In this talk, I will discuss some recent results on the spectrum of the Laplacian and drifted Laplacian on complete noncompact manifolds.

The Lichnerowicz-Obata theorem states that if the Ricci curvature of a complete \( n \)-dimensional Riemannian manifold \( M \) is bounded below by a positive constant \( a \), then the first nonzero eigenvalue of the Laplacian \( \Delta \) on \( M \) satisfies \( \lambda_1 \geq \frac{na}{n-1} \). Moreover the equality holds if and only if the manifold is a round sphere of radius \( \sqrt{\frac{n-1}{a}} \). When \((M^n, g, e^{-f}dv)\) is a complete smooth metric measure space with the Bakry-Emery Ricci curvature tensor \( \text{ric}_f \geq ag \), constant \( a > 0 \), \( M \) may be non-compact. It is known that the spectrum of the drifted Laplacian \( \Delta_f = \Delta - \langle \nabla f, \nabla \cdot \rangle \) on such \( M \) is discrete and the first nonzero eigenvalue of \( \Delta_f \) has lower bound \( a \). We will discuss the rigidity of this lower bound and prove that if it is achieved with multiplicity \( k \), then \( M \) is isometric to \( \Sigma^{n-k} \times \mathbb{R}^k \) for some complete \((n-k)\)-dimensional manifold \( \Sigma \). One special example is gradient shrinking Ricci solitons. We also discuss the case of self-shrinkers for mean curvature flows. This is a joint work with Xu Cheng.