Commit: 6ed92f8527de1b84dd020ae49e70d477b0458f93
algebra.tex, lemma-NAK,

Lemma 0.1. *(Nakayama's lemma.) If M is a finite nonzero module over R, then $\mathfrak{m}M \neq M$.*

Proof. Here is a silly way to prove this: If $\mathfrak{m}M = M$ for M finite then by induction $\mathfrak{m}^n M = M$. Hence the completion of M with respect to the maximal ideal is zero. Hence $M \otimes_R \hat{R} = 0$, see Lemma **??**. But $R \to \hat{R}$ is faithfully flat by Lemma **??** and hence we conclude $M = 0$ by Lemma **??**. □

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Lemma 0.2. *(Nakayama's lemma.) Let R be a ring, let M be an R-module, and let $I \subset R$ be an ideal.*

1. *If M is finite, and $IM = M$, then there exists a $f = 1 + i \in 1 + I$ such that $fM = 0$.*
2. *If M is finite, $IM = M$, and $I \subset rad(R)$ then $M = 0$.*
3. *If $IM = M$, I is nilpotent, then $M = 0$.*

Proof. Proof of 1. Write $M = \sum Rx_j$, $j = 1, \ldots, r$. Write $x_j = \sum i_{jj'} x_{j'}$ with $i_{jj'} \in I$. In other words $\sum (\delta_{jj'} - i_{jj'}) x_{j'} = 0$. Hence the determinant f of the $r \times r$ matrix $(\delta_{jj'} - i_{jj'})$ is a solution. The other parts are easy. □

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algebra.tex, lemma-NAK,

Lemma 0.3. *(Nakayama's lemma.) Let R be a ring, let M be an R-module, and let $I \subset R$ be an ideal.*

1. *If M is finite, and $IM = M$, then there exists a $f = 1 + i \in 1 + I$ such that $fM = 0$.*
2. *If M is finite, $IM = M$, and $I \subset rad(R)$ then $M = 0$.*
3. *If $IM = M$, I is nilpotent, then $M = 0$.*

Proof. Proof of (1). Write $M = \sum Rx_j$, $j = 1, \ldots, r$. Write $x_j = \sum i_{jj'} x_{j'}$ with $i_{jj'} \in I$. In other words $\sum (\delta_{jj'} - i_{jj'}) x_{j'} = 0$. Hence the determinant f of the $r \times r$ matrix $(\delta_{jj'} - i_{jj'})$ is a solution. The other parts are easy. □

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algebra.tex, lemma-NAK,

Lemma 0.4. *(Nakayama's lemma.) Let R be a ring, let M be an R-module, and let $I \subset R$ be an ideal.*

1. *If M is finite, and $IM = M$, then there exists a $f = 1 + i \in 1 + I$ such that $fM = 0$.*
2. *If M is finite, $IM = M$, and $I \subset rad(R)$ then $M = 0$.*
3. *If $IM = M$, I is nilpotent, then $M = 0$.*

Proof. Proof of (1). Write $M = \sum Rx_j$, $j = 1, \ldots, r$. Write $x_j = \sum i_{jj'} x_{j'}$ with $i_{jj'} \in I$. In other words $\sum (\delta_{jj'} - i_{jj'}) x_{j'} = 0$. Hence the determinant f of the $r \times r$ matrix $(\delta_{jj'} - i_{jj'})$ is a solution. The other parts are easy. □

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algebra.tex, lemma-NAK,

Lemma 0.5. *(Nakayama's lemma.) Let R be a ring, let M be an R-module, and let $I \subset R$ be an ideal.*

(1) *If M is finite, and $IM = M$, then there exists a $f = 1 + i \in 1 + I$ such that $fM = 0$.*
(2) *If M is finite, $IM = M$, and $I \subset rad(R)$ then $M = 0$.*
(3) *If $IM = M$, I is nilpotent, then $M = 0$.*

Proof. Proof of (1). Write $M = \sum Rx_j$, $j = 1, \ldots, r$. Write $x_j = \sum i_{jj'} x_{j'}$ with $i_{jj'} \in I$. In other words $\sum(\delta_{jj'} - i_{jj'})x_{j'} = 0$. Hence the determinant f of the $r \times r$ matrix $(\delta_{jj'} - i_{jj'})$ is a solution. The other parts are easy. □

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algebra.tex, lemma-NAK,

Lemma 0.6. *(Nakayama's lemma.) Let R be a ring, let M be an R-module, and let $I \subset R$ be an ideal.*

(1) *If M is finite, and $IM = M$, then there exists a $f = 1 + i \in 1 + I$ such that $fM = 0$.*
(2) *If M is finite, $IM = M$, and $I \subset rad(R)$ then $M = 0$.*
(3) *If $IM = M$, I is nilpotent, then $M = 0$.*

Proof. Proof of (1). Write $M = \sum Rx_j$, $j = 1, \ldots, r$. Write $x_j = \sum i_{jj'} x_{j'}$ with $i_{jj'} \in I$. In other words $\sum(\delta_{jj'} - i_{jj'})x_{j'} = 0$. Hence the determinant f of the $r \times r$ matrix $(\delta_{jj'} - i_{jj'})$ is a solution. The other parts are easy. □

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algebra.tex, lemma-NAK, 00DV

Lemma 0.7. *(Nakayama's lemma.) Let R be a ring, let M be an R-module, and let $I \subset R$ be an ideal.*

(1) *If M is finite, and $IM = M$, then there exists a $f = 1 + i \in 1 + I$ such that $fM = 0$.*
(2) *If M is finite, $IM = M$, and $I \subset rad(R)$ then $M = 0$.*
(3) *If $IM = M$, I is nilpotent, then $M = 0$.*

Proof. Proof of (1). Write $M = \sum Rx_j$, $j = 1, \ldots, r$. Write $x_j = \sum i_{jj'} x_{j'}$ with $i_{jj'} \in I$. In other words $\sum(\delta_{jj'} - i_{jj'})x_{j'} = 0$. Hence the determinant f of the $r \times r$ matrix $(\delta_{jj'} - i_{jj'})$ is a solution. The other parts are easy. □

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algebra.tex, lemma-NAK, 00DV

Lemma 0.8. *(Nakayama's lemma.) Let R be a ring, let M be an R-module, and let $I \subset R$ be an ideal.*

(1) *If M is finite, and $IM = M$, then there exists a $f = 1 + i \in 1 + I$ such that $fM = 0$.*
(2) *If M is finite, $IM = M$, and $I \subset rad(R)$ then $M = 0$.*
(3) *If $IM = M$, I is nilpotent, then $M = 0$.*

Proof. Proof of (1). Write $M = \sum Rx_j$, $j = 1, \ldots, r$. Write $x_j = \sum i_{jj'}x_{j'}$ with $i_{jj'} \in I$. In other words $\sum(\delta_{jj'} - i_{jj'})x_{j'} = 0$. Hence the determinant f of the $r \times r$ matrix $(\delta_{jj'} - i_{jj'})$ is a solution. The other parts are easy. □

Commit: 046ef996f091e082c9abf898cbda171e7d057afd
algebra.tex, lemma-NAK, 00DV

Lemma 0.9. *(Nakayama's lemma.) Let R be a ring, let M be an R-module, and let $I \subset R$ be an ideal.*

1. *If M is finite, and $IM = M$, then there exists a $f = 1 + i \in 1 + I$ such that $fM = 0$.*
2. *If M is finite, $IM = M$, and $I \subset rad(R)$ then $M = 0$.*
3. *If $IM = M$, I is nilpotent, then $M = 0$.*

Proof. Proof of (1). Write $M = \sum Rx_j$, $j = 1, \ldots, r$. Write $x_j = \sum i_{jj'}x_{j'}$ with $i_{jj'} \in I$. In other words $\sum(\delta_{jj'} - i_{jj'})x_{j'} = 0$. Hence the determinant f of the $r \times r$ matrix $(\delta_{jj'} - i_{jj'})$ is a solution. The other parts are easy. □

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algebra.tex, lemma-NAK, 00DV

Lemma 0.10. *(Nakayama's lemma.) Let R be a ring, let M be an R-module, and let $I \subset R$ be an ideal.*

1. *If M is finite, and $IM = M$, then there exists a $f = 1 + i \in 1 + I$ such that $fM = 0$.*
2. *If M is finite, $IM = M$, and $I \subset rad(R)$ then $M = 0$.*
3. *If $IM = M$, I is nilpotent, then $M = 0$.*

Proof. Proof of (1). Write $M = \sum Rx_j$, $j = 1, \ldots, r$. Write $x_j = \sum i_{jj'}x_{j'}$ with $i_{jj'} \in I$. In other words $\sum(\delta_{jj'} - i_{jj'})x_{j'} = 0$. Hence the determinant f of the $r \times r$ matrix $(\delta_{jj'} - i_{jj'})$ is a solution. The other parts are easy. □

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algebra.tex, lemma-NAK, 00DV

Lemma 0.11 (Nakayama's lemma)**.** *Let R be a ring, let M be an R-module, and let $I \subset R$ be an ideal.*

1. *If M is finite and $IM = M$, then there exists a $f \in 1 + I$ such that $fM = 0$.*
2. *If $IM = M$, $I \subset rad(R)$, and M is finite, then $M = 0$.*
3. *If $N, N' \subset M$, $M = N + IN'$, $I \subset rad(R)$, and N' is finite then $M = N$.*
4. *If $x_1, \ldots, x_n \in M$ generate M/IM and M is finite, then there exists an $f \in 1 + I$ such that $x_1, \ldots, x_n$ generate M_f over R_f.*
5. *If $x_1, \ldots, x_n \in M$ generate M/IM, $I \subset rad(R)$, and M is finite, then M is generated by $x_1, \ldots, x_n$.*
6. *If $IM = M$, I is nilpotent, then $M = 0$.*
7. *If $N, N' \subset M$, $M = N + IN'$, and I is nilpotent then $M = N$.*
8. *If $\{x_\alpha\}_{\alpha \in A}$ is a set of elements of M which generate M/IM and I is nilpotent, then M is generated by the x_α.*

Proof. Proof of (1). Choose generators $y_1, \ldots, y_m$ of M over R. For each i we can write $y_i = \sum z_{ij} y_j$ with $z_{ij} \in I$. In other words $\sum_j (\delta_{ij} - z_{ij}) y_j = 0$. Let f be the determinant f of the $m \times m$ matrix $A = (\delta_{ij} - z_{ij})$. Note that $f \in 1 + I$. By Lemma **??** there exists an $m \times m$ matrix B such that $BA = f 1_{m \times m}$. Writing out we see that $f y_j = \sum_{i,j} b_{hi} a_{ij} y_j = 0$ for every j. This implies that f annihilates M.

By Lemma **??** an element of $1 + \text{rad}(R)$ is invertible element of R. Hence we see that (1) implies (2). We obtain (3) by applying (2) to M/N. We obtain (4) by applying (1) to $M/Rx_1 + \ldots + Rx_n$. We obtain (5) from (4) by the first remark of this paragraph.

Part (6) holds because if $M = IM$ then $M = I^n M$ for all $n \geq 0$ and I being nilpotent means $I^n = 0$ for some $n \gg 0$. Parts (7) and (8) follow from (6) by considering the quotient of M by the given submodule. □

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algebra.tex, lemma-NAK, 00DV

Lemma 0.12 (Nakayama's lemma)**.** *Let R be a ring, let M be an R-module, and let $I \subset R$ be an ideal.*

(1) *If $IM = M$ and M is finite, then there exists a $f \in 1+I$ such that $fM = 0$.*
(2) *If $IM = M$, M is finite, and $I \subset \text{rad}(R)$, then $M = 0$.*
(3) *If $N, N' \subset M$, $M = N + IN'$, and N' is finite, then there exists a $f \in 1+I$ such that $M_f = N_f$.*
(4) *If $N, N' \subset M$, $M = N + IN'$, N' is finite, and $I \subset \text{rad}(R)$, then $M = N$.*
(5) *If $N \to M$ is a module map, $N/IN \to M/IM$ is surjective, and M is finite, then there exists a $f \in 1 + I$ such that $N_f \to M_f$ is surjective.*
(6) *If $N \to M$ is a module map, $N/IN \to M/IM$ is surjective, M is finite, and $I \subset \text{rad}(R)$, then $N \to M$ is surjective.*
(7) *If $x_1, \ldots, x_n \in M$ generate M/IM and M is finite, then there exists an $f \in 1 + I$ such that $x_1, \ldots, x_n$ generate M_f over R_f.*
(8) *If $x_1, \ldots, x_n \in M$ generate M/IM, M is finite, and $I \subset \text{rad}(R)$, then M is generated by $x_1, \ldots, x_n$.*
(9) *If $IM = M$, I is nilpotent, then $M = 0$.*
(10) *If $N, N' \subset M$, $M = N + IN'$, and I is nilpotent then $M = N$.*
(11) *If $N \to M$ is a module map, I is nilpotent, and $N/IN \to M/IM$ is surjective, then $N \to M$ is surjective.*
(12) *If $\{x_\alpha\}_{\alpha \in A}$ is a set of elements of M which generate M/IM and I is nilpotent, then M is generated by the x_α.*

Proof. Proof of (1). Choose generators $y_1, \ldots, y_m$ of M over R. For each i we can write $y_i = \sum z_{ij} y_j$ with $z_{ij} \in I$. In other words $\sum_j (\delta_{ij} - z_{ij}) y_j = 0$. Let f be the determinant f of the $m \times m$ matrix $A = (\delta_{ij} - z_{ij})$. Note that $f \in 1 + I$. By Lemma **??** there exists an $m \times m$ matrix B such that $BA = f 1_{m \times m}$. Writing out we see that $f y_j = \sum_{i,j} b_{hi} a_{ij} y_j = 0$ for every j. This implies that f annihilates M.

By Lemma **??** an element of $1+\text{rad}(R)$ is invertible element of R. Hence we see that (1) implies (2). We obtain (3) by applying (1) to M/N which is finite as N' is finite. We obtain (4) by applying (2) to M/N which is finite as N' is finite. We obtain (5) by applying (3) to M and the submodules $\text{Im}(N \to M)$ and M. We obtain (6) by applying (4) to M and the submodules $\text{Im}(N \to M)$ and M. We obtain (7) by applying (5) to the map $R^{\oplus n} \to M$, $(a_1, \ldots, a_n) \mapsto a_1x_1 + \ldots + a_nx_n$. We obtain (8) by applying (6) to the map $R^{\oplus n} \to M$, $(a_1, \ldots, a_n) \mapsto a_1x_1 + \ldots + a_nx_n$.

Part (9) holds because if $M = IM$ then $M = I^nM$ for all $n \geq 0$ and I being nilpotent means $I^n = 0$ for some $n \gg 0$. Parts (10), (11), and (12) follow from (9) by the arguments used above. □

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algebra.tex, lemma-NAK, 00DV

Lemma 0.13 (Nakayama's lemma)**.** *Let R be a ring, let M be an R-module, and let $I \subset R$ be an ideal.*

(1) *If $IM = M$ and M is finite, then there exists a $f \in 1+I$ such that $fM = 0$.*
(2) *If $IM = M$, M is finite, and $I \subset rad(R)$, then $M = 0$.*
(3) *If $N, N' \subset M$, $M = N + IN'$, and N' is finite, then there exists a $f \in 1+I$ such that $M_f = N_f$.*
(4) *If $N, N' \subset M$, $M = N + IN'$, N' is finite, and $I \subset rad(R)$, then $M = N$.*
(5) *If $N \to M$ is a module map, $N/IN \to M/IM$ is surjective, and M is finite, then there exists a $f \in 1 + I$ such that $N_f \to M_f$ is surjective.*
(6) *If $N \to M$ is a module map, $N/IN \to M/IM$ is surjective, M is finite, and $I \subset rad(R)$, then $N \to M$ is surjective.*
(7) *If $x_1, \ldots, x_n \in M$ generate M/IM and M is finite, then there exists an $f \in 1 + I$ such that $x_1, \ldots, x_n$ generate M_f over R_f.*
(8) *If $x_1, \ldots, x_n \in M$ generate M/IM, M is finite, and $I \subset rad(R)$, then M is generated by $x_1, \ldots, x_n$.*
(9) *If $IM = M$, I is nilpotent, then $M = 0$.*
(10) *If $N, N' \subset M$, $M = N + IN'$, and I is nilpotent then $M = N$.*
(11) *If $N \to M$ is a module map, I is nilpotent, and $N/IN \to M/IM$ is surjective, then $N \to M$ is surjective.*
(12) *If $\{x_\alpha\}_{\alpha \in A}$ is a set of elements of M which generate M/IM and I is nilpotent, then M is generated by the x_α.*

Proof. Proof of (1). Choose generators $y_1, \ldots, y_m$ of M over R. For each i we can write $y_i = \sum z_{ij}y_j$ with $z_{ij} \in I$. In other words $\sum_j(\delta_{ij} - z_{ij})y_j = 0$. Let f be the determinant f of the $m \times m$ matrix $A = (\delta_{ij} - z_{ij})$. Note that $f \in 1 + I$. By Lemma **??** there exists an $m \times m$ matrix B such that $BA = f1_{m\times m}$. Writing out we see that $fy_j = \sum_{i,j} b_{hi}a_{ij}y_j = 0$ for every j. This implies that f annihilates M.

By Lemma **??** an element of $1+\text{rad}(R)$ is invertible element of R. Hence we see that (1) implies (2). We obtain (3) by applying (1) to M/N which is finite as N' is finite. We obtain (4) by applying (2) to M/N which is finite as N' is finite. We obtain (5) by applying (3) to M and the submodules $\text{Im}(N \to M)$ and M. We obtain (6) by applying (4) to M and the submodules $\text{Im}(N \to M)$ and M. We obtain (7) by applying (5) to the map $R^{\oplus n} \to M$, $(a_1, \ldots, a_n) \mapsto a_1x_1 + \ldots + a_nx_n$. We obtain (8) by applying (6) to the map $R^{\oplus n} \to M$, $(a_1, \ldots, a_n) \mapsto a_1x_1 + \ldots + a_nx_n$.

Part (9) holds because if $M = IM$ then $M = I^nM$ for all $n \geq 0$ and I being nilpotent means $I^n = 0$ for some $n \gg 0$. Parts (10), (11), and (12) follow from (9) by the arguments used above. □

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algebra.tex, lemma-NAK, 00DV

Lemma 0.14 (Nakayama's lemma)**.** *Let R be a ring, let M be an R-module, and let $I \subset R$ be an ideal.*

(1) *If $IM = M$ and M is finite, then there exists a $f \in 1+I$ such that $fM = 0$.*
(2) *If $IM = M$, M is finite, and $I \subset rad(R)$, then $M = 0$.*
(3) *If $N, N' \subset M$, $M = N + IN'$, and N' is finite, then there exists a $f \in 1+I$ such that $M_f = N_f$.*
(4) *If $N, N' \subset M$, $M = N + IN'$, N' is finite, and $I \subset rad(R)$, then $M = N$.*
(5) *If $N \to M$ is a module map, $N/IN \to M/IM$ is surjective, and M is finite, then there exists a $f \in 1 + I$ such that $N_f \to M_f$ is surjective.*
(6) *If $N \to M$ is a module map, $N/IN \to M/IM$ is surjective, M is finite, and $I \subset rad(R)$, then $N \to M$ is surjective.*
(7) *If $x_1, \ldots, x_n \in M$ generate M/IM and M is finite, then there exists an $f \in 1 + I$ such that $x_1, \ldots, x_n$ generate M_f over R_f.*
(8) *If $x_1, \ldots, x_n \in M$ generate M/IM, M is finite, and $I \subset rad(R)$, then M is generated by $x_1, \ldots, x_n$.*
(9) *If $IM = M$, I is nilpotent, then $M = 0$.*
(10) *If $N, N' \subset M$, $M = N + IN'$, and I is nilpotent then $M = N$.*
(11) *If $N \to M$ is a module map, I is nilpotent, and $N/IN \to M/IM$ is surjective, then $N \to M$ is surjective.*
(12) *If $\{x_\alpha\}_{\alpha \in A}$ is a set of elements of M which generate M/IM and I is nilpotent, then M is generated by the x_α.*

Proof. Proof of (1). Choose generators $y_1, \ldots, y_m$ of M over R. For each i we can write $y_i = \sum z_{ij} y_j$ with $z_{ij} \in I$. In other words $\sum_j (\delta_{ij} - z_{ij}) y_j = 0$. Let f be the determinant f of the $m \times m$ matrix $A = (\delta_{ij} - z_{ij})$. Note that $f \in 1 + I$. By Lemma **??** there exists an $m \times m$ matrix B such that $BA = f 1_{m \times m}$. Writing out we see that $f y_j = \sum_{i,j} b_{hi} a_{ij} y_j = 0$ for every j. This implies that f annihilates M.

By Lemma **??** an element of $1+\text{rad}(R)$ is invertible element of R. Hence we see that (1) implies (2). We obtain (3) by applying (1) to M/N which is finite as N' is finite. We obtain (4) by applying (2) to M/N which is finite as N' is finite. We obtain (5) by applying (3) to M and the submodules $\text{Im}(N \to M)$ and M. We obtain (6) by applying (4) to M and the submodules $\text{Im}(N \to M)$ and M. We obtain (7) by applying (5) to the map $R^{\oplus n} \to M$, $(a_1, \ldots, a_n) \mapsto a_1 x_1 + \ldots + a_n x_n$. We obtain (8) by applying (6) to the map $R^{\oplus n} \to M$, $(a_1, \ldots, a_n) \mapsto a_1 x_1 + \ldots + a_n x_n$.

Part (9) holds because if $M = IM$ then $M = I^n M$ for all $n \geq 0$ and I being nilpotent means $I^n = 0$ for some $n \gg 0$. Parts (10), (11), and (12) follow from (9) by the arguments used above. □

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algebra.tex, lemma-NAK, 00DV

Lemma 0.15 (Nakayama's lemma)**.** *Let R be a ring, let M be an R-module, and let $I \subset R$ be an ideal.*

(1) *If $IM = M$ and M is finite, then there exists a $f \in 1+I$ such that $fM = 0$.*
(2) *If $IM = M$, M is finite, and $I \subset rad(R)$, then $M = 0$.*

(3) *If $N, N' \subset M$, $M = N + IN'$, and N' is finite, then there exists a $f \in 1+I$ such that $M_f = N_f$.*
(4) *If $N, N' \subset M$, $M = N + IN'$, N' is finite, and $I \subset rad(R)$, then $M = N$.*
(5) *If $N \to M$ is a module map, $N/IN \to M/IM$ is surjective, and M is finite, then there exists a $f \in 1 + I$ such that $N_f \to M_f$ is surjective.*
(6) *If $N \to M$ is a module map, $N/IN \to M/IM$ is surjective, M is finite, and $I \subset rad(R)$, then $N \to M$ is surjective.*
(7) *If $x_1, \ldots, x_n \in M$ generate M/IM and M is finite, then there exists an $f \in 1 + I$ such that $x_1, \ldots, x_n$ generate M_f over R_f.*
(8) *If $x_1, \ldots, x_n \in M$ generate M/IM, M is finite, and $I \subset rad(R)$, then M is generated by $x_1, \ldots, x_n$.*
(9) *If $IM = M$, I is nilpotent, then $M = 0$.*
(10) *If $N, N' \subset M$, $M = N + IN'$, and I is nilpotent then $M = N$.*
(11) *If $N \to M$ is a module map, I is nilpotent, and $N/IN \to M/IM$ is surjective, then $N \to M$ is surjective.*
(12) *If $\{x_\alpha\}_{\alpha \in A}$ is a set of elements of M which generate M/IM and I is nilpotent, then M is generated by the x_α.*

Proof. Proof of (1). Choose generators $y_1, \ldots, y_m$ of M over R. For each i we can write $y_i = \sum z_{ij} y_j$ with $z_{ij} \in I$. In other words $\sum_j (\delta_{ij} - z_{ij}) y_j = 0$. Let f be the determinant f of the $m \times m$ matrix $A = (\delta_{ij} - z_{ij})$. Note that $f \in 1 + I$. By Lemma **??** there exists an $m \times m$ matrix B such that $BA = f 1_{m \times m}$. Writing out we see that $f y_j = \sum_{i,j} b_{hi} a_{ij} y_j = 0$ for every j. This implies that f annihilates M.

By Lemma **??** an element of $1+\text{rad}(R)$ is invertible element of R. Hence we see that (1) implies (2). We obtain (3) by applying (1) to M/N which is finite as N' is finite. We obtain (4) by applying (2) to M/N which is finite as N' is finite. We obtain (5) by applying (3) to M and the submodules $\text{Im}(N \to M)$ and M. We obtain (6) by applying (4) to M and the submodules $\text{Im}(N \to M)$ and M. We obtain (7) by applying (5) to the map $R^{\oplus n} \to M$, $(a_1, \ldots, a_n) \mapsto a_1 x_1 + \ldots + a_n x_n$. We obtain (8) by applying (6) to the map $R^{\oplus n} \to M$, $(a_1, \ldots, a_n) \mapsto a_1 x_1 + \ldots + a_n x_n$.

Part (9) holds because if $M = IM$ then $M = I^n M$ for all $n \geq 0$ and I being nilpotent means $I^n = 0$ for some $n \gg 0$. Parts (10), (11), and (12) follow from (9) by the arguments used above. ☐

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Lemma 0.16 (Nakayama's lemma)**.** *Let R be a ring, let M be an R-module, and let $I \subset R$ be an ideal.*

(1) *If $IM = M$ and M is finite, then there exists a $f \in 1+I$ such that $fM = 0$.*
(2) *If $IM = M$, M is finite, and $I \subset rad(R)$, then $M = 0$.*
(3) *If $N, N' \subset M$, $M = N + IN'$, and N' is finite, then there exists a $f \in 1+I$ such that $M_f = N_f$.*
(4) *If $N, N' \subset M$, $M = N + IN'$, N' is finite, and $I \subset rad(R)$, then $M = N$.*
(5) *If $N \to M$ is a module map, $N/IN \to M/IM$ is surjective, and M is finite, then there exists a $f \in 1 + I$ such that $N_f \to M_f$ is surjective.*
(6) *If $N \to M$ is a module map, $N/IN \to M/IM$ is surjective, M is finite, and $I \subset rad(R)$, then $N \to M$ is surjective.*

(7) *If* $x_1, \ldots, x_n \in M$ *generate* M/IM *and* M *is finite, then there exists an* $f \in 1 + I$ *such that* $x_1, \ldots, x_n$ *generate* M_f *over* R_f*.*
(8) *If* $x_1, \ldots, x_n \in M$ *generate* M/IM*,* M *is finite, and* $I \subset rad(R)$*, then* M *is generated by* $x_1, \ldots, x_n$*.*
(9) *If* $IM = M$*,* I *is nilpotent, then* $M = 0$*.*
(10) *If* $N, N' \subset M$*,* $M = N + IN'$*, and* I *is nilpotent then* $M = N$*.*
(11) *If* $N \to M$ *is a module map,* I *is nilpotent, and* $N/IN \to M/IM$ *is surjective, then* $N \to M$ *is surjective.*
(12) *If* $\{x_\alpha\}_{\alpha \in A}$ *is a set of elements of* M *which generate* M/IM *and* I *is nilpotent, then* M *is generated by the* x_α*.*

Proof. Proof of (1). Choose generators $y_1, \ldots, y_m$ of M over R. For each i we can write $y_i = \sum z_{ij} y_j$ with $z_{ij} \in I$. In other words $\sum_j (\delta_{ij} - z_{ij}) y_j = 0$. Let f be the determinant f of the $m \times m$ matrix $A = (\delta_{ij} - z_{ij})$. Note that $f \in 1 + I$. By Lemma **??** there exists an $m \times m$ matrix B such that $BA = f 1_{m \times m}$. Writing out we see that $f y_j = \sum_{i,j} b_{hi} a_{ij} y_j = 0$ for every j. This implies that f annihilates M.

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