The idea of this note is to relate the linear independence (or dependence) of sets of complex numbers to the topological properties of certain spaces.

By the Riemann surface of a holomorphic function \( h \) on some region of \( U \), I mean the connected component of the étale space of the sheaf of holomorphic functions containing the germs of \( h \). One may alternatively construct this surface as follows. Let \( M_h \) be the set of germs of all analytic continuations of \( h \), and given \( h \) defined on \( B_r(x) \), let the set of germs of \( h \) on \( B_r(x) \) be in a basis of open sets. This has an obvious complex analytic structure, via the map identifying the germs of \( h \) over \( B_r(x) \) with \( B_r(x) \) itself. There are two natural holomorphic maps from \( M_h \rightarrow \mathbb{C} \). The first, \( \pi_h : M_h \rightarrow \mathbb{C} \) sends a germ of \( h \) at \( x \) to \( x \), namely, \([h]_x \mapsto x\). The latter, \( e_h : M_h \rightarrow \mathbb{C} \), is given by evaluation. Namely, \( e_h : [h]_x \mapsto [h]_x(x) \).

If \( h \) is a multivalued function, it’s worth thinking of this space as the graph of the relation \( h \); one may embed \( h(z) \) into \( \mathbb{C}^2 \) via \( (\pi_h, e_h) \).

Now consider the following situation.

Let \( f \) be a meromorphic function on \( \mathbb{C} \) with poles only of order 1, say at \( z_1, z_2, \ldots, z_n \), and residues \( \lambda_i = \text{Res}_{z_i}(f) \). That is, \( f \) is a section of the sheaf of holomorphic functions \( \mathcal{H} \) over \( U = \mathbb{C} - \{ z_1, \ldots, z_n \} \), and in particular, \( M_f = \mathbb{C} - \{ z_1, \ldots, z_n \} \). Let \( g \) be an antiderivative of \( f \); that is, \( g' = f \). Note that \( g \) is multivalued. There is a map \( \pi_d : M_g \rightarrow M_f \) given by differentiation, which is an endomorphism of \( \mathcal{H} \) and thus an endomorphism of its étale space, sending \( M_g \) to \( M_f \). As this is a restriction of an endomorphism of the étale space, it is clearly a covering map. (Alternately, one may see that this is a covering map by noting that basis sets of \( M_g \) in our second construction are mapped isomorphically to basis sets of \( M_f \).)

Let \( V \subset \mathbb{C} \) be the \( \mathbb{Q} \)-vector space spanned by the \( \lambda_i \), that is \( V = \sum_i Q \lambda_i \). Then I claim that the following sequence is exact:

\[
H_1(M_g, \mathbb{Q}) \xrightarrow{H_1(\pi_d)} H_1(M_f, \mathbb{Q}) \xrightarrow{f} V \longrightarrow 0
\]

where \( f \) sends \( [\gamma] \mapsto \int_{[\gamma]} f \, dz \).

**Proof.** (1) \( f \) is surjective: \( H_1(M_f, \mathbb{Q}) \simeq \mathbb{Q}^n \), and has a basis given by fundamental cycles of \( M_f \). Let \([v_i]\) be the fundamental cycle corresponding to the puncture at \( z_i \); then \( f([v_i]) = \lambda_i \). But the \( \lambda_i \) span \( V \), so the map is surjective.
(2) \( \int \circ H_1(\pi_d) = 0 \): Consider \([v] \in H_1(M_g, \mathbb{Q})\). Then we have
\[
\int \circ H_1(\pi_d) \circ [v] = \int_{\pi_d([v])} f \, dz \\
= \int_{[v]} \pi_d^*(f) \, dz \\
= \int_{[v]} d(\pi_d^*(f)) \\
= \int_{d([v])} \pi_d^*(f) \\
= 0
\]
as \(d([v]) = 0\).

(3) \( \ker(f) \subseteq \text{im}(H_1(\pi_d)) \): By the Hurewicz theorem we have the following commutative diagram:
\[
\begin{array}{ccc}
\pi_1(M_g) & \xrightarrow{\pi_1(\pi_d)} & \pi_1(M_f) \\
\downarrow{h \otimes \mathbb{Q}} & & \downarrow{h \otimes \mathbb{Q}} \\
H_1(M_g, \mathbb{Q}) & \xrightarrow{H_1(\pi_d)} & H_1(M_f, \mathbb{Q}) \\
& \xrightarrow{f} & V \\
\end{array}
\]
where tensoring with \( \mathbb{Q} \) makes sense as the \( h \) maps are abelianization. Consider some linear combination \( \sum q_i [v_i] \), and multiply by \( N \) so that the coefficients are in \( \mathbb{Z} \); we can then lift to \( \pi_1(M_f) \). By this and the diagram, it suffices to show that \( \text{im}(\pi_1(\pi_d)) \supset \ker(f \circ (h \otimes \mathbb{Q})) \). Consider a loop \([\ell] \in \pi_1(M_f)\) that maps to zero in \( V \), e.g. choosing a representative, we have \( \int_{S^1} \ell^* f \, dz = 0 \). Then we claim it lifts to a loop in \( M_g \). Indeed, let \( \delta_t \) be such that \( f|_{B_{\delta_t}(\ell(t))} \) has no singularities. Then letting \( g_t : B_{\delta_t}(\ell(t)) \rightarrow \mathbb{C} \) be the map
\[
z \mapsto \int_{\ell([0, e^{2\pi it}])} f \, dz + \int_{\ell(t)}^z f \, dz,
\]
we have that \( \ell' : t \mapsto [g_t]_{\ell(t)} \) defines a loop in \( M_g \) (viewed as a space of germs), which is a lift of \( \ell \).

(Note that the integral is well-defined by the choice of \( \delta_t \).

\[\square\]

So why do we care? Well, the kernel of \( f \) is precisely the vector space of relations over \( \mathbb{Q} \) between the \( \lambda_i \)—to see this, note that as e.g. in (1) above, the map \( f \) acts on the basis \([v_i]\) of \( H_1(M_f, \mathbb{Q}) \) as \([v_i] \mapsto \lambda_i \). So the image of \( H_1(\pi_d) \) is the vector space of relations, and this covering map gives a topological description of the linear dependence of the \( \lambda_i \) over \( \mathbb{Q} \).

Now let \( C_0 = \pi_1(M_f) \) and let \( C_i = [C_{i-1}, C_{i-1}] \). Then \( C_i/C_{i+1} \) is abelian for all \( i \geq 0 \); we claim the sequence
\[
\cdots \rightarrow C_{i+1}/C_{i+2} \mathbb{Z} \otimes \mathbb{Q} \rightarrow C_{i}/C_{i+1} \mathbb{Z} \otimes \mathbb{Q} \rightarrow C_{i-1}/C_{i} \mathbb{Z} \otimes \mathbb{Q} \rightarrow \cdots \rightarrow C_0/C_{1} \mathbb{Z} \otimes \mathbb{Q} \rightarrow H_1(M_g, \mathbb{Q}) \rightarrow H_1(M_f, \mathbb{Q}) \rightarrow V \rightarrow 0
\]
is exact. The proof again follows from Hurewicz.