HONORS COMPLEX VARIABLES, MATH W4065
PROBLEM SET 11

DUE MONDAY, DECEMBER 14, 2009

This version (Dec. 8):

- Corrected the definition of μ_U: it should refer to maps to the unit disk, not the whole complex plane.
- Clarified the complement of U in the last problem.

We saw in the proof of the Riemann Mapping Theorem for a simply-connected domain $U \subsetneq \mathbb{C}$ that maximizing the derivative of functions that map to the unit disk $\mathbb{D}$ played a crucial role. Specifically, for $z_0 \in U$, define

$$\mu_U(z_0) = \max\{\, |f'(z_0)| \mid f : U \to \mathbb{D} \text{ holomorphic}, f \text{ injective}\,\}.$$

(The proof of the Riemann Mapping Theorem shows in particular that this maximum is obtained, and the function that achieves the maximum is a conformal equivalence.)

(1) Compute μ for the unit disk. That is, for $z_0 \in \mathbb{D}$, compute $\mu_{\mathbb{D}}(z_0)$.
(2) Compute μ for the upper half-plane. That is, for $z_0 \in \mathbb{H}$, compute $\mu_{\mathbb{H}}(z_0)$.
(3) Suppose we have a conformal equivalence $g : U \to V$. How is $\mu_U(z_0)$ related to $\mu_V(g(z_0))$?
(4) Consider the combination $\mu_U(z)\,ds$, where $ds = \sqrt{dx^2 + dy^2}$ is the standard arc length element, as a way to measure a length of curves. Explicitly, for γ a smooth arc in U defined by the parametrization $z : [a, b] \to U$, define

$$\int_\gamma \mu_U(z)\,ds := \int_a^b \mu_U(z(t))\sqrt{x'(t)^2 + y'(t)^2}\,dt$$

where $x(t)$ and $y(t)$ are the real and imaginary parts of $z(t)$.

If $g : U \to U$ is a conformal automorphism, use the previous problem to show that $\int_\gamma \mu_U(z)\,ds = \int_{g(\gamma)} \mu_U(z)\,ds$. (*Hint:* Review the proof of invariance of the path integral under reparametrization.)
(5) Show that the constraint that f is injective in the definition of μ_U is unnecessary. (*Hint:* first think about the case that U is the disk.)
(6) Let $U' = \mathbb{C} \setminus U$ be the complement of U. For $z_0 \in U$, define

$$\begin{aligned} d(z_0, U') &= \min\{\, |z_0 - w| \mid w \notin U\,\} \\ &= \max\{\, r \mid \text{disk of radius } r \text{ around } z_0 \text{ contained in } U\,\}. \end{aligned}$$

Show that
$$\mu_U(z_0) \le \frac{1}{d(z_0, U')}.$$
(*Hint:* You could also define a constant based on maximizing $|f'(0)|$ among all injective maps from $\mathbb{D}$ to U with $f(0) = z_0$. How does this constant compare to $\mu_U(z_0)$? There is an obvious bound on this second map.)

(7) *Optional:* The *Köbe–Bieberbach Theorem* says that, if $f : \mathbb{D} \to \mathbb{C}$ is an injective function with $f(0) = 0$ and $f'(0) = 1$, then the image of f contains the disk of radius $1/4$ around 0. (You can see a proof outlined in Stein–Shakarchi, Problem 3.9.1, if you are interested. The function $f(z) = z/(1-z)^2$ shows that the bound cannot be improved.) Use this theorem to show that
$$\mu_U(z_0) \ge \frac{4}{d(z_0, U')}.$$

Remark. These properties, and in particular problem (4), show that the combination $\mu_U(z)\,ds$ defines a metric on U that is invariant under conformal automorphisms. In fact, this metric is the standard hyperbolic metric, called the *Poincaré metric* (up to a constant scale factor). For instance, if you study Escher's prints "Circle Limit IV" (left) or "Regular Division of the Plane VI" (right) carefully, you'll see that the size of the figures scales according to your answers to Problems (1) and (2).