{X}_{Zar}) \subset PSh(\mathcal{X})$$

We sometimes write $Sh(\mathcal{X}_{fppf}) = Sh(\mathcal{X})$ and $Ab(\mathcal{X}_{fppf}) = Ab(\mathcal{X})$ in accordance with our terminology that a sheaf on $\mathcal{X}$ is an fppf sheaf on $\mathcal{X}$.

With this setup functoriality of these topoi is straightforward, and moreover, is compatible with the inclusion functors above.

Lemma 58.4.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. The functors ${}_pf$ and f^p of (58.3.1.1) transform τ sheaves into τ sheaves and define a morphism of topoi $f : Sh(\mathcal{X}_\tau) \to Sh(\mathcal{Y}_\tau)$.*

Proof. This follows immediately from Stacks, Lemma 50.10.3. □

In other words, pushforward and pullback of presheaves as defined in Section 58.3 also produces *pushforward* and *pullback* of τ-sheaves. Having said all of the above we see that we can write $f^p = f^{-1}$ and ${}_pf = f_*$ without any possibility of confusion.

Definition 58.4.5. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. We denote

$$f = (f^{-1}, f_*) : Sh(\mathcal{X}_{fppf}) \longrightarrow Sh(\mathcal{Y}_{fppf})$$

the *associated morphism of fppf topoi* constructed above. Similarly for the associated Zariski, étale, smooth, and syntomic topoi.

As discussed in Sites, Section 9.38 the same formula (on the underlying sheaf of sets) defines pushforward and pullback for sheaves (for one of our topologies) of pointed sets, abelian groups, groups, monoids, rings, modules over a fixed ring, and lie algebras over a fixed field, etc.

58.5. Computing pushforward

Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\mathcal{F}$ be a presheaf on $\mathcal{X}$. Let $y \in Ob(\mathcal{Y})$. We can compute $f_*\mathcal{F}(y)$ in the following way. Suppose that y lies over the scheme V and using the 2-Yoneda lemma think of y as a 1-morphism. Consider the projection

$$\text{pr} : (Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X} \longrightarrow \mathcal{X}$$

Then we have a canonical identification

$$f_*\mathcal{F}(y) = \Gamma\Big((Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X},\ \text{pr}^{-1}\mathcal{F}\Big) \tag{58.5.0.1}$$

Namely, objects of the 2-fibre product are triples $(h : U \to V, x, f(x) \to h^*y)$. Dropping the h from the notation we see that this is equivalent to the data of an object x of $\mathcal{X}$ and a morphism $\alpha : f(x) \to y$ of $\mathcal{Y}$. Since $f_*\mathcal{F}(y) = lim_{f(x)\to y}\mathcal{F}(x)$ by definition the equality follows.

As a consequence we have the following ``base change'' result for pushforwards. This result is trivial and hinges on the fact that we are using ``big'' sites.

Lemma 58.5.1. *Let S be a scheme. Let*

$$\begin{tikzcd} \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} \arrow[r, "g'"'] \arrow[d, "f'"'] & \mathcal{X} \arrow[d, "f"] \\ \mathcal{Y}' \arrow[r, "g"] & \mathcal{Y} \end{tikzcd}$$

be a 2-cartesian diagram of categories fibred in groupoids over S. Then we have a canonical isomorphism

$$g^{-1}f_*\mathcal{F} \longrightarrow f'_*(g')^{-1}\mathcal{F}$$

functorial in the presheaf $\mathcal{F}$ on $\mathcal{X}$.

Proof. Given an object y' of $\mathcal{Y}'$ over V there is an equivalence

$$(Sch/V)_{fppf} \times_{g(y'),\mathcal{Y}} \mathcal{X} = (Sch/V)_{fppf} \times_{y',\mathcal{Y}'} (\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X})$$

Hence by (58.5.0.1) a bijection $g^{-1}f_*\mathcal{F}(y') \to f'_*(g')^{-1}\mathcal{F}(y')$. We omit the verification that this is compatible with restriction mappings. $\square$

In the case of a representable morphism of categories fibred in groupoids this formula (58.5.0.1) simplifies. We suggest the reader skip the rest of this section.

Lemma 58.5.2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. The following are equivalent*

1. *f is representable, and*
2. *for every $y \in Ob(\mathcal{Y})$ the functor $\mathcal{X}^{opp} \to Sets$, $x \mapsto Mor_{\mathcal{Y}}(f(x), y)$ is representable.*

Proof. According to the discussion in Algebraic Stacks, Section 57.6 we see that f is representable if and only if for every $y \in Ob(\mathcal{Y})$ lying over U the 2-fibre product $(Sch/U)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$ is representable, i.e., of the form $(Sch/V_y)_{fppf}$ for some scheme V_y over U. Objects in this 2-fibre products are triples $(h : V \to U, x, \alpha : f(x) \to h^*y)$ where α lies over id_V. Dropping the h from the notation we see that this is equivalent to the data of an object x of $\mathcal{X}$ and a morphism $f(x) \to y$. Hence the 2-fibre product is representable by V_y and $f(x_y) \to y$ where x_y is an object of $\mathcal{X}$ over V_y if and only if the functor in (2) is representable by x_y with universal object a map $f(x_y) \to y$. $\square$

Let

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathcal{Y} \arrow[dl, "q"] \\ & (Sch/S)_{fppf} & \end{tikzcd}$$

be a 1-morphism of categories fibred in groupoids. Assume f is representable. For every $y \in Ob(\mathcal{Y})$ we choose an object $u(y) \in Ob(\mathcal{X})$ representing the functor $x \mapsto Mor_{\mathcal{Y}}(f(x), y)$ of Lemma 58.5.2 (this is possible by the axiom of choice). The objects come with canonical morphisms $f(u(y)) \to y$ by construction. For every morphism $\beta : y' \to y$ in $\mathcal{Y}$ we obtain a unique morphism $u(\beta) : u(y') \to u(y)$ in $\mathcal{X}$ such that the diagram

$$\begin{tikzcd} f(u(y')) \arrow[r, "f(u(\beta))"] \arrow[d] & f(u(y)) \arrow[d] \\ y' \arrow[r] & y \end{tikzcd}$$

commutes. In other words, $u : \mathcal{Y} \to \mathcal{X}$ is a functor. In fact, we can say a little bit more. Namely, suppose that $V' = q(y')$, $V = q(y)$, $U' = p(u(y'))$ and $U = p(u(y))$. Then

$$\begin{tikzcd} U' \arrow[r, "p(u(\beta))"] \arrow[d] & U \arrow[d] \\ V' \arrow[r, "q(\beta)"] & V \end{tikzcd}$$

is a fibre product square. This is true because $U' \to U$ represents the base change $(Sch/V')_{fppf} \times_{y', \mathcal{Y}} \mathcal{X} \to (Sch/V)_{fppf} \times_{y, \mathcal{Y}} \mathcal{X}$ of $V' \to V$.

Lemma 58.5.3. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a representable 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. Then the functor $u : \mathcal{Y}_\tau \to \mathcal{X}_\tau$ is continuous and defines a morphism of sites $\mathcal{X}_\tau \to \mathcal{Y}_\tau$ which induces the same morphism of topoi $Sh(\mathcal{X}_\tau) \to Sh(\mathcal{Y}_\tau)$ as the morphism f constructed in Lemma 58.4.4. Moreover, $f_*\mathcal{F}(y) = \mathcal{F}(u(y))$ for any presheaf $\mathcal{F}$ on $\mathcal{X}$.*

Proof. Let $\{y_i \to y\}$ be a τ-covering in $\mathcal{Y}$. By definition this simply means that $\{q(y_i) \to q(y)\}$ is a τ-covering of schemes. By the final remark above the lemma we see that $\{p(u(y_i)) \to p(u(y))\}$ is the base change of the τ-covering $\{q(y_i) \to q(y)\}$ by $p(u(y)) \to q(y)$, hence is itself a τ-covering by the axioms of a site. Hence $\{u(y_i) \to u(y)\}$ is a τ-covering of $\mathcal{X}$. This proves that u is continuous.

Let's use the notation u_p, u_s, u^p, u^s of Sites, Sections 9.5 and 9.13. If we can show the final assertion of the lemma, then we see that $f_* = u^p = u^s$ (by continuity of u seen above) and hence by adjointness $f^{-1} = u_s$ which will prove u_s is exact, hence that u determines a morphism of sites, and the equality will be clear as well. To see that $f_*\mathcal{F}(y) = \mathcal{F}(u(y))$ note that by definition

$$f_*\mathcal{F}(y) = ({}_pf\mathcal{F})(y) = lim_{f(x) \to y} \mathcal{F}(x).$$

Since $u(y)$ is a final object in the category the limit is taken over we conclude. $\square$

58.6. The structure sheaf

Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. The 2-category of categories fibred in groupoids over $(Sch/S)_{fppf}$ has a final object, namely, $\text{id} : (Sch/S)_{fppf} \to (Sch/S)_{fppf}$ and p is a 1-morphism from $\mathcal{X}$ to this final object. Hence any presheaf $\mathcal{G}$ on $(Sch/S)_{fppf}$ gives a presheaf $p^{-1}\mathcal{G}$ on $\mathcal{X}$ defined by the rule $p^{-1}\mathcal{G}(x) = \mathcal{G}(p(x))$. Moreover, the discussion in Section 58.4 shows that $p^{-1}\mathcal{G}$ is a τ sheaf whenever $\mathcal{G}$ is a τ-sheaf.

Recall that the site $(Sch/S)_{fppf}$ is a ringed site with structure sheaf $\mathcal{O}$ defined by the rule

$$(Sch/S)^{opp} \longrightarrow Rings, \quad U/S \longmapsto \Gamma(U, \mathcal{O}_U)$$

see Descent, Definition 31.6.2.

Definition 58.6.1. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. The *structure sheaf of $\mathcal{X}$* is the sheaf of rings $\mathcal{O}_\mathcal{X} = p^{-1}\mathcal{O}$.

For an object x of $\mathcal{X}$ lying over U we have $\mathcal{O}_\mathcal{X}(x) = \mathcal{O}(U) = \Gamma(U, \mathcal{O}_U)$. Needless to say $\mathcal{O}_\mathcal{X}$ is also a Zariski, étale, smooth, and syntomic sheaf, and hence each of the sites $\mathcal{X}_{Zar}$, $\mathcal{X}_{étale}$, $\mathcal{X}_{smooth}$, $\mathcal{X}_{syntomic}$, and $\mathcal{X}_{fppf}$ is a ringed site. This construction is functorial as well.

Lemma 58.6.2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. There is a canonical identification $f^{-1}\mathcal{O}_\mathcal{X} = \mathcal{O}_\mathcal{Y}$ which turns $f : Sh(\mathcal{X}_\tau) \to Sh(\mathcal{Y}_\tau)$ into a morphism of ringed topoi.*

Proof. Denote $p : \mathcal{X} \to (Sch/S)_{fppf}$ and $q : \mathcal{Y} \to (Sch/S)_{fppf}$ the structural functors. Then $q = p \circ f$, hence $q^{-1} = f^{-1} \circ p^{-1}$ by Lemma 58.3.2. The result follows. □

Remark 58.6.3. In the situation of Lemma 58.6.2 the morphism of ringed topoi $f : Sh(\mathcal{X}_\tau) \to Sh(\mathcal{Y}_\tau)$ is flat as is clear from the equality $f^{-1}\mathcal{O}_\mathcal{X} = \mathcal{O}_\mathcal{Y}$. This is a bit counter intuitive, for example because a closed immersion of algebraic stacks is typically not flat (as a morphism of algebraic stacks). However, exactly the same thing happens when taking a closed immersion $i : X \to Y$ of schemes: in this case the associated morphism of big τ-sites $i : (Sch/X)_\tau \to (Sch/Y)_\tau$ also is flat.

58.7. Sheaves of modules

Since we have a structure sheaf we have modules.

Definition 58.7.1. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$.

(1) A *presheaf of modules on $\mathcal{X}$* is a presheaf of $\mathcal{O}_\mathcal{X}$-modules. The category of presheaves of modules is denoted $PMod(\mathcal{O}_\mathcal{X})$.
(2) We say a presheaf of modules $\mathcal{F}$ is an *$\mathcal{O}_\mathcal{X}$-module*, or more precisely a *sheaf of $\mathcal{O}_\mathcal{X}$-modules* if $\mathcal{F}$ is an fppf sheaf. The category of $\mathcal{O}_\mathcal{X}$-modules is denoted $Mod(\mathcal{O}_\mathcal{X})$.

These (pre)sheaves of modules occur in the literature as *(pre)sheaves of $\mathcal{O}_\mathcal{X}$-modules on the big fppf site of $\mathcal{X}$*. We will occasionally use this terminology if we want to distinguish these categories from others. We will also encounter presheaves of modules which are sheaves in the Zariski, étale, smooth, or syntomic topologies (without necessarily being sheaves). If need be these will be denoted $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ and similarly for the other topologies.

Next, we address functoriality -- first for presheaves of modules. Let

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathcal{Y} \arrow[dl, "q"] \\ & (Sch/S)_{fppf} & \end{tikzcd}$$

be a 1-morphism of categories fibred in groupoids. The functors f^{-1}, f_* on abelian presheaves extend to functors

$$(58.7.1.1) \quad f^{-1} : PMod(\mathcal{O}_\mathcal{Y}) \longrightarrow PMod(\mathcal{O}_\mathcal{X}) \quad \text{and} \quad f_* : PMod(\mathcal{O}_\mathcal{Y}) \longrightarrow PMod(\mathcal{O}_\mathcal{X})$$

This is immediate for f^{-1} because $f^{-1}\mathcal{G}(x) = \mathcal{G}(f(x))$ which is a module over $\mathcal{O}_\mathcal{Y}(f(x)) = \mathcal{O}(q(f(x))) = \mathcal{O}(p(x)) = \mathcal{O}_\mathcal{X}(x)$. Afternatively it follows because $f^{-1}\mathcal{O}_\mathcal{Y} = \mathcal{O}_\mathcal{X}$ and because f^{-1} commutes with limits (on presheaves). Since f_* is a right adjoint it commutes with all limits (on presheaves) in particular products. Hence we can extend f_* to a functor on presheaves of modules as in the proof of Modules on Sites, Lemma 16.12.1. We claim that the functors (58.7.1.1) form an adjoint pair of functors:

$$Mor_{PMod(\mathcal{O}_\mathcal{X})}(f^{-1}\mathcal{G}, \mathcal{F}) = Mor_{PMod(\mathcal{O}_\mathcal{Y})}(\mathcal{G}, f_*\mathcal{F}).$$

As $f^{-1}\mathcal{O}_\mathcal{Y} = \mathcal{O}_\mathcal{X}$ this follows from Modules on Sites, Lemma 16.12.3 by endowing $\mathcal{X}$ and $\mathcal{Y}$ with the chaotic topology.

Next, we discuss functoriality for modules, i.e., for sheaves of modules in the fppf topology. Denote by f also the induced morphism of ringed topoi, see Lemma 58.6.2 (for the fppf toplogies right now). Note that the functors f^{-1} and f_* of (58.7.1.1) preserve the subcategories of sheaves of modules, see Lemma 58.4.4. Hence it follows immediately that

$$(58.7.1.2) \quad f^{-1} : Mod(\mathcal{O}_\mathcal{Y}) \longrightarrow Mod(\mathcal{O}_\mathcal{X}) \quad \text{and} \quad f_* : Mod(\mathcal{O}_\mathcal{Y}) \longrightarrow Mod(\mathcal{O}_\mathcal{X})$$

form an adjoint pair of functors:

$$Mor_{Mod(\mathcal{O}_\mathcal{X})}(f^{-1}\mathcal{G}, \mathcal{F}) = Mor_{Mod(\mathcal{O}_\mathcal{Y})}(\mathcal{G}, f_*\mathcal{F}).$$

By uniqueness of adjoints we conclude that $f^* = f^{-1}$ where f^* is as defined in Modules on Sites, Section 16.13 for the morphism of ringed topoi f above. Of course we could have seen this directly because $f^*(-) = f^{-1}(-) \otimes_{f^{-1}\mathcal{O}_\mathcal{Y}} \mathcal{O}_\mathcal{X}$ and because $f^{-1}\mathcal{O}_\mathcal{Y} = \mathcal{O}_\mathcal{X}$.

Similarly for sheaves of modules in the Zariski, étale, smooth, syntomic topology.

58.8. Representable categories

In this short section we compare our definitions with what happens in case the algebraic stacks in question are representable.

Lemma 58.8.1. *Let S be a scheme. Let $\mathcal{X}$ be a category fibred in groupoids over (Sch/S). Assume $\mathcal{X}$ is representable by a scheme X. For $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$ there is a canonical equivalence*

$$(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X}) = ((Sch/X)_\tau, \mathcal{O}_X)$$

of ringed sites.

Proof. This follows by choosing an equivalence $(Sch/X)_\tau \to \mathcal{X}$ of categories fibred in groupoids over $(Sch/S)_{fppf}$ and using the functoriality of the construction $\mathcal{X} \rightsquigarrow \mathcal{X}_\tau$. $\square$

Lemma 58.8.2. *Let S be a scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of categories fibred in groupoids over S. Assume $\mathcal{X}$, $\mathcal{Y}$ are representable by schemes X, Y. Let $f : X \to Y$ be the morphism of schemes corresponding to f. For $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$ the morphism of ringed topoi $f : (Sh(\mathcal{X}_\tau), \mathcal{O}_\mathcal{X}) \to (Sh(\mathcal{X}_\tau), \mathcal{O}_\mathcal{X})$ agrees with the morphisms of ringed topoi $f : (Sh((Sch/X)_\tau), \mathcal{O}_X) \to (Sh((Sch/Y)_\tau), \mathcal{O}_Y)$ via the identifications of Lemma 58.8.1.*

Proof. Follows by unwinding the definitions. □

58.9. Restriction

A trivial but useful observation is that the localization of a category fibred in groupoids at an object is equivalent to the big site of the scheme it lies over.

Lemma 58.9.1. *Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. Let $x \in Ob(\mathcal{X})$ lying over $U = p(x)$. The functor p induces an equivalence of sites $\mathcal{X}_\tau/x \to (Sch/U)_\tau$.*

Proof. Note that $(Sch/U)_\tau$ is the localization of the site $(Sch/S)_{fppf}$ at the object U. It follows from Categories, Definition 4.32.1 that the rule $x'/x \mapsto p(x')/p(x)$ defines an equivalence of categories $\mathcal{X}_\tau/x \to (Sch/U)_\tau$. Whereupon it follows from Stacks, Definition 50.10.2 that coverings of x' in $\mathcal{X}_\tau/x$ are in bijective correspondence with coverings of $p(x')$ in $(Sch/U)_\tau$. □

We use the lemma above to talk about the pullback and the restriction of a (pre)sheaf to a scheme.

Definition 58.9.2. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $x \in Ob(\mathcal{X})$ lying over $U = p(x)$. Let $\mathcal{F}$ be a presheaf on $\mathcal{X}$.

(1) The *pullback $x^{-1}\mathcal{F}$ of $\mathcal{F}$* is the restriction $\mathcal{F}|_{(\mathcal{X}/x)}$ viewed as a presheaf on $(Sch/U)_{fppf}$ via the equivalence $\mathcal{X}/x \to (Sch/U)_{fppf}$ of Lemma 58.9.1.
(2) The *restriction of $\mathcal{F}$ to $U_{étale}$* is $x^{-1}\mathcal{F}|_{U_{étale}}$, abusively written $\mathcal{F}|_{U_{étale}}$.

This notation makes sense because to the object x the 2-Yoneda lemma, see Algebraic Stacks, Section 57.5 associates a 1-morphism $x : (Sch/U)_{fppf} \to \mathcal{X}/x$ which is quasi-inverse to $p : \mathcal{X}/x \to (Sch/U)_{fppf}$. Hence $x^{-1}\mathcal{F}$ truly is the pullback of $\mathcal{F}$ via this 1-morphism. In particular, by the material above, if $\mathcal{F}$ is a sheaf (or a Zariski, étale, smooth, syntomic sheaf), then $x^{-1}\mathcal{F}$ is a sheaf on $(Sch/U)_{fppf}$ (or on $(Sch/U)_{Zar}$, $(Sch/U)_{étale}$, $(Sch/U)_{smooth}$, $(Sch/U)_{syntomic}$).

Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\varphi : x \to y$ be a morphism of $\mathcal{X}$ lying over the morphism of schemes $a : U \to V$. Recall that a induces a morphism of small étale sites $a_{small} : U_{étale} \to V_{étale}$, see Étale Cohomology, Section 38.34. Let $\mathcal{F}$ be a presheaf on $\mathcal{X}$. Let $\mathcal{F}|_{U_{étale}}$ and $\mathcal{F}|_{V_{étale}}$ be the restrictions of $\mathcal{F}$ via x and y. There is a natural *comparison* map

$$c_\varphi : \mathcal{F}|_{V_{étale}} \longrightarrow a_{small,*}(\mathcal{F}|_{U_{étale}}) \tag{58.9.2.1}$$

of presheaves on $U_{\acute{e}tale}$. Namely, if $V' \to V$ is étale, set $U' = V' \times_V U$ and define c_φ on sections over V' via

$$\begin{tikzcd} a_{small,*}(\mathcal{F}|_{U_{\acute{e}tale}})(V') \arrow[r, equal] & \mathcal{F}|_{U_{\acute{e}tale}}(U') \arrow[r, equal] & \mathcal{F}(x') \\ \mathcal{F}|_{V_{\acute{e}tale}}(V') \arrow[u, "c_\varphi"] \arrow[rr, equal] & & \mathcal{F}(y') \arrow[u, "\mathcal{F}(\varphi')"'] \end{tikzcd}$$

Here $\varphi' : x' \to y'$ is a morphism of $\mathcal{X}$ fitting into a commutative diagram

$$\begin{tikzcd} x' \arrow[r] \arrow[d, "\varphi'"'] & x \arrow[d, "\varphi"] \\ y' \arrow[r] & y \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} U' \arrow[r] \arrow[d] & U \arrow[d, "a"] \\ V' \arrow[r] & V \end{tikzcd}$$

The existence and uniqueness of φ' follow from the axioms of a category fibred in groupoids. We omit the verification that c_φ so defined is indeed a map of presheaves (i.e., compatible with restriction mappings) and that it is functorial in $\mathcal{F}$. In case $\mathcal{F}$ is a sheaf for the étale topology we obtain a *comparison* map

$$c_\varphi : a_{small}^{-1}(\mathcal{F}|_{V_{\acute{e}tale}}) \longrightarrow \mathcal{F}|_{U_{\acute{e}tale}} \tag{58.9.2.2}$$

which is also denoted c_φ as indicated (this is the customary abuse of notation in not distinguishing between adjoint maps).

Lemma 58.9.3. *Let $\mathcal{F}$ be an étale sheaf on $\mathcal{X} \to (Sch/S)_{fppf}$.*

(1) *If $\varphi : x \to y$ and $\psi : y \to z$ are morphisms of $\mathcal{X}$ lying over $a : U \to V$ and $b : V \to W$, then the composition*
$$a_{small}^{-1}(b_{small}^{-1}(\mathcal{F}|_{W_{\acute{e}tale}})) \xrightarrow{a_{small}^{-1}c_\psi} a_{small}^{-1}(\mathcal{F}|_{V_{\acute{e}tale}}) \xrightarrow{c_\varphi} \mathcal{F}|_{U_{\acute{e}tale}}$$
is equal to $c_{\psi\circ\varphi}$ via the identification
$$(b \circ a)_{small}^{-1}(\mathcal{F}|_{W_{\acute{e}tale}}) = a_{small}^{-1}(b_{small}^{-1}(\mathcal{F}|_{W_{\acute{e}tale}})).$$
(2) *If $\varphi : x \to y$ lies over an étale morphism of schemes $a : U \to V$, then (58.9.2.2) is an isomorphism.*
(3) *Suppose $f : \mathcal{Y} \to \mathcal{X}$ is a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$ and y is an object of $\mathcal{Y}$ lying over the scheme U with image $x = f(y)$. Then there is a canonical identification $f^{-1}\mathcal{F}|_{U_{\acute{e}tale}} = \mathcal{F}|_{U_{\acute{e}tale}}$.*
(4) *Moreover, given $\psi : y' \to y$ in $\mathcal{Y}$ lying over $a : U' \to U$ the comparison map $c_\psi : a_{small}^{-1}(F^{-1}\mathcal{F}|_{U_{\acute{e}tale}}) \to F^{-1}\mathcal{F}|_{U'_{\acute{e}tale}}$ is equal to the comparison map $c_{f(\psi)} : a_{small}^{-1}\mathcal{F}|_{U_{\acute{e}tale}} \to \mathcal{F}|_{U'_{\acute{e}tale}}$ via the indentifications in (3).*

Proof. The verification of these properties is omitted. ☐

Next, we turn to the restriction of (pre)sheaves of modules.

Lemma 58.9.4. *Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\tau \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\}$. Let $x \in Ob(\mathcal{X})$ lying over $U = p(x)$. The equivalence of Lemma 58.9.1 extends to an equivalence of ringed sites $(\mathcal{X}_\tau/x, \mathcal{O}_\mathcal{X}|_x) \to ((Sch/U)_\tau, \mathcal{O})$.*

Proof. This is immediate from the construction of the structure sheaves. ☐

Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Let $\mathcal{F}$ be a (pre)sheaf of modules on $\mathcal{X}$ as in Definition 58.7.1. Let x be an object of $\mathcal{X}$ lying over U. Then Lemma 58.9.4 guarantees that the restriction $x^{-1}\mathcal{F}$ is a (pre)sheaf of modules on $(Sch/U)_{fppf}$. We will sometimes write $x^*\mathcal{F} = x^{-1}\mathcal{F}$ in this case. Similarly, if $\mathcal{F}$ is a sheaf for the Zariski, étale, smooth, or syntomic topology, then $x^{-1}\mathcal{F}$ is as well. Moreover, the restriction $\mathcal{F}|_{U_{\acute{e}tale}} = x^{-1}\mathcal{F}|_{U_{\acute{e}tale}}$ to U is a presheaf of $\mathcal{O}_{U_{\acute{e}tale}}$-modules. If $\mathcal{F}$ is a sheaf for the étale topology, then $\mathcal{F}|_{U_{\acute{e}tale}}$ is a sheaf of modules. Moreover, if $\varphi : x \to y$ is a morphism of $\mathcal{X}$ lying over $a : U \to V$ then the comparison map (58.9.2.2) is compatible with $a_{small}^\sharp$ (see Descent, Remark 31.6.4) and induces a *comparison* map

$$c_\varphi : a_{small}^*(\mathcal{F}|_{V_{\acute{e}tale}}) \longrightarrow \mathcal{F}|_{U_{\acute{e}tale}} \tag{58.9.4.1}$$

of $\mathcal{O}_{U_{\acute{e}tale}}$-modules. Note that the properties (1), (2), (3), and (4) of Lemma 58.9.3 hold in the setting of étale sheaves of modules as well. We will use this in the following without further mention.

Lemma 58.9.5. *Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. The site $\mathcal{X}_\tau$ has enough points.*

Proof. By Sites, Lemma 9.34.5 we have to show that there exists a family of objects x of $\mathcal{X}$ such that $\mathcal{X}_\tau/x$ has enough points and such that the sheaves $h_x^\#$ cover the final object of the category of sheaves. By Lemma 58.9.1 and Étale Cohomology, Lemma 38.30.1 we see that $\mathcal{X}_\tau/x$ has enough points for every object x and we win. □

58.10. Restriction to algebraic spaces

In this section we consider sheaves on categories representable by algebraic spaces. The following lemma is the analogue of Topologies, Lemma 30.4.13 for algebraic spaces.

Lemma 58.10.1. *Let S be a scheme. Let $\mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Assume $\mathcal{X}$ is representably by an algebraic space F. Then there exists a continuous and cocontinuous functor $F_{\acute{e}tale} \to \mathcal{X}_{\acute{e}tale}$ which induces a morphism of ringed sites*

$$\pi_F : (\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X}) \longrightarrow (F_{\acute{e}tale}, \mathcal{O}_F)$$

and a morphism of ringed topoi

$$i_F : (Sh(F_{\acute{e}tale}), \mathcal{O}_F) \longrightarrow (Sh(\mathcal{X}_{\acute{e}tale}), \mathcal{O}_\mathcal{X})$$

such that $\pi_F \circ i_F = id$. Moreover $\pi_{F,*} = i_F^{-1}$.

Proof. Choose an equivalence $j : \mathcal{S}_F \to \mathcal{X}$, see Algebraic Stacks, Sections 57.7 and 57.8. An object of $F_{\acute{e}tale}$ is a scheme U together with an étale morphism $\varphi : U \to F$. Then φ is an object of $\mathcal{S}_F$ over U. Hence $j(\varphi)$ is an object of $\mathcal{X}$ over U. In this way j induces a functor $u : F_{\acute{e}tale} \to \mathcal{X}$. It is clear that u is continuous and cocontinuous for the étale topology on $\mathcal{X}$. Since j is an equivalence, the functor u is fully faithful. Also, fibre products and equalizers exist in $F_{\acute{e}tale}$ and u commutes with them because these are computed on the level of underlying schemes in $F_{\acute{e}tale}$. Thus Sites, Lemmas 9.19.5, 9.19.6, and 9.19.7 apply. In particular u defines a morphism of topoi $i_F : Sh(F_{\acute{e}tale}) \to Sh(\mathcal{X}_{\acute{e}tale})$ and there exists a left adjoint $i_{F,!}$ of i_F^{-1} which commutes with fibre products and equalizers.

We claim that $i_{F,!}$ is exact. If this is true, then we can define π_F by the rules $\pi_F^{-1} = i_{F,!}$ and $\pi_{F,*} = i_F^{-1}$ and everything is clear. To prove the claim, note that we already know that $i_{F,!}$ is right exact and preserves fibre products. Hence it suffices to show that $i_{F,!}* = *$ where

$*$ indicates the final object in the category of sheaves of sets. Let U be a scheme and let $\varphi : U \to F$ be surjective and étale. Set $R = U \times_F U$. Then

$$h_R \rightrightarrows h_U \longrightarrow *$$

is a coequalizer diagram in $Sh(F_{\acute{e}tale})$. Using the right exactness of $i_{F,!}$, using $i_{F,!} = (u_p\ )^\#$, and using Sites, Lemma 9.5.6 we see that

$$h_{u(R)} \rightrightarrows h_{u(U)} \longrightarrow i_{F,!}*$$

is a coequalizer diagram in $Sh(F_{\acute{e}tale})$. Using that j is an equivalence and that $F = U/R$ it follows that the coequalizer in $Sh(\mathcal{X}_{\acute{e}tale})$ of the two maps $h_{u(R)} \to h_{u(U)}$ is $*$. We omit the proof that these morphisms are compatible with structure sheaves. □

Assume $\mathcal{X}$ is an algebraic stack represented by the algebraic space F. Let $j : \mathcal{S}_F \to \mathcal{X}$ be an equivalence and denote $u : F_{\acute{e}tale} \to \mathcal{X}_{\acute{e}tale}$ the functor of the proof of Lemma 58.10.1 above. Given a sheaf $\mathcal{F}$ on $\mathcal{X}_{\acute{e}tale}$ we have

$$\pi_{F,*}\mathcal{F}(U) = i_F^{-1}\mathcal{F}(U) = \mathcal{F}(u(U)).$$

This is why we often think of i_F^{-1} as a *restriction functor* similarly to Definition 58.9.2 and to the restriction of a sheaf on the big étale site of a scheme to the small étale site of a scheme. We often use the notation

$$\mathcal{F}|_{F_{\acute{e}tale}} = i_F^{-1}\mathcal{F} = \pi_{F,*}\mathcal{F} \tag{58.10.1.1}$$

in this situation.

Lemma 58.10.2. *Let S be a scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume $\mathcal{X}$, $\mathcal{Y}$ are representable by algebraic spaces F, G. Denote $f : F \to G$ the induced morphism of algebraic spaces, and $f_{small} : F_{\acute{e}tale} \to G_{\acute{e}tale}$ the corresponding morphism of ringed topoi. Then*

$$\begin{tikzcd} (Sh(\mathcal{X}_{\acute{e}tale}), \mathcal{O}_\mathcal{X}) \arrow[r, "f"'] \arrow[d, "\pi_F"'] & (Sh(\mathcal{Y}_{\acute{e}tale}), \mathcal{O}_\mathcal{Y}) \arrow[d, "\pi_G"] \\ (Sh(F_{\acute{e}tale}), \mathcal{O}_F) \arrow[r, "f_{small}"] & (Sh(G_{\acute{e}tale}), \mathcal{O}_G) \end{tikzcd}$$

is a commutative diagram of ringed topoi.

Proof. This is similar to Topologies, Lemma 30.4.16 (3) but there is a small snag due to the fact that $F \to G$ may not be representable by schemes. In particular we don't get a commutative diagram of ringed sites, but only a commutative diagram of ringed topoi.

Before we start the proof proper, we choose equivalences $j : \mathcal{S}_F \to \mathcal{X}$ and $j' : \mathcal{S}_G \to \mathcal{Y}$ which induce functors $u : F_{\acute{e}tale} \to \mathcal{X}$ and $u' : G_{\acute{e}tale} \to \mathcal{Y}$ as in the proof of Lemma 58.10.1. Because of the 2-functoriality of sheaves on categories fibred in groupoids over Sch_{fppf} (see discussion in Section 58.3) we may assume that $\mathcal{X} = \mathcal{S}_F$ and $\mathcal{Y} = \mathcal{S}_G$ and that $f : \mathcal{S}_F \to \mathcal{S}_G$ is the functor associated to the morphism $f : F \to G$. Correspondingly we will omit u and u' from the notation, i.e., given an object $U \to F$ of $F_{\acute{e}tale}$ we denote U/F the corresponding object of $\mathcal{X}$. Similarly for G.

Let $\mathcal{G}$ be a sheaf on $\mathcal{X}_{\acute{e}tale}$. To prove (2) we compute $\pi_{G,*}f_*\mathcal{G}$ and $f_{small,*}\pi_{F,*}\mathcal{G}$. To do this let $V \to G$ be an object of $G_{\acute{e}tale}$. Then

$$\pi_{G,*}f_*\mathcal{G}(V) = f_*\mathcal{G}(V/G) = \Gamma\Big((Sch/V)_{fppf} \times_\mathcal{Y} \mathcal{X},\ \mathrm{pr}^{-1}\mathcal{G}\Big)$$

see (58.5.0.1). The fibre product in the formula is

$$(Sch/V)_{fppf} \times_{\mathcal{Y}} \mathcal{X} = (Sch/V)_{fppf} \times_{\mathcal{S}_G} \mathcal{S}_F = \mathcal{S}_{V \times_G F}$$

i.e., it is the split category fibred in groupoids associated to the algebraic space $V \times_G F$. And $\mathrm{pr}^{-1}\mathcal{G}$ is a sheaf on $\mathcal{S}_{V \times_G F}$ for the étale topology.

In particular, if $V \times_G F$ is representable, i.e., if it is a scheme, then $\pi_{G,*} f_* \mathcal{G}(V) = \mathcal{G}(V \times_G F/F)$ and also

$$f_{small,*}\pi_{F,*}\mathcal{G}(V) = \pi_{F,*}\mathcal{G}(V \times_G F) = \mathcal{G}(V \times_G F/F)$$

which proves the desired equality in this special case.

In general, choose a scheme U and a surjective étale morphism $U \to V \times_G F$. Set $R = U \times_{V \times_G F} U$. Then $U/V \times_G F$ and $R/V \times_G F$ are objects of the fibre product category above. Since $\mathrm{pr}^{-1}\mathcal{G}$ is a sheaf for the étale topology on $\mathcal{S}_{V \times_G F}$ the diagram

$$\Gamma\Big((Sch/V)_{fppf} \times_{\mathcal{Y}} \mathcal{X},\ \mathrm{pr}^{-1}\mathcal{G}\Big) \longrightarrow \mathrm{pr}^{-1}\mathcal{G}(U/V \times_G F) \rightrightarrows \mathrm{pr}^{-1}\mathcal{G}(R/V \times_G F)$$

is an equalizer diagram. Note that $\mathrm{pr}^{-1}\mathcal{G}(U/V \times_G F) = \mathcal{G}(U/F)$ and $\mathrm{pr}^{-1}\mathcal{G}(R/V \times_G F) = \mathcal{G}(R/F)$ by the definition of pullbacks. Moreover, by the material in Properties of Spaces, Section 41.15 (especially, Properties of Spaces, Remark 41.15.4 and Lemma 41.15.7) we see that there is an equalizer diagram

$$f_{small,*}\pi_{F,*}\mathcal{G}(V) \longrightarrow \pi_{F,*}\mathcal{G}(U/F) \rightrightarrows \pi_{F,*}\mathcal{G}(R/F)$$

Since we also have $\pi_{F,*}\mathcal{G}(U/F) = \mathcal{G}(U/F)$ and $\pi_{F,*}\mathcal{G}(U/F) = \mathcal{G}(U/F)$ we obtain a canonical identification $f_{small,*}\pi_{F,*}\mathcal{G}(V) = \pi_{G,*}f_*\mathcal{G}(V)$. We omit the proof that this is compatible with restriction mappings and that it is functorial in $\mathcal{G}$. $\square$

Let $f : \mathcal{X} \to \mathcal{Y}$ and $f : F \to G$ be as in the second part of the lemma above. A consequence of the lemma, using (58.10.1.1), is that

(58.10.2.1) $$(f_*\mathcal{F})|_{G_{\acute{e}tale}} = f_{small,*}(\mathcal{F}|_{F_{\acute{e}tale}})$$

for any sheaf $\mathcal{F}$ on $\mathcal{X}_{\acute{e}tale}$. Moreover, if $\mathcal{F}$ is a sheaf of $\mathcal{O}$-modules, then (58.10.2.1) is an isomorphism of $\mathcal{O}_G$-modules on $G_{\acute{e}tale}$.

Finally, suppose that we have a 2-commutative diagram

$$\begin{tikzcd} \mathcal{U} \arrow[r, "a"] \arrow[dr, "f"'] & \mathcal{V} \arrow[d, "g"] \arrow[dl, phantom, "\swarrow\varphi"] \\ & \mathcal{X} \end{tikzcd}$$

of 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$, that $\mathcal{F}$ is a sheaf on $\mathcal{X}_{\acute{e}tale}$, and that $\mathcal{U}, \mathcal{V}$ are representable by algebraic spaces U, V. Then we obtain a comparison map

(58.10.2.2) $$c_\varphi : a_{small}^{-1}(g^{-1}\mathcal{F}|_{V_{\acute{e}tale}}) \longrightarrow f^{-1}\mathcal{F}|_{U_{\acute{e}tale}}$$

where $a : U \to V$ denotes the morphism of algebraic spaces corresponding to a. This is the analogue of (58.9.2.2). We define c_φ as the adjoint to the map

$$g^{-1}\mathcal{F}|_{V_{\acute{e}tale}} \longrightarrow a_{small,*}(f^{-1}\mathcal{F}|_{U_{\acute{e}tale}}) = (a_*f^{-1}\mathcal{F})|_{V_{\acute{e}tale}}$$

(equality by (58.10.2.1)) which is the restriction to V (58.10.1.1) of the map

$$g^{-1}\mathcal{F} \to a_*a^{-1}g^{-1}\mathcal{F} = a_*f^{-1}\mathcal{F}$$

where the last equality uses the 2-commutativity of the diagram above. In case $\mathcal{F}$ is a sheaf of $\mathcal{O}_\mathcal{X}$-modules c_φ induces a *comparison* map

$$c_\varphi : a_{small}^*(g^*\mathcal{F}|_{V_{étale}}) \longrightarrow f^*\mathcal{F}|_{U_{étale}} \tag{58.10.2.3}$$

of $\mathcal{O}_{U_{étale}}$-modules. Note that the properties (1), (2), (3), and (4) of Lemma 58.9.3 hold in this setting as well.

58.11. Quasi-coherent modules

At this point we can apply the general definition of a quasi-coherent module to the situation discussed in this chapter.

Definition 58.11.1. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. A *quasi-coherent module on* $\mathcal{X}$, or a *quasi-coherent* $\mathcal{O}_\mathcal{X}$*-module* is a quasi-coherent module on the ringed site $(\mathcal{X}_{fppf}, \mathcal{O}_\mathcal{X})$ as in Modules on Sites, Definition 16.23.1. The category of quasi-coherent sheaves on $\mathcal{X}$ is denoted $QCoh(\mathcal{O}_\mathcal{X})$ or $QCoh(\mathcal{X})$.

If $\mathcal{X}$ is an algebraic stack, then this definition agrees with all definitions in the literature in the sense that $QCoh(\mathcal{X})$ is equivalent (modulo set theoretic issues) to any variant of this category defined in the literature. For example, we will match our definition with the definition in [**Ols07b**, Definition 6.1] in Cohomology on Stacks, Lemma 58.11.5. We will also see alternative constructions of this category later on.

In general (as is the case for morphisms of schemes) the pushforward of quasi-coherent sheaf along a 1-morphism is not quasi-coherent. Pullback does preserve quasi-coherence.

Lemma 58.11.2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. The pullback functor $f^* = f^{-1} : Mod(\mathcal{O}_\mathcal{Y}) \to Mod(\mathcal{O}_\mathcal{X})$ preserves quasi-coherent sheaves.*

Proof. This is a general fact, see Modules on Sites, Lemma 16.23.4. □

It turns out that quasi-coherent sheaves have a very simple characterization in terms of their pullbacks. See also Lemma 58.11.5 for a characterization in terms of restrictions.

Lemma 58.11.3. *Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_\mathcal{X}$-modules. Then $\mathcal{F}$ is quasi-coherent if and only if $x^*\mathcal{F}$ is a quasi-coherent sheaf on $(Sch/U)_{fppf}$ for every object x of $\mathcal{X}$ with $U = p(x)$.*

Proof. By Lemma 58.11.2 the condition is necessary. Conversely, since $x^*\mathcal{F}$ is just the restriction to $\mathcal{X}_{fppf}/x$ we see that it is sufficient directly from the definition of a quasi-coherent sheaf (and the fact that the notion of being quasi-coherent is an intrinsic property of sheaves of modules, see Modules on Sites, Section 16.18). □

Although there is a variant for the Zariski topology, it seems that the étale topology is the natural topology to use in the following definition.

Definition 58.11.4. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_\mathcal{X}$-modules. We say $\mathcal{F}$ is *locally quasi-coherent*[2] if $\mathcal{F}$ is a sheaf for the étale topology and for every object x of $\mathcal{X}$ the restriction $x^*\mathcal{F}|_{U_{étale}}$ is a quasi-coherent sheaf. Here $U = p(x)$.

[2]This is nonstandard notation.

We use $LQCoh(\mathcal{O}_\mathcal{X})$ to indicate the category of locally quasi-coherent modules. We now have the following diagram of categories of modules

$$\begin{tikzcd} QCoh(\mathcal{O}_\mathcal{X}) \arrow[r] \arrow[d] & Mod(\mathcal{O}_\mathcal{X}) \arrow[d] \\ LQCoh(\mathcal{O}_\mathcal{X}) \arrow[r] & Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X}) \end{tikzcd}$$

where the arrows are strictly full embeddings. It turns out that many results for quasi-coherent sheaves have a counter part for locally quasi-coherent modules. Moreover, from many points of view (as we shall see later) this is a natural category to consider. For example the quasi-coherent sheaves are exactly those locally quasi-coherent modules that are ``cartesian", i.e., satisfy the second condition of the lemma below.

Lemma 58.11.5. *Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_\mathcal{X}$-modules. Then $\mathcal{F}$ is quasi-coherent if and only if the following two conditions hold*

(1) *$\mathcal{F}$ is locally quasi-coherent, and*
(2) *for any morphism $\varphi : x \to y$ of $\mathcal{X}$ lying over $f : U \to V$ the comparison map $c_\varphi : f^*_{small}\mathcal{F}|_{V_{étale}} \to \mathcal{F}|_{U_{étale}}$ of (58.9.4.1) is an isomorphism.*

Proof. Assume $\mathcal{F}$ is quasi-coherent. Then $\mathcal{F}$ is a sheaf for the fppf topology, hence a sheaf for the étale topology. Moreover, any pullback of $\mathcal{F}$ to a ringed topos is quasi-coherent, hence the restrictions $x^*\mathcal{F}|_{U_{étale}}$ are quasi-coherent. This proves $\mathcal{F}$ is locally quasi-coherent. Let y be an object of $\mathcal{X}$ with $V = p(y)$. We have seen that $\mathcal{X}/y = (Sch/V)_{fppf}$. By Descent, Proposition 31.6.11 it follows that $y^*\mathcal{F}$ is the quasi-coherent module associated to a (usual) quasi-coherent module $\mathcal{F}_V$ on the scheme V. Hence certainly the comparison maps (58.9.4.1) are isomorphisms.

Conversely, suppose that $\mathcal{F}$ satisfies (1) and (2). Let y be an object of $\mathcal{X}$ with $V = p(y)$. Denote $\mathcal{F}_V$ the quasi-coherent module on the scheme V corresponding to the restriction $y^*\mathcal{F}|_{V_{étale}}$ which is quasi-coherent by assumption (1), see Descent, Proposition 31.6.11. Condition (2) now signifies that the restrictions $x^*\mathcal{F}|_{U_{étale}}$ for x over y are each isomorphic to the (étale sheaf associated to the) pullback of $\mathcal{F}_V$ via the corresponding morphism of schemes $U \to V$. Hence $y^*\mathcal{F}$ is the sheaf on $(Sch/V)_{fppf}$ associated to $\mathcal{F}_V$. Hence it is quasi-coherent (by Descent, Proposition 31.6.11 again) and we see that $\mathcal{F}$ is quasi-coherent on $\mathcal{X}$ by Lemma 58.11.3. $\square$

Lemma 58.11.6. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. The pullback functor $f^* = f^{-1} : Mod(\mathcal{Y}_{étale}, \mathcal{O}_\mathcal{Y}) \to Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ preserves locally quasi-coherent sheaves.*

Proof. Let $\mathcal{G}$ be locally quasi-coherent on $\mathcal{Y}$. Choose an object x of $\mathcal{X}$ lying over the scheme U. The restriction $x^*f^*\mathcal{G}|_{U_{étale}}$ equals $(f \circ x)^*\mathcal{G}|_{U_{étale}}$ hence is a quasi-coherent sheaf by assumption on $\mathcal{G}$. $\square$

Lemma 58.11.7. *Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids.*

(1) *The category $LQCoh(\mathcal{O}_\mathcal{X})$ has colimits and they agree with colimits in the category $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$.*
(2) *The category $LQCoh(\mathcal{O}_\mathcal{X})$ is abelian with kernels and cokernels computed in $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$, in other words the inclusion functor is exact.*

(3) *Given a short exact sequence* $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ *of* $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$ *if two out of three are locally quasi-coherent so is the third.*
(4) *Given* $\mathcal{F}, \mathcal{G}$ *in* $LQCoh(\mathcal{O}_\mathcal{X})$ *the tensor product* $\mathcal{F} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{G}$ *in* $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$ *is an object of* $LQCoh(\mathcal{O}_\mathcal{X})$.
(5) *Given* $\mathcal{F}, \mathcal{G}$ *in* $LQCoh(\mathcal{O}_\mathcal{X})$ *with* $\mathcal{F}$ *locally of finite presentation on* $\mathcal{X}_{\acute{e}tale}$ *the sheaf* $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})$ *in* $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$ *is an object of* $LQCoh(\mathcal{O}_\mathcal{X})$.

Proof. Each of these statements follows from the corresponding statement of Descent, Lemma 31.6.13. For example, suppose that $\mathcal{I} \to LQCoh(\mathcal{O}_\mathcal{X})$, $i \mapsto \mathcal{F}_i$ is a diagram. Consider the object $\mathcal{F} = colim_i \mathcal{F}_i$ of $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$. For any object x of $\mathcal{X}$ with $U = p(x)$ the pullback functor x^* commutes with all colimits as it is a left adjoint. Hence $x^*\mathcal{F} = colim_i x^*\mathcal{F}_i$. Similarly we have $x^*\mathcal{F}|_{U_{\acute{e}tale}} = colim_i x^*\mathcal{F}_i|_{U_{\acute{e}tale}}$. Now by assumption each $x^*\mathcal{F}_i|_{U_{\acute{e}tale}}$ is quasi-coherent, hence the colimit is quasi-coherent by the aforementioned Descent, Lemma 31.6.13. This proves (1).

It follows from (1) that cokernels exist in $LQCoh(\mathcal{O}_\mathcal{X})$ and agree with the cokernels computed in $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of $LQCoh(\mathcal{O}_\mathcal{X})$ and let $\mathcal{K} = \mathrm{Ker}(\varphi)$ computed in $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$. If we can show that $\mathcal{K}$ is a locally quasi-coherent module, then the proof of (2) is complete. To see this, note that kernels are computed in the category of presheaves (no sheafification necessary). Hence $\mathcal{K}|_{U_{\acute{e}tale}}$ is the kernel of the map $\mathcal{F}|_{U_{\acute{e}tale}} \to \mathcal{G}|_{U_{\acute{e}tale}}$, i.e., is the kernel of a map of quasi-coherent sheaves on $U_{\acute{e}tale}$ whence quasi-coherent by Descent, Lemma 31.6.13. This proves (2).

Parts (3), (4), and (5) follow in exactly the same way. Details omitted. □

In the generality discussed here we don't know how to prove that the category of quasi-coherent sheaves is abelian. Here is what we can prove without any further work.

Lemma 58.11.8. *Let* $p : \mathcal{X} \to (Sch/S)_{fppf}$ *be a category fibred in groupoids.*

(1) *The category* $QCoh(\mathcal{O}_\mathcal{X})$ *has colimits and they agree with colimits in the category* $Mod(\mathcal{O}_\mathcal{X})$ *as well as with colimits in the category* $LQCoh(\mathcal{O}_\mathcal{X})$.
(2) *Given* $\mathcal{F}, \mathcal{G}$ *in* $QCoh(\mathcal{O}_\mathcal{X})$ *the tensor product* $\mathcal{F} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{G}$ *in* $Mod(\mathcal{O}_\mathcal{X})$ *is an object of* $QCoh(\mathcal{O}_\mathcal{X})$.
(3) *Given* $\mathcal{F}, \mathcal{G}$ *in* $QCoh(\mathcal{O}_\mathcal{X})$ *with* $\mathcal{F}$ *locally of finite presentation on* $\mathcal{X}_{fppf}$ *the sheaf* $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})$ *in* $Mod(\mathcal{O}_\mathcal{X})$ *is an object of* $QCoh(\mathcal{O}_\mathcal{X})$.

Proof. Let $\mathcal{I} \to QCoh(\mathcal{O}_\mathcal{X})$, $i \mapsto \mathcal{F}_i$ be a diagram. Consider the object $\mathcal{F} = colim_i \mathcal{F}_i$ of $Mod(\mathcal{O}_\mathcal{X})$. For any object x of $\mathcal{X}$ with $U = p(x)$ the pullback functor x^* commutes with all colimits as it is a left adjoint. Hence $x^*\mathcal{F} = colim_i x^*\mathcal{F}_i$ in $Mod((Sch/U)_{fppf}, \mathcal{O})$. We conclude from Descent, Lemma 31.6.13 that $x^*\mathcal{F}$ is quasi-coherent, hence $\mathcal{F}$ is quasi-coherent, see Lemma 58.11.3. Thus we see that $QCoh(\mathcal{O}_\mathcal{X})$ has colimits and they agree with colimits in the category $Mod(\mathcal{O}_\mathcal{X})$. In particular the (fppf) sheaf $\mathcal{F}$ is also the colimit of the diagram in $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$, hence $\mathcal{F}$ is also the colimit in $LQCoh(\mathcal{O}_\mathcal{X})$. This proves (1).

Parts (2) and (3) are proved in the same way. Details omitted. □

58.12. Stackification and sheaves

It turns out that the category of sheaves on a category fibred in groupoids only ``knows about'' the stackification.

Lemma 58.12.1. *Let* $f : \mathcal{X} \to \mathcal{Y}$ *be a* 1*-morphism of categories fibred in groupoids over* $(Sch/S)_{fppf}$*. If* f *induces an equivalence of stackifications, then the morphism of topoi* $f : Sh(\mathcal{X}_{fppf}) \to Sh(\mathcal{Y}_{fppf})$ *is an equivalence.*

Proof. We may assume $\mathcal{Y}$ is the stackification of $\mathcal{X}$. We claim that $f : \mathcal{X} \to \mathcal{Y}$ is a special cocontinuous functor, see Sites, Definition 9.25.2 which will prove the lemma. By Stacks, Lemma 50.10.3 the functor f is continuous and cocontinuous. By Stacks, Lemma 50.8.1 we see that conditions (3), (4), and (5) of Sites, Lemma 9.25.1 hold. $\square$

Lemma 58.12.2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. If f induces an equivalence of stackifications, then f^* induces equivalences $Mod(\mathcal{O}_\mathcal{X}) \to Mod(\mathcal{O}_\mathcal{Y})$ and $QCoh(\mathcal{O}_\mathcal{X}) \to QCoh(\mathcal{O}_\mathcal{Y})$.*

Proof. We may assume $\mathcal{Y}$ is the stackification of $\mathcal{X}$. The first assertion is clear from Lemma 58.12.1 and $\mathcal{O}_\mathcal{X} = f^{-1}\mathcal{O}_\mathcal{Y}$. Pullback of quasi-coherent sheaves are quasi-coherent, see Lemma 58.11.2. Hence it suffices to show that if $f^*\mathcal{G}$ is quasi-coherent, then $\mathcal{G}$ is. To see this, let y be an object of $\mathcal{Y}$. Translating the condition that $\mathcal{Y}$ is the stackification of $\mathcal{X}$ we see there exists an fppf covering $\{y_i \to y\}$ in $\mathcal{Y}$ such that $y_i \cong f(x_i)$ for some x_i object of $\mathcal{X}$. Say x_i and y_i lie over the scheme U_i. Then $f^*\mathcal{G}$ being quasi-coherent, means that $x_i^*f^*\mathcal{G}$ is quasi-coherent. Since $x_i^*f^*\mathcal{G}$ is isomorphic to $y_i^*\mathcal{G}$ (as sheaves on $(Sch/U_i)_{fppf}$ we see that $y_i^*\mathcal{G}$ is quasi-coherent. It follows from Modules on Sites, Lemma 16.23.3 that the restriction of $\mathcal{G}$ to $\mathcal{Y}/y$ is quasi-coherent. Hence $\mathcal{G}$ is quasi-coherent by Lemma 58.11.3. $\square$

58.13. Quasi-coherent sheaves and presentations

In Groupoids in Spaces, Definition 52.12.1 we have the defined the notion of a quasi-coherent module on an arbitrary groupoid. The following (formal) proposition tells us that we can study quasi-coherent sheaves on quotient stacks in terms of quasi-coherent modules on presentations.

Proposition 58.13.1. *Let (U, R, s, t, c) be a groupoid in algebraic spaces over S. Let $\mathcal{X} = [U/R]$ be the quotient stack. The category of quasi-coherent modules on $\mathcal{X}$ is equivalent to the category of quasi-coherent modules on (U, R, s, t, c).*

Proof. Denote $QCoh(U, R, s, t, c)$ the category of quasi-coherent modules on the groupoid (U, R, s, t, c). We will construct quasi-inverse functors

$$QCoh(\mathcal{O}_\mathcal{X}) \longleftrightarrow QCoh(U, R, s, t, c).$$

According to Lemma 58.12.2 the stackification map $[U/_pR] \to [U/R]$ (see Groupoids in Spaces, Definition 52.19.1) induces an equivalence of categories of quasi-coherent sheaves. Thus it suffices to prove the lemma with $\mathcal{X} = [U/_pR]$.

Recall that an object $x = (T, u)$ of $\mathcal{X} = [U/_pR]$ is given by a scheme T and a morphism $u : T \to U$. A morphism $(T, u) \to (T', u')$ is given by a pair (f, r) where $f : T \to T'$ and $r : T \to R$ with $s \circ r = u$ and $t \circ r = u' \circ f$. Let us call a *special morphism* any morphism of the form $(f, e \circ u' \circ f) : (T, u' \circ f) \to (T', u')$. The category of (T, u) with special morphisms is just the category of schemes over U.

Let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathcal{X}$. Then we obtain for every $x = (T, u)$ a quasi-coherent sheaf $\mathcal{F}_{(T,u)} = x^*\mathcal{F}|_{T_{étale}}$ on T. Moreover, for any morphism $(f, r) : x = (T, u) \to (T', u') = x'$ we obtain a comparison isomorphism

$$c_{(f,r)} : f^*_{small}\mathcal{F}_{(T',u')} \longrightarrow \mathcal{F}_{(T,u)}$$

see Lemma 58.11.5. Moreover, these isomorphisms are compatible with compositions, see Lemma 58.9.3. If U, R are schemes, then we can construct the quasi-coherent sheaf on the groupoid as follows: First the object (U, id) corresponds to a quasi-coherent sheaf $\mathcal{F}_{(U,\mathrm{id})}$ on U. Next, the isomorphism $\alpha : t^*_{small}\mathcal{F}_{(U,\mathrm{id})} \to s^*_{small}\mathcal{F}_{(U,\mathrm{id})}$ comes from

(1) the morphism $(R, \text{id}_R) : (R, s) \to (R, t)$ in the category $[U/_pR]$ which produces an isomorphism $\mathcal{F}_{(R,t)} \to \mathcal{F}_{(R,s)}$,
(2) the special morphism $(R, s) \to (U, \text{id})$ which produces an isomorphism $s_{small}^*\mathcal{F}_{(U,\text{id})} \to \mathcal{F}_{(R,s)}$, and
(3) the special morphism $(R, t) \to (U, \text{id})$ which produces an isomorphism $t_{small}^*\mathcal{F}_{(U,\text{id})} \to \mathcal{F}_{(R,t)}$.

The cocycle condition for α follows from the condition that (U, R, s, t, c) is groupoid, i.e., that composition is associative (details omitted).

To do this in general, i.e., when U and R are algebraic spaces, it suffices to explain how to associate to an algebraic space (W, u) over U a quasi-coherent sheaf $\mathcal{F}_{(W,u)}$ and to construct the comparison maps for morphisms between these. We set $\mathcal{F}_{(W,u)} = x^*\mathcal{F}|_{W_{étale}}$ where x is the 1-morphism $\mathcal{S}_W \to \mathcal{S}_U \to [U/_pR]$ and the comparison maps are explained in (58.10.2.3).

Conversely, suppose that $(\mathcal{G}, \alpha)$ is a quasi-coherent module on (U, R, s, t, c). We are going to define a presheaf of modules $\mathcal{F}$ on $\mathcal{X}$ as follows. Given an object (T, u) of $[U/_pR]$ we set

$$\mathcal{F}(T, u) := \Gamma(T, u_{small}^*\mathcal{G}).$$

Given a morphism $(f, r) : (T, u) \to (T', u')$ we get a map

$$\begin{aligned} \mathcal{F}(T', u') &= \Gamma(T', (u')_{small}^*\mathcal{G}) \\ &\to \Gamma(T, f_{small}^*(u')_{small}^*\mathcal{G}) = \Gamma(T, (u' \circ f)_{small}^*\mathcal{G}) \\ &= \Gamma(T, (t \circ r)_{small}^*\mathcal{G}) = \Gamma(T, r_{small}^*t_{small}^*\mathcal{G}) \\ &\to \Gamma(T, r_{small}^*s_{small}^*\mathcal{G}) = \Gamma(T, (s \circ r)_{small}^*\mathcal{G}) \\ &= \Gamma(T, u_{small}^*\mathcal{G}) \\ &= \mathcal{F}(T, u) \end{aligned}$$

where the first arrow is pullback along f and the second arrow is α. Note that if (T, r) is a special morphism, then this map is just pullback along f as $e_{small}^*\alpha = \text{id}$ by the axioms of a sheaf of quasi-coherent modules on a groupoid. The cocycle condition implies that $\mathcal{F}$ is a presheaf of modules (details omitted). It is immediate from the definition that $\mathcal{F}$ is quasi-coherent when pulled back to $(Sch/T)_{fppf}$ (by the simple description of the restriction maps of $\mathcal{F}$ in case of a special morphism).

We omit the verification that the functors constructed above are quasi-inverse to each other. □

We finish this section with a technical lemma on maps out of quasi-coherent sheaves. It is an analogue of Schemes, Lemma 21.7.1. We will see later (Criteria for Representability, Theorem 59.17.2) that the assumptions on the groupoid imply that $\mathcal{X}$ is an algebraic stack.

Lemma 58.13.2. *Let (U, R, s, t, c) be a groupoid in algebraic spaces over S. Assume s, t are flat and locally of finite presentation. Let $\mathcal{X} = [U/R]$ be the quotient stack. Denote $\pi : \mathcal{S}_U \to \mathcal{X}$ the quotient map. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_\mathcal{X}$-module, and let $\mathcal{H}$ be any object of $Mod(\mathcal{O}_\mathcal{X})$. The map*

$$Hom_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{H}) \longrightarrow Hom_{\mathcal{O}_U}(x^*\mathcal{F}|_{U_{étale}}, x^*\mathcal{H}|_{U_{étale}}), \quad \phi \longmapsto x^*\phi|_{U_{étale}}$$

is injective and its image consists of exactly those $\varphi : x^*\mathcal{F}|_{U_{étale}} \to x^*\mathcal{H}|_{U_{étale}}$ *which give rise to a commutative diagram*

$$\begin{tikzcd} s_{small}^*(x^*\mathcal{F}|_{U_{étale}}) \arrow[r] \arrow[d, "s_{small}^*\varphi"] & (x \circ s)^*\mathcal{F}|_{R_{étale}} = (x \circ t)^*\mathcal{F}|_{R_{étale}} & t_{small}^*(x^*\mathcal{F}|_{U_{étale}}) \arrow[l] \arrow[d, "t_{small}^*\varphi"'] \\ s_{small}^*(x^*\mathcal{H}|_{U_{étale}}) \arrow[r] & (x \circ s)^*\mathcal{H}|_{R_{étale}} = (x \circ t)^*\mathcal{H}|_{R_{étale}} & t_{small}^*(x^*\mathcal{H}|_{U_{étale}}) \arrow[l] \end{tikzcd}$$

of modules on $R_{étale}$ *where the horizontal arrows are the comparison maps (58.10.2.3).*

Proof. According to Lemma 58.12.2 the stackification map $[U/_pR] \to [U/R]$ (see Groupoids in Spaces, Definition 52.19.1) induces an equivalence of categories of quasi-coherent sheaves and of fppf $\mathcal{O}$-modules. Thus it suffices to prove the lemma with $\mathcal{X} = [U/_pR]$. By Proposition 58.13.1 and its proof there exists a quasi-coherent module $(\mathcal{G}, \alpha)$ on (U, R, s, t, c) such that $\mathcal{F}$ is given by the rule $\mathcal{F}(T, u) = \Gamma(T, u^*\mathcal{G})$. In particular $x^*\mathcal{F}|_{U_{étale}} = \mathcal{G}$ and it is clear that the map of the statement of the lemma is injective. Moreover, given a map $\varphi : \mathcal{G} \to x^*\mathcal{H}|_{U_{étale}}$ and given any object $y = (T, u)$ of $[U/_pR]$ we can consider the map

$$\mathcal{F}(y) = \Gamma(T, u^*\mathcal{G}) \xrightarrow{u_{small}^*\varphi} \Gamma(T, u_{small}^* x^*\mathcal{H}|_{U_{étale}}) \to \Gamma(T, y^*\mathcal{H}|_{T_{étale}}) = \mathcal{H}(y)$$

where the second arrow is the comparison map (58.9.4.1) for the sheaf $\mathcal{H}$. This assignment is compatible with the restriction mappings of the sheaves $\mathcal{F}$ and $\mathcal{G}$ for morphisms of $[U/_pR]$ if the cocycle condition of the lemma is satisfied. Proof omitted. Hint: the restriction maps of $\mathcal{F}$ are made explicit in terms of $(\mathcal{G}, \alpha)$ in the proof of Proposition 58.13.1. ☐

58.14. Quasi-coherent sheaves on algebraic stacks

Let $\mathcal{X}$ be an algebraic stack over S. By Algebraic Stacks, Lemma 57.16.2 we can find an equivalence $[U/R] \to \mathcal{X}$ where (U, R, s, t, c) is a smooth groupoid in algebraic spaces. Then

$$QCoh(\mathcal{O}_\mathcal{X}) \cong QCoh(\mathcal{O}_{[U/R]}) \cong QCoh(U, R, s, t, c)$$

where the second equivalence is Proposition 58.13.1. Hence the category of quasi-coherent sheaves on an algebraic stack is equivalent to the category of quasi-coherent modules on a smooth groupoid in algebraic spaces. In particular, by Groupoids in Spaces, Lemma 52.12.5 we see that $QCoh(\mathcal{O}_\mathcal{X})$ is abelian!

There is something slightly disconcerting about our current setup. It is that the fully faithful embedding

$$QCoh(\mathcal{O}_\mathcal{X}) \longrightarrow Mod(\mathcal{O}_\mathcal{X})$$

is in general **not** exact. However, exactly the same thing happens for schemes: for most schemes X the embedding

$$QCoh(\mathcal{O}_X) \cong QCoh((Sch/X)_{fppf}, \mathcal{O}_X) \longrightarrow Mod((Sch/X)_{fppf}, \mathcal{O}_X)$$

isn't exact, see Descent, Lemma 31.6.13. Parenthetically, the example in the proof of Descent, Lemma 31.6.13 shows that in general the strictly full embedding $QCoh(\mathcal{O}_\mathcal{X}) \to LQCoh(\mathcal{O}_\mathcal{X})$ isn't exact either.

We collect all the positive results obtained so far in a single statement.

Lemma 58.14.1. *Let* $\mathcal{X}$ *be an algebraic stack over* S.

(1) *If* $[U/R] \to \mathcal{X}$ *is a presentation of* $\mathcal{X}$ *then there is a canonical equivalence* $QCoh(\mathcal{O}_\mathcal{X}) \cong QCoh(U, R, s, t, c)$.
(2) *The category* $QCoh(\mathcal{O}_\mathcal{X})$ *is abelian.*

(3) *The category* $QCoh(\mathcal{O}_\mathcal{X})$ *has colimits and they agree with colimits in the category* $Mod(\mathcal{O}_\mathcal{X})$.
(4) *Given* $\mathcal{F}, \mathcal{G}$ *in* $QCoh(\mathcal{O}_\mathcal{X})$ *the tensor product* $\mathcal{F} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{G}$ *in* $Mod(\mathcal{O}_\mathcal{X})$ *is an object of* $QCoh(\mathcal{O}_\mathcal{X})$.
(5) *Given* $\mathcal{F}, \mathcal{G}$ *in* $QCoh(\mathcal{O}_\mathcal{X})$ *with* $\mathcal{F}$ *locally of finite presentation on* $\mathcal{X}_{fppf}$ *the sheaf* $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})$ *in* $Mod(\mathcal{O}_\mathcal{X})$ *is an object of* $QCoh(\mathcal{O}_\mathcal{X})$.

Proof. Properties (3), (4), and (5) were proven in Lemma 58.11.8. Part (1) is Proposition 58.13.1. Part (2) follows from Groupoids in Spaces, Lemma 52.12.5 as discussed above. □

Proposition 58.14.2. *Let* $\mathcal{X}$ *be an algebraic stack over* S. *The inclusion functor* $QCoh(\mathcal{O}_\mathcal{X}) \to Mod(\mathcal{O}_\mathcal{X})$ *has a right adjoint*

$$Q^3 : Mod(\mathcal{O}_\mathcal{X}) \longrightarrow QCoh(\mathcal{O}_\mathcal{X})$$

such that for every quasi-coherent sheaf $\mathcal{F}$ *the adjunction mapping* $Q(\mathcal{F}) \to \mathcal{F}$ *is an isomorphism. Moreover, the category* $QCoh(\mathcal{O}_\mathcal{X})$ *has limits and enough injectives.*

Proof. This proof is a repeat of the proof in the case of schemes, see Properties, Proposition 23.21.4 and the case of algebraic spaces, see Properties of Spaces, Proposition 41.29.2. We urge the reader to read either of those proofs first.

The two assertions about $Q(\mathcal{F}) \to \mathcal{F}$ and limits in $QCoh(\mathcal{O}_\mathcal{X})$ are formal consequences of the existence of Q, the fact that the inclusion is fully faithful, and the fact that $Mod(\mathcal{O}_\mathcal{X})$ has limits (see Modules on Sites, Lemma 16.14.2). The existence of injectives follows from the existence of injectives in $Mod(\mathcal{O}_\mathcal{X})$ (see Injectives, Theorem 17.12.4) and Homology, Lemma 10.22.3. Thus it suffices to construct Q.

Choose a presentation $\mathcal{X} = [U/R]$ so that (U, R, s, t, c) is a smooth groupoid in algebraic spaces and in particular s and t are flat morphisms of algebraic spaces. By Lemma 58.14.1 above we have $QCoh(\mathcal{O}_\mathcal{X}) = QCoh(U, R, s, t, c)$. By Groupoids in Spaces, Lemma 52.13.2 there exists a set T and a family $(\mathcal{F}_t)_{t \in T}$ of quasi-coherent sheaves on $\mathcal{X}$ such that every quasi-coherent sheaf on $\mathcal{X}$ is the directed colimit of its subsheaves which are isomorphic to one of the $\mathcal{F}_t$.

Given an object $\mathcal{G}$ of $QCoh(\mathcal{O}_\mathcal{X})$ we set

$$Q(\mathcal{G}) = colim_{(t,\psi)} \mathcal{F}_t$$

The colimit is over the category of pairs (t, ψ) where $t \in T$ and $\psi : \mathcal{F}_t \to \mathcal{G}$ is a morphism of $\mathcal{O}_\mathcal{X}$-modules. A morphism $(t, \psi) \to (t', \psi')$ is given by a morphism $\beta : \mathcal{F}_t \to \mathcal{F}_{t'}$ such that $\psi' \circ \beta = \psi$. By Lemma 58.14.1 the colimit is quasi-coherent. Note that there is a canonical map $Q(\mathcal{G}) \to \mathcal{G}$ by definition of the colimit. The formula

$$Hom(\mathcal{H}, Q(\mathcal{G})) = Hom(\mathcal{H}, \mathcal{G})$$

holds for $\mathcal{H} = \mathcal{F}_t$ by construction. It follows formally from this and the fact that every $\mathcal{H}$ is a directed colimit of $\mathcal{O}_\mathcal{X}$-modules isomorphic to $\mathcal{F}_t$ that this equality holds for any quasi-coherent module $\mathcal{H}$ on X. This finishes the proof. □

[3] This functor is sometimes called the *coherator*.

58.15. Cohomology

Let S be a scheme and let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. For any $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$ the categories $Ab(\mathcal{X}_\tau)$ and $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$ have enough injectives, see Injectives, Theorems 17.11.4 and 17.12.4. Thus we can use the machinery of Cohomology on Sites, Section 19.3 to define the cohomology groups

$$H^p(\mathcal{X}_\tau, \mathcal{F}) = H^p_\tau(\mathcal{X}, \mathcal{F}) \quad \text{and} \quad H^p(x, \mathcal{F}) = H^p_\tau(x, \mathcal{F})$$

for any $x \in Ob(\mathcal{X})$ and any object $\mathcal{F}$ of $Ab(\mathcal{X}_\tau)$ or $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$. Moreover, if $f : \mathcal{X} \to \mathcal{Y}$ is a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$, then we obtain the higher direct images $R^i f_*\mathcal{F}$ in $Ab(\mathcal{Y}_\tau)$ or $Mod(\mathcal{Y}_\tau, \mathcal{O}_\mathcal{Y})$. Of course, as explained in Cohomology on Sites, Section 19.4 there are also derived versions of $H^p(-)$ and $R^i f_*$.

Lemma 58.15.1. *Let S be a scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Let $x \in Ob(\mathcal{X})$ be an object lying over the scheme U. Let $\mathcal{F}$ be an object of $Ab(\mathcal{X}_\tau)$ or $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$. Then*

$$H^p_\tau(x, \mathcal{F}) = H^p((Sch/U)_\tau, x^{-1}\mathcal{F})$$

and if $\tau = étale$, then we also have

$$H^p_{étale}(x, \mathcal{F}) = H^p(U_{étale}, \mathcal{F}|_{U_{étale}}).$$

Proof. The first statement follows from Cohomology on Sites, Lemma 19.8.1 and the equivalence of Lemma 58.9.4. The second statement follows from the first combined with Étale Cohomology, Lemma 38.20.5. □

58.16. Injective sheaves

The pushfoward of an injective abelian sheaf or module is injective.

Lemma 58.16.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$.*

(1) *$f_*\mathcal{I}$ is injective in $Ab(\mathcal{Y}_\tau)$ for $\mathcal{I}$ injective in $Ab(\mathcal{X}_\tau)$, and*
(2) *$f_*\mathcal{I}$ is injective in $Mod(\mathcal{Y}_\tau, \mathcal{O}_\mathcal{Y})$ for $\mathcal{I}$ injective in $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$.*

Proof. This follows formally from the fact that f^{-1} is an exact left adjoint of f_*, see Homology, Lemma 10.22.1. □

In the rest of this section we prove that pullback f^{-1} has a left adjoint $f_!$ on abelian sheaves and modules. If f is representable (by schemes or by algebraic spaces), then it will turn out that $f_!$ is exact and f^{-1} will preserve injectives. We first prove a few preliminary lemmas about fibre products and equalizers in categories fibred in groupoids and their behaviour with respect to morphisms.

Lemma 58.16.2. *Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids.*

(1) *The category $\mathcal{X}$ has fibre products.*
(2) *If the $Isom$-presheaves of $\mathcal{X}$ are representable by algebraic spaces, then $\mathcal{X}$ has equalizers.*
(3) *If $\mathcal{X}$ is an algebraic stack (or more generally a quotient stack), then $\mathcal{X}$ has equalizers.*

Proof. Part (1) follows Categories, Lemma 4.32.13 as $(Sch/S)_{fppf}$ has fibre products.

Let $a, b : x \to y$ be morphisms of $\mathcal{X}$. Set $U = p(x)$ and $V = p(y)$. The category of schemes has equalizers hence we can let $W \to U$ be the equalizer of $p(a)$ and $p(b)$. Denote $c : z \to x$ a morphism of $\mathcal{X}$ lying over $W \to U$. The equalizer of a and b, if it exists, is the equalizer of $a \circ c$ and $b \circ c$. Thus we may assume that $p(a) = p(b) = f : U \to V$. As $\mathcal{X}$ is fibred in groupoids, there exists a unique automorphism $i : x \to x$ in the fibre category of $\mathcal{X}$ over U such that $a \circ i = b$. Again the equalizer of a and b is the equalizer of id_x and i. Recall that the $Isom_{\mathcal{X}}(x)$ is the presheaf on $(Sch/U)_{fppf}$ which to V/U associates the set of automorphisms of $x|_V$ in the fibre category of $\mathcal{X}$ over V, see Stacks, Definition 50.2.2. If $Isom_{\mathcal{X}}(x)$ is representable by an algebraic space $G \to U$, then we see that id_x and i define morphisms $e, i : U \to G$ over U. Set $V = U \times_{e,G,i} U$, which by Morphisms of Spaces, Lemma 42.5.7 is a scheme. Then it is clear that $x|_V \to x$ is the equalizer of the maps id_x and i in $\mathcal{X}$. This proves (2).

If $\mathcal{X} = [U/R]$ for some groupoid in algebraic spaces (U, R, s, t, c) over S, then the hypothesis of (2) holds by Bootstrap, Lemma 54.11.3. If $\mathcal{X}$ is an algebraic stack, then we can choose a presentation $[U/R] \cong \mathcal{X}$ by Algebraic Stacks, Lemma 57.16.2. □

Lemma 58.16.3. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$.*

(1) *The functor f transforms fibre products into fibre products.*
(2) *If f is faithful, then f transforms equalizers into equalizers.*

Proof. By Categories, Lemma 4.32.13 we see that a fibre product in $\mathcal{X}$ is any commutative square lying over a fibre product diagram in $(Sch/S)_{fppf}$. Similarly for $\mathcal{Y}$. Hence (1) is clear.

Let $x \to x'$ be the equalizer of two morphisms $a, b : x' \to x''$ in $\mathcal{X}$. We will show that $f(x) \to f(x')$ is the equalizer of $f(a)$ and $f(b)$. Let $y \to f(x)$ be a morphism of $\mathcal{Y}$ equalizing $f(a)$ and $f(b)$. Say x, x', x'' lie over the schemes U, U', U'' and y lies over V. Denote $h : V \to U'$ the image of $y \to f(x)$ in the category of schemes. The morphism $y \to f(x)$ is isomorphic to $f(h^*x') \to f(x')$ by the axioms of fibred categories. Hence, as f is faithful, we see that $h^*x' \to x'$ equalizes a and b. Thus we obtain a unique morphism $h^*x' \to x$ whose image $y = f(h^*x') \to f(x)$ is the desired morphism in $\mathcal{Y}$. □

Lemma 58.16.4. *Let $f : \mathcal{X} \to \mathcal{Y}$, $g : \mathcal{Z} \to \mathcal{Y}$ be faithful 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$.*

(1) *the functor $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{Y}$ is faithful, and*
(2) *if $\mathcal{X}, \mathcal{Z}$ have equalizers, so does $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$.*

Proof. We think of objects in $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ as quadruples (U, x, z, α) where $\alpha : f(x) \to g(z)$ is an isomorphism over U, see Categories, Lemma 4.29.3. A morphism $(U, x, z, \alpha) \to (U', x', z', \alpha')$ is a pair of morphisms $a : x \to x'$ and $b : z \to z'$ compatible with α and α'. Thus it is clear that if f and g are faithful, so is the functor $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{Y}$. Now, suppose that $(a, b), (a', b') : (U, x, z, \alpha) \to (U', x', z', \alpha')$ are two morphisms of the 2-fibre product. Then consider the equalizer $x'' \to x$ of a and a' and the equalizer $z'' \to z$ of b and b'. Since f commutes with equalizers (by Lemma 58.16.3) we see that $f(x'') \to f(x)$ is the equalizer of $f(a)$ and $f(a')$. Similarly, $g(z'') \to g(z)$ is the equalizer of $g(b)$ and $g(b')$. Picture

$$\begin{tikzcd} f(x'') \arrow[r] \arrow[d, dotted, "\alpha''"'] & f(x) \arrow[r, shift left, "f(a)"] \arrow[r, shift right, "f(a')"'] \arrow[d, "\alpha"] & f(x') \arrow[d, "\alpha'"] \\ g(z'') \arrow[r] & g(z) \arrow[r, shift left, "g(b)"] \arrow[r, shift right, "g(b')"'] & g(z') \end{tikzcd}$$

It is clear that the dotted arrow exists and is an isomorphism. However, it is not a priori the case that the image of α'' in the category of schemes is the identity of its source. On the other hand, the existence of α'' means that we can assume that x'' and z'' are defined over the same scheme and that the morphisms $x'' \to x$ and $z'' \to z$ have the same image in the category of schemes. Redoing the diagram above we see that the dotted arrow now does project to an identity morphism and we win. Some details omitted. $\square$

As we are working with big sites we have the following somewhat counter intuitive result (which also holds for morphisms of big sites of schemes). Warning: This result isn't true if we drop the hypothesis that f is faithful.

Lemma 58.16.5. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. The functor $f^{-1} : Ab(\mathcal{Y}_\tau) \to Ab(\mathcal{X}_\tau)$ has a left adjoint $f_! : Ab(\mathcal{X}_\tau) \to Ab(\mathcal{Y}_\tau)$. If f is faithful and $\mathcal{X}$ has equalizers, then*

(1) *$f_!$ is exact, and*
(2) *$f^{-1}\mathcal{I}$ is injective in $Ab(\mathcal{X}_\tau)$ for $\mathcal{I}$ injective in $Ab(\mathcal{Y}_\tau)$.*

Proof. By Stacks, Lemma 50.10.3 the functor f is continuous and cocontinuous. Hence by Modules on Sites, Lemma 16.16.2 the functor $f^{-1} : Ab(\mathcal{Y}_\tau) \to Ab(\mathcal{X}_\tau)$ has a left adjoint $f_! : Ab(\mathcal{X}_\tau) \to Ab(\mathcal{Y}_\tau)$. To see (1) we apply Modules on Sites, Lemma 16.16.3 and to see that the hypotheses of that lemma are satisfied use Lemmas 58.16.2 and 58.16.3 above. Part (2) follows from this formally, see Homology, Lemma 10.22.1. $\square$

Lemma 58.16.6. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. The functor $f^* : Mod(\mathcal{Y}_\tau, \mathcal{O}_\mathcal{Y}) \to Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$ has a left adjoint $f_! : Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X}) \to Mod(\mathcal{Y}_\tau, \mathcal{O}_\mathcal{Y})$ which agrees with the functor $f_!$ of Lemma 58.16.5 on underlying abelian sheaves. If f is faithful and $\mathcal{X}$ has equalizers, then*

(1) *$f_!$ is exact, and*
(2) *$f^{-1}\mathcal{I}$ is injective in $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$ for $\mathcal{I}$ injective in $Mod(\mathcal{Y}_\tau, \mathcal{O}_\mathcal{X})$.*

Proof. Recall that f is a continuous and cocontinuous functor of sites and that $f^{-1}\mathcal{O}_\mathcal{Y} = \mathcal{O}_\mathcal{X}$. Hence Modules on Sites, Lemma 16.35.1 implies f^* has a left adjoint $f_!^{Mod}$. Let x be an object of $\mathcal{X}$ lying over the scheme U. Then f induces an equivalence of ringed sites

$$\mathcal{X}/x \longrightarrow \mathcal{Y}/f(x)$$

as both sides are equivalent to $(Sch/U)_\tau$, see Lemma 58.9.4. Modules on Sites, Remark 16.35.2 shows that $f_!$ agrees with the functor on abelian sheaves.

Assume now that $\mathcal{X}$ has equalizers and that f is faithful. Lemma 58.16.5 tells us that $f_!$ is exact. Finally, Homology, Lemma 10.22.1 implies the statement on pullbacks of injective modules. $\square$

58.17. The Čech complex

To compute the cohomology of a sheaf on an algebraic stack we compare it to the cohomology of the sheaf restricted to coverings of the given algebraic stack.

Throughout this section the situation will be as follows. We are given a 1-morphism of categories fibred in groupoids

(58.17.0.1)
$$\begin{tikzcd} \mathcal{U} \arrow[rr, "f"] \arrow[dr, "q"'] & & \mathcal{X} \arrow[dl, "p"] \\ & (Sch/S)_{fppf} & \end{tikzcd}$$

We are going to think about $\mathcal{U}$ as a ``covering'' of $\mathcal{X}$. Hence we want to consider the simplicial object

$$\mathcal{U} \times_{\mathcal{X}} \mathcal{U} \times_{\mathcal{X}} \mathcal{U} \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \mathcal{U} \times_{\mathcal{X}} \mathcal{U} \substack{\longrightarrow\\\longrightarrow} \mathcal{U}$$

in the category of categories fibred in groupoids over $(Sch/S)_{fppf}$. However, since this is a $(2,1)$-category and not a category, we should say explicitly what we mean. Namely, we let $\mathcal{U}_n$ be the category with objects $(u_0, \ldots, u_n, x, \alpha_0, \ldots, \alpha_n)$ where $\alpha_i : f(u_i) \to x$ is an isomorphism in $\mathcal{X}$. We denote $f_n : \mathcal{U}_n \to \mathcal{X}$ the 1-morphism which assigns to $(u_0, \ldots, u_n, x, \alpha_0, \ldots, \alpha_n)$ the object x. Note that $\mathcal{U}_0 = \mathcal{U}$ and $f_0 = f$. Given a map $\varphi : [m] \to [n]$ we consider the 1-morphism $\mathcal{U}_\varphi : \mathcal{U}_n \longrightarrow \mathcal{U}_n$ given by

$$(u_0, \ldots, u_n, x, \alpha_0, \ldots, \alpha_n) \longmapsto (u_{\varphi(0)}, \ldots, u_{\varphi(n)}, x, \alpha_{\varphi(0)}, \ldots, \alpha_{\varphi(n)})$$

on objects. All of these 1-morphisms compose correctly on the nose (no 2-morphisms required) and all of these 1-morphisms are 1-morphisms over $\mathcal{X}$. We denote $\mathcal{U}_\bullet$ this simplicial object. If $\mathcal{F}$ is a presheaf of sets on $\mathcal{X}$, then we obtain a cosimplicial set

$$\Gamma(\mathcal{U}_0, f_0^{-1}\mathcal{F}) \substack{\longrightarrow\\\longrightarrow} \Gamma(\mathcal{U}_1, f_1^{-1}\mathcal{F}) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \Gamma(\mathcal{U}_2, f_2^{-1}\mathcal{F})$$

Here the arrows are the pullback maps along the given morphisms of the simplicial object. If $\mathcal{F}$ is a presheaf of abelian groups, this is a cosimplicial abelian group.

Let $\mathcal{U} \to \mathcal{X}$ be as above and let $\mathcal{F}$ be an abelian presheaf on $\mathcal{X}$. The *Čech complex* associated to the situation is denoted $\check{\mathcal{C}}^\bullet(\mathcal{U} \to \mathcal{X}, \mathcal{F})$. It is the cochain complex associated to the cosimplicial abelian group above, see Simplicial, Section 14.23. It has terms

$$\check{\mathcal{C}}^n(\mathcal{U} \to \mathcal{X}, \mathcal{F}) = \Gamma(\mathcal{U}_n, f_n^{-1}\mathcal{F}).$$

The boundary maps are the maps

$$d^n = \sum\nolimits_{i=0}^{n+1} (-1)^i \delta_i^{n+1} : \Gamma(\mathcal{U}_n, f_n^{-1}\mathcal{F}) \longrightarrow \Gamma(\mathcal{U}_{n+1}, f_{n+1}^{-1}\mathcal{F})$$

where δ_i^{n+1} corresponds to the map $[n] \to [n+1]$ omitting the index i. Note that the map $\Gamma(\mathcal{X}, \mathcal{F}) \to \Gamma(\mathcal{U}_0, f_0^{-1}\mathcal{F}_0)$ is in the kernel of the differential d^0. Hence we define the *extended Čech complex* to be the complex

$$\ldots \to 0 \to \Gamma(\mathcal{X}, \mathcal{F}) \to \Gamma(\mathcal{U}_0, f_0^{-1}\mathcal{F}_0) \to \Gamma(\mathcal{U}_1, f_1^{-1}\mathcal{F}_1) \to \ldots$$

with $\Gamma(\mathcal{X}, \mathcal{F})$ placed in degree -1. The extended Čech complex is acyclic if and only if the canonical map

$$\Gamma(\mathcal{X}, \mathcal{F})[0] \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U} \to \mathcal{X}, \mathcal{F})$$

is a quasi-isomorphism of complexes.

Lemma 58.17.1. *Generalities on Čech complexes.*

(1) *If*

$$\begin{tikzcd} \mathcal{V} \arrow[r, "h"] \arrow[d, "g"] & \mathcal{U} \arrow[d, "f"] \\ \mathcal{Y} \arrow[r, "e"] & \mathcal{X} \end{tikzcd}$$

is 2-commutative diagram of categories fibred in groupoids over $(Sch/S)_{fppf}$, then there is a morphism of Čech complexes

$$\check{\mathcal{C}}^\bullet(\mathcal{U} \to \mathcal{X}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{V} \to \mathcal{Y}, e^{-1}\mathcal{F})$$

(2) *if h and e are equivalences, then the map of (1) is an isomorphism,*
(3) *if $f, f' : \mathcal{U} \to \mathcal{X}$ are 2-isomorphic, then the associated Čech complexes are isomorphic,*

Proof. In the situation of (1) let $t : f \circ h \to e \circ g$ be a 2-morphism. The map on complexes is given in degree n by pullback along the 1-morphisms $\mathcal{V}_n \to \mathcal{U}_n$ given by the rule

$$(v_0, \ldots, v_n, y, \beta_0, \ldots, \beta_n) \longmapsto (h(v_0), \ldots, h(v_n), e(y), e(\beta_0) \circ t_{v_0}, \ldots, e(\beta_n) \circ t_{v_n}).$$

For (2), note that pullback on global sections is an isomorphism for any presheaf of sets when the pullback is along an equivalence of categories. Part (3) follows on combining (1) and (2). ☐

Lemma 58.17.2. *If there exists a 1-morphism $s : \mathcal{X} \to \mathcal{U}$ such that $f \circ s$ is 2-isomorphic to $\mathrm{id}_\mathcal{X}$ then the extended Čech complex is homotopic to zero.*

Proof. Set $\mathcal{U}' = \mathcal{U} \times_\mathcal{X} \mathcal{X}$ equal to the fibre product as described in Categories, Lemma 4.29.3. Set $f' : \mathcal{U}' \to \mathcal{X}$ equal to the second projection. Then $\mathcal{U} \to \mathcal{U}'$, $u \mapsto (u, f(x), 1)$ is an equivalence over $\mathcal{X}$, hence we may replace $(\mathcal{U}, f)$ by $(\mathcal{U}', f')$ by Lemma 58.17.1. The advantage of this is that now f' has a section s' such that $f' \circ s' = \mathrm{id}_\mathcal{X}$ on the nose. Namely, if $t : s \circ f \to \mathrm{id}_\mathcal{X}$ is a 2-isomorphism then we can set $s'(x) = (s(x), x, t_x)$. Thus we may assume that $f \circ s = \mathrm{id}_\mathcal{X}$.

In the case that $f \circ s = \mathrm{id}_\mathcal{X}$ the result follows from general principles. We give the homotopy explicitly. Namely, for $n \geq 0$ define $s_n : \mathcal{U}_n \to \mathcal{U}_{n+1}$ to be the 1-morphism defined by the rule on objects

$$(u_0, \ldots, u_n, x, \alpha_0, \ldots, \alpha_n) \longmapsto (u_0, \ldots, u_n, s(x), x, \alpha_0, \ldots, \alpha_n, \mathrm{id}_x).$$

Define

$$h^{n+1} : \Gamma(\mathcal{U}_{n+1}, f_{n+1}^{-1}\mathcal{F}) \longrightarrow \Gamma(\mathcal{U}_n, f_n^{-1}\mathcal{F})$$

as pullback along s_n. We also set $s_{-1} = s$ and $h^0 : \Gamma(\mathcal{U}_0, f_0^{-1}\mathcal{F}) \to \Gamma(\mathcal{X}, \mathcal{F})$ equal to pullback along s_{-1}. Then the family of maps $\{h^n\}_{n \geq 0}$ is a homotopy between 1 and 0 on the extended Čech complex. ☐

58.18. The relative Čech complex

Let $f : \mathcal{U} \to \mathcal{X}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$ as in (58.17.0.1). Consider the associated simplicial object $\mathcal{U}_\bullet$ and the maps $f_n : \mathcal{U}_n \to \mathcal{X}$. Let $\tau \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\}$. Finally, suppose that $\mathcal{F}$ is a sheaf (of sets) on $\mathcal{X}_\tau$. Then

$$f_{0,*}f_0^{-1}\mathcal{F} \rightrightarrows f_{1,*}f_1^{-1}\mathcal{F} \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} f_{2,*}f_2^{-1}\mathcal{F}$$

is a cosimplicial sheaf on $\mathcal{X}_\tau$ where we use the pullback maps introduced in Sites, Section 9.39. If $\mathcal{F}$ is an abelian sheaf, then $f_{n,*}f_n^{-1}\mathcal{F}$ form a cosimplicial abelian sheaf on $\mathcal{X}_\tau$. The associated complex (see Simplicial, Section 14.23)

$$\ldots \to 0 \to f_{0,*}f_0^{-1}\mathcal{F} \to f_{1,*}f_1^{-1}\mathcal{F} \to f_{2,*}f_2^{-1}\mathcal{F} \to \ldots$$

is called the *relative Čech complex* associated to the situation. We will denote this complex $\mathcal{K}^\bullet(f, \mathcal{F})$. The *extended relative Čech complex* is the complex

$$\ldots \to 0 \to \mathcal{F} \to f_{0,*}f_0^{-1}\mathcal{F} \to f_{1,*}f_1^{-1}\mathcal{F} \to f_{2,*}f_2^{-1}\mathcal{F} \to \ldots$$

with $\mathcal{F}$ in degree -1. The extended relative Čech complex is acyclic if and only if the map $\mathcal{F}[0] \to \mathcal{K}^\bullet(f, \mathcal{F})$ is a quasi-isomorphism of complexes of sheaves.

Remark 58.18.1. We can define the complex $\mathcal{K}^\bullet(f, \mathcal{F})$ also if $\mathcal{F}$ is a presheaf, only we cannot use the reference to Sites, Section 9.39 to define the pullback maps. To explain the pullback maps, suppose given a commutative diagram

$$\begin{tikzcd} \mathcal{V} \arrow[rr, "h"] \arrow[dr, "g"'] & & \mathcal{U} \arrow[dl, "f"] \\ & \mathcal{X} & \end{tikzcd}$$

of categories fibred in groupoids over $(Sch/S)_{fppf}$ and a presheaf $\mathcal{G}$ on $\mathcal{U}$ we can define the pullback map $f_*\mathcal{G} \to g_*h^{-1}\mathcal{G}$ as the composition

$$f_*\mathcal{G} \longrightarrow f_*h_*h^{-1}\mathcal{G} = g_*h^{-1}\mathcal{G}$$

where the map comes from the adjunction map $\mathcal{G} \to h_*h^{-1}\mathcal{G}$. This works because in our situation the functors h_* and h^{-1} are adjoint in presheaves (and agree with their counter parts on sheaves). See Sections 58.3 and 58.4.

Lemma 58.18.2. *Generalities on relative Čech complexes.*

(1) *If*

$$\begin{tikzcd} \mathcal{V} \arrow[r, "h"] \arrow[d, "g"'] & \mathcal{U} \arrow[d, "f"] \\ \mathcal{Y} \arrow[r, "e"] & \mathcal{X} \end{tikzcd}$$

is 2-commutative diagram of categories fibred in groupoids over $(Sch/S)_{fppf}$, then there is a morphism $e^{-1}\mathcal{K}^\bullet(f, \mathcal{F}) \to \mathcal{K}^\bullet(g, e^{-1}\mathcal{F})$.

(2) *if h and e are equivalences, then the map of (1) is an isomorphism,*

(3) *if $f, f' : \mathcal{U} \to \mathcal{X}$ are 2-isomorphic, then the associated relative Čech complexes are isomorphic,*

Proof. Literally the same as the proof of Lemma 58.17.1 using the pullback maps of Remark 58.18.1. □

Lemma 58.18.3. *If there exists a 1-morphism $s : \mathcal{X} \to \mathcal{U}$ such that $f \circ s$ is 2-isomorphic to $id_\mathcal{X}$ then the extended relative Čech complex is homotopic to zero.*

Proof. Literally the same as the proof of Lemma 58.17.2. □

Remark 58.18.4. Let us ``compute'' the value of the relative Čech complex on an object x of $\mathcal{X}$. Say $p(x) = U$. Consider the 2-fibre product diagram (which serves to introduce the notation $g : \mathcal{V} \to \mathcal{Y}$)

$$\begin{tikzcd} \mathcal{V} \arrow[r, equal] \arrow[d, "g"] & (Sch/U)_{fppf} \times_{x,\mathcal{X}} \mathcal{U} \arrow[r] \arrow[d] & \mathcal{U} \arrow[d, "f"] \\ \mathcal{Y} \arrow[r, equal] & (Sch/U)_{fppf} \arrow[r, "x"] & \mathcal{X} \end{tikzcd}$$

Note that the morphism $\mathcal{V}_n \to \mathcal{U}_n$ of the proof of Lemma 58.17.1 induces an equivalence $\mathcal{V}_n = (Sch/U)_{fppf} \times_{x,\mathcal{X}} \mathcal{U}_n$. Hence we see from (58.5.0.1) that

$$\Gamma(x, \mathcal{K}^\bullet(f, \mathcal{F})) = \check{\mathcal{C}}^\bullet(\mathcal{V} \to \mathcal{Y}, x^{-1}\mathcal{F})$$

In words: The value of the relative Čech complex on an object x of $\mathcal{X}$ is the Čech complex of the base change of f to $\mathcal{X}/x \cong (Sch/U)_{fppf}$. This implies for example that Lemma 58.17.2 implies Lemma 58.18.3 and more generally that results on the (usual) Čech complex imply results for the relative Čech complex.

Lemma 58.18.5. *Let*

$$\begin{tikzcd} \mathcal{V} \arrow[r, "h"] \arrow[d, "g"] & \mathcal{U} \arrow[d, "f"] \\ \mathcal{Y} \arrow[r, "e"] & \mathcal{X} \end{tikzcd}$$

be a 2-fibre product of categories fibred in groupoids over $(Sch/S)_{fppf}$ and let $\mathcal{F}$ be an abelian presheaf on $\mathcal{X}$. Then the map $e^{-1}\mathcal{K}^\bullet(f, \mathcal{F}) \to \mathcal{K}^\bullet(g, e^{-1}\mathcal{F})$ of Lemma 58.18.2 is an isomorphism of complexes of abelian presheaves.

Proof. Let y be an object of $\mathcal{Y}$ lying over the scheme T. Set $x = e(y)$. We are going to show that the map induces an isomorphism on sections over y. Note that

$$\Gamma(y, e^{-1}\mathcal{K}^\bullet(f, \mathcal{F})) = \Gamma(x, \mathcal{K}^\bullet(f, \mathcal{F})) = \check{\mathcal{C}}^\bullet((Sch/T)_{fppf} \times_{x,\mathcal{X}} \mathcal{U} \to (Sch/T)_{fppf}, x^{-1}\mathcal{F})$$

by Remark 58.18.4. On the other hand,

$$\Gamma(y, \mathcal{K}^\bullet(g, e^{-1}\mathcal{F})) = \check{\mathcal{C}}^\bullet((Sch/T)_{fppf} \times_{y,\mathcal{Y}} \mathcal{V} \to (Sch/T)_{fppf}, y^{-1}e^{-1}\mathcal{F})$$

also by Remark 58.18.4. Note that $y^{-1}e^{-1}\mathcal{F} = x^{-1}\mathcal{F}$ and since the diagram is 2-cartesian the 1-morphism

$$(Sch/T)_{fppf} \times_{y,\mathcal{Y}} \mathcal{V} \to (Sch/T)_{fppf} \times_{x,\mathcal{X}} \mathcal{U}$$

is an equivalence. Hence the map on sections over y is an isomorphism by Lemma 58.17.1. $\square$

Exactness can be checked on a ``covering''.

Lemma 58.18.6. *Let $f : \mathcal{U} \to \mathcal{X}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. Let*

$$\mathcal{F} \to \mathcal{G} \to \mathcal{H}$$

be a complex in $Ab(\mathcal{X}_\tau)$. Assume that

(1) *for every object x of $\mathcal{X}$ there exists a covering $\{x_i \to x\}$ in $\mathcal{X}_\tau$ such that each x_i is isomorphic to $f(u_i)$ for some object u_i of $\mathcal{U}$, and*
(2) *$f^{-1}\mathcal{F} \to f^{-1}\mathcal{G} \to f^{-1}\mathcal{H}$ is exact.*

Then the sequence $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is exact.

Proof. Let x be an object of $\mathcal{X}$ lying over the scheme T. Consider the sequence $x^{-1}\mathcal{F} \to x^{-1}\mathcal{G} \to x^{-1}\mathcal{H}$ of abelian sheaves on $(Sch/T)_\tau$. It suffices to show this sequence is exact. By assumption there exists a τ-covering $\{T_i \to T\}$ such that $x|_{T_i}$ is isomorphic to $f(u_i)$ for some object u_i of $\mathcal{U}$ over T_i and moreover the sequence $u_i^{-1}f^{-1}\mathcal{F} \to u_i^{-1}f^{-1}\mathcal{G} \to u_i^{-1}f^{-1}\mathcal{H}$ of abelian sheaves on $(Sch/T_i)_\tau$ is exact. Since $u_i^{-1}f^{-1}\mathcal{F} = x^{-1}\mathcal{F}|_{(Sch/T_i)_\tau}$ we conclude that the sequence $x^{-1}\mathcal{F} \to x^{-1}\mathcal{G} \to x^{-1}\mathcal{H}$ become exact after localizing at each of the members of a covering, hence the sequence is exact. $\square$

Proposition 58.18.7. *Let $f : \mathcal{U} \to \mathcal{X}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. If*

1. *$\mathcal{F}$ is an abelian sheaf on $\mathcal{X}_\tau$, and*
2. *for every object x of $\mathcal{X}$ there exists a covering $\{x_i \to x\}$ in $\mathcal{X}_\tau$ such that each x_i is isomorphic to $f(u_i)$ for some object u_i of $\mathcal{U}$,*

then the extended relative Čech complex

$$\ldots \to 0 \to \mathcal{F} \to f_{0,*}f_0^{-1}\mathcal{F} \to f_{1,*}f_1^{-1}\mathcal{F} \to f_{2,*}f_2^{-1}\mathcal{F} \to \ldots$$

is exact in $Ab(\mathcal{X}_\tau)$.

Proof. By Lemma 58.18.6 it suffices to check exactness after pulling back to $\mathcal{U}$. By Lemma 58.18.5 the pullback of the extended relative Čech complex is isomorphic to the extend relative Čech complex for the morphism $\mathcal{U} \times_\mathcal{X} \mathcal{U} \to \mathcal{U}$ and an abelian sheaf on $\mathcal{U}_\tau$. Since there is a section $\Delta_{\mathcal{U}/\mathcal{X}} : \mathcal{U} \to \mathcal{U} \times_\mathcal{X} \mathcal{U}$ exactness follows from Lemma 58.18.3. $\square$

Using this we can construct the Čech-to-cohomology spectral sequence as follows. We first give a technical, precise version. In the next section we give a version that applies only to algebraic stacks.

Lemma 58.18.8. *Let $f : \mathcal{U} \to \mathcal{X}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. Assume*

1. *$\mathcal{F}$ is an abelian sheaf on $\mathcal{X}_\tau$,*
2. *for every object x of $\mathcal{X}$ there exists a covering $\{x_i \to x\}$ in $\mathcal{X}_\tau$ such that each x_i is isomorphic to $f(u_i)$ for some object u_i of $\mathcal{U}$,*
3. *the category $\mathcal{U}$ has equalizers, and*
4. *the functor f is faithful.*

Then there is a first quadrant spectral sequence of abelian groups

$$E_2^{p,q} = H^q((\mathcal{U}_p)_\tau, f_p^{-1}\mathcal{F}) \Rightarrow H^{p+q}(\mathcal{X}_\tau, \mathcal{F})$$

converging to the cohomology of $\mathcal{F}$ in the τ-topology.

Proof. Before we start the proof we make some remarks. By Lemma 58.16.4 (and induction) all of the categories fibred in groupoids $\mathcal{U}_p$ have equalizers and all of the morphisms $f_p : \mathcal{U}_p \to \mathcal{X}$ are faithful. Let $\mathcal{I}$ be an injective object of $Ab(\mathcal{X}_\tau)$. By Lemma 58.16.5 we see $f_p^{-1}\mathcal{I}$ is an injective object of $Ab((\mathcal{U}_p)_\tau)$. Hence $f_{p,*}f_p^{-1}\mathcal{I}$ is an injective object of $Ab(\mathcal{X}_\tau)$ by Lemma 58.16.1. Hence Proposition 58.18.7 shows that the extended relative Čech complex

$$\ldots \to 0 \to \mathcal{I} \to f_{0,*}f_0^{-1}\mathcal{I} \to f_{1,*}f_1^{-1}\mathcal{I} \to f_{2,*}f_2^{-1}\mathcal{I} \to \ldots$$

is an exact complex in $Ab(\mathcal{X}_\tau)$ all of whose terms are injective. Taking global sections of this complex is exact and we see that the Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{U} \to \mathcal{X}, \mathcal{I})$ is quasi-isomorphic to $\Gamma(\mathcal{X}_\tau, \mathcal{I})[0]$.

With these preliminaries out of the way consider the two spectral sequences associated to the double complex (see Homology, Section 10.19)

$$\check{\mathcal{C}}^\bullet(\mathcal{U} \to \mathcal{X}, \mathcal{I}^\bullet)$$

where $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution in $Ab(\mathcal{X}_\tau)$. The discussion above shows that Homology, Lemma 10.19.6 applies which shows that $\Gamma(\mathcal{X}_\tau, \mathcal{I}^\bullet)$ is quasi-isomorphic to the total complex associated to the double complex. By our remarks above the complex $f_p^{-1}\mathcal{I}^\bullet$ is an injective resolution of $f_p^{-1}\mathcal{F}$. Hence the other spectral sequence is as indicated in the lemma. □

To be sure there is a version for modules as well.

Lemma 58.18.9. *Let $f : \mathcal{U} \to \mathcal{X}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. Assume*

1. *$\mathcal{F}$ is an object of $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$,*
2. *for every object x of $\mathcal{X}$ there exists a covering $\{x_i \to x\}$ in $\mathcal{X}_\tau$ such that each x_i is isomorphic to $f(u_i)$ for some object u_i of $\mathcal{U}$,*
3. *the category $\mathcal{U}$ has equalizers, and*
4. *the functor f is faithful.*

Then there is a first quadrant spectral sequence of $\Gamma(\mathcal{O}_\mathcal{X})$-modules

$$E_2^{p,q} = H^q((\mathcal{U}_p)_\tau, f_p^*\mathcal{F}) \Rightarrow H^{p+q}(\mathcal{X}_\tau, \mathcal{F})$$

converging to the cohomology of $\mathcal{F}$ in the τ-topology.

Proof. The proof of this lemma is identical to the proof of Lemma 58.18.8 except that it uses an injective resolution in $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$ and it uses Lemma 58.16.6 instead of Lemma 58.16.5. □

Here is a lemma that translates a more usual kind of covering in the kinds of coverings we have encountered above.

Lemma 58.18.10. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$.*

1. *Assume that f is representable by algebraic spaces, surjective, flat, and locally of finite presentation. Then for any object y of $\mathcal{Y}$ there exists an fppf covering $\{y_i \to y\}$ and objects x_i of $\mathcal{X}$ such that $f(x_i) \cong y_i$ in $\mathcal{Y}$.*
2. *Assume that f is representable by algebraic spaces, surjective, and smooth. Then for any object y of $\mathcal{Y}$ there exists an étale covering $\{y_i \to y\}$ and objects x_i of $\mathcal{X}$ such that $f(x_i) \cong y_i$ in $\mathcal{Y}$.*

Proof. Proof of (1). Suppose that y lies over the scheme V. We may think of y as a morphism $(Sch/V)_{fppf} \to \mathcal{Y}$. By definition the 2-fibre product $\mathcal{X} \times_\mathcal{Y} (Sch/V)_{fppf}$ is representable by an algebraic space W and the morphism $W \to V$ is surjective, flat, and locally of finite presentation. Choose a scheme U and a surjective étale morphism $U \to W$. Then $U \to V$ is also surjective, flat, and locally of finite presentation (see Morphisms of Spaces, Lemmas 42.35.7, 42.35.8, 42.6.4, 42.26.2, and 42.27.2). Hence $\{U \to V\}$ is an fppf covering. Denote x the object of $\mathcal{X}$ over U corresponding to the 1-morphism $(Sch/U)_{fppf} \to \mathcal{X}$. Then $\{f(x) \to y\}$ is the desired fppf covering of $\mathcal{Y}$.

Proof of (1). Suppose that y lies over the scheme V. We may think of y as a morphism $(Sch/V)_{fppf} \to \mathcal{Y}$. By definition the 2-fibre product $\mathcal{X} \times_{\mathcal{Y}} (Sch/V)_{fppf}$ is representable by an algebraic space W and the morphism $W \to V$ is surjective and smooth. Choose a scheme U and a surjective étale morphism $U \to W$. Then $U \to V$ is also surjective and smooth (see Morphisms of Spaces, Lemmas 42.35.6, 42.6.4, and 42.33.2). Hence $\{U \to V\}$ is a smooth covering. By More on Morphisms, Lemma 33.26.7 there exists an étale covering $\{V_i \to V\}$ such that each $V_i \to V$ factors through U. Denote x_i the object of $\mathcal{X}$ over V_i corresponding to the 1-morphism

$$(Sch/V_i)_{fppf} \to (Sch/U)_{fppf} \to \mathcal{X}.$$

Then $\{f(x_i) \to y\}$ is the desired étale covering of $\mathcal{Y}$. □

Lemma 58.18.11. *Let $f : \mathcal{U} \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{Y}$ be composable 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. Assume*

1. *$\mathcal{F}$ is an abelian sheaf on $\mathcal{X}_\tau$,*
2. *for every object x of $\mathcal{X}$ there exists a covering $\{x_i \to x\}$ in $\mathcal{X}_\tau$ such that each x_i is isomorphic to $f(u_i)$ for some object u_i of $\mathcal{U}$,*
3. *the category $\mathcal{U}$ has equalizers, and*
4. *the functor f is faithful.*

Then there is a first quadrant spectral sequence of abelian sheaves on $\mathcal{Y}_\tau$

$$E_2^{p,q} = R^q(g \circ f_p)_* f_p^{-1}\mathcal{F} \Rightarrow R^{p+q}g_*\mathcal{F}$$

where all higher direct images are computed in the τ-topology.

Proof. Note that the assumptions on $f : \mathcal{U} \to \mathcal{X}$ and $\mathcal{F}$ are indentical to those in Lemma 58.18.8. Hence the preliminary remarks made in the proof of that lemma hold here also. These remarks imply in particular that

$$0 \to g_*\mathcal{I} \to (g \circ f_0)_* f_0^{-1}\mathcal{I} \to (g \circ f_1)_* f_1^{-1}\mathcal{I} \to \ldots$$

is exact if $\mathcal{I}$ is an injective object of $Ab(\mathcal{X}_\tau)$. Having said this, consider the two spectral sequences of Homology, Section 10.19 associated to the double complex $\mathcal{C}^{\bullet,\bullet}$ with terms

$$\mathcal{C}^{p,q} = (g \circ f_p)_*\mathcal{I}^q$$

where $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution in $Ab(\mathcal{X}_\tau)$. The first spectral sequence implies, via Homology, Lemma 10.19.6, that $g_*\mathcal{I}^\bullet$ is quasi-isomorphic to the total complex associated to $\mathcal{C}^{\bullet,\bullet}$. Since $f_p^{-1}\mathcal{I}^\bullet$ is an injective resolution of $f_p^{-1}\mathcal{F}$ (see Lemma 58.16.5) the second spectral sequence has terms $E_2^{p,q} = R^q(g \circ f_p)_* f_p^{-1}\mathcal{F}$ as in the statement of the lemma. □

Lemma 58.18.12. *Let $f : \mathcal{U} \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{Y}$ be composable 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. Assume*

1. *$\mathcal{F}$ is an object of $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$,*
2. *for every object x of $\mathcal{X}$ there exists a covering $\{x_i \to x\}$ in $\mathcal{X}_\tau$ such that each x_i is isomorphic to $f(u_i)$ for some object u_i of $\mathcal{U}$,*
3. *the category $\mathcal{U}$ has equalizers, and*
4. *the functor f is faithful.*

Then there is a first quadrant spectral sequence in $Mod(\mathcal{Y}_\tau, \mathcal{O}_\mathcal{Y})$

$$E_2^{p,q} = R^q(g \circ f_p)_* f_p^{-1}\mathcal{F} \Rightarrow R^{p+q}g_*\mathcal{F}$$

where all higher direct images are computed in the τ-topology.

Proof. The proof is identical to the proof of Lemma 58.18.11 except that it uses an injective resolution in $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$ and it uses Lemma 58.16.6 instead of Lemma 58.16.5. □

58.19. Cohomology on algebraic stacks

Let $\mathcal{X}$ be an algebraic stack over S. In the sections above we have seen how to define sheaves for the étale, ..., fppf topologies on $\mathcal{X}$. In fact, we have constructed a site $\mathcal{X}_\tau$ for each $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. There is a notion of an abelian sheaf $\mathcal{F}$ on these sites. In the chapter on cohomology of sites we have explained how to define cohomology. Putting all of this together, let's define the *derived global sections*

$$R\Gamma_{Zar}(\mathcal{X}, \mathcal{F}), R\Gamma_{étale}(\mathcal{X}, \mathcal{F}), \ldots, R\Gamma_{fppf}(\mathcal{X}, \mathcal{F})$$

as $\Gamma(\mathcal{X}_\tau, \mathcal{I}^\bullet)$ where $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution in $Ab(\mathcal{X}_\tau)$. The ith cohomology group is the ith cohomology of the total derived cohomology. We will denote this

$$H^i_{Zar}(\mathcal{X}, \mathcal{F}), H^i_{étale}(\mathcal{X}, \mathcal{F}), \ldots, H^i_{fppf}(\mathcal{X}, \mathcal{F}).$$

It will turn out that $H^i_{étale} = H^i_{smooth}$ because of More on Morphisms, Lemma 33.26.7. If $\mathcal{F}$ is a presheaf of $\mathcal{O}_\mathcal{X}$-modules which is a sheaf in the τ-topology, then we use injective resolutions in $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$ to compute total derived global sections and cohomology groups; of course the end result is quasi-isomorphic resp. isomorphic by the general fact Cohomology on Sites, Lemma 19.12.4.

Sofar our only tool to compute cohomology groups is the result on Čech complexes proved above. We rephrase it here in the language of algebraic stacks for the étale and the fppf topology. Let $f : \mathcal{U} \to \mathcal{X}$ be a 1-morphism of algebraic stacks. Recall that

$$f_p : \mathcal{U}_p = \mathcal{U} \times_\mathcal{X} \ldots \times_\mathcal{X} \mathcal{U} \longrightarrow \mathcal{X}$$

is the structure morphism where there are $(p+1)$-factors. Also, recall that a sheaf on $\mathcal{X}$ is a sheaf for the fppf topology. Note that if $\mathcal{U}$ is an algebraic space, then $f : \mathcal{U} \to \mathcal{X}$ is representable by algebraic spaces, see Algebraic Stacks, Lemma 57.10.11. Thus the proposition applies in particular to a smooth cover of the algebraic stack $\mathcal{X}$ by a scheme.

Proposition 58.19.1. *Let $f : \mathcal{U} \to \mathcal{X}$ be a 1-morphism of algebraic stacks.*

(1) *Let $\mathcal{F}$ be an abelian étale sheaf on $\mathcal{X}$. Assume that f is representable by algebraic spaces, surjective, and smooth. Then there is a spectral sequence*

$$E_2^{p,q} = H^q_{étale}(\mathcal{U}_p, f_p^{-1}\mathcal{F}) \Rightarrow H^{p+q}_{étale}(\mathcal{X}, \mathcal{F})$$

(2) *Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{X}$. Assume that f is representable by algebraic spaces, surjective, flat, and locally of finite presentation. Then there is a spectral sequence*

$$E_2^{p,q} = H^q_{fppf}(\mathcal{U}_p, f_p^{-1}\mathcal{F}) \Rightarrow H^{p+q}_{fppf}(\mathcal{X}, \mathcal{F})$$

Proof. To see this we will check the hypotheses (1) -- (4) of Lemma 58.18.8. The 1-morphism f is faithful by Algebraic Stacks, Lemma 57.15.2. This proves (4). Hypothesis (3) follows from the fact that $\mathcal{U}$ is an algebraic stack, see Lemma 58.16.2. To see (2) apply Lemma 58.18.10. Condition (1) is satisfied by fiat. □

58.20. Higher direct images and algebraic stacks

Let $g : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of algebraic stacks over S. In the sections above we have constructed a morphism of ringed topoi $g : Sh(\mathcal{X}_\tau) \to Sh(\mathcal{Y}_\tau)$ for each $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. In the chapter on cohomology of sites we have explained how to define higher direct images. Hence the *derived direct image* $Rg_*\mathcal{F}$ is defined as $g_*\mathcal{I}^\bullet$ where $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution in $Ab(\mathcal{X}_\tau)$. The ith higher direct image $R^ig_*\mathcal{F}$ is the ith cohomology of the derived direct image. Important: it matters which topology τ is used here!

If $\mathcal{F}$ is a presheaf of $\mathcal{O}_\mathcal{X}$-modules which is a sheaf in the τ-topology, then we use injective resolutions in $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$ to compute derived direct image and higher direct images.

Sofar our only tool to compute the higher direct images of g_* is the result on Čech complexes proved above. This requires the choice of a ``covering'' $f : \mathcal{U} \to \mathcal{X}$. If $\mathcal{U}$ is an algebraic space, then $f : \mathcal{U} \to \mathcal{X}$ is representable by algebraic spaces, see Algebraic Stacks, Lemma 57.10.11. Thus the proposition applies in particular to a smooth cover of the algebraic stack $\mathcal{X}$ by a scheme.

Proposition 58.20.1. *Let $f : \mathcal{U} \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{Y}$ be composable 1-morphisms of algebraic stacks.*

(1) *Assume that f is representable by algebraic spaces, surjective and smooth.*
   (a) *If $\mathcal{F}$ is in $Ab(\mathcal{X}_{étale})$ then there is a spectral sequence*
$$E_2^{p,q} = R^q(g \circ f_p)_* f_p^{-1}\mathcal{F} \Rightarrow R^{p+q}g_*\mathcal{F}$$
   in $Ab(\mathcal{Y}_{étale})$ with higher direct images computed in the étale topology.
   (b) *If $\mathcal{F}$ is in $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ then there is a spectral sequence*
$$E_2^{p,q} = R^q(g \circ f_p)_* f_p^{-1}\mathcal{F} \Rightarrow R^{p+q}g_*\mathcal{F}$$
   in $Mod(\mathcal{Y}_{étale}, \mathcal{O}_\mathcal{Y})$.

(2) *Assume that f is representable by algebraic spaces, surjective, flat, and locally of finite presentation.*
   (a) *If $\mathcal{F}$ is in $Ab(\mathcal{X})$ then there is a spectral sequence*
$$E_2^{p,q} = R^q(g \circ f_p)_* f_p^{-1}\mathcal{F} \Rightarrow R^{p+q}g_*\mathcal{F}$$
   in $Ab(\mathcal{Y})$ with higher direct images computed in the fppf topology.
   (b) *If $\mathcal{F}$ is in $Mod(\mathcal{O}_\mathcal{X})$ then there is a spectral sequence*
$$E_2^{p,q} = R^q(g \circ f_p)_* f_p^{-1}\mathcal{F} \Rightarrow R^{p+q}g_*\mathcal{F}$$
   in $Mod(\mathcal{O}_\mathcal{Y})$.

Proof. To see this we will check the hypotheses (1) -- (4) of Lemma 58.18.11 and Lemma 58.18.12. The 1-morphism f is faithful by Algebraic Stacks, Lemma 57.15.2. This proves (4). Hypothesis (3) follows from the fact that $\mathcal{U}$ is an algebraic stack, see Lemma 58.16.2. To see (2) apply Lemma 58.18.10. Condition (1) is satisfied by fiat in all four cases. □

Here is a description of higher direct images for a morphism of algebraic stacks.

Lemma 58.20.2. *Let S be a scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of algebraic stacks*[4] *over S. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Let $\mathcal{F}$ be an object of $Ab(\mathcal{X}_\tau)$*

[4] This result should hold for any 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$.

or $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$. Then the sheaf $R^if_*\mathcal{F}$ is the sheaf associated to the presheaf

$$y \longmapsto H^i_\tau\Big((Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X},\ pr^{-1}\mathcal{F}\Big)$$

Here y is a typical object of $\mathcal{Y}$ lying over the scheme V.

Proof. Choose an injective resolution $\mathcal{F}[0] \to \mathcal{I}^\bullet$. By the formula for pushforward (58.5.0.1) we see that $R^if_*\mathcal{F}$ is the sheaf associated to the presheaf which associates to y the cohomology of the complex

$$\begin{matrix} \Gamma\Big((Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X},\ \mathrm{pr}^{-1}\mathcal{I}^{i-1}\Big) \\ \downarrow \\ \Gamma\Big((Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X},\ \mathrm{pr}^{-1}\mathcal{I}^{i}\Big) \\ \downarrow \\ \Gamma\Big((Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X},\ \mathrm{pr}^{-1}\mathcal{I}^{i+1}\Big) \end{matrix}$$

Since pr^{-1} is exact, it suffices to show that pr^{-1} preserves injectives. This follows from Lemmas 58.16.5 and 58.16.6 as well as the fact that pr is a representable morphism of algebraic stacks (so that pr is faithful by Algebraic Stacks, Lemma 57.15.2 and that $(Sch/V)_{fppf}\times_{y,\mathcal{Y}}\mathcal{X}$ has equalizers by Lemma 58.16.2). □

Here is a trivial base change result.

Lemma 58.20.3. *Let S be a scheme. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Let*

$$\begin{tikzcd} \mathcal{Y}' \times_\mathcal{Y} \mathcal{X} \arrow[r, "g'"'] \arrow[d, "f'"'] & \mathcal{X} \arrow[d, "f"] \\ \mathcal{Y}' \arrow[r, "g"] & \mathcal{Y} \end{tikzcd}$$

be a 2-cartesian diagram of algebraic stacks over S. Then the base change map is an isomorphism

$$g^{-1}Rf_*\mathcal{F} \longrightarrow Rf'_*(g')^{-1}\mathcal{F}$$

functorial for $\mathcal{F}$ in $Ab(\mathcal{X}_\tau)$ or $\mathcal{F}$ in $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$.

Proof. The isomorphism $g^{-1}f_*\mathcal{F} = f'_*(g')^{-1}\mathcal{F}$ is Lemma 58.5.1 (and it holds for arbitrary presheaves). For the derived direct images, there is a base change map because the morphisms g and g' are flat, see Cohomology on Sites, Section 19.15. To see that this map is a quasi-isomorphism we can use that for an object y' of $\mathcal{Y}'$ over a scheme V there is an equivalence

$$(Sch/V)_{fppf} \times_{g(y'),\mathcal{Y}} \mathcal{X} = (Sch/V)_{fppf} \times_{y',\mathcal{Y}'} (\mathcal{Y}' \times_\mathcal{Y} \mathcal{X})$$

We conclude that the induced map $g^{-1}R^if_*\mathcal{F} \to R^if'_*(g')^{-1}\mathcal{F}$ is an isomorphism by Lemma 58.20.2. □

58.21. Comparison

In this section we collect some results on comparing cohomology defined using stacks and using algebraic spaces.

Lemma 58.21.1. *Let S be a scheme. Let $\mathcal{X}$ be an algebraic stack over S representable by the algebraic space F.*

(1) *$\mathcal{I}|_{F_{étale}}$ is injective in $Ab(F_{étale})$ for $\mathcal{I}$ injective in $Ab(\mathcal{X}_{étale})$, and*

(2) $\mathcal{I}|_{F_{\acute{e}tale}}$ *is injective in* $Mod(F_{\acute{e}tale}, \mathcal{O}_F)$ *for* $\mathcal{I}$ *injective in* $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O})$.

Proof. This follows formally from the fact that the restriction functor $\pi_{F,*} = i_F^{-1}$ (see Lemma 58.10.1) is an exact left adjoint of $i_{F,*}$, see Homology, Lemma 10.22.1. □

Lemma 58.21.2. *Let S be a scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks over S. Assume $\mathcal{X}$, $\mathcal{Y}$ are representable by algebraic spaces F, G. Denote $f : F \to G$ the induced morphism of algebraic spaces.*

(1) *For any $\mathcal{F} \in Ab(\mathcal{X}_{\acute{e}tale})$ we have*
$$(Rf_*\mathcal{F})|_{G_{\acute{e}tale}} = Rf_{small,*}(\mathcal{F}|_{F_{\acute{e}tale}})$$
in $D(G_{\acute{e}tale})$.

(2) *For any object $\mathcal{F}$ of $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_{\mathcal{X}})$ we have*
$$(Rf_*\mathcal{F})|_{G_{\acute{e}tale}} = Rf_{small,*}(\mathcal{F}|_{F_{\acute{e}tale}})$$
in $D(\mathcal{O}_G)$.

Proof. Follows immediately from Lemma 58.21.1 and (58.10.2.1) on choosing an injective resolution of $\mathcal{F}$. □

Lemma 58.21.3. *Let S be a scheme. Consider a 2-fibre product square*

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "g'"] \arrow[d, "f'"'] & \mathcal{X} \arrow[d, "f"] \\ \mathcal{Y}' \arrow[r, "g"] & \mathcal{Y} \end{tikzcd}$$

of algebraic stacks over S. Assume that f is representable by algebraic spaces and that $\mathcal{Y}'$ is representable by an algebraic space G'. Then $\mathcal{X}'$ is representable by an algebraic space F' and denoting $f' : F' \to G'$ the induced morphism of algebraic spaces we have

$$g^{-1}(Rf_*\mathcal{F})|_{G'_{\acute{e}tale}} = Rf'_{small,*}((g')^{-1}\mathcal{F}|_{F'_{\acute{e}tale}})$$

for any $\mathcal{F}$ in $Ab(\mathcal{X}_{\acute{e}tale})$ or in $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_{\mathcal{X}})$

Proof. Follows formally on combining Lemmas 58.20.3 and 58.21.2. □

58.22. Change of topology

Here is a technical lemma which tells us that the fppf cohomology of a locally quasi-coherent sheaf is equal to its étale cohomology provided the comparison maps are isomorphisms for morphisms of $\mathcal{X}$ lying over flat morphisms.

Lemma 58.22.1. *Let S be a scheme. Let $\mathcal{X}$ be an algebraic stack over S. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_{\mathcal{X}}$-modules. Assume*

(a) *$\mathcal{F}$ is locally quasi-coherent, and*

(b) *for any morphism $\varphi : x \to y$ of $\mathcal{X}$ which lies over a morphism of schemes $f : U \to V$ which is flat and locally of finite presentation the comparison map $c_\varphi : f^*_{small}\mathcal{F}|_{V_{\acute{e}tale}} \to \mathcal{F}|_{U_{\acute{e}tale}}$ of (58.9.4.1) is an isomorphism.*

Then $\mathcal{F}$ is a sheaf for the fppf topology.

Proof. Let $\{x_i \to x\}$ be an fppf covering of $\mathcal{X}$ lying over the fppf covering $\{f_i : U_i \to U\}$ of schemes over S. By assumption the restriction $\mathcal{G} = \mathcal{F}|_{U_{étale}}$ is quasi-coherent and the comparison maps $f_{i,small}^*\mathcal{G} \to \mathcal{F}|_{U_{i,étale}}$ are isomorphisms. Hence the sheaf condition for $\mathcal{F}$ and the covering $\{x_i \to x\}$ is equivalent to the sheaf condition for $\mathcal{G}^a$ on $(Sch/U)_{fppf}$ and the covering $\{U_i \to U\}$ which holds by Descent, Lemma 31.6.1. □

Lemma 58.22.2. *Let S be a scheme. Let $\mathcal{X}$ be an algebraic stack over S. Let $\mathcal{F}$ be a presheaf $\mathcal{O}_\mathcal{X}$-module such that*

(a) *$\mathcal{F}$ is locally quasi-coherent, and*
(b) *for any morphism $\varphi : x \to y$ of $\mathcal{X}$ which lies over a morphism of schemes $f : U \to V$ which is flat and locally of finite presentation, the comparison map $c_\varphi : f_{small}^*\mathcal{F}|_{V_{étale}} \to \mathcal{F}|_{U_{étale}}$ of (58.9.4.1) is an isomorphism.*

Then $\mathcal{F}$ is an $\mathcal{O}_\mathcal{X}$-module and we have the following

(1) *If $\epsilon : \mathcal{X}_{fppf} \to \mathcal{X}_{étale}$ is the comparison morphism, then $R\epsilon_*\mathcal{F} = \epsilon_*\mathcal{F}$.*
(2) *The cohomology groups $H^p_{fppf}(\mathcal{X}, \mathcal{F})$ are equal to the cohomology groups computed in the étale topology on $\mathcal{X}$. Similarly for the cohomology groups $H^p_{fppf}(x, \mathcal{F})$ and the derived versions $R\Gamma(\mathcal{X}, \mathcal{F})$ and $R\Gamma(x, \mathcal{F})$.*
(3) *If $f : \mathcal{X} \to \mathcal{Y}$ is a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$ then $R^if_*\mathcal{F}$ is equal to the fppf-sheafification of the higher direct image computed in the étale cohomology. Similarly for derived pushforward.*

Proof. The assertion that $\mathcal{F}$ is an $\mathcal{O}_\mathcal{X}$-module follows from Lemma 58.22.1. Note that ϵ is a morphism of sites given by the identity functor on $\mathcal{X}$. The sheaf $R^p\epsilon_*\mathcal{F}$ is therefore the sheaf associated to the presheaf $x \mapsto H^p_{fppf}(x, \mathcal{F})$, see Cohomology on Sites, Lemma 19.8.4. To prove (1) it suffices to show that $H^p_{fppf}(x, \mathcal{F}) = 0$ for $p > 0$ whenever x lies over an affine scheme U. By Lemma 58.15.1 we have $H^p_{fppf}(x, \mathcal{F}) = H^p((Sch/U)_{fppf}, x^{-1}\mathcal{F})$. Combining Descent, Lemma 31.7.4 with Coherent, Lemma 25.2.2 we see that these cohomology groups are zero.

We have seen above that $\epsilon_*\mathcal{F}$ and $\mathcal{F}$ are the sheaves on $\mathcal{X}_{étale}$ and $\mathcal{X}_{fppf}$ corresponding to the same presheaf on $\mathcal{X}$ (and this is true more generally for any sheaf in the fppf topology on $\mathcal{X}$). We often abusively identify $\mathcal{F}$ and $\epsilon_*\mathcal{F}$ and this is the sense in which parts (2) and (3) of the lemma should be understood. Thus part (2) follows formally from (1) and the Leray spectral sequence, see Cohomology on Sites, Lemma 19.14.5.

Finally we prove (3). The sheaf $R^if_*\mathcal{F}$ (resp. $Rf_{étale,*}\mathcal{F}$) is the sheaf associated to the presheaf

$$y \longmapsto H^i_\tau\Big((Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X},\ \mathrm{pr}^{-1}\mathcal{F}\Big)$$

where τ is $fppf$ (resp. $étale$), see Lemma 58.20.2. Note that $\mathrm{pr}^{-1}\mathcal{F}$ satisfies properties (a) and (b) also (by Lemmas 58.11.6 and 58.9.3), hence these two presheaves are equal by (2). This immediately implies (3). □

We will use the following lemma to compare étale cohomology of sheaves on algebraic stacks with cohomology on the lisse-étale topos.

Lemma 58.22.3. *Let S be a scheme. Let $\mathcal{X}$ be an algebraic stack over S. Let $\tau = étale$ (resp. $\tau = fppf$). Let $\mathcal{X}' \subset \mathcal{X}$ be a full subcategory with the following properties*

(1) *if $x \to x'$ is a morphism of $\mathcal{X}$ which lies over a smooth (resp. flat and locally finitely presented) morphism of schemes and $x' \in Ob(\mathcal{X}')$, then $x \in Ob(\mathcal{X}')$, and*
(2) *there exists an object $x \in Ob(\mathcal{X}')$ lying over a scheme U such that the associated 1-morphism $x : (Sch/U)_{fppf} \to \mathcal{X}$ is smooth and surjective.*

We get a site $\mathcal{X}'_\tau$ by declaring a covering of $\mathcal{X}'$ to be any family of morphisms $\{x_i \to x\}$ in $\mathcal{X}'$ which is a covering in $\mathcal{X}_\tau$. Then the inclusion functor $\mathcal{X}' \to \mathcal{X}_\tau$ is fully faithful, cocontinuous, and continuous, whence defines a morphism of topoi

$$g : Sh(\mathcal{X}'_\tau) \longrightarrow Sh(\mathcal{X}_\tau)$$

and $H^p(\mathcal{X}'_\tau, g^{-1}\mathcal{F}) = H^p(\mathcal{X}_\tau, \mathcal{F})$ for all $p \geq 0$ and all $\mathcal{F} \in Ab(\mathcal{X}_\tau)$.

Proof. Note that assumption (1) implies that if $\{x_i \to x\}$ is a covering of $\mathcal{X}_\tau$ and $x \in Ob(\mathcal{X}')$, then we have $x_i \in Ob(\mathcal{X}')$. Hence we see that $\mathcal{X}' \to \mathcal{X}$ is continuous and cocontinuous as the coverings of objects of $\mathcal{X}'_\tau$ agree with their coverings seen as objects of $\mathcal{X}_\tau$. We obtain the morphism g and the functor g^{-1} is identified with the restriction functor, see Sites, Lemma 9.19.5.

In particular, if $\{x_i \to x\}$ is a covering in $\mathcal{X}'_\tau$, then for any abelian sheaf $\mathcal{F}$ on $\mathcal{X}$ then

$$\check{H}^p(\{x_i \to x\}, g^{-1}\mathcal{F}) = \check{H}^p(\{x_i \to x\}, \mathcal{F})$$

Thus if $\mathcal{I}$ is an injective abelian sheaf on $\mathcal{X}_\tau$ then we see that the higher Čech cohomology groups are zero (Cohomology on Sites, Lemma 19.11.2). Hence $H^p(x, g^{-1}\mathcal{I}) = 0$ for all objects x of $\mathcal{X}'$ (Cohomology on Sites, Lemma 19.11.8). In other words injective abelian sheaves on $\mathcal{X}_\tau$ are right acyclic for the functor $H^0(x, g^{-1}-)$. It follows that $H^p(x, g^{-1}\mathcal{F}) = H^p(x, \mathcal{F})$ for all $\mathcal{F} \in Ab(\mathcal{X})$ and all $x \in Ob(\mathcal{X}')$.

Choose an object $x \in \mathcal{X}'$ lying over a scheme U as in assumption (2). In particular $\mathcal{X}/x \to \mathcal{X}$ is a morphism of algebraic stacks which representable by algebraic spaces, surjective, and smooth. (Note that $\mathcal{X}/x$ is equivalent to $(Sch/U)_{fppf}$, see Lemma 58.9.1.) The map of sheaves

$$h_x \longrightarrow *$$

in $Sh(\mathcal{X}_\tau)$ is surjective. Namely, for any object x' of $\mathcal{X}$ there exists a τ-covering $\{x'_i \to x'\}$ such that there exist morphisms $x'_i \to x$, see Lemma 58.18.10. Since g is exact, the map of sheaves

$$g^{-1}h_x \longrightarrow * = g^{-1}*$$

in $Sh(\mathcal{X}'_\tau)$ is surjective also. Let $h_{x,n}$ be the $(n+1)$-fold product $h_x \times \ldots \times h_x$. Then we have spectral sequences

$$E_1^{p,q} = H^q(h_{x,p}, \mathcal{F}) \Rightarrow H^{p+q}(\mathcal{X}_\tau, \mathcal{F}) \tag{58.22.3.1}$$

and

$$E_1^{p,q} = H^q(g^{-1}h_{x,p}, g^{-1}\mathcal{F}) \Rightarrow H^{p+q}(\mathcal{X}'_\tau, g^{-1}\mathcal{F}) \tag{58.22.3.2}$$

see Cohomology on Sites, Lemma 19.13.2.

Case I: $\mathcal{X}$ has a final object x which is also an object of $\mathcal{X}'$. This case follows immediately from the discussion in the second paragraph above.

Case II: $\mathcal{X}$ is representable by an algebraic space F. In this case the sheaves $h_{x,n}$ are representable by an object x_n in $\mathcal{X}$. (Namely, if $\mathcal{S}_F = \mathcal{X}$ and $x : U \to F$ is the given object, then $h_{x,n}$ is representable by the object $U \times_F \ldots \times_F U \to F$ of $\mathcal{S}_F$.) It follows that $H^q(h_{x,p}, \mathcal{F}) = H^q(x_p, \mathcal{F})$. The morphisms $x_n \to x$ lie over smooth morphisms of schemes, hence $x_n \in \mathcal{X}'$ for all n. Hence $H^q(g^{-1}h_{x,p}, g^{-1}\mathcal{F}) = H^q(x_p, g^{-1}\mathcal{F})$. Thus in the two spectral sequences (58.22.3.1) and (58.22.3.2) above the $E_1^{p,q}$ terms agree by the discussion in the second paragraph. The lemma follows in Case II as well.

Case III: $\mathcal{X}$ is an algebraic stack. We claim that in this case the cohomology groups $H^q(h_{x,p}, \mathcal{F})$ and $H^q(g^{-1}h_{x,n}, g^{-1}\mathcal{F})$ agree by Case II above. Once we have proved this the result will follow as before.

Namely, consider the category $\mathcal{X}/h_{x,n}$, see Sites, Lemma 9.26.3. Since $h_{x,n}$ is the $(n+1)$-fold product of h_x an object of this category is an $(n+2)$-tuple $(y, s_0, \ldots, s_n)$ where y is an object of $\mathcal{X}$ and each $s_i : y \to x$ is a morphism of $\mathcal{X}$. This is a category over $(Sch/S)_{fppf}$. There is an equivalence

$$\mathcal{X}/h_{x,n} \longrightarrow (Sch/U)_{fppf} \times_{\mathcal{X}} \ldots \times_{\mathcal{X}} (Sch/U)_{fppf} =: \mathcal{U}_n$$

over $(Sch/S)_{fppf}$. Namely, if $x : (Sch/U)_{fppf} \to \mathcal{X}$ also denotes the 1-morphism associated with x and $p : \mathcal{X} \to (Sch/S)_{fppf}$ the structure functor, then we can think of $(y, s_0, \ldots, s_n)$ as $(y, f_0, \ldots, f_n, \alpha_0, \ldots, \alpha_n)$ where y is an object of $\mathcal{X}$, $f_i : p(y) \to p(x)$ is a morphism of schemes, and $\alpha_i : y \to x(f_i)$ an isomorphism. The category of $2n+3$-tuples $(y, f_0, \ldots, f_n, \alpha_0, \ldots, \alpha_n)$ is an incarnation of the $(n+1)$-fold fibred product $\mathcal{U}_n$ of algebraic stacks displayed above, as we discussed in Section 58.17. By Cohomology on Sites, Lemma 19.13.3 we have

$$H^p(\mathcal{U}_n, \mathcal{F}|_{\mathcal{U}_n}) = H^p(\mathcal{X}/h_{x,n}, \mathcal{F}|_{\mathcal{X}/h_{x,n}}) = H^p(h_{x,n}, \mathcal{F}).$$

Finally, we discuss the ``primed'' analogue of this. Namely, $\mathcal{X}'/h_{x,n}$ corresponds, via the equivalence above to the full subcategory $\mathcal{U}_n' \subset \mathcal{U}_n$ consisting of those tuples $(y, f_0, \ldots, f_n, \alpha_0, \ldots, \alpha_n)$ with $y \in \mathcal{X}'$. Hence certainly property (1) of the statement of the lemma holds for the inclusion $\mathcal{U}_n' \subset \mathcal{U}_n$. To see property (2) choose an object $\xi = (y, s_0, \ldots, s_n)$ which lies over a scheme W such that $(Sch/W)_{fppf} \to \mathcal{U}_n$ is smooth and surjective (this is possible as $\mathcal{U}_n$ is an algebraic stack). Then $(Sch/W)_{fppf} \to \mathcal{U}_n \to (Sch/U)_{fppf}$ is smooth as a composition of base changes of the morphism $x : (Sch/U)_{fppf} \to \mathcal{X}$, see Algebraic Stacks, Lemmas 57.10.6 and 57.10.5. Thus axiom (1) for $\mathcal{X}$ implies that y is an object of $\mathcal{X}'$ whence ξ is an object of $\mathcal{U}_n'$. Using again

$$H^p(\mathcal{U}_n', \mathcal{F}|_{\mathcal{U}_n'}) = H^p(\mathcal{X}'/h_{x,n}, \mathcal{F}|_{\mathcal{X}'/h_{x,n}}) = H^p(g^{-1}h_{x,n}, g^{-1}\mathcal{F}).$$

we now can use Case II for $\mathcal{U}_n' \subset \mathcal{U}_n$ to conclude. □

58.23. Other chapters

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Commutative Algebra
(8) Brauer Groups
(9) Sites and Sheaves
(10) Homological Algebra
(11) Derived Categories
(12) More on Algebra
(13) Smoothing Ring Maps
(14) Simplicial Methods
(15) Sheaves of Modules
(16) Modules on Sites
(17) Injectives
(18) Cohomology of Sheaves
(19) Cohomology on Sites
(20) Hypercoverings
(21) Schemes
(22) Constructions of Schemes
(23) Properties of Schemes
(24) Morphisms of Schemes

(25) Coherent Cohomology
(26) Divisors
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(28) Varieties
(29) Chow Homology
(30) Topologies on Schemes
(31) Descent
(32) Adequate Modules
(33) More on Morphisms
(34) More on Flatness
(35) Groupoid Schemes
(36) More on Groupoid Schemes
(37) Étale Morphisms of Schemes
(38) Étale Cohomology
(39) Crystalline Cohomology
(40) Algebraic Spaces
(41) Properties of Algebraic Spaces
(42) Morphisms of Algebraic Spaces
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(46) More on Morphisms of Spaces
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CHAPTER 59

Criteria for Representability

59.1. Introduction

The purpose of this chapter is to find criteria guaranteeing that a a stack in groupoids over the category of schemes with the fppf topology is an algebraic stack. Historically, this often involved proving that certain functors were representable, see Grothendieck's lectures [**Gro95a**], [**Gro95b**], [**Gro95e**], [**Gro95f**], [**Gro95c**], and [**Gro95d**]. This explains the title of this chapter. Another important source of this material comes from the work of Artin, see [**Art69c**], [**Art70a**], [**Art73a**], [**Art71c**], [**Art71a**], [**Art69a**], [**Art69e**], and [**Art74a**].

59.2. Conventions

The conventions we use in this chapter are the same as those in the chapter on algebraic stacks, see Algebraic Stacks, Section 57.2.

59.3. What we already know

The analogue of this chapter for algebraic spaces is the chapter entitled ``Bootstrap'', see Bootstrap, Section 54.1. That chapter already contains some representability results. Moreover, some of the preliminary material treated there we already have worked out in the chapter on algebraic stacks. Here is a list:

(1) We discuss morphisms of presheaves representable by algebraic spaces in Bootstrap, Section 54.3. In Algebraic Stacks, Section 57.9 we discuss the notion of a 1-morphism of categories fibred in groupoids being representable by algebraic spaces.
(2) We discuss properties of morphisms of presheaves representable by algebraic spaces in Bootstrap, Section 54.4. In Algebraic Stacks, Section 57.10 we discuss the notion of a 1-morphism of categories fibred in groupoids being representable by algebraic spaces.
(3) We proved that if F is a sheaf whose diagonal is representable by algebraic spaces and which has an étale covering by an algebraic space, then F is an algebraic space, see Bootstrap, Theorem 54.6.1. (This is a weak version of the result in the next item on the list.)
(4) We proved that if F is a sheaf and if there exists an algebraic space U and a morphism $U \to F$ which is representable by algebraic spaces, surjective, flat, and locally of finite presentation, then F is an algebraic space, see Bootstrap, Theorem 54.10.1.
(5) We have also proved the ``smooth'' analogue of (4) for algebraic stacks: If $\mathcal{X}$ is a stack in groupoids over $(Sch/S)_{fppf}$ and if there exists a stack in groupoids $\mathcal{U}$ over $(Sch/S)_{fppf}$ which is representably by an algebraic space and a 1-morphism $u : \mathcal{U} \to \mathcal{X}$ which is representable by algebraic spaces, surjective, and smooth then $\mathcal{X}$ is an algebraic stack, see Algebraic Stacks, Lemma 57.15.3.

Our first task now is to prove the analogue of (4) for algebraic stacks in general; it is Theorem 59.16.1.

59.4. Morphisms of stacks in groupoids

This section is preliminary and should be skipped on a first reading.

Lemma 59.4.1. *Let $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If $\mathcal{X} \to \mathcal{Z}$ and $\mathcal{Y} \to \mathcal{Z}$ are representable by algebraic spaces and étale so is $\mathcal{X} \to \mathcal{Y}$.*

Proof. Let $\mathcal{U}$ be a representable category fibred in groupoids over S. Let $f : \mathcal{U} \to \mathcal{Y}$ be a 1-morphism. We have to show that $\mathcal{X} \times_{\mathcal{Y}} \mathcal{U}$ is representable by an algebraic space and étale over $\mathcal{U}$. Consider the composition $h : \mathcal{U} \to \mathcal{Z}$. Then

$$\mathcal{X} \times_{\mathcal{Z}} \mathcal{U} \longrightarrow \mathcal{Y} \times_{\mathcal{Z}} \mathcal{U}$$

is a 1-morphism between categories fibres in groupoids which are both representable by algebraic spaces and both étale over $\mathcal{U}$. Hence by Properties of Spaces, Lemma 41.13.6 this is represented by an étale morphism of algebraic spaces. Finally, we obtain the result we want as the morphism f induces a morphism $\mathcal{U} \to \mathcal{Y} \times_{\mathcal{Z}} \mathcal{U}$ and we have

$$\mathcal{X} \times_{\mathcal{Y}} \mathcal{U} = (\mathcal{X} \times_{\mathcal{Z}} \mathcal{U}) \times_{(\mathcal{Y} \times_{\mathcal{Z}} \mathcal{U})} \mathcal{U}.$$

□

Lemma 59.4.2. *Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be stacks in groupoids over $(Sch/S)_{fppf}$. Suppose that $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Z} \to \mathcal{Y}$ are 1-morphisms. If*

1. *$\mathcal{Y}$, $\mathcal{Z}$ are representable by algebraic spaces Y, Z over S,*
2. *the associated morphism of algebraic spaces $Y \to Z$ is surjective, flat and locally of finite presentation, and*
3. *$\mathcal{Y} \times_{\mathcal{Z}} \mathcal{X}$ is a stack in setoids,*

then $\mathcal{X}$ is a stack in setoids.

Proof. This is a special case of Stacks, Lemma 50.6.10. □

The following lemma is the analogue of Algebraic Stacks, Lemma 57.15.3 and will be superseded by the stronger Theorem 59.16.1.

Lemma 59.4.3. *Let S be a scheme. Let $u : \mathcal{U} \to \mathcal{X}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. If*

1. *$\mathcal{U}$ is representable by an algebraic space, and*
2. *u is representable by algebraic spaces, surjective, flat and locally of finite presentation,*

then $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ representable by algebraic spaces.

Proof. Given two schemes T_1, T_2 over S denote $\mathcal{T}_i = (Sch/T_i)_{fppf}$ the associated representable fibre categories. Suppose given 1-morphisms $f_i : \mathcal{T}_i \to \mathcal{X}$. According to Algebraic Stacks, Lemma 57.10.11 it suffices to prove that the 2-fibered product $\mathcal{T}_1 \times_{\mathcal{X}} \mathcal{T}_2$ is representable by an algebraic space. By Stacks, Lemma 50.6.8 this is in any case a stack in setoids. Thus $\mathcal{T}_1 \times_{\mathcal{X}} \mathcal{T}_2$ corresponds to some sheaf F on $(Sch/S)_{fppf}$, see Stacks, Lemma 50.6.3. Let U be the algebraic space which represents $\mathcal{U}$. By assumption

$$\mathcal{T}_i' = \mathcal{U} \times_{u, \mathcal{X}, f_i} \mathcal{T}_i$$

is representable by an algebraic space T_i' over S. Hence $\mathcal{T}_1' \times_{\mathcal{U}} \mathcal{T}_2'$ is representable by the algebraic space $T_1' \times_U T_2'$. Consider the commutative diagram

$$\begin{tikzcd}
& \mathcal{T}_1 \times_{\mathcal{X}} \mathcal{T}_2 \arrow[rr] \arrow[dd] & & \mathcal{T}_1 \arrow[dd] \\
\mathcal{T}_1' \times_{\mathcal{U}} \mathcal{T}_2' \arrow[ur] \arrow[rr] \arrow[dd] & & \mathcal{T}_1' \arrow[ur] \arrow[dd] & \\
& \mathcal{T}_2 \arrow[rr] & & \mathcal{X} \\
\mathcal{T}_2' \arrow[ur] \arrow[rr] & & \mathcal{U} \arrow[ur] &
\end{tikzcd}$$

In this diagram the bottom square, the right square, the back square, and the front square are 2-fibre products. A formal argument then shows that $\mathcal{T}_1' \times_{\mathcal{U}} \mathcal{T}_2' \to \mathcal{T}_1 \times_{\mathcal{X}} \mathcal{T}_2$ is the ``base change" of $\mathcal{U} \to \mathcal{X}$, more precisely the diagram

$$\begin{tikzcd}
\mathcal{T}_1' \times_{\mathcal{U}} \mathcal{T}_2' \arrow[r] \arrow[d] & \mathcal{U} \arrow[d] \\
\mathcal{T}_1 \times_{\mathcal{X}} \mathcal{T}_2 \arrow[r] & \mathcal{X}
\end{tikzcd}$$

is a 2-fibre square. Hence $T_1' \times_U T_2' \to F$ is representable by algebraic spaces, flat, locally of finite presentation and surjective, see Algebraic Stacks, Lemmas 57.9.6, 57.9.7, 57.10.4, and 57.10.6. Therefore F is an algebraic space by Bootstrap, Theorem 54.10.1 and we win. □

59.5. Limit preserving on objects

Let S be a scheme. Let $p : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. We will say that p is *limit preserving on objects* if the following condition holds: Given any data consisting of

(1) an affine scheme $U = lim_{i \in I} U_i$ which is written as the directed limit of affine schemes U_i over S,
(2) an object y_i of $\mathcal{Y}$ over U_i for some i,
(3) an object x of $\mathcal{X}$ over U, and
(4) an isomorphism $\gamma : p(x) \to y_i|_U$,

then there exists an $i' \geq i$, an object $x_{i'}$ of $\mathcal{X}$ over $U_{i'}$, an isomorphism $\beta : x_{i'}|_U \to x$, and an isomorphism $\gamma_{i'} : p(x_{i'}) \to y_i|_{U_{i'}}$ such that

$$\begin{tikzcd}
p(x_{i'}|_U) \arrow[r, "\gamma_{i'}|_U"'] \arrow[d, "p(\beta)"'] & (y_i|_{U_{i'}})|_U \arrow[d, equal] \\
p(x) \arrow[r, "\gamma"] & y_i|_U
\end{tikzcd} \tag{59.5.0.1}$$

commutes. In this situation we say that ``$(i', x_{i'}, \beta, \gamma_{i'})$ is a *solution* to the problem posed by our data (1), (2), (3), (4)". The motivation for this definition comes from More on Morphisms of Spaces, Lemma 46.4.2.

Lemma 59.5.1. *Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Z} \to \mathcal{Y}$ be 1-morphisms of stacks in groupoids over $(Sch/S)_{fppf}$. If $p : \mathcal{X} \to \mathcal{Y}$ is limit preserving on objects, then so is the base change $p' : \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{Z}$ of p by q.*

Proof. This is formal. Let $U = lim_{i \in I} U_i$ be the directed limit of affine schemes U_i over S, let z_i be an object of $\mathcal{Z}$ over U_i for some i, let w be an object of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ over U, and let $\delta : p'(w) \to z_i|_U$ be an isomorphism. We may write $w = (U, x, z, \alpha)$ for some object x of $\mathcal{X}$ over U and object z of $\mathcal{Z}$ over U and isomorphism $\alpha : p(x) \to q(z)$. Note that $p'(w) = z$ hence $\delta : z \to z_i|_U$. Set $y_i = q(z_i)$ and $\gamma = q(\delta) \circ \alpha : p(x) \to y_i|_U$. As p is limit preserving on objects there exists an $i' \geq i$ and an object $x_{i'}$ of $\mathcal{X}$ over $U_{i'}$ as well as isomorphisms $\beta : x_{i'}|_U \to x$ and $\gamma_{i'} : p(x_{i'}) \to y_i|_{U_{i'}}$ such that (59.5.0.1) commutes. Then we consider the object $w_{i'} = (U_{i'}, x_{i'}, z_i|_{U_{i'}}, \gamma_{i'})$ of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ over $U_{i'}$ and define isomorphisms

$$w_{i'}|_U = (U, x_{i'}|_U, z_i|_U, \gamma_{i'}|_U) \xrightarrow{(\beta, \delta^{-1})} (U, x, z, \alpha) = w$$

and

$$p'(w_{i'}) = z_i|_{U_{i'}} \xrightarrow{\text{id}} z_i|_{U_{i'}}.$$

These combine to give a solution to the problem. □

Lemma 59.5.2. *Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms of stacks in groupoids over $(Sch/S)_{fppf}$. If p and q are limit preserving on objects, then so is the composition $q \circ p$.*

Proof. This is formal. Let $U = lim_{i \in I} U_i$ be the directed limit of affine schemes U_i over S, let z_i be an object of $\mathcal{Z}$ over U_i for some i, let x be an object of $\mathcal{X}$ over U, and let $\gamma : q(p(x)) \to z_i|_U$ be an isomorphism. As q is limit preserving on objects there exist an $i' \geq i$, an object $y_{i'}$ of $\mathcal{Y}$ over $U_{i'}$, an isomorphism $\beta : y_{i'}|_U \to p(x)$, and an isomorphism $\gamma_{i'} : q(y_{i'}) \to z_i|_{U_{i'}}$ such that (59.5.0.1) is commutative. As p is limit preserving on objects there exist an $i'' \geq i'$, an object $x_{i''}$ of $\mathcal{X}$ over $U_{i''}$, an isomorphism $\beta' : x_{i''}|_U \to x$, and an isomorphism $\gamma'_{i''} : p(x_{i''}) \to y_{i'}|_{U_{i''}}$ such that (59.5.0.1) is commutative. The solution is to take $x_{i''}$ over $U_{i''}$ with isomorphism

$$q(p(x_{i''})) \xrightarrow{q(\gamma'_{i''})} q(y_{i'})|_{U_{i''}} \xrightarrow{\gamma_{i'}|_{U_{i''}}} z_i|_{U_{i''}}$$

and isomorphism $\beta' : x_{i''}|_U \to x$. We omit the verification that (59.5.0.1) is commutative. □

Lemma 59.5.3. *Let $p : \mathcal{X} \to \mathcal{Y}$ be a 1-morphisms of stacks in groupoids over $(Sch/S)_{fppf}$. If p is representable by algebraic spaces, then the following are equivalent:*

(1) *p is limit preserving on objects, and*
(2) *p is locally of finite presentation (see Algebraic Stacks, Definition 57.10.1).*

Proof. Assume (2). Let $U = lim_{i \in I} U_i$ be the directed limit of affine schemes U_i over S, let y_i be an object of $\mathcal{Y}$ over U_i for some i, let x be an object of $\mathcal{X}$ over U, and let $\gamma : p(x) \to y_i|_U$ be an isomorphism. Let X_{y_i} denote an algebraic space over U_i representing the 2-fibre product

$$(Sch/U_i)_{fppf} \times_{y_i, \mathcal{Y}, p} \mathcal{X}.$$

Note that $\xi = (U, U \to U_i, x, \gamma^{-1})$ defines an object of this 2-fibre product over U. Via the 2-Yoneda lemma ξ corresponds to a morphism $f_\xi : U \to X_{y_i}$ over U_i. By More on Morphisms of Spaces, Proposition 46.4.9 there exists an $i' \geq i$ and a morphism $f_{i'} : U_{i'} \to X_{y_i}$ such that f_ξ is the composition of $f_{i'}$ and the projection morphism $U \to U_{i'}$. Also, the 2-Yoneda lemma tells us that $f_{i'}$ corresponds to an object $\xi_{i'} = (U_{i'}, U_{i'} \to U_i, x_{i'}, \alpha)$ of the displayed 2-fibre product over $U_{i'}$ whose restriction to U recovers ξ. In particular we obtain an isomorphism $\gamma : x_{i'}|U \to x$. Note that $\alpha : y_i|_{U_{i'}} \to p(x_{i'})$. Hence we see that taking $x_{i'}$, the isomorphism $\gamma : x_{i'}|U \to x$, and the isomorphism $\beta = \alpha^{-1} : p(x_{i'}) \to y_i|_{U_{i'}}$ is a solution to the problem.

Assume (1). Choose a scheme T and a 1-morphism $y : (Sch/T)_{fppf} \to \mathcal{Y}$. Let X_y be an algebraic space over T representing the 2-fibre product $(Sch/T)_{fppf} \times_{y,\mathcal{Y},p} \mathcal{X}$. We have to show that $X_y \to T$ is locally of finite presentation. To do this we may use More on Morphisms of Spaces, Proposition 46.4.9 in the form described in More on Morphisms of Spaces, Remark 46.4.10. Hence it suffices to show that given an affine scheme $U = lim_{i \in I} U_i$ written as the directed limit of affine schemes over T, then $X_y(U) = colim_i X_y(U_i)$. Pick any $i \in I$ and set $y_i = y|_{U_i}$. Also denote i' an element of I which is bigger than or equal to i. By the 2-Yoneda lemma morphisms $U \to X_y$ over T correspond bijectively to isomorphism classes of pairs (x, α) where x is an object of $\mathcal{X}$ over U and $\alpha : y|_U \to p(x)$ is an isomorphism. Of course giving α is, up to an inverse, the same thing as giving an isomorphism $\gamma : p(x) \to y_i|_U$. Similarly for morphisms $U_{i'} \to X_y$ over T. Hence (1) guarantees that

$$X_y(U) = colim_{i' \geq i} X_y(U_{i'})$$

in this situation and we win. □

Lemma 59.5.4. *Let $p : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. Assume p is representable by algebraic spaces and an open immersion. Then p is limit preserving on objects.*

Proof. This follows from Lemma 59.5.3 and (via the general principle Algebraic Stacks, Lemma 57.10.9) from the fact that an open immersion of algebraic spaces is locally of finite presentation, see Morphisms of Spaces, Lemma 42.26.10. □

59.6. Formally smooth on objects

Let S be a scheme. Let $p : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. We will say that p is *formally smooth on objects* if the following condition holds: Given any data consisting of

(1) a first order thickening $U \subset U'$ of affine schemes over S,
(2) an object y' of $\mathcal{Y}$ over U',
(3) an object x of $\mathcal{X}$ over U, and
(4) an isomorphism $\gamma : p(x) \to y'|_U$,

then there exists an object x' of $\mathcal{X}$ over U' with an isomorphism $\beta : x'|_U \to x$ and an isomorphism $\gamma' : p(x') \to y'$ such that

(59.6.0.1)
$$\begin{tikzcd} p(x'|_U) \arrow[r, "\gamma'|_U"] \arrow[d, "p(\beta)"] & y'|_U \arrow[d, equal] \\ p(x) \arrow[r, "\gamma"] & y'|_U \end{tikzcd}$$

commutes. In this situation we say that ``(x', β, γ') is a *solution* to the problem posed by our data (1), (2), (3), (4)".

Lemma 59.6.1. *Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Z} \to \mathcal{Y}$ be 1-morphisms of stacks in groupoids over $(Sch/S)_{fppf}$. If $p : \mathcal{X} \to \mathcal{Y}$ is formally smooth on objects, then so is the base change $p' : \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{Z}$ of p by q.*

Proof. This is formal. Let $U \subset U'$ be a first order thickening of affine schemes over S, let z' be an object of $\mathcal{Z}$ over U', let w be an object of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ over U, and let $\delta : p'(w) \to z'|_U$ be an isomorphism. We may write $w = (U, x, z, \alpha)$ for some object x of $\mathcal{X}$ over U and object z of $\mathcal{Z}$ over U and isomorphism $\alpha : p(x) \to q(z)$. Note that $p'(w) = z$ hence $\delta : z \to z|_U$. Set $y' = q(z')$ and $\gamma = q(\delta) \circ \alpha : p(x) \to y'|_U$. As p is formally smooth on objects there exists an object x' of $\mathcal{X}$ over U' as well as isomorphisms $\beta : x'|_U \to x$ and $\gamma' : p(x') \to y'$ such that (59.6.0.1) commutes. Then we consider the object $w = (U', x', z', \gamma')$ of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ over U' and define isomorphisms

$$w'|_U = (U, x'|_U, z'|_U, \gamma'|_U) \xrightarrow{(\beta, \delta^{-1})} (U, x, z, \alpha) = w$$

and

$$p'(w') = z' \xrightarrow{\text{id}} z'.$$

These combine to give a solution to the problem. □

Lemma 59.6.2. *Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms of stacks in groupoids over $(Sch/S)_{fppf}$. If p and q are formally smooth on objects, then so is the composition $q \circ p$.*

Proof. This is formal. Let $U \subset U'$ be a first order thickening of affine schemes over S, let z' be an object of $\mathcal{Z}$ over U', let x be an object of $\mathcal{X}$ over U, and let $\gamma : q(p(x)) \to z'|_U$ be an isomorphism. As q is formally smooth on objects there exist an object y' of $\mathcal{Y}$ over U', an isomorphism $\beta : y'|_U \to p(x)$, and an isomorphism $\gamma' : q(y') \to z'$ such that (59.6.0.1) is commutative. As p is formally smooth on objects there exist an object x' of $\mathcal{X}$ over U', an isomorphism $\beta' : x'|_U \to x$, and an isomorphism $\gamma'' : p(x') \to y'$ such that (59.6.0.1) is commutative. The solution is to take x' over U' with isomorphism

$$q(p(x')) \xrightarrow{q(\gamma'')} q(y') \xrightarrow{\gamma'} z'$$

and isomorphism $\beta' : x'|_U \to x$. We omit the verification that (59.6.0.1) is commutative. □

Note that the class of formally smooth morphisms of algebraic spaces is stable under arbitrary base change and local on the target in the fpqc topology, see More on Morphisms of Spaces, Lemma 46.16.3 and 46.16.10. Hence condition (2) in the lemma below makes sense.

Lemma 59.6.3. *Let $p : \mathcal{X} \to \mathcal{Y}$ be a 1-morphisms of stacks in groupoids over $(Sch/S)_{fppf}$. If p is representable by algebraic spaces, then the following are equivalent:*

1. *p is formally smooth on objects, and*
2. *p is formally smooth (see Algebraic Stacks, Definition 57.10.1).*

Proof. Assume (2). Let $U \subset U'$ be a first order thickening of affine schemes over S, let y' be an object of $\mathcal{Y}$ over U', let x be an object of $\mathcal{X}$ over U, and let $\gamma : p(x) \to y'|_U$ be an isomorphism. Let $X_{y'}$ denote an algebraic space over U' representing the 2-fibre product

$$(Sch/U')_{fppf} \times_{y', \mathcal{Y}, p} \mathcal{X}.$$

Note that $\xi = (U, U \to U', x, \gamma^{-1})$ defines an object of this 2-fibre product over U. Via the 2-Yoneda lemma ξ corresponds to a morphism $f_\xi : U \to X_{y'}$ over U'. As $X_{y'} \to U'$ is formally smooth by assumption there exists a morphism $f' : U' \to X_{y'}$ such that f_ξ is the composition of f' and the morphism $U \to U'$. Also, the 2-Yoneda lemma tells us that f' corresponds to an object $\xi' = (U', U' \to U', x', \alpha)$ of the displayed 2-fibre product over U' whose restriction to U recovers ξ. In particular we obtain an isomorphism $\gamma : x'|U \to x$.

Note that $\alpha : y' \to p(x')$. Hence we see that taking x', the isomorphism $\gamma : x'|U \to x$, and the isomorphism $\beta = \alpha^{-1} : p(x') \to y'$ is a solution to the problem.

Assume (1). Choose a scheme T and a 1-morphism $y : (Sch/T)_{fppf} \to \mathcal{Y}$. Let X_y be an algebraic space over T representing the 2-fibre product $(Sch/T)_{fppf} \times_{y,\mathcal{Y},p} \mathcal{X}$. We have to show that $X_y \to T$ is formally smooth. Hence it suffices to show that given a first order thickening $U \subset U'$ of affine schemes over T, then $X_y(U') \to X_y(U')$ is surjective (morphisms in the category of algebraic spaces over T). Set $y' = y|_{U'}$. By the 2-Yoneda lemma morphisms $U \to X_y$ over T correspond bijectively to isomorphism classes of pairs (x, α) where x is an object of $\mathcal{X}$ over U and $\alpha : y|_U \to p(x)$ is an isomorphism. Of course giving α is, up to an inverse, the same thing as giving an isomorphism $\gamma : p(x) \to y'|_U$. Similarly for morphisms $U' \to X_y$ over T. Hence (1) guarantees the surjectivity of $X_y(U') \to X_y(U')$ in this situation and we win. □

59.7. Surjective on objects

Let S be a scheme. Let $p : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. We will say that p is *surjective on objects* if the following condition holds: Given any data consisting of

(1) a field k over S, and
(2) an object y of $\mathcal{Y}$ over $Spec(k)$,

then there exists an extension $k \subset K$ of fields over S, an object x of $\mathcal{X}$ over $Spec(K)$ such that $p(x) \cong y|_{Spec(K)}$.

Lemma 59.7.1. *Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Z} \to \mathcal{Y}$ be 1-morphisms of stacks in groupoids over $(Sch/S)_{fppf}$. If $p : \mathcal{X} \to \mathcal{Y}$ is surjective on objects, then so is the base change $p' : \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{Z}$ of p by q.*

Proof. This is formal. Let z be an object of $\mathcal{Z}$ over a field k. As p is surjective on objects there exists an extension $k \subset K$ and an object x of $\mathcal{X}$ over K and an isomorphism $\alpha : p(x) \to q(z)|_{Spec(K)}$. Then $w = (Spec(K), x, z|_{Spec(K)}, \alpha)$ is an object of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ over K with $p'(w) = z|_{Spec(K)}$. □

Lemma 59.7.2. *Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms of stacks in groupoids over $(Sch/S)_{fppf}$. If p and q are surjective on objects, then so is the composition $q \circ p$.*

Proof. This is formal. Let z be an object of $\mathcal{Z}$ over a field k. As q is surjective on objects there exists a field extension $k \subset K$ and an object y of $\mathcal{Y}$ over K such that $q(y) \cong x|_{Spec(K)}$. As p is surjective on objects there exists a field extension $K \subset L$ and an object x of $\mathcal{X}$ over L such that $p(x) \cong y|_{Spec(L)}$. Then the field extension $k \subset L$ and the object x of $\mathcal{X}$ over L satisfy $q(p(x)) \cong z|_{Spec(L)}$ as desired. □

Lemma 59.7.3. *Let $p : \mathcal{X} \to \mathcal{Y}$ be a 1-morphisms of stacks in groupoids over $(Sch/S)_{fppf}$. If p is representable by algebraic spaces, then the following are equivalent:*

(1) *p is surjective on objects, and*
(2) *p is surjective (see Algebraic Stacks, Definition 57.10.1).*

Proof. Assume (2). Let k be a field and let y be an object of $\mathcal{Y}$ over k. Let X_y denote an algebraic space over k representing the 2-fibre product

$$(Sch/\, Spec(k))_{fppf} \times_{y,\mathcal{Y},p} \mathcal{X}.$$

As we've assumed that p is surjective we see that X_y is not empty. Hence we can find a field extension $k \subset K$ and a K-valued point x of X_y. Via the 2-Yoneda lemma this corresponds to an object x of $\mathcal{X}$ over K together with an isomorphism $p(x) \cong y|_{Spec(K)}$ and we see that (1) holds.

Assume (1). Choose a scheme T and a 1-morphism $y : (Sch/T)_{fppf} \to \mathcal{Y}$. Let X_y be an algebraic space over T representing the 2-fibre product $(Sch/T)_{fppf} \times_{y,\mathcal{Y},p} \mathcal{X}$. We have to show that $X_y \to T$ is surjective. By Morphisms of Spaces, Definition 42.6.2 we have to show that $|X_y| \to |T|$ is surjective. This means exactly that given a field k over T and a morphism $t : Spec(k) \to T$ there exists a field extension $k \subset K$ and a morphism $x : Spec(K) \to X_y$ such that

$$\begin{tikzcd} Spec(K) \arrow[r, "x"'] \arrow[d] & X_y \arrow[d] \\ Spec(k) \arrow[r, "t"] & T \end{tikzcd}$$

commutes. By the 2-Yoneda lemma this means exactly that we have to find $k \subset K$ and an object x of $\mathcal{X}$ over K such that $p(x) \cong t^*y|_{Spec(K)}$. Hence (1) guarantees that this is the case and we win. □

59.8. Algebraic morphisms

The following notion is occasionally useful.

Definition 59.8.1. Let S be a scheme. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. We say that F is *algebraic* if for every scheme T and every object ξ of $\mathcal{Y}$ over T the 2-fibre product

$$(Sch/T)_{fppf} \times_{\xi,\mathcal{Y}} \mathcal{X}$$

is an algebraic stack over S.

With this terminology in place we have the following result that generalizes Algebraic Stacks, Lemma 57.15.4.

Lemma 59.8.2. *Let S be a scheme. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. If*

(1) *$\mathcal{Y}$ is an algebraic stack, and*
(2) *F is algebraic (see above),*

then $\mathcal{X}$ is an algebraic stack.

Proof. By assumption (1) there exists a scheme T and an object ξ of $\mathcal{Y}$ over T such that the corresponding 1-morphism $\xi : (Sch/T)_{fppf} \to \mathcal{Y}$ is smooth an surjective. Then $\mathcal{U} = (Sch/T)_{fppf} \times_{\xi,\mathcal{Y}} \mathcal{X}$ is is an algebraic stack by assumption (2). Choose a scheme U and a surjective smooth 1-morphism $(Sch/U)_{fppf} \to \mathcal{U}$. The projection $\mathcal{U} \longrightarrow \mathcal{X}$ is, as the base change of the morphism $\xi : (Sch/T)_{fppf} \to \mathcal{Y}$, surjective and smooth, see Algebraic Stacks, Lemma 57.10.6. Then the composition $(Sch/U)_{fppf} \to \mathcal{U} \to \mathcal{X}$ is surjective and smooth as a composition of surjective and smooth morphisms, see Algebraic Stacks, Lemma 57.10.5. Hence $\mathcal{X}$ is an algebraic stack by Algebraic Stacks, Lemma 57.15.3. □

Lemma 59.8.3. *Let S be a scheme. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. If $\mathcal{X}$ is an algebraic stack and $\Delta : \mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ is representable by algebraic spaces, then F is algebraic.*

Proof. Choose a representable stack in groupoids $\mathcal{U}$ and a surjective smooth 1-morphism $\mathcal{U} \to \mathcal{X}$. Let T be a scheme and let ξ be an object of $\mathcal{Y}$ over T. The morphism of 2-fibre products

$$(Sch/T)_{fppf} \times_{\xi,\mathcal{Y}} \mathcal{U} \longrightarrow (Sch/T)_{fppf} \times_{\xi,\mathcal{Y}} \mathcal{X}$$

is representable by algebraic spaces, surjective, and smooth as a base change of $\mathcal{U} \to \mathcal{X}$, see Algebraic Stacks, Lemmas 57.9.7 and 57.10.6. By our condition on the diagonal of $\mathcal{Y}$ we see that the source of this morphism is representable by an algebraic space, see Algebraic Stacks, Lemma 57.10.11. Hence the target is an algebraic stack by Algebraic Stacks, Lemma 57.15.3. $\square$

59.9. Spaces of sections

Given morphisms $W \to Z \to U$ we can consider the functor that associates to a scheme U' over U the set of sections $\sigma : Z_{U'} \to W_{U'}$ of the base change $W_{U'} \to Z_{U'}$ of the morphism $W \to Z$. In this section we prove some preliminary lemmas on this functor.

Lemma 59.9.1. *Let $Z \to U$ be a finite morphism of schemes. Let W be an algebraic space and let $W \to Z$ be a surjective étale morphism. Then there exists a surjective étale morphism $U' \to U$ and a section*

$$\sigma : Z_{U'} \to W_{U'}$$

of the morphism $W_{U'} \to Z_{U'}$.

Proof. We may choose a separated scheme W' and a surjective étale morphism $W' \to W$. Hence after replacing W by W' we may assume that W is a separated scheme. Write $f : W \to Z$ and $\pi : Z \to U$. Note that $f \circ \pi : W \to U$ is separated as W is separated (see Schemes, Lemma 21.21.14). Let $u \in U$ be a point. Clearly it suffices to find an étale neighbourhood (U', u') of (U, u) such that a section σ exists over U'. Let $z_1, \ldots, z_r$ be the points of Z lying above u. For each i choose a point $w_i \in W$ which maps to z_i. We may pick an étale neighbourhood $(U', u') \to (U, u)$ such that the conclusions of More on Morphisms, Lemma 33.28.5 hold for both $Z \to U$ and the points $z_1, \ldots, z_r$ and $W \to U$ and the points $w_1, \ldots, w_r$. Hence, after replacing (U, u) by (U', u') and relabeling, we may assume that all the field extensions $\kappa(u) \subset \kappa(z_i)$ and $\kappa(u) \subset \kappa(w_i)$ are purely inseparable, and moreover that there exist disjoint union decompositions

$$Z = V_1 \amalg \ldots \amalg V_r \amalg A, \quad W = W_1 \amalg \ldots \amalg W_r \amalg B$$

by open and closed subschemes with $z_i \in V_i$, $w_i \in W_i$ and $V_i \to U$, $W_i \to U$ finite. After replacing U by $U \setminus \pi(A)$ we may assume that $A = \emptyset$, i.e., $Z = V_1 \amalg \ldots \amalg V_r$. After replacing W_i by $W_i \cap f^{-1}(V_i)$ and B by $B \cup \bigcup W_i \cap f^{-1}(Z \setminus V_i)$ we may assume that f maps W_i into V_i. Then $f_i = f|_{W_i} : W_i \to V_i$ is a morphism of schemes finite over U, hence finite (see Morphisms, Lemma 24.42.12). It is also étale (by assumption), $f_i^{-1}(\{z_i\}) = w_i$, and induces an isomorphism of residue fields $\kappa(z_i) = \kappa(w_i)$ (because both are purely inseparable extensions of $\kappa(u)$ and $\kappa(z_i) \subset \kappa(w_i)$ is separable as f is étale). Hence by Étale Morphisms, Lemma 37.14.2 we see that f_i is an isomorphism in a neighbourhood V_i' of z_i. Since $\pi : Z \to U$ is closed, after shrinking U, we may assume that $W_i \to V_i$ is an isomorphism. This proves the lemma. $\square$

Lemma 59.9.2. *Let $Z \to U$ be a finite locally free morphism of schemes. Let W be an algebraic space and let $W \to Z$ be an étale morphism. Then the functor*

$$F : (Sch/U)_{fppf}^{opp} \longrightarrow Sets,$$

defined by the rule

$$U' \longmapsto F(U') = \{\sigma : Z_{U'} \to W_{U'} \text{ section of } W_{U'} \to Z_{U'}\}$$

is an algebraic space and the morphism $F \to U$ is étale.

Proof. Assume first that $W \to Z$ is also separated. Let U' be a scheme over U and let $\sigma \in F(U')$. By Morphisms of Spaces, Lemma 42.5.7 the morphism σ is a closed immersion. Moreover, σ is étale by Properties of Spaces, Lemma 41.13.6. Hence σ is also an open immersion, see Morphisms of Spaces, Lemma 42.40.2. In other words, $Z_\sigma = \sigma(Z_{U'}) \subset W_{U'}$ is an open subspace such that the morphism $Z_\sigma \to Z_{U'}$ is an isomorphism. In particular, the morphism $Z_\sigma \to U'$ is finite. Hence we obtain a transformation of functors

$$F \longrightarrow (W/U)_{fin}, \quad \sigma \longmapsto (U' \to U, Z_\sigma)$$

where $(W/U)_{fin}$ is the finite part of the morphism $W \to U$ introduced in More on Groupoids in Spaces, Section 53.8. It is clear that this transformation of functors is injective (since we can recover σ from Z_σ as the inverse of the isomorphism $Z_\sigma \to Z_{U'}$). By More on Groupoids in Spaces, Proposition 53.8.11 we know that $(W/U)_{fin}$ is an algebraic space étale over U. Hence to finish the proof in this case it suffices to show that $F \to (W/U)_{fin}$ is representable and an open immersion. To see this suppose that we are given a morphism of schemes $U' \to U$ and an open subspace $Z' \subset W_{U'}$ such that $Z' \to U'$ is finite. Then it suffices to show that there exists an open subscheme $U'' \subset U'$ such that a morphism $T \to U'$ factors through U'' if and only if $Z' \times_{U'} T$ maps isomorphically to $Z \times_{U'} T$. This follows from Quot, Lemma 47.3.6 (here we use that $Z \to B$ is flat and locally of finite presentation as well as finite). Hence we have proved the lemma in case $W \to Z$ is separated as well as étale.

In the general case we choose a separated scheme W' and a surjective étale morphism $W' \to W$. Note that the morphisms $W' \to W$ and $W \to Z$ are separated as their source is separated. Denote F' the functor associated to $W' \to Z \to U$ as in the lemma. In the first paragraph of the proof we showed that F' is representable by an algebraic space étale over U. By Lemma 59.9.1 the map of functors $F' \to F$ is surjective for the étale topology on Sch/U. Moreover, if U' and $\sigma : Z_{U'} \to W_{U'}$ define a point $\xi \in F(U')$, then the fibre product

$$F'' = F' \times_{F,\xi} U'$$

is the functor on Sch/U' associated to the morphisms

$$W'_{U'} \times_{W_{U'},\sigma} Z_{U'} \to Z_{U'} \to U'.$$

Since the first morphism is separated as a base change of a separated morphism, we see that F'' is an algebraic space étale over U' by the result of the first paragraph. It follows that $F' \to F$ is a surjective étale transformation of functors, which is representable by algebraic spaces. Hence F is an algebraic space by Bootstrap, Theorem 54.10.1. Since $F' \to F$ is an étale surjective morphism of algebraic spaces it follows that $F \to U$ is étale because $F' \to U$ is étale. $\square$

59.10. Relative morphisms

Let S be a scheme. Let $Z \to B$ and $X \to B$ be morphisms of algebraic spaces over S. Given a scheme T we can consider pairs (a, b) where $a : T \to B$ is a morphism and

$b : T\times_{a,B} Z \to T\times_{a,B} X$ is a morphism over T. Picture

(59.10.0.1)
$$\begin{tikzcd} T\times_{a,B} Z \arrow[rr, "b"] \arrow[dr] & & T\times_{a,B} X \arrow[dl] & Z \arrow[dr] & & X \arrow[dl] \\ & T \arrow[rrrr, "a"] & & & B & \end{tikzcd}$$

Of course, we can also think of b as a morphism $b : T\times_{a,B} Z \to X$ such that

$$\begin{tikzcd} T\times_{a,B} Z \arrow[r] \arrow[d] \arrow[rr, bend left, "b"] & Z \arrow[dr] & X \arrow[d] \\ T \arrow[rr, "a"] & & B \end{tikzcd}$$

commutes. In this situation we can define a functor

(59.10.0.2) $$Mor_B(Z, X) : (Sch/S)^{opp} \longrightarrow Sets, \quad T \longmapsto \{(a, b) \text{ as above}\}$$

Sometimes we think of this as a functor defined on the category of schemes over B, in which case we drop a from the notation.

Lemma 59.10.1. *Let S be a scheme. Let $Z \to B$ and $X \to B$ be morphisms of algebraic spaces over S. Then*

(1) *$Mor_B(Z, X)$ is a sheaf on $(Sch/S)_{fppf}$.*
(2) *If T is an algebraic space over S, then there is a canonical bijection*

$$Mor_{Sh((Sch/S)_{fppf})}(T, Mor_B(Z, X)) = \{(a, b) \text{ as in } (59.10.0.1)\}$$

Proof. Let T be an algebraic space over S. Let $\{T_i \to T\}$ be an fppf covering of T (as in Topologies on Spaces, Section 44.4). Suppose that $(a_i, b_i) \in Mor_B(Z, X)(T_i)$ such that $(a_i, b_i)|_{T_i\times_T T_j} = (a_j, b_j)|_{T_i\times_T T_j}$ for all i, j. Then by Descent on Spaces, Lemma 45.6.2 there exists a unique morphism $a : T \to B$ such that a_i is the composition of $T_i \to T$ and a. Then $\{T_i \times_{a_i,B} Z \to T\times_{a,B} Z\}$ is an fppf covering too and the same lemma implies there exists a unique morphism $b : T\times_{a,B} Z \to T\times_{a,B} X$ such that b_i is the composition of $T_i \times_{a_i,B} Z \to T\times_{a,B} Z$ and b. Hence $(a, b) \in Mor_B(Z, X)(T)$ restricts to (a_i, b_i) over T_i for all i.

Note that the result of the preceding paragraph in particular implies (1).

Let T be an algebraic space over S. In order to prove (2) we will construct mutually inverse maps between the displayed sets. In the following when we say ``pair'' we mean a pair (a, b) fitting into (59.10.0.1).

Let $v : T \to Mor_B(Z, X)$ be a natural transformation. Choose a scheme U and a surjective étale morphism $p : U \to T$. Then $v(p) \in Mor_B(Z, X)(U)$ corresponds to a pair (a_U, b_U) over U. Let $R = U \times_T U$ with projections $t, s : R \to U$. As v is a transformation of functors we see that the pullbacks of (a_U, b_U) by s and t agree. Hence, since $\{U \to T\}$ is an fppf covering, we may apply the result of the first paragraph that deduce that there exists a unique pair (a, b) over T.

Conversely, let (a, b) be a pair over T. Let $U \to T$, $R = U\times_T U$, and $t, s : R \to U$ be as above. Then the restriction $(a, b)|_U$ gives rise to a tranformation of functors $v : h_U \to Mor_B(Z, X)$ by the Yoneda lemma (Categories, Lemma 4.3.5). As the two pullbacks $s^*(a, b)|_U$ and $t^*(a, b)|_U$ are equal, we see that v coequalizes the two maps $h_t, h_s : h_R \to h_U$. Since

$T = U/R$ is the fppf quotient sheaf by Spaces, Lemma 40.9.1 and since $Mor_B(Z, X)$ is an fppf sheaf by (1) we conclude that v factors through a map $T \to Mor_B(Z, X)$.

We omit the verification that the two constructions above are mutually inverse. □

Lemma 59.10.2. *Let S be a scheme. Let $Z \to B$, $X \to B$, and $B' \to B$ be morphisms of algebraic spaces over S. Set $Z' = B' \times_B Z$ and $X' = B' \times_B X$. Then*

$$Mor_{B'}(Z', X') = B' \times_B Mor_B(Z, X)$$

in $Sh((Sch/S)_{fppf})$.

Proof. The equality as functors follows immediately from the definitions. The equality as sheaves follows from this because both sides are sheaves according to Lemma 59.10.1 and the fact that a fibre product of sheaves is the same as the corresponding fibre product of pre-sheaves (i.e., functors). □

Lemma 59.10.3. *Let S be a scheme. Let $Z \to B$ and $X' \to X \to B$ be morphisms of algebraic spaces over S. Assume*

1. *$X' \to X$ is étale, and*
2. *$Z \to B$ is finite locally free.*

Then $Mor_B(Z, X') \to Mor_B(Z, X)$ is representable by algebraic spaces and étale. If $X' \to X$ is also surjective, then $Mor_B(Z, X') \to Mor_B(Z, X)$ is surjective.

Proof. Let U be a scheme and let $\xi = (a, b)$ be an element of $Mor_B(Z, X)(U)$. We have to prove that the functor

$$h_U \times_{\xi, Mor_B(Z,X)} Mor_B(Z, X')$$

is representable by an algebaic space étale over U. Set $Z_U = U \times_{a,B} Z$ and $W = Z_U \times_{b,X} X'$. Then $W \to Z_U \to U$ is as in Lemma 59.9.2 and the sheaf F defined there is identified with the fibre product displayed above. Hence the first assertion of the lemma. The second assertion follows from this and Lemma 59.9.1 which guarantees that $F \to U$ is surjective in the situation above. □

Lemma 59.10.4. *Let $Z \to B$ and $X \to B$ be morphisms of affine schemes. Assume $\Gamma(Z, \mathcal{O}_Z)$ is a finite free $\Gamma(B, \mathcal{O}_B)$-module. Then $Mor_B(Z, X)$ is representable by an affine scheme over B.*

Proof. Write $B = Spec(R)$. Choose a basis $\{e_1, \ldots, e_m\}$ for $\Gamma(Z, \mathcal{O}_Z)$. Finally, choose a presentation

$$\Gamma(X, \mathcal{O}_X) = R[\{x_i\}_{i\in I}]/(\{f_k\}_{k\in K}).$$

We will denote $\overline{x}_i$ the image of x_i in this quotient. Write

$$P = R[\{a_{ij}\}_{i\in I, 1\leq j\leq m}].$$

Consider the R-algebra map

$$\Psi : R[\{x_i\}_{i\in I}] \longrightarrow P \otimes_R \Gamma(Z, \mathcal{O}_Z), \quad x_i \longmapsto \sum\nolimits_j a_{ij} \otimes e_j.$$

Write $\Psi(f_k) = \sum c_{kj} \otimes e_j$ with $c_{kj} \in P$. Finally, denote $J \subset P$ the ideal generated by the elements c_{kj}, $k \in K$, $1 \leq j \leq m$. We claim that $W = Spec(P/J)$ represents the functor $Mor_B(Z, X)$.

First, note that by construction P/J is an R-algebra, hence a morphism $a_{univ} : W \to B$. Second, by construction the map Ψ factors through $\Gamma(X, \mathcal{O}_X)$, hence we obtain an P/J-algebra homomorphism

$$P/J \otimes_R \Gamma(X, \mathcal{O}_X) \longrightarrow P/J \otimes_R \Gamma(Z, \mathcal{O}_Z)$$

which determines a morphism $b_{univ} : W \times_{a_{univ}, B} Z \to W \times_{a_{univ}, B} X$. By the Yoneda lemma the pair (a_{univ}, b_{univ}) determines a transformation of functors $W \to Mor_B(Z, X)$ which we claim is an isomorphism. To show that it is an isomorphism it suffices to show that it induces a bijection of sets $W(T) \to Mor_B(Z, X)(T)$ over any affine scheme T.

Suppose $T = Spec(R')$ is an affine scheme and $(a, b) \in Mor_B(Z, X)(T)$, then a defines an R-algebra structure on R' and b defines an R'-algebra map

$$b^\sharp : R' \otimes_R \Gamma(X, \mathcal{O}_X) \longrightarrow R' \otimes_R \Gamma(Z, \mathcal{O}_Z).$$

In particular we can write $b^\sharp(1 \otimes \overline{x}_i) = \sum \alpha_{ij} \otimes e_j$ for some $\alpha_{ij} \in R'$. This corresponds to an R-algebra map $P \to R'$ determined by the rule $a_{ij} \mapsto \alpha_{ij}$. This map factors through the quotient P/J by the construction of the ideal J to give a map $P/J \to R'$. This in turn corresponds to a morphism $T \to W$ such that (a, b) is the pullback of (a_{univ}, b_{univ}). Some details omitted. □

Proposition 59.10.5. *Let S be a scheme. Let $Z \to B$ and $X \to B$ be morphisms of algebraic spaces over S. If $Z \to B$ is finite locally free then $Mor_B(Z, X)$ is an algebraic space.*

Proof. Choose a scheme $B' = \coprod B'_i$ which is a disjoint union of affine schemes B'_i and an étale surjective morphism $B' \to B$. We may also assume that $B'_i \times_B Z$ is the spectrum of a ring which is finite free as a $\Gamma(B'_i, \mathcal{O}_{B'_i})$-module. By Lemma 59.10.2 and Spaces, Lemma 40.5.5 the morphism $Mor_{B'}(Z', X') \to Mor_B(Z, X)$ is surjective étale. Hence by Bootstrap, Theorem 54.10.1 it suffices to prove the proposition when $B = B'$ is a disjoint union of affine schemes B'_i so that each $B'_i \times_B Z$ is finite free over B'_i. Then it actually suffices to prove the result for the restriction to each B'_i. Thus we may assume that B is affine and that $\Gamma(Z, \mathcal{O}_Z)$ is a finite free $\Gamma(B, \mathcal{O}_B)$-module.

Choose a scheme X' which is a disjoint union of affine schemes and a surjective étale morphism $X' \to X$. By Lemma 59.10.3 the morphism $Mor_B(Z, X') \to Mor_B(Z, X)$ is representable by algebraic spaces, étale, and surjective. Hence by Bootstrap, Theorem 54.10.1 it suffices to prove the proposition when X is a disjoint union of affine schemes. This reduces us to the case discussed in the next paragraph.

Assume $X = \coprod_{i \in I} X_i$ is a disjoint union of affine schemes, B is affine, and that $\Gamma(Z, \mathcal{O}_Z)$ is a finite free $\Gamma(B, \mathcal{O}_B)$-module. For any finite subset $E \subset I$ set

$$F_E = Mor_B(Z, \coprod\nolimits_{i \in E} X_i).$$

By Lemma 59.10.4 we see that F_E is an algebraic space. Consider the morphism

$$\coprod\nolimits_{E \subset I \text{ finite}} F_E \longrightarrow Mor_B(Z, X)$$

Each of the morphisms $F_E \to Mor_B(Z, X)$ is an open immersion, because it is simply the locus parametrizing pairs (a, b) where b maps into the open subscheme $\coprod_{i \in E} X_i$ of X. Moreover, if T is quasi-compact, then for any pair (a, b) the image of b is contained in $\coprod_{i \in E} X_i$ for some $E \subset I$ finite. Hence the displayed arrow is in fact an open covering and we win[1] by Spaces, Lemma 40.8.4. □

[1] Modulo some set theoretic arguments. Namely, we have to show that $\coprod F_E$ is an algebraic space. This follows because $|I| \leq \text{size}(X)$ and $\text{size}(F_E) \leq \text{size}(X)$ as follows from the explicit description of F_E in the proof of Lemma 59.10.4. Some details omitted.

59.11. Restriction of scalars

Suppose $X \to Z \to B$ are morphisms of algebraic spaces over S. Given a scheme T we can consider pairs (a, b) where $a : T \to B$ is a morphism and $b : T \times_{a,B} Z \to X$ is a morphism over Z. Picture

(59.11.0.1)
$$\begin{tikzcd} & X \arrow[d] \\ T \times_{a,B} Z \arrow[r] \arrow[d] \arrow[ru, "b"] & Z \arrow[d] \\ T \arrow[r, "a"] & B \end{tikzcd}$$

In this situation we can define a functor

(59.11.0.2) $$\mathrm{Res}_{Z/B}(X) : (Sch/S)^{opp} \longrightarrow Sets, \quad T \longmapsto \{(a, b) \text{ as above}\}$$

Sometimes we think of this as a functor defined on the category of schemes over B, in which case we drop a from the notation.

Lemma 59.11.1. *Let S be a scheme. Let $X \to Z \to B$ be morphisms of algebraic spaces over S. Then*

(1) *$Res_{Z/B}(X)$ is a sheaf on $(Sch/S)_{fppf}$.*
(2) *If T is an algebraic space over S, then there is a canonical bijection*

$$Mor_{Sh((Sch/S)_{fppf})}(T, Res_{Z/B}(X)) = \{(a, b) \textit{ as in } (59.11.0.1)\}$$

Proof. Let T be an algebraic space over S. Let $\{T_i \to T\}$ be an fppf covering of T (as in Topologies on Spaces, Section 44.4). Suppose that $(a_i, b_i) \in \mathrm{Res}_{Z/B}(X)(T_i)$ such that $(a_i, b_i)|_{T_i \times_T T_j} = (a_j, b_j)|_{T_i \times_T T_j}$ for all i, j. Then by Descent on Spaces, Lemma 45.6.2 there exists a unique morphism $a : T \to B$ such that a_i is the composition of $T_i \to T$ and a. Then $\{T_i \times_{a_i,B} Z \to T \times_{a,B} Z\}$ is an fppf covering too and the same lemma implies there exists a unique morphism $b : T \times_{a,B} Z \to X$ such that b_i is the composition of $T_i \times_{a_i,B} Z \to T \times_{a,B} Z$ and b. Hence $(a, b) \in \mathrm{Res}_{Z/B}(X)(T)$ restricts to (a_i, b_i) over T_i for all i.

Note that the result of the preceding paragraph in particular implies (1).

Let T be an algebraic space over S. In order to prove (2) we will construct mutually inverse maps between the displayed sets. In the following when we say ``pair'' we mean a pair (a, b) fitting into (59.11.0.1).

Let $v : T \to \mathrm{Res}_{Z/B}(X)$ be a natural transformation. Choose a scheme U and a surjective étale morphism $p : U \to T$. Then $v(p) \in \mathrm{Res}_{Z/B}(X)(U)$ corresponds to a pair (a_U, b_U) over U. Let $R = U \times_T U$ with projections $t, s : R \to U$. As v is a transformation of functors we see that the pullbacks of (a_U, b_U) by s and t agree. Hence, since $\{U \to T\}$ is an fppf covering, we may apply the result of the first paragraph that deduce that there exists a unique pair (a, b) over T.

Conversely, let (a, b) be a pair over T. Let $U \to T$, $R = U \times_T U$, and $t, s : R \to U$ be as above. Then the restriction $(a, b)|_U$ gives rise to a tranformation of functors $v : h_U \to \mathrm{Res}_{Z/B}(X)$ by the Yoneda lemma (Categories, Lemma 4.3.5). As the two pullbacks $s^*(a, b)|_U$ and $t^*(a, b)|_U$ are equal, we see that v coequalizes the two maps $h_t, h_s : h_R \to h_U$. Since $T = U/R$ is the fppf quotient sheaf by Spaces, Lemma 40.9.1 and since $\mathrm{Res}_{Z/B}(X)$ is an fppf sheaf by (1) we conclude that v factors through a map $T \to \mathrm{Res}_{Z/B}(X)$.

We omit the verification that the two constructions above are mutually inverse. □

Of course the sheaf $\text{Res}_{Z/B}(X)$ comes with a natural transformation of functors $\text{Res}_{Z/B}(X) \to B$. We will use this without further mention in the following.

Lemma 59.11.2. *Let S be a scheme. Let $X \to Z \to B$ and $B' \to B$ be morphisms of algebraic spaces over S. Set $Z' = B' \times_B Z$ and $X' = B' \times_B X$. Then*

$$Res_{Z'/B'}(X') = B' \times_B Res_{Z/B}(X)$$

in $Sh((Sch/S)_{fppf})$.

Proof. The equality as functors follows immediately from the definitions. The equality as sheaves follows from this because both sides are sheaves according to Lemma 59.11.1 and the fact that a fibre product of sheaves is the same as the corresponding fibre product of pre-sheaves (i.e., functors). □

Lemma 59.11.3. *Let S be a scheme. Let $X' \to X \to Z \to B$ be morphisms of algebraic spaces over S. Assume*

1. *$X' \to X$ is étale, and*
2. *$Z \to B$ is finite locally free.*

Then $Res_{Z/B}(X') \to Res_{Z/B}(X)$ is representable by algebraic spaces and étale. If $X' \to X$ is also surjective, then $Res_{Z/B}(X') \to Res_{Z/B}(X)$ is surjective.

Proof. Let U be a scheme and let $\xi = (a, b)$ be an element of $\text{Res}_{Z/B}(X)(U)$. We have to prove that the functor

$$h_U \times_{\xi, \text{Res}_{Z/B}(X)} \text{Res}_{Z/B}(X')$$

is representable by an algebaic space étale over U. Set $Z_U = U \times_{a,B} Z$ and $W = Z_U \times_{b,X} X'$. Then $W \to Z_U \to U$ is as in Lemma 59.9.2 and the sheaf F defined there is identified with the fibre product displayed above. Hence the first assertion of the lemma. The second assertion follows from this and Lemma 59.9.1 which guarantees that $F \to U$ is surjective in the situation above. □

At this point we can use the lemmas above to prove that $\text{Res}_{Z/B}(X)$ is an algebraic space whenever $Z \to B$ is finite locally free in almost exactly the same way as in the proof that $Mor_B(Z, X)$ is an algebraic spaces, see Proposition 59.10.5. Instead we will directly deduce this result from the following lemma and the fact that $Mor_B(Z, X)$ is an algebraic space.

Lemma 59.11.4. *Let S be a scheme. Let $X \to Z \to B$ be morphisms of algebraic spaces over S. The following diagram*

$$\begin{tikzcd} Mor_B(Z, X) \arrow[r] & Mor_B(Z, Z) \\ Res_{Z/B}(X) \arrow[r] \arrow[u] & B \arrow[u, "id_Z"'] \end{tikzcd}$$

is a cartesian diagram of sheaves on $(Sch/S)_{fppf}$.

Proof. Omitted. Hint: Exercise in the functorial point of view in algebraic geometry. □

Proposition 59.11.5. *Let S be a scheme. Let $X \to Z \to B$ be morphisms of algebraic spaces over S. If $Z \to B$ is finite locally free then $Res_{Z/B}(X)$ is an algebraic space.*

Proof. By Proposition 59.10.5 the functors $Mor_B(Z, X)$ and $Mor_B(Z, Z)$ are algebraic spaces. Hence this follows from the cartesian diagram of Lemma 59.11.4 and the fact that fibre products of algebraic spaces exist and are given by the fibre product in the underlying category of sheaves of sets (see Spaces, Lemma 40.7.2). $\square$

59.12. Finite Hilbert stacks

In this section we prove some results concerning the finite Hilbert stacks $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ introduced in Examples of Stacks, Section 55.17.

Lemma 59.12.1. *Consider a 2-commutative diagram*

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "G"'] \arrow[d, "F'"'] & \mathcal{X} \arrow[d, "F"] \\ \mathcal{Y}' \arrow[r, "H"] & \mathcal{Y} \end{tikzcd}$$

of stacks in groupoids over $(Sch/S)_{fppf}$ with a given 2-isomorphism $\gamma : H \circ F' \to F \circ G$. In this situation we obtain a canonical 1-morphism $\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}') \to \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$. This morphism is compatible with the forgetful 1-morphisms of Examples of Stacks, Equation (55.17.2.1).

Proof. We map the object (U, Z, y', x', α') to the object $(U, Z, H(y'), G(x'), \gamma \star \mathrm{id}_H \star \alpha')$ where $\star$ denotes horizontal composition of 2-morphisms, see Categories, Definition 4.25.1. To a morphism $(f, g, b, a) : (U_1, Z_1, y'_1, x'_1, \alpha'_1) \to (U_2, Z_2, y'_2, x'_2, \alpha'_2)$ we assign $(f, g, H(b), G(a))$. We omit the verification that this defines a functor between categories over $(Sch/S)_{fppf}$. $\square$

Lemma 59.12.2. *In the situation of Lemma 59.12.1 assume that the given square is 2-cartesian. Then the diagram*

$$\begin{tikzcd} \mathcal{H}_d(\mathcal{X}'/\mathcal{Y}') \arrow[r] \arrow[d] & \mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \arrow[d] \\ \mathcal{Y}' \arrow[r] & \mathcal{Y} \end{tikzcd}$$

is 2-cartesian.

Proof. We get a 2-commutative diagram by Lemma 59.12.1 and hence we get a 1-morphism (i.e., a functor)

$$\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}') \longrightarrow \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$$

We indicate why this functor is essentially surjective. Namely, an object of the category on the right hand side is given by a scheme U over S, an object y' of $\mathcal{Y}'_U$, an object (U, Z, y, x, α) of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ over U and an isomorphism $H(y') \to y$ in $\mathcal{Y}_U$. The assumption means exactly that there exists an object x' of $\mathcal{X}'_Z$ such that there exist isomorphisms $G(x') \cong x$ and $\alpha' : y'|_Z \to F'(x')$ compatible with α. Then we see that (U, Z, y', x', α') is an object of $\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}')$ over U. Details omitted. $\square$

Lemma 59.12.3. *In the situation of Lemma 59.12.1 assume*

(1) $\mathcal{Y}' = \mathcal{Y}$ *and* $H = id_{\mathcal{Y}}$,
(2) G *is representable by algebraic spaces and étale.*

Then $\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}) \to \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ is representable by algebraic spaces and étale. If G is also surjective, then $\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}) \to \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ is surjective.

Proof. Let U be a scheme and let $\xi = (U, Z, y, x, \alpha)$ be an object of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ over U. We have to prove that the 2-fibre product

$$(Sch/U)_{fppf} \times_{\xi, \mathcal{H}_d(\mathcal{X}/\mathcal{Y})} \mathcal{H}_d(\mathcal{X}'/\mathcal{Y}) \tag{59.12.3.1}$$

is representable by an algebraic space étale over U. An object of this over U' corresponds to an object x' in the fibre category of $\mathcal{X}'$ over $Z_{U'}$ such that $G(x') \cong x|_{Z_{U'}}$. By assumption the 2-fibre product

$$(Sch/Z)_{fppf} \times_{x, \mathcal{X}} \mathcal{X}'$$

is representable by an algebraic space W such that the projection $W \to Z$ is étale. Then (59.12.3.1) is representable by the algebraic space F parametrizing sections of $W \to Z$ over U introduced in Lemma 59.9.2. Since $F \to U$ is étale we conclude that $\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}) \to \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ is representable by algebraic spaces and étale. Finally, if $\mathcal{X}' \to \mathcal{X}$ is surjective also, then $W \to Z$ is surjective, and hence $F \to U$ is surjective by Lemma 59.9.1. Thus in this case $\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}) \to \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ is also surjective. □

Lemma 59.12.4. *In the situation of Lemma 59.12.1. Assume that G, H are representable by algebraic spaces and étale. Then $\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}') \to \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ is representable by algebraic spaces and étale. If also H is surjective and the induced functor $\mathcal{X}' \to \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}$ is surjective, then $\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}') \to \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ is surjective.*

Proof. Set $\mathcal{X}'' = \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}$. By Lemma 59.4.1 the 1-morphism $\mathcal{X}' \to \mathcal{X}''$ is representable by algebraic spaces and étale (in particular the condition in the second statement of the lemma that $\mathcal{X}' \to \mathcal{X}''$ be surjective makes sense). We obtain a 2-commutative diagram

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d] & \mathcal{X}'' \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \mathcal{Y}' \arrow[r] & \mathcal{Y}' \arrow[r] & \mathcal{Y} \end{tikzcd}$$

It follows from Lemma 59.12.2 that $\mathcal{H}_d(\mathcal{X}''/\mathcal{Y}')$ is the base change of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ by $\mathcal{Y}' \to \mathcal{Y}$. In particular we see that $\mathcal{H}_d(\mathcal{X}''/\mathcal{Y}') \to \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ is representable by algebraic spaces and étale, see Algebraic Stacks, Lemma 57.10.6. Moreover, it is also surjective if H is. Hence if we can show that the result holds for the left square in the diagram, then we're done. In this way we reduce to the case where $\mathcal{Y}' = \mathcal{Y}$ which is the content of Lemma 59.12.3. □

Lemma 59.12.5. *Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. Assume that $\Delta : \mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ is representable by algebraic spaces. Then*

$$\mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \longrightarrow \mathcal{H}_d(\mathcal{X}) \times \mathcal{Y}$$

see Examples of Stacks, Equation (55.17.2.1) is representably by algebraic spaces.

Proof. Let U be a scheme and let $\xi = (U, Z, p, x, 1)$ be an object of $\mathcal{H}_d(\mathcal{X}) = \mathcal{H}_d(\mathcal{X}/S)$ over U. Here p is just the structure morphism of U. The fifth component 1 exists and is unique since everything is over S. Also, let y be an object of $\mathcal{Y}$ over U. We have to show the 2-fibre product

$$(Sch/U)_{fppf} \times_{\xi \times y, \mathcal{H}_d(\mathcal{X}) \times \mathcal{Y}} \mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \tag{59.12.5.1}$$

is representable by an algebraic space. To explain why this is so we introduce

$$I = Isom_{\mathcal{Y}}(y|_Z, F(x))$$

which is an algebraic space over Z by assumption. Let $a : U' \to U$ be a scheme over U. What does it mean to give an object of the fibre category of (59.12.5.1) over U'? Well, it means that we have an object $\xi' = (U', Z', y', x', \alpha')$ of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ over U' and isomorphisms $(U', Z', p', x', 1) \cong (U, Z, p, x, 1)|_{U'}$ and $y' \cong y|_{U'}$. Thus ξ' is isomorphic to $(U', U' \times_{a,U} Z, a^*y, x|_{U' \times_{a,U} Z}, \alpha)$ for some morphism

$$\alpha : a^*y|_{U' \times_{a,U} Z} \longrightarrow F(x|_{U' \times_{a,U} Z})$$

in the fibre category of $\mathcal{Y}$ over $U' \times_{a,U} Z$. Hence we can view α as a morphism $b : U' \times_{a,U} Z \to I$. In this way we see that (59.12.5.1) is representable by $\mathrm{Res}_{Z/U}(I)$ which is an algebraic space by Proposition 59.11.5. □

The following lemma is a (partial) generalization of Lemma 59.12.3.

Lemma 59.12.6. *Let $F : \mathcal{X} \to \mathcal{Y}$ and $G : \mathcal{X}' \to \mathcal{X}$ be 1-morphisms of stacks in groupoids over $(Sch/S)_{fppf}$. If G is representable by algebraic spaces, then the 1-morphism*

$$\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}) \longrightarrow \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$$

is representable by algebraic spaces.

Proof. Let U be a scheme and let $\xi = (U, Z, y, x, \alpha)$ be an object of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ over U. We have to prove that the 2-fibre product

$$(Sch/U)_{fppf} \times_{\xi, \mathcal{H}_d(\mathcal{X}/\mathcal{Y})} \mathcal{H}_d(\mathcal{X}'/\mathcal{Y}) \tag{59.12.6.1}$$

is representable by an algebraic space étale over U. An object of this over $a : U' \to U$ corresponds to an object x' of $\mathcal{X}'$ over $U' \times_{a,U} Z$ such that $G(x') \cong x|_{U' \times_{a,U} Z}$. By assumption the 2-fibre product

$$(Sch/Z)_{fppf} \times_{x, \mathcal{X}} \mathcal{X}'$$

is representable by an algebraic space X over Z. It follows that (59.12.6.1) is representable by $\mathrm{Res}_{Z/U}(X)$, which is an algebraic space by Proposition 59.11.5. □

Lemma 59.12.7. *Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. Assume F is representable by algebraic spaces and locally of finite presentation. Then*

$$p : \mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \to \mathcal{Y}$$

is limit preserving on objects.

Proof. This means we have to show the following: Given

(1) an affine scheme $U = lim_i U_i$ which is written as the directed limit of affine schemes U_i over S,
(2) an object y_i of $\mathcal{Y}$ over U_i for some i, and
(3) an object $\Xi = (U, Z, y, x, \alpha)$ of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ over U such that $y = y_i|_U$,

then there exists an $i' \geq i$ and an object $\Xi_{i'} = (U_{i'}, Z_{i'}, y_{i'}, x_{i'}, \alpha_{i'})$ of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ over $U_{i'}$ with $\Xi_{i'}|_U = \Xi$ and $y_{i'} = y_i|_{U_{i'}}$. Namely, the last two equalities will take care of the commutativity of (59.5.0.1).

Let $X_{y_i} \to U_i$ be an algebraic space representing the 2-fibre product

$$(Sch/U_i)_{fppf} \times_{y_i, \mathcal{Y}, F} \mathcal{X}.$$

Note that $X_{y_i} \to U_i$ is locally of finite presentation by our assumption on F. Write Ξ. It is clear that $\xi = (Z, Z \to U_i, x, \alpha)$ is an object of the 2-fibre product displayed above, hence ξ gives rise to a morphism $f_\xi : Z \to X_{y_i}$ of algebraic spaces over U_i (since X_{y_i} is the functor of isomorphisms classes of objects of $(Sch/U_i)_{fppf} \times_{y,\mathcal{Y},F} \mathcal{X}$, see Algebraic Stacks, Lemma 57.8.2). By Limits, Lemmas 27.6.1 and 27.6.7 there exists an $i' \geq i$ and a finite locally free morphism $Z_{i'} \to U_{i'}$ of degree d whose base change to U is Z. By More on Morphisms of Spaces, Proposition 46.4.9 we may, after replacing i' by a bigger index, assume there exists a morphism $f_{i'} : Z_{i'} \to X_{y_i}$ such that

$$\begin{tikzcd} Z \arrow[r] \arrow[d] \arrow[rr, bend left, "f_\xi"] & Z_{i'} \arrow[r, "f_{i'}"'] \arrow[d] & X_{y_i} \arrow[d] \\ U \arrow[r] & U_{i'} \arrow[r] & U_i \end{tikzcd}$$

is commutative. We set $\Xi_{i'} = (U_{i'}, Z_{i'}, y_{i'}, x_{i'}, \alpha_{i'})$ where

(1) $y_{i'}$ is the object of $\mathcal{Y}$ over $U_{i'}$ which is the pullback of y_i to $U_{i'}$,
(2) $x_{i'}$ is the object of $\mathcal{X}$ over $Z_{i'}$ corresponding via the 2-Yoneda lemma to the 1-morphism

$$(Sch/Z_{i'})_{fppf} \to \mathcal{S}_{X_{y_i}} \to (Sch/U_i)_{fppf} \times_{y_i,\mathcal{Y},F} \mathcal{X} \to \mathcal{X}$$

where the middle arrow is the equivalence which defines X_{y_i} (notation as in Algebraic Stacks, Sections 57.8 and 57.7).
(3) $\alpha_{i'} : y_{i'}|_{Z_{i'}} \to F(x_{i'})$ is the isomorphism coming from the 2-commutativity of the diagram

$$\begin{tikzcd} (Sch/Z_{i'})_{fppf} \arrow[r] \arrow[dr] & (Sch/U_i)_{fppf} \times_{y_i,\mathcal{Y},F} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "F"] \\ & (Sch/U_{i'})_{fppf} \arrow[r] & \mathcal{Y} \end{tikzcd}$$

Recall that $f_\xi : Z \to X_{y_i}$ was the morphism corresponding to the object $\xi = (Z, Z \to U_i, x, \alpha)$ of $(Sch/U_i)_{fppf} \times_{y_i,\mathcal{Y},F} \mathcal{X}$ over Z. By construction $f_{i'}$ is the morphism corresponding to the object $\xi_{i'} = (Z_{i'}, Z_{i'} \to U_i, x_{i'}, \alpha_{i'})$. As $f_\xi = f_{i'} \circ (Z \to Z_{i'})$ we see that the object $\xi_{i'} = (Z_{i'}, Z_{i'} \to U_i, x_{i'}, \alpha_{i'})$ pulls back to ξ over Z. Thus $x_{i'}$ pulls back to x and $\alpha_{i'}$ pulls back to α. This means that $\Xi_{i'}$ pulls back to Ξ over U and we win. □

59.13. The finite Hilbert stack of a point

Let $d \geq 1$ be an integer. In Examples of Stacks, Definition 55.17.2 we defined a stack in groupoids $\mathcal{H}_d$. In this section we prove that $\mathcal{H}_d$ is an algebraic stack. We will throughout assume that $S = Spec(\mathbf{Z})$. The general case will follow from this by base change. Recall that the fibre category of $\mathcal{H}_d$ over a scheme T is the category of finite locally free morphisms $\pi : Z \to T$ of degree d. Instead of classifying these directly we first study the quasi-coherent sheaves of algebras $\pi_*\mathcal{O}_Z$.

Let R be a ring. Let us temporarily make the following definition: A *free d-dimensional algebra over R* is given by a commutative R-algebra structure m on $R^{\oplus d}$ such that $e_1 = (1, 0, \ldots, 0)$ is a unit[2]. We think of m as an R-linear map

$$m : R^{\oplus d} \otimes_R R^{\oplus d} \longrightarrow R^{\oplus d}$$

[2]It may be better to think of this as a pair consisting of a multiplication map $m : R^{\oplus d} \otimes_R R^{\oplus d} \to R^{\oplus d}$ and a ring map $\psi : R \to R^{\oplus d}$ satisfying a bunch of axioms.

such that $m(e_1, x) = m(x, e_1) = x$ and such that m defines a commutative and associative ring structure. If we write $m(e_i, e_j) = \sum a_{ij}^k e_k$ then we see this boils down to the conditions

$$\begin{cases} \sum_l a_{ij}^l a_{lk}^m = \sum_l a_{il}^m a_{jk}^l & \forall i, j, k, m \\ a_{ij}^k = a_{ji}^k & \forall i, j, k \\ a_{i1}^j = \delta_{ij} & \forall i, j \end{cases}$$

where δ_{ij} is the Kronecker δ-function. OK, so let's define

$$R_{univ} = \mathbf{Z}[a_{ij}^k]/J$$

where the ideal J is the ideal generated by the relations displayed above. Denote

$$m_{univ} : R_{univ}^{\oplus d} \otimes_{R_{univ}} R_{univ}^{\oplus d} \longrightarrow R_{univ}^{\oplus d}$$

the free d-dimensional algebra m over R_{univ} whose structure constants are the classes of a_{ij}^k modulo J. Then it is clear that given any free d-dimensional algebra m over a ring R there exists a unique $\mathbf{Z}$-algebra homomorphism $\psi : R_{univ} \to R$ such that $\psi_* m_{univ} = m$ (this means that m is what you get by applying the base change functor $- \otimes_{R_{univ}} R$ to m_{univ}). In other words, setting $X = Spec(R_{univ})$ we obtain a canonical identification

$$X(T) = \{\text{free } d\text{-dimensional algebras } m \text{ over } R\}$$

for varying $T = Spec(R)$. By Zariski localization we obtain the following seemingly more general identification

$$X(T) = \{\text{free } d\text{-dimensional algebras } m \text{ over } \Gamma(T, \mathcal{O}_T)\} \tag{59.13.0.1}$$

for any scheme T.

Next we talk a little bit about *isomorphisms of free d-dimensional R-algebras*. Namely, suppose that m, m' are two free d-dimensional algebras over a ring R. An *isomorphism from m to m'* is given by an invertible R-linear map

$$\varphi : R^{\oplus d} \longrightarrow R^{\oplus d}$$

such that $\varphi(e_1) = e_1$ and such that

$$m \circ \varphi \otimes \varphi = \varphi \circ m'.$$

Note that we can compose these so that the collection of free d-dimensional algebras over R becomes a category. In this way we optain a functor

$$FA_d : Sch_{fppf}^{opp} \longrightarrow Groupoids \tag{59.13.0.2}$$

from the category of schemes to groupoids: to a scheme T we associate the set of free d-dimensional algebras over $\Gamma(T, \mathcal{O}_T)$ endowed with the structure of a category using the notion of isomorphisms just defined.

The above suggests we consider the functor G in groups which associates to any scheme T the group

$$G(T) = \{g \in \mathrm{GL}_d(\Gamma(T, \mathcal{O}_T)) \mid g(e_1) = e_1\}$$

It is clear that $G \subset \mathrm{GL}_d$ (see Groupoids, Example 35.5.4) is the closed subgroup scheme cut out by the equations $x_{11} = 1$ and $x_{i1} = 0$ for $i > 1$. Hence G is a smooth affine group scheme over $Spec(\mathbf{Z})$. Consider the action

$$a : G \times_{Spec(\mathbf{Z})} X \longrightarrow X$$

which associates to a T-valued point (g, m) with $T = Spec(R)$ on the left hand side the free d-dimensional algebra over R given by

$$a(g, m) = g^{-1} \circ m \circ g \otimes g.$$

Note that this means that g defines an isomorphism $m \to a(g, m)$ of d-dimensional free R-algebras. We omit the verification that a indeed defines an action of the group scheme G on the scheme X.

Lemma 59.13.1. *The functor in groupoids FA_d defined in (59.13.0.2) is isomorphic (!) to the functor in groupouids which associates to a scheme T the category with*

1. *set of objects is $X(T)$,*
2. *set of morphhisms is $G(T) \times X(T)$,*
3. *$s : G(T) \times X(T) \to X(T)$ is the projection map,*
4. *$t : G(T) \times X(T) \to X(T)$ is $a(T)$, and*
5. *composition $G(T) \times X(T) \times_{s,X(T),t} G(T) \times X(T) \to G(T) \times X(T)$ is given by $((g, m), (g', m')) \mapsto (gg', m')$.*

Proof. We have seen the rule on objects in (59.13.0.1). We have also seen above that $g \in G(T)$ can be viewed as a morphism from m to $a(g, m)$ for any free d-dimensional algebra m. Conversely, any morphism $m \to m'$ is given by an invertible linear map φ which corresponds to an element $g \in G(T)$ such that $m' = a(g, m)$. □

In fact the groupoid $(X, G \times X, s, t, c)$ described in the lemma above is the groupoid associated to the action $a : G \times X \to X$ as defined in Groupoids, Lemma 35.13.1. Since G is smooth over $Spec(\mathbf{Z})$ we see that the two morphisms $s, t : G \times X \to X$ are smooth: by symmetry it suffices to prove that one of them is, and s is the base change of $G \to Spec(\mathbf{Z})$. Hence $(G \times X, X, s, t, c)$ is a smooth groupoid scheme, and the quotient stack $[X/G]$ is an algebraic stack by Algebraic Stacks, Theorem 57.17.3.

Proposition 59.13.2. *The stack $\mathcal{H}_d$ is equivalent to the quotient stack $[X/G]$ described above. In particular $\mathcal{H}_d$ is an algebraic stack.*

Proof. Note that by Groupoids in Spaces, Definition 52.19.1 the quotient stack $[X/G]$ is the stackification of the category fibred in groupoids associated to the ``presheaf in groupoids'' which associates to a scheme T the groupoid

$$(X(T), G(T) \times X(T), s, t, c).$$

Since this ``presheaf in groupoids'' is isomorphic to $FA_d$ by Lemma 59.13.1 it suffices to prove that the $\mathcal{H}_d$ is the stackification of (the category fibred in groupoids associated to the ``presheaf in groupoids'') FA_d. To do this we first define a functor

$$Spec : FA_d \longrightarrow \mathcal{H}_d$$

Recall that the fibre category of $\mathcal{H}_d$ over a scheme T is the category of finite locally free morphisms $Z \to T$ of degree d. Thus given a scheme T and a free d-dimensional $\Gamma(T, \mathcal{O}_T)$-algebra m we may assign to this the object

$$Z = \underline{Spec}_T(\mathcal{A})$$

of $\mathcal{H}_{d,T}$ where $\mathcal{A} = \mathcal{O}_T^{\oplus d}$ endowed with a $\mathcal{O}_T$-algebra structure via m. Moreover, if m' is a second such free d-dimensional $\Gamma(T, \mathcal{O}_T)$-algebra and if $\varphi : m \to m'$ is an isomorphism of these, then the induced $\mathcal{O}_T$-linear map $\varphi : \mathcal{O}_T^{\oplus d} \to \mathcal{O}_T^{\oplus d}$ induces an isomorphism

$$\varphi : \mathcal{A}' \longrightarrow \mathcal{A}$$

of quasi-coherent $\mathcal{O}_T$-algebras. Hence

$$\underline{Spec}_T(\varphi) : \underline{Spec}_T(\mathcal{A}) \longrightarrow \underline{Spec}_T(\mathcal{A}')$$

is a morphism in the fibre category $\mathcal{H}_{d,T}$. We omit the verification that this construction is compatible with base change so we get indeed a functor $Spec : FA_d \to \mathcal{H}_d$ as claimed above.

To show that $Spec : FA_d \to \mathcal{H}_d$ induces an equivalence between the stackification of FA_d and $\mathcal{H}_d$ it suffices to check that

(1) $Isom(m, m') = Isom(Spec(m), Spec(m'))$ for any $m, m' \in FA_d(T)$.
(2) for any scheme T and any object $Z \to T$ of $\mathcal{H}_{d,T}$ there exists a covering $\{T_i \to T\}$ such that $Z|_{T_i}$ is isomorphic to $Spec(m)$ for some $m \in FA_d(T_i)$, and

see Stacks, Lemma 50.9.1. The first statement follows from the observation that any isomorphism

$$\underline{Spec}_T(\mathcal{A}) \longrightarrow \underline{Spec}_T(\mathcal{A}')$$

is necessarily given by a global invertible matrix g when $\mathcal{A} = \mathcal{A}' = \mathcal{O}_T^{\oplus d}$ as modules. To prove the second statement let $\pi : Z \to T$ be a finite locally free morphism of degree d. Then $\mathcal{A}$ is a locally free sheaf $\mathcal{O}_T$-modules of rank d. Consider the element $1 \in \Gamma(T, \mathcal{A})$. This element is nonzero in $\mathcal{A} \otimes_{\mathcal{O}_{T,t}} \kappa(t)$ for every $t \in T$ since the scheme $Z_t = Spec(\mathcal{A} \otimes_{\mathcal{O}_{T,t}} \kappa(t))$ is nonempty being of degree $d > 0$ over $\kappa(t)$. Thus $1 : \mathcal{O}_T \to \mathcal{A}$ can locally be used as the first basis element (for example you can use Algebra, Lemma 7.73.3 parts (1) and (2) to see this). Thus, after localizing on T we may assume that there exists an isomorphism $\varphi : \mathcal{A} \to \mathcal{O}_T^{\oplus d}$ such that $1 \in \Gamma(\mathcal{A})$ corresponds to the first basis element. In this situation the multiplication map $\mathcal{A} \otimes_{\mathcal{O}_T} \mathcal{A} \to \mathcal{A}$ translates via φ into a free d-dimensional algebra m over $\Gamma(T, \mathcal{O}_T)$. This finishes the proof. $\square$

59.14. Finite Hilbert stacks of spaces

The finite Hilbert stack of an algebraic space is an algebraic stack.

Lemma 59.14.1. *Let S be a scheme. Let X be an algebraic space over S. Then $\mathcal{H}_d(X)$ is an algebraic stack.*

Proof. The 1-morphism

$$\mathcal{H}_d(X) \longrightarrow \mathcal{H}_d$$

is representably by algebraic spaces according to Lemma 59.12.6. The stack $\mathcal{H}_d$ is an algebraic stack according to Proposition 59.13.2. Hence $\mathcal{H}_d(X)$ is an algebraic stack by Algebraic Stacks, Lemma 57.15.4. $\square$

This lemma allows us to bootstrap.

Lemma 59.14.2. *Let S be a scheme. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$ such that*

(1) *$\mathcal{X}$ is representable by an algebraic space, and*
(2) *F is representable by algebraic spaces, surjective, flat, and locally of finite presentation.*

Then $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ is an algebraic stack.

Proof. Choose a representable stack in groupoids $\mathcal{U}$ over S and a 1-morphism $f : \mathcal{U} \to \mathcal{H}_d(\mathcal{X})$ which is representable by algebraic spaces, smooth, and surjective. This is possible because $\mathcal{H}_d(\mathcal{X})$ is an algebraic stack by Lemma 59.14.1. Consider the 2-fibre product

$$\mathcal{W} = \mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \times_{\mathcal{H}_d(\mathcal{X}), f} \mathcal{U}.$$

Since $\mathcal{U}$ is representable (in particular a stack in setoids) it follows from Examples of Stacks, Lemma 55.17.3 and Stacks, Lemma 50.6.7 that $\mathcal{W}$ is a stack in setoids. The 1-morphism $\mathcal{W} \to \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ is representable by algebraic spaces, smooth, and surjective as a base change of the morphism f (see Algebraic Stacks, Lemmas 57.9.7 and 57.10.6). Thus, if we can show that $\mathcal{W}$ is representable by an algebraic space, then the lemma follows from Algebraic Stacks, Lemma 57.15.3.

The diagonal of $\mathcal{Y}$ is representable by algebraic spaces according to Lemma 59.4.3. We may apply Lemma 59.12.5 to see that the 1-morphism

$$\mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \longrightarrow \mathcal{H}_d(\mathcal{X}) \times \mathcal{Y}$$

is representable by algebraic spaces. Consider the 2-fibre product

$$\mathcal{V} = \mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \times_{(\mathcal{H}_d(\mathcal{X}) \times \mathcal{Y}), f \times F} (\mathcal{U} \times \mathcal{X}).$$

The projection morphism $\mathcal{V} \to \mathcal{U} \times \mathcal{X}$ is representable by algebraic spaces as a base change of the last displayed morphism. Hence $\mathcal{V}$ is an algebraic space (see Bootstrap, Lemma 54.3.6 or Algebraic Stacks, Lemma 57.9.8). The 1-morphism $\mathcal{V} \to \mathcal{U}$ fits into the following 2-cartesian diagram

$$\begin{tikzcd} \mathcal{V} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "F"] \\ \mathcal{W} \arrow[r] & \mathcal{Y} \end{tikzcd}$$

because

$$\mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \times_{(\mathcal{H}_d(\mathcal{X}) \times \mathcal{Y}), f \times F} (\mathcal{U} \times \mathcal{X}) = (\mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \times_{\mathcal{H}_d(\mathcal{X}), f} \mathcal{U}) \times_{\mathcal{Y}, F} \mathcal{X}.$$

Hence $\mathcal{V} \to \mathcal{W}$ is representable by algebraic spaces, surjective, flat, and locally of finite presentation as a base change of F. It follows that the same thing is true for the corresponding sheaves of sets associated to $\mathcal{V}$ and $\mathcal{W}$, see Algebraic Stacks, Lemma 57.10.4. Thus we conclude that the sheaf associated to $\mathcal{W}$ is an algebraic space by Bootstrap, Theorem 54.10.1. $\square$

59.15. LCI locus in the Hilbert stack

Please consult Examples of Stacks, Section 55.17 for notation. Fix a 1-morphism $F : \mathcal{X} \longrightarrow \mathcal{Y}$ of stacks in groupoids over $(Sch/S)_{fppf}$. Assume that F is representable by algebraic spaces. Fix $d \geq 1$. Consider an object (U, Z, y, x, α) of $\mathcal{H}_d$. There is an induced 1-morphism

$$(Sch/Z)_{fppf} \longrightarrow (Sch/U)_{fppf} \times_{y, \mathcal{Y}, F} \mathcal{X}$$

(by the universal property of 2-fibre products) which is representable by a morphism of algebraic spaces over U. Namely, since F is representable by algebraic spaces, we may choose an algebraic space X_y over U which represents the 2-fibre product $(Sch/U)_{fppf} \times_{y, \mathcal{Y}, F} \mathcal{X}$. Since $\alpha : y|_Z \to F(x)$ is an isomorphism we see that $\xi = (Z, Z \to U, x, \alpha)$ is an object of the 2-fibre product $(Sch/U)_{fppf} \times_{y, \mathcal{Y}, F} \mathcal{X}$ over Z. Hence ξ gives rise to a morphism

$x_\alpha : Z \to X_y$ of algebraic spaces over U as X_y is the functor of isomorphisms classes of objects of $(Sch/U)_{fppf} \times_{y,\mathcal{Y},F} \mathcal{X}$, see Algebraic Stacks, Lemma 57.8.2. Here is a picture

$$
\begin{tikzcd}
Z \arrow[r, "x_\alpha"] \arrow[dr] & X_y \arrow[d] \\
& U
\end{tikzcd}
\qquad
\begin{tikzcd}
(Sch/Z)_{fppf} \arrow[r, "{x,\alpha}"] \arrow[dr] & (Sch/U)_{fppf} \times_{y,\mathcal{Y},F} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "F"] \\
& (Sch/U)_{fppf} \arrow[r, "y"] & \mathcal{Y}
\end{tikzcd}
\tag{59.15.0.1}
$$

We remark that if $(f, g, b, a) : (U, Z, y, x, \alpha) \to (U', Z', y', x', \alpha')$ is a morphism between objects of $\mathcal{H}_d$, then the morphism $x'_{\alpha'} : Z' \to X'_{y'}$ is the base change of the morphism x_α by the morphism $g : U' \to U$ (details omitted).

Now assume moreover that F is flat and locally of finite presentation. In this situation we define a full subcategory

$$\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y}) \subset \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$$

consisting of those objects (U, Z, y, x, α) of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ such that the corresponding morphism $x_\alpha : Z \to X_y$ is unramified and a local complete intersection morphism (see Morphisms of Spaces, Definition 42.34.1 and More on Morphisms of Spaces, Definition 46.24.1 for definitions).

Lemma 59.15.1. *Let S be a scheme. Fix a 1-morphism $F : \mathcal{X} \longrightarrow \mathcal{Y}$ of stacks in groupoids over $(Sch/S)_{fppf}$. Assume F is representable by algebraic spaces, flat, and locally of finite presentation. Then $\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$ is a stack in groupoids and the inclusion functor*

$$\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y}) \longrightarrow \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$$

is representable and an open immersion.

Proof. Let $\Xi = (U, Z, y, x, \alpha)$ be an object of $\mathcal{H}_d$. It follows from the remark following (59.15.0.1) that the pullback of Ξ by $U' \to U$ belongs to $\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$ if and only if the base change of x_α is unramified and a local complete intersection morphism. Note that $Z \to U$ is finite locally free (hence flat, locally of finite presentation and universally closed) and that $X_y \to U$ is flat and locally of finite presentation by our assumption on F. Then Quot, Lemmas 47.3.1 and 47.3.7 imply exists an open subscheme $W \subset U$ such that a morphism $U' \to U$ factors through W if and only if the base change of x_α via $U' \to U$ is unramified and a local complete intersection morphism. This implies that

$$(Sch/U)_{fppf} \times_{\Xi, \mathcal{H}_d(\mathcal{X}/\mathcal{Y})} \mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$$

is representable by W. Hence the final statement of the lemma holds. The first statement (that $\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$ is a stack in groupoids) follows from this an Algebraic Stacks, Lemma 57.15.5. $\square$

Local complete intersection morphisms are ``locally unobstructed". This holds in much greater generality than the special case that we need in this chapter here.

Lemma 59.15.2. *Let $U \subset U'$ be a first order thickening of affine schemes. Let X' be an algebraic space flat over U'. Set $X = U \times_{U'} X'$. Let $Z \to U$ be finite locally free of degree d. Finally, let $f : Z \to X$ be unramified and a local complete intersection morphism. Then* *there exists a commutative diagram*

$$\begin{tikzcd}
(Z \subset Z') \arrow[rr, "{(f,f')}"'] \arrow[dr] & & (X \subset X') \arrow[dl] \\
& (U \subset U') &
\end{tikzcd}$$

of algebraic spaces over U' such that $Z' \to U'$ is finite locally free of degree d and $Z = U \times_{U'} Z'$.

Proof. By More on Morphisms of Spaces, Lemma 46.24.8 the conormal sheaf $\mathcal{C}_{Z/X}$ of the unramified morphism $Z \to X$ is a finite locally free $\mathcal{O}_Z$-module and by More on Morphisms of Spaces, Lemma 46.24.9 we have an exact sequence

$$0 \to i^*\mathcal{C}_{X/X'} \to \mathcal{C}_{Z/X'} \to \mathcal{C}_{Z/X} \to 0$$

of conormal sheaves. Since Z is affine this sequence is split. Choose a splitting

$$\mathcal{C}_{Z/X'} = i^*\mathcal{C}_{X/X'} \oplus \mathcal{C}_{Z/X}$$

Let $Z \subset Z''$ be the universal first order thickening of of Z over X' (see More on Morphisms of Spaces, Section 46.12). Denote $\mathcal{I} \subset \mathcal{O}_{Z''}$ the quasi-coherent sheaf of ideals corresponding to $Z \subset Z''$. By definition we have $\mathcal{C}_{Z/X'}$ is $\mathcal{I}$ viewed as a sheaf on Z. Hence the splitting above determines a splitting

$$\mathcal{I} = i^*\mathcal{C}_{X/X'} \oplus \mathcal{C}_{Z/X}$$

Let $Z' \subset Z''$ be the closed subscheme cut out by $\mathcal{C}_{Z/X} \subset \mathcal{I}$ viewed as a quasi-coherent sheaf of ideals on Z''. It is clear that Z' is a first order thickening of Z and that we obtain a commutative diagram of first order thickenings as in the statement of the lemma.

Since $X' \to U'$ is flat and since $X = U \times_{U'} X'$ we see that $\mathcal{C}_{X/X'}$ is the pullback of $\mathcal{C}_{U/U'}$ to X, see More on Morphisms of Spaces, Lemma 46.15.1. Note that by construction $\mathcal{C}_{Z/Z'} = i^*\mathcal{C}_{X/X'}$ hence we conclude that $\mathcal{C}_{Z/Z'}$ is isomorphic to the pullback of $\mathcal{C}_{U/U'}$ to Z. Applying More on Morphisms of Spaces, Lemma 46.15.1 once again (or its analogue for schemes, see More on Morphisms, Lemma 33.8.1) we conclude that $Z' \to U'$ is flat and that $Z = U \times_{U'} Z'$. Finally, More on Morphisms, Lemma 33.8.3 shows that $Z' \to U'$ is finite locally free of degree d. □

Lemma 59.15.3. *Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. Assume F is representable by algebraic spaces, flat, and locally of finite presentation. Then*

$$p : \mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y}) \to \mathcal{Y}$$

is formally smooth on objects.

Proof. We have to show the following: Given

(1) an object (U, Z, y, x, α) of $\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$ over an affine scheme U,
(2) a first order thickening $U \subset U'$, and
(3) an object y' of $\mathcal{Y}$ over U' such that $y'|_U = y$,

then there exists an object $(U', Z', y', x', \alpha')$ of $\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$ over U' with $Z = U \times_{U'} Z'$, with $x = x'|_Z$, and with $\alpha = \alpha'|_U$. Namely, the last two equalities will take care of the commutativity of (59.6.0.1).

Consider the morphism $x_\alpha : Z \to X_y$ constructed in Equation (59.15.0.1). Denote similarly $X'_{y'}$ the algebraic space over U' representing the 2-fibre product $(Sch/U')_{fppf} \times_{y',\mathcal{Y},F} \mathcal{X}$.

By assumption the morphism $X'_{y'} \to U'$ is flat (and locally of finite presentation). As $y'|_U = y$ we see that $X_y = U \times_{U'} X'_{y'}$. Hence we may apply Lemma 59.15.2 to find $Z' \to U'$ finite locally free of degree d with $Z = U \times_{U'} Z'$ and with $Z' \to X'_{y'}$ extending x_α. By construction the morphism $Z' \to X'_{y'}$ corresponds to a pair (x', α'). It is clear that $(U', Z', y', x', \alpha')$ is an object of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ over U' with $Z = U \times_{U'} Z'$, with $x = x'|_Z$, and with $\alpha = \alpha'|_U$. As we've seen in Lemma 59.15.1 that $\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y}) \subset \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ is an ``open substack" it follows that $(U', Z', y', x', \alpha')$ is an object of $\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$ as desired. □

Lemma 59.15.4. *Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. Assume F is representable by algebraic spaces, flat, surjective, and locally of finite presentation. Then*

$$\coprod\nolimits_{d \geq 1} \mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y}) \longrightarrow \mathcal{Y}$$

is surjective on objects.

Proof. It suffices to prove the following: For any field k and object y of $\mathcal{Y}$ over $Spec(k)$ there exists an integer $d \geq 1$ and an object (U, Z, y, x, α) of $\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$ with $U = Spec(k)$. Namely, in this case we see that p is surjective on objects in the strong sense that an extension of the field is not needed.

Denote X_y the algebraic space over $U = Spec(k)$ representing the 2-fibre product $(Sch/U')_{fppf} \times_{y', \mathcal{Y}, F} \mathcal{X}$. By assumption the morphism $X_y \to Spec(k)$ is surjective and locally of finite presentation (and flat). In particular X_y is nonempty. Choose a nonempty affine scheme V and an étale morphism $V \to X_y$. Note that $V \to Spec(k)$ is (flat), surjective, and locally of finite presentation (by Morphisms of Spaces, Definition 42.26.1). Pick a closed point $v \in V$ where $V \to Spec(k)$ is Cohen-Macaulay (i.e., V is Cohen-Macaulay at v), see More on Morphisms, Lemma 33.15.4. Applying More on Morphisms, Lemma 33.16.4 we find a regular immersion $Z \to V$ with $Z = \{v\}$. This implies $Z \to V$ is a closed immersion. Moreover, it follows that $Z \to Spec(k)$ is finite (for example by Algebra, Lemma 7.113.1). Hence $Z \to Spec(k)$ is finite locally free of some degree d. Now $Z \to X_y$ is unramified as the composition of an closed immersion followed by an étale morphism (see Morphisms of Spaces, Lemmas 42.34.3, 42.35.10, and 42.34.8). Finally, $Z \to X_y$ is a local complete intersection morphism as a composition of a regular immersion of schemes and an étale morphism of algebraic spaces (see More on Morphisms, Lemma 33.38.9 and Morphisms of Spaces, Lemmas 42.35.6 and 42.33.8 and More on Morphisms of Spaces, Lemmas 46.24.6 and 46.24.5). The morphism $Z \to X_y$ corresponds to an object x of $\mathcal{X}$ over Z together with an isomorphism $\alpha : y|_Z \to F(x)$. We obtain an object (U, Z, y, x, α) of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$. By what was said above about the morphism $Z \to X_y$ we see that it actually is an object of the subcategory $\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$ and we win. □

59.16. Bootstrapping algebraic stacks

The following theorem is one of the main results of this chapter.

Theorem 59.16.1. *Let S be a scheme. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. If*

1. *$\mathcal{X}$ is representable by an algebraic space, and*
2. *F is representable by algebraic spaces, surjective, flat and locally of finite presentation,*

then $\mathcal{Y}$ is an algebraic stack.

Proof. By Lemma 59.4.3 we see that the diagonal of $\mathcal{Y}$ is representable by algebraic spaces. Hence we only need to verify the existence of a 1-morphism $f : \mathcal{V} \to \mathcal{Y}$ of stacks in groupouds over $(Sch/S)_{fppf}$ with $\mathcal{V}$ representable and f surjective and smooth. By Lemma 59.14.2 we know that

$$\coprod\nolimits_{d\geq 1} \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$$

is an algebraic stack. It follows from Lemma 59.15.1 and Algebraic Stacks, Lemma 57.15.5 that

$$\coprod\nolimits_{d\geq 1} \mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$$

is an algebraic stack as well. Choose a representable stack in groupoids $\mathcal{V}$ over $(Sch/S)_{fppf}$ and a surjective and smooth 1-morphism

$$\mathcal{V} \longrightarrow \coprod\nolimits_{d\geq 1} \mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y}).$$

We claim that the composition

$$\mathcal{V} \longrightarrow \coprod\nolimits_{d\geq 1} \mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y}) \longrightarrow \mathcal{Y}$$

is smooth and surjective which finishes the proof of the theorem. In fact, the smoothness will be a consequence of Lemmas 59.12.7 and 59.15.3 and the surjectivity a consequence of Lemma 59.15.4. We spell out the details in the following paragraph.

By construction $\mathcal{V} \to \coprod_{d\geq 1} \mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$ is representable by algebraic spaces, surjective, and smooth (and hence also locally of finite presentation and formally smooth by the general principle Algebraic Stacks, Lemma 57.10.9 and More on Morphisms of Spaces, Lemma 46.16.6). Applying Lemmas 59.5.3, 59.6.3, and 59.7.3 we see that $\mathcal{V} \to \coprod_{d\geq 1} \mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$ is limit preserving on objects, formally smooth on objects, and surjective on objects. The 1-morphism $\coprod_{d\geq 1} \mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y}) \to \mathcal{Y}$ is

(1) limit preserving on objects: this is Lemma 59.12.7 for $\mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \to \mathcal{Y}$ and we combine it with Lemmas 59.15.1, 59.5.4, and 59.5.2 to get it for $\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y}) \to \mathcal{Y}$,
(2) formally smooth on objects by Lemma 59.15.3, and
(3) surjective on objects by Lemma 59.15.4.

Using Lemmas 59.5.2, 59.6.2, and 59.7.2 we conclude that the composition $\mathcal{V} \to \mathcal{Y}$ is limit preserving on objects, formally smooth on objects, and surjective on objects. Using Lemmas 59.5.3, 59.6.3, and 59.7.3 we see that $\mathcal{V} \to \mathcal{Y}$ is locally of finite presentation, formally smooth, and surjective. Finally, using (via the general principle Algebraic Stacks, Lemma 57.10.9) the infinitesimal lifting criterion (More on Morphisms of Spaces, Lemma 46.16.6) we see that $\mathcal{V} \to \mathcal{Y}$ is smooth and we win. □

59.17. Applications

Our first task is to show that the quotient stack $[U/R]$ associated to a ``flat and locally finitely presented groupoid" is an algebraic stack. See Groupoids in Spaces, Definition 52.19.1 for the definition of the quotient stack. The following lemma is preliminary and is the analogue of Algebraic Stacks, Lemma 57.17.2.

Lemma 59.17.1. *Let S be a scheme contained in Sch_{fppf}. Let (U, R, s, t, c) be a groupoid in algebraic spaces over S. Assume s, t are flat and locally of finite presentation. Then the morphism $\mathcal{S}_U \to [U/R]$ is flat, locally of finite presentation, and surjective.*

Proof. Let T be a scheme and let $x : (Sch/T)_{fppf} \to [U/R]$ be a 1-morphism. We have to show that the projection

$$\mathcal{S}_U \times_{[U/R]} (Sch/T)_{fppf} \longrightarrow (Sch/T)_{fppf}$$

is surjective and smooth. We already know that the left hand side is representable by an algebraic space F, see Algebraic Stacks, Lemmas 57.17.1 and 57.10.11. Hence we have to show the corresponding morphism $F \to T$ of algebraic spaces is surjective, locally of finite presentation, and flat. Since we are working with properties of morphisms of algebraic spaces which are local on the target in the fppf topology we may check this fppf locally on T. By construction, there exists an fppf covering $\{T_i \to T\}$ of T such that $x|_{(Sch/T_i)_{fppf}}$ comes from a morphism $x_i : T_i \to U$. (Note that $F \times_T T_i$ represents the 2-fibre product $\mathcal{S}_U \times_{[U/R]} (Sch/T_i)_{fppf}$ so everything is compatible with the base change via $T_i \to T$.) Hence we may assume that x comes from $x : T \to U$. In this case we see that

$$\mathcal{S}_U \times_{[U/R]} (Sch/T)_{fppf} = (\mathcal{S}_U \times_{[U/R]} \mathcal{S}_U) \times_{\mathcal{S}_U} (Sch/T)_{fppf} = \mathcal{S}_R \times_{\mathcal{S}_U} (Sch/T)_{fppf}$$

The first equality by Categories, Lemma 4.28.10 and the second equality by Groupoids in Spaces, Lemma 52.21.2. Clearly the last 2-fibre product is represented by the algebraic space $F = R \times_{s,U,x} T$ and the projection $R \times_{s,U,x} T \to T$ is flat and locally of finite presentation as the base change of the flat locally finitely presented morphism of algebraic spaces $s : R \to U$. It is also surjective as s has a section (namely the identity $e : U \to R$ of the groupoid). This proves the lemma. $\square$

Here is the first main result of this section.

Theorem 59.17.2. *Let S be a scheme contained in Sch_{fppf}. Let (U, R, s, t, c) be a groupoid in algebraic spaces over S. Assume s, t are flat and locally of finite presentation. Then the quotient stack $[U/R]$ is an algebraic stack over S.*

Proof. We check the two conditions of Theorem 59.16.1 for the morphism

$$(Sch/U)_{fppf} \longrightarrow [U/R].$$

The first is trivial (as U is an algebraic space). The second is Lemma 59.17.1. $\square$

59.18. When is a quotient stack algebraic?

In Groupoids in Spaces, Section 52.19 we have defined the quotient stack $[U/R]$ associated to a groupoid (U, R, s, t, c) in algebraic spaces. Note that $[U/R]$ is a stack in groupoids whose diagonal is representable by algebraic spaces (see Bootstrap, Lemma 54.11.3 and Algebraic Stacks, Lemma 57.10.11) and such that there exists an algebraic space U and a 1-morphism $(Sch/U)_{fppf} \to [U/R]$ which is an ``fppf surjection" in the sense that it induces a map on presheaves of isomorphism classes of objects which becomes surjective after sheafification. However, it is not the case that that $[U/R]$ is an algebraic stack in general. This is not a contradiction with Theorem 59.16.1 as the 1-morphism $(Sch/U)_{fppf} \to [U/R]$ is not representable by algebraic spaces in general, and if it is it may not be flat and locally of finite presentation.

The easiest way to make examples of non-algebraic quotient stacks is to look at quotients of the form $[S/G]$ where S is a scheme and G is a group scheme over S acting trivially on S. Namely, we will see below (Lemma 59.18.3) that if $[S/G]$ is algebraic, then $G \to S$ has to be flat and locally of finite presentation. An explicit example can be found in Examples, Section 64.32.

Lemma 59.18.1. *Let S be a scheme and let B be an algebraic space over S. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. The quotient stack $[U/R]$ is an algebraic stack if and only if there exists a morphism of algebraic spaces $g : U' \to U$ such that*

1. *the composition $U' \times_{g,U,t} R \to R \xrightarrow{s} U$ is a surjection of sheaves, and*
2. *the morphisms $s', t' : R' \to U'$ are flat and locally of finite presentation where (U', R', s', t', c') is the restriction of (U, R, s, t, c) via g.*

Proof. First, assume that $g : U' \to U$ satisfies (1) and (2). Property (1) implies that $[U'/R'] \to [U/R]$ is an equivalence, see Groupoids in Spaces, Lemma 52.24.2. By Theorem 59.17.2 the quotient stack $[U'/R']$ is an algebraic stack. Hence $[U/R]$ is an algebraic stack too, see Algebraic Stacks, Lemma 57.12.4.

Conversely, assume that $[U/R]$ is an algebraic stack. We may choose a scheme W and a surjective smooth 1-morphism

$$f : (Sch/W)_{fppf} \longrightarrow [U/R].$$

By the 2-Yoneda lemma (Algebraic Stacks, Section 57.5) this corresponds to an object ξ of $[U/R]$ over W. By the description of $[U/R]$ in Groupoids in Spaces, Lemma 52.23.1 we can find a surjective, flat, locally finitely presented morphism $b : U' \to W$ of schemes such that $\xi' = b^*\xi$ corresponds to a morphism $g : U' \to U$. Note that the 1-morphism

$$f' : (Sch/U')_{fppf} \longrightarrow [U/R].$$

corresponding to ξ' is surjective, flat, and locally of finite presentation, see Algebraic Stacks, Lemma 57.10.5. Hence $(Sch/U')_{fppf} \times_{[U/R]} (Sch/U')_{fppf}$ which is represented by the algebraic space

$$Isom_{[U/R]}(\text{pr}_0^*\xi', \text{pr}_1^*\xi') = (U' \times_S U') \times_{(g\circ \text{pr}_0, g\circ \text{pr}_1), U\times_S U} R = R'$$

(see Groupoids in Spaces, Lemma 52.21.1 for the first equality; the second is the definition of restriction) is flat and locally of finite presentation over U' via both s' and t' (by base change, see Algebraic Stacks, Lemma 57.10.6). By this description of R' and by Algebraic Stacks, Lemma 57.16.1 we obtain a canonical fully faithful 1-morphism $[U'/R'] \to [U/R]$. This 1-morphism is essentially surjective because f' is flat, locally of finite presentation, and surjective (see Stacks, Lemma 50.4.8); another way to prove this is to use Algebraic Stacks, Remark 57.16.3. Finally, we can use Groupoids in Spaces, Lemma 52.24.2 to conclude that the composition $U' \times_{g,U,t} R \to R \xrightarrow{s} U$ is a surjection of sheaves. □

Lemma 59.18.2. *Let S be a scheme and let B be an algebraic space over S. Let G be a group algebraic space over B. Let X be an algebraic space over B and let $a : G \times_B X \to X$ be an action of G on X over B. The quotient stack $[X/G]$ is an algebraic stack if and only if there exists a morphism of algebraic spaces $\varphi : X' \to X$ such that*

1. *$G \times_B X' \to X$, $(g, x') \mapsto a(g, \varphi(x'))$ is a surjection of sheaves, and*
2. *the two projections $X'' \to X'$ of the algebraic space X'' given by the rule*
$$T \longmapsto \{(x_1', g, x_2') \in (X' \times_B G \times_B X')(T) \mid \varphi(x_1') = a(g, \varphi(x_2'))\}$$
are flat and locally of finite presentation.

Proof. This lemma is a special case of Lemma 59.18.1. Namely, the quotient stack $[X/G]$ is by Groupoids in Spaces, Definition 52.19.1 equal to the quotient stack $[X/G \times_B X]$ of the groupoid in algebraic spaces $(X, G \times_B X, s, t, c)$ associated to the group action in Groupoids in Spaces, Lemma 52.14.1. There is one small observation that is needed to get condition (1). Namely, the morphism $s : G \times_B X \to X$ is the second projection and the morphism

$t : G \times_B X \to X$ is the action morphism a. Hence the morphism $h : U' \times_{g,U,t} R \to R \xrightarrow{s} U$ from Lemma 59.18.1 corresponds to the morphism

$$X' \times_{\varphi,X,a} (G \times_B X) \xrightarrow{\mathrm{pr}_1} X$$

in the current setting. However, because of the symmetry given by the inverse of G this morphism is isomorphic to the morphism

$$(G \times_B X) \times_{\mathrm{pr}_1,X,\varphi} X' \xrightarrow{a} X$$

of the statement of the lemma. Details omitted. □

Lemma 59.18.3. *Let S be a scheme and let B be an algebraic space over S. Let G be a group algebraic space over B. Endow B with the trivial action of G. Then the quotient stack $[B/G]$ is an algebraic stack if and only if G is flat and locally of finite presentation over B.*

Proof. If G is flat and locally of finite presentation over B, then $[B/G]$ is an algebraic stack by Theorem 59.17.2.

Conversely, assume that $[B/G]$ is an algebraic stack. By Lemma 59.18.2 and because the action is trivial, we see there exists an algebraic space B' and a morphism $B' \to B$ such that (1) $B' \to B$ is a surjection of sheaves and (2) the projections

$$B' \times_B G \times_B B' \to B'$$

are flat and locally of finite presentation. Note that the base change $B' \times_B G \times_B B' \to G \times_B B'$ of $B' \to B$ is a surjection of sheaves also. Thus it follows from Descent on Spaces, Lemma 45.7.1 that the projection $G \times_B B' \to B'$ is flat and locally of finite presentation. By (1) we can find an fppf covering $\{B_i \to B\}$ such that $B_i \to B$ factors through $B' \to B$. Hence $G \times_B B_i \to B_i$ is flat and locally of finite presentation by base change. By Descent on Spaces, Lemmas 45.10.11 and 45.10.8 we conclude that $G \to B$ is flat and locally of finite presentation. □

59.19. Algebraic stacks in the étale topology

Let S be a scheme. Instead of working with stacks in groupoids over the big fppf site $(Sch/S)_{fppf}$ we could work with stacks in groupoids over the big étale site $(Sch/S)_{étale}$. All of the material in Algebraic Stacks, Sections 57.4, 57.5, 57.6, 57.7, 57.8, 57.9, 57.10, and 57.11 makes sense for categories fibred in groupouds over $(Sch/S)_{étale}$. Thus we get a second notion of an algebraic stack by working in the étale topology. This notion is (a priori) weaker then the notion introduced in Algebraic Stacks, Definition 57.12.1 since a stack in the fppf topology is certainly a stack in the étale topology. However, the notions are equivalent as is shown by the following lemma.

Lemma 59.19.1. *Denote the common underlying category of Sch_{fppf} and $Sch_{étale}$ by Sch_α (see Sheaves on Stacks, Section 58.4 and Topologies, Remark 30.9.1). Let S be an object of Sch_α. Let*

$$p : \mathcal{X} \to Sch_\alpha/S$$

be a category fibred in groupoids with the following properties:

1. *$\mathcal{X}$ is a stack in groupoids over $(Sch/S)_{étale}$,*
2. *the diagonal $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces[3], and*

[3] Here we can either mean sheaves in the étale topology whose diagonal is representable and which have an étale surjective covering by a scheme or algebraic spaces as defined in Algebraic Spaces, Definition 40.6.1. Namely, by Bootstrap, Lemma 54.12.1 there is no difference.

(3) *there exists $U \in Ob(Sch_\alpha/S)$ and a 1-morphism $(Sch/U)_{étale} \to \mathcal{X}$ which is surjective and smooth.*

Then $\mathcal{X}$ is an algebraic stack in the sense of Algebraic Stacks, Definition 57.12.1.

Proof. Note that properties (2) and (3) of the lemma and the corresponding properties (2) and (3) of Algebraic Stacks, Definition 57.12.1 are independent of the topology. This is true because these properties involve only the notion of a 2-fibre product of categories fibred in groupoids, 1- and 2-morphisms of categories fibred in groupoids, the notion of a 1-morphism of categories fibred in groupoids representable by algebraic spaces, and what it means for such a 1-morphism to be surjective and smooth. Thus all we have to prove is that an étale stack in groupoids $\mathcal{X}$ with properties (2) and (3) is also an fppf stack in groupoids.

Using (2) let R be an algebraic space representing

$$(Sch_\alpha/U) \times_{\mathcal{X}} (Sch_\alpha/U)$$

By (3) the projections $s, t : R \to U$ are smooth. Exactly as in the proof of Algebraic Stacks, Lemma 57.16.1 there exists a groupoid in spaces (U, R, s, t, c) and a canonical fully faithful 1-morphism $[U/R]_{étale} \to \mathcal{X}$ where $[U/R]_{étale}$ is the étale stackification of presheaf in groupoids

$$T \longmapsto (U(T), R(T), s(T), t(T), c(T))$$

Claim: If $V \to T$ is a surjective smooth morphism from an algebraic space V to a scheme T, then there exists an étale covering $\{T_i \to T\}$ refining the covering $\{V \to T\}$. This follows from More on Morphisms, Lemma 33.26.7 or the more general Sheaves on Stacks, Lemma 58.18.10. Using the claim and arguing exactly as in Algebraic Stacks, Lemma 57.16.2 it follows that $[U/R]_{étale} \to \mathcal{X}$ is an equivalence.

Next, let $[U/R]$ denote the quotient stack in the fppf topology which is an algebraic stack by Algebraic Stacks, Theorem 57.17.3. Thus we have 1-morphisms

$$U \to [U/R]_{étale} \to [U/R].$$

Both $U \to [U/R]_{étale} \cong \mathcal{X}$ and $U \to [U/R]$ are surjective and smooth (the first by assumption and the second by the theorem) and in both cases the fibre product $U \times_{\mathcal{X}} U$ and $U \times_{[U/R]} U$ is representable by R. Hence the 1-morphism $[U/R]_{étale} \to [U/R]$ is fully faithful (since morphisms in the quotient stacks are given by morphisms into R, see Groupoids in Spaces, Section 52.23).

Finally, for any scheme T and morphism $t : T \to [U/R]$ the fibre product $V = T \times_{U/R} U$ is an algebraic space surjective and smooth over T. By the claim above there exists an étale covering $\{T_i \to T\}_{i \in I}$ and morphisms $T_i \to V$ over T. This proves that the object t of $[U/R]$ over T comes étale locally from U. We conclude that $[U/R]_{étale} \to [U/R]$ is an equivalence of stacks in groupoids over $(Sch/S)_{étale}$ by Stacks, Lemma 50.4.8. This concludes the proof. □

59.20. Other chapters

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(7) Commutative Algebra
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CHAPTER 60

Properties of Algebraic Stacks

60.1. Introduction

Please see Algebraic Stacks, Section 57.1 for a brief introduction to algebraic stacks, and please read some of that chapter for our foundations of algebraic stacks. The intent is that in that chapter we are careful to distinguish between schemes, algebraic spaces, algebraic stacks, and starting with this chapter we employ the customary abuse of language where all of these concepts are used interchangeably.

The goal of this chapter is to introduce some basic notions and properties of algebraic stacks. A fundamental reference for the case of quasi-separated algebraic stacks with representable diagonal is [**LMB00a**].

60.2. Conventions and abuse of language

We choose a big fppf site Sch_{fppf}. All schemes are contained in Sch_{fppf}. And all rings A considered have the property that $Spec(A)$ is (isomorphic) to an object of this big site.

We also fix a base scheme S, by the conventions above an element of Sch_{fppf}. The reader who is only interested in the absolute case can take $S = Spec(\mathbf{Z})$.

Here are our conventions regarding algebraic stacks:

(1) When we say *algebraic stack* we will mean an algebraic stacks over S, i.e., a category fibred in groupoids $p : \mathcal{X} \to (Sch/S)_{fppf}$ which satisfies the conditions of Algebraic Stacks, Definition 57.12.1.
(2) We will say $f : \mathcal{X} \to \mathcal{Y}$ is a *morphism of algebraic stacks* to indicate a 1-morphism of algebraic stacks over S, i.e., a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$, see Algebraic Stacks, Definition 57.12.3.
(3) A 2-*morphism* $\alpha : f \to g$ will indicate a 2-morphism in the 2-category of algebraic stacks over S, see Algebraic Stacks, Definition 57.12.3.
(4) Given morphisms $\mathcal{X} \to \mathcal{Z}$ and $\mathcal{Y} \to \mathcal{Z}$ of algebraic stacks we abusively call the 2-fibre product $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ the *fibre product*.
(5) We will write $\mathcal{X} \times_S \mathcal{Y}$ for the product of the algebraic stacks $\mathcal{X}$, $\mathcal{Y}$.
(6) We will often abuse notation and say two algebraic stacks $\mathcal{X}$ and $\mathcal{Y}$ are *isomorphic* if they are equivalent in this 2-category.

Here are our conventions regarding algebraic spaces.

(1) If we say X is an *algebraic space* then we mean that X is an algebraic space over S, i.e., X is a presheaf on $(Sch/S)_{fppf}$ which satisfies the conditions of Spaces, Definition 40.6.1.
(2) A *morphism of algebraic spaces* $f : X \to Y$ is a morphism of algebraic spaces over S as defined in Spaces, Definition 40.6.3.

(3) We will **not** distinguish between an algebraic space X and the algebraic stack $\mathcal{S}_X \to (Sch/S)_{fppf}$ it gives rise to, see Algebraic Stacks, Lemma 57.13.1.
(4) In particular, a *morphism* $f : X \to \mathcal{Y}$ from X to an algebraic stack $\mathcal{Y}$ means a morphism $f : \mathcal{S}_X \to \mathcal{Y}$ of algebraic stacks. Similarly for morphisms $\mathcal{Y} \to X$.
(5) Moreover, given an algebraic stack $\mathcal{X}$ we say $\mathcal{X}$ *is an algebraic space* to indicate that $\mathcal{X}$ is representable by an algebraic space, see Algebraic Stacks, Definition 57.8.1.
(6) We will use the following notational convention: If we indicate an algebraic stack by a roman capital (such as $X, Y, Z, A, B, \ldots$) then it will be the case that its inertia stack is trivial, and hence it is an algebraic space, see Algebraic Stacks, Proposition 57.13.3.

Here are our conventions regarding schemes.

(1) If we say X is a *scheme* then we mean that X is a scheme over S, i.e., X is an object of $(Sch/S)_{fppf}$.
(2) By a *morphism of schemes* we mean a morphism of schemes over S.
(3) We will **not** distinguish between a scheme X and the algebraic stack $\mathcal{S}_X \to (Sch/S)_{fppf}$ it gives rise to, see Algebraic Stacks, Lemma 57.13.1.
(4) In particular, a *morphism* $f : X \to \mathcal{Y}$ from a scheme X to an algebraic stack $\mathcal{Y}$ means a morphism $f : \mathcal{S}_X \to \mathcal{Y}$ of algebraic stacks. Similarly for morphisms $\mathcal{Y} \to X$.
(5) Moreover, given an algebraic stack $\mathcal{X}$ we say $\mathcal{X}$ *is a scheme* to indicate that $\mathcal{X}$ is representable, see Algebraic Stacks, Section 57.4.

Here are our conventions regarding morphism of algebraic stacks:

(1) A morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *representable*, or *representable by schemes* if for every scheme T and morphism $T \to \mathcal{Y}$ the fibre product $T \times_{\mathcal{Y}} \mathcal{X}$ is a scheme. See Algebraic Stacks, Section 57.6.
(2) A morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *representable by algebraic spaces* if for every scheme T and morphism $T \to \mathcal{Y}$ the fibre product $T \times_{\mathcal{Y}} \mathcal{X}$ is an algebraic space. See Algebraic Stacks, Definition 57.9.1. In this case $Z \times_{\mathcal{Y}} \mathcal{X}$ is an algebraic space whenever $Z \to \mathcal{Y}$ is a morphism whose source is an algebraic space, see Algebraic Stacks, Lemma 57.9.8.

Note that every morphism $X \to \mathcal{Y}$ from an algebraic space to an algebraic stack is representable by algebraic spaces, see Algebraic Stacks, Lemma 57.10.11. We will use this basic result without further mention.

60.3. Properties of morphisms representable by algebraic spaces

We will study properties of (arbitrary) morphisms of algebraic stacks in its own chapter. For morphisms representable by algebraic spaces we know what it means to be surjective, smooth, or étale, etc. This applies in particular to morphisms $X \to \mathcal{Y}$ from algebraic spaces to algebraic stacks. In this section, we recall how this works, we list the properties to which this applies, and we prove a few easy lemmas.

Our first lemma says a morphism is representable by algebraic spaces if it is so after a base change by a flat, locally finitely presented, surjective morphism.

Lemma 60.3.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let W be an algebraic space and let $W \to \mathcal{Y}$ be surjective, locally of finite presentation, and flat. The following are equivalent*

(1) *f is representable by algebraic spaces, and*
(2) *$W \times_{\mathcal{Y}} \mathcal{X}$ is an algebraic space.*

Proof. The implication (1) $\Rightarrow$ (2) is Algebraic Stacks, Lemma 57.9.8. Conversely, let $W \to \mathcal{Y}$ be as in (2). To prove (1) it suffices to show that f is faithful on fibre categories, see Algebraic Stacks, Lemma 57.15.2. Assumption (2) implies in particular that $W \times_{\mathcal{Y}} \mathcal{X} \to W$ is faithful. Hence the faithfulness of f follows from Stacks, Lemma 50.6.9. $\square$

Let P be a property of morphisms of algebraic spaces which is fppf local on the target and preserved by arbitrary base change. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks representable by algebraic spaces. Then we say *f has property P* if and only if for every scheme T and morphism $T \to \mathcal{Y}$ the morphism of algebraic spaces $T \times_{\mathcal{Y}} \mathcal{X} \to T$ has property P, see Algebraic Stacks, Definition 57.10.1.

It turns out that if $f : \mathcal{X} \to \mathcal{Y}$ is representable by algebraic spaces and has property P, then for any morphism of algebraic stacks $\mathcal{Y}' \to \mathcal{Y}$ the base change $\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Y}'$ has property P, see Algebraic Stacks, Lemmas 57.9.7 and 57.10.6. If the property P is preserved under compositions, then this holds also in the setting of morphisms of algebraic stacks representable by algebraic spaces, see Algebraic Stacks, Lemmas 57.9.9 and 57.10.5. Moreover, in this case products $\mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{Y}_1 \times \mathcal{Y}_2$ of morphisms representable by algebraic spaces having property $\mathcal{P}$ have property $\mathcal{P}$, see Algebraic Stacks, Lemma 57.10.8.

Finally, if we have two properties P, P' of morphisms of algebraic spaces which are fppf local on the target and preserved by arbitrary base change and if $P(f) \Rightarrow P'(f)$ for every morphism f, then the same implication holds for the corresponding property of morphisms of algebraic stacks representable by algebraic spaces, see Algebraic Stacks, Lemma 57.10.9. We will use this *without further mention* in the following and in the following chapters.

The discussion above applies to each of the following properties of morphisms of algebraic spaces

(1) quasi-compact, see Morphisms of Spaces, Lemma 42.9.3 and Descent on Spaces, Lemma 45.10.1,
(2) quasi-separated, see Morphisms of Spaces, Lemma 42.5.4 and Descent on Spaces, Lemma 45.10.2,
(3) universally closed, see Morphisms of Spaces, Lemma 42.10.3 and Descent on Spaces, Lemma 45.10.3,
(4) universally open, see Morphisms of Spaces, Lemma 42.7.3 and Descent on Spaces, Lemma 45.10.4,
(5) surjective, see Morphisms of Spaces, Lemma 42.6.5 and Descent on Spaces, Lemma 45.10.5,
(6) universally injective, see Morphisms of Spaces, Lemma 42.18.5 and Descent on Spaces, Lemma 45.10.6,
(7) locally of finite type, see Morphisms of Spaces, Lemma 42.22.3 and Descent on Spaces, Lemma 45.10.7,
(8) locally of finite presentation, see Morphisms of Spaces, Lemma 42.26.3 and Descent on Spaces, Lemma 45.10.8,
(9) finite type, see Morphisms of Spaces, Lemma 42.22.3 and Descent on Spaces, Lemma 45.10.9,
(10) finite presentation, see Morphisms of Spaces, Lemma 42.26.3 and Descent on Spaces, Lemma 45.10.10,

(11) flat, see Morphisms of Spaces, Lemma 42.27.3 and Descent on Spaces, Lemma 45.10.11,
(12) open immersion, see Morphisms of Spaces, Section 42.4 and Descent on Spaces, Lemma 45.10.12,
(13) isomorphism, see Descent on Spaces, Lemma 45.10.13,
(14) affine, see Morphisms of Spaces, Lemma 42.19.5 and Descent on Spaces, Lemma 45.10.14,
(15) closed immersion, see Morphisms of Spaces, Section 42.4 and Descent on Spaces, Lemma 45.10.15,
(16) separated, see Morphisms of Spaces, Lemma 42.5.4 and Descent on Spaces, Lemma 45.10.16,
(17) proper, see Morphisms of Spaces, Lemma 42.36.2 and Descent on Spaces, Lemma 45.10.17,
(18) quasi-affine, see Morphisms of Spaces, Lemma 42.20.5 and Descent on Spaces, Lemma 45.10.18,
(19) integral, see Morphisms of Spaces, Lemma 42.37.5 and Descent on Spaces, Lemma 45.10.20,
(20) finite, see Morphisms of Spaces, Lemma 42.37.5 and Descent on Spaces, Lemma 45.10.21,
(21) (locally) quasi-finite, see Morphisms of Spaces, Lemma 42.25.3 and Descent on Spaces, Lemma 45.10.22,
(22) syntomic, see Morphisms of Spaces, Lemma 42.32.3 and Descent on Spaces, Lemma 45.10.23,
(23) smooth, see Morphisms of Spaces, Lemma 42.33.3 and Descent on Spaces, Lemma 45.10.24,
(24) unramified, see Morphisms of Spaces, Lemma 42.34.4 and Descent on Spaces, Lemma 45.10.25,
(25) étale, see Morphisms of Spaces, Lemma 42.35.4 and Descent on Spaces, Lemma 45.10.26,
(26) finite locally free, see Morphisms of Spaces, Lemma 42.38.5 and Descent on Spaces, Lemma 45.10.27,
(27) monomorphism, see Morphisms of Spaces, Lemma 42.14.5 and Descent on Spaces, Lemma 45.10.28,
(28) immersion, see Morphisms of Spaces, Section 42.4 and Descent on Spaces, Lemma 45.11.1,
(29) locally separated, see Morphisms of Spaces, Lemma 42.5.4 and Descent on Spaces, Lemma 45.11.2,

Lemma 60.3.2. *Let P be a property of morphisms of algebraic spaces as above. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks representable by algebraic spaces. The following are equivalent:*

(1) *f has P,*
(2) *for every algebraic space Z and morphism $Z \to \mathcal{Y}$ the morphism $Z \times_{\mathcal{Y}} \mathcal{X} \to Z$ has P.*

Proof. The implication (2) $\Rightarrow$ (1) is immediate. Assume (1). Let $Z \to \mathcal{Y}$ be as in (2). Choose a scheme U and a surjective étale morphism $U \to Z$. By assumption the morphism

$U \times_{\mathcal{Y}} \mathcal{X} \to U$ has P. But the diagram

$$\begin{tikzcd} U \times_{\mathcal{Y}} \mathcal{X} \arrow[r] \arrow[d] & Z \times_{\mathcal{Y}} \mathcal{X} \arrow[d] \\ U \arrow[r] & Z \end{tikzcd}$$

is cartesian, hence the right vertical arrow has P as $\{U \to Z\}$ is an fppf covering. $\square$

The following lemma tells us it suffices to check P after a base change by a surjective, flat, locally finitely presented morphism.

Lemma 60.3.3. *Let P be a property of morphisms of algebraic spaces as above. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks representable by algebraic spaces. Let W be an algebraic space and let $W \to \mathcal{Y}$ be surjective, locally of finite presentation, and flat. Set $V = W \times_{\mathcal{Y}} \mathcal{X}$. Then*

$$(f \textit{ has } P) \Leftrightarrow (\textit{the projection } V \to W \textit{ has } P).$$

Proof. The implication from left to right follows from Lemma 60.3.2. Assume $V \to W$ has P. Let T be a scheme, and let $T \to \mathcal{Y}$ be a morphism. Consider the commutative diagram

$$\begin{tikzcd} T \times_{\mathcal{Y}} \mathcal{X} \arrow[d] & T \times_{\mathcal{Y}} W \arrow[l] \arrow[r] \arrow[d] & W \arrow[d] \\ T & T \times_{\mathcal{Y}} V \arrow[l] \arrow[r] & V \end{tikzcd}$$

of algebraic spaces. The squares are cartesian. The bottom left morphism is a surjective, flat morphism which is locally of finite presentation, hence $\{T \times_{\mathcal{Y}} V \to T\}$ is an fppf covering. Hence the fact that the right vertical arrow has property P implies that the left vertical arrow has property P. $\square$

Lemma 60.3.4. *Let P be a property of morphisms of algebraic spaces as above. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks representable by algebraic spaces. Let $\mathcal{Z} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces, surjective, flat, and locally of finite presentation. Set $\mathcal{W} = \mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}$. Then*

$$(f \textit{ has } P) \Leftrightarrow (\textit{the projection } \mathcal{W} \to \mathcal{Z} \textit{ has } P).$$

Proof. Choose an algebraic space W and a morphism $W \to \mathcal{Z}$ which is surjective, flat, and locally of finite presentation. By the discussion above the composition $W \to \mathcal{Y}$ is also surjective, flat, and locally of finite presentation. Denote $V = W \times_{\mathcal{Z}} \mathcal{W} = V \times_{\mathcal{Y}} \mathcal{X}$. By Lemma 60.3.3 we see that f has $\mathcal{P}$ if and only if $V \to W$ does and that $\mathcal{W} \to \mathcal{Z}$ has $\mathcal{P}$ if and only if $V \to W$ does. The lemma follows. $\square$

Lemma 60.3.5. *Let P be a property of morphisms of algebraic spaces as above. Let $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks representable by algebraic spaces. Assume*

1. *$\mathcal{X} \to \mathcal{Y}$ is surjective, flat, and locally of finite presentation,*
2. *the composition has P, and*
3. *P is local on the source in the fppf topology.*

Then $\mathcal{Y} \to \mathcal{Z}$ has property P.

Proof. Let Z be a scheme and let $Z \to \mathcal{Z}$ be a morphism. Set $X = \mathcal{X} \times_{\mathcal{Z}} Z$, $Y = \mathcal{Y} \times_{\mathcal{Z}} Z$. By (1) $\{X \to Y\}$ is an fppf covering of algebraic spaces and by (2) $X \to Z$ has property P. By (3) this implies that $Y \to Z$ has property P and we win. $\square$

Lemma 60.3.6. *Let $g : \mathcal{X}' \to \mathcal{X}$ be a morphism of algebraic stacks which is representable by algebraic spaces. Let $[U/R] \to \mathcal{X}$ be a presentation. Set $U' = U \times_{\mathcal{X}} \mathcal{X}'$, and $R' = R \times_{\mathcal{X}} \mathcal{X}'$. Then there exists a groupoid in algebraic spaces of the form (U', R', s', t', c'), a presentation $[U'/R'] \to \mathcal{X}'$, and the diagram*

$$\begin{tikzcd} {[U'/R']} \arrow[r] \arrow[d, "{[pr]}"'] & \mathcal{X}' \arrow[d, "g"] \\ {[U/R]} \arrow[r] & \mathcal{X} \end{tikzcd}$$

is 2-commutative where the morphism $[pr]$ comes from a morphism of groupoids $pr : (U', R', s', t', c') \to (U, R, s, t, c)$.

Proof. Since $U \to \mathcal{Y}$ is surjective and smooth, see Algebraic Stacks, Lemma 57.17.2 the base change $U' \to \mathcal{X}'$ is also surjective and smooth. Hence, by Algebraic Stacks, Lemma 57.16.2 it suffices to show that $R' = U' \times_{\mathcal{X}'} U'$ in order to get a smooth groupoid (U', R', s', t', c') and a presentation $[U'/R'] \to \mathcal{X}'$. Using that $R = V \times_{\mathcal{Y}} V$ (see Groupoids in Spaces, Lemma 52.21.2) this follows from

$$R' = U \times_{\mathcal{X}} U \times_{\mathcal{X}} \mathcal{X}' = (U \times_{\mathcal{X}} \mathcal{X}') \times_{\mathcal{X}'} (U \times_{\mathcal{X}} \mathcal{X}')$$

see Categories, Lemmas 4.28.8 and 4.28.10. Clearly the projection morphisms $U' \to U$ and $R' \to R$ give the desired morphism of groupoids $\text{pr} : (U', R', s', t', c') \to (U, R, s, t, c)$. Hence the morphism [pr] of quotient stacks by Groupoids in Spaces, Lemma 52.20.1.

We still have to show that the diagram 2-commutes. It is clear that the diagram

$$\begin{tikzcd} U' \arrow[r, "f'"'] \arrow[d, "\text{pr}_U"'] & \mathcal{X}' \arrow[d, "g"] \\ U \arrow[r, "f"] & \mathcal{X} \end{tikzcd}$$

2-commutes where $\text{pr}_U : U' \to U$ is the projection. There is a canonical 2-arrow $\tau : f \circ t \to f \circ s$ in $Mor(R, \mathcal{X})$ coming from $R = U \times_{\mathcal{X}} U$, $t = \text{pr}_0$, and $s = \text{pr}_1$. Using the isomorphism $R' \to U' \times_{\mathcal{X}'} U'$ we get similarly an isomorphism $\tau' : f' \circ t' \to f' \circ s'$. Note that $g \circ f' \circ t' = f \circ t \circ \text{pr}_R$ and $g \circ f' \circ s' = f \circ s \circ \text{pr}_R$, where $\text{pr}_R : R' \to R$ is the projection. Thus it makes sense to ask if

$$\tau \star \text{id}_{\text{pr}_R} = \text{id}_g \star \tau'. \qquad (60.3.6.1)$$

Now we make two claims: (1) if Equation (60.3.6.1) holds, then the diagram 2-commutes, and (2) Equation (60.3.6.1) holds. We omit the proof of both claims. Hints: part (1) follows from the construction of $f = f_{can}$ and $f' = f'_{can}$ in Algebraic Stacks, Lemma 57.16.1. Part (2) follows by carefully working through the definitions. □

Remark 60.3.7. Let $\mathcal{Y}$ be an algebraic stack. Consider the following 2-category:

(1) An object is a morphism $f : \mathcal{X} \to \mathcal{Y}$ which is representable by algebraic spaces,
(2) a 1-morphism $(g, \beta) : (f_1 : \mathcal{X}_1 \to \mathcal{Y}) \to (f_2 : \mathcal{X}_2 \to \mathcal{Y})$ consists of a morphism $g : \mathcal{X}_1 \to \mathcal{X}_2$ and a 2-morphism $\beta : f_1 \to f_2 \circ g$, and
(3) a 2-morphism between $(g, \beta), (g', \beta') : (f_1 : \mathcal{X}_1 \to \mathcal{Y}) \to (f_2 : \mathcal{X}_2 \to \mathcal{Y})$ is a 2-morphism $\alpha : g \to g'$ such that $(\text{id}_{f_2} \star \alpha) \circ \beta = \beta'$.

Let us denote this 2-category $Spaces/\mathcal{Y}$ by analogy with the notation of Topologies on Spaces, Section 44.2. Now we claim that in this 2-category the morphism categories

$$Mor_{Spaces/\mathcal{Y}}((f_1 : \mathcal{X}_1 \to \mathcal{Y}), (f_2 : \mathcal{X}_2 \to \mathcal{Y}))$$

are all setoids. Namely, a 2-morphism α is a rule which to each object x_1 of $\mathcal{X}_1$ assigns an isomorphism $\alpha_{x_1} : g(x_1) \longrightarrow g'(x_1)$ in the relevant fibre category of $\mathcal{X}_2$ such that the diagram

$$\begin{tikzcd} & f_2(x_1) \arrow[dl, "\beta_{x_1}"'] \arrow[dr, "\beta'_{x_1}"] & \\ f_2(g(x_1)) \arrow[rr, "f_2(\alpha_{x_1})"] & & f_2(g'(x_1)) \end{tikzcd}$$

commutes. But since f_2 is faithful (see Algebraic Stacks, Lemma 57.15.2) this means that if α_{x_1} exists, then it is unique! In other words the 2-category $Spaces/\mathcal{Y}$ is very close to being a category. Namely, if we replace 1-morphisms by isomorphism classes of 1-morphisms we obtain a category. We will often perform this replacement without further mention.

60.4. Points of algebraic stacks

Let $\mathcal{X}$ be an algebraic stack. Let K, L be two fields and let $p : Spec(K) \to \mathcal{X}$ and $q : Spec(L) \to \mathcal{X}$ be morphisms. We say that p and q are *equivalent* if there exists a field Ω and a 2-commutative diagram

$$\begin{tikzcd} Spec(\Omega) \arrow[r] \arrow[d] & Spec(L) \arrow[d, "q"] \\ Spec(K) \arrow[r, "p"] & \mathcal{X}. \end{tikzcd}$$

Lemma 60.4.1. *The notion above does indeed define an equivalence relation on morphisms from spectra of fields into the algebraic stack $\mathcal{X}$.*

Proof. It is clear that the relation is reflexive and symmetric. Hence we have to prove that it is transitive. This comes down to the following: Given a diagram

$$\begin{tikzcd} Spec(\Omega) \arrow[r, "b"'] \arrow[d, "a"'] & Spec(L) \arrow[d, "q"] & Spec(\Omega') \arrow[l, "b'"] \arrow[d, "a'"] \\ Spec(K) \arrow[r, "p"] & \mathcal{X} & Spec(K') \arrow[l, "p'"'] \end{tikzcd}$$

with both squares 2-commutative we have to show that p is equivalent to p'. By the 2-Yoneda lemma (see Algebraic Stacks, Section 57.5) the morphisms p, p', and q are given by objects x, x', and y in the fibre categories of $\mathcal{X}$ over $Spec(K)$, $Spec(K')$, and $Spec(L)$. The 2-commutativity of the squares means that there are isomorphisms $\alpha : a^*x \to b^*y$ and $\alpha' : (a')^*x' \to (b')^*y$ in the fibre categories of $\mathcal{X}$ over $Spec(\Omega)$ and $Spec(\Omega')$. Choose any field Ω'' and embeddings $\Omega \to \Omega''$ and $\Omega' \to \Omega''$ agreeing on L. Then we can extend the diagram above to

$$\begin{tikzcd} & Spec(\Omega'') \arrow[dl, "c"'] \arrow[d, "q'"] \arrow[dr, "c'"] & \\ Spec(\Omega) \arrow[r, "b"'] \arrow[d, "a"'] & Spec(L) \arrow[d, "q"] & Spec(\Omega') \arrow[l, "b'"] \arrow[d, "a'"] \\ Spec(K) \arrow[r, "p"] & \mathcal{X} & Spec(K') \arrow[l, "p'"'] \end{tikzcd}$$

with commutative triangles and

$$(q')^*(\alpha')^{-1} \circ (q')^*\alpha : (a \circ c)^*x \longrightarrow (a' \circ c')^*x'$$

is an isomorphism in the fibre category over $Spec(\Omega'')$. Hence p is equivalent to p' as desired. $\square$

Definition 60.4.2. Let $\mathcal{X}$ be an algebraic stack. A *point* of $\mathcal{X}$ is an equivalence class of morphisms from spectra of fields into $\mathcal{X}$. The set of points of $\mathcal{X}$ is denoted $|\mathcal{X}|$.

This agrees with our definition of points of algebraic spaces, see Properties of Spaces, Definition 41.4.1. Moreover, for a scheme we recover the usual notion of points, see Properties of Spaces, Lemma 41.4.2. If $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks then there is an induced map $|f| : |\mathcal{X}| \to |\mathcal{Y}|$ which maps a representative $x : Spec(K) \to \mathcal{X}$ to the representative $f \circ x : Spec(K) \to \mathcal{Y}$. This is well defined: namely 2-isomorphic 1-morphisms remain 2-isomorphic after pre- or post-composing by a 1-morphism because you can horizontally pre- or post-compose by the identity of the given 1-morphism. This holds in any (strict) $(2, 1)$-category. If

$$\begin{tikzcd}
\mathcal{X} \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d] \\
\mathcal{W} \arrow[r] & \mathcal{Z}
\end{tikzcd}$$

is a 2-commutative diagram of algebraic stacks, then the diagram of sets

$$\begin{tikzcd}
|\mathcal{X}| \arrow[r] \arrow[d] & |\mathcal{Y}| \arrow[d] \\
|\mathcal{W}| \arrow[r] & |\mathcal{Z}|
\end{tikzcd}$$

is commutative. In particular, if $\mathcal{X} \to \mathcal{Y}$ is an equivalence then $|\mathcal{X}| \to |\mathcal{Y}|$ is a bijection.

Lemma 60.4.3. *Let*

$$\begin{tikzcd}
\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\
\mathcal{Z} \arrow[r] & \mathcal{Y}
\end{tikzcd}$$

be a fibre product of algebraic stacks. Then the map of sets of points

$$|\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}| \longrightarrow |\mathcal{Z}| \times_{|\mathcal{Y}|} |\mathcal{X}|$$

is surjective.

Proof. Namely, suppose given fields K, L and morphisms $Spec(K) \to \mathcal{X}$, $Spec(L) \to \mathcal{Z}$, then the assumption that they agree as elements of $|\mathcal{Y}|$ means that there is a common extension $K \subset M$ and $L \subset M$ such that $Spec(M) \to Spec(K) \to \mathcal{X} \to \mathcal{Y}$ and $Spec(M) \to Spec(L) \to \mathcal{Z} \to \mathcal{Y}$ are 2-isomorphic. And this is exactly the condition which says you get a morphism $Spec(M) \to \mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}$. $\square$

Lemma 60.4.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. The following are equivalent:*

(1) $|f| : |\mathcal{X}| \to |\mathcal{Y}|$ *is surjective, and*
(2) f *is surjective.*

Proof. Assume (1). Let $T \to \mathcal{Y}$ be a morphism whose source is a scheme. To prove (2) we have to show that the morphism of algebraic spaces $T \times_{\mathcal{Y}} \mathcal{X} \to T$ is surjective. By Morphisms of Spaces, Definition 42.6.2 this means we have to show that $|T \times_{\mathcal{Y}} \mathcal{X}| \to |T|$ is surjective. Applying Lemma 60.4.3 we see that this follows from (1).

Conversely, assume (2). Let $y : Spec(K) \to \mathcal{Y}$ be a morphism from the spectrum of a field into $\mathcal{Y}$. By assumption the morphism $Spec(K) \times_{y,\mathcal{Y}} \mathcal{X} \to Spec(K)$ of algebraic spaces is surjective. By Morphisms of Spaces, Definition 42.6.2 this means there exists a field extension $K \subset K'$ and a morphism $Spec(K') \to Spec(K) \times_{y,\mathcal{Y}} \mathcal{X}$ such that the left square of the diagram

$$\begin{tikzcd} Spec(K') \arrow[r] \arrow[d] & Spec(K) \times_{y,\mathcal{Y}} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ Spec(K) \arrow[r, equal] & Spec(K) \arrow[r, "y"] & \mathcal{Y} \end{tikzcd}$$

is commutative. This shows that $|X| \to |\mathcal{Y}|$ is surjective. □

Here is a lemma explaining how to compute the set of points in terms of a presentation.

Lemma 60.4.5. *Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{X} = [U/R]$ be a presentation of $\mathcal{X}$, see Algebraic Stacks, Definition 57.16.5. Then the image of $|R| \to |U| \times |U|$ is an equivalence relation and $|\mathcal{X}|$ is the quotient of $|U|$ by this equivalence relation.*

Proof. The assumption means that we have a smooth groupoid (U, R, s, t, c) in algebraic spaces, and an equivalence $f : [U/R] \to \mathcal{X}$. We may assume $\mathcal{X} = [U/R]$. The induced morphism $p : U \to \mathcal{X}$ is smooth and surjective, see Algebraic Stacks, Lemma 57.17.2. Hence $|U| \to |\mathcal{X}|$ is surjective by Lemma 60.4.4. Note that $R = U \times_{\mathcal{X}} U$, see Groupoids in Spaces, Lemma 52.21.2. Hence Lemma 60.4.3 implies the map

$$|R| \longrightarrow |U| \times_{|\mathcal{X}|} |U|$$

is surjective. Hence the image of $|R| \to |U| \times |U|$ is exactly the set of pairs $(u_1, u_2) \in |U| \times |U|$ such that u_1 and u_2 have the same image in $|\mathcal{X}|$. Combining these two statements we get the result of the lemma. □

Remark 60.4.6. The result of Lemma 60.4.5 can be generalized as follows. Let $\mathcal{X}$ be an algebraic stack. Let U be an algebraic space and let $f : U \to \mathcal{X}$ be a surjective morphism (which makes sense by Section 60.3). Let $R = U \times_{\mathcal{X}} U$, let (U, R, s, t, c) be the groupoid in algebraic spaces, and let $f_{can} : [U/R] \to \mathcal{X}$ be the canonical morphism as constructed in Algebraic Stacks, Lemma 57.16.1. Then the image of $|R| \to |U| \times |U|$ is an equivalence relation and $|\mathcal{X}| = |U|/|R|$. The proof of Lemma 60.4.5 works without change. (Of course in general $[U/R]$ is not an algebraic stack, and in general f_{can} is not an isomorphism.)

Lemma 60.4.7. *There exists a unique topology on the sets of points of algebraic stacks with the following properties:*

1. *for every morphism of algebraic stacks $\mathcal{X} \to \mathcal{Y}$ the map $|\mathcal{X}| \to |\mathcal{Y}|$ is continuous, and*
2. *for every morphism $U \to \mathcal{X}$ which is flat and locally of finite presentation with U an algebraic space the map of topological spaces $|U| \to |\mathcal{X}|$ is continuous and open.*

Proof. Choose a morphism $p : U \to \mathcal{X}$ which is surjective, flat, and locally of finite presentation with U an algebraic space. Such exist by the definition of an algebraic stack, as a smooth morphism is flat and locally of finite presentation (see Morphisms of Spaces, Lemmas 42.33.5 and 42.33.7). We define a topology on $|\mathcal{X}|$ by the rule: $W \subset |\mathcal{X}|$ is open if and only if $|p|^{-1}(W)$ is open in $|U|$. To show that this is independent of the choice of p, let $p' : U' \to \mathcal{X}$ be another morphism which is surjective, flat, locally of finite presentation from an algebraic space to $\mathcal{X}$. Set $U'' = U \times_{\mathcal{X}} U'$ so that we have a 2-commutative diagram

$$\begin{tikzcd} U'' \arrow[r] \arrow[d] & U' \arrow[d] \\ U \arrow[r] & \mathcal{X} \end{tikzcd}$$

As $U \to \mathcal{X}$ and $U' \to \mathcal{X}$ are surjective, flat, locally of finite presentation we see that $U'' \to U'$ and $U'' \to U$ are surjective, flat and locally of finite presentation, see Lemma 60.3.2. Hence the maps $|U''| \to |U'|$ and $|U''| \to |U|$ are continuous, open and surjective, see Morphisms of Spaces, Definition 42.6.2 and Lemma 42.27.5. This clearly implies that our definition is independent of the choice of $p : U \to \mathcal{X}$.

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. By Algebraic Stacks, Lemma 57.15.1 we can find a 2-commutative diagram

$$\begin{tikzcd} U \arrow[r, "a"'] \arrow[d, "x"'] & V \arrow[d, "y"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}$$

with surjective smooth vertical arrows. Consider the associated commutative diagram

$$\begin{tikzcd} {|U|} \arrow[r, "|a|"'] \arrow[d, "|x|"'] & {|V|} \arrow[d, "|y|"] \\ {|\mathcal{X}|} \arrow[r, "|f|"] & {|\mathcal{Y}|} \end{tikzcd}$$

of sets. If $W \subset |\mathcal{Y}|$ is open, then by the definition above this means exactly that $|y|^{-1}(W)$ is open in $|V|$. Since $|a|$ is continuous we conclude that $|a|^{-1}|y|^{-1}(W) = |x|^{-1}|f|^{-1}(W)$ is open in $|W|$ which means by definition that $|f|^{-1}(W)$ is open in $|\mathcal{X}|$. Thus $|f|$ is continuous.

Finally, we have to show that if U is an algebraic space, and $U \to \mathcal{X}$ is flat and locally of finite presentation, then $|U| \to |\mathcal{X}|$ is open. Let $V \to \mathcal{X}$ be surjective, flat, and locally of finite presentation with V an algebraic space. Consider the commutative diagram

$$\begin{tikzcd} {|U \times_{\mathcal{X}} V|} \arrow[r, "e"'] \arrow[dr, "f"'] & {|U| \times_{|\mathcal{X}|} |V|} \arrow[r, "d"'] \arrow[d, "c"'] & {|V|} \arrow[d, "b"] \\ & {|U|} \arrow[r, "a"] & {|\mathcal{X}|} \end{tikzcd}$$

Now the morphism $U \times_{\mathcal{X}} V \to U$ is surjective, i.e, $f : |U \times_{\mathcal{X}} V| \to |U|$ is surjective. The left top horizontal arrow is surjective, see Lemma 60.4.3. The morphism $U \times_{\mathcal{X}} V \to V$ is flat and locally of finite presentation, hence $d \circ e : |U \times_{\mathcal{X}} V| \to |V|$ is open, see Morphisms of Spaces, Lemma 42.27.5. Pick $W \subset |U|$ open. The properties above imply that $b^{-1}(a(W)) = (d \circ e)(f^{-1}(W))$ is open, which by construction means that $a(W)$ is open as desired. $\square$

Definition 60.4.8. Let $\mathcal{X}$ be an algebraic stack. The underlying *topological space* of $\mathcal{X}$ is the set of points $|\mathcal{X}|$ endowed with the topology constructed in Lemma 60.4.7.

This definition does not conflict with the already existing topology on $|\mathcal{X}|$ if $\mathcal{X}$ is an algebraic space.

Lemma 60.4.9. *Let $\mathcal{X}$ be an algebraic stack. Every point of $|\mathcal{X}|$ has a fundamental system of quasi-compact open neighbourhoods. In particular $|\mathcal{X}|$ is locally quasi-compact in the sense of Topology, Definition 5.18.1.*

Proof. This follows formally from the fact that there exists a scheme U and a surjective, open, continuous map $U \to |\mathcal{X}|$ of topological spaces. Namely, if $U \to \mathcal{X}$ is surjective and smooth, then Lemma 60.4.7 garantees that $|U| \to |\mathcal{X}|$ is continuous, surjective, and open. $\square$

60.5. Surjective morphisms

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. In Section 60.3 we have already defined what it means for f to be surjective. In Lemma 60.4.4 we have seen that this is equivalent to requiring $|f| : |\mathcal{X}| \to |\mathcal{Y}|$ to be surjective. This clears the way for the following definition.

Definition 60.5.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is *surjective* if the map $|f| : |\mathcal{X}| \to |\mathcal{Y}|$ of associated topological spaces is surjective.

Here are some lemmas.

Lemma 60.5.2. *The composition of surjective morphisms is surjective.*

Proof. Omitted. $\square$

Lemma 60.5.3. *The base change of a surjective morphism is surjective.*

Proof. Omitted. Hint: Use Lemma 60.4.3. $\square$

Lemma 60.5.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $\mathcal{Y}' \to \mathcal{Y}$ be a surjective morphism of algebraic stacks. If the base change $f' : \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Y}'$ of f is surjective, then f is surjective.*

Proof. Immediate from Lemma 60.4.3. $\square$

Lemma 60.5.5. *Let $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. If $\mathcal{X} \to \mathcal{Z}$ is surjective so is $\mathcal{Y} \to \mathcal{Z}$.*

Proof. Immediate. $\square$

60.6. Quasi-compact algebraic stacks

The following definition is equivalent with the definition for algebraic spaces by Properties of Spaces, Lemma 41.5.2.

Definition 60.6.1. Let $\mathcal{X}$ be an algebraic stack. We say $\mathcal{X}$ is *quasi-compact* if and only if $|\mathcal{X}|$ is quasi-compact.

Lemma 60.6.2. *Let $\mathcal{X}$ be an algebraic stack. The following are equivalent:*

(1) *$\mathcal{X}$ is quasi-compact,*
(2) *there exists a surjective smooth morphism $U \to \mathcal{X}$ with U a quasi-compact scheme,*
(3) *there exists a surjective smooth morphism $U \to \mathcal{X}$ with U a quasi-compact algebraic space, and*

(4) *there exists a surjective morphism $\mathcal{U} \to \mathcal{X}$ of algebraic stacks such that $\mathcal{U}$ is quasi-compact.*

Proof. We will use Lemma 60.4.4. Suppose $\mathcal{U}$ and $\mathcal{U} \to \mathcal{X}$ are as in (4). Then since $|\mathcal{U}| \to |\mathcal{X}|$ is surjective and continuous we conclude that $|\mathcal{X}|$ is quasi-compact. Thus (4) implies (1). The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are immediate. Assume (1), i.e., $\mathcal{X}$ is quasi-compact, i.e., that $|\mathcal{X}|$ is quasi-compact. Choose a scheme U and a surjective smooth morphism $U \to \mathcal{X}$. Then since $|U| \to |\mathcal{X}|$ is open we see that there exists a quasi-compact open $U' \subset U$ such that $|U'| \to |X|$ is surjective (and still smooth). Hence (2) holds. $\square$

Lemma 60.6.3. *A finite disjoint union of quasi-compact algebraic stacks is a quasi-compact algebraic stack.*

Proof. This is clear from the corresponding topological fact. $\square$

60.7. Properties of algebraic stacks defined by properties of schemes

Any smooth local property of schemes gives rise to a corresponding property of algebraic stacks via the following lemma. Note that a property of schemes which is smooth local is also étale local as any étale covering is also a smooth covering. Hence for a smooth local property P of schemes we know what it means to say that an algebraic space has P, see Properties of Spaces, Section 60.7.

Lemma 60.7.1. *Let $\mathcal{P}$ be a property of schemes which is local in the smooth topology, see Descent, Definition 31.11.1. Let $\mathcal{X}$ be an algebraic stack. The following are equivalent*

(1) *for some scheme U and some surjective smooth morphism $U \to \mathcal{X}$ the scheme U has property $\mathcal{P}$,*
(2) *for every scheme U and every smooth morphism $U \to \mathcal{X}$ the scheme U has property $\mathcal{P}$,*
(3) *for some algebraic space U and some surjective smooth morphism $U \to \mathcal{X}$ the algebraic space U has property $\mathcal{P}$, and*
(4) *for every algebraic space U and every smooth morphism $U \to \mathcal{X}$ the algebraic space U has property $\mathcal{P}$.*

If $\mathcal{X}$ is a scheme this is equivalent to $\mathcal{P}(U)$. If $\mathcal{X}$ is an algebraic space this is equivalent to X having property $\mathcal{P}$.

Proof. Let $U \to \mathcal{X}$ surjective and smooth with U an algebraic space. Let $V \to \mathcal{X}$ be a smooth morphism with V an algebraic space. Choose schemes U' and V' and surjective étale morphisms $U' \to U$ and $V' \to V$. Finally, choose a scheme W and a surjective étale morphism $W \to V' \times_{\mathcal{X}} U'$. Then $W \to V'$ and $W \to U'$ are smooth morphisms of schemes as compositions of étale and smooth morphisms of algebraic spaces, see Morphisms of Spaces, Lemmas 42.35.6 and 42.33.2. Moreover, $W \to V'$ is surjective as $U' \to \mathcal{X}$ is surjective. Hence, we have

$$\mathcal{P}(U) \Leftrightarrow \mathcal{P}(U') \Rightarrow \mathcal{P}(W) \Rightarrow \mathcal{P}(V') \Leftrightarrow \mathcal{P}(V)$$

where the equivalences are by definition of property $\mathcal{P}$ for algebraic spaces, and the two implications come from Descent, Definition 31.11.1. This proves (3) $\Rightarrow$ (4).

The implications (2) $\Rightarrow$ (1), (1) $\Rightarrow$ (3), and (4) $\Rightarrow$ (2) are immediate. $\square$

Definition 60.7.2. Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{P}$ be a property of schemes which is local in the smooth topology. We say $\mathcal{X}$ *has property $\mathcal{P}$* if any of the equivalent conditions of Lemma 60.7.1 hold.

Remark 60.7.3. Here is a list of properties which are local for the smooth topology (keep in mind that the fpqc, fppf, and syntomic topologies are stronger than the smooth topology):

(1) locally Noetherian, see Descent, Lemma 31.12.1,
(2) Jacobson, see Descent, Lemma 31.12.2,
(3) locally Noetherian and (S_k), see Descent, Lemma 31.13.1,
(4) Cohen-Macaulay, see Descent, Lemma 31.13.2,
(5) reduced, see Descent, Lemma 31.14.1,
(6) normal, see Descent, Lemma 31.14.2,
(7) locally Noetherian and (R_k), see Descent, Lemma 31.14.3,
(8) regular, see Descent, Lemma 31.14.4,
(9) Nagata, see Descent, Lemma 31.14.5.

Any smooth local property of germs of schemes gives rise to a corresponding property of algebraic stacks. Note that a property of germs which is smooth local is also étale local. Hence for a smooth local property of germs of schemes P we know what it means to say that an algebraic space X has property P at $x \in |X|$, see Properties of Spaces, Section 60.7.

Lemma 60.7.4. *Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$ be a point of $\mathcal{X}$. Let $\mathcal{P}$ be a property of germs of schemes which is smooth local, see Descent, Definition 31.17.1. The following are equivalent*

(1) *for any smooth morphism $U \to \mathcal{X}$ with U a scheme and $u \in U$ with $a(u) = x$ we have $\mathcal{P}(U, u)$,*
(2) *for some smooth morphism $U \to \mathcal{X}$ with U a scheme and some $u \in U$ with $a(u) = x$ we have $\mathcal{P}(U, u)$,*
(3) *for any smooth morphism $U \to \mathcal{X}$ with U an algebraic space and $u \in |U|$ with $a(u) = x$ the algebraic space U has property $\mathcal{P}$ at u, and*
(4) *for some smooth morphism $U \to \mathcal{X}$ with U a an algebraic space and some $u \in |U|$ with $a(u) = x$ the algebraic space U has property $\mathcal{P}$ at u.*

If $\mathcal{X}$ is representable, then this is equivalent to $\mathcal{P}(\mathcal{X}, x)$. If $\mathcal{X}$ is an algebraic space then this is equivalent to $\mathcal{X}$ having property $\mathcal{P}$ at x.

Proof. Let $a : U \to \mathcal{X}$ and $u \in |U|$ as in (3). Let $b : V \to \mathcal{X}$ be another smooth morphism with V an algebraic space and $v \in |V|$ with $b(v) = x$ also. Choose a scheme U', an étale morphism $U' \to U$ and $u' \in U'$ mapping to u. Choose a scheme V', an étale morphism $V' \to V$ and $v' \in V'$ mapping to v. By Lemma 60.4.3 there exists a point $\overline{w} \in |V' \times_{\mathcal{X}} U'|$ mapping to u' and v'. Choose a scheme W and a surjective étale morphism $W \to V' \times_{\mathcal{X}} U'$. We may choose a $w \in |W|$ mapping to $\overline{w}$ (see Properties of Spaces, Lemma 41.4.4). Then $W \to V'$ and $W \to U'$ are smooth morphisms of schemes as compositions of étale and smooth morphisms of algebraic spaces, see Morphisms of Spaces, Lemmas 42.35.6 and 42.33.2. Hence

$$\mathcal{P}(U, u) \Leftrightarrow \mathcal{P}(U', u') \Leftrightarrow \mathcal{P}(W, w) \Leftrightarrow \mathcal{P}(V', v') \Leftrightarrow \mathcal{P}(V, v)$$

The outer two equivalences by Properties of Spaces, Definition 41.7.5 and the other two by what it means to be a smooth local property of germs of schemes. This proves (4) ⇒ (3).

The implications (1) ⇒ (2), (2) ⇒ (4), and (3) ⇒ (1) are immediate. □

Definition 60.7.5. Let $\mathcal{P}$ be a property of germs of schemes which is smooth local. Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$. We say $\mathcal{X}$ *has property $\mathcal{P}$ at x* if any of the equivalent conditions of Lemma 60.7.4 holds.

60.8. Monomorphisms of algebraic stacks

We define a monomorphism of algebraic stacks in the following way. We will see in Lemma 60.8.4 that this is compatible with the corresponding 2-category theoretic notion.

Definition 60.8.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is a *monomorphism* if it is representable by algebraic spaces and a monomorphism in the sense of Section 60.3.

First some basic lemmas.

Lemma 60.8.2. *Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $\mathcal{Z} \to \mathcal{Y}$ be a monomorphism. Then $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X}$ is a monomorphism.*

Proof. This follows from the general discussion in Section 60.3. □

Lemma 60.8.3. *Compositions of monomorphisms of algebraic stacks are monomorphisms.*

Proof. This follows from the general discussion in Section 60.3 and Morphisms of Spaces, Lemma 42.14.4. □

Lemma 60.8.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent:*

1. *f is a monomorphism,*
2. *f is fully faithful,*
3. *the diagonal $\Delta_f : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is an equivalence, and*
4. *there exists an algebraic space W and a surjective, flat morphism $W \to \mathcal{Y}$ which is locally of finite presentation such that $V = \mathcal{X} \times_{\mathcal{Y}} W$ is an algebraic space, and the morphism $V \to W$ is a monomorphism of algebraic spaces.*

Proof. The equivalence of (1) and (4) follows from the general discussion in Section 60.3 and in particular Lemmas 60.3.1 and 60.3.3.

The equivalence of (2) and (3) is Categories, Lemma 4.32.9.

Assume the equivalent conditions (2) and (3). Then f is representable by algebraic spaces according to Algebraic Stacks, Lemma 57.15.2. Moreover, the 2-Yoneda lemma combined with the fully faithfullness implies that for every scheme T the functor

$$Mor(T, \mathcal{X}) \longrightarrow Mor(T, \mathcal{Y})$$

is fully faithful. Hence given a morphism $y : T \to \mathcal{Y}$ there exists up to unique 2-isomorphism at most one morphism $x : T \to \mathcal{X}$ such that $y \cong f \circ x$. In particular, given a morphism of schemes $h : T' \to T$ there exists at most one lift $\tilde{h} : T' \to T \times_{\mathcal{Y}} \mathcal{X}$ of h. Thus $T \times_{\mathcal{Y}} \mathcal{X} \to T$ is a monomorphism of algebraic spaces, which proves that (1) holds.

Finally, assume that (1) holds. Then for any scheme T and morphism $y : T \to \mathcal{Y}$ the fibre product $T \times_{\mathcal{Y}} \mathcal{X}$ is an algebraic space, and $T \times_{\mathcal{Y}} \mathcal{X} \to T$ is a monomorphism. Hence there exists up to unique isomorphism exactly one pair (x, α) where $x : T \to \mathcal{X}$ is a morphism and $\alpha : f \circ x \to y$ is a 2-morphism. Applying the 2-Yoneda lemma this says exactly that f is fully faithful, i.e., that (2) holds. □

Lemma 60.8.5. *A monomorphism of algebraic stacks induces an injective map of sets of points.*

Proof. Let $f : \mathcal{X} \to \mathcal{Y}$ be a monomorphism of algebraic stacks. Suppose that $x_i : Spec(K_i) \to \mathcal{X}$ be morphisms such that $f \circ x_1$ and $f \circ x_2$ define the same element of $|\mathcal{Y}|$. Applying the definition we find a common extension Ω with corresponding morphisms $c_i : Spec(\Omega) \to Spec(K_i)$ and a 2-isomorphism $\beta : f \circ x_1 \circ c_1 \to f \circ x_1 \circ c_2$. As f is fully faithful, see Lemma 60.8.4, we can lift β to an isomorphism $\alpha : f \circ x_1 \circ c_1 \to f \circ x_1 \circ c_2$. Hence x_1 and x_2 define the same point of $|\mathcal{X}|$ as desired. □

60.9. Immersions of algebraic stacks

Immersions of algebraic stacks are defined as follows.

Definition 60.9.1. Immersions.

(1) A morphism of algebraic stacks is called an *open immersion* if it is representable, and an open immersion in the sense of Section 60.3.
(2) A morphism of algebraic stacks is called a *closed immersion* if it is representable, and a closed immersion in the sense of Section 60.3.
(3) A morphism of algebraic stacks is called an *immersion* if it is representable, and an immersion in the sense of Section 60.3.

This is not the most convenient way to think about immersions for us. For us it is a little bit more convenient to think of an immersion as a morphism of algebraic stacks which is representable by algebraic spaces and is an immersion in the sense of Section 60.3. Similarly for closed and open immersions. Since this is clearly equivalent to the notion just defined we shall use this characterization without further mention. We prove a few simple lemmas about this notion.

Lemma 60.9.2. *Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $\mathcal{Z} \to \mathcal{Y}$ be a (closed, resp. open) immersion. Then $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X}$ is a (closed, resp. open) immersion.*

Proof. This follows from the general discussion in Section 60.3. □

Lemma 60.9.3. *Compositions of immersions of algebraic stacks are immersions. Similarly for closed immersions and open immersions.*

Proof. This follows from the general discussion in Section 60.3 and Spaces, Lemma 40.12.2. □

Lemma 60.9.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. let W be an algebraic space and let $W \to \mathcal{Y}$ be a surjective, flat morphism which is locally of finite presentation. The following are equivalent:*

(1) *f is an (open, resp. closed) immersion, and*
(2) *$V = W \times_{\mathcal{Y}} \mathcal{X}$ is an algebraic space, and $V \to W$ is an (open, resp. closed) immersion.*

Proof. This follows from the general discussion in Section 60.3 and in particular Lemmas 60.3.1 and 60.3.3. □

Lemma 60.9.5. *An immersion is a monomorphism.*

Proof. See Morphisms of Spaces, Lemma 42.14.7. □

The following two lemmas explain how to think about immersions in terms of presentations.

Lemma 60.9.6. *Let (U, R, s, t, c) be a smooth groupoid in algebraic spaces. Let $i : \mathcal{Z} \to [U/R]$ be an immersion. Then there exists an R-invariant locally closed subspace $Z \subset U$ and a presentation $[Z/R_Z] \to \mathcal{Z}$ where R_Z is the restriction of R to Z such that*

$$\begin{tikzcd} {[Z/R_Z]} \arrow[rr] \arrow[dr] & & \mathcal{Z} \arrow[dl, "i"] \\ & {[U/R]} & \end{tikzcd}$$

is 2-commutative. If i is a closed (resp. open) immersion then Z is a closed (resp. open) subspace of U.

Proof. By Lemma 60.3.6 we get a commutative diagram

$$\begin{tikzcd} {[U'/R']} \arrow[rr] \arrow[dr] & & \mathcal{Z} \arrow[dl] \\ & {[U/R]} & \end{tikzcd}$$

where $U' = \mathcal{Z} \times_{[U/R]} U$ and $R' = \mathcal{Z} \times_{[U/R]} R$. Since $\mathcal{Z} \to [U/R]$ is an immersion we see that $U' \to U$ is an immersion of algebraic spaces. Let $Z \subset U$ be the locally closed subspace such that $U' \to U$ factors through Z and induces an isomorphism $U' \to Z$. It is clear from the construction of R' that $R' = U' \times_{U,t} R = R \times_{s,U} U'$. This implies that $Z \cong U'$ is R-invariant and that the image of $R' \to R$ identifies R' with the restriction $R_Z = s^{-1}(Z) = t^{-1}(Z)$ of R to Z. Hence the lemma holds. $\square$

Lemma 60.9.7. *Let (U, R, s, t, c) be a smooth groupoid in algebraic spaces. Let $\mathcal{X} = [U/R]$ be the associated algebraic stack, see Algebraic Stacks, Theorem 57.17.3. Let $Z \subset U$ be an R-invariant locally closed subspace. Then*

$$[Z/R_Z] \longrightarrow [U/R]$$

is an immersion of algebraic stacks, where R_Z is the restriction of R to Z. If $Z \subset U$ is open (resp. closed) then the morphism is an open (resp. closed) immersion of algebraic stacks.

Proof. Recall that by Groupoids in Spaces, Definition 52.17.1 (see also discussion following the definition) we have $R_Z = s^{-1}(Z) = t^{-1}(Z)$ as locally closed subspaces of R. Hence the two morphisms $R_Z \to Z$ are smooth as base changes of s and t. Hence $(Z, R_Z, s|_{R_Z}, t|_{R_Z}, c|_{R_Z \times_{s,Z,t} R_Z})$ is a smooth groupoid in algebraic spaces, and we see that $[Z/R_Z]$ is an algebraic stack, see Algebraic Stacks, Theorem 57.17.3. The assumptions of Groupoids in Spaces, Lemma 52.24.3 are all satisfied and it follows that we have a 2-fibre square

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & {[Z/R_Z]} \arrow[d] \\ U \arrow[r] & {[U/R]} \end{tikzcd}$$

It follows from this and Lemma 60.3.1 that $[Z/R_Z] \to [U/R]$ is representable by algebraic spaces, whereupon it follows from Lemma 60.3.3 that the right vertical arrow is an immersion (resp. closed immersion, resp. open immersion) if and only if the left vertical arrow is. $\square$

We can define open, closed, and locally closed substacks as follows.

Definition 60.9.8. Let $\mathcal{X}$ be an algebraic stack.

(1) An *open substack* of $\mathcal{X}$ is a strictly full subcategory $\mathcal{X}' \subset \mathcal{X}$ such that $\mathcal{X}'$ is an algebraic stack and $\mathcal{X}' \to \mathcal{X}$ is an open immersion.
(2) A *closed substack* of $\mathcal{X}$ is a strictly full subcategory $\mathcal{X}' \subset \mathcal{X}$ such that $\mathcal{X}'$ is an algebraic stack and $\mathcal{X}' \to \mathcal{X}$ is a closed immersion.
(3) A *locally closed substack* of $\mathcal{X}$ is a strictly full subcategory $\mathcal{X}' \subset \mathcal{X}$ such that $\mathcal{X}'$ is an algebraic stack and $\mathcal{X}' \to \mathcal{X}$ is an immersion.

This definition should be used with caution. Namely, if $f : \mathcal{X} \to \mathcal{Y}$ is an equivalence of algebraic stacks and $\mathcal{X}' \subset \mathcal{X}$ is an open substack, then it is not necessarily the case that the subcategory $f(\mathcal{X}')$ is an open substack of $\mathcal{Y}$. The problem is that it may not be a *strictly* full subcategory; but this is also the only problem. Here is a formal statement.

Lemma 60.9.9. *For any immersion $i : \mathcal{Z} \to \mathcal{X}$ there exists a unique locally closed substack $\mathcal{X}' \subset \mathcal{X}$ such that i factors as the composition of an equivalence $i' : \mathcal{Z} \to \mathcal{X}'$ followed by the inclusion morphism $\mathcal{X}' \to \mathcal{X}$. If i is a closed (resp. open) immersion, then $\mathcal{X}'$ is a closed (resp. open) substack of $\mathcal{X}$.*

Proof. Omitted. □

Lemma 60.9.10. *Let $[U/R] \to \mathcal{X}$ be a presentation of an algebraic stack. There is a canonical bijection*

$$R\text{-invariant locally closed subspaces } Z \text{ of } U \leftrightarrow \text{locally closed substacks } \mathcal{Z} \text{ of } \mathcal{X}$$

where if Z corresponds to $\mathcal{Z}$, then $[Z/R_Z] \to \mathcal{Z}$ is a presentation fitting into a 2-commutative diagram with the given presentation of $\mathcal{X}$. Similarly for closed substacks and open substacks.

Proof. Omitted. Hints: Use Lemma 60.9.6 to go from right to left and Lemma 60.9.7 from left to right. □

Lemma 60.9.11. *Let $\mathcal{X}$ be an algebraic stack. The rule $\mathcal{U} \mapsto |\mathcal{U}|$ defines an inclusion preserving bijection between open substacks of $\mathcal{X}$ and open subsets of $|\mathcal{X}|$.*

Proof. Choose a presentation $[U/R] \to \mathcal{X}$, see Algebraic Stacks, Lemma 57.16.2. By Lemma 60.9.10 we see that open substacks correspond to R-invariant open subschemes of U. On the other hand Lemmas 60.4.5 and 60.4.7 guarantee these correspond bijectively to open subsets of $|\mathcal{X}|$. □

Lemma 60.9.12. *Let $\mathcal{X}$ be an algebraic stack. Let U be an algebraic space and $U \to \mathcal{X}$ a surjective smooth morphism. For an open immersion $V \hookrightarrow U$, there exists an algebraic stack $\mathcal{Y}$, an open immersion $\mathcal{Y} \to \mathcal{X}$, and a surjective smooth morphism $V \to \mathcal{Y}$.*

Proof. We define a category fibred in groupoids $\mathcal{Y}$ by letting the fiber category $\mathcal{Y}_T$ over an object T of $(Sch/S)_{fppf}$ be the full subcategory of $\mathcal{X}_T$ consisting of all $y \in Ob(\mathcal{X}_T)$ such that the projection morphism $V \times_{\mathcal{X},y} T \to T$ surjective. Now for any morphism $x : T \to \mathcal{X}$, the 2-fibred product $T \times_{x,\mathcal{X}} \mathcal{Y}$ has fiber category over T' consisting of triples $(f : T' \to T, y \in \mathcal{X}_{T'}, f^*x \simeq y)$ such that $V \times_{\mathcal{X},y} T' \to T'$ is surjective. Note that $T \times_{x,\mathcal{X}} \mathcal{Y}$ is fibered in setoids since $\mathcal{Y} \to \mathcal{X}$ is faithful (see Stacks, Lemma 50.6.7). Now the isomorphism

$f^*x \simeq y$ gives the diagram

$$\begin{tikzcd} V\times_{\mathcal{X},y} T' \arrow[r] \arrow[d] & V\times_{\mathcal{X},x} T \arrow[r] \arrow[d] & V \arrow[d] \\ T' \arrow[r, "f"] & T \arrow[r, "x"] & \mathcal{X} \end{tikzcd}$$

where both squares are cartesian. The morphism $V\times_{\mathcal{X},x} T \to T$ is smooth by base change, and hence open. Let $T_0 \subset T$ be its image. From the cartesian squares we deduce that $V\times_{\mathcal{X},y} T' \to T'$ is surjective if and only if f lands in T_0. Therefore $T\times_{x,\mathcal{X}} \mathcal{Y}$ is representable by T_0, so the inclusion $\mathcal{Y} \to \mathcal{X}$ is an open immersion. By Algebraic Stacks, Lemma 57.15.5 we conclude that $\mathcal{Y}$ is an algebraic stack. Lastly if we denote the morphism $V \to \mathcal{X}$ by g, we have $V\times_{\mathcal{X}} V \to V$ is surjective (the diagonal gives a section). Hence g is in the image of $\mathcal{Y}_V \to \mathcal{X}_V$, i.e., we obtain a morphism $g' : V \to \mathcal{Y}$ fitting into the commutative diagram

$$\begin{tikzcd} V \arrow[r] \arrow[d, "g'"] & U \arrow[d] \\ \mathcal{Y} \arrow[r] & \mathcal{X} \end{tikzcd}$$

Since $V\times_{g,\mathcal{X}} \mathcal{Y} \to V$ is a monomorphism, it is in fact an isomorphism since $(1, g')$ defines a section. Therefore $g' : V \to \mathcal{Y}$ is a smooth morphism, as it is the base change of the smooth morphism $g : V \to \mathcal{X}$. It is surjective by our construction of $\mathcal{Y}$ which finishes the proof of the lemma. □

Lemma 60.9.13. *Let $\mathcal{X}$ be an algebraic stack and $\mathcal{X}_i \subset \mathcal{X}$ a collection of open substacks indexed by $i \in I$. Then there exists an open substack, which we denote $\bigcup_{i\in I} \mathcal{X}_i \subset \mathcal{X}$, such that the $\mathcal{X}_i$ are open substacks covering it.*

Proof. We define a fibred subcategory $\mathcal{X}' = \bigcup_{i\in I} \mathcal{X}_i$ by letting the fiber category over an object T of $(Sch/S)_{fppf}$ be the full subcategory of $\mathcal{X}_T$ consisting of all $x \in Ob(\mathcal{X}_T)$ such that the morphism $\coprod_{i\in I}(\mathcal{X}_i \times_{\mathcal{X}} T) \to T$ is surjective. Let $x_i \in Ob((\mathcal{X}_i)_T)$. Then $(x_i, 1)$ gives a section of $\mathcal{X}_i \times_{\mathcal{X}} T \to T$, so we have an isomorphism. Thus $\mathcal{X}_i \subset \mathcal{X}'$ is a full subcategory. Now let $x \in Ob(\mathcal{X}_T)$. Then $\mathcal{X}_i \times_{\mathcal{X}} T$ is representable by an open subscheme $T_i \subset T$. The 2-fibred product $\mathcal{X}' \times_{\mathcal{X}} T$ has fiber over T' consisting of $(y \in \mathcal{X}_{T'}, f : T' \to T, f^*x \simeq y)$ such that $\coprod(\mathcal{X}_i \times_{\mathcal{X},y} T') \to T'$ is surjective. The isomorphism $f^*x \simeq y$ induces an isomorphism $\mathcal{X}_i \times_{\mathcal{X},y} T' \simeq T_i \times_T T'$. Then the $T_i \times_T T'$ cover T' if and only if f lands in $\bigcup T_i$. Therefore we have a diagram

$$\begin{tikzcd} T_i \arrow[r] \arrow[d] & \bigcup T_i \arrow[r] \arrow[d] & T \arrow[d] \\ \mathcal{X}_i \arrow[r] & \mathcal{X}' \arrow[r] & \mathcal{X} \end{tikzcd}$$

with both squares cartesian. By Algebraic Stacks, Lemma 57.15.5 we conclude that $\mathcal{X}' \subset \mathcal{X}$ is algebraic and an open substack. It is also clear from the cartesian squares above that the morphism $\coprod_{i\in I} \mathcal{X}_i \to \mathcal{X}'$ which finishes the proof of the lemma. □

Lemma 60.9.14. *Let $\mathcal{X}$ be an algebraic stack and $\mathcal{X}' \subset \mathcal{X}$ a quasi-compact open substack. Suppose that we have a collection of open substacks $\mathcal{X}_i \subset \mathcal{X}$ indexed by $i \in I$ such that $\mathcal{X}' \subset \bigcup_{i\in I} \mathcal{X}_i$, where we define the union as in Lemma 60.9.13. Then there exists a finite subset $I' \subset I$ such that $\mathcal{X}' \subset \bigcup_{i\in I'} \mathcal{X}_i$.*

Proof. Since $\mathcal{X}$ is algebraic, there exists a scheme U with a surjective smooth morphism $U \to \mathcal{X}$. Let $U_i \subset U$ be the open subscheme representing $\mathcal{X}_i \times_{\mathcal{X}} U$ and $U' \subset U$ the open subscheme representing $\mathcal{X}' \times_{\mathcal{X}} U$. By hypothesis, $U' \subset \bigcup_{i \in I} U_i$. From the proof of Lemma 60.6.2, there is a quasi-compact open $V \subset U'$ such that $V \to \mathcal{X}'$ is a surjective smooth morphism. Therefore there exists a finite subset $I' \subset I$ such that $V \subset \bigcup_{i \in I'} U_i$. We claim that $\mathcal{X}' \subset \bigcup_{i \in I'} \mathcal{X}_i$. Take $x \in Ob(\mathcal{X}'_T)$ for $T \in Ob((Sch/S)_{fppf})$. Since $\mathcal{X}' \to \mathcal{X}$ is a monomorphism, we have cartesian squares

$$\begin{tikzcd} V \times_{\mathcal{X}} T \arrow[r] \arrow[d] & T \arrow[r, equal] \arrow[d, "x"] & T \arrow[d, "x"] \\ V \arrow[r] & \mathcal{X}' \arrow[r] & \mathcal{X} \end{tikzcd}$$

By base change, $V \times_{\mathcal{X}} T \to T$ is surjective. Therefore $\bigcup_{i \in I'} U_i \times_{\mathcal{X}} T \to T$ is also surjective. Let $T_i \subset T$ be the open subscheme representing $\mathcal{X}_i \times_{\mathcal{X}} T$. By a formal argument, we have a Cartesian square

$$\begin{tikzcd} U_i \times_{\mathcal{X}_i} T_i \arrow[r] \arrow[d] & U \times_{\mathcal{X}} T \arrow[d] \\ T_i \arrow[r] & T \end{tikzcd}$$

where the vertical arrows are surjective by base change. Since $U_i \times_{\mathcal{X}_i} T_i \simeq U_i \times_{\mathcal{X}} T$, we find that $\bigcup_{i \in I'} T_i = T$. Hence x is an object of $(\bigcup_{i \in I'} \mathcal{X}_i)_T$ by definition of the union. Observe that the inclusion $\mathcal{X}' \subset \bigcup_{i \in I'} \mathcal{X}_i$ is automatically an open substack. □

Lemma 60.9.15. *Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{X}_i$, $i \in I$ be a set of open substacks of $\mathcal{X}$. Assume*

1. *$\mathcal{X} = \bigcup_{i \in I} \mathcal{X}_i$, and*
2. *each $\mathcal{X}_i$ is an algebraic space.*

Then $\mathcal{X}$ is an algebraic space.

Proof. Apply Stacks, Lemma 50.6.10 to the morphism $\coprod_{i \in I} \mathcal{X}_i \to \mathcal{X}$ and the morphism $\text{id} : \mathcal{X} \to \mathcal{X}$ to see that $\mathcal{X}$ is a stack in setoids. Hence $\mathcal{X}$ is an algebraic space, see Algebraic Stacks, Proposition 57.13.3. □

Lemma 60.9.16. *Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{X}_i$, $i \in I$ be a set of open substacks of $\mathcal{X}$. Assume*

1. *$\mathcal{X} = \bigcup_{i \in I} \mathcal{X}_i$, and*
2. *each $\mathcal{X}_i$ is a scheme*

Then $\mathcal{X}$ is a scheme.

Proof. By Lemma 60.9.15 we see that $\mathcal{X}$ is an algebraic space. Since any algebraic space has a largest open subspace which is a scheme, see Properties of Spaces, Lemma 41.10.1 we see that $\mathcal{X}$ is a scheme. □

The following lemma is the analogue of More on Groupoids, Lemma 36.5.1.

Lemma 60.9.17. *Let $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ be properties of morphisms of algebraic spaces. Assume*

1. *$\mathcal{P}, \mathcal{Q}, \mathcal{R}$ are fppf local on the target and stable under arbitrary base change,*
2. *smooth $\Rightarrow \mathcal{R}$,*
3. *for any morphism $f : X \to Y$ which has $\mathcal{Q}$ there exists a largest open subspace $W(\mathcal{P}, f) \subset X$ such that $f|_{W(\mathcal{P}, f)}$ has $\mathcal{P}$, and*

(4) *for any morphism $f : X \to Y$ which has $\mathcal{Q}$, and any morphism $Y' \to Y$ which has $\mathcal{R}$ we have $Y' \times_Y W(\mathcal{P}, f) = W(\mathcal{P}, f')$, where $f' : X_{Y'} \to Y'$ is the base change of f.*

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphisms of algebraic stacks representable by algebraic spaces. Assume f has $\mathcal{Q}$. Then

(A) *there exists a largest open substack $\mathcal{X}' \subset \mathcal{X}$ such that $f|_{\mathcal{X}'}$ has $\mathcal{P}$, and*
(B) *if $\mathcal{Z} \to \mathcal{Y}$ is a morphism of algebraic stacks representable by algebraic spaces which has $\mathcal{R}$ then $\mathcal{Z} \times_\mathcal{Y} \mathcal{X}'$ is the largest open substack of $\mathcal{Z} \times_\mathcal{Y} \mathcal{X}$ over which the base change $\mathrm{id}_\mathcal{Z} \times f$ has property $\mathcal{P}$.*

Proof. Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Y}$. Set $U = V \times_\mathcal{Y} \mathcal{X}$ and let $f' : U \to V$ be the base change of f. The morphism of algebraic spaces $f' : U \to V$ has property $\mathcal{Q}$. Thus we obtain the open $W(\mathcal{P}, f') \subset U$ by assumption (3). Note that $U \times_\mathcal{X} U = (V \times_\mathcal{Y} V) \times_\mathcal{Y} \mathcal{X}$ hence the morphism $f'' : U \times_\mathcal{X} U \to V \times_\mathcal{Y} V$ is the base change of f via either projection $V \times_\mathcal{Y} V \to V$. By our choice of V these projections are smooth, hence have property $\mathcal{R}$ by (2). Thus by (4) we see that the inverse images of $W(\mathcal{P}, f')$ under the two projections $\mathrm{pr}_i : U \times_\mathcal{X} U \to U$ agree. In other words, $W(\mathcal{P}, f')$ is an R-invariant subspace of U (where $R = U \times_\mathcal{X} U$). Let $\mathcal{X}'$ be the open substack of $\mathcal{X}$ corresponding to $W(\mathcal{P}, f)$ via Lemma 60.9.6. By construction $W(\mathcal{P}, f') = \mathcal{X}' \times_\mathcal{Y} V$ hence $f|_{\mathcal{X}'}$ has property $\mathcal{P}$ by Lemma 60.3.3. Also, $\mathcal{X}'$ is the largest open substack such that $f|_{\mathcal{X}'}$ has $\mathcal{P}$ as the same maximality holds for $W(\mathcal{P}, f)$. This proves (A).

Finally, if $\mathcal{Z} \to \mathcal{Y}$ is a morphism of algebraic stacks representable by algebraic spaces which has $\mathcal{R}$ then we set $T = V \times_\mathcal{Y} \mathcal{Z}$ and we see that $T \to V$ is a morphism of algebraic spaces having property $\mathcal{R}$. Set $f'_T : T \times_V U \to T$ the base change of f'. By (4) again we see that $W(\mathcal{P}, f'_T)$ is the inverse image of $W(\mathcal{P}, f)$ in $T \times_V U$. This implies (B); some details omitted. □

Remark 60.9.18. Warning: Lemma 60.9.17 should be used with care. For example, it applies to $\mathcal{P}$=``flat'', $\mathcal{Q}$ =``empty'', and $\mathcal{R}$ =``flat and locally of finite presentation''. But given a morphism of algebraic spaces $f : X \to Y$ the largest open subspace $W \subset X$ such that $f|_W$ is flat is *not* the set of points where f is flat!

Remark 60.9.19. Notwithstanding the warning in Remark 60.9.18 there are some cases where Lemma 60.9.17 can be used without causing ambiguity. We give a list. In each case we omit the verification of assumptions (1) and (2) and we give references which imply (3) and (4). Here is the list:

(1) $\mathcal{Q}$ =``locally of finite type'', $\mathcal{R} = \emptyset$, and $\mathcal{P}$ =``relative dimension $\leq d$''. See Morphisms of Spaces, Definition 42.30.2 and Morphisms of Spaces, Lemmas 42.31.4 and 42.31.3.
(2) $\mathcal{Q}$ =``locally of finite type'', $\mathcal{R} = \emptyset$, and $\mathcal{P}$ =``locally quasi-finite''. This is the case $d = 0$ of the previous item, see Morphisms of Spaces, Lemma 42.31.6. On the other hand, properties (3) and (4) are spelled out in Morphisms of Spaces, Lemma 42.31.7.
(3) $\mathcal{Q}$ =``locally of finite type'', $\mathcal{R} = \emptyset$, and $\mathcal{P}$ =``unramified''. This is Morphisms of Spaces, Lemma 42.34.10.
(4) $\mathcal{Q}$ =``locally of finite presentation'', $\mathcal{R}$ =``flat and locally of finite presentation'', and $\mathcal{P}$=``flat''. See More on Morphisms of Spaces, Theorem 46.17.1 and Lemma 46.17.2. Note that here $W(\mathcal{P}, f)$ is always exactly the set of points where the morphism f is flat because we only consider this open when f has $\mathcal{Q}$ (see loc.cit.).

(5) $\mathcal{Q}$ =``locally of finite presentation'', $\mathcal{R}$ =``flat and locally of finite presentation'', and $\mathcal{P}$ =``étale''. This follows on combining (3) and (4) because an unramified morphism which is flat and locally of finite presentation is étale, see Morphisms of Spaces, Lemma 42.35.12.
(6) Add more here as needed (compare with the longer list at More on Groupoids, Remark 36.5.3).

60.10. Reduced algebraic stacks

We have already defined reduced algebraic stacks in Section 60.7.

Lemma 60.10.1. *Let $\mathcal{X}$ be an algebraic stack. Let $T \subset |\mathcal{X}|$ be a closed subset. There exists a unique closed substack $\mathcal{Z} \subset \mathcal{X}$ with the following properties: (a) we have $|\mathcal{Z}| = T$, and (b) $\mathcal{Z}$ is reduced.*

Proof. Let $U \to \mathcal{X}$ be a surjective smooth morphism, where U is an algebraic space. Set $R = U \times_{\mathcal{X}} U$, so that there is a presentation $[U/R] \to \mathcal{X}$, see Algebraic Stacks, Lemma 57.16.2. As usual we denote $s, t : R \to U$ the two smooth projection morphisms. By Lemma 60.4.5 we see that T corresponds to a closed subset $T' \subset |U|$ such that $|s|^{-1}(T') = |t|^{-1}(T')$. Let $Z \subset U$ be the reduced induced algebraic space structure on T', see Properties of Spaces, Definition 41.9.3. The fibre products $Z \times_{U,t} R$ and $R \times_{s,U} Z$ are closed subspaces of R (Spaces, Lemma 40.12.3). The projections $Z \times_{U,t} R \to Z$ and $R \times_{s,U} Z \to Z$ are smooth by Morphisms of Spaces, Lemma 42.33.3. Thus as Z is reduced, it follows that $Z \times_{U,t} R$ and $R \times_{s,U} Z$ are reduced, see Remark 60.7.3. Since

$$|Z \times_{U,t} R| = |t|^{-1}(T') = |s|^{-1}(T') = R \times_{s,U} Z$$

we conclude from the uniqueness in Properties of Spaces, Lemma 41.9.1 that $Z \times_{U,t} R = R \times_{s,U} Z$. Hence Z is an R-invariant closed subspace of U. By the corresondence of Lemma 60.9.10 (and its proof) we obtain a closed substack $\mathcal{Z} \subset \mathcal{Z}$ with a presentation $[Z/R_Z] \to \mathcal{Z}$. Then $|\mathcal{Z}| = |Z|/|R_Z| = |T'|/\sim$ is the given closed subset T. We omit the proof of unicity. □

Lemma 60.10.2. *Let $\mathcal{X}$ be an algebraic stack. If $\mathcal{X}' \subset \mathcal{X}$ is a closed substack, $\mathcal{X}$ is reduced and $|\mathcal{X}'| = |\mathcal{X}|$, then $\mathcal{X}' = \mathcal{X}$.*

Proof. Choose a presentation $[U/R] \to \mathcal{X}$ with U a scheme. As $\mathcal{X}$ is reduced, we see that U is reduced (by definition of reduced algebraic stacks). By Lemma 60.9.10 $\mathcal{X}'$ corresponds to an R-invariant closed subscheme $Z \subset U$. But now $|Z| \subset |U|$ is the inverse image of $|\mathcal{X}'|$, and hence $|Z| = |U|$. Hence Z is a closed subscheme of U whose underlying sets of points agree. By Schemes, Lemma 21.12.6 the map $\mathrm{id}_U : U \to U$ factors through $Z \to U$, and hence $Z = U$, i.e., $\mathcal{X}' = \mathcal{X}$. □

Lemma 60.10.3. *Let $\mathcal{X}$, $\mathcal{Y}$ be algebraic stacks. Let $\mathcal{Z} \subset \mathcal{X}$ be a closed substack Assume $\mathcal{Y}$ is reduced. A morphism $f : \mathcal{Y} \to \mathcal{X}$ factors through $\mathcal{Z}$ if and only if $f(|\mathcal{Y}|) \subset |\mathcal{Z}|$.*

Proof. Assume $f(|\mathcal{Y}|) \subset |\mathcal{Z}|$. Consider $\mathcal{Y} \times_{\mathcal{X}} \mathcal{Z} \to \mathcal{Y}$. There is an equivalence $\mathcal{Y} \times_{\mathcal{X}} \mathcal{Z} \to \mathcal{Y}'$ where $\mathcal{Y}'$ is a closed substack of $\mathcal{Y}$, see Lemmas 60.9.2 and 60.9.9. Using Lemmas 60.4.3, 60.8.5, and 60.9.5 we see that $|\mathcal{Y}'| = |\mathcal{Y}|$. Hence we have reduced the lemma to Lemma 60.10.2. □

Definition 60.10.4. Let $\mathcal{X}$ be an algebraic stack. Let $Z \subset |\mathcal{X}|$ be a closed subset. An *algebraic stack structure on Z* is given by a closed substack $\mathcal{Z}$ of $\mathcal{X}$ with $|\mathcal{Z}|$ equal to

Z. The *reduced induced algebraic stack structure* on Z is the one constructed in Lemma 60.10.1. The *reduction* $\mathcal{X}_{red}$ *of* $\mathcal{X}$ is the reduced induced algebraic stack structure on $|\mathcal{X}|$.

In fact we can use this to define the reduced induced algebraic stack structure on a locally closed subset.

Remark 60.10.5. Let X be an algebraic stack. Let $T \subset |\mathcal{X}|$ be a locally closed subset. Let ∂T be the boundary of T in the topological space $|\mathcal{X}|$. In a formula

$$\partial T = \overline{T} \setminus T.$$

Let $\mathcal{U} \subset \mathcal{X}$ be the open substack of X with $|\mathcal{U}| = |\mathcal{X}| \setminus \partial T$, see Lemma 60.9.11. Let $\mathcal{Z}$ be the reduced closed substack of $\mathcal{U}$ with $|\mathcal{Z}| = T$ obtained by taking the reduced induced closed subspace structure, see Definition 60.10.4. By construction $\mathcal{Z} \to \mathcal{U}$ is a closed immersion of algebraic stacks and $\mathcal{U} \to \mathcal{X}$ is an open immersion, hence $\mathcal{Z} \to \mathcal{X}$ is an immersion of algebraic stacks by Lemma 60.9.3. Note that $\mathcal{Z}$ is a reduced algebraic stack and that $|\mathcal{Z}| = T$ as subsets of $|X|$. We sometimes say $\mathcal{Z}$ is the *reduced induced substack structure* on T.

60.11. Residual gerbes

In the stacks project we would like to define the *residual gerbe* of an algebraic stack $\mathcal{X}$ at a point $x \in |\mathcal{X}|$ to be a monomorphism of algebraic stacks $m_x : \mathcal{Z}_x \to \mathcal{X}$ where $\mathcal{Z}_x$ is a reduced algebraic stack having a unique point which is mapped by m_x to x. It turns out that there are many issues with this notion; existence is not clear in general and neither is uniqueness. We resolve the uniqueness issue by imposing a slightly stronger condition on the algebraic stacks $\mathcal{Z}_x$. We discuss this in more detail by working through a few simple lemmas regarding reduced algebraic stacks having a unique point.

Lemma 60.11.1. *Let $\mathcal{Z}$ be an algebraic stack. Let k be a field and let $Spec(k) \to \mathcal{Z}$ be surjective and flat. Then any morphism $Spec(k') \to \mathcal{Z}$ where k' is a field is surjective and flat.*

Proof. Consider the fibre square

$$\begin{tikzcd} T \arrow[r] \arrow[d] & Spec(k) \arrow[d] \\ Spec(k') \arrow[r] & \mathcal{Z} \end{tikzcd}$$

Note that $T \to Spec(k')$ is flat and surjective hence T is not empty. On the other hand $T \to Spec(k)$ is flat as k is a field. Hence $T \to \mathcal{Z}$ is flat and surjective. It follows from Morphisms of Spaces, Lemma 42.28.5 (via the discussion in Section 60.3) that $Spec(k') \to \mathcal{Z}$ is flat. It is clear that it is surjective as by assumption $|\mathcal{Z}|$ is a singleton. $\square$

Lemma 60.11.2. *Let $\mathcal{Z}$ be an algebraic stack. The following are equivalent*

(1) *$\mathcal{Z}$ is reduced and $|\mathcal{Z}|$ is a singleton,*
(2) *there exists a surjective flat morphism $Spec(k) \to \mathcal{Z}$ where k is a field, and*
(3) *there exists a locally of finite type, surjective, flat morphism $Spec(k) \to \mathcal{Z}$ where k is a field.*

Proof. Assume (1). Let W be a scheme and let $W \to \mathcal{Z}$ be a surjective smooth morphism. Then W is a reduced scheme. Let $\eta \in W$ be a generic point of an irreducible component of W. Since W is reduced we have $\mathcal{O}_{W,\eta} = \kappa(\eta)$. It follows that the canonical morphism

$\eta = Spec(\kappa(\eta)) \to W$ is flat. We see that the composition $\eta \to \mathcal{X}$ is flat (see Morphisms of Spaces, Lemma 42.27.2). It is also surjective as $|\mathcal{X}|$ is a singleton. In other words (2) holds.

Assume (2). Let W be a scheme and let $W \to \mathcal{X}$ be a surjective smooth morphism. Choose a field k and a surjective flat morphism $Spec(k) \to \mathcal{X}$. Then $W \times_{\mathcal{X}} Spec(k)$ is an algebraic space smooth over k, hence regular (see Spaces over Fields, Lemma 48.5.1) and in particular reduced. Since $W \times_{\mathcal{X}} Spec(k) \to W$ is surjective and flat we conclude that W is reduced (Descent on Spaces, Lemma 45.8.2). In other words (1) holds.

It is clear that (3) implies (2). Finally, assume (2). Pick a nonempty affine scheme W and a smooth morphism $W \to \mathcal{X}$. Pick a closed point $w \in W$ and set $k = \kappa(w)$. The composition

$$Spec(k) \xrightarrow{w} W \longrightarrow \mathcal{X}$$

is locally of finite type by Morphisms of Spaces, Lemmas 42.22.2 and 42.33.6. It is also flat and surjective by Lemma 60.11.1. Hence (3) holds. $\square$

The following lemma singles out a slightly better class of singleton algebraic stacks than the preceding lemma.

Lemma 60.11.3. *Let $\mathcal{X}$ be an algebraic stack. The following are equivalent*

1. *$\mathcal{X}$ is reduced, locally Noetherian, and $|\mathcal{X}|$ is a singleton, and*
2. *there exists a locally finitely presented, surjective, flat morphism $Spec(k) \to \mathcal{X}$ where k is a field.*

Proof. Assume (2) holds. By Lemma 60.11.2 we see that $\mathcal{X}$ is reduced and $|\mathcal{X}|$ is a singleton. Let W be a scheme and let $W \to \mathcal{X}$ be a surjective smooth morphism. Choose a field k and a locally finitely presented, surjective, flat morphism $Spec(k) \to \mathcal{X}$. Then $W \times_{\mathcal{X}} Spec(k)$ is an algebraic space smooth over k, hence locally Noetherian (see Morphisms of Spaces, Lemma 42.22.5). Since $W \times_{\mathcal{X}} Spec(k) \to W$ is flat, surjective, and locally of finite presentation, we see that $\{W \times_{\mathcal{X}} Spec(k) \to W\}$ is an fppf covering and we conclude that W is locally Noetherian (Descent on Spaces, Lemma 45.8.3). In other words (1) holds.

Assume (1). Pick a nonempty affine scheme W and a smooth morphism $W \to \mathcal{X}$. Pick a closed point $w \in W$ and set $k = \kappa(w)$. Because W is locally Noetherian the morphism $w : Spec(k) \to W$ is of finite presentation, see Morphisms, Lemma 24.20.7. Hence the composition

$$Spec(k) \xrightarrow{w} W \longrightarrow \mathcal{X}$$

is locally of finite presentation by Morphisms of Spaces, Lemmas 42.26.2 and 42.33.5. It is also flat and surjective by Lemma 60.11.1. Hence (2) holds. $\square$

Lemma 60.11.4. *Let $\mathcal{X}' \to \mathcal{X}$ be a monomorphism of algebraic stacks. Assume there exists a field k and a locally finitely presented, surjective, flat morphism $Spec(k) \to \mathcal{X}$. Then either $\mathcal{X}'$ is empty or $\mathcal{X}' \to \mathcal{X}$ is an equivalence.*

Proof. We may assume that $\mathcal{X}'$ is nonempty. In this case the fibre product $T = \mathcal{X}' \times_{\mathcal{X}} Spec(k)$ is nonempty, see Lemma 60.4.3. Now T is an algebraic space and the projection $T \to Spec(k)$ is a monomorphism. Hence $T = Spec(k)$, see Morphisms of Spaces, Lemma 42.14.8. We conclude that $Spec(k) \to \mathcal{X}$ factors through $\mathcal{X}'$. Suppose the morphism $z : Spec(k) \to \mathcal{X}$ is given by the object ξ over $Spec(k)$. We have just seen that ξ is isomorphic to an object ξ' of $\mathcal{X}'$ over $Spec(k)$. Since z is is surjective, flat, and locally of finite presentation we see that every object of $\mathcal{X}$ over any scheme is fppf locally isomorphic to a pullback of ξ, hence also to a pullback of ξ'. By descent of objects for stacks in groupoids this implies that $\mathcal{Z}' \to \mathcal{Z}$ is essentially surjective (as well as fully faithful, see Lemma 60.8.4). Hence we win. $\square$

Lemma 60.11.5. *Let $\mathcal{Z}$ be an algebraic stack. Assume $\mathcal{Z}$ satisfies the equivalent conditions of Lemma 60.11.2. Then there exists a unique strictly full subcategory $\mathcal{Z}' \subset \mathcal{Z}$ such that $\mathcal{Z}'$ is an algebraic stack which satisfies the equivalent conditions of Lemma 60.11.3. The inclusion morphism $\mathcal{Z}' \to \mathcal{Z}$ is a monomorphism of algebraic stacks.*

Proof. The last part is immediate from the first part and Lemma 60.8.4. Pick a field k and a morphism $Spec(k) \to \mathcal{Z}$ which is surjective, flat, and locally of finite type. Set $U = Spec(k)$ and $R = U \times_{\mathcal{Z}} U$. The projections $s, t : R \to U$ are locally of finite type. Since U is the spectrum of a field, it follows that s, t are flat and locally of finite presentation (by Morphisms of Spaces, Lemma 42.26.7). We see that $\mathcal{Z}' = [U/R]$ is an algebraic stack by Criteria for Representability, Theorem 59.17.2. By Algebraic Stacks, Lemma 57.16.1 we obtain a canonical morphism

$$f : \mathcal{Z}' \longrightarrow \mathcal{Z}$$

which is fully faithful. Hence this morphism is representable by algebraic spaces, see Algebraic Stacks, Lemma 57.15.2 and a monomorphism, see Lemma 60.8.4. By Criteria for Representability, Lemma 59.17.1 the morphism $U \to \mathcal{Z}'$ is surjective, flat, and locally of finite presentation. Hence $\mathcal{Z}'$ is an algebraic stack which satisfies the equivalent conditions of Lemma 60.11.3. By Algebraic Stacks, Lemma 57.12.4 we may replace $\mathcal{Z}'$ by its essential image in $\mathcal{Z}$. Hence we have proved all the assertions of the lemma except for the uniqueness of $\mathcal{Z}' \subset \mathcal{Z}$. Suppose that $\mathcal{Z}'' \subset \mathcal{Z}$ is a second such algebraic stack. Then the projections

$$\mathcal{Z}' \longleftarrow \mathcal{Z}' \times_{\mathcal{Z}} \mathcal{Z}'' \longrightarrow \mathcal{Z}''$$

are monomorphisms. The algebraic stack in the middle is nonempty by Lemma 60.4.3. Hence the two projections are isomorphisms by Lemma 60.11.4 and we win. $\square$

Example 60.11.6. Here is an example where the morphism constructed in Lemma 60.11.5 isn't an isomorphism. This example shows that imposing that residual gerbes are locally Noetherian is necessary in Definition 60.11.8. In fact, the example is even an algebraic space! Let $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ be the absolute Galois group of $\mathbf{Q}$ with the pro-finite topology. Let

$$U = Spec(\overline{\mathbf{Q}}) \times_{Spec(\mathbf{Q})} Spec(\overline{\mathbf{Q}}) = \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \times Spec(\overline{\mathbf{Q}})$$

(we omit a precise explanation of the meaning of the last equal sign). Let G denote the absolute Galois group $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ with the discrete topology viewed as a constant group scheme over $Spec(\overline{\mathbf{Q}})$, see Groupoids, Example 35.5.6. Then G acts freely and transitively on U. Let $X = U/G$, see Spaces, Definition 40.14.4. Then X is a non-noetherian reduced algebraic space with exactly one point. Furthermore, X has a (locally) finite type point:

$$x : Spec(\overline{\mathbf{Q}}) \longrightarrow U \longrightarrow X$$

Indeed, every point of U is actually closed! As X is an algebraic space over $\overline{\mathbf{Q}}$ it follows that x is a monomorphism. So x is the morphism constructed in Lemma 60.11.5 but x is not an isomorphism. In fact $Spec(\overline{\mathbf{Q}}) \to X$ is the residual gerbe of X at x.

It will turn out later that under mild assumptions on the algebraic stack $\mathcal{X}$ the equivalent conditions of the following lemma are satisfied for every point $x \in |\mathcal{X}|$ (insert future reference here).

Lemma 60.11.7. *Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$ be a point. The following are equivalent*

(1) *there exists an algebraic stack $\mathcal{Z}$ and a monomorphism $\mathcal{Z} \to \mathcal{X}$ such that $|\mathcal{Z}|$ is a singleton and such that the image of $|\mathcal{Z}|$ in $|\mathcal{X}|$ is x,*
(2) *there exists a reduced algebraic stack $\mathcal{Z}$ and a monomorphism $\mathcal{Z} \to \mathcal{X}$ such that $|\mathcal{Z}|$ is a singleton and such that the image of $|\mathcal{Z}|$ in $|\mathcal{X}|$ is x,*
(3) *there exists an algebraic stack $\mathcal{Z}$, a monomorphism $f : \mathcal{Z} \to \mathcal{X}$, and a surjective flat morphism $z : Spec(k) \to \mathcal{X}$ where k is a field such that $x = f(z)$.*

Moreover, if these conditions hold, then there exists a unique strictly full subcategory $\mathcal{Z}_x \subset \mathcal{X}$ such that $\mathcal{Z}_x$ is a reduced, locally Noetherian algebraic stack and $|\mathcal{Z}_x|$ is a singleton which maps to x via the map $|\mathcal{Z}_x| \to |\mathcal{X}|$.

Proof. If $\mathcal{Z} \to \mathcal{X}$ is as in (1), then $\mathcal{Z}_{red} \to \mathcal{X}$ is as in (2). (See Section 60.10 for the notion of the reduction of an algebraic stack.) Hence (1) implies (2). It is immediate that (2) implies (1). The equivalence of (2) and (3) is immediate from Lemma 60.11.2.

At this point we've seen the equivalence of (1) -- (3). Pick a monomorphism $f : \mathcal{Z} \to \mathcal{X}$ as in (2). Note that this implies that f is fully faithful, see Lemma 60.8.4. Denote $\mathcal{Z}' \subset \mathcal{X}$ the essential image of the functor f. Then $f : \mathcal{Z} \to \mathcal{Z}'$ is an equivalence and hence $\mathcal{Z}'$ is an algebraic stack, see Algebraic Stacks, Lemma 57.12.4. Apply Lemma 60.11.5 to get a strictly full subcategory $\mathcal{Z}_x \subset \mathcal{Z}'$ as in the statement of the lemma. This proves all the statements of the lemma except for uniqueness.

In order to prove the uniqueness suppose that $\mathcal{Z}_x \subset \mathcal{X}$ and $\mathcal{Z}'_x \subset \mathcal{X}$ are two stricly full subcategories as in the statement of the lemma. Then the projections

$$\mathcal{Z}'_x \longleftarrow \mathcal{Z}'_x \times_{\mathcal{X}} \mathcal{Z}_x \longrightarrow \mathcal{Z}_x$$

are monomorphisms. The algebraic stack in the middle is nonempty by Lemma 60.4.3. Hence the two projections are isomorphisms by Lemma 60.11.4 and we win. □

Having explained the above we can now make the following definition.

Definition 60.11.8. Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$.

(1) We say the *residual gerbe of $\mathcal{X}$ at x exists* if the equivalent conditions (1), (2), and (3) of Lemma 60.11.7 hold.
(2) If the residual gerbe of $\mathcal{X}$ at x exists, then the *residual gerbe of $\mathcal{X}$ at x*[1] is the strictly full subcategory $\mathcal{Z}_x \subset \mathcal{X}$ constructed in Lemma 60.11.7.

In particular we know that $\mathcal{Z}_x$ (if it exists) is a locally Noetherian, reduced algebraic stack and that there exists a field and a surjective, flat, locally finitely presented morphism

$$Spec(k) \longrightarrow \mathcal{Z}_x.$$

We will see in Morphisms of Stacks, Lemma 61.19.10 that $\mathcal{Z}_x$ is a gerbe. It turns out that $\mathcal{Z}_x$ is a regular algebraic stack as follows from the following lemma.

Lemma 60.11.9. *A reduced, locally Noetherian algebraic stack $\mathcal{Z}$ such that $|\mathcal{Z}|$ is a singleton is regular.*

[1]This clashes with [**LMB00a**] in spirit, but not in fact. Namely, in Chapter 11 they associate to any point on any quasi-separated algebraic stack a gerbe (not necessarily algebraic) which they call the residual gerbe. We will see in Morphisms of Stacks, Lemma 61.21.1 that on a quasi-separated algebraic stack every point has a residual gerbe in our sense which is then equivalent to theirs. For more information on this topic see [**Ryd10**, Appendix B].

Proof. Let $W \to \mathcal{Z}$ be a surjective smooth morphism where W is a scheme. Let k be a field and let $Spec(k) \to \mathcal{Z}$ be surjective, flat, and locally of finite presentation (see Lemma 60.11.3). The algebraic space $T = W \times_{\mathcal{Z}} Spec(k)$ is smooth over k in particular regular, see Spaces over Fields, Lemma 48.5.1. Since $T \to W$ is locally of finite presentation, flat, and surjective it follows that W is regular, see Descent on Spaces, Lemma 45.8.4. By definition this means that $\mathcal{Z}$ is regular. □

Lemma 60.11.10. *Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$. Assume that the residual gerbe $\mathcal{Z}_x$ of $\mathcal{X}$ exists. Let $f : Spec(K) \to \mathcal{X}$ be a morphism where K is a field in the equivalence class of x. Then f factors through the inclusion morphism $\mathcal{Z}_x \to \mathcal{X}$.*

Proof. Choose a field k and a surjective flat locally finite presentation morphism $Spec(k) \to \mathcal{Z}_x$. Set $T = Spec(K) \times_{\mathcal{X}} \mathcal{Z}_x$. By Lemma 60.4.3 we see that T is nonempty. As $\mathcal{Z}_x \to \mathcal{X}$ is a monomorphism we see that $T \to Spec(K)$ is a monomorphism. Hence by Morphisms of Spaces, Lemma 42.14.8 we see that $T = Spec(K)$ which proves the lemma. □

Lemma 60.11.11. *Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$. Let $\mathcal{Z}$ be an algebraic stack satisfying the equivalent conditions of Lemma 60.11.3 and let $\mathcal{Z} \to \mathcal{X}$ be a monomorphism such that the image of $|\mathcal{Z}| \to |\mathcal{X}|$ is x. Then the residual gerbe $\mathcal{Z}_x$ of $\mathcal{X}$ at x exists and $\mathcal{Z} \to \mathcal{X}$ factors as $\mathcal{Z} \to \mathcal{Z}_x \to \mathcal{X}$ where the first arrow is an equivalence.*

Proof. Let $\mathcal{Z}_x \subset \mathcal{X}$ be the full subcategory corresponding to the essential image of the functor $\mathcal{Z} \to \mathcal{X}$. Then $\mathcal{Z} \to \mathcal{Z}_x$ is an equivalence, hence $\mathcal{Z}_x$ is an algebraic stack, see Algebraic Stacks, Lemma 57.12.4. Since $\mathcal{Z}_x$ inherits all the properties of $\mathcal{Z}$ from this equivalence it is clear from the uniqueness in Lemma 60.11.7 that $\mathcal{Z}_x$ is the residual gerbe of $\mathcal{X}$ at x. □

60.12. Other chapters

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CHAPTER 61

Morphisms of Algebraic Stacks

61.1. Introduction

In this chapter we introduce some types of morphisms of algebraic stacks. A reference in the case of quasi-separated algebraic stacks with representable diagonal is [**LMB00a**].

The goal is to extend the definition of each of the types of morphisms of algebraic spaces to morphisms of algebraic stacks. Each case is slightly different and it seems best to treat them all separately.

For morphisms of algebraic stacks which are representable by algebraic spaces we have already defined a large number of types of morphisms, see Properties of Stacks, Section 60.3. For each corresponding case in this chapter we have to make sure the definition in the general case is compatible with the definition given there.

61.2. Conventions and abuse of language

We continue to use the conventions and the abuse of language introduced in Properties of Stacks, Section 60.2.

61.3. Properties of diagonals

The diagonal of an algebraic stack is closely related to the $Isom$-sheaves, see Algebraic Stacks, Lemma 57.10.11. By the second defining property of an algebraic stack these $Isom$-sheaves are always algebraic spaces.

Lemma 61.3.1. *Let $\mathcal{X}$ be an algebraic stack. Let T be a scheme and let x, y be objects of the fibre category of $\mathcal{X}$ over T. Then the morphism $Isom_{\mathcal{X}}(x, y) \to T$ is locally of finite type.*

Proof. By Algebraic Stacks, Lemma 57.16.2 we may assume that $\mathcal{X} = [U/R]$ for some smooth groupoid in algebraic spaces. By Descent on Spaces, Lemma 45.10.7 it suffices to check the property fppf locally on T. Thus we may assume that x, y come from morphisms $x', y' : T \to U$. By Groupoids in Spaces, Lemma 52.21.1 we see that in this case $Isom_{\mathcal{X}}(x, y) = T \times_{(y',x'),U\times_S U} R$. Hence it suffices to prove that $R \to U \times_S U$ is locally of finite type. This follows from the fact that the composition $s : R \to U \times_S U \to U$ is smooth (hence locally of finite type, see Morphisms of Spaces, Lemmas 42.33.5 and 42.26.5) and Morphisms of Spaces, Lemma 42.22.6. $\square$

Lemma 61.3.2. *Let $\mathcal{X}$ be an algebraic stack. Let T be a scheme and let x, y be objects of the fibre category of $\mathcal{X}$ over T. Then*

1. *$Isom_{\mathcal{X}}(y, y)$ is a group algebraic space over T, and*
2. *$Isom_{\mathcal{X}}(x, y)$ is a pseudo torsor for $Isom_{\mathcal{X}}(y, y)$ over T.*

Proof. See Groupoids in Spaces, Definitions 52.5.1 and 52.9.1. The lemma follows immediately from the fact that $\mathcal{X}$ is a stack in groupoids. $\square$

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The *diagonal of f* is the morphism

$$\Delta_f : \mathcal{X} \longrightarrow \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$$

Here are two properties that every diagonal morphism has.

Lemma 61.3.3. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Then*

(1) *Δ_f is representable by algebraic spaces, and*
(2) *Δ_f is locally of finite type.*

Proof. Let T be a scheme and let $a : T \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ be a morphism. By definition of the fibre product and the 2-Yoneda lemma the morphism a is given by a triple $a = (x, x', \alpha)$ where x, x' are objects of $\mathcal{X}$ over T, and $\alpha : f(x) \to f(x')$ is a morphism in the fibre category of $\mathcal{Y}$ over T. By definition of an algebraic stack the sheaves $Isom_{\mathcal{X}}(x, x')$ and $Isom_{\mathcal{Y}}(f(x), f(x'))$ are algebraic spaces over T. In this language α defines a section of the morphism $Isom_{\mathcal{X}}(x, x') \to T$. A T'-valued point of $\mathcal{X} \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}, a} T$ for $T' \to T$ a scheme over T is the same thing as an isomorphism $x|_{T'} \to x'|_{T'}$ whose image under f is $\alpha|_{T'}$. Thus we see that

(61.3.3.1)
$$\begin{tikzcd} \mathcal{X} \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}, a} T \arrow[r] \arrow[d] & Isom_{\mathcal{X}}(x, x') \arrow[d] \\ T \arrow[r, "\alpha"] & Isom_{\mathcal{Y}}(f(x), f(x')) \end{tikzcd}$$

is a fibre square of sheaves over T. In particular we see that $\mathcal{X} \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}, a} T$ is an algebraic space which proves part (1) of the lemma.

To prove the second statement we have to show that the left vertical arrow of Diagram (61.3.3.1) is locally of finite type. By Lemma 61.3.1 the algebraic space $Isom_{\mathcal{X}}(x, x')$ and is locally of finite type over T. Hence the right vertical arrow of Diagram (61.3.3.1) is locally of finite type, see Morphisms of Spaces, Lemma 42.22.6. We conclude by Morphisms of Spaces, Lemma 42.22.3. □

Lemma 61.3.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. Then*

(1) *Δ_f is representable (by schemes),*
(2) *Δ_f is locally of finite type,*
(3) *Δ_f is a monomorphism,*
(4) *Δ_f is separated, and*
(5) *Δ_f is locally quasi-finite.*

Proof. We have already seen in Lemma 61.3.3 that Δ_f is representable by algebraic spaces. Hence the statements (2) -- (5) make sense, see Properties of Stacks, Section 60.3. Also Lemma 61.3.3 garantees (2) holds. Let $T \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ be a morphism and contemplate Diagram (61.3.3.1). By Algebraic Stacks, Lemma 57.9.2 the right vertical arrow is injective as a map of sheaves, i.e., a monomorphism of algebraic spaces. Hence also the morphism $T \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}} \mathcal{X} \to T$ is a monomorphism. Thus (3) holds. We already know that $T \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}} \mathcal{X} \to T$ is locally of finite type. Thus Morphisms of Spaces, Lemma 42.25.8 allows us to conclude that $T \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}} \mathcal{X} \to T$ is locally quasi-finite and separated. This proves (4) and (5). Finally, Morphisms of Spaces, Proposition 42.39.2 implies that $T \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}} \mathcal{X}$ is a scheme which proves (1). □

Lemma 61.3.5. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks representable by algebraic spaces. Then the following are equivalent*

(1) *f is separated,*
(2) *Δ_f is a closed immersion,*
(3) *Δ_f is proper, or*
(4) *Δ_f is universally closed.*

Proof. The statements ``$f$ is separated", ``Δ_f is a closed immersion", ``$\Delta_f$ is universally closed", and ``Δ_f is proper" refer to the notions defined in Properties of Stacks, Section 60.3. Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Y}$. Set $U = \mathcal{X} \times_{\mathcal{Y}} V$ which is an algebraic space by assumption, and the morphism $U \to \mathcal{X}$ is surjective and smooth. By Categories, Lemma 4.28.14 and Properties of Stacks, Lemma 60.3.3 we see that for any property P (as in that lemma) we have: Δ_f has P if and only if $\Delta_{U/V} : U \to U \times_V U$ has P. Hence the equivalence of (2), (3) and (4) follows from Morphisms of Spaces, Lemma 42.36.6 applied to $U \to V$. Moreover, if (1) holds, then $U \to V$ is separated and we see that $\Delta_{U/V}$ is a closed immersion, i.e., (2) holds. Finally, assume (2) holds. Let T be a scheme, and $a : T \to \mathcal{Y}$ a morphism. Set $T' = \mathcal{X} \times_{\mathcal{Y}} T$. To prove (1) we have to show that the morphism of algebraic spaces $T' \to T$ is separated. Using Categories, Lemma 4.28.14 once more we see that $\Delta_{T'/T}$ is the base change of Δ_f. Hence our assumption (2) implies that $\Delta_{T'/T}$ is a closed immersion, hence $T' \to T$ is separated as desired. $\square$

Lemma 61.3.6. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks representable by algebraic spaces. Then the following are equivalent*

(1) *f is quasi-separated,*
(2) *Δ_f is quasi-compact, or*
(3) *Δ_f is finite type.*

Proof. The statements ``$f$ is quasi-separated", ``Δ_f is quasi-compact", and ``Δ_f is finite type" refer to the notions defined in Properties of Stacks, Section 60.3. Note that (2) and (3) are equivalent in view of the fact that Δ_f is locally of finite type by Lemma 61.3.4 (and Algebraic Stacks, Lemma 57.10.9). Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Y}$. Set $U = \mathcal{X} \times_{\mathcal{Y}} V$ which is an algebraic space by assumption, and the morphism $U \to \mathcal{X}$ is surjective and smooth. By Categories, Lemma 4.28.14 and Properties of Stacks, Lemma 60.3.3 we see that we have: Δ_f is quasi-compact if and only if $\Delta_{U/V} : U \to U \times_V U$ is quasi-compact. If (1) holds, then $U \to V$ is quasi-separated and we see that $\Delta_{U/V}$ is quasi-compact, i.e., (2) holds. Assume (2) holds. Let T be a scheme, and $a : T \to \mathcal{Y}$ a morphism. Set $T' = \mathcal{X} \times_{\mathcal{Y}} T$. To prove (1) we have to show that the morphism of algebraic spaces $T' \to T$ is quasi-separated. Using Categories, Lemma 4.28.14 once more we see that $\Delta_{T'/T}$ is the base change of Δ_f. Hence our assumption (2) implies that $\Delta_{T'/T}$ is quasi-compact, hence $T' \to T$ is quasi-separated as desired. $\square$

Lemma 61.3.7. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks representable by algebraic spaces. Then the following are equivalent*

(1) *f is locally separated, and*
(2) *Δ_f is an immersion.*

Proof. The statements ``$f$ is quasi-separated", and ``Δ_f is an immersion" refer to the notions defined in Properties of Stacks, Section 60.3. Proof omitted. Hint: Argue as in the proofs of Lemmas 61.3.5 and 61.3.6. $\square$

61.4. Separation axioms

Let $\mathcal{X} = [U/R]$ be a presentation of an algebraic stack. Then the properties of the diagonal of $\mathcal{X}$ over S, are the properties of the morphism $j : R \to U \times_S U$. For example, if $\mathcal{X} = [S/G]$ for some smooth group G in algebraic spaces over S then j is the structure morphism $G \to U$. Hence the diagonal is not automatically separated itself (contrary to what happens in the case of schemes and algebraic spaces). To say that $[S/G]$ is quasi-separated over S should certainly imply that $G \to S$ is quasi-compact, but we hesitate to say that $[S/G]$ is quasi-separated over S without also requiring the morphism $G \to S$ to be quasi-separated. In other words, requiring the diagonal morphism to be quasi-compact does not really agree with our intuition for a ``quasi-separated algebraic stack'', and we should also require the diagonal itself to be quasi-separated.

What about ``separated algebraic stacks''? We have seen in Morphisms of Spaces, Lemma 42.36.6 that an algebraic space is separated if and only if the diagonal is proper. This is the condition that is usually used to define separated algebraic stacks too. In the example $[S/G] \to S$ above this means that $G \to S$ is a proper group scheme. This means algebraic stacks of the form $[Spec(k)/E]$ are proper over k where E is an elliptic curve over k (insert future reference here). In certain situations it may be more natural to assume the diagonal is finite.

Definition 61.4.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.

(1) We say f is *DM* if Δ_f is unramified[1].
(2) We say f is *quasi-DM* if Δ_f is locally quasi-finite[2].
(3) We say f is *separated* if Δ_f is proper.
(4) We say f is *quasi-separated* if Δ_f is quasi-compact and quasi-separated.

In this definition we are using that Δ_f is representable by algebraic spaces and we are using Properties of Stacks, Section 60.3 to make sense out of imposing conditions on Δ_f. We note that these definitions do not conflict with the already existing notions if f is representable by algebraic spaces, see Lemmas 61.3.6 and 61.3.5. There is an interesting way to characterize these conditions by looking at higher diagonals, see Lemma 61.6.3.

Definition 61.4.2. Let $\mathcal{X}$ be an algebraic stack over the base scheme S. Denote $p : \mathcal{X} \to S$ the structure morphism.

(1) We say $\mathcal{X}$ is *DM over S* if $p : \mathcal{X} \to S$ is DM.
(2) We say $\mathcal{X}$ is *quasi-DM over S* if $p : \mathcal{X} \to S$ is quasi-DM.
(3) We say $\mathcal{X}$ is *separated over S* if $p : \mathcal{X} \to S$ is separated.
(4) We say $\mathcal{X}$ is *quasi-separated over S* if $p : \mathcal{X} \to S$ is quasi-separated.
(5) We say $\mathcal{X}$ is *DM* if $\mathcal{X}$ is DM[3] over $Spec(\mathbf{Z})$.
(6) We say $\mathcal{X}$ is *quasi-DM* if $\mathcal{X}$ is quasi-DM over $Spec(\mathbf{Z})$.
(7) We say $\mathcal{X}$ is *separated* if $\mathcal{X}$ is separated over $Spec(\mathbf{Z})$.

[1]The letters DM stand for Deligne-Mumford. If f is DM then given any scheme T and any morphism $T \to \mathcal{Y}$ the fibre product $\mathcal{X}_T = \mathcal{X} \times_{\mathcal{Y}} T$ is an algebraic stack over T whose diagonal is unramified, i.e., $\mathcal{X}_T$ is DM. This implies $\mathcal{X}_T$ is a Deligne-Mumford stack, see Theorem 61.15.6. In other words a DM morphism is one whose ``fibres'' are Deligne-Mumford stacks. This hopefully at least motivates the terminology.

[2]If f is quasi-DM, then the ``fibres'' $\mathcal{X}_T$ of $\mathcal{X} \to \mathcal{Y}$ are quasi-DM. An algebraic stack $\mathcal{X}$ is quasi-DM exactly if there exists a scheme U and a surjective flat morphism $U \to \mathcal{X}$ of finite presentation which is locally quasi-finite, see Theorem 61.15.3. Note the similarity to being Deligne-Mumford, which is defined in terms of having an étale covering by a scheme.

[3]Theorem 61.15.6 shows that this is equivalent to $\mathcal{X}$ being a Deligne-Mumford stack.

(8) We say $\mathcal{X}$ is *quasi-separated* if $\mathcal{X}$ is quasi-separated over $Spec(\mathbf{Z})$.

In the last 4 definitions we view $\mathcal{X}$ as an algebraic stack over $Spec(\mathbf{Z})$ via Algebraic Stacks, Definition 57.19.2.

Thus in each case we have an absolute notion and a notion relative to our given base scheme (mention of which is usually surpressed by our abuse of notation introduced in Properties of Stacks, Section 60.2). We will see that (1) $\Leftrightarrow$ (5) and (2) $\Leftrightarrow$ (6) in Lemma 61.4.13. We spend some time proving some standard results on these notions.

Lemma 61.4.3. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.*

(1) *If f is separated, then f is quasi-separated.*
(2) *If f is DM, then f is quasi-DM.*
(3) *If f is representable by algebraic spaces, then f is DM.*

Proof. To see (1) note that a proper morphism of algebraic spaces is quasi-compact and quasi-separated, see Morphisms of Spaces, Definition 42.36.1. To see (2) note that an unramified morphism of algebraic spaces is locally quasi-finite, see Morphisms of Spaces, Lemma 42.34.7. Finally (3) follows from Lemma 61.3.4. □

Lemma 61.4.4. *All of the separation axioms listed in Definition 61.4.1 are stable under base change.*

Proof. Let $f : \mathcal{X} \to \mathcal{Y}$ and $\mathcal{Y}' \to \mathcal{Y}$ be morphisms of algebraic stacks. Let $f' : \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Y}'$ be the base change of f by $\mathcal{Y}' \to \mathcal{Y}$. Then $\Delta_{f'}$ is the base change of Δ_f by the morphism $\mathcal{X}' \times_{\mathcal{Y}'} \mathcal{X}' \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$, see Categories, Lemma 4.28.14. By the results of Properties of Stacks, Section 60.3 each of the properties of the diagonal used in Definition 61.4.1 is stable under base change. Hence the lemma is true. □

Lemma 61.4.5. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $W \to \mathcal{Y}$ be a surjective, flat, and locally of finite presentation where W is an algebraic space. If the base change $W \times_{\mathcal{Y}} \mathcal{X} \to W$ has one of the separation properties of Definition 61.4.1 then so does f.*

Proof. Denote $g : W \times_{\mathcal{Y}} \mathcal{X} \to W$ the base change. Then Δ_g is the base change of Δ_f by the morphism $q : W \times_{\mathcal{Y}} (\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}) \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$. Since q is the base change of $W \to \mathcal{Y}$ we see that q is representable by algebraic spaces, surjective, flat, and locally of finite presentation. Hence the result follows from Properties of Stacks, Lemma 60.3.4. □

Lemma 61.4.6. *Let S be a scheme. The property of being quasi-DM over S, quasi-separated over S, or separated over S (see Definition 61.4.2) is stable under change of base scheme, see Algebraic Stacks, Definition 57.19.3.*

Proof. Follows immediately from Lemma 61.4.4. □

Lemma 61.4.7. *Let $f : \mathcal{X} \to \mathcal{Z}$, $g : \mathcal{Y} \to \mathcal{Z}$ and $\mathcal{Z} \to \mathcal{T}$ be morphisms of algebraic stacks. Consider the induced morphism $i : \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{X} \times_{\mathcal{T}} \mathcal{Y}$. Then*

(1) *i is representable by algebraic spaces and locally of finite type,*
(2) *if $\Delta_{\mathcal{Z}/\mathcal{T}}$ is quasi-separated, then i is quasi-separated,*
(3) *if $\Delta_{\mathcal{Z}/\mathcal{T}}$ is separated, then i is separated,*
(4) *if $\mathcal{Z} \to \mathcal{T}$ is DM, then i is unramified,*
(5) *if $\mathcal{Z} \to \mathcal{T}$ is quasi-DM, then i is locally quasi-finite,*
(6) *if $\mathcal{Z} \to \mathcal{T}$ is separated, then i is proper, and*
(7) *if $\mathcal{Z} \to \mathcal{T}$ is quasi-separated, then i is quasi-compact and quasi-separated.*

Proof. The following diagram

$$\begin{tikzcd}
\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \arrow[r, "i"'] \arrow[d] & \mathcal{X} \times_{\mathcal{T}} \mathcal{Y} \arrow[d] \\
\mathcal{Z} \arrow[r, "\Delta_{\mathcal{Z}/\mathcal{T}}"] & \mathcal{Z} \times_{\mathcal{T}} \mathcal{Z}
\end{tikzcd}$$

is a 2-fibre product diagram, see Categories, Lemma 4.28.13. Hence i is the base change of the diagonal morphism $\Delta_{\mathcal{Z}/\mathcal{T}}$. Thus the lemma follows from Lemma 61.3.3, and the material in Properties of Stacks, Section 60.3. □

Lemma 61.4.8. *Let $\mathcal{T}$ be an algebraic stack. Let $g : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks over $\mathcal{T}$. Consider the graph $i : \mathcal{X} \to \mathcal{X} \times_{\mathcal{T}} \mathcal{Y}$ of g. Then*

(1) *i is representable by algebraic spaces and locally of finite type,*
(2) *if $\mathcal{Y} \to \mathcal{T}$ is DM, then i is unramified,*
(3) *if $\mathcal{Y} \to \mathcal{T}$ is quasi-DM, then i is locally quasi-finite,*
(4) *if $\mathcal{Y} \to \mathcal{T}$ is separated, then i is proper, and*
(5) *if $\mathcal{Y} \to \mathcal{T}$ is quasi-separated, then i is quasi-compact and quasi-separated.*

Proof. This is a special case of Lemma 61.4.7 applied to the morphism $\mathcal{X} = \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y} \to \mathcal{X} \times_{\mathcal{T}} \mathcal{Y}$. □

Lemma 61.4.9. *Let $f : \mathcal{X} \to \mathcal{T}$ be a morphism of algebraic stacks. Let $s : \mathcal{T} \to \mathcal{X}$ be a morphism such that $f \circ s$ is 2-isomorphic to $id_{\mathcal{T}}$. Then*

(1) *s is representable by algebraic spaces and locally of finite type,*
(2) *if f is DM, then s is unramified,*
(3) *if f is quasi-DM, then s is locally quasi-finite,*
(4) *if f is separated, then s is proper, and*
(5) *if f is quasi-separated, then s is quasi-compact and quasi-separated.*

Proof. This is a special case of Lemma 61.4.8 applied to $g = s$ and $\mathcal{Y} = \mathcal{T}$ in which case $i : \mathcal{T} \to \mathcal{T} \times_{\mathcal{T}} \mathcal{X}$ is 2-isomorphic to s. □

Lemma 61.4.10. *All of the separation axioms listed in Definition 61.4.1 are stable under composition of morphisms.*

Proof. Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks to which the axiom in question applies. The diagonal $\Delta_{\mathcal{X}/\mathcal{Z}}$ is the composition

$$\mathcal{X} \longrightarrow \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \longrightarrow \mathcal{X} \times_{\mathcal{Z}} \mathcal{X}.$$

Our separation axiom is defined by requiring the diagonal to have some property $\mathcal{P}$. By Lemma 61.4.7 above we see that the second arrow also has this property. Hence the lemma follows since the composition of morphisms which are representable by algebraic spaces with property $\mathcal{P}$ also is a morphism with property $\mathcal{P}$, see our general discussion in Properties of Stacks, Section 60.3 and Morphisms of Spaces, Lemmas 42.34.3, 42.25.2, 42.36.3, 42.9.4, and 42.5.8. □

Lemma 61.4.11. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks over the base scheme S.*

(1) *If $\mathcal{Y}$ is DM over S and f is DM, then $\mathcal{X}$ is DM over S.*
(2) *If $\mathcal{Y}$ is quasi-DM over S and f is quasi-DM, then $\mathcal{X}$ is quasi-DM over S.*
(3) *If $\mathcal{Y}$ is separated over S and f is separated, then $\mathcal{X}$ is separated over S.*

(4) *If $\mathcal{Y}$ is quasi-separated over S and f is quasi-separated, then $\mathcal{X}$ is quasi-separated over S.*
(5) *If $\mathcal{Y}$ is DM and f is DM, then $\mathcal{X}$ is DM.*
(6) *If $\mathcal{Y}$ is quasi-DM and f is quasi-DM, then $\mathcal{X}$ is quasi-DM.*
(7) *If $\mathcal{Y}$ is separated and f is separated, then $\mathcal{X}$ is separated.*
(8) *If $\mathcal{Y}$ is quasi-separated and f is quasi-separated, then $\mathcal{X}$ is quasi-separated.*

Proof. Parts (1), (2), (3), and (4) follow immediately from Lemma 61.4.10 and Definition 61.4.2. For (5), (6), (7), and (8) think of $\mathcal{X}$ and $\mathcal{Y}$ as algebraic stacks over $Spec(\mathbf{Z})$ and apply Lemma 61.4.10. Details omitted. □

The following lemma is a bit different to the analogue for algebraic spaces. To compare take a look at Morphisms of Spaces, Lemma 42.5.10.

Lemma 61.4.12. *Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks.*
(1) *If $g \circ f$ is DM then so is f.*
(2) *If $g \circ f$ is quasi-DM then so is f.*
(3) *If $g \circ f$ is separated and Δ_g is separated, then f is separated.*
(4) *If $g \circ f$ is quasi-separated and Δ_g is quasi-separated, then f is quasi-separated.*

Proof. Consider the factorization

$$\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X} \times_{\mathcal{Z}} \mathcal{X}$$

of the diagonal morphism of $g \circ f$. Both morphisms are representable by algebraic spaces, see Lemmas 61.3.3 and 61.4.7. Hence for any scheme T and morphism $T \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ we get morphisms of algebraic spaces

$$A = \mathcal{X} \times_{(\mathcal{X} \times_{\mathcal{Z}} \mathcal{X})} T \longrightarrow B = (\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}) \times_{(\mathcal{X} \times_{\mathcal{Z}} \mathcal{X})} T \longrightarrow T.$$

If $g \circ f$ is DM (resp. quasi-DM), then the composition $A \to T$ is unramified (resp. locally quasi-finite). Hence (1) (resp. (2)) follows on applying Morphisms of Spaces, Lemma 42.34.11 (resp. Morphisms of Spaces, Lemma 42.25.7). This proves (1) and (2).

Proof of (3). Assume $g \circ f$ is quasi-separated and Δ_g is quasi-separated. Consider the factorization

$$\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X} \times_{\mathcal{Z}} \mathcal{X}$$

of the diagonal morphism of $g \circ f$. Both morphisms are representable by algebraic spaces and the second one is quasi-separated, see Lemmas 61.3.3 and 61.4.7. Hence for any scheme T and morphism $T \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ we get morphisms of algebraic spaces

$$A = \mathcal{X} \times_{(\mathcal{X} \times_{\mathcal{Z}} \mathcal{X})} T \longrightarrow B = (\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}) \times_{(\mathcal{X} \times_{\mathcal{Z}} \mathcal{X})} T \longrightarrow T$$

such that $B \to T$ is quasi-separated. The composition $A \to T$ is quasi-compact and quasi-separated as we have assumed that $g \circ f$ is quasi-separated. Hence $A \to B$ is quasi-separated by Morphisms of Spaces, Lemma 42.5.10. And $A \to B$ is quasi-compact by Morphisms of Spaces, Lemma 42.9.8. Thus f is quasi-separated.

Proof of (4). Assume $g \circ f$ is separated and Δ_g is separated. Consider the factorization

$$\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X} \times_{\mathcal{Z}} \mathcal{X}$$

of the diagonal morphism of $g \circ f$. Both morphisms are representable by algebraic spaces and the second one is separated, see Lemmas 61.3.3 and 61.4.7. Hence for any scheme T and morphism $T \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ we get morphisms of algebraic spaces

$$A = \mathcal{X} \times_{(\mathcal{X} \times_{\mathcal{Z}} \mathcal{X})} T \longrightarrow B = (\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}) \times_{(\mathcal{X} \times_{\mathcal{Z}} \mathcal{X})} T \longrightarrow T$$

such that $B \to T$ is separated. The composition $A \to T$ is proper as we have assumed that $g \circ f$ is quasi-separated. Hence $A \to B$ is proper by Morphisms of Spaces, Lemma 42.36.5 which means that f is separated. □

Lemma 61.4.13. *Let $\mathcal{X}$ be an algebraic stack over the base scheme S.*

1. *$\mathcal{X}$ is DM $\Leftrightarrow$ $\mathcal{X}$ is DM over S.*
2. *$\mathcal{X}$ is quasi-DM $\Leftrightarrow$ $\mathcal{X}$ is quasi-DM over S.*
3. *If $\mathcal{X}$ is separated, then $\mathcal{X}$ is separated over S.*
4. *If $\mathcal{X}$ is quasi-separated, then $\mathcal{X}$ is quasi-separated over S.*

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks over the base scheme S.

5. *If $\mathcal{X}$ is DM over S, then f is DM.*
6. *If $\mathcal{X}$ is quasi-DM over S, then f is quasi-DM.*
7. *If $\mathcal{X}$ is separated over S and $\Delta_{\mathcal{Y}/S}$ is separated, then f is separated.*
8. *If $\mathcal{X}$ is quasi-separated over S and $\Delta_{\mathcal{Y}/S}$ is quasi-separated, then f is quasi-separated.*

Proof. Parts (5), (6), (7), and (8) follow immediately from Lemma 61.4.12 and Spaces, Definition 40.13.2. To prove (3) and (4) think of X and Y as algebraic stacks over $Spec(\mathbf{Z})$ and apply Lemma 61.4.12. Similarly, to prove (1) and (2), think of $\mathcal{X}$ as an algebraic stack over $Spec(\mathbf{Z})$ consider the morphisms

$$\mathcal{X} \longrightarrow \mathcal{X} \times_S \mathcal{X} \longrightarrow \mathcal{X} \times_{Spec(\mathbf{Z})} \mathcal{X}$$

Both arrows are representable by algebraic spaces. The second arrow is unramified and locally quasi-finite as the base change of the immersion $\Delta_{S/\mathbf{Z}}$. Hence the composition is unramified (resp. locally quasi-finite) if and only if the first arrow is unramified (resp. locally quasi-finite), see Morphisms of Spaces, Lemmas 42.34.3 and 42.34.11 (resp. Morphisms of Spaces, Lemmas 42.25.2 and 42.25.7). □

Lemma 61.4.14. *Let $\mathcal{X}$ be an algebraic stack. Let W be an algebraic space, and let $f : W \to \mathcal{X}$ be a surjective, flat, locally finitely presented morphism.*

1. *If f is unramified (i.e., étale, i.e., $\mathcal{X}$ is Deligne-Mumford), then $\mathcal{X}$ is DM.*
2. *If f is locally quasi-finite, then $\mathcal{X}$ is quasi-DM.*

Proof. Note that if f is unramified, then it is étale by Morphisms of Spaces, Lemma 42.35.12. This explains the parenthetical remark in (1). Assume f is unramified (resp. locally quasi-finite). We have to show that $\Delta_{\mathcal{X}} : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is unramified (resp. locally quasi-finite). Note that $W \times W \to \mathcal{X} \times \mathcal{X}$ is also surjective, flat, and locally of finite presentation. Hence it suffices to show that

$$W \times_{\mathcal{X} \times \mathcal{X}, \Delta_{\mathcal{X}}} \mathcal{X} = W \times_{\mathcal{X}} W \longrightarrow W \times W$$

is unramified (resp. locally quasi-finite), see Properties of Stacks, Lemma 60.3.3. By assumption the morphism $\text{pr}_i : W \times_{\mathcal{X}} W \to W$ is unramified (resp. locally quasi-finite). Hence the displayed arrow is unramified (resp. locally quasi-finite) by Morphisms of Spaces, Lemma 42.34.11 (resp. Morphisms of Spaces, Lemma 42.25.7). □

Lemma 61.4.15. *A monomorphism of algebraic stacks is separated and DM. The same is true for immersions of algebraic stacks.*

Proof. If $f : \mathcal{X} \to \mathcal{Y}$ is a monomorphism of algebraic stacks, then Δ_f is an isomorphism, see Properties of Stacks, Lemma 60.8.4. Since an isomorphism of algebraic spaces is proper and unramified we see that f is separated and DM. The second assertion follows from the first as an immersion is a monomorphism, see Properties of Stacks, Lemma 60.9.5. □

Lemma 61.4.16. *Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$. Assume the residual gerbe $\mathcal{Z}_x$ of $\mathcal{X}$ at x exists. If $\mathcal{X}$ is DM, resp. quasi-DM, resp. separated, resp. quasi-separated, then so is $\mathcal{Z}_x$.*

Proof. This is true because $\mathcal{Z}_x \to \mathcal{X}$ is a monomorphism hence DM and separated by Lemma 61.4.15. Apply Lemma 61.4.11 to conclude. □

61.5. Inertia stacks

The (relative) inertia stack of a stack in groupoids is defined in Stacks, Section 50.7. The actual construction, in the setting of fibred categories, and some of its properties is in Categories, Section 4.31.

Lemma 61.5.1. *Let $\mathcal{X}$ be an algebraic stack. Then the inertia stack $\mathcal{I}_\mathcal{X}$ is an algebraic stack as well. The morphism*

$$\mathcal{I}_\mathcal{X} \longrightarrow \mathcal{X}$$

is representable by algebraic spaces and locally of finite type. More generally, let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Then the morphism

$$\mathcal{I}_{\mathcal{X}/\mathcal{Y}} \longrightarrow \mathcal{X}$$

is representable by algebraic spaces and locally of finite type.

Proof. By Categories, Lemma 4.31.1 there are equivalences

$$\mathcal{I}_\mathcal{X} \to \mathcal{X} \times_{\Delta, \mathcal{X} \times_S \mathcal{X}, \Delta} \mathcal{X} \quad \text{and} \quad \mathcal{I}_{\mathcal{X}/\mathcal{Y}} \to \mathcal{X} \times_{\Delta, \mathcal{X} \times_\mathcal{Y} \mathcal{X}, \Delta} \mathcal{X}$$

which shows that the inertia stacks are algebraic stacks. Let $T \to \mathcal{X}$ be a morphism given by the object x of the fibre category of $\mathcal{X}$ over T. Then we get a 2-fibre product square

$$\begin{tikzcd} Isom_\mathcal{X}(x, x) \arrow[r] \arrow[d] & \mathcal{I}_\mathcal{X} \arrow[d] \\ T \arrow[r, "x"] & \mathcal{X} \end{tikzcd}$$

This follows immediately from the definition of $\mathcal{I}_\mathcal{X}$. Since $Isom_\mathcal{X}(x, x)$ is always an algebraic space locally of finite type over T (see Lemma 61.3.1) we conclude that $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is representable by algebraic spaces and locally of finite type. Finally, for the relative inertia we get

$$\begin{tikzcd} Isom_\mathcal{X}(x, x) \arrow[d] & K \arrow[l] \arrow[r] \arrow[d] & \mathcal{I}_{\mathcal{X}/\mathcal{Y}} \arrow[d] \\ Isom_\mathcal{Y}(f(x), f(x)) & T \arrow[l, "e"'] \arrow[r, "x"] & \mathcal{X} \end{tikzcd}$$

with both squares 2-fibre products. This follows from Categories, Lemma 4.31.3. The left vertical arrow is a morphism of algebraic spaces locally of finite type over T, and hence is locally of finite type, see Morphisms of Spaces, Lemma 42.22.6. Thus K is an algebraic space and $K \to T$ is locally of finite type. This proves the assertion on the relative inertia. □

Remark 61.5.2. Let $\mathcal{X}$ be an algebraic stack. In Properties of Stacks, Remark 60.3.7 we have seen that the 2-category of morphisms $\mathcal{X}' \to \mathcal{X}$ representable by algebraic spaces with target $\mathcal{X}$ forms a category. In this category the inertia stack of $\mathcal{X}$ is a *group object*. Recall that an object of $\mathcal{I}_\mathcal{X}$ is just a pair (x, α) where x is an object of $\mathcal{X}$ and α is an automorphism of x in the fibre category of $\mathcal{X}$ that x lives in. The composition

$$c : \mathcal{I}_\mathcal{X} \times_\mathcal{X} \mathcal{I}_\mathcal{X} \longrightarrow \mathcal{I}_\mathcal{X}$$

is given by the rule on objects

$$((x, \alpha), (x', \alpha'), \beta) \mapsto (x, \alpha \circ \beta^{-1} \circ \alpha' \circ \beta)$$

which makes sense as $\beta : x \to x'$ is an isomorphism in the fibre category by our definition of fibre products. The neutral element $e : \mathcal{X} \to \mathcal{I}_\mathcal{X}$ is given by the functor $x \mapsto (x, \mathrm{id}_x)$. We omit the proof that the axioms of a group object hold. There is a variant of this remark for relative inertia stacks.

Let $\mathcal{X}$ be an algebraic stack and let $\mathcal{I}_\mathcal{X}$ be its inertia stack. We have seen in the proof of Lemma 61.5.1 that for any scheme T and object x of $\mathcal{X}$ over T there is a canonical cartesian square

$$\begin{tikzcd} Isom_\mathcal{X}(x, x) \arrow[r] \arrow[d] & \mathcal{I}_\mathcal{X} \arrow[d] \\ T \arrow[r, "x"] & \mathcal{X} \end{tikzcd}$$

The group structure on $\mathcal{I}_\mathcal{X}$ discussed in Remark 61.5.2 induces the group structure on $Isom_\mathcal{X}(x, x)$ of Lemma 61.3.2. This allows us to define the sheaf $Isom_\mathcal{X}$ also for morphisms from algebraic spaces to $\mathcal{X}$. We formalize this in the following definition.

Definition 61.5.3. Let $\mathcal{X}$ be an algebraic stack and let X be an algebraic space. Let $x : X \to \mathcal{X}$ be a morphism. We set

$$Isom_\mathcal{X}(x, x) = X \times_{x, \mathcal{X}} \mathcal{I}_\mathcal{X}$$

We endow it with the structure of a group algebraic space over X by pulling back the composition law discussed in Remark 61.5.2. We will sometimes refer to $Isom_\mathcal{X}(x, x)$ as the *sheaf of automorphisms of x*.

As a variant we may occasionally use the notation $Isom_\mathcal{X}(x, y)$ when given two morphisms $x, y : X \to \mathcal{X}$. This will mean simply the algebraic space

$$(X \times_{x, \mathcal{X}, y} X) \times_{X \times X, \Delta_X} X.$$

Then it is true, as in Lemma 61.3.2, that $Isom_\mathcal{X}(x, y)$ is a pseudo torsor for $Isom_\mathcal{X}(x, x)$ over X. We omit the verification.

Lemma 61.5.4. *Let $\pi : \mathcal{X} \to X$ be a morphism from an algebraic stack to an algebaic space. Let $f : X' \to X$ be a morphism of algebraic spaces. Set $\mathcal{X}' = X' \times_X \mathcal{X}$. Then both squares in the diagram*

$$\begin{tikzcd} \mathcal{I}_{\mathcal{X}'} \arrow[r] \arrow[d, "\textit{Categories, Equation } (4.31.2.1)"'] & \mathcal{X}' \arrow[r] \arrow[d] & X' \arrow[d] \\ \mathcal{I}_\mathcal{X} \arrow[r] & \mathcal{X} \arrow[r] & X \end{tikzcd}$$

are fibre product squares.

Proof. The inertia stack $\mathcal{I}_{\mathcal{X}'}$ is the defined as the category of pairs (x', α') where x' is an object of $\mathcal{X}'$ and α' is an automorphism of x' in its fibre category over $(Sch/S)_{fppf}$, see Categories, Section 4.31. Suppose that x' lies over the scheme U and maps to the object x

of $\mathcal{X}$. By the construction of the 2-fibre product in Categories, Lemma 4.29.3 we see that $x' = (U, a', x, 1)$ where $a' : U \to X'$ is a morphism and 1 indicates that $f \circ a' = \pi \circ x$ as morphisms $U \to X$. Moreover we have $Isom_{\mathcal{X}'}(x', x') = Isom_{\mathcal{X}}(x, x)$ as sheaves on U (by the very construction of the 2-fibre product). This implies that the left square is a fibre product square (details omitted). □

Lemma 61.5.5. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a monomorphism of algebraic stacks. Then the diagram*

$$\begin{tikzcd} \mathcal{I}_\mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \mathcal{I}_\mathcal{Y} \arrow[r] & \mathcal{Y} \end{tikzcd}$$

is a fibre product square.

Proof. This follows immediately from the fact that f is fully faithful (see Properties of Stacks, Lemma 60.8.4) and the definition of the inertia in Categories, Section 4.31. Namely, an object of $\mathcal{I}_\mathcal{X}$ over a scheme T is the same thing as a pair (x, α) consisting of an object x of $\mathcal{X}$ over T and a morphism $\alpha : x \to x$ in the fibre category of $\mathcal{X}$ over T. As f is fully faithful we see that α is the same thing as a morphism $\beta : f(x) \to f(x)$ in the fibre category of $\mathcal{Y}$ over T. Hence we can think of objects of $\mathcal{I}_\mathcal{X}$ over T as triples $((y, \beta), x, \gamma)$ where y is an object of $\mathcal{Y}$ over T, $\beta : y \to y$ in $\mathcal{Y}_T$ and $\gamma : y \to f(x)$ is an isomorhism over T, i.e., an object of $\mathcal{I}_\mathcal{Y} \times_\mathcal{Y} \mathcal{X}$ over T. □

Lemma 61.5.6. *Let $\mathcal{X}$ be an algebraic stack. Let $[U/R] \to \mathcal{X}$ be a presentation. Let G/U be the stabilizer group algebraic space associated to the groupoid (U, R, s, t, c). Then*

$$\begin{tikzcd} G \arrow[r] \arrow[d] & U \arrow[d] \\ \mathcal{I}_\mathcal{X} \arrow[r] & \mathcal{X} \end{tikzcd}$$

is a fibre product diagram.

Proof. Immediate from Groupoids in Spaces, Lemma 52.25.2. □

61.6. Higher diagonals

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. In this situation it makes sense to consider not only the diagonal

$$\Delta_f : \mathcal{X} \to \mathcal{X} \times_\mathcal{Y} \mathcal{X}$$

but also the diagonal of the diagonal, i.e., the morphism

$$\Delta_{\Delta_f} : \mathcal{X} \longrightarrow \mathcal{X} \times_{(\mathcal{X} \times_\mathcal{Y} \mathcal{X})} \mathcal{X}$$

Because of this we sometimes use the following terminology. We denote $\Delta_{f,0} = f$ the *zeroth diagonal*, we denote $\Delta_{f,1} = \Delta_f$ the *first diagonal*, and we denote $\Delta_{f,2} = \Delta_{\Delta_f}$ the *second diagonal*. Note that $\Delta_{f,1}$ is representable by algebraic spaces and locally of finite type, see Lemma 61.3.3. Hence $\Delta_{f,2}$ is representable, a monomorphism, locally of finite type, separated, and locally quasi-finite, see Lemma 61.3.4.

We can describe the second diagonal using the relative inertia stack. Namely, the fibre product $\mathcal{X} \times_{(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})} \mathcal{X}$ is equivalent to the relative inertia stack $\mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ by Categories, Lemma 4.31.1. Moreover, via this identification the second diagonal becomes the *neutral section*

$$e : \mathcal{X} \to \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$$

of the relative inertia stack. Moreover, recall from the proof of Lemma 61.5.1 that given a morphism $x : T \to \mathcal{X}$ the fibre product $T \times_{x,\mathcal{X}} \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ is given as the *kernel* K of the homomorphism of group algebraic spaces

$$Isom_{\mathcal{X}}(x, x) \longrightarrow Isom_{\mathcal{Y}}(f(x), f(x))$$

over T. The morphism e corresponds to the neutral section $e : T \to K$ in this situation.

Lemma 61.6.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Then f is representable by algebraic spaces if and only if the second diagonal is an isomorphism.*

Proof. Namely, f is representable by algebraic spaces if and only if f is faithful, see Algebraic Stacks, Lemma 57.15.2. On the other hand, f is faithful if and only if for every object x of $\mathcal{X}$ over a scheme T the functor f induces an injection $Isom_{\mathcal{X}}(x, x) \to Isom_{\mathcal{Y}}(f(x), f(x))$, which happens if and only if the kernel K is trivial, which happens if and only if $e : T \to K$ is an isomorphism for every $x : T \to \mathcal{X}$. Since $K = T \times_{x,\mathcal{X}} \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ as discussed above, this proves the lemma. □

Lemma 61.6.2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Then*

1. *$\Delta_{f,1}$ separated $\Leftrightarrow$ $\Delta_{f,2}$ closed immersion $\Leftrightarrow$ $\Delta_{f,2}$ proper $\Leftrightarrow$ $\Delta_{f,2}$ universally closed,*
2. *$\Delta_{f,1}$ quasi-separated $\Leftrightarrow$ $\Delta_{f,2}$ finite type $\Leftrightarrow$ $\Delta_{f,2}$ quasi-compact, and*
3. *$\Delta_{f,1}$ locally separated $\Leftrightarrow$ $\Delta_{f,2}$ immersion.*

Proof. Follows from Lemmas 61.3.5, 61.3.6, and 61.3.7 applied to $\Delta_{f,1}$. □

The following lemma is kind of cute and it may suggest a generalization of these conditions to higher algebraic stacks.

Lemma 61.6.3. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Then*

1. *f is separated if and only if $\Delta_{f,1}$ and $\Delta_{f,2}$ are universally closed, and*
2. *f is quasi-separated if and only if $\Delta_{f,1}$ and $\Delta_{f,2}$ are quasi-compact.*
3. *f is quasi-DM if and only if $\Delta_{f,1}$ and $\Delta_{f,2}$ are locally quasi-finite.*
4. *f is DM if and only if $\Delta_{f,1}$ and $\Delta_{f,2}$ are unramified.*

Proof. Proof of (1). Assume that $\Delta_{f,2}$ and $\Delta_{f,1}$ are universally closed. Then $\Delta_{f,1}$ is separated and universally closed by Lemma 61.6.2. By Morphisms of Spaces, Lemma 42.10.7 and Algebraic Stacks, Lemma 57.10.9 we see that $\Delta_{f,1}$ is quasi-compact. Hence it is quasi-compact, separated, universally closed and locally of finite type (by Lemma 61.3.3) so proper. This proves "$\Leftarrow$" of (1). The proof of the implication in the other direction is omitted.

Proof of (2). This follows immediately from Lemma 61.6.2.

Proof of (3). This follows from the fact that $\Delta_{f,2}$ is always locally quasi-finite by Lemma 61.3.4 applied to $\Delta_f = \Delta_{f,1}$.

Proof of (4). This follows from the fact that $\Delta_{f,2}$ is always unramified as Lemma 61.3.4 applied to $\Delta_f = \Delta_{f,1}$ shows that $\Delta_{f,2}$ is locally of finite type and a monomorphism. See More on Morphisms of Spaces, Lemma 46.11.8. □

61.7. Quasi-compact morphisms

Let f be a morphism of algebraic stacks which is representable by algebraic spaces. In Properties of Stacks, Section 60.3 we have defined what it means for f to be quasi-compact. Here is another characterization.

Lemma 61.7.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. The following are equivalent:*

(1) *f is quasi-compact, and*
(2) *for every quasi-compact algebraic stack $\mathcal{Z}$ and any morphism $\mathcal{Z} \to \mathcal{Y}$ the algebraic stack $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}$ is quasi-compact.*

Proof. Assume (1), and let $\mathcal{Z} \to \mathcal{Y}$ be a morphism of algebraic stacks with $\mathcal{Z}$ quasi-compact. By Properties of Stacks, Lemma 60.6.2 there exists a quasi-compact scheme U and a surjective smooth morphism $U \to \mathcal{Z}$. Since f is representable by algebraic spaces and quasi-compact we see by definition that $U \times_{\mathcal{Y}} \mathcal{X}$ is an algebraic space, and that $U \times_{\mathcal{Y}} \mathcal{X} \to U$ is quasi-compact. Hence $U \times_Y X$ is a quasi-compact algebraic space. The morphism $U \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}$ is smooth and surjective (as the base change of the smooth and surjective morphism $U \to \mathcal{Z}$). Hence $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}$ is quasi-compact by another application of Properties of Stacks, Lemma 60.6.2

Assume (2). Let $Z \to \mathcal{Y}$ be a morphism, where Z is a scheme. We have to show that the morphism of algebraic spaces $p : Z \times_{\mathcal{Y}} \mathcal{X} \to Z$ is quasi-compact. Let $U \subset Z$ be affine open. Then $p^{-1}(U) = U \times_{\mathcal{Y}} \mathcal{Z}$ and the algebraic space $U \times_{\mathcal{Y}} \mathcal{Z}$ is quasi-compact by assumption (2). Hence p is quasi-compact, see Morphisms of Spaces, Lemma 42.9.7. □

This motivates the following definition.

Definition 61.7.2. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is *quasi-compact* if for every quasi-compact algebraic stack $\mathcal{Z}$ and morphism $\mathcal{Z} \to \mathcal{Y}$ the fibre product $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}$ is quasi-compact.

By Lemma 61.7.1 above this agrees with the already existing notion for morphisms of algebraic stacks representable by algebraic spaces. In particular this notion agrees with the notions already defined for morphisms between algebraic stacks and schemes.

Lemma 61.7.3. *The base change of a quasi-compact morphism of algebraic stacks by any morphism of algebraic stacks is quasi-compact.*

Proof. Omitted. □

Lemma 61.7.4. *The composition of a pair of quasi-compact morphisms of algebraic stacks is quasi-compact.*

Proof. Omitted. □

Lemma 61.7.5. *Let*

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathcal{Y} \arrow[dl, "q"] \\ & \mathcal{Z} & \end{tikzcd}$$

be a 2-commutative diagram of morphisms of algebraic stacks. If f is surjective and p is quasi-compact, then q is quasi-compact.

Proof. Let $\mathcal{T}$ be a quasi-compact algebraic stack, and let $\mathcal{T} \to \mathcal{Z}$ be a morphism. By Properties of Stacks, Lemma 60.5.3 the morphism $\mathcal{T} \times_{\mathcal{Z}} \mathcal{X} \to \mathcal{T} \times_{\mathcal{Z}} \mathcal{Y}$ is surjective and by assumption $\mathcal{T} \times_{\mathcal{Z}} \mathcal{X}$ is quasi-compact. Hence $\mathcal{T} \times_{\mathcal{Z}} \mathcal{Y}$ is quasi-compact by Properties of Stacks, Lemma 60.6.2. □

Lemma 61.7.6. *Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. If $g \circ f$ is quasi-compact and g is quasi-separated then f is quasi-compact.*

Proof. This is true because f equals the composition $(1, f) : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{Y}$. The first map is quasi-compact by Lemma 61.4.9 because it is a section of the quasi-separated morphism $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{X}$ (a base change of g, see Lemma 61.4.4). The second map is quasi-compact as it is the base change of f, see Lemma 61.7.3. And compositions of quasi-compact morphisms are quasi-compact, see Lemma 61.7.4. □

Lemma 61.7.7. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.*

1. *If $\mathcal{X}$ is quasi-compact and $\mathcal{Y}$ is quasi-separated, then f is quasi-compact.*
2. *If $\mathcal{X}$ is quasi-compact and quasi-separated and $\mathcal{Y}$ is quasi-separated, then f is quasi-compact and quasi-separated.*
3. *A fibre product of quasi-compact and quasi-separated algebraic stacks is quasi-compact and quasi-separated.*

Proof. Part (1) follows from Lemma 61.7.6. Part (2) follows from (1) and Lemma 61.4.12. For (3) let $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Z} \to \mathcal{Y}$ be morphisms of quasi-compact and quasi-separated algebraic stacks. Then $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{Z}$ is quasi-compact and quasi-separated as a base change of $\mathcal{X} \to \mathcal{Y}$ using (2) and Lemmas 61.7.3 and 61.4.4. Hence $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ is quasi-compact and quasi-separated as an algebraic stack quasi-compact and quasi-separated over $\mathcal{Z}$, see Lemmas 61.4.11 and 61.7.4. □

61.8. Noetherian algebraic stacks

We have already defined locally Noetherian algebraic stacks in Properties of Stacks, Section 60.7.

Definition 61.8.1. Let $\mathcal{X}$ be an algebraic stack. We say $\mathcal{X}$ is *Noetherian* if $\mathcal{X}$ is quasi-compact, quasi-separated and locally Noetherian.

Note that a Noetherian algebraic stack $\mathcal{X}$ is not just quasi-compact and locally Noetherian, but also quasi-separated. In the language of Section 61.6 if we denote $p : \mathcal{X} \to Spec(\mathbf{Z})$ the ``absolute'' structure morphism (i.e., the structure morphism of $\mathcal{X}$ viewed as an algebraic stack over $\mathbf{Z}$), then

$$\mathcal{X} \text{ Noetherian} \Leftrightarrow \mathcal{X} \text{ locally Noetherian and } \Delta_{p,0}, \Delta_{p,1}, \Delta_{p,2} \text{ quasi-compact.}$$

This will later mean that an algebraic stack of finite type over a Noetherian algebraic stack is not automatically Noetherian.

61.9. Open morphisms

Let f be a morphism of algebraic stacks which is representable by algebraic spaces. In Properties of Stacks, Section 60.3 we have defined what it means for f to be universally open. Here is another characterization.

Lemma 61.9.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. The following are equivalent*

1. *f is universally open, and*

(2) *for every morphism of algebraic stacks $\mathcal{Z} \to \mathcal{Y}$ the morphism of topological spaces $|\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}| \to |\mathcal{Z}|$ is open.*

Proof. Assume (1), and let $\mathcal{Z} \to \mathcal{Y}$ be as in (2). Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Z}$. By assumption the morphism $V \times_{\mathcal{Y}} \mathcal{X} \to V$ of algebraic spaces is universally open, in particular the map $|V \times_{\mathcal{Y}} \mathcal{X}| \to |V|$ is open. By Properties of Stacks, Section 60.4 in the commutative diagram

$$\begin{tikzcd} |V \times_{\mathcal{Y}} \mathcal{X}| \arrow[r] \arrow[d] & |\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}| \arrow[d] \\ |V| \arrow[r] & |\mathcal{Z}| \end{tikzcd}$$

the horizontal arrows are open and surjective, and moreover

$$|V \times_{\mathcal{Y}} \mathcal{X}| \longrightarrow |V| \times_{|\mathcal{Z}|} |\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}|$$

is surjective. Hence as the left vertical arrow is open it follows that the right vertical arrow is open. This proves (2). The implication (2) $\Rightarrow$ (1) follows from the definitions. □

Thus we may use the following natural definition.

Definition 61.9.2. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.

(1) We say f is *open* if the map of topological spaces $|\mathcal{X}| \to |\mathcal{Y}|$ is open.
(2) We say f is *universally open* if for every morphism of algebraic stacks $\mathcal{Z} \to \mathcal{Y}$ the morphism of topological spaces

$$|\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}| \to |\mathcal{Z}|$$

is open, i.e., the base change $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z}$ is open.

Lemma 61.9.3. *The base change of a universally open morphism of algebraic stacks by any morphism of algebraic stacks is universally open.*

Proof. This is immediate from the definition. □

Lemma 61.9.4. *The composition of a pair of (universally) open morphisms of algebraic stacks is (universally) open.*

Proof. Omitted. □

61.10. Submersive morphisms

Definition 61.10.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.

(1) We say f is *submersive*[4] if the continuous map $|\mathcal{X}| \to |\mathcal{Y}|$ is submersive, see Topology, Definition 5.15.1.
(2) We say f is *universally submersive* if for every morphism of algebraic stacks $\mathcal{Y}' \to \mathcal{Y}$ the base change $\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Y}'$ is submersive.

We note that a submersive morphism is in particular surjective.

[4]This is very different from the notion of a submersion of differential manifolds.

61.11. Universally closed morphisms

Let f be a morphism of algebraic stacks which is representable by algebraic spaces. In Properties of Stacks, Section 60.3 we have defined what it means for f to be universally closed. Here is another characterization.

Lemma 61.11.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. The following are equivalent*

1. *f is universally closed, and*
2. *for every morphism of algebraic stacks $\mathcal{Z} \to \mathcal{Y}$ the morphism of topological spaces $|\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}| \to |\mathcal{Z}|$ is closed.*

Proof. Assume (1), and let $\mathcal{Z} \to \mathcal{Y}$ be as in (2). Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Z}$. By assumption the morphism $V \times_{\mathcal{Y}} \mathcal{X} \to V$ of algebraic spaces is universally closed, in particular the map $|V \times_{\mathcal{Y}} \mathcal{X}| \to |V|$ is closed. By Properties of Stacks, Section 60.4 in the commutative diagram

$$\begin{tikzcd} |V \times_{\mathcal{Y}} \mathcal{X}| \arrow[r] \arrow[d] & |\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}| \arrow[d] \\ |V| \arrow[r] & |\mathcal{Z}| \end{tikzcd}$$

the horizontal arrows are open and surjective, and moreover

$$|V \times_{\mathcal{Y}} \mathcal{X}| \longrightarrow |V| \times_{|\mathcal{Z}|} |\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}|$$

is surjective. Hence as the left vertical arrow is closed it follows that the right vertical arrow is closed. This proves (2). The implication (2) ⇒ (1) follows from the definitions. ☐

Thus we may use the following natural definition.

Definition 61.11.2. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.

1. We say f is *closed* if the map of topological spaces $|\mathcal{X}| \to |\mathcal{Y}|$ is closed.
2. We say f is *universally closed* if for every morphism of algebraic stacks $\mathcal{Z} \to \mathcal{Y}$ the morphism of topological spaces
$$|\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}| \to |\mathcal{Z}|$$
is closed, i.e., the base change $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z}$ is closed.

Lemma 61.11.3. *The base change of a universally closed morphism of algebraic stacks by any morphism of algebraic stacks is universally closed.*

Proof. This is immediate from the definition. ☐

Lemma 61.11.4. *The composition of a pair of (universally) closed morphisms of algebraic stacks is (universally) closed.*

Proof. Omitted. ☐

61.12. Types of morphisms smooth local on source-and-target

Given a property of morphisms of algebraic spaces which is *smooth local on the source-and-target*, see Descent on Spaces, Definition 45.18.1 we may use it to define a corresponding property of morphisms of algebraic stacks, namely by imposing either of the equivalent conditions of the lemma below.

Lemma 61.12.1. *Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is smooth local on the source-and-target. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Consider commutative diagrams*

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "a"'] & V \arrow[d, "b"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}$$

where U and V are algebraic spaces and the vertical arrows are smooth. The following are equivalent

(1) *for any diagram as above such that in addition $U \to \mathcal{X} \times_{\mathcal{Y}} V$ is smooth the morphism h has property $\mathcal{P}$, and*
(2) *for some diagram as above with $a : U \to \mathcal{X}$ surjective the morphism h has property $\mathcal{P}$.*

If $\mathcal{X}$ and $\mathcal{Y}$ are representable by algebraic spaces, then this is also equivalent to f (as a morphism of algebraic spaces) having property $\mathcal{P}$. If $\mathcal{P}$ is also preserved under any base change, and fppf local on the base, then for morphisms f which are representable by algebraic spaces this is also equivalent to f having property $\mathcal{P}$ in the sense of Properties of Stacks, Section 60.3.

Proof. Let us prove the implication (1) ⇒ (2). Pick an algebraic space V and a surjective and smooth morphism $V \to \mathcal{Y}$. Pick an algebraic space U and a surjective and smooth morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$. Note that $U \to \mathcal{X}$ is surjective and smooth as well, as a composition of the base change $\mathcal{X} \times_{\mathcal{Y}} V \to \mathcal{X}$ and the chosen map $U \to \mathcal{X} \times_{\mathcal{Y}} V$. Hence we obtain a diagram as in (1). Thus if (1) holds, then $h : U \to V$ has property $\mathcal{P}$, which means that (2) holds as $U \to \mathcal{X}$ is surjective.

Conversely, assume (2) holds and let U, V, a, b, h be as in (2). Next, let U', V', a', b', h' be any diagram as in (1). Picture

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d] & V \arrow[d] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd} \qquad \begin{tikzcd} U' \arrow[r, "h'"] \arrow[d] & V' \arrow[d] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}$$

To show that (2) implies (1) we have to prove that h' has $\mathcal{P}$. To do this consider the commutative diagram

$$\begin{tikzcd} U \arrow[dd, "h"] & U \times_{\mathcal{X}} U' \arrow[l] \arrow[r] \arrow[d] \arrow[dd, bend left, "(h,h')"] & U' \arrow[dd, "h'"] \\ & U \times_{\mathcal{Y}} V' \arrow[ul] \arrow[d] & \\ V & V \times_{\mathcal{Y}} V' \arrow[l] \arrow[r] & V' \end{tikzcd}$$

of algebraic spaces. Note that the horizontal arrows are smooth as base changes of the smooth morphisms $V \to \mathcal{Y}$, $V' \to \mathcal{Y}$, $U \to \mathcal{X}$, and $U' \to \mathcal{X}$. Note that

$$\begin{tikzcd} U \times_{\mathcal{X}} U' \arrow[r] \arrow[d] & U' \arrow[d] \\ U \times_{\mathcal{Y}} V' \arrow[r] & \mathcal{X} \times_{\mathcal{Y}} V' \end{tikzcd}$$

is cartesian, hence the left vertical arrow is smooth as U', V', a', b', h' is as in (1). Since $\mathcal{P}$ is local on the target we see that the base change $U \times_{\mathcal{Y}} V' \to V \times_{\mathcal{Y}} V'$ has $\mathcal{P}$ and hence after precomposing by the smooth morphism $U \times_{\mathcal{X}} U' \to U \times_{\mathcal{Y}} V'$ the morphism we conclude (h, h') has $\mathcal{P}$. Finally, since $U \times_X U' \to U'$ is surjective this implies that h' has $\mathcal{P}$ as $\mathcal{P}$ is local on the source-and-target. This finishes the proof of the equivalence of (1) and (2).

If $\mathcal{X}$ and $\mathcal{Y}$ are representable, then Descent on Spaces, Lemma 45.18.3 applies which shows that (1) and (2) are equivalent to f having $\mathcal{P}$.

Finally, suppose f is representable, and U, V, a, b, h are as in part (2) of the lemma, and that $\mathcal{P}$ is preserved under arbitrary base change. We have to show that for any scheme Z and morphism $Z \to \mathcal{X}$ the base change $Z \times_{\mathcal{Y}} \mathcal{X} \to Z$ has property $\mathcal{P}$. Consider the diagram

$$\begin{tikzcd} Z \times_{\mathcal{Y}} U \arrow[r] \arrow[d] & Z \times_{\mathcal{Y}} V \arrow[d] \\ Z \times_{\mathcal{Y}} \mathcal{X} \arrow[r] & Z \end{tikzcd}$$

Note that the top horizontal arrow is a base change of h and hence has property $\mathcal{P}$. The left vertical arrow is smooth and surjective and the right vertical arrow is smooth. Thus Descent on Spaces, Lemma 45.18.3 kicks in and shows that $Z \times_{\mathcal{Y}} \mathcal{X} \to Z$ has property $\mathcal{P}$. $\square$

Definition 61.12.2. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is smooth local on the source-and-target. We say a morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks *has property* $\mathcal{P}$ if the equivalent conditions of Lemma 61.12.1 hold.

Remark 61.12.3. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is smooth local on the source-and-target and stable under composition. Then the property of morphisms of algebraic stacks defined in Definition 61.12.2 is stable under composition. Namely, let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks having propery $\mathcal{P}$. Choose an algebraic space W and a surjective smooth morphism $W \to \mathcal{Z}$. Choose an algebraic space V and a surjective smooth morphism $V \to \mathcal{Y} \times_{\mathcal{Z}} W$. Finally, choose an algebraic space U and a surjective and smooth morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$. Then the morphisms $V \to W$ and $U \to V$ have property $\mathcal{P}$ by definition. Whence $U \to W$ has property $\mathcal{P}$ as we assumed that $\mathcal{P}$ is stable under composition. Thus, by definition again, we see that $g \circ f : \mathcal{X} \to \mathcal{Z}$ has property $\mathcal{P}$.

Remark 61.12.4. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is smooth local on the source-and-target and stable under base change. Then the property of morphisms of algebraic stacks defined in Definition 61.12.2 is stable under base change. Namely, let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y}' \to \mathcal{Y}$ be morphisms of algebraic stacks and assume f has propery $\mathcal{P}$. Choose an algebraic space V and a surjective smooth morphism $V \to \mathcal{Y}$. Choose an algebraic space U and a surjective smooth morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$. Finally, choose an algebraic space V' and a surjective and smooth morphism $V' \to \mathcal{Y}' \times_{\mathcal{Y}} V$. Then the morphism

$U \to V$ has property $\mathcal{P}$ by definition. Whence $V' \times_V U \to V'$ has property $\mathcal{P}$ as we assumed that $\mathcal{P}$ is stable under base change. Considering the diagram

$$\begin{tikzcd} V' \times_V U \arrow[r] \arrow[d] & \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ V' \arrow[r] & \mathcal{Y}' \arrow[r] & \mathcal{Y} \end{tikzcd}$$

we see that the left top horizontal arrow is smooth and surjective, whence by definition we see that the projection $\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Y}'$ has property $\mathcal{P}$.

Remark 61.12.5. Let $\mathcal{P}, \mathcal{P}'$ be properties of morphisms of algebraic spaces which are smooth local on the source-and-target and stable under base change. Suppose that we have $\mathcal{P} \Rightarrow \mathcal{P}'$ for morphisms of algebraic spaces. Then we also have $\mathcal{P} \Rightarrow \mathcal{P}'$ for the properties of morphisms of algebraic stacks defined in Definition 61.12.2 using $\mathcal{P}$ and $\mathcal{P}'$. This is clear from the definition.

61.13. Morphisms of finite type

The property ``locally of finite type" of morphisms of algebraic spaces is smooth local on the source-and-target, see Descent on Spaces, Remark 45.18.5. It is also stable under base change and fpqc local on the target, see Morphisms of Spaces, Lemma 42.22.3 and Descent on Spaces, Lemma 45.10.7. Hence, by Lemma 61.12.1 above, we may define what it means for a morphism of algebraic spaces to be locally of finite type as follows and it agrees with the already existing notion defined in Properties of Stacks, Section 60.3 when the morphism is representable by algebraic spaces.

Definition 61.13.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.

(1) We say f *locally of finite type* if the equivalent conditions of Lemma 61.12.1 hold with $\mathcal{P}=$ locally of finite type.
(2) We say f is *of finite type* if it is locally of finite type and quasi-compact.

Lemma 61.13.2. *The composition of finite type morphisms is of finite type. The same holds for locally of finite type.*

Proof. Combine Remark 61.12.3 with Morphisms of Spaces, Lemma 42.22.2. □

Lemma 61.13.3. *A base change of a finite type morphism is finite type. The same holds for locally of finite type.*

Proof. Combine Remark 61.12.4 with Morphisms of Spaces, Lemma 42.22.3. □

Lemma 61.13.4. *An immersion is locally of finite type.*

Proof. Follows from Morphisms of Spaces, Lemma 42.22.7. □

Lemma 61.13.5. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If f is locally of finite type and $\mathcal{Y}$ is locally Noetherian, then $\mathcal{X}$ is locally Noetherian.*

Proof. Let

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{Y} \end{tikzcd}$$

be a commutative diagram where U, V are schemes, $V \to \mathcal{Y}$ is surjective and smooth, and $U \to V \times_{\mathcal{Y}} \mathcal{X}$ is surjective and smooth. Then $U \to V$ is locally of finite type. If $\mathcal{Y}$ is locally Noetherian, then V is locally Noetherian. By Morphisms, Lemma 24.14.6 we see that U is locally Noetherian, which means that $\mathcal{X}$ is locally Noetherian. $\square$

The following two lemmas will be improved on later (after we have discussed morphisms of algebraic stacks which are locally of finite presentation).

Lemma 61.13.6. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $W \to \mathcal{Y}$ be a surjective, flat, and locally of finite presentation where W is an algebraic space. If the base change $W \times_{\mathcal{Y}} \mathcal{X} \to W$ is locally of finite type, then f is locally of finite type.*

Proof. Choose an algebraic space V and a surjective smooth morphism $V \to \mathcal{Y}$. Choose an algebraic space U and a surjective smooth morphism $U \to V \times_{\mathcal{Y}} \mathcal{X}$. We have to show that $U \to V$ is locally of finite presentation. Now we base change everything by $W \to \mathcal{Y}$: Set $U' = W \times_{\mathcal{Y}} U$, $V' = W \times_{\mathcal{Y}} V$, $\mathcal{X}' = W \times_{\mathcal{Y}} \mathcal{X}$, and $\mathcal{Y}' = W \times_{\mathcal{Y}} \mathcal{Y} = W$. Then it is still true that $U' \to V' \times_{\mathcal{Y}'} \mathcal{X}'$ is smooth by base change. Hence by our definition of locally finite type morphisms of algebraic stacks and the assumption that $\mathcal{X}' \to \mathcal{Y}'$ is locally of finite type, we see that $U' \to V'$ is locally of finite type. Then, since $V' \to V$ is surjective, flat, and locally of finite presentation as a base change of $W \to \mathcal{Y}$ we see that $U \to V$ is locally of finite type by Descent on Spaces, Lemma 45.10.7 and we win. $\square$

Lemma 61.13.7. *Let $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. Assume $\mathcal{X} \to \mathcal{Z}$ is locally of finite type and that $\mathcal{X} \to \mathcal{Y}$ is representable by algebraic spaces, surjective, flat, and locally of finite presentation. Then $\mathcal{Y} \to \mathcal{Z}$ is locally of finite type.*

Proof. Choose an algebraic space W and a surjective smooth morphism $W \to \mathcal{Z}$. Choose an algebraic space V and a surjective smooth morphism $V \to W \times_{\mathcal{Z}} \mathcal{Y}$. Set $U = V \times_{\mathcal{Y}} \mathcal{X}$ which is an algebraic space. We know that $U \to V$ is surjective, flat, and locally of finite presentation and that $U \to W$ is locally of finite type. Hence the lemma reduces to the case of morphisms of algebraic spaces. The case of morphisms of algebraic spaces is Descent on Spaces, Lemma 45.14.2. $\square$

Lemma 61.13.8. *Let $f : \mathcal{X} \to \mathcal{Y}$, $g : \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. If $g \circ f : \mathcal{X} \to \mathcal{Z}$ is locally of finite type, then $f : \mathcal{X} \to \mathcal{Y}$ is locally of finite type.*

Proof. We can find a diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & W \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{Y} \arrow[r] & \mathcal{Z} \end{tikzcd}$$

where U, V, W are schemes, the vertical arrow $W \to \mathcal{Z}$ is surjective and smooth, the arrow $V \to \mathcal{Y} \times_{\mathcal{Z}} W$ is surjective and smooth, and the arrow $U \to \mathcal{X} \times_{\mathcal{Y}} V$ is surjective and smooth. Then also $U \to \mathcal{X} \times_{\mathcal{Z}} V$ is surjective and smooth (as a composition of a surjective and smooth morphism with a base change of such). By definition we see that $U \to W$ is locally of finite type. Hence $U \to V$ is locally of finite type by Morphisms, Lemma 24.14.8 which in turn means (by definition) that $\mathcal{X} \to \mathcal{Y}$ is locally of finite type. $\square$

61.14. Points of finite type

Let $\mathcal{X}$ be an algebraic stack. A finite type point $x \in |\mathcal{X}|$ is a point which can be represented by a morphism $Spec(k) \to \mathcal{X}$ which is locally of finite type. Finite type points are a suitable replacement of closed points for algebraic spaces and algebraic stacks. There are always ``enough of them'' for example.

Lemma 61.14.1. *Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$. The following are equivalent:*

(1) *There exists a morphism $Spec(k) \to \mathcal{X}$ which is locally of finite type and represents x.*
(2) *There exists a scheme U, a closed point $u \in U$, and a smooth morphism $\varphi : U \to \mathcal{X}$ such that $\varphi(u) = x$.*

Proof. Let $u \in U$ and $U \to \mathcal{X}$ be as in (2). Then $Spec(\kappa(u)) \to U$ is of finite type, and $U \to \mathcal{X}$ is representable and locally of finite type (by Morphisms of Spaces, Lemmas 42.35.8 and 42.26.5). Hence we see (1) holds by Lemma 61.13.2.

Conversely, assume $Spec(k) \to \mathcal{X}$ is locally of finite type and represents x. Let $U \to \mathcal{X}$ be a surjective smooth morphism where U is a scheme. By assumption $U \times_{\mathcal{X}} Spec(k) \to U$ is a morphism of algebraic spaces which is locally of finite type. Pick a finite type point v of $U \times_{\mathcal{X}} Spec(k)$ (there exists at least one, see Morphisms of Spaces, Lemma 42.24.3). By Morphisms of Spaces, Lemma 42.24.4 the image $u \in U$ of v is a finite type point of U. Hence by Morphisms, Lemma 24.15.4 after shrinking U we may assume that u is a closed point of U, i.e., (2) holds. □

Definition 61.14.2. Let $\mathcal{X}$ be an algebraic stack. We say a point $x \in |\mathcal{X}|$ is a *finite type point*[5] if the equivalent conditions of Lemma 61.14.1 are satisfied. We denote $\mathcal{X}_{\text{ft-pts}}$ the set of finite type points of $\mathcal{X}$.

We can describe the set of finite type points as follows.

Lemma 61.14.3. *Let $\mathcal{X}$ be an algebraic stack. We have*

$$\mathcal{X}_{ft\text{-}pts} = \bigcup\nolimits_{\varphi : U \to X \ smooth} |\varphi|(U_0)$$

where U_0 is the set of closed points of U. Here we may let U range over all schemes smooth over $\mathcal{X}$ or over all affine schemes smooth over $\mathcal{X}$.

Proof. Immediate from Lemma 61.14.1. □

Lemma 61.14.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If f is locally of finite type, then $f(\mathcal{X}_{ft\text{-}pts}) \subset \mathcal{Y}_{ft\text{-}pts}$.*

Proof. Take $x \in \mathcal{X}_{\text{ft-pts}}$. Represent x by a locally finite type morphism $x : Spec(k) \to \mathcal{X}$. Then $f \circ x$ is locally of finite type by Lemma 61.13.2. Hence $f(x) \in \mathcal{Y}_{\text{ft-pts}}$. □

Lemma 61.14.5. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If f is locally of finite type and surjective, then $f(\mathcal{X}_{ft\text{-}pts}) = \mathcal{Y}_{ft\text{-}pts}$.*

Proof. We have $f(\mathcal{X}_{\text{ft-pts}}) \subset \mathcal{Y}_{\text{ft-pts}}$ by Lemma 61.14.4. Let $y \in |\mathcal{Y}|$ be a finite type point. Represent y by a morphism $Spec(k) \to \mathcal{Y}$ which is locally of finite type. As f is surjective the algebraic stack $\mathcal{X}_k = Spec(k) \times_{\mathcal{Y}} \mathcal{X}$ is nonempty, therefore has a finite type point $x \in |\mathcal{X}_k|$ by Lemma 61.14.3. Now $\mathcal{X}_k \to \mathcal{X}$ is a morphism which is locally of finite type as a base change of $Spec(k) \to \mathcal{Y}$ (Lemma 61.13.3). Hence the image of x in $\mathcal{X}$ is a finite type point by Lemma 61.14.4 which maps to y by construction. □

[5] This is a slight abuse of language as it would perhaps be more correct to say ``locally finite type point''.

Lemma 61.14.6. *Let $\mathcal{X}$ be an algebraic stack. For any locally closed subset $T \subset |\mathcal{X}|$ we have*

$$T \neq \emptyset \Rightarrow T \cap \mathcal{X}_{ft\text{-}pts} \neq \emptyset.$$

In particular, for any closed subset $T \subset |\mathcal{X}|$ we see that $T \cap \mathcal{X}_{ft\text{-}pts}$ is dense in T.

Proof. Let $i : \mathcal{Z} \to \mathcal{X}$ be the reduced induced substack structure on T, see Properties of Stacks, Remark 60.10.5. An immersion is locally of finite type, see Lemma 61.13.4. Hence by Lemma 61.14.4 we see $\mathcal{Z}_{\text{ft-pts}} \subset \mathcal{X}_{\text{ft-pts}} \cap T$. Finally, any nonempty affine scheme U with a smooth morphism towards $\mathcal{Z}$ has at least one closed point, hence $\mathcal{Z}$ has at least one finite type point by Lemma 61.14.3. The lemma follows. □

Here is another, more technical, characterization of a finite type point on an algebraic stack. It tells us in particular that the residual gerbe of $\mathcal{X}$ at x exists whenever x is a finite type point!

Lemma 61.14.7. *Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$. The following are equivalent:*

(1) *x is a finite type point,*
(2) *there exists an algebraic stack $\mathcal{Z}$ whose underlying topological space $|\mathcal{Z}|$ is a singleton, and a morphism $f : \mathcal{Z} \to \mathcal{X}$ which is locally of finite type such that $\{x\} = |f|(|\mathcal{Z}|)$, and*
(3) *the residual gerbe $\mathcal{Z}_x$ of $\mathcal{X}$ at x exists and the inclusion morphism $\mathcal{Z}_x \to \mathcal{X}$ is locally of finite type.*

Proof. (All of the morphisms occuring in this paragraph are representable by algebraic spaces, hence the conventions and results of Properties of Stacks, Section 60.3 are applicable.) Assume x is a finite type point. Choose an affine scheme U, a closed point $u \in U$, and a smooth morphism $\varphi : U \to \mathcal{X}$ with $\varphi(u) = x$, see Lemma 61.14.3. Set $u = Spec(\kappa(u))$ as usual. Set $R = u \times_{\mathcal{X}} u$ so that we obtain a groupoid in algebraic spaces (u, R, s, t, c), see Algebraic Stacks, Lemma 57.16.1. The projection morphisms $R \to u$ are the compositions

$$R = u \times_{\mathcal{X}} u \to u \times_{\mathcal{X}} U \to u \times_{\mathcal{X}} X = u$$

where the first arrow is of finite type (a base change of the closed immersion of schemes $u \to U$) and the second arrow is smooth (a base change of the smooth morphism $U \to \mathcal{X}$). Hence $s, t : R \to u$ are locally of finite type (as compositions, see Morphisms of Spaces, Lemma 42.22.2). Since u is the spectrum of a field, it follows that s, t are flat and locally of finite presentation (by Morphisms of Spaces, Lemma 42.26.7). We see that $\mathcal{Z} = [u/R]$ is an algebraic stack by Criteria for Representability, Theorem 59.17.2. By Algebraic Stacks, Lemma 57.16.1 we obtain a canonical morphism

$$f : \mathcal{Z} \longrightarrow \mathcal{X}$$

which is fully faithful. Hence this morphism is representable by algebraic spaces, see Algebraic Stacks, Lemma 57.15.2 and a monomorphism, see Properties of Stacks, Lemma 60.8.4. It follows that the residual gerbe $\mathcal{Z}_x \subset \mathcal{X}$ of $\mathcal{X}$ at x exists and that f factors through an equivalence $\mathcal{Z} \to \mathcal{Z}_x$, see Properties of Stacks, Lemma 60.11.11. By construction the diagram

$$\begin{tikzcd} u \arrow[r] \arrow[d] & U \arrow[d] \\ \mathcal{Z} \arrow[r, "f"] & \mathcal{X} \end{tikzcd}$$

is commutative. By Criteria for Representability, Lemma 59.17.1 the left vertical arrow is surjective, flat, and locally of finite presentation. Consider

$$\begin{tikzcd} u \times_{\mathcal{X}} U \arrow[r] \arrow[d] & \mathcal{Z} \times_{\mathcal{X}} U \arrow[r] \arrow[d] & U \arrow[d] \\ u \arrow[r] & \mathcal{Z} \arrow[r, "f"] & \mathcal{X} \end{tikzcd}$$

As $u \to \mathcal{X}$ is locally of finite type, we see that the base change $u \times_{\mathcal{X}} U \to U$ is locally of finite type. Moreover, $u \times_{\mathcal{X}} U \to \mathcal{Z} \times_{\mathcal{X}} U$ is surjective, flat, and locally of finite presentation as a base change of $u \to \mathcal{Z}$. Thus $\{u \times_{\mathcal{X}} U \to \mathcal{Z} \times_{\mathcal{X}} U\}$ is an fppf covering of algebraic spaces, and we conclude that $\mathcal{Z} \times_{\mathcal{X}} U \to U$ is locally of finite type by Descent on Spaces, Lemma 45.14.1. By definition this means that f is locally of finite type (because the vertical arrow $\mathcal{Z} \times_{\mathcal{X}} U \to \mathcal{Z}$ is smooth as a base change of $U \to \mathcal{X}$ and surjective as $\mathcal{Z}$ has only one point). Since $\mathcal{Z} = \mathcal{Z}_x$ we see that (3) holds.

It is clear that (3) implies (2). If (2) holds then x is a finite type point of $\mathcal{X}$ by Lemma 61.14.4 and Lemma 61.14.6 to see that $\mathcal{Z}_{\text{ft-pts}}$ is nonempty, i.e., the unique point of $\mathcal{Z}$ is a finite type point of $\mathcal{Z}$. $\square$

61.15. Special presentations of algebraic stacks

The following lemma gives a criterion for when a ``slice'' of a presentation is still flat over the algebraic stack.

Lemma 61.15.1. *Let $\mathcal{X}$ be an algebraic stack. Consider a cartesian diagram*

$$\begin{tikzcd} U \arrow[d] & F \arrow[l, "p"'] \arrow[d] \\ \mathcal{X} & Spec(k) \arrow[l] \end{tikzcd}$$

where U is an algebraic space, k is a field, and $U \to \mathcal{X}$ is flat and locally of finite presentation. Let $f_1, \ldots, f_r \in \Gamma(U, \mathcal{O}_U)$ and $z \in |F|$ such that $f_1, \ldots, f_r$ map to a regular sequence in the local ring $\mathcal{O}_{F,\overline{z}}$. Then, after replacing U by an open subspace containing $p(z)$, the morphism

$$V(f_1, \ldots, f_r) \longrightarrow \mathcal{X}$$

is flat and locally of finite presentation.

Proof. Choose a scheme W and a surjective smooth morphism $W \to \mathcal{X}$. Choose an extension of fields $k \subset k'$ and a morphism $w : Spec(k') \to W$ such that $Spec(k') \to W \to \mathcal{X}$ is 2-isomorphic to $Spec(k') \to Spec(k) \to \mathcal{X}$. This is possible as $W \to \mathcal{X}$ is surjective. Consider the commutative diagram

$$\begin{tikzcd} U \arrow[d] & U \times_{\mathcal{X}} W \arrow[l, "\text{pr}_0"'] \arrow[d] & F' \arrow[l, "p'"'] \arrow[d] \\ \mathcal{X} & W \arrow[l] & Spec(k') \arrow[l] \end{tikzcd}$$

both of whose squares are cartesian. By our choice of w we see that $F' = F \times_{Spec(k)} Spec(k')$. Thus $F' \to F$ is surjective and we can choose a point $z' \in |F'|$ mapping to z. Since $F' \to F$ is flat we see that $\mathcal{O}_{F,\overline{z}} \to \mathcal{O}_{F',\overline{z}'}$ is flat, see Morphisms of Spaces, Lemma 42.27.7. Hence $f_1, \ldots, f_r$ map to a regular sequence in $\mathcal{O}_{F',\overline{z}'}$, see Algebra, Lemma 7.65.7. Note that $U \times_{\mathcal{X}} W \to W$ is a morphism of algebraic spaces which is flat and locally of finite presentation. Hence by More on Morphisms of Spaces, Lemma 46.19.1 we see that there exists an open subspace U' of $U \times_{\mathcal{X}} W$ containing $p(z')$ such that the intersection $U' \cap (V(f_1, \ldots, f_r) \times_{\mathcal{X}} W)$ is flat and locally of finite presentation over W. Note that $\text{pr}_0(U')$ is an open subspace of U containing $p(z)$ as pr_0 is smooth hence open. Now we see that $U' \cap (V(f_1, \ldots, f_r) \times_{\mathcal{X}} W) \to \mathcal{X}$ is flat and locally of finite presentation as the composition

$$U' \cap (V(f_1, \ldots, f_r) \times_{\mathcal{X}} W) \to W \to \mathcal{X}.$$

Hence Properties of Stacks, Lemma 60.3.5 implies $\text{pr}_0(U') \cap V(f_1, \ldots, f_r) \to \mathcal{X}$ is flat and locally of finite presentation as desired. $\square$

Lemma 61.15.2. *Let $\mathcal{X}$ be an algebraic stack. Consider a cartesian diagram*

$$\begin{tikzcd} U \arrow[d] & F \arrow[l, "p"'] \arrow[d] \\ \mathcal{X} & Spec(k) \arrow[l] \end{tikzcd}$$

where U is an algebraic space, k is a field, and $U \to \mathcal{X}$ is locally of finite type. Let $z \in |F|$ be such that $\dim_z(F) = 0$. Then, after replacing U by an open subspace containing $p(z)$, the morphism

$$U \longrightarrow \mathcal{X}$$

is locally quasi-finite.

Proof. Since $f : U \to \mathcal{X}$ is locally of finite type there exists a maximal open $W(f) \subset U$ such that the restriction $f|_{W(f)} : W(f) \to \mathcal{X}$ is locally quasi-finite, see Properties of Stacks, Remark 60.9.19 (2). Hence all we need to do is prove that $p(z)$ is a point of $W(f)$. Moreover, the remark referenced above also shows the formation of $W(f)$ commutes with arbitrary base change by a morphism which is representable by algebraic spaces. Hence it suffices to show that the morphism $F \to Spec(k)$ is locally quasi-finite at z. This follows immediately from Morphisms of Spaces, Lemma 42.31.6. $\square$

A quasi-DM stack has a locally quasi-finite ``covering'' by a scheme.

Theorem 61.15.3. *Let $\mathcal{X}$ be an algebraic stack. The following are equivalent*

1. *$\mathcal{X}$ is quasi-DM, and*
2. *there exists a scheme W and a surjective, flat, locally finitely presented, locally quasi-finite morphism $W \to \mathcal{X}$.*

Proof. The implication (2) ⇒ (1) is Lemma 61.4.14. Assume (1). Let $x \in |\mathcal{X}|$ be a finite type point. We will produce a scheme over $\mathcal{X}$ which ``works'' in a neighbourhood of x. At the end of the proof we will take the disjoint union of all of these to conclude.

Let U be an affine scheme, $U \to \mathcal{X}$ a smooth morphism, and $u \in U$ a closed point which maps to x, see Lemma 61.14.1. Denote $u = Spec(\kappa(u))$ as usual. Consider the following commutative diagram

$$\begin{tikzcd} u \arrow[d] & R \arrow[l] \arrow[d] \\ U \arrow[d] & F \arrow[l, "p"'] \arrow[d] \\ \mathcal{X} & u \arrow[l] \end{tikzcd}$$

with both squares fibre product squares, in particular $R = u \times_{\mathcal{X}} u$. In the proof of Lemma 61.14.7 we have seen that (u, R, s, t, c) is a groupoid in algebraic spaces with s, t locally of finite type. Let $G \to u$ be the stabilizer group algebraic space (see Groupoids in Spaces, Definition 52.15.2). Note that

$$G = R \times_{(u \times u)} u = (u \times_{\mathcal{X}} u) \times_{(u \times u)} u = \mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} u.$$

As $\mathcal{X}$ is quasi-DM we see that G is locally quasi-finite over u. By More on Groupoids in Spaces, Lemma 53.7.10 we have $\dim(R) = 0$.

Let $e : u \to R$ be the identity of the groupoid. Thus both compositions $u \to R \to u$ are equal to the identity morphism of u. Note that $R \subset F$ is a closed subspace as $u \subset U$ is a closed subscheme. Hence we can also think of e as a point of F. Consider the maps of étale local rings

$$\mathcal{O}_{U,u} \xrightarrow{p^\sharp} \mathcal{O}_{F,\overline{e}} \longrightarrow \mathcal{O}_{R,\overline{e}}$$

Note that $\mathcal{O}_{R,\overline{e}}$ has dimension 0 by the result of the first paragraph. On the other hand, the kernel of the second arrow is $p^\sharp(\mathfrak{m}_u)\mathcal{O}_{F,\overline{e}}$ as R is cut out in F by $\mathfrak{m}_u$. Thus we see that

$$\mathfrak{m}_{\overline{z}} = \sqrt{p^\sharp(\mathfrak{m}_u)\mathcal{O}_{F,\overline{e}}}$$

On the other hand, as the morphism $U \to \mathcal{X}$ is smooth we see that $F \to u$ is a smooth morphism of algebraic spaces. This means that F is a regular algebraic space (Spaces over Fields, Lemma 48.5.1). Hence $\mathcal{O}_{F,\overline{e}}$ is a regular local ring (Properties of Spaces, Lemma 41.22.1). Note that a regular local ring is Cohen-Macaulay (Algebra, Lemma 7.98.3). Let $d = \dim(\mathcal{O}_{F,\overline{e}})$. By Algebra, Lemma 7.96.11 we can find $f_1, \ldots, f_d \in \mathcal{O}_{U,u}$ whose images $\varphi(f_1), \ldots, \varphi(f_d)$ form a regular sequence in $\mathcal{O}_{F,\overline{z}}$. By Lemma 61.15.1 after shrinking U we may assume that $Z = V(f_1, \ldots, f_d) \to \mathcal{X}$ is flat and locally of finite presentation. Note that by construction $F_Z = Z \times_{\mathcal{X}} u$ is a closed subspace of $F = U \times_{\mathcal{X}} u$, that e is a point of this closed subspace, and that

$$\dim(\mathcal{O}_{F_Z,\overline{e}}) = 0.$$

By Morphisms of Spaces, Lemma 42.31.1 it follows that $\dim_e(F_Z) = 0$ because the transcendece degree of e relative to u is zero. Hence it follows from Lemma 61.15.2 that after possibly shrinking U the morphism $Z \to \mathcal{X}$ is locally quasi-finite.

We conclude that for every finite type point x of $\mathcal{X}$ there exists a locally quasi-finite, flat, locally finitely presented morphism $f_x : Z_x \to \mathcal{X}$ with x in the image of $|f_x|$. Set $W = \coprod_x Z_x$ and $f = \coprod f_x$. Then f is flat, locally of finite presentation, and locally quasi-finite. In particular the image of $|f|$ is open, see Properties of Stacks, Lemma 60.4.7. By construction the image contains all finite type points of $\mathcal{X}$, hence f is surjective by Lemma 61.14.6 (and Properties of Stacks, Lemma 60.4.4). □

Lemma 61.15.4. *Let $\mathcal{Z}$ be a DM, locally Noetherian, reduced algebraic stack with $|\mathcal{Z}|$ a singleton. Then there exists a field k and a surjective étale morphism $Spec(k) \to \mathcal{Z}$.*

Proof. By Properties of Stacks, Lemma 60.11.3 there exists a field k and a surjective, flat, locally finitely presented morphism $Spec(k) \to \mathcal{Z}$. Set $U = Spec(k)$ and $R = U \times_{\mathcal{Z}} U$ so we obtain a groupoid in algebraic spaces (U, R, s, t, c), see Algebraic Stacks, Lemma 57.9.2. Note that by Algebraic Stacks, Remark 57.16.3 we have an equivalence

$$f_{can} : [U/R] \longrightarrow \mathcal{Z}$$

The projections $s, t : R \to U$ are locally of finite presentation. As $\mathcal{X}$ is DM we see that the stabilizer group algebraic space

$$G = U \times_{U \times U} R = U \times_{U \times U} (U \times_{\mathcal{X}} U) = U \times_{\mathcal{X} \times \mathcal{X}, \Delta_{\mathcal{X}}} \mathcal{X}$$

is unramified over U. In particular $\dim(G) = 0$ and by More on Groupoids in Spaces, Lemma 53.7.10 we have $\dim(R) = 0$. This implies that R is a scheme, see Spaces over Fields, Lemma 48.4.1. By Varieties, Lemma 28.13.2 we see that R (and also G) is the disjoint union of spectra of Artinian local rings finite over k via either s or t. Let $P = Spec(A) \subset R$ be the open and closed subscheme whose underlying point is the identity e of the groupoid scheme (U, R, s, t, c). As $s \circ e = t \circ e = \mathrm{id}_{Spec(k)}$ we see that A is an Artinian local ring whose residue field is identified with k via either $s^\sharp : k \to A$ or $t^\sharp : k \to A$. Note that $s, t : Spec(A) \to Spec(k)$ are finite (by the lemma referenced above). Since $G \to Spec(k)$ is unramified we see that

$$G \cap P = P \times_{U \times U} U = Spec(A \otimes_{k \otimes k} k)$$

is unramified over k. On the other hand $A \otimes_{k \otimes k} k$ is local as a quotient of A and surjects onto k. We conclude that $A \otimes_{k \otimes k} k = k$. It follows that $P \to U \times U$ is universally injective (as P has only one point with residue field k, unramified (by the computation of the fibre over the unique image point above), and of finite type (because s, t are) hence a monomorphism (see Étale Morphisms, Lemma 37.7.1). Thus $s|_P, t|_P : P \to U$ define a finite flat equivalence relation. Thus we may apply Groupoids, Proposition 35.19.8 to conclude that U/P exists and is a scheme $\overline{U}$. Moreover, $U \to \overline{U}$ is finite locally free and $P = U \times_{\overline{U}} U$. In fact $\overline{U} = Spec(k_0)$ where $k_0 \subset k$ is the ring of R-invariant functions. As k is a field it follows from the definition Groupoids, Equation (35.19.0.1) that k_0 is a field.

We claim that

$$Spec(k_0) = \overline{U} = U/P \to [U/R] = \mathcal{X} \tag{61.15.4.1}$$

is the desired surjective étale morphism. It follows from Properties of Stacks, Lemma 60.11.1 that this morphism is surjective. Thus it suffices to show that (61.15.4.1) is étale[6]. Instead of proving the étaleness directly we first apply Bootstrap, Lemma 54.9.1 to see that there exists a groupoid scheme $(\overline{U}, \overline{R}, \overline{s}, \overline{t}, \overline{c})$ such that (U, R, s, t, c) is the restriction of $(\overline{U}, \overline{R}, \overline{s}, \overline{t}, \overline{c})$ via the quotient morphism $U \to \overline{U}$. (We verified all the hypothesis of the lemma above except for the assertion that $j : R \to U \times U$ is separated and locally quasi-finite which follows from the fact that R is a separated scheme locally quasi-finite over k.) Since $U \to \overline{U}$ is finite locally free we see that $[U/R] \to [\overline{U}/\overline{R}]$ is an equivalence, see Groupoids in Spaces, Lemma 52.24.2.

Note that s, t are the base changes of the morphsms $\overline{s}, \overline{t}$ by $U \to \overline{U}$. As $\{U \to \overline{U}\}$ is an fppf covering we conclude $\overline{s}, \overline{t}$ are flat, locally of finite presentation, and locally quasi-finite, see Descent, Lemmas 31.19.13, 31.19.9, and 31.19.22. Consider the commutative diagram

$$\begin{tikzcd}
U \times_{\overline{U}} U \arrow[r, equal] \arrow[dr] & P \arrow[r] \arrow[d] & R \arrow[d] \\
 & \overline{U} \arrow[r, "\overline{e}"] & \overline{R}
\end{tikzcd}$$

[6] We urge the reader to find his/her own proof of this fact. In fact the argument has a lot in common with the final argument of the proof of Bootstrap, Theorem 54.10.1 hence probably should be isolated into its own lemma somewhere.

It is a general fact about restrictions that the outer four corners form a cartesian diagram. By the equality we see the inner square is cartesian. Since P is open in R we conclude that $\overline{e}$ is an open immersion by Descent, Lemma 31.19.14.

But of course, if $\overline{e}$ is an open immersion and $\overline{s}, \overline{t}$ are flat and locally of finite presentation then the morphisms $\overline{t}, \overline{s}$ are étale. For example you can see this by applying More on Groupoids, Lemma 36.4.1 which shows that $\Omega_{\overline{R}/\overline{U}} = 0$ implies that $\overline{s}, \overline{t} : \overline{R} \to \overline{U}$ is unramified (see Morphisms, Lemma 24.34.2), which in turn implies that $\overline{s}, \overline{t}$ are étale (see Morphisms, Lemma 24.35.16). Hence $\mathcal{Z} = [\overline{U}/\overline{R}]$ is an étale presentation of the algebraic stack $\mathcal{Z}$ and we conlude that $\overline{U} \to \mathcal{Z}$ is étale by Properties of Stacks, Lemma 60.3.3. $\square$

Lemma 61.15.5. *Let $\mathcal{X}$ be an algebraic stack. Consider a cartesian diagram*

$$\begin{tikzcd} U \arrow[d] & F \arrow[l, "p"'] \arrow[d] \\ \mathcal{X} & Spec(k) \arrow[l] \end{tikzcd}$$

where U is an algebraic space, k is a field, and $U \to \mathcal{X}$ is flat and locally of finite presentation. Let $z \in |F|$ be such that $F \to Spec(k)$ is unramified at z. Then, after replacing U by an open subspace containing $p(z)$, the morphism

$$U \longrightarrow \mathcal{X}$$

is étale.

Proof. Since $f : U \to \mathcal{X}$ is flat and locally of finite presentation there exists a maximal open $W(f) \subset U$ such that the restriction $f|_{W(f)} : W(f) \to \mathcal{X}$ is étale, see Properties of Stacks, Remark 60.9.19 (5). Hence all we need to do is prove that $p(z)$ is a point of $W(f)$. Moreover, the remark referenced above also shows the formation of $W(f)$ commutes with arbitrary base change by a morphism which is representable by algebraic spaces. Hence it suffices to show that the morphism $F \to Spec(k)$ is étale at z. Since it is flat and locally of finite presentation as a base change of $U \to \mathcal{X}$ and since $F \to Spec(k)$ is unramified at z by assumption, this follows from Morphisms of Spaces, Lemma 42.35.12. $\square$

A DM stack is a Deligne-Mumford stack.

Theorem 61.15.6. *Let $\mathcal{X}$ be an algebraic stack. The following are equivalent*

(1) *$\mathcal{X}$ is DM,*
(2) *$\mathcal{X}$ is Deligne-Mumford, and*
(3) *there exists a scheme W and a surjective étale morphism $W \to \mathcal{X}$.*

Proof. Recall that (3) is the definition of (2), see Algebraic Stacks, Definition 57.12.2. The implication (3) $\Rightarrow$ (1) is Lemma 61.4.14. Assume (1). Let $x \in |\mathcal{X}|$ be a finite type point. We will produce a scheme over $\mathcal{X}$ which ``works'' in a neighbourhood of x. At the end of the proof we will take the disjoint union of all of these to conclude.

By Lemma 61.14.7 the residual gerbe $\mathcal{Z}_x$ of $\mathcal{X}$ at x exists and $\mathcal{Z}_x \to \mathcal{X}$ is locally of finite type. By Lemma 61.4.16 the algebraic stack $\mathcal{Z}_x$ is DM. By Lemma 61.15.4 there exists a field k and a surjective étale morphism $z : Spec(k) \to \mathcal{Z}_x$. In particular the composition $x : Spec(k) \to \mathcal{X}$ is locally of finite type (by Morphisms of Spaces, Lemmas 42.22.2 and 42.35.9).

Pick a scheme U and a smooth morphism $U \to \mathcal{X}$ such that x is in the image of $|U| \to |\mathcal{X}|$. Consider the following fibre square

$$\begin{tikzcd} U \arrow[d] & F \arrow[l] \arrow[d] \\ \mathcal{X} & Spec(k) \arrow[l, "x"'] \end{tikzcd}$$

in other words $F = U \times_{\mathcal{X},x} Spec(k)$. By Properties of Stacks, Lemma 60.4.3 we see that F is nonempty. As $\mathcal{Z}_x \to \mathcal{X}$ is a monomorphism we have

$$Spec(k) \times_{z,\mathcal{Z}_x,z} Spec(k) = Spec(k) \times_{x,\mathcal{X},x} Spec(k)$$

with étale projection maps to $Spec(k)$ by construction of z. Since

$$F \times_U F = (Spec(k) \times_{\mathcal{X}} Spec(k)) \times_{Spec(k)} F$$

we see that the projections maps $F \times_U F \to F$ are étale as well. It follows that $\Delta_{F/U} : F \to F \times_U F$ is étale (see Morphisms of Spaces, Lemma 42.35.11). By Morphisms of Spaces, Lemma 42.40.2 this implies that $\Delta_{F/U}$ is an open immersion, which finally implies by Morphisms of Spaces, Lemma 42.34.9 that $F \to U$ is unramified.

Pick a nonempty affine scheme V and an étale morphism $V \to F$. (This could be avoided by working directly with F, but it seems easier to explain what's going on by doing so.) Picture

$$\begin{tikzcd} U \arrow[d] & F \arrow[l] \arrow[d] & V \arrow[l] \arrow[dl] \\ \mathcal{X} & Spec(k) \arrow[l, "x"'] & \end{tikzcd}$$

Then $V \to Spec(k)$ is a smooth morphism of schemes and $V \to U$ is an unramified morphism of schemes (see Morphisms of Spaces, Lemmas 42.33.2 and 42.34.3). Pick a closed point $v \in V$ with $k \subset \kappa(v)$ finite separable, see Varieties, Lemma 28.15.6. Let $u \in U$ be the image point. The local ring $\mathcal{O}_{V,v}$ is regular (see Varieties, Lemma 28.15.3) and the local ring homomorphism

$$\varphi : \mathcal{O}_{U,u} \longrightarrow \mathcal{O}_{V,v}$$

coming from the morphism $V \to U$ is such that $\varphi(\mathfrak{m}_u)\mathcal{O}_{V,v} = \mathfrak{m}_v$, see Morphisms, Lemma 24.34.14. Hence we can find $f_1, \ldots, f_d \in \mathcal{O}_{U,u}$ such that the images $\varphi(f_1), \ldots, \varphi(f_d)$ form a basis for $\mathfrak{m}_v/\mathfrak{m}_v^2$ over $\kappa(v)$. Since $\mathcal{O}_{V,v}$ is a regular local ring this implies that $\varphi(f_1), \ldots, \varphi(f_d)$ form a regular sequence in $\mathcal{O}_{V,v}$ (see Algebra, Lemma 7.98.3). After replacing U by an open neighbourhood of u we may assume $f_1, \ldots, f_d \in \Gamma(U, \mathcal{O}_U)$. After replacing U by a possibly even smaller open neighbourhood of u we may assume that $V(f_1, \ldots, f_d) \to \mathcal{X}$ is flat and locally of finite presentation, see Lemma 61.15.1. By construction

$$V(f_1, \ldots, f_d) \times_{\mathcal{X}} Spec(k) \longleftarrow V(f_1, \ldots, f_d) \times_{\mathcal{X}} V$$

is étale and $V(f_1, \ldots, f_d) \times_{\mathcal{X}} V$ is the closed subscheme $T \subset V$ cut out by $f_1|_V, \ldots, f_d|_V$. Hence by construction $v \in T$ and

$$\mathcal{O}_{T,v} = \mathcal{O}_{V,v}/(\varphi(f_1), \ldots, \varphi(f_d)) = \kappa(v)$$

a finite separable extension of k. It follows that $T \to Spec(k)$ is unramified at v, see Morphisms, Lemma 24.34.14. By definition of an unramified morphism of algebraic spaces this means that $V(f_1, \ldots, f_d) \times_{\mathcal{X}} Spec(k) \to Spec(k)$ is unramified at the image of v in

$V(f_1, \ldots, f_d) \times_{\mathcal{X}} Spec(k)$. Applying Lemma 61.15.5 we see that on shrinking U to yet another open neighbourhood of u the morphism $V(f_1, \ldots, f_d) \to \mathcal{X}$ is étale.

We conclude that for every finite type point x of $\mathcal{X}$ there exists an étale morphism $f_x : W_x \to \mathcal{X}$ with x in the image of $|f_x|$. Set $W = \coprod_x W_x$ and $f = \coprod f_x$. Then f is étale. In particular the image of $|f|$ is open, see Properties of Stacks, Lemma 60.4.7. By construction the image contains all finite type points of $\mathcal{X}$, hence f is surjective by Lemma 61.14.6 (and Properties of Stacks, Lemma 60.4.4). □

61.16. Quasi-finite morphisms

The property ``locally quasi-finite" of morphisms of algebraic spaces is not smooth local on the source-and-target so we cannot use the material in Section 61.12 to define locally quasi-finite morphisms of algebraic stacks. We do already know what it means for a morphism of algebraic stacks representable by algebraic spaces to be locally quasi-finite, see Properties of Stacks, Section 60.3. To find a condition suitable for general morphisms we make the following observation.

Lemma 61.16.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Assume f is representable by algebraic spaces. The following are equivalent*

(1) *f is locally quasi-finite, and*
(2) *f is locally of finite type and for every morphism $Spec(k) \to \mathcal{Y}$ where k is a field the space $|Spec(k) \times_{\mathcal{Y}} \mathcal{X}|$ is discrete.*

Proof. Assume (1). In this case the morphism of algebraic spaces $\mathcal{X}_k \to Spec(k)$ is locally quasi-finite as a base change of f. Hence $|\mathcal{X}_k|$ is discrete by Morphisms of Spaces, Lemma 42.25.4. Conversely, assume (2). Pick a surjective smooth morphism $V \to \mathcal{Y}$ where V is a scheme. It suffices to show that the morphism of algebraic spaces $V \times_{\mathcal{Y}} \mathcal{X} \to V$ is locally quasi-finite, see Properties of Stacks, Lemma 60.3.3. The morphism $V \times_{\mathcal{Y}} \mathcal{X} \to V$ is locally of finite type by assumption. For any morphism $Spec(k) \to V$ where k is a field

$$Spec(k) \times_V (V \times_{\mathcal{Y}} \mathcal{X}) = Spec(k) \times_{\mathcal{Y}} \mathcal{X}$$

has a discrete space of points by assumption. Hence we conclude that $V \times_{\mathcal{Y}} \mathcal{X} \to V$ is locally quasi-finite by Morphisms of Spaces, Lemma 42.25.4. □

A morphism of algebraic stacks which is representable by algebraic spaces is quasi-DM, see Lemma 61.4.3. Combined with the lemma above we see that the following definition does not conflict with all of the already existing notion in the case of morphisms representable by algebraic spaces.

Definition 61.16.2. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is *locally quasi-finite* if f is quasi-DM, locally of finite type, and for every morphism $Spec(k) \to \mathcal{Y}$ where k is a field the space $|\mathcal{X}_k|$ is discrete.

The condition that f be quasi-DM is natural. For example, let k be a field and consider the morphism $\pi : [Spec(k)/\mathbf{G}_m] \to Spec(k)$ which has singleton fibres and is locally of finite type. As we will see later this morphism is smooth of relative dimension -1, and we'd like our locally quasi-finite morphisms to have relative dimension 0. Also, note that the section $Spec(k) \to [Spec(k)/\mathbf{G}_m]$ does not have discrete fibres, hence is not locally quasi-finite, and we'd like to have the following permanence property for locally quasi-finite morphisms: If $f : \mathcal{X} \to \mathcal{X}'$ is a morphism of algebraic stacks locally quasi-finite over the algebraic stack $\mathcal{Y}$, then f is locally quasi-finite (in fact something a bit stronger holds, see Lemma 61.16.8).

Another justification for the definition above is Lemma 61.16.7 below which characterizes being locally quasi-finite in terms of the existence of suitable ``presentations'' or ``coverings'' of $\mathcal{X}$ and $\mathcal{Y}$.

Lemma 61.16.3. *A base change of a locally quasi-finite morphism is locally quasi-finite.*

Proof. We have seen this for quasi-DM morphisms in Lemma 61.4.4 and for locally finite type morphisms in Lemma 61.13.3. It is immediate that the condition on fibres is inherited by a base change. $\square$

Lemma 61.16.4. *Let $\mathcal{X} \to Spec(k)$ be a locally quasi-finite morphism where $\mathcal{X}$ is an algebraic stack and k is a field. Let $f : V \to \mathcal{X}$ be a locally quasi-finite morphism where V is a scheme. Then $V \to Spec(k)$ is locally quasi-finite.*

Proof. By Lemma 61.13.2 we see that $V \to Spec(k)$ is locally of finite type. Assume, to get a contradiction, that $V \to Spec(k)$ is not locally quasi-finite. Then there exists a nontrivial specialization $v \leadsto v'$ of points of V, see Morphisms, Lemma 24.19.6. In particular $\text{trdeg}_k(\kappa(v)) > \text{trdeg}_k(\kappa(v'))$, see Morphisms, Lemma 24.27.6. Because $|\mathcal{X}|$ is discrete we see that $|f|(v) = |f|(v')$. Consider $R = V \times_{\mathcal{X}} V$. Then R is an algebraic space and the projections $s, t : R \to V$ are locally quasi-finite as base changes of $V \to \mathcal{X}$ (which is representable by algebraic spaces so this follows from the discussion in Properties of Stacks, Section 60.3). By Properties of Stacks, Lemma 60.4.3 we see that there exists an $r \in |R|$ such that $s(r) = v$ and $t(r) = v'$. By Morphisms of Spaces, Lemma 42.30.3 we see that the transcendence degree of v/k is equal to the transcendence degree of r/k is equal to the transcendence degree of v'/k. This contradiction proves the lemma. $\square$

Lemma 61.16.5. *A composition of a locally quasi-finite morphisms is locally quasi-finite.*

Proof. We have seen this for quasi-DM morphisms in Lemma 61.4.10 and for locally finite type morphisms in Lemma 61.13.2. Let $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Y} \to \mathcal{Z}$ be locally quasi-finite. Let k be a field and let $Spec(k) \to \mathcal{Z}$ be a morphism. It suffices to show that $|\mathcal{X}_k|$ is discrete. By Lemma 61.16.3 the morphisms $\mathcal{X}_k \to \mathcal{Y}_k$ and $\mathcal{Y}_k \to Spec(k)$ are locally quasi-finite. In particular we see that $\mathcal{Y}_k$ is a quasi-DM algebraic stack, see Lemma 61.4.13. By Theorem 61.15.3 we can find a scheme V and a surjective, flat, locally finitely presented, locally quasi-finite morphism $V \to \mathcal{Y}_k$. By Lemma 61.16.4 we see that V is locally quasi-finite over k, in particular $|V|$ is discrete. The morphism $V \times_{\mathcal{Y}_k} \mathcal{X}_k \to \mathcal{X}_k$ is surjective, flat, and locally of finite presentation hence $|V \times_{\mathcal{Y}_k} \mathcal{X}_k| \to |\mathcal{X}_k|$ is surjective and open. Thus it suffices to show that $|V \times_{\mathcal{Y}_k} \mathcal{X}_k|$ is discrete. Note that V is a disjoint union of spectra of Artinian local k-algebras A_i with residue fields k_i, see Varieties, Lemma 28.13.2. Thus it suffices to show that each

$$|\, Spec(A_i) \times_{\mathcal{Y}_k} \mathcal{X}_k| = |\, Spec(k_i) \times_{\mathcal{Y}_k} \mathcal{X}_k| = |\, Spec(k_i) \times_{\mathcal{Y}} \mathcal{X}|$$

is discrete, which follows from the assumption that $\mathcal{X} \to \mathcal{Y}$ is locally quasi-finite. $\square$

Before we characterize locally quasi-finite morphisms in terms of coverings we do it for quasi-DM morphisms.

Lemma 61.16.6. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent*

(1) *f is quasi-DM,*
(2) *for any morphism $V \to \mathcal{Y}$ with V an algebraic space there exists a surjective, flat, locally finitely presented, locally quasi-finite morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$ where U is an algebraic space, and*

(3) *there exist algebraic spaces U, V and a morphism $V \to \mathcal{Y}$ which is surjective, flat, and locally of finite presentation, and a morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$ which is surjective, flat, locally of finite presentation, and locally quasi-finite.*

Proof. The implication (2) $\Rightarrow$ (3) is immediate.

Assume (1) and let $V \to \mathcal{Y}$ be as in (2). Then $\mathcal{X} \times_{\mathcal{Y}} V \to V$ is quasi-DM, see Lemma 61.4.4. By Lemma 61.4.3 the algebraic space V is DM, hence quasi-DM. Thus $\mathcal{X} \times_{\mathcal{Y}} V$ is quasi-DM by Lemma 61.4.11. Hence we may apply Theorem 61.15.3 to get the morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$ as in (2).

Assume (3). Let $V \to \mathcal{Y}$ and $U \to \mathcal{X} \times_{\mathcal{Y}} V$ be as in (3). To prove that f is quasi-DM it suffices to show that $\mathcal{X} \times_{\mathcal{Y}} V \to V$ is quasi-DM, see Lemma 61.4.5. By Lemma 61.4.14 we see that $\mathcal{X} \times_{\mathcal{Y}} V$ is quasi-DM. Hence $\mathcal{X} \times_{\mathcal{Y}} V \to V$ is quasi-DM by Lemma 61.4.13 and (1) holds. This finishes the proof of the lemma. $\square$

Lemma 61.16.7. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent*

(1) *f is locally quasi-finite,*
(2) *f is quasi-DM and for any morphism $V \to \mathcal{Y}$ with V an algebraic space and any locally quasi-finite morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$ where U is an algebraic space the morphism $U \to V$ is locally quasi-finite,*
(3) *for any morphism $V \to \mathcal{Y}$ from an algebraic space V there exists a surjective, flat, locally finitely presented, and locally quasi-finite morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$ where U is an algebraic space such that $U \to V$ is locally quasi-finite,*
(4) *there exists algebraic spaces U, V, a surjective, flat, and locally of finitely presented morphism $V \to \mathcal{Y}$, and a morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$ which is surjective, flat, locally of finite presentation, and locally quasi-finite such that $U \to V$ is locally quasi-finite.*

Proof. Assume (1). Then f is quasi-DM by assumption. Let $V \to \mathcal{Y}$ and $U \to \mathcal{X} \times_{\mathcal{Y}} V$ be as in (2). By Lemma 61.16.5 the composition $U \to \mathcal{X} \times_{\mathcal{Y}} V \to V$ is locally quasi-finite. Thus (1) implies (2).

Assume (2). Let $V \to \mathcal{Y}$ be as in (3). By Lemma 61.16.6 we can find an algebraic space U and a surjective, flat, locally finitely presented, locally quasi-finite morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$. By (2) the composition $U \to V$ is locally quasi-finite. Thus (2) implies (3).

It is immediate that (3) implies (4).

Assume (4). We will prove (1) holds, which finishes the proof. By Lemma 61.16.6 we see that f is quasi-DM. To prove that f is locally of finite type it suffices to prove that $g : \mathcal{X} \times_{\mathcal{Y}} V \to V$ is locally of finite type, see Lemma 61.13.6. Then it suffices to check that g precomposed with $h : U \to \mathcal{X} \times_{\mathcal{Y}} V$ is locally of finite type, see Lemma 61.13.7. Since $g \circ h : U \to V$ was assumed to be locally quasi-finite this holds, hence f is locally of finite type. Finally, let k be a field and let $Spec(k) \to \mathcal{Y}$ be a morphism. Then $V \times_{\mathcal{Y}} Spec(k)$ is a nonempty algebraic space which is locally of finite presentation over k. Hence we can find a finite extension $k \subset k'$ and a morphism $Spec(k') \to V$ such that

$$\begin{tikzcd} Spec(k') \arrow[r] \arrow[d] & V \arrow[d] \\ Spec(k) \arrow[r] & \mathcal{Y} \end{tikzcd}$$

commutes (details omitted). Then $\mathcal{X}_{k'} \to \mathcal{X}_k$ is representable (by schemes), surjective, and finite locally free. In particular $|\mathcal{X}_{k'}| \to |\mathcal{X}_k|$ is surjective and open. Thus it suffices to prove that $|\mathcal{X}_{k'}|$ is discrete. Since

$$U \times_V Spec(k') = U \times_{\mathcal{X} \times_{\mathcal{Y}} V} \mathcal{X}_{k'}$$

we see that $U \times_V Spec(k') \to \mathcal{X}_{k'}$ is surjective, flat, and locally of finite presentation (as a base change of $U \to \mathcal{X} \times_{\mathcal{Y}} V$). Hence $|U \times_V Spec(k')| \to |\mathcal{X}_{k'}|$ is surjective and open. Thus it suffices to show that $|U \times_V Spec(k')|$ is discrete. This follows from the fact that $U \to V$ is locally quasi-finite (either by our definition above or from the original definition for morphisms of algebraic spaces, via Morphisms of Spaces, Lemma 42.25.4). □

Lemma 61.16.8. *Let $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. Assume that $\mathcal{X} \to \mathcal{Z}$ is locally quasi-finite and $\mathcal{Y} \to \mathcal{Z}$ is quasi-DM. Then $\mathcal{X} \to \mathcal{Y}$ is locally quasi-finite.*

Proof. Write $\mathcal{X} \to \mathcal{Y}$ as the composition

$$\mathcal{X} \longrightarrow \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \longrightarrow \mathcal{Y}$$

The second arrow is locally quasi-finite as a base change of $\mathcal{X} \to \mathcal{Z}$, see Lemma 61.16.3. The first arrow is locally quasi-finite by Lemma 61.4.8 as $\mathcal{Y} \to \mathcal{Z}$ is quasi-DM. Hence $\mathcal{X} \to \mathcal{Y}$ is locally quasi-finite by Lemma 61.16.5. □

61.17. Flat morphisms

The property ``being flat'' of morphisms of algebraic spaces is smooth local on the source-and-target, see Descent on Spaces, Remark 45.18.5. It is also stable under base change and fpqc local on the target, see Morphisms of Spaces, Lemma 42.27.3 and Descent on Spaces, Lemma 45.10.11. Hence, by Lemma 61.12.1 above, we may define what it means for a morphism of algebraic spaces to be flat as follows and it agrees with the already existing notion defined in Properties of Stacks, Section 60.3 when the morphism is representable by algebraic spaces.

Definition 61.17.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is *flat* if the equivalent conditions of Lemma 61.12.1 hold with $\mathcal{P}=$ flat.

Lemma 61.17.2. *The composition of flat morphisms is flat.*

Proof. Combine Remark 61.12.3 with Morphisms of Spaces, Lemma 42.27.2. □

Lemma 61.17.3. *A base change of a flat morphism is flat.*

Proof. Combine Remark 61.12.4 with Morphisms of Spaces, Lemma 42.27.3. □

Lemma 61.17.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $\mathcal{Z} \to \mathcal{Y}$ be a surjective flat morphism of algebraic stacks. If the base change $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z}$ is flat, then f is flat.*

Proof. Choose an algebraic space W and a surjective smooth morphism $W \to \mathcal{Z}$. Then $W \to \mathcal{Z}$ is surjective and flat (Morphisms of Spaces, Lemma 42.33.7) hence $W \to \mathcal{Y}$ is surjective and flat (by Properties of Stacks, Lemma 60.5.2 and Lemma 61.17.2). Since the base change of $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z}$ by $W \to \mathcal{Z}$ is a flat morphism (Lemma 61.17.3) we may replace $\mathcal{Z}$ by W.

Choose an algebraic space V and a surjective smooth morphism $V \to \mathcal{Y}$. Choose an algebraic space U and a surjective smooth morphism $U \to V \times_{\mathcal{Y}} \mathcal{X}$. We have to show that $U \to V$ is flat. Now we base change everything by $W \to \mathcal{Y}$: Set $U' = W \times_{\mathcal{Y}} U$, $V' = W \times_{\mathcal{Y}} V$,

$\mathcal{X}' = W \times_{\mathcal{Y}} \mathcal{X}$, and $\mathcal{Y}' = W \times_{\mathcal{Y}} \mathcal{Y} = W$. Then it is still true that $U' \to V' \times_{\mathcal{Y}'} \mathcal{X}'$ is smooth by base change. Hence by our definition of flat morphisms of algebraic stacks and the assumption that $\mathcal{X}' \to \mathcal{Y}'$ is flat, we see that $U' \to V'$ is flat. Then, since $V' \to V$ is surjective as a base change of $W \to \mathcal{Y}$ we see that $U \to V$ is flat by Morphisms of Spaces, Lemma 42.28.3 (2) and we win. □

Lemma 61.17.5. *Let $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. If $\mathcal{X} \to \mathcal{Z}$ is flat and $\mathcal{X} \to \mathcal{Y}$ is surjective and flat, then $\mathcal{Y} \to \mathcal{Z}$ is flat.*

Proof. Choose an algebraic space W and a surjective smooth morphism $W \to \mathcal{Z}$. Choose an algebraic space V and a surjective smooth morphism $V \to W \times_{\mathcal{Z}} \mathcal{Y}$. Choose an algebraic space U and a surjective smooth morphism $U \to V \times_{\mathcal{Y}} \mathcal{X}$. We know that $U \to V$ is flat and that $U \to W$ is flat. Also, as $\mathcal{X} \to \mathcal{Y}$ is surjective we see that $U \to V$ is surjective (as a composition of surjective morphisms). Hence the lemma reduces to the case of morphisms of algebraic spaces. The case of morphisms of algebraic spaces is Morphisms of Spaces, Lemma 42.28.5. □

61.18. Morphisms of finite presentation

The property ``locally of finite presentation'' of morphisms of algebraic spaces is smooth local on the source-and-target, see Descent on Spaces, Remark 45.18.5. It is also stable under base change and fpqc local on the target, see Morphisms of Spaces, Lemma 42.26.3 and Descent on Spaces, Lemma 45.10.8. Hence, by Lemma 61.12.1 above, we may define what it means for a morphism of algebraic spaces to be locally of finite presentation as follows and it agrees with the already existing notion defined in Properties of Stacks, Section 60.3 when the morphism is representable by algebraic spaces.

Definition 61.18.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.

(1) We say f *locally of finite presentation* if the equivalent conditions of Lemma 61.12.1 hold with $\mathcal{P}$= locally of finite presentation.
(2) We say f is *of finite presentation* if it is locally of finite presentation, quasi-compact, and quasi-separated.

Note that a morphism of finite presentation is **not** just a quasi-compact morphism which is locally of finite presentation.

Lemma 61.18.2. *The composition of finitely presented morphisms is of finite presentation. The same holds for morphisms which are locally of finite presentation.*

Proof. Combine Remark 61.12.3 with Morphisms of Spaces, Lemma 42.26.2. □

Lemma 61.18.3. *A base change of a finitely presented morphism is of finite presentation. The same holds for morphisms which are locally of finite presentation.*

Proof. Combine Remark 61.12.4 with Morphisms of Spaces, Lemma 42.26.3. □

Lemma 61.18.4. *A morphism which is locally of finite presentation is locally of finite type. A morphism of finite presentation is of finite type.*

Proof. Combine Remark 61.12.5 with Morphisms of Spaces, Lemma 42.26.5. □

Lemma 61.18.5. *Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks If $g \circ f$ is locally of finite presentation and g is locally of finite type, then f is locally of finite presentation.*

Proof. Choose an algebraic space W and a surjective smooth morphism $W \to \mathcal{Z}$. Choose an algebraic space V and a surjective smooth morphism $V \to \mathcal{Y} \times_{\mathcal{Z}} W$. Choose an algebraic space U and a surjective smooth morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$. The lemma follows upon applying Morphisms of Spaces, Lemma 42.26.9 to the morphisms $U \to V \to W$. □

Lemma 61.18.6. *An open immersion is locally of finite presentation.*

Proof. Follows from Morphisms of Spaces, Lemma 42.26.10. □

Lemma 61.18.7. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $\mathcal{Z} \to \mathcal{Y}$ be a surjective, flat, locally finitely presented morphism of algebraic stacks. If the base change $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z}$ is locally of finite presentation, then f is locally of finite presentation.*

Proof. Choose an algebraic space W and a surjective smooth morphism $W \to \mathcal{Z}$. Then $W \to \mathcal{Z}$ is surjective, flat, and locally of finite presentation (Morphisms of Spaces, Lemmas 42.33.7 and 42.33.5) hence $W \to \mathcal{Y}$ is surjective, flat, and locally of finite presentation (by Properties of Stacks, Lemma 60.5.2 and Lemmas 61.17.2 and 61.18.2). Since the base change of $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z}$ by $W \to \mathcal{Z}$ is locally of finite presentation (Lemma 61.17.3) we may replace $\mathcal{Z}$ by W.

Choose an algebraic space V and a surjective smooth morphism $V \to \mathcal{Y}$. Choose an algebraic space U and a surjective smooth morphism $U \to V \times_{\mathcal{Y}} \mathcal{X}$. We have to show that $U \to V$ is locally of finite presentation. Now we base change everything by $W \to \mathcal{Y}$: Set $U' = W \times_{\mathcal{Y}} U$, $V' = W \times_{\mathcal{Y}} V$, $\mathcal{X}' = W \times_{\mathcal{Y}} \mathcal{X}$, and $\mathcal{Y}' = W \times_{\mathcal{Y}} \mathcal{Y} = W$. Then it is still true that $U' \to V' \times_{\mathcal{Y}'} \mathcal{X}'$ is smooth by base change. Hence by our definition of locally finitely presented morphisms of algebraic stacks and the assumption that $\mathcal{X}' \to \mathcal{Y}'$ is locally of finite presentation, we see that $U' \to V'$ is locally of finite presentation. Then, since $V' \to V$ is surjective, flat, and locally of finite presentation as a base change of $W \to \mathcal{Y}$ we see that $U \to V$ is locally of finite presentation by Descent on Spaces, Lemma 45.10.8 and we win. □

Lemma 61.18.8. *Let $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. If $\mathcal{X} \to \mathcal{Z}$ is locally of finite presentation and $\mathcal{X} \to \mathcal{Y}$ is surjective, flat, and locally of finite presentation, then $\mathcal{Y} \to \mathcal{Z}$ is locally of finite presentation.*

Proof. Choose an algebraic space W and a surjective smooth morphism $W \to \mathcal{Z}$. Choose an algebraic space V and a surjective smooth morphism $V \to W \times_{\mathcal{Z}} \mathcal{Y}$. Choose an algebraic space U and a surjective smooth morphism $U \to V \times_{\mathcal{Y}} \mathcal{X}$. We know that $U \to V$ is flat and locally of finite presentation and that $U \to W$ is locally of finite presentation. Also, as $\mathcal{X} \to \mathcal{Y}$ is surjective we see that $U \to V$ is surjective (as a composition of surjective morphisms). Hence the lemma reduces to the case of morphisms of algebraic spaces. The case of morphisms of algebraic spaces is Descent on Spaces, Lemma 45.14.1. □

Lemma 61.18.9. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is surjective, flat, and locally of finite presentation. Then for every scheme U and object y of $\mathcal{Y}$ over U there exists an fppf covering $\{U_i \to U\}$ and objects x_i of $\mathcal{X}$ over U_i such that $f(x_i) \cong y|_{U_i}$ in $\mathcal{Y}_{U_i}$.*

Proof. We may think of y as a morphism $U \to \mathcal{Y}$. By Properties of Stacks, Lemma 60.5.3 and Lemmas 61.18.3 and 61.17.3 we see that $\mathcal{X} \times_{\mathcal{Y}} U \to U$ is surjective, flat, and locally of finite presentation. Let V be a scheme and let $V \to \mathcal{X} \times_{\mathcal{Y}} U$ smooth and surjective. Then $V \to \mathcal{X} \times_{\mathcal{Y}} U$ is also surjective, flat, and locally of finite presentation (see Morphisms of Spaces, Lemmas 42.33.7 and 42.33.5). Hence also $V \to U$ is surjective, flat, and locally of finite presentation, see Properties of Stacks, Lemma 60.5.2 and Lemmas 61.18.2, and 61.17.2. Hence $\{V \to U\}$ is the desired fppf covering and $x : V \to \mathcal{X}$ is the desired object. $\square$

Lemma 61.18.10. *Let $f_j : \mathcal{X}_j \to \mathcal{X}$, $j \in J$ be a family of morphisms of algebraic stacks which are each flat and locally of finite presentation and which are jointly surjective, i.e., $|\mathcal{X}| = \bigcup |\mathcal{X}_i|$. Then for every scheme U and object x of $\mathcal{X}$ over U there exists an fppf covering $\{U_i \to U\}_{i\in I}$, a map $a : I \to J$, and objects x_i of $\mathcal{X}_{a(i)}$ over U_i such that $f_{a(i)}(x_i) \cong y|_{U_i}$ in $\mathcal{X}_{U_i}$.*

Proof. Apply Lemma 61.18.9 to the morphism $\coprod_{j\in J} \mathcal{X}_j \to \mathcal{X}$. (There is a slight set theoretic issue here -- due to our setup of things -- which we ignore.) To finish, note that a morphism $x_i : U_i \to \coprod_{j\in J} \mathcal{X}_j$ is given by a disjoint union decomposition $U_i = \coprod U_{i,j}$ and morphisms $U_{i,j} \to \mathcal{X}_j$. Then the fppf covering $\{U_{i,j} \to U\}$ and the morphisms $U_{i,j} \to \mathcal{X}_j$ do the job. $\square$

Lemma 61.18.11. *Let $f : \mathcal{X} \to \mathcal{Y}$ be flat and locally of finite presentation. Then $|f| : |\mathcal{X}| \to |\mathcal{Y}|$ is open.*

Proof. Choose a scheme V and a smooth surjective morphism $V \to \mathcal{Y}$. Choose a scheme U and a smooth surjective morphism $U \to V \times_{\mathcal{Y}} \mathcal{X}$. By assumption the morphism of schemes $U \to V$ is flat and locally of finite presentation. Hence $U \to V$ is open by Morphisms, Lemma 24.24.9. By construction of the topology on $|\mathcal{Y}|$ the map $|V| \to |\mathcal{Y}|$ is open. The map $|U| \to |\mathcal{X}|$ is surjective. The result follows from these facts by elementary topology. $\square$

61.19. Gerbes

An important type of algebraic stack are the stacks of the form $[B/G]$ where B is an algebraic space and G is a flat and locally finitely presented group algebraic space over B (acting trivially on B), see Criteria for Representability, Lemma 59.18.3. It turns out that an algebraic stack is a gerbe when it locally in the fppf topology is of this form, see Lemma 61.19.8. In this section we briefly discuss this notion and the corresponding relative notion.

Definition 61.19.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say $\mathcal{X}$ is a *gerbe over $\mathcal{Y}$* if $\mathcal{X}$ is a gerbe over $\mathcal{Y}$ as stacks in groupoids over $(Sch/S)_{fppf}$, see Stacks, Definition 50.11.4. We say an algebraic stack $\mathcal{X}$ is a *gerbe* if there exists a morphism $\mathcal{X} \to X$ where X is an algebraic space which turns $\mathcal{X}$ into a gerbe over X.

The condition that $\mathcal{X}$ be a gerbe over $\mathcal{Y}$ is defined purely in terms of the topology and category theory underlying the given algebraic stacks; but as we will see later this condition has geometric consequences. For example it implies that $\mathcal{X} \to \mathcal{Y}$ is surjective, flat, and locally of finite presentation, see Lemma 61.19.7. The absolute notion is trickier to parse, because it may not be at first clear that X is well determined. Actually, it is.

Lemma 61.19.2. *Let $\mathcal{X}$ be an algebraic stack. If $\mathcal{X}$ is a gerbe, then the sheafification of the presheaf*

$$(Sch/S)_{fppf}^{opp} \to Sets, \quad U \mapsto Ob(\mathcal{X}_U)/\cong$$

is an algebraic space and $\mathcal{X}$ is a gerbe over it.

Proof. (In this proof the abuse of language introduced in Section 61.2 really pays off.) Choose a morphism $\pi : \mathcal{X} \to X$ where X is an algebraic space which turns $\mathcal{X}$ into a gerbe over X. It suffices to prove that X is the sheafification of the presheaf $\mathcal{F}$ displayed in the lemma. It is clear that there is a map $c : \mathcal{F} \to X$. We will use Stacks, Lemma 50.11.3 properties (2)(a) and (2)(b) to see that the map $c^\# : \mathcal{F}^\# \to X$ is surjective and injective, hence an isomorphism, see Sites, Lemma 9.11.2. Surjective: Let T be a scheme and let $f : T \to X$. By property (2)(a) there exists an fppf covering $\{h_i : T_i \to T\}$ and morphisms $x_i : T_i \to \mathcal{X}$ such that $f \circ h_i$ corresponds to $\pi \circ x_i$. Hence we see that $f|_{T_i}$ is in the image of c. Injective: Let T be a scheme and let $x, y : T \to \mathcal{X}$ be morphisms such that $c \circ x = c \circ y$. By (2)(b) we can find a covering $\{T_i \to T\}$ and morphisms $x|_{T_i} \to y|_{T_i}$ in the fibre category $\mathcal{X}_{T_i}$. Hence the restrictions $x|_{T_i}, y|_{T_i}$ are equal in $\mathcal{F}(T_i)$. This proves that x, y give the same section of $\mathcal{F}^\#$ over T as desired. □

Lemma 61.19.3. *Let*

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \mathcal{Y}' \arrow[r] & \mathcal{Y} \end{tikzcd}$$

be a fibre product of algebraic stacks. If $\mathcal{X}$ is a gerbe over $\mathcal{Y}$, then $\mathcal{X}'$ is a gerbe over $\mathcal{Y}'$.

Proof. Immediate from the definitions and Stacks, Lemma 50.11.5. □

Lemma 61.19.4. *Let $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. If $\mathcal{X}$ is a gerbe over $\mathcal{Y}$ and $\mathcal{Y}$ is a gerbe over $\mathcal{Z}$, then $\mathcal{X}$ is a gerbe over $\mathcal{Z}$.*

Proof. Immediate from Stacks, Lemma 50.11.6. □

Lemma 61.19.5. *Let*

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \mathcal{Y}' \arrow[r] & \mathcal{Y} \end{tikzcd}$$

be a fibre product of algebraic stacks. If $\mathcal{Y}' \to \mathcal{Y}$ is surjective, flat, and locally of finite presentation and $\mathcal{X}'$ is a gerbe over $\mathcal{Y}'$, then $\mathcal{X}$ is a gerbe over $\mathcal{Y}$.

Proof. Follows immediately from Lemma 61.18.9 and Stacks, Lemma 50.11.7. □

Lemma 61.19.6. *Let $\pi : \mathcal{X} \to U$ be a morphism from an algebraic stack to an algebraic space and let $x : U \to \mathcal{X}$ be a section of π. Set $G = Isom_{\mathcal{X}}(x, x)$, see Definition 61.5.3. If $\mathcal{X}$ is a gerbe over U, then*

(1) *there is a canonical equivalence of stacks in groupoids*

$$x_{can} : [U/G] \longrightarrow \mathcal{X}.$$

where $[U/G]$ is the quotient stack for the trivial action of G on U,

(2) *$G \to U$ is flat and locally of finite presentation, and*

(3) *$U \to \mathcal{X}$ is surjective, flat, and locally of finite presentation.*

Proof. Set $R = U \times_{x,\mathcal{X},x} U$. The morphism $R \to U \times U$ factors through the diagonal $\Delta_U : U \to U \times U$ as it factors through $U \times_U U = U$. Hence $R = G$ because

$$
\begin{aligned}
G &= Isom_{\mathcal{X}}(x, x) \\
&= U \times_{x,\mathcal{X}} \mathcal{I}_{\mathcal{X}} \\
&= U \times_{x,\mathcal{X}} (\mathcal{X} \times_{\Delta,\mathcal{X}\times_S\mathcal{X},\Delta} \mathcal{X}) \\
&= (U \times_{x,\mathcal{X},x} U) \times_{U\times U,\Delta_U} U \\
&= R \times_{U\times U,\Delta_U} U \\
&= R
\end{aligned}
$$

for the fourth equality use Categories, Lemma 4.28.12. Let $t, s : R \to U$ be the projections. The composition law $c : R \times_{s,U,t} R \to R$ constructed on R in Algebraic Stacks, Lemma 57.16.1 agrees with the group law on G (proof omitted). Thus Algebraic Stacks, Lemma 57.16.1 shows we obtain a canonical fully faithful 1-morphism

$$x_{can} : [U/G] \longrightarrow \mathcal{X}$$

of stacks in groupoids over $(Sch/S)_{fppf}$. To see that it is an equivalence it suffices to show that it is essentially surjective. To do this it suffices to show that any object of $\mathcal{X}$ over a scheme T comes fppf locally from x via a morphism $T \to U$, see Stacks, Lemma 50.4.8. However, this follows the condition that π turns $\mathcal{X}$ into a gerbe over X, see property (2)(a) of Stacks, Lemma 50.11.3.

By Criteria for Representability, Lemma 59.18.3 we conclude that $G \to U$ is flat and locally of finite presentation. Finally, $U \to \mathcal{X}$ is surjective, flat, and locally of finite presentation by Criteria for Representability, Lemma 59.17.1. □

Lemma 61.19.7. *Let $\pi : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent*

(1) *$\mathcal{X}$ is a gerbe over $\mathcal{Y}$, and*
(2) *there exists an algebraic space U, a group algebraic space G flat and locally of finite presentation over U, and a surjective, flat, and locally finitely presented morphism $U \to \mathcal{Y}$ such that $\mathcal{X} \times_{\mathcal{Y}} U \cong [U/G]$ over U.*

Proof. Assume (2). By Lemma 61.19.5 to prove (1) it suffices to show that $[U/G]$ is a gerbe over U. This is immediate from Groupoids in Spaces, Lemma 52.26.2.

Assume (1). Any base change of π is a gerbe, see Lemma 61.19.3. As a first step we choose a scheme V and a surjective smooth morphism $V \to \mathcal{Y}$. Thus we may assume that $\pi : \mathcal{X} \to V$ is a gerbe over a scheme. This means that there exists an fppf covering $\{V_i \to V\}$ such that the fibre category $\mathcal{X}_{V_i}$ is nonempty, see Stacks, Lemma 50.11.3 (2)(a). Note that $U = \coprod V_i \to U$ is surjective, flat, and locally of finite presentation. Hence we may replace V by U and assume that $\pi : \mathcal{X} \to U$ is a gerbe over a scheme U and that there exists an object x of $\mathcal{X}$ over U. By Lemma 61.19.6 we see that $\mathcal{X} = [U/G]$ over U for some flat and locally finitely presented group algebraic space G over U. □

Lemma 61.19.8. *Let $\pi : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If $\mathcal{X}$ is a gerbe over $\mathcal{Y}$, then π is surjective, flat, and locally of finite presentation.*

Proof. By Properties of Stacks, Lemma 60.5.4 and Lemmas 61.17.4 and 61.18.7 it suffices to prove to the lemma after replacing π by a base change with a surjective, flat, locally finitely presented morphism $\mathcal{Y}' \to \mathcal{Y}$. By Lemma 61.19.7 we may assume $\mathcal{Y} = U$ is an algebraic space and $\mathcal{X} = [U/G]$ over U. Then $U \to [U/G]$ is surjective, flat, and locally of finite presentation, see Lemma 61.19.6. This implies that π is surjective, flat, and locally of finite presentation by Properties of Stacks, Lemma 60.5.5 and Lemmas 61.17.5 and 61.18.8. ☐

Proposition 61.19.9. *Let $\mathcal{X}$ be an algebraic stack. The following are equivalent*

(1) *$\mathcal{X}$ is a gerbe, and*
(2) *$\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is flat and locally of finite presentation.*

Proof. Assume (1). Choose a morphism $\mathcal{X} \to X$ into an algebraic space X which turns $\mathcal{X}$ into a gerbe over X. Let $X' \to X$ is a surjective, flat, locally finitely presented morphism and set $\mathcal{X}' = X' \times_X \mathcal{X}$. Note that $\mathcal{X}'$ is a gerbe over X' by Lemma 61.19.3. Then both squares in

$$\begin{tikzcd} \mathcal{I}_{\mathcal{X}'} \arrow[r] \arrow[d] & \mathcal{X}' \arrow[r] \arrow[d] & X' \arrow[d] \\ \mathcal{I}_\mathcal{X} \arrow[r] & \mathcal{X} \arrow[r] & X \end{tikzcd}$$

are fibre product squares, see Lemma 61.5.4. Hence to prove $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is flat and locally of finite presentation it suffices to do so after such a base change by Lemmas 61.17.4 and 61.18.7. Thus we can apply Lemma 61.19.7 to assume that $\mathcal{X} = [U/G]$. By Lemma 61.19.6 we see G is flat and locally of finite presentation over U and that $x : U \to [U/G]$ is surjective, flat, and locally of finite presentation. Moreover, the pullback of $\mathcal{I}_\mathcal{X}$ by x is G and we conclude that (2) holds by descent again, i.e., by Lemmas 61.17.4 and 61.18.7.

Conversely, assume (2). Choose a smooth presentation $\mathcal{X} = [U/R]$, see Algebraic Stacks, Section 57.16. Denote $G \to U$ the stabilizer group algebraic space of the groupoid (U, R, s, t, c, e, i), see Groupoids in Spaces, Definition 52.15.2. By Lemma 61.5.6 we see that $G \to U$ is flat and locally of finite presentation as a base change of $\mathcal{I}_\mathcal{X} \to \mathcal{X}$, see Lemmas 61.17.3 and 61.18.3. Consider the following action

$$a : G \times_{U,t} R \to R, \quad (g, r) \mapsto c(g, r)$$

of G on R. This action is free on T-valued points for any scheme T as R is a groupoid. Hence $R' = R/G$ is an algebraic space and the quotient morphism $\pi : R \to R'$ is surjective, flat, and locally of finite presentation by Bootstrap, Lemma 54.11.5. The projections $s, t : R \to U$ are G-invariant, hence we obtain morphisms $s', t' : R' \to U$ such that $s = s' \circ \pi$ and $t = t' \circ \pi$. Since $s, t : R \to U$ are flat and locally of finite presentation we conclude that s', t' are flat and locally of finite presentation, see Morphisms of Spaces, Lemmas 42.28.5 and Descent on Spaces, Lemma 45.14.1. Consider the morphism

$$j' = (t', s') : R' \longrightarrow U \times U.$$

We claim this is a monomorphism. Namely, suppose that T is a scheme and that $a, b : T \to R'$ are morphisms which have the same image in $U \times U$. By definition of the quotient $R' = R/G$ there exists an fppf covering $\{h_j : T_j \to T\}$ such that $a \circ h_j = \pi \circ a_j$ and $b \circ h_j = \pi \circ b_j$ for some morphisms $a_j, b_j : T_j \to R$. Since a_j, b_j have the same image in $U \times U$ we see that $g_j = c(a_j, i(b_j))$ is a T_j-valued point of G such that $c(g_j, b_j) = a_j$. In other words, a_j and b_j have the same image in R' and the claim is proved. Since $j : R \to U \times U$ is a pre-equivalence relation (see Groupoids in Spaces, Lemma 52.11.2) and $R \to R'$ is surjective (as a map of sheaves) we see that $j' : R' \to U \times U$ is an equivalence relation. Hence Bootstrap, Theorem 54.10.1 shows that $X = U/R'$ is an algebraic space. Finally, we claim that the morphism

$$\mathcal{X} = [U/R] \longrightarrow X = U/R'$$

turns $\mathcal{X}$ into a gerbe over X. This follows from Groupoids in Spaces, Lemma 52.26.1 as $R \to R'$ is surjective, flat, and locally of finite presentation (if needed use Bootstrap, Lemma 54.4.5 to see this implies the required hypothesis). $\square$

At this point we have developed enough machinery to prove that residual gerbes (when they exist) are gerbes.

Lemma 61.19.10. *Let $\mathcal{Z}$ be a reduced, locally Noetherian algebraic stack such that $|\mathcal{Z}|$ is a singleton. Then $\mathcal{Z}$ is a gerbe over a reduced, locally Noetherian algebraic space Z with $|Z|$ a singleton.*

Proof. By Properties of Stacks, Lemma 60.11.3 there exists a surjective, flat, locally finitely presented morphism $Spec(k) \to \mathcal{Z}$ where k is a field. Then $\mathcal{I}_Z \times_{\mathcal{Z}} Spec(k) \to Spec(k)$ is representable by algebraic spaces and locally of finite type (as a base change of $\mathcal{I}_{\mathcal{Z}} \to \mathcal{Z}$, see Lemmas 61.5.1 and 61.13.3). Therefore it is locally of finite presentation, see Morphisms of Spaces, Lemma 42.26.7. Of course it is also flat as k is a field. Hence we may apply Lemmas 61.17.4 and 61.18.7 to see that $\mathcal{I}_{\mathcal{Z}} \to \mathcal{Z}$ is flat and locally of finite presentation. We conclude that $\mathcal{Z}$ is a gerbe by Proposition 61.19.9. Let $\pi : \mathcal{Z} \to Z$ be a morphism to an algebraic space such that $\mathcal{Z}$ is a gerbe over Z. Then π is surjective, flat, and locally of finite presentation by Lemma 61.19.8. Hence $Spec(k) \to Z$ is surjective, flat, and locally of finite presentation as a composition, see Properties of Stacks, Lemma 60.5.2 and Lemmas 61.17.2 and 61.18.2. Hence by Properties of Stacks, Lemma 60.11.3 we see that $|Z|$ is a singleton and that Z is locally Noetherian and reduced. $\square$

Lemma 61.19.11. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If $\mathcal{X}$ is a gerbe over $\mathcal{Y}$ then the map $|\mathcal{X}| \to |\mathcal{Y}|$ is a homeomorphism of topological spaces.*

Proof. Let k be a field and let y be an object of $\mathcal{Y}$ over $Spec(k)$. By Stacks, Lemma 50.11.3 property (2)(a) there exists an fppf covering $\{T_i \to Spec(k)\}$ and objects x_i of $\mathcal{X}$ over T_i with $f(x_i) \cong y|_{T_i}$. Choose an i such that $T_i \neq \emptyset$. Choose a morphism $Spec(K) \to T_i$ for some field K. Then $k \subset K$ and $x_i|_K$ is an object of $\mathcal{X}$ lying over $y|_K$. Thus we see that $|\mathcal{Y}| \to |\mathcal{X}|$. is surjective. The map $|\mathcal{Y}| \to |\mathcal{X}|$ is also injective. Namely, if x, x' are objects of $\mathcal{X}$ over $Spec(k)$ whose images $f(x), f(x')$ become isomorphic (over an extension) in $\mathcal{Y}$, then Stacks, Lemma 50.11.3 property (2)(b) garantees the existence of an extension of k over which x and x' become isomorphic (details omitted). Hence $|\mathcal{X}| \to |\mathcal{Y}|$ is continuous and bijective and it suffices to show that it is also open. This follows from Lemmas 61.19.8 and 61.18.11. $\square$

The following lemma tells us that residual gerbes exist for all points on any algebraic stack which is a gerbe.

Lemma 61.19.12. *Let $\mathcal{X}$ be an algebraic stack. If $\mathcal{X}$ is a gerbe then for every $x \in |\mathcal{X}|$ the residual gerbe of $\mathcal{X}$ at x exists.*

Proof. Let $\pi : \mathcal{X} \to X$ be a morphism from $\mathcal{X}$ into an algebraic space X which turns $\mathcal{X}$ into a gerbe over X. Let $Z_x \to X$ be the residual space of X at x, see Decent Spaces, Definition 43.10.6. Let $\mathcal{Z} = \mathcal{X} \times_X Z_x$. By Lemma 61.19.3 the algebraic stack $\mathcal{Z}$ is a gerbe over Z_x. Hence $|\mathcal{Z}| = |Z_x|$ (Lemma 61.19.11) is a singleton. Since $\mathcal{Z} \to Z_x$ is locally of finite presentation as a base change of π (see Lemmas 61.19.8 and 61.18.3) we see that $\mathcal{Z}$ is locally Noetherian, see Lemma 61.13.5. Thus the residual gerbe $\mathcal{Z}_x$ of $\mathcal{X}$ at x exists and is equal to $\mathcal{Z}_x = \mathcal{Z}_{red}$ the reduction of the algebraic stack $\mathcal{Z}$. Namely, we have seen above that $|\mathcal{Z}_{red}|$ is a singleton mapping to $x \in |\mathcal{X}|$, it is reduced by construction, and it is locally

Noetherian (as the reduction of a locally Noetherian algebraic stack is locally Noetherian, details omitted). ☐

61.20. Stratification by gerbes

The goal of this section is to show that many algebraic stacks $\mathcal{X}$ have a ``stratification'' by locally closed substacks $\mathcal{X}_i \subset \mathcal{X}$ such that each $\mathcal{X}_i$ is a gerbe. This shows that in some sense gerbes are the building blocks out of which any algebraic stack is constructed. Note that by stratification we only mean that

$$|\mathcal{X}| = \bigcup\nolimits_i |\mathcal{X}_i|$$

and nothing more (in general). Hence it is harmless to replace $\mathcal{X}$ by its reduction (see Properties of Stacks, Section 60.10) in order to study this stratification.

The following proposition tells us there is (almost always) a dense open substack of the reduction of $\mathcal{X}$

Proposition 61.20.1. *Let $\mathcal{X}$ be a reduced algebraic stack such that $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is quasi-compact. Then there exists a dense open substack $\mathcal{U} \subset \mathcal{X}$ which is a gerbe.*

Proof. According to Proposition 61.19.9 it is enough to find a dense open substack $\mathcal{U}$ such that $\mathcal{I}_\mathcal{U} \to \mathcal{U}$ is flat and locally of finite presentation. Note that $\mathcal{I}_\mathcal{U} = \mathcal{I}_\mathcal{X} \times_\mathcal{X} \mathcal{U}$, see Lemma 61.5.4.

Choose a presentation $\mathcal{X} = [U/R]$. Let $G \to U$ be the stabilizer group algebraic space of the groupoid R. By Lemma 61.5.6 we see that $G \to U$ is the base change of $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ hence quasi-compact (by assumption) and locally of finite type (by Lemma 61.5.1). Let $W \subset U$ be the largest open (possibly empty) subscheme such that the restriction $G_W \to W$ is flat and locally of finite presentation (we omit the proof that W exists; hint: use that the properties are local). By Morphisms of Spaces, Proposition 42.29.1 we see that $W \subset U$ is dense. Note that $W \subset U$ is R-invariant by More on Groupoids in Spaces, Lemma 53.4.2. Hence W corresponds to an open substack $\mathcal{U} \subset \mathcal{X}$ by Properties of Stacks, Lemma 60.9.10. Since $|U| \to |\mathcal{X}|$ is open and $|W| \subset |U|$ is dense we conclude that $\mathcal{U}$ is dense in $\mathcal{X}$. Finally, the morphism $\mathcal{I}_\mathcal{U} \to \mathcal{U}$ is flat and locally of finite presentation because the base change by the surjective smooth morphism $W \to \mathcal{U}$ is the morphism $G_W \to W$ which is flat and locally of finite presentation by construction. See Lemmas 61.17.4 and 61.18.7. ☐

The above proposition immediately implies that any point has a residual gerbe on an algebraic stack with quasi-compact inertia, as we will show in Lemma 61.21.1. It turns out that there doesn't always exist a finite stratification by gerbes. Here is an example.

Example 61.20.2. Let k be a field. Take $U = Spec(k[x_0, x_1, x_2, \ldots])$ and let $\mathbf{G}_m$ act by $t(x_0, x_1, x_2, \ldots) = (tx_0, t^p x_1, t^{p^2} x_2, \ldots)$ where p is a prime number. Let $\mathcal{X} = [U/\mathbf{G}_m]$. This is an algebraic stack. There is a stratification of $\mathcal{X}$ by strata

1. $\mathcal{X}_0$ is where x_0 is not zero,
2. $\mathcal{X}_1$ is where x_0 is zero but x_1 is not zero,
3. $\mathcal{X}_2$ is where x_0, x_1 are zero, but x_2 is not zero,
4. and so on, and
5. $\mathcal{X}_\infty$ is where all the x_i are zero.

Each stratum is a gerbe over a scheme with group μ_{p^i} for $\mathcal{X}_i$ and $\mathbf{G}_m$ for $\mathcal{X}_\infty$. The strata are reduced locally closed substacks. There is no coarser stratification with the same properties.

Nonetheless, using transfinite induction we can use Proposition 61.20.1 find possibly infinite stratifications by gerbes...!

Lemma 61.20.3. *Let $\mathcal{X}$ be an algebraic stack such that $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is quasi-compact. Then there exists a well ordered index set I and for every $i \in I$ a reduced locally closed substack $\mathcal{U}_i \subset \mathcal{X}$ such that*

1. *each $\mathcal{U}_i$ is a gerbe,*
2. *we have $|\mathcal{X}| = \bigcup_{i \in I} |\mathcal{U}_i|$,*
3. *$T_i = |\mathcal{X}| \setminus \bigcup_{i' < i} |\mathcal{U}_{i'}|$ is closed in $|\mathcal{X}|$ for all $i \in I$, and*
4. *$|\mathcal{U}_i|$ is open in T_i.*

We can moreover arrange it so that either (a) $|\mathcal{U}_i| \subset T_i$ is dense, or (b) $\mathcal{U}_i$ is quasi-compact. In case (a), if we choose $\mathcal{U}_i$ as large as possible (see proof for details), then the stratification is canonical.

Proof. Let $T \subset |\mathcal{X}|$ be a nonempty closed subset. We are going to find (resp. choose) for every such T a reduced locally closed substack $\mathcal{U}(T) \subset \mathcal{X}$ with $|\mathcal{U}(T)| \subset T$ open dense (resp. nonempty quasi-compact). Namely, by Properties of Stacks, Lemma 60.10.1 there exists a unique reduced closed substack $\mathcal{X}' \subset \mathcal{X}$ such that $T = |\mathcal{X}'|$. Note that $\mathcal{I}_{\mathcal{X}'} = \mathcal{I}_\mathcal{X} \times_\mathcal{X} \mathcal{X}'$ by Lemma 61.5.5. Hence $\mathcal{I}_{\mathcal{X}'} \to \mathcal{X}'$ is quasi-compact as a base change, see Lemma 61.7.3. Therefore Proposition 61.20.1 implies there exists a dense maximal (see proof proposition) open substack $\mathcal{U} \subset \mathcal{X}'$ which is a gerbe. In case (a) we set $\mathcal{U}(T) = \mathcal{U}$ (this is canonical) and in case (b) we simply choose a nonempty quasi-compact open $\mathcal{U}(T) \subset \mathcal{U}$, see Properties of Stacks, Lemma 60.4.9 (we can do this for all T simultaneously by the axiom of choice).

By transfinite induction we construct for every ordinal α a closed subset $T_\alpha \subset |\mathcal{X}|$. For $\alpha = 0$ we set $T_0 = |\mathcal{X}|$. Given T_α set

$$T_{\alpha+1} = T_\alpha \setminus |\mathcal{U}(T_\alpha)|.$$

If β is a limit ordinal we set

$$T_\beta = \bigcap\nolimits_{\alpha < \beta} T_\alpha.$$

We claim that $T_\alpha = \emptyset$ for all α large enough. Namely, assume that $T_\alpha \neq \emptyset$ for all α. Then we obtain an injective map from the class of ordinals into the set of subsets of $|\mathcal{X}|$ which is a contradiction.

The claim implies the lemma. Namely, let

$$I = \{\alpha \mid \mathcal{U}_\alpha \neq \emptyset\}.$$

This is a well ordered set by the claim. For $i = \alpha \in I$ we set $\mathcal{U}_i = \mathcal{U}_\alpha$. So $\mathcal{U}_i$ is a reduced locally closed substack and a gerbe, i.e., (1) holds. By construction $T_i = T\alpha$ if $i = \alpha \in I$, hence (3) holds. Also, (4) and (a) or (b) hold by our choice of $\mathcal{U}(T)$ as well. Finally, to see (2) let $x \in |\mathcal{X}|$. There exists a smallest ordinal β with $x \notin T_\beta$ (because the ordinals are well-ordered). In this case β has to be a successor ordinal by the definition of T_β for limit ordinals. Hence $\beta = \alpha + 1$ and $x \in |\mathcal{U}(T_\alpha)|$ and we win. $\square$

Remark 61.20.4. We can wonder about the order type of the canonical stratifications which occur as output of the stratifications of type (a) constructed in Lemma 61.20.3. A natural guess is that the well ordered set I has *cardinality* at most $\aleph_0$. We have no idea if this is true or false. If you do please email stacks.project@gmail.com.

61.21. Existence of residual gerbes

In this section we prove that residual gerbes (as defined in Properties of Stacks, Definition 60.11.8) exist on many algebraic stacks. First, here is the promised application of Proposition 61.20.1.

Lemma 61.21.1. *Let $\mathcal{X}$ be an algebraic stack such that $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is quasi-compact. Then the residual gerbe of $\mathcal{X}$ at x exists for every $x \in |\mathcal{X}|$.*

Proof. Let $T = \overline{\{x\}} \subset |\mathcal{X}|$ be the closure of x. By Properties of Stacks, Lemma 60.10.1 there exists a reduced closed substack $\mathcal{X}' \subset \mathcal{X}$ such that $T = |\mathcal{X}'|$. Note that $\mathcal{I}_{\mathcal{X}'} = \mathcal{I}_\mathcal{X} \times_\mathcal{X} \mathcal{X}'$ by Lemma 61.5.5. Hence $\mathcal{I}_{\mathcal{X}'} \to \mathcal{X}'$ is quasi-compact as a base change, see Lemma 61.7.3. Therefore Proposition 61.20.1 implies there exists a dense open substack $\mathcal{U} \subset \mathcal{X}'$ which is a gerbe. Note that $x \in |\mathcal{U}|$ because $\{x\} \subset T$ is a dense subset too. Hence a residual gerbe $\mathcal{Z}_x \subset \mathcal{U}$ of $\mathcal{U}$ at x exists by Lemma 61.19.12. It is immediate from the definitions that $\mathcal{Z}_x \to \mathcal{X}$ is a residual gerbe of $\mathcal{X}$ at x. $\square$

If the stack is quasi-DM then residual gerbes exist too. In particular, residual gerbes always exist for Delinge-Mumford stacks.

Lemma 61.21.2. *Let $\mathcal{X}$ be a quasi-DM algebraic stack. Then the residual gerbe of $\mathcal{X}$ at x exists for every $x \in |\mathcal{X}|$.*

Proof. Choose a scheme U and a surjective, flat, locally finite presented, and locally quasi-finite morphism $U \to \mathcal{X}$, see Theorem 61.15.3. Set $R = U \times_\mathcal{X} U$. The projections $s, t : R \to U$ are surjective, flat, locally of finite presentation, and locally quasi-finite as base changes of the morphism $U \to \mathcal{X}$. There is a canonical morphism $[U/R] \to \mathcal{X}$ (see Algebraic Stacks, Lemma 57.16.1) which is an equivalence because $U \to \mathcal{X}$ is surjective, flat, and locally of finite presentation, see Algebraic Stacks, Remark 57.16.3. Thus we may assume that $\mathcal{X} = [U/R]$ where (U, R, s, t, c) is a groupoid in algebraic spaces such that $s, t : R \to U$ are surjective, flat, locally of finite presentation, and locally quasi-finite. Set

$$U' = \coprod\nolimits_{u \in U \text{ lying over } x} Spec(\kappa(u)).$$

The canonical morphism $U' \to U$ is a monomorphism. Let

$$R' = U' \times_\mathcal{X} U' = R \times_{(U \times U)} (U' \times U')$$

Because $U' \to U$ is a monomorphism we see that both projections $s', t' : R' \to U'$ factor as a monomorphism followed by a locally quasi-finite morphism. Hence, as U' is a disjoint union of spectra of fields, using Spaces over Fields, Lemma 48.4.3 we conclude that the morphisms $s', t' : R' \to U'$ are locally quasi-finite. Again since U' is a disjoint union of spectra of fields, the morphisms s', t' are also flat. Finally, s', t' locally quasi-finite implies s', t' locally of finite type, hence s', t' locally of finite presentation (because U' is a disjoint union of spectra of fields in particular locally Noetherian, so that Morphisms of Spaces, Lemma 42.26.7 applies). Hence $\mathcal{Z} = [U'/R']$ is an algebraic stack by Criteria for Representability, Theorem 59.17.2. As R' is the restriction of R by $U' \to U$ we see $\mathcal{Z} \to \mathcal{X}$ is a monomorphism by Groupoids in Spaces, Lemma 52.24.1 and Properties of Stacks, Lemma 60.8.4. Since $\mathcal{Z} \to \mathcal{X}$ is a monomorphism we see that $|\mathcal{Z}| \to |\mathcal{X}|$ is injective, see Properties of Stacks, Lemma 60.8.5. By Properties of Stacks, Lemma 60.4.3 we see that

$$|U'| = |\mathcal{Z} \times_\mathcal{X} U'| \longrightarrow |\mathcal{Z}| \times_{|\mathcal{X}|} |U'|$$

is surjective which implies (by our choice of U') that $|\mathcal{Z}| \to |\mathcal{X}|$ has image $\{x\}$. We conclude that $|\mathcal{Z}|$ is a singleton. Finally, by construction U' is locally Noetherian and reduced, i.e., $\mathcal{Z}$ is reduced and locally Noetherian. This means that the essential image of $\mathcal{Z} \to \mathcal{X}$ is the residual gerbe of $\mathcal{X}$ at x, see Properties of Stacks, Lemma 60.11.11. □

61.22. Smooth morphisms

The property ``being smooth'' of morphisms of algebraic spaces is smooth local on the source-and-target, see Descent on Spaces, Remark 45.18.5. It is also stable under base change and fpqc local on the target, see Morphisms of Spaces, Lemma 42.33.3 and Descent on Spaces, Lemma 45.10.24. Hence, by Lemma 61.12.1 above, we may define what it means for a morphism of algebraic spaces to be smooth as follows and it agrees with the already existing notion defined in Properties of Stacks, Section 60.3 when the morphism is representable by algebraic spaces.

Definition 61.22.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is *smooth* if the equivalent conditions of Lemma 61.12.1 hold with $\mathcal{P}$= smooth.

Lemma 61.22.2. *The composition of smooth morphisms is smooth.*

Proof. Combine Remark 61.12.3 with Morphisms of Spaces, Lemma 42.33.2. □

Lemma 61.22.3. *A base change of a smooth morphism is smooth.*

Proof. Combine Remark 61.12.4 with Morphisms of Spaces, Lemma 42.33.3. □

61.23. Other chapters

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CHAPTER 62

Cohomology of Algebraic Stacks

62.1. Introduction

In this chapter we write about cohomology of algebraic stacks. This mean in particular cohomology of quasi-coherent sheaves, i.e., we prove analogues of the results in the chapter entitled ``Coherent Cohomology''. The results in this chapter are different from those in [**LMB00a**] mainly because we consistently use the ``big sites''. Before reading this chapter please take a quick look at the chapter ``Sheaves on Algebraic Stacks'' in order to become familiar with the terminology introduced there, see Sheaves on Stacks, Section 58.1.

62.2. Conventions and abuse of language

We continue to use the conventions and the abuse of language introduced in Properties of Stacks, Section 60.2.

62.3. Notation

Different topologies. If we indicate an algebraic stack by a calligraphic letter, such as $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$, then the notation $\mathcal{X}_{Zar}, \mathcal{X}_{étale}, \mathcal{X}_{smooth}, \mathcal{X}_{syntomic}, \mathcal{X}_{fppf}$ indicates the site introduced in Sheaves on Stacks, Definition 58.4.1. (Think ``big site''.) Correspondingly the structure sheaf of $\mathcal{X}$ is a sheaf on $\mathcal{X}_{fppf}$. On the other hand, algebraic spaces and schemes are usually indicated by roman capitals, such as X, Y, Z, and in this case $X_{étale}$ indicates the small étale site of X (as defined in Topologies, Definition 30.4.8 or Properties of Spaces, Definition 41.15.1). It seems that the distinction should be clear enough.

The default topology is the fppf topology. Hence we will sometimes say ``sheaf on $\mathcal{X}$'' or ``sheaf of $\mathcal{O}_\mathcal{X}$'' modules when we mean sheaf on $\mathcal{X}_{fppf}$ or object of $Mod(\mathcal{X}_{fppf}, \mathcal{O}_\mathcal{X})$.

If $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks, then the functors f_* and f^{-1} defined on presheaves preserves sheaves for any of the topologies mentioned above. In particular when we discuss the pushforward or pullback of a sheaf we don't have to mention which topology we are working with. The same isn't true when we compute cohomology groups and/or higher direct images. In this case we will always mention which topology we are working with.

Suppose that $f : X \to \mathcal{Y}$ is a morphism from an algebraic space X to an algebraic stack $\mathcal{Y}$. Let $\mathcal{G}$ be a sheaf on $\mathcal{Y}_\tau$ for some topology τ. In this case $f^{-1}\mathcal{G}$ is a sheaf for the τ topology on $\mathcal{S}_X$ (the algebraic stack associated to X) because (by our conventions) f really is a 1-morphism $f : \mathcal{S}_X \to \mathcal{Y}$. If $\tau = étale$ or stronger, then we write $f^{-1}\mathcal{G}|_{X_{étale}}$ to denote the restriction to the étale site of X, see Sheaves on Stacks, Section 58.21. If $\mathcal{G}$ is an $\mathcal{O}_\mathcal{X}$-module we sometimes write $f^*\mathcal{G}$ and $f^*\mathcal{G}|_{X_{étale}}$ instead.

62.4. Pullback of quasi-coherent modules

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. It is a very general fact that quasi-coherent modules on ringed topoi are compatible with pullbacks. In particular the pullback f^* preserves quasi-coherent modules and we obtain a functor

$$f^* : QCoh(\mathcal{O}_\mathcal{Y}) \longrightarrow QCoh(\mathcal{O}_\mathcal{X}),$$

see Sheaves on Stacks, Lemma 58.11.2. In general this functor isn't exact, but if f is flat then it is.

Lemma 62.4.1. *If $f : \mathcal{X} \to \mathcal{Y}$ is a flat morphism of algebraic stacks then $f^* : QCoh(\mathcal{O}_\mathcal{Y}) \to QCoh(\mathcal{O}_\mathcal{X})$ is an exact functor.*

Proof. Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Y}$. Choose a scheme U and a surjective smooth morphism $U \to V \times_\mathcal{Y} \mathcal{X}$. Then $U \to \mathcal{X}$ is still smooth and surjective as a composition of two such morphisms. From the commutative diagram

$$\begin{tikzcd} U \arrow[r, "f'"'] \arrow[d] & V \arrow[d] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}$$

we obtain a commutative diagram

$$\begin{tikzcd} QCoh(\mathcal{O}_U) & QCoh(\mathcal{O}_V) \arrow[l] \\ QCoh(\mathcal{O}_\mathcal{X}) \arrow[u] & QCoh(\mathcal{O}_\mathcal{Y}) \arrow[l] \arrow[u] \end{tikzcd}$$

of abelian categories. Our proof that the bottom two categories in this diagram are abelian showed that the vertical functors are faithful exact functors (see proof of Sheaves on Stacks, Lemma 58.14.1). Since f' is a flat morphism of schemes (by our definition of flat morphisms of algebraic stacks) we see that $(f')^*$ is an exact functor on quasi-coherent sheaves on V. Thus we win. $\square$

62.5. The key lemma

The following lemma is the basis for our understanding of higher direct images of certain types of sheaves of modules. There are two verions: one for the étale topology and one for the fppf topology.

Lemma 62.5.1. *Let $\mathcal{M}$ be a rule which associates to every algebraic stack $\mathcal{X}$ a subcategory $\mathcal{M}_\mathcal{X}$ of $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ such that*

1. *$\mathcal{M}_\mathcal{X}$ is a weak Serre subcategory of $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ (see Homology, Definition 10.7.1) for all algebraic stacks $\mathcal{X}$,*
2. *for a smooth morphism of algebraic stacks $f : \mathcal{Y} \to \mathcal{X}$ the functor f^* maps $\mathcal{M}_\mathcal{X}$ into $\mathcal{M}_\mathcal{Y}$,*
3. *if $f_i : \mathcal{X}_i \to \mathcal{X}$ is a family of smooth morphisms of algebraic stacks with $|\mathcal{X}| = \bigcup |f_i|(|\mathcal{X}_i|)$, then an object $\mathcal{F}$ of $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ is in $\mathcal{M}_\mathcal{X}$ if and only if $f_i^*\mathcal{F}$ is in $\mathcal{M}_{\mathcal{X}_i}$ for all i, and*
4. *if $f : \mathcal{Y} \to \mathcal{X}$ is a morphism of algebraic stacks such that $\mathcal{X}$ and $\mathcal{Y}$ are representable by affine schemes, then $R^i f_*$ maps $\mathcal{M}_\mathcal{Y}$ into $\mathcal{M}_\mathcal{X}$.*

Then for any quasi-compact and quasi-separated morphism $f : \mathcal{Y} \to \mathcal{X}$ of algebraic stacks $R^i f_*$ maps $\mathcal{M}_\mathcal{Y}$ into $\mathcal{M}_\mathcal{X}$. (Higher direct images computed in étale topology.)

Proof. Let $f : \mathcal{Y} \to \mathcal{X}$ be a quasi-compact and quasi-separated morphism of algebraic stacks and let $\mathcal{F}$ be an object of $\mathcal{M}_\mathcal{Y}$. Choose a surjective smooth morphism $\mathcal{U} \to \mathcal{X}$ where $\mathcal{U}$ is representable by a scheme. By Sheaves on Stacks, Lemma 58.20.3 taking higher direct images commutes with base change. Assumption (2) shows that the pullback of $\mathcal{F}$ to $\mathcal{U} \times_\mathcal{X} \mathcal{Y}$ is in $\mathcal{M}_{\mathcal{U} \times_\mathcal{X} \mathcal{Y}}$ because the projection $\mathcal{U} \times_\mathcal{X} \mathcal{Y} \to \mathcal{Y}$ is smooth as a base change of a smooth morphism. Hence (3) shows we may replace $\mathcal{Y} \to \mathcal{X}$ by the projection $\mathcal{U} \times_\mathcal{X} \mathcal{Y} \to \mathcal{U}$. In other words, we may assume that $\mathcal{X}$ is representable by a scheme. Using (3) once more, we see that the question is Zariski local on $\mathcal{X}$, hence we may assume that $\mathcal{X}$ is representable by an affine scheme. Since f is quasi-compact this implies that also $\mathcal{Y}$ is quasi-compact. Thus we may choose a surjective smooth morphism $g : \mathcal{V} \to \mathcal{Y}$ where $\mathcal{V}$ is representable by an affine scheme.

In this situation we have the spectral sequence

$$E_2^{p,q} = R^q(f \circ g_p)_* g_p^* \mathcal{F} \Rightarrow R^{p+q} f_* \mathcal{F}$$

of Sheaves on Stacks, Proposition 58.20.1. Recall that this is the spectral sequence associated to a double complex. By assumption (1) we may use Homology, Remark 10.19.7. Note that the morphisms

$$g_p : \mathcal{V}_p = \mathcal{V} \times_\mathcal{Y} \ldots \times_\mathcal{Y} \mathcal{V} \longrightarrow \mathcal{Y}$$

are smooth as compositions of base changes of the smooth morphism g. Thus the sheaves $g_p^*\mathcal{F}$ are in $\mathcal{M}_{\mathcal{V}_p}$ by (2). Hence it suffices to prove that the higher direct images of objects of $\mathcal{M}_{\mathcal{V}_p}$ under the morphisms

$$\mathcal{V}_p = \mathcal{V} \times_\mathcal{Y} \ldots \times_\mathcal{Y} \mathcal{V} \longrightarrow \mathcal{X}$$

are in $\mathcal{M}_\mathcal{X}$. The algebraic stacks $\mathcal{V}_p$ are quasi-compact and quasi-separated by Morphisms of Stacks, Lemma 61.7.7. Of course each $\mathcal{V}_p$ is representable by an algebraic space (the diagonal of the algebraic stack $\mathcal{Y}$ is representable by algebraic spaces). This reduces us to the case where $\mathcal{Y}$ is representable by an algebraic space and $\mathcal{X}$ is representable by an affine scheme.

In the situation where $\mathcal{Y}$ is representable by an algebraic space and $\mathcal{X}$ is representable by an affine scheme, we choose anew a surjective smooth morphism $\mathcal{V} \to \mathcal{Y}$ where $\mathcal{V}$ is representable by an affine scheme. Going through the argument above once again we once again reduce to the morphisms $\mathcal{V}_p \to \mathcal{X}$. But in the current situation the algebraic stacks $\mathcal{V}_p$ are representable by quasi-compact and quasi-separated schemes (becase the diagonal of an algebraic space is representable by schemes).

Thus we may assume $\mathcal{Y}$ is representable by a scheme and $\mathcal{X}$ is representable by an affine scheme. Choose (again) a surjective smooth morphism $\mathcal{V} \to \mathcal{Y}$ where $\mathcal{V}$ is representable by an affine scheme. In this case all the algebraic stacks $\mathcal{V}_p$ are representable by separated schemes (because the diagonal of a scheme is separated).

Thus we may assume $\mathcal{Y}$ is representable by a separated scheme and $\mathcal{X}$ is representable by an affine scheme. Choose (yet again) a surjective smooth morphism $\mathcal{V} \to \mathcal{Y}$ where $\mathcal{V}$ is representable by an affine scheme. In this case all the algebraic stacks $\mathcal{V}_p$ are representable by affine schemes (because the diagonal of a separated scheme is a closed immersion hence affine) and this case is handled by assumption (4). This finishes the proof. $\square$

Here is the version for the fppf topology.

Lemma 62.5.2. *Let $\mathcal{M}$ be a rule which associates to every algebraic stack $\mathcal{X}$ a subcategory $\mathcal{M}_\mathcal{X}$ of $Mod(\mathcal{O}_\mathcal{X})$ such that*

1. *$\mathcal{M}_\mathcal{X}$ is a weak Serre subcategory of $Mod(\mathcal{O}_\mathcal{X})$ for all algebraic stacks $\mathcal{X}$,*
2. *for a smooth morphism of algebraic stacks $f : \mathcal{Y} \to \mathcal{X}$ the functor f^* maps $\mathcal{M}_\mathcal{X}$ into $\mathcal{M}_\mathcal{Y}$,*
3. *if $f_i : \mathcal{X}_i \to \mathcal{X}$ is a family of smooth morphisms of algebraic stacks with $|\mathcal{X}| = \bigcup |f_i|(|\mathcal{X}_i|)$, then an object $\mathcal{F}$ of $Mod(\mathcal{O}_\mathcal{X})$ is in $\mathcal{M}_\mathcal{X}$ if and only if $f_i^*\mathcal{F}$ is in $\mathcal{M}_{\mathcal{X}_i}$ for all i, and*
4. *if $f : \mathcal{Y} \to \mathcal{X}$ is a morphism of algebraic stacks and $\mathcal{X}$ and $\mathcal{Y}$ are representable by affine schemes, then R^if_* maps $\mathcal{M}_\mathcal{Y}$ into $\mathcal{M}_\mathcal{X}$.*

Then for any quasi-compact and quasi-separated morphism $f : \mathcal{Y} \to \mathcal{X}$ of algebraic stacks R^if_* maps $\mathcal{M}_\mathcal{Y}$ into $\mathcal{M}_\mathcal{X}$. (Higher direct images computed in fppf topology.)

Proof. Identical to the proof of Lemma 62.5.1. □

62.6. Locally quasi-coherent modules

Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_\mathcal{X}$-modules. We can ask whether $\mathcal{F}$ is *locally quasi-coherent*, see Sheaves on Stacks, Definition 58.11.4. Briefly, this means $\mathcal{F}$ is an $\mathcal{O}_\mathcal{X}$-module for the étale topology such that for any morphism $f : U \to \mathcal{X}$ the restriction $f^*\mathcal{F}|_{U_{étale}}$ is quasi-coherent on $U_{étale}$. (The actual definition is slightly different, but equivalent.) A useful fact is that

$$LQCoh(\mathcal{O}_\mathcal{X}) \subset Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$$

is a weak Serre subcategory, see Sheaves on Stacks, Lemma 58.11.7.

Lemma 62.6.1. *Let $\mathcal{X}$ be an algebraic stack. Let $f_j : \mathcal{X}_j \to \mathcal{X}$ be a family of smooth morphisms of algebraic stacks with $|\mathcal{X}| = \bigcup |f_j|(|\mathcal{X}_j|)$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_\mathcal{X}$-modules on $\mathcal{X}_{étale}$. If each $f_j^{-1}\mathcal{F}$ is locally quasi-coherent, then so is $\mathcal{F}$.*

Proof. We may replace each of the algebraic stacks $\mathcal{X}_j$ by a scheme U_j (using that any algebraic stack has a smooth covering by a scheme and that compositions of smooth morphisms are smooth, see Morphisms of Stacks, Lemma 61.22.2). The pullback of $\mathcal{F}$ to $(Sch/U_j)_{étale}$ is still locally quasi-coherent, see Sheaves on Stacks, Lemma 58.11.6. Then $f = \coprod f_j : U = \coprod U_j \to \mathcal{X}$ is a surjective smooth morphism. Let x be an object of $\mathcal{X}$. By Sheaves on Stacks, Lemma 58.18.10 there exists an étale covering $\{x_i \to x\}_{i\in I}$ such that each x_i lifts to an object u_i of $(Sch/U)_{étale}$. This just means that x, x_i live over schemes V, V_i, that $\{V_i \to V\}$ is an étale covering, and that x_i comes from a morphism $u_i : V_i \to U$. The restriction $x_i^*\mathcal{F}|_{V_{i,étale}}$ is equal to the restriction of $f^*\mathcal{F}$ to $V_{i,étale}$, see Sheaves on Stacks, Lemma 58.9.3. Hence $x^*\mathcal{F}|_{V_{étale}}$ is a sheaf on the small étale site of V which is quasi-coherent when restricted to $V_{i,étale}$ for each i. This implies that it is quasi-coherent (as desired), for example by Properties of Spaces, Lemma 41.26.6. □

Lemma 62.6.2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. Let $\mathcal{F}$ be a locally quasi-coherent $\mathcal{O}_\mathcal{X}$-module on $\mathcal{X}_{étale}$. Then $R^if_*\mathcal{F}$ (computed in the étale topology) is a locally quasi-coherent on $\mathcal{Y}_{étale}$.*

Proof. We will use Lemma 62.5.1 to prove this. We will check its assumptions (1) -- (4). Parts (1) and (2) follows from Sheaves on Stacks, Lemma 58.11.7. Part (3) follows from Lemma 62.6.1. Thus it suffices to show (4).

Suppose $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacs such that $\mathcal{X}$ and $\mathcal{Y}$ are representable by affine schemes X and Y. Choose any object y of $\mathcal{Y}$ lying over a scheme V. For clarity, denote $\mathcal{V} = (Sch/V)_{fppf}$ the algebraic stack corresponding to V. Consider the cartesian diagram

$$\begin{tikzcd} \mathcal{Z} \arrow[r, "g"'] \arrow[d, "f'"'] & \mathcal{X} \arrow[d, "f"] \\ \mathcal{V} \arrow[r, "y"] & \mathcal{Y} \end{tikzcd}$$

Thus $\mathcal{Z}$ is representable by the scheme $Z = V \times_Y X$ and f' is quasi-compact and separated (even affine). By Sheaves on Stacks, Lemma 58.21.3 we have

$$R^i f_* \mathcal{F}|_{V_{\acute{e}tale}} = R^i f'_{small,*}\big(g^*\mathcal{F}|_{Z_{\acute{e}tale}}\big)$$

The right hand side is a quasi-coherent sheaf on $V_{\acute{e}tale}$ by Cohomology of Spaces, Lemma 49.4.1. This implies the left hand side is quasi-coherent which is what we had to prove. $\square$

Lemma 62.6.3. *Let $\mathcal{X}$ be an algebraic stack. Let $f_j : \mathcal{X}_j \to \mathcal{X}$ be a family of flat and locally finitely presented morphisms of algebraic stacks with $|\mathcal{X}| = \bigcup |f_j|(|\mathcal{X}_j|)$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_\mathcal{X}$-modules on $\mathcal{X}_{fppf}$. If each $f_j^{-1}\mathcal{F}$ is locally quasi-coherent, then so is $\mathcal{F}$.*

Proof. First, suppose there is a morphism $a : \mathcal{U} \to \mathcal{X}$ which is surjective, flat, locally of finite presentation, quasi-compact, and quasi-separated such that $a^*\mathcal{F}$ is locally quasi-coherent. Then there is an exact sequence

$$0 \to \mathcal{F} \to a_*a^*\mathcal{F} \to b_*b^*\mathcal{F}$$

where b is the morphism $b : \mathcal{U} \times_\mathcal{X} \mathcal{U} \to \mathcal{X}$, see Sheaves on Stacks, Proposition 58.18.7 and Lemma 58.18.10. Moreover, the pullback $b^*\mathcal{F}$ is the pullback of $a^*\mathcal{F}$ via one of the projection morphisms, hence is locally quasi-coherent (Sheaves on Stacks, Lemma 58.11.6). The modules $a_*a^*\mathcal{F}$ and $b_*b^*\mathcal{F}$ are locally quasi-coherent by Lemma 62.6.2. (Note that a_* and b_* don't care about which topology is used to calculate them.) We conclude that $\mathcal{F}$ is locally quasi-coherent, see Sheaves on Stacks, Lemma 58.11.7.

We are going to reduce the proof of the general case the the situation in the first paragraph. Let x be an object of $\mathcal{X}$ lying over the scheme U. We have to show that $\mathcal{F}|_{U_{\acute{e}tale}}$ is a quasi-coherent $\mathcal{O}_U$-module. It suffices to do this (Zariski) locally on U, hence we may assume that U is affine. By Morphisms of Stacks, Lemma 61.18.10 there exists an fppf covering $\{a_i : U_i \to U\}$ such that each $x \circ a_i$ factors through some f_j. Hence $a_i^*\mathcal{F}$ is locally quasi-coherent on $(Sch/U_i)_{fppf}$. After refining the covering we may assume $\{U_i \to U\}_{i=1,\ldots,n}$ is a standard fppf covering. Then $x^*\mathcal{F}$ is an fppf module on $(Sch/U)_{fppf}$ whose pullback by the morphism $a : U_1 \amalg \ldots \amalg U_n \to U$ is locally quasi-coherent. Hence by the first paragraph we see that $x^*\mathcal{F}$ is locally quasi-coherent, which certainly implies that $\mathcal{F}|_{U_{\acute{e}tale}}$ is quasi-coherent. $\square$

62.7. Flat comparison maps

Let $\mathcal{X}$ be an algebraic stack and let $\mathcal{F}$ be an object of $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$. Given an object x of $\mathcal{X}$ lying over the scheme U the restriction $\mathcal{F}|_{U_{\acute{e}tale}}$ is the restriction of $x^{-1}\mathcal{F}$ to the small étale site of U, see Sheaves on Stacks, Definition 58.9.2. Next, let $\varphi : x \to x'$ be a morphism of

$\mathcal{X}$ lying over a morphism of schemes $f : U \to U'$. Thus a 2-commutative diagram

$$\begin{tikzcd} U \arrow[rr, "f"] \arrow[dr, "x"'] & & U' \arrow[dl, "x'"] \\ & \mathcal{X} & \end{tikzcd}$$

Associated to φ we obtain a comparison map between restrictions

$$c_\varphi : f_{small}^*(\mathcal{F}|_{U'_{\acute{e}tale}}) \longrightarrow \mathcal{F}|_{U_{\acute{e}tale}} \quad (62.7.0.1)$$

see Sheaves on Stacks, Equation (58.9.4.1). In this situation we can consider the following property of $\mathcal{F}$.

Definition 62.7.1. Let $\mathcal{X}$ be an algebraic stack and let $\mathcal{F}$ in $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$. We say $\mathcal{F}$ has the *flat base change property*[1] if and only if c_φ is an isomorphism whenever f is flat.

Here is a lemma with some properties of this notion.

Lemma 62.7.2. *Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{F}$ be an $\mathcal{O}_\mathcal{X}$-module on $\mathcal{X}_{\acute{e}tale}$.*

(1) *If $\mathcal{F}$ has the flat base change property then for any morphism $g : \mathcal{Y} \to \mathcal{X}$ of algebraic stacks, the pullback $g^*\mathcal{F}$ does too.*
(2) *The full subcategory of $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$ consisting of modules with the flat base change property is a weak Serre subcategory.*
(3) *Let $f_i : \mathcal{X}_i \to \mathcal{X}$ be a family of smooth morphisms of algebraic stacks such that $|\mathcal{X}| = \bigcup_i |f_i|(|\mathcal{X}_i|)$. If each $f_i^*\mathcal{F}$ has the flat base change property then so does $\mathcal{F}$.*
(4) *The category of $\mathcal{O}_\mathcal{X}$-modules on $\mathcal{X}_{\acute{e}tale}$ with the flat base change property has colimits and they agree with colimits in $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$.*

Proof. Let $g : \mathcal{Y} \to \mathcal{X}$ be as in (1). Let y be an object of $\mathcal{Y}$ lying over a scheme V. By Sheaves on Stacks, Lemma 58.9.3 we have $(g^*\mathcal{F})|_{V_{\acute{e}tale}} = \mathcal{F}|_{V_{\acute{e}tale}}$. Moreover a comparison mapping for the sheaf $g^*\mathcal{F}$ on $\mathcal{Y}$ is a special case of a comparison map for the sheaf $\mathcal{F}$ on $\mathcal{X}$, see Sheaves on Stacks, Lemma 58.9.3. In this way (1) is clear.

Proof of (2). We use the characterization of weak Serre subcategories of Homology, Lemma 10.7.3. Kernels and cokernels of maps between sheaves having the flat base change property also have the flat base change property. This is clear because f_{small}^* is exact for a flat morphism of schemes and since the restriction functors $(-)|_{U_{\acute{e}tale}}$ are exact (because we are working in the étale topology). Finally, if $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is a short exact sequence of $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$ and the outer two sheaves have the flat base change property then the middle one does as well, again because of the exactness of f_{small}^* and the restriction functors (and the 5 lemma).

Proof of (3). Let $f_i : \mathcal{X}_i \to \mathcal{X}$ be a jointly surjective family of smooth morphisms of algebraic stacks and assume each $f_i^*\mathcal{F}$ has the flat base change property. By part (1), the definition of an algebraic stack, and the fact that compositions of smooth morphisms are smooth (see Morphisms of Stacks, Lemma 61.22.2) we may assume that each $\mathcal{X}_i$ is representable by a scheme. Let $\varphi : x \to x'$ be a morphism of $\mathcal{X}$ lying over a flat morphism $a : U \to U'$ of schemes. By Sheaves on Stacks, Lemma 58.18.10 there exists a jointly

[1] This may be nonstandard notation.

surjective family of étale morphisms $U'_i \to U'$ such that $U' \to U' \to \mathcal{X}$ factors through $\mathcal{X}_i$. Thus we obtain commutative diagrams

$$\begin{tikzcd} U_i = U \times_{U'} U'_i \arrow[r, "a_i"] \arrow[d] & U'_i \arrow[r, "x'_i"] \arrow[d] & \mathcal{X}_i \arrow[d, "f_i"] \\ U \arrow[r, "a"] & U' \arrow[r, "x'"] & \mathcal{X} \end{tikzcd}$$

Note that each a_i is a flat morphism of schemes as a base change of a. Denote $\psi_i : x_i \to x'_i$ the morphism of $\mathcal{X}_i$ lying over a_i with target x'_i. By assumption the comparison maps $c_{\psi_i} : (a_i)^*_{small}\big(f_i^*\mathcal{F}|_{(U'_i)_{étale}}\big) \to f_i^*\mathcal{F}|_{(U_i)_{étale}}$ is an isomorphism. Because the vertical arrows $U'_i \to U'$ and $U_i \to U$ are étale, the sheaves $f_i^*\mathcal{F}|_{(U'_i)_{étale}}$ and $f_i^*\mathcal{F}|_{(U_i)_{étale}}$ are the restrictions of $\mathcal{F}|_{U'_{étale}}$ and $\mathcal{F}|_{U_{étale}}$ and the map c_{ψ_i} is the restriction of c_φ to $(U_i)_{étale}$, see Sheaves on Stacks, Lemma 58.9.3. Since $\{U_i \to U\}$ is an étale covering, this implies that the comparison map c_φ is an isomorphism which is what we wanted to prove.

Proof of (4). Let $\mathcal{I} \to Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$, $i \mapsto \mathcal{F}_i$ be a diagram and assume each $\mathcal{F}_i$ has the flat base change property. Recall that $colim_i\, \mathcal{F}_i$ is the sheafification of the presheaf colimit. As we are using the étale topology, it is clear that

$$(colim_i\, \mathcal{F}_i)|_{U_{étale}} = colim_i\, \mathcal{F}_i|_{U_{étale}}$$

As f^*_{small} commutes with colimits (as a left adjoint) we see that (4) holds. ☐

Lemma 62.7.3. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. Let $\mathcal{F}$ be an object of $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ which is locally quasi-coherent and has the flat base change property. Then each $R^ig_*\mathcal{F}$ (computed in the étale topology) has the flat base change property.*

Proof. We will use Lemma 62.5.1 to prove this. For every algebraic stack $\mathcal{X}$ let $\mathcal{M}_\mathcal{X}$ denote the full subcategory of $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ consisting of locally quasi-coherent sheaves with the flat base change property. Once we verify conditions (1) -- (4) of Lemma 62.5.1 the lemma will follow. Properties (1), (2), and (3) follow from Sheaves on Stacks, Lemmas 58.11.6 and 58.11.7 and Lemmas 62.6.1 and 62.7.2. Thus it suffices to show part (4).

Suppose $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacs such that $\mathcal{X}$ and $\mathcal{Y}$ are representable by affine schemes X and Y. In this case, suppose that $\psi : y \to y'$ is a morphism of $\mathcal{Y}$ lying over a flat morphism $b : V \to V'$ of schemes. For clarity denote $\mathcal{V} = (Sch/V)_{fppf}$ and $\mathcal{V}' = (Sch/V')_{fppf}$ the corresponding algebraic stacks. Consider the diagram of algebraic stacks

$$\begin{tikzcd} \mathcal{Z} \arrow[r, "a"] \arrow[d, "f''"'] & \mathcal{Z}' \arrow[r, "x'"] \arrow[d, "f'"'] & \mathcal{X} \arrow[d, "f"] \\ \mathcal{V} \arrow[r, "b"] & \mathcal{V}' \arrow[r, "y'"] & \mathcal{Y} \end{tikzcd}$$

with both squares cartesian. As f is representable by schemes (and quasi-compact and separated -- even affine) we see that $\mathcal{Z}$ and $\mathcal{Z}'$ are representable by schemes Z and Z' and in fact $Z = V \times_{V'} Z'$. Since $\mathcal{F}$ has the flat base change property we see that

$$a^*_{small}\big(\mathcal{F}|_{Z'_{étale}}\big) \longrightarrow \mathcal{F}|_{Z_{étale}}$$

is an isomorphism. Moreover,

$$R^if_*\mathcal{F}|_{V'_{étale}} = R^i(f')_{small,*}\big(\mathcal{F}|_{Z'_{étale}}\big)$$

and

$$R^i f_*\mathcal{F}|_{V_{\acute{e}tale}} = R^i(f'')_{small,*}(\mathcal{F}|_{Z_{\acute{e}tale}})$$

by Sheaves on Stacks, Lemma 58.21.3. Hence we see that the comparision map

$$c_\psi : b^*_{small}(R^i f_*\mathcal{F}|_{V'_{\acute{e}tale}}) \longrightarrow R^i f_*\mathcal{F}|_{V_{\acute{e}tale}}$$

is an isomorphism by Cohomology of Spaces, Lemma 49.9.1. Thus $R^i f_*\mathcal{F}$ has the flat base change property. Since $R^i f_*\mathcal{F}$ is locally quasi-coherent by Lemma 62.6.2 we win. □

Proposition 62.7.4. *Summary of results on locally quasi-coherent modules having the flat base change property.*

(1) *Let $\mathcal{X}$ be an algebraic stack. If $\mathcal{F}$ is an object of $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$ which is locally quasi-coherent and has the flat base change property, then $\mathcal{F}$ is a sheaf for the fppf topology, i.e., it is an object of $Mod(\mathcal{O}_\mathcal{X})$.*
(2) *The category of modules which are locally quasi-coherent and have the flat base change property is a weak Serre subcategory $\mathcal{M}_\mathcal{X}$ of both $Mod(\mathcal{O}_\mathcal{X})$ and $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$.*
(3) *Pullback f^* along any morphism of algebraic stacks $f : \mathcal{X} \to \mathcal{Y}$ induces a functor $f^* : \mathcal{M}_\mathcal{Y} \to \mathcal{M}_\mathcal{X}$.*
(4) *If $f : \mathcal{X} \to \mathcal{Y}$ is a quasi-compact and quasi-separated morphism of algebraic stacks and $\mathcal{F}$ is an object of $\mathcal{M}_\mathcal{X}$, then*
   (a) *the derived direct image $Rf_*\mathcal{F}$ and the higher direct images $R^i f_*\mathcal{F}$ can be computed in either the étale or the fppf topology with the same result, and*
   (b) *each $R^i f_*\mathcal{F}$ is an object of $\mathcal{M}_\mathcal{Y}$.*
(5) *The category $\mathcal{M}_\mathcal{X}$ has colimits and they agree with colimits in $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$ as well as in $Mod(\mathcal{O}_\mathcal{X})$.*

Proof. Part (1) is Sheaves on Stacks, Lemma 58.22.1.

Part (2) for the embedding $\mathcal{M}_\mathcal{X} \subset Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$ we have seen in the proof of Lemma 62.7.3. Let us prove (2) for the embedding $\mathcal{M}_\mathcal{X} \subset Mod(\mathcal{O}_\mathcal{X})$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism between objects of $\mathcal{M}_\mathcal{X}$. Since $\mathrm{Ker}(\varphi)$ is the same whether computed in the étale or the fppf topology, we see that $\mathrm{Ker}(\varphi)$ is in $\mathcal{M}_\mathcal{X}$ by the étale case. On the other hand, the cokernel computed in the fppf topology is the fppf sheafification of the cokernel computed in the étale topology. However, this étale cokernel is in $\mathcal{M}_\mathcal{X}$ hence an fppf sheaf by (1) and we see that the cokernel is in $\mathcal{M}_\mathcal{X}$. Finally, suppose that

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

is an exact sequence in $Mod(\mathcal{O}_\mathcal{X})$ (i.e., using the fppf topology) with $\mathcal{F}_1$, $\mathcal{F}_2$ in $\mathcal{M}_\mathcal{X}$. In order to show that $\mathcal{F}_2$ is an object of $\mathcal{M}_\mathcal{X}$ it suffices to show that the sequence is also exact in the étale topology. To do this it suffices to show that any element of $H^1_{fppf}(x, \mathcal{F}_1)$ becomes zero on the members of an étale covering of x (for any object x of $\mathcal{X}$). This is true because $H^1_{fppf}(x, \mathcal{F}_1) = H^1_{\acute{e}tale}(x, \mathcal{F}_1)$ by Sheaves on Stacks, Lemma 58.22.2 and because of locality of cohohomology, see Cohomology on Sites, Lemma 19.8.3. This proves (2).

Part (3) follows from Lemma 62.7.2 and Sheaves on Stacks, Lemma 58.11.6.

Part (4)(b) for $R^i f_*\mathcal{F}$ computed in the étale cohomology follows from Lemma 62.7.3. Whereupon part (4)(a) follows from Sheaves on Stacks, Lemma 58.22.2 combined with (1) above.

Part (5) for the étale topology follows from Sheaves on Stacks, Lemma 58.11.7 and Lemma 62.7.2. The fppf version then follows as the colimit in the étale topology is already an fppf sheaf by part (1). □

Lemma 62.7.5. *Let $\mathcal{X}$ be an algebraic stack. With $\mathcal{M}_\mathcal{X}$ the category of locally quasi-coherent modules with the flat base change property.*

(1) *Let $f_j : \mathcal{X}_j \to \mathcal{X}$ be a family of smooth morphisms of algebraic stacks with $|\mathcal{X}| = \bigcup |f_j|(|\mathcal{X}_j|)$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_\mathcal{X}$-modules on $\mathcal{X}_{étale}$. If each $f_j^{-1}\mathcal{F}$ is in $\mathcal{M}_{\mathcal{X}_i}$, then $\mathcal{F}$ is in $\mathcal{M}_\mathcal{X}$.*
(2) *Let $f_j : \mathcal{X}_j \to \mathcal{X}$ be a family of flat and locally finitely presented morphisms of algebraic stacks with $|\mathcal{X}| = \bigcup |f_j|(|\mathcal{X}_j|)$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_\mathcal{X}$-modules on $\mathcal{X}_{fppf}$. If each $f_j^{-1}\mathcal{F}$ is in $\mathcal{M}_{\mathcal{X}_i}$, then $\mathcal{F}$ is in $\mathcal{M}_\mathcal{X}$.*

Proof. Part (1) follows from a combination of Lemmas 62.6.1 and 62.7.2. The proof of (2) is analogous to the proof of Lemma 62.6.3. Let $\mathcal{F}$ of a sheaf of $\mathcal{O}_\mathcal{X}$-modules on $\mathcal{X}_{fppf}$.

First, suppose there is a morphism $a : \mathcal{U} \to \mathcal{X}$ which is surjective, flat, locally of finite presentation, quasi-compact, and quasi-separated such that $a^*\mathcal{F}$ is locally quasi-coherent and has the flat base change property. Then there is an exact sequence

$$0 \to \mathcal{F} \to a_*a^*\mathcal{F} \to b_*b^*\mathcal{F}$$

where b is the morphism $b : \mathcal{U} \times_\mathcal{X} \mathcal{U} \to \mathcal{X}$, see Sheaves on Stacks, Proposition 58.18.7 and Lemma 58.18.10. Moreover, the pullback $b^*\mathcal{F}$ is the pullback of $a^*\mathcal{F}$ via one of the projection morphisms, hence is locally quasi-coherent and has the flat base change property, see Proposition 62.7.4. The modules $a_*a^*\mathcal{F}$ and $b_*b^*\mathcal{F}$ are locally quasi-coherent and have the flat base change property by Proposition 62.7.4. We conclude that $\mathcal{F}$ is locally quasi-coherent and has the flat base change property by Proposition 62.7.4.

Choose a scheme U and a surjective smooth morphism $x : U \to \mathcal{X}$. By part (1) it suffices to show that $x^*\mathcal{F}$ is locally quasi-coherent and has the flat base change property. Again by part (1) it suffices to do this (Zariski) locally on U, hence we may assume that U is affine. By Morphisms of Stacks, Lemma 61.18.10 there exists an fppf covering $\{a_i : U_i \to U\}$ such that each $x \circ a_i$ factors through some f_j. Hence the module $a_i^*\mathcal{F}$ on $(Sch/U_i)_{fppf}$ is locally quasi-coherent and has the flat base change property. After refining the covering we may assume $\{U_i \to U\}_{i=1,\ldots,n}$ is a standard fppf covering. Then $x^*\mathcal{F}$ is an fppf module on $(Sch/U)_{fppf}$ whose pullback by the morphism $a : U_1 \amalg \ldots \amalg U_n \to U$ is locally quasi-coherent and has the flat base change property. Hence by the previous paragraph we see that $x^*\mathcal{F}$ is locally quasi-coherent and has the flat base change property as desired. □

62.8. Parasitic modules

The following definition is compatible with Descent, Definition 31.7.1.

Definition 62.8.1. Let $\mathcal{X}$ be an algebraic stack. A presheaf of $\mathcal{O}_\mathcal{X}$-modules $\mathcal{F}$ is *parasitic* if we have $\mathcal{F}(x) = 0$ for any object x of $\mathcal{X}$ which lies over a scheme U such that the corresponding morphism $x : U \to \mathcal{X}$ is flat.

Here is a lemma with some properties of this notion.

Lemma 62.8.2. *Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_\mathcal{X}$-modules.*

(1) *If $\mathcal{F}$ is parasitic and $g : \mathcal{Y} \to \mathcal{X}$ is a flat morphism of algebraic stacks, then $g^*\mathcal{F}$ is parasitic.*
(2) *For $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$ we have*
    (a) *the τ sheafification of a parasitic presheaf of modules is parasitic, and*
    (b) *the full subcategory of $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$ consisting of parasitic modules is a Serre subcategory.*

(3) *Suppose $\mathcal{F}$ is a sheaf for the étale topology. Let $f_i : \mathcal{X}_i \to \mathcal{X}$ be a family of smooth morphisms of algebraic stacks such that $|\mathcal{X}| = \bigcup_i |f_i|(|\mathcal{X}_i|)$. If each $f_i^*\mathcal{F}$ is parasitic then so is $\mathcal{F}$.*

(4) *Suppose $\mathcal{F}$ is a sheaf for the fppf topology. Let $f_i : \mathcal{X}_i \to \mathcal{X}$ be a family of flat and locally finitely presented morphisms of algebraic stacks such that $|\mathcal{X}| = \bigcup_i |f_i|(|\mathcal{X}_i|)$. If each $f_i^*\mathcal{F}$ is parasitic then so is $\mathcal{F}$.*

Proof. To see part (1) let y be an object of $\mathcal{Y}$ which lies over a scheme V such that the corresponding morphism $y : V \to \mathcal{Y}$ is flat. Then $g(y) : V \to \mathcal{Y} \to \mathcal{X}$ is flat as a composition of flat morphisms (see Morphisms of Stacks, Lemma 61.17.2) hence $\mathcal{F}(g(y))$ is zero by assumption. Since $g^*\mathcal{F} = g^{-1}\mathcal{F}(y) = \mathcal{F}(g(y))$ we conclude $g^*\mathcal{F}$ is parasitic.

To see part (2)(a) note that if $\{x_i \to x\}$ is a τ-covering of $\mathcal{X}$, then each of the morphisms $x_i \to x$ lies over a flat morphism of schemes. Hence if x lies over a scheme U such that $x : U \to \mathcal{X}$ is flat, so do all of the objects x_i. Hence the presheaf $\mathcal{F}^+$ (see Sites, Section 9.10) is parasitic if the presheaf $\mathcal{F}$ is parasitic. This proves (2)(a) as the sheafification of $\mathcal{F}$ is $(\mathcal{F}^+)^+$.

Let $\mathcal{F}$ be a parasitic τ-module. It is immediate from the definitions that any submodule of $\mathcal{F}$ is parasitic. On the other hand, if $\mathcal{F}' \subset \mathcal{F}$ is a submodule, then it is equally clear that the presheaf $x \mapsto \mathcal{F}(x)/\mathcal{F}'(x)$ is parasitic. Hence the quotient $\mathcal{F}/\mathcal{F}'$ is a parasitic module by (2)(a). Finally, we have to show that given a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ with $\mathcal{F}_1$ and $\mathcal{F}_3$ parasitic, then $\mathcal{F}_2$ is parasitic. This follows immediately on evaluating on x lying over a scheme flat over $\mathcal{X}$. This proves (2)(b), see Homology, Lemma 10.7.2.

Let $f_i : \mathcal{X}_i \to \mathcal{X}$ be a jointly surjective family of smooth morphisms of algebraic stacks and assume each $f_i^*\mathcal{F}$ is parasitic. Let x be an object of $\mathcal{X}$ which lies over a scheme U such that $x : U \to \mathcal{X}$ is flat. Consider a surjective smooth covering $W_i \to U \times_{x,\mathcal{X}} \mathcal{X}_i$. Denote $y_i : W_i \to \mathcal{X}_i$ the projection. It follows that $\{f_i(y_i) \to x\}$ is a covering for the smooth topology on $\mathcal{X}$. Since a composition of flat morphisms is flat we see that $f_i^*\mathcal{F}(y_i) = 0$. On the other hand, as we saw in the proof of (1), we have $f_i^*\mathcal{F}(y_i) = \mathcal{F}(f_i(y_i))$. Hence we see that for some smooth covering $\{x_i \to x\}_{i \in I}$ in $\mathcal{X}$ we have $\mathcal{F}(x_i) = 0$. This implies $\mathcal{F}(x) = 0$ because the smooth topology is the same as as the étale topology, see More on Morphisms, Lemma 33.26.7. Namely, $\{x_i \to x\}_{i \in I}$ lies over a smooth covering $\{U_i \to U\}_{i \in I}$ of schemes. By the lemma just referenced there exists an étale covering $\{V_j \to U\}_{j \in J}$ which refines $\{U_i \to U\}_{i \in I}$. Denote $x'_j = x|_{V_j}$. Then $\{x'_j \to x\}$ is an étale covering in $\mathcal{X}$ refining $\{x_i \to x\}_{i \in I}$. This means the map $\mathcal{F}(x) \to \prod_{j \in J} \mathcal{F}(x'_j)$, which is injective as $\mathcal{F}$ is a sheaf in the étale topology, factors through $\mathcal{F}(x) \to \prod_{i \in I} \mathcal{F}(x_i)$ which is zero. Hence $\mathcal{F}(x) = 0$ as desired.

Proof of (4): omitted. Hint: similar, but simpler, than the proof of (3). □

Parasitic modules are preserved under absolutely any pushforward.

Lemma 62.8.3. *Let $\tau \in \{étale, fppf\}$. Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{F}$ be a parasitic object of $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$.*

(1) *$H^i_\tau(\mathcal{X}, \mathcal{F}) = 0$ for all i.*

(2) *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Then $R^i f_*\mathcal{F}$ (computed in τ-topology) is a parasitic object of $Mod(\mathcal{Y}_\tau, \mathcal{O}_\mathcal{Y})$.*

Proof. We first reduce (2) to (1). By Sheaves on Stacks, Lemma 58.20.2 we see that $R^i f_*\mathcal{F}$ is the sheaf associated to the presheaf

$$y \longmapsto H^i_\tau\Big(V \times_{y,\mathcal{Y}} \mathcal{X},\ \mathrm{pr}^{-1}\mathcal{F}\Big)$$

Here y is a typical object of $\mathcal{Y}$ lying over the scheme V. By Lemma 62.8.2 it suffices to show that these cohomology groups are zero when $y : V \to \mathcal{Y}$ is flat. Note that $\mathrm{pr} : V \times_{y,\mathcal{Y}} \mathcal{X} \to \mathcal{X}$ is flat as a base change of y. Hence by Lemma 62.8.2 we see that $\mathrm{pr}^{-1}\mathcal{F}$ is parasitic. Thus it suffices to prove (1).

To see (1) we can use the spectral sequence of Sheaves on Stacks, Proposition 58.19.1 to reduce this to the the case where $\mathcal{X}$ is an algebraic stack representable by an algebraic space. Note that in the spectral sequence each $f_p^{-1}\mathcal{F} = f_p^*\mathcal{F}$ is a parasitic module by Lemma 62.8.2 because the morphisms $f_p : \mathcal{U}_p = \mathcal{U} \times_{\mathcal{X}} \ldots \times_{\mathcal{X}} \mathcal{U} \to \mathcal{X}$ are flat. Reusing this spectral sequence one more time (as in the proof of the key Lemma 62.5.1) we reduce to the case where the algebraic stack $\mathcal{X}$ is representable by a scheme X. Then $H^i_\tau(\mathcal{X}, \mathcal{F}) = H^i((Sch/X)_\tau, \mathcal{F})$. In this case the vanishing follows easily from an argument with Čech coverings, see Descent, Lemma 31.7.2. □

The following lemma is one of the major reasons we care about parasitic modules. To understand the statement, recall that the functors $QCoh(\mathcal{O}_\mathcal{X}) \to Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ and $QCoh(\mathcal{O}_\mathcal{X}) \to Mod(\mathcal{O}_\mathcal{X})$ aren't exact in general.

Lemma 62.8.4. *Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{F}^\bullet$ be an exact complex in $QCoh(\mathcal{O}_\mathcal{X})$. Then the cohomology sheaves of $\mathcal{F}^\bullet$ in either the étale or the fppf topology are parasitic $\mathcal{O}_\mathcal{X}$-modules.*

Proof. Let $x : U \to \mathcal{X}$ be a flat morphism where U is a scheme. Then $x^*\mathcal{F}^\bullet$ is exact by Lemma 62.4.1. Hence the restriction $x^*\mathcal{F}^\bullet|_{U_{étale}}$ is exact which is what we had to prove. □

62.9. Quasi-coherent modules, I

We have seen that the category of quasi-coherent modules on an algebraic stack is equivalent to the category of quasi-coherent modules on a presentation, see Sheaves on Stacks, Section 58.14. This fact is the basis for the following.

Lemma 62.9.1. *Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{M}_\mathcal{X}$ be the category of locally quasi-coherent modules with the flat base change property, see Proposition 62.7.4. The inclusion functor $i : QCoh(\mathcal{O}_\mathcal{X}) \to \mathcal{M}_\mathcal{X}$ has a right adjoint*

$$Q : \mathcal{M}_\mathcal{X} \to QCoh(\mathcal{O}_\mathcal{X})$$

such that $Q \circ i$ is the identity functor.

Proof. Choose a scheme U and a surjective smooth morphism $f : U \to \mathcal{X}$. Set $R = U \times_\mathcal{X} U$ so that we obtain a smooth groupoid (U, R, s, t, c) in algebraic spaces with the property that $\mathcal{X} = [U/R]$, see Algebraic Stacks, Lemma 57.16.2. We may and do replace $\mathcal{X}$ by $[U/R]$. In the proof of Sheaves on Stacks, Proposition 58.13.1 we constructed a functor

$$q_1 : QCoh(U, R, s, t, c) \longrightarrow QCoh(\mathcal{O}_\mathcal{X}).$$

The construction of the inverse functor in the proof of Sheaves on Stacks, Proposition 58.13.1 works for objects of $\mathcal{M}_\mathcal{X}$ and induces a functor

$$q_2 : \mathcal{M}_\mathcal{X} \longrightarrow QCoh(U, R, s, t, c).$$

Namely, if $\mathcal{F}$ is an object of $\mathcal{M}_\mathcal{X}$ the we set

$$q_2(\mathcal{F}) = (f^*\mathcal{F}|_{U_{\acute{e}tale}}, \alpha)$$

where α is the isomorphism

$$t_{small}^*(f^*\mathcal{F}|_{U_{\acute{e}tale}}) \to t^*f^*\mathcal{F}|_{R_{\acute{e}tale}} \to s^*f^*\mathcal{F}|_{R_{\acute{e}tale}} \to s_{small}^*(f^*\mathcal{F}|_{U_{\acute{e}tale}})$$

where the outer two morphisms are the comparison maps. Note that $q_2(\mathcal{F})$ is quasi-coherent precisely because $\mathcal{F}$ is locally quasi-coherent (and we used the flat base change property in the construction of the descent datum α). We omit the verification that the cocycle condition (see Groupoids in Spaces, Definition 52.12.1) holds. We define $Q = q_1 \circ q_2$. Let $\mathcal{F}$ be an object of $\mathcal{M}_\mathcal{X}$ and let $\mathcal{G}$ be an object of $QCoh(\mathcal{O}_\mathcal{X})$. We have

$$\begin{aligned} Mor_{\mathcal{M}_\mathcal{X}}(i(\mathcal{G}), \mathcal{F}) &= Mor_{QCoh(U,R,s,t,c)}(q_2(\mathcal{G}), q_2(\mathcal{F})) \\ &= Mor_{QCoh(\mathcal{O}_\mathcal{X})}(\mathcal{G}, Q(\mathcal{F})) \end{aligned}$$

where the first equality is Sheaves on Stacks, Lemma 58.13.2 and the second equality holds because q_1 and q_2 are inverse equivalences of categories. The assertion $Q \circ i \cong \text{id}$ is a formal consequence of the fact that i is fully faithful. $\square$

Lemma 62.9.2. *Let $\mathcal{X}$ be an algebraic stack. Let $Q : \mathcal{M}_\mathcal{X} \to QCoh(\mathcal{O}_\mathcal{X})$ be the functor constructed in Lemma 62.9.1.*

(1) *The kernel of Q is exactly the collection of parasitic objects of $\mathcal{M}_\mathcal{X}$.*
(2) *For any object $\mathcal{F}$ of $\mathcal{M}_\mathcal{X}$ both the kernel and the cokernel of the adjunction map $Q(\mathcal{F}) \to \mathcal{F}$ are parasitic.*
(3) *The functor Q is exact.*

Proof. Write $\mathcal{X} = [U/R]$ as in the proof of Lemma 62.9.1. Let $\mathcal{F}$ be an object of $\mathcal{M}_\mathcal{X}$. It is clear from the proof of Lemma 62.9.1 that $\mathcal{F}$ is in the kernel of Q if and only if $\mathcal{F}|_{U_{\acute{e}tale}} = 0$. In particular, if $\mathcal{F}$ is parasitic then $\mathcal{F}$ is in the kernel. Next, let $x : V \to \mathcal{X}$ be a flat morphism, where V is a scheme. Set $W = V \times_\mathcal{X} U$ and consider the diagram

$$\begin{tikzcd} W \arrow[r, "q"] \arrow[d, "p"] & V \arrow[d] \\ U \arrow[r] & \mathcal{X} \end{tikzcd}$$

Note that the projection $p : W \to U$ is flat and the projection $q : W \to V$ is smooth and surjective. This implies that q_{small}^* is a faithful functor on quasi-coherent modules. By assumption $\mathcal{F}$ has the flat base change property so that we obtain $p_{small}^*\mathcal{F}|_{U_{\acute{e}tale}} \cong q_{small}^*\mathcal{F}|_{V_{\acute{e}tale}}$. Thus if $\mathcal{F}$ is in the kernel of Q, then $\mathcal{F}|_{V_{\acute{e}tale}} = 0$ which completes the proof of (1).

Part (2) follows from the discussion above and the fact that the map $Q(\mathcal{F}) \to \mathcal{F}$ becomes an isomorphism after restricting to $U_{\acute{e}tale}$.

To see part (3) note that Q is left exact as a right adjoint. Suppose that $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is a short exact sequence in $\mathcal{M}_\mathcal{X}$. Let $\mathcal{E} = \text{Coker}(Q(\mathcal{G}) \to Q(\mathcal{H}))$ in $QCoh(\mathcal{O}_\mathcal{X})$. Since $QCoh(\mathcal{O}_\mathcal{X}) \to \mathcal{M}_\mathcal{X}$ is a left adjoint it is right exact. Hence we see that $Q(\mathcal{G}) \to Q(\mathcal{H}) \to \mathcal{E} \to 0$ is exact in $\mathcal{M}_\mathcal{X}$. Using Lemma 62.8.4 we find that the top row of the following commutative diagram has parasitic cohomology sheaves at $Q(\mathcal{F})$ and $Q(\mathcal{G})$:

$$\begin{tikzcd} 0 \arrow[r] & Q(\mathcal{F}) \arrow[r] \arrow[d, "a"] & Q(\mathcal{G}) \arrow[r] \arrow[d, "b"] & Q(\mathcal{H}) \arrow[r] \arrow[d, "c"] & \mathcal{E} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{H} \arrow[r] & 0 & \end{tikzcd}$$

The bottom row is exact and the vertical arrows a, b, c have parasitic kernel and cokernels by part (2). It follows that $\mathcal{E}$ is parasitic: in the quotient category of $Mod(\mathcal{O}_\mathcal{X})$/Parasitic (see Homology, Lemma 10.7.6 and Lemma 62.8.2) we see that a, b, c are isomorphisms and that the top row becomes exact. As it is also quasi-coherent, we conclude that $\mathcal{E}$ is zero because $\mathcal{E} = Q(\mathcal{E}) = 0$ by part (1). $\square$

62.10. Pushforward of quasi-coherent modules

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Consider the pushforward

$$f_* : Mod(\mathcal{O}_\mathcal{X}) \longrightarrow Mod(\mathcal{O}_\mathcal{Y})$$

It turns out that this functor almost never preserves the subcategories of quasi-coherent sheaves. For example, consider the morphism of schemes

$$j : X = \mathbf{A}^2_k \setminus \{0\} \longrightarrow \mathbf{A}^2_k = Y.$$

Associated to this we have the corresponding morphism of algebraic stacks

$$f = j_{big} : \mathcal{X} = (Sch/X)_{fppf} \to (Sch/Y)_{fppf} = \mathcal{Y}$$

The pushforward $f_*\mathcal{O}_\mathcal{X}$ of the structure sheaf has global sections $k[x, y]$. Hence if $f_*\mathcal{O}_\mathcal{X}$ is quasi-coherent on $\mathcal{Y}$ then we would have $f_*\mathcal{O}_\mathcal{X} = \mathcal{O}_\mathcal{Y}$. However, consider $T = Spec(k) \to \mathbf{A}^2_k = Y$ mapping to 0. Then $\Gamma(T, f_*\mathcal{O}_\mathcal{X}) = 0$ because $X \times_Y T = \emptyset$ whereas $\Gamma(T, \mathcal{O}_\mathcal{Y}) = k$. On the positive side, we know from Coherent, Lemma 25.6.2 that for any flat morphism $T \to Y$ we have the equality $\Gamma(T, f_*\mathcal{O}_\mathcal{X}) = \Gamma(T, \mathcal{O}_\mathcal{Y})$ (this uses that j is quasi-compact and quasi-separated).

Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. We work around the problem mentioned above using the following three observations:

(1) f_* does preserve locally quasi-coherent modules (Lemma 62.6.2),
(2) f_* transforms a quasi-coherent sheaf into a locally quasi-coherent sheaf whose flat comparison maps are isomorphisms (Lemma 62.7.3), and
(3) locally quasi-coherent $\mathcal{O}_\mathcal{Y}$-modules with the flat base change property give rise to quasi-coherent modules on a presentation of $\mathcal{Y}$ and hence quasi-coherent modules on $\mathcal{Y}$, see Sheaves on Stacks, Section 58.14.

Thus we obtain a functor

$$f_{QCoh,*} : QCoh(\mathcal{O}_\mathcal{X}) \longrightarrow QCoh(\mathcal{O}_\mathcal{Y})$$

which is a right adjoint to $f^* : QCoh(\mathcal{O}_\mathcal{Y}) \to QCoh(\mathcal{O}_\mathcal{X})$ such that moreover

$$\Gamma(y, f_*\mathcal{F}) = \Gamma(y, f_{QCoh,*}\mathcal{F})$$

for any $y \in Ob(\mathcal{Y})$ such that the associated 1-morphism $y : V \to \mathcal{Y}$ is flat, see (insert future reference here). Moreover, a similar construction will produce functors $R^i f_{QCoh,*}$. However, these results will not be sufficient to produce a total direct image functor (of complexes with quasi-coherent cohomology sheaves).

Proposition 62.10.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. The functor $f^* : QCoh(\mathcal{O}_\mathcal{Y}) \to QCoh(\mathcal{O}_\mathcal{X})$ has a right adjoint*

$$f_{QCoh,*} : QCoh(\mathcal{O}_\mathcal{X}) \longrightarrow QCoh(\mathcal{O}_\mathcal{Y})$$

which can be defined as the composition

$$QCoh(\mathcal{O}_\mathcal{X}) \to \mathcal{M}_\mathcal{X} \xrightarrow{f_*} \mathcal{M}_\mathcal{Y} \xrightarrow{Q} QCoh(\mathcal{O}_\mathcal{Y})$$

where the functors f_* and Q are as in Proposition 62.7.4 and Lemma 62.9.1. Moreover, if we define $R^i f_{QCoh,*}$ as the composition

$$QCoh(\mathcal{O}_\mathcal{X}) \to \mathcal{M}_\mathcal{X} \xrightarrow{R^i f_*} \mathcal{M}_\mathcal{Y} \xrightarrow{Q} QCoh(\mathcal{O}_\mathcal{Y})$$

then the sequence of functors $\{R^i f_{QCoh,*}\}_{i \geq 0}$ forms a cohomological δ-functor.

Proof. This is a combination of the results mentioned in the statement. The adjointness can be shown as follows: Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_\mathcal{X}$-module and let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_\mathcal{Y}$-module. Then we have

$$\begin{aligned} Mor_{QCoh(\mathcal{O}_\mathcal{X})}(f^*\mathcal{G}, \mathcal{F}) &= Mor_{\mathcal{M}_\mathcal{Y}}(\mathcal{G}, f_*\mathcal{F}) \\ &= Mor_{QCoh(\mathcal{O}_\mathcal{Y})}(\mathcal{G}, Q(f_*\mathcal{F})) \\ &= Mor_{QCoh(\mathcal{O}_\mathcal{Y})}(\mathcal{G}, f_{QCoh,*}\mathcal{F}) \end{aligned}$$

the first equality by adjointness of f_* and f^* (for arbitrary sheaves of modules). By Proposition 62.7.4 we see that $f_*\mathcal{F}$ is an object of $\mathcal{M}_\mathcal{Y}$ (and can be computed in either the fppf or étale topology) and we obtain the second equality by Lemma 62.9.1. The third equality is the definition of $f_{QCoh,*}$.

To see that $\{R^i f_{QCoh,*}\}_{i \geq 0}$ is a cohomological δ-functor as defined in Homology, Definition 10.9.1 let

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

be a short exact sequence of $QCoh(\mathcal{O}_\mathcal{X})$. This sequence may not be an exact sequence in $Mod(\mathcal{O}_\mathcal{X})$ but we know that it is up to parasitic modules, see Lemma 62.8.4. Thus we may break up the sequence into short exact sequences

$$\begin{aligned} 0 \to \mathcal{P}_1 \to \mathcal{F}_1 \to \mathcal{I}_2 \to 0 \\ 0 \to \mathcal{I}_2 \to \mathcal{F}_2 \to \mathcal{Q}_2 \to 0 \\ 0 \to \mathcal{P}_2 \to \mathcal{Q}_2 \to \mathcal{I}_3 \to 0 \\ 0 \to \mathcal{I}_3 \to \mathcal{F}_3 \to \mathcal{P}_3 \to 0 \end{aligned}$$

of $Mod(\mathcal{O}_\mathcal{X})$ with $\mathcal{P}_i$ parasitic. Note that each of the sheaves $\mathcal{P}_j$, $\mathcal{I}_j$, $\mathcal{Q}_j$ is an object of $\mathcal{M}_\mathcal{X}$, see Proposition 62.7.4. Applying $R^i f_*$ we obtain long exact sequences

$$\begin{aligned} 0 \to f_*\mathcal{P}_1 \to f_*\mathcal{F}_1 \to f_*\mathcal{I}_2 \to R^1 f_*\mathcal{P}_1 \to \ldots \\ 0 \to f_*\mathcal{I}_2 \to f_*\mathcal{F}_2 \to f_*\mathcal{Q}_2 \to R^1 f_*\mathcal{I}_2 \to \ldots \\ 0 \to f_*\mathcal{P}_2 \to f_*\mathcal{Q}_2 \to f_*\mathcal{I}_3 \to R^1 f_*\mathcal{P}_2 \to \ldots \\ 0 \to f_*\mathcal{I}_3 \to f_*\mathcal{F}_3 \to f_*\mathcal{P}_3 \to R^1 f_*\mathcal{I}_3 \to \ldots \end{aligned}$$

where are the terms are objects of $\mathcal{M}_\mathcal{Y}$ by Proposition 62.7.4. By Lemma 62.8.3 the sheaves $R^i f_*\mathcal{P}_j$ are parasitic, hence vanish on applying the functor Q, see Lemma 62.9.2. Since Q is exact the maps

$$Q(R^i f_*\mathcal{F}_3) \cong Q(R^i f_*\mathcal{I}_3) \cong Q(R^i f_*\mathcal{Q}_2) \to Q(R^{i+1} f_*\mathcal{I}_2) \cong Q(R^{i+1} f_*\mathcal{F}_1)$$

can serve as the connecting map which turns the family of functors $\{R^i f_{QCoh,*}\}_{i \geq 0}$ into a cohomological δ-functor. $\square$

Lemma 62.10.2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. Let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathcal{X}$. Then there exists a spectral sequence with E_2-page*

$$E_2^{p,q} = H^p(\mathcal{Y}, R^q f_{QCoh,*}\mathcal{F})$$

converging to $H^{p+q}(\mathcal{X}, \mathcal{F})$.

Proof. By Cohomology on Sites, Lemma 19.14.4 the Leray spectral sequence with

$$E_2^{p,q} = H^p(\mathcal{Y}, R^q f_*\mathcal{F})$$

converges to $H^{p+q}(\mathcal{X}, \mathcal{F})$. The kernel and cokernel of the adjunction map

$$R^q f_{QCoh,*}\mathcal{F} \longrightarrow R^q f_*\mathcal{F}$$

are parasitic modules on $\mathcal{Y}$ (Lemma 62.9.2) hence have vanishing cohomology (Lemma 62.8.3). It follows formally that $H^p(\mathcal{Y}, R^q f_{QCoh,*}\mathcal{F}) = H^p(\mathcal{Y}, R^q f_*\mathcal{F})$ and we win. $\square$

Lemma 62.10.3. *Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be quasi-compact and quasi-separated morphisms of algebraic stacks. Let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathcal{X}$. Then there exists a spectral sequence with E_2-page*

$$E_2^{p,q} = R^p g_{QCoh,*}(R^q f_{QCoh,*}\mathcal{F})$$

converging to $R^{p+q}(g \circ f)_{QCoh,*}\mathcal{F}$.

Proof. By Cohomology on Sites, Lemma 19.14.6 the Leray spectral sequence with

$$E_2^{p,q} = R^p g_*(R^q f_*\mathcal{F})$$

converges to $R^{p+q}(g \circ f)_*\mathcal{F}$. By the results of Proposition 62.7.4 all the terms of this spectral sequence are objects of $\mathcal{M}_\mathcal{Z}$. Applying the exact functor $Q_\mathcal{Z} : \mathcal{M}_\mathcal{Z} \to QCoh(\mathcal{O}_\mathcal{Z})$ we obtain a spectral sequence in $QCoh(\mathcal{O}_\mathcal{Z})$ convering to $R^{p+q}(g \circ f)_{QCoh,*}\mathcal{F}$. Hence the result follows if we can show that

$$Q_\mathcal{Z}(R^p g_*(R^q f_*\mathcal{F})) = Q_\mathcal{Z}(R^p g_*(Q_\mathcal{X}(R^q f_*\mathcal{F}))$$

This follows from the fact that the kernel and cokernel of the map

$$Q_\mathcal{X}(R^q f_*\mathcal{F}) \longrightarrow R^q f_*\mathcal{F}$$

are parasitic (Lemma 62.9.2) and that $R^p g_*$ transforms parasitic modules into parasitic modules (Lemma 62.8.3). $\square$

To end this section we make explicit the spectral sequences associated to a smooth covering by a scheme. Please compare with Sheaves on Stacks, Sections 58.19 and 58.20.

Proposition 62.10.4. *Let $f : \mathcal{U} \to \mathcal{X}$ be a morphism of algebraic stacks. Assume f is representable by algebraic spaces, surjective, flat, and locally of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_\mathcal{X}$-module. Then there is a spectral sequence*

$$E_2^{p,q} = H^q(\mathcal{U}_p, f_p^*\mathcal{F}) \Rightarrow H^{p+q}(\mathcal{X}, \mathcal{F})$$

where f_p is the morphism $\mathcal{U} \times_\mathcal{X} \ldots \times_\mathcal{X} \mathcal{U} \to \mathcal{X}$ ($p + 1$ factors).

Proof. This is a special case of Sheaves on Stacks, Proposition 58.19.1. $\square$

Proposition 62.10.5. *Let $f : \mathcal{U} \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{Y}$ be composable morphisms of algebraic stacks. Assume that*

1. *f is representable by algebraic spaces, surjective, flat, locally of finite presentation, quasi-compact, and quasi-separated, and*
2. *g is quasi-compact and quasi-separated.*

If $\mathcal{F}$ is in $QCoh(\mathcal{O}_\mathcal{X})$ then there is a spectral sequence

$$E_2^{p,q} = R^q(g \circ f_p)_{QCoh,*} f_p^*\mathcal{F} \Rightarrow R^{p+q} g_{QCoh,*}\mathcal{F}$$

in $QCoh(\mathcal{O}_\mathcal{Y})$.

Proof. Note that each of the morphisms $f_p : \mathcal{U} \times_{\mathcal{X}} \ldots \times_{\mathcal{X}} \mathcal{U} \to \mathcal{X}$ is quasi-compact and quasi-separated, hence $g \circ f_p$ is quasi-compact and quasi-separated, hence the assertion makes sense (i.e., the functors $R^q(g \circ f_p)_{QCoh,*}$ are defined). There is a spectral sequence

$$E_2^{p,q} = R^q(g \circ f_p)_* f_p^{-1}\mathcal{F} \Rightarrow R^{p+q}g_*\mathcal{F}$$

by Sheaves on Stacks, Proposition 58.20.1. Applying the exact functor $Q_{\mathcal{Y}} : \mathcal{M}_{\mathcal{Y}} \to QCoh(\mathcal{O}_{\mathcal{Y}})$ gives the desired spectral sequence in $QCoh(\mathcal{O}_{\mathcal{Y}})$. □

62.11. The lisse-étale and the flat-fppf sites

In the book [**LMB00a**] many of the results above are proved using the lisse-étale site of an algebraic stack. We define this site here. In Examples, Section 64.37 we show that the lisse-étale site isn't functorial. We also define its analogue, the flat-fppf site, which is better suited to the development of algebraic stacks as given in the stacks project (because we use the fppf topology as our base topology). Of course the flat-fppf site isn't functorial either.

Definition 62.11.1. Let $\mathcal{X}$ be an algebraic stack.

1. The *lisse-étale site* of $\mathcal{X}$ is the full subcategory $\mathcal{X}_{lisse,étale}$[2] of $\mathcal{X}$ whose objects are those $x \in Ob(\mathcal{X})$ lying over a scheme U such that $x : U \to \mathcal{X}$ is smooth. A covering of $\mathcal{X}_{lisse,étale}$ is a family of morphisms $\{x_i \to x\}_{i\in I}$ of $\mathcal{X}_{lisse,étale}$ which forms a covering of $\mathcal{X}_{étale}$.
2. The *flat-fppf site* of $\mathcal{X}$ is the full subcategory $\mathcal{X}_{flat,fppf}$ of $\mathcal{X}$ whose objects are those $x \in Ob(\mathcal{X})$ lying over a scheme U such that $x : U \to \mathcal{X}$ is flat. A covering of $\mathcal{X}_{flat,fppf}$ is a family of morphisms $\{x_i \to x\}_{i\in I}$ of $\mathcal{X}_{flat,fppf}$ which forms a covering of $\mathcal{X}_{fppf}$.

We denote $\mathcal{O}_{\mathcal{X}_{lisse,étale}}$ the restriction of $\mathcal{O}_{\mathcal{X}}$ to the lisse-étale site and similarly for $\mathcal{O}_{\mathcal{X}_{flat,fppf}}$. The relationship between the lisse-étale site and the étale site is as follows (we mainly stick to ``topological'' properties in this lemma).

Lemma 62.11.2. *Let $\mathcal{X}$ be an algebraic stack.*

1. *The inclusion functor $\mathcal{X}_{lisse,étale} \to \mathcal{X}_{étale}$ is fully faithful, continuous and cocontinuous. It follows that*
   (a) *there is a morphism of topoi*
   $$g : Sh(\mathcal{X}_{lisse,étale}) \longrightarrow Sh(\mathcal{X}_{étale})$$
   with g^{-1} given by restriction,
   (b) *the functor g^{-1} has a left adjoint $g_!^{Sh}$ on sheaves of sets,*
   (c) *the adjunction maps $g^{-1}g_* \to id$ and $id \to g^{-1}g_!^{Sh}$ are isomorphisms,*
   (d) *the functor g^{-1} has a left adjoint $g_!$ on abelian sheaves,*
   (e) *the adjunction map $id \to g^{-1}g_!$ is an isomorphism, and*
   (f) *we have $g^{-1}\mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}_{lisse,étale}}$ hence g induces a flat morphism of ringed topoi such that $g^{-1} = g^*$.*
2. *The inclusion functor $\mathcal{X}_{flat,fppf} \to \mathcal{X}_{fppf}$ is fully faithful, continuous and cocontinuous. It follows that*

[2] In the literature the site is denoted Lis-ét($\mathcal{X}$) or Lis-Et($\mathcal{X}$) and the associated topos is denoted $\mathcal{X}_{\text{lis-ét}}$ or $\mathcal{X}_{\text{lis-et}}$. In the stacks project our convention is to name the site and denote the corresponding topos by $Sh(\mathcal{C})$.

(a) *there is a morphism of topoi*
$$g : Sh(\mathcal{X}_{flat,fppf}) \longrightarrow Sh(\mathcal{X}_{fppf})$$
with g^{-1} given by restriction,
(b) *the functor g^{-1} has a left adjoint $g_!^{Sh}$ on sheaves of sets,*
(c) *the adjunction maps $g^{-1}g_* \to id$ and $id \to g^{-1}g_!^{Sh}$ are isomorphisms,*
(d) *the functor g^{-1} has a left adjoint $g_!$ on abelian sheaves,*
(e) *the adjunction map $id \to g^{-1}g_!$ is an isomorphism, and*
(f) *we have $g^{-1}\mathcal{O}_\mathcal{X} = \mathcal{O}_{\mathcal{X}_{flat,fppf}}$ hence g induces a flat morphism of ringed topoi such that $g^{-1} = g^*$.*

Proof. In both cases it is immediate that the functor is fully faithful, continuous, and cocontinuous (see Sites, Definitions 9.13.1 and 9.18.1). Hence properties (a), (b), (c) follow from Sites, Lemmas 9.19.5 and 9.19.7. Parts (d), (e) follow from Modules on Sites, Lemmas 16.16.2 and 16.16.4. Part (f) is immediate. ☐

Lemma 62.11.3. *Let $\mathcal{X}$ be an algebraic stack. Notation as in Lemma 62.11.2.*

(1) *There exists a functor*
$$g_! : Mod(\mathcal{X}_{lisse,étale}, \mathcal{O}_{\mathcal{X}_{lisse,étale}}) \longrightarrow Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$$
which is left adjoint to g^*. Moreover it agrees with the functor $g_!$ on abelian sheaves and $g^*g_! = id$.
(2) *There exists a functor*
$$g_! : Mod(\mathcal{X}_{flat,fppf}, \mathcal{O}_{\mathcal{X}_{flat,fppf}}) \longrightarrow Mod(\mathcal{X}_{fppf}, \mathcal{O}_\mathcal{X})$$
which is left adjoint to g^*. Moreover it agrees with the functor $g_!$ on abelian sheaves and $g^*g_! = id$.

Proof. In both cases, the existence of the functor $g_!$ follows from Modules on Sites, Lemma 16.35.1. To see that $g_!$ agrees with the functor on abelian sheaves we will show the maps (16.35.2.1) of Modules on Sites, Remark 16.35.2 are isomorphisms.

Lisse-étale case. Let $x \in Ob(\mathcal{X}_{lisse,étale})$ lying over a scheme U with $x : U \to \mathcal{X}$ smooth. Consider the induced fully faithful functor
$$g' : \mathcal{X}_{lisse,étale}/x \longrightarrow \mathcal{X}_{étale}/x$$
The right hand side is identified with $(Sch/U)_{étale}$ and the left hand side with the full subcategory of schemes U'/U such that the composition $U' \to U \to \mathcal{X}$ is smooth. Thus Étale Cohomology, Lemma 38.49.2 applies.

Flat-fppf case. Let $x \in Ob(\mathcal{X}_{flat,fppf})$ lying over a scheme U with $x : U \to \mathcal{X}$ flat. Consider the induced fully faithful functor
$$g' : \mathcal{X}_{flat,fppf}/x \longrightarrow \mathcal{X}_{fppf}/x$$
The right hand side is identified with $(Sch/U)_{fppf}$ and the left hand side with the full subcategory of schemes U'/U such that the composition $U' \to U \to \mathcal{X}$ is flat. Thus Étale Cohomology, Lemma 38.49.2 applies.

In both cases the equality $g^*g_! = \text{id}$ follows from $g^* = g^{-1}$ and the equality for abelian sheaves in Lemma 62.11.2. ☐

Lemma 62.11.4. *Let $\mathcal{X}$ be an algebraic stack. Notation as in Lemmas 62.11.2 and 62.11.3.*

(1) *We have* $g_!\mathcal{O}_{\mathcal{X}_{lisse,étale}} = \mathcal{O}_\mathcal{X}$.
(2) *We have* $g_!\mathcal{O}_{\mathcal{X}_{flat,fppf}} = \mathcal{O}_\mathcal{X}$.

Proof. In this proof we write $\mathcal{C} = \mathcal{X}_{étale}$ (resp. $\mathcal{C} = \mathcal{X}_{fppf}$) and we denote $\mathcal{C}' = \mathcal{X}_{lisse,étale}$ (resp. $\mathcal{C}' = \mathcal{X}_{flat,fppf}$). Then $\mathcal{C}'$ is a full subcategory of $\mathcal{C}$. In this proof we will think of objects V of $\mathcal{C}$ as schemes over $\mathcal{X}$ and objects U of $\mathcal{C}'$ as schemes smooth (resp. flat) over $\mathcal{X}$. Finally, we write $\mathcal{O} = \mathcal{O}_\mathcal{X}$ and $\mathcal{O}' = \mathcal{O}_{\mathcal{X}_{lisse,étale}}$ (resp. $\mathcal{O}' = \mathcal{O}_{\mathcal{X}_{flat,fppf}}$). In the notation above we have $\mathcal{O}(V) = \Gamma(V, \mathcal{O}_V)$ and $\mathcal{O}'(U) = \Gamma(U, \mathcal{O}_U)$. Consider the $\mathcal{O}$-module homomorphism $g_!\mathcal{O}' \to \mathcal{O}$ adjoint to the identification $\mathcal{O}' = g^{-1}\mathcal{O}$.

To see that $g_!\mathcal{O}' \to \mathcal{O}$ is surjective it suffices to show that $1 \in \Gamma(\mathcal{C}, \mathcal{O})$ is locally in the image. Choose an object U of $\mathcal{C}'$ corresponding to a surjective smooth morphism $U \to \mathcal{X}$. Then viewing U both as an object of $\mathcal{C}'$ and $\mathcal{C}$ we have $g_!\mathcal{O}'(U) = \mathcal{O}'(U) = \mathcal{O}(U)$ whence $1 \in \mathcal{O}(U)$ is in the image. Since U surjects onto the final object of $Sh(\mathcal{C})$ we conclude $g_!\mathcal{O}' \to \mathcal{O}$ is surjective.

Suppose that $s \in g_!\mathcal{O}'(V)$ is a section mapping to zero in $\mathcal{O}(V)$. To finish the proof we have to show that s is zero. After replacing V by the members of a covering we may assume s is an element of the colimit

$$colim_{V\to U}\,\mathcal{O}'(U)$$

Say $s = \sum(\varphi_i, s_i)$ is a finite sum with $\varphi_i : V \to U_i$, U_i smooth (resp. flat) over $\mathcal{X}$, and $s_i \in \Gamma(U_i, \mathcal{O}_{U_i})$. Choose a scheme W surjective étale over the algebraic space $U = U_1 \times_\mathcal{X} \ldots \times_\mathcal{X} U_n$. Note that W is still smooth (resp. flat) over $\mathcal{X}$, i.e., defines an object of $\mathcal{C}'$. The fibre product

$$V' = V \times_{(\varphi_1,\ldots,\varphi_n),U} W$$

is surjective étale over V, hence it suffices to show that s maps to zero in $g_!\mathcal{O}'(V')$. Note that the restriction $\sum(\varphi_i, s_i)|_{V'}$ corresponds to the sum of the pullbacks of the functions s_i to W. In other words, we have reduced to the case of (φ, s) where $\varphi : V \to U$ is a morphism with U in $\mathcal{C}'$ and $s \in \mathcal{O}'(U)$ restricts to zero in $\mathcal{O}(V)$. By the commutative diagram

$$\begin{tikzcd} V \arrow[r, "(\varphi,0)"'] \arrow[dr, "\varphi"'] & U \times \mathbf{A}^1 \\ & U \arrow[u, "(\mathrm{id},0)"'] \end{tikzcd}$$

we see that $((\varphi, 0) : V \to U \times \mathbf{A}^1, \mathrm{pr}_2^*x)$ represents zero in the colimit above. Hence we may replace U by $U \times \mathbf{A}^1$, φ by $(\varphi, 0)$ and s by $\mathrm{pr}_1^*s + \mathrm{pr}_2^*x$. Thus we may assume that the vanishing locus $Z : s = 0$ in U of s is smooth (resp. flat) over $\mathcal{X}$. Then we see that $(V \to Z, 0)$ and (φ, s) have the same value in the colimit, i.e., we see that the element s is zero as desired. $\square$

The lisse-étale and the flat-fppf sites can be used to characterize parasitic modules as follows.

Lemma 62.11.5. *Let $\mathcal{X}$ be an algebraic stack.*

(1) *Let $\mathcal{F}$ be an $\mathcal{O}_\mathcal{X}$-module with the flat base change property on $\mathcal{X}_{étale}$. The following are equivalent*
   (a) *$\mathcal{F}$ is parasitic, and*
   (b) *$g^*\mathcal{F} = 0$ where $g : Sh(\mathcal{X}_{lisse,étale}) \to Sh(\mathcal{X}_{étale})$ is as in Lemma 62.11.2.*
(2) *Let $\mathcal{F}$ be an $\mathcal{O}_\mathcal{X}$-module on $\mathcal{X}_{fppf}$. The following are equivalent*
   (a) *$\mathcal{F}$ is parasitic, and*

(b) $g^*\mathcal{F} = 0$ *where* $g : Sh(\mathcal{X}_{flat,fppf}) \to Sh(\mathcal{X}_{fppf})$ *is as in Lemma 62.11.2.*

Proof. Part (2) is immediate from the definitions (this is one of the advantages of the flat-fppf site over the lisse-étale site). The implication (1)(a) ⇒ (1)(b) is immediate as well. To see (1)(b) ⇒ (1)(a) let U be a scheme and let $x : U \to \mathcal{X}$ be a surjective smooth morphism. Then x is an object of the lisse-étale site of $\mathcal{X}$. Hence we see that (1)(b) implies that $\mathcal{F}|_{U_{\acute{e}tale}} = 0$. Let $V \to \mathcal{X}$ be an flat morphism where V is a scheme. Set $W = U \times_{\mathcal{X}} V$ and consider the diagram

$$\begin{tikzcd} W \arrow[r, "q"] \arrow[d, "p"] & V \arrow[d] \\ U \arrow[r] & \mathcal{X} \end{tikzcd}$$

Note that the projection $p : W \to U$ is flat and the projection $q : W \to V$ is smooth and surjective. This implies that q_{small}^* is a faithful functor on quasi-coherent modules. By assumption $\mathcal{F}$ has the flat base change property so that we obtain $p_{small}^*\mathcal{F}|_{U_{\acute{e}tale}} \cong q_{small}^*\mathcal{F}|_{V_{\acute{e}tale}}$. Thus if $\mathcal{F}$ is in the kernel of g^*, then $\mathcal{F}|_{V_{\acute{e}tale}} = 0$ as desired. □

Lemma 62.11.6. *Let $\mathcal{X}$ be an algebraic stack. Notation as in Lemmas 62.11.2 and 62.11.3.*

(1) *The functor $g_! : Ab(\mathcal{X}_{lisse,\acute{e}tale}) \to Ab(\mathcal{X}_{\acute{e}tale})$ has a left derived functor*

$$Lg_! : D(\mathcal{X}_{lisse,\acute{e}tale}) \longrightarrow D(\mathcal{X}_{\acute{e}tale})$$

which is left adjoint to g^{-1} and such that $g^{-1}Lg_! = id$.

(2) *The functor $g_! : Mod(\mathcal{X}_{lisse,\acute{e}tale}, \mathcal{O}_{\mathcal{X}_{lisse,\acute{e}tale}}) \to Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_{\mathcal{X}})$ has a left derived functor*

$$Lg_! : D(\mathcal{O}_{\mathcal{X}_{lisse,\acute{e}tale}}) \longrightarrow D(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_{\mathcal{X}})$$

which is left adjoint to g^* and such that $g^*Lg_! = id$.

(3) *The functor $g_! : Ab(\mathcal{X}_{flat,fppf}) \to Ab(\mathcal{X}_{fppf})$ has a left derived functor*

$$Lg_! : D(\mathcal{X}_{flat,fppf}) \longrightarrow D(\mathcal{X}_{fppf})$$

which is left adjoint to g^{-1} and such that $g^{-1}Lg_! = id$.

(4) *The functor $g_! : Mod(\mathcal{X}_{flat,fppf}, \mathcal{O}_{\mathcal{X}_{flat,fppf}}) \to Mod(\mathcal{X}_{fppf}, \mathcal{O}_{\mathcal{X}})$ has a left derived functor*

$$Lg_! : D(\mathcal{O}_{\mathcal{X}_{flat,fppf}}) \longrightarrow D(\mathcal{O}_{\mathcal{X}})$$

which is left adjoint to g^* and such that $g^*Lg_! = id$.

Warning: It is not clear (a priori) that $Lg_!$ on modules agrees with $Lg_!$ on abelian sheaves, see Cohomology on Sites, Remark 19.22.2.

Proof. The existence of the functor $Lg_!$ and adjointess to g^* is Cohomology on Sites, Lemma 19.22.1. (For the case of abelian sheaves use the constant sheaf $\mathbf{Z}$ as the structure sheaves.) Moreover, it is computed on a complex $\mathcal{H}^\bullet$ by taking a suitable left resolution $\mathcal{K}^\bullet \to \mathcal{H}^\bullet$ and applying the functor $g_!$ to $\mathcal{K}^\bullet$. Since $g^{-1}g_!\mathcal{K}^\bullet = \mathcal{K}^\bullet$ by Lemmas 62.11.3 and 62.11.2 we see that the final assertion holds in each case. □

The lisse-étale site is functorial for smooth morphisms of algebraic stacks and the flat-fppf site is functorial for flat morphisms of algebraic stacks.

Lemma 62.11.7. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.*

(1) *If f is smooth, then f restricts to a continuous and cocontinuous functor $\mathcal{X}_{lisse,étale} \to \mathcal{Y}_{lisse,étale}$ which gives a morphism of ringed topoi fitting into the following commutative diagram*

$$\begin{tikzcd} Sh(\mathcal{X}_{lisse,étale}) \arrow[r, "g'"'] \arrow[d, "f'"'] & Sh(\mathcal{X}_{étale}) \arrow[d, "f"] \\ Sh(\mathcal{Y}_{lisse,étale}) \arrow[r, "g"] & Sh(\mathcal{Y}_{étale}) \end{tikzcd}$$

We have $f'_*(g')^{-1} = g^{-1}f_*$ and $g'_!(f')^{-1} = f^{-1}g_!$.

(2) *If f is flat, then f restricts to a continuous and cocontinuous functor $\mathcal{X}_{flat,fppf} \to \mathcal{Y}_{flat,fppf}$ which gives a morphism of ringed topoi fitting into the following commutative diagram*

$$\begin{tikzcd} Sh(\mathcal{X}_{flat,fppf}) \arrow[r, "g'"'] \arrow[d, "f'"'] & Sh(\mathcal{X}_{fppf}) \arrow[d, "f"] \\ Sh(\mathcal{Y}_{flat,fppf}) \arrow[r, "g"] & Sh(\mathcal{Y}_{fppf}) \end{tikzcd}$$

We have $f'_*(g')^{-1} = g^{-1}f_*$ and $g'_!(f')^{-1} = f^{-1}g_!$.

Proof. The initial statement comes from the fact that if $x \in Ob(\mathcal{X})$ lies over a scheme U such that $x : U \to \mathcal{X}$ is smooth (resp. flat) and if f is smooth (resp. flat) then $f(x) : U \to \mathcal{Y}$ is smooth (resp. flat), see Morphisms of Stacks, Lemmas 61.22.2 and 61.17.2. The induced functor $\mathcal{X}_{lisse,étale} \to \mathcal{Y}_{lisse,étale}$ (resp. $\mathcal{X}_{flat,fppf} \to \mathcal{Y}_{flat,fppf}$) is continuous and cocontinuous by our definition of coverings in these categories. Finally, the commutativity of the diagram is a consequence of the fact that the horizontal morphisms are given by the inclusion functors (see Lemma 62.11.2) and Sites, Lemma 9.19.2.

To show that $f'_*(g')^{-1} = g^{-1}f_*$ let $\mathcal{F}$ be a sheaf on $\mathcal{X}_{étale}$ (resp. $\mathcal{X}_{fppf}$). There is a canonical pullback map

$$g^{-1}f_*\mathcal{F} \longrightarrow f'_*(g')^{-1}\mathcal{F}$$

see Sites, Section 9.39. We claim this map is an isomorphism. To prove this pick an object y of $\mathcal{Y}_{lisse,étale}$ (resp. $\mathcal{Y}_{flat,fppf}$). Say y lies over the scheme V such that $y : V \to \mathcal{Y}$ is smooth (resp. flat). Since g^{-1} is the restriction we find that

$$\left(g^{-1}f_*\mathcal{F}\right)(y) = \Gamma(V \times_{y,\mathcal{Y}} \mathcal{X}, \ \mathrm{pr}^{-1}\mathcal{F})$$

by Sheaves on Stacks, Equation (58.5.0.1). Let $(V \times_{y,\mathcal{Y}} \mathcal{X})' \subset V \times_{y,\mathcal{Y}} \mathcal{X}$ be the full subcategory consisting of objects $z : W \to V \times_{y,\mathcal{Y}} \mathcal{X}$ such that the induced morphism $W \to \mathcal{X}$ is smooth (resp. flat). Denote

$$\mathrm{pr}' : (V \times_{y,\mathcal{Y}} \mathcal{X})' \longrightarrow \mathcal{X}_{lisse,étale} \text{ (resp. } \mathcal{X}_{flat,fppf})$$

the restriction of the functor pr used in the formula above. Exactly the same argument that proves Sheaves on Stacks, Equation (58.5.0.1) shows that for any sheaf $\mathcal{H}$ on $\mathcal{X}_{lisse,étale}$ (resp. $\mathcal{X}_{flat,fppf}$) we have

$$f'_*\mathcal{H}(y) = \Gamma((V \times_{y,\mathcal{Y}} \mathcal{X})', \ (\mathrm{pr}')^{-1}\mathcal{H}) \tag{62.11.7.1}$$

Since $(g')^{-1}$ is restriction we see that

$$\left(f'_*(g')^{-1}\mathcal{F}\right)(y) = \Gamma((V \times_{y,\mathcal{Y}} \mathcal{X})', \ \mathrm{pr}^{-1}\mathcal{F}|_{(V \times_{y,\mathcal{Y}} \mathcal{X})'})$$

By Sheaves on Stacks, Lemma 58.22.3 we see that

$$\Gamma((V \times_{y, \mathcal{Y}} \mathcal{X})', \ \mathrm{pr}^{-1}\mathcal{F}|_{(V \times_{y, \mathcal{Y}} \mathcal{X})'}) = \Gamma(V \times_{y, \mathcal{Y}} \mathcal{X}, \ \mathrm{pr}^{-1}\mathcal{F})$$

are equal as desired; although we omit the verification of the assumptions of the lemma we note that the fact that $V \to \mathcal{Y}$ is smooth (resp. flat) is used to verify the second condition.

Finally, the equality $g'_!(f')^{-1} = f^{-1}g_!$ follows formally from the equality $f'_*(g')^{-1} = g^{-1}f_*$ by the adjointness of f^{-1} and f_*, the adjointness of $g_!$ and g^{-1}, and their ``primed'' versions. □

Lemma 62.11.8. *With assumptions and notation as in Lemma 62.11.7. We have*

$$g^{-1} \circ Rf_* = Rf'_* \circ (g')^{-1} \quad \textit{and} \quad L(g')_! \circ (f')^{-1} = f^{-1} \circ Lg_!$$

on unbounded derived categories (both for the case of modules and for the case of abelian sheaves).

Proof. Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{X}_{\acute{e}tale}$ (resp. $\mathcal{X}_{fppf}$). We first show that the canonical (base change) map

$$g^{-1}Rf_*\mathcal{F} \longrightarrow Rf'_*(g')^{-1}\mathcal{F}$$

is an isomorphism. To do this let y be an object of $\mathcal{Y}_{lisse,\acute{e}tale}$ (resp. $\mathcal{Y}_{flat,fppf}$). Say y lies over the scheme V such that $y : V \to \mathcal{Y}$ is smooth (resp. flat). Since g^{-1} is the restriction we find that

$$\left(g^{-1}R^pf_*\mathcal{F}\right)(y) = H^p_\tau(V \times_{y, \mathcal{Y}} \mathcal{X}, \ \mathrm{pr}^{-1}\mathcal{F})$$

where $\tau = \acute{e}tale$ (resp. $\tau = fppf$), see Sheaves on Stacks, Lemma 58.20.2. By (62.11.7.1) for any sheaf $\mathcal{H}$ on $\mathcal{X}_{lisse,\acute{e}tale}$ (resp. $\mathcal{X}_{flat,fppf}$)

$$f'_*\mathcal{H}(y) = \Gamma((V \times_{y, \mathcal{Y}} \mathcal{X})', \ (\mathrm{pr}')^{-1}\mathcal{H})$$

An object of $(V \times_{y, \mathcal{Y}} \mathcal{X})'$ can be seen as a pair (x, φ) where x is an object of $\mathcal{X}_{lisse,\acute{e}tale}$ (resp. $\mathcal{X}_{flat,fppf}$) and $\varphi : f(x) \to y$ is a morphism in $\mathcal{Y}$. We can also think of φ as a section of $(f')^{-1}h_y$ over x. Thus $(V \times_{\mathcal{Y}} \mathcal{X})'$ is the localization of the site $\mathcal{X}_{lisse,\acute{e}tale}$ (resp. $\mathcal{X}_{flat,fppf}$) at the sheaf of sets $(f')^{-1}h_y$, see Sites, Lemma 9.26.3. The morphism

$$\mathrm{pr}' : (V \times_{y, \mathcal{Y}} \mathcal{X})' \to \mathcal{X}_{lisse,\acute{e}tale} \text{ (resp. } \mathrm{pr}' : (V \times_{y, \mathcal{Y}} \mathcal{X})' \to \mathcal{X}_{flat,fppf})$$

is the localization morphism. In particular, the pullback $(\mathrm{pr}')^{-1}$ preserves injective abelian sheaves, see Cohomology on Sites, Lemma 19.13.3. At this point exactly the same argument as in Sheaves on Stacks, Lemma 58.20.2 shows that

$$R^pf'_*\mathcal{H}(y) = H^p_\tau((V \times_{y, \mathcal{Y}} \mathcal{X})', \ (\mathrm{pr}')^{-1}\mathcal{H}) \tag{62.11.8.1}$$

where $\tau = \acute{e}tale$ (resp. $\tau = fppf$). Since $(g')^{-1}$ is given by restriction we conclude that

$$\left(R^pf'_*(g')^*\mathcal{F}\right)(y) = H^p_\tau((V \times_{y, \mathcal{Y}} \mathcal{X})', \ \mathrm{pr}^{-1}\mathcal{F}|_{(V \times_{y, \mathcal{Y}} \mathcal{X})'})$$

Finally, we can apply Sheaves on Stacks, Lemma 58.22.3 to see that

$$H^p_\tau((V \times_{y, \mathcal{Y}} \mathcal{X})', \ \mathrm{pr}^{-1}\mathcal{F}|_{(V \times_{y, \mathcal{Y}} \mathcal{X})'}) = H^p_\tau(V \times_{y, \mathcal{Y}} \mathcal{X}, \ \mathrm{pr}^{-1}\mathcal{F})$$

are equal as desired; although we omit the verification of the assumptions of the lemma we note that the fact that $V \to \mathcal{Y}$ is smooth (resp. flat) is used to verify the second condition.

The rest of the proof is formal. Since cohomology of abelian groups and sheaves of modules agree we also conclude that $g^{-1}Rf_*\mathcal{F} = Rf'_*(g')^{-1}\mathcal{F}$ when $\mathcal{F}$ is a sheaf of modules on $\mathcal{X}_{\acute{e}tale}$ (resp. $\mathcal{X}_{fppf}$).

Next we show that for $\mathcal{G}$ (either sheaf of modules or abelian groups) on $\mathcal{Y}_{lisse,étale}$ (resp. $\mathcal{Y}_{flat,fppf}$) the canonical map

$$L(g')_!(f')^{-1}\mathcal{G} \to f^{-1}Lg_!\mathcal{G}$$

is an isomorphism. To see this it is enough to prove for any injective sheaf $\mathcal{I}$ on $\mathcal{X}_{étale}$ (resp. $\mathcal{X}_{fppf}$) that the induced map

$$Hom(L(g')_!(f')^{-1}\mathcal{G}, \mathcal{I}[n]) \leftarrow Hom(f^{-1}Lg_!\mathcal{G}, \mathcal{I}[n])$$

is an isomorphism for all $n \in \mathbf{Z}$. (Hom's taken in suitable derived categories.) By the adjointness of f^{-1} and Rf_*, the adjointness of $Lg_!$ and g^{-1}, and their ``primed'' versions this follows from the isomorphism $g^{-1}Rf_*\mathcal{I} \to Rf'_*(g')^{-1}\mathcal{I}$ proved above.

In the case of a bounded complex $\mathcal{G}^\bullet$ (of modules or abelian groups) on $\mathcal{Y}_{lisse,étale}$ (resp. $\mathcal{Y}_{fppf}$) the canonical map

$$L(g')_!(f')^{-1}\mathcal{G}^\bullet \to f^{-1}Lg_!\mathcal{G}^\bullet \tag{62.11.8.2}$$

is an isomorphism as follows from the case of a sheaf by the usual arguments involving truncations and the fact that the functors $L(g')_!(f')^{-1}$ and $f^{-1}Lg_!$ are exact functors of triangulated categories.

Suppose that $\mathcal{G}^\bullet$ is a bounded above complex (of modules or abelian groups) on $\mathcal{Y}_{lisse,étale}$ (resp. $\mathcal{Y}_{fppf}$). The canonical map (62.11.8.2) is an isomorphism because we can use the stupid truncations $\sigma_{\geq -n}$ (see Homology, Section 10.11) to write $\mathcal{G}^\bullet$ as a colimit $\mathcal{G}^\bullet = colim\, \mathcal{G}^\bullet_n$ of bounded complexes. This gives a distinguished triangle

$$\bigoplus\nolimits_{n\geq 1} \mathcal{G}^\bullet_n \to \bigoplus\nolimits_{n\geq 1} \mathcal{G}^\bullet_n \to \mathcal{G}^\bullet \to \ldots$$

and each of the functors $L(g')_!$, $(f')^{-1}$, f^{-1}, $Lg_!$ commutes with direct sums (of complexes).

If $\mathcal{G}^\bullet$ is an arbitrary complex (of modules or abelian groups) on $\mathcal{Y}_{lisse,étale}$ (resp. $\mathcal{Y}_{fppf}$) then we use the canonical truncations $\tau_{\leq n}$ (see Homology, Section 10.11) to write $\mathcal{G}^\bullet$ as a colimit of bounded above complexes and we repeat the argument of the paragraph above.

Finally, by the adjointness of f^{-1} and Rf_*, the adjointness of $Lg_!$ and g^{-1}, and their ``primed'' versions we conclude that the first identity of the lemma follows from the second in full generality. □

62.12. Quasi-coherent modules, II

In this section we explain how to think of quasi-coherent modules on an algebraic stack in terms of its lisse-étale or flat-fppf site.

Lemma 62.12.1. *Let $\mathcal{X}$ be an algebraic stack.*

(1) *Let $f_j : \mathcal{X}_j \to \mathcal{X}$ be a family of smooth morphisms of algebraic stacks with $|\mathcal{X}| = \bigcup |f_j|(|\mathcal{X}_j|)$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_\mathcal{X}$-modules on $\mathcal{X}_{étale}$. If each $f_j^{-1}\mathcal{F}$ is quasi-coherent, then so is $\mathcal{F}$.*
(2) *Let $f_j : \mathcal{X}_j \to \mathcal{X}$ be a family of flat and locally finitely presented morphisms of algebraic stacks with $|\mathcal{X}| = \bigcup |f_j|(|\mathcal{X}_j|)$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_\mathcal{X}$-modules on $\mathcal{X}_{fppf}$. If each $f_j^{-1}\mathcal{F}$ is quasi-coherent, then so is $\mathcal{F}$.*

Proof. Proof of (1). We may replace each of the algebraic stacks $\mathcal{X}_j$ by a scheme U_j (using that any algebraic stack has a smooth covering by a scheme and that compositions of smooth morphisms are smooth, see Morphisms of Stacks, Lemma 61.22.2). The pullback of $\mathcal{F}$ to $(Sch/U_j)_{étale}$ is still locally quasi-coherent, see Sheaves on Stacks, Lemma 58.11.2. Then $f = \coprod f_j : U = \coprod U_j \to \mathcal{X}$ is a smooth surjective morphism. Let $x : V \to \mathcal{X}$ be an object of $\mathcal{X}$. By Sheaves on Stacks, Lemma 58.18.10 there exists an étale covering $\{x_i \to x\}_{i \in I}$ such that each x_i lifts to an object u_i of $(Sch/U)_{étale}$. This just means that x_i lives over a scheme V_i, that $\{V_i \to V\}$ is an étale covering, and that x_i comes from a morphism $u_i : V_i \to U$. Then $x_i^*\mathcal{F} = u_i^* f^*\mathcal{F}$ is quasi-coherent. This implies that $x^*\mathcal{F}$ on $(Sch/V)_{étale}$ is quasi-coherent, for example by Modules on Sites, Lemma 16.23.3. By Sheaves on Stacks, Lemma 58.11.3 we see that $\mathcal{F}$ is quasi-coherent.

Proof of (2). This is proved using exactly the same argument, which we fully write out here. We may replace each of the algebraic stacks $\mathcal{X}_j$ by a scheme U_j (using that any algebraic stack has a smooth covering by a scheme and that flat and locally finite presented morphisms are preserved by composition, see Morphisms of Stacks, Lemmas 61.17.2 and 61.18.2). The pullback of $\mathcal{F}$ to $(Sch/U_j)_{étale}$ is still locally quasi-coherent, see Sheaves on Stacks, Lemma 58.11.2. Then $f = \coprod f_j : U = \coprod U_j \to \mathcal{X}$ is a surjective, flat, and locally finitely presented morphism. Let $x : V \to \mathcal{X}$ be an object of $\mathcal{X}$. By Sheaves on Stacks, Lemma 58.18.10 there exists an fppf covering $\{x_i \to x\}_{i \in I}$ such that each x_i lifts to an object u_i of $(Sch/U)_{étale}$. This just means that x_i lives over a scheme V_i, that $\{V_i \to V\}$ is an fppf covering, and that x_i comes from a morphism $u_i : V_i \to U$. Then $x_i^*\mathcal{F} = u_i^* f^*\mathcal{F}$ is quasi-coherent. This implies that $x^*\mathcal{F}$ on $(Sch/V)_{étale}$ is quasi-coherent, for example by Modules on Sites, Lemma 16.23.3. By Sheaves on Stacks, Lemma 58.11.3 we see that $\mathcal{F}$ is quasi-coherent. □

We recall that we have defined the notion of a quasi-coherent module on any ringed topos in Modules on Sites, Section 16.23.

Lemma 62.12.2. *Let $\mathcal{X}$ be an algebraic stack. Notation as in Lemma 62.11.2.*

1. *Let $\mathcal{H}$ be a quasi-coherent $\mathcal{O}_{\mathcal{X}_{lisse,étale}}$-module on the lisse-étale site of $\mathcal{X}$. Then $g_!\mathcal{H}$ is a quasi-coherent module on $\mathcal{X}$.*
2. *Let $\mathcal{H}$ be a quasi-coherent $\mathcal{O}_{\mathcal{X}_{flat,fppf}}$-module on the flat-fppf site of $\mathcal{X}$. Then $g_!\mathcal{H}$ is a quasi-coherent module on $\mathcal{X}$.*

Proof. Pick a scheme U and a surjective smooth morphism $x : U \to \mathcal{X}$. By Modules on Sites, Definition 16.23.1 there exists an étale (resp. fppf) covering $\{U_i \to U\}_{i \in I}$ such that each pullback $f_i^{-1}\mathcal{H}$ has a global presentation (see Modules on Sites, Definition 16.17.1). Here $f_i : U_i \to \mathcal{X}$ is the composition $U_i \to U \to \mathcal{X}$ which is a morphism of algebraic stacks. (Recall that the pullback ``is'' the restriction to $\mathcal{X}/f_i$, see Sheaves on Stacks, Definition 58.9.2 and the discussion following.) Since each f_i is smooth (resp. flat) by Lemma 62.11.7 we see that $f_i^{-1}g_!\mathcal{H} = g_{i,!}(f_i')^{-1}\mathcal{H}$. Using Lemma 62.12.1 we reduce the statement of the lemma to the case where $\mathcal{H}$ has a global presentation. Say we have

$$\bigoplus\nolimits_{j \in J} \mathcal{O} \longrightarrow \bigoplus\nolimits_{i \in I} \mathcal{O} \longrightarrow \mathcal{H} \longrightarrow 0$$

of $\mathcal{O}$-modules where $\mathcal{O} = \mathcal{O}_{\mathcal{X}_{lisse,étale}}$ (resp. $\mathcal{O} = \mathcal{O}_{\mathcal{X}_{flat,fppf}}$). Since $g_!$ commutes with arbitrary colimits (as a left adjoint functor, see Lemma 62.11.3 and Categories, Lemma 4.22.2) we conclude that there exists an exact sequence

$$\bigoplus\nolimits_{j \in J} g_!\mathcal{O} \longrightarrow \bigoplus\nolimits_{i \in I} g_!\mathcal{O} \longrightarrow g_!\mathcal{H} \longrightarrow 0$$

Finally, Lemma 62.11.4 shows that $g_!\mathcal{O} = \mathcal{O}_\mathcal{X}$ and we win. □

Lemma 62.12.3. *Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{M}_\mathcal{X}$ be the category of locally quasi-coherent $\mathcal{O}_\mathcal{X}$-modules with the flat base change property.*

(1) *With g as in Lemma 62.11.2 for the lisse-étale site we have*
   (a) *the functors g^{-1} and $g_!$ define mutually inverse functors*

$$QCoh(\mathcal{O}_\mathcal{X}) \underset{g_!}{\overset{g^{-1}}{\rightleftarrows}} QCoh(\mathcal{X}_{lisse,étale}, \mathcal{O}_{\mathcal{X}_{lisse,étale}})$$

   (b) *if $\mathcal{F}$ is in $\mathcal{M}_\mathcal{X}$ then $g^{-1}\mathcal{F}$ is in $QCoh(\mathcal{X}_{lisse,étale}, \mathcal{O}_{\mathcal{X}_{lisse,étale}})$ and*
   (c) *$Q(\mathcal{F}) = g_!g^{-1}\mathcal{F}$ where Q is as in Lemma 62.9.1.*

(2) *With g as in Lemma 62.11.2 for the flat-fppf site we have*
   (a) *the functors g^{-1} and $g_!$ define mutually inverse functors*

$$QCoh(\mathcal{O}_\mathcal{X}) \underset{g_!}{\overset{g^{-1}}{\rightleftarrows}} QCoh(\mathcal{X}_{flat,fppf}, \mathcal{O}_{\mathcal{X}_{flat,fppf}})$$

   (b) *if $\mathcal{F}$ is in $\mathcal{M}_\mathcal{X}$ then $g^{-1}\mathcal{F}$ is in $QCoh(\mathcal{X}_{flat,fppf}, \mathcal{O}_{\mathcal{X}_{flat,fppf}})$ and*
   (c) *$Q(\mathcal{F}) = g_!g^{-1}\mathcal{F}$ where Q is as in Lemma 62.9.1.*

Proof. Pullback by any morphism of ringed topoi preserves categories of quasi-coherent modules, see Modules on Sites, Lemma 16.23.4. Hence g^{-1} preserves the categories of quasi-coherent modules. The same is true for $g_!$ by Lemma 62.12.2. We know that $\mathcal{H} \to g^{-1}g_!\mathcal{H}$ is an isomorphism by Lemma 62.11.2. Conversely, if $\mathcal{F}$ is in $QCoh(\mathcal{O}_\mathcal{X})$ then the map $g_!g^{-1}\mathcal{F} \to \mathcal{F}$ is a map of quasi-coherent modules on $\mathcal{X}$ whose restriction to any scheme smooth over $\mathcal{X}$ is an isomorphism. Then the discussion in Sheaves on Stacks, Sections 58.13 and 58.14 (comparing with quasi-coherent modules on presentations) shows it is an isomorphism. This proves (1)(a) and (2)(a).

Let $\mathcal{F}$ be an object of $\mathcal{M}_\mathcal{X}$. By Lemma 62.9.2 the kernel and cokernel of the map $Q(\mathcal{F}) \to \mathcal{F}$ are parasitic. Hence by Lemma 62.11.5 and since $g^* = g^{-1}$ is exact, we conclude $g^*Q(\mathcal{F}) \to g^*\mathcal{F}$ is an isomorphism. Thus $g^*\mathcal{F}$ is quasi-coherent. This proves (1)(b) and (2)(b). Finally, (1)(c) and (2)(c) follow because $g_!g^*Q(\mathcal{F}) \to Q(\mathcal{F})$ is an isomorphism by our arguments above. □

Remark 62.12.4. Let $\mathcal{X}$ be an algebraic stack. The results of Lemmas 62.9.1 and 62.9.2 imply that

$$QCoh(\mathcal{O}_\mathcal{X}) = \mathcal{M}_\mathcal{X}/\text{Parasitic} \cap \mathcal{M}_\mathcal{X}$$

in words: the category of quasi-coherent modules is the category of locally quasi-coherent modules with the flat base change property divided out by the Serre subcategory consisting of parasitic objects. See Homology, Lemma 10.7.6. The existence of the inclusion functor $i : QCoh(\mathcal{O}_\mathcal{X}) \to \mathcal{M}_\mathcal{X}$ which is left adjoint to the quotient functor means that $\mathcal{M}_\mathcal{X} \to QCoh(\mathcal{O}_\mathcal{X})$ is a *Bousfield colocalization* or a *right Bousfield localization* (insert future reference here). Our next goal is to show a similar result holds on the level of derived categories.

Lemma 62.12.5. *Let $\mathcal{X}$ be an algebraic stack. Notation as in Lemma 62.11.2.*

(1) *Let $\mathcal{H}$ be a quasi-coherent $\mathcal{O}_{\mathcal{X}_{lisse,étale}}$-module on the lisse-étale site of $\mathcal{X}$. For all $p \in \mathbf{Z}$ the sheaf $H^p(Lg_!\mathcal{H})$ is a locally quasi-coherent module with the flat base change property on $\mathcal{X}$.*
(2) *Let $\mathcal{H}$ be a quasi-coherent $\mathcal{O}_{\mathcal{X}_{flat,fppf}}$-module on the flat-fppf site of $\mathcal{X}$. For all $p \in \mathbf{Z}$ the sheaf $H^p(Lg_!\mathcal{H})$ is a locally quasi-coherent module with the flat base change property on $\mathcal{X}$.*

Proof. Pick a scheme U and a surjective smooth morphism $x : U \to \mathcal{X}$. By Modules on Sites, Definition 16.23.1 there exists an étale (resp. fppf) covering $\{U_i \to U\}_{i\in I}$ such that each pullback $f_i^{-1}\mathcal{H}$ has a global presentation (see Modules on Sites, Definition 16.17.1). Here $f_i : U_i \to \mathcal{X}$ is the composition $U_i \to U \to \mathcal{X}$ which is a morphism of algebraic stacks. (Recall that the pullback ``is" the restriction to $\mathcal{X}/f_i$, see Sheaves on Stacks, Definition 58.9.2 and the discussion following.) After refining the covering we may assume each U_i is an affine scheme. Since each f_i is smooth (resp. flat) by Lemma 62.11.8 we see that $f_i^{-1}Lg_!\mathcal{H} = Lg_{i,!}(f_i')^{-1}\mathcal{H}$. Using Lemma 62.7.5 we reduce the statement of the lemma to the case where $\mathcal{H}$ has a global presentation and where $\mathcal{X} = (Sch/X)_{fppf}$ for some affine scheme $X = Spec(A)$.

Say our presentation looks like

$$\bigoplus\nolimits_{j\in J}\mathcal{O} \longrightarrow \bigoplus\nolimits_{i\in I}\mathcal{O} \longrightarrow \mathcal{H} \longrightarrow 0$$

where $\mathcal{O} = \mathcal{O}_{\mathcal{X}_{lisse,étale}}$ (resp. $\mathcal{O} = \mathcal{O}_{\mathcal{X}_{flat,fppf}}$). Note that the site $\mathcal{X}_{lisse,étale}$ (resp. $\mathcal{X}_{flat,fppf}$) has a final object, namely X/X which is quasi-compact (see Cohomology on Sites, Section 19.16). Hence we have

$$\Gamma(\bigoplus\nolimits_{i\in I}\mathcal{O}) = \bigoplus\nolimits_{i\in I} A$$

by Cohomology on Sites, Lemma 19.16.1. Hence the map in the presentation corresponds to a similar presentation

$$\bigoplus\nolimits_{j\in J} A \longrightarrow \bigoplus\nolimits_{i\in I} A \longrightarrow M \longrightarrow 0$$

of an A-module M. Moreover, $\mathcal{H}$ is equal to the restriction to the lisse-étale (resp. flat-fppf) site of the quasi-coherent sheaf M^a associated to M. Choose a resolution

$$\ldots \to F_2 \to F_1 \to F_0 \to M \to 0$$

by free A-modules. The complex

$$\ldots \mathcal{O} \otimes_A F_2 \to \mathcal{O} \otimes_A F_1 \to \mathcal{O} \otimes_A F_0 \to \mathcal{H} \to 0$$

is a resolution of $\mathcal{H}$ by free $\mathcal{O}$-modules because for each object U/X of $\mathcal{X}_{lisse,étale}$ (resp. $\mathcal{X}_{flat,fppf}$) the structure morphism $U \to X$ is flat. Hence by construction the value of $Lg_!\mathcal{H}$ is

$$\ldots \to \mathcal{O}_{\mathcal{X}} \otimes_A F_2 \to \mathcal{O}_{\mathcal{X}} \otimes_A F_1 \to \mathcal{O}_{\mathcal{X}} \otimes_A F_0 \to 0 \to \ldots$$

Since this is a complex of quasi-coherent modules on $\mathcal{X}_{étale}$ (resp. $\mathcal{X}_{fppf}$) it follows from Proposition 62.7.4 that $H^p(Lg_!\mathcal{H})$ is quasi-coherent. $\square$

Lemma 62.12.6. *Let $\mathcal{X}$ be an algebraic stack.*

(1) *$QCoh(\mathcal{O}_{\mathcal{X}_{lisse,étale}})$ is a weak Serre subcategory of $Mod(\mathcal{O}_{\mathcal{X}_{lisse,étale}})$.*
(2) *$QCoh(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$ is a weak Serre subcategory of $Mod(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$.*

Proof. We will verify conditions (1), (2), (3), (4) of Homology, Lemma 10.7.3. Since 0 is a quasi-coherent module on any ringed site we see that (1) holds. By definition $QCoh(\mathcal{O})$ is a strictly full subcategory $Mod(\mathcal{O})$, so (2) holds. Let $\varphi : \mathcal{G} \to \mathcal{F}$ be a morphism of quasi-coherent modules on $\mathcal{X}_{lisse,étale}$ or $\mathcal{X}_{flat,fppf}$. We have $g^*g_!\mathcal{F} = \mathcal{F}$ and similarly for $\mathcal{G}$ and φ, see Lemma 62.11.3. By Lemma 62.12.2 we see that $g_!\mathcal{F}$ and $g_!\mathcal{G}$ are quasi-coherent $\mathcal{O}_\mathcal{X}$-modules. Hence we see that $\text{Ker}(g_!\varphi)$ and $\text{Coker}(g_!\varphi)$ are quasi-coherent modules on $\mathcal{X}$. Since g^* is exact (see Lemma 62.11.2) we see that $g^*\text{Ker}(g_!\varphi) = \text{Ker}(g^*g_!\varphi) = \text{Ker}(\varphi)$ and $g^*\text{Coker}(g_!\varphi) = \text{Coker}(g^*g_!\varphi) = \text{Coker}(\varphi)$ are quasi-coherent too (see Lemma 62.12.3). This proves (3). Finally, suppose that

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0$$

is an extension of $\mathcal{O}_{\mathcal{X}_{lisse,étale}}$-modules (resp. $\mathcal{O}_{\mathcal{X}_{flat,fppf}}$-modules) with $\mathcal{F}$ and $\mathcal{G}$ quasi-coherent. Then

$$H^{-1}(Lg_!\mathcal{G}) \to g_!\mathcal{F} \to g_!\mathcal{E} \to g_!\mathcal{G} \to 0$$

is an exact sequence with $g_!\mathcal{F}$, $g_!\mathcal{G}$, and $H^{-1}(Lg_!\mathcal{G})$ locally quasi-coherent with the flat base change property, see Lemma 62.12.5. By Proposition 62.7.4 it follows that $g_!\mathcal{E}$ is locally quasi-coherent with the flat base change property. Finally, Lemma 62.12.3 implies that $\mathcal{E} = g^{-1}g_!\mathcal{E}$ is quasi-coherent as desired. □

62.13. Derived categories of quasi-coherent modules

Let $\mathcal{X}$ be an algebraic stack. As the inclusion functor $QCoh(\mathcal{O}_\mathcal{X}) \to Mod(\mathcal{O}_\mathcal{X})$ isn't exact, we cannot define $D_{QCoh}(\mathcal{O}_\mathcal{X})$ as the full subcategory of $D(\mathcal{O}_\mathcal{X})$ consisting of complexes with quasi-coherent cohomology sheaves. In stead we define the category as follows.

Definition 62.13.1. Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{M}_\mathcal{X} \subset Mod(\mathcal{O}_\mathcal{X})$ denote the category of locally quasi-coherent $\mathcal{O}_\mathcal{X}$-modules with the flat base change property. Let $\mathcal{P}_\mathcal{X} \subset \mathcal{M}_\mathcal{X}$ be the full subcategory consisting of parasitic objects. We define the *derived category of $\mathcal{O}_\mathcal{X}$-modules with quasi-coherent cohomology sheaves* as the Verdier quotient[3]

$$D_{QCoh}(\mathcal{O}_\mathcal{X}) = D_{\mathcal{M}_\mathcal{X}}(\mathcal{O}_\mathcal{X})/D_{\mathcal{P}_\mathcal{X}}(\mathcal{O}_\mathcal{X})$$

This definition makes sense: By Proposition 62.7.4 we see that $\mathcal{M}_\mathcal{X}$ is a weak Serre subcategory of $Mod(\mathcal{O}_\mathcal{X})$ hence $D_{\mathcal{M}_\mathcal{X}}(\mathcal{O}_\mathcal{X})$ is a strictly full, saturated triangulated subcategory of $D(\mathcal{O}_\mathcal{X})$, see Derived Categories, Lemma 11.12.1. Since parasitic modules form a Serre subcategory of $Mod(\mathcal{O}_\mathcal{X})$ (by Lemma 62.8.2) we see that $\mathcal{P}_\mathcal{X} = \text{Parasitic} \cap \mathcal{M}_\mathcal{X}$ is a weak Serre subcategory of $Mod(\mathcal{O}_\mathcal{X})$ and hence $D_{\mathcal{P}_\mathcal{X}}(\mathcal{O}_\mathcal{X})$ is a strictly full, saturated triangulated subcategory of $D(\mathcal{O}_\mathcal{X})$. Since clearly

$$D_{\mathcal{P}_\mathcal{X}}(\mathcal{O}_\mathcal{X}) \subset D_{\mathcal{M}_\mathcal{X}}(\mathcal{O}_\mathcal{X})$$

we conclude that the first is a strictly full, saturated triangulated subcategory of the second. Hence the Verdier quotient exists. A morphism $a : E \to E'$ of $D_{\mathcal{M}_\mathcal{X}}(\mathcal{O}_\mathcal{X})$ becomes an isomorphism in $D_{QCoh}(\mathcal{O}_\mathcal{X})$ if and only if the cone $C(a)$ has parasitic cohomology sheaves, see Derived Categories, Section 11.6 and especially Lemma 11.6.10.

Consider the functors

$$D_{\mathcal{M}_\mathcal{X}}(\mathcal{O}_\mathcal{X}) \xrightarrow{H^i} \mathcal{M}_\mathcal{X} \xrightarrow{Q} QCoh(\mathcal{O}_\mathcal{X})$$

[3] This definition is different from the one in the literature, see [**Ols07b**, 6.3], but it agrees with that definition by Lemma 62.13.3.

Note that Q annihilates the subcategory $\mathcal{P}_\mathcal{X}$, see Lemma 62.9.2. By Derived Categories, Lemma 11.6.8 we obtain a cohomological functor

$$H^i : D_{QCoh}(\mathcal{O}_\mathcal{X}) \longrightarrow QCoh(\mathcal{O}_\mathcal{X}) \tag{62.13.1.1}$$

Moreover, note that $E \in D_{QCoh}(\mathcal{O}_\mathcal{X})$ is zero if and only if $H^i(E) = 0$ for all $i \in \mathbf{Z}$.

Note that the categories $\mathcal{P}_\mathcal{X}$ and $\mathcal{M}_\mathcal{X}$ are also weak Serre subcategories of the abelian category $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ of modules in the étale topology, see Proposition 62.7.4 and Lemma 62.8.2. Hence the statement of the following lemma makes sense.

Lemma 62.13.2. *Let $\mathcal{X}$ be an algebraic stack. The comparision morphism $\epsilon : \mathcal{X}_{fppf} \to \mathcal{X}_{étale}$ induces a commutative diagram*

$$\begin{tikzcd} D_{\mathcal{P}_\mathcal{X}}(\mathcal{O}_\mathcal{X}) \arrow[r] & D_{\mathcal{M}_\mathcal{X}}(\mathcal{O}_\mathcal{X}) \arrow[r] & D(\mathcal{O}_\mathcal{X}) \\ D_{\mathcal{P}_\mathcal{X}}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X}) \arrow[r] \arrow[u, "\epsilon^*"] & D_{\mathcal{M}_\mathcal{X}}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X}) \arrow[r] \arrow[u, "\epsilon^*"] & D(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X}) \arrow[u, "\epsilon^*"] \end{tikzcd}$$

Moreover, the left two vertical arrows are equivalences of triangulated categories, hence we also obtain an equivalence

$$D_{\mathcal{M}_\mathcal{X}}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})/D_{\mathcal{P}_\mathcal{X}}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X}) \longrightarrow D_{QCoh}(\mathcal{O}_\mathcal{X})$$

Proof. Since ϵ^* is exact it is clear that we obtain a diagram as in the statement of the lemma. We will show the middle vertical arrow is an equivalence by applying Cohomology on Sites, Lemma 19.20.2 to the following situation: $\mathcal{C} = \mathcal{X}$, $\tau = fppf$, $\tau' = étale$, $\mathcal{O} = \mathcal{O}_\mathcal{X}$, $\mathcal{A} = \mathcal{M}_\mathcal{X}$, and $\mathcal{B}$ is the set of objects of $\mathcal{X}$ lying over affine schemes. To see the lemma applies we have to check conditions (1), (2), (3), (4). Conditions (1) and (2) are clear from the discussion above (explicitly this follows from Proposition 62.7.4). Condition (3) holds because every scheme has a Zariski open covering by affines. Condition (4) follows from Descent, Lemma 31.7.4.

We omit the verification that the equivalence of categories $\epsilon^* : D_{\mathcal{M}_\mathcal{X}}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X}) \to D_{\mathcal{M}_\mathcal{X}}(\mathcal{O}_\mathcal{X})$ induces an equivalence of the subcategories of complexes with parasitic cohomology sheaves. $\square$

It turns out that $D_{QCoh}(\mathcal{O}_\mathcal{X})$ is the same as the derived category of complexes of modules with quasi-coherent cohomology sheaves on the lisse-étale or flat-fppf site.

Lemma 62.13.3. *Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{F}^\bullet$ be an object of $D_{\mathcal{M}_\mathcal{X}}(\mathcal{O}_\mathcal{X})$.*

1. *With g as in Lemma 62.11.2 for the lisse-étale site we have*
   - (a) *$g^{-1}\mathcal{F}^\bullet$ is in $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{lisse,étale}})$,*
   - (b) *$g^{-1}\mathcal{F}^\bullet = 0$ if and only if $\mathcal{F}^\bullet$ is in $D_{\mathcal{P}_\mathcal{X}}(\mathcal{O}_\mathcal{X})$,*
   - (c) *$Lg_!\mathcal{H}^\bullet$ is in $D_{\mathcal{M}_\mathcal{X}}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ for $\mathcal{H}^\bullet$ in $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{lisse,étale}})$, and*
   - (d) *the functors g^{-1} and $Lg_!$ define mutually inverse functors*

$$\begin{tikzcd} D_{QCoh}(\mathcal{O}_\mathcal{X}) \arrow[r, shift left, "g^{-1}"] & D_{QCoh}(\mathcal{O}_{\mathcal{X}_{lisse,étale}}) \arrow[l, shift left, "Lg_!"] \end{tikzcd}$$

2. *With g as in Lemma 62.11.2 for the flat-fppf site we have*
   - (a) *$g^{-1}\mathcal{F}^\bullet$ is in $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{lisse,étale}})$,*
   - (b) *$g^{-1}\mathcal{F}^\bullet = 0$ if and only if $\mathcal{F}^\bullet$ is in $D_{\mathcal{P}_\mathcal{X}}(\mathcal{O}_\mathcal{X})$,*

(c) *$Lg_!\mathcal{H}^\bullet$ is in $D_{\mathcal{M}_\mathcal{X}}(\mathcal{O}_\mathcal{X})$ for $\mathcal{H}^\bullet$ in $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$, and*

(d) *the functors g^{-1} and $Lg_!$ define mutually inverse functors*

$$D_{QCoh}(\mathcal{O}_\mathcal{X}) \underset{Lg_!}{\overset{g^{-1}}{\rightleftarrows}} D_{QCoh}(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$$

Proof. The functor g^{-1} is exact, hence (a) and (b) follow from Lemmas 62.12.3 and 62.11.5.

The construction of $Lg_!$ in Lemma 62.11.6 (via Cohomology on Sites, Lemma 19.22.1 which in turn uses Derived Categories, Proposition 11.27.2) shows that $Lg_!$ on any object $\mathcal{H}^\bullet$ of $D(\mathcal{O}_{\mathcal{X}_{lisse,étale}})$ is computed as

$$Lg_!\mathcal{H}^\bullet = colim\, g_!\mathcal{K}_n^\bullet = g_!\, colim\, \mathcal{K}_n^\bullet$$

(termwise colimits) where the quasi-isomorphism $colim\, \mathcal{K}_n^\bullet \to \mathcal{H}^\bullet$ induces quasi-isomorphisms $\mathcal{K}_n^\bullet \to \tau_{\leq n}\mathcal{H}^\bullet$. Since $\mathcal{M}_\mathcal{X} \subset Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ (resp. $\mathcal{M}_\mathcal{X} \subset Mod(\mathcal{O}_\mathcal{X})$) is preserved under colimits we see that it suffices to prove (c) on bounded above complexes $\mathcal{H}^\bullet$ in $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{lisse,étale}})$ (resp. $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$). In this case to show that $H^n(Lg_!\mathcal{H}^\bullet)$ is in $\mathcal{M}_\mathcal{X}$ we can argue by induction on the integer m such that $\mathcal{H}^i = 0$ for $i > m$. If $m < n$, then $H^n(Lg_!\mathcal{H}^\bullet) = 0$ and the result holds. In general consider the distinguished triangle

$$\tau_{\leq m-1}\mathcal{H}^\bullet \to \mathcal{H}^\bullet \to H^m(\mathcal{H}^\bullet)[-m] \to \ldots$$

and apply the functor $Lg_!$. Since $\mathcal{M}_\mathcal{X}$ is a weak Serre subcategory of the module category it suffices to prove (c) for two out of three. We have the result for $Lg_!\tau_{\leq m-1}\mathcal{H}^\bullet$ by induction and we have the result for $Lg_!H^m(\mathcal{H}^\bullet)[-m]$ by Lemma 62.12.5. Whence (c) holds.

Let us prove (2)(d). By (a) and (b) the functor $g^{-1} = g^*$ induces a functor

$$c : D_{QCoh}(\mathcal{O}_\mathcal{X}) \longrightarrow D_{QCoh}(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$$

see Derived Categories, Lemma 11.6.8. Thus we have the following diagram of triangulated categories

$$\begin{tikzcd} D_{\mathcal{M}_\mathcal{X}}(\mathcal{O}_\mathcal{X}) \arrow[rr, "q"] \arrow[dr, shift left, "g^{-1}"] & & D_{QCoh}(\mathcal{O}_\mathcal{X}) \arrow[dl, "c"] \\ & D_{QCoh}(\mathcal{O}_{\mathcal{X}_{flat,fppf}}) \arrow[ul, shift left, "Lg_!"] & \end{tikzcd}$$

where q is the quotient functor, the inner triangle is commutative, and $g^{-1}Lg_! = \text{id}$. For any object of E of $D_{\mathcal{M}_\mathcal{X}}(\mathcal{O}_\mathcal{X})$ the map $a : Lg_!g^{-1}E \to E$ maps to a quasi-isomorphism in $D(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$. Hence the cone on a maps to zero under g^{-1} and by (b) we see that $q(a)$ is an isomorphism. Thus $q \circ Lg_!$ is a quasi-inverse to c.

In the case of the lisse-étale site exactly the same argument as above proves that

$$D_{\mathcal{M}_\mathcal{X}}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})/D_{\mathcal{P}_\mathcal{X}}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$$

is equivalent to $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{lisse,étale}})$. Applying the last equivalence of Lemma 62.13.2 finishes the proof. □

The following lemma tells us that the quotient functor $D_{\mathcal{M}_\mathcal{X}}(\mathcal{O}_\mathcal{X}) \to D_{QCoh}(\mathcal{O}_\mathcal{X})$ is a Bousfield colocalization (insert future reference here).

Lemma 62.13.4. *Let $\mathcal{X}$ be an algebraic stack. Let E be an object of $D_{\mathcal{M}_\mathcal{X}}(\mathcal{O}_\mathcal{X})$. There exists a canonical distinguished triangle*

$$E' \to E \to P \to E'[1]$$

in $D_{\mathcal{M}_\mathcal{X}}(\mathcal{O}_\mathcal{X})$ such that P is in $D_{\mathcal{P}_\mathcal{X}}(\mathcal{O}_\mathcal{X})$ and

$$Hom_{D(\mathcal{O}_\mathcal{X})}(E', P') = 0$$

for all P' in $D_{\mathcal{P}_\mathcal{X}}(\mathcal{O}_\mathcal{X})$.

Proof. Consider the morphism of ringed topoi $g : Sh(\mathcal{X}_{flat,fppf}) \longrightarrow Sh(\mathcal{X}_{fppf})$. Set $E' = Lg_!g^{-1}E$ and let P be the cone on the adjunction map $E' \to E$. Since $g^{-1}E' \to g^{-1}E$ is an isomorphism we see that P is an object of $D_{\mathcal{P}_\mathcal{X}}(\mathcal{O}_\mathcal{X})$ by Lemma 62.13.3 (2)(b). Finally, $Hom(E', P') = Hom(Lg_!g^{-1}E, P') = Hom(g^{-1}E, g^{-1}P') = 0$ as $g^{-1}P' = 0$.

Uniqueness. Suppose that $E'' \to E \to P'$ is a second distinguished triangle as in the statement of the lemma. Since $Hom(E', P') = 0$ the morphism $E' \to E$ factors as $E' \to E'' \to E$, see Derived Categories, Lemma 11.4.2. Similarly, the morphism $E'' \to E$ factors as $E'' \to E' \to E$. Consider the composition $\varphi : E' \to E'$ of the maps $E' \to E''$ and $E'' \to E'$. Note that $\varphi - 1 : E' \to E'$ fits into the commutative diagram

$$\begin{tikzcd} E' \arrow[r] \arrow[d, "\varphi-1"] & E \arrow[d, "0"] \\ E' \arrow[r] & E \end{tikzcd}$$

hence factors through $P[-1] \to E$. Since $Hom(E', P[-1]) = 0$ we see that $\varphi = 1$. Whence the maps $E' \to E''$ and $E'' \to E'$ are inverse to each other. □

62.14. Derived pushforward of quasi-coherent modules

Here is a first application of the material above.

Proposition 62.14.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. The functor Rf_* induces a commutative diagram*

$$\begin{tikzcd} D^+_{\mathcal{P}_\mathcal{X}}(\mathcal{O}_\mathcal{X}) \arrow[r] \arrow[d, "Rf_*"] & D^+_{\mathcal{M}_\mathcal{X}}(\mathcal{O}_\mathcal{X}) \arrow[r] \arrow[d, "Rf_*"] & D(\mathcal{O}_\mathcal{X}) \arrow[d, "Rf_*"] \\ D^+_{\mathcal{P}_\mathcal{Y}}(\mathcal{O}_\mathcal{Y}) \arrow[r] & D^+_{\mathcal{M}_\mathcal{Y}}(\mathcal{O}_\mathcal{Y}) \arrow[r] & D(\mathcal{O}_\mathcal{Y}) \end{tikzcd}$$

and hence induces a functor

$$Rf_{QCoh,*} : D^+_{QCoh}(\mathcal{X}) \longrightarrow D^+_{QCoh}(\mathcal{Y})$$

on quotient categories. Moreover, the functor R^if_{QCoh} of Proposition 62.10.1 are equal to $H^i \circ Rf_{QCoh,*}$ with H^i as in (62.13.1.1).

Proof. We have to show that Rf_*E is an object of $D^+_{\mathcal{M}_\mathcal{Y}}(\mathcal{O}_\mathcal{Y})$ for E in $D^+_{\mathcal{M}_\mathcal{X}}(\mathcal{O}_\mathcal{X})$. This follows from Proposition 62.7.4 and the spectral sequence $R^if_*H^j(E) \Rightarrow R^{i+j}f_*E$. The case of parasitic modules works the same way using Lemma 62.8.3. The final statement is clear from the definition of H^i in (62.13.1.1). □

62.15. Derived pullback of quasi-coherent modules

Derived pullback of complexes with quasi-coherent cohomology sheaves exists in general.

Proposition 62.15.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The exact functor f^* induces a commutative diagram*

$$\begin{tikzcd} D_{\mathcal{M}_\mathcal{X}}(\mathcal{O}_\mathcal{X}) \arrow[r] & D(\mathcal{O}_\mathcal{X}) \\ D_{\mathcal{M}_\mathcal{Y}}(\mathcal{O}_\mathcal{Y}) \arrow[r] \arrow[u, "f^*"] & D(\mathcal{O}_\mathcal{Y}) \arrow[u, "f^*"'] \end{tikzcd}$$

The composition

$$D_{\mathcal{M}_\mathcal{Y}}(\mathcal{O}_\mathcal{Y}) \xrightarrow{f^*} D_{\mathcal{M}_\mathcal{X}}(\mathcal{O}_\mathcal{X}) \xrightarrow{q_\mathcal{X}} D_{QCoh}(\mathcal{O}_\mathcal{X})$$

is left deriveable with respect to the localization $D_{\mathcal{M}_\mathcal{Y}}(\mathcal{O}_\mathcal{Y}) \to D_{QCoh}(\mathcal{O}_\mathcal{X})$ and we may define Lf^*_{QCoh} as its left derived functor

$$Lf^*_{QCoh} : D_{QCoh}(\mathcal{O}_\mathcal{Y}) \longrightarrow D_{QCoh}(\mathcal{O}_\mathcal{X})$$

(see Derived Categories, Definitions 11.14.2 and 11.14.9). If f is quasi-compact and quasi-separated, then Lf^*_{QCoh} and $Rf_{QCoh,*}$ satisfy the following adjointness:

$$Hom_{D_{QCoh}(\mathcal{O}_\mathcal{X})}(Lf^*_{QCoh}A, B) = Hom_{D_{QCoh}(\mathcal{O}_\mathcal{Y})}(A, Rf_{QCoh,*}B)$$

for $A \in D_{QCoh}(\mathcal{O}_\mathcal{Y})$ and $B \in D^+_{QCoh}(\mathcal{O}_\mathcal{X})$.

Proof. We have to show that f^*E is an object of $D_{\mathcal{M}_\mathcal{X}}(\mathcal{O}_\mathcal{X})$ for E in $D_{\mathcal{M}_\mathcal{Y}}(\mathcal{O}_\mathcal{Y})$. Since $f^* = f^{-1}$ is exact this follows immediately from the fact that f^* maps $\mathcal{M}_\mathcal{Y}$ into $\mathcal{M}_\mathcal{X}$.

Set $\mathcal{D} = D_{\mathcal{M}_\mathcal{Y}}(\mathcal{O}_\mathcal{Y})$. Let S be the collection of morphisms in $\mathcal{D}$ whose cone is an object of $D_{\mathcal{P}_\mathcal{Y}}(\mathcal{O}_\mathcal{Y})$. Set $\mathcal{D}' = D_{QCoh}(\mathcal{O}_\mathcal{X})$. Set $F = q \circ f^* : \mathcal{D} \to \mathcal{D}'$. Then $\mathcal{D}, S, \mathcal{D}', F$ are as in Derived Categories, Situation 11.14.1 and Definition 11.14.2. Let us prove that $LF(E)$ is defined for any object E of $\mathcal{D}$. Namely, consider the triangle

$$E' \to E \to P \to E'[1]$$

constructed in Lemma 62.13.4. Note that $s : E' \to E$ is an element of S. We claim that E' computes LF. Namely, suppose that $s' : E'' \to E$ is another element of S, i.e., fits into a triangle $E'' \to E \to P' \to E''[1]$ with P' in $D_{\mathcal{P}_\mathcal{Y}}(\mathcal{O}_\mathcal{Y})$. By Lemma 62.13.4 (and its proof) we see that $E' \to E$ factors through $E'' \to E$. Thus we see that $E' \to E$ is cofinal in the system S/E. Hence it is clear that E' computes LF.

To see the final statement, write $B = q_\mathcal{X}(H)$ and $A = q_\mathcal{Y}(E)$. Choose $E' \to E$ as above. We will use on the one hand that $Rf_{QCoh,*}(B) = q_\mathcal{Y}(Rf_*H)$ and on the other that $Lf^*_{QCoh}(A) = q_\mathcal{X}(f^*E')$.

$$\begin{aligned} Hom_{D_{QCoh}(\mathcal{O}_\mathcal{X})}(Lf^*_{QCoh}A, B) &= Hom_{D_{QCoh}(\mathcal{O}_\mathcal{X})}(q_\mathcal{X}(f^*E'), q_\mathcal{X}(H)) \\ &= colim_{H \to H'} Hom_{D(\mathcal{O}_\mathcal{X})}(f^*E', H') \\ &= colim_{H \to H'} Hom_{D(\mathcal{O}_\mathcal{Y})}(E', Rf_*H') \\ &= Hom_{D(\mathcal{O}_\mathcal{Y})}(E', Rf_*H) \\ &= Hom_{D_{QCoh}(\mathcal{O}_\mathcal{Y})}(A, Rf_{QCoh,*}B) \end{aligned}$$

Here the colimit is over morphisms $s : H \to H'$ in $D^+_{\mathcal{M}_\mathcal{X}}(\mathcal{O}_\mathcal{X})$ whose cone $P(s)$ is an object of $D^+_{\mathcal{P}_\mathcal{X}}(\mathcal{O}_\mathcal{X})$. The first equality holds by construction. The second equality holds by construction of the Verdier quotient. The third equality holds by Cohomology on Sites, Lemma 19.19.1. Since $Rf_*P(s)$ is an object of $D^+_{\mathcal{P}_\mathcal{Y}}(\mathcal{O}_\mathcal{Y})$ by Proposition 62.14.1 we see that $Hom_{D(\mathcal{O}_\mathcal{Y})}(E', Rf_*P(s)) = 0$. Thus the fourth equality holds. The final equality holds by construction of E'. $\square$

62.16. Other chapters

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(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Commutative Algebra
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CHAPTER 63

Introducing Algebraic Stacks

63.1. Why read this?

We give a very informal introduction to algebraic stacks aimed at graduate students and advanced undergraduates. The goal is to quickly introduce a simple language which you can use to think about local and global properties of your favorite moduli problem. Having done this it should be possible to ask yourself well-posed questions about moduli problems and to start solving them, whilst assuming a general theory exists. If you end up with an interesting result, you can go back to the general theory in the other parts of the stacks project and fill in the gaps as needed.

The point of view we take here is very close to the point of view taken in [**KM85**] and [**Mum65**].

63.2. Preliminary

Let S be a scheme. An *elliptic curve* over S is a triple $(E, f, 0)$ where E is a scheme and $f : E \to S$ and $0 : S \to E$ are morphisms of schemes such that

1. (1) $f : E \to S$ is proper, smooth of relative dimension 1,
2. (2) for every $s \in S$ the fibre E_s is a connected curve of genus 1, i.e., $H^0(E_s, \mathcal{O})$ and $H^1(E_s, \mathcal{O})$ both are 1-dimensional $\kappa(s)$-vector spaces, and
3. (3) 0 is a section of f.

Given elliptic curves $(E, f, 0)/S$ and $(E', f', 0')/S'$ a *morphism of elliptic curves over* $a : S \to S'$ is a morphism $\alpha : E \to E'$ such that the diagram

$$\begin{tikzcd}
E \arrow[r, "\alpha"] \arrow[d, "f"] & E' \arrow[d, "f'"'] \\
S \arrow[r, "a"] \arrow[u, bend left=60, "0"] & S' \arrow[u, bend right=60, "0'"']
\end{tikzcd}$$

is commutative. We are going to define the stack of elliptic curves $\mathcal{M}_{1,1}$. In the rest of the stacks project we work out the method introduced in Deligne and Mumford's paper [**DM69a**] which consists in presenting $\mathcal{M}_{1,1}$ as a category endowed with a functor

$$p : \mathcal{M}_{1,1} \longrightarrow Sch, \quad (E, f, 0)/S \longmapsto S$$

This means you work with fibred categories over the categories of schemes, topologies, stacks fibred in groupoids, coverings, etc, etc. In this chapter we throw all of that out of the window and we think about it a bit differently -- probably closer to how the initiators of the theory started thinking about it themselves.

63.3. The moduli stack of elliptic curves

Here is what we are going to do:

(1) Start with your favorite category of schemes *Sch*.
(2) Add a new symbol $\mathcal{M}_{1,1}$.
(3) A morphism $S \to \mathcal{M}_{1,1}$ **is** an elliptic curve $(E, f, 0)$ over S.
(4) A diagram

$$\begin{tikzcd} S \arrow[rr, "a"'] \arrow[dr, "{(E,f,0)}"'] & & S' \arrow[dl, "{(E',F',0')}"] \\ & \mathcal{M}_{1,1} & \end{tikzcd}$$

is commutative if and only if there exists a morphism $\alpha : E \to E'$ of elliptic curves over $a : S \to S'$. We say α *witnesses* the commutativity of the diagram.
(5) Note that commutative diagrams glue as follows

$$\begin{tikzcd} S \arrow[r, "a"'] \arrow[dr, "{(E,f,0)}"'] & S' \arrow[r, "a'"'] \arrow[d, "{(E',F',0')}"'] & S'' \arrow[dl, "{(E'',F'',0'')}"] \\ & \mathcal{M}_{1,1} & \end{tikzcd}$$

because $\alpha' \circ \alpha$ witnesses the commutativity of the outer triangle if α and α' witness the commutativity of the left and right triangles.
(6) The composition

$$S' \xrightarrow{a} S' \xrightarrow{(E',f',0')} \mathcal{M}_{1,1}$$

is given by $(E' \times_{S'} S, f' \times_{S'} S, 0' \times_{S'} S)$.

At the end of this procedure we have enlarged the category *Sch* of schemes with exactly one object...

Except that we haven't defined what a morphism from $\mathcal{M}_{1,1}$ to a scheme T is. The answer is that it is the weakest possible notion such that compositions make sense. Thus a morphism $F : \mathcal{M}_{1,1} \to T$ is a rule which to every elliptic curve $(E, f, 0)/S$ associates a morphism $F(E, f, 0) : S \to T$ such that given any commutative diagram

$$\begin{tikzcd} S \arrow[rr, "a"'] \arrow[dr, "{(E,f,0)}"'] & & S' \arrow[dl, "{(E',F',0')}"] \\ & \mathcal{M}_{1,1} & \end{tikzcd}$$

the diagram

$$\begin{tikzcd} S \arrow[rr, "a"'] \arrow[dr, "{F(E,f,0)}"'] & & S' \arrow[dl, "{F(E',F',0')}"] \\ & T & \end{tikzcd}$$

is commutative also. An example is the j-invariant

$$j : \mathcal{M}_{1,1} \longrightarrow \mathbf{A}^1_{\mathbf{Z}}$$

which you may have heard of. Aha, so now we're done...

Except, no we're not! We still have to define a notion of morphisms $\mathcal{M}_{1,1} \to \mathcal{M}_{1,1}$. This we do in exactly the same way as before, i.e., a morphism $F : \mathcal{M}_{1,1} \to \mathcal{M}_{1,1}$ is a rule which to every elliptic curve $(E, f, 0)/S$ associates another elliptic curve $F(E, f, 0)$ preserving commutativity of diagrams as above. However, since I don't know of a nontrivial example of such a functor, I'll just define the set of morphisms from $\mathcal{M}_{1,1}$ to itself to consist of the identity for now.

I hope you see how to add other objects to this enlarged category. Somehow it seems intuitively clear that given any ``well-behaved'' moduli problem we can perform the construction above and add an object to our category. In fact, much of modern day algebraic geometry takes place in such a universe where *Sch* is enlarged with countably many (explicitly constructed) moduli stacks.

You may object that the category we obtain isn't a category because there is a ``vagueness'' about when diagrams commute and which combinations of diagrams continue to commute as we have to produce a witness to the commutativity. However, it turns out that this, the idea of having witnesses to commutativity, is a valid approach to 2-categories! Thus we stick with it.

63.4. Fibre products

The question we pose here is what should be the fibre product

$$\begin{tikzcd} & ? \arrow[dl, dotted] \arrow[dr, dotted] & \\ S \arrow[dr, "{(E,f,0)}"'] & & S' \arrow[dl, "{(E',f',0')}"] \\ & \mathcal{M}_{1,1} & \end{tikzcd}$$

The answer: A morphism from a scheme T into ? should be a triple (a, a', α) where $a : T \to S$, $a' : T \to S'$ are morphisms of schemes and where $\alpha : E \times_{S,a} T \to E' \times_{S',a'} T$ is a morphism of elliptic curves over id_T. This makes sense because of our definition of composition and commutative diagrams earlier in the discussion.

Lemma 63.4.1 (Key fact)**.** *The functor $Sch^{opp} \to Sets$, $T \mapsto \{(a, a', \alpha)$ as above$\}$ is representable by a scheme $S \times_{\mathcal{M}_{1,1}} S'$.*

Proof. Idea of proof. Relate this functor to

$$Isom_{S\times S'}(E \times S', S \times E')$$

and use Grothendieck's theory of Hilbert schemes. $\square$

Remark 63.4.2. We have the formula $S \times_{\mathcal{M}_{1,1}} S' = (S \times S') \times_{\mathcal{M}_{1,1}\times\mathcal{M}_{1,1}} \mathcal{M}_{1,1}$. Hence the key fact is a property of the diagonal $\Delta_{\mathcal{M}_{1,1}}$ of $\mathcal{M}_{1,1}$.

In any case the key fact allows us to make the following definition.

Definition 63.4.3. We say a morphism $S \to \mathcal{M}_{1,1}$ is *smooth* if for every morphism $S' \to \mathcal{M}_{1,1}$ the projection morphism

$$S \times_{\mathcal{M}_{1,1}} S' \longrightarrow S'$$

is smooth.

Note that this is compatible with the notion of a smooth morphism of schemes as the base change of a smooth morphism is smooth. Moreover, it is clear how to extend this definition to other properties of morphisms into $\mathcal{M}_{1,1}$ (or your own favorite moduli stack). In particular we will use it below for *surjective* morphisms.

63.5. The definition

We'll formulate it as a definition and not as a result since we expect the reader to try out other cases (not just the stack $\mathcal{M}_{1,1}$ and not just Sch the category of all schemes).

Definition 63.5.1. We say $\mathcal{M}_{1,1}$ is an *algebraic stack* if and only if

(1) We have descent for objects for the étale topology on Sch.
(2) The key fact holds.
(3) there exists a surjective and smooth morphism $S \to \mathcal{M}_{1,1}$.

The first condition is a ``sheaf property''. We're going to spell it out since there is a technical point we should make. Suppose given a scheme S and an étale covering $\{S_i \to S\}$ and morphisms $e_i : S_i \to \mathcal{M}_{1,1}$ such that the diagrams

$$\begin{tikzcd} S_i \times_S S_j \arrow[rr, "\mathrm{id}"'] \arrow[dr, "e_i \circ \mathrm{pr}_1"'] & & S_i \times_S S_j \arrow[dl, "e_j \circ \mathrm{pr}_2"] \\ & \mathcal{M}_{1,1} & \end{tikzcd}$$

commute. The sheaf condition does *not* guarantee the existence of a morphism $e : S \to \mathcal{M}_{1,1}$ in this situation. Namely, we need to pick witnesses α_{ij} for the diagrams above and require that

$$\mathrm{pr}_{02}^*\alpha_{ik} = \mathrm{pr}_{12}\alpha_{jk} \circ \mathrm{pr}_{01}^*\alpha_{ij}$$

as witnesses over $S_i \times_S S_j \times_S S_k$. I think it is clear what this means... If not, then I'm afraid you'll have to read some of the material on categories fibred in groupoids, etc. In any case, the displayed equation is often called the *cocycle condition*. A more precise statement of the ``sheaf property'' is: given $\{S_i \to S\}$, $e_i : S_i \to \mathcal{M}_{1,1}$ and witnesses α_{ij} satisfying the cocycle condition, there exists a unique (up to unique isomorphism) $e : S \to \mathcal{M}_{1,1}$ with $e_i \cong e|_{S_i}$ recovering the α_{ij}.

As you can see even formulating a precise statement takes a bit of work. The proof of this ``sheaf property'' relies on a fundamental technique in algebraic geometry, namely descent theory. My suggestion is to initially simply accept the ``sheaf property'' holds, and see what it implies in practice. In fact, a certain amount of mental agility is required to boil the ``sheaf property'' down to a manageable statement that you can fit on a napkin. Perhaps the simplest variant which is already a bit interesting is the following: Suppose we have a Galois extension $K \subset L$ of fields with Galois group $G = \mathrm{Gal}(L/K)$. Set $T = Spec(L)$ and $S = Spec(K)$. Then $\{T \to S\}$ is an étale covering. Let $(E, f, 0)$ be an elliptic curve over L. (Yes, this just means that $E \subset \mathbf{P}^2_L$ is given by a Weierstrass equation and 0 is the usual point at infinity.) Denote $E_\sigma = E \times_{T, Spec(\sigma)} T$ the base change. (Yes, this corresponds to applying σ to the coefficients of the Weierstrass equation, or is it σ^{-1}?) Now, suppose moreover that for every $\sigma \in G$ we are given an isomorphism

$$\alpha_\sigma : E \longrightarrow E_\sigma$$

over T. The cocycle condition above means in this situation that

$$(\alpha_\tau)^\sigma \circ \alpha_\sigma = \alpha_{\tau\sigma}$$

for $\sigma, \tau \in G$. If you've ever done any group cohomology then this should be familiar. Anyway, the ``glueing'' condition on $\mathcal{M}_{1,1}$ says that if you have a solution to this set of equations, then there exists an elliptic curve E' over S such that $E \cong E \times_S T$ (it says a little bit more because it also tells you how to recover the α_σ).

Challenge: Can you prove this entirely using only elliptic curves defined in terms of Weierstrass equations?

63.6. A smooth cover

The last thing we have to do is find a smooth cover of $\mathcal{M}_{1,1}$. In fact, in some sense the existence of a smooth cover *implies*[1] the key fact! In the case of elliptic curves we use the Weierstrass equation to construct one.

Set

$$W = Spec(\mathbf{Z}[a_1, a_2, a_3, a_4, a_6, 1/\Delta])$$

where $\Delta \in \mathbf{Z}[a_1, a_2, a_3, a_4, a_6]$ is a certain polynomial (see below). Set

$$\mathbf{P}^2_W \supset E_W : zy^2 + a_1xyz + a_3yz^2 = x^3 + a_2x^2z + a_4xz^3 + a_6z^3.$$

Denote $f_W : E_W \to W$ the projection. Finally, denote $0_W : W \to E_W$ the section of f_W given by $(0 : 1 : 0)$. It turns out that there is a degree 12 homogeneous polynomial Δ in a_i where $\deg(a_i) = i$ such that $E_W \to W$ is smooth. You can find it explicitly by computing partials of the Weierstrass equation -- of course you can also look it up. You can also use pari/gp to compute it for you. Here it is

$$\begin{aligned}\Delta = &-a_6a_1^6 + a_4a_3a_1^5 + ((-a_3^2 - 12a_6)a_2 + a_4^2)a_1^4 + \\ &(8a_4a_3a_2 + (a_3^3 + 36a_6a_3))a_1^3 + \\ &((-8a_3^2 - 48a_6)a_2^2 + 8a_4^2a_2 + (-30a_4a_3^2 + 72a_6a_4))a_1^2 + \\ &(16a_4a_3a_2^2 + (36a_3^3 + 144a_6a_3)a_2 - 96a_4^2a_3)a_1 + \\ &(-16a_3^2 - 64a_6)a_2^3 + 16a_4^2a_2^2 + (72a_4a_3^2 + 288a_6a_4)a_2 + \\ &- 27a_3^4 - 216a_6a_3^2 - 64a_4^3 - 432a_6^2\end{aligned}$$

You may recognize the last two terms from the case $y^2 = x^3 + Ax + B$ having discriminant $-64A^3 - 432B^2 = -16(4A^3 + 27B^2)$.

Lemma 63.6.1. *The morphism* $W \xrightarrow{(E_W, f_W, 0_W)} \mathcal{M}_{1,1}$ *is smooth and surjective.*

Proof. Surjectivity follows from the fact that every elliptic curve over a field has a Weierstrass equation. We give a very rough sketch of one way to prove smoothness. Consider the sub group scheme

$$H = \left\{ \begin{pmatrix} u^2 & s & 0 \\ 0 & u^3 & 0 \\ r & t & 1 \end{pmatrix} \middle| \begin{matrix} u \text{ unit} \\ s, r, t \text{ arbitrary} \end{matrix} \right\} \subset \mathrm{GL}_{3,\mathbf{Z}}$$

There is an action $H \times W \to W$ of H on the Weierstrass scheme W. To find the equations for this action write out what a coordinate change given by a matrix in H does to the general Weierstrass equation. Then it turns out the following statements hold

(1) any elliptic curve $(E, f, 0)/S$ has Zariski locally on S a Weierstrass equation,
(2) any two Weierstrass equations for $(E, f, 0)$ differ (Zariski locally) by an element of H.

[1]This is a bit of a cheat because in checking the smoothness you have to prove something very close to the key fact -- after all smoothness is defined in terms of fibre products. The advantage is that you only have to prove the existence of these fibre products in the case that on one side you have the morphism that you are trying to show provides the smooth cover.

Considering the fibre product $S \times_{\mathcal{M}_{1,1}} W = Isom_{S\times W}(E \times W, S \times E_W)$ we conclude that this means that the morphism $W \to \mathcal{M}_{1,1}$ is an H-torsor. Since $H \to \mathrm{Spec}(\mathbf{Z})$ is smooth, and since torsors over smooth group schemes are smooth we win. □

Remark 63.6.2. The argument sketched above actually shows that $\mathcal{M}_{1,1} = [W/H]$ is a global quotient stack. It is true about 50% of the time that an argument proving a moduli stack is algebraic will show that it is a global quotient stack.

63.7. Properties of algebraic stacks

Ok, so now we know that $\mathcal{M}_{1,1}$ is an algebraic stack. What can we do with this? Well, it isn't so much the fact that it is an algebraic stack that helps us here, but more the point of view that properties of $\mathcal{M}_{1,1}$ should be encoded in the properties of morphisms $S \to \mathcal{M}_{1,1}$, i.e., in families of elliptic curves. We list some examples

Local properties:

$$\mathcal{M}_{1,1} \to \mathrm{Spec}(\mathbf{Z}) \text{ is smooth} \Leftrightarrow W \to \mathrm{Spec}(\mathbf{Z}) \text{ is smooth}$$

Idea. Local properties of an algebraic stack are encoded in the local properties of its smooth cover.

Global properties:

$$\mathcal{M}_{1,1} \text{ is quasi-compact} \Leftarrow W \text{ is quasi-compact}$$
$$\mathcal{M}_{1,1} \text{ is irreducible} \Leftarrow W \text{ is irreducible}$$

Idea. Some global properties of an algebraic stack can be read off from the corresponding property of a *suitable*[2] smooth cover.

Quasi-coherent sheaves:

$$QCoh(\mathcal{M}_{1,1}) = H\text{-equivariant quasi-coherent modules on } W$$

Idea. On the one hand a quasi-coherent module on $\mathcal{M}_{1,1}$ should correspond to a quasi-coherent sheaf $\mathcal{F}_{S,e}$ on S for each morphism $e : S \to \mathcal{M}_{1,1}$. In particular for the morphism $(E_W, f_W, 0_W) : W \to \mathcal{M}_{1,1}$. Since this morphism is H-equivariant we see the quasi-coherent module $\mathcal{F}_W$ we obtain is H-equivariant. Conversely, given an H-equivariant module we can recover the sheaves $\mathcal{F}_{S,e}$ by descent theory starting with the observation that $S \times_{e,\mathcal{M}_{1,1}} W$ is an H-torsor.

Picard group:

$$\mathrm{Pic}(\mathcal{M}_{1,1}) = \mathrm{Pic}_H(W) = \mathbf{Z}/12\mathbf{Z}$$

Idea. We have seen the first equality above. Note that $\mathrm{Pic}(W) = 0$ because the ring $\mathbf{Z}[a_1, a_2, a_3, a_4, a_6, 1/\Delta]$ has trivial class group. There is an exact sequence

$$\mathbf{Z}\Delta \to \mathrm{Pic}_H(\mathbf{A}^5_{\mathbf{Z}}) \to \mathrm{Pic}_H(W) \to 0$$

The middle group equals $Hom(H, \mathbf{G}_m) = \mathbf{Z}$. The image Δ is 12 because Δ has degree 12. This argument is roughly correct, see [**FO10**].

Étale cohomology: Let Λ be a ring. There is a first quadrant spectral sequence converging to $H^{p+q}_{étale}(\mathcal{M}_{1,1}, \Lambda)$ with E_2-page

$$E_2^{p,q} = H^q_{étale}(W \times H \times \ldots \times H, \Lambda) \quad (p \text{ factors } H)$$

[2] I suppose that it is possible an irreducible algebraic stack exists which doesn't have an irreducible smooth cover -- but if so it is going to be quite nasty!

Idea. Note that

$$W \times_{\mathcal{M}_{1,1}} W \times_{\mathcal{M}_{1,1}} \ldots \times_{\mathcal{M}_{1,1}} W = W \times H \times \ldots \times H$$

because $W \to \mathcal{M}_{1,1}$ is a H-torsor. The spectral sequence is the Čech-to-cohomology spectral sequence for the smooth cover $\{W \to \mathcal{M}_{1,1}\}$. For example we see that $H^0_{\acute{e}tale}(\mathcal{M}_{1,1}, \Lambda) = \Lambda$ because W is connected, and $H^1_{\acute{e}tale}(\mathcal{M}_{1,1}, \Lambda) = 0$ because $H^1_{\acute{e}tale}(W, \Lambda) = 0$ (of course this requires a proof). Of course, the smooth covering $W \to \mathcal{M}_{1,1}$ may not be ``optimal'' for the computation of étale cohomology.

63.8. Other chapters

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CHAPTER 64

Examples

64.1. Introduction

This chapters will contain examples which illuminate the theory.

64.2. Noncomplete completion

Let R be a ring and let $\mathfrak{m}$ be a maximal ideal. Consider the completion

$$R^\wedge = lim\, R/\mathfrak{m}^n.$$

Note that $R^\wedge$ is a local ring with maximal ideal $\mathfrak{m}' = \text{Ker}(R^\wedge \to R/\mathfrak{m})$. Namely, if $x = (x_n) \in R^\wedge$ is not in $\mathfrak{m}'$, then $y = (x_n^{-1}) \in R^\wedge$ satisfies $xy = 1$, whence $R^\wedge$ is local by Algebra, Lemma 7.17.2. Now it is always true that $R^\wedge$ complete in its limit topology (see the discussion in More on Algebra, Section 12.27). But beyond that, we have the following questions:

(1) Is it true that $\mathfrak{m}R^\wedge = \mathfrak{m}'$?
(2) Is $R^\wedge$ viewed as an $R^\wedge$-module $\mathfrak{m}'$-adically complete?
(3) Is $R^\wedge$ viewed as an R-module $\mathfrak{m}$-adically complete?

It turns out that of these questions all have a negative answer. The example below was taken from an upublished note of Bart de Smit and Hendrik Lenstra. It also is discussed in [**Bou61**, Exercise III.2.12] over a finite field. It is further discussed in [**Yek11**, Example 1.8]

Let k be a field, $R = k[x_1, x_2, x_3, \ldots]$, and $\mathfrak{m} = (x_1, x_2, x_3, \ldots)$. We will think of an element f of $R^\wedge$ as a (possibly) infinite sum

$$f = \sum a_I x^I$$

(using multi-index notation) such that for each $d \geq 0$ there are only finitely many nonzero a_I for $|I| = d$. The maximal ideal $\mathfrak{m}' \subset R^\wedge$ is the collection of f with zero constant term. In particular, the element

$$f = x_1 + x_2^2 + x_3^3 + \ldots$$

is in $\mathfrak{m}'$ but not in $\mathfrak{m}R^\wedge$ which shows that (1) is false in this example. Note that we do have $\mathfrak{m}R^\wedge \subset \mathfrak{m}'$. Hence, $R^\wedge$ is not $\mathfrak{m}$-adically complete as an R-module, then it is also not $\mathfrak{m}'$-adically complete. To show that $R^\wedge$ is not $\mathfrak{m}$-adically complete (as an R-module) it suffices to show that $K_2 = \text{Ker}(R^\wedge \to R/\mathfrak{m}^2)$ is not equal to $\mathfrak{m}^2 R^\wedge$, see Algebra, Lemma 7.90.6. Note that an element of $\mathfrak{m}^2 R^\wedge \subset (\mathfrak{m}')^2$ can be written as a finite sum

$$\sum\nolimits_{i=1,\ldots,t} f_i g_i \qquad (64.2.0.1)$$

with $f_i, g_i \in R^\wedge$ having vanishing constant terms. To get an example we are going to choose an $z \in K_2$ of the form

$$z = z_1 + z_2 + z_3 + \ldots$$

with the following properties

(1) there exist sequences $1 < d_1 < d_2 < d_3 < \ldots$ and $0 < n_1 < n_2 < n_3 < \ldots$ such that $z_i \in k[x_{n_i}, x_{n_i+1}, \ldots, x_{n_{i+1}-1}]$ homogeneous of degree d_i, and
(2) in the ring $k[[x_{n_i}, x_{n_i+1}, \ldots, x_{n_{i+1}-1}]]$ the element z_i cannot be written as a sum (64.2.0.1) with $t \leq i$.

Clearly this implies that z is not in $(\mathfrak{m}')^2$ because the image of the relation (64.2.0.1) in the ring $k[[x_{n_i}, x_{n_i+1}, \ldots, x_{n_{i+1}-1}]]$ for i large enough would produce a contradiction. Hence it suffices to prove that for all $t > 0$ there exists a $d \gg 0$ and an integer n such that we can find an homogeneous element $z \in k[x_1, \ldots, x_n]$ of degree d which cannot be written as a sum (64.2.0.1) for the given t in $k[[x_1, \ldots, x_n]]$. Take $n > 2t$ and any $d > 1$ prime to the characteristic of p and set $z = \sum_{i=1,\ldots,n} x_i^d$. Then the vanishing locus of the ideal

$$(\frac{\partial z}{\partial x_1}, \ldots, \frac{\partial z}{\partial x_n}) = (dx_1^{d-1}, \ldots, dx_n^{d-1})$$

consists of one point. On the other hand,

$$\frac{\partial(\sum_{i=1,\ldots,t} f_i g_i)}{\partial x_j} \in (f_1, \ldots, f_t, g_1, \ldots, g_t)$$

by the Leibniz rule and hence the vanishing locus of these derivatives contains at least

$$V(f_1, \ldots, f_t, g_1, \ldots, g_t) \subset Spec(k[[x_1, \ldots, x_n]]).$$

Hence this is a contradition as the dimension of $V(f_1, \ldots, f_t, g_1, \ldots, g_t)$ is at least $n - 2t \geq 1$.

Lemma 64.2.1. *There exists a local ring R and a maximal ideal $\mathfrak{m}$ such that the completion $R^\wedge$ of R with respect to $\mathfrak{m}$ has the following properties*

(1) *$R^\wedge$ is local, but its maximal ideal is not equal to $\mathfrak{m}R^\wedge$,*
(2) *$R^\wedge$ is not a complete local ring, and*
(3) *$R^\wedge$ is not $\mathfrak{m}$-adically complete as an R-module.*

Proof. This follows from the discussion above as (with $R = k[x_1, x_2, x_3, \ldots]$) the completion of the localization $R_\mathfrak{m}$ is equal to the completion of R. □

64.3. Noncomplete quotient

Let k be a field. Let

$$R = k[t, z_1, z_2, z_3, \ldots, w_1, w_2, w_3, \ldots, x]/(z_i t - x^i w_i, z_i w_j)$$

Note that in particular $z_i z_j t = 0$ in this ring. Any element f of R can be uniquely written as a finite sum

$$f = \sum\nolimits_{i=0,\ldots,d} f_i x^i$$

where each $f_i \in k[t, z_i, w_j]$ has no terms involving the products $z_i t$ or $z_i w_j$. Moreover, if f is written in this way, then $f \in (x^n)$ if and only if $f_i = 0$ for $i < n$. So x is a nonzero divisor and $\bigcap (x^n) = 0$. Let $R^\wedge$ be the completion of R with respect to the ideal (x). Note that $R^\wedge$ is (x)-adically complete, see Algebra, Lemma 7.90.7. By the above we see that an element of $R^\wedge$ can be uniquely written as an infinite sum

$$f = \sum\nolimits_{i=0}^{\infty} f_i x^i$$

where each $f_i \in k[t, z_i, w_j]$ has no terms involving the products $z_i t$ or $z_i w_j$. Consider the element

$$f = \sum\nolimits_{i=1}^{\infty} x^{i-1} w_i = x w_1 + x^2 w_2 + x^3 w_3 + \ldots$$

i.e., we have $f_n = w_n$. Note that $f \in (t, x^n)$ for every n because $x^m w_m \in (t)$ for all m. We claim that $f \not\in (t)$. To prove this assume that $tg = f$ where $g = \sum g_l x^l$ in canonical form as above. Since $tz_iz_j = 0$ we may as well assume that none of the g_l have terms involving the products z_iz_j. Examining the process to get tg in canonical form we see the following: Given any term cm of g_l where $c \in k$ and m is a monomial in t, z_i, w_j and we make the following replacement

(1) if the monomial m does not involve any z_i, then ctm is a term of f_l, and
(2) if the monomial m does involve a z_i then it is eqal to $m = z_i$ and we see that cw_i is term of f_{l+i}.

Since g_0 is a polynomial only finitely many of the variables z_i occur in it. Pick n such that z_n does not occur in g_0. Then the rules above show that w_n does not occur in f_n which is a contradiction. It follows that $R^\wedge/(t)$ is not complete, see Algebra, Lemma 7.90.13.

Lemma 64.3.1. *There exists a ring R complete with respect to a principal ideal I and a principal ideal J such that R/J is not I-adically complete.*

Proof. See discussion above. □

64.4. Completion is not exact

A quick example is the following. Suppose that $R = k[t]$. Let $P = K = \bigoplus_{n \in \mathbf{N}} R$ and $M = \bigoplus_{n \in \mathbf{N}} R/(t^n)$. Then there is a short exact sequence $0 \to K \to P \to M \to 0$ where the first map is given by multiplication by t^n on the nth summand. We claim that $0 \to K^\wedge \to P^\wedge \to M^\wedge \to 0$ is not exact in the middle. Namely, $\xi = (t^2, t^3, t^4, \ldots) \in P^\wedge$ maps to zero in $M^\wedge$ but is not in the image of $K^\wedge \to P^\wedge$, because it would be the image of $(t, t, t, \ldots)$ which is not an element of $K^\wedge$.

A ``smaller'' example is the following. In the situation of Lemma 64.3.1 the short exact sequence $0 \to J \to R \to R/J \to 0$ does not remain exact after completion. Namely, if $f \in J$ is a generator, then $f : R \to J$ is surjective, hence $R \to J^\wedge$ is surjective, hence the image of $J^\wedge \to R$ is $(f) = J$ but the fact that R/J is noncomplete means that the kernel of the surjection $R \to (R/J)^\wedge$ is strictly bigger than J, see Algebra, Lemmas 7.90.1 and 7.90.13. By the same token the sequence $R \to R \to R/(f) \to 0$ does not remain exact on completion.

Lemma 64.4.1. *Completion is not an exact functor in general; it is not even right exact in general. This holds even when I is finitely generated on the category of finitely presented modules.*

Proof. See discussion above. □

64.5. The category of complete modules is not abelian

Let R be a ring and let $I \subset R$ be a finitely generated ideal. Consider the category $\mathcal{A}$ of I-adically complete R-modules, see Algebra, Definition 7.90.5. Let $\varphi : M \to N$ be a morphism of $\mathcal{A}$. The cokernel of φ in $\mathcal{A}$ is the completion $(\text{Coker}(\varphi))^\wedge$ of the usual cokernel (as I is finitely generated this completion is complete, see Algebra, Lemma 7.90.7). Let $K = \text{Ker}(\varphi)$. We claim that K is complete and hence is the kernel of φ in $\mathcal{A}$. Namely, let $K^\wedge$ be the completion. As M is complete we obtain a factorization

$$K \to K^\wedge \to M \xrightarrow{\varphi} N$$

Since φ is continuous for the I-adic topology, $K \to K^\wedge$ has dense image, and $K = \mathrm{Ker}(\varphi)$ we conclude that $K^\wedge$ maps into K. Thus $K^\wedge = K \oplus C$ and K is a direct sum of a complete module, hence complete.

We will give an example that shows that Im≠Coim in general. We take $R = \mathbf{Z}_p = lim_n \mathbf{Z}/p^n\mathbf{Z}$ to be the ring of p-adic integers and we take $I = (p)$. Consider the map

$$\mathrm{diag}(1, p, p^2, \ldots) : \left(\bigoplus\nolimits_{n\geq 1} \mathbf{Z}_p\right)^\wedge \longrightarrow \prod\nolimits_{n \geq 1} \mathbf{Z}_p$$

where the left hand side is the p-adic completion of the direct sum. Hence an element of the left hand side is a vector $(x_1, x_2, x_3, \ldots)$ with $x_i \in \mathbf{Z}_p$ with p-adic valuation $v_p(x_i) \to \infty$ as $i \to \infty$. This maps to $(x_1, px_2, p^2x_3, \ldots)$. Hence we see that $(1, p, p^2, \ldots)$ is in the closure of the image but not in the image. By our description of kernels and cokernels above it is clear that Im≠Coim for this map.

Lemma 64.5.1. *Let R be a ring and let $I \subset R$ be a finitely generated ideal. The category of I-adically complete R-modules has kernels and cokernels but is not abelian in general.*

Proof. See above. $\square$

64.6. Regular sequences and base change

We are going to construct a ring R with a regular sequence (x, y, z) such that there exists a nonzero element $\delta \in R/zR$ with $x\delta = y\delta = 0$.

To construct our example we first construct a peculiar module E over the ring $k[x, y, z]$ where k is any field. Namely, E will be a push-out as in the following diagram

$$\begin{tikzcd}
\frac{xk[x,y,z,y^{-1}]}{xyk[x,y,z]} \arrow[r] \arrow[d, "z/x"] & \frac{k[x,y,z,x^{-1},y^{-1}]}{yk[x,y,z,x^{-1}]} \arrow[r] \arrow[d] & \frac{k[x,y,z,x^{-1},y^{-1}]}{yk[x,y,z,x^{-1}]+xk[x,y,z,y^{-1}]} \arrow[d] \\
\frac{k[x,y,z,y^{-1}]}{yzk[x,y,z]} \arrow[r] & E \arrow[r] & \frac{k[x,y,z,x^{-1},y^{-1}]}{yk[x,y,z,x^{-1}]+xk[x,y,z,y^{-1}]}
\end{tikzcd}$$

where the rows are short exact sequences (we dropped the outer zeros due to typesetting problems). Another way to describe E is as

$$E = \{(f, g) \mid f \in k[x, y, z, x^{-1}, y^{-1}], g \in k[x, y, z, y^{-1}]\}/\sim$$

where $(f, g) \sim (f', g')$ if and only if there exists a $h \in k[x, y, z, y^{-1}]$ such that

$$f = f' + xh \bmod yk[x, y, z, x^{-1}], \quad g = g' - zh \bmod yzk[x, y, z]$$

We claim: (a) $x : E \to E$ is injective, (b) $y : E/xE \to E/xE$ is injective, (c) $E/(x, y)E = 0$, (d) there exists a nonzero element $\delta \in E/zE$ such that $x\delta = y\delta = 0$.

To prove (a) suppose that (f, g) is a pair that gives rise to an element of E and that $(xf, xg) \sim 0$. Then there exists a $h \in k[x, y, z, y^{-1}]$ such that $xf + xh \in yk[x, y, z, x^{-1}]$ and $xg - zh \in yzk[x, y, z]$. We may assume that $h = \sum a_{i,j,k}x^iy^jz^k$ is a sum of monomials where only $j \leq 0$ occurs. Then $xg - zh \in yzk[x, y, z]$ implies that only $i > 0$ occurs, i.e., $h = xh'$ for some $h' \in k[x, y, z, y^{-1}]$. Then $(f, g) \sim (f + xh', g - zh')$ and we see that we may assume that $g = 0$ and $h = 0$. In this case $xf \in yk[x, y, z, x^{-1}]$ implies $f \in yk[x, y, z, x^{-1}]$ and we see that $(f, g) \sim 0$. Thus $x : E \to E$ is injective.

Since multiplication by x is an isomorphism on $\frac{k[x,y,z,x^{-1},y^{-1}]}{yk[x,y,z,x^{-1}]}$ we see that E/xE is isomorphic to

$$\frac{k[x,y,z,y^{-1}]}{yzk[x,y,z] + xk[x,y,z,y^{-1}] + zk[x,y,z,y^{-1}]} = \frac{k[x,y,z,y^{-1}]}{xk[x,y,z,y^{-1}] + zk[x,y,z,y^{-1}]}$$

and hence multiplication by y is an isomorphism on E/xE. This clearly implies (b) and (c).

Let $e \in E$ be the equivalence class of $(1,0)$. Suppose that $e \in zE$. Then there exist $f \in k[x,y,z,x^{-1},y^{-1}]$, $g \in k[x,y,z,y^{-1}]$, and $h \in k[x,y,z,y^{-1}]$ such that

$$1 + zf + xh \in yk[x,y,z,x^{-1}], \quad 0 + zg - zh \in yzk[x,y,z].$$

This is impossible: the monomial 1 cannot occur in zf, nor in xh. On the other hand, we have $ye = 0$ and $xe = (x,0) \sim (0,-z) = z(0,-1)$. Hence settting δ equal to the congruence class of e in E/zE we obtain (d).

Lemma 64.6.1. *There exists a local ring R and a regular sequence x, y, z (in the maximal ideal) such that there exists a nonzero element $\delta \in R/zR$ with $x\delta = y\delta = 0$.*

Proof. Let $R = k[x,y,z] \oplus E$ where E is the module above considered as a square zero ideal. Then it is clear that x, y, z is a regular sequence in R, and that the element $\delta \in E/zE \subset R/zR$ gives an element with the desired properties. To get a local example we may localize R at the maximal ideal $\mathfrak{m} = (x,y,z,E)$. The sequence x, y, z remains a regular sequence (as localization is exact), and the element δ remains nonzero as it is supported at $\mathfrak{m}$. □

Lemma 64.6.2. *There exists a local homomorphism of local rings $A \to B$ and a regular sequence x, y in the maximal ideal of B such that $B/(x,y)$ is flat over A, but such that the images $\overline{x}, \overline{y}$ of x, y in $B/\mathfrak{m}_A B$ do not form a regular sequence, nor even a Koszul-regular sequence.*

Proof. Set $A = k[z]_{(z)}$ and let $B = (k[x,y,z] \oplus E)_{(x,y,z,E)}$. Since x, y, z is a regular sequence in B, see proof of Lemma 64.6.1, we see that x, y is a regular sequence in B and that $B/(x,y)$ is a torsion free A-module, hence flat. On the other hand, there exists a nonzero element $\delta \in B/\mathfrak{m}_A B = B/zB$ which is annihilated by $\overline{x}, \overline{y}$. Hence $H_2(K_\bullet(B/\mathfrak{m}_A B, \overline{x}, \overline{y})) \neq 0$. Thus $\overline{x}, \overline{y}$ is not Koszul-regular, in particular it is not a regular sequence, see More on Algebra, Lemma 12.22.2. □

64.7. A Noetherian ring of infinite dimension

A Noetherian local ring has finite dimension as we saw in Algebra, Proposition 7.57.8. But there exist Noetherian rings of infinite dimension. See [**Nag62**, Appendix, Example 1].

Namely, let k be a field, and consider the ring

$$R = k[x_1, x_2, x_3, \ldots].$$

Let $\mathfrak{p}_i = (x_{2^{i-1}}, x_{2^{i-1}+1}, \ldots, x_{2^i-1})$ for $i = 1, 2, \ldots$ which are prime ideals of R. Let S be the multiplicative subset

$$S = \bigcap\nolimits_{i \geq 1} (R \setminus \mathfrak{p}_i).$$

Consider the ring $A = S^{-1}R$. We claim that

(1) The maximal ideals of the ring A are the ideals $\mathfrak{m}_i = \mathfrak{p}_i A$.
(2) We have $A_{\mathfrak{m}_i} = R_{\mathfrak{p}_i}$ which is a Noetherian local ring of dimension 2^i.
(3) The ring A is Noetherian.

Hence it is clear that this is the example we are looking for. Details omitted.

64.8. Local rings with nonreduced completion

In Algebra, Example 7.110.4 we gave an example of a characteristic p Noetherian local domain R of dimension 1 whose completion is nonreduced. In this section we present the example of [**FR70**, Proposition 3.1] which gives a similar ring in characteristic zero.

Let $\mathbf{C}\{x\}$ be the ring of convergent power series over the field $\mathbf{C}$ of complex numbers. The ring of all power series $\mathbf{C}[[x]]$ is its completion. Let $K = \mathbf{C}\{x\}[1/x] = f.f.(B)$ be the field of convergent Laurent series. The K-module $\Omega_{K/\mathbf{C}}$ of algebraic differentials of K over $\mathbf{C}$ is an infinite dimensional K-vector space (proof omitted). We may choose $f_n \in x\mathbf{C}\{x\}$, $n \geq 1$ such that $\mathrm{d}x, \mathrm{d}f_1, \mathrm{d}f_2, \ldots$ are part of a basis of $\Omega_{K/\mathbf{C}}$. Thus we can find a $\mathbf{C}$-derivation

$$D : \mathbf{C}\{x\} \longrightarrow \mathbf{C}((x))$$

such that $D(x) = 0$ and $D(f_i) = x^{-n}$. Let

$$A = \{f \in \mathbf{C}\{x\} \mid D(f) \in \mathbf{C}[[x]]\}$$

We claim that

(1) $\mathbf{C}\{x\}$ is integral over A,
(2) A is a local domain,
(3) $\dim(A) = 1$,
(4) the maximal ideal of A is generated by x and xf_1,
(5) A is Noetherian, and
(6) the completion of A is equal to the ring of dual numbers over $\mathbf{C}[[x]]$.

Since the dual numbers are nonreduced the ring A gives the example.

Note that if $0 \neq f \in x\mathbf{C}\{x\}$ then we may write $D(f) = h/f^n$ for some $n \geq 0$ and $h \in \mathbf{C}[[x]]$. Hence $D(f^{n+1}/(n+1)) \in \mathbf{C}[[x]]$ and $D(f^{n+2}/(n+2)) \in \mathbf{C}[[x]]$. Thus we see $f^{n+1}, f^{n+2} \in A$! In particular we see (1) holds. We also conclude that the fraction field of A is equal to the fraction field of $\mathbf{C}\{x\}$. It also follows immediately that $A \cap x\mathbf{C}\{x\}$ is the set of nonunits of A, hence A is a local domain of dimension 1. If we can show (4) then it will follow that A is Noetherian (proof omitted). Suppose that $f \in A \cap x\mathbf{C}\{x\}$. Write $D(f) = h$, $h \in \mathbf{C}[[x]]$. Write $h = c + xh'$ with $c \in \mathbf{C}$, $h' \in \mathbf{C}[[x]]$. Then $D(f - cxf_1) = c + xh' - c = xh'$. On the other hand $f - cxf_1 = xg$ with $g \in \mathbf{C}\{x\}$, but by the computation above we have $D(g) = h' \in \mathbf{C}[[x]]$ and hence $g \in A$. Thus $f = cxf_1 + xg \in (x, xf_1)$ as desired.

Finally, why is the completion of A nonreduced? Denote $\hat{A}$ the completion of A. Of course this maps surjectively to the completion $\mathbf{C}[[x]]$ of $\mathbf{C}\{x\}$ because $x \in A$. Denote this map $\psi : \hat{A} \to \mathbf{C}[[x]]$. Above we saw that $\mathfrak{m}_A = (x, xf_1)$ and hence $D(\mathfrak{m}_A^n) \subset (x^{n-1})$ by an easy computation. Thus $D : A \to \mathbf{C}[[x]]$ is continuous and gives rise to a continuous derivation $\hat{D} : \hat{A} \to \mathbf{C}[[x]]$ over ψ. Hence we get a ring map

$$\psi + \epsilon\hat{D} : \hat{A} \longrightarrow \mathbf{C}[[x]][\epsilon].$$

Since $\hat{A}$ is a one dimensional Noetherian complete local ring, if we can show this arrow is surjective then it will follow that $\hat{A}$ is nonreduced. Actually the map is an isomorphism but we omit the verification of this. The subring $\mathbf{C}[x]_{(x)} \subset A$ gives rise to a map $i : \mathbf{C}[[x]] \to \hat{A}$ on completions such that $i \circ \psi = \mathrm{id}$ and such that $D \circ i = 0$ (as $D(x) = 0$ by construction). Consider the elements $x^n f_n \in A$. We have

$$(\psi + \epsilon D)(x^n f_n) = x^n f_n + \epsilon$$

for all $n \geq 1$. Surjectivity easily follows from these remarks.

64.9. A non catenary Noetherian local ring

Even though there is a succesful dimension theory of Noetherian local rings there are non-catenary Noetherian local rings. An example may be found in [**Nag62**, Appendix, Example 2]. In fact, we will present this example in the simplest case. Namely, we will construct a local Noetherian domain A of dimension 2 which is not universally catenary. (Note that A is automatically catenary, see Exercises, Exercise 65.12.2.) The existence of a Noetherian local ring which is not universally catenary implies the existence of a Noetherian local ring which is not catenary -- and we spell this out at the end of this section in the particular example at hand.

Let k be a field, and consider the formal power series ring $k[[x]]$ in one variable over k. Let

$$z = \sum\nolimits_{i=1}^{\infty} a_i x^i$$

be a formal power series. We assume z as an element of the Laurent series field $k((x)) = f.f.(k[[x]])$ is transcendental over $k(x)$. Put

$$z_j = x^{-j}(z - \sum\nolimits_{i=1,\ldots,j-1} a_i x^i) = \sum\nolimits_{i=j}^{\infty} a_i x^{i-j} \in k[[x]].$$

Note that $Z = z_1$. Let R be the subring of $k[[x]]$ generated by x, z and all of the z_j, in other words

$$R = k[x, z_1, z_2, z_3, \ldots] \subset k[[x]].$$

Consider the ideals $\mathfrak m = (x)$ and $\mathfrak n = (x - 1, z_1, z_2, \ldots)$ of R.

We have $x(z_{j+1} + a_j) = z_j$. Hence $R/\mathfrak m = k$ and $\mathfrak m$ is a maximal ideal. Moreover, any element of R not in $\mathfrak m$ maps to a unit in $k[[x]]$ and hence $R_{\mathfrak m} \subset k[[x]]$. In fact it is easy to deduce that $R_{\mathfrak m}$ is a discrete valuation ring and residue field k.

We claim that

$$R/(x - 1) = k[x, z_1, z_2, z_3, \ldots]/(x - 1) \cong k[z].$$

Namely, the relation above implies that $(x-1)(z_{j+1} + a_j) = -z_{j+1} - a_j + z_j$, and hence we may express the class of z_{j+1} in terms of z_j in the quotient $R/(x-1)$. Since the fraction field of R has transcendence degree 2 over k by construction we see that z is transcendental over k in $R/(x - 1)$, whence the desired isomorphism. Hence $\mathfrak n = (x - 1, z)$ and is a maximal ideal. In fact the map

$$k[x, x^{-1}, z]_{(x-1,z)} \longrightarrow R_{\mathfrak n}$$

is an isomorphism (since x^{-1} is invertible in $R_{\mathfrak n}$ and since $z_{j+1} = x^{-1}z_j - a_j = \ldots = f_j(x, x^{-1}, z)$). This shows that $R_{\mathfrak n}$ is a regular local ring of dimension 2 and residue field k.

Let S be the multiplicative subset

$$S = (R \setminus \mathfrak m) \cap (R \setminus \mathfrak n) = R \setminus (\mathfrak m \cup \mathfrak n)$$

and set $B = S^{-1}R$. We claim that

(1) The ring B is a k-algebra.
(2) The maximal ideals of the ring B are the two ideals $\mathfrak m B$ and $\mathfrak n B$.
(3) The residue fields at these maximal ideals is k.
(4) We have $B_{\mathfrak m B} = R_{\mathfrak m}$ and $B_{\mathfrak n B} = R_{\mathfrak n}$ which are Noetherian regular local rings of dimensions 1 and 2.
(5) The ring B is Noetherian.

We omit the details of the verifications.

Whenever given a k-algebra B with the properties listed above we get an example as follows. Take $A = k + \text{rad}(B) \subset B$, in our case $\text{rad}(B) = \mathfrak{m}B + \mathfrak{n}B$. It is easy to see that B is finite over A and hence A is Noetherian by Eakin's theorem (see [**Eak68**], or [**Nag62**, Appendix A1], or insert future reference here). Also A is a local domain with the same fraction field as B and residue field k. Since the dimension of B is 2 we see that A has dimension 2 as well, by Algebra, Lemma 7.103.4.

If A were universally catenary then the dimension formula, Algebra, Lemma 7.104.1 would give $\dim(B_{\mathfrak{m}B}) = 2$ contradiction.

Note that B is generated by one element over A. Hence $B = A[x]/\mathfrak{p}$ for some prime $\mathfrak{p}$ of $A[x]$. Let $\mathfrak{m}' \subset A[x]$ be the maximal ideal corresponding to $\mathfrak{m}B$. Then on the one hand $\dim(A[x]_{\mathfrak{m}'}) = 3$ and on the other hand

$$(0) \subset \mathfrak{p}A[x]_{\mathfrak{m}'} \subset \mathfrak{m}'A[x]_{\mathfrak{m}'}$$

is a maximal chain of primes. Hence $A[x]_{\mathfrak{m}'}$ is an example of a non catenary Noetherian local ring.

64.10. Non-quasi-affine variety with quasi-affine normalization

The existence of an example of this kind is mentioned in [**DG67**, II Remark 6.6.13]. They refer to the fifth volume of EGA for such an example, but the fifth volume did not appear.

Let k be a field. Let $Y = \mathbf{A}^2_k \setminus \{(0,0)\}$. We are going to construct a finite surjective birational morphism $\pi : Y \longrightarrow X$ with X a variety over k such that X is not quasi-affine. Namely, consider the following curves in Y:

$$\begin{array}{lcl} C_1 & : & x = 0 \\ C_2 & : & y = 0 \end{array}$$

Note that $C_1 \cap C_2 = \emptyset$. We choose the isomorphism $\varphi : C_1 \to C_2$, $(0, y) \mapsto (y^{-1}, 0)$. We claim there is a unique morphism $\pi : Y \to X$ as above such that

$$\begin{tikzcd} C_1 \arrow[r, shift left, "\mathrm{id}"] \arrow[r, shift right, "\varphi"'] & Y \arrow[r, "\pi"] & X \end{tikzcd}$$

is a coequalizer diagram in the category of varieties (and even in the category of schemes). Accepting this for the moment let us show that such an X cannot be quasi-affine. Namely, it is clear that we would get

$$\Gamma(X, \mathcal{O}_X) = \{f \in k[x, y] \mid f(0, y) = f(y^{-1}, 0)\} = k \oplus (xy) \subset k[x, y].$$

In particular these functions do not separate the points $(1, 0)$ and $(-1, 0)$ whose images in X (we will see below) are distinct (if the characteristic of k is not 2).

To show that X exists consider the Zariski open $D(x + y) \subset Y$ of Y. This is the spectrum of the ring $k[x, y, 1/(x + y)]$ and the curves C_1, C_2 are completely contained in $D(x + y)$. Moreover the morphism

$$C_1 \coprod C_2 \longrightarrow D(x + y) \cap Y = Spec(k[x, y, 1/(x + y)])$$

is a closed immersion. It follows from Algebra, Lemma 7.47.10 that the ring

$$A = \{f \in k[x, y, 1/(x + y)] \mid f(0, y) = f(y^{-1}, 0)\}$$

is of finite type over k. On the other hand we have the open $D(xy) \subset Y$ of Y which is disjoint from the curves C_1 and C_2. It is the spectrum of the ring

$$B = k[x, y, 1/xy].$$

Note that we have $A_{xy} \cong B_{x+y}$ (since A clearly contains the elements $xyP(x, y)$ any polynomial P and the element $xy/(x+y)$). The scheme X is obtained by glueing the affine schemes $Spec(A)$ and $Spec(B)$ using the isomorphism $A_{xy} \cong B_{x+y}$ and hence is clearly of finite type over k. To see that it is separated one has to show that the ring map $A \otimes_k B \to B_{x+y}$ is surjective. To see this use that $A \otimes_k B$ contains the element $xy/(x + y) \otimes 1/xy$ which maps to $1/(x + y)$. The morphism $X \to Y$ is given by the natural maps $D(x + y) \to Spec(A)$ and $D(xy) \to Spec(B)$. Since these are both finite we deduce that $X \to Y$ is finite as desired. We omit the verification that X is indeed the coequalizer of the displayed diagram above, however, see (insert future reference for push outs in the category of schemes here). Note that the morphism $\pi : Y \to X$ does map the points $(1, 0)$ and $(-1, 0)$ to distinct points in X because the function $(x + y^3)/(x + y)^2 \in A$ has value $1/1$, resp. $-1/(-1)^2 = -1$ which are always distinct (unless the characteristic is 2 -- please find your own points for characteristic 2). We summarize this discussion in the form of a lemma.

Lemma 64.10.1. *Let k be a field. There exists a variety X whose normalization is quasi-affine but which is itself not quasi-affine.*

Proof. See discussion above and (insert future reference on normalization here). □

64.11. A locally closed subscheme which is not open in closed

This is a copy of Morphisms, Example 24.2.10. Here is an example of an immersion which is not a composition of an open immersion followed by a closed immersion. Let k be a field. Let $X = Spec(k[x_1, x_2, x_3, \ldots])$. Let $U = \bigcup_{n=1}^{\infty} D(x_n)$. Then $U \to X$ is an open immersion. Consider the ideals

$$I_n = (x_1^n, x_2^n, \ldots, x_{n-1}^n, x_n - 1, x_{n+1}, x_{n+2}, \ldots) \subset k[x_1, x_2, x_3, \ldots][1/x_n].$$

Note that $I_n k[x_1, x_2, x_3, \ldots][1/x_n x_m] = (1)$ for any $m \neq n$. Hence the quasi-coherent ideals $\tilde{I}_n$ on $D(x_n)$ agree on $D(x_n x_m)$, namely $\tilde{I}_n|_{D(x_n x_m)} = \mathcal{O}_{D(x_n x_m)}$ if $n \neq m$. Hence these ideals glue to a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_U$. Let $Z \subset U$ be the closed subscheme corresponding to $\mathcal{I}$. Thus $Z \to X$ is an immersion.

We claim that we cannot factor $Z \to X$ as $Z \to \overline{Z} \to X$, where $\overline{Z} \to X$ is closed and $Z \to \overline{Z}$ is open. Namely, $\overline{Z}$ would have to be defined by an ideal $I \subset k[x_1, x_2, x_3, \ldots]$ such that $I_n = Ik[x_1, x_2, x_3, \ldots][1/x_n]$. But the only element $f \in k[x_1, x_2, x_3, \ldots]$ which ends up in all I_n is 0! Hence I does not exist.

64.12. Pushforward of quasi-coherent modules

In Schemes, Lemma 21.24.1 we proved that f_* transformes quasi-coherent modules into quasi-coherent modules when f is quasi-compact and quasi-separated. Here are some examples to show that these conditions are both necessary.

Suppose that $Y = \mathrm{Spec}(A)$ is an affine scheme and that $X = \coprod_{n \in \mathbf{N}} Y$. We claim that $f_*\mathcal{O}_X$ is not quasi-coherent where $f : X \to Y$ is the obvious morphism. Namely, for $a \in A$ we have

$$f_*\mathcal{O}_X(D(a)) = \prod\nolimits_{n \in \mathbf{N}} A_a$$

Hence, in order for $f_*\mathcal{O}_X$ to be quasi-coherent we would need

$$\prod\nolimits_{n\in\mathbf{N}} A_a = \left(\prod\nolimits_{n\in\mathbf{N}} A\right)_a$$

for all $a \in A$. This isn't true in general, for example if $A = \mathbf{Z}$ and $a = 2$, then $(1, 1/2, 1/4, 1/8, \ldots)$ is an element of the left hand side which is not in the right hand side. Note that f is a non-quasi-compact separated morphism.

Let k be a field. Set

$$A = k[t, z, x_1, x_2, x_3, \ldots]/(tx_1z, t^2x_2^2z, t^3x_3^3z, \ldots)$$

Let $Y = Spec(A)$. Let $V \subset Y$ be the open subscheme $V = D(x_1) \cup D(x_2) \cup \ldots$. Let X be two copies of Y glued along V. Let $f : X \to Y$ be the obvious morphism. Then we have an exact sequence

$$0 \to f_*\mathcal{O}_X \to \mathcal{O}_Y \oplus \mathcal{O}_Y \xrightarrow{(1,-1)} j_*\mathcal{O}_V$$

where $j : V \to Y$ is the inclusion morphism. Since

$$A \longrightarrow \prod A_{x_n}$$

is injective (details omitted) we see that $\Gamma(Y, f_*\mathcal{O}_X) = A$. On the other hand, the kernel of the map

$$A_t \longrightarrow \prod A_{tx_n}$$

is nonzero because it contains the element z. Hence $\Gamma(D(t), f_*\mathcal{O}_X)$ is strictly bigger than A_t because it contains $(z, 0)$. Thus we see that $f_*\mathcal{O}_X$ is not quasi-coherent. Note that f is quasi-compact but non-quasi-separated.

Lemma 64.12.1. *Schemes, Lemma 21.24.1 is sharp in the sense that one can neither drop the assumption of quasi-compactness nor the assumption of quasi-separatedness.*

Proof. See discussion above. □

64.13. A nonfinite module with finite free rank 1 stalks

Let $R = \mathbf{Q}[x]$. Set $M = \sum_{n\in\mathbf{N}} \frac{1}{x-n} R$ as a submodule of the fraction field of R. Then M is not finitely generated, but for every prime $\mathfrak{p}$ of R we have $M_\mathfrak{p} \cong R_\mathfrak{p}$ as an $R_\mathfrak{p}$-module.

64.14. A finite flat module which is not projective

This is a copy of Algebra, Remark 7.72.3. It is not true that a finite R-module which is R-flat is automatically projective. A counter example is where $R = \mathcal{C}^\infty(\mathbf{R})$ is the ring of infinitely differentiable functions on $\mathbf{R}$, and $M = R_\mathfrak{m} = R/I$ where $\mathfrak{m} = \{f \in R \mid f(0) = 0\}$ and $I = \{f \in R \mid \exists\epsilon, \epsilon > 0 : f(x) = 0\ \forall x, |x| < \epsilon\}$.

The morphism $Spec(R/I) \to Spec(R)$ is also an example of a flat closed immersion which is not open.

Lemma 64.14.1. *Strange flat modules.*

1. *There exists a ring R and a finite flat R-module M which is not projective.*
2. *There exists a closed immersion which is flat but not open.*

Proof. See discussion above. □

64.15. A projective module which is not locally free

We give two examples. One where the rank is between 0 and 1 and one where the rank is $\aleph_0$.

Lemma 64.15.1. *Let R be a ring. Let $I \subset R$ be an ideal generated by a countable collection of idempotents. Then I is projective as an R-module.*

Proof. Say $I = (e_1, e_2, e_3, \ldots)$ with e_n an idempotent of R. After inductively replacing e_{n+1} by $e_n + (1 - e_n)e_{n+1}$ we may assume that $(e_1) \subset (e_2) \subset (e_3) \subset \ldots$ and hence $I = \bigcup_{n\geq 1}(e_n) = colim_n\, e_nR$. In this case

$$Hom_R(I, M) = Hom_R(colim_n\, e_nR, M) = lim_n\, Hom_R(e_nR, M) = lim_n\, e_nM$$

Note that the transition maps $e_{n+1}M \to e_nM$ are given by multiplication by e_n are are surjective. Hence by Algebra, Lemma 7.80.4 the functor $Hom_R(I, M)$ is exact, i.e., I is a projective R-module. □

Lemma 64.15.2. *Let R be a ring. Let $n \geq 1$. Let M be an R-module generated by $< n$ elements. Then any R-module map $f : R^{\oplus n} \to M$ has a nonzero kernel.*

Proof. Choose a surjection $R^{\oplus n-1} \to M$. We may lift the map f to a map $f' : R^{\oplus n} \to R^{\oplus n-1}$. It suffices to prove f' has a nonzero kernel. The map $f' : R^{\oplus n} \to R^{\oplus n-1}$ is given by a matrix $A = (a_{ij})$. If one of the a_{ij} is not nilpotent, say $a = a_{ij}$ is not, then we can replace A by the localization A_a and we may assume a_{ij} is a unit. Since if we find a nonzero kernel after localization then there was a nonzero kernel to start with as localization is exact, see Algebra, Proposition 7.9.12. In this case we can do a base change on both $R^{\oplus n}$ and $R^{\oplus n-1}$ and reduce to the case where

$$A = \begin{pmatrix} 1 & 0 & 0\ldots & \\ 0 & a_{22} & a_{23} & \ldots \\ 0 & a_{32} & \ldots & \\ \ldots & \ldots & & \end{pmatrix}$$

Hence in this case we win by induction on n. If not then each a_{ij} is nilpotent. Set $I = (a_{ij}) \subset R$. Note that $I^{m+1} = 0$ for some $m \geq 0$. Let m be the largest integer such that $I^m \neq 0$. Then we see that $(I^m)^{\oplus n}$ is contained in the kernel of the map and we win. □

Suppose that $P \subset Q$ is an inclusion of R-modules with Q a finite R-module and P locally free, see Algebra, Definition 7.72.1. Suppose that Q can be generated by N elements as an R-module. Then it follows from Lemma 64.15.2 that P is finite locally free (with the free parts having rank at most N). And in this case P is a finite R-module, see Algebra, Lemma 7.72.2.

Combining this with the above we see that a non-finitely-generated ideal which is generated by a countable collection of idempotents is projective but not locally free. An explicit example is $R = \prod_{n\in\mathbf{N}} \mathbf{F}_2$ and I the ideal generated by the idempotents

$$e_n = (1, 1, \ldots, 1, 0, \ldots)$$

where the sequence of 1's has length n.

Lemma 64.15.3. *There exists a ring R and an ideal I such that I is projective as an R-module but not locally free as an R-module.*

Proof. See above. □

Lemma 64.15.4. *Let K be a field. Let C_i, $i = 1, \ldots, n$ be smooth, projective, geometrically irreducible curves over K. Let $P_i \in C_i(K)$ be a rational point and let $Q_i \in C_i$ be a point such that $[\kappa(Q_i) : K] = 2$. Then $[P_1 \times \ldots \times P_n]$ is nonzero in $A_0(U_1 \times_K \ldots \times_K U_n)$ where $U_i = C_i \setminus \{Q_i\}$.*

Proof. There is a degree map $\deg : A_0(C_1 \times_K \ldots \times_K C_n) \to \mathbf{Z}$ Because each Q_i has degree 2 over K we see that any zero cycle supported on the ``boundary"

$$C_1 \times_K \ldots \times_K C_n \setminus U_1 \times_K \ldots \times_K U_n$$

has degree divisible by 2. □

We can construct another example of a projective but not locally free module using the lemma above as follows. Let C_n, $n = 1, 2, 3, \ldots$ be smooth, projective, geometrically irreducible curves over $\mathbf{Q}$ each with a pair of points $P_n, Q_n \in C_n$ such that $\kappa(P_n) = \mathbf{Q}$ and $\kappa(Q_n)$ is a quadratic extension of $\mathbf{Q}$. Set $U_n = C_n \setminus \{Q_n\}$; this is an affine curve. Let $\mathcal{L}_n$ be the inverse of the ideal sheaf of P_n on U_n. Note that $c_1(\mathcal{L}_n) = [P_n]$ in the group of zero cycles $A_0(U_n)$. Set $A_n = \Gamma(U_n, \mathcal{O}_{U_n})$. Let $L_n = \Gamma(U_n, \mathcal{L}_n)$ which is a locally free module of rank 1 over A_n. Set

$$B_n = A_1 \otimes_{\mathbf{Q}} A_2 \otimes_{\mathbf{Q}} \ldots \otimes_{\mathbf{Q}} A_n$$

so that $Spec(B_n) = U_1 \times \ldots \times U_n$ all products over $Spec(\mathbf{Q})$. For $i \leq n$ we set

$$L_{n,i} = A_1 \otimes_{\mathbf{Q}} \ldots \otimes_{\mathbf{Q}} M_i \otimes_{\mathbf{Q}} \ldots \otimes_{\mathbf{Q}} A_n$$

which is a locally free B_n-module of rank 1. Note that this is also the global sections of $\text{pr}_i^*\mathcal{L}_n$. Set

$$B_\infty = colim_n B_n \quad \text{and} \quad L_{\infty,i} = colim_n L_{n,i}$$

Finally, set

$$M = \bigoplus\nolimits_{i \geq 1} L_{\infty,i}.$$

This is a direct sum of finite locally free modules, hence projective. We claim that M is not locally free. Namely, suppose that $f \in B_\infty$ is a nonzero function such that M_f is free over $(B_\infty)_f$. Let $e_1, e_2, \ldots$ be a basis. Choose $n \geq 1$ such that $f \in B_n$. Choose $m \geq n + 1$ such that $e_1, \ldots, e_{n+1}$ are in

$$\bigoplus\nolimits_{1 \leq i \leq m} L_{m,i}.$$

Because the elements $e_1, \ldots, e_{n+1}$ are part of a basis after a faithfully flat base change we conclude that the chern classes

$$c_i(\text{pr}_1^*\mathcal{L}_1 \oplus \ldots \oplus \text{pr}_m^*\mathcal{L}_m), \quad i = m, m-1, \ldots, m-n$$

are zero in the chow group of

$$D(f) \subset U_1 \times \ldots \times U_m$$

Since f is the pullback of a function on $U_1 \times \ldots \times U_n$ this implies in particular that

$$c_{m-n}(\mathcal{O}_W^{\oplus n} \oplus \text{pr}_1^*\mathcal{L}_{n+1} \oplus \ldots \oplus \text{pr}_{m-n}^*\mathcal{L}_m) = 0.$$

on the variety

$$W = (C_{n+1} \times \ldots \times C_m)_K$$

over the field $K = \mathbf{Q}(C_1 \times \ldots \times C_n)$. In other words the cycle

$$[(P_{n+1} \times \ldots \times P_m)_K]$$

is zero in the chow group of zero cycles on W. This contradicts Lemma 64.15.4 above because the points Q_i, $n + 1 \leq i \leq m$ induce corresponding points Q_i' on $(C_n)_K$ and as $K/\mathbf{Q}$ is geometrically irreducible we have $[\kappa(Q_i') : K] = 2$.

Lemma 64.15.5. *There exists a countable ring R and a projective module M which is a direct sum of countably many locally free rank 1 modules such that M is not locally free.*

Proof. See above. $\square$

64.16. Zero dimensional local ring with nonzero flat ideal

In [**Laz67**] there is an example of a zero dimensional local ring with a nonzero flat ideal. Here is the construction. Let k be a field. Let X_i, Y_i, $i \geq 1$ be variables. Take $R = k[X_i, Y_i]/(X_i - Y_iX_{i+1}, Y_i^2)$. Denote x_i, resp. y_i the image of X_i, resp. Y_i in this ring. Note that

$$x_i = y_ix_{i+1} = y_iy_{i+1}x_{i+2} = y_iy_{i+1}y_{i+2}x_{i+3} = \ldots$$

in this ring. The ring R has only one prime ideal, namely $\mathfrak{m} = (x_i, y_i)$. We claim that the ideal $I = (x_i)$ is flat as an R-module.

Note that the annihilator of x_i in R is the ideal $(x_1, x_2, x_3, \ldots, y_i, y_{i+1}, y_{i+2}, \ldots)$. Consider the R-module M generated by elements e_i, $i \geq 1$ and relations $e_i = y_ie_{i+1}$. Then M is flat as it is the colimit $colim_i\ R$ of copies of R with transition maps

$$R \xrightarrow{y_1} R \xrightarrow{y_2} R \xrightarrow{y_3} \ldots$$

Note that the annilator of e_i in M is the ideal $(x_1, x_2, x_3, \ldots, y_i, y_{i+1}, y_{i+2}, \ldots)$. Since every element of M, resp. I can be written as fe_i, resp. hx_i for some $f, h \in R$ we see that the map $M \to I$, $e_i \to x_i$ is an isomorphism and I is flat.

Lemma 64.16.1. *There exists a local ring R with a unique prime ideal and a nonzero ideal $I \subset R$ which is a flat R-module*

Proof. See discussion above. $\square$

64.17. An epimorphism of zero-dimensional rings which is not surjective

In [**Laz69**] one can find the following example. Let k be a field. Consider the ring homomorphism

$$k[x_1, x_2, \ldots, z_1, z_2, \ldots]/(x_i^{4^i}, z_i^{4^i}) \longrightarrow k[x_1, x_2, \ldots, y_1, y_2, \ldots]/(x_i^{4^i}, y_i - x_{i+1}y_{i+1}^2)$$

which maps x_i to x_i and z_i to x_iy_i. Note that $y_i^{4^{i+1}}$ is zero in the right hand side but that y_1 is not zero (details omitted). This map is not surjective: we can think of the above as a map of $\mathbf{Z}$-graded algebras by setting $\deg(x_i) = -1$, $\deg(z_i) = 0$, and $\deg(y_i) = 1$ and then it is clear that y_1 is not in the image. Finally, the map is an epimorphism because

$$y_{i-1} \otimes 1 = x_iy_i^2 \otimes 1 = y_i \otimes x_iy_i = x_iy_i \otimes y_i = 1 \otimes x_iy_i^2.$$

hence the tensor product of the target over the source is isomorphic to the target.

Lemma 64.17.1. *There exists an epimorphism of local rings of dimension 0 which is not a surjection.*

Proof. See discussion above. $\square$

64.18. Finite type, not finitely presented, flat at prime

Let k be a field. Consider the local ring $A_0 = k[x,y]_{(x,y)}$. Denote $\mathfrak{p}_{0,n} = (y + x^n + x^{n+1})$. This is a prime ideal. Set

$$A = A_0[z_1, z_2, z_3, \ldots]/(z_n z_m, z_n(y + x^n + x^{2n+1}))$$

Note that $A \to A_0$ is a surjection whose kernel is an ideal of square zero. Hence A is also a local ring and the prime ideals of A are in one-to-one correspondence with the prime ideals of A_0. Denote $\mathfrak{p}_n$ the prime ideal of A corresponding to $\mathfrak{p}_{0,n}$. Observe that $\mathfrak{p}_n$ is the annihilator of z_n in A. Let

$$C = A[z]/(xz^2 + z + y)[\frac{1}{2zx + 1}].$$

Note that $A \to C$ is an étale ring map, see Algebra, Example 7.126.8. Let $\mathfrak{q} \subset C$ be the maximal ideal generated by x, y, z and all z_n. As $A \to C$ is flat we see that the annihilator of z_n in C is $\mathfrak{p}_n C$. We compute

$$\begin{aligned} C/\mathfrak{p}_n C &= A_0/(y + x^n + x^{2n+1}) \\ &= k[x]_{(x)}[z]/(xz^2 + z - x^n - x^{2n+1}) \\ &= k[x]_{(x)}[z]/(z - x^n) \times k[x]_{(x)}[z]/(xz + x^{n+1} + 1) \\ &= k[x]_{(x)} \times k(x) \end{aligned}$$

because $(z - x^n)(xz + x^{n+1} + 1) = xz^2 + z - x^n - x^{2n+1}$. Hence we see that $\mathfrak{p}_n C = \mathfrak{r}_n \cap \mathfrak{q}_n$ with $\mathfrak{r}_n = \mathfrak{p}_n C + (z - x^n)C$ and $\mathfrak{q}_n = \mathfrak{p}_n C + (xz + x^{n+1} + 1)C$. Since $\mathfrak{q}_n + \mathfrak{r}_n = C$ we also get $\mathfrak{p}_n C = \mathfrak{r}_n \mathfrak{q}_n$, It follows that $\mathfrak{q}_n$ is the annihilator of $\xi_n = (z - x^n)z_n$. Observe that on the one hand $\mathfrak{r}_n \subset \mathfrak{q}$, and on the other hand $\mathfrak{q}_n + \mathfrak{q} = C$. This follows for example because $\mathfrak{q}_n$ is a maximal ideal of C distinct from $\mathfrak{q}$. Similarly we have $\mathfrak{q}_n + \mathfrak{q}_m = C$. At this point we let

$$B = \operatorname{Im}(C \longrightarrow C_{\mathfrak{q}})$$

We observe that the elements ξ_n map to zero in B as $xz + x^{n+1} + 1$ is not in $\mathfrak{q}$. Denote $\mathfrak{q}' \subset B$ the image of $\mathfrak{q}$. By construction B is a finite type A-algebra, with $B_{\mathfrak{q}'} \cong C_{\mathfrak{q}}$. In particular we see that $B_{\mathfrak{q}'}$ is flat over A.

We claim there does not exist an element $g' \in B$, $g' \notin \mathfrak{q}'$ such that $B_{g'}$ is of finite presentation over A. We sketch a proof of this claim. Choose an element $g \in C$ which maps to $g' \in B$. Consider the map $C_g \to B_{g'}$. By Algebra, Lemma 7.6.3 we see that B_g is finitely presented over A if and only if the kernel of $C_g \to B_{g'}$ is finitely generated. But the element $g \in C$ is not contained in $\mathfrak{q}$, hence maps to a nonzero element of $A_0[z]/(xz^2 + z + y)$. Hence g can only be contained in finitely many of the prime ideals $\mathfrak{q}_n$, because the primes $(y + x^n + x^{2n+1}, xz + x^{n+1} + 1)$ are an infinite collection of codimension 1 points of the 2-dimensional irreducible Noetherian space $Spec(k[x,y,z]/(xz^2 + z + y))$. The map

$$\bigoplus\nolimits_{g \notin \mathfrak{q}_n} C/\mathfrak{q}_n \longrightarrow C_g, \quad (c_n) \longrightarrow \sum c_n \xi_n$$

is injective and its image is the kernel of $C_g \to B_{g'}$. We omit the proof of this statement. (Hint: Write $A = A_0 \oplus I$ as an A_0-module where I is the kernel of $A \to A_0$. Similarly, write $C = C_0 \oplus IC$. Write $IC = \bigoplus Cz_n \cong \bigoplus (C/\mathfrak{r}_n \oplus C/\mathfrak{q}_n)$ and study the effect of multiplication by g on the summands.) This concludes the sketch of the proof of the claim. This also proves that $B_{g'}$ is not flat over A for any g' as above. Namely, if it were flat, then the annihilator of the image of z_n in $B_{g'}$ would be $\mathfrak{p}_n B_{g'}$, and would not contain $z - x^n$.

As a consequence we can answer (negatively) a question posed in [**GR71**, Part I, Remarques (3.4.7) (v)]. Here is a precise statement.

Lemma 64.18.1. *There exists a local ring A, a finite type ring map $A \to B$ and a prime $\mathfrak{q}$ lying over $\mathfrak{m}_A$ such that $B_\mathfrak{q}$ is flat over A, and for any element $g \in B$, $g \notin \mathfrak{q}$ the ring B_g is neither finitely presented over A nor flat over A.*

Proof. See discussion above. □

64.19. Finite type, flat and not of finite presentation

In this section we give some examples of ring maps and morphisms which are of finite type and flat but not of finite presentation.

Let R be a ring which has an ideal I such that R/I is a finite flat module but not projective, see Section 64.14 for an explicit example. Note that this means that I is not finitely generated, see Algebra, Lemma 7.100.5. Note that $I = I^2$, see Algebra, Lemma 7.100.2. The base ring in our examples will be R and correspondingly the base scheme $S = Spec(R)$.

Consider the ring map $R \to R \oplus R/I\epsilon$ where $\epsilon^2 = 0$ by convention. This is a finite, flat ring map which is not of finite presentation. All the fibre rings are complete intersections and geometrically irreducible.

Let $A = R[x, y]/(xy, ay; a \in I)$. Note that as an R-module we have $A = \bigoplus_{i \geq 0} Ry^i \oplus \bigoplus_{j>0} R/Ix^j$. Hence $R \to A$ is a flat finite type ring map which is not of finite presentation. Each fibre ring is isomorphic to either $\kappa(\mathfrak{p})[x, y]/(xy)$ or $\kappa(\mathfrak{p})[x]$.

We can turn the previous example into a projective morphism by taking $B = R[X_0, X_1, X_2]/(X_1X_2, aX_2; a \in I)$. In this case $X = \text{Proj}(B) \to S$ is a proper flat morphism which is not of finite presentation such that for each $s \in S$ the fibre X_s is isomorphic either to $\mathbf{P}^1_s$ or to the closed subscheme of $\mathbf{P}^2_s$ defined by the vanishing of X_1X_2 (this is a projective nodal curve of arithmetic genus 0).

Let $M = R \oplus R \oplus R/I$. Set $B = \text{Sym}_R(M)$ the symmetric algebra on M. Set $X = \text{Proj}(B)$. Then $X \to S$ is a proper flat morphism, not of finite presentation such that for $s \in S$ the geometric fibre is isomorphic to either $\mathbf{P}^1_s$ or $\mathbf{P}^2_s$. In particular these fibres are smooth and geometrically irreducible.

Lemma 64.19.1. *There exist examples of*

1. *a flat finite type ring map with geometrically irreducible complete intersection fibre rings which is not of finite presentation,*
2. *a flat finite type ring map with geometrically connected, geometrically reduced, dimension 1, complete intersection fibre rings which is not of finite presentation,*
3. *a proper flat morphism of schemes $X \to S$ each of whose fibres is isomorphic to either $\mathbf{P}^1_s$ or to the vanishing locus of X_1X_2 in $\mathbf{P}^2_s$ which is not of finite presentation, and*
4. *a proper flat morphism of schemes $X \to S$ each of whose fibres is isomorphic to either $\mathbf{P}^1_s$ or $\mathbf{P}^2_s$ which is not of finite presentation.*

Proof. See discussion above. □

64.20. Topology of a finite type ring map

Let $A \to B$ be a local map of local domains. If A is Noetherian, $A \to B$ is essentially of finite type, and $A/\mathfrak{m}_A \subset B/\mathfrak{m}_B$ is finite then there exists a prime $\mathfrak{q} \subset B$, $\mathfrak{q}\neq\mathfrak{m}_B$ such that $A \to B/\mathfrak{q}$ is the localization of a quasi-finite ring map. See More on Morphisms, Lemma 33.33.6.

In this section we give an example that shows this result is false A is no longer Noetherian. Namely, let k be a field and set

$$A = \{a_0 + a_1x + a_2x^2 + \ldots \mid a_0 \in k, a_i \in k((y)) \text{ for } i \geq 1\}$$

and

$$C = \{a_0 + a_1x + a_2x^2 + \ldots \mid a_0 \in k[y], a_i \in k((y)) \text{ for } i \geq 1\}.$$

The inclusion $A \to C$ is of finite type as C is generated by y over A. We claim that A is a local ring with maximal ideal $\mathfrak{m} = \{a_1x + a_2x^2 + \ldots \in A\}$ and no prime ideals besides (0) and $\mathfrak{m}$. Namely, an element $f = a_0 + a_1x + a_2x^2 + \ldots$ of A is invertible as soon as $a_0\neq 0$. If $\mathfrak{q} \subset A$ is a nonzero prime ideal, and $f = a_ix^i + \ldots \in \mathfrak{q}$, then using properties of power series one sees that for any $g \in k((y))$ the element $g^{i+1}x^{i+1} \in \mathfrak{q}$, i.e., $gx \in \mathfrak{q}$. This proves that $\mathfrak{q} = \mathfrak{m}$.

As to the spectrum of the ring C, arguing in the same way as above we see that any nonzero prime ideal contains the prime $\mathfrak{p} = \{a_1x + a_2x^2 + \ldots \in C\}$ which lies over $\mathfrak{m}$. Thus the only prime of C which lies over (0) is (0). Set $\mathfrak{m}_C = yC + \mathfrak{p}$ and $B = C_{\mathfrak{m}_C}$. Then $A \to B$ is the desired example.

Lemma 64.20.1. *There exists a local homomorphism $A \to B$ of local domains which is essentially of finite type and such that $A/\mathfrak{m}_A \to B/\mathfrak{m}_B$ is finite such that for every prime $\mathfrak{q}\neq\mathfrak{m}_B$ of B the ring map $A \to B/\mathfrak{q}$ is not the localization of a quasi-finite ring map.*

Proof. See the discussion above. □

64.21. Pure not universally pure

Let k be a field. Let

$$R = k[[x, xy, xy^2, \ldots]] \subset k[[x, y]].$$

In other words, a power series $f \in k[[x, y]]$ is in R if and only if $f(0, y)$ is a constant. In particular $R[1/x] = k[[x, y]][1/x]$ and R/xR is a local ring with a maximal ideal whose square is zero. Denote $R[y] \subset k[[x, y]]$ the set of power series $f \in k[[x, y]]$ such that $f(0, y)$ is a polynomial in y. Then $R \to R[y]$ is a finite type but not finitely presented ring map which induces an isomorphism after inverting x. Also there is a surjection $R[y]/xR[y] \to k[y]$ whose kernel has square zero. Consider the finitely presented ring map $R \to S = R[t]/(xt-xy)$. Again $R[1/x] \to S[1/x]$ is an isomorphism and in this case $S/xS \cong (R/xR)[t]/(xy)$ maps onto $k[t]$ with nilpotent kernel. There is a surjection $S \to R[y]$, $t \longmapsto y$ which induces an isomorphism on inverting x and a surjection with nilpotent kernel modulo x. Hence the kernel of $S \to R[y]$ is locally nilpotent. In particular $S \to R[y]$ is a universal homeomorphism.

First we claim that S is an S-module which is relatively pure over R. Since on inverting x we obtain an isomorphism we only need to check this at the maximal ideal $\mathfrak{m} \subset R$. Since R is complete with respect to its maximal ideal it is henselian hence we need only check that every prime $\mathfrak{p} \subset R$, $\mathfrak{p}\neq\mathfrak{m}$, the unique prime $\mathfrak{q}$ of S lying over $\mathfrak{p}$ satisfies $\mathfrak{m}S + \mathfrak{q}\neq S$. Since $\mathfrak{p}\neq\mathfrak{m}$ it corresponds to a unique prime ideal of $k[[x, y]][1/x]$. Hence either $\mathfrak{p} = (0)$

or $\mathfrak{p} = (f)$ for some irreducible element $f \in k[[x, y]]$ which is not associated to x (here we use that $k[[x, y]]$ is a UFD -- insert future reference here). In the first case $\mathfrak{q} = (0)$ and the result is clear. In the second case we may multiply f by a unit so that $f \in R[y]$ (Weierstrass preparation; details omitted). Then it is easy to see that $R[y]/fR[y] \cong k[[x, y]]/(f)$ hence f defines a prime ideal of $R[y]$ and $\mathfrak{m}R[y] + fR[y] \neq R[y]$. Since $S \to R[y]$ is a universal homeomorphism we deduce the desired result for S also.

Second we claim that S is not universally relatively pure over R. Namely, to see this it sufffices to find a valuation ring $\mathcal{O}$ and a local ring map $R \to \mathcal{O}$ such that $Spec(R[y] \otimes_R \mathcal{O}) \to Spec(\mathcal{O})$ does not hit the closed point of $Spec(\mathcal{O})$. Equivalently, we have to find $\varphi : R \to \mathcal{O}$ such that $\varphi(x) \neq 0$ and $v(\varphi(x)) > v(\varphi(xy))$ where v is the valuation of $\mathcal{O}$. (Because this means that the valuation of y is negative.) To do this consider the ring map

$$R \longrightarrow \{a_0 + a_1x + a_2x^2 + \ldots \mid a_0 \in k[y^{-1}], a_i \in k((y))\}$$

defined in the obvious way. We can find a valuation ring $\mathcal{O}$ dominating the localization of the right hand side at the maximal ideal (y^{-1}, x) and we win.

Lemma 64.21.1. *There exists a morphism of affine schemes of finite presentation $X \to S$ and an $\mathcal{O}_X$-module $\mathcal{F}$ of finite presentation such that $\mathcal{F}$ is pure relative to S, but not universally pure relative to S.*

Proof. See discussion above. □

64.22. A formally smooth non-flat ring map

Let k be a field. Consider the k-algebra $k[\mathbf{Q}]$. This is the k-algebra with basis x_α, $\alpha \in \mathbf{Q}$ and multiplication determined by $x_\alpha x_\beta = x_{\alpha+\beta}$. (In particular $x_0 = 1$.) Consider the k-algebra homomorphism

$$k[\mathbf{Q}] \longrightarrow k, \quad x_\alpha \longmapsto 1.$$

It is surjective with kernel J generated by the elements $x_\alpha - 1$. Let us compute J/J^2. Note that multiplication by x_α on J/J^2 is the identity map. Denote z_α the class of $x_\alpha - 1$ modulo J^2. These classes generate J/J^2. Since

$$(x_\alpha - 1)(x_\beta - 1) = x_{\alpha+\beta} - x_\alpha - x_\beta + 1 = (x_{\alpha+\beta} - 1) - (x_\alpha - 1) - (x_\beta - 1)$$

we see that $z_{\alpha+\beta} = z_\alpha + z_\beta$ in J/J^2. A general element of J/J^2 is of the form $\sum \lambda_\alpha z_\alpha$ with $\lambda_\alpha \in k$ (only finitely many nonzero). Note that if the characteristic of k is $p > 0$ then

$$0 = pz_{\alpha/p} = z_{\alpha/p} + \ldots + z_{\alpha/p} = z_\alpha$$

and we see that $J/J^2 = 0$. If the characteristic of k is zero, then

$$J/J^2 = \mathbf{Q} \otimes_{\mathbf{Z}} k \cong k$$

(details omitted) is not zero.

We claim that $k[\mathbf{Q}] \to k$ is a formally smooth ring map if the characteristic of k is positive. Namely, suppose given a solid commutative diagram

$$\begin{tikzcd}
k \arrow[r] \arrow[rd, dotted] & A \\
k[\mathbf{Q}] \arrow[u] \arrow[r, "\varphi"] & A' \arrow[u]
\end{tikzcd}$$

with $A' \to A$ a surjection whose kernel I has square zero. To show that $k[\mathbf{Q}] \to k$ is formally smooth we have to prove that φ factors through k. Since $\varphi(x_\alpha - 1)$ maps to zero in A we see that φ induces a map $\overline{\varphi} : J/J^2 \to I$ whose vanishing is the obstruction to the desired factorization. Since $J/J^2 = 0$ if the characteristic is $p > 0$ we get the result we want, i.e., $k[\mathbf{Q}] \to k$ is formally smooth in this case. Finally, this ring map is not flat, for example as the nonzero divisor $x_2 - 1$ is mapped to zero.

Lemma 64.22.1. *There exists a formally smooth ring map which is not flat.*

Proof. See discussion above. □

64.23. A formally étale non-flat ring map

In this section we give a counterexample to the final sentence in [**DG67**, 0_{IV}, Example 19.10.3(i)] (this was not one of the items caught in their later errata lists). Consider $A \to A/J$ for a local ring A and a nonzero proper ideal J such that $J^2 = J$ (so J isn't finitely generated); the valuation ring of an algebraically closed non-archimedean field with J its maximal ideal is a source of such (A, J). These non-flat quotient maps are formally étale. Namely, suppose given a commutative diagram

$$\begin{tikzcd} A/J \arrow[r] & R/I \\ A \arrow[u] \arrow[r, "\varphi"] & R \arrow[u] \end{tikzcd}$$

where I is an ideal of the ring R with $I^2 = 0$. Then $A \to R$ factors uniquely through A/J because

$$\varphi(J) = \varphi(J^2) \subset (\varphi(J)A)^2 \subset I^2 = 0.$$

Hence this also provides a counterexample to the formally étale case of the ``structure theorem'' for locally finite type and formally étale morphisms in [**DG67**, IV, Theorem 18.4.6(i)] (but not a counterexample to part (ii), which is what people actually use in practice). The error in the proof of the latter is that the very last step of the proof is to invoke the incorrect [**DG67**, 0_{IV}, Example 19.3.10(i)], which is how the counterexample just mentioned creeps in.

Lemma 64.23.1. *There exist formally étale nonflat ring maps.*

Proof. See discussion above. □

64.24. A formally étale ring map with nontrivial cotangent complex

Let k be a field. Consider the ring

$$R = k[\{x_n\}_{n\geq 1}, \{y_n\}_{n\geq 1}]/(x_1y_1, x_{nm}^m - x_n, y_{nm}^m - y_n)$$

Let A be the localization at the maximal ideal generated by all x_n, y_n and denote $J \subset A$ the maximal ideal. Set $B = A/J$. By construction $J^2 = J$ and hence $A \to B$ is formally étale (see Section 64.23). We claim that the element $x_1 \otimes y_1$ is a nonzero element in the kernel of

$$J \otimes_A J \longrightarrow J.$$

Namely, (A, J) is the colimit of the localizations (A_n, J_n) of the rings

$$R_n = k[x_n, y_n]/(x_n^n y_n^n)$$

at their corresponding maximal ideals. Then $x_1 \otimes y_1$ corresponds to the element $x_n^n \otimes y_n^n \in J_n \otimes_{A_n} J_n$ and is nonzero (by an explicit computation which we omit). Since $\otimes$ commutes with colimits we conclude. By [**Ill72**, III Section 3.3] we see that J is not weakly regular.

Hence by [**Ill72**, III Proposition 3.3.3] we see that the cotangent complex $L_{B/A}$ is not zero. In fact, we can be more precise. We have $H_0(L_{B/A}) = \Omega_{B/A}$ and $H_1(L_{B/A}) = 0$ because $J/J^2 = 0$. But from the five-term exact sequence of Quillen's fundamental spectral sequence [**Rei**, Corollary 8.2.6] and the nonvanishing of $\text{Tor}_2^A(B, B) = \text{Ker}(J \otimes_A J \to J)$ we conclude that $H_2(L_{B/A})$ is nonzero.

Lemma 64.24.1. *There exists a formally étale surjective ring map $A \to B$ with $L_{B/A}$ not equal to zero.*

Proof. See discussion above. □

64.25. Ideals generated by sets of idempotents and localization

Let R be a ring. Consider the ring

$$B(R) = R[x_n; n \in \mathbf{Z}]/(x_n(x_n - 1), x_n x_m; n \neq m)$$

It is easy to show that every prime $\mathfrak{q} \subset B(R)$ is either of the form

$$\mathfrak{q} = \mathfrak{p}B(R) + (x_n; n \in \mathbf{Z})$$

or of the form

$$\mathfrak{q} = \mathfrak{p}B(R) + (x_n - 1) + (x_m; n \neq m, m \in \mathbf{Z}).$$

Hence we see that

$$Spec(B(R)) = Spec(R) \amalg \coprod\nolimits_{n \in \mathbf{Z}} Spec(R)$$

where the topology is not just the disjoint union topology. It has the following properties: Each of the copies indexed by $n \in \mathbf{Z}$ is an open subscheme, namely it is the standard open $D(x_n)$. The "central" copy of $Spec(R)$ is in the closure of the union of any infinitely many of the other copies of $Spec(R)$. Note that this last copy of $Spec(R)$ is cut out by the ideal $(x_n, n \in \mathbf{Z})$ which is generated by the idempotents x_n. Hence we see that if $Spec(R)$ is connected, then the decomposition above is exactly the decomposition of $Spec(B(R))$ into connected components.

Next, let $A = \mathbf{C}[x, y]/((y - x^2 + 1)(y + x^2 - 1))$. The spectrum of A consists of two irreducible components $C_1 = Spec(A_1)$, $C_2 = Spec(A_2)$ with $A_1 = \mathbf{C}[x, y]/(y - x^2 + 1)$ and $A_2 = \mathbf{C}[x, y]/(y + x^2 - 1)$. Note that these are parametrized by $(x, y) = (t, t^2 - 1)$ and $(x, y) = (t, -t^2 + 1)$ which meet in $P = (-1, 0)$ and $Q = (1, 0)$. We can make a twisted version of $B(A)$ where we glue $B(A_1)$ to $B(A_2)$ in the following way: Above P we let $x_n \in B(A_1) \otimes \kappa(P)$ correspond to $x_n \in B(A_2) \otimes \kappa(P)$, but above Q we let $x_n \in B(A_1) \otimes \kappa(P)$ correspond to $x_{n+1} \in B(A_2) \otimes \kappa(P)$. Let $B^{twist}(A)$ denote the resulting A-algebra. Details omitted. By construction $B^{twist}(A)$ is Zariski locally over A isomorphic to the untwisted version. Namely, this happens over both the principal open $Spec(A) \setminus \{P\}$ and the principal open $Spec(A) \setminus \{Q\}$. However, our choice of glueing produces enough "monodromy" such that $Spec(B^{twist}(A))$ is connected (details omitted). Finally, there is a central copy of $Spec(A) \to Spec(B^{twist}(A))$ which gives a closed subscheme whose ideal is Zariski locally on $B^{twist}(A)$ cut out by ideals generated by idempotents, but not globally (as $B^{twist}(A)$ has no nontrivial idempotents).

Lemma 64.25.1. *There exists an affine scheme $X = Spec(A)$ and a closed subscheme $T \subset X$ such that T is Zariski locally on X cut out by ideals generated by idempotents, but T is not cut out by an ideal generated by idempotents.*

Proof. See above. □

64.26. Non flasque quasi-coherent sheaf associated to injective module

For more examples of this type see [**BGI71**, Exposé II, Appendix I] where Illusie explains some examples due to Verdier.

Consider the affine scheme $X = Spec(A)$ where

$$A = k[f, g, x, y, \{a_n, b_n\}_{n\geq 1}]/(fy - gx, \{a_n f^n + b_n g^n\}_{n\geq 1})$$

is the ring from Properties, Example 23.22.2. Set $I = (f, g) \subset A$. Consider the quasi-compact open $U = D(f) \cup D(g)$ of X. We have seen in loc. cit. that there is a section $s \in \mathcal{O}_X(U)$ which does not come from an A-module map $I^n \to A$ for any $n \geq 0$.

Let $\alpha : A \to J$ be the embedding of A into an injective A-module. Let $Q = J/\alpha(A)$ and denote $\beta : J \to Q$ the quotient map. We claim that the map

$$\Gamma(X, \tilde{J}) \longrightarrow \Gamma(U, \tilde{J})$$

is not surjective. Namely, we claim that $\alpha(s)$ is not in the image. To see this, we argue by contradiction. So assume that $x \in J$ is an element which restricts to $\alpha(s)$ over U. Then $\beta(x) \in Q$ is an element which restricts to 0 over U. Hence we know that $I^n\beta(x) = 0$ for some n, see Properties, Lemma 23.22.1. This implies that we get a morphism $\varphi : I^n \to A$, $h \mapsto \alpha^{-1}(hx)$. It is easy to see that this morphism φ gives rise to the section s via the map of Properties, Lemma 23.22.1 which is a contradiction.

Lemma 64.26.1. *There exists an affine scheme $X = Spec(A)$ and an injective A-module J such that $\tilde{J}$ is not a flasque sheaf on X. Even the restriction $\Gamma(X, \tilde{J}) \to \Gamma(U, \tilde{J})$ with U quasi-compact open need not be surjective.*

Proof. See above. □

64.27. A non-separated flat group scheme

Every group scheme over a field is separated, see Groupoids, Lemma 35.7.2. This is not true for group schemes over a base.

Let k be a field. Let $S = Spec(k[x]) = \mathbf{A}^1_k$. Let G be the affine line with 0 doubled (see Schemes, Example 21.14.3) seen as a scheme over S. Thus a fibre of $G \to S$ is either a singleton or a set with two elements (one in U and one in V). Thus we can endow these fibres with the structure of a group (by letting the element in U be the zero of the group structure). More precisely, G has two opens U, V which map isomorphically to S such that $U \cap V$ is mapped isomorphically to $S \setminus \{0\}$. Then

$$G \times_S G = U \times_S U \cup V \times_S U \cup U \times_S V \cup V \times_S V$$

where each piece is isomorphic to S. Hence we can define a multiplication $m : G \times_S G \to G$ as the unique S-morphism which maps the first and the last piece into U and the two middle pieces into V. This matches the pointwise description given above. We omit the verification that this defines a group scheme structure.

Lemma 64.27.1. *There exists a flat group scheme of finite type over the affine line which is not separated.*

Proof. See discussion above. □

64.28. A non-flat group scheme with flat identity component

Let $X \to S$ be a monomorphism of schemes. Let $G = S \amalg X$. Let $m : G \times_S G \to G$ be the S-morphism

$$G \times_S G = X \times_S X \amalg X \amalg X \amalg S \longrightarrow G = X \amalg S$$

which maps the summands $X \times_S X$ and S into S and maps the summands X into X by the identity morphism. This defines a group law. To see this we have to show that $m \circ (m \times \text{id}_G) = m \circ (\text{id}_G \times m)$ as maps $G \times_S G \times_S G \to G$. Decomposing $G \times_S G \times_S G$ into components as above, we see that we need to verify this for the restriction to each of the 8-pieces. Each piece is isomorphic to either S, X, $X \times_S X$, or $X \times_S X \times_S X$. Moreover, both maps map these pieces to S, X, S, X respectively. Having said this, the fact that $X \to S$ is a monomorphism implies that $X \times_S X \cong X$ and $X \times_S X \times_S X \cong X$ and that there is in each case exactly one S-morphism $S \to S$ or $X \to X$. Thus we see that $m \circ (m \times \text{id}_G) = m \circ (\text{id}_G \times m)$. Thus taking $X \to S$ to be any nonflat monomorphism of schemes (e.g., a closed immersion) we get an example of a group scheme over a base S whose identity component is S (hence flat) but which is not flat.

Lemma 64.28.1. *There exists a group scheme G over a base S whose identity component is flat over S but which is not flat over S.*

Proof. See discussion above. $\square$

64.29. A non-separated group algebraic space over a field

Every group scheme over a field is separated, see Groupoids, Lemma 35.7.2. This is not true for group algebraic spaces over a field.

Let k be a field of characteristic zero. Consider the algebraic space $G = \mathbf{A}^1_k/\mathbf{Z}$ from Spaces, Example 40.14.8. By construction G is the fppf sheaf associated to the presheaf

$$T \longmapsto \Gamma(T, \mathcal{O}_T)/\mathbf{Z}$$

on the category of schemes over k. The obvious addition rule on the presheaf induces an addtion $m : G \times G \to G$ which turns G into a group algebraic space over $Spec(k)$. Note that G is not separated (and not even quasi-separated or locally separated). On the other hand $G \to Spec(k)$ is of finite type!

Lemma 64.29.1. *There exists a group algebraic space of finite type over a field which is not separated (and not even quasi-separated or locally separated).*

Proof. See discussion above. $\square$

64.30. Specializations between points in fibre étale morphism

If $f : X \to Y$ is an étale, or more generally a locally quasi-finite morphism of schemes, then there are no specializations between points of fibres, see Morphisms, Lemma 24.19.8. However, for morphisms of algebraic spaces this doesn't hold in general.

To give an example, let k be a field. Set

$$P = k[u, u^{-1}, y, \{x_n\}_{n \in \mathbf{Z}}].$$

Consider the action of $\mathbf{Z}$ on P by k-algebra maps generated by the automorphism τ given by the rules $\tau(u) = u$, $\tau(y) = uy$, and $\tau(x_n) = x_{n+1}$. For $d \geq 1$ set $I_d = ((1-u^d)y, x_n - x_{n+d}, n \in \mathbf{Z})$. Then $V(I_d) \subset Spec(P)$ is the fix point locus of τ^d. Let $S \subset P$ be the multiplicative subset generated by y and all $1 - u^d$, $d \in \mathbf{N}$. Then we see that $\mathbf{Z}$ acts freely on $U = Spec(S^{-1}P)$. Let $X = U/\mathbf{Z}$ be the quotient algebraic space, see Spaces, Definition 40.14.4.

Consider the prime ideals $\mathfrak{p}_n = (x_n, x_{n+1}, \ldots)$ in $S^{-1}P$. Note that $\tau(\mathfrak{p}_n) = \mathfrak{p}_{n+1}$. Hence each of these define point $\xi_n \in U$ whose image in X is the same point x of X. Moreover we have the specializations

$$\ldots \rightsquigarrow \xi_n \rightsquigarrow \xi_{n-1} \rightsquigarrow \ldots$$

We conclude that $U \to X$ is an example of the promised type.

Lemma 64.30.1. *There exists an étale morphism of algebraic spaces $f : X \to Y$ and a nontrivial specialization of points $x \rightsquigarrow x'$ in $|X|$ with $f(x) = f(x')$ in $|Y|$.*

Proof. See discussion above. $\square$

64.31. A torsor which is not an fppf torsor

In Groupoids, Remark 35.9.5 we raise the question whether any G-torsor is a G-torsor for the fppf topology. In this section we show that this is not always the case.

Let k be a field. All schemes and stacks are over k in what follows. Let $G \to Spec(k)$ be the group scheme

$$G = (\mu_{2,k})^\infty = \mu_{2,k} \times_k \mu_{2,k} \times_k \mu_{2,k} \times_k \ldots = lim_n (\mu_{2,k})^n$$

where $\mu_{2,k}$ is the group scheme of second roots of unity over $Spec(k)$, see Groupoids, Example 35.5.2. As an inverse limit of affine schemes we see that G is an affine group scheme. In fact it is the spectrum of the ring $k[t_1, t_2, t_3, \ldots]/(t_i^2 - 1)$. The multiplication map $m : G \times_k G \to G$ is on the algebra level given by $t_i \mapsto t_i \otimes t_i$.

We claim that any G-torsor over k is of the form

$$P = Spec(k[x_1, x_2, x_3, \ldots]/(x_i^2 - a_i))$$

for certain $a_i \in k^*$ and with G-action $G \times_k P \to P$ given by $x_i \to t_i \otimes x_i$ on the algebra level. We omit the proof. Actually for the example we only need that P is a G-torsor which is clear since over $k' = k(\sqrt{a_1}, \sqrt{a_2}, \ldots)$ the scheme P becomes isomorphic to G in a G-equivariant manner. Note that P is trivial if and only if $k' = k$ since if P has a k-rational point then all of the a_i are squares.

We claim that P is an fppf torsor if and only if the field extension $k \subset k' = k(\sqrt{a_1}, \sqrt{a_2}, \ldots)$ is finite. If k' is finite over k, then $\{Spec(k') \to Spec(k)\}$ is an fppf covering which trivializes P and we see that P is indeed an fppf torsor. Conversely, suppose that P is a G-torsor for the fppf topology. This means that there exists an fppf covering $\{S_i \to Spec(k)\}$ such that each P_{S_i} is trivial. Pick an i such that S_i is not empty. Let $s \in S_i$ be a closed point. By Varieties, Lemma 28.12.1 the field extension $k \subset \kappa(s)$ is finite, and by construction $P_{\kappa(s)}$ has a $\kappa(s)$-rational point. Thus we see that $k \subset k' \subset \kappa(s)$ and k' is finite over k.

To get an explicit example take $k = \mathbf{Q}$ and $a_i = i$ for example (or a_i is the ith prime if you like).

Lemma 64.31.1. *Let S be a scheme. Let G be a group scheme over S. The stack G-Principal classifying principal homogeneous G-spaces (see Examples of Stacks, Subsection 55.13.5) and the stack G-Torsors classifying fppf G-torsors (see Examples of Stacks, Subsection 55.13.8) are not equivalent in general.*

Proof. The discussion above shows that the functor *G-Torsors* → *G-Principal* isn't essentially surjective in general. □

64.32. Stack with quasi-compact flat covering which is not algebraic

In this section we briefly describe an example due to Brian Conrad. You can find the example online at this location. Our example is slightly different.

Let k be an algebraically closed field. All schemes and stacks are over k in what follows. Let $G \to Spec(k)$ be an affine group scheme. In Examples of Stacks, Proposition 55.14.4 we have seen that $\mathcal{X} = [Spec(k)/G]$ is a stack in groupoids over $(Sch/Spec(k))_{fppf}$ which can be described as follows. A 1-morphism $T \to \mathcal{X}$ corresponds by definition to an fppf G_T-torsor P over T. The diagonal 1-morphism

$$\Delta : \mathcal{X} \longrightarrow \mathcal{X} \times_{Spec(k)} \mathcal{X}$$

is representable and affine. The reason for this is that given any pair of G_T-torsors P_1, P_2 in the fppf topology over a scheme S/k the scheme $Isom(P_1, P_2)$ is affine over T. The trivial G-torsor over $Spec(k)$ defines a 1-morphism

$$f : Spec(k) \longrightarrow \mathcal{X}.$$

We claim that this is a surjective 1-morphism. The reason is simply that by definition for any 1-morphism $T \to \mathcal{X}$ there exists a fppf covering $\{T_i \to T\}$ such that P_{T_i} is isomorphic to the trivial G_{T_i}-torsor. Hence the compositions $T_i \to T \to \mathcal{X}$ factor through f. Thus it is clear that the projection $T \times_{\mathcal{X}} Spec(k) \to \mathcal{X}$ is surjective (which is how we define the property that f is surjective, see Algebraic Stacks, Definition 57.10.1). In a similar way you show that f is quasi-compact and flat (details omitted). We also record here the observation that

$$Spec(k) \times_{\mathcal{X}} Spec(k) \cong G$$

as schemes over k.

Suppose there exists a surjective smooth morphism $p : U \to \mathcal{X}$ where U is a scheme. Consider the fibre product

$$\begin{tikzcd} W \arrow[r] \arrow[d] & U \arrow[d] \\ Spec(k) \arrow[r] & \mathcal{X} \end{tikzcd}$$

Then we see that W is a nonempty smooth scheme over k which hence has a k-point. This means that we can factor f through U. Hence we obtain

$$G \cong Spec(k) \times_{\mathcal{X}} Spec(k) \cong (Spec(k) \times_k Spec(k)) \times_{(U \times_k U)} (U \times_{\mathcal{X}} U)$$

and since the projections $U \times_{\mathcal{X}} U \to U$ were assumed smooth we conclude that $U \times_{\mathcal{X}} U \to U \times_k U$ is locally of finite type, see Morphisms, Lemma 24.14.8. It follows that in this case G is locally of finite type over k. Alltogether we have proved the following lemma (which can be significantly generalized).

Lemma 64.32.1. *Let k be a field. Let G be an affine group scheme over k. If the stack $[Spec(k)/G]$ has a smooth covering by a scheme, then G is of finite type over k.*

Proof. See discussion above. □

To get an explicit example as in the title of this section, take for example $G = (\mu_{2,k})^\infty$ the group scheme of Section 64.31, which is not locally of finite type over k. By the discussion above we see that $\mathcal{X} = [Spec(k)/G]$ has properties (1) and (2) of Algebraic Stacks, Definition 57.12.1, but not property (3). Hence $\mathcal{X}$ is not an algebraic stack. On the other hand, there does exists a scheme U an a surjective, flat, quasi-compact morphism $U \to \mathcal{X}$, namely the morphism $f : Spec(k) \to \mathcal{X}$ we studied above.

64.33. A non-algebraic classifying stack

Let $S = \mathrm{Spec}(\mathbf{F}_p)$ and let μ_p denote the group scheme of pth roots of unity over S. In Groupoids in Spaces, Section 52.19 we have introduced the quotient stack $[S/\mu_p]$ and in Examples of Stacks, Section 55.14 we have shown $[S/\mu_p]$ is the classifying stack for fppf μ_p-torsors: Given a scheme T over S the category $Mor_S(T, [S/\mu_p])$ is canonically equivalent to the category of fppf μ_p-torsors over T. Finally, in Criteria for Representability, Theorem 59.17.2 we have seen that $[S/\mu_p]$ is an algebraic stack.

Now we can ask the question: ``How about the category fibred in groupoids $\mathcal{S}$ classifying étale μ_p-torsors?'' (In other words $\mathcal{S}$ is a category over Sch/S whose fibre category over a scheme T is the category of étale μ_p-torsors over T.)

The first objection is that this isn't a stack for the fppf topology, because descent for objects isn't going to hold. For example the μ_p-torsor $\mathrm{Spec}(\mathbf{F}_p(t)[x]/(x^p - t))$ over $T = \mathrm{Spec}(\mathbf{F}_p(T))$ is fppf locally trivial, but not étale locally trivial.

A fix for this first problem is to work with the étale topology and in this case descent for objects does work. Indeed it is true that $\mathcal{S}$ is a stack in groupoids over $(Sch/S)_{étale}$. Moreover, it is also the case that the diagonal $\Delta : \mathcal{S} \to \mathcal{S} \times \mathcal{S}$ is representable (by schemes). This is true because given two μ_p-torsors (whether they be étale locally trivial or not) the sheaf of isomorphisms between them is representable by a scheme.

Thus we can finally ask if there exists a scheme U and a smooth and surjective 1-morphism $U \to \mathcal{S}$. We will show in two ways that this is impossible: by a direct argument (which we advise the reader to skip) and by an argument using a general result.

Direct argument (sketch): Note that the 1-morphism $\mathcal{S} \to \mathrm{Spec}(\mathbf{F}_p)$ satisfies the infinitesimal lifting criterion for formal smoothness. This is true because given a first order infinitesimal thickening of schemes $T \to T'$ the kernel of $\mu_p(T') \to \mu_p(T)$ is isomorphic to the sections of the ideal sheaf of T in T', and hence $H^1_{étale}(T, \mu_p) = H^1_{étale}(T', \mu_p)$. Moreover, $\mathcal{S}$ is a limit preserving stack. Hence if $U \to \mathcal{S}$ is smooth, then $U \to \mathrm{Spec}(\mathbf{F}_p)$ is limit preserving and satisfies the infinitesimal lifting criterion for formal smoothness. This implies that U is smooth over $\mathbf{F}_p$. In particular U is reduced, hence $H^1_{étale}(U, \mu_p) = 0$. Thus $U \to \mathcal{S}$ factors as $U \to \mathrm{Spec}(\mathbf{F}_p) \to \mathcal{S}$ and the first arrow is smooth. By descent of smoothness, we see that $U \to \mathcal{S}$ being smooth would imply $\mathrm{Spec}(\mathbf{F}_p) \to \mathcal{S}$ is smooth. However, this is not the case as $\mathrm{Spec}(\mathbf{F}_p) \times_{\mathcal{S}} \mathrm{Spec}(\mathbf{F}_p)$ is μ_p which is not smooth over $\mathrm{Spec}(\mathbf{F}_p)$.

Structural argument: In Criteria for Representability, Section 59.19 we have seen that we can think of algebraic stacks as those stacks in groupoids for the étale topology with diagonal representable by algebraic spaces having a smooth covering. Hence if a smooth surjective $U \to \mathcal{S}$ exists then $\mathcal{S}$ is an algebraic stack, and in particular satisfies descent in the fppf topology. But we've seen above that $\mathcal{S}$ does not satisfies descent in the fppf topology.

Loosely speaking the arguments above show that the classifying stack in the étale topology for étale locally trivial torsors for a group scheme G over a base B is algebraic if and only if G is smooth over B. One of the advantages of working with the fppf topology is that it suffices to assume that $G \to B$ is flat and locally of finite presentation. In fact the quotient stack (for the fppf topology) $[B/G]$ is algebraic if and only if $G \to B$ is flat and locally of finite presentation, see Criteria for Representability, Lemma 59.18.3.

64.34. Sheaf with quasi-compact flat covering which is not algebraic

Consider the functor $F = (\mathbf{P}^1)^\infty$, i.e., for a scheme T the value $F(T)$ is the set of $f = (f_1, f_2, f_3, \ldots)$ where each $f_i : T \to \mathbf{P}^1$ is a morphism of schemes. Note that $\mathbf{P}^1$ satisfies the sheaf property for fpqc coverings, see Descent, Lemma 31.9.3. A product of sheaves is a sheaf, so F also satisfies the sheaf property for the fpqc toplogy. The diagonal of F is representable: if $f : T \to F$ and $g : S \to F$ are morphisms, then $T \times_F S$ is the scheme theoretic intersection of the closed subschemes $T \times_{f_i, \mathbf{P}^1, g_i} S$ inside the scheme $T \times S$. Consider the group scheme SL_2 which comes with a surjective smooth affine morphism $\mathrm{SL}_2 \to \mathbf{P}^1$. Next, consider $U = (\mathrm{SL}_2)^\infty$ with its canonical (product) morphism $U \to F$. Note that U is an affine scheme. We claim the morphism $U \to F$ is flat, surjective, and universally open. Namely, suppose $f : T \to F$ is a morphism. Then $Z = T \times_F U$ is the infinite fibre product of the schemes $Z_i = T \times_{f_i, \mathbf{P}^1} \mathrm{SL}_2$ over T. Each of the morphisms $Z_i \to T$ is surjective smooth and affine which implies that

$$Z = Z_1 \times_T Z_2 \times_T Z_3 \times_T \ldots$$

is a scheme flat and affine over Z. A simple limit argument shows that $Z \to T$ is open as well.

On the other hand, we claim that F isn't an algebraic space. Namely, if F where an algebraic space it would be a quasi-compact and separated (by our description of fibre products over F) algebraic space. Hence cohomology of quasi-coherent sheaves would vanish above a certain cutoff (see Cohomology of Spaces, Proposition 49.7.2 and remarks preceding it). But clearly by taking the pullback of $\mathcal{O}(-2, -2, \ldots, -2)$ under the projection

$$(\mathbf{P}^1)^\infty \longrightarrow (\mathbf{P}^1)^n$$

(which has a section) we can obtain a quasi-coherent sheaf whose cohomology is nonzero in degree n. Altogether we obtain an answer to a question asked by Anton Geraschenko on mathoverflow.

Lemma 64.34.1. *There exists a functor $F : Sch^{opp} \to Sets$ which satisfies the sheaf condition for the fpqc topology, has representable diagonal $\Delta : F \to F \times F$, and such that there exists a surjective, flat, universally open, quasi-compact morphism $U \to F$ where U is a scheme, but such that F is not an algebraic space.*

Proof. See discussion above. □

64.35. Sheaves and specializations

In the following we fix a big étale site $Sch_{étale}$ as constructed in Topologies, Definition 30.4.6. Moreover, a scheme will be an object of this site. Recall that if x, x' are points of a scheme X we say x is a *specialization* of x' or we write $x' \leadsto x$ if $x \in \overline{\{x'\}}$. This is true in particular if $x = x'$.

Consider the functor $F : Sch_{étale} \to Ab$ defined by the following rules:

$$F(X) = \prod\nolimits_{x \in X} \prod\nolimits_{x' \in X, x' \leadsto x, x' \neq x} \mathbf{Z}/2\mathbf{Z}$$

Given a scheme X we denote $|X|$ the underlying set of points. An element $a \in F(X)$ will be viewed as a map of sets $|X| \times |X| \to \mathbf{Z}/2\mathbf{Z}$, $(x, x') \mapsto a(x, x')$ which is zero if $x = x'$ or if x is not a specialization of x'. Given a morphism of schemes $f : X \to Y$ we define

$$F(f) : F(Y) \longrightarrow F(X)$$

by the rule that for $b \in F(Y)$ we set

$$F(f)(b)(x, x') = \begin{cases} 0 & \text{if } x \text{ is not a specialization of } x' \\ b(f(x), f(x')) & \text{else.} \end{cases}$$

Note that this really does define an element of $F(X)$. We claim that if $f : X \to Y$ and $g : Y \to Z$ are composable morphisms then $F(f) \circ F(g) = F(g \circ f)$. Namely, let $c \in F(Z)$ and let $x' \rightsquigarrow x$ be a specialization of points in X, then

$$F(g \circ f)(x, x') = c(g(f(x)), g(f(x'))) = F(g)(F(f)(c))(x, x')$$

because $f(x') \rightsquigarrow f(x)$. (This also works if $f(x) = f(x')$.)

Let G be the sheafification of F in the étale topology.

I claim that if X is a scheme and $x' \rightsquigarrow x$ is a specialization and $x' \neq x$, then $G(X) \neq 0$. Namely, let $a \in F(X)$ be an element such that when we think of a as a function $|X| \times |X| \to \mathbf{Z}/2\mathbf{Z}$ it is nonzero at (x, x'). Let $\{f_i : U_i \to X\}$ be an étale covering of X. Then we can pick an i and a point $u_i \in U_i$ with $f_i(u_i) = x$. Since generalizations lift along flat morphisms (see Morphisms, Lemma 24.24.8) we can find a specialization $u_i' \rightsquigarrow u_i$ with $f_i(u_i') = x'$. By our construction above we see that $F(f_i)(a) \neq 0$. Hence a determines a nonzero element of $G(X)$.

Note that if $X = Spec(k)$ where k is a field (or more generally a ring all of whose prime ideals are maximal), then $F(X) = 0$ and for every étale morphism $U \to X$ we have $F(U) = 0$ because there are no specializations between distinct points in fibres of an étale morphism. Hence $G(X) = 0$.

Suppose that $X \subset X'$ is a thickening, see More on Morphisms, Definition 33.2.1. Then the category of schemes étale over X' is equivalent to the category of schemes étale over X by the base change functor $U' \mapsto U = U' \times_{X'} X$, see Étale Cohomology, Theorem 38.45.1. Since it is always the case that $F(U) = F(U')$ in this situation we see that also $G(X) = G(X')$.

As a variant we can consider the presheaf F_n which associates to a scheme X the collection of maps $a : |X|^{n+1} \to \mathbf{Z}/2\mathbf{Z}$ where $a(x_0, \ldots, x_n)$ is nonzero only if $x_n \rightsquigarrow \ldots \rightsquigarrow x_0$ is a sequence of specializations and $x_n \neq x_{n-1} \neq \ldots \neq x_0$. Let G_n be the sheaf associated to F_n. In exactly the same way as above one shows that G_n is nonzero if $\dim(X) \geq n$ and is zero if $\dim(X) < n$.

Lemma 64.35.1. *There exists a sheaf of abelian groups G on $Sch_{étale}$ with the following properties*

(1) $G(X) = 0$ *whenever* $\dim(X) < n$,
(2) $G(X)$ *is not zero if* $\dim(X) \geq n$, *and*
(3) *if $X \subset X'$ is a thickening, then $G(X) = G(X')$.*

Proof. See the discussion above. $\square$

Remark 64.35.2. Here are some remarks:

(1) The presheaves F and F_n are separated presheaves.
(2) It turns out that F, F_n are not sheaves.

(3) One can show that G, G_n is actually a sheaf for the fppf topology.

We will prove these results if we need them.

64.36. Sheaves and constructible functions

In the following we fix a big étale site $Sch_{étale}$ as constructed in Topologies, Definition 30.4.6. Moreover, a scheme will be an object of this site. A *constructible stratification* of a scheme X is a locally finite disjoint union decomposition $X = \coprod_{i \in I} X_i$ such that each $X_i \subset X$ is a locally constructible subset of X. Locally finite means that for any quasi-compact open $U \subset X$ there are only finitely many $i \in I$ such that $X_i \cap U$ is not empty. Note that if $f : X \to Y$ is a morphism of schemes and $Y = \coprod Y_j$ is a constructible stratification, then $X = \coprod f^{-1}(Y_j)$ is a constructible stratification of X. Given a set S and a scheme X a *constructible function* $f : |X| \to S$ is a map such that $X = \coprod_{s \in S} f^{-1}(s)$ is a constructible stratification of X. If G is an (abstract group) and $a, b : |X| \to G$ are constructible functions, then $ab : |X| \to G, x \mapsto a(x)b(x)$ is a constructible function too. The reason is that given any two constructible stratifications there is a third one refining both.

Let A be any abelian group. For any scheme X we define

$$F(X) = \frac{\{a : |X| \to A \mid a \text{ is a constructible function}\}}{\text{locally constant functions } |X| \to A}$$

We think of an element a of $F(X)$ simply as a function well defined up to adding a locally constant one. Given a morphism of schemes $f : X \to Y$ and an element $b \in F(Y)$, then we define $F(f)(b) = b \circ f$. Thus F is a presheaf on $Sch_{étale}$.

Note that if $\{f_i : U_i \to X\}$ is an fppf covering, and $a \in F(X)$ is such that $F(f_i)(a) = 0$ in $F(U_i)$, then $a \circ f_i$ is a locally constant function for each i. This means in turn that a is a locally constant function as the morphisms f_i are open. Hence $a = 0$ in $F(X)$. Thus we see that F is a separated presheaf (in the fppf topology hence a fortiori in the étale topology).

Let G be the sheafification of F in the étale topology. Since F is separated, and since $F(X) \neq 0$ for example when X is the spectrum of a discrete valuation ring, we see that G is not zero.

Let $X = Spec(k)$ where k is a field. Then any étale covering of X can be dominated by a covering $\{Spec(k') \to Spec(k)\}$ with $k \subset k'$ a finite separable extension of fields. Since $F(Spec(k')) = 0$ we see that $G(X) = 0$.

Suppose that $X \subset X'$ is a thickening, see More on Morphisms, Definition 33.2.1. Then the category of schemes étale over X' is equivalent to the category of schemes étale over X by the base change functor $U' \mapsto U = U' \times_{X'} X$, see Étale Cohomology, Theorem 38.45.1. Since $F(U) = F(U')$ in this situation we see that also $G(X) = G(X')$.

The sheaf G is limit preserving, see More on Morphisms of Spaces, Definition 46.4.1. Namely, let R be a ring which is written as a directed colimit $R = colim_i R_i$ of rings. Set $X = Spec(R)$ and $X_i = Spec(R_i)$, so that $X = lim_i X_i$. Then $G(X) = colim_i G(X_i)$. To prove this one first proves that a constructible stratification of $Spec(R)$ comes from a constructible stratifications of some $Spec(R_i)$. Hence the result for F. To get the result for the sheafification, use that any étale ring map $R \to R'$ comes from an étale ring map $R_i \to R_i'$ for some i. Details omitted.

Lemma 64.36.1. *There exists a sheaf of abelian groups G on $Sch_{étale}$ with the following properties*

(1) *$G(Spec(k)) = 0$ whenever k is a field,*
(2) *G is limit preserving,*
(3) *if $X \subset X'$ is a thickening, then $G(X) = G(X')$, and*
(4) *G is not zero.*

Proof. See discussion above. □

64.37. The lisse-étale site is not functorial

The *lisse-étale* site $X_{lisse,étale}$ of X is the category of schemes smooth over X endowed with (usual) étale coverings, see Cohomology of Stacks, Section 62.11. Let $f : X \to Y$ be a morphism of schemes. There is a functor

$$u : Y_{lisse,étale} \longrightarrow X_{lisse,étale}, \quad V/Y \longmapsto V \times_Y X$$

which is continuous. Hence we obtain an adjoint pair of functors

$$u^s : Sh(X_{lisse,étale}) \longrightarrow Sh(Y_{lisse,étale}), \quad u_s : Sh(Y_{lisse,étale}) \longrightarrow Sh(X_{lisse,étale}),$$

see Sites, Section 9.13. We claim that, in general, u does **not** define a morphism of sites, see Sites, Definition 9.14.1. In other words, we claim that u_s is not left exact in general. Note that representable presheaves are sheaves on lisse-étale sites. Hence, by Sites, Lemma 9.13.5 we see that $u_s h_V = h_{V \times_Y X}$. Now consider two morphisms

$$\begin{tikzcd} V_1 \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] \arrow[dr] & V_2 \arrow[d] \\ & Y \end{tikzcd}$$

of schemes V_1, V_2 smooth over Y. Now if u_s is left exact, then we would have

$$u_s\text{Equalizer}(h_a, h_b : h_{V_1} \to h_{V_2}) = \text{Equalizer}(h_{a\times 1}, h_{b\times 1} : h_{V_1\times_Y X} \to h_{V_2\times_Y X})$$

We will take the morphisms $a, b : V_1 \to V_2$ such that there exists no morphism from a scheme smooth over Y into $(a = b) \subset V_1$, i.e., such that the left hand side is the empty sheaf, but such that after base change to X the equalizer is nonempty and smooth over X. A silly example is to take $X = Spec(\mathbf{F}_p)$, $Y = Spec(\mathbf{Z})$ and $V_1 = V_2 = \mathbf{A}^1_{\mathbf{Z}}$ with morphisms $a(x) = x$ and $b(x) = x + p$. Note that the equalizer of a and b is the fibre of $\mathbf{A}^1_{\mathbf{Z}}$ over (p).

Lemma 64.37.1. *The lisse-étale site is not functorial, even for morphisms of schemes.*

Proof. See discussion above. □

64.38. Derived pushforward of quasi-coherent modules

Let k be a field of characteristic $p > 0$. Let $S = Spec(k[x])$. Let $G = \mathbf{Z}/p\mathbf{Z}$ viewed either as an abstract group or as a constant group scheme over S. Consider the algebraic stack $\mathcal{X} = [S/G]$ where G acts trivially on S, see Examples of Stacks, Remark 55.14.3 and Criteria for Representability, Lemma 59.18.3. Consider the structure morphism

$$f : \mathcal{X} \longrightarrow S$$

This morphism is quasi-compact and quasi-separated. Hence we get a functor

$$Rf_{QCoh,*} : D^+_{QCoh}(\mathcal{O}_{\mathcal{X}}) \longrightarrow D^+_{QCoh}(\mathcal{O}_S),$$

see Cohomology of Stacks, Proposition 62.14.1. Let's compute $Rf_{QCoh,*}\mathcal{O}_{\mathcal{X}}$. Since $D_{QCoh}(\mathcal{O}_S)$ is equivalent to the derived category of $k[x]$-modules (see Coherent, Lemma 25.4.1) this is equivalent to computing $R\Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$. For this we can use the covering $S \to \mathcal{X}$ and the spectral sequence

$$H^q(S \times_\mathcal{X} \ldots \times_\mathcal{X} S, O) \Rightarrow H^{p+q}(\mathcal{X}, \mathcal{O}_\mathcal{X})$$

see Cohomology of Stacks, Proposition 62.10.4. Note that

$$S \times_\mathcal{X} \ldots \times_\mathcal{X} S = S \times G^p$$

which is affine. Thus the complex

$$k[x] \to \mathrm{Map}(G, k[x]) \to \mathrm{Map}(G^2, k[x]) \to \ldots$$

computes $R\Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$. Here for $\varphi \in \mathrm{Map}(G^{p-1}, k[x])$ its differential is the map which sends $(g_1, \ldots, g_p)$ to

$$\varphi(g_2, \ldots, g_p) + \sum\nolimits_{i=1}^{p-1} (-1)^i \varphi(g_1, \ldots, g_i + g_{i+1}, \ldots, g_p) + (-1)^p \varphi(g_1, \ldots, g_{p-1}).$$

This is just the complex computing the group cohomology of G acting trivially on $k[x]$ (insert future reference here). The cohomology of the cyclic group G on $k[x]$ is exactly one copy of $k[x]$ in each cohomological degree ≥ 0 (insert future reference here). We conclude that

$$Rf_*\mathcal{O}_\mathcal{X} = \bigoplus\nolimits_{n \geq 0} \mathcal{O}_S[-n]$$

Now, consider the complex

$$E = \bigoplus\nolimits_{m \geq 0} \mathcal{O}_\mathcal{X}[m]$$

This is an object of $D_{QCoh}(\mathcal{X})$. Note that in the derived category we have

$$E = \prod\nolimits_{m \geq 0} \mathcal{O}_\mathcal{X}[m]$$

because this is true on affine objects over $\mathcal{X}$ by Injectives, Remark 17.17.5 (details omitted). Since cohomology commutes with limits we see that

$$Rf_*E = \prod\nolimits_{m \geq 0} \left(\bigoplus\nolimits_{n \geq 0} \mathcal{O}_S[m-n] \right)$$

Note that this complex is not an object of $D_{QCoh}(\mathcal{O}_S)$.

Lemma 64.38.1. *A quasi-compact and quasi-separated morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks need not induce a functor $Rf_* : D_{QCoh}(\mathcal{O}_\mathcal{X}) \to D_{QCoh}(\mathcal{O}_\mathcal{Y})$.*

Proof. See discussion above. □

64.39. A big abelian category

The purpose of this section is to give an example of a ``big'' abelian category $\mathcal{A}$ and objects M, N such that the collection of isomorphism classes of extensions $\mathrm{Ext}_\mathcal{A}(M, N)$ is not a set. The example is due to Freyd, see [**Fre64**, page 131, Exercise A].

We define $\mathcal{A}$ as follows. An object of $\mathcal{A}$ consists of a triple (M, α, f) where M is an abelian group and α is an ordinal and $f : \alpha \to \mathrm{End}(M)$ is a map. A morphism $(M, \alpha, f) \to (M', \alpha', f')$ is given by a homomorphism of abelian groups $\varphi : M \to M'$ such that for *any* ordinal β we have

$$\varphi \circ f(\beta) = f'(\beta) \circ \varphi$$

Here the rule is that we set $f(\beta) = 0$ if β is not in α and similarly we set $f'(\beta)$ equal to zero if β is not an element of α'. We omit the verification that the category so defined is abelian.

Consider the object $Z = (\mathbf{Z}, \emptyset, f)$, i.e., all the operators are zero. The observation is that computed in $\mathcal{A}$ the group $\mathrm{Ext}^1_\mathcal{A}(Z, Z)$ is a proper class and not a set. Namely, for each ordinal α we can find an extension $(M, \alpha + 1, f)$ of Z by Z whose underlying group is $M = \mathbf{Z} \oplus \mathbf{Z}$ and where the value of f is always zero except for

$$f(\alpha) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

This clearly produces a proper class of isomorphism classes of extensions. In particular, the derived category of $\mathcal{A}$ has proper classes for its collections of morphism, see Derived Categories, Lemma 11.26.6. This means that some care has to be exercised when defining Verdier quotients of triangulated categories.

Lemma 64.39.1. *There exists a ``big'' abelian category $\mathcal{A}$ whose Ext-groups are proper classes.*

Proof. See discussion above. $\square$

64.40. Other chapters

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(7) Commutative Algebra
(8) Brauer Groups
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(36) More on Groupoid Schemes
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CHAPTER 65

Exercises

65.1. Algebra

This first section just contains some assorted questions.

Exercise 65.1.1. Let A be a ring, and $\mathfrak{m}$ a maximal ideal. In $A[X]$ let $\tilde{\mathfrak{m}}_1 = (\mathfrak{m}, X)$ and $\tilde{\mathfrak{m}}_2 = (\mathfrak{m}, X-1)$. Show that

$$A[X]_{\tilde{\mathfrak{m}}_1} \cong A[X]_{\tilde{\mathfrak{m}}_2}.$$

Exercise 65.1.2. Find an example of a non Noetherian ring R such that every finitely generated ideal of R is finitely presented as an R-module. (A ring is said to be *coherent* if the last property holds.)

Exercise 65.1.3. Suppose that $(A, \mathfrak{m}, k)$ is a Noetherian local ring. For any finite A-module M define $r(M)$ to be the minimum number of generators of M as an A-module. This number equals $\dim_k M/\mathfrak{m}M = \dim_k M \otimes_A k$ by NAK.

(1) Show that $r(M \otimes_A N) = r(M)r(N)$.
(2) Let $I \subset A$ be an ideal with $r(I) > 1$. Show that $r(I^2) < r(I)^2$.
(3) Conclude that if every ideal in A is a flat module, then A is a PID (or a field).

Exercise 65.1.4. Let k be a field. Show that the following pairs of k-algebras are not isomorphic:

(1) $k[x_1, \ldots, x_n]$ and $k[x_1, \ldots, x_{n+1}]$ for any $n \geq 1$.
(2) $k[a, b, c, d, e, f]/(ab + cd + ef)$ and $k[x_1, \ldots, x_n]$ for $n = 5$.
(3) $k[a, b, c, d, e, f]/(ab + cd + ef)$ and $k[x_1, \ldots, x_n]$ for $n = 6$.

Remark 65.1.5. Of course the idea of this exercise is to find a simple argument in each case rather than applying a ``big'' theorem. Nonetheless it is good to be guided by general principles.

Exercise 65.1.6. Algebra. (Silly and should be easy.)

(1) Give an example of a ring A and a nonsplit short exact sequence of A-modules
$$0 \to M_1 \to M_2 \to M_3 \to 0.$$
(2) Give an example of a nonsplit sequence of A-modules as above and a faithfully flat $A \to B$ such that
$$0 \to M_1 \otimes_A B \to M_2 \otimes_A B \to M_3 \otimes_A B \to 0.$$
is split as a sequence of B-modules.

Exercise 65.1.7. Suppose that k is a field having a primitive nth root of unity ζ. This means that $\zeta^n = 1$, but $\zeta^m \neq 1$ for $0 < m < n$.

(1) Show that the characteristic of k is prime to n.

(2) Suppose that $a \in k$ is an element of k which is not an dth power in k for any divisor d of n, $in \geq d > 1$. Show that $k[x]/(x^n - a)$ is a field. (Hint: Consider a splitting field for $x^n - a$ and use Galois theory.)

Exercise 65.1.8. Let $\nu : k[x] \setminus \{0\} \to \mathbf{Z}$ be a map with the following properties: $\nu(fg) = \nu(f) + \nu(g)$ whenever f, g not zero, and $\nu(f + g) \geq min(\nu(f), \nu(g))$ whenever f, g, $f + g$ are not zero, and $\nu(c) = 0$ for all $c \in k^*$.

(1) Show that if f, g, and $f + g$ are nonzero and $\nu(f) \neq \nu(g)$ then we have equality $\nu(f + g) = min(\nu(f), \nu(g))$.
(2) Show that if $f = \sum a_i x^i$, $f \neq 0$, then $\nu(f) \geq min(\{i\nu(x)\}_{a_i \neq 0})$. When does equality hold?
(3) Show that if ν attains a negative value then $\nu(f) = -n\deg(f)$ for some $n \in \mathbf{N}$.
(4) Suppose $\nu(x) \geq 0$. Show that $\{f \mid f = 0,\ or\ \nu(f) > 0\}$ is a prime ideal of $k[x]$.
(5) Describe all possible ν.

Let A be a ring. An *idempotent* is an element $e \in A$ such that $e^2 = e$. The elements 1 and 0 are always idempotent. A *nontrivial idempotent* is an idempotent which is not equal to zero. Two idempotents $e, e' \in A$ are called *orthogonal* if $ee' = 0$.

Exercise 65.1.9. Let A be a ring. Show that A is a product of two nonzero rings if and only if A has a nontrivial idempotent.

Exercise 65.1.10. Let A be a ring and let $I \subset A$ be a locally nilpotent ideal. Show that the map $A \to A/I$ induces a bijection on idempotents. (Hint: It may be easier to prove this when I is nilpotent. Do this first. Then use ``absolute Noetherian reduction'' to reduce to the nilpotent case.)

65.2. Colimits

Definition 65.2.1. A *directed partially ordered set* is a nonempty set I endowed with a partial ordering $\leq$ such that given any pair $i, j \in I$ there exists a $k \in I$ such that $i \leq k$ and $j \leq k$. A *system of rings* over I is given by a ring A_i for each $i \in I$ and a map of rings $\varphi_{ij} : A_i \to A_j$ whenever $i \leq j$ such that the composition $A_i \to A_j \to A_k$ is equal to $A_i \to A_k$ whenever $i \leq j \leq k$.

One similarly defines systems of groups, modules over a fixed ring, vector spaces over a field, etc.

Exercise 65.2.2. Let I be a directed partially ordered set and let (A_i, φ_{ij}) be a system of rings over I. Show that there exists a ring A and maps $\varphi_i : A_i \to A$ such that $\varphi_j \circ \varphi_{ij} = \varphi_i$ for all $i \leq j$ with the following universal property: Given any ring B and maps $\psi_i : A_i \to B$ such that $\psi_j \circ \varphi_{ij} = \psi_i$ for all $i \leq j$, then there exists a unique ring map $\psi : A \to B$ such that $\psi_i = \psi \circ \varphi_i$.

Definition 65.2.3. The ring A constructed in Exercise 65.2.2 is called the *colimit* of the system. Notation $colim\, A_i$.

Exercise 65.2.4. Let $(I, \geq)$ be a directed partially ordered set and let (A_i, φ_{ij}) be a system of rings over I with colimit A. Prove that there is a bijection

$$\mathrm{Spec}(A) = \{(\mathfrak{p}_i)_{i \in I} \mid \ \mathfrak{p}_i \subset A_i \text{ and } \mathfrak{p}_i = \varphi_{ij}^{-1}(\mathfrak{p}_j)\ \forall i \leq j\} \subset \prod\nolimits_{i \in I} Spec(A_i)$$

The set on the right hand side is the limit of the sets $Spec(A_i)$. Notation $lim\, Spec(A_i)$.

Exercise 65.2.5. Let $(I, \geq)$ be a directed partially ordered set and let (A_i, φ_{ij}) be a system of rings over I with colimit A. Suppose that $\mathrm{Spec}(A_j) \to \mathrm{Spec}(A_i)$ is surjective for all $i \leq j$. Show that $\mathrm{Spec}(A) \to \mathrm{Spec}(A_i)$ is surjective for all i. (Hint: You can try to use Tychonoff, but there is also a basically trivial direct algebraic proof based on Algebra, Lemma 7.16.9.)

Exercise 65.2.6. Let $A \subset B$ be an integral ring extension. Prove that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective. Use the exercises above, the fact that this holds for a finite ring extension (proved in the lectures), and by proving that $B = colim\, B_i$ is a directed colimit of finite extensions $A \subset B_i$.

Exercise 65.2.7. Let $(I, \geq)$ be a partially ordered set which is directed. Let A be a ring and let $(N_i, \varphi_{i,i'})$ be a directed system of A-modules indexed by I. Suppose that M is another A-module. Prove that

$$colim_{i\in I}\, M \otimes_A N_i \cong M \otimes_A \Big(colim_{i\in I}\, N_i \Big).$$

Definition 65.2.8. A module M over R is said to be of *finite presentation* over R if it is isomorphic to the cokernel of a map of finite free modules $R^{\oplus n} \to R^{\oplus m}$.

Exercise 65.2.9. Prove that any module over any ring is

(1) the colimit of its finitely generated submodules, and
(2) in some way a colimit of finitely presented modules.

65.3. Additive and abelian categories

Exercise 65.3.1. Let k be a field. Let $\mathscr{C}$ be the category of filtered vector spaces over k, see Homology, Definition 10.13.1 for the definition of a filtered object of any category.

(1) Show that this is an additive category (explain carefully what the direct sum of two objects is).
(2) Let $f : (V, F) \to (W, F)$ be a morphism of $\mathscr{C}$. Show that f has a kernel and cokernel (explain precisely what the kernel and cokernel of f are).
(3) Give an example of a map of $\mathscr{C}$ such that the canonical map $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is not an isomorphism.

Exercise 65.3.2. Let R be a Noetherian domain. Let $\mathscr{C}$ be the category of finitely generated torsion free R-modules.

(1) Show that this is an additive category.
(2) Let $f : N \to M$ be a morphism of $\mathscr{C}$. Show that f has a kernel and cokernel (make sure you define precisely what the kernel and cokernel of f are).
(3) Give an example of a Noetherian domain R and a map of $\mathscr{C}$ such that the canonical map $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is not an isomorphism.

Exercise 65.3.3. Give an example of a category which is additive and has kernels and cokernels but which is not as in Exercises 65.3.1 and 65.3.2.

65.4. Flat ring maps

Exercise 65.4.1. Let S be a multiplicative subset of the ring A.

(1) For an A-module M show that $S^{-1}M = S^{-1}A \otimes_A M$.
(2) Show that $S^{-1}A$ is flat over A.

Exercise 65.4.2. Find an injection $M_1 \to M_2$ of A-modules such that $M_1 \otimes N \to M_2 \otimes N$ is not injective in the following cases:

(1) $A = k[x, y]$ and $N = (x, y) \subset A$. (Here and below k is a field.)
(2) $A = k[x, y]$ and $N = A/(x, y)$.

Exercise 65.4.3. Give an example of a ring A and a finite A-module M which is a flat but not a projective A-module.

Remark 65.4.4. If M is of finite presentation and flat over A, then M is projective over A. Thus your example will have to involve a ring A which is not Noetherian. I know of an example where A is the ring of $\mathcal{C}^\infty$-functions on $\mathbf{R}$.

Exercise 65.4.5. Find a flat but not free module over $\mathbf{Z}_{(2)}$.

Exercise 65.4.6. Flat deformations.

(1) Suppose that k is a field and $k[\epsilon]$ is the ring of dual numbers $k[\epsilon] = k[x]/(x^2)$ and $\epsilon = \bar{x}$. Show that for any k-algebra A there is a flat $k[\epsilon]$-algebra B such that A is isomorphic to $B/\epsilon B$.
(2) Suppose that $k = \mathbf{F}_p = \mathbf{Z}/p\mathbf{Z}$ and
$$A = k[x_1, x_2, x_3, x_4, x_5, x_6]/(x_1^p, x_2^p, x_3^p, x_4^p, x_5^p, x_6^p).$$
Show that there exists a flat $\mathbf{Z}/p^2\mathbf{Z}$-algebra B such that B/pB is isomorphic to A. (So here p plays the role of ϵ.)
(3) Now let $p = 2$ and consider the same question for $k = \mathbf{F}_2 = \mathbf{Z}/2\mathbf{Z}$ and
$$A = k[x_1, x_2, x_3, x_4, x_5, x_6]/(x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2, x_1x_2 + x_3x_4 + x_5x_6).$$
However, in this case show that there does *not* exist a flat $\mathbf{Z}/4\mathbf{Z}$-algebra B such that $B/2B$ is isomorphic to A. (Find the trick! The same example works in arbitrary characteristic $p > 0$, except that the computation is more difficult.)

Exercise 65.4.7. Let $(A, \mathfrak{m}, k)$ be a local ring and let $k \subset k'$ be a finite field extension. Show there exists a flat, local map of local rings $A \to B$ such that $\mathfrak{m}_B = \mathfrak{m}B$ and $B/\mathfrak{m}B$ is isomorphic to k' as k-algebra. (Hint: first do the case where $k \subset k'$ is generated by a single element.)

Remark 65.4.8. The same result holds for arbitrary field extensions $k \subset K$.

65.5. The Spectrum of a ring

Exercise 65.5.1. Compute $Spec(\mathbf{Z})$ as a set and describe its topology.

Exercise 65.5.2. Let A be any ring. For $f \in A$ we define $D(f) := \{\mathfrak{p} \subset A \mid f \not\in \mathfrak{p}\}$. Prove that the open subsets $D(f)$ form a basis of the topology of $Spec(A)$.

Exercise 65.5.3. Prove that the map $I \mapsto V(I)$ defines a natural bijection
$$\{I \subset A \text{ with } I = \sqrt{I}\} \longrightarrow \{T \subset Spec(A) \text{ closed}\}$$

Definition 65.5.4. A topological space X is called *quasi-compact* if for any open covering $X = \bigcup_{i \in I} U_i$ there is a finite subset $\{i_1, \ldots, i_n\} \subset I$ such that $X = U_{i_1} \cup \ldots U_{i_n}$.

Exercise 65.5.5. Prove that $Spec(A)$ is quasi-compact for any ring A.

Definition 65.5.6. A topological space X is said to verify the separation axiom T_0 if for any pair of points $x, y \in X$, $x \neq y$ there is an open subset of X containing one but not the other. We say that X is *Hausdorff* if for any pair $x, y \in X$, $x \neq y$ there are disjoint open subsets U, V such that $x \in U$ and $y \in V$.

Exercise 65.5.7. Show that $Spec(A)$ is **not** Hausdorff in general. Prove that $Spec(A)$ is T_0. Give an example of a topological space X that is not T_0.

Remark 65.5.8. Usually the word compact is reserved for quasi-compact and Hausdorff spaces.

Definition 65.5.9. A topological space X is called *irreducible* if X is not empty and if $X = Z_1 \cup Z_2$ with $Z_1, Z_2 \subset X$ closed, then either $Z_1 = X$ or $Z_2 = X$. A subset $T \subset X$ of a topological space is called *irreducible* if it is an irreducible topological space with the topology induced from X. This definition implies T is irreducible if and only if the closure $\bar{T}$ of T in X is irreducible.

Exercise 65.5.10. Prove that $Spec(A)$ is irreducible if and only if $Nil(A)$ is a prime ideal and that in this case it is the unique minimal prime ideal of A.

Exercise 65.5.11. Prove that a closed subset $T \subset Spec(A)$ is irreducible if and only if it is of the form $T = V(\mathfrak{p})$ for some prime ideal $\mathfrak{p} \subset A$.

Definition 65.5.12. A point x of an irreducible topological space X is called a *generic point* of X if X is equal to the closure of the subset $\{x\}$.

Exercise 65.5.13. Show that in a T_0 space X every irreducible closed subset has at most one generic point.

Exercise 65.5.14. Prove that in $Spec(A)$ every irreducible closed subset *does* have a generic point. In fact show that the map $\mathfrak{p} \mapsto \overline{\{\mathfrak{p}\}}$ is a bijection of $Spec(A)$ with the set of irreducible closed subsets of X.

Exercise 65.5.15. Give an example to show that an irreducible subset of $Spec(\mathbf{Z})$ does not neccesarily have a generic point.

Definition 65.5.16. A topological space X is called *Noetherian* if any decreasing sequence $Z_1 \supset Z_2 \supset Z_3 \supset \ldots$ of closed subsets of X stabilizes. (It is called *Artinian* if any increasing sequence of closed subsets stabilizes.)

Exercise 65.5.17. Show that if the ring A is Noetherian then the topological space $Spec(A)$ is Noetherian. Give an example to show that the converse is false. (The same for Artinian if you like.)

Definition 65.5.18. A maximal irreducible subset $T \subset X$ is called an *irreducible component* of the space X. Such an irreducible component of X is automatically a closed subset of X.

Exercise 65.5.19. Prove that any irreducible subset of X is contained in an irreducible component of X.

Exercise 65.5.20. Prove that a Noetherian topological space X has only finitely many irreducible components, say $X_1, \ldots, X_n$, and that $X = X_1 \cup X_2 \cup \ldots \cup X_n$. (Note that any X is always the union of its irreducible components, but that if $X = \mathbf{R}$ with its usual topology for instance then the irreducible components of X are the one point subsets. This is not terribly interesting.)

Exercise 65.5.21. Show that irreducible components of $Spec(A)$ correspond to minimal primes of A.

Definition 65.5.22. A point $x \in X$ is called *closed* if $\overline{\{x\}} = \{x\}$. Let x, y be points of X. We say that x is a *specialization* of y, or that y is a *generalization* of x if $x \in \overline{\{y\}}$.

Exercise 65.5.23. Show that closed points of $Spec(A)$ correspond to maximal ideals of A.

Exercise 65.5.24. Show that $\mathfrak{p}$ is a generalization of $\mathfrak{q}$ in $Spec(A)$ if and only if $\mathfrak{p} \subset \mathfrak{q}$. Characterize closed points, maximal ideals, generic points and minimal prime ideals in terms of generalization and specialization. (Here we use the terminology that a point of a possibly reducible topological space X is called a generic point if it is a generic points of one of the irreducible components of X.)

Exercise 65.5.25. Let I and J be ideals of A. What is the condition for $V(I)$ and $V(J)$ to be disjoint?

Definition 65.5.26. A topological space X is called *connected* if it is not the union of two nonempty disjoint open subsets. A *connected component* of X is a (nonempty) maximal connected subset. Any point of X is contained in a connected component of X and any connected component of X is closed in X. (But in general a connected component need not be open in X.)

Exercise 65.5.27. Show that $Spec(A)$ is disconnected iff $A \cong B \times C$ for certain nonzero rings B, C.

Exercise 65.5.28. Let T be a connected component of $Spec(A)$. Prove that T is stable under generalization. Prove that T is an open subset of $Spec(A)$ if A is Noetherian. (Remark: This is wrong when A is an infinite product of copies of $\mathbf{F}_2$ for example. The spectrum of this ring consists of infinitely many closed points.)

Exercise 65.5.29. Compute $Spec(k[x])$, i.e., describe the prime ideals in this ring, describe the possible specializations, and describe the topology. (Work this out when k is algebraically closed but also when k is not.)

Exercise 65.5.30. Compute $Spec(k[x, y])$, where k is algebraically closed. [Hint: use the morphism $\varphi : Spec(k[x, y]) \to Spec(k[x])$; if $\varphi(\mathfrak{p}) = (0)$ then localize with respect to $S = \{f \in k[x] \mid f \neq 0\}$ and use result of lecture on localization and $Spec$.] (Why do you think algebraic geometers call this affine 2-space?)

Exercise 65.5.31. Compute $Spec(\mathbf{Z}[y])$. [Hint: as above.] (Affine 1-space over $\mathbf{Z}$.)

65.6. Localization

Exercise 65.6.1. Let A be a ring. Let $S \subset A$ be a multiplicative subset. Let M be an A-module. Let $N \subset S^{-1}M$ be an $S^{-1}A$-submodule. Show that there exists an A-submodule $N' \subset M$ such that $N = S^{-1}N'$. (This useful result applies in particular to ideals of $S^{-1}A$.)

Exercise 65.6.2. Let A be a ring. Let M be an A-module. Let $m \in M$.

(1) Show that $I = \{a \in A \mid am = 0\}$ is an ideal of A.
(2) For a prime $\mathfrak{p}$ of A show that the image of m in $M_{\mathfrak{p}}$ is zero if and only if $I \not\subset \mathfrak{p}$.
(3) Show that m is zero if and only if the image of m is zero in $M_{\mathfrak{p}}$ for all primes $\mathfrak{p}$ of A.
(4) Show that m is zero if and only if the image of m is zero in $M_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ of A.
(5) Show that $M = 0$ if and only if $M_{\mathfrak{m}}$ is zero for all maximal ideals $\mathfrak{m}$.

Exercise 65.6.3. Find a pair (A, f) where A is a domain with three or more pairwise distinct primes and $f \in A$ is an element such that the principal localization $A_f = \{1, f, f^2, \ldots\}^{-1}A$ is a field.

Exercise 65.6.4. Let A be a ring. Let M be a finite A-module. Let $S \subset A$ be a multiplicative set. Assume that $S^{-1}M = 0$. Show that there exists an $f \in S$ such that the principal localization $M_f = \{1, f, f^2, \ldots\}^{-1}M$ is zero.

Exercise 65.6.5. Give an example of a triple (A, I, S) where A is a ring, $0 \neq I \neq A$ is a proper nonzero ideal, and $S \subset A$ is a multiplicative subset such that $A/I \cong S^{-1}A$ as A-algebras.

65.7. Nakayama's Lemma

Exercise 65.7.1. Let A be a ring. Let I be an ideal of A. Let M be an A-module. Let $x_1, \ldots, x_n \in M$. Assume that

(1) M/IM is generated by $x_1, \ldots, x_n$,
(2) M is a finite A-module,
(3) I is contained in every maximal ideal of A.

Show that $x_1, \ldots, x_n$ generate M. (Suggested solution: Reduce to a localization at a maximal ideal of A using Exercise 65.6.2 and exactness of localization. Then reduce to the statement of Nakayama's lemma in the lectures by looking at the quotient of M by the submodule generated by $x_1, \ldots, x_n$.)

65.8. Length

Definition 65.8.1. Let A be a ring. Let M be an A-module. The *length* of M as an R-module is

$$\text{length}_A(M) = \sup\{n \mid \exists\, 0 = M_0 \subset M_1 \subset \ldots \subset M_n = M,\ M_i \neq M_{i+1}\}.$$

In other words, the supremum of the lengths of chains of submodules.

Exercise 65.8.2. Show that a module M over a ring A has length 1 if and only if it is isomorphic to $A/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ in A.

Exercise 65.8.3. Compute the length of the following modules over the following rings. Briefly(!) explain your answer. (Please feel free to use additivity of the length function in short exact sequences, see Algebra, Lemma 7.48.3).

(1) The length of $\mathbf{Z}/120\mathbf{Z}$ over $\mathbf{Z}$.
(2) The length of $\mathbf{C}[x]/(x^{100} + x + 1)$ over $\mathbf{C}[x]$.
(3) The length of $\mathbf{R}[x]/(x^4 + 2x^2 + 1)$ over $\mathbf{R}[x]$.

Exercise 65.8.4. Let $A = k[x, y]_{(x,y)}$ be the local ring of the affine plane at the origin. Make any assumption you like about the field k. Suppose that $f = x^3 + x^2y^2 + y^{100}$ and $g = y^3 - x^{999}$. What is the length of $A/(f, g)$ as an A-module? (Possible way to proceed: think about the ideal that f and g generate in quotients of the form $A/\mathfrak{m}_A^n = k[x, y]/(x, y)^n$ for varying n. Try to find n such that $A/(f, g) + \mathfrak{m}_A^n \cong A/(f, g) + \mathfrak{m}_A^{n+1}$ and use NAK.)

65.9. Singularities

Exercise 65.9.1. Let k be any field. Suppose that $A = k[[x, y]]/(f)$ and $B = k[[u, v]]/(g)$, where $f = xy$ and $g = uv + \delta$ with $\delta \in (u, v)^3$. Show that A and B are isomorphic rings.

Remark 65.9.2. A singularity on a curve over a field k is called an ordinary double point if the complete local ring of the curve at the point is of the form $k'[[x, y]]/(f)$, where (a) k' is a finite separable extension of k, (b) the initial term of f has degree two, i.e., it looks like $q = ax^2 + bxy + cy^2$ for some $a, b, c \in k'$ not all zero, and (c) q is a nondegenerate quadratic form over k' (in char 2 this means that b is not zero). In general there is one isomorphism class of such rings for each isomophism class of pairs (k', q).

65.10. Hilbert Nullstellensatz

Exercise 65.10.1. *A silly argument using the complex numbers!* Let $\mathbf{C}$ be the complex number field. Let V be a vector space over $\mathbf{C}$. The spectrum of a linear operator $T : V \to V$ is the set of complex numbers $\lambda \in \mathbf{C}$ such that the operator $T - \lambda \mathrm{id}_V$ is not invertible.

(1) Show that $\mathbf{C}(X) = f.f.(\mathbf{C}[X])$ has uncountable dimension over $\mathbf{C}$.
(2) Show that any linear operator on V has a nonempty spectrum if the dimension of V is finite or countable.
(3) Show that if a finitely generated $\mathbf{C}$-algebra R is a field, then the map $\mathbf{C} \to R$ is an isomorphism.
(4) Show that any maximal ideal $\mathfrak{m}$ of $\mathbf{C}[x_1, \ldots, x_n]$ is of the form $(x_1 - \alpha_1, \ldots, x_n - \alpha_n)$ for some $\alpha_i \in \mathbf{C}$.

Remark 65.10.2. Let k be a field. Then for every integer $n \in \mathbf{N}$ and every maximal ideal $\mathfrak{m} \subset k[x_1, \ldots, x_n]$ the quotient $k[x_1, \ldots, x_n]/\mathfrak{m}$ is a finite field extension of k. This will be shown later in the course. Of course (please check this) it implies a similar statement for maximal ideals of finitely generated k-algebras. The exercise above proves it in the case $k = \mathbf{C}$.

Exercise 65.10.3. Let k be a field. Please use Remark 65.10.2.

(1) Let R be a k-algebra. Suppose that $\dim_k R < \infty$ and that R is a domain. Show that R is a field.
(2) Suppose that R is a finitely generated k-algebra, and $f \in R$ not nilpotent. Show that there exists a maximal ideal $\mathfrak{m} \subset R$ with $f \not\in \mathfrak{m}$.
(3) Show by an example that this statement fails when R is not of finite type over a field.
(4) Show that any radical ideal $I \subset \mathbf{C}[x_1, \ldots, x_n]$ is the intersection of the maximal ideals containing it.

Remark 65.10.4. This is the Hilbert Nullstellensatz. Namely it says that the closed subsets of $Spec(k[x_1, \ldots, x_n])$ (which correspond to radical ideals by a previous exercise) are determined by the closed points contained in them.

Exercise 65.10.5. Let $A = \mathbf{C}[x_{11}, x_{12}, x_{21}, x_{22}, y_{11}, y_{12}, y_{21}, y_{22}]$. Let I be the ideal of A generated by the entries of the matrix XY, with

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}.$$

Find the irreducible components of the closed subset $V(I)$ of $Spec(A)$. (I mean describe them and give equations for each of them. You do not have to prove that the equations you write down define prime ideals.) Hints:

(1) You may use the Hilbert Nullstellensatz, and it suffices to find irreducible locally closed subsets which cover the set of closed points of $V(I)$.
(2) There are two easy components.
(3) An image of an irreducible set under a continuous map is irreducible.

65.11. Dimension

Exercise 65.11.1. Construct a ring A with finitely many prime ideals having dimension > 1.

Exercise 65.11.2. Let $f \in \mathbf{C}[x, y]$ be a nonconstant polynomial. Show that $\mathbf{C}[x, y]/(f)$ has dimension 1.

Exercise 65.11.3. Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let $n \geq 1$. Let $\mathfrak{m}' = (\mathfrak{m}, x_1, \ldots, x_n)$ in the polynomial ring $R[x_1, \ldots, x_n]$. Show that

$$\dim(R[x_1, \ldots, x_n]_{\mathfrak{m}'}) = \dim(R) + n.$$

65.12. Catenary rings

Definition 65.12.1. A Noetherian ring A is said to be *catenary* if for any triple of prime ideals $\mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \mathfrak{p}_3$ we have

$$ht(\mathfrak{p}_3/\mathfrak{p}_1) = ht(\mathfrak{p}_3/\mathfrak{p}_2) + ht(\mathfrak{p}_2/\mathfrak{p}_1).$$

Here $ht(\mathfrak{p}/\mathfrak{q})$ means the height of $\mathfrak{p}/\mathfrak{q}$ in the ring $A/\mathfrak{q}$.

Exercise 65.12.2. Show that a Noetherian local domain of dimension 2 is catenary.

Exercise 65.12.3. Let k be a field. Show that a finite type k-algebra is catenary.

65.13. Fraction fields

Exercise 65.13.1. Consider the domain

$$\mathbf{Q}[r, s, t]/(s^2 - (r - 1)(r - 2)(r - 3), t^2 - (r + 1)(r + 2)(r + 3)).$$

Find a domain of the form $\mathbf{Q}[x, y]/(f)$ with isomorphic field of fractions.

65.14. Transcendence degree

Exercise 65.14.1. Let $k \subset K \subset K'$ be field extensions with K' algebraic over K. Prove that $\text{trdeg}_k(K) = \text{trdeg}_k(K')$. (Hint: Show that if $x_1, \ldots, x_d \in K$ are algebraically independent over k and $d < \text{trdeg}_k(K')$ then $k(x_1, \ldots, x_d) \subset K$ cannot be algebraic.)

65.15. Finite locally free modules

Definition 65.15.1. Let A be a ring. Recall that a *finite locally free* A-module M is a module such that for every $\mathfrak{p} \in Spec(A)$ there exists an $f \in A$, $f \not\in \mathfrak{p}$ such that M_f is a finite free A_f-module. We say M is an *invertible module* if M is finite locally free of rank 1, i.e., for every $\mathfrak{p} \in Spec(A)$ there exists an $f \in A$, $f \not\in \mathfrak{p}$ such that $M_f \cong A_f$ as an A_f-module.

Exercise 65.15.2. Prove that the tensor product of finite locally free modules is finite locally free. Prove that the tensor product of two invertible modules is invertible.

Definition 65.15.3. Let A be a ring. The *class group of* A, sometimes called the *Picard group of* A is the set $\text{Pic}(A)$ of isomomorphism classes of invertible A-modules endowed with a group operation defined by tensor product (see Exercise 65.15.2).

Note that the class group of A is trivial exactly when every invertible module is isomorphic to a free module of rank 1.

Exercise 65.15.4. Show that the class groups of the following rings are trivial

(1) a polynomial ring $A = k[x]$ where k is a field,

(2) the integers $A = \mathbf{Z}$,
(3) a polynomial ring $A = k[x, y]$ where k is a field, and
(4) the quotient $k[x, y]/(xy)$ where k is a field.

Exercise 65.15.5. Show that the class group of the the ring $A = k[x, y]/(y^2 - f(x))$ where k is a field of characteristic not 2 and where $f(x) = (x - t_1) \ldots (x - t_n)$ with $t_1, \ldots, t_n \in k$ distinct and $n \geq 3$ an odd integer is not trivial. (Hint: Show that the ideal $(y, x - t_1)$ defines a nontrivial element of $\text{Pic}(A)$.)

Exercise 65.15.6. Let A be a ring.

(1) Suppose that M is a finite locally free A-module, and suppose that $\varphi : M \to M$ is an endomorphism. Define/construct the *trace* and *determinant* of φ and prove that your construction is ``functorial in the triple (A, M, φ)".
(2) Show that if M, N are finite locally free A-modules, and if $\varphi : M \to N$ and $\psi : N \to M$ then $Trace(\varphi \circ \psi) = Trace(\psi \circ \varphi)$ and $Det(\varphi \circ \psi) = Det(\psi \circ \varphi)$.
(3) In case M is finite locally free show that Det defines a multiplicative map $End_A(M) \to A$.

Exercise 65.15.7. Now suppose that B is an A-algebra which is finite locally free as an A-module, in other words B is a finite locally free A-algebra.

(1) Define $Trace_{B/A}$ and $Norm_{B/A}$ using $Trace$ and Det as defined above.
(2) Let $b \in B$ and let $\pi : Spec(B) \to Spec(A)$ be the induced morphism. Show that $\pi(V(b)) = V(Norm_{B/A}(b))$. (Recall that $V(f) = \{\mathfrak{p} \mid f \in \mathfrak{p}\}$.)
(3) (Base change.) Suppose that $i : A \to A'$ is a ring map. Set $B' = B \otimes_A A'$. Indicate why $i(Norm_{B/A}(b))$ equals $Norm_{B'/A'}(b \otimes 1)$.
(4) Compute $Norm_{B/A}(b)$ when $B = A \times A \times A \times \ldots \times A$ and $b = (a_1, \ldots, a_n)$.
(5) Compute the norm of $y - y^3$ under the finite flat map $\mathbf{Q}[x] \to \mathbf{Q}[y]$, $x \to y^n$. (Hint: use the ``base change" $A = \mathbf{Q}[x] \subset A' = \mathbf{Q}(\zeta_n)(x^{1/n})$.)

65.16. Glueing

Exercise 65.16.1. Suppose that A is a ring and M is an A-module. Let f_i, $i \in I$ be a collection of elements of A such that

$$Spec(A) = \bigcup D(f_i).$$

(1) Show that if M_{f_i} is a finite A_{f_i}-module, then M is a finite A-module.
(2) Show that if M_{f_i} is a flat A_{f_i}-module, then M is a flat A-module. (This is kind of silly if you think about it right.)

Remark 65.16.2. In algebraic geometric language this means that the property of ``being finitely generated" or ``being flat" is local for the Zariski topology (in a suitable sense). You can also show this for the property ``being of finite presentation".

Exercise 65.16.3. Suppose that $A \to B$ is a ring map. Let $f_i \in A$, $i \in I$ and $g_j \in B$, $j \in J$ be collections of elements such that

$$Spec(A) = \bigcup D(f_i) \quad \text{and} \quad Spec(B) = \bigcup D(g_j).$$

Show that if $A_{f_i} \to B_{f_i g_j}$ is of finite type for all i, j then $A \to B$ is of finite type.

65.17. Going up and going down

Definition 65.17.1. Let $\phi : A \to B$ be a homomorphism of rings. We say that the *going-up theorem* holds for ϕ if the following condition is satisfied:

(GU) for any $\mathfrak{p}, \mathfrak{p}' \in Spec(A)$ such that $\mathfrak{p} \subset \mathfrak{p}'$, and for any $P \in Spec(B)$ lying over $\mathfrak{p}$, there exists $P' \in Spec(B)$ lying over $\mathfrak{p}'$ such that $P \subset P'$.

Similarly, we say that the *going-down theorem* holds for ϕ if the following condition is satisfied:

(GD) for any $\mathfrak{p}, \mathfrak{p}' \in Spec(A)$ such that $\mathfrak{p} \subset \mathfrak{p}'$, and for any $P' \in Spec(B)$ lying over $\mathfrak{p}'$, there exists $P \in Spec(B)$ lying over $\mathfrak{p}$ such that $P \subset P'$.

Exercise 65.17.2. In each of the following cases determine whether (GU), (GD) holds, and explain why. (Use any Prop/Thm/Lemma you can find, but check the hypotheses in each case.)

(1) k is a field, $A = k$, $B = k[x]$.
(2) k is a field, $A = k[x]$, $B = k[x, y]$.
(3) $A = \mathbf{Z}$, $B = \mathbf{Z}[1/11]$.
(4) k is an algebraically closed field, $A = k[x, y]$, $B = k[x, y, z]/(x^2 - y, z^2 - x)$.
(5) $A = \mathbf{Z}$, $B = \mathbf{Z}[i, 1/(2 + i)]$.
(6) $A = \mathbf{Z}$, $B = \mathbf{Z}[i, 1/(14 + 7i)]$.
(7) k is an algebraically closed field, $A = k[x]$, $B = k[x, y, 1/(xy - 1)]/(y^2 - y)$.

Exercise 65.17.3. Let k be an algebraically closed field. Compute the image in $Spec(k[x, y])$ of the following maps:

(1) $Spec(k[x, yx^{-1}]) \to Spec(k[x, y])$, where $k[x, y] \subset k[x, yx^{-1}] \subset k[x, y, x^{-1}]$. (Hint: To avoid confusion, give the element yx^{-1} another name.)
(2) $Spec(k[x, y, a, b]/(ax - by - 1)) \to Spec(k[x, y])$.
(3) $Spec(k[t, 1/(t - 1)]) \to Spec(k[x, y])$, induced by $x \mapsto t^2$, and $y \mapsto t^3$.
(4) $k = \mathbf{C}$ (complex numbers), $Spec(k[s, t]/(s^3 + t^3 - 1)) \to Spec(k[x, y])$, where $x \mapsto s^2$, $y \mapsto t^2$.

Remark 65.17.4. Finding the image as above usually is done by using elimination theory.

65.18. Fitting ideals

Exercise 65.18.1. Let R be a ring and let M be a finite R-module. Choose a presentation

$$\bigoplus\nolimits_{j\in J} R \longrightarrow R^{\oplus n} \longrightarrow M \longrightarrow 0.$$

of M. Note that the map $R^{\oplus n} \to M$ is given by a sequence of elements $x_1, \ldots, x_n$ of M. The elements x_i are *generators* of M. The map $\bigoplus_{j\in J} R \to R^{\oplus n}$ is given by a $n \times J$ matrix A with coefficients in R. In other words, $A = (a_{ij})_{i=1,\ldots,n, j\in J}$. The columns $(a_{1j}, \ldots, a_{nj})$, $j \in J$ of A are said to be the *relations*. Any vector $(r_i) \in R^{\oplus n}$ such that $\sum r_i x_i = 0$ is a linear combination of the columns of A. Of course any finite R-module has a lot of different presentations.

(1) Show that the ideal generated by the $(n - k) \times (n - k)$ minors of A is independent of the choice of the presentation. This ideal is the *kth fitting ideal of M*. Notation $Fit_k(M)$.
(2) Show that $Fit_0(M) \subset Fit_1(M) \subset Fit_2(M) \subset \ldots$. (Hint: Use that a determinant can be computed by expanding along a column.)
(3) Show that the following are equivalent:

(a) $Fit_{r-1}(M) = (0)$ and $Fit_r(M) = R$, and
(b) M is locally free of rank r.

65.19. Hilbert functions

Definition 65.19.1. A *numerical polynomial* is a polynomial $f(x) \in \mathbf{Q}[x]$ such that $f(n) \in \mathbf{Z}$ for every integer n.

Definition 65.19.2. A *graded module* M over a ring A is an A-module M endowed with a direct sum decomposition $\bigoplus_{n \in \mathbf{Z}} M_n$ into A-submodules. We will say that M is *locally finite* if all of the M_n are finite A-modules. Suppose that A is a Noetherian ring and that φ is a *Euler-Poincaré function* on finite A-modules. This means that for every finitely generated A-module M we are given an integer $\varphi(M) \in \mathbf{Z}$ and for every short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

we have $\varphi(M) = \varphi(M') + \varphi(M')$. The *Hilbert function* of a locally finite graded module M (with respect to φ) is the function $\chi_\varphi(M, n) = \varphi(M_n)$. We say that M has a *Hilbert polynomial* if there is some numerical polynomial P_φ such that $\chi_\varphi(M, n) = P_\varphi(n)$ for all sufficiently large integers n.

Definition 65.19.3. A *graded A-algebra* is a graded A-module $B = \bigoplus_{n \geq 0} B_n$ together with an A-bilinear map

$$B \times B \longrightarrow B, \ (b, b') \longmapsto bb'$$

that turns B into an A-algebra so that $B_n \cdot B_m \subset B_{n+m}$. Finally, a *graded module M over a graded A-algebra B* is given by a graded A-module M together with a (compatible) B-module structure such that $B_n \cdot M_d \subset M_{n+d}$. Now you can define *homomorphisms of graded modules/rings*, *graded submodules*, *graded ideals*, *exact sequences of graded modules*, etc, etc.

Exercise 65.19.4. Let $A = k$ a field. What are all possible Euler-Poincaré functions on finite A-modules in this case?

Exercise 65.19.5. Let $A = \mathbf{Z}$. What are all possible Euler-Poincaré functions on finite A-modules in this case?

Exercise 65.19.6. Let $A = k[x, y]/(xy)$ with k algebraically closed. What are all possible Euler-Poincaré functions on finite A-modules in this case?

Exercise 65.19.7. Suppose that A is Noetherian. Show that the kernel of a map of locally finite graded A-modules is locally finite.

Exercise 65.19.8. Let k be a field and let $A = k$ and $B = k[x, y]$ with grading determined by $\deg(x) = 2$ and $\deg(y) = 3$. Let $\varphi(M) = \dim_k(M)$. Compute the Hilbert function of B as a graded k-module. Is there a Hilbert polynomial in this case?

Exercise 65.19.9. Let k be a field and let $A = k$ and $B = k[x, y]/(x^2, xy)$ with grading determined by $\deg(x) = 2$ and $\deg(y) = 3$. Let $\varphi(M) = \dim_k(M)$. Compute the Hilbert function of B as a graded k-module. Is there a Hilbert polynomial in this case?

Exercise 65.19.10. Let k be a field and let $A = k$. Let $\varphi(M) = \dim_k(M)$. Fix $d \in \mathbf{N}$. Consider the graded A-algebra $B = k[x, y, z]/(x^d + y^d + z^d)$, where x, y, z each have degree 1. Compute the Hilbert function of B. Is there a Hilbert polynomial in this case?

65.20. Proj of a ring

Definition 65.20.1. Let R be a graded ring. A *homogeneous* ideal is simply an ideal $I \subset R$ which is also a graded submodule of R. Equivalently, it is an ideal generated by homogeneous elements. Equivalently, if $f \in I$ and

$$f = f_0 + f_1 + \ldots + f_n$$

is the decomposition of f into homogenous pieces in R then $f_i \in I$ for each i.

Definition 65.20.2. We define the *homogeneous spectrum Proj*(R) of the graded ring R to be the set of homogenous, prime ideals $\mathfrak{p}$ of R such that $R_+ \not\subset \mathfrak{p}$. Note that Proj($R$) is a subset of *Spec*($R$) and hence has a natural induced topology.

Definition 65.20.3. Let $R = \bigoplus_{d \geq 0} R_d$ be a graded ring, let $f \in R_d$ and assume that $d \geq 1$. We define $R_{(f)}$ to be the subring of R_f consisting of elements of the form r/f^n with r homogenous and $\deg(r) = nd$. Furthermore, we define

$$D_+(f) = \{\mathfrak{p} \in \text{Proj}(R) | f \notin \mathfrak{p}\}.$$

Finally, for a homogenous ideal $I \subset R$ we define $V_+(I) = V(I) \cap \text{Proj}(R)$.

Exercise 65.20.4. On the topology on Proj(R). With definitions and notation as above prove the following statements.

1. Show that $D_+(f)$ is open in Proj(R).
2. Show that $D_+(ff') = D_+(f) \cap D_+(f')$.
3. Let $g = g_0 + \ldots + g_m$ be an element of R with $g_i \in R_i$. Express $D(g) \cap \text{Proj}(R)$ in terms of $D_+(g_i)$, $i \geq 1$ and $D(g_0) \cap \text{Proj}(R)$. No proof necessary.
4. Let $g \in R_0$ be a homogenous element of degree 0. Express $D(g) \cap \text{Proj}(R)$ in terms of $D_+(f_\alpha)$ for a suitable family $f_\alpha \in R$ of homogenous elements of positive degree.
5. Show that the collection $\{D_+(f)\}$ of opens forms a basis for the topology of Proj(R).
6. Show that there is a canonical bijection $D_+(f) \to Spec(R_{(f)})$. (Hint: Imitate the proof for *Spec* but at some point thrown in the radical of an ideal.)
7. Show that the map from (6) is a homeomorphism.
8. Give an example of an R such that Proj(R) is not quasi-compact. No proof necessary.
9. Show that any closed subset $T \subset \text{Proj}(R)$ is of the form $V_+(I)$ for some homogenous ideal $I \subset R$.

Remark 65.20.5. There is a continuous map $\text{Proj}(R) \longrightarrow Spec(R_0)$.

Exercise 65.20.6. If $R = A[X]$ with $\deg(X) = 1$, show that the natural map $\text{Proj}(R) \to Spec(A)$ is a bijection and in fact a homeomorphism.

Exercise 65.20.7. Blowing up: part I. In this exercise $R = Bl_I(A) = A \oplus I \oplus I^2 \oplus \ldots$. Consider the natural map $b : \text{Proj}(R) \to Spec(A)$. Set $U = Spec(A) - V(I)$. Show that

$$b : b^{-1}(U) \longrightarrow U$$

is a homeomorphism. Thus we may think of U as an open subset of Proj(R). Let $Z \subset Spec(A)$ be an irreducible closed subscheme with generic point $\xi \in Z$. Assume that $\xi \notin V(I)$, in other words $Z \not\subset V(I)$, in other words $\xi \in U$, in other words $Z \cap U \neq \emptyset$. We define the *strict transform* Z' of Z to be the closure of the unique point ξ' lying above ξ. Another way to say this is that Z' is the closure in Proj(R) of the locally closed subset $Z \cap U \subset U \subset \text{Proj}(R)$.

Exercise 65.20.8. Blowing up: Part II. Let $A = k[x, y]$ where k is a field, and let $I = (x, y)$. Let R be the blow up algebra for A and I.

(1) Show that the strict transforms of $Z_1 = V(\{x\})$ and $Z_2 = V(\{y\})$ are disjoint.
(2) Show that the strict transforms of $Z_1 = V(\{x\})$ and $Z_2 = V(\{x - y^2\})$ are not disjoint.
(3) Find an ideal $J \subset A$ such that $V(J) = V(I)$ and such that the strict transforms of $Z_1 = V(\{x\})$ and $Z_2 = V(\{x - y^2\})$ are disjoint.

Exercise 65.20.9. Let R be a graded ring.

(1) Show that Proj(R) is empty if $R_n = (0)$ for all $n >> 0$.
(2) Show that Proj(R) is an irreducible topological space if R is a domain and R_+ is not zero. (Recall that the empty topological space is not irreducible.)

Exercise 65.20.10. Blowing up: Part III. Consider A, I and U, Z as in the definition of strict transform. Let $Z = V(\mathfrak{p})$ for some prime ideal $\mathfrak{p}$. Let $\bar{A} = A/\mathfrak{p}$ and let $\bar{I}$ be the image of I in $\bar{A}$.

(1) Show that there exists a surjective ring map $R := Bl_I(A) \to \bar{R} := Bl_{\bar{I}}(\bar{A})$.
(2) Show that the ring map above induces a bijective map from Proj($\bar{R}$) onto the strict transform Z' of Z. (This is not so easy. Hint: Use 5(b) above.)
(3) Conclude that the strict transform $Z' = V_+(P)$ where $P \subset R$ is the homogenous ideal defined by $P_d = I^d \cap \mathfrak{p}$.
(4) Suppose that $Z_1 = V(\mathfrak{p})$ and $Z_2 = V(\mathfrak{q})$ are irreducible closed subsets defined by prime ideals such that $Z_1 \not\subset Z_2$, and $Z_2 \not\subset Z_1$. Show that blowing up the ideal $I = \mathfrak{p} + \mathfrak{q}$ separates the strict transforms of Z_1 and Z_2, i.e., $Z_1' \cap Z_2' = \emptyset$. (Hint: Consider the homogenous ideal P and Q from part (c) and consider $V(P + Q)$.)

65.21. Cohen-Macaulay rings of dimension 1

Definition 65.21.1. A Noetherian local ring A is said to be *Cohen-Macaulay* of dimension d if it has dimension d and there exists a system of parameters $x_1, \ldots, x_d$ for A such that x_i is a nonzero divisor in $A/(x_1, \ldots, x_{i-1})$ for $i = 1, \ldots, d$.

Exercise 65.21.2. Cohen-Macaulay rings of dimension 1. Part I: Theory.

(1) Let $(A, \mathfrak{m})$ be a local Noetherian with $\dim A = 1$. Show that if $x \in \mathfrak{m}$ is not a zero divisor then
   (a) $\dim A/xA = 0$, in other words A/xA is Artinian, in other words $\{x\}$ is a system of parameters for A.
   (b) A is has no embedded prime.
(2) Conversely, let $(A, \mathfrak{m})$ be a local Noetherian ring of dimension 1. Show that if A has no embedded prime then there exists a nonzero divisor in $\mathfrak{m}$.

Exercise 65.21.3. Cohen-Macaulay rings of dimension 1. Part II: Examples.

(1) Let A be the local ring at (x, y) of $k[x, y]/(x^2, xy)$.
   (a) Show that A has dimension 1.
   (b) Prove that every element of $\mathfrak{m} \subset A$ is a zero divisor.
   (c) Find $z \in \mathfrak{m}$ such that $\dim A/zA = 0$ (no proof required).
(2) Let A be the local ring at (x, y) of $k[x, y]/(x^2)$. Find a nonzero divisor in $\mathfrak{m}$ (no proof required).

Exercise 65.21.4. Local rings of embedding dimension 1. Suppose that $(A, \mathfrak{m}, k)$ is a Noetherian local ring of embedding dimension 1, i.e.,

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 = 1.$$

Show that the function $f(n) = \dim_k \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is either constant with value 1, or its values are

$$1, 1, \ldots, 1, 0, 0, 0, 0, 0, \ldots$$

Exercise 65.21.5. Regular local rings of dimension 1. Suppose that $(A, \mathfrak{m}, k)$ is a regular Noetherian local ring of dimension 1. Recall that this means that A has dimension 1 and embedding dimension 1, i.e.,

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 = 1.$$

Let $x \in \mathfrak{m}$ be any element whose class in $\mathfrak{m}/\mathfrak{m}^2$ is not zero.

(1) Show that for every element y of $\mathfrak{m}$ there exists an integer n such that y can be written as $y = ux^n$ with $u \in A^*$ a unit.
(2) Show that x is a nonzero divisor in A.
(3) Conclude that A is a domain.

Exercise 65.21.6. Let $(A, \mathfrak{m}, k)$ be a Noetherian local ring with associated graded $Gr_{\mathfrak{m}}(A)$.

(1) Suppose that $x \in \mathfrak{m}^d$ maps to a nonzero divisor $\bar{x} \in \mathfrak{m}^d/\mathfrak{m}^{d+1}$ in degree d of $Gr_{\mathfrak{m}}(A)$. Show that x is a nonzero divisor.
(2) Suppose the depth of A is at least 1. Namely, suppose that there exists a nonzero divisor $y \in \mathfrak{m}$. In this case we can do better: assume just that $x \in \mathfrak{m}^d$ maps to the element $\bar{x} \in \mathfrak{m}^d/\mathfrak{m}^{d+1}$ in degree d of $Gr_{\mathfrak{m}}(A)$ which is a nonzero divisor on sufficiently high degrees: $\exists N$ such that for all $n \geq N$ the map of multiplication by $\bar{x}$
$$\mathfrak{m}^n/\mathfrak{m}^{n+1} \longrightarrow \mathfrak{m}^{n+d}/\mathfrak{m}^{n+d+1}$$
is injective. Then show that x is a nonzero divisor.

Exercise 65.21.7. Suppose that $(A, \mathfrak{m}, k)$ is a Noetherian local ring of dimension 1. Assume also that the embedding dimension of A is 2, i.e., assume that

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 = 2.$$

Notation: $f(n) = \dim_k \mathfrak{m}^n/\mathfrak{m}^{n+1}$. Pick generators $x, y \in \mathfrak{m}$ and write $Gr_{\mathfrak{m}}(A) = k[\bar{x}, \bar{y}]/I$ for some homogenous ideal I.

(1) Show that there exists a homogenous element $F \in k[\bar{x}, \bar{y}]$ such that $I \subset (F)$ with equality in all sufficiently high degrees.
(2) Show that $f(n) \leq n + 1$.
(3) Show that if $f(n) < n + 1$ then $n \geq \deg(F)$.
(4) Show that if $f(n) < n + 1$, then $f(n + 1) \leq f(n)$.
(5) Show that $f(n) = \deg(F)$ for all $n >> 0$.

Exercise 65.21.8. Cohen-Macaulay rings of dimension 1 and embedding dimension 2. Suppose that $(A, \mathfrak{m}, k)$ is a Noetherian local ring which is Cohen-Macaulay of dimension 1. Assume also that the embedding dimension of A is 2, i.e., assume that

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 = 2.$$

Notations: f, F, $x, y \in \mathfrak{m}$, I as in Ex. 6 above. Please use any results from the problems above.

(1) Suppose that $z \in \mathfrak{m}$ is an element whose class in $\mathfrak{m}/\mathfrak{m}^2$ is a linear form $\alpha\bar{x}+\beta\bar{y} \in k[\bar{x}, \bar{y}]$ which is coprime with f.
   (a) Show that z is a nonzero divisor on A.
   (b) Let $d = \deg(F)$. Show that $\mathfrak{m}^n = z^{n+1-d}\mathfrak{m}^{d-1}$ for all sufficiently large n. (Hint: First show $z^{n+1-d}\mathfrak{m}^{d-1} \to \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is surjective by what you know about $Gr_{\mathfrak{m}}(A)$. Then use NAK.)

(2) What condition on k garantees the existence of such a z? (No proof required; it's too easy.)
Now we are going to assume there exists a z as above. This turns out to be a harmless assumption (in the sense that you can reduce to the situation where it holds in order to obtain the results in parts (d) and (e) below).

(3) Now show that $\mathfrak{m}^\ell = z^{\ell-d+1}\mathfrak{m}^{d-1}$ for all $\ell \geq d$.

(4) Conclude that $I = (F)$.

(5) Conclude that the function f has values

$$2, 3, 4, \ldots, d-1, d, d, d, d, d, d, d, \ldots$$

Remark 65.21.9. This suggests that a local Noetherian Cohen-Macaulay ring of dimension 1 and embedding dimension 2 is of the form B/FB, where B is a 2-dimensional regular local ring. This is more or less true (under suitable ``niceness" properties of the ring).

65.22. Infinitely many primes

A section with a collection of strange questions on rings where infinitely many primes are not invertible.

Exercise 65.22.1. Give an example of a finite type $\mathbf{Z}$-algebra R with the following two properties:
(a) There is no ring map $R \to \mathbf{Q}$.
(b) For every prime p there exists a maximal ideal $\mathfrak{m} \subset R$ such that $R/\mathfrak{m} \cong \mathbf{F}_p$.

Exercise 65.22.2. For $f \in \mathbf{Z}[x, u]$ we define $f_p(x) = f(x, x^p) \bmod p \in \mathbf{F}_p[x]$. Give an example of an $f \in \mathbf{Z}[x, u]$ such that the following two properties hold:
(a) There exist infinitely many p such that f_p does not have a zero in $\mathbf{F}_p$.
(b) For all $p >> 0$ the polynomial f_p either has a linear or a quadratic factor.

Exercise 65.22.3. For $f \in \mathbf{Z}[x, y, u, v]$ we define $f_p(x, y) = f(x, y, x^p, y^p) \bmod p \in \mathbf{F}_p[x, y]$. Give an ``interesting" example of an $f$ such that $f_p$ is reducible for all $p >> 0$. For example, $f = xv - yu$ with $f_p = xy^p - x^p y = xy(x^{p-1} - y^{p-1})$ is ``uninteresting"; any f depeding only on x, u is ``uninteresting", etc.

Remark 65.22.4. Let $h \in \mathbf{Z}[y]$ be a monic polynomial of degree d. Then:

(1) The map $A = \mathbf{Z}[x] \to B = \mathbf{Z}[y]$, $x \mapsto h$ is finite locally free of rank d.
(2) For all primes p the map $A_p = \mathbf{F}_p[x] \to B_p = \mathbf{F}_p[y]$, $y \mapsto h(y) \bmod p$ is finite locally free of rank d.

Exercise 65.22.5. Let h, A, B, A_p, B_p be as in the remark. For $f \in \mathbf{Z}[x, u]$ we define $f_p(x) = f(x, x^p) \bmod p \in \mathbf{F}_p[x]$. For $g \in \mathbf{Z}[y, v]$ we define $g_p(y) = g(y, y^p) \bmod p \in \mathbf{F}_p[y]$.

(1) Give an example of a h and g such that there does not exist a f with the property

$$f_p = Norm_{B_p/A_p}(g_p).$$

(2) Show that for any choice of h and g as above there exists a nonzero f such that for all p we have
$$Norm_{B_p/A_p}(g_p) \text{ divides } f_p.$$
If you want you can restrict to the case $h = y^n$, even with $n = 2$, but it is true in general.

(3) Discuss the relevance of this to Exercises 6 & 7 of the previous set.

Exercise 65.22.6. Unsolved problems. They may be really hard or they may be easy. I don't know.

(1) Is there any $f \in \mathbf{Z}[x, u]$ such that f_p is irreducible for an infinite number of p? (Hint: Yes, this happens for $f(x, u) = u - x - 1$ and also for $f(x, u) = u^2 - x^2 + 1$.)
(2) Let $f \in \mathbf{Z}[x, u]$ nonzero, and suppose $\deg_x(f_p) = dp + d'$ for all large p. (In other words $\deg_u(f) = d$ and the coefficient c of u^d in f has $\deg_x(c) = d'$.) Suppose we can write $d = d_1 + d_2$ and $d' = d'_1 + d'_2$ with $d_1, d_2 > 0$ and $d'_1, d'_2 \geq 0$ such that for all sufficiently large p there exists a factorization
$$f_p = f_{1,p} f_{2,p}$$
with $\deg_x(f_{1,p}) = d_1 p + d'_1$. Is it true that f comes about via a norm construction as in Exercise 4? (More precisely, are there a h and g such that $Norm_{B_p/A_p}(g_p)$ divides f_p for all $p >> 0$.)
(3) Analogous question to the one in (b) but now with $f \in \mathbf{Z}[x_1, x_2, u_1, u_2]$ irreducible and just assuming that $f_p(x_1, x_2) = f(x_1, x_2, x_1^p, x_2^p) \bmod p$ factors for all $p >> 0$.

65.23. Filtered derived category

In order to do the exercises in this section, please read the material in Homology, Section 10.13. We will say A is a filtered object of $\mathcal{A}$, to mean that A comes endowed with a filtration F which we omit from the notation.

Exercise 65.23.1. Let $\mathcal{A}$ be an abelian category. Let I be a filtered object of $\mathcal{A}$. Assume that the filtration on I is finite and that each $\text{gr}^p(I)$ is an injective object of $\mathcal{A}$. Show that there exists an isomorphism $I \cong \bigoplus \text{gr}^p(I)$ with filtration $F^p(I)$ corresponding to $\bigoplus_{p' \geq p} \text{gr}^p(I)$.

Exercise 65.23.2. Let $\mathcal{A}$ be an abelian category. Let I be a filtered object of $\mathcal{A}$. Assume that the filtration on I is finite. Show the following are equivalent:

(1) For any solid diagram
$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d] & B \arrow[dl, dashed] \\ I & \end{tikzcd}$$
of filtered objects with (i) the filtrations on A and B are finite, and (ii) $\text{gr}(\alpha)$ injective the dotted arrow exists making the diagram commute.
(2) Each $\text{gr}^p I$ is injective.

Note that given a morphism $\alpha : A \to B$ of filtered objects with finite filtrations to say that $\text{gr}(\alpha)$ injective is the same thing as saying that α is a *strict monomorphism* in the category $\text{Fil}(\mathcal{A})$. Namely, being a monomorphism means $\text{Ker}(\alpha) = 0$ and strict means that this also implies $\text{Ker}(\text{gr}(\alpha)) = 0$. See Homology, Lemma 10.13.15. (We only use the term ``injective'' for a morphism in an abelian category, allthough it makes sense in any additive category having kernels.) The exercises above justifies the following definition.

Definition 65.23.3. Let $\mathcal{A}$ be an abelian category. Let I be a filtered object of $\mathcal{A}$. Assume the filtration on I is finite. We say I is *filtered injective* if each $\mathrm{gr}^p(I)$ is an injective object of $\mathcal{A}$.

We make the following definition to avoid having to keep saying ``with a finite filtration'' everywhere.

Definition 65.23.4. Let $\mathcal{A}$ be an abelian category. We denote $\mathit{Fil}^f(\mathcal{A})$ the full subcategory of $\mathrm{Fil}(\mathcal{A})$ whose objects consist of those $A \in \mathit{Ob}(\mathrm{Fil}(\mathcal{A}))$ whose filtration is finite.

Exercise 65.23.5. Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has enough injectives. Let A be an object of $\mathrm{Fil}^f(\mathcal{A})$. Show that there exists a strict monomorphism $\alpha : A \to I$ of A into a filtered injective object I of $\mathrm{Fil}^f(\mathcal{A})$.

Definition 65.23.6. Let $\mathcal{A}$ be an abelian category. Let $\alpha : K^\bullet \to L^\bullet$ be a morphism of complexes of $\mathrm{Fil}(\mathcal{A})$. We say that α is a *filtered quasi-isomorphism* if for each $p \in \mathbf{Z}$ the morphism $\mathrm{gr}^p(K^\bullet) \to \mathrm{gr}^p(L^\bullet)$ is a quasi-isomorphism.

Definition 65.23.7. Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$ be a complex of $\mathrm{Fil}^f(\mathcal{A})$. We say that $K^\bullet$ is *filtered acyclic* if for each $p \in \mathbf{Z}$ the complex $\mathrm{gr}^p(K^\bullet)$ is acyclic.

Exercise 65.23.8. Let $\mathcal{A}$ be an abelian category. Let $\alpha : K^\bullet \to L^\bullet$ be a morphism of bounded below complexes of $\mathrm{Fil}^f(\mathcal{A})$. (Note the superscript f.) Show that the following are equivalent:

(1) α is a filtered quasi-isomorphism,
(2) for each $p \in \mathbf{Z}$ the map $\alpha : F^pK^\bullet \to F^pL^\bullet$ is a quasi-isomorphism,
(3) for each $p \in \mathbf{Z}$ the map $\alpha : K^\bullet/F^pK^\bullet \to L^\bullet/F^pL^\bullet$ is a quasi-isomorphism, and
(4) the cone of α (see Derived Categories, Definition 11.8.1) is a filtered acyclic complex.

Moreover, show that if α is a filtered quasi-isomorphism then α is also a usual quasi-isomorphism.

Exercise 65.23.9. Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has enough injectives. Let A be an object of $\mathrm{Fil}^f(\mathcal{A})$. Show there exists a complex $I^\bullet$ of $\mathrm{Fil}^f(\mathcal{A})$, and a morphism $A[0] \to I^\bullet$ such that

(1) each I^p is filtered injective,
(2) $I^p = 0$ for $p < 0$, and
(3) $A[0] \to I^\bullet$ is a filtered quasi-isomorphism.

Exercise 65.23.10. Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has enough injectives. Let $K^\bullet$ be a bounded below complex of objects of $\mathrm{Fil}^f(\mathcal{A})$. Show there exists a filtered quasi-isomorphism $\alpha : K^\bullet \to I^\bullet$ with $I^\bullet$ a complex of $\mathrm{Fil}^f(\mathcal{A})$ having filtered injective terms I^n, and bounded below. In fact, we may choose α such that each α^n is a strict monomorphism.

Exercise 65.23.11. Let $\mathcal{A}$ be an abelian category. Consider a solid diagram

$$\begin{tikzcd} K^\bullet \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & L^\bullet \arrow[dl, dashed, "\beta"] \\ I^\bullet & \end{tikzcd}$$

of complexes of $\mathrm{Fil}^f(\mathcal{A})$. Assume $K^\bullet$, $L^\bullet$ and $I^\bullet$ are bounded below and assume each I^n is a filtered injective object. Also assume that α is a filtered quasi-isomorphism.

(1) There exists a map of complexes β making the diagram commute up to homotopy.
(2) If α is a strict monomorphism in every degree then we can find a β which makes the diagram commute.

Exercise 65.23.12. Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$, $K^\bullet$ be complexes of $\mathrm{Fil}^f(\mathcal{A})$. Assume

(1) $K^\bullet$ bounded below and filtered acyclic, and
(2) $I^\bullet$ bounded below and consisting of filtered injective objects.

Then any morphism $K^\bullet \to I^\bullet$ is homotopic to zero.

Exercise 65.23.13. Let $\mathcal{A}$ be an abelian category. Consider a solid diagram

$$\begin{tikzcd} K^\bullet \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & L^\bullet \arrow[dl, dashed, "\beta_i"] \\ I^\bullet & \end{tikzcd}$$

of complexes of $\mathrm{Fil}^f(\mathcal{A})$. Assume $K^\bullet$, $L^\bullet$ and $I^\bullet$ bounded below and each I^n a filtered injective object. Aslo assume α a filtered quasi-isomorphism. Any two morphisms β_1, β_2 making the diagram commute up to homotopy are homotopic.

65.24. Regular functions

Exercise 65.24.1. In this exercise we try to see what happens with regular functions over non-algebraically closed fields. Let k be a field. Let $Z \subset k^n$ be a Zariski locally closed subset, i.e., there exist ideals $I \subset J \subset k[x_1, \ldots, x_n]$ such that

$$Z = \{a \in k^n \mid f(a) = 0 \ \forall f \in I, \ \exists g \in J, \ g(a) \neq 0\}.$$

A function $\varphi : Z \to k$ is said to be *regular* if for every $z \in Z$ there exists a Zariski open neighbourhood $z \in U \subset Z$ and polynomials $f, g \in k[x_1, \ldots, x_n]$ such that $g(u) \neq 0$ for all $u \in U$ and such that $\varphi(u) = f(u)/g(u)$ for all $u \in U$.

(1) If $k = \bar{k}$ and $Z = k^n$ show that regular functions are given by polynomials. (Only do this if you haven't seen this argument before.)
(2) If k is finite show that (a) every function φ is regular, (b) the ring of regular functions is finite dimensional over k. (If you like you can take $Z = k^n$ and even $n = 1$.)
(3) If $k = \mathbf{R}$ give an example of a regular function on $Z = \mathbf{R}$ which is not given by a polynomial.
(4) If $k = \mathbf{Q}_p$ give an example of a regular function on $Z = \mathbf{Q}_p$ which is not given by a polynomial.

65.25. Sheaves

A morphism $f : X \to Y$ of a category $\mathcal{C}$ is an *monomorphism* if for every pair of morphisms $a, b : W \to X$ we have $f \circ a = f \circ b \Rightarrow a = b$. A monomorphism in the category of sets is an injective map of sets.

Exercise 65.25.1. Carefully prove that a map of sheaves of sets is an monomorphism (in the category of sheaves of sets) if and only if the induced maps on all the stalks are injective.

A morphism $f : X \to Y$ of a category $\mathcal{C}$ is an *isomorphism* if there exists a morphsm $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. An isomorphism in the category of sets is a bijective map of sets.

Exercise 65.25.2. Carefully prove that a map of sheaves of sets is an isomorphism (in the category of sheaves of sets) if and only if the induced maps on all the stalks are bijective.

A morphism $f : X \to Y$ of a category $\mathcal{C}$ is an *epimorphism* if for every pair of morphisms $a, b : Y \to Z$ we have $a \circ f = b \circ f \Rightarrow a = b$. An epimorphism in the category of sets is a surjective map of sets.

Exercise 65.25.3. Carefully prove that a map of sheaves of sets is an epimorphism (in the category of sheaves of sets) if and only if the induced maps on all the stalks are surjective.

Exercise 65.25.4. Let $f : X \to Y$ be a map of topological spaces. Prove pushforward f_* and pullback f^{-1} for sheaves of **sets** form an adjoint pair of functors.

Exercise 65.25.5. Let $j : U \to X$ be an open immersion. Show that j^{-1} has a left adjoint $j_!$ on the category of sheaves of sets. Characterize the stalks of $j_!(\mathcal{G})$. (Hint: $j_!$ is called extension by zero when you do this for abelian sheaves...)

Exercise 65.25.6. Let $X = \mathbf{R}$ with the usual topology. Let $\mathcal{O}_X = \underline{\mathbf{Z}/2\mathbf{Z}}_X$. Let $i : Z = \{0\} \to X$ be the inclusion and let $\mathcal{O}_Z = \underline{\mathbf{Z}/2\mathbf{Z}}_Z$. Prove the following (the first three follow from the definitions but if you are not clear on the definitions you should elucidate them):

(1) $i_*\mathcal{O}_Z$ is a skyscraper sheaf.
(2) There is a canonical surjective map from $\underline{\mathbf{Z}/2\mathbf{Z}}_X \to i_*\underline{\mathbf{Z}/2\mathbf{Z}}_Z$. Denote the kernel $\mathcal{I} \subset \mathcal{O}_X$.
(3) $\mathcal{I}$ is an ideal sheaf of $\mathcal{O}_X$.
(4) The sheaf $\mathcal{I}$ on X cannot be locally generated by sections (as in Modules, Definition 15.8.1.)

Exercise 65.25.7. Let X be a topological space. Let $\mathcal{F}$ be an abelian sheaf on X. Show that $\mathcal{F}$ is the quotient of a (possibly very large) direct sum of sheaves all of whose terms are of the form

$$j_!(\underline{\mathbf{Z}}_U)$$

where $U \subset X$ is open and $\underline{\mathbf{Z}}_U$ denotes the constant sheaf with value $\mathbf{Z}$ on U.

Remark 65.25.8. Let X be a topological space. In the category of abelian sheaves the direct sum of a family of sheaves $\{\mathcal{F}_i\}_{i \in I}$ is the sheaf associated to the presheaf $U \mapsto \bigoplus \mathcal{F}_i(U)$. Consequently the stalk of the direct sum at a point x is the direct sum of the stalks of the $\mathcal{F}_i$ at x.

Exercise 65.25.9. Let X be a topological space. Suppose we are given a collection of abelian groups A_x indexed by $x \in X$. Show that the rule $U \mapsto \prod_{x \in U} A_x$ with obvious restriction mappings defines a sheaf $\mathcal{G}$ of abelian groups. Show, by an example, that usually it is not the case that $\mathcal{G}_x = A_x$ for $x \in X$.

Exercise 65.25.10. Let X, A_x, $\mathcal{G}$ be as in Exercise 65.25.9. Let $\mathcal{B}$ be a basis for the topology of X, see Topology, Definition 5.3.1. For $U \in \mathcal{B}$ let A_U be a subgroup $A_U \subset \mathcal{G}(U) = \prod_{x \in U} A_x$. Assume that for $U \subset V$ with $U, V \in \mathcal{B}$ the restriction maps A_V into A_U. For $U \subset X$ open set

$$\mathcal{F}(U) = \left\{ (s_x)_{x \in U} \;\middle|\; \begin{array}{l} \text{for every } x \text{ in } U \text{ there exists } V \in \mathcal{B} \\ x \in V \subset U \text{ such that } (s_y)_{y \in V} \in A_V \end{array} \right\}$$

Show that $\mathcal{F}$ defines a sheaf of abelian groups on X. Show, by an example, that it is usually not the case that $\mathcal{F}(U) = A_U$ for $U \in \mathcal{B}$.

65.26. Schemes

Let LRS be the category of locally ringed spaces. An affine scheme is an object in LRS isomorphic in LRS to a pair of the form $(Spec(A), \mathcal{O}_{Spec(A)})$. A scheme is an object $(X, \mathcal{O}_X)$ of LRS such that every point $x \in X$ has an open neighbourhood $U \subset X$ such that the pair $(U, \mathcal{O}_X|_U)$ is an affine scheme.

Exercise 65.26.1. Find a 1-point locally ringed space which is not a scheme.

Exercise 65.26.2. Suppose that X is a scheme whose underlying topological space has 2 points. Show that X is an affine scheme.

Exercise 65.26.3. Suppose that X is a scheme whose underlying topological space is a finite discrete set. Show that X is an affine scheme.

Exercise 65.26.4. Show that there exists a non-affine scheme having three points.

Exercise 65.26.5. Suppose that X is a quasi-compact scheme. Show that X has a closed point.

Remark 65.26.6. When $(X, \mathcal{O}_X)$ is a ringed space and $U \subset X$ is an open subset then $(U, \mathcal{O}_X|_U)$ is a ringed space. Notation: $\mathcal{O}_U = \mathcal{O}_X|_U$. There is a canonical morphisms of ringed spaces

$$j : (U, \mathcal{O}_U) \longrightarrow (X, \mathcal{O}_X).$$

If $(X, \mathcal{O}_X)$ is a locally ringed space, so is $(U, \mathcal{O}_U)$ and j is a morphism of locally ringed spaces. If $(X, \mathcal{O}_X)$ is a scheme so is $(U, \mathcal{O}_U)$ and j is a morphism of schemes. We say that $(U, \mathcal{O}_U)$ is an *open subscheme* of $(X, \mathcal{O}_X)$ and that j is an *open immersion*. More generally, any morphism $j' : (V, \mathcal{O}_V) \to (X, \mathcal{O}_X)$ that is *isomorphic* to a morphism $j : (U, \mathcal{O}_U) \to (X, \mathcal{O}_X)$ as above is called an open immersion.

Exercise 65.26.7. Give an example of an affine scheme $(X, \mathcal{O}_X)$ and an open $U \subset X$ such that $(U, \mathcal{O}_X|U)$ is not an affine scheme.

Exercise 65.26.8. Given an example of a pair of affine schemes $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$, an open subscheme $(U, \mathcal{O}_X|_U)$ of X and a morphism of schemes $(U, \mathcal{O}_X|_U) \to (Y, \mathcal{O}_Y)$ that does not extend to a morphism of schemes $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$.

Exercise 65.26.9. (This is pretty hard.) Given an example of a scheme X, and open subscheme $U \subset X$ and a closed subscheme $Z \subset U$ such that Z does not extend to a closed subscheme of X.

Exercise 65.26.10. Give an example of a scheme X, a field K, and a morphism of ringed spaces $Spec(K) \to X$ which is NOT a morphism of schemes.

Exercise 65.26.11. Do all the exercises in Hartshorne, [**Har77**, Chapter II], Sections 1 and 2... Just kidding!

Definition 65.26.12. A scheme X is called *integral* if for every nonempty affine open $U \subset X$ the ring $\Gamma(U, \mathcal{O}_X) = \mathcal{O}_X(U)$ is a domain.

Exercise 65.26.13. Give an example of a morphism of *integral* schemes $f : X \to Y$ such that the induced maps $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ are surjective for all $x \in X$, but f is not a closed immersion.

Exercise 65.26.14. Give an example of a fibre product $X \times_S Y$ such that X and Y are affine but $X \times_S Y$ is not.

Remark 65.26.15. It turns out this cannot happen with S separated. Do you know why?

Exercise 65.26.16. Give an example of a scheme V which is integral 1-dimensional scheme of finite type over $\mathbf{Q}$ such that $Spec(\mathbf{C}) \times_{Spec(\mathbf{Q})} V$ is not integral.

Exercise 65.26.17. Give an example of a scheme V which is integral 1-dimensional scheme of finite type over a field k such that $Spec(k') \times_{Spec(k)} V$ is not reduced for some finite field extension $k \subset k'$.

Remark 65.26.18. If your scheme is affine then dimension is the same as the Krull dimension of the underlying ring. So you can use last semesters results to compute dimension.

65.27. Morphisms

An important question is, given a morphism $\pi : X \to S$, whether the morphism has a section or a rational section. Here are some example exercises.

Exercise 65.27.1. Consider the morphism of schemes

$$\pi : X = Spec(\mathbf{C}[x, t, 1/xt]) \longrightarrow S = Spec(\mathbf{C}[t]).$$

(1) Show there does not exist a morphism $\sigma : S \to X$ such that $\pi \circ \sigma = \mathrm{id}_U$.
(2) Show there does exist a nonempty open $U \subset S$ and a morphism $\sigma : U \to X$ such that $\pi \circ \sigma = \mathrm{id}_U$.

Exercise 65.27.2. Consider the morphism of schemes

$$\pi : X = Spec(\mathbf{C}[x, t]/(x^2 + t)) \longrightarrow S = Spec(\mathbf{C}[t]).$$

Show there does not exist a nonempty open $U \subset S$ and a morphism $\sigma : U \to X$ such that $\pi \circ \sigma = \mathrm{id}_U$.

Exercise 65.27.3. Let $A, B, C \in \mathbf{C}[t]$ be nonzero polynomials. Consider the morphism of schemes

$$\pi : X = Spec(\mathbf{C}[x, y, t]/(A + Bx^2 + Cy^2)) \longrightarrow S = Spec(\mathbf{C}[t]).$$

Show there does exist a nonempty open $U \subset S$ and a morphism $\sigma : U \to X$ such that $\pi \circ \sigma = \mathrm{id}_U$. (Hint: Symbolically, write $x = X/Z$, $y = Y/Z$ for some $X, Y, Z \in \mathbf{C}[t]$ of degree $\leq d$ for some d, and work out the condition that this solves the equation. Then show, using dimension theory, that if $d >> 0$ you can find nonzero X, Y, Z solving the equation.)

Remark 65.27.4. Exercise 65.27.3 is a special case of ``Tsen's theorem''. Exercise 65.27.5 shows that the method is limited to low degree equations (conics when the base and fibre have dimension 1).

Exercise 65.27.5. Consider the morphism of schemes

$$\pi : X = Spec(\mathbf{C}[x, y, t]/(1 + tx^3 + t^2y^3)) \longrightarrow S = Spec(\mathbf{C}[t])$$

Show there does not exist a nonempty open $U \subset S$ and a morphism $\sigma : U \to X$ such that $\pi \circ \sigma = \mathrm{id}_U$.

Exercise 65.27.6. Consider the schemes

$$X = Spec(\mathbf{C}[\{x_i\}_{i=1}^8, s, t]/(1 + sx_1^3 + s^2x_2^3 + tx_3^3 + stx_4^3 + s^2tx_5^3 + t^2x_6^3 + st^2x_7^3 + s^2t^2x_8^3))$$

and

$$S = Spec(\mathbf{C}[s, t])$$

and the morphism of schemes

$$\pi : X \longrightarrow S$$

Show there does not exist a nonempty open $U \subset S$ and a morphism $\sigma : U \to X$ such that $\pi \circ \sigma = \mathrm{id}_U$.

Exercise 65.27.7. (For the number theorists.) Give an example of a closed subscheme

$$Z \subset Spec\left(\mathbf{Z}[x, \frac{1}{x(x-1)(2x-1)}]\right)$$

such that the morphism $Z \to Spec(\mathbf{Z})$ is finite and surjective.

Exercise 65.27.8. If you do not like number theory, you can try the variant where you look at

$$Spec\left(\mathbf{F}_p[t, x, \frac{1}{x(x-t)(tx-1)}]\right) \longrightarrow Spec(\mathbf{F}_p[t])$$

and you try to find a closed subscheme of the top scheme which maps finite surjectively to the bottom one. (There is a theoretical reason for having a finite ground field here; allthough it may not be necessary in this particular case.)

Remark 65.27.9. The interpretation of the results of Exercise 65.27.7 and 65.27.8 is that given the morphism $X \to S$ all of whose fibres are nonempty, there exists a finite surjective morphism $S' \to S$ such that the base change $X_{S'} \to S'$ does have a section. This is not a general fact, but it holds if the base is the spectrum of a dedekind ring with finite residue fields at closed points, and the morphism $X \to S$ is flat with geometrically irreducible generic fibre. See Exercise 65.27.10 below for an example where it doesn't work.

Exercise 65.27.10. Prove there exist a $f \in \mathbf{C}[x, t]$ which is not divisible by $t - \alpha$ for any $\alpha \in \mathbf{C}$ such that there does not exist any $Z \subset Spec(\mathbf{C}[x, t, 1/f])$ which maps finite surjectively to $Spec(\mathbf{C}[t])$. (I think that $f(x, t) = (xt - 2)(x - t + 3)$ works. To show any candidate has the required property is not so easy I think.)

65.28. Tangent Spaces

Definition 65.28.1. For any ring R we denote $R[\epsilon]$ the ring of *dual numbers*. As an R-module it is free with basis $1, \epsilon$. The ring structure comes from setting $\epsilon^2 = 0$.

Exercise 65.28.2. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point, let $s = f(x)$. Consider the solid commutative diagram

$$\begin{tikzcd}
Spec(\kappa(x)) \arrow[r] \arrow[rr, bend left] \arrow[dr] & Spec(\kappa(x)[\epsilon]) \arrow[r, dotted] \arrow[d] & X \arrow[d] \\
 & Spec(\kappa(s)) \arrow[r] & S
\end{tikzcd}$$

with the curved arrow being the canonical morphism of $Spec(\kappa(x))$ into X. If $\kappa(x) = \kappa(s)$ show that the set of dotted arrows which make the diagram commute are in one to one correspondence with the set of linear maps

$$Hom_{\kappa(x)}(\frac{\mathfrak{m}_x}{\mathfrak{m}_x^2 + \mathfrak{m}_s \mathcal{O}_{X,x}}, \kappa(x))$$

In other words: describe such a bijection. (This works more generally if $\kappa(x) \supset \kappa(s)$ is a separable algebraic extension.)

Definition 65.28.3. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. We dub the set of dotted arrows of Exercise 65.28.2 the *tangent space of X over S* and we denote it $T_{X/S,x}$. An element of this space is called a *tangent vector* of X/S at x.

Exercise 65.28.4. For any field K prove that the diagram

$$\begin{tikzcd} Spec(K) \arrow[r] \arrow[d] & Spec(K[\epsilon_1]) \arrow[d] \\ Spec(K[\epsilon_2]) \arrow[r] & Spec(K[\epsilon_1, \epsilon_2]/(\epsilon_1\epsilon_2)) \end{tikzcd}$$

is a push out diagram in the category of schemes. (Here $\epsilon_i^2 = 0$ as before.)

Exercise 65.28.5. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. Define addition of tangent vectors, using Exercise 65.28.4 and a suitable morphism

$$Spec(K[\epsilon]) \longrightarrow Spec(K[\epsilon_1, \epsilon_2]/(\epsilon_1\epsilon_2)).$$

Similarly, define scalar multiplication of tangent vectors (this is easier). Show that $T_{X/S,x}$ becomes a $\kappa(x)$-vector space with your constructions.

Exercise 65.28.6. Let k be a field. Consider the structure morphism $f : X = \mathbf{A}^1_k \to Spec(k) = S$.

(1) Let $x \in X$ be a closed point. What is the dimension of $T_{X/S,x}$?
(2) Let $\eta \in X$ be the generic point. What is the dimension of $T_{X/S,\eta}$?
(3) Consider now X as a scheme over $Spec(\mathbf{Z})$. What are the dimensions of $T_{X/\mathbf{Z},x}$ and $T_{X/\mathbf{Z},\eta}$?

Remark 65.28.7. Exercise 65.28.6 explains why it is necessary to consider the tangent space of X over S to get a good notion.

Exercise 65.28.8. Consider the morphism of schemes

$$f : X = Spec(\mathbf{F}_p(t)) \longrightarrow Spec(\mathbf{F}_p(t^p)) = S$$

Compute the tangent space of X/S at the unique point of X. Isn't that weird? What do you think happens if you take the morphism of schemes corresponding to $\mathbf{F}_p[t^p] \to \mathbf{F}_p[t]$?

Exercise 65.28.9. Let k be a field. Compute the tangent space of X/k at the point $x = (0, 0)$ where $X = Spec(k[x, y]/(x^2 - y^3))$.

Exercise 65.28.10. Let $f : X \to Y$ be a morphism of schemes over S. Let $x \in X$ be a point. Set $y = f(x)$. Assume that the natural map $\kappa(y) \to \kappa(x)$ is bijective. Show, using the definition, that f induces a natural linear map

$$\mathrm{d}f : T_{X/S,x} \longrightarrow T_{Y/S,y}.$$

Match it with what happens on local rings via Exercise 65.28.2 in case $\kappa(x) = \kappa(s)$.

Exercise 65.28.11. Let k be an algebraically closed field. Let

$$\begin{array}{rcl} f : \mathbf{A}^n_k & \longrightarrow & \mathbf{A}^m_k \\ (x_1, \ldots, x_n) & \longmapsto & (f_1(x_i), \ldots, f_m(x_i)) \end{array}$$

be a morphism of schemes over k. This is given by m polynomials $f_1, \ldots, f_m$ in n variables. Consider the matrix

$$A = \left(\frac{\partial f_j}{\partial x_i}\right)$$

Let $x \in \mathbf{A}^n_k$ be a closed point. Set $y = f(x)$. Show that the map on tangent spaces $T_{\mathbf{A}^n_k/k,x} \to T_{\mathbf{A}^m_k/k,y}$ is given by the value of the matrix A at the point x.

65.29. Quasi-coherent Sheaves

Definition 65.29.1. Let X be a scheme. A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is *quasi-coherent* if for every affine open $Spec(R) = U \subset X$ the restriction $\mathcal{F}|_U$ is of the form $\widetilde{M}$ for some R-module M.

It is enough to check this conditions on the members of an affine open covering of X. See Schemes, Section 21.24 for more results.

Definition 65.29.2. Let X be a topological space. Let $x, x' \in X$. We say x is a *specialization* of x' if and only if $x \in \overline{\{x'\}}$.

Exercise 65.29.3. Let X be a scheme. Let $x, x' \in X$. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. Suppose that (a) x is a specialization of x' and (b) $\mathcal{F}_{x'} \neq 0$. Show that $\mathcal{F}_x \neq 0$.

Exercise 65.29.4. Find an example of a scheme X, points $x, x' \in X$, a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ such that (a) x is a specialization of x' and (b) $\mathcal{F}_{x'} \neq 0$ and $\mathcal{F}_x = 0$.

Definition 65.29.5. A scheme X is called *locally Noetherian* if and only if for every point $x \in X$ there exists an affine open $Spec(R) = U \subset X$ such that R is Noetherian. A scheme is *Noetherian* if it is locally Noetherian and quasi-compact.

If X is locally Noetherian then any affine open of X is the spectrum of a Noetherian ring, see Properties, Lemma 23.5.2.

Definition 65.29.6. Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. We say $\mathcal{F}$ is *coherent* if for every point $x \in X$ there exists an affine open $Spec(R) = U \subset X$ such that $\mathcal{F}|_U$ is isomorphic to $\widetilde{M}$ for some finite R-module M.

Exercise 65.29.7. Let $X = Spec(R)$ be an affine scheme.

(1) Let $f \in R$. Let $\mathcal{G}$ be a quasi-coherent sheaf of $\mathcal{O}_{D(f)}$-modules on the open subscheme $D(f)$. Show that $\mathcal{G} = \mathcal{F}|_U$ for some quasi-coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$.
(2) Let $I \subset R$ be an ideal. Let $i : Z \to X$ be the closed subscheme of X corresponding to I. Let $\mathcal{G}$ be a quasi-coherent sheaf of $\mathcal{O}_Z$-modules on the closed subscheme Z. Show that $\mathcal{G} = i^*\mathcal{F}$ for some quasi-coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$. (Why is this silly?)
(3) Assume that R is Noetherian. Let $f \in R$. Let $\mathcal{G}$ be a coherent sheaf of $\mathcal{O}_{D(f)}$-modules on the open subscheme $D(f)$. Show that $\mathcal{G} = \mathcal{F}|_U$ for some coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$.

Remark 65.29.8. If $U \to X$ is a quasi-compact immersion then any quasi-coherent sheaf on U is the restriction of a quasi-coherent sheaf on X. If X is a Noetherian scheme, and $U \subset X$ is open, then any coherent sheaf on U is the restriction of a coherent sheaf on X. Of course the exercise above is easier, and shouldn't use these general facts.

65.30. Proj and projective schemes

Exercise 65.30.1. Give examples of graded rings S such that

(1) $\text{Proj}(S)$ is affine and nonempty, and
(2) $\text{Proj}(S)$ is integral, nonempty but not isomorphic to $\mathbf{P}^n_A$ for any $n \geq 0$, any ring A.

Exercise 65.30.2. Give an example of a nonconstant morphism of schemes $\mathbf{P}^1_{\mathbf{C}} \to \mathbf{P}^5_{\mathbf{C}}$ over $Spec(\mathbf{C})$.

Exercise 65.30.3. Give an example of an isomorphism of schemes

$$\mathbf{P}^1_{\mathbf{C}} \to \text{Proj}(\mathbf{C}[X_0, X_1, X_2]/(X_0^2 + X_1^2 + X_2^2))$$

Exercise 65.30.4. Give an example of a morphism of schemes $f : X \to \mathbf{A}^1_{\mathbf{C}} = Spec(\mathbf{C}[T])$ such that the (scheme theoretic) fibre X_t of f over $t \in \mathbf{A}^1_{\mathbf{C}}$ is (a) isomorphic to $\mathbf{P}^1_{\mathbf{C}}$ when t is a closed point not equal to 0, and (b) not isomorphic to $\mathbf{P}^1_{\mathbf{C}}$ when $t = 0$. We will call X_0 the *special fibre* of the morphism. This can be done in many, many ways. Try to give examples that satisfy (each of) the following additional restraints (unless it isn't possible):

(1) Can you do it with special fibre projective?
(2) Can you do it with special fibre irreducible and projective?
(3) Can you do it with special fibre integral and projective?
(4) Can you do it with special fibre smooth and projective?
(5) Can you do it with f a flat morphism? This just means that for every affine open $Spec(A) \subset X$ the induced ring map $\mathbf{C}[t] \to A$ is flat, which in this case means that any nonzero polynomial in t is a nonzero divisor on A.
(6) Can you do it with f a flat and projective morphism?
(7) Can you do it with f flat, projective and special fibre reduced?
(8) Can you do it with f flat, projective and special fibre irreducible?
(9) Can you do it with f flat, projective and special fibre integral?

What do you think happens when you replace $\mathbf{P}^1_{\mathbf{C}}$ with another variety over $\mathbf{C}$? (This can get very hard depending on which of the variants above you ask for.)

Exercise 65.30.5. Let $n \geq 1$ be any positive integer. Give an example of a surjective morphism $X \to \mathbf{P}^n_{\mathbf{C}}$ with X affine.

Exercise 65.30.6. Maps of Proj. Let R and S be graded rings. Suppose we have a ring map

$$\psi : R \to S$$

and an integer $e \geq 1$ such that $\psi(R_d) \subset S_{de}$ for all $d \geq 0$. (By our conventions this is not a homomorphism of graded rings, unless $e = 1$.)

(1) For which elements $\mathfrak{p} \in \text{Proj}(S)$ is there a well-defined corresponding point in $\text{Proj}(R)$? In other words, find a suitable open $U \subset \text{Proj}(S)$ such that ψ defines a continuous map $r_\psi : U \to \text{Proj}(R)$.
(2) Give an example where $U \neq \text{Proj}(S)$.
(3) Give an example where $U = \text{Proj}(S)$.

(4) (Do not write this down.) Convince yourself that the continuous map $U \to \mathrm{Proj}(R)$ comes canonically with a map on sheaves so that r_ψ is a morphism of schemes:
$$\mathrm{Proj}(S) \supset U \longrightarrow \mathrm{Proj}(R).$$
(5) What can you say about this map if $R = \bigoplus_{d \geq 0} S_{de}$ (as a graded ring with S_e, S_{2e}, etc in degree 1, 2, etc) and ψ is the inclusion mapping?

Notation. Let R be a graded ring as above and let $n \geq 0$ be an integer. Let $X = \mathrm{Proj}(R)$. Then there is a unique quasi-coherent $\mathcal{O}_X$-module $\mathcal{O}_X(n)$ on X such that for every homogeneous element $f \in R$ of positive degree we have $\mathcal{O}_X|_{D_+(f)}$ is the quasi-coherent sheaf associated to the $R_{(f)} = (R_f)_0$-module $(R_f)_n$ (=elements homogenous of degree n in $R_f = R[1/f]$). See Hartshorne, page 116+. Note that there are natural maps
$$\mathcal{O}_X(n_1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(n_2) \longrightarrow \mathcal{O}_X(n_1 + n_2)$$

Exercise 65.30.7. Pathologies in Proj. Give examples of R as above such that
(1) $\mathcal{O}_X(1)$ is not an invertible $\mathcal{O}_X$-module.
(2) $\mathcal{O}_X(1)$ is invertible, but the natural map $\mathcal{O}_X(1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(1) \to \mathcal{O}_X(2)$ is NOT an isomorphism.

Exercise 65.30.8. Let S be a graded ring. Let $X = \mathrm{Proj}(S)$. Show that any finite set of points of X is contained in a standard affine open.

Exercise 65.30.9. Let S be a graded ring. Let $X = \mathrm{Proj}(S)$. Let $Z, Z' \subset X$ be two closed subschemes. Let $\varphi : Z \to Z'$ be an isomorphism. Assume $Z \cap Z' = \emptyset$. Show that for any $z \in Z$ there exists an affine open $U \subset X$ such that $z \in U$, $\varphi(z) \in U$ and $\varphi(Z \cap U) = Z' \cap U$. (Hint: Use Exercise 65.30.8 and something akin to Schemes, Lemma 21.11.5.)

65.31. Morphisms from surfaces to curves

Exercise 65.31.1. Let R be a ring. Let $R \to k$ be a map from R to a field. Let $n \geq 0$. Show that
$$Mor_{Spec(R)}(Spec(k), \mathbf{P}^n_R) = (k^{n+1} \setminus \{0\})/k^*$$
where k^* acts via scalar multiplication on k^{n+1}. From now on we denote $(x_0 : \ldots : x_n)$ the morphism $Spec(k) \to \mathbf{P}^n_k$ corresponding to the equivalence class of the element $(x_0, \ldots, x_n) \in k^{n+1} \setminus \{0\}$.

Exercise 65.31.2. Let k be a field. Let $Z \subset \mathbf{P}^2_k$ be a irreducible closed subscheme. Show that either (a) Z is a closed point, or (b) there exists an homogeneous irreducible $F \in k[X_0, X_1, X_2]$ of degree > 0 such that $Z = V_+(F)$, or (c) $Z = \mathbf{P}^2_k$. (Hint: Look on a standard affine open.)

Exercise 65.31.3. Let k be a field. Let $Z_1, Z_2 \subset \mathbf{P}^2_k$ be irreducible closed subschemes of the form $V_+(F)$ for some homogeneous irreducible $F_i \in k[X_0, X_1, X_2]$ of degree > 0. Show that $Z_1 \cap Z_2$ is not empty. (Hint: Use dimension theory to estimate the dimension of the local ring of $k[X_0, X_1, X_2]/(F_1, F_2)$ at 0.)

Exercise 65.31.4. Show there does not exist a nonconstant morphism of schemes $\mathbf{P}^2_{\mathbf{C}} \to \mathbf{P}^1_{\mathbf{C}}$ over $Spec(\mathbf{C})$. Here a *constant morphism* is one whose image is a single point. (Hint: If the morphism is not constant consider the fibres over 0 and ∞ and argue that they have to meet to get a contradiction.)

Exercise 65.31.5. Let k be a field. Suppose that $X \subset \mathbf{P}^3_k$ is a closed subscheme given by a single homogeneous equation $F \in k[X_0, X_1, X_2, X_3]$. In other words,

$$X = \mathrm{Proj}(k[X_0, X_1, X_2, X_3]/(F)) \subset \mathbf{P}^3_k$$

as explained in the course. Assume that

$$F = X_0 G + X_1 H$$

for some homogeneous polynomials $G, H \in k[X_0, X_1, X_2, X_3]$ of positive degree. Show that if X_0, X_1, G, H have no common zeros then there exists a nonconstant morphism

$$X \longrightarrow \mathbf{P}^1_k$$

of schemes over $Spec(k)$ which on field points (see Exercise 65.31.1) looks like $(x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : x_1)$ whenever x_0 or x_1 is not zero.

65.32. Invertible sheaves

Definition 65.32.1. Let X be a locally ringed space. An *invertible $\mathcal{O}_X$-module* on X is a sheaf of $\mathcal{O}_X$-modules $\mathcal{L}$ such that every point has an open neighbourhood $U \subset X$ such that $\mathcal{L}|_U$ is isomorphic to $\mathcal{O}_U$ as $\mathcal{O}_U$-module. We say that $\mathcal{L}$ is trivial if it is isomorphic to $\mathcal{O}_X$ as a $\mathcal{O}_X$-module.

Exercise 65.32.2. General facts.

(1) Show that an invertible $\mathcal{O}_X$-module on a scheme X is quasi-coherent.
(2) Suppose $X \to Y$ is a morphism of ringed spaces, and $\mathcal{L}$ an invertible $\mathcal{O}_Y$-module. Show that $f^*\mathcal{L}$ is an invertible $\mathcal{O}_X$ module.

Exercise 65.32.3. Algebra.

(1) Show that an invertible $\mathcal{O}_X$-module on an affine scheme $Spec(A)$ corresponds to an A-module M which is (i) finite, (ii) projective, (iii) locally free of rank 1, and hence (iv) flat, and (v) finitely presented. (Feel free to quote things from last semesters course; or from algebra books.)
(2) Suppose that A is a domain and that M is a module as in (a). Show that M is isomorphic as an A-module to an ideal $I \subset A$ such that $IA_{\mathfrak{p}}$ is principal for every prime $\mathfrak{p}$.

Definition 65.32.4. Let R be a ring. An *invertible module* M is an R-module M such that $\widetilde{M}$ is an invertible sheaf on the spectrum of R. We say M is *trivial* if $M \cong R$ as an R-module.

In other words, M is invertible if and only if it satisfies all of the following conditions: it is flat, of finite presentation, projective, and locally free of rank 1. (Of course it suffices for it to be locally free of rank 1).

Exercise 65.32.5. Simple examples.

(1) Let k be a field. Let $A = k[x]$. Show that $X = Spec(A)$ has only trivial invertible $\mathcal{O}_X$-modules. In other words, show that every invertible A-module is free of rank 1.
(2) Let A be the ring

$$A = \{f \in k[x] \mid f(0) = f(1)\}.$$

Show there exists a nontrivial invertible A-module, unless $k = \mathbf{F}_2$. (Hint: Think about $Spec(A)$ as identifying 0 and 1 in $\mathbf{A}^1_k = Spec(k[x])$.)

(3) Same question as in (2) for the ring $A = k[x^2, x^3] \subset k[x]$ (except now $k = \mathbf{F}_2$ works as well).

Exercise 65.32.6. Higher dimensions.

(1) Prove that every invertible sheaf on two dimensional affine space is trivial. More precisely, let $\mathbf{A}^2_k = Spec(k[x, y])$ where k is a field. Show that every invertible sheaf on $\mathbf{A}^2_k$ is trivial. (Hint: One way to do this is to consider the corresponding module M, to look at $M \otimes_{k[x,y]} k(x)[y]$, and then use Exercise 65.32.5 (1) to find a generator for this; then you still have to think. Another way to is to use Exercise 65.32.3 and use what we know about ideals of the polynomial ring: primes of height one are generated by an irreducible polynomial; then you still have to think.)
(2) Prove that every invertible sheaf on any open subscheme of two dimensional affine space is trivial. More precisely, let $U \subset \mathbf{A}^2_k$ be an open subscheme where k is a field. Show that every invertible sheaf on U is trivial. Hint: Show that every invertible sheaf on U extends to one on $\mathbf{A}^2_k$. Not easy; but you can find it in Hartshorne.
(3) Find an example of a nontrivial invertible sheaf on a punctured cone over a field. More precisely, let k be a field and let $C = Spec(k[x, y, z]/(xy - z^2))$. Let $U = C \setminus \{(x, y, z)\}$. Find a nontrivial invertible sheaf on U. Hint: It may be easier to compute the group of isomorphism classes of invertible sheaves on U than to just find one. Note that U is covered by the opens $Spec(k[x, y, z, 1/x]/(xy - z^2))$ and $Spec(k[x, y, z, 1/y]/(xy - z^2))$ which are ``easy'' to deal with.

Definition 65.32.7. Let X be a locally ringed space. The *Picard group of X* is the set $\text{Pic}(X)$ of isomorphism classes of invertible $\mathcal{O}_X$-modules with addition given by tensor product. See Modules, Definition 15.21.6. For a ring R we set $\text{Pic}(R) = \text{Pic}(Spec(R))$.

Exercise 65.32.8. Let R be a ring.

(1) Show that if R is a Noetherian normal domain, then $\text{Pic}(R) = \text{Pic}(R[t])$. [Hint: There is a map $R[t] \to R$, $t \mapsto 0$ which is a left inverse to the map $R \to R[t]$. Hence it suffices to show that any invertible $R[t]$-module M such that $M/tM \cong R$ is free of rank 1. Let $K = f.f.(R)$. Pick a trivialization $K[t] \to M \otimes_{R[t]} K[t]$ which is possible by Exercise 65.32.5 (1). Adjust it so it agrees with the trivialization of M/tM above. Show that it is in fact a trivialization of M over $R[t]$ (this is where normality comes in).]
(2) Let k be a field. Show that $\text{Pic}(k[x^2, x^3, t]) \neq \text{Pic}(k[x^2, x^3])$.

65.33. Čech Cohomology

Exercise 65.33.1. Čech cohomology. Here k is a field.

(1) Let X be a scheme with an open covering $\mathcal{U} : X = U_1 \cup U_2$, with $U_1 = Spec(k[x])$, $U_2 = Spec(k[y])$ with $U_1 \cap U_2 = Spec(k[z, 1/z])$ and with open immersions $U_1 \cap U_2 \to U_1$ resp. $U_1 \cap U_2 \to U_2$ determined by $x \mapsto z$ resp. $y \mapsto z$ (and I really mean this). (We've seen in the lectures that such an X exists; it is the affine line zith zero doubled.) Compute $H^1(\mathcal{U}, \mathcal{O})$; eg. give a basis for it as a k-vectorspace.
(2) For each element in $H^1(\mathcal{U}, \mathcal{O})$ construct an exact sequence of sheaves of $\mathcal{O}_X$-modules

$$0 \to \mathcal{O}_X \to E \to \mathcal{O}_X \to 0$$

such that the boundary $\delta(1) \in H^1(\mathcal{U}, \mathcal{O})$ equals the given element. (Part of the problem is to make sense of this. See also below. It is also OK to show abstractly such a thing has to exist.)

Definition 65.33.2. (Definition of delta.) Suppose that

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

is a short exact sequence of abelian sheaves on any topological space X. The boundary map $\delta : H^0(X, \mathcal{F}_3) \to H^1(X, \mathcal{F}_1)$ is defined as follows. Take an element $\tau \in H^0(X, \mathcal{F}_3)$. Choose an open covering $\mathcal{U} : X = \bigcup_{i \in I} U_i$ such that for each i there exists a section $\tilde{\tau}_i \in \mathcal{F}_2$ lifting the restriction of τ to U_i. Then consider the assignment

$$(i_0, i_1) \longmapsto \tilde{\tau}_{i_0}|_{U_{i_0 i_1}} - \tilde{\tau}_{i_1}|_{U_{i_0 i_1}}.$$

This is clearly a 1-coboundary in the Čech complex $C^*(\mathcal{U}, \mathcal{F}_2)$. But we observe that (thinking of $\mathcal{F}_1$ as a subsheaf of $\mathcal{F}_2$) the RHS always is a section of $\mathcal{F}_1$ over $U_{i_0 i_1}$. Hence we see that the assignment defines a 1-cochain in the complex $C^*(\mathcal{U}, \mathcal{F}_2)$. The cohomology class of this 1-cochain is by definition $\delta(\tau)$.

65.34. Divisors

We collect all relevant definitions here in one spot for convenience.

Definition 65.34.1. Throughout, let S be any scheme and let X be a Noetherian, integral scheme.

(1) A *Weil divisor* on X is a formal linear combination $\Sigma n_i[Z_i]$ of prime divisors Z_i with integer coefficients.
(2) A *prime divisor* is a closed subscheme $Z \subset X$, which is integral with generic point $\xi \in Z$ such that $\mathcal{O}_{X,\xi}$ has dimension 1. We will use the notation $\mathcal{O}_{X,Z} = \mathcal{O}_{X,\xi}$ when $\xi \in Z \subset X$ is as above. Note that $\mathcal{O}_{X,Z} \subset K(X)$ is a subring of the function field of X.
(3) The *Weil divisor associated to a rational function* $f \in K(X)^*$ is the sum $\Sigma v_Z(f)[Z]$. Here $v_Z(f)$ is defined as follows
   (a) If $f \in \mathcal{O}_{X,Z}^*$ then $v_Z(f) = 0$.
   (b) If $f \in \mathcal{O}_{X,Z}$ then
   $$v_Z(f) = \text{length}_{\mathcal{O}_{X,Z}}(\mathcal{O}_{X,Z}/(f)).$$
   (c) If $f = \frac{a}{b}$ with $a, b \in \mathcal{O}_{X,Z}$ then
   $$v_Z(f) = \text{length}_{\mathcal{O}_{X,Z}}(\mathcal{O}_{X,Z}/(a)) - \text{length}_{\mathcal{O}_{X,Z}}(\mathcal{O}_{X,Z}/(b)).$$
(4) An *effective Cartier divisor* on a scheme S is a closed subscheme $D \subset S$ such that every point $d \in D$ has an affine open neighbourhood $Spec(A) = U \subset S$ in S so that $D \cap U = Spec(A/(f))$ with $f \in A$ a nonzero divisor.
(5) The *Weil divisor* $[D]$ *associated to an effective Cartier divisor* $D \subset X$ of our Noetherian integral scheme X is defined as the sum $\Sigma v_Z(D)[Z]$ where $v_Z(D)$ is defined as follows
   (a) If the generic point ξ of Z is not in D then $v_Z(D) = 0$.
   (b) If the generic point ξ of Z is in D then
   $$v_Z(D) = \text{length}_{\mathcal{O}_{X,Z}}(\mathcal{O}_{X,Z}/(f))$$
   where $f \in \mathcal{O}_{X,Z} = \mathcal{O}_{X,\xi}$ is the nonzero divisor which defines D in an affine neighbourhood of ξ (as in (4) above).

(6) Let S be a scheme. The *sheaf of total quotient rings* $\mathcal{K}_S$ is the sheaf of $\mathcal{O}_S$-algebras which is the sheafification of the pre-sheaf $\mathcal{K}'$ defined as follows. For $U \subset S$ open we set $\mathcal{K}'(U) = S_U^{-1}\mathcal{O}_S(U)$ where $S_U \subset \mathcal{O}_S(U)$ is the multiplicative subset consisting of sections $f \in \mathcal{O}_S(U)$ such that the germ of f in $\mathcal{O}_{S,u}$ is a nonzero divisor for every $u \in U$. In particular the elements of S_U are all nonzero divisors. Thus $\mathcal{O}_S$ is a subsheaf of $\mathcal{K}_S$, and we get a short exact sequence

$$0 \to \mathcal{O}_S^* \to \mathcal{K}_S^* \to \mathcal{K}_S^*/\mathcal{O}_S^* \to 0.$$

(7) A *Cartier divisor* on a scheme S is a global section of the quotient sheaf $\mathcal{K}_S^*/\mathcal{O}_S^*$.

(8) The *Weil divisor associated to a Cartier divisor* $\tau \in \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ over our Noetherian integral scheme X is the sum $\Sigma v_Z(\tau)[Z]$ where $v_Z(\tau)$ is defined as by the following recipe
   (a) If the germ of τ at the generic point ξ of Z is zero -- in other words the image of τ in the stalk $(\mathcal{K}^*/\mathcal{O}^*)_\xi$ is ``zero'' -- then $v_Z(\tau) = 0$.
   (b) Find an affine open neighbourhood $Spec(A) = U \subset X$ so that $\tau|_U$ is the image of a section $f \in \mathcal{K}(U)$ and moreover $f = a/b$ with $a, b \in A$. Then we set

$$v_Z(f) = \text{length}_{\mathcal{O}_{X,Z}}(\mathcal{O}_{X,Z}/(a)) - \text{length}_{\mathcal{O}_{X,Z}}(\mathcal{O}_{X,Z}/(b)).$$

Remarks 65.34.2. Here are some trivial remarks.

(1) On a Noetherian integral scheme X the sheaf $\mathcal{K}_X$ is constant with value the function field $K(X)$.
(2) To make sense out of the definitions above one needs to show that

$$\text{length}_{\mathcal{O}}(\mathcal{O}/(ab)) = \text{length}_{\mathcal{O}}(\mathcal{O}/(a)) + \text{length}_{\mathcal{O}}(\mathcal{O}/(b))$$

for any pair (a, b) of nonzero elements of a Noetherian 1-dimensional local domain $\mathcal{O}$. This will be done in the lectures.

Exercise 65.34.3. (On any scheme.) Describe how to assign a Cartier divisor to an effective Cartier divisor.

Exercise 65.34.4. (On an integral scheme.) Describe how to assign a Cartier divisor D to a rational function f such that the Weil divisor associated to D and to f agree. (This is silly.)

Exercise 65.34.5. Give an example of a Weil divisor on a variety which is not the Weil divisor associated to any Cartier divisor.

Exercise 65.34.6. Give an example of a Weil divisor D on a variety which is not the Weil divisor associated to any Cartier divisor but such that nD is the Weil divisor associated to a Cartier divisor for some $n > 1$.

Exercise 65.34.7. Give an example of a Weil divisor D on a variety which is not the Weil divisor associated to any Cartier divisor and such that nD is NOT the Weil divisor associated to a Cartier divisor for any $n > 1$. (Hint: Consider a cone, for example $X : xy - zw = 0$ in $\mathbf{A}_k^4$. Try to show that $D = [x = 0, z = 0]$ works.)

Exercise 65.34.8. On a separated scheme X of finite type over a field: Give an example of a Cartier divisor which is not the difference of two effective Cartier divisors. Hint: Find some X which does not have any nonempty effective Cartier Cartier divisors for example the scheme constructed in Harthorne, III Exercise 5.9. There is even an example with X a variety -- namely the variety of Exercise 65.34.9.

Exercise 65.34.9. Example of a nonprojective proper variety. Let k be a field. Let $L \subset \mathbf{P}^3_k$ be a line and let $C \subset \mathbf{P}^3_k$ be a nonsingular conic. Assume that $C \cap L = \emptyset$. Choose an isomorphism $\varphi : L \to C$. Let X be the k-variety obtained by glueing C to L via φ. In other words there is a surjective proper birational morphism

$$\pi : \mathbf{P}^3_k \longrightarrow X$$

and an open $U \subset X$ such that $\pi : \pi^{-1}(U) \to U$ is an isomorphism, $\pi^{-1}(U) = \mathbf{P}^3_k \setminus (L \cup C)$ and such that $\pi|_L = \pi|_C \circ \varphi$. (These conditions do not yet uniquely define X. In order to do this you need to specify the structure sheaf of X along points of $Z = X \setminus U$.) Show X exists, is a proper variety, but is not projective. (Hint: For existence use the result of Exercise 65.30.9. For non-projectivity use that $\text{Pic}(\mathbf{P}^3_k) = \mathbf{Z}$ to show that X cannot have an ample invertible sheaf.)

65.35. Differentials

Definitions and results. Kähler differentials.

(1) Let $R \to A$ be a ring map. The *module of Kähler differentials of A over R* is
$$\Omega_{A/R} = \bigoplus\nolimits_{a \in A} A \cdot da \Big/ \langle d(a_1a_2) - a_1da_2 - a_2da_1, dr\rangle.$$
The canonical universal R-derivation $d : A \to \Omega_{A/R}$ maps $a \mapsto da$.

(2) Consider the short exact sequence
$$0 \to I \to A \otimes_R A \to A \to 0$$
which defines the ideal I. There is a canonical derivation $d : A \to I/I^2$ which maps a to the class of $a \otimes 1 - 1 \otimes a$. This is another presentation of the module of derivations of A over R, in other words
$$(I/I^2, d) \cong (\Omega_{A/R}, d).$$

(3) For multiplicative subsets $S_R \subset R$ and $S_A \subset A$ such that S_R maps into S_A we have
$$\Omega_{S_A^{-1}A/S_R^{-1}R} = S_A^{-1}\Omega_{A/R}.$$

(4) If A is a finitely presented R-algebra then $\Omega_{A/R}$ is a finitely presented A-module. Hence in this case the *fitting* ideals of $\Omega_{A/R}$ are defined. (See exercise set 6 of last semester.)

(5) Let $f : X \to S$ be a morphism of schemes. There is a quasi-coherent sheaf of $\mathcal{O}_X$-modules $\Omega_{X/S}$ and a $\mathcal{O}_S$-linear derivation
$$d : \mathcal{O}_X \longrightarrow \Omega_{X/S}$$
such that for any affine opens $Spec(A) = U \subset X$, $Spec(R) = V \subset S$ with $f(U) \subset V$ we have
$$\Gamma(Spec(A), \Omega_{X/S}) = \Omega_{A/R}$$
compatibly with d.

Exercise 65.35.1. Let $k[\epsilon]$ be the ring of dual numbers over the field k, i.e., $\epsilon^2 = 0$.

(1) Consider the ring map
$$R = k[\epsilon] \to A = k[x, \epsilon]/(\epsilon x)$$
Show that the fitting ideals of $\Omega_{A/R}$ are (starting with the zeroth fitting ideal)
$$(\epsilon), A, A, \ldots$$

(2) Consider the map $R = k[t] \to A = k[x, y, t]/(x(y-t)(y-1), x(x-t))$. Show that the fitting ideals of of $\Omega_{A/R}$ in A are (assume characteristic k is zero for simplicity)

$$x(2x - t)(2y - t - 1)A,\ (x, y, t) \cap (x, y - 1, t),\ A,\ A, \ldots$$

So the 0-the fitting ideal is cut out by a single element of A, the 1st fitting ideal defines two closed points of $Spec(A)$, and the others are all trivial.

(3) Consider the map $R = k[t] \to A = k[x, y, t]/(xy-t^n)$. Compute the fitting ideals of $\Omega_{A/R}$.

Remark 65.35.2. The kth fitting ideal of $\Omega_{X/S}$ is commonly used to define the singular scheme of the morphism $X \to S$ when X has relative dimension k over S. But as part (a) shows, you have to be careful doing this when your family does not have ``constant'' fibre dimension, e.g., when it is not flat. As part (b) shows, flatness doesn't garantee it works either (and yes this is a flat family). In ``good cases'' -- such as in (c) -- for families of curves you expect the 0-th fitting ideal to be zero and the 1st fitting ideal to define (scheme-theoretically) the singular locus.

Exercise 65.35.3. Suppose that R is a ring and

$$A = k[x_1, \ldots , x_n]/(f_1, \ldots , f_n).$$

Note that we are assuming that A is presented by the same number of equations as variables. Thus the matrix of partial derivatives

$$(\partial f_i/\partial x_j)$$

is $n \times n$, i.e., a square matrix. Assume that its determinant is invertible as an element in A. Note that this is exactly the condition that says that $\Omega_{A/R} = (0)$ in this case of n-generators and n relations. Let $\pi : B' \to B$ be a surjection of R-algebras whose kernel J has square zero (as an ideal in B'). Let $\varphi : A \to B$ be a homomorphism of R-algebras. Show there exists a unique homomorphism of R-algebras $\varphi' : A \to B'$ such that $\varphi = \pi \circ \varphi'$.

Exercise 65.35.4. Find a generalization of the result of the previous exercise to the case where $A = R[x, y]/(f)$.

65.36. Schemes, Final Exam, Fall 2007

These were the questions in the final exam of a course on Schemes, in the Spring of 2007 at Columbia University.

Exercise 65.36.1. Definitions. Provide definitions of the following concepts.

(1) X is a *scheme*
(2) the morphism of schemes $f : X \to Y$ is *finite*
(3) the morphisms of schemes $f : X \to Y$ is *of finite type*
(4) the scheme X is *Noetherian*
(5) the $\mathcal{O}_X$-module $\mathcal{L}$ on the scheme X is *invertible*
(6) the *genus* of a nonsingular projective curve over an algebraically closed field

Exercise 65.36.2. Let $X = Spec(\mathbf{Z}[x, y])$, and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Suppose that $\mathcal{F}$ is zero when restricted to the standard affine open $D(x)$.

(1) Show that every global section s of $\mathcal{F}$ is killed by some power of x, i.e., $x^n s = 0$ for some $n \in \mathbf{N}$.
(2) Do you think the same is true if we do not assume that $\mathcal{F}$ is quasi-coherent?

Exercise 65.36.3. Suppose that $X \to Spec(R)$ is a proper morphism and that R is a discrete valuation ring with residue field k. Suppose that $X \times_{Spec(R)} Spec(k)$ is the empty scheme. Show that X is the empty scheme.

Exercise 65.36.4. Consider the projective[1] variety

$$\mathbf{P}^1 \times \mathbf{P}^1 = \mathbf{P}^1_{\mathbf{C}} \times_{Spec(\mathbf{C})} \mathbf{P}^1_{\mathbf{C}}$$

over the field of complex numbers $\mathbf{C}$. It is covered by four affine pieces, corresponding to pairs of standard affine pieces of $\mathbf{P}^1_{\mathbf{C}}$. For example, suppose we use homogenous coordinates X_0, X_1 on the first factor and Y_0, Y_1 on the second. Set $x = X_1/X_0$, and $y = Y_1/Y_0$. Then the 4 affine open pieces are the spectra of the rings

$$\mathbf{C}[x, y], \quad \mathbf{C}[x^{-1}, y], \quad \mathbf{C}[x, y^{-1}], \quad \mathbf{C}[x^{-1}, y^{-1}].$$

Let $X \subset \mathbf{P}^1 \times \mathbf{P}^1$ be the closed subscheme which is the closure of the closed subset of the first affine piece given by the equation

$$y^3(x^4 + 1) = x^4 - 1.$$

(1) Show that X is contained in the union of the first and the last of the 4 affine open pieces.
(2) Show that X is a nonsingular projective curve.
(3) Consider the morphism $pr_2 : X \to \mathbf{P}^1$ (projection onto the first factor). On the first affine piece it is the map $(x, y) \mapsto x$. Briefly explain why it has degree 3.
(4) Compute the ramification points and ramification indices for the map $pr_2 : X \to \mathbf{P}^1$.
(5) Compute the genus of X.

Exercise 65.36.5. Let $X \to Spec(\mathbf{Z})$ be a morphism of finite type. Suppose that there is an infinite number of primes p such that $X \times_{Spec(\mathbf{Z})} Spec(\mathbf{F}_p)$ is not empty.

(1) Show that $X \times_{Spec(\mathbf{Z})} Spec(\mathbf{Q})$ is not empty.
(2) Do you think the same is true if we replace the condition ``finite type" by the condition ``locally of finite type"?

65.37. Schemes, Final Exam, Spring 2009

These were the questions in the final exam of a course on Schemes, in the Spring of 2009 at Columbia University.

Exercise 65.37.1. Let X be a Noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf on X. Let $x \in X$ be a point. Assume that $\text{Supp}(\mathcal{F}) = \{x\}$.

(1) Show that x is a closed point of X.
(2) Show that $H^0(X, \mathcal{F})$ is not zero.
(3) Show that $\mathcal{F}$ is generated by global sections.
(4) Show that $H^p(X, \mathcal{F}) = 0$ for $p > 0$.

Remark 65.37.2. Let k be a field. Let $\mathbf{P}^2_k = \text{Proj}(k[X_0, X_1, X_2])$. Any invertible sheaf on $\mathbf{P}^2_k$ is isomorphic to $\mathcal{O}_{\mathbf{P}^2_k}(n)$ for some $n \in \mathbf{Z}$. Recall that

$$\Gamma(\mathbf{P}^2_k, \mathcal{O}_{\mathbf{P}^2_k}(n)) = k[X_0, X_1, X_2]_n$$

[1]The projective embedding is $((X_0, X_1), (Y_0, Y_1)) \mapsto (X_0Y_0, X_0Y_1, X_1Y_0, X_1Y_1)$ in other words $(x, y) \mapsto (1, y, x, xy)$.

is the degree n part of the polynomial ring. For a quasi-coherent sheaf $\mathcal{F}$ on $\mathbf{P}^2_k$ set $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_{\mathbf{P}^2_k}} \mathcal{O}_{\mathbf{P}^2_k}(n)$ as usual.

Exercise 65.37.3. Let k be a field. Let $\mathcal{E}$ be a vector bundle on $\mathbf{P}^2_k$, i.e., a finite locally free $\mathcal{O}_{\mathbf{P}^2_k}$-module. We say $\mathcal{E}$ is *split* if $\mathcal{E}$ is isomorphic to a direct sum invertible $\mathcal{O}_{\mathbf{P}^2_k}$-modules.

(1) Show that $\mathcal{E}$ is split if and only if $\mathcal{E}(n)$ is split.
(2) Show that if $\mathcal{E}$ is split then $H^1(\mathbf{P}^2_k, \mathcal{E}(n)) = 0$ for all $n \in \mathbf{Z}$.
(3) Let
$$\varphi : \mathcal{O}_{\mathbf{P}^2_k} \longrightarrow \mathcal{O}_{\mathbf{P}^2_k}(1) \oplus \mathcal{O}_{\mathbf{P}^2_k}(1) \oplus \mathcal{O}_{\mathbf{P}^2_k}(1)$$
be given by linear forms $L_0, L_1, L_2 \in \Gamma(\mathbf{P}^2_k, \mathcal{O}_{\mathbf{P}^2_k}(1))$. Assume $L_i \neq 0$ for some i. What is the condition on L_0, L_1, L_2 such that the cokernel of φ is a vector bundle? Why?
(4) Given an example of such a φ.
(5) Show that $\text{Coker}(\varphi)$ is not split (if it is a vector bundle).

Remark 65.37.4. Freely use the following facts on dimension theory (and add more if you need more).

(1) The dimension of a scheme is the supremum of the length of chains of irreducible closed subsets.
(2) The dimension of a finite type scheme over a field is the maximum of the dimensions of its affine opens.
(3) The dimension of a Noetherian scheme is the maximum of the dimensions of its irreducible components.
(4) The dimension of an affine scheme coincides with the dimension of the corresponding ring.
(5) Let k be a field and let A be a finite type k-algebra. If A is a domain, and $x \neq 0$, then $\dim(A) = \dim(A/xA) + 1$.

Exercise 65.37.5. Let k be a field. Let X be a projective, reduced scheme over k. Let $f : X \to \mathbf{P}^1_k$ be a morphism of schemes over k. Assume there exists an integer $d \geq 0$ such that for every point $t \in \mathbf{P}^1_k$ the fibre $X_t = f^{-1}(t)$ is irreducible of dimension d. (Recall that an irreducible space is not empty.)

(1) Show that $\dim(X) = d + 1$.
(2) Let $X_0 \subset X$ be an irreducible component of X of dimension $d+1$. Prove that for every $t \in \mathbf{P}^1_k$ the fibre $X_{0,t}$ has dimension d.
(3) What can you conclude about X_t and $X_{0,t}$ from the above?
(4) Show that X is irreducible.

Remark 65.37.6. Given a projective scheme X over a field k and a coherent sheaf $\mathcal{F}$ on X we set
$$\chi(X, \mathcal{F}) = \sum\nolimits_{i \geq 0} (-1)^i \dim_k H^i(X, \mathcal{F}).$$

Exercise 65.37.7. Let k be a field. Write $\mathbf{P}^3_k = \text{Proj}(k[X_0, X_1, X_2, X_3])$. Let $C \subset \mathbf{P}^3_k$ be a *type* (5, 6) *complete intersection curve*. This means that there exist $F \in k[X_0, X_1, X_2, X_3]_5$ and $G \in k[X_0, X_1, X_2, X_3]_6$ such that
$$C = \text{Proj}(k[X_0, X_1, X_2, X_3]/(F, G))$$
is a variety of dimension 1. (Variety implies reduced and irreducible, but feel free to assume C is nonsingular if you like.) Let $i : C \to \mathbf{P}^3_k$ be the corresponding closed immersion. Being

a complete intersection also implies that

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^3_k}(-11) \xrightarrow{\begin{pmatrix} -G \\ F \end{pmatrix}} \mathcal{O}_{\mathbf{P}^3_k}(-5) \oplus \mathcal{O}_{\mathbf{P}^3_k}(-6) \xrightarrow{(F,G)} \mathcal{O}_{\mathbf{P}^3_k} \longrightarrow i_*\mathcal{O}_C \longrightarrow 0$$

is an exact sequence of sheaves. Please use these facts to:

(1) compute $\chi(C, i^*\mathcal{O}_{\mathbf{P}^3_k}(n))$ for any $n \in \mathbf{Z}$, and
(2) compute the dimension of $H^1(C, \mathcal{O}_C)$.

Exercise 65.37.8. Let k be a field. Consider the rings

$$A = k[x, y]/(xy)$$
$$B = k[u, v]/(uv)$$
$$C = k[t, t^{-1}] \times k[s, s^{-1}]$$

and the k-algebra maps

$$\begin{array}{llll} A \longrightarrow C, & x \mapsto (t, 0), & y \mapsto (0, s) \\ B \longrightarrow C, & u \mapsto (t^{-1}, 0), & v \mapsto (0, s^{-1}) \end{array}$$

It is a true fact that these maps induce isomorphisms $A_{x+y} \to C$ and $B_{u+v} \to C$. Hence the maps $A \to C$ and $B \to C$ identify $Spec(C)$ with open subsets of $Spec(A)$ and $Spec(B)$. Let X be the scheme obtained by glueing $Spec(A)$ and $Spec(B)$ along $Spec(C)$:

$$X = Spec(A) \coprod\nolimits_{Spec(C)} Spec(B).$$

As we saw in the course such a scheme exists and there are affine opens $Spec(A) \subset X$ and $Spec(B) \subset X$ whose overlap is exactly $Spec(C)$ identified with an open of each of these using the maps above.

(1) Why is X separated?
(2) Why is X of finite type over k?
(3) Compute $H^1(X, \mathcal{O}_X)$, or what is its dimension?
(4) What is a more geometric way to describe X?

65.38. Schemes, Final Exam, Fall 2010

These were the questions in the final exam of a course on Schemes, in the Fall of 2010 at Columbia University.

Exercise 65.38.1. Definitions. Provide definitions of the following concepts.

(1) a separated scheme,
(2) a quasi-compact morphism of schemes,
(3) an affine morphism of schemes,
(4) a multiplicative subset of a ring,
(5) a Noetherian scheme,
(6) a variety.

Exercise 65.38.2. Prime avoidance.

(1) Let A be a ring. Let $I \subset A$ be an ideal and let $\mathfrak{q}_1$, $\mathfrak{q}_2$ be prime ideals such that $I \not\subset \mathfrak{q}_i$. Show that $I \not\subset \mathfrak{q}_1 \cup \mathfrak{q}_2$.
(2) What is a geometric interpretation of (1)?
(3) Let $X = \text{Proj}(S)$ for some graded ring S. Let $x_1, x_2 \in X$. Show that there exists a standard open $D_+(F)$ which contains both x_1 and x_2.

Exercise 65.38.3. Why is a composition of affine morphisms affine?

Exercise 65.38.4. Examples. Give examples of the following:

(1) A reducible projective scheme over a field k.
(2) A scheme with 100 points.
(3) A non-affine morphism of schemes.

Exercise 65.38.5. Chevalley's theorem and the Hilbert Nullstellensatz.

(1) Let $\mathfrak{p} \subset \mathbf{Z}[x_1, \ldots , x_n]$ be a maximal ideal. What does Chevalley's theorem imply about $\mathfrak{p} \cap \mathbf{Z}$?
(2) In turn, what does the Hilbert Nullstellensatz imply about $\kappa(\mathfrak{p})$?

Exercise 65.38.6. Let A be a ring. Let $S = A[X]$ as a graded A-algebra where X has degree 1. Show that $\text{Proj}(S) \cong Spec(A)$ as schemes over A.

Exercise 65.38.7. Let $A \to B$ be a finite ring map. Show that $Spec(B)$ is a H-projective scheme over $Spec(A)$.

Exercise 65.38.8. Give an example of a scheme X over a field k such that X is irreducible and such that for some finite extension $k \subset k$ the base change $X_{k'} = X \times_{Spec(k)} Spec(k')$ is connected but reducible.

65.39. Schemes, Final Exam, Spring 2011

These were the questions in the final exam of a course on Schemes, in the Spring of 2011 at Columbia University.

Exercise 65.39.1. Definitions. Provide definitions of the italicized concepts.

(1) a *separated* scheme,
(2) a *universally closed* morphism of schemes,
(3) *A dominates B* for local rings A, B contained in a common field,
(4) the *dimension* of a scheme X,
(5) the *codimension* of an irreducible closed subscheme Y of a scheme X,

Exercise 65.39.2. Results. State something formally equivalent to the fact discussed in the course.

(1) The valuative criterion of properness for a morphism $X \to Y$ of varieties for example.
(2) The relationship between $\dim(X)$ and the function field $k(X)$ of X for a variety X over a field k.
(3) Fill in the blank: The category of nonsingular projective curves over k and non-constant morphisms is anti-equivalent to … … ….
(4) Noether normalization.
(5) Jacobian criterion.

Exercise 65.39.3. Let k be a field. Let $F \in k[X_0, X_1, X_2]$ be a homogeneous form of degree d. Assume that $C = V_+(F) \subset \mathbf{P}^2_k$ is a smooth curve over k. Denote $i : C \to \mathbf{P}^2_k$ the corresponding closed immersion.

(1) Show that there is a short exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^2_k}(-d) \to \mathcal{O}_{\mathbf{P}^2_k} \to i_*\mathcal{O}_C \to 0$$

of coherent sheaves on $\mathbf{P}^2_k$: tell me what the maps are and briefly why it is exact.

(2) Conclude that $H^0(C, \mathcal{O}_C) = k$.
(3) Compute the genus of C.
(4) Assume now that $P = (0 : 0 : 1)$ is not on C. Prove that $\pi : C \to \mathbf{P}^1_k$ given by $(a_0 : a_1 : a_2) \mapsto (a_0 : a_1)$ has degree d.
(5) Assume k is algebraically closed, assume all ramification indices (the ``e_i") are 1 or 2, and assume the characteristic of k is not equal to 2. How many ramification points does $\pi : C \to \mathbf{P}^1_k$ have?
(6) In terms of F, what do you think is a set of equations of the set of ramification points of π?
(7) Can you guess K_C?

Exercise 65.39.4. Let k be a field. Let X be a ``triangle" over k, i.e., you get X by glueing three copies of $\mathbf{A}^1_k$ to each other by identifying 0 on the first copy to 1 on the second copy, 0 on the second copy to 1 on the first copy, and 0 on the third copy to 1 on the first copy. It turns out that X is isomorphic to $Spec(k[x, y]/(xy(x+y+1)))$; feel free to use this. Compute the Picard group of X.

Exercise 65.39.5. Let k be a field. Let $\pi : X \to Y$ be a finite birational morphism of curves with X a projective nonsingular curve over k. It follows from the material in the course that Y is a proper curve and that π is the normalization morphism of Y. We have also seen in the course that there exists a dense open $V \subset Y$ such that $U = \pi^{-1}(V)$ is a dense open in X and $\pi : U \to V$ is an isomorphism.

(1) Show that there exists an effective Cartier divisor $D \subset X$ such that $D \subset U$ and such that $\mathcal{O}_X(D)$ is ample on X.
(2) Let D be as in (1). Show that $E = \pi(D)$ is an effective Cartier divisor on Y.
(3) Briefly indicate why
    (a) the map $\mathcal{O}_Y \to \pi_*\mathcal{O}_X$ has a coherent cokernel Q which is supported in $Y \setminus V$, and
    (b) for every n there is a corresponding map $\mathcal{O}_Y(nE) \to \pi_*\mathcal{O}_X(nD)$ whose cokernel is isomorphic to Q.
(4) Show that $\dim_k H^0(X, \mathcal{O}_X(nD)) - \dim_k H^0(Y, \mathcal{O}_Y(nE))$ is bounded (by what?) and conclude that the invertible sheaf $\mathcal{O}_Y(nE)$ has lots of sections for large n (why?).

65.40. Schemes, Final Exam, Fall 2011

These were the questions in the final exam of a course on Commutative Algebra, in the Fall of 2011 at Columbia University.

Exercise 65.40.1. Definitions. Provide definitions of the italicized concepts.

(1) a *Noetherian* ring,
(2) a *Noetherian* scheme,
(3) a *finite* ring homomorphism,
(4) a *finite* morphism of schemes,
(5) the *dimension* of a ring.

Exercise 65.40.2. Results. State something formally equivalent to the fact discussed in the course.

(1) Zariski's Main Theorem.
(2) Noether normalization.
(3) Chinese remainder theorem.
(4) Going up for finite ring maps.

Exercise 65.40.3. Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring whose residue field has characteristic not 2. Suppose that $\mathfrak{m}$ is generated by three elements x, y, z and that $x^2 + y^2 + z^2 = 0$ in A.

(1) What are the possible values of $\dim(A)$?
(2) Give an example to show that each value is possible.
(3) Show that A is a domain if $\dim(A) = 2$. (Hint: look at $\bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$.)

Exercise 65.40.4. Let A be a ring. Let $S \subset T \subset A$ be multiplicative subsets. Assume that

$$\{\mathfrak{q} \mid \mathfrak{q} \cap S = \emptyset\} = \{\mathfrak{q} \mid \mathfrak{q} \cap T = \emptyset\}.$$

Show that $S^{-1}A \to T^{-1}A$ is an isomorphism.

Exercise 65.40.5. Let k be an algebraically closed field. Let

$$V_0 = \{A \in \text{Mat}(3 \times 3, k) \mid \text{rank}(A) = 1\} \subset \text{Mat}(3 \times 3, k) = k^9.$$

(1) Show that V_0 is the set of closed points of a (Zariski) locally closed subset $V \subset \mathbf{A}^9_k$.
(2) Is V irreducible?
(3) What is $\dim(V)$?

Exercise 65.40.6. Prove that the ideal (x^2, xy, y^2) in $\mathbf{C}[x, y]$ cannot be generated by 2 elements.

Exercise 65.40.7. Let $f \in \mathbf{C}[x, y]$ be a nonconstant polynomial. Show that for some $\alpha, \beta \in \mathbf{C}$ the $\mathbf{C}$-algebra map

$$\mathbf{C}[t] \longrightarrow \mathbf{C}[x, y]/(f), \quad t \longmapsto \alpha x + \beta y$$

is finite.

Exercise 65.40.8. Show that given finitely many points $p_1, \ldots, p_n \in \mathbf{C}^2$ the scheme $\mathbf{A}^2_{\mathbf{C}} \setminus \{p_1, \ldots, p_n\}$ is a union of two affine opens.

Exercise 65.40.9. Show that there exists a surjective morphism of schemes $\mathbf{A}^1_{\mathbf{C}} \to \mathbf{P}^1_{\mathbf{C}}$. (Surjective just means surjective on underlying sets of points.)

Exercise 65.40.10. Let k be an algebraically closed field. Let $A \subset B$ be an extension of domains which are both finite type k-algebras. Prove that the image of $\text{Spec}(B) \to \text{Spec}(A)$ contains a nonempty open subset of $\text{Spec}(A)$ using the following steps:

(1) Prove it if $A \to B$ is also finite.
(2) Prove it in case the fraction field of B is a finite extension of the fraction field of A.
(3) Reduce the statement to the previous case.

65.41. Other chapters

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Commutative Algebra
(8) Brauer Groups
(9) Sites and Sheaves
(10) Homological Algebra
(11) Derived Categories
(12) More on Algebra
(13) Smoothing Ring Maps
(14) Simplicial Methods
(15) Sheaves of Modules
(16) Modules on Sites

(17) Injectives
(18) Cohomology of Sheaves
(19) Cohomology on Sites
(20) Hypercoverings
(21) Schemes
(22) Constructions of Schemes
(23) Properties of Schemes
(24) Morphisms of Schemes
(25) Coherent Cohomology
(26) Divisors
(27) Limits of Schemes
(28) Varieties
(29) Chow Homology
(30) Topologies on Schemes
(31) Descent
(32) Adequate Modules
(33) More on Morphisms
(34) More on Flatness
(35) Groupoid Schemes
(36) More on Groupoid Schemes
(37) Étale Morphisms of Schemes
(38) Étale Cohomology
(39) Crystalline Cohomology
(40) Algebraic Spaces
(41) Properties of Algebraic Spaces
(42) Morphisms of Algebraic Spaces
(43) Decent Algebraic Spaces
(44) Topologies on Algebraic Spaces
(45) Descent and Algebraic Spaces
(46) More on Morphisms of Spaces
(47) Quot and Hilbert Spaces
(48) Spaces over Fields
(49) Cohomology of Algebraic Spaces
(50) Stacks
(51) Formal Deformation Theory
(52) Groupoids in Algebraic Spaces
(53) More on Groupoids in Spaces
(54) Bootstrap
(55) Examples of Stacks
(56) Quotients of Groupoids
(57) Algebraic Stacks
(58) Sheaves on Algebraic Stacks
(59) Criteria for Representability
(60) Properties of Algebraic Stacks
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CHAPTER 66

A Guide to the Literature

66.1. Short introductory articles

- Barbara Fantechi: *Stacks for Everybody* [**Fan01**]
- Dan Edidin: *What is a stack?* [**Edi03**]
- Dan Edidin: *Notes on the construction of the moduli space of curves* [**Edi00**]
- Angelo Vistoli: *Intersection theory on algebraic stacks and on their moduli spaces*, and especially the appendix. [**Vis89**]

66.2. Classic references

- Mumford: *Picard groups of moduli problems* [**Mum65**]
  Mumford never uses the term ``stack'' here but the concept is implicit in the paper; he computes the picard group of the moduli stack of elliptic curves.
- Deligne, Mumford: *The irreducibility of the space of curves of given genus* [**DM69b**]
  This influential paper introduces ``algebraic stacks'' in the sense which are now universally called Deligne-Mumford stacks (stacks with representable diagonal which admit étale presentations by schemes). There are many foundational results *without proof*. The paper uses stacks to give two proofs of the irreducibility of the moduli space of curves of genus g.
- Artin: *Versal deformations and algebraic stacks* [**Art74b**]
  This paper introduces ``algebraic stacks'' which generalize Deligne-Mumford stacks and are now commonly referred to as *Artin stacks*, stacks with representable diagonal which admit smooth presentations by schemes. This paper gives deformation-theoretic criterion known as Artin's criterion which allows one to prove that a given moduli stack is an Artin stack without explicitly exhibiting a presentation.

66.3. Books and online notes

- Laumon, Moret-Bailly: *Champs Algébriques* [**LMB00b**]
  This book is currently the most exhaustive reference on stacks containing many foundational results. It assumes the reader is familiar with algebraic spaces and frequently references Knutson's book [**Knu71b**]. There is an error in chapter 12 concerning the functoriality of the lisse-étale site of an algebraic stack. One doesn't need to worry about this as the error has been patched by Martin Olsson (see [**Ols07b**]) and the results in the remaining chapters (after perhaps slight modification) are correct.
- The Stacks Project Authors: *Stacks Project* [**Aut**].
  You are reading it!
- Anton Geraschenko: *Lecture notes for Martin Olsson's class on stacks* [**Ols07a**]

This course systematically develops the theory of algebraic spaces before introducing algebraic stacks (first defined in Lecture 27!). In addition to basic properties, the course covers the equivalence between being Deligne-Mumford and having unramified diagonal, the lisse-étale site on an Artin stack, the theory of quasi-coherent sheaves, the Keel-Mori theorem, cohomological descent, and gerbes (and their relation to the Brauer group). There are also some exercises.

- Behrend, Conrad, Edidin, Fantechi, Fulton, Göttsche, and Kresch: *Algebraic stacks*, online notes for a book being currently written [**BCE$^+$**]
  The aim of this book is to give a friendly introduction to stacks without assuming a sophisticated background with a focus on examples and applications. Unlike [**LMB00b**], it is not assumed that the reader has digested the theory of algebraic spaces. Instead, Deligne-Mumford stacks are introduced with algebraic spaces being a special case with part of the goal being to develop enough theory to prove the assertions in [**DM69b**]. The general theory of Artin stacks is to be developed in the second part. Only a fraction of the book is now available on Kresch's website.

66.4. Related references on foundations of stacks

- Vistoli: *Notes on Grothendieck topologies, fibered categories and descent theory* [**Vis05**]
  Contains useful facts on fibered categories, stacks and descent theory in the fpqc topology as well as rigorous proofs.
- Knutson: *Algebraic Spaces* [**Knu71b**]
  This book, which evolved from his PhD thesis under Michael Artin, contains the foundations of the theory of algebraic spaces. The book [**LMB00b**] frequently references this text. See also Artin's papers on algebraic spaces: [**Art69b**], [**Art69d**], [**Art69f**], [**Art70b**], [**Art71d**], [**Art71b**], [**Art73b**], and [**Art74b**]
- Grothendieck et al, *Théorie des Topos et Cohomologie Étale des Schémas I, II, III* also known as SGA4 [**MA71**]
  Volume 1 contains many general facts on universes, sites and fibered categories. The word ``champ'' (French for ``stack'') appears in Deligne's Exposé XVIII.
- Jean Giraud: *Cohomologie non abélienne* [**Gir65**]
  The book discusses fibered categories, stacks, torsors and gerbes over general sites but does not discuss algebraic stacks. For instance, if G is a sheaf of abelian groups on X, then in the same way $H^1(X, G)$ can be identified with G-torsors, $H^2(X, G)$ can be identified with an appropriately defined set of G-gerbes. When G is not abelian, then $H^2(X, G)$ is defined as the set of G-gerbes.
- Kelly and Street: *Review of the elements of 2-categories* [**KS74**]
  The category of stacks form a 2-category although a simple type of 2-category where are 2-morphisms are invertible. This is a reference on general 2-categories. I have never used this so I cannot say how useful it is. Also note that [**Aut**] contains some basics on 2-categories.

66.5. Papers in the literature

Below is a list of research papers which contain fundamental results on stacks and algebraic spaces. The intention of the summaries is to indicate only the results of the paper which contribute toward stack theory; in many cases these results are subsidiary to the main goals of the paper. We divide the papers into categories with some papers falling into multiple categories.

66.5.1. Deformation theory and algebraic stacks. The first three papers by Artin do not contain anything on stacks but they contain powerful results with the first two papers being essential for [**Art74b**].

- Artin: *Algebraic approximation of structures over complete local rings* [**Art69b**]
  It is proved that under mild hypotheses any effective formal deformation can be approximated: if $F : (Sch/S) \to (Sets)$ is a contravariant functor locally of finite presentation with S finite type over a field or excellent DVR, $s \in S$, and $\hat{\xi} \in F(\hat{\mathcal{O}}_{S,s})$ is an effective formal deformation, then for any $n > 0$, there exists an residually trivial étale neighborhood $(S', s') \to (S, s)$ and $\xi' \in F(S')$ such that ξ' and $\hat{\xi}$ agree up to order n (ie. have the same restriction in $F(\mathcal{O}_{S,s}/\mathfrak{m}^n)$).
- Artin: *Algebraization of formal moduli I* [**Art69d**]
  It is proved that under mild hypotheses any effective formal versal deformation is algebraizable. Let $F : (Sch/S) \to (Sets)$ be a contravariant functor locally of finite presentation with S finite type over a field or excellent DVR, $s \in S$ be a locally closed point , $\hat{A}$ be a complete noetherian local $\mathcal{O}_S$-algebra with residue field k' a finite extension of $k(s)$, and $\hat{\xi} \in F(\hat{A})$ be an effective formal versal deformation of an element $\xi_0 \in F(k')$. Then there is a scheme X finite type over S and a closed point $x \in X$ with residue field $k(x) = k'$ and an element $\xi \in F(X)$ such that there is an isomorphism $\hat{\mathcal{O}}_{X,x} \cong \hat{A}$ identifying the restrictions of ξ and $\hat{\xi}$ in each $F(\hat{A}/\mathfrak{m}^n)$. The algebraization is unique if $\hat{\xi}$ is a universal deformation. Applications are given to the representability of the Hilbert and Picard schemes.
- Artin: *Algebraization of formal moduli. II* [**Art70b**]
  Vaguely, it is shown that if one can contract a closed subset $Y' \subseteq X'$ formally locally around Y', then exists a global morphism $X' \to X$ contracting Y with X an algebraic space.
- Artin: *Versal deformations and algebraic stacks* [**Art74b**]
  This momentous paper builds on his work in [**Art69b**] and [**Art69d**]. This paper introduces Artin's criterion which allows one to prove algebraicity of a stack by verifying deformation-theoretic properties. More precisely (but not very precisely), Artin constructs a presentation of a limit preserving stack $\mathcal{X}$ locally around a point $x \in \mathcal{X}(k)$ as follows: assuming the stack $\mathcal{X}$ satisfies Schlessinger's criterion([**Sch68**]), there exists a formal versal deformation $\hat{\xi} \in lim\, \mathcal{X}(\hat{A}/\mathfrak{m}^n)$ of x. Assuming that formal deformations are effective (i.e., $\mathcal{X}(\hat{A}) \to lim\, \mathcal{X}(\hat{A}/\mathfrak{m}^n)$ is bijective), then one obtains an effective formal versal deformation $\xi \in \mathcal{X}(\hat{A})$. Using results in [**Art69d**], one produces a finite type scheme U and an element $\xi_U : U \to \mathcal{X}$ which is formally versal at a point $u \in U$ over x. Then if we assume $\mathcal{X}$ admits a deformation and obstruction theory satisfying certain conditions (ie. compatibility with étale localization and completion as well as constructibility condition), then it is shown in section 4 that formal versality is an open condition so that after shrinking U, $U \to \mathcal{X}$ is smooth. Artin also presents a proof that any stack admitting an fppf presentation by a scheme admits a smooth presentation by a scheme so that in particular one can form quotient stacks by flat, separated, finitely presented group schemes.
- Conrad, de Jong: *Approximation of Versal Deformations* [**CdJ02b**]

This paper offers an approach to Artin's algebraization result by applying Popescu's powerful result: if A is a noetherian ring and B a noetherian A-algebra, then the map $A \to B$ is a regular morphism if and only if B is a direct limit of smooth A-algebras. It is not hard to see that Popescu's result implies Artin's approximation over an arbitrary excellent scheme (the excellence hypothesis implies that for a local ring A, the map $A^{\mathrm{h}} \to \hat{A}$ from the henselization to the completion is regular). The paper uses Popescu's result to give a ``groupoid'' generalization of the main theorem in [**Art69d**] which is valid over arbitrary excellent base schemes and for arbitrary points $s \in S$. In particular, the results in [**Art74b**] hold under an arbitrary excellent base. They discuss the étale-local uniqueness of the algebraization and whether the automorphism group of the object acts naturally on the henselization of the algebraization.

- Jason Starr: *Artin's axioms, composition, and moduli spaces* [**Sta06**]
  The paper establishes that Artin's axioms for algebraization are compatible with the composition of 1-morphisms.
- Martin Olsson: *Deformation theory of representable morphism of algebraic stacks* [**Ols06a**]
  This generalizes standard deformation theory results for morphisms of schemes to representable morphisms of algebraic stacks in terms of the cotangent complex. These results cannot be viewed as consequences of Illusie's general theory as the cotangent complex of a representable morphism $X \to \mathcal{X}$ is not defined in terms of cotangent complex of a morphism of ringed topoi (because the lisse-étale site is not functorial).

66.5.2. Coarse moduli spaces.

- Keel, Mori: *Quotients in Groupoids* [**KM97b**]
  It had apparently long been ``folklore'' that separated Deligne-Mumford stacks admitted coarse moduli spaces. A rigorous (although terse) proof of the following theorem is presented here: if $\mathcal{X}$ is an Artin stack locally of finite type over a noetherian base scheme such that the inertia stack $I_{\mathcal{X}} \to \mathcal{X}$ is finite, then there exists a coarse moduli space $\phi : \mathcal{X} \to Y$ with ϕ separated and Y an algebraic space locally of finite type over S. The hypothesis that the inertia is finite is precisely the right condition: there exists a coarse moduli space $\phi : \mathcal{X} \to Y$ with ϕ separated if and only if the inertia is finite.
- Conrad: *The Keel-Mori Theorem via Stacks* [**Con05**]
  Keel and Mori's paper [**KM97b**] is written in the groupoid language and some find it challenging to grasp. Brian Conrad presents a stack-theoretic version of the proof which is quite transparent although it uses the sophisticated language of stacks. Conrad also removes the noetherian hypothesis.
- Rydh: *Existence of quotients by finite groups and coarse moduli spaces* [**Ryd07a**]
  Rydh removes the hypothesis from [**KM97b**] and [**Con05**] that $\mathcal{X}$ be finitely presented over some base.
- Abramovich, Olsson, Vistoli: *Tame stacks in positive characteristic* [**AOV08**]
  They define a *tame Artin stack* as an Artin stack with finite inertia such that if $\phi : \mathcal{X} \to Y$ is the coarse moduli space, ϕ_* is exact on quasi-coherent sheaves. They prove that for an Artin stack with finite inertia, the following are equivalent: $\mathcal{X}$ is tame $\iff$ the stabilizers of $\mathcal{X}$ are linearly reductive $\iff$ $\mathcal{X}$ is étale locally on the coarse moduli space a quotient of an affine scheme by a linearly reductive group scheme. For a tame Artin stack, the coarse moduli space is particularly

nice. For instance, the coarse moduli space commutes with arbitrary base change while a general coarse moduli space for an Artin stack with finite inertia will only commute with flat base change.

- Alper: *Good moduli spaces for Artin stacks* [**Alp08**]
  For general Artin stacks with infinite affine stabilizer groups (which are necessarily non-separated), coarse moduli spaces often do not exist. The simplest example is $[\mathbf{A}^1/\mathbf{G}_m]$. It is defined here that a quasi-compact morphism $\phi : \mathcal{X} \to Y$ is a *good moduli space* if $\mathcal{O}_Y \to \phi_*\mathcal{O}_\mathcal{X}$ is an isomorphism and ϕ_* is exact on quasi-coherent sheaves. This notion generalizes a tame Artin stack in [**AOV08**] as well as encapsulates Mumford's geometric invariant theory: if G is a reductive group acting linearly on $X \subseteq \mathbf{P}^n$, then the morphism from the quotient stack of the semi-stable locus to the GIT quotient $[X^{ss}/G] \to X//G$ is a good moduli space. The notion of a good moduli space has many nice geometric properties: (1) ϕ is surjective, universally closed, and universally submersive, (2) ϕ identifies points in Y with points in $\mathcal{X}$ up to closure equivalence, (3) ϕ is universal for maps to algebraic spaces, (4) good moduli spaces are stable under arbitrary base change, and (5) a vector bundle on an Artin stack descends to the good moduli space if and only if the representations are trivial at closed points.

66.5.3. Intersection theory.

- Vistoli: *Intersection theory on algebraic stacks and on their moduli spaces* [**Vis89**]
  This paper develops the foundations for intersection theory with rational coefficients for Deligne-Mumford stacks. If $\mathcal{X}$ is a separated Deligne-Mumford stack, the chow group $A_*(\mathcal{X})$ with rational coefficients is defined as the free abelian group of integral closed substacks of dimension k up to rational equivalence. There is a flat pullback, a proper push-forward and a generalized Gysin homomorphism for regular local embeddings. If $\phi : \mathcal{X} \to Y$ is a moduli space (ie. a proper morphism with is bijective on geometric points), there is an induced push-forward $A_*(\mathcal{X}) \to A_k(Y)$ which is an isomorphism.
- Edidin, Graham: *Equivariant Intersection Theory* [**EG98**]
  The purpose of this article is to develop intersection theory with integral coefficients for a quotient stack $[X/G]$ of an action of an algebraic group G on an algebraic space X or, in other words, to develop a G-equivariant intersection theory of X. Equivariant chow groups defined using only invariant cycles does not produce a theory with nice properties. Instead, generalizing Totaro's definition in the case of BG and motivated by the fact that if $V \to X$ is a vector bundle then $A_i(X) \cong A_i(V)$ naturally, the authors define $A_i^G(X)$ as follows: Let $\dim(X) = n$ and $\dim(G) = g$. For each i, choose a l-dimensional G-representation V where G acts freely on an open subset $U \subseteq V$ whose complement as codimension $d > n - i$. So $X_G = [X \times U/G]$ is an algebraic space (it can even be chosen to be a scheme). Then they define $A_i^G(X) = A_{i+l-g}(X_G)$. For the quotient stack, one defines $A_i([X/G]) := A_{i+g}^G(X) = A_{i+l}(X_G)$. In particular, $A_i([X/G]) = 0$ for $i > \dim[X/G] = n - g$ but can be non-zero for $i << 0$ (eg. $A_i(B\mathbf{G}_m) = \mathbf{Z}$ for $i \leq 0$). They establish that these equivariant Chow groups enjoy the same functorial properties as ordinary Chow groups. Furthermore, they establish that if $[X/G] \cong [Y/H]$ that $A_i([X/G]) = A_i([Y/H])$ so that the definition is independent on how the stack is presented as a quotient stack.
- Kresch: *Cycle Groups for Artin Stacks* [**Kre99**]

Kresch defines Chow groups for arbitrary Artin stacks agreeing with Edidin and Graham's definition in [**EG98**] in the case of quotient stack. For algebraic stacks with affine stabilizer groups, the theory satisfies the usual properties.

- Behrend and Fantechi: *The intrinsic normal cone* [**BF97**]
  Generalizing a construction due to Li and Tian, Behrend and Fantechi construct a virtual fundamental class for a Deligne-Mumford stack.

66.5.4. Quotient stacks. Quotient stacks[1] form a very important subclass of Artin stacks which include almost all moduli stacks studied by algebraic geometers. The geometry of a quotient stack $[X/G]$ is the G-equivariant geometry of X. It is often easier to show properties are true for quotient stacks and some results are only known to be true for quotient stacks. The following papers address: When is an algebraic stack a global quotient stack? Is an algebraic stack ``locally'' a quotient stack?

- Laumon, Moret-Bailly: [**LMB00b**, Chapter 6]
  Chapter 6 contains several facts about the local and global structure of algebraic stacks. It is proved that an algebraic stack $\mathcal{X}$ over S is a quotient stack $[Y/G]$ with Y an algebraic space (resp. scheme, resp. affine scheme) and G a finite group if and only if there exists an algebraic space (resp. scheme, resp. affine scheme) Y' and an finite étale morphism $Y' \to \mathcal{X}$. It is shown that any Deligne-Mumford stack over S and $x : Spec(K) \to \mathcal{X}$ admits an representable, étale and separated morphism $\phi : [X/G] \to \mathcal{X}$ where G is a finite group acting on an affine scheme over S such that $Spec(K) = [X/G] \times_{\mathcal{X}} Spec(K)$. The existence of presentations with geometrically connected fibers is also discussed in detail.
- Edidin, Hassett, Kresch, Vistoli: *Brauer Groups and Quotient stacks* [**EHKV01**]
  First, they establish some fundamental (although not very difficult) facts concerning when a given algebraic stack (always assumed finite type over a noetherian scheme in this paper) is a quotient stack. For an algebraic stack $\mathcal{X}$: $\mathcal{X}$ is a quotient stack $\iff$ there exists a vector bundle $V \to \mathcal{X}$ such that for every geometric point, the stabilizer acts faithfully on the fiber $\iff$ there exists a vector bundle $V \to \mathcal{X}$ and a locally closed substack $V^0 \subseteq V$ such that V^0 is representable and surjects onto F. They establish that an algebraic stack is a quotient stack if there exists finite flat cover by an algebraic space. Any smooth Deligne-Mumford stack with generically trivial stabilizer is a quotient stack. They show that a $\mathbf{G}_m$-gerbe over a noetherian scheme X corresponding to $\beta \in H^2(X, \mathbf{G}_m)$ is a quotient stack if and only if β is in the image of the Brauer map $\mathrm{Br}(X) \to \mathrm{Br}'(X)$. They use this to produce a non-separated Deligne-Mumford stack that is not a quotient stack.
- Totaro: *The resolution property for schemes and stacks* [**Tot04**]
  A stack has the resolution property if every coherent sheaf is the quotient of a vector bundle. The first main theorem is that if $\mathcal{X}$ is a normal noetherian algebraic stack with affine stabilizer groups at closed points, then the following are equivalent: (1) $\mathcal{X}$ has the resolution property and (2) $\mathcal{X} = [Y/\mathrm{GL}_n]$ with Y quasi-affine. In the case $\mathcal{X}$ is finite type over a field, then (1) and (2) are equivalent to: (3) $\mathcal{X} = [Spec(A)/G]$ with G an affine group scheme finite type over k. The implication that quotient stacks have the resolution property was proven by Thomason. The second main theorem is that if $\mathcal{X}$ is a smooth Deligne-Mumford stack over a

[1] In the literature, *quotient stack* often means a stack of the form $[X/G]$ with X an algebraic space and G a subgroup scheme of GL_n rather than an arbitrary flat group scheme.

field which has a finite and generically trivial stabilizer group $I_{\mathcal{X}} \to \mathcal{X}$ and whose coarse moduli space is a scheme with affine diagonal, then $\mathcal{X}$ has the resolution property. Another cool result states that if $\mathcal{X}$ is a noetherian algebraic stack satisfying the resolution property, then $\mathcal{X}$ has affine diagonal if and only if the closed points have affine stabilizer.

- Kresch: *On the Geometry of Deligne-Mumford Stacks* [**Kre09**]
  This article summarizes general structure results of Deligne-Mumford stacks (of finite type over a field) and contains some interesting results concerning quotient stacks. It is shown that any smooth, separated, generically tame Deligne-Mumford stack with quasi-projective coarse moduli space is a quotient stack $[Y/G]$ with Y quasi-projective and G an algebraic group. If $\mathcal{X}$ is a Deligne-Mumford stack whose coarse moduli space is a scheme, then $\mathcal{X}$ is Zariski-locally a quotient stack if and only if it admits a Zariski-open covering by stack quotients of schemes by finite groups. If $\mathcal{X}$ is a Deligne-Mumford stack proper over a field of characteristic 0 with coarse moduli space Y, then: Y is projective and $\mathcal{X}$ is a quotient stack $\iff$ Y is projective and $\mathcal{X}$ possesses a generating sheaf $\iff$ $\mathcal{X}$ admits a closed embedding into a smooth proper DM stack with projective coarse moduli space. This motivates a definition that a Deligne-Mumford stack is *projective* if there exists a closed embedding into a smooth, proper Deligne-Mumford stack with projective coarse moduli space.
- Kresch, Vistoli *On coverings of Deligne-Mumford stacks and surjectivity of the Brauer map* [**KV04**]
  It is shown that in characteristic 0 and for a fixed n, the following two statements are equivalent: (1) every smooth Deligne-Mumford stack of dimension n is a quotient stack and (2) the Azumaya Brauer group coincides with the cohomological Brauer group for smooth schemes of dimension n.
- Kresch: *Cycle Groups for Artin Stacks* [**Kre99**]
  It is shown that a reduced Artin stack finite type over a field with affine stabilizer groups admits a stratification by quotient stacks.
- Abramovich-Vistoli: *Compactifying the space of stable maps* [**AV02**]
  Lemma 2.2.3 establishes that for any separated Deligne-Mumford stack is étale-locally on the coarse moduli space a quotient stack $[U/G]$ where U affine and G a finite group. [**Ols06b**, Theorem 2.12] shows in this argument G is even the stabilizer group.
- Abramovich, Olsson, Vistoli: *Tame stacks in positive characteristic* [**AOV08**]
  This paper shows that a tame Artin stack is étale locally on the coarse moduli space a quotient stack of an affine by the stabilizer group.
- Alper: *On the local quotient structure of Artin stacks* [**Alp09**]
  It is conjectured that for an Artin stack $\mathcal{X}$ and a closed point $x \in \mathcal{X}$ with linearly reductive stabilizer, then there is an étale morphism $[V/G_x] \to \mathcal{X}$ with V an algebraic space. Some evidence for this conjecture is given. A simple deformation theory argument (based on ideas in [**AOV08**]) shows that it is true formally locally. A stack-theoretic proof of Luna's étale slice theorem is presented proving that for stacks $\mathcal{X} = [Spec(A)/G]$ with G linearly reductive, then étale locally on the GIT quotient $Spec(A^G)$, $\mathcal{X}$ is a quotient stack by the stabilizer.

66.5.5. Cohomology.

- Olsson: *Sheaves on Artin stacks* [**Ols07b**]

This paper develops the theory of quasi-coherent and constructible sheaves proving basic cohomological properties. This paper corrects a mistake in [**LMB00b**] in the functorirality of the lisse-étale site. The cotangent complex is constructed. In addition, the following theorems are proved: Grothendieck's Fundamental Theorem for proper morphisms, Grothendieck's Existence Theorem, Zariski's Connectness Theorem and finiteness theorem for proper pushoforwards of coherent and constructible sheaves.

- Behrend: *Derived l-adic categories for algebraic stacks* [**Beh03**]
  Proves the Lefschetz trace formula for algebraic stacks.
- Behrend: *Cohomology of stacks* [**Beh04**]
  Defines the de Rham cohomology for differentiable stacks and singular cohomology for topological stacks.
- Faltings: *Finiteness of coherent cohomology for proper fppf stacks* [**Fal03**]
  Proves coherence for direct images of coherent sheaves for proper morphisms.
- Abramovich, Corti, Vistoli: *Twisted bundles and admissible covers* [**ACV03**]
  The appendix contains the proper base change theorem for étale cohomology for tame Deligne-Mumford stacks.

66.5.6. Existence of finite covers by schemes. The existence of finite covers of Deligne-Mumford stacks by schemes is an important result. In intersection theory on Deligne-Mumford stacks, it is an essential ingredient in defining proper push-forward for non-representable morphisms. There are several results about $\overline{\mathscr{M}}_g$ relying on the existence of a finite cover by a *smooth* scheme which was proven by Looijenga. Perhaps the first result in this direction is [**Ses72**, Theorem 6.1] which treats the equivariant setting.

- Vistoli: *Intersection theory on algebraic stacks and on their moduli spaces* [**Vis89**]
  If $\mathcal{X}$ is a Deligne-Mumford stack with a moduli space (ie. a proper morphism which is bijective on geometric points), then there exists a finite morphism $X \to \mathcal{X}$ from a scheme X.
- Laumon, Moret-Bailly: [**LMB00b**, Chapter 16]
  As an application of Zariski's main theorem, Theorem 16.6 establishes: if $\mathcal{X}$ is a Deligne-Mumford stack finite type over a noetherian scheme, then there exists a finite, surjective, generically étale morphism $Z \to \mathcal{X}$ with Z a scheme. It is also shown in Corollary 16.6.2 that any noetherian normal algebraic space is isomorphic to the algebraic space quotient X'/G for a finite group G acting a normal scheme X.
- Edidin, Hassett, Kresch, Vistoli: *Brauer Groups and Quotient stacks* [**EHKV01**]
  Theorem 2.7 states: if $\mathcal{X}$ is an algebraic stack of finite type over a noetherian ground scheme S, then the diagonal $\mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is quasi-finite if and only if there exists a finite surjective morphism $X \to F$ from a scheme X.
- Kresch, Vistoli: *On coverings of Deligne-Mumford stacks and surjectivity of the Brauer map* [**KV04**]
  It is proved here that any smooth, separated Deligne-Mumford stack finite type over a field with quasi-projective coarse moduli space admits a finite, flat cover by a smooth quasi-projective scheme.
- Olsson: *On proper coverings of Artin stacks* [**Ols05**]
  Proves that if $\mathcal{X}$ is an Artin stack separated and finite type over S, then there exists a proper surjective morphism $X \to \mathcal{X}$ from a scheme X quasi-projective over S.

As an application, Olsson proves coherence and constructibility of direct image sheaves under proper morphisms. As an application, he proves Grothendieck's existence theorem for proper Artin stacks.

66.5.7. Rigidification. Rigidification is a process for removing a flat subgroup from the inertia. For example, if X is a projective variety, the morphism from the Picard stack to the Picard scheme is a rigidification of the group of automorphism $\mathbf{G}_m$.

- Abramovich, Corti, Vistoli: *Twisted bundles and admissible covers* [**ACV03**]
  Let $\mathcal{X}$ be an algebraic stack over S and H be a flat, finitely presented separated group scheme over S. Assume that for every object $\xi \in \mathcal{X}(T)$ there is an embedding $H(S) \hookrightarrow \mathrm{Aut}_{\mathcal{X}(T)}(\xi)$ which is compatible under pullbacks in the sense that for every arrow $\phi : \xi \to \xi'$ over $f : T \to T'$ and $g \in H(T)$, $g \circ \phi = \phi \circ f^* g$. Then there exists an algebraic stack $\mathcal{X}/H$ and a morphism $\rho : \mathcal{X} \to \mathcal{X}/H$ which is an fppf gerbe such that for every $\xi \in \mathcal{X}(T)$, the morphism $\mathrm{Aut}_{\mathcal{X}(T)}(\xi) \to \mathrm{Aut}_{\mathcal{X}/H(T)}(\xi)$ is surjective with kernel $H(T)$.
- Romagny: *Group actions on stacks and applications* [**Rom05**]
  Discusses how group actions behave with respect to rigidifications.
- Abramovich, Graber, Visolit: *Gromov-Witten theory for Deligne-Mumford stacks* [**AGV08**]
  The appendix gives a summary of rigidification as in [**ACV03**] with two alternative interpretations. This paper also contains constructions for gluing algebraic stacks along closed substacks and for taking roots of line bundles.
- Abramovich, Olsson, Vistoli: *Tame stacks in positive characteristic* ([**AOV08**])
  The appendix handles the more complicated situation where the flat subgroup stack of the inertia $H \subseteq I_{\mathcal{X}}$ is normal but not necessarily central.

66.5.8. Stacky curves.

- Abramovich, Vistoli: *Compactifying the space of stable maps* [**AV02**]
  This paper introduces *twisted curves*. If one defines the moduli space of stable maps into a stacks using stable curves, the result is not compact. By using maps from twisted curves, the authors define a proper moduli stack.
- Behrend, Noohi: *Uniformization of Deligne-Mumford curves* [**BN06**]
  Proves a uniformization theorem of Deligne-Mumford analytic curves.

66.5.9. Hilbert, Quot, Hom and branchvariety stacks.

- Vistoli: *The Hilbert stack and the theory of moduli of families* [**Vis91**]
  If $\mathcal{X}$ is a algebraic stack separated and locally of finite type over a locally noetherian and locally separated algebraic space S, Vistoli defines the Hilbert stack $\mathscr{H}\mathrm{ilb}(\mathscr{F}/S)$ parameterizing finite and unramifed morphisms from proper schemes. It is claimed without proof that $\mathscr{H}\mathrm{ilb}(\mathscr{F}/S)$ is an algebraic stack. As a consequence, it is proved that with $\mathcal{X}$ as above, the Hom stack $\mathscr{H}\mathrm{om}_S(T, \mathcal{X})$ is an algebraic stack if T is proper and flat over S.
- Olsson, Starr: *Quot functors for Deligne-Mumford stacks* [**OS03b**]
  If $\mathcal{X}$ is a Deligne-Mumford stack separated and locally of finite presentation over an algebraic space S and $\mathscr{F}$ is a locally finitely-presented $\mathcal{O}_{\mathcal{X}}$-module, the quot functor $\mathrm{Quot}(\mathscr{F}/\mathcal{X}/S)$ is represented by an algebraic space separated and locally of finite presentation over S. This paper also defines generating sheaves and proves existence of a generating sheaf for tame, separated Deligne-Mumford stacks which are global quotient stacks of a scheme by a finite group.

- Olsson: *Hom-stacks and Restrictions of Scalars* [**Ols06b**]
  Suppose $\mathcal{X}$ and $\mathcal{Y}$ are Artin stacks locally of finite presentation over an algebraic space S with finite diagonal with $\mathcal{X}$ proper and flat over S such that fppf-locally on S, $\mathcal{X}$ admits a finite finitely presented flat cover by an algebraic space (eg. $\mathcal{X}$ is Deligne-Mumford or a tame Artin stack). Then $Hom_S(\mathcal{X}, \mathcal{Y})$ is an Artin stack locally of finite presentation over S.
- Alexeev and Knutson: *Complete moduli spaces of branchvarieties* ([**AK06**])
  They define a branchvariety of $\mathbf{P}^n$ as a finite morphism $X \to \mathbf{P}^n$ from a *reduced* scheme X. They prove that the moduli stack of branchvarieties with fixed Hilbert polynomial and total degrees of i-dimensional components is a proper Artin stack with finite stabilizer. They compare the stack of branchvarieties with the Hilbert scheme, Chow scheme and moduli space of stable maps.
- Lieblich: *Remarks on the stack of coherent algebras* [**Lie06**]
  This paper constructs a generalization of Alexeev and Knutson's stack of branchvarieties over a scheme Y by building the stack as a stack of algebras over the structure sheaf of Y. Existence proofs of Quot and *Hom* spaces are given.
- Starr: *Artin's axioms, composition, and moduli spaces* [**Sta06**]
  As an application of the main result, a common generalization of Vistoli's Hilbert stack [**Vis91**] and Alexeev and Knutson's stack of branchvarieties [**AK06**] is provided. If $\mathcal{X}$ is an algebraic stack locally of finite type over an an excellent scheme S with finite diagonal, then the stack $\mathcal{H}$ parameterizing morphisms $g : T \to \mathcal{X}$ from a proper algebraic space T with a G-ample line bundle L is an Artin stack locally of finite type over S.
- Lundkvist and Skjelnes: *Non-effective deformations of Grothendieck's Hilbert functor* [**LS08**]
  Shows that the Hilbert functor of a non-separated scheme is not represented since there are non-effective deformations.

66.5.10. Toric stacks. Toric stacks provide a great class of examples and a natural testing ground for conjectures due to the dictionary between the geometry of a toric stack and the combinatorics of its stacky fan in a similar way that toric varieties provide examples and counterexamples in scheme theory.

- Borisov, Chen and Smith: *The orbifold Chow ring of toric Deligne-Mumford stacks* [**BCS05**]
  Inspired by Cox's construction for toric varieties, this paper defines smooth toric DM stacks as explicit quotient stacks associated to a combinatorial object called a *stacky fan*.
- Iwanari: *The category of toric stacks* [**Iwa09**]
  This paper defines a *toric triple* as a smooth Deligne-Mumford stack $\mathcal{X}$ with an open immersion $\mathbf{G}_m \hookrightarrow \mathcal{X}$ with dense image (and therefore $\mathcal{X}$ is an orbifold) and an action $\mathcal{X} \times \mathbf{G}_m \to \mathcal{X}$. It is shown that there is an equivalence between the 2-category of toric triples and the 1-category of stacky fans. The relationship between toric triples and the definition of smooth toric DM stacks in [**BCS05**] is discussed.
- Iwanari: *Integral Chow rings for toric stacks* [**Iwa07**]
  Generalizes Cox's Δ-collections for toric varieties to toric orbifolds.
- Perroni: *A note on toric Deligne-Mumford stacks* [**Per08**]
  Generalizes Cox's Δ-collections and Iwanari's paper [**Iwa07**] to general smooth toric DM stacks.

- Fantechi, Mann, and Nironi: *Smooth toric DM stacks* [**BF07**]
  This paper defines a smooth toric DM stack as a smooth DM stack $\mathcal{X}$ with the action of a DM torus $\mathcal{T}$ (ie. a Picard stack isomorphic to $T \times BG$ with G finite) having an open dense orbit isomorphic to $\mathcal{T}$. They give a ``bottom-up description" and prove an equivalence between smooth toric DM stacks and stacky fans.

66.5.11. Theorem on formal functions and Grothendieck's Existence Theorem. These papers give generalizations of the theorem on formal functions [**DG67**, III.4.1.5] (sometimes referred to Grothendieck's Fundamental Theorem for proper morphisms) and Grothendieck's Existence Theorem [**DG67**, III.5.1.4].

- Knutson: *Algebraic spaces* [**Knu71b**, Chapter V]
  Generalizes these theorems to algebraic spaces.
- Abramovich-Vistoli: *Compactifying the space of stable maps* [**AV02**, A.1.1]
  Generalizes these theorems to tame Deligne-Mumford stacks
- Olsson and Starr: *Quot functors for Deligne-Mumford stacks* [**OS03b**]
  Generalizes these theorems to separated Deligne-Mumford stacks.
- Olsson: *On proper coverings of Artin stacks* [**Ols05**]
  Provides a generalization to proper Artin stacks.
- Conrad: *Formal GAGA on Artin stacks* [**Con**]
  Provides a generalization to proper Artin stacks and proves a formal GAGA theorem.
- Olsson: *Sheaves on Artin stacks* [**Ols07b**]
  Provides another proof for the generalization to proper Artin stacks.

66.5.12. Group actions on stacks. Actions of groups on algebraic stacks naturally appear. For instance, symmetric group S_n acts on $\overline{\mathcal{M}}_{g,n}$ and for an action of a group G on a scheme X, the normalizer of G in $\text{Aut}(X)$ acts on $[X/G]$. Furthermore, torus actions on stacks often appear in Gromov-Witten theory.

- Romagny: *Group actions on stacks and applications* [**Rom05**]
  This paper makes precise what it means for a group to act on an algebraic stack and proves existence of fixed points as well as existence of quotients for actions of group schemes on algebraic stacks. See also Romagny's earlier note [**Rom03**].

66.5.13. Taking roots of line bundles. This useful construction was discovered independently by Cadman and by Abramovich, Graber and Vistoli. Given a scheme X with an effective Cartier divisor D, the rth root stack is an Artin stack branched over X at D with a μ_r stabilizer over D and scheme-like away from D.

- Charles Cadman *Using Stacks to Impose Tangency Conditions on Curves* [**Cad07**]
- Abramovich, Graber, Vistoli: *Gromov-Witten theory for Deligne-Mumford stacks* [**AGV08**]

66.5.14. Other papers.

- Lieblich: *Moduli of twisted sheaves* [**Lie07**]
  This paper contains a summary of gerbes and twisted sheaves. If $\mathcal{X} \to X$ is a μ_n-gerbe with X a projective relative surface with smooth connected geometric fibers, it is shown that the stack of semistable $\mathcal{X}$-twisted sheaves is an Artin stack locally of finite presentation over S. This paper also develops the theory of associated points and purity of sheaves on Artin stacks.
- Lieblich, Osserman: *Functorial reconstruction theorem for stacks* [**LO08**]

Proves some surprising and interesting results on when an algebraic stack can be reconstructed from its associated functor.

- David Rydh: *Noetherian approximation of algebraic spaces and stacks* [**Ryd08**]
  This paper shows that every quasi-compact algebraic stack with quasi-finite diagonal can be approximated by a noetherian stack. There are applications to removing the noetherian hypothesis in results of Chevalley, Serre, Zariski and Chow.

66.6. Stacks in other fields

- Behrend and Noohi: *Uniformization of Deligne-Mumford curves* [**BN06**]
  Gives an overview and comparison of topological, analytic and algebraic stacks.
- Behrang Noohi: *Foundations of topological stacks I* [**Noo05**]
- David Metzler: *Topological and smooth stacks* [**Met05**]

66.7. Higher stacks

- Lurie: *Higher topos theory* [**Lur09f**]
- Lurie: *Derived Algebraic Geometry I - V* [**Lur09a**], [**Lur09b**], [**Lur09c**], [**Lur09d**], [**Lur09e**]
- Toën: *Higher and derived stacks: a global overview* [**Toë09**]
- Toën and Vezzosi: *Homotopical algebraic geometry I, II* [**TV05**], [**TV08**]

66.8. Other chapters

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CHAPTER 67

Desirables

67.1. Introduction

This is basically just a list of things that we want to put in the stacks project. As we add material to the project continuously this is always somewhat behind the current state of the project.

67.2. Conventions

We should have a chapter with a short list of conventions used in the document. This chapter already exists, see Conventions, Section 2.1, but a lot more could be added there. Especially useful would be to find ``hidden'' conventions and tacit assumptions and put those there.

67.3. Sites and Topoi

We have a chapter on sites and sheaves, see Sites, Section 9.1. We have a chapter on ringed sites (and topoi) and modules on them, see Modules on Sites, Section 16.1. We have a chapter on cohomology in this setting, see Cohomology on Sites, Section 19.1. But a lot more could be added, especially in the chapter on cohomology.

67.4. Stacks

We have a chapter on (abstract) stacks, see Stacks, Section 50.1. It would be nice if

(1) improve the discussion on ``stackyfication'',
(2) give examples of stackyfication,
(3) more examples in general,
(4) a discussion of gerbes is missing sofar.

Example result: Given a sheaf of abelian groups $\mathcal{F}$ over $\mathcal{C}$ the set of equivalence classes of gerbes with ``group'' $\mathcal{F}$ is bijective to $H^2(\mathcal{C}, \mathcal{F})$.

67.5. Simplicial methods

We have a chapter on simplicial methods, see Simplicial, Section 14.1. This has to be reviewed and improved. Moreover, there should be a chapter on ``simplicial algebraic geometry'', where we discuss simplicial schemes, and how to think of their geometry, cohomology, etc. Then this should be tied into the chapter on hypercoverings to ``explain'' the results of this chapter in the new language.

67.6. Cohomology of schemes

There is already a chapter on cohomology of quasi-coherent sheaves, see Coherent, Section 25.1. What is missing are chapters on étale cohomology and flat cohomology of schemes, the relation with Galois cohomology etc.

67.7. Deformation theory a la Schlessinger

What is needed is a discussion of Schlessinger's paper first and foremost. It would be nice to discuss this a tiny bit more generally than in Schlessinger's paper, but it is easy to fix this up later also. For example we could discuss what happens if you have automorphisms (e.g., functor in groupoids). After all this is usually why you have hulls and not actual prorepresentability.

67.8. Definition of algebraic stacks

An algebraic stack is a stack that has a diagonal representable by algebraic spaces, that is the target of a surjective smooth morphism from a scheme.

The notion ``Deligne-Mumford stack'' will be reserved for a stack as in [**DM69a**]. We will reserve the term ``Artin stack'' for a stack such as in the papers by Artin [**Art69c**], and [**Art74a**]. (See also [**CdJ02a**].) In other words, and Artin stack will be an algebraic stack with some reasonable finiteness and separatedness conditions.

67.9. Examples of schemes, algebraic spaces, algebraic stacks

It really is not that hard to show that $\mathcal{M}_g$ is an algebraic stack for $g \geq 2$. We should have $[X/G]$ here. We should really have a long list of moduli problems here and prove they are all algebraic stacks. (Some of them we can postpone the proof until after Artin approximation.) For example the Kontsevich moduli space in characteristic $p > 0$.

Here are some items for the list of moduli problems mentioned above.

(1) $\mathcal{M}_g$, i.e., moduli of smooth projective curves of genus g,
(2) $\overline{\mathcal{M}}_g$, i.e., moduli of stable genus g curves,
(3) $\mathcal{A}_g$, i.e., principally polarised abelian schemes of genus g,
(4) $\mathcal{M}_{1,1}$, i.e., 1-pointed smooth projective genus 1 curves,
(5) $\mathcal{M}_{g,n}$, i.e., smooth projective genus g-curves with n pairwise distinct labeled points,
(6) $\overline{\mathcal{M}}_{g,n}$, i.e., stable n-pointed nodal projective genus g-curves,
(7) $\mathcal{H}om_S(\mathcal{X}, \mathcal{Y})$, moduli of morphisms (with suitable conditions on the stacks $\mathcal{X}$, $\mathcal{Y}$ and the base scheme S),
(8) $Bun_G(X) = \mathcal{H}om_S(X, BG)$, the stack of G-bundles of the geometric Langlands programme (with suitable conditions on the scheme X, the group scheme G, and the base scheme S),
(9) $Pic_{\mathcal{X}/S}$, i.e., the Picard stack associated to an algebraic stack over a base scheme (or space).

How about the algebraic space you get from the deformation theory of a general surface in $\mathbf{P}^3$ with a node? (I mean where you deform it to a general smooth surface in $\mathbf{P}^3$.)

Perhaps we can talk about some small dimensional examples here too. For example the stack where you have $\mathbf{A}^1$ with a $B(\mathbf{Z}/2)$ sitting at 0. Bugeyed covers. You name it.

67.10. Properties of algebraic stacks

Such as the various ways of defining what a proper algebraic stack is. Of course these things are really properties of morphisms of stacks.

We can define singularities (up to smooth factors) etc. Prove that a connected normal stack is irreducible, etc.

67.11. Lisse étale site of an algebraic stack

This has to be explained and introduced. Explain it is not functorial with respect to 1-morphisms of algebraic stacks. Define étale cohomology of an algebraic stack with coefficients in a sheaf on the lisse-étale site. Prove a Leray spectral sequence exists (always?). Explain about cohomology of quasi-coherent sheaves.

67.12. Things you always wanted to know but were afraid to ask

There are going to be lots of lemmas that you use over and over again that are useful but aren't really mentioned specifically in the literature, or it isn't easy to find references for. Bag of tricks.

Example: Given two groupoids in schemes $R \Rightarrow U$ and $R' \Rightarrow U'$ what does it mean to have a 1-morphism $[U/R] \to [U'/R']$ purely in terms of groupoids in schemes. (This is bad because surely this is in the lit somewhere.) More anybody?

67.13. Quasi-coherent sheaves on stacks

Define them and explain how you get them. You can define them as living on the lisse-étale site or on all of the stack and show the two notions are equivalent. Cohomology of quasi-coherent sheaves.

67.14. Flat and smooth

Artin's theorem that having a flat surjection from a scheme is a replacement for the smooth surjective condition.

67.15. Artin's representability theorem

Title is clear enough. Perhaps we can reformulate the condition of having a deformation theory a little to adapt it more to the examples we know about, especially those where there is a perfect obstruction theory (discussions with Jason)?

67.16. DM stacks are finitely covered by schemes

This all begins with Gabber's lemma I think. Somewhere in Asterisque about Faltings proof of Mordell?

67.17. Martin Olson's paper on properness

This proves two notions of proper are the same. We can also discuss Faltings result that it suffices to use DVR's in certain cases.

67.18. Proper pushforward of coherent sheaves

No comments yet.

67.19. Keel and Mori

See [**KM97a**]. The steps in this article also give a good way of looking at what an algebraic stack locally looks like.

67.20. Add more here

Please.

67.21. Other chapters

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CHAPTER 68

Coding Style

68.1. List of style comments

These will be changed over time, but having some here now will hopefully encourage a consistent LaTeX style. We will call ``code[1]'' the contents of the source files.

(1) Keep all lines in all tex files to at most 80 characters.
(2) Do not use indentation in the tex file. Use syntax highlighting in your editor, instead of indentation, to visualize environments, etc.
(3) Use

```
\medskip\noindent
```

to start a new paragraph, and use

```
\noindent
```

to start a new paragraph just after an environment.
(4) Do not break the code for mathematical formulas across lines if possible. If the complete code complete with enclosing dollar signs does not fit on the line, then start with the first dollar sign on the first character of the next line. If it still does not fit, find a mathematically reasonable spot to break the code.
(5) Displayed math equations should be coded as follows

```
$$
...
...
$$
```

In other words, start with a double dollar sign on a line by itself and end similarly.
(6) *Do not use any macros*. Rationale: This makes it easier to read the tex file, and start editing an arbitrary part without having to learn inumerable macros. And it doesn't make it harder or more timeconsuming to write. Of course the disadvantage is that the same mathematical object may be TeXed differently in different places in the text, but this should be easy to spot.
(7) The theorem environments we use are: ``theorem'', ``proposition'', ``lemma'' (plain), ``definition'', ``example'', ``exercise'', ``situation'' (definition), ``remark'', ``remarks'' (remark). Of course there is also a ``proof'' environment.
(8) An environment ``foo'' should be coded as follows

```
\begin{foo}
...
...
\end{foo}
```

similarly to the way displayed equations are coded.

[1] It is all Knuth's fault. See [**Knu79**].

(9) Instead of a ``corollary", just use ``lemma" environment since likely the result will be used to prove the next bigger theorem anyway.
(10) Directly following each lemma, proposition, or theorem is the proof of said lemma, proposition, or theorem. No nested proofs please.
(11) The files preamble.tex, chapters.tex and fdl.tex are special tex files. Apart from these, each tex file has the following structure

```
\input{preamble}
\begin{document}
\title{Title}
\maketitle
\tableofcontents
...
...
\input{chapters}
\bibliography{my}
\bibliographystyle{amsalpha}
\end{document}
```

(12) Try to add labels to lemmas, propositions, theorems, and even remarks, exercise, and other environments. If labelling a lemma use something like

```
\begin{lemma}
\label{lemma-bar}
...
\end{lemma}
```

Similarly for all other environments. In other words, the label of a environment named ``foo" starts with ``foo-". In addition to this please make all labels consist only of lower case letters, digits, and the symbol ``-".
(13) Never refer to ``the lemma above" (or proposition, etc). Instead use:

```
Lemma \ref{lemma-bar} above
```

This means that later moving lemmas around is basically harmless.
(14) Cross-file referencing. To reference a lemma labeled ``lemma-bar" in the file foo.tex which has title ``Foo", please use the following code

```
Foo, Lemma \ref{foo-lemma-bar}
```

If this does not work, then take a look at the file preamble.tex to find the correct expression to use. This will produce the ``Foo, Lemma <link>" in the output file so it will be clear that the link points out of the file.
(15) If at all possible avoid forward references in proof environments. (It should be possible to write an automated test for this.)
(16) Do not start any sentence with a mathematical symbol.
(17) Do not have a sentence of the type ``This follows from the following" just before a lemma, proposition, or theorem. Every sentence ends with a period.
(18) State all hypotheses in each lemma, proposition, theorem. This makes it easier for readers to see if a given lemma, proposition, or theorem applies to their particular problem.
(19) Keep proofs short; less than 1 page in pdf or dvi. You can always achieve this by splitting out the proof in lemmas etc.
(20) In a defining property foobar use

```
{\it foobar}
```

in the code inside the definition evironment. Similarly if the definition occurs in the text of the document. This will make it easier for the reader to see what it is that is being defined.

(21) Put any definition that will be used outside the section it is in, in its own definition evironment. Temporary definitions may be made in the text. A tricky case is that of mathematical constructions (which are often definitions involving interrelated lemmas). Maybe a good solution is to have them in their own short section so users can refer to the section instead of a definition.

(22) Do not number equations unless they are actually being referenced somewhere in the text. We can always add labels later.

(23) In statements of lemmas, propositions and theorems and in proofs keep the sentences short. For example, instead of ``Let $R$ be a ring and let $M$ be an $R$-module." write ``Let R be a ring. Let M be an R-module.". Rationale: This makes it easier to parse the trickier parts of proofs and statements.

(24) Use the

```
\section
```

command to make sections, but try to avoid using subsections and subsubsections.

(25) Avoid using complicated latex constructions.

68.2. Other chapters

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CHAPTER 69

Obsolete

69.1. Introduction

In this chapter we put some lemmas that have become ``obsolete'' (see [**Mil17**]).

69.2. Lemmas related to ZMT

The lemmas in this section were originally used in the proof of the (algebraic version of) Zariski's Main Theorem, Algebra, Theorem 7.114.13.

Lemma 69.2.1. *Let $\varphi : R \to S$ be a ring map. Suppose $t \in S$ satisfies the relation $\varphi(a_0) + \varphi(a_1)t + \ldots + \varphi(a_n)t^n = 0$. Set $u_n = \varphi(a_n)$, $u_{n-1} = u_n t + \varphi(a_{n-1})$, and so on till $u_1 = u_2 t + \varphi(a_1)$. Then all of $u_n, u_{n-1}, \ldots, u_1$ and $u_n t, u_{n-1} t, \ldots, u_1 t$ are integral over R, and the ideals $(\varphi(a_0), \ldots, \varphi(a_n))$ and $(u_n, \ldots, u_1)$ of S are equal.*

Proof. We prove this by induction on n. As $u_n = \varphi(a_n)$ we conclude from Algebra, Lemma 7.114.1 that $u_n t$ is integral over R. Of course $u_n = \varphi(a_n)$ is integral over R. Then $u_{n-1} = u_n t + \varphi(a_{n-1})$ is integral over R (see Algebra, Lemma 7.32.7) and we have

$$\varphi(a_0) + \varphi(a_1)t + \ldots + \varphi(a_{n-1})t^{n-1} + u_{n-1}t^{n-1} = 0.$$

Hence by the induction hypothesis applied to the map $S' \to S$ where S' is the integral closure of R in S and the displayed equation we see that $u_{n-1}, \ldots, u_1$ and $u_{n-1}t, \ldots, u_1 t$ are all in S' too. The statement on the ideals is immediate from the shape of the elements and the fact that $u_1 t + \varphi(a_0) = 0$. ☐

Lemma 69.2.2. *Let $\varphi : R \to S$ be a ring map. Suppose $t \in S$ satisfies the relation $\varphi(a_0) + \varphi(a_1)t + \ldots + \varphi(a_n)t^n = 0$. Let $J \subset S$ be an ideal such that for at least one i we have $\varphi(a_i) \notin J$. Then there exists a $u \in S$, $u \notin J$ such that both u and ut are integral over R.*

Proof. This is immediate from Lemma 69.2.1 since one of the elements u_i will not be in J. ☐

The following two lemmas are a way of describing closed subschemes of $\mathbf{P}^1_R$ cut out by one (nondegenerate) equation.

Lemma 69.2.3. *Let R be a ring. Let $F(X, Y) \in R[X, Y]$ be homogenous of degree d. Assume that for every prime $\mathfrak{p}$ of R at least one coefficient of F is not in $\mathfrak{p}$. Let $S = R[X, Y]/(F)$ as a graded ring. Then for all $n \geq d$ the R-module S_n is finite locally free of rank d.*

Proof. The R-module S_n has a presentation

$$R[X, Y]_{n-d} \to R[X, Y]_n \to S_n \to 0.$$

Thus by Algebra, Lemma 7.73.3 it is enough to show that multiplication by F induces an injective map $\kappa(\mathfrak{p})[X, Y] \to \kappa(\mathfrak{p})[X, Y]$ for all primes $\mathfrak{p}$. This is clear from the assumption that F does not map to the zero polynomial mod $\mathfrak{p}$. The assertion on ranks is clear from this as well. $\square$

Lemma 69.2.4. *Let k be a field. Let $F, G \in k[X, Y]$ be homogeneous of degrees d, e. Assume F, G relatively prime. Then multiplication by G is injective on $S = k[X, Y]/(F)$.*

Proof. This is one way to define ``relatively prime''. If you have another definition, then you can show it is equivalent to this one. $\square$

Lemma 69.2.5. *Let R be a ring. Let $F(X, Y) \in R[X, Y]$ be homogenous of degree d. Let $S = R[X, Y]/(F)$ as a graded ring. Let $\mathfrak{p} \subset R$ be a prime such that some coefficient of F is not in $\mathfrak{p}$. There exists an $f \in R$ $f \notin \mathfrak{p}$, an integer e, and a $G \in R[X, Y]_e$ such that multiplication by G induces isomorphisms $(S_n)_f \to (S_{n+e})_f$ for all $n \geq d$.*

Proof. During the course of the proof we may replace R by R_f for $f \in R$, $f \notin \mathfrak{p}$ (finitely often). As a first step we do such a replacement such that some coefficient of F is invertible in R. In particular the modules S_n are now locally free of rank d for $n \geq d$ by Lemma 69.2.3. Pick any $G \in R[X, Y]_e$ such that the image of G in $\kappa(\mathfrak{p})[X, Y]$ is relatively prime to the image of $F(X, Y)$ (this is possible for some e). Apply Algebra, Lemma 7.73.3 to the map induced by multiplication by G from $S_d \to S_{d+e}$. By our choice of G and Lemma 69.2.4 we see $S_d \otimes \kappa(\mathfrak{p}) \to S_{d+e} \otimes \kappa(\mathfrak{p})$ is bijective. Thus, after replacing R by R_f for a suitable f we may assume that $G : S_d \to S_{d+e}$ is bijective. This in turn implies that the image of G in $\kappa(\mathfrak{p}')[X, Y]$ is relatively prime to the image of F for all primes $\mathfrak{p}'$ of R. And then by Algebra, Lemma 7.73.3 again we see that all the maps $G : S_d \to S_{d+e}$, $n \geq d$ are isomorphisms. $\square$

Remark 69.2.6. Let R be a ring. Suppose that we have $F \in R[X, Y]_d$ and $G \in R[X, Y]_e$ such that, setting $S = R[X, Y]/(F)$ we have (1) S_n is finite locally free of rank d for all $n \geq d$, and (2) multiplication by G defines ismorphisms $S_n \to S_{n+e}$ for all $n \geq d$. In this case we may define a finite, locally free R-algebra A as follows:

(1) as an R-module $A = S_{ed}$, and
(2) multiplication $A \times A \to A$ is given by the rule that $H_1 H_2 = H_3$ if and only if $G^d H_3 = H_1 H_2$ in S_{2ed}.

This makes sense because multiplication by G^d induces a bijective map $S_{de} \to S_{2de}$. It is easy to see that this defines a ring structure. Note the confusing fact that the element G^d defines the unit element of the ring A.

Lemma 69.2.7. *Let R be a ring, let $f \in R$. Suppose we have S, S' and the solid arrows forming the following commutative diagram of rings*

$$\begin{tikzcd}
 & S'' \arrow[rd, dashed] \arrow[dd, dashed] & \\
R \arrow[ru, dashed] \arrow[rr] \arrow[d] & & S \arrow[d] \\
R_f \arrow[r] & S' \arrow[r] & S_f
\end{tikzcd}$$

Assume that $R_f \to S'$ is finite. Then we can find a finite ring map $R \to S''$ and dotted arrows as in the diagram such that $S' = (S'')_f$.

Proof. Namely, suppose that S' is generated by x_i over R_f, $i = 1, \ldots, w$. Let $P_i(t) \in R_f[t]$ be a monic polynomial such that $P_i(x_i) = 0$. Say P_i has degree $d_i > 0$. Write $P_i(t) = t^{d_i} + \sum_{j<d_i}(a_{ij}/f^n)t^j$ for some uniform n. Also write the image of x_i in S_f as g_i/f^n for suitable $g_i \in S$. Then we know that the element $\xi_i = f^{nd_i}g_i^{d_i} + \sum_{j<d_i} f^{n(d_i-j)}a_{ij}g_i^j$ of S is killed by a power of f. Hence upon increasing n to n', which replaces g_i by $f^{n'-n}g_i$ we may assume $\xi_i = 0$. Then S' is generated by the elements $f^n x_i$, each of which is a zero of the monic polynomial $Q_i(t) = t^{d_i} + \sum_{j<d_i} f^{n(d_i-j)}a_{ij}t^j$ with coefficients in R. Also, by construction $Q_i(f^n g_i) = 0$ in S. Thus we get a finite R-algebra $S'' = R[z_1, \ldots, z_w]/(Q_1(z_1), \ldots, Q_w(z_w))$ which fits into a commutative diagram as above. The map $\alpha : S'' \to S$ maps z_i to $f^n g_i$ and the map $\beta : S'' \to S'$ maps z_i to $f^n x_i$. It may not yet be the case that β induces an isomorphism $(S'')_f \cong S'$. For the moment we only know that this map is surjective. The problem is that there could be elements $h/f^n \in (S'')_f$ which map to zero in S' but are not zero. In this case $\beta(h)$ is an element of S such that $f^N\beta(h) = 0$ for some N. Thus $f^N h$ is an element ot the ideal $J = \{h \in S'' \mid \alpha(h) = 0 \text{ and } \beta(h) = 0\}$ of S''. OK, and it is easy to see that S''/J does the job. $\square$

69.3. Formally smooth ring maps

Lemma 69.3.1. *Let R be a ring. Let S be a R-algebra. If S is of finite presentation and formally smooth over R then S is smooth over R.*

Proof. See Algebra, Proposition 7.127.13. $\square$

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CHAPTER 70

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CHAPTER 71

Auto generated index

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71.2. Definitions listed per chapter

Introduction

Conventions

Set Theory

Categories

In 4.2.1: *category*
In 4.2.4: *isomorphism*
In 4.2.5: *groupoid*
In 4.2.8: *functor*
In 4.2.9: *faithful*, *fully faithful*, *essentially surjective*
In 4.2.10: *subcategory*, *full subcategory*, *strictly full*
In 4.2.15: *natural transformation*, *morphism of functors*
In 4.2.17: *equivalence of categories*, *quasi-inverse*
In 4.2.20: *product category*
In 4.3.1: *opposite category*
In 4.3.2: *contravariant*
In 4.3.3: *presheaf of sets on $\mathscr{C}$*, *presheaf*
In 4.3.6: *representable*
In 4.4.1: *product*
In 4.4.2: *has products of pairs of objects*
In 4.5.1: *coproduct*, *amalgamated sum*
In 4.5.2: *has coproducts of pairs of objects*
In 4.6.1: *fibre product*
In 4.6.2: *has fibre products*
In 4.6.3: *representable*
In 4.8.2: *representable*, *F is relatively representable over G*
In 4.9.1: *push out*
In 4.10.1: *equalizer*
In 4.11.1: *coequalizer*
In 4.12.1: *initial*, *final*
In 4.13.1: *limit*
In 4.13.2: *colimit*
In 4.13.5: *product*
In 4.13.6: *coproduct*
In 4.15.1: *connected*
In 4.17.1: *directed*, *filtered*, *directed*, *filtered*
In 4.17.5: *$\mathscr{I}$ is cofinal in $\mathscr{J}$*
In 4.18.1: *codirected*, *cofiltered*, *codirected*, *cofiltered*
In 4.19.1: *system over I in $\mathscr{C}$*, *inductive system over I in $\mathscr{C}$*, *inverse system over I in*

$\mathscr{C}$, *projective system over* I *in* $\mathscr{C}$, *transition maps*
In 4.19.2: *directed system*, *directed inverse system*, *directed*
In 4.20.1: *is essentially constant*, *value*, *essentially constant*, *value*
In 4.20.2: *essentially constant system*, *essentially constant inverse system*
In 4.21.1: *left exact*, *right exact*, *exact*
In 4.22.1: *left adjoint*, *right adjoint*
In 4.23.1: *monomorphism*, *epimorphism*
In 4.24.1: *left multiplicative system*, *right multiplicative system*, *multiplicative system*
In 4.24.17: *saturated*
In 4.25.1: *horizontal*
In 4.26.1: 2-*category*, 1-*morphisms*, 2-*morphisms*, *vertical*, *composition*, *horizontal*
In 4.26.2: *sub* 2-*category*
In 4.26.4: *equivalent*
In 4.26.5: *functor*, *weak functor*, *pseudo functor*
In 4.27.1: $(2,1)$-*category*
In 4.28.1: *final object*
In 4.28.2: *2-fibre product of* f *and* g
In 4.29.1: 2-*category of categories over* $\mathscr{C}$
In 4.29.2: *fibre category*, *lift*, x *lies over* U, *lift*, ϕ *lies over* f
In 4.30.1: *strongly cartesian morphism*, *strongly* $\mathscr{C}$*-cartesian morphism*
In 4.30.4: *fibred category over* $\mathscr{C}$
In 4.30.5: *choice of pullbacks*, *pullback functor*
In 4.30.8: 2-*category of fibred categories over* $\mathscr{C}$
In 4.31.2: *relative inertia of* $\mathscr{S}$ *over* $\mathscr{S}'$, *inertia fibred category* $\mathscr{I}_{\mathscr{S}}$ *of* $\mathscr{S}$
In 4.32.1: *fibred in groupoids*
In 4.32.6: 2-*category of categories fibred in groupoids over* $\mathscr{C}$
In 4.33.2: *split fibred category*, $\mathscr{S}_F$
In 4.34.2: *split category fibred in groupoids*, $\mathscr{S}_F$
In 4.35.1: *discrete*
In 4.35.2: *category fibred in sets*, *category fibred in discrete categories*
In 4.35.3: 2-*category of categories fibred in sets over* $\mathscr{C}$
In 4.36.1: *setoid*
In 4.36.2: *category fibred in setoids*
In 4.36.3: 2-*category of categories fibred in setoids over* $\mathscr{C}$
In 4.37.1: *representable*
In 4.38.5: *representable*, $\mathscr{X}$ *is relatively representable over* $\mathscr{Y}$

Topology

In 5.3.1: *base for the topology on* X, *basis for the topology on* X
In 5.4.1: *connected*, *connected component*
In 5.4.6: *totally disconnected*
In 5.4.7: *locally connected*
In 5.5.1: *irreducible*, *irreducible component*
In 5.5.4: *generic point*, *Kolmogorov*, *sober*
In 5.6.1: *Noetherian*, *locally Noetherian*
In 5.7.1: *chain of irreducible closed subsets*, *length*, *dimension*, *Krull dimension*, *Krull dimension of* X *at* x
In 5.7.4: *equidimensional*
In 5.8.1: *catenary*
In 5.8.3: *codimension*
In 5.9.1: *quasi-compact*, *quasi-compact*, *retrocompact*
In 5.10.1: *constructible*, *locally construcible*
In 5.12.2: *closed*, *proper*, *quasi-proper*, *universally closed*
In 5.13.1: *Jacobson*
In 5.14.1: *specialization*, *generalization*, *stable under specialization*, *stable under generalization*
In 5.14.3: *specializations lift along* f, *specializing*, *generalizations lift along* f, *generalizing*
In 5.15.1: *submersive*
In 5.16.1: *immediate specialization*, *dimension function*
In 5.17.1: *interior*, *nowhere dense*
In 5.18.1: *locally quasi-compact*
In 5.18.3: *isolated point*

Sheaves on Spaces

In 6.3.1: *presheaf* $\mathscr{F}$ *of sets on* X, *morphism* $\varphi : \mathscr{F} \to \mathscr{G}$ *of presheaves of sets on* X
In 6.3.2: *constant presheaf with value* A
In 6.4.4: *presheaf of abelian groups on* X, *abelian presheaf over* X, *morphism of abelian presheaves over* X

In 6.5.1: *presheaf $\mathcal{F}$ on X with values in $\mathcal{C}$*, *morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves with value in $\mathcal{C}$*
In 6.5.2: *underlying presheaf of sets of $\mathcal{F}$*
In 6.6.1: *presheaf of $\mathcal{O}$-modules*, *morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves of $\mathcal{O}$-modules*
In 6.7.1: *sheaf $\mathcal{F}$ of sets on X*, *morphism of sheaves of sets*
In 6.7.4: *constant sheaf with value A*
In 6.8.1: *abelian sheaf on X*, *sheaf of abelian groups on X*
In 6.9.1: *sheaf*
In 6.10.1: *sheaf of $\mathcal{O}$-modules*, *morphism of sheaves of $\mathcal{O}$-modules*
In 6.11.2: *separated*
In 6.15.1: *type of algebraic structure*
In 6.16.2: *subpresheaf*, *subsheaf*, *injective*, *surjective*, *injective*, *surjective*
In 6.21.7: *f-map $\xi : \mathcal{G} \to \mathcal{F}$*
In 6.21.9: *composition of φ and ψ*
In 6.25.1: *ringed space*, *morphism of ringed spaces*
In 6.25.3: *composition of morphisms of ringed spaces*
In 6.26.1: *pushforward*, *pullback*
In 6.27.1: *skyscraper sheaf at x with value A*, *skyscraper sheaf*, *skyscraper sheaf*, *skyscraper sheaf*, *skyscraper sheaf*
In 6.30.1: *presheaf $\mathcal{F}$ of sets on $\mathcal{B}$*, *morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves of sets on $\mathcal{B}$*
In 6.30.2: *sheaf $\mathcal{F}$ of sets on $\mathcal{B}$*, *morphism of sheaves of sets on $\mathcal{B}$*
In 6.30.8: *presheaf $\mathcal{F}$ with values in $\mathcal{C}$ on $\mathcal{B}$*, *morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves with values in $\mathcal{C}$ on $\mathcal{B}$*, *sheaf $\mathcal{F}$ with values in $\mathcal{C}$ on $\mathcal{B}$*
In 6.30.11: *presheaf of $\mathcal{O}$-modules $\mathcal{F}$ on $\mathcal{B}$*, *morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves of $\mathcal{O}$-modules on $\mathcal{B}$*, *sheaf $\mathcal{F}$ of $\mathcal{O}$-modules on $\mathcal{B}$*
In 6.31.2: *restriction of $\mathcal{G}$ to U*, *restriction of $\mathcal{G}$ to U*, *open subspace of $(X, \mathcal{O})$ associated to U*, *restriction of $\mathcal{G}$ to U*
In 6.31.3: *extension of $\mathcal{F}$ by the empty set $j_{p!}\mathcal{F}$*, *extension of $\mathcal{F}$ by the empty set $j_!\mathcal{F}$*
In 6.31.5: *extension $j_{p!}\mathcal{F}$ of $\mathcal{F}$ by 0*, *extension $j_!\mathcal{F}$ of $\mathcal{F}$ by 0*, *extension $j_{p!}\mathcal{F}$ of $\mathcal{F}$ by e*, *extension $j_!\mathcal{F}$ of $\mathcal{F}$ by e*, *extension by 0*, *extension by 0*

Commutative Algebra

In 7.5.1: *finite R-module*, *finitely generated R-module*, *finitely presented R-module*, *R-module of finite presentation*
In 7.6.1: *finite type*, *S is a finite type R-algebra*, *finite presentation*
In 7.7.1: *finite*
In 7.8.1: *partially ordered set*, *directed set*
In 7.8.2: *system (M_i, μ_{ij}) of R-modules over I*, *directed system*
In 7.8.7: *homomorphism of systems*
In 7.8.12: *relation*
In 7.9.1: *multiplicative subset of R*
In 7.9.2: *localization of A with respect to S*
In 7.9.6: *localization*
In 7.11.1: *R-bilinear*
In 7.11.6: *(A, B)-bimodule*
In 7.13.1: *base change*, *base change*
In 7.16.1: *spectrum*
In 7.16.3: *Zariski*, *standard opens*
In 7.17.1: *local ring*, *local homomorphism of local rings*, *local ring map $\varphi : R \to S$*
In 7.25.2: *Oka family*
In 7.31.1: *Jacobson ring*
In 7.32.1: *integral over R*, *integral*
In 7.32.8: *integral closure*, *integrally closed*
In 7.33.1: *normal*
In 7.33.3: *almost integral over R*, *completely normal*
In 7.33.10: *normal*
In 7.34.1: *integral over I*
In 7.35.1: *flat*, *faithfully flat*, *flat*, *faithfully flat*
In 7.36.1: *going up*, *going down*
In 7.37.1: *algebraically independent*, *purely transcendental extension*, *transcendence basis*
In 7.37.3: *transcendence degree*
In 7.38.1: *algebraic*, *separable*, *purely inseparable*, *normal*, *Galois*
In 7.38.3: *separable degree*, *inseparable degree*, *degree of inseparability*
In 7.38.6: *algebraic closure of k in K*, *algebraically closed in K*
In 7.39.1: *separably generated over k*, *separable over k*

In 7.40.1: *geometrically reduced over k*
In 7.42.1: *perfect*
In 7.42.5: *perfect closure*
In 7.43.6: *geometrically irreducible over k*
In 7.44.3: *geometrically connected over k*
In 7.45.1: *geometrically integral over k*
In 7.46.1: *dominates*, *valuation ring*, *centered*
In 7.46.8: *value group*, *valuation*, *discrete valuation ring*
In 7.47.2: *locally nilpotent*, *nilpotent*
In 7.48.1: *length*
In 7.48.9: *simple*
In 7.49.1: *Artinian*
In 7.50.1: *essentially of finite type*, *essentially of finite presentation*
In 7.53.1: *homogeneous spectrum*
In 7.54.1: *blowup algebra*, *Rees algebra*, *affine blowup algebra*
In 7.55.2: *numerical polynomial*
In 7.56.1: *an ideal of definition of R*
In 7.56.7: $d(M)$
In 7.57.1: *Krull dimension*
In 7.57.2: *height*
In 7.57.9: *system of parameters of R*, *regular local ring*, *regular system of parameters*
In 7.59.2: *support of M*
In 7.60.1: *associated*
In 7.61.1: *symbolic power*
In 7.62.2: *relative assassin of N over S/R*
In 7.63.1: *weakly associated*
In 7.64.1: *embedded associated primes*, *embedded primes of R*
In 7.65.1: *M-regular*, *M-regular sequence in I*, *regular sequence*
In 7.65.4: *I-depth*, *depth*
In 7.66.1: *M-quasi-regular*, *quasi-regular sequence*
In 7.67.2: *resolution*, *resolution of M by free R-modules*, *resolution of M by finite free R-modules*
In 7.71.1: *projective*
In 7.72.1: *locally free*, *finite locally free*
In 7.76.1: *universally injective*, *universally exact*
In 7.78.1: *direct sum dévissage*, *Kaplansky dévissage*
In 7.80.1: *Mittag-Leffler inverse system*
In 7.82.1: *Mittag-Leffler directed system of modules*
In 7.82.2: *dominates*
In 7.82.6: *Mittag-Leffler*
In 7.84.1: *coherent module*, *coherent ring*
In 7.90.5: *I-adically complete*, *I-adically complete*
In 7.94.4: *rank*
In 7.95.1: *Cohen-Macaulay*
In 7.96.1: *Cohen-Macaulay*
In 7.96.6: *Cohen-Macaulay*
In 7.96.9: *maximal Cohen-Macaulay*
In 7.97.1: *catenary*
In 7.97.5: *universally catenary*
In 7.100.1: *pure*
In 7.101.2: *finite projective dimension*, *projective dimension*
In 7.101.6: *finite global dimension*, *global dimension*
In 7.102.6: *regular*
In 7.103.5: *local ring of the fibre at $\mathfrak{q}$*
In 7.111.1: *associates*, *irreducible*, *prime*
In 7.111.4: *unique factorization domain*, *UFD*
In 7.111.6: *principal ideal domain*, *PID*
In 7.111.8: *Dedekind domain*
In 7.112.2: *order of vanishing along R*
In 7.112.3: *lattice in V*
In 7.112.5: *distance between M and M'*
In 7.113.3: *quasi-finite at $\mathfrak{q}$*, *quasi-finite*
In 7.114.8: *strongly transcendental over R*
In 7.116.1: *relative dimension of S/R at $\mathfrak{q}$*, *relative dimension of*
In 7.122.1: *derivation*, *R-derivation*, *Leibniz rule*
In 7.122.2: *module of Kähler diffentials*, *module of differentials*
In 7.123.1: *naive cotangent complex*
In 7.124.1: *global complete intersection over k*, *local complete intersection over k*
In 7.124.5: *complete intersection (over k)*
In 7.125.1: *syntomic*, *flat local complete intersection over R*
In 7.125.5: *relative global complete intersection*
In 7.126.1: *smooth*
In 7.126.6: *standard smooth algebra over R*
In 7.126.11: *smooth at $\mathfrak{q}$*

In 7.127.1: *formally smooth over R*
In 7.130.1: *small extension*
In 7.132.1: *étale*, *étale at $\mathfrak{q}$*
In 7.132.13: *standard étale*
In 7.135.1: *formally unramified over R*
In 7.136.2: *universal first order thickening*, *conormal module*, $C_{S/R}$
In 7.137.1: *formally étale over R*
In 7.138.1: *unramified*, *G-unramified*, *unramified at $\mathfrak{q}$*, *G-unramified at $\mathfrak{q}$*
In 7.139.1: *henselian*, *strictly henselian*
In 7.139.14: *henselization*, *strict henselization of R with respect to $\kappa \subset \kappa^{sep}$*, *strict henselization*
In 7.140.1: (R_k), *regular in codimension $\leq k$*, (S_k)
In 7.143.1: *complete local ring*
In 7.143.4: *coefficient ring*
In 7.143.5: *Cohen ring*
In 7.144.1: *N-1*, *N-2*, *Japanese*
In 7.144.15: *universally Japanese*, *Nagata ring*
In 7.144.23: *analytically unramified*, *analytically unramified*
In 7.147.2: *geometrically normal*
In 7.148.2: *geometrically regular*

Brauer groups

In 8.2.1: *finite*
In 8.2.2: *skew field*
In 8.2.3: *simple*, *simple*
In 8.2.4: *central*
In 8.2.5: *opposite algebra*
In 8.5.2: *Brauer group*
In 8.8.1: *splits*, *splitting field*

Sites and Sheaves

In 9.2.1: *presheaf of sets*, *Morphisms of presheaves*
In 9.2.2: *presheaf*, *morphism*
In 9.3.1: *injective*, *surjective*
In 9.3.3: *subpresheaf*
In 9.3.5: *image of φ*
In 9.6.1: *family of morphisms with fixed target*
In 9.6.2: *site*, *coverings of $\mathcal{C}$*
In 9.7.1: *sheaf*
In 9.7.5: *$Sh(\mathcal{C})$*
In 9.7.6: *sheaf*
In 9.8.1: *morphism of families of maps with fixed target of $\mathcal{C}$ from $\mathcal{U}$ to $\mathcal{V}$*, *morphism from $\mathcal{U}$ to $\mathcal{V}$*, *refinement*
In 9.8.2: *combinatorially equivalent*, *tautologically equivalent*
In 9.10.9: *separated*
In 9.10.11: *sheaf associated to $\mathcal{F}$*
In 9.11.1: *injective*, *surjective*
In 9.12.1: *effective epimorphism*, *universal effective epimorphism*
In 9.12.2: *weaker than the canonical topology*, *subcanonical*
In 9.12.3: $\underline{U} = h_U$, U, *representable sheaf*
In 9.13.1: *continuous*
In 9.14.1: *morphism of sites*
In 9.14.4: *composition*
In 9.15.1: *topos*, *morphism of topoi*, *composition $f \circ g$*
In 9.18.1: *cocontinuous*
In 9.21.1: *localization of the site $\mathcal{C}$ at the object U*, *localization morphism*, *direct image functor*, *restriction of $\mathcal{F}$ to $\mathcal{C}/U$*, *extension of $\mathcal{G}$ by the empty set*
In 9.25.2: *special cocontinuous functor u from $\mathcal{C}$ to $\mathcal{D}$*
In 9.26.4: *localization of the topos $Sh(\mathcal{C})$ at $\mathcal{F}$*, *localization morphism*
In 9.28.1: *point of the topos $Sh(\mathcal{C})$*
In 9.28.2: *point p of the site $\mathcal{C}$*
In 9.28.6: *skyscraper sheaf*
In 9.32.1: *2-morphism from f to g*
In 9.33.2: *morphism $f : p \to p'$*
In 9.34.1: *conservative*, *has enough points*
In 9.37.1: *sheaf theoretically empty*
In 9.37.3: *almost cocontinuous*
In 9.38.1: *pushforward*
In 9.39.1: *global sections*
In 9.40.1: *sieve S on U*
In 9.40.3: *sieve on U generated by the morphisms f_i*
In 9.40.4: *pullback of S by f*
In 9.40.6: *topology on $\mathcal{C}$*
In 9.40.8: *finer*
In 9.40.10: *sheaf*
In 9.40.12: *canonical topology*
In 9.41.2: *topology associated to $\mathcal{C}$*
In 9.42.2: *separated*
In 9.42.4: *sheaf associated to $\mathcal{F}$*

In 9.45.1: *point p*

Homological Algebra

In 10.3.1: *preadditive*, *additive*
In 10.3.3: *zero object*
In 10.3.5: *direct sum*
In 10.3.8: *additive*
In 10.3.9: *kernel*, *cokernel*, *coimage of f*, *image of f*
In 10.3.12: *abelian*
In 10.3.14: *injective*, *surjective*, *subobject*, *quotient*
In 10.3.18: *complex*, *exact at y*, *exact*, *short exact sequence*
In 10.3.20: *split*
In 10.4.1: *extension E of B by A*
In 10.4.2: *Ext-group*
In 10.7.1: *Serre subcategory*, *weak Serre subcategory*
In 10.7.5: *kernel of the functor F*
In 10.8.1: *zeroth K-group of $\mathcal{A}$*
In 10.9.1: *cohomological δ-functor*, *δ-functor*
In 10.9.2: *morphism of δ-functors from F to G*
In 10.9.3: *universal δ-functor*
In 10.10.2: *homotopy equivalence*, *homotopy equivalent*
In 10.10.4: *quasi-isomorphism*, *acyclic*
In 10.10.8: *homotopy equivalence*, *homotopy equivalent*
In 10.10.10: *quasi-isomorphism*, *acyclic*
In 10.12.1: *k-shifted chain complex $A[k]_\bullet$*
In 10.12.2: $H_{i+k}(A_\bullet) \to H_i(A[k]_\bullet)$
In 10.12.7: *k-shifted cochain complex $A[k]^\bullet$*
In 10.12.8: $H^{i+k}(A^\bullet) \longrightarrow H^i(A[k]^\bullet)$
In 10.13.1: *decreasing filtration*, *filtered object of $\mathcal{A}$*, *morphism $(A, F) \to (B, F)$ of filtered objects*, *induced filtration*, *quotient filtration*, *finite*, *separated*, *exhaustive*
In 10.13.3: *strict*
In 10.13.12: *graded object of $\mathcal{A}$*, *morphism $(A, k) \to (B, k)$ of graded objects*
In 10.14.1: *spectral sequence in $\mathcal{A}$*, *morphism of spectral sequences*
In 10.14.2: *limit*, *collapses at E_r*, *degenerates at E_r*
In 10.15.1: *exact couple*, *morphism of exact couples*
In 10.15.3: *spectral sequence associated to the exact couple*
In 10.16.1: *differential object*, *morphism of differential objects*
In 10.16.3: *homology*
In 10.16.5: *spectral sequence associated to (A, d, α)*
In 10.17.1: *filtered differential object*
In 10.17.4: *induced filtration*
In 10.17.6: *converges*, *abuts to*, *converges to*
In 10.18.1: *filtered complex $K^\bullet$ of $\mathcal{A}$*
In 10.18.5: *induced filtration*
In 10.18.7: *converges*
In 10.19.1: *double complex*
In 10.19.2: *associated simple complex $sA^\bullet$*, *associated total complex*
In 10.19.4: *converges*, *converges*
In 10.20.1: *injective*
In 10.20.4: *enough injectives*
In 10.20.5: *functorial injective embeddings*
In 10.21.1: *projective*
In 10.21.4: *enough projectives*
In 10.21.5: *functorial projective surjections*
In 10.23.2: *Mittag-Leffler condition*, *ML*
In 10.25.1: *differential graded algebra*
In 10.25.2: *homomorphism of differential graded algebras*
In 10.25.3: *commutative*, *strictly commutative*
In 10.25.4: *tensor product differential graded algebra*

Derived Categories

In 11.3.1: *triangle*, *morphism of triangles*
In 11.3.2: *triangulated category*, *distinguished triangles*, *pre-triangulated category*
In 11.3.3: *exact functor*, *triangulated functor*
In 11.3.4: *pre-triangulated subcategory*, *triangulated subcategory*
In 11.3.5: *homological*, *cohomological*
In 11.3.6: *δ-functor from $\mathcal{A}$ to $\mathcal{D}$*, *image of the short exact sequence under the given δ-functor*
In 11.5.1: *compatible with the triangulated structure*
In 11.6.1: *saturated*
In 11.6.5: *kernel of F*, *kernel of H*

In 11.6.7: *quotient category $\mathcal{D}/\mathcal{B}$*, *quotient functor*
In 11.7.1: *category of (cochain) complexes*, *bounded below*, *bounded above*, *bounded*
In 11.8.1: *cone*
In 11.8.3: *termwise split injection $\alpha : A^\bullet \to B^\bullet$*, *termwise split surjection $\beta : B^\bullet \to C^\bullet$*
In 11.8.8: *termwise split sequence of complexes of $\mathcal{A}$*, *triangle associated to the termwise split sequence of complexes*
In 11.9.1: *distinguished triangle of $K(\mathcal{A})$*
In 11.10.3: *derived category of $\mathcal{A}$*, *bounded derived category*
In 11.13.1: *category of finite filtered objects of $\mathcal{A}$*
In 11.13.2: *filtered quasi-isomorphism*, *filtered acyclic*
In 11.13.5: *filtered derived category of $\mathcal{A}$*
In 11.13.7: *bounded filtered derived category*
In 11.14.2: *right derived functor RF is defined at*, *value of RF at X*, *left derived functor LF is defined at*, *value of LF at X*
In 11.14.9: *right deriveable*, *everywhere defined*, *left deriveable*, *everywhere defined*
In 11.14.10: *computes*, *computes*
In 11.15.3: *right derived functors of F*, *left derived functors of F*, *right acyclic for F*, *acyclic for RF*, *left acyclic for F*, *acyclic for LF*
In 11.16.2: *ith right derived functor R^iF of F*
In 11.17.1: *injective resolution of A*, *injective resolution of $K^\bullet$*
In 11.18.1: *projective resolution of A*, *projective resolution of $K^\bullet$*
In 11.20.1: *Cartan-Eilenberg resolution*
In 11.22.2: *resolution functor*
In 11.25.1: *filtered injective*
In 11.26.1: *ith extension group*
In 11.26.4: *Yoneda extension*, *equivalent*
In 11.28.1: *K-injective*

More on Algebra

In 12.3.3: *K-flat*
In 12.3.12: *derived tensor product*
In 12.5.1: *Tor independent over R*
In 12.8.1: *I-power torsion module*, *an f-power torsion module*
In 12.10.1: *auto-associated*
In 12.17.1: *torsion*, *torsion free*
In 12.19.1: *strict transform of M along $R \to R'$*
In 12.21.1: *Koszul complex*
In 12.21.2: *Koszul complex on $f_1, \ldots, f_r$*
In 12.22.1: *Koszul-regular*, *H_1-regular*
In 12.23.1: *regular ideal*, *Koszul-regular ideal*, *H_1-regular ideal*, *quasi-regular ideal*
In 12.24.2: *local complete intersection*
In 12.27.1: *topological ring*, *topological module*, *homomorphism of topological modules*, *homomorphism of topological rings*, *linearly topologized*, *linearly topologized*, *ideal of definition*, *pre-admissible*, *admissible*, *pre-adic*, *adic*
In 12.28.1: *formally smooth over R*
In 12.28.3: *formally smooth for the $\mathfrak{n}$-adic topology*
In 12.31.1: *regular*
In 12.34.1: *p-independent over k*, *p-basis of K over k*
In 12.35.1: *J-0*, *J-1*, *J-2*
In 12.38.1: *G-ring*
In 12.39.1: *quasi-excellent*, *excellent*
In 12.40.1: *m-pseudo-coherent*, *pseudo-coherent*, *m-pseudo-coherent*, *pseudo-coherent*
In 12.41.1: *tor-amplitude in $[a, b]$*, *finite tor dimension*, *tor dimension $\leq d$*, *finite tor dimension*
In 12.42.1: *perfect*, *perfect*
In 12.43.2: *compact object*
In 12.44.2: *an A-module finitely presented relative to R*
In 12.45.4: *m-pseudo-coherent relative to R*, *pseudo-coherent relative to R*, *m-pseudo-coherent relative to R*, *pseudo-coherent relative to R*
In 12.46.1: *pseudo-coherent ring map*, *perfect ring map*

Smoothing Ring Maps

In 13.3.1: *singular ideal of A over R*
In 13.3.3: *elementary standard in A over R*, *strictly standard in A over R*

Simplicial Methods

In 14.2.1: $\delta^n_j : [n-1] \to [n]$, $\sigma^n_j : [n+1] \to [n]$
In 14.3.1: *simplicial object U of $\mathcal{C}$*, *simplicial set*, *simplicial abelian group*, *morphism of simplicial objects $U \to U'$*, *category of simplicial objects of $\mathcal{C}$*
In 14.5.1: *cosimplicial object U of $\mathcal{C}$*, *cosimplicial set*, *cosimplicial abelian group*, *morphism of cosimplicial objects $U \to U'$*, *category of cosimplicial objects of $\mathcal{C}$*
In 14.6.1: *product of U and V*
In 14.7.1: *fibre product of V and W over U*
In 14.8.1: *push out of V and W over U*
In 14.9.1: *product of U and V*
In 14.10.1: *fibre product of V and W over U*
In 14.11.1: *n-simplex of U*, *face of x*, *degeneracy of x*, *degenerate*
In 14.12.1: *product $U \times V$ of U and V*, *product $U \times V$ exists*
In 14.13.1: $\mathit{Hom}(U, V)$
In 14.15.1: $\mathit{Hom}(U, V)$
In 14.16.1: *split*
In 14.17.1: *n-truncated simplicial object of $\mathcal{C}$*, *morphism of n-truncated simplicial objects*
In 14.18.1: *augmentation $\epsilon : U \to X$ of U towards an object X of $\mathcal{C}$*
In 14.20.3: *Eilenberg-Maclane object $K(A, k)$*
In 14.24.1: *homotopic*, *homotopy connecting a and b*
In 14.24.5: *homotopy equivalence*, *homotopy equivalent*
In 14.26.1: *homotopic*, *homotopy connecting a and b*

Sheaves of Modules

In 15.4.1: *generated by global sections*, *generate*
In 15.4.5: *subsheaf generated by the s_i*
In 15.5.1: *support of $\mathcal{F}$*, *support of s*
In 15.8.1: *locally generated by sections*
In 15.9.1: *finite type*
In 15.10.1: *quasi-coherent sheaf of $\mathcal{O}_X$-modules*
In 15.10.6: *sheaf associated to the module M and the ring map α*, *sheaf associated to the module M*
In 15.11.1: *finite presentation*
In 15.12.1: *coherent $\mathcal{O}_X$-module*
In 15.13.1: *closed immersion of ringed spaces*
In 15.14.1: *locally free*, *finite locally free*
In 15.16.1: *flat*
In 15.16.3: *flat at x*
In 15.17.1: *flat at x*, *flat*
In 15.20.1: *Koszul complex*
In 15.20.2: *Koszul complex on $f_1, \ldots, f_r$*
In 15.21.1: *invertible $\mathcal{O}_X$-module*, *trivial*
In 15.21.3: *tensor power of $\mathcal{L}$*
In 15.21.4: *associated graded ring*
In 15.21.6: *Picard group*

Modules on Sites

In 16.4.1: *free abelian presheaf*
In 16.5.1: *free abelian sheaf*
In 16.6.1: *ringed site*, *structure sheaf*, *morphism of ringed sites*, *composition of morphisms of ringed sites*
In 16.7.1: *ringed topos*, *structure sheaf*, *morphism of ringed topoi*, *composition of morphisms of ringed topoi*
In 16.8.1: *2-morphism from f to g*
In 16.9.1: *presheaf of $\mathcal{O}$-modules*, *morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves of $\mathcal{O}$-modules*
In 16.10.1: *sheaf of $\mathcal{O}$-modules*, *morphism of sheaves of $\mathcal{O}$-modules*
In 16.13.1: *pushforward*, *pullback*
In 16.16.1: $g_{p!}\mathcal{F}$, $g_!\mathcal{F} = (g_{p!}\mathcal{F})^\#$
In 16.17.1: *free $\mathcal{O}$-module*, *finite free*, *generated by global sections*, *generated by finitely many global sections*, *global presentation*, *global finite presentation*
In 16.19.1: *localization of the ringed site $(\mathcal{C}, \mathcal{O})$ at the object U*, *localization morphism*, *direct image functor*, *restriction of $\mathcal{F}$ to $\mathcal{C}/U$*, *extension by zero*
In 16.21.2: *localization of the ringed topos $(Sh(\mathcal{C}), \mathcal{O})$ at $\mathcal{F}$*, *localization morphism*
In 16.23.1: *locally free*, *finite locally free*, *locally generated by sections*, *of finite type*, *quasi-coherent*, *of finite presentation*, *coherent*
In 16.26.1: *flat*, *flat*, *flat*, *flat*
In 16.27.1: *flat*, *flat*
In 16.28.1: *rank r*, *invertible $\mathcal{O}$-module*, $\mathcal{O}^*$
In 16.28.4: *Picard group*

In 16.29.1: *$\mathcal{O}_1$-derivation*, *φ-derivation*, *Leibniz rule*
In 16.29.3: *module of differentials*, *universal φ-derivation*
In 16.29.6: *Y-derivation*, *sheaf of differentials $\Omega_{X/Y}$ of X over Y*, *universal Y-derivation*
In 16.34.4: *locally ringed site*
In 16.34.6: *locally ringed*
In 16.34.8: *morphism of locally ringed topoi*, *morphism of locally ringed sites*

Injectives

In 17.3.1: *$M \mapsto M^\vee$*, *free module*
In 17.6.4: *α-small with respect to I*
In 17.14.1: *generator*, *Grothendieck abelian category*
In 17.15.1: *size*

Cohomology of Sheaves

In 18.5.1: *torsor*, *$\mathcal{G}$-torsor*, *morphism of $\mathcal{G}$-torsors*, *trivial $\mathcal{G}$-torsor*
In 18.9.1: *Čech complex*, *Čech cohomology groups*
In 18.17.1: *alternating Čech complex*
In 18.17.2: *ordered Čech complex*
In 18.18.1: *locally finite*
In 18.20.2: *K-flat*
In 18.20.13: *derived tensor product*

Cohomology on Sites

In 19.5.1: *pseudo torsor*, *pseudo $\mathcal{G}$-torsor*, *morphism of pseudo $\mathcal{G}$-torsors*, *torsor*, *$\mathcal{G}$-torsor*, *morphism of G-torsors*, *trivial $\mathcal{G}$-torsor*
In 19.9.1: *Cech complex*, *Cech cohomology groups*
In 19.13.4: *limp*
In 19.17.2: *K-flat*
In 19.17.11: *derived tensor product*

Hypercoverings

In 20.2.1: *semi-representable objects*
In 20.2.2: *which associates a sheaf to a semi-representable object*
In 20.2.4: *covering*
In 20.2.6: *hypercovering*
In 20.3.1: *homology of K*

Schemes

In 21.2.1: *locally ringed space $(X, \mathcal{O}_X)$*, *local ring of X at x*, *residue field of X at x*, *morphism of locally ringed spaces*
In 21.3.1: *open immersion*
In 21.3.3: *open subspace of X associated to U*
In 21.4.1: *closed immersion*
In 21.4.4: *closed subspace of X associated to the sheaf of ideals $\mathcal{I}$*
In 21.5.2: *standard open covering*, *standard open covering*
In 21.5.3: *structure sheaf $\mathcal{O}_{Spec(R)}$ of the spectrum of R*, *spectrum*
In 21.5.5: *affine scheme*, *morphism of affine schemes*
In 21.9.1: *scheme*, *morphism of schemes*
In 21.10.2: *open immersion*, *open subscheme*, *closed immersion*, *closed subscheme*, *immersion*, *locally closed immersion*
In 21.12.1: *reduced*
In 21.12.5: *scheme structure on Z*, *reduced induced scheme structure*, *reduction X_{red} of X*
In 21.15.1: *representable by a scheme*, *representable*
In 21.15.3: *satisfies the sheaf property for the Zariski topology*, *subfunctor $H \subset F$*, *representable by open immersions*, *covers F*
In 21.17.1: *fibre product*
In 21.17.7: *inverse image $f^{-1}(Z)$ of the closed subscheme Z*
In 21.18.1: *scheme over S*, *structure morphism*, *scheme over R*, *morphism $f : X \to Y$ of schemes over S*, *base change*, *base change*, *base change*
In 21.18.3: *preserved under arbitrary base change*, *preserved under base change*, *preserved under arbitrary base change*, *preserved under base change*
In 21.18.4: *scheme theoretic fibre X_s of f over s*, *fibre of f over s*
In 21.19.1: *quasi-compact*
In 21.20.1: *universally closed*
In 21.20.3: *satisfies the existence part of the valuative criterion*, *satisfies the uniqueness part of the valuative criterion*

In 21.21.3: *separated*, *quasi-separated*, *separated*, *quasi-separated*
In 21.23.1: *monomorphism*

Constructions of Schemes

In 22.4.5: *relative spectrum of $\mathcal{A}$ over S*, *spectrum of $\mathcal{A}$ over S*
In 22.5.1: *affine n-space over S*, *affine n-space over R*
In 22.6.1: *vector bundle associated to $\mathcal{E}$*
In 22.6.2: *vector bundle $\pi : V \to S$ over S*, *morphism of vector bundles over S*
In 22.7.1: *cone associated to $\mathcal{A}$*, *affine cone associated to $\mathcal{A}$*
In 22.7.2: *cone $\pi : C \to S$ over S*, *morphism of cones*
In 22.8.2: *standard open covering*
In 22.8.3: *structure sheaf $\mathcal{O}_{Proj(S)}$ of the homogeneous spectrum of S*, *homogeneous spectrum*
In 22.10.1: *twist of the structure sheaf of $Proj(S)$*
In 22.13.2: *projective n-space over $\mathbf{Z}$*, *projective n-space over S*, *projective n-space over R*
In 22.16.7: *relative homogeneous spectrum of $\mathcal{A}$ over S*, *homogeneous spectrum of $\mathcal{A}$ over S*, *relative Proj of $\mathcal{A}$ over S*
In 22.20.1: *projective bundle associated to $\mathcal{E}$*, *twist of the structure sheaf*
In 22.21.1: *blowing up of X along Z*, *blowing up of X in the ideal sheaf $\mathcal{I}$*

Properties of Schemes

In 23.3.1: *integral*
In 23.4.1: *local*
In 23.4.2: *locally P*
In 23.5.1: *locally Noetherian*, *Noetherian*
In 23.6.1: *Jacobson*
In 23.7.1: *normal*
In 23.8.1: *Cohen-Macaulay*
In 23.9.1: *regular*, *nonsingular*
In 23.10.1: *dimension*, *dimension of X at x*
In 23.11.1: *catenary*
In 23.12.1: *regular in codimension k*, (R_k), (S_k)
In 23.13.1: *Japanese*, *universally Japanese*, *Nagata*
In 23.14.1: *regular locus*, *singular locus*
In 23.15.1: *quasi-affine*
In 23.19.1: *locally projective*
In 23.21.1: *κ-generated*
In 23.22.4: *subsheaf of sections annihilated by $\mathcal{I}$*
In 23.23.1: *ample*

Morphisms of Schemes

In 24.4.2: *scheme theoretic image*
In 24.5.1: *scheme theoretic closure of U in X*, *scheme theoretically dense in X*
In 24.6.1: *dominant*
In 24.7.1: *birational*
In 24.8.1: *equivalent*, *rational map from X to Y*, *S-rational map from X to Y*
In 24.8.2: *rational function on X*
In 24.8.3: *ring of rational functions on X*
In 24.8.5: *function field*, *field of rational functions*
In 24.9.1: *surjective*
In 24.10.1: *universally injective*, *radicial*
In 24.11.1: *affine*
In 24.12.1: *quasi-affine*
In 24.13.1: *local*, *stable under base change*, *stable under composition*
In 24.13.2: *locally of type P*
In 24.14.1: *finite type at $x \in X$*, *locally of finite type*, *finite type*
In 24.15.3: *finite type point*
In 24.16.1: *universally catenary*
In 24.18.1: *J-2*
In 24.19.1: *quasi-finite at a point $x \in X$*, *locally quasi-finite*, *quasi-finite*
In 24.20.1: *finite presentation at $x \in X$*, *locally of finite presentation*, *finite presentation*
In 24.22.1: *open*, *universally open*
In 24.23.1: *submersive*, *universally submersive*
In 24.24.1: *flat at a point $x \in X$*, *flat over S at a point $x \in X$*, *flat*, *flat over S*
In 24.25.2: *canonical scheme structure on T*
In 24.28.1: *relative dimension $\leq d$ at x*, *relative dimension $\leq d$*, *relative dimension d*
In 24.30.1: *syntomic at $x \in X$*, *syntomic*, *local complete intersection over k*, *standard syntomic*
In 24.30.15: *syntomic of relative dimension d*

In 24.31.1: *conormal sheaf* $\mathcal{C}_{Z/X}$ *of* Z *in* X, *conormal sheaf of* i
In 24.32.1: *derivation*, S*-derivation*, *Leibniz rule*
In 24.32.4: *sheaf of differentials* $\Omega_{X/S}$ *of* X *over* S, *universal* S*-derivation*
In 24.33.1: *smooth at* $x \in X$, *smooth*, *standard smooth*
In 24.33.13: *smooth of relative dimension* d
In 24.34.1: *unramified at* $x \in X$, G*-unramified at* $x \in X$, *unramified*, G*-unramified*
In 24.35.1: *étale at* $x \in X$, *étale*, *standard étale*
In 24.36.1: *relatively ample*, f*-relatively ample*, *ample on* X/S, f*-ample*
In 24.37.1: *relatively very ample*, f*-relatively very ample*, *very ample on* X/S, f*-very ample*
In 24.39.1: *quasi-projective*, *H-quasi-projective*, *locally quasi-projective*
In 24.40.1: *proper*
In 24.41.1: *projective*, *H-projective*, *locally projective*
In 24.42.1: *integral*, *finite*
In 24.43.1: *universal homeomorphism*
In 24.44.1: *finite locally free*, *rank*, *degree*
In 24.45.5: *degree of* X *over* Y
In 24.46.2: *integral closure of* $\mathcal{O}_X$ *in* $\mathcal{A}$
In 24.46.3: *normalization of* X *in* Y
In 24.46.12: *normalization*
In 24.48.1: *bounds the degrees of the fibres of* f, *fibres of* f *are universally bounded*

Coherent Cohomology

In 25.10.5: *scheme theoretic support of* $\mathcal{F}$
In 25.13.1: *depth* k *at a point*, *depth* k *at a point*, (S_k), (S_k)
In 25.13.2: *Cohen-Macaulay*

Divisors

In 26.2.1: *associated*, *associated points of* X
In 26.4.1: *embedded associated point*, *embedded point*, *embedded component*
In 26.5.1: *weakly associated*, *weakly associated points of* X
In 26.7.1: *relative assassin of* $\mathcal{F}$ *in* X *over* S
In 26.8.1: *relative weak assassin of* $\mathcal{F}$ *in* X *over* S
In 26.9.1: *locally principal closed subscheme*, *effective Cartier divisor*
In 26.9.4: *sum of the effective Cartier divisors* D_1 *and* D_2
In 26.9.8: *pullback of* D *by* f *is defined*, *pullback of the effective Cartier divisor*
In 26.9.11: *invertible sheaf* $\mathcal{O}_S(D)$ *associated to* D
In 26.9.13: *regular section*
In 26.9.15: *zero scheme*
In 26.10.2: *relative effective Cartier divisor*
In 26.11.1: *conormal algebra* $\mathcal{C}_{Z/X,*}$ *of* Z *in* X, *conormal algebra of* f
In 26.11.5: *normal cone* C_ZX, *normal bundle*
In 26.12.2: *regular*, *Koszul-regular*, H_1*-regular*, *quasi-regular*
In 26.13.1: *regular immersion*, *Koszul-regular immersion*, H_1*-regular immersion*, *quasi-regular immersion*
In 26.14.2: *relative quasi-regular immersion*, *relative* H_1*-regular immersion*
In 26.15.1: *sheaf of meromorphic functions on* X, $\mathcal{K}_X$, *meromorphic function*
In 26.15.3: *pulbacks of meromorphic functions are defined for* f
In 26.15.5: *meromorphic section of* $\mathcal{F}$
In 26.15.10: *regular*
In 26.15.14: *ideal sheaf of denominators of* s

Limits of Schemes

Varieties

In 28.3.1: *variety*
In 28.4.1: *geometrically reduced at* x, *geometrically reduced*
In 28.5.1: *geometrically connected*
In 28.6.1: *geometrically irreducible*
In 28.7.1: *geometrically pointwise integral at* x, *geometrically pointwise integral*, *geometrically integral*
In 28.8.1: *geometrically normal at* x, *geometrically normal*
In 28.10.1: *geometrically regular at* x, *geometrically regular over* k
In 28.13.1: *algebraic* k*-scheme*, *locally algebraic* k*-scheme*

In 28.16.1: *affine variety*, *projective variety*, *quasi-projective variety*, *proper variety*

Chow Homology and Chern Classes

In 29.2.1: *admissible*, *symbol*, *admissible relation*, *determinant of the finite length R-module*
In 29.3.1: 2-*periodic complex*, *cohomology modules*, *exact*, (2, 1)-*periodic complex*, *cohomology modules*
In 29.3.2: *multiplicity*, *Herbrand quotient*
In 29.3.4: *determinant of* (M, φ, ψ)
In 29.4.3: *symbol associated to* M, a, b
In 29.4.5: *tame symbol*
In 29.7.5: *δ-dimension of Z*
In 29.8.1: *locally finite*, *cycle on X*, *k-cycle*
In 29.9.2: *multiplicity of Z' in Z*, *k-cycle associated to Z*
In 29.10.2: *multiplicity of Z' in $\mathcal{F}$*, *k-cycle associated to $\mathcal{F}$*
In 29.12.1: *pushforward*
In 29.14.1: *flat pullback of α by f*
In 29.16.1: *order of vanishing of f along Z*
In 29.17.1: *principal divisor associated to f*
In 29.19.1: *rationally equivalent to zero*, *rationally equivalent*, *Chow group of k-cycles on X*, *Chow group of k-cycles module rational equivalence on X*
In 29.23.1: *order of vanishing of s along Z*
In 29.24.1: *Weil divisor associated to s*, *Weil divisor associated to $\mathcal{L}$*
In 29.25.1: *intersection with the first chern class of $\mathcal{L}$*
In 29.27.3: *ϵ-invariant*
In 29.27.5: *sum of the effective Cartier divisors*
In 29.28.1: *Gysin homomorphism*
In 29.34.1: *chern classes of $\mathcal{E}$ on X*, *total chern class of $\mathcal{E}$ on X*
In 29.35.1: *intersection with the jth chern class of $\mathcal{E}$*
In 29.36.1: *polynomial relation among the chern classes*

Topologies on Schemes

In 30.3.1: *Zariski covering of T*
In 30.3.4: *standard Zariski covering*
In 30.3.5: *big Zariski site*
In 30.3.7: *big Zariski site of S*, *small Zariski site of S*, *big affine Zariski site of S*
In 30.3.14: *restriction to the small Zariski site*
In 30.4.1: *étale covering of T*
In 30.4.5: *standard étale covering*
In 30.4.6: *big étale site*
In 30.4.8: *big étale site of S*, *small étale site of S*, *big affine étale site of S*
In 30.4.14: *restriction to the small étale site*
In 30.5.1: *smooth covering of T*
In 30.5.5: *standard smooth covering*
In 30.5.6: *big smooth site*
In 30.5.8: *big smooth site of S*, *big affine smooth site of S*
In 30.6.1: *syntomic covering of T*
In 30.6.5: *standard syntomic covering*
In 30.6.6: *big syntomic site*
In 30.6.8: *big syntomic site of S*, *big affine syntomic site of S*
In 30.7.1: *fppf covering of T*
In 30.7.5: *standard fppf covering*
In 30.7.6: *big fppf site*
In 30.7.8: *big fppf site of S*, *big affine fppf site of S*
In 30.8.1: *fpqc covering of T*
In 30.8.9: *standard fpqc covering*
In 30.8.12: *satisfies the sheaf property for the given family*, *satisfies the sheaf property for the fpqc topology*

Descent

In 31.2.1: *descent datum $(\mathcal{F}_i, \varphi_{ij})$ for quasi-coherent sheaves*, *cocycle condition*, *morphism $\psi : (\mathcal{F}_i, \varphi_{ij}) \to (\mathcal{F}'_i, \varphi'_{ij})$ of descent data*
In 31.2.3: *trivial descent datum*, *canonical descent datum*, *effective*
In 31.3.1: *descent datum (N, φ) for modules with respect to $R \to A$*, *cocycle condition*, *morphism $(N, \varphi) \to (N', \varphi')$ of descent data*
In 31.3.4: *effective*
In 31.6.2: *structure sheaf of the big site $(Sch/S)_\tau$*, *sheaf of $\mathcal{O}$-modules associated to $\mathcal{F}$*, *sheaf of $\mathcal{O}$-modules associated to $\mathcal{F}$*
In 31.7.1: *parasitic*, *parasitic for the τ-topology*
In 31.11.1: *local in the τ-topology*

In 31.16.1: *germ of X at x*, *morphism of germs*, *composition of morphisms of germs*
In 31.16.2: *étale*, *smooth*
In 31.17.1: *étale local*, *smooth local*
In 31.18.1: *τ local on the base*, *τ local on the target*, *local on the base for the τ-topology*
In 31.22.1: *τ local on the source*, *local on the source for the τ-topology*
In 31.28.3: *étale local on source-and-target*
In 31.29.1: *étale local on the source-and-target*
In 31.30.1: *descent datum for $V/X/S$*, *cocycle condition*, *descent datum relative to $X \to S$*, *morphism $f : (V/X, \varphi) \to (V'/X, \varphi')$ of descent data relative to $X \to S$*
In 31.30.3: *descent datum (V_i, φ_{ij}) relative to the family $\{X_i \to S\}$*, *morphism $\psi : (V_i, \varphi_{ij}) \to (V'_i, \varphi'_{ij})$ of descent data*
In 31.30.7: *pullback functor*
In 31.30.9: *pullback functor*
In 31.30.10: *trivial descent datum*, *canonical descent datum*, *effective*
In 31.30.11: *canonical descent datum*, *effective*
In 31.32.1: *morphisms of type $\mathcal{P}$ satisfy descent for τ-coverings*
In 31.36.1: *cartesian*, *$V_\bullet$ is cartesian over $X_\bullet$*
In 31.36.2: *simplicial scheme associated to f*

Adequate Modules

In 32.3.1: *module-valued functor*, *morphism of module-valued functors*
In 32.3.2: *adequate*, *linearly adequate*
In 32.5.1: *adequate*
In 32.5.7: *$Adeq(\mathcal{O})$*, *$Adeq((Sch/S)_\tau, \mathcal{O})$*, *$Adeq(S)$*
In 32.8.1: *pure projective*, *pure injective*
In 32.8.5: *pure projective resolution*, *pure injective resolution*
In 32.8.8: *pure extension module*

More on Morphisms

In 33.2.1: *thickening*, *first order thickening*, *morphism of thickenings*, *thickenings over S*, *morphisms of thickenings over S*
In 33.3.1: *first order infinitesimal neighbourhood*
In 33.4.1: *formally unramified*
In 33.5.2: *universal first order thickening*, *conormal sheaf of Z over X*
In 33.6.1: *formally étale*
In 33.9.1: *formally smooth*
In 33.13.1: *normal at x*, *normal morphism*
In 33.14.1: *regular at x*, *regular morphism*
In 33.15.1: *Cohen-Macaulay at x*, *Cohen-Macaulay morphism*
In 33.25.1: *étale neighbourhood of (S, s)*, *morphism of étale neighbourhoods*, *elementary étale neighbourhood*
In 33.36.2: *pseudo-coherent*
In 33.37.2: *perfect*
In 33.38.2: *Koszul at x*, *Koszul morphism*, *local complete intersection morphism*

More on flatness

In 34.2.1: *$\mathcal{F}$ is locally finitely presented relative to S*
In 34.5.1: *one step dévissage of $\mathcal{F}/X/S$ over s*
In 34.5.2: *one step dévissage of $\mathcal{F}/X/S$ at x*
In 34.5.6: *standard shrinking*
In 34.6.1: *complete dévissage of $\mathcal{F}/X/S$ over s*
In 34.6.2: *complete dévissage of $\mathcal{F}/X/S$ at x*
In 34.6.5: *standard shrinking*
In 34.7.1: *elementary étale localization of the ring map $R \to S$ at $\mathfrak{q}$*
In 34.7.2: *complete dévissage of $N/S/R$ over $\mathfrak{r}$*
In 34.7.4: *complete dévissage of $N/S/R$ at $\mathfrak{q}$*
In 34.15.2: *impurity of $\mathcal{F}$ above s*
In 34.16.1: *pure along X_s*, *universally pure along X_s*, *pure along X_s*, *universally S-pure*, *universally pure relative to S*, *S-pure*, *pure relative to S*, *S-pure*, *pure relative to S*
In 34.21.1: *universal flattening of $\mathcal{F}$ exists*, *universal flattening of X exists*
In 34.21.2: *flattening stratification*, *flattening stratification*

Groupoid Schemes

In 35.3.1: *pre-relation*, *relation*, *pre-equivalence relation*, *equivalence relation on U over S*
In 35.3.3: *restriction*, *pullback*

In 35.4.1: *group scheme over S*, *morphism $\psi : (G, m) \to (G', m')$ of group schemes over S*
In 35.4.3: *closed subgroup scheme*, *open subgroup scheme*
In 35.4.4: *smooth group scheme*, *flat group scheme*, *separated group scheme*
In 35.8.1: *action of G on the scheme X/S*, *equivariant*, *G-equivariant*
In 35.9.1: *pseudo G-torsor*, *formally principally homogeneous under G*, *trivial*
In 35.9.3: *principal homogeneous space*, *G-torsor*, *G-torsor in the τ topology*, *τ G-torsor*, *τ torsor*, *quasi-isotrivial*, *locally trivial*
In 35.10.1: *G-equivariant quasi-coherent $\mathcal{O}_X$-module*, *equivariant quasi-coherent $\mathcal{O}_X$-module*
In 35.11.1: *groupoid scheme over S*, *groupoid over S*, *morphism $f : (U, R, s, t, c) \to (U', R', s', t', c')$ of groupoid schemes over S*
In 35.12.1: *quasi-coherent module on (U, R, s, t, c)*
In 35.14.2: *stabilizer of the groupoid scheme (U, R, s, t, c)*
In 35.15.2: *restriction of (U, R, s, t, c) to U'*
In 35.16.1: *R-invariant*, *R-invariant*, *R-invariant*
In 35.17.1: *quotient sheaf U/R*
In 35.17.2: *representable quotient*, *representable quotient*

More on Groupoid Schemes

Étale Morphisms of Schemes

In 37.3.1: *unramified homomorphism of local rings*
In 37.3.5: *unramified at x*, *unramified*
In 37.9.1: *flat*, *faithfully flat*, *flat (resp. faithfully flat)*
In 37.9.3: *flat over Y at $x \in X$*, *flat at $x \in X$*, *flat*, *faithfully flat*
In 37.11.1: *étale homomorphism of local rings*
In 37.11.4: *étale at $x \in X$*, *étale*

Étale Cohomology

In 38.4.1: *étale covering*
In 38.9.1: *presheaf of sets*, *abelian presheaf*
In 38.10.1: *family of morphisms with fixed target*
In 38.10.2: *site*, *coverings*
In 38.11.1: *separated presheaf*, *sheaf*
In 38.11.4: *category of sheaves of sets*, *abelian sheaves*
In 38.13.1: *zeroth Cech cohomology group*
In 38.15.1: *fpqc covering*
In 38.15.5: *satisfies the sheaf property for the fpqc topology*
In 38.16.1: *descent datum*, *effective*
In 38.16.5: *descent datum*
In 38.16.6: *effective*
In 38.17.2: *ringed site*, *quasi-coherent*
In 38.18.1: *Cech complex*, *Cech cohomology groups*
In 38.18.4: *free abelian presheaf on $\mathcal{G}$*
In 38.20.1: *τ-covering*
In 38.20.2: *standard τ-covering*
In 38.20.4: *big τ-site of S*, *small τ-site of S*
In 38.21.1: *étale topos*, *small étale topos*, *Zariski topos*, *small Zariski topos*, *big τ-topos*
In 38.23.1: *constant sheaf*
In 38.23.3: *structure sheaf*
In 38.26.1: *étale*
In 38.26.3: *standard étale*
In 38.27.1: *étale covering*
In 38.27.3: *big étale site over S*, *small étale site over S*, *big*, *small Zariski sites*
In 38.29.1: *geometric point*, *lies over*, *étale neighborhood*, *morphism of étale neighborhoods*
In 38.29.6: *stalk*
In 38.31.3: *support of $\mathcal{F}$*, *support of σ*
In 38.32.2: *henselian*
In 38.32.6: *strictly henselian*
In 38.33.2: *étale local ring of S at $\overline{s}$*, *strict henselization of $\mathcal{O}_{S,s}$*, *henselization of $\mathcal{O}_{S,s}$*, *strict henselization of S at $\overline{s}$*, *henselization of S at s*
In 38.35.1: *direct image*, *pushforward*
In 38.35.3: *direct image*, *pushforward*
In 38.35.4: *higher direct images*
In 38.36.1: *inverse image*, *pullback*
In 38.55.1: *G-set*, *discrete G-set*, *morphism of G-sets*, *G-Sets*

In 38.56.1: *absolute Galois group*, *algebraic*
In 38.57.1: *G-module*, *discrete G-module*, *morphism of G-modules*, *Mod_G*
In 38.57.3: *continuous group cohomology groups*, *group cohomology groups*, *Galois cohomology groups*, *Galois cohomology groups of K with coefficients in M*
In 38.59.3: *similar*, *equivalent*
In 38.59.4: *Brauer group*
In 38.59.8: *C_r*, *nontrivial solution*
In 38.59.13: *variety*, *curve*
In 38.61.1: *abelian variety*
In 38.62.1: *finite locally constant*
In 38.62.3: *constructible*
In 38.63.1: *extension by zero*
In 38.64.3: *restriction*, *trace*
In 38.66.1: *absolute frobenius*
In 38.66.5: *geometric frobenius*
In 38.66.9: *arithmetic frobenius*
In 38.66.11: *geometric frobenius*
In 38.67.1: *trace*
In 38.69.4: *total right derived functor of F*, *total right derived functor of G*
In 38.70.1: *filtered injective*, *projective*, *filtered quasi-isomorphism*
In 38.71.1: *filtered derived functor*
In 38.73.1: *perfect*
In 38.75.1: *finite Tor-dimension*
In 38.75.4: *perfect complexes*
In 38.76.1: *global Lefschetz number*
In 38.76.2: *local Lefschetz number*
In 38.77.2: *G-trace of f on P*
In 38.80.1: *$\mathbf{Z}_\ell$-sheaf*, *lisse*, *morphism*
In 38.80.6: *torsion*, *stalk*
In 38.80.8: *ℓ-adic cohomology*
In 38.81.1: *L-function of $\mathcal{F}$*
In 38.81.3: *L-function of $\mathcal{F}$*
In 38.89.1: *open*
In 38.93.1: *unramified cusp form on $GL_2(\mathbf{A})$ with values in Λ*

Crystalline Cohomology

In 39.2.1: *divided power structure*
In 39.3.1: *divided power ring*, *homomorphism of divided power rings*
In 39.4.1: *extends*
In 39.6.2: *divided power envelope of J in B relative to (A, I, γ)*
In 39.8.1: *δ is compatible with γ*
In 39.9.2: *divided power thickening*, *homomorphism of divided power thickenings*
In 39.10.1: *divided power A-derivation*
In 39.11.1: *divided power structure γ*
In 39.11.2: *divided power scheme*, *morphism of divided power schemes*
In 39.11.3: *divided power thickening*
In 39.12.1: *divided power thickening of X relative to $(S, \mathcal{I}, \gamma)$*, *morphism of divided power thickenings of X relative to $(S, \mathcal{I}, \gamma)$*
In 39.12.4: *Zariski, étale, smooth, syntomic, or fppf covering*, *big crystalline site*
In 39.13.1: *crystalline site*
In 39.15.1: *locally quasi-coherent*, *quasi-coherent*, *crystal in $\mathcal{O}_{X/S}$-modules*
In 39.15.3: *crystal in quasi-coherent modules*, *crystal in finite locally free modules*
In 39.16.1: *S-derivation $D : \mathcal{O}_{X/S} \to \mathcal{F}$*
In 39.31.2: *F-crystal on X/S (relative to σ)*, *nondegenerate*

Algebraic Spaces

In 40.5.1: *property $\mathcal{P}$*
In 40.6.1: *algebraic space over S*
In 40.6.3: *morphism $f : F \to F'$ of algebraic spaces over S*
In 40.9.2: *étale equivalence relation*
In 40.9.3: *presentation*
In 40.12.1: *open immersion*, *open subspace*, *closed immersion*, *closed subspace*, *immersion*, *locally closed subspace*
In 40.12.5: *Zariski covering*
In 40.12.6: *small Zariski site F_{Zar}*
In 40.13.2: *separated over S*, *locally separated over S*, *quasi-separated over S*, *Zariski locally quasi-separated over S*
In 40.14.4: *acts freely*, *quotient of U by G*
In 40.16.2: *base change of F' to S*, *viewed as an algebraic space over S'*

Properties of Algebraic Spaces

In 41.3.1: *separated*, *locally separated*, *quasi-separated*, *Zariski locally quasi-separated*, *separated*, *locally separated*, *quasi-separated*, *Zariski locally quasi-separated*
In 41.4.1: *point*
In 41.4.7: *topological space*
In 41.5.1: *quasi-compact*

In 41.7.2: *has property* $\mathcal{P}$
In 41.7.5: *has property* $\mathcal{P}$ *at* x
In 41.8.1: *dimension of* X *at* x
In 41.8.2: *dimension*
In 41.9.3: *algebraic space structure on* Z, *reduced induced algebraic space structure*, *reduction* X_{red} *of* X
In 41.12.1: *Noetherian*
In 41.13.2: *étale*
In 41.15.1: *small étale site* $X_{étale}$
In 41.15.2: $X_{spaces,étale}$
In 41.15.6: *étale topos*, *small étale topos*
In 41.15.8: f*-map* $\varphi : \mathcal{G} \to \mathcal{F}$
In 41.16.1: *geometric point*, *geometric point lying over* x
In 41.16.2: *étale neighborhood*, *morphism of étale neighborhoods*
In 41.16.6: *stalk*
In 41.17.3: *support of* $\mathcal{F}$, *support of* σ
In 41.18.2: *structure sheaf*
In 41.19.2: *étale local ring of* X *at* $\overline{x}$, *strict henselization of* X *at* $\overline{x}$
In 41.20.2: *dimension of the local ring of* X *at* x
In 41.21.2: *geometrically unibranch at* x, *geometrically unibranch*
In 41.26.1: *quasi-coherent*
In 41.28.2: *locally projective*

Morphisms of Algebraic Spaces

In 42.5.2: *separated*, *locally separated*, *quasi-separated*
In 42.6.2: *surjective*
In 42.7.2: *open*, *universally open*
In 42.8.1: *submersive*, *universally submersive*
In 42.9.2: *quasi-compact*
In 42.10.2: *closed*, *universally closed*
In 42.11.1: *satisfies the uniqueness part of the valuative criterion*, *satisfies the existence part of the valuative criterion*, *satisfies the valuative criterion*
In 42.14.1: *monomorphism*
In 42.18.3: *universally injective*
In 42.19.2: *affine*
In 42.20.2: *quasi-affine*
In 42.21.2: *has property* $\mathcal{P}$
In 42.21.4: *has property* $\mathcal{Q}$ *at* x
In 42.22.1: *locally of finite type*, *finite type at* x, *of finite type*
In 42.24.2: *finite type point*
In 42.25.1: *locally quasi-finite*, *quasi-finite at* x, *quasi-finite*
In 42.26.1: *locally of finite presentation*, *finite presentation at* x, *of finite presentation*
In 42.27.1: *flat*, *flat at* x
In 42.28.2: *flat at* x *over* Y, *flat over* Y
In 42.30.1: *dimension of the local ring of the fibre of* f *at* x, *transcendence degree of* $x/f(x)$, f *has relative dimension* d *at* x
In 42.30.2: *relative dimension* $\leq d$, *relative dimension* d
In 42.32.1: *syntomic*, *syntomic at* x
In 42.33.1: *smooth*, *smooth at* x
In 42.34.1: *unramified*, *unramified at* x, *G-unramified*, *G-unramified at* x
In 42.35.1: *étale at* x
In 42.36.1: *proper*
In 42.37.2: *integral*, *finite*
In 42.38.2: *finite locally free*, *rank*, *degree*
In 42.41.2: *universal homeomorphism*

Decent Algebraic Spaces

In 43.3.1: *fibres of* f *are universally bounded*
In 43.6.1: *decent*, *reasonable*, *very reasonable*
In 43.10.6: *residual space of* X *at* x
In 43.13.1: *has property* (β), *has property* (β), *decent*, *reasonable*, *very reasonable*

Topologies on Algebraic Spaces

In 44.3.1: *fpqc covering of* X
In 44.4.1: *fppf covering of* X
In 44.5.1: *syntomic covering of* X
In 44.6.1: *smooth covering of* X
In 44.7.1: *étale covering of* X
In 44.8.1: *Zariski covering of* X

Descent and Algebraic Spaces

In 45.3.1: *descent datum* $(\mathcal{F}_i, \varphi_{ij})$ *for quasi-coherent sheaves*, *cocycle condition*, *morphism* $\psi : (\mathcal{F}_i, \varphi_{ij}) \to (\mathcal{F}'_i, \varphi'_{ij})$ *of descent data*
In 45.3.3: *trivial descent datum*, *canonical descent datum*, *effective*
In 45.9.1: τ *local on the base*, τ *local on the target*, *local on the base for the* τ*-topology*

In 45.12.1: *τ local on the source*, *local on the source for the τ-topology*
In 45.18.1: *smooth local on source-and-target*

More on Morphisms of Spaces

In 46.3.1: *radicial*
In 46.4.1: *locally of finite presentation*, *limit preserving*, *locally of finite presentation over S*, *locally of finite presentation*, *relatively limit preserving*
In 46.5.1: *conormal sheaf $\mathcal{C}_{Z/X}$ of Z in X*, *conormal sheaf of i*
In 46.6.2: *sheaf of differentials $\Omega_{X/Y}$ of X over Y*, *universal Y-derivation*
In 46.8.1: *thickening*, *first order thickening*, *morphism of thickenings*, *thickenings over B*, *morphisms of thickenings over B*
In 46.9.1: *first order infinitesimal neighbourhood*
In 46.10.1: *formally smooth*, *formally étale*, *formally unramified*
In 46.11.1: *formally unramified*
In 46.12.5: *universal first order thickening*, *conormal sheaf of Z over X*
In 46.13.1: *formally étale*
In 46.16.1: *formally smooth*
In 46.18.2: *the restriction of $\mathcal{F}$ to its fibre over z is flat at x over the fibre of Y over z*, *the fibre of X over z is flat at x over the fibre of Y over z*, *the fibre of X over z is flat over the fibre of Y over z*
In 46.21.2: *Koszul-regular immersion*, *H_1-regular immersion*, *quasi-regular immersion*
In 46.22.1: *pseudo-coherent*, *pseudo-coherent at x*
In 46.23.1: *perfect*, *perfect at x*
In 46.24.1: *Koszul morphism*, *local complete intersection morphism*, *Koszul at x*

Quot and Hilbert Spaces

Algebraic Spaces over Fields

Cohomology of Algebraic Spaces

In 49.3.1: *derived category of $\mathcal{O}_X$-modules with quasi-coherent cohomology sheaves*
In 49.6.2: *alternating Čech complex*

Stacks

In 50.2.2: *presheaf of morphisms from x to y*, *presheaf of isomorphisms from x to y*
In 50.3.1: *descent datum (X_i, φ_{ij}) in $\mathcal{S}$ relative to the family $\{f_i : U_i \to U\}$*, *cocycle condition*, *morphism $\psi : (X_i, \varphi_{ij}) \to (X'_i, \varphi'_{ij})$ of descent data*
In 50.3.4: *pullback functor*
In 50.3.5: *trivial descent datum*, *canonical descent datum*, *effective*
In 50.4.1: *stack*
In 50.4.5: *2-category of stacks over $\mathcal{C}$*
In 50.5.1: *stack in groupoids*
In 50.5.5: *2-category of stacks in groupoids over $\mathcal{C}$*
In 50.6.1: *stack in setoids*, *stack in sets*, *stack in discrete categories*
In 50.6.5: *2-category of stacks in setoids over $\mathcal{C}$*
In 50.10.2: *structure of site on $\mathcal{S}$ inherited from $\mathcal{C}$*, *$\mathcal{S}$ is endowed with the topology inherited from $\mathcal{C}$*
In 50.11.1: *gerbe*
In 50.11.4: *gerbe over*
In 50.12.4: *$f_*\mathcal{S}$*, *pushforward of $\mathcal{S}$ along f*
In 50.12.9: *$f^{-1}\mathcal{S}$*, *pullback of $\mathcal{S}$ along f*

Formal Deformation Theory

In 51.3.1: *$\mathcal{C}_\Lambda$*, *classical case*
In 51.3.2: *small extension*
In 51.3.6: *relative cotangent space*
In 51.3.9: *essential surjection*
In 51.4.1: *$\widehat{\mathcal{C}}_\Lambda$*
In 51.5.1: *category cofibered in groupoids over $\mathcal{C}$*
In 51.6.1: *prorepresentable*
In 51.6.2: *predeformation category*, *morphism of predeformation categories*
In 51.7.1: *category $\widehat{\mathcal{F}}$ of formal objects of $\mathcal{F}$*, *formal object $\xi = (R, \xi_n, f_n)$ of $\mathcal{F}$*, *morphism $a : \xi \to \eta$ of formal objects*
In 51.7.3: *completion of $\mathcal{F}$*
In 51.8.1: *smooth*
In 51.8.13: *versal*
In 51.9.1: *conditions (S1) and (S2)*
In 51.10.1: *R-linear*
In 51.10.9: *tangent space TF of F*
In 51.11.1: *tangent space $T\mathcal{F}$ of $\mathcal{F}$*

In 51.11.3: *differential* $d\varphi : T\mathcal{F} \to T\mathcal{G}$ *of* φ
In 51.13.4: *minimal*, *miniversal*
In 51.15.1: *condition (RS)*
In 51.15.8: *deformation category*
In 51.16.1: *lift of* x *along* f, *morphism of lifts*
In 51.18.1: *group of infinitesimal automorphisms of* x' *over* x
In 51.18.2: *group of infinitesimal automorphisms of* x_0
In 51.18.5: *automorphism functor of* x
In 51.19.1: *category of groupoids in functors on* $\mathcal{C}$, *groupoid in functors on* $\mathcal{C}$, *morphism* $(U, R, s, t, c) \to (U', R', s', t', c')$ *of groupoids in functors on* $\mathcal{C}$
In 51.19.4: *representable*
In 51.19.7: *restriction* $(U, R, s, t, c)|_{\mathcal{C}'}$ *of* (U, R, s, t, c) *to* $\mathcal{C}'$
In 51.19.9: *quotient category cofibered in groupoids* $[U/R] \to \mathcal{C}$, *quotient morphism* $U \to [U/R]$
In 51.20.1: *prorepresentable*
In 51.20.2: *completion* $(U, R, s, t, c)^\wedge$ *of* (U, R, s, t, c)
In 51.21.1: *smooth*
In 51.23.1: *presentation of* $\mathcal{F}$ *by* (U, R, s, t, c)
In 51.25.1: *normalized*, *minimal*

Groupoids in Algebraic Spaces

In 52.4.1: *pre-relation*, *relation*, *pre-equivalence relation*, *equivalence relation on* U *over* B
In 52.4.3: *restriction*, *pullback*
In 52.5.1: *group algebraic space over* B, *morphism* $\psi : (G, m) \to (G', m')$ *of group algebraic spaces over* B
In 52.8.1: *action of* G *on the algebraic space* X/B, *equivariant*, G*-equivariant*
In 52.8.2: *free*
In 52.9.1: *pseudo* G*-torsor*, *formally principally homogeneous under* G, *trivial*
In 52.9.3: *principal homogeneous space*, *principal homogeneous* G*-space over* B, G*-torsor in the* τ *topology*, τ G*-torsor*, τ *torsor*, *quasi-isotrivial*, *locally trivial*
In 52.10.1: G*-equivariant quasi-coherent* $\mathcal{O}_X$*-module*, *equivariant quasi-coherent* $\mathcal{O}_X$*-module*
In 52.11.1: *groupoid in algebraic spaces over* B, *morphism* $f : (U, R, s, t, c) \to (U', R', s', t', c')$ *of groupoids in algebraic spaces over* B
In 52.12.1: *quasi-coherent module on* (U, R, s, t, c)
In 52.15.2: *stabilizer of the groupoid in algebraic spaces* (U, R, s, t, c)
In 52.16.2: *restriction of* (U, R, s, t, c) *to* U'
In 52.17.1: R*-invariant*, R*-invariant*, R*-invariant*
In 52.18.1: *quotient sheaf* U/R
In 52.18.3: *quotient representable by an algebraic space*, *representable quotient*, *representable quotient*, *quotient representable by an algebraic space*
In 52.19.1: *quotient stack*, *quotient stack*

More on Groupoids in Spaces

In 53.11.1: *split over* u, *splitting of* R *over* u, *quasi-split over* u, *quasi-splitting of* R *over* u

Bootstrap

In 54.3.1: *representable by algebraic spaces*
In 54.4.1: *property* $\mathcal{P}$

Examples of Stacks

In 55.17.2: *degree* d *finite Hilbert stack of* $\mathcal{X}$ *over* $\mathcal{Y}$

Quotients of Groupoids

In 56.3.1: R*-invariant*, G*-invariant*
In 56.3.4: *pullback*, *flat pullback*
In 56.4.1: *categorical quotient*, *categorical quotient in* $\mathcal{C}$, *categorical quotient in the category of schemes*, *categorical quotient in schemes*
In 56.4.4: *universal categorical quotient*, *uniform categorical quotient*
In 56.5.1: *orbit*, R*-orbit*
In 56.5.4: *weakly* R*-equivalent*, R*-equivalent*, *weak orbit*, *weak* R*-orbit*, *orbit*, R*-orbit*
In 56.5.8: *set-theoretically* R*-invariant*, *separates orbits*, *separates* R*-orbits*
In 56.5.13: *set-theoretic pre-equivalence relation*, *set-theoretic equivalence relation*
In 56.5.18: *orbit space for* R

In 56.6.1: *coarse quotient*, *coarse quotient in schemes*
In 56.7.1: *uniformly*, *universally*
In 56.8.1: *sheaf of R-invariant functions on X*, *the functions on X are the R-invariant functions on U*
In 56.9.1: *good quotient*
In 56.10.1: *geometric quotient*

Algebraic Stacks

In 57.8.1: *representable by an algebraic space over S*
In 57.9.1: *representable by algebraic spaces*
In 57.10.1: *property $\mathcal{P}$*
In 57.12.1: *algebraic stack over S*
In 57.12.2: *Deligne-Mumford stack*
In 57.12.3: *2-category of algebraic stacks over S*
In 57.16.4: *smooth groupoid*
In 57.16.5: *presentation*
In 57.19.2: *viewed as an algebraic stack over S'*
In 57.19.3: *change of base of $\mathcal{X}'$*

Sheaves on Algebraic Stacks

In 58.3.1: *presheaf on $\mathcal{X}$*, *morphism of presheaves on $\mathcal{X}$*
In 58.4.1: *associated Zariski site*, *associated étale site*, *associated smooth site*, *associated syntomic site*, *associated fppf site*
In 58.4.3: *Zariski sheaf*, *sheaf for the Zariski topology*, *étale sheaf*, *sheaf for the étale topology*, *smooth sheaf*, *sheaf for the smooth topology*, *syntomic sheaf*, *sheaf for the syntomic topology*, *fppf sheaf*, *sheaf*, *sheaf for the fppf topology*
In 58.4.5: *associated morphism of fppf topoi*
In 58.6.1: *structure sheaf of $\mathcal{X}$*
In 58.7.1: *presheaf of modules on $\mathcal{X}$*, *$\mathcal{O}_\mathcal{X}$-module*, *sheaf of $\mathcal{O}_\mathcal{X}$-modules*
In 58.9.2: *pullback $x^{-1}\mathcal{F}$ of $\mathcal{F}$*, *restriction of $\mathcal{F}$ to $U_{étale}$*
In 58.11.1: *quasi-coherent module on $\mathcal{X}$*, *quasi-coherent $\mathcal{O}_\mathcal{X}$-module*
In 58.11.4: *locally quasi-coherent*

Criteria for Representability

In 59.8.1: *algebraic*

Properties of Algebraic Stacks

In 60.4.2: *point*
In 60.4.8: *topological space*
In 60.5.1: *surjective*
In 60.6.1: *quasi-compact*
In 60.7.2: *has property $\mathcal{P}$*
In 60.7.5: *has property $\mathcal{P}$ at x*
In 60.8.1: *monomorphism*
In 60.9.1: *open immersion*, *closed immersion*, *immersion*
In 60.9.8: *open substack*, *closed substack*, *locally closed substack*
In 60.10.4: *algebraic stack structure on Z*, *reduced induced algebraic stack structure*, *reduction $\mathcal{X}_{red}$ of $\mathcal{X}$*
In 60.11.8: *residual gerbe of $\mathcal{X}$ at x exists*, *residual gerbe of $\mathcal{X}$ at x*

Morphisms of Algebraic Stacks

In 61.4.1: *DM*, *quasi-DM*, *separated*, *quasi-separated*
In 61.4.2: *DM over S*, *quasi-DM over S*, *separated over S*, *quasi-separated over S*, *DM*, *quasi-DM*, *separated*, *quasi-separated*
In 61.5.3: *sheaf of automorphisms of x*
In 61.7.2: *quasi-compact*
In 61.8.1: *Noetherian*
In 61.9.2: *open*, *universally open*
In 61.10.1: *submersive*, *universally submersive*
In 61.11.2: *closed*, *universally closed*
In 61.12.2: *has property $\mathcal{P}$*
In 61.13.1: *locally of finite type*, *of finite type*
In 61.14.2: *finite type point*
In 61.16.2: *locally quasi-finite*
In 61.17.1: *flat*
In 61.18.1: *locally of finite presentation*, *of finite presentation*
In 61.19.1: *gerbe over*, *gerbe*
In 61.22.1: *smooth*

Cohomology of Algebraic Stacks

In 62.7.1: *flat base change property*
In 62.8.1: *parasitic*
In 62.11.1: *lisse-étale site*, *flat-fppf site*
In 62.13.1: *derived category of $\mathcal{O}_\mathcal{X}$-modules with quasi-coherent cohomology sheaves*

Introducing Algebraic Stacks

In 63.4.3: *smooth*

In 63.5.1: *algebraic stack*

Examples

Exercises

In 65.2.1: *directed partially ordered set*, *system of rings*
In 65.2.3: *colimit*
In 65.2.8: *finite presentation*
In 65.5.4: *quasi-compact*
In 65.5.6: *Hausdorff*
In 65.5.9: *irreducible*, *irreducible*
In 65.5.12: *generic point*
In 65.5.16: *Noetherian*, *Artinian*
In 65.5.18: *irreducible component*
In 65.5.22: *closed*, *specialization*, *generalization*
In 65.5.26: *connected*, *connected component*
In 65.8.1: *length*
In 65.12.1: *catenary*
In 65.15.1: *finite locally free*, *invertible module*
In 65.15.3: *class group of A*, *Picard group of A*
In 65.17.1: *going-up theorem*, *going-down theorem*
In 65.19.1: *numerical polynomial*
In 65.19.2: *graded module*, *locally finite*, *Euler-Poincaré function*, *Hilbert function*, *Hilbert polynomial*
In 65.19.3: *graded A-algebra*, *graded module M over a graded A-algebra B*, *homomorphisms of graded modules/rings*, *graded submodules*, *graded ideals*, *exact sequences of graded modules*
In 65.20.1: *homogeneous*
In 65.20.2: *homogeneous spectrum $Proj(R)$*
In 65.20.3: $R_{(f)}$
In 65.21.1: *Cohen-Macaulay*
In 65.23.3: *filtered injective*
In 65.23.4: $Fil^f(\mathcal{A})$
In 65.23.6: *filtered quasi-isomorphism*
In 65.23.7: *filtered acyclic*
In 65.26.12: *integral*
In 65.28.1: *dual numbers*
In 65.28.3: *tangent space of X over S*, *tangent vector*
In 65.29.1: *quasi-coherent*
In 65.29.2: *specialization*
In 65.29.5: *locally Noetherian*, *Noetherian*
In 65.29.6: *coherent*
In 65.32.1: *invertible $\mathcal{O}_X$-module*
In 65.32.4: *invertible module M*, *trivial*
In 65.32.7: *Picard group of X*
In 65.33.2: $\delta(\tau)$
In 65.34.1: *Weil divisor*, *prime divisor*, *Weil divisor associated to a rational function $f \in K(X)^*$*, *effective Cartier divisor*, *Weil divisor $[D]$ associated to an effective Cartier divisor $D \subset X$*, *sheaf of total quotient rings $\mathcal{K}_S$*, *Cartier divisor*, *Weil divisor associated to a Cartier divisor*

A Guide to the Literature

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