We investigate quantitative properties of nonnegative solutions $u(t, x) \geq 0$ to the nonlinear fractional diffusion equation, $\partial_t u + L(u^m) = 0$, posed in a bounded domain, $x \in \Omega \subset \mathbb{R}^N$ for $t > 0$ and $m > 1$. As $L$ we can take the most common definitions of the fractional Laplacian $(-\Delta)^s$, $0 < s < 1$, in a bounded domain with zero Dirichlet boundary conditions, as well as more general classes of operators. We consider a class of very weak solutions for the equation at hand, that we call weak dual solutions, and we obtain a priori estimates in the form of smoothing effects, absolute upper bounds, lower bounds, and Harnack inequalities. We also investigate the boundary behaviour and we obtain sharp estimates from above and below. The standard Laplacian case $s = 1$ or the linear case $m = 1$ are recovered as limits. The method is quite general, suitable to be applied to a number of similar problems that will be briefly discussed as examples. As a consequence, we can prove existence and uniqueness of minimal weak dual solutions with data in $L^1_{\Phi_1}$, where $\Phi_1$ is the first eigenfunction of $L$. We also briefly show existence and uniqueness of $H^{-s}$ solutions with a different approach. As a byproduct, we derive similar estimates for the elliptic semilinear equation $L S^m = S$ and we prove existence and uniqueness of $H^{-s}(\Omega)$ solutions via parabolic techniques. Solutions to this elliptic problem represents the asymptotic profiles of the rescaled solutions, namely the stationary states of the rescaled equation $\partial_t v = -L(v^m) + v$. Finally, we will study the asymptotic behaviour. We will prove sharp rates of decay of the rescaled solution to the unique stationary profile $S$ and also for the relative error $v/S - 1$. The sharp rates of convergence can be obtained with two different methods: one is based on the above estimates, that guarantee existence of the "friendly giant". Another approach is given by a new entropy method, based on the so-called Caffarelli-Silvestre extension. This is a joint work with J. L. Vazquez (UAM, Madrid, Spain) and Y. Sire (Univ. Marseille, France).

References:
