Lemma 0.1. (Nakayama’s lemma.) If $M$ is a finite nonzero module over $R$, then $\text{m} M \neq M$.

Proof. Here is a silly way to prove this: If $\text{m} M = M$ for $M$ finite then by induction $\text{m}^n M = M$. Hence the completion of $M$ with respect to the maximal ideal is zero. Hence $M \otimes_R \hat{R} = 0$, see Lemma ???. But $R \rightarrow \hat{R}$ is faithfully flat by Lemma ?? and hence we conclude $M = 0$ by Lemma ???. □

Lemma 0.2. (Nakayama’s lemma.) Let $R$ be a ring, let $M$ be an $R$-module, and let $I \subset R$ be an ideal.

1. If $M$ is finite, and $IM = M$, then there exists a $f = 1 + i \in 1 + I$ such that $fM = 0$.
2. If $M$ is finite, $IM = M$, and $I \subset \text{rad}(R)$ then $M = 0$.
3. If $IM = M$, $I$ is nilpotent, then $M = 0$.

Proof. Proof of (1). Write $M = \sum Rx_j$, $j = 1, \ldots, r$. Write $x_j = \sum i_{jj'} x_{jj'}$ with $i_{jj'} \in I$. In other words $\sum (\delta_{jj'} - i_{jj'}) x_{jj'} = 0$. Hence the determinant $f$ of the $r \times r$ matrix $(\delta_{jj'} - i_{jj'})$ is a solution. The other parts are easy. □

Lemma 0.3. (Nakayama’s lemma.) Let $R$ be a ring, let $M$ be an $R$-module, and let $I \subset R$ be an ideal.

1. If $M$ is finite, and $IM = M$, then there exists a $f = 1 + i \in 1 + I$ such that $fM = 0$.
2. If $M$ is finite, $IM = M$, and $I \subset \text{rad}(R)$ then $M = 0$.
3. If $IM = M$, $I$ is nilpotent, then $M = 0$.

Proof. Proof of (1). Write $M = \sum Rx_j$, $j = 1, \ldots, r$. Write $x_j = \sum i_{jj'} x_{jj'}$ with $i_{jj'} \in I$. In other words $\sum (\delta_{jj'} - i_{jj'}) x_{jj'} = 0$. Hence the determinant $f$ of the $r \times r$ matrix $(\delta_{jj'} - i_{jj'})$ is a solution. The other parts are easy. □

Lemma 0.4. (Nakayama’s lemma.) Let $R$ be a ring, let $M$ be an $R$-module, and let $I \subset R$ be an ideal.

1. If $M$ is finite, and $IM = M$, then there exists a $f = 1 + i \in 1 + I$ such that $fM = 0$.
2. If $M$ is finite, $IM = M$, and $I \subset \text{rad}(R)$ then $M = 0$.
3. If $IM = M$, $I$ is nilpotent, then $M = 0$.

Proof. Proof of (1). Write $M = \sum Rx_j$, $j = 1, \ldots, r$. Write $x_j = \sum i_{jj'} x_{jj'}$ with $i_{jj'} \in I$. In other words $\sum (\delta_{jj'} - i_{jj'}) x_{jj'} = 0$. Hence the determinant $f$ of the $r \times r$ matrix $(\delta_{jj'} - i_{jj'})$ is a solution. The other parts are easy. □
Lemma 0.5. (Nakayama’s lemma.) Let $R$ be a ring, let $M$ be an $R$-module, and let $I \subset R$ be an ideal.

1. If $M$ is finite, and $IM = M$, then there exists a $f = 1 + i \in 1 + I$ such that $fM = 0$.
2. If $M$ is finite, $IM = M$, and $I \subset \text{rad}(R)$ then $M = 0$.
3. If $IM = M$, $I$ is nilpotent, then $M = 0$.

Proof. Proof of (1). Write $M = \sum Rx_j$, $j = 1, \ldots, r$. Write $x_j = \sum i_{jj'}x_{j'}$, with $i_{jj'} \in I$. In other words $\sum(\delta_{jj'} - i_{jj'})x_{j'} = 0$. Hence the determinant $f$ of the $r \times r$ matrix $(\delta_{jj'} - i_{jj'})$ is a solution. The other parts are easy. \qed

Lemma 0.6. (Nakayama’s lemma.) Let $R$ be a ring, let $M$ be an $R$-module, and let $I \subset R$ be an ideal.

1. If $M$ is finite, and $IM = M$, then there exists a $f = 1 + i \in 1 + I$ such that $fM = 0$.
2. If $M$ is finite, $IM = M$, and $I \subset \text{rad}(R)$ then $M = 0$.
3. If $IM = M$, $I$ is nilpotent, then $M = 0$.

Proof. Proof of (1). Write $M = \sum Rx_j$, $j = 1, \ldots, r$. Write $x_j = \sum i_{jj'}x_{j'}$, with $i_{jj'} \in I$. In other words $\sum(\delta_{jj'} - i_{jj'})x_{j'} = 0$. Hence the determinant $f$ of the $r \times r$ matrix $(\delta_{jj'} - i_{jj'})$ is a solution. The other parts are easy. \qed

Lemma 0.7. (Nakayama’s lemma.) Let $R$ be a ring, let $M$ be an $R$-module, and let $I \subset R$ be an ideal.

1. If $M$ is finite, and $IM = M$, then there exists a $f = 1 + i \in 1 + I$ such that $fM = 0$.
2. If $M$ is finite, $IM = M$, and $I \subset \text{rad}(R)$ then $M = 0$.
3. If $IM = M$, $I$ is nilpotent, then $M = 0$.

Proof. Proof of (1). Write $M = \sum Rx_j$, $j = 1, \ldots, r$. Write $x_j = \sum i_{jj'}x_{j'}$, with $i_{jj'} \in I$. In other words $\sum(\delta_{jj'} - i_{jj'})x_{j'} = 0$. Hence the determinant $f$ of the $r \times r$ matrix $(\delta_{jj'} - i_{jj'})$ is a solution. The other parts are easy. \qed
Proof. Proof of (1). Write $M = \sum Rx_j$, $j = 1, \ldots, r$. Write $x_j = \sum i_{jj'}x_{j'}$ with $i_{jj'} \in I$. In other words $\sum (\delta_{jj'} - i_{jj'})x_{j'} = 0$. Hence the determinant $f$ of the $r \times r$ matrix $(\delta_{jj'} - i_{jj'})$ is a solution. The other parts are easy. \qed

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Lemma 0.9. (Nakayama’s lemma.) Let $R$ be a ring, let $M$ be an $R$-module, and let $I \subset R$ be an ideal.

1. If $M$ is finite, and $IM = M$, then there exists a $f = 1 + i \in 1 + I$ such that $fM = 0$.
2. If $M$ is finite, $IM = M$, and $I \subset \text{rad}(R)$ then $M = 0$.
3. If $IM = M$, $I$ is nilpotent, then $M = 0$.

Proof. Proof of (1). Write $M = \sum Rx_j$, $j = 1, \ldots, r$. Write $x_j = \sum i_{jj'}x_{j'}$ with $i_{jj'} \in I$. In other words $\sum (\delta_{jj'} - i_{jj'})x_{j'} = 0$. Hence the determinant $f$ of the $r \times r$ matrix $(\delta_{jj'} - i_{jj'})$ is a solution. The other parts are easy. \qed

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Lemma 0.10. (Nakayama’s lemma.) Let $R$ be a ring, let $M$ be an $R$-module, and let $I \subset R$ be an ideal.

1. If $M$ is finite, and $IM = M$, then there exists a $f = 1 + i \in 1 + I$ such that $fM = 0$.
2. If $M$ is finite, $IM = M$, and $I \subset \text{rad}(R)$ then $M = 0$.
3. If $IM = M$, $I$ is nilpotent, then $M = 0$.

Proof. Proof of (1). Write $M = \sum Rx_j$, $j = 1, \ldots, r$. Write $x_j = \sum i_{jj'}x_{j'}$ with $i_{jj'} \in I$. In other words $\sum (\delta_{jj'} - i_{jj'})x_{j'} = 0$. Hence the determinant $f$ of the $r \times r$ matrix $(\delta_{jj'} - i_{jj'})$ is a solution. The other parts are easy. \qed

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Lemma 0.11. (Nakayama’s lemma.) Let $R$ be a ring, let $M$ be an $R$-module, and let $I \subset R$ be an ideal.

1. If $M$ is finite and $IM = M$, then there exists a $f \in 1 + I$ such that $fM = 0$.
2. If $IM = M$, $I \subset \text{rad}(R)$, and $M$ is finite, then $M = 0$.
3. If $N, N' \subset M$, $M = N + IN'$, $I \subset \text{rad}(R)$, and $N'$ is finite then $M = N$.
4. If $x_1, \ldots, x_n \in M$ generate $M/IM$ and $M$ is finite, then there exists an $f \in 1 + I$ such that $x_1, \ldots, x_n$ generate $M_f$ over $R_f$.
5. If $x_1, \ldots, x_n \in M$ generate $M/IM$, $I \subset \text{rad}(R)$, and $M$ is finite, then $M$ is generated by $x_1, \ldots, x_n$.
6. If $IM = M$, $I$ is nilpotent, then $M = 0$.
7. If $N, N' \subset M$, $M = N + IN'$, and $I$ is nilpotent then $M = N$.
8. If $\{x_\alpha\}_{\alpha \in A}$ is a set of elements of $M$ which generate $M/IM$ and $I$ is nilpotent, then $M$ is generated by the $x_\alpha$. 

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Proof. Proof of (1). Choose generators \( y_1, \ldots, y_m \) of \( M \) over \( R \). For each \( i \) we can write \( y_i = \sum z_{ij} y_j \) with \( z_{ij} \in I \). In other words \( \sum_j (\delta_{ij} - z_{ij}) y_j = 0 \). Let \( f \) be the determinant of the \( m \times m \) matrix \( A = (\delta_{ij} - z_{ij}) \). Note that \( f \in 1 + I \). By Lemma ?? there exists an \( m \times m \) matrix \( B \) such that \( BA = f^1_{m \times m} \). Writing out we see that \( f y_j = \sum_i b_{ij} a_{ij} y_j = 0 \) for every \( j \). This implies that \( f \) annihilates \( M \).

By Lemma ?? an element of \( 1 + \text{rad}(R) \) is invertible element of \( R \). Hence we see that (1) implies (2). We obtain (3) by applying (2) to \( M/N \). We obtain (4) by applying (1) to \( M/Rx_1 + \cdots + Rx_n \). We obtain (5) from (4) by the first remark of this paragraph.

Part (6) holds because if \( M = IM \) then \( M = I^n M \) for all \( n \geq 0 \) and \( I \) being nilpotent means \( I^n = 0 \) for some \( n \gg 0 \). Parts (7) and (8) follow from (6) by considering the quotient of \( M \) by the given submodule. \( \square \)

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algebra.tex, lemma-NAK, 00DV

Lemma 0.12 (Nakayama’s lemma). Let \( R \) be a ring, let \( M \) be an \( R \)-module, and let \( I \subset R \) be an ideal.

1. If \( IM = M \) and \( M \) is finite, then there exists a \( f \in 1 + I \) such that \( fM = 0 \).
2. If \( IM = M \), \( M \) is finite, and \( I \subset \text{rad}(R) \), then \( M = 0 \).
3. If \( N, N' \subset M, M = N + IN', \) and \( N' \) is finite, then there exists a \( f \in 1 + I \) such that \( M_f = N_f \).
4. If \( N, N' \subset M, M = N + IN', \) \( N' \) is finite, and \( I \subset \text{rad}(R) \), then \( M = N \).
5. If \( N \to M \) is a module map, \( N/IN \to M/IM \) is surjective, and \( M \) is finite, then there exists a \( f \in 1 + I \) such that \( N_f \to M_f \) is surjective.
6. If \( N \to M \) is a module map, \( N/IN \to M/IM \) is surjective, \( M \) is finite, and \( I \subset \text{rad}(R) \), then \( N \to M \) is surjective.
7. If \( x_1, \ldots, x_n \in M \) generate \( M/IM \) and \( M \) is finite, then there exists an \( f \in 1 + I \) such that \( x_1, \ldots, x_n \) generate \( M_f \) over \( R_f \).
8. If \( x_1, \ldots, x_n \in M \) generate \( M/IM \), \( M \) is finite, and \( I \subset \text{rad}(R) \), then \( M \) is generated by \( x_1, \ldots, x_n \).
9. If \( IM = M, I \) is nilpotent, then \( M = 0 \).
10. If \( N, N' \subset M, M = N + IN' \), and \( I \) is nilpotent then \( M = N \).
11. If \( N \to M \) is a module map, \( I \) is nilpotent, and \( N/IN \to M/IM \) is surjective, then \( N \to M \) is surjective.
12. If \( \{x_\alpha\}_{\alpha \in A} \) is a set of elements of \( M \) which generate \( M/IM \) and \( I \) is nilpotent, then \( M \) is generated by the \( x_\alpha \).

Proof. Proof of (1). Choose generators \( y_1, \ldots, y_m \) of \( M \) over \( R \). For each \( i \) we can write \( y_i = \sum z_{ij} y_j \) with \( z_{ij} \in I \). In other words \( \sum_j (\delta_{ij} - z_{ij}) y_j = 0 \). Let \( f \) be the determinant \( f \) of the \( m \times m \) matrix \( A = (\delta_{ij} - z_{ij}) \). Note that \( f \in 1 + I \). By Lemma ?? there exists an \( m \times m \) matrix \( B \) such that \( BA = f^1_{m \times m} \). Writing out we see that \( f y_j = \sum_i b_{ij} a_{ij} y_j = 0 \) for every \( j \). This implies that \( f \) annihilates \( M \).

By Lemma ?? an element of \( 1 + \text{rad}(R) \) is invertible element of \( R \). Hence we see that (1) implies (2). We obtain (3) by applying (1) to \( M/N \) which is finite as \( N' \) is finite. We obtain (4) by applying (2) to \( M/N \) which is finite as \( N' \) is finite. We obtain (5) by applying (3) to \( M \) and the submodules \( \text{Im}(N \to M) \) and \( M \). We obtain (6) by applying (4) to \( M \) and the submodules \( \text{Im}(N \to M) \) and \( M \). We obtain (7) by
applying (5) to the map \( R^{\oplus n} \rightarrow M, (a_1, \ldots, a_n) \mapsto a_1 x_1 + \ldots + a_n x_n \). We obtain (8) by applying (6) to the map \( R^{\oplus n} \rightarrow M, (a_1, \ldots, a_n) \mapsto a_1 x_1 + \ldots + a_n x_n \).

Part (9) holds because if \( M = IM \) then \( M = I^n M \) for all \( n \geq 0 \) and \( I \) being nilpotent means \( I^n = 0 \) for some \( n \gg 0 \). Parts (10), (11), and (12) follow from (9) by the arguments used above.

\[ \square \]

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**Lemma 0.13** (Nakayama’s lemma). Let \( R \) be a ring, let \( M \) be an \( R \)-module, and let \( I \subset R \) be an ideal.

1. If \( IM = M \) and \( M \) is finite, then there exists a \( f \in 1 + I \) such that \( fM = 0 \).
2. If \( IM = M \), \( M \) is finite, and \( I \subset \text{rad}(R) \), then \( M = 0 \).
3. If \( N, N' \subset M \), \( M = N + IN' \), and \( N' \) is finite, then there exists a \( f \in 1 + I \) such that \( Mf = Nf \).
4. If \( N, N' \subset M \), \( M = N + IN' \), \( N' \) is finite, and \( I \subset \text{rad}(R) \), then \( M = N \).
5. If \( N \rightarrow M \) is a module map, \( N/I \rightarrow M/IM \) is surjective, and \( M \) is finite, then there exists a \( f \in 1 + I \) such that \( Mf \rightarrow Mf \) is surjective.
6. If \( N \rightarrow M \) is a module map, \( N/I \rightarrow M/IM \) is surjective, \( M \) is finite, and \( I \subset \text{rad}(R) \), then \( N \rightarrow M \) is surjective.
7. If \( x_1, \ldots, x_n \in M \) generate \( M/IM \) and \( M \) is finite, then there exists an \( f \in 1 + I \) such that \( x_1, \ldots, x_n \) generate \( Mf \) over \( Rf \).
8. If \( x_1, \ldots, x_n \in M \) generate \( M/IM \) and \( M \) is finite, and \( I \subset \text{rad}(R) \), then \( M \) is generated by \( x_1, \ldots, x_n \).
9. If \( IM = M \), \( I \) is nilpotent, then \( M = 0 \).
10. If \( N, N' \subset M \), \( M = N + IN' \), and \( I \) is nilpotent then \( M = N \).
11. If \( N \rightarrow M \) is a module map, \( I \) is nilpotent, and \( N/I \rightarrow M/IM \) is surjective, then \( N \rightarrow M \) is surjective.
12. If \( \{ x_\alpha \}_{\alpha \in A} \) is a set of elements of \( M \) which generate \( M/IM \) and \( I \) is nilpotent, then \( M \) is generated by the \( x_\alpha \).

**Proof.** Proof of (1). Choose generators \( y_1, \ldots, y_n \) of \( M \) over \( R \). For each \( i \) we can write \( y_i = \sum z_{ij} y_j \) with \( z_{ij} \in I \). In other words \( \sum (\delta_{ij} - z_{ij}) y_j = 0 \). Let \( f \) be the determinant of a \( m \times m \) matrix \( A = (\delta_{ij} - z_{ij}) \). Note that \( f \in 1 + I \). By Lemma ?? there exists an \( m \times m \) matrix \( B \) such that \( BA = f I_{m \times m} \). Writing out we see that \( f y_j = \sum_{i,j} b_{ij} a_{ij} y_i = 0 \) for every \( j \). This implies that \( f \) annihilates \( M \).

By Lemma ?? an element of \( 1 + \text{rad}(R) \) is invertible element of \( R \). Hence we see that (1) implies (2). We obtain (3) by applying (1) to \( M/N \) which is finite as \( N' \) is finite. We obtain (4) by applying (2) to \( M/N \) which is finite as \( N' \) is finite. We obtain (5) by applying (3) to \( M \) and the submodules \( \text{Im}(N \rightarrow M) \) and \( M \). We obtain (6) by applying (4) to \( M \) and the submodules \( \text{Im}(N \rightarrow M) \) and \( M \). We obtain (7) by applying (5) to the map \( R^{\oplus n} \rightarrow M, (a_1, \ldots, a_n) \mapsto a_1 x_1 + \ldots + a_n x_n \). We obtain (8) by applying (6) to the map \( R^{\oplus n} \rightarrow M, (a_1, \ldots, a_n) \mapsto a_1 x_1 + \ldots + a_n x_n \).

Part (9) holds because if \( M = IM \) then \( M = I^n M \) for all \( n \geq 0 \) and \( I \) being nilpotent means \( I^n = 0 \) for some \( n \gg 0 \). Parts (10), (11), and (12) follow from (9) by the arguments used above. \[ \square \]
Lemma 0.14 (Nakayama’s lemma). Let $R$ be a ring, let $M$ be an $R$-module, and let $I \subseteq R$ be an ideal.

1. If $IM = M$ and $M$ is finite, then there exists a $f \in 1 + I$ such that $fM = 0$.
2. If $IM = M$, $M$ is finite, and $I \subseteq \text{rad}(R)$, then $M = 0$.
3. If $N, N' \subseteq M$, $M = N + IN'$, and $N'$ is finite, then there exists a $f \in 1 + I$ such that $M_f = N_f$.
4. If $N, N' \subseteq M$, $M = N + IN'$, $N'$ is finite, and $I \subseteq \text{rad}(R)$, then $M = N$.
5. If $N \rightarrow M$ is a module map, $N/IN \rightarrow M/IM$ is surjective, and $M$ is finite, then there exists a $f \in 1 + I$ such that $N_f \rightarrow M_f$ is surjective.
6. If $N \rightarrow M$ is a module map, $N/IN \rightarrow M/IM$ is surjective, $M$ is finite, and $I \subseteq \text{rad}(R)$, then $N \rightarrow M$ is surjective.
7. If $x_1, \ldots, x_n \in M$ generate $M/IM$ and $M$ is finite, then there exists an $f \in 1 + I$ such that $x_1, \ldots, x_n$ generate $M_f$ over $R_f$.
8. If $x_1, \ldots, x_n \in M$ generate $M/IM$ and $M$ is finite, and $I \subseteq \text{rad}(R)$, then $M$ is generated by $x_1, \ldots, x_n$.
9. If $IM = M$, $I$ is nilpotent, then $M = 0$.
10. If $N, N' \subseteq M$, $M = N + IN'$, and $I$ is nilpotent then $M = N$.
11. If $N \rightarrow M$ is a module map, $I$ is nilpotent, and $N/IN \rightarrow M/IM$ is surjective, then $N \rightarrow M$ is surjective.
12. If $\{x_\alpha\}_{\alpha \in A}$ is a set of elements of $M$ which generate $M/IM$ and $I$ is nilpotent, then $M$ is generated by the $x_\alpha$.

Proof. Proof of (1). Choose generators $y_1, \ldots, y_m$ of $M$ over $R$. For each $i$ we can write $y_i = \sum z_{ij}y_j$ with $z_{ij} \in I$. In other words $\sum_j (\delta_{ij} - z_{ij})y_j = 0$. Let $f$ be the determinant $f$ of the $m \times m$ matrix $A = (\delta_{ij} - z_{ij})$. Note that $f \in 1 + I$. By Lemma ?? there exists an $m \times m$ matrix $B$ such that $BA = f1_{m \times m}$. Writing out we see that $f y_j = \sum_{i,j} b_{ij} a_{ij} y_j = 0$ for every $j$. This implies that $f$ annihilates $M$.

By Lemma ?? an element of $1 + \text{rad}(R)$ is invertible element of $R$. Hence we see that (1) implies (2). We obtain (3) by applying (1) to $M/N$ which is finite as $N'$ is finite. We obtain (4) by applying (2) to $M/N$ which is finite as $N'$ is finite. We obtain (5) by applying (3) to $M$ and the submodules $\text{Im}(N \rightarrow M)$ and $M$. We obtain (6) by applying (4) to $M$ and the submodules $\text{Im}(N \rightarrow M)$ and $M$. We obtain (7) by applying (5) to the map $M^\oplus n \rightarrow M$, $(a_1, \ldots, a_n) \mapsto a_1x_1 + \ldots + a_n x_n$. We obtain (8) by applying (6) to the map $M^\oplus n \rightarrow M$, $(a_1, \ldots, a_n) \mapsto a_1 x_1 + \ldots + a_n x_n$.

Part (9) holds because if $M = IM$ then $M = I^n M$ for all $n \geq 0$ and $I$ being nilpotent means $I^n = 0$ for some $n \gg 0$. Parts (10), (11), and (12) follow from (9) by the arguments used above. \qed
(3) If \( N, N' \subseteq M, M = N + IN' \), and \( N' \) is finite, then there exists a \( f \in 1+I \) such that \( M_f = N_f \).

(4) If \( N, N' \subseteq M, M = N + IN' \), \( N' \) is finite, and \( I \subseteq \text{rad}(R) \), then \( M = N \).

(5) If \( N \to M \) is a module map, \( N/IN \to M/IM \) is surjective, and \( M \) is finite, then there exists a \( f \in 1+I \) such that \( N_f \to M_f \) is surjective.

(6) If \( N \to M \) is a module map, \( N/IN \to M/IM \) is surjective, \( M \) is finite, and \( I \subseteq \text{rad}(R) \), then \( N \to M \) is surjective.

(7) If \( x_1, \ldots, x_n \in M \) generate \( M/IM \) and \( M \) is finite, then there exists an \( f \in 1+I \) such that \( x_1, \ldots, x_n \) generate \( M_f \over R_f \).

(8) If \( x_1, \ldots, x_n \in M \) generate \( M/IM \), \( M \) is finite, and \( I \subseteq \text{rad}(R) \), then \( M \) is generated by \( x_1, \ldots, x_n \).

(9) If \( IM = M, I \) is nilpotent, then \( M = 0 \).

(10) If \( N, N' \subseteq M, M = N + IN' \), and \( I \) is nilpotent then \( M = N \).

(11) If \( N \to M \) is a module map, \( I \) is nilpotent, and \( N/IN \to M/IM \) is surjective, then \( N \to M \) is surjective.

(12) If \( \{x_\alpha\}_{\alpha \in A} \) is a set of elements of \( M \) which generate \( M/IM \) and \( I \) is nilpotent, then \( M \) is generated by the \( x_\alpha \).

Proof. Proof of (1). Choose generators \( y_1, \ldots, y_m \) of \( M \) over \( R \). For each \( i \) we can write \( y_i = \sum z_{ij} y_j \) with \( z_{ij} \in I \). In other words \( \sum_j (\delta_{ij} - z_{ij}) y_j = 0 \). Let \( f \) be the determinant \( f \) of the \( m \times m \) matrix \( A = (\delta_{ij} - z_{ij}) \). Note that \( f \in 1+I \). By Lemma ?? there exists an \( m \times m \) matrix \( B \) such that \( BA = f \text{I}_{m \times m} \). Writing out we see that \( f y_j = \sum_{i,j} b_{ij} a_i y_j = 0 \) for every \( j \). This implies that \( f \) annihilates \( M \).

By Lemma ?? an element of \( 1 + \text{rad}(R) \) is invertible element of \( R \). Hence we see that (1) implies (2). We obtain (3) by applying (1) to \( M/N \) which is finite as \( N' \) is finite. We obtain (4) by applying (2) to \( M/N \) which is finite as \( N' \) is finite. We obtain (5) by applying (3) to \( M \) and the submodules \( \text{Im}(N \to M) \) and \( M \). We obtain (6) by applying (4) to \( M \) and the submodules \( \text{Im}(N \to M) \) and \( M \). We obtain (7) by applying (5) to the map \( R^{\oplus n} \to M, (a_1, \ldots, a_n) \mapsto a_1 x_1 + \ldots + a_n x_n \). We obtain (8) by applying (6) to the map \( R^{\oplus n} \to M, (a_1, \ldots, a_n) \mapsto a_1 x_1 + \ldots + a_n x_n \).

Part (9) holds because if \( M = IM \) then \( M = I^n M \) for all \( n \geq 0 \) and \( I \) being nilpotent means \( I^n = 0 \) for some \( n \gg 0 \). Parts (10), (11), and (12) follow from (9) by the arguments used above.

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\textbf{Lemma 0.16 (Nakayama’s lemma).} \textit{Let} \( R \) \textit{be a ring, let} \( M \) \textit{be an} \( R \)-\textit{module, and let} \( I \subseteq R \) \textit{be an ideal.}

(1) If \( IM = M \) and \( M \) is finite, then there exists a \( f \in 1+I \) such that \( fM = 0 \).

(2) If \( IM = M \), \( M \) is finite, and \( I \subseteq \text{rad}(R) \), then \( M = 0 \).

(3) If \( N, N' \subseteq M, M = N + IN' \), and \( N' \) is finite, then there exists a \( f \in 1+I \) such that \( M_f = N_f \).

(4) If \( N, N' \subseteq M, M = N + IN' \), \( N' \) is finite, and \( I \subseteq \text{rad}(R) \), then \( M = N \).

(5) If \( N \to M \) is a module map, \( N/IN \to M/IM \) is surjective, and \( M \) is finite, then there exists a \( f \in 1+I \) such that \( N_f \to M_f \) is surjective.

(6) If \( N \to M \) is a module map, \( N/IN \to M/IM \) is surjective, \( M \) is finite, and \( I \subseteq \text{rad}(R) \), then \( N \to M \) is surjective.
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(7) If \(x_1, \ldots, x_n \in M\) generate \(M/IM\) and \(M\) is finite, then there exists an \(f \in 1+I\) such that \(x_1, \ldots, x_n\) generate \(M_f\) over \(R_f\).

(8) If \(x_1, \ldots, x_n \in M\) generate \(M/IM\), \(M\) is finite, and \(I \subset \text{rad}(R)\), then \(M\) is generated by \(x_1, \ldots, x_n\).

(9) If \(IM = M\), \(I\) is nilpotent, then \(M = 0\).

(10) If \(N, N' \subset M\), \(M = N + IN'\), and \(I \subset \text{rad}(R)\), then \(M\) is generated by \(x_1, \ldots, x_n\).

(11) If \(IM = M\), \(I\) is nilpotent, then \(M = 0\).

(12) If \(\{x_\alpha\}_{\alpha \in A}\) is a set of elements of \(M\) which generate \(M/IM\) and \(I\) is nilpotent, then \(M\) is generated by the \(x_\alpha\).

Proof. Proof of (1). Choose generators \(y_1, \ldots, y_m\) of \(M\) over \(R\). For each \(i\) we can write \(y_i = \sum z_{ij}y_j\) with \(z_{ij} \in I\). In other words \(\sum_j (\delta_{ij} - z_{ij})y_j = 0\). Let \(f\) be the determinant \(f\) of the \(m \times m\) matrix \(A = (\delta_{ij} - z_{ij})\). Note that \(f \in 1+I\). By Lemma ?? there exists an \(m \times m\) matrix \(B\) such that \(BA = f1_{m \times m}\). Writing out we see that \(fy_j = \sum a_{ij}y_j = 0\) for every \(j\). This implies that \(f\) annihilates \(M\).

By Lemma ?? an element of \(1+\text{rad}(R)\) is invertible element of \(R\). Hence we see that (1) implies (2). We obtain (3) by applying (1) to \(M/N\) which is finite as \(N'\) is finite. We obtain (4) by applying (2) to \(M/N\) which is finite as \(N'\) is finite. We obtain (5) by applying (3) to \(M\) and the submodules \(\text{Im}(N \to M)\) and \(M\). We obtain (6) by applying (4) to \(M\) and the submodules \(\text{Im}(N \to M)\) and \(M\). We obtain (7) by applying (5) to the map \(R^\oplus n \to M\), \((a_1, \ldots, a_n) \mapsto a_1x_1 + \ldots + a_nx_n\). We obtain (8) by applying (6) to the map \(R^\oplus n \to M\), \((a_1, \ldots, a_n) \mapsto a_1x_1 + \ldots + a_nx_n\).

Part (9) holds because if \(M = IM\) then \(M = I^nM\) for all \(n \geq 0\) and \(I\) being nilpotent means \(I^n = 0\) for some \(n \gg 0\). Parts (10), (11), and (12) follow from (9) by the arguments used above. \(\square\)