1. Introduction

The goal of this seminar is to understand the extra structure imposed on the de Rham cohomology groups of a smooth manifold $M$ by the data of a Kähler structure on $M$, how that data varies in families, and its interaction with the structure of the complex submanifolds of $M$.

1.1. Smooth Manifolds. Let $M$ be a smooth manifold and $T_M$ its tangent bundle. Let $\Lambda^k(M)$ be the sheaf of smooth sections to $\bigwedge^k(T^*_M)$, with $\Lambda^0(M) := C^\infty(M)$. Then there is a chain complex of sheaves $\Lambda^\bullet(M) : 0 \to C^\infty(M) \xrightarrow{d} \Lambda^1(M) \to \Lambda^2(M) \to \cdots$ exact away from degree zero (by the Poincare lemma—NB: remark on this), where the kernel of the map $d : C^\infty(M) \to \Lambda^1(M)$ is the constant sheaf $\mathbb{R}$. We define the de Rham cohomology of $M$ to be $H^k_{dR}(M) := H^k(\Gamma(M, \Lambda^\bullet(M)))$.

The sheaves in the complex $\Lambda^\bullet(M)$ are manifestly fine (e.g. they admit partitions of unity), and so the de Rham cohomology of $M$ is the derived functor cohomology of the constant sheaf $\mathbb{R}$.

Now if $A$ is any Abelian group and $\underline{A}$ is the constant $A$-valued sheaf on $M$, we define a resolution of $\underline{A}$ as follows: for any open set $V \subset M$ and any cover $\mathcal{U}$ of $V$ by open sets, we define the group of $\mathcal{U}$-small singular chains $C^i_\mathcal{U}(V)$ to be the free abelian group on singular chains $\iota : \Delta^i \to U$ whose image is wholly contained in some $U \in \mathcal{U}$; we let $C^i_\mathcal{U}(V, A) = \text{Hom}_{\text{Ab}}(C^i_\mathcal{U}(V), A)$. Finally, we define a presheaf $C^i(M, A)$ on $M$ by $\Gamma(V, C^i(M, A)) = \lim_{\mathcal{U}} C^i_\mathcal{U}(V, A)$.

These presheaves are sheaves (barycentric subdivision), are soft (check this) and have global sections weakly-equivalent to the complex of $i$-th singular co-chains of $M$ (explain this). They fit together into an exact sequence $C^\bullet(M, A) : 0 \to C^0(M, A) \xrightarrow{d} C^1(M, A) \to \cdots$ via the usual singular co-chain maps, which is exact away from degree zero as $M$ is locally contractible. The kernel of the map $d : C^0(M, A) \to C^1(M, A)$ is the constant sheaf $\underline{A}$. Thus the derived functor cohomology of $\underline{A}$ is the singular cohomology of $M$ with coefficients in $A$.

Putting these two results together we have

**Theorem 1** (de Rham’s Theorem). The de Rham cohomology of a manifold $M$ is naturally isomorphic to its singular cohomology with coefficients in $\mathbb{R}$ and is thus a topological invariant.

In general, given a flat vector bundle $E$ on $M$, and letting $\underline{E}$ be the sheaf of flat sections to $E$, an identical argument gives that the complex $\underline{E} \otimes \Lambda^\bullet(M)$ computes the derived functor cohomology of $\underline{E}$. Furthermore, this is naturally isomorphic to the singular cohomology of $M$ with coefficients in $E$, viewed as a local system.
1.2. Complex Manifolds. Now, let us consider the case where $M$ has a complex structure. In this case, there is a natural endomorphism $I : T_M \to T_M$ satisfying $I^2 = -1$ (that is, an almost-complex structure) given by viewing $T_M$ as a complex vector bundle and multiplying by $i$. Upon complexification, one finds that

$$T_M \otimes \mathbb{C} = T_M^{1,0} \oplus T_M^{0,1}$$

where $T^{1,0}$ is the eigenspace of $I$ with eigenvalue $i$, and $T^{0,1}$ is its conjugate (that is, the eigenspace for $-i$). As a complex sub-bundle of $T_M \otimes \mathbb{C}$, $T_M^{1,0}$ is isomorphic to $T^{1,0}$. Upon complexification, one finds that $T_M \otimes \mathbb{R} C = T^{1,0}_{M,1} \oplus T_{0,1}^M$ where $T^{1,0}_{1,0}$ is the eigenspace of $I$ with eigenvalue $i$, and $T^{0,1}_{0,1}$ is its conjugate (that is, the eigenspace for $-i$).

As a complex sub-bundle of $T_M \otimes \mathbb{R} C$, $T^{1,0}_{1,0}$ is isomorphic to $T^{1,0}_M$.

Now note that $\Lambda^k(M) \otimes \mathbb{C}$ (which by our remarks above, computes the derived-functor cohomology of $\mathbb{C}$ splits naturally as

$$\Lambda^k(M) \otimes \mathbb{C} = \bigoplus_{p+q=k} \Gamma \left( \bigwedge^p (T^1_M)^* \otimes \bigwedge^q (T^{0,1}_M)^* \right).$$

Thus we set $A^{p,q}(M)$ to be the sheaf of smooth sections to

$$\bigwedge^p (T^1_M)^* \otimes \bigwedge^q (T^{0,1}_M)^*.$$ 

Note that $d(A^{p,q}(M)) \subset A^{p+1,q}(M) \oplus A^{p,q+1}(M)$ so we may write $d = \partial + \overline{\partial}$ where

$$\partial : A^{p,q} \to A^{p+1,q}, \overline{\partial} : A^{p,q} \to A^{p,q+1}.$$ 

Furthermore, $d \circ d = 0$ implies $\partial^2 = 0, \overline{\partial}^2 = 0, \partial \overline{\partial} + \overline{\partial} \partial = 0$, so the $A^{p,q}$ fit into a double complex $A^{\bullet,\bullet}$:

\[
\begin{array}{ccccccc}
\cdots & & A^{q-1,p+1} & \xrightarrow{\partial} & A^{q,p+1} & \xrightarrow{\overline{\partial}} & A^{q+1,p+1} & \cdots \\
& & \uparrow \partial & & \uparrow \overline{\partial} & & \uparrow \partial & \\
\cdots & & A^{q-1,p} & \xrightarrow{\partial} & A^{q,p} & \xrightarrow{\overline{\partial}} & A^{q+1,p} & \cdots \\
& & \uparrow \partial & & \uparrow \overline{\partial} & & \uparrow \partial & \\
\cdots & & A^{q-1,p-1} & \xrightarrow{\partial} & A^{q,p-1} & \xrightarrow{\overline{\partial}} & A^{q+1,p-1} & \cdots \\
& & \uparrow \partial & & \uparrow \overline{\partial} & & \uparrow \partial & \\
& & \vdots & & \vdots & & \vdots & \\
\end{array}
\]

Note that the total complex of this double complex satisfies

$$\text{Tot}(A^{\bullet,\bullet}) \simeq \Lambda^k(M) \otimes \mathbb{R} \mathbb{C}.$$ 

The Dolbeault Lemma (or $\overline{\partial}$-lemma) states that the rows of this double complex, i.e. the complexes $A^{\bullet,p}$ are exact complexes of sheaves away from degree 0. The kernel of the map

$$\overline{\partial} : A^{p,0} \to A^{p,1}$$

is the sheaf of holomorphic $p$-forms

$$\Omega^p(M) : \Gamma(U, \Omega^p(M)) = \{ \text{holomorphic sections to } \bigwedge^p (T_M^*) \}.$$ 

We define the Dolbeault cohomology

$$H^{p,q}(M) = H^q(\Gamma(M, A^{p,\bullet})).$$ 

By the Dolbeault Lemma and the fact that the $A^{p,q}$ are manifestly fine sheaves, this agrees with the derived functor cohomology $H^q(M, \Omega^p(M))$. Furthermore, if $E$ is any holomorphic vector bundle on $M$ and $E$ its
sheaf of holomorphic sections, we have that $E \otimes A^{0,\bullet}$ is a fine resolution of $E$, and so computes its sheaf cohomology.

In general, the cohomology

$$H^{p,q}(E) := H^q(A^{p,\bullet} \otimes E)$$

is referred to as the Dolbeault cohomology of $E$; we will refer to the differentials in the complex $A^{p,\bullet} \otimes E$ as $\overline{\partial}_E$.

1.3. Kähler Manifolds. Let $V$ be a complex vector space; let $W_R$ be its real dual $W_R \cong \text{Hom}_{\mathbb{R}}(V, \mathbb{R})$ and $W_C$ the complexification $\text{Hom}_{\mathbb{R}}(V, \mathbb{C}) \cong W_R \otimes_{\mathbb{R}} \mathbb{C}$. Then as before $W_R$ has a natural endomorphism $I$ satisfying $I^2 = -1$, induced by the complex structure on $V$, and so

$$W_C \cong W^{1,0} \otimes W^{0,1}$$

where $W^{1,0}$ is the eigenspace for $I \otimes \mathbb{C}$ with eigenvalue $i$, and $W^{0,1}$ is its complex conjugate. Let

$$W^{1,1} := W^{1,0} \otimes W^{0,1} \subset \bigwedge^2(W_C).$$

We have the following important lemma from linear algebra:

**Proposition 1.** Hermitian forms on $V$ are naturally in bijection with elements of

$$W^{1,1}_R := W^{1,1} \cap \bigwedge^2(W_R).$$

The map is defined as follows: given a hermitian form $h$, we send

$$h \mapsto -\text{Im}(h).$$

Likewise, Hermitian metrics on a complex manifold are in bijection with elements of

$$A^{1,1}(M) \cap \Lambda^2(M).$$

We say a Hermitian metric $h$ is Kähler if the associated 2-form $-\text{Im}(h)$ is closed; a Kähler manifold is a complex manifold equipped with a Kähler metric, or equivalently a closed real 2-form of type $(1,1)$ as above. Some examples:

- $\mathbb{C}^n$ with the standard metric is Kähler.
- $\mathbb{C}^n/L$ with $L$ a lattice is Kähler, where the metric is induced from the standard metric on $\mathbb{C}^n$.
- $\mathbb{CP}^n$ admits a Kähler metric; one can see the associated 2-form as e.g. Poincaré dual to a representative of a top dimensional (integral) homology class of a hyperplane; alternately, one may view the form as a representative of the Chern class of $\mathcal{O}(1)$.
- Any complex submanifold of a Kähler manifold inherits a Kähler metric; in particular, smooth projective varieties admit a Kähler structure.

2. Main Results

Now that we’ve introduced some of the major players, we’re ready to go over the main results for the next few weeks.

2.1. Harmonic Forms and Cohomology. For the rest of the lecture, we’ll assume $M$ is compact and orientable. If $M$ is smooth and $g$ is a Riemannian metric on $M$, $g$ will induce an $L^2$-inner product $g_k$ on $\bigwedge^k(T^*_M)$ and thus on $\Gamma(M, \Lambda^k(M))$, via

$$\langle \gamma, \eta \rangle := \int_M g_k(\gamma, \eta) \text{Vol}$$

where $\text{Vol}$ is a volume form; analogously, an inner (resp. Hermitian) product on a flat vector bundle $E$ (resp. holomorphic vector bundle $F$) will induce an $L^2$-inner product on $\Lambda^k \otimes E$ (resp. $A^{p,\bullet} \otimes F$).

If the $\Lambda^k(M)$ had finite-dimensional spaces of global sections, this inner product would allow us to pick out canonical representatives for $H^k_{dR}(M)$, simply by taking the orthogonal complement of $\text{im}(d)$ in $\ker(d)$. Unfortunately, this is not the case, but we can still use the finite-dimensional setting for inspiration:
Proposition 2. Let $V$ be a finite dimensional inner product space and $\Delta : V \to V$ a self-adjoint operator. Then there is a natural (orthogonal) decomposition

$$V \simeq \text{Ker}(\Delta) \oplus \text{Im}(\Delta).$$

Proof. By dimension-counting, it suffices to show that $\text{Ker}(\Delta) \perp \text{Im}(\Delta)$. Consider $v \in \text{Im}(\Delta), v = \Delta w$. Then we have for $u \in \text{Ker}(\Delta)$ that

$$(v, u) = (\Delta w, u) = (w, \Delta u) = 0$$

completing the proof. $\square$

Now, consider the following model situation:

Proposition 3. Let

$$U \xrightarrow{d'} V \xrightarrow{d} W$$

be a sequence of finite dimensional inner product spaces such that $d^2 = 0$ and $d$ is formally adjoint to $d^*$ in the sense that

$$(dv, w) = (v, d^* w)$$

for $v \in U, w \in V$ or $v \in V, w \in W$. Let $\Delta : V \to V$ equal $dd^* + d^* d$ and let $H = \text{Ker}(\Delta)$. Then

$$H \simeq \text{Ker}(d) \cap \text{Ker}(d^*)$$

and $V$ has a natural orthogonal decomposition as

$$V \simeq H \oplus \text{Im}(\Delta)$$

with orthogonal decompositions

$$\text{Ker}(d) \simeq H \oplus \text{Im}(d)$$

and

$$\text{Ker}(d^*) \simeq H \oplus \text{Im}(d^*).$$

Thus, there are natural isomorphisms

$$H \simeq \text{Ker}(d)/\text{Im}(d) \simeq \text{Ker}(d^*)/\text{Im}(d^*).$$

Proof. We first show that $H \simeq \text{Ker}(d) \cap \text{Ker}(d^*)$. Clearly $H \subset \text{Ker}(d) \cap \text{Ker}(d^*)$. For the other inclusion, consider $v \in H$, that is, $\Delta v = 0$. Then we have

$$0 = (\Delta v, v) = (dd^* v, v) + (d^* dv, v) = (d^* v, d^* v) + (dv, dv)$$

which, as desired, implies that $d^* v = dv = 0$ by the non-degeneracy of the inner product.

Now $\Delta$ is clearly self-adjoint, so by the Lemma, there is an orthogonal decomposition

$$V \simeq H \oplus \text{Im}(\Delta).$$

So to show that $V \simeq H \oplus \text{Im}(d) \oplus \text{Im}(d^*)$ it suffices to show that

$$\text{Im}(\Delta) \simeq \text{Im}(d) \oplus \text{Im}(d^*).$$

Indeed, it is clear that $\text{Im}(\Delta) \subset \text{Im}(d) + \text{Im}(d^*)$; furthermore,

$$(d\alpha, d^*\beta) = (d^2 \alpha, \beta) = 0$$

so $\text{Im}(d) \perp \text{Im}(d^*)$. So we need only show that $\text{Im}(d) \oplus \text{Im}(d^*) \subset \text{Im}(\Delta)$. But $\text{Im}(d) \oplus \text{Im}(d^*) \perp H$ as for $v \in H$ we have

$$(d\alpha + d^*\beta, v) = (\alpha, d^* v) + (\beta, dv) = 0$$

so we must have that $\text{Im}(d) \oplus \text{Im}(d^*) \subset \text{Im}(\Delta)$, as $\text{Im}(\Delta)$ is the orthogonal complement of $H$ by the Lemma.

The rest of the proof follows by noting that $\text{Im}(d^*) \perp \text{Ker}(d)$ as for $v \in \text{Ker}(d)$ we have

$$(v, d^* \alpha) = (dv, \alpha) = 0$$

and that $H \oplus \text{Im}(d) \subset \text{Ker}(d)$ trivially, so we have

$$H \oplus \text{Im}(d) \subset \text{Ker}(d) \subset \text{Im}(d^*)^\perp = H \oplus \text{Im}(d)$$

and thus $H \oplus \text{Im}(d) \simeq \text{Ker}(d)$. The statement $\text{Ker}(d^*) \simeq H \oplus \text{Im}(d^*)$ follows identically. $\square$
In the de Rham and Dolbeault settings, and for the cohomology of flat or (resp., Dolbeault cohomology
of holomorphic) vector bundles, we will be able to construct formal adjoints \( d^*, \partial^*, \partial^*E \) to \( d, \partial, \partial^E, \partial \), with
the Laplacians
\[
\Delta_d := d^*d + dd^*, \quad \Delta_\partial := \partial^*\partial + \partial\partial^*E, \quad \Delta_\partial^E := \partial^*E\partial + \partial E\partial^*, \quad \Delta_\partial = \partial^*\partial + \partial\partial^*
\]
elliptic.

We say a form is harmonic if it is in the kernel of one of these operators; in general, which operator we
are referring to will be clear from context.

The ellipticity of these Laplacians will imply

Theorem 2. Let \( M \) be a compact orientable Riemannian manifold, with metric \( g \). Let \( E \) be a flat vector
bundle on \( M \) with a fixed metric, with sheaf of flat sections \( E \). Then if \( \mathcal{H}^k(E) \) denotes the harmonic \( E \)-valued
differential forms on \( M \), the natural map
\[
\mathcal{H}^k(E) \to H^k(M, E)
\]
is an isomorphism, where the latter is the derived functor cohomology of \( E \).

Theorem 3. Let \( M \) be a compact complex manifold with Hermitian metric \( g \). Let \( E \) be a holomorphic vector
bundle with a fixed Hermitian metric, with sheaf of holomorphic sections \( E \). Then if \( \mathcal{H}^{p,q}(E) \) is the space \( \Delta_E \)-harmonic \( E \)-valued \((p,q)\)-forms (e.g. forms in \( A^{p,q} \otimes E \)) then the natural map
\[
\mathcal{H}^{p,q}(E) \to H^{p,q}(E)
\]
is an isomorphism.

Taking \( E \) to be the trivial real or complex line bundle over \( M \) gives these results for de Rham and
Dolbeault cohomology. Very cheaply, the ellipticity of the Laplacian will also give

Corollary 1. All of the cohomology groups above are finite-dimensional vector spaces.

2.2. The Kähler Case. We let \( M \) be a Kähler manifold of complex dimension \( n \). Remarkably, the existence
of a Kähler metric will imply the following identities:

\[
\Delta_d = 2\Delta_\partial = 2\Delta_\partial^E.
\]

Thus in particular
\[
\mathcal{H}^k_{dR}(M) \otimes \mathbb{C} \simeq \bigoplus_{p+q=k} \mathcal{H}^{p,q}(M)
\]
and
\[
\mathcal{H}^{p,q}(M) \simeq \mathcal{H}^{q,p}(M).
\]

Putting this together with the above theorems gives

Theorem 4 (Hodge Decomposition).

\[
H^k_{dR}(M, \mathbb{C}) \simeq \bigoplus_{p+q=k} H^{p,q}(M)
\]
and
\[
\overline{H}^{p,q}(M) \simeq H^{q,p}(M).
\]

This is an important constraint on the cohomology of Kähler manifolds; e.g. their betti numbers of odd
degree, \( b_{2i+1} \), must be even.

Note that the Hodge Decomposition is a formal consequence of the Kähler identities between our various
Laplacians, and thus does not depend on the Kähler metric. Furthermore, while the de Rham cohomology is
a topological invariant, the integral or rational Hodge structure (e.g. the intersections \( H^{p,q}(M) \cap H^{p+1,q}_\text{sing}(M, \mathbb{Z}) \)
or \( H^{p,q}(M) \cap H^{p+1,q}_\text{sing}(M, \mathbb{Q}) \)) depend on the complex structure of \( M \). For example, a complex torus over \( \mathbb{C} \) is
entirely determined by its integral Hodge structure.

The cohomology of a Kähler manifold admits other decompositions, which are sensitive to the Kähler
form. Namely, let \( [\omega] \) be the class of the Kähler form in \( H^2(M, \mathbb{R}) \), and let
\[
L : H^k(M, \mathbb{R}) \stackrel{[\omega]}{\to} H^{k+2}(M, \mathbb{R})
\]
be the map induced by cupping with $[\omega]$. $L$ is induced by a map $L^k : \Lambda^k(M) \to \Lambda^{k+2}(M)$, with formal adjoint $\Lambda$.

We will find that $L, \Lambda$ induce an $\mathfrak{sl}_2$-representation on the cohomology of $M$. As a consequence

**Theorem 5** (Hard Lefschetz). For $k \leq n$, the map

$$L^{n-k} : H^k(M, \mathbb{R}) \to H^{2n-k}(M, \mathbb{R})$$

is an isomorphism.

As a corollary,

**Corollary 2.** If $k < n$,

$$L : H^k(M, \mathbb{R}) \to H^{k+2}(M, \mathbb{R})$$

is injective. Thus the odd degree betti numbers $b_{2i-1}$ increase for $2i-1 \leq n$ and the even degree betti numbers $b_{2i}$ increase for $2i \leq n$.

Furthermore, the representation theory of $\mathfrak{sl}_2$ implies that if we define the primitive part of the cohomology of $M$ via

$$P^k(M) := \ker(L^{n-k+1} : H^k(M, \mathbb{R}) \to H^{2n-k+2}(M, \mathbb{R})$$

for $k \leq n$, one has

**Theorem 6** (Lefschetz Decomposition). The natural map

$$i : \bigoplus_{k-2r \geq 0} P^{k-2r}(M) \xrightarrow{L^r} H^k(M, \mathbb{R})$$

is an isomorphism. Furthermore, the $P^k$ are compatible with the Hodge structure in the sense that setting

$$P^{p,q}(M) = P^k(M) \otimes \mathbb{C} \cap H^{p,q}(M)$$

we have

$$P^k(M) \otimes \mathbb{C} \simeq \bigoplus_{p+q=k} P^{p,q}(M)$$

and

$$P^{p,q}(M) \simeq P^{q,p}(M).$$