1. Introduction

The goal of this talk is to compute the first couple stable homotopy groups of spheres, via strictly ad hoc methods. At some point \( \pi^3_3 \) I won’t be able to do this anymore, so I’ll turn the talk into a discussion, with maybe half an idea of how to compute \( \pi^3_3 \).

The main tool here will be the Barratt-Priddy-Quillen Theorem, which asserts that the following four spaces are weakly equivalent:

1. \( \Omega^\infty S^\infty := \lim_{\longrightarrow} \Omega^n S^n \), where the maps are given by the adjuncton of \( \Sigma \) and \( \Omega \) (aka \( QS^0 \));
2. \( (\mathbb{Z} \times \Sigma_\infty)_+ \), where \( \Sigma_\infty \) is the group of automorphisms of a countable set having finite support, and + is the Quillen + construction, which I’ll discuss in a bit.
3. The group completion of \( B(\bigsqcup_n \Sigma_n) \). Here \( \bigsqcup_n \Sigma_n \) is the category whose objects are the integers, with \( \text{Hom}(m,n) = \emptyset \) if \( m \neq n \) and \( \text{Hom}(n,n) = \Sigma_n \). This category is a monoid under the concatenation map \( \Sigma_n \times \Sigma_m \to \Sigma_{n+m} \); it is this operation with respect to which we group complete.
4. \( K(\text{FinSet}) := \Omega S^\bullet \text{FinSet} | \), where this last is the Waldhausen \( S^\bullet \) construction and \( \text{FinSet} \) is the category of pointed finite sets. I won’t say too much about this unless people really want me to.

I’ll give a proof of this theorem from Segal’s paper “Categories and Cohomology Theories.” The non-technical content is in proving that the first space \( \Omega^\infty S^\infty \) is weakly equivalent to any of the others—while the other equivalences are not easy (in particular, \( (2) \simeq (3) \) is basically the group completion theorem) and are far from contentless, they are boring. Also, everything I’ll need will be contained in the equivalence \( (1) \simeq (2) \).

After computing \( \pi^0_0, \pi^1_1, \pi^2_2 \) using the Barratt-Priddy-Quillen theorem and some group theory, I’ll discuss a relationship between homology spheres and the homotopy groups of spheres.

2. The Barratt-Priddy-Quillen Theorem

We begin with some discussion of a certain model for topological monoids, which Segal calls \( \Gamma \)-spaces. Here \( \Gamma \) is the following category:

- The objects are all finite sets.
- \( \text{Hom}(S,T) = \{ \theta : S \to 2^T \mid \theta(i) \cap \theta(j) = \emptyset \text{ if } i \neq j \} \).
- Composition is given by \( \theta \circ \psi(i) = \bigcup_{x \in \psi(i)} \theta(x) \).

A \( \Gamma \)-space is a contravariant functor \( A : \Gamma \to \text{Top} \) so that

- \( A(\emptyset) \) is contractible
- The map \( A([n]) \to A([1]) \times \cdots \times ([1]) \) (induced by the maps \( i_k : 1 \mapsto \{ k \}, k \in [n] \)) is a homotopy equivalence.

This should be thought of as a monoid as follows—the “underlying set” is \( A([1]) \), and \( A([n]) \) is identified with \( n \)-tuples of elements. Then the map \( A([n]) \to A([1]) \), “multiplication” is induced by the map \( 1 \mapsto \{ 1, \ldots, n \} \subset 2^{[n]} \). This in fact turns \( A(1) \) into a particularly nice \( H \)-space.

Note that \( \Gamma \)-spaces may be thought of as simplicial spaces, through the natural embedding \( \Delta \to \Gamma \). In particular, we may take geometric realizations. Note also that “level 1” of \( |A| \) is homotopy equivalent to \( SA(1) \).

If \( A \) is a \( \Gamma \)-space, we may set \( BA(S) = [T \mapsto A(S \times T)] \). I will leave checking that this is a \( \Gamma \)-space as a tedious exercise. Note that \( BA(1) = |A| \). In particular, there is a natural map from \( SA(1) \to |A| \simeq BA(1) \), so the \( B^k A(1) \) form a (pre)-spectrum; call the functor associating to a \( \Gamma \)-space a pre-spectrum \( B \).
Now if \( \mathcal{C} \) is a category in which sums exist (note that they exist only up to isomorphism!), we may use it to construct a \( \Gamma \)-space as above. Let \( S \) be a finite set, and let \( \mathcal{P}(S) \) be the category of subsets of \( S \), with maps given by inclusion. Let \( \mathcal{C}(S) \) be the following category:

- The objects are functors \( \mathcal{P}(S) \to \mathcal{C} \) which take disjoint unions to sums;
- The morphisms are natural *iso*morphisms*.

Then \( \Gamma_{\mathcal{C}} : S \mapsto |\mathcal{C}(S)| \) is a \( \Gamma \)-space. If \( S \) is the skeleton of the category of finite sets,

\[
|\Gamma_{S}(1)| = B \left( \bigcup_n \Sigma_n \right)
\]

Now, we have a way \( B \) of associating to each \( \Gamma \)-space a spectrum; let us see that we have a way of associating to each \( (\Omega) \)-spectrum a \( \Gamma \)-space.

Now for each spectrum \( X \), we define the category \( \Gamma_{\mathcal{C}} = \text{Mor}(S^n, X) \). \( \mathcal{C} \) is the category of \( \Gamma \)-spaces with level-wise acyclic Hurewicz fibrations inverted.

I’ll provide one of the adjoint maps, \( BA(X) \to X \). For each \( m, n \), we must give

\[
(S^n)^m \times \text{Mor}(S^n, X) \to (\omega X)_n;
\]

this is given by evaluation.

Now we may prove the Barratt-Priddy-Quillen theorem, which we’ll prove as the equivalent statement:

**Theorem 1.** \( B\Gamma_S \simeq \mathbb{S} \).

**Proof.** By Yoneda, it is enough to give a natural isomorphism

\[
\text{Hom}_{Sp}(B\Gamma_S, X) \simeq \text{Hom}_{Sp}(\mathbb{S}, X) \simeq \pi_0(X).
\]

By adjointness, it suffices to show

\[
\text{Hom}_{\Gamma}(\Gamma_S, A(X)) = \pi_0(X);
\]

furthermore, \( \pi_0(X) = \pi_0(A X(1)) \) by construction.

There is a natural map, given by looking at the component containing the image of the point \( B\Sigma_1 \subset \Gamma_S(1) \) in \( \mathbb{A}X(1) \). Let’s construct an inverse.

To do this, we’ll need an auxiliary construction. Let \( F \) be a functor from the category of finite sets with inclusions to \( \text{Top} \), and let \( S_F \) be the topological category whose objects are pairs \((S, x) \in \mathcal{F}(S)\) and whose morphisms are maps \( \phi : S \to T \) so that \( F(\phi)(x) = y \). Then one can form \( \Gamma_{S,F} \) as before; this is a \( \Gamma \)-space if \( F(S \sqcup T) \simeq F(S) \times F(T) \).

Now choose \( a \in X(1) \), and let \( F_n \) be the homotopy fiber of the map \( \mathbb{A}X(n) \to \mathbb{A}X(1) \times \cdots \times \mathbb{A}X(1) \) at the point \((a, \ldots, a)\); this is contractible by the definition of a \( \Gamma \)-space; \( n \mapsto F_n \) is a functor as before. There is obviously a map \( \Gamma_{F,S} \to \mathbb{A}X \).

But by contractibility of the \( F_n \), the (forgetful) map \( \Gamma_{S,F} \to \Gamma_F \), induced by sending \((S, x)\) to \( S \) is an isomorphism in \( \Gamma \). This proves the theorem.

Now applying \( \omega \) gives \( \Omega BB \left( \bigcup_n \Sigma_n \right) \simeq \Omega B |\Gamma_S(1)| = \Omega^\infty S^\infty \). The term on the left is the group completion of \( B \left( \bigcup_n \Sigma_n \right) \), giving BPQ.

Now the group completion theorem implies that the group completion of \( B \left( \bigcup_n \Sigma_n \right) \) has identity component with the same homology as \( B \left( \bigcup_n \Sigma_n \right) \), but with abelianized fundamental group (that is \( \mathbb{Z}/2\mathbb{Z} \) rather than \( \Sigma_\infty \)). Now, for any CW complex \( X \) and a perfect normal subgroup \( N \subset \pi_1(X) \), one may associate a space \( X^+ \) with a map

\[
q_N : X \to X^+ \n\]

such that

- \( \ker(\pi_1(q_N)) = N \)
- The homotopy fiber of \( q_N \) is acyclic (in particular, \( q_N \) is a homology equivalence).
- \( q_N \) is homotopy-universal for maps \( f : X \to Y \) with \( N \subset \ker(\pi_1(f)) \).

I’ll describe a construction of this space in a bit—but in particular, note that \( q_{X^+} : B \left( \bigcup_n \Sigma_n \right) \to \Omega BB \left( \bigcup_n \Sigma_n \right) \) exhibits \( \Omega BB \left( \bigcup_n \Sigma_n \right)^+ \) as \( B \left( \bigcup_n \Sigma_n \right)^+ \) by Whitehead.

Now we construct \( X^+ \) in the most naive way possible. As \( N \) is perfect, it maps to zero in \( H_1 \); choose a set of generators. Kill them by attaching 2-cells; in \( H_2 \) this introduces a \( \mathbb{Z} \) for each relation between
the generators—this doesn’t alter $H_1$ by the perfection of $N$; however, these can be chosen so there are no syzygies, so one may fix the homology by only adding 3-cells. This can easily be seen to be universal—any map from $X \to Y$ killing $N$ extends to the new 2-cells as the map kills $N$; of course it similarly kills the relations so the map extends to the 3-cells.

3. Stable Homotopy Groups

We’re ready to start computing. Manifestly

\[ \pi^n_0 = \mathbb{Z}. \]

By the + construction description,

\[ \pi^+_1 = \Sigma_\infty^1 = \mathbb{Z}/2\mathbb{Z}. \]

Now note that $\widetilde{B\Sigma_\infty}$ is homology-equivalent to the classifying space of the alternating group, and thus $\widetilde{B\Sigma_\infty}^+ \simeq BA^+$. In particular

\[ \pi^2_2 \simeq H_2(A_\infty, \mathbb{Z}). \]

So let’s discuss this group a little bit. Because of the short exact sequences

\[ 0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^* \to 0 \]

and

\[ 0 \to \mathbb{C}^* \to GL(n, \mathbb{C}) \to PGL(n, \mathbb{C}) \to 0 \]

this group measures the failure of projective representations to lift to honest-to-god representations. In particular (by e.g. inflation-restriction), $H_2(G, \mathbb{Z})$ is the kernel of a central extension

\[ 0 \to H_2(G, \mathbb{Z}) \to \hat{G} \to G \to 0 \]

where $\hat{G}$ is the so-called universal covering group of $G$—namely, a group to which projective reps of $G$ lift uniquely. One way to construct such groups is to take a faithful projective rep of $G$, and pull back to $GL$; then for large enough representations (if $G$ is finite) this will contain the universal covering group.

For $A_n$ with large $n$, one may take the permutation representations $S_n \to GL(n, \mathbb{R})$, and let $V$ be the larger irreducible sub-rep. This gives a map $S_n \to O(n - 1)$, which restricts to a map $A_n \to SO(n - 1)$. Pulling back to Spin exhibits the requisite covering, and the kernel is $\mathbb{Z}/2\mathbb{Z}$.

Let’s talk about the higher stable homotopy groups now. If $M$ is a homology sphere of dimension $n$ with homotopy group $\pi_1(M)$ (which is necessarily perfect, by Hurewicz), then any action of $\pi_1(M)$ on a finite set gives an element of $\pi^n_1$ as follows. Namely, there is a natural map

\[ M \to K(\pi_1(M), 1) \]

(which can be viewed either as classifying the universal cover of $M$ or via building $K(\pi_1(M), 1)$ by killing higher homotopy groups of $M$). Now a map $\pi_1(M) \to \Sigma_N$ gives a map $K(\pi_1(M), 1) \to K(\Sigma_N, 1)$; the composition gives a map

\[ S^n \to \Omega^\infty S^n \]

after applying the + construction. (Open to audience participation)