PICARD GROUPS OF MODULI PROBLEMS II

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1. Recap

Let's briefly recall what we did last time. I discussed the stack $B\mathbb{G}_m$, as classifying line bundles—by analyzing the sense in which line bundles may be specified locally (e.g., descent data), we arrived at an implicit definition of a stack. I also described the definition of a quotient stack, by analogy with the case of an étale categorical quotient $X \to X/G$. Let's recall a few equivalent ways of thinking about the definition of a stack quotient. Throughout, G is a finite group.

Recall that a map $X \to Y/G$ was specified by an étale cover $U \to X$, a map $\phi : U \to Y$, and an element $g \in \Gamma(U \times_X U, \underline{G})$ so that $g\pi_1^*\phi = \pi_2^*\phi$. The element g was required to satisfy a cocycle condition—that is, $\pi_{12}^* g \cdot \pi^* 23^* g = \pi_{13}^* g$, where $\pi_{ij} : U \times_X U \times_X U \to U \times_X U$ are the projections to the (i, j)-factors. Note that this data is the same as specifying an étale-locally trivial G-torsor $\mathcal{G}$ on X and a G-equivariant map $\mathcal{G} \to Y$. [Exercise: Convince yourself that this is true!] An isomorphism (U, ϕ, g) and (U', ϕ', g') is given by étale covers $V \to U, V \to U'$ so that the pullbacks of ϕ, ϕ', g, g' agree on V (with an evident equivalence relation on isomorphisms for refinements of étale covers).

Another way of thinking about this is as follows—Y represents some sheaf h_Y. We can define a pre-sheaf of groupoids $h_{Y/G}$ by setting the objects of $h_{Y/G}(T) = h_Y(T)$, with morphisms between two T-points x and y given by

$$\mathrm{Hom}(x, y) := \{g \in G \mid gx = y\}.$$

The stack $h_{Y/G}$ is given by sheafifying $h_{Y/G}$ in the étale topology (where I mean sheafification in the *homotopical* sense of a sheaf of groupoids).

Example 1. Consider the stack $\mathbb{A}^1/\mathbb{G}_m$, where $\mathbb{G}_m$ acts by scaling (this is a stack in the *smooth* topology, not the étale topology). A map $X \to \mathbb{A}^1/\mathbb{G}_m$ is given by a line bundle $\mathcal{L}$ on X and a section $s \in \Gamma(X, \mathcal{L})$. Note that $B\mathbb{G}_m = \mathrm{pt}/\mathbb{G}_m$ is a closed substack of $\mathbb{A}^1/\mathbb{G}_m$ (via $\mathrm{pt} \mapsto 0$). Thus the map $X \to \mathbb{A}^1/\mathbb{G}_m$ induces a map $V(s) \to B\mathbb{G}_m$, which classifies the conormal bundle of $V(s)$.

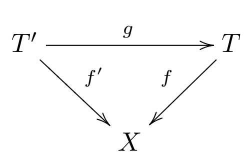
Example 2. $BG = \mathrm{pt}/G$ classifies étale-locally trivial G-torsors, for G a finite group (or étale group scheme, really).

2. The Picard Group of BG

The ultimate goal of this note is to computer $\mathrm{Pic}(\mathcal{M}_{1,1})$, following Mumford. We will now do a warm-up, by defining and computing $\mathrm{Pic}(BG)$, for a finite group G. [Can you guess what the answer is?] We work over a field k.

Suppose G is a finite group. Let us discuss what it means to give a line bundle on BG. If X is a scheme, and $\mathcal{L}$ is a line bundle on X, a line bundle on X is specified by giving a line bundle $\mathcal{L}$ on any fpqc (or fppf, or étale) cover $U \to X$, as well as descent data—namely, if $\pi_i : U \times_X U \to U$ are the projections, an isomorphism $f : \pi_1^*\mathcal{L} \xrightarrow{\sim} \pi_2^*\mathcal{L}$, satisfying the cocycle conditon.

Remark 1. Note that we may view a line bundle as giving a lot more data than the above—namely if $\mathcal{L}$ is a line bundle on X, and $f : T \to X$ is any morphism, we get a line bundle $f^*\mathcal{L}$ on T. Furthermore, if we have a commutative triangle



there is a canonical isomorphism ${f'}^*\mathcal{L} \xrightarrow{\sim} g^* f^* \mathcal{L}$; these isomorphisms are compatible with compositions in the obvious sense.

Now, here are two (equivalent) ways of thinking of a line bundle on BG. As above, we may view such a line bundle as associating to each map $T \to BG$ (e.g. a G-torsor $\mathcal{G}$ on T) a line bundle $\mathcal{L}_\mathcal{G}$—furthermore, for each $h : T' \to T$ over BG, we need to specify an isomorphism $h^*\mathcal{L}_G \xrightarrow{\sim} \mathcal{L}_{G'}$, (where $T' \to BG$ is specified by a G-torsor $\mathcal{G}'$), compatible with compositions. Note that there are automorphisms $T \to T$ over BG, which are the identity on T—namely, automorphisms of $\mathcal{G}$, aka G itself! So in particular, we have a map $\chi_\mathcal{G} : G \to \mathrm{Aut}(\mathcal{L}_\mathcal{G}) = \mathbb{G}_m$ for each $\mathcal{G}$.

On the other hand, we may specify a line bundle $\mathcal{L}$ on an étale cover; that is, *every* line bundle on BG comes from a line bundle and descent data on pt. Let us think about what it means to specify a line bundle with descent data on pt, which is an étale cover of BG. Of course, there is only one choice of line bundle on pt, namely $\mathcal{O}_{\mathrm{pt}}$. Descent data is the same as an automorphism of $\mathcal{O}_G$ (here we view G as a discrete scheme), namely, a (set-theoretic) map $G \to k^*$, which satisfies the cocycle conditon. An unwinding of the cocycle condition, which I will omit, shows that this map is a cocycle if the map $G \to k^*$ it is a homomorphism! In particular, a line bundle is the same as a homomorphism $G \to k^*$! That is, $\mathrm{Pic}(BG) = \mathrm{Hom}(G, k^*) = H^1(G, k^*)$.

Note that the homomorphism $G \to k^*$ associated to a line bundle $\mathcal{L}$ is precisely the same as $\chi_\mathcal{G}$, where $\mathcal{G}$ is the trivial $\mathcal{G}$-torsor over pt. We may be even more explicit: if $\chi : G \to k^*$ is a character, the total space of $\mathcal{L}_\mathcal{G}$ is given by

$$\mathrm{Tot}\, \mathcal{L}_\mathcal{G} = \mathcal{G} \times_G \mathbb{A}^1$$

where G acts on $\mathbb{A}^1$ via χ.

This result should not be totally unexptected, from topology—if $k = \mathbb{C}$, note that $H^1(G, k^*) = H^2(G, \mathbb{Z})$, which precisely classifies line bundles on the (topological) space BG!

Remark 2. An identical argument shows that

$$\mathrm{Pic}(B\mathbb{G}_m) = H^1(\mathbb{G}_m, \mathbb{G}_m) = \mathbb{Z} = H^2(\mathbb{CP}^\infty, \mathbb{Z})(!)$$

as one might expect from the discussion above.

Remark 3. These results hold true even if k is not algebraically closed—one must apply Hilbert 90 in the argument, however. Note that the computation of the Picard group depends on the number of roots of unity k contains, and so varies depending on the characteristic of k and whether or not it is algebraically closed.

3. $\mathcal{M}_{1,1}$

Like Mumford, I'll work over a field k of characteristic different from 2 or 3. (Olsson and Fulton work out the Picard group of $\mathcal{M}_{1,1}$ over a quite general base, if you're interested.) We define:

Definition 4. A *family of elliptic curves* over S is a smooth projective morphism $\pi : \mathcal{X} \to S$ whose geometric fibers are curves of genus 1, and with a section $\epsilon : S \to \mathcal{X}$ (the *identity section*).

The moduli stack $\mathcal{M}_{1,1}$ is described as follows: $\mathcal{M}_{1,1}(T)$ is the groupoid of families of elliptic curves over T, with the evident notion of isomorphism.

At this point, it is not at all obvious that $\mathcal{M}_{1,1}$ is algebraic—that is, we wish to find an étale cover of $\mathcal{M}_{1,1}$. We note first that the map

$$\mathcal{M}_{1,1} \to \mathcal{M}_{1,1} \times \mathcal{M}_{1,1}$$

is representable. This follows from the representability of the Isom functor, due to Grothendieck. Indeed, if T is a scheme, and $\mathcal{X}_1, \mathcal{X}_2$ are families of elliptic curves over T, the pullback of the diagram

$$\begin{array}{ccc} & & T \\ & & \big\downarrow{\scriptstyle (\mathcal{X}_1, \mathcal{X}_2)} \\ \mathcal{M}_{1,1} & \xrightarrow{\Delta} & \mathcal{M}_{1,1} \times \mathcal{M}_{1,1} \end{array}$$

has T'-points given by $\mathrm{Isom}_T(\mathcal{X}_1, \mathcal{X}_2)(T')$.

Remark 5. Note that in the analytic setting (that is, if we work in the analytic topology, where covers are given by surjective local homeomorphisms), $\mathcal{M}_{1,1}$ admits a presentation as a quotient stack. Let $\mathbb{H}$ be the upper-half plane. Then the functor of points of $\mathbb{H}$ (in complex-analytic spaces) is given by

$$\mathbb{H}(T) = \{\phi : \mathcal{Z} \to \mathbb{C}_T; s_1, s_2 \in \Gamma(T, \mathcal{Z})\}/\simeq$$

where $\mathbb{C}_T$ is the trivial $\mathbb{C}$-local system on T, $\mathcal{Z}$ is a rank 2 $\mathbb{Z}$, local system, s_1 and s_2 are trivializing global sections to $\mathcal{Z}$ with $\phi(s_1)/\phi(s_2) \in \mathbb{H}$, and ϕ is the inclusion of a lattice in $\mathbb{C}$ on each fiber. The universal family over $\mathbb{H}$ is given by the map over $\mathbb{H}$

$$\mathbb{Z}^2 \times \mathbb{H} \to \mathbb{C} \times \mathbb{H}$$

$$(n, m) \mapsto n + \tau m$$

where τ is the coordinate on $\mathbb{H}$.

Taking the cokernel of this map (over $\mathbb{H}$) gives an elliptic curve over $\mathbb{H}$, with identity section given by the zero section. $SL(2, \mathbb{Z})$ acts naturally on $\mathbb{H}$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d}.$$

(This action is induced by the natural action of $SL(2, \mathbb{Z})$ on the moduli problem, given by moving s_1, s_2 around.)

I claim that the quotient stack $\mathbb{H}/SL(2, \mathbb{Z})$ is $\mathcal{M}_{1,1}$. Namely, we may view $\mathbb{H}$ as representing the functor $\mathbb{H}(T) = \{$families of elliptic curves on T with trivialized homology basis$\}/\simeq$, where by "trivialized homology basis" for a family of elliptic curves $\mathcal{X} \to T$, I mean, a choice of trivialization of the homology local system $\mathcal{H}_1(\mathcal{X}/T, \mathbb{Z})$. Now a map $X \to \mathcal{M}_{1,1}$ lifts analytically-locally to a map to $\mathbb{H}$, (by choosing an open cover of X where the homology local systems are trivial), and the lifts are a torsor for $SL(2, \mathbb{Z})$, giving the claim.

Now we may compute $\mathrm{Pic}(\mathcal{M}_{1,1})$ analytically. Namely, $\mathrm{Pic}(\mathbb{H}) = \{1\}$, so $\mathrm{Pic}(\mathcal{M}_{1,1}) = \mathrm{Hom}(SL(2, \mathbb{Z}), \mathbb{C}^*)$. The abelianization of $SL(2, \mathbb{Z})$ is $\mathbb{Z}/12\mathbb{Z}$, so $\mathrm{Pic}(\mathcal{M}_{1,1}) = \mathbb{Z}/12\mathbb{Z}$.

Let's get back to the algebraic setting. Now we ask: what does it mean for a map $U \to \mathcal{M}_{1,1}$, induced by a family of elliptic curves $\mathcal{X} \to U$, to be an étale cover? Last time, we said that this meant that for any pullback diagram

$$\begin{array}{ccc} U \times_{\mathcal{M}_{1,1}} T & \longrightarrow & U \\ \downarrow & & \downarrow \mathcal{X} \\ T & \xrightarrow{\mathcal{Y}} & \mathcal{M}_{1,1} \end{array}$$

the map $U \times_{\mathcal{M}_{1,1}} T \to T$ is an étale surjection. Now a T'-point of $U \times_{\mathcal{M}_{1,1}} T$ is given by a maps $f : T' \to U, g : T' \to T$, and an isomorphism $f^*\mathcal{X} \overset{\sim}{\to} g^*\mathcal{Y}$ over T'. There exist families $\mathcal{Y}$ containing every isomorphism class of elliptic curve, e.g. the family over $\mathbb{A}^1 \setminus \{0, 1\}$ (with parameter λ) defined by

$$y^2 = x(x - 1)(x - \lambda).$$

So $U \overset{\mathcal{X}}{\to} \mathcal{M}_{1,1}$ is a surjection if and only if the the family $\mathcal{X} \to U$ has every isomorphism class of elliptic curve among its fibers.

Now recall that a morphism is étale if and only if it is locally of finite presentation and *formally* étale—that is, a lfp map $X \to Y$ is étale if and only if every diagram

$$\begin{array}{ccc} \mathrm{Spec}(A/I) & \longrightarrow & X \\ \downarrow & \overset{\exists !}{\nearrow} & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

admits a unique dotted arrow as above, making the diagram commute, where A is an Artinian-local k-algebra and I is a square-zero ideal. Thus we say $U \overset{\mathcal{X}}{\to} \mathcal{M}_{1,1}$ is (finite) étale if it satisfies the diagram above, and if each isomorphism-class of elliptic curve appears at most finitely-many times in $\mathcal{X}$.

Let's unwind this a bit more (that is, we'll give a more explicit description of the small étale site of $\mathcal{M}_{1,1}$. Suppose an elliptic curve is given by the equation

$$y^2 = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$$

where $\alpha_1 \neq \alpha_2 \neq \alpha_3$. We set $\lambda = \frac{\alpha_3 - \alpha_1}{\alpha_2 - \alpha_1}$ and

$$j = -64 \left(\frac{(\lambda - 2)(2\lambda - 1)(\lambda + 1)}{\lambda(\lambda - 1)} \right)^2 .$$

What does this mean? The elliptic curve as presented admits a 2-to-1 map to $\mathbb{P}^1$, given by projection to the x coordinate. This map is ramified over $\alpha_1, \alpha_2, \alpha_3, \infty$; composing with a fractional linear transformation we may arrange that it is ramified over $0, 1, \infty, \lambda$. Permuting the α_i's leads to a different λ—namely one of

$$\lambda, 1 - \lambda, \frac{1}{\lambda}, \frac{\lambda - 1}{\lambda}, \frac{1}{1 - \lambda}, \frac{\lambda}{\lambda - 1}.$$

Note that the values of λ for which this permutation action has non-trivial stabilizer are $2, 1/2, -1, -\omega, -\omega^2$, where ω is a primitive cube root of 1. In this case $j = 0, 1728$.

Now every elliptic curve has at least one non-trivial automorphism, given by inversion—in the presentation above, this is given by sending y to $-y$. In the cases where $j = 0, 1728$, there are extra automorphisms, given as follows. We may represent the case $j = 0$ by the curve

$$y^2 = x(x^2 - 1),$$

which has automorphism group $\mathbb{Z}/4\mathbb{Z}$, generated by the automorphism

$$x \mapsto -x$$

$$y \mapsto iy.$$

The second case is represented by the curve

$$y^2 = x^3 - 1$$

which has automorphism group $\mathbb{Z}/6\mathbb{Z}$, generated by

$$x \mapsto \omega x$$

$$y \mapsto -y.$$

Let $A = \frac{27}{4} \frac{1728 - j}{j}$, and consider the family

$$y^2 = x^3 + A(x + 1)$$

over $\mathbb{A}^1_j \setminus \{0, 1\}$. The fiber over some point s is the curve with j-invariant s. This is étale of degree 2 over $\mathbb{M}_{1,1}$, as elliptic curves admit no infinitesimal deformations (fixing the identity) and each curve in this family has automorphism group $\mathbb{Z}/2\mathbb{Z}$. But it is not a cover, as it misses the curves with j-invariant $0, 1728$.

That said, if $\mathcal{X}/S$ is any family of elliptic curves, there is a natural map $\mathcal{X} \to \mathbb{A}^1_j$ sending each closed point $s \in S$ to the j-invariant of $\mathcal{X}_s$.

Remark 6. It is not obvious that this map is algebraic; here's a sketch. Namely, let $\epsilon : S \to \mathcal{X}$ be the identity section. Then there is a diagram

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathbb{P}(\pi_* \mathcal{O}(2\epsilon)) \\ & \searrow \quad \swarrow & \\ & S & \end{array} .$$

We may choose a Zariski-cover of S so that $\mathbb{P}(\pi_* \mathcal{O}(2\epsilon))$ is trivialized over each open in our cover; then the j-invariant may be defined as above on each open set. It's easy to see that it glues together to a map on all of S.

Unfortunately the map $j : S \to \mathbb{A}^1_j$ does not determine $\mathcal{X}/S$. Note that if S is connected and has varying modulus, the map $S \to \mathbb{A}^1_j$ is étale over $\mathbb{A}^1_j \setminus \{0, 1728\}$, and ramified to degree 2 if $j = 0$, and degree 3 if $j = 1728$. Let's add another invariant of $\mathcal{X}/S$ which, along with the j-invariant, does determine the family. Namely, suppose $S \to \mathcal{M}_{1,1}$ is étale and S is connected. Then we may take the fiber product

$$\begin{array}{ccc} T & \longrightarrow & \mathcal{A}^1_j \setminus \{0, 1728\} \\ \downarrow & & \downarrow \\ S & \longrightarrow & \mathcal{M}_{1,1} \end{array}$$

Over the locus $\{s \in S \mid j(\mathcal{X}_s) \neq 0, 1728\}$, $T \to S$ is étale of degree 2. So normalizing S in the function field of T (if T is connected), we obtain a $2:1$ cover of S, ramified exactly where $j = 0, 1728$ (if T is not connected, we replace it with two copies of S). We claim that the map $j : S \to \mathbb{A}^1_j$ and T/S determine the family $\mathcal{X}$.

Sketch Proof. Let S_0 be the locus where T/S is étale, and let $T' \to S_0$ be an étale cover trivializing T/S (which as before extends uniquely to a ramified map $T'' \to S$. Then the map $T' \to \mathbb{A}^1_j \setminus \{0, 1728\}$ determines a family of elliptic curves over T', with trivial monodromy—thus this family extends uniquely to a family with trivial monodromy on T''. Furthermore, the descent data associated to T'/S_0 extends uniquely, so we obtain a natural family of elliptic curves on S associated to $j, T/S$. □

Now we may give a more concrete description of the small étale site over $\mathcal{M}_{1,1}$:

- An étale map $S \to \mathcal{M}_{1,1}$, where S is a connected smooth curve, is a dominant map $j : S \to \mathcal{A}^1_j$, étale over $\mathbb{A}^1_j \setminus \{0, 1728\}$, and ramified to degree 2 if $j = 0$, and degree 3 if $j = 1728$, plus a double covering T/S ramified exactly where $j = 0, 1728$.
- A cover is a collection of such maps, so that the associated j maps are jointly surjective.

4. Algebraic Computation of the Picard Group

We now compute $\mathrm{Pic}(\mathcal{M}_{1,1})$. Recall that, as before, an element of $\mathrm{Pic}(\mathcal{M}_{1,1})$ is a way of (functorially) associating, to each family of elliptic curves $\mathcal{X}/S$, a line bundle $\mathcal{L}(\mathcal{X}/S)$ over S. Here is a natural, and important, example:

Example 3 (The Hodge Bundle). To each $\pi : \mathcal{X} \to S$ a family of elliptic curves, we associate the line bundle $R^1\pi_*\mathcal{O}_X$. This is a line bundle by cohomology and base change, and is obviously functorial under pullback.

Now we wish to associate some invariant to each line bundle on $\mathcal{M}_{1,1}$. Now note that each elliptic curve admits an automorphism—namely inversion—so given a line bundle $\mathcal{L}$ on $\mathcal{M}_{1,1}$, and a map $f : S \to \mathcal{M}_{1,1}$, we obtain a natural automorphism $\mathcal{L}(\text{inversion})$ on $f^*\mathcal{L}$, which squares to the identity. (Indeed, it is easy to see that this is independent of f, as this homomorphism is locally constant.) Thus, we obtain a map $\mathbb{Z}/2\mathbb{Z} \to \mathbb{G}_m$. Furthermore, our two distinguished elliptic curves, with j-invariant $0, 1728$, yield analogous homomorphisms $\mu_4 \to \mathbb{G}_m, \mu_6 \to \mathbb{G}_m$. Putting these all together, we obtain a natural homomorphism

$$\mu_{12} = (\mu_4) \times_{\mu_2} \mu_6 \to \mathbb{G}_m,$$

for each line bundle—that is, a canonical map

$$\mathrm{Pic}(\mathcal{M}_{1,1}) \to \mathrm{Hom}(\mu_{12}, \mathbb{G}_m).$$

We claim this map is an isomorphism (in characteristics different from $2, 3$). Furthermore, we claim that the Hodge bundle above maps to generator of $\mathrm{Hom}(\mu_{12}, \mathbb{G}_m)$.

It's easy to check that the Hodge bundle is a generator. We may compute the homomorphism $\mu_{12} \to \mathbb{G}_m$ induced by the Hodge bundle as follows. Let $C_0/\operatorname{Spec}(k)$ be the curve with j-invariant 0, defined by the equation

$$y^2 = x^3 - x$$

and $C_{1728}/\operatorname{Spec}(k)$ the curve defined by

$$y^2 = x^3 - 1.$$

We must compute the action of the automorphism groups of these curves on the Hodge bundle, which in this case is just $H^1(C, \mathcal{O}_X)$, which by Serre duality is dual to $H^0(C, \omega_C)$, which has generator dx/y.

Now the automorphism group of C_0 is generated by $x \mapsto -x, y \mapsto iy$, which thus has action $dx/y \mapsto idx/y$ (that is, multiplication by a primitive 4-th root of unity). The automorphism group of C_{1728} is generated by $x \mapsto \omega x, y \mapsto -y$, which has action $dx/y \mapsto -\omega dx/y$ (that is, multiplication by a primitive 6-th root of unity). Thus the induced map $\mu_{12} \to \mathbb{G}_m$ hits a primitive 12-th root of unity, and is thus a generator of the character group of μ_{12}. We have shown that the map $\mathrm{Pic}(\mathcal{M}_{1,1}) \to \mathrm{Hom}(\mu_{12}, \mathbb{G}_m)$ is surjective.

We now show that the map is injective. Suppose $\mathcal{L}$ is a line bundle mapping to zero in $\mathrm{Hom}(\mu_{12}, \mathbb{G}_m)$. It suffices to trivialize $\mathcal{L}$ on any surjective étale cover of $\mathcal{M}_{1,1}$, so that the associated descent data is trivial (by which I mean, the descent data commutes with our chosen isomorphism $\mathcal{L}|_S \simeq \mathcal{O}_S$). Just to be explicit, let's use the λ-family over $S = \mathbb{A}^1_\lambda \setminus \{0, 1\}$, defined by

$$y^2 = x(x-1)(x-\lambda),$$

which is a degree 12 cover of $\mathcal{M}_{1,1}$. All line bundles on S are trivial, so we choose an isomorphism $\phi : \mathcal{O}_S \to \mathcal{L}|_S$, which is in $\Gamma(S, \mathcal{L}|_S)^G$, where G is the group acting on S generated by $\lambda \mapsto \lambda^{-1}, \lambda \mapsto 1 - \lambda$ (this fixed space is non-empty by the existence of the j-invariant). Now we have descent data $\pi_1^*\mathcal{L} \xrightarrow{\sim} \pi_2^*\mathcal{L}$, where $\pi_i : S \times_{\mathcal{M}_{1,1}} S \to S$ are the projections. We need to check that

$$\begin{array}{ccc} \pi_1^*\mathcal{L} & \longrightarrow & \pi_2^*\mathcal{L} \\ \downarrow{\scriptstyle \pi_1^*\phi} & & \downarrow{\scriptstyle \pi_2^*\phi} \\ \pi_1^*\mathcal{O}_S & \longrightarrow & \pi_2^*\mathcal{O}_S \end{array}$$

commutes.

But we may check this on the level of stalks, whence it is precisely the triviality of the map $\mathrm{Hom}(\mu_{12}, \mathbb{G}_m)$.