ÉTALE MORPHISMS OF SCHEMES

024J

Contents

1. Introduction

024K In this Chapter, we discuss étale morphisms of schemes. We illustrate some of the more important concepts by working with the Noetherian case. Our principal goal is to collect for the reader enough commutative algebra results to start reading a treatise on étale cohomology. An auxiliary goal is to provide enough evidence to ensure that the reader stops calling the phrase "the étale topology of schemes" an exercise in general nonsense, if (s)he does indulge in such blasphemy.

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We will refer to the other chapters of the stacks project for standard results in algebraic geometry (on schemes and commutative algebra). We will provide detailed proofs of the new results that we state here.

2. Conventions

039F In this chapter, frequently schemes will be assumed locally Noetherian and frequently rings will be assumed Noetherian. But in all the statements we will reiterate this when necessary, and make sure we list all the hypotheses! On the other hand, here are some general facts that we will use often and are useful to keep in mind:

(1) A ring homomorphism $A \to B$ of finite type with A Noetherian is of finite presentation. See Algebra, Lemma 30.4.
(2) A morphism (locally) of finite type between locally Noetherian schemes is automatically (locally) of finite presentation. See Morphisms, Lemma 21.9.
(3) Add more like this here.

3. Unramified morphisms

024L We first define the notion of unramified morphisms for local rings, and then globalize it to get one for arbitrary schemes.

024M **Definition 3.1.** Let A, B be Noetherian local rings. A local homomorphism $A \to B$ is said to be *unramified homomorphism of local rings* if

(1) $\mathfrak{m}_A B = \mathfrak{m}_B$,
(2) $\kappa(\mathfrak{m}_B)$ is a finite separable extension of $\kappa(\mathfrak{m}_A)$, and
(3) B is essentially of finite type over A (this means that B is the localization of a finite type A-algebra at a prime).

This is the local version of the definition in Algebra, Section 146. In that section a ring map $R \to S$ is defined to be unramified if and only if it is of finite type, and $\Omega_{S/R} = 0$. It is shown in Algebra, Lemmas 146.5 and 146.7 that given a ring map $R \to S$ of finite type, and a prime $\mathfrak{q}$ of S lying over $\mathfrak{p} \subset R$, then we have

$$R \to S \text{ is unramified at } \mathfrak{q} \Leftrightarrow \mathfrak{p}S_{\mathfrak{q}} = \mathfrak{q}S_{\mathfrak{q}} \text{ and } \kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q}) \text{ finite separable}$$

Thus we see that for a local homomorphism of local rings the properties of our definition above are closely related to the question of being unramified. In fact, we have proved the following lemma.

039G **Lemma 3.2.** *Let $A \to B$ be of finite type with A a Noetherian ring. Let $\mathfrak{q}$ be a prime of B lying over $\mathfrak{p} \subset A$. Then $A \to B$ is unramified at $\mathfrak{q}$ if and only if $A_{\mathfrak{p}} \to B_{\mathfrak{q}}$ is an unramified homomorphism of local rings.*

Proof. See discussion above. □

We will characterize the property of being unramified in terms of completions. For a Noetherian local ring A we denote $A^\wedge$ the completion of A with respect to the maximal ideal. It is also a Noetherian local ring, see Algebra, Lemma 95.6.

039H **Lemma 3.3.** *Let A, B be Noetherian local rings. Let $A \to B$ be a local homomorphism.*

(1) *if $A \to B$ is an unramified homomorphism of local rings, then $B^\wedge$ is a finite $A^\wedge$ module,*

(2) *if $A \to B$ is an unramified homomorphism of local rings and $\kappa(\mathfrak{m}_A) = \kappa(\mathfrak{m}_B)$, then $A^\wedge \to B^\wedge$ is surjective,*
(3) *if $A \to B$ is an unramified homomorphism of local rings and $\kappa(\mathfrak{m}_A)$ is separably closed, then $A^\wedge \to B^\wedge$ is surjective,*
(4) *if A and B are complete discrete valuation rings, then $A \to B$ is an unramified homomorphism of local rings if and only the uniformizer for A maps to a uniformizer for B, and the residue field extension is finite separable (and B is essentially of finite type over A).*

Proof. Part (1) is a special case of Algebra, Lemma 95.7. For part (2), note that the $\kappa(\mathfrak{m}_A)$-vector space $B^\wedge/\mathfrak{m}_{A^\wedge}B^\wedge$ is generated by 1. Hence by Nakayama's lemma (Algebra, Lemma 19.1) the map $A^\wedge \to B^\wedge$ is surjective. Part (3) is a special case of part (2). Part (4) is immediate from the definitions. □

039I **Lemma 3.4.** *Let A, B be Noetherian local rings. Let $A \to B$ be a local homomorphism such that B is essentially of finite type over A. The following are equivalent*

(1) *$A \to B$ is an unramified homomorphism of local rings*
(2) *$A^\wedge \to B^\wedge$ is an unramified homomorphism of local rings, and*
(3) *$A^\wedge \to B^\wedge$ is unramified.*

Proof. The equivalence of (1) and (2) follows from the fact that $\mathfrak{m}_A A^\wedge$ is the maximal ideal of $A^\wedge$ (and similarly for B) and faithful flatness of $B \to B^\wedge$. For example if $A^\wedge \to B^\wedge$ is unramified, then $\mathfrak{m}_A B^\wedge = (\mathfrak{m}_A B)B^\wedge = \mathfrak{m}_B B^\wedge$ and hence $\mathfrak{m}_A B = \mathfrak{m}_B$.

Assume the equivalent conditions (1) and (2). By Lemma 3.3 we see that $A^\wedge \to B^\wedge$ is finite. Hence $A^\wedge \to B^\wedge$ is of finite presentation, and by Algebra, Lemma 146.7 we conclude that $A^\wedge \to B^\wedge$ is unramified at $\mathfrak{m}_{B^\wedge}$. Since $B^\wedge$ is local we conclude that $A^\wedge \to B^\wedge$ is unramified.

Assume (3). By Algebra, Lemma 146.5 we conclude that $A^\wedge \to B^\wedge$ is an unramified homomorphism of local rings, i.e., (2) holds. □

024N **Definition 3.5.** (See Morphisms, Definition 35.1 for the definition in the general case.) Let Y be a locally Noetherian scheme. Let $f : X \to Y$ be locally of finite type. Let $x \in X$.

(1) We say f is *unramified at x* if $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is an unramified homomorphism of local rings.
(2) The morphism $f : X \to Y$ is said to be *unramified* if it is unramified at all points of X.

Let us prove that this definition agrees with the definition in the chapter on morphisms of schemes. This in particular guarantees that the set of points where a morphism is unramified is open.

039J **Lemma 3.6.** *Let Y be a locally Noetherian scheme. Let $f : X \to Y$ be locally of finite type. Let $x \in X$. The morphism f is unramified at x in the sense of Definition 3.5 if and only if it is unramified in the sense of Morphisms, Definition 35.1.*

Proof. This follows from Lemma 3.2 and the definitions. □

Here are some results on unramified morphisms. The formulations as given in this list apply only to morphisms locally of finite type between locally Noetherian schemes. In each case we give a reference to the general result as proved earlier in the project, but in some cases one can prove the result more easily in the Noetherian case. Here is the list:

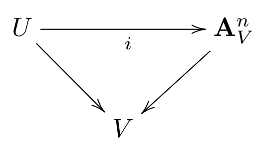
(1) Unramifiedness is local on the source and the target in the Zariski topology.
(2) Unramified morphisms are stable under base change and composition. See Morphisms, Lemmas 35.5 and 35.4.
(3) Unramified morphisms of schemes are locally quasi-finite and quasi-compact unramified morphisms are quasi-finite. See Morphisms, Lemma 35.10
(4) Unramified morphisms have relative dimension 0. See Morphisms, Definition 29.1 and Morphisms, Lemma 29.5.
(5) A morphism is unramified if and only if all its fibres are unramified. That is, unramifiedness can be checked on the scheme theoretic fibres. See Morphisms, Lemma 35.12.
(6) Let X and Y be unramified over a base scheme S. Any S-morphism from X to Y is unramified. See Morphisms, Lemma 35.16.

4. Three other characterizations of unramified morphisms

024O The following theorem gives three equivalent notions of being unramified at a point. See Morphisms, Lemma 35.14 for (part of) the statement for general schemes.

024P **Theorem 4.1.** *Let Y be a locally Noetherian scheme. Let $f : X \to Y$ be a morphism of schemes which is locally of finite type. Let x be a point of X. The following are equivalent*

(1) *f is unramified at x,*
(2) *the stalk $\Omega_{X/Y,x}$ of the module of relative differentials at x is trivial,*
(3) *there exist open neighbourhoods U of x and V of $f(x)$, and a commutative diagram*



where i is a closed immersion defined by a quasi-coherent sheaf of ideals $\mathcal{I}$ such that the differentials dg for $g \in \mathcal{I}_{i(x)}$ generate $\Omega_{\mathbf{A}^n_V/V,i(x)}$, and
(4) *the diagonal $\Delta_{X/Y} : X \to X \times_Y X$ is a local isomorphism at x.*

Proof. The equivalence of (1) and (2) is proved in Morphisms, Lemma 35.14.

If f is unramified at x, then f is unramified in an open neighbourhood of x; this does not follow immediately from Definition 3.5 of this chapter but it does follow from Morphisms, Definition 35.1 which we proved to be equivalent in Lemma 3.6. Choose affine opens $V \subset Y$, $U \subset X$ with $f(U) \subset V$ and $x \in U$, such that f is unramified on U, i.e., $f|_U : U \to V$ is unramified. By Morphisms, Lemma 35.13 the morphism $U \to U \times_V U$ is an open immersion. This proves that (1) implies (4).

If $\Delta_{X/Y}$ is a local isomorphism at x, then $\Omega_{X/Y,x} = 0$ by Morphisms, Lemma 33.7. Hence we see that (4) implies (2). At this point we know that (1), (2) and (4) are all equivalent.

Assume (3). The assumption on the diagram combined with Morphisms, Lemma 33.15 show that $\Omega_{U/V,x} = 0$. Since $\Omega_{U/V,x} = \Omega_{X/Y,x}$ we conclude (2) holds.

Finally, assume that (2) holds. To prove (3) we may localize on X and Y and assume that X and Y are affine. Say $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$. The point $x \in X$ corresponds to a prime $\mathfrak{q} \subset B$. Our assumption is that $\Omega_{B/A,\mathfrak{q}} = 0$ (see Morphisms, Lemma 33.5 for the relationship between differentials on schemes and modules of differentials in commutative algebra). Since Y is locally Noetherian and f locally of finite type we see that A is Noetherian and $B \cong A[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$, see Properties, Lemma 5.2 and Morphisms, Lemma 15.2. In particular, $\Omega_{B/A}$ is a finite B-module. Hence we can find a single $g \in B$, $g \not\in \mathfrak{q}$ such that the principal localization $(\Omega_{B/A})_g$ is zero. Hence after replacing B by B_g we see that $\Omega_{B/A} = 0$ (formation of modules of differentials commutes with localization, see Algebra, Lemma 129.8). This means that $\mathrm{d}(f_j)$ generate the kernel of the canonical map $\Omega_{A[x_1, \ldots, x_n]/A} \otimes_A B \to \Omega_{B/A}$. Thus the surjection $A[x_1, \ldots, x_n] \to B$ of A-algebras gives the commutative diagram of (3), and the theorem is proved. □

How can we use this theorem? Well, here are a few remarks:

(1) Suppose that $f : X \to Y$ and $g : Y \to Z$ are two morphisms locally of finite type between locally Noetherian schemes. There is a canonical short exact sequence
$$f^*(\Omega_{Y/Z}) \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$$
see Morphisms, Lemma 33.9. The theorem therefore implies that if $g \circ f$ is unramified, then so is f. This is Morphisms, Lemma 35.16.

(2) Since $\Omega_{X/Y}$ is isomorphic to the conormal sheaf of the diagonal morphism (Morphisms, Lemma 33.7) we see that if $X \to Y$ is a monomorphism of locally Noetherian schemes and locally of finite type, then $X \to Y$ is unramified. In particular, open and closed immersions of locally Noetherian schemes are unramified. See Morphisms, Lemmas 35.7 and 35.8.

(3) The theorem also implies that the set of points where a morphism $f : X \to Y$ (locally of finite type of locally Noetherian schemes) is not unramified is the support of the coherent sheaf $\Omega_{X/Y}$. This allows one to give a scheme theoretic definition to the "ramification locus".

5. The functorial characterization of unramified morphisms

024Q In basic algebraic geometry we learn that some classes of morphisms can be characterized functorially, and that such descriptions are quite useful. Unramified morphisms too have such a characterization.

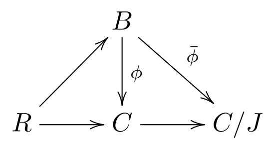
024R **Theorem 5.1.** *Let $f : X \to S$ be a morphism of schemes. Assume S is a locally Noetherian scheme, and f is locally of finite type. Then the following are equivalent:*

(1) *f is unramified,*

(2) *the morphism f is formally unramified: for any affine S-scheme T and subscheme T_0 of T defined by a square-zero ideal, the natural map*
$$\mathrm{Hom}_S(T, X) \longrightarrow \mathrm{Hom}_S(T_0, X)$$
is injective.

Proof. See More on Morphisms, Lemma 4.8 for a more general statement and proof. What follows is a sketch of the proof in the current case.

Firstly, one checks both properties are local on the source and the target. This we may assume that S and X are affine. Say $X = \mathrm{Spec}(B)$ and $S = \mathrm{Spec}(R)$. Say $T = \mathrm{Spec}(C)$. Let J be the square-zero ideal of C with $T_0 = \mathrm{Spec}(C/J)$. Assume that we are given the diagram



Secondly, one checks that the association $\phi' \mapsto \phi' - \phi$ gives a bijection between the set of liftings of $\bar{\phi}$ and the module $\mathrm{Der}_R(B, J)$. Thus, we obtain the implication (1) $\Rightarrow$ (2) via the description of unramified morphisms having trivial module of differentials, see Theorem 4.1.

To obtain the reverse implication, consider the surjection $q : C = (B \otimes_R B)/I^2 \to B = C/J$ defined by the square zero ideal $J = I/I^2$ where I is the kernel of the multiplication map $B \otimes_R B \to B$. We already have a lifting $B \to C$ defined by, say, $b \mapsto b \otimes 1$. Thus, by the same reasoning as above, we obtain a bijective correspondence between liftings of $\mathrm{id} : B \to C/J$ and $\mathrm{Der}_R(B, J)$. The hypothesis therefore implies that the latter module is trivial. But we know that $J \cong \Omega_{B/R}$. Thus, B/R is unramified. ☐

6. Topological properties of unramified morphisms

024S The first topological result that will be of utility to us is one which says that unramified and separated morphisms have "nice" sections. The material in this section does not require any Noetherian hypotheses.

024T **Proposition 6.1.** *Sections of unramified morphisms.*

(1) *Any section of an unramified morphism is an open immersion.*
(2) *Any section of a separated morphism is a closed immersion.*
(3) *Any section of an unramified separated morphism is open and closed.*

Proof. Fix a base scheme S. If $f : X' \to X$ is any S-morphism, then the graph $\Gamma_f : X' \to X' \times_S X$ is obtained as the base change of the diagonal $\Delta_{X/S} : X \to X \times_S X$ via the projection $X' \times_S X \to X \times_S X$. If $g : X \to S$ is separated (resp. unramified) then the diagonal is a closed immersion (resp. open immersion) by Schemes, Definition 21.3 (resp. Morphisms, Lemma 35.13). Hence so is the graph as a base change (by Schemes, Lemma 18.2). In the special case $X' = S$, we obtain (1), resp. (2). Part (3) follows on combining (1) and (2). ☐

We can now explicitly describe the sections of unramified morphisms.

024U **Theorem 6.2.** *Let Y be a connected scheme. Let $f : X \to Y$ be unramified and separated. Every section of f is an isomorphism onto a connected component. There exists a bijective correspondence*

$$\text{sections of } f \leftrightarrow \left\{ \begin{matrix} \text{connected components } X' \text{ of } X \text{ such that} \\ \text{the induced map } X' \to Y \text{ is an isomorphism} \end{matrix} \right\}$$

In particular, given $x \in X$ there is at most one section passing through x.

Proof. Direct from Proposition 6.1 part (3). ☐

The preceding theorem gives us some idea of the "rigidity" of unramified morphisms. Further indication is provided by the following proposition which, besides being intrinsically interesting, is also useful in the theory of the algebraic fundamental group (see [Gro71, Exposé V]). See also the more general Morphisms, Lemma 35.17.

024V **Proposition 6.3.** *Let S is be a scheme. Let $\pi : X \to S$ be unramified and separated. Let Y be an S-scheme and $y \in Y$ a point. Let $f, g : Y \to X$ be two S-morphisms. Assume*

1. *Y is connected*
2. *$x = f(y) = g(y)$, and*
3. *the induced maps $f^\sharp, g^\sharp : \kappa(x) \to \kappa(y)$ on residue fields are equal.*

Then $f = g$.

Proof. The maps $f, g : Y \to X$ define maps $f', g' : Y \to X_Y = Y \times_S X$ which are sections of the structure map $X_Y \to Y$. Note that $f = g$ if and only if $f' = g'$. The structure map $X_Y \to Y$ is the base change of π and hence unramified and separated also (see Morphisms, Lemmas 35.5 and Schemes, Lemma 21.13). Thus according to Theorem 6.2 it suffices to prove that f' and g' pass through the same point of X_Y. And this is exactly what the hypotheses (2) and (3) guarantee, namely $f'(y) = g'(y) \in X_Y$. ☐

0AKI **Lemma 6.4.** *Let S be a Noetherian scheme. Let $X \to S$ be a quasi-compact unramified morphism. Let $Y \to S$ be a morphism with Y Noetherian. Then $\mathrm{Mor}_S(Y, X)$ is a finite set.*

Proof. Assume first $X \to S$ is separated (which is often the case in practice). Since Y is Noetherian it has finitely many connected components. Thus we may assume Y is connected. Choose a point $y \in Y$ with image $s \in S$. Since $X \to S$ is unramified and quasi-compact then fibre X_s is finite, say $X_s = \{x_1, \ldots, x_n\}$ and $\kappa(s) \subset \kappa(x_i)$ is a finite field extension. See Morphisms, Lemma 35.10, 20.5, and 20.10. For each i there are at most finitely many $\kappa(s)$-algebra maps $\kappa(x_i) \to \kappa(y)$ (by elementary field theory). Thus $\mathrm{Mor}_S(Y, X)$ is finite by Proposition 6.3.

General case. There exists a nonempty open $U \subset X$ such that $X_U \to U$ is finite (in particular separated), see Morphisms, Lemma 47.1 (the lemma applies since we've already seen above that a quasi-compact unramified morphism is quasi-finite and since $X \to S$ is quasi-separated by Morphisms, Lemma 15.7). Let $Z \subset S$ be the reduced closed subscheme supported on the complement of U. By Noetherian induction, we see that $\mathrm{Mor}_Z(Y_Z, X_Z)$ is finite (details omitted). By the result of the first paragraph the set $\mathrm{Mor}_U(Y_U, X_U)$ is finite. Thus it suffices to show that

$$\mathrm{Mor}_S(Y, X) \longrightarrow \mathrm{Mor}_Z(Y_Z, X_Z) \times \mathrm{Mor}_U(Y_U, X_U)$$

is injective. This follows from the fact that the set of points where two morphisms $a, b : Y \to X$ agree is open in Y, due to the fact that $\Delta : X \to X \times_S X$ is open, see Morphisms, Lemma 35.13. ☐

7. Universally injective, unramified morphisms

06ND

Recall that a morphism of schemes $f : X \to Y$ is universally injective if any base change of f is injective (on underlying topological spaces), see Morphisms, Definition 11.1. Universally injective and unramified morphisms can be characterized as follows.

05VH **Lemma 7.1.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

1. *f is unramified and a monomorphism,*
2. *f is unramified and universally injective,*
3. *f is locally of finite type and a monomorphism,*
4. *f is universally injective, locally of finite type, and formally unramified,*
5. *f is locally of finite type and X_y is either empty or $X_y \to y$ is an isomorphism for all $y \in Y$.*

Proof. We have seen in More on Morphisms, Lemma 4.8 that being formally unramified and locally of finite type is the same thing as being unramified. Hence (4) is equivalent to (2). A monomorphism is certainly universally injective and formally unramified hence (3) implies (4). It is clear that (1) implies (3). Finally, if (2) holds, then $\Delta : X \to X \times_S X$ is both an open immersion (Morphisms, Lemma 35.13) and surjective (Morphisms, Lemma 11.2) hence an isomorphism, i.e., f is a monomorphism. In this way we see that (2) implies (1).

Condition (3) implies (5) because monomorphisms are preserved under base change (Schemes, Lemma 23.5) and because of the description of monomorphisms towards the spectra of fields in Schemes, Lemma 23.10. Condition (5) implies (4) by Morphisms, Lemmas 11.2 and 35.12. ☐

This leads to the following useful characterization of closed immersions.

04XV **Lemma 7.2.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

1. *f is a closed immersion,*
2. *f is a proper monomorphism,*
3. *f is proper, unramified, and universally injective,*
4. *f is universally closed, unramified, and a monomorphism,*
5. *f is universally closed, unramified, and universally injective,*
6. *f is universally closed, locally of finite type, and a monomorphism,*
7. *f is universally closed, universally injective, locally of finite type, and formally unramified.*

Proof. The equivalence of (4) – (7) follows immediately from Lemma 7.1.

Let $f : X \to S$ satisfy (6). Then f is separated, see Schemes, Lemma 23.3 and has finite fibres. Hence More on Morphisms, Lemma 31.4 shows f is finite. Then Morphisms, Lemma 43.13 implies f is a closed immersion, i.e., (1) holds.

Note that (1) $\Rightarrow$ (2) because a closed immersion is proper and a monomorphism (Morphisms, Lemma 41.6 and Schemes, Lemma 23.7). By Lemma 7.1 we see that (2) implies (3). It is clear that (3) implies (5). ☐

Here is another result of a similar flavor.

04DG **Lemma 7.3.** *Let $\pi : X \to S$ be a morphism of schemes. Let $s \in S$. Assume that*

1. *π is finite,*

(2) *π is unramified,*
(3) *$\pi^{-1}(\{s\}) = \{x\}$, and*
(4) *$\kappa(s) \subset \kappa(x)$ is purely inseparable*[1].

Then there exists an open neighbourhood U of s such that $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is a closed immersion.

Proof. The question is local on S. Hence we may assume that $S = \text{Spec}(A)$. By definition of a finite morphism this implies $X = \text{Spec}(B)$. Note that the ring map $\varphi : A \to B$ defining π is a finite unramified ring map. Let $\mathfrak{p} \subset A$ be the prime corresponding to s. Let $\mathfrak{q} \subset B$ be the prime corresponding to x. By Conditions (2), (3) and (4) imply that $B_\mathfrak{q}/\mathfrak{p}B_\mathfrak{q} = \kappa(\mathfrak{p})$. Algebra, Lemma 40.11 we have $B_\mathfrak{q} = B_\mathfrak{p}$ (note that a finite ring map satisfies going up, see Algebra, Section 40.) Hence we see that $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p} = \kappa(\mathfrak{p})$. As B is a finite A-module we see from Nakayama's lemma (see Algebra, Lemma 19.1) that $B_\mathfrak{p} = \varphi(A_\mathfrak{p})$. Hence (using the finiteness of B as an A-module again) there exists a $f \in A$, $f \notin \mathfrak{p}$ such that $B_f = \varphi(A_f)$ as desired. $\square$

The topological results presented above will be used to give a functorial characterization of étale morphisms similar to Theorem 5.1.

8. Examples of unramified morphisms

024W Here are a few examples.

024X **Example 8.1.** Let k be a field. Unramified quasi-compact morphisms $X \to \text{Spec}(k)$ are affine. This is true because X has dimension 0 and is Noetherian, hence is a finite discrete set, and each point gives an affine open, so X is a finite disjoint union of affines hence affine. Noether normalization forces X to be the spectrum of a finite k-algebra A. This algebra is a product of finite separable field extensions of k. Thus, an unramified quasi-compact morphism to $\text{Spec}(k)$ corresponds to a finite number of finite separable field extensions of k. In particular, an unramified morphism with a connected source and a one point target is forced to be a finite separable field extension. As we will see later, $X \to \text{Spec}(k)$ is étale if and only if it is unramified. Thus, in this case at least, we obtain a very easy description of the étale topology of a scheme. Of course, the cohomology of this topology is another story.

024Y **Example 8.2.** Property (3) in Theorem 4.1 gives us a canonical source of examples for unramified morphisms. Fix a ring R and an integer n. Let $I = (g_1, \ldots, g_m)$ be an ideal in $R[x_1, \ldots, x_n]$. Let $\mathfrak{q} \subset R[x_1, \ldots, x_n]$ be a prime. Assume $I \subset \mathfrak{q}$ and that the matrix

$$\left(\frac{\partial g_i}{\partial x_j}\right) \bmod \mathfrak{q} \quad \in \quad \text{Mat}(n \times m, \kappa(\mathfrak{q}))$$

has rank n. Then the morphism $f : Z = \text{Spec}(R[x_1, \ldots, x_n]/I) \to \text{Spec}(R)$ is unramified at the point $x \in Z \subset \mathbf{A}^n_R$ corresponding to $\mathfrak{q}$. Clearly we must have $m \geq n$. In the extreme case $m = n$, i.e., the differential of the map $\mathbf{A}^n_R \to \mathbf{A}^n_R$ defined by the g_i's is an isomorphism of the tangent spaces, then f is also flat x and, hence, is an étale map (see Algebra, Definition 134.6, Lemma 134.7 and Example 134.8).

[1]In view of condition (2) this is equivalent to $\kappa(s) = \kappa(x)$.

024Z **Example 8.3.** Fix an extension of number fields L/K with rings of integers $\mathcal{O}_L$ and $\mathcal{O}_K$. The injection $K \to L$ defines a morphism $f : \text{Spec}(\mathcal{O}_L) \to \text{Spec}(\mathcal{O}_K)$. As discussed above, the points where f is unramified in our sense correspond to the set of points where f is unramified in the conventional sense. In the conventional sense, the locus of ramification in $\text{Spec}(\mathcal{O}_L)$ can be defined by vanishing set of the different; this is an ideal in $\mathcal{O}_L$. In fact, the different is nothing but the annihilator of the module $\Omega_{\mathcal{O}_L/\mathcal{O}_K}$. Similarly, the discriminant is an ideal in $\mathcal{O}_K$, namely it is the norm of the different. The vanishing set of the discriminant is precisely the set of points of K which ramify in L. Thus, denoting by X the complement of the closed subset defined by the different in $\text{Spec}(\mathcal{O}_L)$, we obtain a morphism $X \to \text{Spec}(\mathcal{O}_L)$ which is unramified. Furthermore, this morphism is also flat, as any local homomorphism of discrete valuation rings is flat, and hence this morphism is actually étale. If L/K is Galois, then denoting by Y the complement of the closed subset defined by the discriminant in $\text{Spec}(\mathcal{O}_K)$, we see that we get even a finite étale morphism $X \to Y$. Thus, this is an example of a finite étale covering.

9. Flat morphisms

0250 This section simply exists to summarize the properties of flatness that will be useful to us. Thus, we will be content with stating the theorems precisely and giving references for the proofs.

After briefly recalling the necessary facts about flat modules over Noetherian rings, we state a theorem of Grothendieck which gives sufficient conditions for “hyperplane sections” of certain modules to be flat.

0251 **Definition 9.1.** Flatness of modules and rings.

(1) A module N over a ring A is said to be *flat* if the functor $M \mapsto M \otimes_A N$ is exact.
(2) If this functor is also faithful, we say that N is *faithfully flat* over A.
(3) A morphism of rings $f : A \to B$ is said to be *flat (resp. faithfully flat)* if the functor $M \mapsto M \otimes_A B$ is exact (resp. faithful and exact).

Here is a list of facts with references to the algebra chapter.

(1) Free and projective modules are flat. This is clear for free modules and follows for projective modules as they are direct summands of free modules and $\otimes$ commutes with direct sums.
(2) Flatness is a local property, that is, M is flat over A if and only if $M_{\mathfrak{p}}$ is flat over $A_{\mathfrak{p}}$ for all $\mathfrak{p} \in \text{Spec}(A)$. See Algebra, Lemma 38.19.
(3) If M is a flat A-module and $A \to B$ is a ring map, then $M \otimes_A B$ is a flat B-module. See Algebra, Lemma 38.7.
(4) Finite flat modules over local rings are free. See Algebra, Lemma 76.4.
(5) If $f : A \to B$ is a morphism of arbitrary rings, f is flat if and only if the induced maps $A_{f^{-1}(\mathfrak{q})} \to B_{\mathfrak{q}}$ are flat for all $\mathfrak{q} \in \text{Spec}(B)$. See Algebra, Lemma 38.19
(6) If $f : A \to B$ is a local homomorphism of local rings, f is flat if and only if it is faithfully flat. See Algebra, Lemma 38.17.
(7) A map $A \to B$ of rings is faithfully flat if and only if it is flat and the induced map on spectra is surjective. See Algebra, Lemma 38.16.
(8) If A is a noetherian local ring, the completion $A^\wedge$ is faithfully flat over A. See Algebra, Lemma 95.3.

(9) Let A be a Noetherian local ring and M an A-module. Then M is flat over A if and only if $M \otimes_A A^\wedge$ is flat over $A^\wedge$. (Combine the previous statement with Algebra, Lemma 38.8.)

Before we move on to the geometric category, we present Grothendieck's theorem, which provides a convenient recipe for producing flat modules.

0252 **Theorem 9.2.** *Let A, B be Noetherian local rings. Let $f : A \to B$ be a local homomorphism. If M is a finite B-module that is flat as an A-module, and $t \in \mathfrak{m}_B$ is an element such that multiplication by t is injective on $M/\mathfrak{m}_A M$, then M/tM is also A-flat.*

Proof. See Algebra, Lemma 97.1. See also [Mat70, Section 20]. □

0253 **Definition 9.3.** (See Morphisms, Definition 25.1). Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module.

(1) Let $x \in X$. We say $\mathcal{F}$ is *flat over Y at $x \in X$* if $\mathcal{F}_x$ is a flat $\mathcal{O}_{Y,f(x)}$-module. This uses the map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ to think of $\mathcal{F}_x$ as a $\mathcal{O}_{Y,f(x)}$-module.
(2) Let $x \in X$. We say f is *flat at $x \in X$* if $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat.
(3) We say f is *flat* if it is flat at all points of X.
(4) A morphism $f : X \to Y$ that is flat and surjective is sometimes said to be *faithfully flat*.

Once again, here is a list of results:

(1) The property (of a morphism) of being flat is, by fiat, local in the Zariski topology on the source and the target.
(2) Open immersions are flat. (This is clear because it induces isomorphisms on local rings.)
(3) Flat morphisms are stable under base change and composition. Morphisms, Lemmas 25.7 and 25.5.
(4) If $f : X \to Y$ is flat, then the pullback functor $QCoh(\mathcal{O}_Y) \to QCoh(\mathcal{O}_X)$ is exact. This is immediate by looking at stalks.
(5) Let $f : X \to Y$ be a morphism of schemes, and assume Y is quasi-compact and quasi-separated. In this case if the functor f^* is exact then f is flat. (Proof omitted. Hint: Use Properties, Lemma 21.1 to see that Y has "enough" ideal sheaves and use the characterization of flatness in Algebra, Lemma 38.5.)

10. Topological properties of flat morphisms

0254 We "recall" below some openness properties that flat morphisms enjoy.

0255 **Theorem 10.1.** *Let Y be a locally Noetherian scheme. Let $f : X \to Y$ be a morphism which is locally of finite type. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The set of points in X where $\mathcal{F}$ is flat over S is an open set. In particular the set of points where f is flat is open in X.*

Proof. See More on Morphisms, Theorem 12.1. □

039K **Theorem 10.2.** *Let Y be a locally Noetherian scheme. Let $f : X \to Y$ be a morphism which is flat and locally of finite type. Then f is (universally) open.*

Proof. See Morphisms, Lemma 25.9. □

0256 **Theorem 10.3.** *A faithfully flat quasi-compact morphism is a quotient map for the Zariski topology.*

Proof. See Morphisms, Lemma 25.10. □

An important reason to study flat morphisms is that they provide the adequate framework for capturing the notion of a family of schemes parametrized by the points of another scheme. Naively one may think that any morphism $f : X \to S$ should be thought of as a family parametrized by the points of S. However, without a flatness restriction on f, really bizarre things can happen in this so-called family. For instance, we aren't guaranteed that relative dimension (dimension of the fibres) is constant in a family. Other numerical invariants, such as the Hilbert polynomial, too may change from fibre to fibre. Flatness prevents such things from happening and, therefore, provides some "continuity" to the fibres.

11. Étale morphisms

0257 In this section, we will define étale morphisms and prove a number of important properties about them. The most important one, no doubt, is the functorial characterization presented in Theorem 16.1. Following this, we will also discuss a few properties of rings which are insensitive to an étale extension (properties which hold for a ring if and only if they hold for all its étale extensions) to motivate the basic tenet of étale cohomology – étale morphisms are the algebraic analogue of local isomorphisms.

As the title suggests, we will define the class of étale morphisms – the class of morphisms (whose surjective families) we shall deem to be coverings in the category of schemes over a base scheme S in order to define the étale site $S_{étale}$. Intuitively, an étale morphism is supposed to capture the idea of a covering space and, therefore, should be close to a local isomorphism. If we're working with varieties over algebraically closed fields, this last statement can be made into a definition provided we replace "local isomorphism" with "formal local isomorphism" (isomorphism after completion). One can then give a definition over any base field by asking that the base change to the algebraic closure be étale (in the aforementioned sense). But, rather than proceeding via such aesthetically displeasing constructions, we will adopt a cleaner, albeit slightly more abstract, algebraic approach.

0258 **Definition 11.1.** Let A, B be Noetherian local rings. A local homomorphism $f : A \to B$ is said to be a *étale homomorphism of local rings* if it is flat and unramified homomorphism of local rings (please see Definition 3.1).

This is the local version of the definition of an étale ring map in Algebra, Section 140. The exact definition given in that section is that it is a smooth ring map of relative dimension 0. It is shown (in Algebra, Lemma 140.2) that an étale R-algebra S always has a presentation

$$S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$$

such that

$$g = \det \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_2/\partial x_1 & \ldots & \partial f_n/\partial x_1 \\ \partial f_1/\partial x_2 & \partial f_2/\partial x_2 & \ldots & \partial f_n/\partial x_2 \\ \ldots & \ldots & \ldots & \ldots \\ \partial f_1/\partial x_n & \partial f_2/\partial x_n & \ldots & \partial f_n/\partial x_n \end{pmatrix}$$

maps to an invertible element in S. The following two lemmas link the two notions.

039L **Lemma 11.2.** *Let $A \to B$ be of finite type with A a Noetherian ring. Let $\mathfrak{q}$ be a prime of B lying over $\mathfrak{p} \subset A$. Then $A \to B$ is étale at $\mathfrak{q}$ if and only if $A_{\mathfrak{p}} \to B_{\mathfrak{q}}$ is an étale homomorphism of local rings.*

Proof. See Algebra, Lemmas 140.3 (flatness of étale maps), 140.5 (étale maps are unramified) and 140.7 (flat and unramified maps are étale). □

039M **Lemma 11.3.** *Let A, B be Noetherian local rings. Let $A \to B$ be a local homomorphism such that B is essentially of finite type over A. The following are equivalent*

1. *$A \to B$ is an étale homomorphism of local rings*
2. *$A^\wedge \to B^\wedge$ is an étale homomorphism of local rings, and*
3. *$A^\wedge \to B^\wedge$ is étale.*

Moreover, in this case $B^\wedge \cong (A^\wedge)^{\oplus n}$ as $A^\wedge$-modules for some $n \geq 1$.

Proof. To see the equivalences of (1), (2) and (3), as we have the corresponding results for unramified ring maps (Lemma 3.4) it suffices to prove that $A \to B$ is flat if and only if $A^\wedge \to B^\wedge$ is flat. This is clear from our lists of properties of flat maps since the ring maps $A \to A^\wedge$ and $B \to B^\wedge$ are faithfully flat. For the final statement, by Lemma 3.3 we see that $B^\wedge$ is a finite flat $A^\wedge$ module. Hence it is finite free by our list of properties on flat modules in Section 9. □

The integer n which occurs in the lemma above is nothing other than the degree $[\kappa(\mathfrak{m}_B) : \kappa(\mathfrak{m}_A)]$ of the residue field extension. In particular, if $\kappa(\mathfrak{m}_A)$ is separably closed, we see that $A^\wedge \to B^\wedge$ is an isomorphism, which vindicates our earlier claims.

0259 **Definition 11.4.** (See Morphisms, Definition 36.1.) Let Y be a locally Noetherian scheme. Let $f : X \to Y$ be a morphism of schemes which is locally of finite type.

1. Let $x \in X$. We say f is *étale at $x \in X$* if $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is an étale homomorphism of local rings.
2. The morphism is said to be *étale* if it is étale at all its points.

Let us prove that this definition agrees with the definition in the chapter on morphisms of schemes. This in particular guarantees that the set of points where a morphism is étale is open.

039N **Lemma 11.5.** *Let Y be a locally Noetherian scheme. Let $f : X \to Y$ be locally of finite type. Let $x \in X$. The morphism f is étale at x in the sense of Definition 11.4 if and only if it is unramified at x in the sense of Morphisms, Definition 36.1.*

Proof. This follows from Lemma 11.2 and the definitions. □

Here are some results on étale morphisms. The formulations as given in this list apply only to morphisms locally of finite type between locally Noetherian schemes. In each case we give a reference to the general result as proved earlier in the project, but in some cases one can prove the result more easily in the Noetherian case. Here is the list:

1. An étale morphism is unramified. (Clear from our definitions.)
2. Étaleness is local on the source and the target in the Zariski topology.
3. Étale morphisms are stable under base change and composition. See Morphisms, Lemmas 36.4 and 36.3.

(4) Étale morphisms of schemes are locally quasi-finite and quasi-compact étale morphisms are quasi-finite. (This is true because it holds for unramified morphisms as seen earlier.)
(5) Étale morphisms have relative dimension 0. See Morphisms, Definition 29.1 and Morphisms, Lemma 29.5.
(6) A morphism is étale if and only if it is flat and all its fibres are étale. See Morphisms, Lemma 36.8.
(7) Étale morphisms are open. This is true because an étale morphism is flat, and Theorem 10.2.
(8) Let X and Y be étale over a base scheme S. Any S-morphism from X to Y is étale. See Morphisms, Lemma 36.18.

12. The structure theorem

025A We present a theorem which describes the local structure of étale and unramified morphisms. Besides its obvious independent importance, this theorem also allows us to make the transition to another definition of étale morphisms that captures the geometric intuition better than the one we've used so far.

To state it we need the notion of a *standard étale ring map*, see Algebra, Definition 140.14. Namely, suppose that R is a ring and $f, g \in R[t]$ are polynomials such that

(a) f is a monic polynomial, and
(b) $f' = \mathrm{d}f/\mathrm{d}t$ is invertible in the localization $R[t]_g/(f)$.

Then the map
$$R \longrightarrow R[t]_g/(f) = R[t, 1/g]/(f)$$
is a standard étale algebra, and any standard étale algebra is isomorphic to one of these. It is a pleasant exercise to prove that such a ring map is flat, and unramified and hence étale (as expected of course). A special case of a standard étale ring map is any ring map
$$R \longrightarrow R[t]_{f'}/(f) = R[t, 1/f']/(f)$$
with f a monic polynomial, and any standard étale algebra is (isomorphic to) a principal localization of one of these.

025B **Theorem 12.1.** *Let $f : A \to B$ be an étale homomorphism of local rings. Then there exist $f, g \in A[t]$ such that*

(1) *$B' = A[t]_g/(f)$ is standard étale – see (a) and (b) above, and*
(2) *B is isomorphic to a localization of B' at a prime.*

Proof. Write $B = B'_{\mathfrak q}$ for some finite type A-algebra B' (we can do this because B is essentially of finite type over A). By Lemma 11.2 we see that $A \to B'$ is étale at $\mathfrak q$. Hence we may apply Algebra, Proposition 140.17 to see that a principal localization of B' is standard étale. □

Here is the version for unramified homomorphisms of local rings.

039O **Theorem 12.2.** *Let $f : A \to B$ be an unramified morphism of local rings. Then there exist $f, g \in A[t]$ such that*

(1) *$B' = A[t]_g/(f)$ is standard étale – see (a) and (b) above, and*
(2) *B is isomorphic to a quotient of a localization of B' at a prime.*

Proof. Write $B = B'_{\mathfrak q}$ for some finite type A-algebra B' (we can do this because B is essentially of finite type over A). By Lemma 3.2 we see that $A \to B'$ is unramified at $\mathfrak q$. Hence we may apply Algebra, Proposition 146.8 to see that a principal localization of B' is a quotient of a standard étale A-algebra. ☐

Via standard lifting arguments, one then obtains the following geometric statement which will be of essential use to us.

025C **Theorem 12.3.** *Let $\varphi : X \to Y$ be a morphism of schemes. Let $x \in X$. If φ is étale at x, then there exist exist affine opens $V \subset Y$ and $U \subset X$ with $x \in U$ and $\varphi(U) \subset V$ such that we have the following diagram*

$$\begin{array}{ccccc} X & \longleftarrow & U & \xrightarrow{\ j\ } & \mathrm{Spec}(R[t]_{f'}/(f)) \\ \downarrow & & \downarrow & & \downarrow \\ Y & \longleftarrow & V & = & \mathrm{Spec}(R) \end{array}$$

where j is an open immersion, and $f \in R[t]$ is monic.

Proof. This is equivalent to Morphisms, Lemma 36.14 although the statements differ slightly. ☐

13. Étale and smooth morphisms

039P An étale morphism is smooth of relative dimension zero. The projection $\mathbf{A}^n_S \to S$ is a standard example of a smooth morphism of relative dimension n. It turns out that any smooth morphism is étale locally of this form. Here is the precise statement.

039Q **Theorem 13.1.** *Let $\varphi : X \to Y$ be a morphism of schemes. Let $x \in X$. If φ is smooth at x, then there exist exist and integer $n \geq 0$ and affine opens $V \subset Y$ and $U \subset X$ with $x \in U$ and $\varphi(U) \subset V$ such that there exists a commutative diagram*

$$\begin{array}{ccccccc} X & \longleftarrow & U & \xrightarrow{\ \pi\ } & \mathbf{A}^n_R & = & \mathrm{Spec}(R[x_1, \ldots, x_n]) \\ \downarrow & & \downarrow & & \downarrow & \swarrow & \\ Y & \longleftarrow & V & = & \mathrm{Spec}(R) & & \end{array}$$

where π is étale.

Proof. See Morphisms, Lemma 36.20. ☐

14. Topological properties of étale morphisms

025F We present a few of the topological properties of étale and unramified morphisms. First, we give what Grothendieck calls the *fundamental property of étale morphisms*, see [Gro71, Exposé I.5].

025G **Theorem 14.1.** *Let $f : X \to Y$ be a morphism of schemes. The following are equivalent:*

1. *f is an open immersion,*
2. *f is universally injective and étale, and*
3. *f is a flat monomorphism, locally of finite presentation.*

Proof. An open immersion is universally injective since any base change of an open immersion is an open immersion. Moreover, it is étale by Morphisms, Lemma 36.9. Hence (1) implies (2).

Assume f is universally injective and étale. Since f is étale it is flat and locally of finite presentation, see Morphisms, Lemmas 36.12 and 36.11. By Lemma 7.1 we see that f is a monomorphism. Hence (2) implies (3).

Assume f is flat, locally of finite presentation, and a monomorphism. Then f is open, see Morphisms, Lemma 25.9. Thus we may replace Y by $f(X)$ and we may assume f is surjective. Then f is open and bijective hence a homeomorphism. Hence f is quasi-compact. Hence Descent, Lemma 21.1 shows that f is an isomorphism and we win. □

Here is another result of a similar flavor.

04DH **Lemma 14.2.** *Let $\pi : X \to S$ be a morphism of schemes. Let $s \in S$. Assume that*

(1) *π is finite,*
(2) *π is étale,*
(3) *$\pi^{-1}(\{s\}) = \{x\}$, and*
(4) *$\kappa(s) \subset \kappa(x)$ is purely inseparable*[2].

Then there exists an open neighbourhood U of s such that $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is an isomorphism.

Proof. By Lemma 7.3 there exists an open neighbourhood U of s such that $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is a closed immersion. But a morphism which is étale and a closed immersion is an open immersion (for example by Theorem 14.1). Hence after shrinking U we obtain an isomorphism. □

15. Topological invariance of the étale topology

06NE Next, we present an extremely crucial theorem which, roughly speaking, says that étaleness is a topological property.

025H **Theorem 15.1.** *Let X and Y be two schemes over a base scheme S. Let S_0 be a closed subscheme of S whose ideal sheaf has square zero. Denote X_0 (resp. Y_0) the base change $S_0 \times_S X$ (resp. $S_0 \times_S Y$). If X is étale over S, then the map*

$$\mathrm{Mor}_S(Y, X) \longrightarrow \mathrm{Mor}_{S_0}(Y_0, X_0)$$

is bijective.

Proof. After base changing via $Y \to S$, we may assume that $Y = S$. In this case the theorem states that any S-morphism $\sigma_0 : S_0 \to X$ actually factors uniquely through a section $S \to X$ of the étale structure morphism $X \to S$.

Existence. Since we have equality of underlying topological spaces $|S_0| = |S|$ and $|X_0| = |X|$, by Theorem 6.2, the section σ_0 is uniquely determined by a connected component X' of X such that the base change $X'_0 = S_0 \times_S X'$ maps isomorphically to S_0. In particular, $X' \to S$ is a universal homeomorphism and therefore universally injective. Since $X' \to S$ is étale, it follows from Theorem 14.1 that $X' \to S$ is an isomorphism and, therefore, it has an inverse σ which is the required section.

[2]In view of condition (2) this is equivalent to $\kappa(s) = \kappa(x)$.

Uniqueness. This follows from Theorem 5.1, or directly from Theorem 6.2, or, if one carefuly observes, from our proof itself. □

From the proof of preceeding theorem, we also obtain one direction of the promised functorial characterization of étale morphisms. The following theorem will be strengthened in Étale Cohomology, Theorem 46.1.

039R **Theorem 15.2** (Une equivalence remarquable de catégories)**.** *Let S be a scheme. Let $S_0 \subset S$ be a closed subscheme defined by an ideal with square zero. The functor*

$$X \longmapsto X_0 = S_0 \times_S X$$

defines an equivalence of categories

$$\{\textit{schemes } X \textit{ étale over } S\} \leftrightarrow \{\textit{schemes } X_0 \textit{ étale over } S_0\}$$

Proof. By Theorem 15.1 we see that this functor is fully faithful. It remains to show that the functor is essentially surjective. Let $Y \to S_0$ be an étale morphism of schemes.

Suppose that the result holds if S and Y are affine. In that case, we choose an affine open covering $Y = \bigcup V_j$ such that each V_j maps into an affine open of S. By assumption (affine case) we can find étale morphisms $W_j \to S$ such that $W_{j,0} \cong V_j$ (as schemes over S_0). Let $W_{j,j'} \subset W_j$ be the open subscheme whose underlying topological space corresponds to $V_j \cap V_{j'}$. Because we have isomorphisms

$$W_{j,j',0} \cong V_j \cap V_{j'} \cong W_{j',j,0}$$

as schemes over S_0 we see by fully faithfulness that we obtain isomorphisms $\theta_{j,j'} : W_{j,j'} \to W_{j',j}$ of schemes over S. We omit the verification that these isomorphisms satisfy the cocycle condition of Schemes, Section 14. Applying Schemes, Lemma 14.2 we obtain a scheme $X \to S$ by glueing the schemes W_j along the identifications $\theta_{j,j'}$. It is clear that $X \to S$ is étale and $X_0 \cong Y$ by construction.

Thus it suffices to show the lemma in case S and Y are affine. Say $S = \text{Spec}(R)$ and $S_0 = \text{Spec}(R/I)$ with $I^2 = 0$. By Algebra, Lemma 140.2 we know that Y is the spectrum of a ring $\overline{A}$ with

$$\overline{A} = (R/I)[x_1, \ldots, x_n]/(\overline{f}_1, \ldots, \overline{f}_n)$$

such that

$$\overline{g} = \det \begin{pmatrix} \partial\overline{f}_1/\partial x_1 & \partial\overline{f}_2/\partial x_1 & \ldots & \partial\overline{f}_n/\partial x_1 \\ \partial\overline{f}_1/\partial x_2 & \partial\overline{f}_2/\partial x_2 & \ldots & \partial\overline{f}_n/\partial x_2 \\ \ldots & \ldots & \ldots & \ldots \\ \partial\overline{f}_1/\partial x_n & \partial\overline{f}_2/\partial x_n & \ldots & \partial\overline{f}_n/\partial x_n \end{pmatrix}$$

maps to an invertible element in A. Choose any lifts $f_i \in R[x_1, \ldots, x_n]$. Since I is nilpotent it follows that the determinant of the matrix of partials of the f_i is invertible in the algebra A defined by

$$A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$$

Hence $R \to A$ is étale and $(R/I) \otimes_R A \cong \overline{A}$. To prove the general case one argues with glueing affine pieces. □

16. The functorial characterization

025J We finally present the promised functorial characterization. Thus there are four ways to think about étale morphisms of schemes:

(1) as a smooth morphism of relative dimension 0,
(2) as locally finitely presented, flat, and unramified morphisms,
(3) using the structure theorem, and
(4) using the functorial characterization.

025K **Theorem 16.1.** *Let $f : X \to S$ be a morphism that is locally of finite presentation. The following are equivalent*

(1) *f is étale,*
(2) *for all affine S-schemes Y, and closed subschemes $Y_0 \subset Y$ defined by square-zero ideals, the natural map*

$$\mathrm{Mor}_S(Y, X) \longrightarrow \mathrm{Mor}_S(Y_0, X)$$

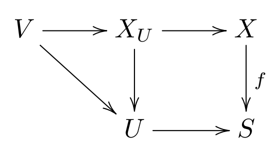
is bijective.

Proof. This is More on Morphisms, Lemma 6.9. □

This characterization says that solutions to the equations defining X can be lifted uniquely through nilpotent thickenings.

17. Étale local structure of unramified morphisms

04HG In the chapter More on Morphisms, Section 30 the reader can find some results on the étale local structure of quasi-finite morphisms. In this section we want to combine this with the topological properties of unramified morphisms we have seen in this chapter. The basic overall picture to keep in mind is



see More on Morphisms, Equation (30.0.1). We start with a very general case.

04HH **Lemma 17.1.** *Let $f : X \to S$ be a morphism of schemes. Let $x_1, \ldots, x_n \in X$ be points having the same image s in S. Assume f is unramified at each x_i. Then there exists an étale neighbourhood $(U, u) \to (S, s)$ and opens $V_{i,j} \subset X_U$, $i = 1, \ldots, n$, $j = 1, \ldots, m_i$ such that*

(1) *$V_{i,j} \to U$ is a closed immersion passing through u,*
(2) *u is not in the image of $V_{i,j} \cap V_{i',j'}$ unless $i = i'$ and $j = j'$, and*
(3) *any point of $(X_U)_u$ mapping to x_i is in some $V_{i,j}$.*

Proof. By Morphisms, Definition 35.1 there exists an open neighbourhood of each x_i which is locally of finite type over S. Replacing X by an open neighbourhood of $\{x_1, \ldots, x_n\}$ we may assume f is locally of finite type. Apply More on Morphisms, Lemma 30.3 to get the étale neighbourhood (U, u) and the opens $V_{i,j}$ finite over U. By Lemma 7.3 after possibly shrinking U we get that $V_{i,j} \to U$ is a closed immersion. □

04HI **Lemma 17.2.** *Let $f : X \to S$ be a morphism of schemes. Let $x_1, \ldots, x_n \in X$ be points having the same image s in S. Assume f is separated and f is unramified at each x_i. Then there exists an étale neighbourhood $(U, u) \to (S, s)$ and a disjoint union decomposition*

$$X_U = W \amalg \coprod\nolimits_{i,j} V_{i,j}$$

such that

1. *$V_{i,j} \to U$ is a closed immersion passing through u,*
2. *the fibre W_u contains no point mapping to any x_i.*

In particular, if $f^{-1}(\{s\}) = \{x_1, \ldots, x_n\}$, then the fibre W_u is empty.

Proof. Apply Lemma 17.1. We may assume U is affine, so X_U is separated. Then $V_{i,j} \to X_U$ is a closed map, see Morphisms, Lemma 41.7. Suppose $(i, j) \neq (i', j')$. Then $V_{i,j} \cap V_{i',j'}$ is closed in $V_{i,j}$ and its image in U does not contain u. Hence after shrinking U we may assume that $V_{i,j} \cap V_{i',j'} = \emptyset$. Moreover, $\bigcup V_{i,j}$ is a closed and open subscheme of X_U and hence has an open and closed complement W. This finishes the proof. □

The following lemma is in some sense much weaker than the preceding one but it may be useful to state it explicitly here. It says that a finite unramified morphism is étale locally on the base a closed immersion.

04HJ **Lemma 17.3.** *Let $f : X \to S$ be a finite unramified morphism of schemes. Let $s \in S$. There exists an étale neighbourhood $(U, u) \to (S, s)$ and a disjoint union decomposition*

$$X_U = \coprod\nolimits_j V_j$$

such that each $V_j \to U$ is a closed immersion.

Proof. Since $X \to S$ is finite the fibre over S is a finite set $\{x_1, \ldots, x_n\}$ of points of X. Apply Lemma 17.2 to this set (a finite morphism is separated, see Morphisms, Section 43). The image of W in U is a closed subset (as $X_U \to U$ is finite, hence proper) which does not contain u. After removing this from U we see that $W = \emptyset$ as desired. □

18. Étale local structure of étale morphisms

04HK This is a bit silly, but perhaps helps form intuition about étale morphisms. We simply copy over the results of Section 17 and change "closed immersion" into "isomorphism".

04HL **Lemma 18.1.** *Let $f : X \to S$ be a morphism of schemes. Let $x_1, \ldots, x_n \in X$ be points having the same image s in S. Assume f is étale at each x_i. Then there exists an étale neighbourhood $(U, u) \to (S, s)$ and opens $V_{i,j} \subset X_U$, $i = 1, \ldots, n$, $j = 1, \ldots, m_i$ such that*

1. *$V_{i,j} \to U$ is an isomorphism,*
2. *u is not in the image of $V_{i,j} \cap V_{i',j'}$ unless $i = i'$ and $j = j'$, and*
3. *any point of $(X_U)_u$ mapping to x_i is in some $V_{i,j}$.*

Proof. An étale morphism is unramified, hence we may apply Lemma 17.1. Now $V_{i,j} \to U$ is a closed immersion and étale. Hence it is an open immersion, for example by Theorem 14.1. Replace U by the intersection of the images of $V_{i,j} \to U$ to get the lemma. □

04HM **Lemma 18.2.** *Let $f : X \to S$ be a morphism of schemes. Let $x_1, \ldots, x_n \in X$ be points having the same image s in S. Assume f is separated and f is étale at each x_i. Then there exists an étale neighbourhood $(U, u) \to (S, s)$ and a disjoint union decomposition*

$$X_U = W \amalg \coprod\nolimits_{i,j} V_{i,j}$$

such that

1. *$V_{i,j} \to U$ is an isomorphism,*
2. *the fibre W_u contains no point mapping to any x_i.*

In particular, if $f^{-1}(\{s\}) = \{x_1, \ldots, x_n\}$, then the fibre W_u is empty.

Proof. An étale morphism is unramified, hence we may apply Lemma 17.2. As in the proof of Lemma 18.1 the morphisms $V_{i,j} \to U$ are open immersions and we win after replacing U by the intersection of their images. □

The following lemma is in some sense much weaker than the preceding one but it may be useful to state it explicitly here. It says that a finite étale morphism is étale locally on the base a "topological covering space", i.e., a finite product of copies of the base.

04HN **Lemma 18.3.** *Let $f : X \to S$ be a finite étale morphism of schemes. Let $s \in S$. There exists an étale neighbourhood $(U, u) \to (S, s)$ and a disjoint union decomposition*

$$X_U = \coprod\nolimits_j V_j$$

such that each $V_j \to U$ is an isomorphism.

Proof. An étale morphism is unramified, hence we may apply Lemma 17.3. As in the proof of Lemma 18.1 we see that $V_{i,j} \to U$ is an open immersion and we win after replacing U by the intersection of their images. □

19. Permanence properties

025L In what follows, we present a few "permanence" properties of étale homomorphisms of Noetherian local rings (as defined in Definition 11.1). See More on Algebra, Sections 34 and 36 for the analogue of this material for the completion and henselization of a Noetherian local ring.

039S **Lemma 19.1.** *Let A, B be Noetherian local rings. Let $A \to B$ be a étale homomorphism of local rings. Then $\dim(A) = \dim(B)$.*

Proof. See for example Algebra, Lemma 110.7. □

039T **Proposition 19.2.** *Let A, B be Noetherian local rings. Let $f : A \to B$ be an étale homomorphism of local rings. Then $depth(A) = depth(B)$*

Proof. See Algebra, Lemma 154.2. □

025Q **Proposition 19.3.** *Let A, B be Noetherian local rings. Let $f : A \to B$ be an étale homomorphism of local rings. Then A is Cohen-Macaulay if and only if B is so.*

Proof. A local ring A is Cohen-Macaulay if and only $\dim(A) = \operatorname{depth}(A)$. As both of these invariants is preserved under an étale extension, the claim follows. □

025N **Proposition 19.4.** *Let A, B be Noetherian local rings. Let $f : A \to B$ be an étale homomorphism of local rings. Then A is regular if and only if B is so.*

Proof. If B is regular, then A is regular by Algebra, Lemma 108.9. Assume A is regular. Let $\mathfrak{m}$ be the maximal ideal of A. Then $\dim_{\kappa(\mathfrak{m})} \mathfrak{m}/\mathfrak{m}^2 = \dim(A) = \dim(B)$ (see Lemma 19.1). On the other hand, $\mathfrak{m}B$ is the maximal ideal of B and hence $\mathfrak{m}_B/\mathfrak{m}_B = \mathfrak{m}B/\mathfrak{m}^2B$ is generated by at most $\dim(B)$ elements. Thus B is regular. (You can also use the slightly more general Algebra, Lemma 110.8.) □

025O **Proposition 19.5.** *Let A, B be Noetherian local rings. Let $f : A \to B$ be an étale homomorphism of local rings. Then A is reduced if and only if B is so.*

Proof. It is clear from the faithful flatness of $A \to B$ that if B is reduced, so is A. See also Algebra, Lemma 155.2. Conversely, assume A is reduced. By assumption B is a localization of a finite type A-algebra B' at some prime $\mathfrak{q}$. After replacing B' by a localization we may assume that B' is étale over A, see Lemma 11.2. Then we see that Algebra, Lemma 154.6 applies to $A \to B'$ and B' is reduced. Hence B is reduced. □

039U **Remark 19.6.** The result on "reducedness" does not hold with a weaker definition of étale local ring maps $A \to B$ where one drops the assumption that B is essentially of finite type over A. Namely, it can happen that a Noetherian local domain A has nonreduced completion $A^\wedge$, see Examples, Section 15. But the ring map $A \to A^\wedge$ is flat, and $\mathfrak{m}_A A^\wedge$ is the maximal ideal of $A^\wedge$ and of course A and $A^\wedge$ have the same residue fields. This is why it is important to consider this notion only for ring extensions which are essentially of finite type (or essentially of finite presentation if A is not Noetherian).

025P **Proposition 19.7.** *Let A, B be Noetherian local rings. Let $f : A \to B$ be an étale homomorphism of local rings. Then A is a normal domain if and only if B is so.*

Proof. See Algebra, Lemma 155.3 for descending normality. Conversely, assume A is normal. By assumption B is a localization of a finite type A-algebra B' at some prime $\mathfrak{q}$. After replacing B' by a localization we may assume that B' is étale over A, see Lemma 11.2. Then we see that Algebra, Lemma 154.7 applies to $A \to B'$ and we conclude that B' is normal. Hence B is a normal domain. □

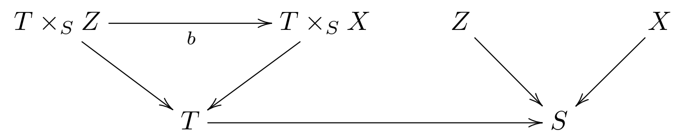
The preceeding propositions give some indication as to why we'd like to think of étale maps as "local isomorphisms". Another property that gives an excellent indication that we have the "right" definition is the fact that for **C**-schemes of finite type, a morphism is étale if and only if the associated morphism on analytic spaces (the **C**-valued points given the complex topology) is a local isomorphism in the analytic sense (open embedding locally on the source). This fact can be proven with the aid of the structure theorem and the fact that the analytification commutes with the formation of the completed local rings – the details are left to the reader.

20. Relative morphisms

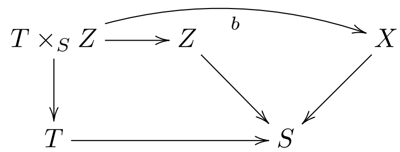
0BL0 We interrupt the discussion of étale morphisms to prove a representability result which we will use in the next section to discuss the category of finite étale coverings. The material in this section is discussed in the correct generality in Criteria for Representability, Section 10.

Let S be a scheme. Let Z and X be schemes over S. Given a scheme T over S we can consider morphisms $b : T \times_S Z \to T \times_S X$ over S. Picture

0BL1 (20.0.1)



Of course, we can also think of b as a morphism $b : T \times_S Z \to X$ such that



commutes. In this situation we can define a functor

0BL2 (20.0.2) $$Mor_S(Z, X) : (Sch/S)^{opp} \longrightarrow Sets, \quad T \longmapsto \{b \text{ as above}\}$$

Here is a basic representability result.

05Y6 **Lemma 20.1.** *Let $Z \to S$ and $X \to S$ be morphisms of affine schemes. Assume $\Gamma(Z, \mathcal{O}_Z)$ is a finite free $\Gamma(S, \mathcal{O}_S)$-module. Then $Mor_S(Z, X)$ is representable by an affine scheme over S.*

Proof. Write $S = \text{Spec}(R)$. Choose a basis $\{e_1, \ldots, e_m\}$ for $\Gamma(Z, \mathcal{O}_Z)$ over R. Choose a presentation

$$\Gamma(X, \mathcal{O}_X) = R[\{x_i\}_{i \in I}]/(\{f_k\}_{k \in K}).$$

We will denote $\overline{x}_i$ the image of x_i in this quotient. Write

$$P = R[\{a_{ij}\}_{i \in I, 1 \leq j \leq m}].$$

Consider the R-algebra map

$$\Psi : R[\{x_i\}_{i \in I}] \longrightarrow P \otimes_R \Gamma(Z, \mathcal{O}_Z), \quad x_i \longmapsto \sum\nolimits_j a_{ij} \otimes e_j.$$

Write $\Psi(f_k) = \sum c_{kj} \otimes e_j$ with $c_{kj} \in P$. Finally, denote $J \subset P$ the ideal generated by the elements c_{kj}, $k \in K$, $1 \leq j \leq m$. We claim that $W = \text{Spec}(P/J)$ represents the functor $Mor_S(Z, X)$.

First, note that by construction P/J is an R-algebra, hence a morphism $W \to S$. Second, by construction the map Ψ factors through $\Gamma(X, \mathcal{O}_X)$, hence we obtain an P/J-algebra homomorphism

$$P/J \otimes_R \Gamma(X, \mathcal{O}_X) \longrightarrow P/J \otimes_R \Gamma(Z, \mathcal{O}_Z)$$

which determines a morphism $b_{univ} : W \times_S Z \to W \times_S X$. By the Yoneda lemma b_{univ} determines a transformation of functors $W \to Mor_S(Z, X)$ which we claim is an isomorphism. To show that it is an isomorphism it suffices to show that it induces a bijection of sets $W(T) \to Mor_S(Z, X)(T)$ over any affine scheme T.

Suppose $T = \text{Spec}(R')$ is an affine scheme over S and $b \in Mor_S(Z, X)(T)$. The structure morphism $T \to S$ defines an R-algebra structure on R' and b defines an R'-algebra map

$$b^\sharp : R' \otimes_R \Gamma(X, \mathcal{O}_X) \longrightarrow R' \otimes_R \Gamma(Z, \mathcal{O}_Z).$$

In particular we can write $b^\sharp(1 \otimes \overline{x}_i) = \sum \alpha_{ij} \otimes e_j$ for some $\alpha_{ij} \in R'$. This corresponds to an R-algebra map $P \to R'$ determined by the rule $a_{ij} \mapsto \alpha_{ij}$.

This map factors through the quotient P/J by the construction of the ideal J to give a map $P/J \to R'$. This in turn corresponds to a morphism $T \to W$ such that b is the pullback of b_{univ}. Some details omitted. □

0BL3 **Lemma 20.2.** *Let $Z \to S$ and $X \to S$ be morphisms of schemes. If $Z \to S$ is finite locally free and $X \to S$ is affine, then $Mor_S(Z, X)$ is representable by a scheme affine over S.*

Proof. Choose an affine open covering $S = \bigcup U_i$ such that $\Gamma(Z \times_S U_i, \mathcal{O}_{Z \times_S U_i})$ is finite free over $\mathcal{O}_S(U_i)$. Let $F_i \subset Mor_S(Z, X)$ be the subfunctor which assigns to T/S the empty set if $T \to S$ does not factor through U_i and $Mor_S(Z, X)(T)$ otherwise. Then the collection of these subfunctors satisfy the conditions (2)(a), (2)(b), (2)(c) of Schemes, Lemma 15.4 which proves the lemma. Condition (2)(a) follows from Lemma 20.1 and the other two follow from straightforward arguments. □

The condition on the morphism $f : X \to S$ in the lemma below is very useful to prove statements like it. It holds if one of the following is true: X is quasi-affine, f is quasi-affine, f is quasi-projective, f is locally projective, there exists an ample invertible sheaf on X, there exists an f-ample invertible sheaf on X, or there exists an f-very ample invertible sheaf on X.

0BL4 **Lemma 20.3.** *Let $Z \to S$ and $X \to S$ be morphisms of schemes. Assume*

1. *$Z \to S$ is finite locally free, and*
2. *for all $(s, x_1, \ldots, x_d)$ where $s \in S$ and $x_1, \ldots, x_d \in X_s$ there exists an affine open $U \subset X$ with $x_1, \ldots, x_d \in U$.*

Then $Mor_S(Z, X)$ is representable by a scheme.

Proof. Consider the set I of pairs (U, V) where $U \subset X$ and $V \subset S$ are affine open and $U \to S$ factors through V. For $i \in I$ denote (U_i, V_i) the corresponding pair. Set $F_i = Mor_{V_i}(Z_{V_i}, U_i)$. It is immediate that F_i is a subfunctor of $Mor_S(Z, X)$. Then we claim that conditions (2)(a), (2)(b), (2)(c) of Schemes, Lemma 15.4 which proves the lemma.

Condition (2)(a) follows from Lemma 20.2.

To check condition (2)(b) consider T/S and $b \in Mor_S(Z, X)$. Thinking of b as a morphism $T \times_S Z \to X$ we find an open $b^{-1}(U_i) \subset T \times_S Z$. Clearly, $b \in F_i(T)$ if and only if $b^{-1}(U_i) = T \times_S Z$. Since the projection $p : T \times_S Z \to T$ is finite hence closed, the set $U_{i,b} \subset T$ of points $t \in T$ with $p^{-1}(\{t\}) \subset b^{-1}(U_i)$ is open. Then $f : T' \to T$ factors through $U_{i,b}$ if and only if $b \circ f \in F_i(T')$ and we are done checking (2)(b).

Finally, we check condition (2)(c) and this is where our condition on $X \to S$ is used. Namely, consider T/S and $b \in Mor_S(Z, X)$. It suffices to prove that every $t \in T$ is contained in one of the opens $U_{i,b}$ defined in the previous paragraph. This is equivalent to the condition that $b(p^{-1}(\{t\})) \subset U_i$ for some i where $p : T \times_S Z \to T$ is the projection and $b : T \times_S Z \to X$ is the given morphism. Since p is finite, the set $b(p^{-1}(\{t\})) \subset X$ is finite and contained in the fibre of $X \to S$ over the image s of t in S. Thus our condition on $X \to S$ exactly shows a suitable pair exists. □

0BL5 **Lemma 20.4.** *Let $Z \to S$ and $X \to S$ be morphisms of schemes. Assume $Z \to S$ is finite locally free and $X \to S$ is separated and locally quasi-finite. Then $Mor_S(Z, X)$ is representable by a scheme.*

Proof. This follows from Lemma 20.3 and More on Morphisms, Lemma 31.12. □

21. Schemes étale over a point

04JI In this section we describe schemes étale over the spectrum of a field. Before we state the result we introduce the category of G-sets for a topological group G.

04JJ **Definition 21.1.** Let G be a topological group. A *G-set*, sometime called a *discrete G-set*, is a set X endowed with a left action $a : G \times X \to X$ such that a is continuous when X is given the discrete topology and $G \times X$ the product topology. A *morphism of G-sets* $f : X \to Y$ is simply any G-equivariant map from X to Y. The category of G-sets is denoted *G-Sets*.

The condition that $a : G \times X \to X$ is continuous signifies simply that the stabilizer of any $x \in X$ is open in G. If G is an abstract group G (i.e., a group but not a topological group) then this agrees with our preceding definition (see for example Sites, Example 6.5) provided we endow G with the discrete topology.

Recall that if $K \subset L$ is an infinite Galois extension then the Galois group $G = \text{Gal}(L/K)$ comes endowed with a canonical topology, see Fields, Section 21.

03QR **Lemma 21.2.** *Let K be a field. Let K^{sep} a separable closure of K. Consider the profinite group $G = Gal(K^{sep}/K)$. The functor*

$$\begin{array}{ccc} \textit{schemes étale over } K & \longrightarrow & G\textit{-Sets} \\ X/K & \longmapsto & \text{Mor}_{\text{Spec}(K)}(\text{Spec}(K^{sep}), X) \end{array}$$

is an equivalence of categories.

Proof. A scheme X over K is étale over K if and only if $X \cong \coprod_{i \in I} \text{Spec}(K_i)$ with each K_i a finite separable extension of K (Morphisms, Lemma 36.7). The functor of the lemma associates to X the G-set

$$\coprod\nolimits_i \text{Hom}_K(K_i, K^{sep})$$

with its natural left G-action. Each element has an open stabilizer by definition of the topology on G. Conversely, any G-set S is a disjoint union of its orbits. Say $S = \coprod S_i$. Pick $s_i \in S_i$ and denote $G_i \subset G$ its open stabilizer. By Galois theory (Fields, Theorem 21.3) the fields $(K^{sep})^{G_i}$ are finite separable field extensions of K, and hence the scheme

$$\coprod\nolimits_i \text{Spec}((K^{sep})^{G_i})$$

is étale over K. This gives an inverse to the functor of the lemma. Some details omitted. □

03QS **Remark 21.3.** Under the correspondence of Lemma 21.2, the coverings in the small étale site $\text{Spec}(K)_{étale}$ of K correspond to surjective families of maps in *G-Sets*.

22. Galois categories

0BMQ In this section we discuss some of the material the reader can find in [Gro71, Exposé V, Sections 4, 5, and 6]. Recall that by our conventions categories have a set of objects and for any pair of objects a set of morphisms. The following lemma tells us that the group of automorphisms of a functor to the category of finite sets is automatically a profinite topological group.

0BMR **Lemma 22.1.** *Let $\mathcal{C}$ be a category and let $F : \mathcal{C} \to Sets$ be a functor. Then*

$$Aut(F) = \lim_{I \subset \mathrm{Ob}(\mathcal{C}) \text{ finite}} Im(Aut(F) \to \prod\nolimits_{X \in I} Aut(F(X)))$$

If $F(X)$ is finite for all X, then $Aut(F) \subset \prod_{X \in \mathrm{Ob}(\mathcal{C})} Aut(F(X))$ is a closed subgroup.

Proof. The lemma proves itself. $\square$

0BMS **Example 22.2.** Let G be a topological group. An important example will be the forgetful functor

0BMT (22.2.1)
$$\textit{Finite-G-Sets} \longrightarrow \textit{Sets}$$

where *Finite-G-Sets* is the full subcategory of *G-Sets* whose objects are the finite G-sets. The category *G-Sets* of G-sets is defined in Definition 21.1.

Let G be a topological group. The *profinite completion* of G will be the profinite group

$$G^\wedge = \lim_{U \subset G \text{ open, normal, finite idex}} G/U$$

with its profinite topology. Observe that the limit is cofiltered as a finite intersection of open, normal subgroups of finite index is another.

0BMU **Lemma 22.3.** *Let G be a topological group. The automorphism group of the functor (22.2.1) endowed with its profinite topology from Lemma 22.1 is the profinite completion of G.*

Proof. Denote F_G the functor (22.2.1). Any morphism $X \to Y$ in *Finite-G-Sets* commutes with the action of G. Thus any $g \in G$ defines an automorphism of F_G and we obtain a canonical homomorphism $G \to \mathrm{Aut}(F_G)$ of groups. Observe that any finite G-set X is a finite disjoint union of G-sets of the form G/H_i with canonical G-action where $H_i \subset G$ is an open subgroup of finite index. Then $U_i = \bigcap gH_ig^{-1}$ is open, normal, and has finite index. Moreover U_i acts trivially on G/H_i hence $U = \bigcap U_i$ acts trivially on $F(X)$. From Lemma 22.1 we conclude there is an induced continuous group homomorphism

$$G^\wedge \longrightarrow \mathrm{Aut}(F_G)$$

Moreover, since G/U acts faithfully on G/U this map is injective. If the image is dense, then the map is surjective and hence a homeomorphism by Topology, Lemma 16.8.

Let $\gamma \in \mathrm{Aut}(F_G)$ and let $X \in \mathrm{Ob}(\mathcal{C})$. We will show there is a $g \in G$ such that γ and g induce the same action on $F_G(X)$. This will finish the proof. As before we see that X is a finite disjoint union of G/H_i. With U_i and U as above, the finite G-set $Y = G/U$ surjects onto G/H_i for all i and hence it suffices to find $g \in G$ such that γ and g induce the same action on $F_G(G/U) = G/U$. Let $e \in G$ be the neutral element and say that $\gamma(eU) = g_0U$ for some $g_0 \in G$. For any $g_1 \in G$ the morphism

$$R_{g_1} : G/U \longrightarrow G/U, \quad gU \longmapsto gg_1U$$

of *Finite-G-Sets* commutes with the action of γ. Hence

$$\gamma(g_1U) = \gamma(R_{g_1}(eU)) = R_{g_1}(\gamma(eU)) = R_{g_1}(g_0U) = g_0g_1U$$

Thus we see that $g = g_0$ works. □

Recall that an exact functor is one which commutes with all finite limits and finite colimits. In particular such a functor commutes with equalizers, coequalizers, fibred products, pushouts, etc.

0BMV **Lemma 22.4.** *Let G be a topological group. Let F : Finite-G-Sets $\to$ Sets be an exact functor with $F(X)$ finite for all X. Then F is isomorphic to the functor (22.2.1).*

Proof. Let X be a nonempty object of *Finite-G-Sets*. The diagram

$$\begin{array}{ccc} X & \longrightarrow & \{*\} \\ \downarrow & & \downarrow \\ \{*\} & \longrightarrow & \{*\} \end{array}$$

is cocartesian. Hence we conclude that $F(X)$ is nonempty. Let $U \subset G$ be an open, normal subgroup with finite index. Observe that

$$G/U \times G/U = \coprod\nolimits_{gU \in G/U} G/U$$

where the summand corresponding to gU corresponds to the orbit of (eU, gU) on the left hand side. Then we see that

$$F(G/U) \times F(G/U) = F(G/U \times G/U) = \coprod\nolimits_{gU \in G/U} F(G/U)$$

Hence $|F(G/U)| = |G/U|$ as $F(G/U)$ is nonempty. Thus we see that

$$\lim\nolimits_{U \subset G \text{ open, normal, finite idex}} F(G/U)$$

is nonempty (Categories, Lemma 21.5). Pick $\gamma = (\gamma_U)$ an element in this limit. Denote F_G the functor (22.2.1). We can identify F_G with the functor

$$X \longmapsto \text{colim}_U \text{Mor}(G/U, X)$$

where $f : G/U \to X$ corresponds to $f(eU) \in X = F_G(X)$ (details omitted). Hence the element γ determines a well defined map

$$t : F_G \longrightarrow F$$

Namely, given $x \in X$ choose U and $f : G/U \to X$ sending eU to x and then set $t_X(x) = F(f)(\gamma_U)$. We will show that t induces a bijective map $t_{G/U} : F_G(G/U) \to F(G/U)$ for any U. This implies in a straightforward manner that t is an isomorphism (details omitted). Since $|F_G(G/U)| = |F(G/U)|$ it suffices to show that $t_{G/U}$ is surjective. The image contains at least one element, namely $t_{G/U}(eU) = F(\text{id}_{G/U})(\gamma_U) = \gamma_U$. For $g \in G$ denote $R_g : G/U \to G/U$ right multiplication. Then set of fixed points of $F(R_g) : F(G/U) \to F(G/U)$ is equal to $F(\emptyset) = \emptyset$ if $g \notin U$ because F commutes with equalizers. It follows that if

$g_1, \ldots, g_{|G/U|}$ is a system of representatives for G/U, then the elements $F(R_{g_i})(\gamma_U)$ are pairwise distinct and hence fill out $F(G/U)$. Then

$$t_{G/U}(g_iU) = F(R_{g_i})(\gamma_U)$$

and the proof is complete. □

0BMW **Example 22.5.** Let $\mathcal{C}$ be a category and let $F : \mathcal{C} \to \textit{Sets}$ be a functor such that $F(X)$ is finite for all $X \in \text{Ob}(\mathcal{C})$. By Lemma 22.1 we see that $G = \text{Aut}(F)$ comes endowed with the structure of a profinite topological group in a canonical manner. We obtain a functor

0BMX (22.5.1) $$\mathcal{C} \longrightarrow \textit{Finite-G-Sets}, \quad X \longmapsto F(X)$$

where $F(X)$ is endowed with the induced action of G. This action is continuous because the kernel of $G \to \text{Aut}(F(X))$ is open in G by construction.

The purpose of defining Galois categories is to single out those pairs $(\mathcal{C}, F)$ for which the functor (22.5.1) is an equivalence. Our definition of a Galois category is as follows.

0BMY **Definition 22.6.** Let $\mathcal{C}$ be a category and let $F : \mathcal{C} \to \textit{Sets}$ be a functor. The pair $(\mathcal{C}, F)$ is a *Galois category* if

Different from the definition in [Gro71, Exposé V, Definition 5.1]. Compare with [BS13, Definition 7.2.1].

(1) $\mathcal{C}$ has finite limits and finite colimits,
0BMZ (2) every object of $\mathcal{C}$ is a finite (possibly empty) coproduct of connected objects,
(3) $F(X)$ is finite for all $X \in \text{Ob}(\mathcal{C})$, and
(4) F reflects isomorphisms and is exact.

Here we say $X \in \text{Ob}(\mathcal{C})$ is connected if it is not initial and for any monomorphism $Y \to X$ either Y is initial or $Y \to X$ is an isomorphism.

Warning: This definition is not the same (although eventually we'll see it is equivalent) as the definition given in most references. Namely, in [Gro71, Exposé V, Definition 5.1] a Galois category is defined to be a category equivalent to *Finite-G-Sets* for some profinite group G. Then Grothendieck characterizes Galois categories by a list of axioms (G1) – (G6) which are weaker than our axioms above. The motivation for our choice is to stress the existence of finite limits and finite colimits and exactness of the functor F. The price we'll pay for this later is that we'll have to work a bit harder to apply the results of this section.

0BN0 **Lemma 22.7.** *Let $(\mathcal{C}, F)$ be a Galois category. Let $X \to Y \in \textit{Arrows}(\mathcal{C})$. Then*

(1) *F is faithful,*
(2) *$X \to Y$ is a monomorphism $\Leftrightarrow F(X) \to F(Y)$ is injective,*
(3) *$X \to Y$ is an epimorphism $\Leftrightarrow F(X) \to F(Y)$ is surjective,*
(4) *an object A of $\mathcal{C}$ is initial if and only if $F(A) = \emptyset$,*
(5) *an object Z of $\mathcal{C}$ is final if and only if $F(Z)$ is a singleton,*
(6) *if X and Y are connected, then $X \to Y$ is an epimorphism,*
0BN1 (7) *if X is connected and $a, b : X \to Y$ are two morphisms then $a = b$ as soon as $F(a)$ and $F(b)$ agree on one element of $F(X)$,*
(8) *if $X = \coprod_{i=1,\ldots,n} X_i$ and $Y = \coprod_{j=1,\ldots,m} Y_j$ where X_i, Y_j are connected, then there is map $\alpha : \{1, \ldots, n\} \to \{1, \ldots, m\}$ such that $X \to Y$ comes from a collection of morphisms $X_i \to Y_{\alpha(i)}$.*

Proof. Proof of (1). Suppose $a, b : X \to Y$ with $F(a) = F(b)$. Let E be the equalizer of a and b. Then $F(E) = F(X)$ and we see that $E = X$ because F reflects isomorphisms.

Proof of (2). This is true because F turns the morphism $X \to X \times_Y X$ into the map $F(X) \to F(X) \times_{F(Y)} F(X)$ and F reflects isomorphisms.

Proof of (3). This is true because F turns the morphism $Y \amalg_X Y \to Y$ into the map $F(Y) \amalg_{F(X)} F(Y) \to F(Y)$ and F reflects isomorphisms.

Proof of (4). There exists an initial object A and certainly $F(A) = \emptyset$. On the other hand, if X is an object with $F(X) = \emptyset$, then the unique map $A \to X$ induces a bijection $F(A) \to F(X)$ and hence $A \to X$ is an isomorphism.

Proof of (5). There exists a final object Z and certainly $F(Z)$ is a singleton. On the other hand, if X is an object with $F(X)$ a singleton, then the unique map $X \to Z$ induces a bijection $F(X) \to F(Z)$ and hence $X \to Z$ is an isomorphism.

Proof of (6). The equalizer E of the two maps $Y \to Y \amalg_X Y$ is not an initial object of $\mathcal{C}$ because $X \to Y$ factors through E and $F(X) \neq \emptyset$. Hence $E = Y$ and we conclude.

Proof of (7). The equalizer E of a and b comes with a monomorphism $E \to X$ and $F(E) \subset F(X)$ is the set of elements where $F(a)$ and $F(b)$ agree. To finish use that either E is initial or $E = X$.

Proof of (8). For each i, j we see that $E_{ij} = X_i \times_Y Y_j$ is either initial or equal to X_i. Picking $s \in F(X_i)$ we see that $E_{ij} = X_i$ if and only if s maps to an element of $F(Y_j) \subset F(Y)$, hence this happens for a unique $j = \alpha(i)$. □

By the lemma above we see that, given a connected object X of a Galois categoey $(\mathcal{C}, F)$, the automorphism group $\text{Aut}(X)$ has order at most $|F(X)|$. Namely, given $s \in F(X)$ and $g \in \text{Aut}(X)$ we see that $g(s) = s$ if and only if $g = \text{id}_X$ by (7). We say X is *Galois* if equality holds. Equivalently, X is Galois if it is connected and $\text{Aut}(X)$ acts transitively on $F(X)$.

0BN2 **Lemma 22.8.** *Let $(\mathcal{C}, F)$ be a Galois category. For any connected object X of $\mathcal{C}$ there exists a Galois object Y and a morphism $Y \to X$.*

Proof. We will use the results of Lemma 22.7 without further mention. Let $n = |F(X)|$. Consider X^n endowed with its natural action of S_n. Let

$$X^n = \coprod\nolimits_{t \in T} Z_t$$

be the decomposition into connected objects. Pick a t such that $F(Z_t)$ contains $(s_1, \ldots, s_n)$ with s_i pairwise distinct. If $(s'_1, \ldots, s'_n) \in F(Z_t)$ is another element, then we claim s'_i are pairwise distinct as well. Namely, if not, say $s'_i = s'_j$, then Z_t is the image of an connected component of X^{n-1} under the diagonal morphism

$$\Delta_{ij} : X^{n-1} \longrightarrow X^n$$

Since morphisms of connected objects are epimorphisms and induce surjections after applying F it would follow that $s_i = s_j$ which is not the case.

Let $G \subset S_n$ be the subgroup of elements with $g(Z_t) = Z_t$. Looking at the action of S_n on

$$F(X)^n = F(X^n) = \coprod\nolimits_{t' \in T} F(Z_{t'})$$

we see that $G = \{g \in S_n \mid g(s_1, \ldots, s_n) \in F(Z_t)\}$. Now pick a second element $(s'_1, \ldots, s'_n) \in F(Z_t)$. Above we have seen that s'_i are pairwise distinct. Thus we can find a $g \in S_n$ with $g(s_1, \ldots, s_n) = (s'_1, \ldots, s'_n)$. In other words, the action of G on $F(Z_t)$ is transitive and the proof is complete. □

Here is a key lemma.

0BN3 **Lemma 22.9.** *Let $(\mathcal{C}, F)$ be a Galois category. Let $G = Aut(F)$ be as in Example 22.5. For any connected X in $\mathcal{C}$ the action of G on $F(X)$ is transitive.*

Compare with [BS13, Definition 7.2.4].

Proof. We will use the results of Lemma 22.7 without further mention. Let I be the set of isomorphism classes of Galois objects in $\mathcal{C}$. For each $i \in I$ let X_i be a representative of the isomorphism class. Choose $\gamma_i \in F(X_i)$ for each $i \in I$. We define a partial ordering on I by setting $i \geq i'$ if and only if there is a morphism $f_{ii'} : X_i \to X_{i'}$. Given such a morphism we can post-compose by an automorphism $X_{i'} \to X_{i'}$ to assure that $F(f_{ii'})(\gamma_i) = \gamma_{i'}$. With this normalization the morphsm $f_{ii'}$ is unique.

We claim that the functor F is isomorphic to the functor F' which sends X to

$$F'(X) = \text{colim}_I \, \text{Mor}_\mathcal{C}(X_i, X)$$

via the transformation of functors $t : F' \to F$ defined as follows: given $f : X_i \to X$ we set $t_X(f) = F(f)(\gamma_i)$. Using (7) we find that t_X is injective. To show surjectivity, let $\gamma \in F(X)$. Then we can immediately reduce to the case where X is connected by the definition of a Galois category. Then we may assume X is Galois by Lemma 22.8. In this case X is isomorphic to X_i for some i and we can choose the isomorphism $X_i \to X$ such that γ_i maps to γ (by definition of Galois objects). We conclude that t is an isomorphism.

Set $A_i = \text{Aut}(X_i)$. We claim that for $i \geq i'$ there is a canonical map $h_{ii'} : A_i \to A_{i'}$ such that for all $a \in A_i$ the diagram

$$\begin{array}{ccc} X_i & \xrightarrow{f_{ii'}} & X_{i'} \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle h_{ii'}(a)} \\ X_i & \xrightarrow{f_{ii'}} & X_{i'} \end{array}$$

commutes. Namely, just let $h_{ii'}(a) = a' : X_{i'} \to X_{i'}$ be the unique automorphism such that $F(a')(\gamma_{i'}) = F(f_{ii'} \circ a)(\gamma_i)$. As before this makes the diagram commute and moreover the choice is unique. It follows that $h_{i'i''} \circ h_{ii'} = h_{ii''}$ if $i \geq i' \geq i''$. Since $F(X_i) \to F(X_{i'})$ is surjective we see that $A_i \to A_{i'}$ is surjective. Taking the inverse limit we obtain a group

$$A = \lim_I A_i$$

This is a profinite group since the automorphism groups are finite and moreover $A \to A_i$ is surjective for all i.

Since elements of A act on the inverse system X_i we get an action of A (on the right) on F' by pre-composing. In other words, we get a homomorphism $A^{opp} \to G$. Since $A \to A_i$ is surjective we conclude that G acts transitively on $F(X_i)$ for all i. Since every connected object is dominated by one of the X_i we conclude the lemma is true. □

0BN4 **Proposition 22.10.** *Let $(\mathcal{C}, F)$ be a Galois category. Let $G = Aut(F)$ be as in Example 22.5. The functor $F : \mathcal{C} \to$ Finite-G-Sets (22.5.1) an equivalence.*

This is a weak version of [Gro71, Exposé V]. The proof is borrowed from [BS13, Theorem 7.2.5].

Proof. We will use the results of Lemma 22.7 without further mention. In particular we know the functor is faithful. By Lemma 22.9 we know that for any connected X the action of G on $F(X)$ is transitive. Hence F preserves the decomposition into connected components (existence of which is an axioms of a Galois category). Let X and Y be objects and let $s : F(X) \to F(Y)$ be a map. Then the graph $\Gamma_s \subset F(X) \times F(Y)$ of s is a union of connected components. Hence there exists a union of connected components Z of $X \times Y$, which comes equipped with a monomorphism $Z \to X \times Y$, with $F(Z) = \Gamma_s$. Since $F(Z) \to F(X)$ is bijective we see that $Z \to X$ is an isomorphism and we conclude that $s = F(f)$ where $f : X \cong Z \to Y$ is the composition. Hence F is fully faithful.

To finish the proof we show that F is essentially surjective. It suffices to show that G/H is in the essential image for any open subgroup $H \subset G$ of finite index. By definition of the topology on G there exists a finite collection of objects X_i such that

$$\mathrm{Ker}(G \longrightarrow \prod\nolimits_i \mathrm{Aut}(F(X_i)))$$

is contained in H. We may assume X_i is connected for all i. We can choose a Galois object Y mapping to a connected component of $\prod X_i$ using Lemma 22.8. Then $U = \mathrm{Ker}(G \to \mathrm{Aut}(Y))$ is contained in H. In fact $F(Y) = G/U$ by our definition of Galois objects. Finally, we get an action of the finite group $M = H/U$ on Y and we set $X = Y/M$, i.e., X is the coequalizer of all the arrows $m : Y \to Y$, $m \in M$. Since F is exact we see that $F(X) = G/H$ and the proof is complete. □

0BN5 **Lemma 22.11.** *Let $(\mathcal{C}, F)$ and $(\mathcal{C}', F')$ be Galois categories. Let $H : \mathcal{C} \to \mathcal{C}'$ be an exact functor. There exists an isomorphism $t : F' \circ H \to F$. The choice of t determines a continuous homomorphism $h : G' = Aut(F') \to Aut(F) = G$ and a 2-commutative diagram*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{H} & \mathcal{C}' \\ \downarrow & & \downarrow \\ \textit{Finite-G-Sets} & \xrightarrow{h} & \textit{Finite-G}'\textit{-Sets} \end{array}$$

The map h is independent of t up to an inner automorphism of G. Conversely, given a continuous homomorphism $h : G' \to G$ there is an exact functor $H : \mathcal{C} \to \mathcal{C}'$ and an isomorphism t recovering h as above.

Proof. By Proposition 22.10 and Lemma 22.3 we may assume $\mathcal{C} =$ *Finite-G-Sets* and F is the forgetful functor and similarly for $\mathcal{C}'$. Thus the existence of t follows from Lemma 22.4. The map h comes from transport of structure via t. The commutativity of the diagram is obvious. Uniqueness of h up to innner conjugation by an element of G comes from the fact that the choice of t is unique up to an element of G. The final statement is straightforward. □

0BN6 **Lemma 22.12.** *Let $(\mathcal{C}, F)$ and $(\mathcal{C}', F')$ be Galois categories. Let $H : \mathcal{C} \to \mathcal{C}'$ be an exact functor. Let $h : G' = Aut(F') \to Aut(F) = G$ be the corresponding continuous homomorphism as in Lemma 22.11. The following are equivalent*

(1) *h is surjective, and*
(2) *H is fully faithful.*

Proof. Here we are just saying that given a continuous group homomorphism $h : G \to G'$ of profinite groups the corresponding functor *Finite-G-Sets* $\to$ *Finite-G'-Sets* is fully faithful if and only if h is surjective. This is clear because h is not surjective if and only if there exists a finite discrete G'-set M with a nontrivial action such that G acts trivially on M. □

0BN7 **Lemma 22.13.** *Let $(\mathcal{C}, F)$ and $(\mathcal{C}', F')$ be Galois categories. Let $H : \mathcal{C} \to \mathcal{C}'$ be an exact functor. Let $h : G' = Aut(F') \to Aut(F) = G$ be the corresponding continuous homomorphism as in Lemma 22.11. The following are equivalent*

(1) *h is injective, and*
(2) *for every connected object X' of $\mathcal{C}'$ there exists an object X of $\mathcal{C}$ and a diagram*
$$X' \leftarrow Y' \to H(X)$$
in $\mathcal{C}'$ where $Y' \to X'$ is an epimorphism and $Y' \to H(X)$ is a monomorphism.

Proof. Using the lemma we translate this into a question for the corresponding functor between the categories of finite G-sets and finite G'-sets.

Let $h : G' \to G$ be an injective continuous group homomorphism of profinite groups. Let $H' \subset G'$ be an open subgroup. Since the topology on G' is the induced topology from G there exists an open subgroup $H \subset G$ such that $h^{-1}H \subset H'$. Then the desired diagram is
$$G'/H' \leftarrow G'/h^{-1}H \to G/H$$

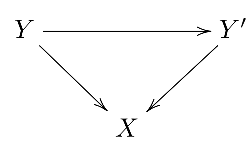
Conversely, assume (2) holds for the functor *Finite-G-Sets* $\to$ *Finite-G'-Sets*. Let $g' \in \text{Ker}(h)$. Pick any open subgroup $H' \subset G'$. By assumption there exists a finite G-set X and a diagram
$$G'/H' \leftarrow Y' \to X$$
of G'-sets with the left arrow surjective and the right arrow injective. Since g' is in the kernel of h we see that g' acts trivally on X. Hence g' acts trivially on Y' and hence trivially on G'/H'. Thus $g' \in H'$. As this holds for all open subgroups we conclude that g' is the identity element as desired. □

23. Finite étale morphisms

0BL6 In this section we prove enough basic results on finite étale morphisms to be able to construct the étale fundamental group.

Let X be a scheme. We will use the notation $F\acute{E}t_X$ to denote the category of scheme finite and étale over X. Thus

(1) an object of $F\acute{E}t_X$ is a finite étale morphism $Y \to X$ with target X, and
(2) a morphism in $F\acute{E}t_X$ from $Y \to X$ to $Y' \to X$ is a morphism $Y \to Y'$ making the diagram



commute.

We will often call an object of $F\acute{E}t_X$ a *finite étale cover* of X (even if Y is empty). It turns out that there is a stack $p : F\acute{E}t \to Sch$ over the category of schemes whose fibre over X is the category $F\acute{E}t_X$ just defined. See Examples of Stacks, Section 6.

0BN8 **Example 23.1.** Let k be an algebraically closed field and $X = \text{Spec}(k)$. In this case $F\acute{E}t_X$ is equivalent to the category of finite sets. This works more generally when k is separably algebraically closed. The reason is that a scheme étale over k is the disjoint union of spectra of fields finite separable over k, see Morphisms, Lemma 36.7.

0BN9 **Lemma 23.2.** *Let X be a scheme. The category $F\acute{E}t_X$ has finite limits and finite colimits and for any morphism $X' \to X$ the base change functor $F\acute{E}t_X \to F\acute{E}t_{X'}$ is exact.*

Proof. Finite limits and left exactness. By Categories, Lemma 18.4 it suffices to show that $F\acute{E}t_X$ has a final object and fibred products. This is clear because the category of all schemes over X has a final object (namely X) and fibred products and fibred products of schemes finite étale over X are finite étale over X. Moreover, it is clear that base change commutes with these operations and hence base change is left exact (Categories, Lemma 23.2).

Finite colimits and right exactness. By Categories, Lemma 18.7 it suffices to show that $F\acute{E}t_X$ has finite coproducts and coequalizers. Finite coproducts are given by disjoint unions (the empty coproduct is the empty scheme). Let $a, b : Z \to Y$ be two morphisms of $F\acute{E}t_X$. Since $Z \to X$ and $Y \to X$ are finite étale we can write $Z = \underline{\text{Spec}}(\mathcal{C})$ and $Y = \underline{\text{Spec}}(\mathcal{B})$ for some finite locally free $\mathcal{O}_X$-algebras $\mathcal{C}$ and $\mathcal{B}$. The morphisms a, b induce two maps $a^\sharp, b^\sharp : \mathcal{B} \to \mathcal{C}$. Let $\mathcal{A} = \text{Eq}(a^\sharp, b^\sharp)$ be their equalizer. If

$$\underline{\text{Spec}}(\mathcal{A}) \longrightarrow X$$

is finite étale, then it is clear that this is the coequalizer (after all we can write any object of $F\acute{E}t_X$ as the relative spectrum of a sheaf of $\mathcal{O}_X$-algebras). This we may do after replacing X by the members of an étale covering (Descent, Lemmas 19.21 and 19.5). Thus by Lemma 18.3 we may assume that $Y = \coprod_{i=1,\ldots,n} X$ and $Z = \coprod_{j=1,\ldots,m} X$. Then

$$\mathcal{C} = \prod\nolimits_{1 \leq j \leq m} \mathcal{O}_X \quad \text{and} \quad \mathcal{B} = \prod\nolimits_{1 \leq i \leq n} \mathcal{O}_X$$

After a further replacement by the members of an open covering we may assume that a, b correspond to maps $a_s, b_s : \{1, \ldots, m\} \to \{1, \ldots, n\}$, i.e., the summand X of Z corresponding to the index j maps into the summand X of Y corresponding to the index $a_s(j)$, resp. $b_s(j)$ under the morphism a, resp. b. Let $\{1, \ldots, n\} \to T$ be the coequalizer of a_s, b_s. Then we see that

$$\mathcal{A} = \prod\nolimits_{t \in T} \mathcal{O}_X$$

whose spectrum is certainly finite étale over X. We omit the verification that this is compatible with base change. Thus base change is a right exact functor. □

0BNA **Remark 23.3.** Let X be a scheme. Consider the natural functors $F_1 : F\acute{E}t_X \to Sch$ and $F_2 : F\acute{E}t_X \to Sch/X$. Then

(1) The functors F_1 and F_2 commute with finite colimits.

(2) The functor F_2 commutes with finite limits,
(3) The functor F_1 commutes with connected finite limits, i.e., with equalizers and fibre products.

The results on limits are immediate from the discussion in the proof of Lemma 23.2 and Categories, Lemma 16.2. It is clear that F_1 and F_2 commute with finite coproducts. By the dual of Categories, Lemma 23.2 we need to show that F_1 and F_2 commute with coequalizers. In the proof of Lemma 23.2 we saw that coequalizers in $F\acute{E}t_X$ look étale locally like this

$$\coprod_{j\in J} U \overset{a}{\underset{b}{\rightrightarrows}} \coprod_{i\in I} U \longrightarrow \coprod_{t\in \text{Coeq}(a,b)} U$$

which is certainly a coequalizer in the category of schemes. Hence the statement follows from the fact that being a coequalizer is fpqc local as formulate precisely in Descent, Lemma 9.4.

0BL7 **Lemma 23.4.** *Let X be a scheme. Given U, V finite étale over X there exists a scheme W finite étale over X such that*

$$\text{Mor}_X(X, W) = \text{Mor}_X(U, V)$$

and such that the same remains true after any base change.

Proof. By Lemma 20.4 there exists a scheme W representing $Mor_X(U, V)$. (Use that an étale morphism is locally quasi-finite by Morphisms, Lemmas 36.6 and that a finite morphism is separated.) This scheme clearly satisfies the formula after any base change. To finish the proof we have to show that $W \to X$ is finite étale. This we may do after replacing X by the members of an étale covering (Descent, Lemmas 19.21 and 19.5). Thus by Lemma 18.3 we may assume that $U = \coprod_{i=1,\ldots,n} X$ and $V = \coprod_{j=1,\ldots,m} X$. In this case $W = \coprod_{\alpha:\{1,\ldots,n\}\to\{1,\ldots,m\}} X$ by inspection (details omitted) and the proof is complete. □

Let X be a scheme. A *geometric point* of X is a morphism $\text{Spec}(k) \to X$ where k is algebraically closed. Such a point is usually denoted $\overline{x}$, i.e., by an overlined small case letter. We often use $\overline{x}$ to denote the scheme $\text{Spec}(k)$ as well as the morphism, and we use $\kappa(\overline{x})$ to denote k. We say $\overline{x}$ *lies over* x to indicate that $x \in X$ is the image of $\overline{x}$. We will discuss this further in Étale Cohomology, Section 29. Given $\overline{x}$ and an étale morphism $U \to X$ we can consider

$$|U_{\overline{x}}| : \text{the underlying set of points of the scheme } U_{\overline{x}} = U \times_X \overline{x}$$

Since $U_{\overline{x}}$ as a scheme over $\overline{x}$ is a disjoint union of copies of $\overline{x}$ (Morphisms, Lemma 36.7) we can also describe this set as

$$|U_{\overline{x}}| = \left\{ \begin{matrix}\text{commutative}\\ \text{diagrams}\end{matrix} \quad \begin{array}{ccc} \overline{x} & \xrightarrow{\overline{u}} & U \\ & \underset{\overline{x}}{\searrow} & \downarrow \\ & & X \end{array} \right\}$$

The assignement $U \mapsto |U_{\overline{x}}|$ is a functor which is often denoted $F_{\overline{x}}$.

0BNB **Lemma 23.5.** *Let X be a connected scheme. Let $\overline{x}$ be a geometric point. The functor*

$$F_{\overline{x}} : F\acute{E}t_X \longrightarrow Sets, \quad Y \longmapsto |Y_{\overline{x}}|$$

defines a Galois category (Definition 22.6).

Proof. After identifying $F\acute{E}t_{\overline{x}}$ with the category of finite sets (Example 23.1) we see that our functor $F_{\overline{x}}$ is nothing but the base change functor for the morphism $\overline{x} \to X$. Thus we see that $F\acute{E}t_X$ has finite limits and finite colimits and that $F_{\overline{x}}$ is exact by Lemma 23.2. We will also use that finite limits in $F\acute{E}t_X$ agree with the corresponding finite limits in the category of schemes over X, see Remark 23.3.

If $Y' \to Y$ is a monomorphism in $F\acute{E}t_X$ then we see that $Y' \to Y' \times_Y Y'$ is an isomorphism, and hence $Y' \to Y$ is a monomorphism of schemes. It follows that $Y' \to Y$ is an open immersion (Theorem 14.1). Since Y' is finite over X and Y separated over X, the morphism $Y' \to Y$ is finite (Morphisms, Lemma 43.12), hence closed (Morphisms, Lemma 43.10), hence it is the inclusion of an open and closed subscheme of Y. It follows that Y is a connected objects of the category $F\acute{E}t_X$ (as in Definition 22.6) if and only if Y is connected as a scheme. Then it follows from Topology, Lemma 6.6 that Y is a finite coproduct of its connected components both as a scheme and in the sense of Definition 22.6.

Let $Y \to Z$ be a morphism in $F\acute{E}t_X$ which induces a bijection $F_{\overline{x}}(Y) \to F_{\overline{x}}(Z)$. We have to show that $Y \to Z$ is an isomorphism. By the above we may assume Z is connected. Since $Y \to Z$ is finite étale and hence finite locally free it suffices to show that $Y \to Z$ is finite locally free of degree 1. This is true in a neighbourhood of any point of Z lying over $\overline{x}$ and since Z is connected and the degree is locally constant we conclude. □

Next we define Grothendieck's algebraic fundamental group.

0BNC **Definition 23.6.** Let X be a connected scheme. Let $\overline{x}$ be a geometric point of X. The *fundamental group* of X with *base point* $\overline{x}$ is the group

$$\pi_1(X, \overline{x}) = \text{Aut}(F_{\overline{x}})$$

of automorphisms of the fibre functor $F_{\overline{x}} : F\acute{E}t_X \to \textit{Sets}$ endowed with its canonical profinite topology from Lemma 22.1.

Combining the above with the material from Section 22 we obtain the following theorem.

0BND **Theorem 23.7.** *Let X be a connected scheme. Let $\overline{x}$ be a geometric point of X.*

(1) *The fibre functor $F_{\overline{x}}$ defines an equivalence of categories*

$$F\acute{E}t_X \longrightarrow \textit{Finite-}\pi_1(X, \overline{x})\textit{-Sets}$$

(2) *Given a second geometric point $\overline{x}'$ of X there exists an isomorphism $t : F_{\overline{x}} \to F_{\overline{x}'}$. This gives an isomorphism $\pi_1(X, \overline{x}) \to \pi_1(X, \overline{x}')$ compatible with the equivalences in (1). This isomorphism is independent of t up to innner conjugation.*

(3) *Given a morphism $f : X \to Y$ of connected schemes denote $\overline{y} = f \circ \overline{x}$. There is a canonical continuous homomorphism*

$$f_* : \pi_1(X, \overline{x}) \to \pi_1(Y, \overline{y})$$

such that the diagram

$$\begin{array}{ccc} F\acute{E}t_Y & \xrightarrow[\text{base change}]{} & F\acute{E}t_X \\ \downarrow{\scriptstyle F_{\overline{y}}} & & \downarrow{\scriptstyle F_{\overline{x}}} \\ \textit{Finite-}\pi_1(Y, \overline{y})\textit{-Sets} & \xrightarrow{f_*} & \textit{Finite-}\pi_1(X, \overline{x})\textit{-Sets} \end{array}$$

is commutative.

Proof. Part (1) follows from Lemma 23.5 and Proposition 22.10. Part (2) is a special case of Lemma 22.11. For part (3) observe that the diagram

$$\begin{array}{ccc} F\acute{E}t_Y & \longrightarrow & F\acute{E}t_X \\ {\scriptstyle F_{\overline{y}}}\downarrow & & \downarrow{\scriptstyle F_{\overline{x}}} \\ Sets & = & Sets \end{array}$$

is commutative (actually commutative, not just 2-commutative) because $\overline{y} = f \circ \overline{x}$. Hence we can apply Lemma 22.11 with the implied transformation of functors to get (3). □

0BNE **Lemma 23.8.** *Let k be a field and let $\overline{k}$ be an algebraic closure. Set $X = \mathrm{Spec}(k)$ and denote $\overline{x} : \mathrm{Spec}(\overline{k}) \to X$ be the geometric point corresponding to our chose algebraic closure. Let $k \subset k^{sep} \subset \overline{k}$ be the separable algebraic closure. There is a canonical isomorphism*

$$Gal(k^{sep}/k) \longrightarrow \pi_1(X, \overline{x})$$

of profinite topological groups.

Proof. We first carefully construct the map. Observe that $\mathrm{Gal}(k^{sep}/k) = \mathrm{Aut}(\overline{k}/k)$ as $\overline{k}$ is the perfection of k^{sep}. Then recall that $\pi_1(X, \overline{x}) = \mathrm{Aut}(F_{\overline{x}})$ where $F_{\overline{x}}$ is the functor

$$Y \longmapsto F_{\overline{x}}(Y) = \mathrm{Mor}_X(\mathrm{Spec}(\overline{k}), Y)$$

Consider the map

$$\mathrm{Aut}(\overline{k}/k) \times F_{\overline{x}}(Y) \to F_{\overline{x}}(Y), \quad (\sigma, \overline{y}) \mapsto \sigma \cdot \overline{y} = \overline{y} \circ \mathrm{Spec}(\sigma)$$

This is an action because

$$\sigma\tau \cdot \overline{y} = \overline{y} \circ \mathrm{Spec}(\sigma\tau) = \overline{y} \circ \mathrm{Spec}(\tau) \circ \mathrm{Spec}(\sigma) = \sigma \cdot (\tau \cdot \overline{y})$$

The action is functorial in $Y \in F\acute{E}t_X$ and we obtain the canonical map.

Using our map above for every object Y in $F\acute{E}t_X$ the finite set $F_{\overline{x}}(Y)$ gets a canonical $\mathrm{Gal}(k^{sep}/k)$-action. To finish the proof it suffices to show that each $F_{\overline{x}}(Y)$ is an object of *Finite*-$\mathrm{Gal}(k^{sep}/k)$-*Sets* and that in this way we obtain an equivalence of categories $F\acute{E}t_X \to$ *Finite*-$\mathrm{Gal}(k^{sep}/k)$-*Sets*. This is sufficient by the recognition results in Proposition 22.10 and Lemma 22.3. To see this one shows that the construction given here is the same as the construction in the equivalence Lemma 21.2 and that the equivalence of that lemma induces an equivalence between the category of finite étale schemes over $\mathrm{Spec}(K)$ and finite G-sets. We omit the details. □

24. Other chapters

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