Given a compact Lie group and a principal bundle over a closed Riemannian manifold, the quotient space of connections, modulo the action of the group of gauge transformations, has fundamental significance for algebraic geometry, low-dimensional topology, the classification of smooth four-dimensional manifolds, and high-energy physics.

The quotient space of connections is equipped with the Yang-Mills energy functional and Atiyah and Bott (1983) had proposed that its gradient flow with respect to the natural Riemannian metric on the quotient space should prove to be an important tool for understanding the topology of the quotient space via an infinite-dimensional Morse theory. The critical points of the energy functional are gauge-equivalence classes of Yang-Mills connections. However, thus far, smooth solutions to the Yang-Mills gradient flow have only been known to exist for all time and converge to critical points, as time tends to infinity, in relatively few cases, including (1) when the base manifold has dimension two or three (Rade, 1991 and 1992, in dimension two and three; G. Daskalopoulos, 1989 and 1992, in dimension two), (2) when the base manifold is a complex algebraic surface (Donaldson, 1985), and (3) in the presence of rotational symmetry in dimension four (Schlatter, Tahvildar-Zadeh, and Struwe, 1998). Global existence of solutions with up to finitely many possible singularities (caused by the “bubbling” phenomenon) was proved independently by Struwe (1994) and Kozono, Maeda, and Naito (1995). However, the question of global existence of smooth solutions over general compact, Riemannian, four-dimensional base manifolds has remained unresolved.

In this talk we shall describe our proof of the following result: Given a compact Lie group and a smooth initial connection on a principal bundle over a compact, Riemannian, four-dimensional manifold, there is a unique, smooth solution to the Yang-Mills gradient flow which exists for all time and converges to a smooth Yang-Mills connection on the given principal bundle as time tends to infinity.