Admissible subcategories of $\mathbb{P}^2$

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Derived categories of coherent sheaves

Definition (not really a definition)

Let X be an algebraic variety. The (bounded) *derived category of coherent sheaves* on X is a category $D^b_{\mathrm{coh}}(X)$ whose objects are bounded complexes of coherent sheaves on X, but the morphisms between two complexes are defined only up to a choice of resolutions on both sides.

Derived category of sheaves is a *very large* invariant of a variety, but still not completely opaque. Many smaller invariants may be recovered from it, such as algebraic K-theory or Hochschild (co)homology.

Admissible subcategories

Definition

A full (triangulated) subcategory $\mathcal{A} \subset D^b_{\mathrm{coh}}(X)$ is *admissible* if the inclusion functor has both left and right adjoints.

Definition

A *semiorthogonal decomposition* $\langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle$ of $D^b_{\mathrm{coh}}(X)$ is a collection of admissible subcategories which jointly generate $D^b_{\mathrm{coh}}(X)$ in a certain minimal way.

Remark

Any admissible subcategory leads to a semiorthogonal decomposition $\langle \mathcal{A}, {}^\perp\mathcal{A} \rangle = D^b_{\mathrm{coh}}(X)$ for a certain admissible subcategory ${}^\perp\mathcal{A}$. Also: $\langle \mathcal{A}^\perp, \mathcal{A} \rangle = D^b_{\mathrm{coh}}(X)$ similarly.

About admissible subcategories

Semiorthogonal decompositions, and hence admissible subcategories, let us understand larger categories in terms of the smaller components.

- **Many examples** of SODs, many tools of various complexity to produce new examples.
  - Pullback $\pi^*\colon D^b_{\mathrm{coh}}(X) \hookrightarrow D^b_{\mathrm{coh}}(Y)$ for a blow-up $\pi\colon Y \to X$;
  - Exceptional collections;
  - Homological projective duality;
  - Mutations of existing SODs;
  - etc.
- **Few general properties** known, even for subcategories of the most familiar varieties such as $\mathbb{P}^n$.

Some basic open questions

Let X be a smooth and proper variety.

Conjecture (Kuznetsov '09)
Any increasing chain of admissible subcategories $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \dots$ *in the category* $D^b_{\mathrm{coh}}(X)$ *stabilizes.*

Conjecture
Let $\mathcal{A} \subset D^b_{\mathrm{coh}}(X)$ *be an admissible subcategory. If there exists a variety* Y *such that* $\mathcal{A} \simeq D^b_{\mathrm{coh}}(Y)$*, then* $\dim Y \leq \dim X$.

Other open questions: some related to indecomposability of some categories, or to so-called *Fano visitors*, or to *phantom* subcategories, etc.

Easy cases

We know how admissible subcategories behave in the following two situations:

- X is a variety such that $D^b_{\mathrm{coh}}(X)$ has **no admissible subcategories** at all. For example, X Calabi-Yau, or more generally with a globally generated canonical bundle.
- X is $\mathbb{P}^1$. Here a **complete classification** of admissible subcategories is possible and easy.

The goal of this talk: sketch the full classification of admissible subcategories in $\mathbb{P}^2$.

Topological rigidity of admissible subcategories

Theorem (Kawatani–Okawa)
Let X be a smooth and proper variety. Any admissible subcategory in $D^b_{\mathrm{coh}}(X)$ is closed under small deformations of objects, and it is invariant under the action of $\mathrm{Aut}^\circ(X)$.

Here ”small” means: in any family of objects whose central fiber is in the admissible subcategory, a Zariski-open set of fibers lie in that subcategory as well.

This theorem follows from the existence of generators for derived categories of coherent sheaves and the semicontinuity of the dimensions of Ext-groups.

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Skyscrapers in $\mathbb{P}^2$

How can we study arbitrary admissible subcategories in $D^b_{\mathrm{coh}}(\mathbb{P}^2)$?

Lemma

Let $\mathcal{B} \subset D^b_{\mathrm{coh}}(\mathbb{P}^2)$ be an admissible subcategory. Let B be the (right) projection $\mathcal{B}_R(\mathcal{O}_p)$ of a skyscraper sheaf at some point $p \in \mathbb{P}^2$ to $\mathcal{B}$.

- *The object B is invariant under the action of* $\mathrm{Stab}(p) \subset \mathrm{PGL}(3)$.
- *The object B uniquely determines the category $\mathcal{B}$.*

Proof sketch.

Note that for $g \in \mathrm{PGL}(3)$ we have $\mathcal{B}_R(g^*\mathcal{O}_p) \simeq g^*B$ by the **uniqueness** of projection functors, since $g^*B \in \mathcal{B}$ by the Kawatani–Okawa theorem. This implies the first part. Similarly, B determines the projections of all skyscrapers to $\mathcal{B}$, and this is enough to recover B. □

Projections of the skyscraper sheaves

How can we study the projection of a skyscraper sheaf into an admissible subcategory $\mathcal{B} \subset D^b_{\mathrm{coh}}(\mathbb{P}^2)$?

- Projections of arbitrary objects are hard to understand.
- Skyscrapers are far from being arbitrary. In particular, a skyscraper sheaf can be considered as a pushforward of a sheaf from various subvarieties of $\mathbb{P}^2$.
- Next slide, I will explain why this helps.

Addington's theorem

General statement in terms of spherical functors:

Theorem (Addington '11)

*Let X be smooth and projective, and let $\mathcal{B} \subset D^b_{\mathrm{coh}}(X)$ be an admissible subcategory. Then for any **anticanonical divisor** $j\colon D \hookrightarrow X$ the composition*

$$\mathcal{B} \hookrightarrow D^b_{\mathrm{coh}}(X) \xrightarrow{j^*} \mathrm{Perf}(D)$$

is a spherical functor.

We only use the following corollary about the projection functor $\mathcal{B}_R\colon D^b_{\mathrm{coh}}(X) \to \mathcal{B}$:

Corollary

Let F be an object in $\mathrm{Perf}(D)$. Then there is a (functorial) triangle

$$j^*\mathcal{B}_R(j_*F) \to F \to T(F)$$

*where T is some **autoequivalence** of $\mathrm{Perf}(D)$.*

Corollary (from the previous slide)
Let F be an object in $\mathrm{Perf}(D)$. Then there is a (functorial) triangle

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- Autoequivalences are **easier to control** and study than admissible subcategories.

Motivation

Corollary (from the previous slide)

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- Autoequivalences are **easier to control** and study than admissible subcategories.
- If we want to **understand the object** $\mathcal{B}_R(j_*F)$ and we know a lot about autoequivalences of $\mathrm{Perf}(D)$, the triangle above may be useful.

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- If we want to **understand the object** $\mathcal{B}_R(j_*F)$ and we know a lot about autoequivalences of $\mathrm{Perf}(D)$, the triangle above may be useful.
- On $\mathbb{P}^2$ this is **exactly what we want**: we can study the object $B := \mathcal{B}_R(\mathcal{O}_p)$ since autoequivalences of (say, smooth) elliptic curves are known and the skyscraper sheaf $\mathcal{O}_p$ is a pushforward from any smooth cubic curve passing through p.

Consequences on surfaces

Assume that X is a **surface**. Pick a smooth anticanonical curve $D \subset X$, and pick a point $p \in D$. Let B be the projection $\mathcal{B}_R(\mathcal{O}_p)$. Then we have a distinguished triangle in $\mathrm{Perf}(D)$:

$$B|_D \to \mathcal{O}_p \to T_p, \qquad \mathrm{Ext}^\bullet_D(T_p, T_p) \simeq \mathrm{Ext}^\bullet_D(\mathcal{O}_p, \mathcal{O}_p)$$

Lemma

Any "skyscraper-like" object $T_p \in \mathrm{Perf}(D)$ on a smooth curve is, up to a shift, either a skyscraper sheaf or a simple vector bundle.

(the same answer is proved for **irreducible** reduced projective curves of arithmetic genus one in the paper [Burban–Kreussler '05].)

Consequences on surfaces, explicitly

Upshot

If X is a surface, then for any admissible subcategory $\mathcal{B}$ and any smooth anticanonical divisor D passing through $p \in X$, the object $B|_D$ is either

- built out of two skyscrapers; or
- built out of a skyscraper and one (simple) vector bundle.

This is a strong structural result:

- $B|_D$ has **at most two cohomology** sheaves.
- If $B|_D$ is not a torsion object, its **class in $K_0(D)$ is nonzero**: the alternating sum of ranks of cohomology sheaves of B is, up to a sign, a rank of the simple vector bundle.

No phantoms in the plane

Theorem
Let $\mathcal{B} \subset D^b_{\mathrm{coh}}(\mathbb{P}^2)$ be a nonzero admissible subcategory. Then $K_0(\mathcal{B}) \neq 0$.

Proof. Pick a point $p \in \mathbb{P}^2$. Let $B := \mathcal{B}_R(\mathcal{O}_p)$ be the (right) projection of a skyscraper sheaf, as above. The object B is $\mathrm{Stab}(p)$-invariant. This group has only two orbits in $\mathbb{P}^2$. Thus there are **three options** for $\mathrm{supp}(B)$:

- $\mathrm{supp}(B) = \emptyset$; this implies $\mathcal{B} = 0$.
- $\mathrm{supp}(B) = \{p\}$; this implies $\mathcal{B} = D^b_{\mathrm{coh}}(\mathbb{P}^2)$.
- $\mathrm{supp}(B) = \mathbb{P}^2$, the nontrivial case.

No phantoms in the plane: proof

Suppose now that $\mathrm{supp}(B) = \mathbb{P}^2$.

- Pick a **smooth cubic curve** $D \subset \mathbb{P}^2$ passing through p. Then D is an anticanonical divisor.
- Consider the restriction $B|_D$. Note that $\mathrm{supp}(B|_D) = D$.
- By the **method of Section 2** we know that $B|_D$ is built either out of two skyscrapers, or out of a skyscraper and a simple vector bundle.

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- By the **method of Section 2** we know that $B|_D$ is built either out of two skyscrapers, or out of a skyscraper and a simple vector bundle.
- $B|_D$ is not a torsion object: thus $B|_D$ has exactly one cohomology sheaf which has nonzero rank on D.
- But then the class of $B|_D$ is **nonzero in** $K_0(D)$. It is the image of the class of B in $K_0(\mathbb{P}^2)$ under the restriction $K_0(\mathbb{P}^2) \to K_0(D)$.
- Then the class of B in $K_0(\mathbb{P}^2)$ is nonzero. $\square$

Admissible subcategories of projective plane

With additional work, the result for $\mathbb{P}^2$ may be improved to a complete classification.

Theorem (P.)

Any admissible subcategory in $D^b_{\mathrm{coh}}(\mathbb{P}^2)$ is one of the known examples, i.e., it comes from a full exceptional collection of $\mathbb{P}^2$.

Explicitly: if $\mathcal{A} \subset D^b_{\mathrm{coh}}(\mathbb{P}^2)$ is admissible, then there exists some full exceptional collection

$$\langle E_1, E_2, E_3 \rangle = D^b_{\mathrm{coh}}(\mathbb{P}^2)$$

obtained from the standard collection by a sequence of mutations, such that the category $\mathcal{A}$ is either 0, $\langle E_1 \rangle$, $\langle E_1, E_2 \rangle$, or $D^b_{\mathrm{coh}}(\mathbb{P}^2)$.

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Idea of the proof: recognize which of $\mathcal{A}$ and $\mathcal{A}^\perp$ is "simpler", prove that it is generated by a single exceptional bundle, closely related to the simple vector bundle on the elliptic curve we say above, then rely on Gorodentsev–Rudakov's classification. $\square$

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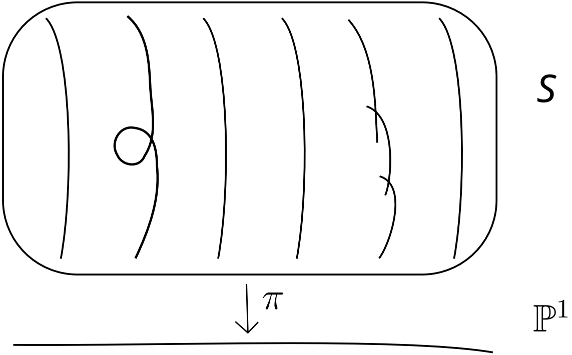
Rational elliptic surfaces



- Fibers $\leftrightarrow$ anticanonical divisors of S.
- General fiber is a smooth elliptic curve.
- Sections of π are (-1)-curves in S (and they exist).

Phantoms and rational elliptic surfaces

Suppose $\mathcal{B} \subset D^b_{\mathrm{coh}}(S)$ is a nonzero admissible subcategory. Can it happen than $K_0(\mathcal{B}) = 0$?

- Pick a point $p \in S$ and let B be the projection $\mathcal{B}_R(\mathcal{O}_p)$.
- That point p lies on a unique anticanonical divisor $j\colon F \subset S$.
- Can consider the pullback j^*B and try the same argument as for $\mathbb{P}^2$.

Suppose $\mathcal{B} \subset D^b_{\mathrm{coh}}(S)$ is a nonzero admissible subcategory. Can it happen than $K_0(\mathcal{B}) = 0$?

- Pick a point $p \in S$ and let B be the projection $\mathcal{B}_R(\mathcal{O}_p)$.
- That point p lies on a unique anticanonical divisor $j \colon F \subset S$.
- Can consider the pullback j^*B and try the same argument as for $\mathbb{P}^2$.
- The proof works **unless** the object j^*B is "built out of two skyscrapers".
- Need to deal with the case where $\mathrm{supp}(B)$ is a **subset of** S intersecting F in at most two points.

No phantoms in del Pezzo surfaces

- If the fibration π has no reducible fibers, this case can be studied explicitly.
- Nothing exotic appears: all such cases arise from the well-known exceptional objects supported on the (-1)-curves.
- This is much more difficult than the case of $\mathbb{P}^2$. Heavily relies on Addington's theorem internally.

Corollary (P.)
Let Y be a del Pezzo surface or a rational elliptic surface without reducible fibers. Let $\mathcal{B} \subset D^b_{\mathrm{coh}}(Y)$ be an admissible subcategory. Then $K_0(\mathcal{B}) \neq 0$.

Thanks for your attention!