LECTURE NOTES: HONORS COMPLEX VARIABLES

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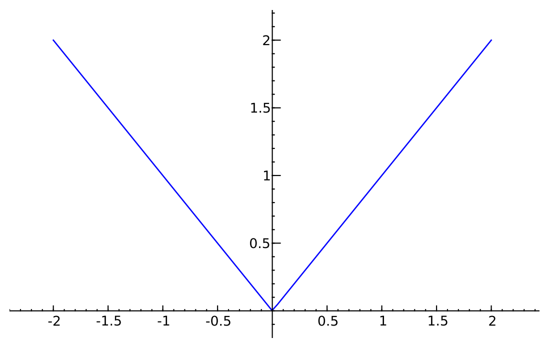
Contents

1. Overview

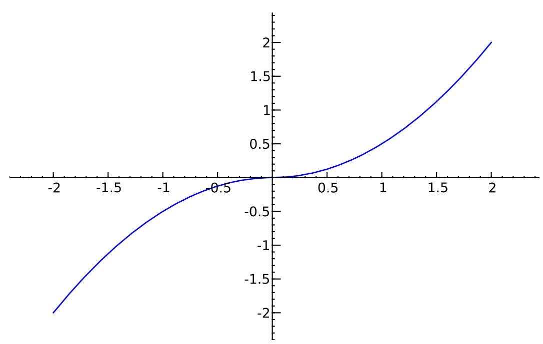
In this course, we will return to single-variable calculus: single-valued functions of a single variable. The difference will be that our single variable will be a *complex* number. This turns out to make a huge difference, on a number of fronts.

- *Harder to visualize:* Complex numbers can be conveniently laid out in the plane. A complex function of a complex argument is therefore a (special) map from the plane to itself. The graph of such a function would be 4-dimensional, and is rather hard to draw. However, we can get much mileage by just thinking about the map.
- *Fewer pathologies:* Functions of a real variable can be not differentiable, like $|x|$ …



…differentiable once, like

$$\int |x|\,dx = \begin{cases} \frac{x^2}{2} & x \geq 0 \\ -\frac{x^2}{2} & x \leq 0 \end{cases}$$



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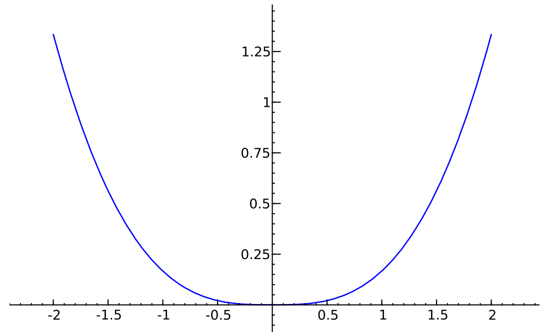
...or twice ...

$$\iint |x|\,dx = \begin{cases} \frac{x^3}{6} & x \geq 0 \\ -\frac{x^3}{6} & x \leq 0 \end{cases}$$



...or have much worse pathologies; for instance, the set of points where the function fails to be differentiable could be dense.

By contrast, if a complex function is differentiable once, it is automatically differentiable infinitely often! Furthermore, the Taylor series converges to the function itself.

- *More rigid:* For a real function, knowing what happens in one region tells you nothing about what happens elsewhere. For a complex differentiable function, it turns out that the value of a function anywhere in a disk in the complex plane is determined by the values on the boundary of the disk, something that is obviously false for real functions.

  Alternatively, for complex differentiable functions there is a method of *analytic continuation*, that lets you transport behavior in one region all over the plane. For instance, consider the *Riemann zeta function*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

  For s real, it's easy to see this sum converges for $s > 1$ and diverges otherwise. For instance,

$$\zeta(2) = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}.$$

  Using analytic continuation, we can give a well-defined value for this function for all values of s, including such oddities as

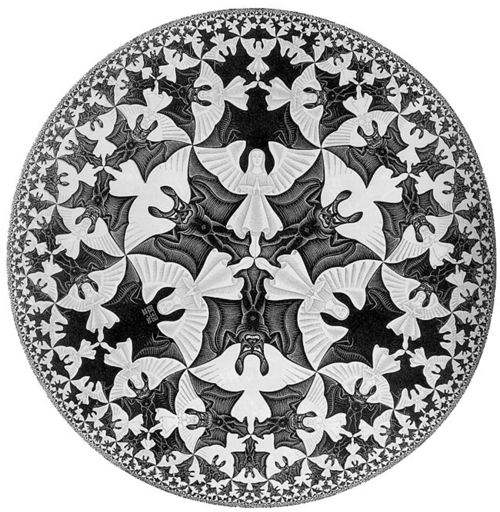
$$\zeta(-1) = 1 + 2 + 3 + 4 + \cdots = ??\,.$$

- *More rigid, 2:* Another instance of the extra rigidity of complex differentiable functions is the following remarkable theorem:

Theorem 1.1 (Liouville)**.** *If a complex differentiable function is bounded in absolute value by a constant everywhere in $\mathbb{C}$, then it is constant.*

This is obviously false for real functions; just consider something like $\cos x$, which just oscillates. It turns out that $\cos x$ can be extended uniquely to a complex function on all of $\mathbb{C}$, but by the theorem above it is bounded on $\mathbb{C}$.

- *Useful to evaluate integrals:* Integrals over complex differentiable functions are determined by the singularities of the function on the interior of the region enclosed. This can be used to evaluate even ordinary integrals. E.g., can prove
$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \pi$$
just from knowing that the denominator blows up when $x = i$.
  This you may know how to do in closed form, but there are many other examples where complex is the most useful technique.
- *Conformal mappings:* As maps from the plane to itself, complex differentiable maps have a beautiful property: they are *conformal*, preserving angles and (infinitesimal) shapes. This has been used in art by, for instance, M. C. Escher:



  (There is a complex differentiable map taking each angel to each other one.)
- *Useful:* Finally, complex analysis arises in many contexts, ranging from electricity and magnetism to fluid flow over airplane wings.

2. Complex arithmetic and its geometry

2.1. **Algebra.** You probably know that complex numbers can be defined taking the real numbers (called $\mathbb{R}$) by adding a new element, called i, with the property that $i^2 = -1$. We can then manipulate expressions using only this one relation and the familiar properties of arithmetic (namely, commutativity and associativity of addition and multiplication and distributivity of multiplication over addition).

For instance,
$$(1+2i) + (5+6i) = (1+5) + (2i+6i) = (1+5) + (2+6)i = 6+8i$$
$$(1+2i)\cdot(5+6i) = 1\cdot(5+6i) + 2i\cdot(5+6i) = (5+6i) + (10i+12i^2) =$$
$$= (5-12) + (6+10)i = -7+16i.$$

2.2. **Complex plane.** Real number line, complex number plane

2.3. **Geometry of addition.** Geometry of complex addition: Vector addition

2.4. **Geometry of multiplication.** Finally, let's look at the geometry of complex multiplication.

First consider multiplication by i. Plot $1+2i$, $i(1+2i)$. What does it look like is happening?

To see this is what happens, let's consider *polar coordinates.* We can write any complex number in polar coordinates:

$$x + iy = r(\cos\theta + i\sin\theta) = r(\operatorname{cis}\theta).$$

(The notation $\operatorname{cis}\theta$ is temporary. The reason will become clear next week...)

Check:

$$i \cdot r\operatorname{cis}\theta = r\operatorname{cis}(\theta + \pi/2).$$

More generally, we have:

Proposition 2.1. *For any r_i, θ_i,*

$$(r_1 \operatorname{cis}\theta_1) \cdot (r_2 \operatorname{cis}\theta_2) = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$$

Draw similar triangle.

Prove it.

2.5. **Division, conjugation, and norm.**

3. Complex differentiable functions

3.1. **Definition.** A function $f : \mathbb{C} \to \mathbb{C}$ is said to be *complex differentiable* at a point $z \in \mathbb{C}$ if

$$\lim_{\substack{h\to 0\\ h\in\mathbb{C}}} \frac{f(z+h) - f(z}{h} \tag{3.1}$$

exists.

Note the similarity to the definition of ordinary differentiability for real-valued functions. Note also the difference: if we set $g(h) = \frac{f(z+h)-f(z)}{h}$, then $g(h)$ is a function on the complex plane (defined when $h \neq 0$); we can think of it as a function from $\mathbb{R}^2$ to itself. The limit must exist whatever direction h approaches 0 from.

For $U \subset \mathbb{C}$, f is said to be *complex differentiable on U* if it is differentiable at every point in U. We say f is *complex differentiable near $z \in \mathbb{C}$* if there is an open neighborhood U of z so that f is complex differentiable on U.

(Note: Bak & Newman call this last notion *analytic at $z \in \mathbb{C}$*. This is really a theorem.)

Reminder: Open sets vs. balls

Example 3.2. The function $f(z) = z$ is complex differentiable. So is $f(z) = c$ for any constant $c \in \mathbb{C}$.

Non-example 3.3. The function $f(z) = \overline{z}$ is not complex differentiable.

3.2. **Cauchy-Riemann equations.** Let h approach zero along real axis or along imaginary axis.

These two must be equal.

Two forms:

$$f_y = i f_x$$

or

$$\begin{aligned} u_y + iv_y &= i(u_x + iv_x) \\ u_y &= -v_x \\ v_y &= u_x \end{aligned}$$

This is necessary, but have not yet proved sufficiency.

3.3. **Analytic Polynomials.**

Definition 3.4. An *analytic polynomial of degree k on* $\mathbb{C}$ is a function

$$P(x, y) = P(x + iy) = a_0 + a_1(x + iy) + \cdots + a_k(x + iy)^k.$$

Example 3.5. $P(x, y) = x^2 - y^2 + 2i \cdot xy$ is analytic.

Non-example 3.6. $P(x, y) = x^2 + y^2 + 2i \cdot xy$ is not analytic.

Proposition 3.7. *P is analytic if and only if $P_y = iP_x$ (i.e., P satisfies the Cauchy-Riemann equations.*

Proof. Sufficiency: Compare coefficients degree-by-degree.

Necessity: Run the argument backwards. □

So analytic polynomials satisfy the Cauchy-Riemann equations. But are they complex differentiable?

Proposition 3.8. *If f and g are both differentiable at z, so are $f + g$, $f \cdot g$, and, if $g(z) \neq 0$, f/g.*

Proof. Exercise. □

Corollary 3.9. *An analytic polynomial is differentiable on all of $\mathbb{C}$. Derivative given by usual rules.*

Proof. Exercise. □

3.4. **Geometry of polynomials, I.**

Let's take a quick look at what a polynomial does as a mapping. Consider $f(z) = z^2$. Can think of as

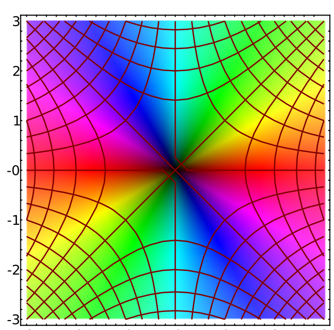
$$f(x, y) = (x^2 - y^2, 2xy).$$

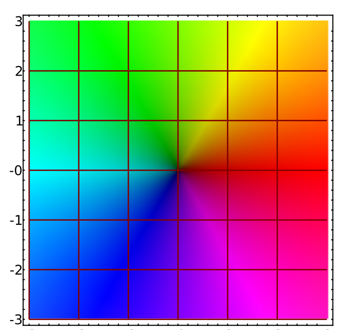
That's complicated. In polar coordinates, get

$$f(r \operatorname{cis} \theta) = (r^2 \operatorname{cis} 2\theta).$$

Geometrically, something happens to the radius, and angle gets multiplied by two.

Graphically, have





(On the right is the identity function for comparison.)

We'll return to polynomials in bigger generality later.

3.5. **Different ways to see differentiation.** There are many different ways to see complex differentiability.

3.5.1. *Calculus definition.* The definition we took was straight from calculus, just generalized a little.

3.5.2. *Cauchy-Riemann equations.* This implies the Cauchy-Riemann equations: if $f(x, y) = u(x, y) + iv(x, y)$, then

$$v_x = -u_y$$
$$v_y = u_x.$$

Later we will prove that if the function satisfies the Cauchy-Riemann equations in a region, it is complex differentiable.

3.5.3. *Conformal map.* We can think of a function from $\mathbb{C}$ to $\mathbb{C}$ as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$. Recall from multi-variable calculus that a function $f : \mathbb{R}^2 \to \mathbb{R}$ is *continuously differentiable* on $U \subset \mathbb{R}^2$ if

- The limits

$$f_x(x, y) = \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{f(x+h, y) - f(x)}{h}$$
$$f_y(x, y) = \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{f(x, y+h) - f(x)}{h}$$

  both exist on U.
- Both f_x and f_y are continuous on U.

If f is continuously differentiable on a neighborhood of (x_0, y_0), it is well-approximated by a linear function:

$$f(x, y) \approx f(x_0, y_0) + (x - x_0)f_x(x_0, y_0) + (y - y_0)f_y(x_0, y_0)$$

or, more precisely,

$$f(x, y) = f(x_0, y_0) + (x - x_0)f_x(x_0, y_0) + (y - y_0)f_y(x_0, y_0) + \varepsilon(x, y)$$

where $\varepsilon(x, y)$ is a function that goes to zero sufficiently fast.

(Will prove an appropriate version of this.)

What's special about complex differentiable functions is that this linear approximation is an amplitwist; the map is *conformal.*

What happens when derivative is zero?

3.5.4. *Function of z.* Intuitively, only z appears in the definition. How to make this precise? Suppose we had a function of x and y.

What does it mean to say a function $f(x, y)$ is a function of x? Answer: $\partial f/\partial y = 0$ (holding x fixed).

What does it mean to say a function $f(x, y)$ is a function of $x+y$? Answer: $\partial f/\partial(x-y) = 0$ (holding $x + y$ fixed), or alternatively

$$f_y = f_x$$

What does it mean to say a function $f(x, y)$ is a function of $x + 2y$? Answer:

$$f_y = 2f_x$$

What does it mean to say a function $f(x, y)$ is a function of $x + iy$? Answer:

$$f_y = if_x$$

These are Cauchy-Riemann equations.

Also written

$$\frac{\partial f}{\partial \overline{z}} = 0$$

(where we hold z fixed.) Here we think about z, $\overline{z}$ as coordinates: note that

$$x = \frac{z + \overline{z}}{2}$$

$$y = \frac{z - \overline{z}}{2i}$$

so they work as coordinates, in some sense.

3.6. **Power series.** For more interesting functions, we turn to *power series.* A power series is an infinite formal sum of the form

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_k z^k + \cdots .$$

It may converge for all values of $z \in \mathbb{C}$, for no values (except $z = 0$), or for something in between.

(A series of complex numbers *converges* if its real and imaginary parts do separately.)

Example 3.10. Consider

$$f(z) = \frac{1}{1-z} = 1 + z + z^2 + \cdots + z^k + \cdots .$$

(Strictly speaking, we have to prove a little more to justify the name "$1/(1 - z)$". But that equality is true at least for real x with $-1 < x < 1$.)

If $|z| > 1$, this series diverges: each term is bigger than the previous one.

If $|z| < 1$, this series converges: compare to an exponential series. (Use: $|1/z^k| = 1/|z|^k$, and $|\mathrm{Re}(w)| < |w|$.)

This series obviously diverges at $z = 1$. Question: What happens elsewhere with $|z| = 1$?

Example 3.11. Consider

$$\exp(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^k}{k!} + \cdots .$$

(Here we take the series as a definition of $\exp(z)$. Later we will prove it satisfies other definitions of $\exp(z)$.)

Lemma 3.12. *The series* $\exp(z)$ *converges for any* $z \in \mathbb{C}$.

Proof. Pick some integer N so that $|z| < N$. Then

$$\begin{aligned}\exp(z) &= C_1 + \frac{z^N}{N!}\sum_{k=0}^{\infty}\frac{z^k}{(N+1)\cdots(N+k)} \\ &\leq C_1 + C_2\sum_{k=0}^{\infty}\left(\frac{z}{N}\right)^k\end{aligned}$$

and the last sum converges, as in the previous example. (That is, we're using the comparison test.) □

Theorem 3.13. *For any power series*

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_k z^k + \cdots,$$

there is a number $R \in [0, \infty]$ *so that* $f(z)$ *converges for* $|z| < R$ *and diverges for* $|z| > R$.

The behavior for $|z| = R$ is indeterminate, and many things can happen. R is called the *radius of convergence.*

We can be more specific about R. First, some definitions.

Definition 3.14. For any set of real numbers A, the *supremum* $\sup X$ is the smallest element of $[-\infty, \infty]$ so that, for each $x \in X$, $\sup X > x$.

(Basic theorem of real analysis: This exists)

Definition 3.15. For a sequence of real numbers $\vec{X} = (x_k)_{k=0}^{\infty}$, define

$$\limsup \vec{X} = \lim_{N\to\infty} \sup\{\, x_k \mid k \geq N \,\}.$$

Alternatively, $\limsup \vec{X}$ is the smallest number C so that, for each $\varepsilon > 0$, there are only finitely many elements of the sequence bigger than $C + \varepsilon$.

This limit necessarily exists: $\sup\{\, x_k \mid k \geq N \,\}$ is a decreasing function of N.
(Pictures)

Theorem 3.16. *For any power series*

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_k z^k + \cdots,$$

there is a number $R \in [0, \infty]$ *so that* $f(z)$ *converges for* $|z| < R$ *and diverges for* $|z| > R$. *Furthermore,*

$$1/R = \limsup_{k\to\infty} |a_k|^{1/k}.$$

Intuitively: Size of k'th term is $|a_k||z|^k = (|a_k|^{1/k}|z|)^k$. If $|z| < R$, only finitely many terms are bigger than 1. Need to show: they go to zero fast enough.

Exercise 3.17. Pick some at least two (infinitely differentiable) real functions, find their power series, and consider the corresponding complex power series. Find the radius of convergence. Can you explain why the radius of convergence is what it is?

Proof. Let's use the comparison test to prove convergence when $|z| < R$. Other direction: Exercise.

Pick z with $|z| < R$. Then $1/|z| > \limsup |a_k|^{1/k}$, so there is an N_0 so that, for $k \geq N_0$, $|a_k|^{1/k} < 1/|z|$.

Do slightly better than this: pick $\varepsilon > 0$ and N so that, for $k \geq N$, $|a_k|^{1/k} < 1/|z| - \varepsilon$.

Then

$$\begin{aligned}\sum_{k=0}^{\infty} |a_k z^k| &= C_1 + \sum_{k=N}^{\infty} |a_k z^k| \\ &= C_1 + \sum_{k=N}^{\infty} (|a_k|^{1/k}|z|)^k \\ &< C_1 + \sum_{k=N}^{\infty} \Big(\Big(\frac{1}{|z|} - \varepsilon\Big)|z|\Big)^k \\ &= C_1 + \sum_{k=N}^{\infty} (C_2)^k\end{aligned}$$

for some constants C_1 and C_2, with $C_2 < 1$. But this last series converges absolutely. (This implies its real and imaginary parts both converge absolutely.) □

In fact, the proof shows slightly more: if R is the radius of convergence of $\sum a_k z^k$, then for any $r < R$, the series $\sum a_k z^k$ converges *uniformly* on the disk of radius r around 0. (The necessary size of C_2 depends only on $|z|$, and the same choice will work for all z within the disk of radius r.)

Example 3.18. The power series $\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k$ has radius of convergence 1.

Example 3.19. The power series $\exp(z) = \sum_k \frac{z^k}{k!}$ has infinite radius of convergence. There are several ways to see this:

- Use Stirling's formula:

$$\begin{aligned}\log n! &\approx n(\log n - 1) + \frac{\log n}{2} + c + O(1/n) \\ n! &\approx C\Big(\frac{n}{e}\Big)^n \sqrt{n}(1 + O(1/n)) \\ \Big(\frac{1}{n!}\Big)^{1/n} &\approx C^{-1/n}\Big(\frac{e}{n}\Big) n^{-1/(2n)}(1 + O(1/n))\end{aligned}$$

  which tends to zero. (This can be made precise by more carefully using Stirling's formula.)
- Apply Theorem 3.16 backwards.
- Directly, using elementary estimates, as in the proof that $\exp(z)$ converges.

Example 3.20. The two series

$$\begin{aligned}f(z) &= \frac{1}{1-z^2} = 1 + z^2 + z^4 + \cdots + z^{2k} + \cdots \\ f(z) &= \frac{1}{1+z^2} = 1 - z^2 + z^4 + \cdots + (-1)^k z^{2k} \cdots\end{aligned}$$

both have radius of convergence equal to 1. From the point of view of a real function, the first makes sense: it blows up at $z = \pm 1$. The second looks mysterious from the real point of view, but from the complex point of view it again makes sense: it blows up at $z = \pm i$.

Principle 3.21. A complex power series has a singularity on the boundary of its disk of convergence.

In a few weeks we will make this principle precise and prove it.

3.7. Differentiability of power series.

Theorem 3.22. *If the power series $f(z) = \sum a_k z^k$ has radius of convergence R, then $f(z)$ is complex differentiable inside of the disk of convergence, and the derivative is given by term-by-term differentiation.*

Lemma 3.23. *If $f(z) = \sum a_k z^k$ has radius of convergence R, then $g(z) = \sum k a_k z^{k-1}$ also has radius of convergence R.*

Proof. Compute the limits:

$$\begin{aligned}\limsup_{k\to\infty} |k a_k|^{1/(k-1)} &= \limsup_{k\to\infty} |k a_k|^{1/k} \\ &= \lim_{k\to\infty} |k|^{1/k} \limsup_{k\to\infty} |a_k|^{1/k} \\ &= 1/R.\end{aligned}$$

(Several assertions here are exercises.) □

Example 3.24. Consider

$$f(z) = 1 + 2z + 3z^2 + \cdots + kz^{k-1} + \cdots$$

We can recognize this as $\frac{1}{(1-z)^2}$ either by differentiating or by squaring the power series for $\frac{1}{1-z}$.

Example 3.25. If we differentiate $\exp(z)$ term-by-term we get the same power series back. This is another of the definitions of $\exp(z)$.

Example 3.26. More generally, differentiation shows that

$$\frac{d}{dz}\exp(kz) = k\exp(kz).$$

One interesting case is when $k = i$. We then find

$$\frac{d^2}{(dz)^2}\exp(iz) = -\exp(iz).$$

The last example is suggestive, as the same formula holds for $\sin(z)$ and $\cos(z)$. A little thought gives the following.

Lemma 3.27. $\cos(z) = \frac{1}{2}(\exp(iz) + \exp(-iz))$ *and* $\sin(z) = \frac{1}{2i}(\exp(iz) - \exp(-iz))$.
For θ real, $\operatorname{cis}(\theta) = \exp(i\theta)$, $\cos(\theta) = \operatorname{Re}(\exp(i\theta))$, *and* $\sin(\theta) = \operatorname{Im}(\exp(i\theta))$.

Here we define $\cos(z)$ and $\sin(z)$ by their power series:

$$\cos(z) = 1 - \frac{z^2}{2} + \frac{z^4}{24} - \cdots + (-1)^k \frac{z^{2k}}{(2k)!} + \cdots$$
$$\sin(z) = z - \frac{z^3}{6} + \frac{z^5}{120} - \cdots + (-1)^k \frac{z^{2k+1}}{(2k+1)!} + \cdots$$

Proof. Compare the power series term-by-term. □

Proof of Theorem 3.22. Let $g(z)$ be the term-by-term differentiation of $f(z)$, and let $S_N(z)$ be the partial sum of $f(z)$:

$$g(z) = \sum_{k=1}^{\infty} k a_k z^{k-1}$$
$$S_N(z) = \sum_{k=0}^{N} a_k z^k.$$

Note that $S_N(z)$ is a polynomial, so it is differentiable, and in fact its differential is the partial sum of $g(z)$.

We'll use the following chain of approximations:

$$\frac{f(z+h)-f(z)}{h} \underset{\substack{\text{close when}\\ N \text{ large}}}{\longleftrightarrow} \frac{S_N(z+h)-S_N(z)}{h} \underset{\substack{\text{close when}\\ h \text{ small}}}{\longleftrightarrow} S'_N(z) \underset{\substack{\text{close when}\\ N \text{ large}}}{\longleftrightarrow} g(z).$$

More formally, let $\varepsilon > 0$ be fixed. We wish to show there is a $\delta > 0$ so that, if $|h| < \delta$,

$$\left| \frac{f(z+h)-f(z)}{h} - g(z) \right| < \varepsilon. \tag{3.28}$$

Now restrict h to a disk around 0 so that $z+h$ is entirely contained in the disk of convergence. Since $\lim_{N\to\infty} S_N(z) = f(z)$ and similarly for $f(z+h)$ and the quotient, there is an N_1 so that for $N > N_1$,

$$\left| \frac{f(z+h)-f(z)}{h} - \frac{S_N(z+h)-S_N(z)}{h} \right| < \varepsilon/3.$$

(Here we use uniform convergence.) Similarly, there is an N_2 so that for $N > N_2$,

$$|S'_N(z) - g(z)| < \varepsilon/3.$$

Now pick some N greater than both N_1 and N_2. Since $S_N(z)$ is differentiable, by definition

$$\lim_{h\to 0} \frac{S_N(z+h)-S_N(z)}{h} = S'_N(z)$$

so there is a $\delta > 0$ so that, for $|h| < \delta$,

$$\left| \frac{S_N(z+h)-S_N(z)}{h} - S'_N(z) \right| < \varepsilon/3.$$

Putting these three inequalities together, we find Equation (3.28) as desired. □

3.8. Equivalence of definitions.

Proposition 3.29. *If $f(z)$ satisfies the Cauchy-Riemann equations at z_0 and the partial derivatives f_x and f_y of $f(x+iy)$ are continuous on an open set U containing z_0, it is also complex differentiable at z_0.*

Proof. Pick $z_0 \in U$. We may as well assume that U is a disk around z_0, since only the behavior close to z_0 matters.

To show: the limit

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists, or equivalently

$$f(z_0 + h) = \underbrace{f(z_0) + hf'(z_0)}_{\text{linear approximation}} + \underbrace{h\varepsilon(h)}_{\text{error term}}$$

where $\varepsilon(h) \to 0$ as $h \to 0$.

There is a standard theorem from real analysis that proves essentially this:

Theorem 3.30. *If a function $f(x, y)$ has continuous partial derivatives in a neighborhood of (x_0, y_0), then*

$$f(x_0 + \Delta x, y_0 + \Delta y) = \underbrace{f(x_0, y_0) + \Delta x f_x(x_0, y_0) + \Delta y f_y(x_0, y_0)}_{\textit{linear approximation}} + \underbrace{\|(\Delta x, \Delta y)\| \varepsilon(\Delta x, \Delta y)}_{\textit{error term}}$$

where $\varepsilon(\Delta x, \Delta y) \to 0$ as $(\Delta x, \Delta y) \to 0$.

(More generally, replace the partial derivatives with the Jacobian matrix applied to the displacement.)

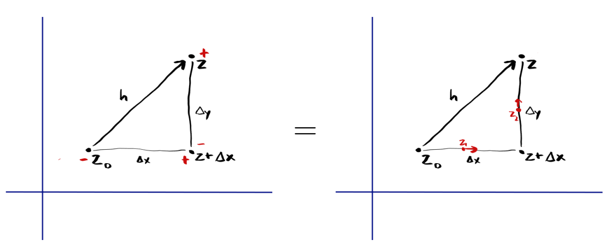
Since this is important, we'll do it explicitly.

We know that if the limit exists, it must be equal to $f_x(z)$ or equivalently $f_y(z)/i$ (since those are the values when we approach along the x or y axis).

Write $f(z) = u(z) + iv(z)$ as usual. Let $z = z_0 + h$ and $h = \Delta x + i\Delta y$. Then

$$\begin{aligned} u(z) - u(z_0) &= (u(z + \Delta x) - u(z)) + (u(z + \Delta x + i\Delta y) - u(z + \Delta x)) \\ &= u_x(z_1) \cdot \Delta x + u_y(z_2) \cdot \Delta y \end{aligned}$$

for some points z_1 between z and $z + \Delta x$ and z_2 between $z + \Delta x$ and $z + \Delta x + i\Delta y$ by the usual, 1-variable, mean value theorem.



Similarly,

$$v(z) - v(z_0) = v_x(z_3) \cdot \Delta x + v_y(z_4) \cdot \Delta y.$$

(Combine these. . .) □

Example 3.31. The function $f(x + iy) = |x||y|$ satisfies the Cauchy-Riemann equations at 0, but is not complex differentiable there.

Exercise 3.32. Check that in the following example, the result is false without the assumption of continuity of f_x, f_y on an open set:

$$f(z) = f(x+iy) = \begin{cases} 0 & x = y = 0 \\ \frac{xy(x+iy)}{x^2+y^2} & \text{otherwise.} \end{cases}$$

You can check that f is continuous, has partial derivatives f_x, f_y everywhere, satisfies the Cauchy-Riemann equations at $z = 0$, but yet is not complex differentiable at 0.

3.9. **Holomorphic functions.** The function $f(x, y) = x^2+y^2$ satisfies the Cauchy-Riemann equations and is complex differentiable at 0, but only at 0. It is not very interesting for our purposes; we instead are concerned with functions that are complex differentiable on an open set.

Definition 3.33. Given an open set $U \subset \mathbb{C}$ (a *domain*), we say that $f : U \to \mathbb{C}$ is *holomorphic* on U if f has continuous partial derivatives and satisfies the Cauchy-Riemann equations at every point in U. (In particular, f is complex differentiable.)

We say that a function $f : \mathbb{C} \to \mathbb{C}$ is *entire* if it is holomorphic on all of $\mathbb{C}$.

(Contrast $\exp(z)$, which is entire, with $\frac{1}{1+z}$, which is holomorphic in $\mathbb{C} \setminus \{-1\}$.)

Proposition 3.34. *If a function $f(z) = u(z) + iv(z)$ is holomorphic on a connected domain U, then*

- *if u is constant, f is constant, and*
- *if $|f|$ is constant, f is constant.*

Proof. We use the Cauchy-Riemann equations:

$$\begin{aligned} v_y &= u_x \\ v_x &= -u_y. \end{aligned}$$

- The CR equations tell us the partial derivatives of v in terms of those of u. Since the partial derivatives of u are 0, so are those of u.
- Assume that $|f| \neq 0$. (Otherwise we are done.) Then $|f|^2 = u^2 + v^2$ is constant. Take the partial derivatives with respect to x and y to find

  $$uu_x + vv_x = 0 \qquad uu_y + vv_y = 0$$

  or

  $$uu_x - vu_y = 0 \qquad uu_y + vu_x = 0.$$

  By adding v times the first equation to u times the second, we find

  $$(u^2 + v^2)u_x = 0$$

  so $u_x = 0$. Similarly, $u_y = 0$.

□

This is the first in a line of successively stronger theorems we will prove. In fact, to conclude that f is constant, it is enough to assume that u is constant on any small disk in U; or even that u is constant on any small line segment in the small disk.

3.10. Inverse functions.

Proposition 3.35. *If g is complex differentiable at z and f is complex differentiable at $g(z)$, then $f \circ g$ is complex differentiable at z, and $(f \circ g)'(z) = f'(g(z)) \cdot g'(z)$.*

Proof. Exercise. □

Proposition 3.36. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is a function and $g : U \to \mathbb{C}$ is a continuous inverse to f in some neighborhood U of z_0; that is, $f(g(z)) = z$ when it is defined. If f is complex differentiable at $g(z_0)$ with $f'(g(z_0)) \neq 0$, then g is complex differentiable at z_0 and*

$$g'(z_0) = \frac{1}{f'(g(z_0))}.$$

(In fact, if $f'(w) = 0$, there cannot be a continuous inverse to f near w; think about what happens to $f(z) = z^2$.)

Proof. We have

$$\begin{aligned} g'(z_0) &= \lim_{z \to z_0} \frac{g(z) - g(z_0)}{z - z_0} \\ &= \lim_{z \to z_0} \frac{g(z) - g(z_0)}{f(g(z)) - f(g(z_0))} \\ &= \lim_{z \to z_0} \frac{1}{\left(\frac{f(g(z)) - f(g(z_0))}{g(z) - g(z_0)} \right)} \\ &= \frac{1}{f'(g(z_0))}. \end{aligned}$$

Here we used the fact that g is an inverse to f, and then divided both top and bottom by $g(z) - g(z_0)$. (This is allowed: $g(z) \neq g(z_0)$ when $z \neq z_0$ since f is an inverse to g.) Finally, observe that since g is continuous, $g(z) \to g(z_0)$ when $z \to z_0$, so the expression in the denominator is just the derivative of f. □

Example 3.37. Let's see what is meant by $\sqrt{z}$ for z a complex number. Every non-zero complex number has two square roots (one the negative of the other). A function "$\sqrt{z}$" should be a consistent choice of one of the two square roots. However, there is no way to do this globally; after you have made one full turn around 0, the two choices have switched places.

Alternatively, just consider what happens in polar coordinates:

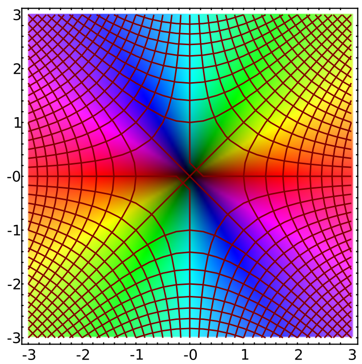
$$(re^{i\theta})^2 = r^2 e^{2i\theta}$$

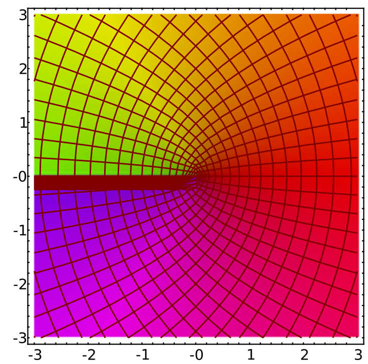
so

$$\sqrt{re^{i\theta}} = \sqrt{r} e^{i\theta/2}.$$

But θ is only defined up to multiples of 2π, so $\theta/2$ is defined up to multiples of π; that is, the expression is defined up to negation.

See below for an illustration. On the left is a plot of $f(z) = z^2$, which wraps twice around 0 as z wraps once around 0; on the right is a plot of $f(z) = \sqrt{z}$, which only makes it half-way around 0 when z wraps once around 0. Note that $\sqrt{z}$ is not continuous on all of $\mathbb{C}$; by convention, we choose to make it not continuous on the negative real axis.





Example 3.38. We can similarly consider the inverse to $\exp(z)$, called $\log(z)$. Recall that $\exp(z)$ essentially converts from Cartesian to polar coordinates:

$$\exp(x + iy) = e^x \operatorname{cis} y$$

so, dually, $\log(z)$ converts from polar to Cartesian coordinates:

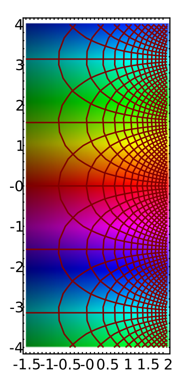
$$\log(re^{i\theta}) = \log(r) + i\theta.$$

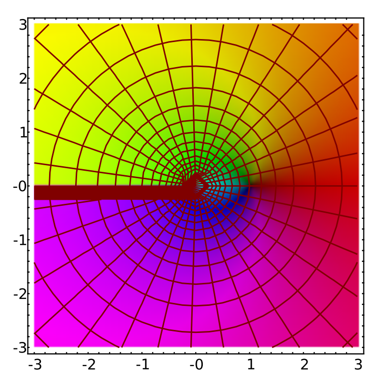
Again, θ in polar coordinates is only defined up to adding multiples of 2π, so the possible values for log (to be an inverse to exp are

$$\log(re^{i\theta}) = \log(r) + i(\theta + 2\pi k)$$

for $k \in \mathbb{Z}$.

Again, let's look at the plots. On the left is $\exp(z)$. Note that it is periodic when we add $2\pi \approx 6.28$ to the imaginary component. On the right is $\log(z)$. Again, we have to make a choice about where to make the function discontinuous, and again the standard choice is along the negative real axis.





4. Line integrals

We now turn from differentiation to integration. In a twist, this will turn out to give us more information about differentiation! In particular we will show that any holomorphic function is infinitely differentiable using these techniques.

4.1. **Definition of the integral.** Suppose we are given a function $f : \mathbb{C} \to \mathbb{C}$ (which we are not yet assuming to be holomorphic), and a *curve* C in the complex plane. By a curve we mean an interval $[a, b]$ in $\mathbb{R}$ and a continuous map (usually denoted z) from $[a, b]$ to $\mathbb{C}$. Then we could consider the integral

$$\int_a^b f(z(t))\, dt.$$

This is not, however, what we are interested in. Instead, we consider a new type of integral defined by

$$\int_C f(z)\, dz := \int_a^b f(z(t)) z'(t)\, dt. \tag{4.1}$$

(We need to assume that $z(t)$ is differentiable for this to make sense; see Definition 4.3 below.)

For some motivation for this extra factor of $z'(t)$, recall that one definition of an ordinary integral

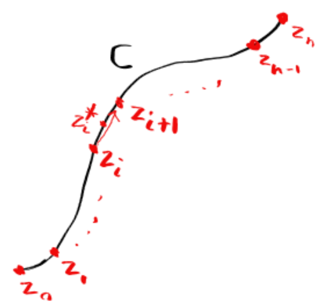
$$\int_a^b g(t)\, dt$$

is to chop up the interval from a to b into many small intervals $[t_i, t_{i+1}]$ and consider

$$\sum_i g(t_i^*) \cdot (t_{i+1} - t_i)$$

for some $t_i^* \in [t_i, t_{i+1}]$. The integral is the limit of this sum as the subdivision gets finer and finer.

When we integrate a complex function f over a curve C in the complex plane, a natural thing to do is to chop up C into small pieces at some points z_i on C and consider the sum



$$\sum_i f(z_i^*) \cdot (z_{i+1} - z_i) \tag{4.2}$$

for some z_i^* between z_i and z_{i+1} on C. Note here that the difference $z_{i+1} - z_i$ is a *complex* displacement, and the multiplication is complex multiplication. The integral could then be defined to be the limit of (4.2) as the subdivision gets finer and finer.

If we write (4.2) in terms of the parameters t_i, which are close to each other, it turns into

$$\sum_i f(z(t_i^*)) \cdot (z(t_{i+1}) - z(t_i)) \approx \sum_i f(z(t_i^*)) \cdot z'(t_i^*) \cdot (t_{i+1} - t_i).$$

It is not too hard to continue this line of argument and show that the limit of (4.2) is (4.1); however, we will just take (4.1) as a definition, since it is somewhat easier technically.

Definition 4.3. A curve $C = ([a, b], z)$ is *differentiable* if z is differentiable as a function of t on the open interval (a, b). It is *piecewise differentiable* if the interval $[a, b]$ can be divided up into subintervals $[t_i, t_{i+1}]$ so that z is differentiable on each subinterval. (However, z is allowed to turn corners at the boundaries between intervals.) C is *piecewise smooth* if furthermore $z'(t) \neq 0$ on the interior of each subinterval.

Definition 4.4. A curve $([a,b],z)$ is *closed* if its initial point equals its endpoint: $z(a)=z(b)$. A curve is *simple* if it does not intersect itself, except possibly at the endpoints: $z(t)\neq z(t')$ when $t<t'$ and $t\neq a$ or $t'\neq b$.

For C a piecewise smooth curve with an interval $[a,b]$, divided into intervals $[t_i,t_{i+1}]$ with $t_0=a$ and $t_N=b$, and $f:\mathbb{C}\to\mathbb{C}$, define

$$\int_C f(z)\,dz := \sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}} f(z(t))z'(t)\,dt. \tag{4.5}$$

Note that if we divide up the curve more finely into smooth pieces, we get the same answer.

By convention, we will assume that all curves are piecewise smooth unless otherwise stated.

For emphasis, we will write $\oint_C f(z)\,dz$ when C is a closed curve.

Exercise 4.6. Let's consider $\oint_C f(z)\,dz$, where C is a circle once counterclockwise around 0: C is defined by $z(\theta)=Re^{i\theta}$ for $\theta\in[0,2\pi]$. Compute:

(1) $\oint_C z\,dz$
(2) $\oint_C 1\,dz$
(3) $\oint_C \frac{1}{z}\,dz$
(4) $\oint_C \frac{1}{z^2}\,dz$
(5) $\oint_C f(z)\,dz$ where $f(x+iy)=x$.

Example 4.7. We compute:

$$\begin{aligned}\oint_C z^n\,dz &= \int_0^{2\pi}(z(\theta))^n z'(\theta)\,d\theta\\ &= \int_0^{2\pi} R^n e^{ni\theta}\cdot iRe^{i\theta}\,d\theta\\ &= iR^{n+1}\int_0^{2\pi} e^{i(n+1)\theta}\,d\theta\\ &= \begin{cases}2\pi i & n=-1\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

For the last step, notice that

$$\int_0^{2\pi} e^{ik\theta}\,d\theta = \int_0^{2\pi}(\cos(k\theta)+i\sin(k\theta))\,d\theta.$$

Geometrically, the vector $e^{ik\theta}$ rotates k times counterclockwise around 0 when θ runs from 0 to 2π; the average value is 0 unless $k=0$ (so the vector does not rotate).

Despite its simplicity, this is one of the most important calculations of the course! Note the importance of the factor $z'(\theta)$: without that factor, we would get a non-zero answer for $n=0$, not $n=-1$.

4.2. **Elementary properties.** We would like to be able to draw a curve in $\mathbb{C}$ and say we integrate along the curve. But does the definition depend not just on the curve, but on how fast we draw it? For instance, if we parametrize the circle C from Example 4.7 by $z(t)=e^{it^2}$ for $t\in[0,\sqrt{2\pi}]$, do we get the same answer?

Definition 4.8. Two curves $C_1 = ([a,b], z_1)$ and $C_2 = ([c,d], z_2)$ differ by a *smooth reparametrization* if there is a function $\lambda : [a,b] \to [c,d]$ with $\lambda(a) = c$, $\lambda(b) = d$, $\lambda'(t) > 0$, and $z_2(\lambda(t)) = z_1(t)$.

For example, the curves $([0, 2\pi], z(t) = e^{it})$ and $([0, \sqrt{2\pi}], z(t) = e^{it^2})$ are related by reparametrization with $\lambda(t) = \sqrt{t}$.

Proposition 4.9. *$\int_C f(z)\,dz$ is invariant under smooth reparametrization of C.*

This follows if you believe the description in Equation (4.2), for instance. It is not true if you consider the integral with respect to t, $\int f(z(t))\,dt$.

Proof. Recall the usual change of variables formula: For λ as above,

$$\int_c^d g(\lambda)\,d\lambda = \int_a^b g(\lambda(t))\frac{d\lambda}{dt}dt.$$

We apply this in our setting:

$$\begin{aligned}\int_c^d f(z_2(\lambda))z_2'(\lambda)\,d\lambda &= \int_a^b f(z_2(\lambda(t)))\frac{dz_2(\lambda)}{d\lambda}\frac{d\lambda(t)}{dt}dt \\ &= \int_a^b f(z_1(t))\frac{dz_1(t)}{dt}dt.\end{aligned}$$

We apply the chain rule in the last step. In this step and throughout, we are working with *complex-valued* functions of a *real* variable, for which almost all standard calculus rules apply, since they are defined to work for the real and imaginary parts separately. □

However, it is more than just the geometric curve that matters for the integral. For a curve $C = ([a,b], z)$, let $-C$ be C run backwards: the curve $([-b, -a], \tilde{z})$ with $\tilde{z}(t) = z(-t)$.

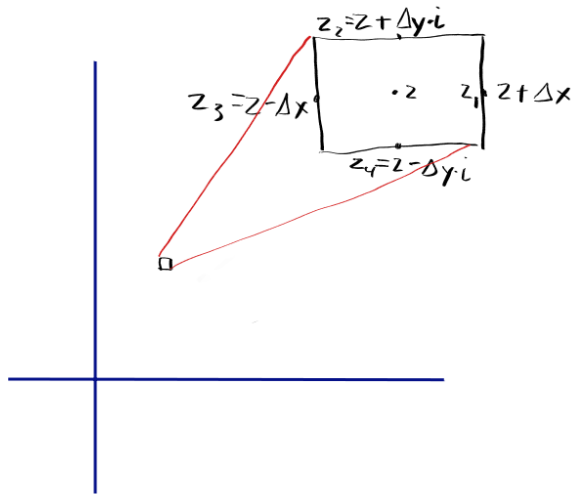
Proposition 4.10. *$\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$.*

Proof.

$$\int_{-b}^{-a} f(\tilde{z}(t))\tilde{z}'(t)\,dt = \int_{-b}^{-a} -f(z(-t))z'(t)\,dt = \int_a^b -f(z(t))z'(t)\,dt.$$ □

Exercise 4.11. Check that if you integrate $1/z$ around a small circle running *clockwise* around 0, you get $-2\pi i$.

4.3. **Motivation: Connection to Cauchy-Riemann equations.** So far everything we have said is true for an arbitrary function $f : \mathbb{C} \to \mathbb{C}$, whether or not it is holomorphic. What's special about holomorphic functions? For a first glimpse, let's consider the integral of a function f around a small rectangle R in the complex plane, as indicated below.



Let's approximate the integral along each edge of the rectangle as the value at the middle of the edge times the complex displacement. We find

$$\begin{aligned}\oint_R f(z)\,dz &\approx f(z_1)\cdot(2i\Delta y) + f(z_2)\cdot(-2\Delta x) + f(z_3)\cdot(-2i\Delta y) + f(z_4)\cdot(2\Delta x)\\ &\approx (2i\Delta y)((f+\Delta x f_x) - (f-\Delta x f_x))\\ &\qquad + (2\Delta x)(-(f+\Delta y f_y) + (f-\Delta y f_y)) \qquad \text{(all evaluated at } z\text{)}\\ &= 4\Delta x\Delta y(if_x - f_y)\end{aligned}$$

which is 0 exactly when the Cauchy-Riemann equations hold.

(More formally, all these approximations can be justified as finding the lowest-order term that does not vanish as $\Delta x, \Delta y \to 0$.)

We will extend and prove this more precisely; the result is called the *Cauchy integral theorem*: the integral of a holomorphic function around a closed curve that does not enclose any singularities (in a sense to be made precise) is 0.

5. Integrals and anti-derivatives

5.1. **Anti-derivatives.** As a start on the Cauchy integral theorem, suppose that we have a holomorphic function f defined on a domain U, and that f has an anti-derivative F on U; that is, F is holomorphic, and

$$F'(z) = f(z)$$

for all $z \in U$.

Proposition 5.1. *If a holomorphic function f has an anti-derivative F on U, then for any curve C in U running from z_0 to z_1,*

$$\int_C f(z)\,dz = F(z_1) - F(z_0).$$

Corollary 5.2. *For f as above and a closed curve C,*

$$\oint_C f(z)\,dz = 0.$$

Proof. In this case, $z_0 = z_1$ by assumption. □

Corollary 5.3. *The function $f(z) = 1/z$ has no anti-derivative on the punctured complex plane $\mathbb{C} \setminus \{0\}$.*

Proof. Example 4.7 gives an explicit closed curve C in $\mathbb{C} \setminus \{0\}$ so that the integral of $1/z$ on C is not 0. □

Note that z^n for $n \neq -1$ *does* have an anti-derivative on $\mathbb{C} \setminus \{0\}$, namely $\frac{1}{n+1} z^{n+1}$.

Proof of Proposition 5.1. If we can show that

$$\frac{dF(z(t))}{dt} = f(z(t))z'(t) \tag{5.4}$$

we will be done, by the fundamental theorem of calculus (applied to the real and imaginary parts separately), as the right hand side is the integrand in the line integral.

To see (5.4), we can either use the chain rule for multi-variable functions: if $F(x + iy) = U(x + iy) + iV(x + iy)$, then

$$\frac{dF(z(t))}{dt} = \frac{d}{dt}\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} U_x & U_y \\ V_x & V_y \end{pmatrix}\begin{pmatrix} x_t \\ y_t \end{pmatrix} = f \cdot z'.$$

More explicitly, let's assume for simplicity that $z'(t) \neq 0$. Then

$$\begin{aligned} \lim_{h \to 0} \frac{F(z(t+h)) - F(z(t))}{h} &= \lim_{h \to 0} \frac{F(z(t+h)) - F(z(t))}{z(t+h) - z(t)} \frac{z(t+h) - z(t)}{h} \\ &= f(z(t))z'(t). \end{aligned}$$

(We are allowed to multiply and divide as above since $z(t+h) - z(t) \neq 0$ for h small enough; this is a consequence of $z'(t) \neq 0$.) □

In fact, we will use these results "backwards": Given a holomorphic function f defined on a connected domain U so that the integral of f along any closed curve is 0, define a function F by picking a basepoint z_0 and defining

$$F(z) = \int_{C_z} f(z)\, dz$$

for any curve C_z running from z_0 to z. The condition that $\oint_C f(z)\, dz = 0$ means that this definition is independent of the choice of C_z: if C_1 and C_2 are two paths from z_0 to z, let $C_1 - C_2$ be the path C_1 followed by the reverse of C_2. Then $C_1 - C_2$ is closed, so

$$0 = \oint_{C_1 - C_2} f(z)\, dz = \int_{C_1} f(z)\, dz - \int_{C_2} f(z)\, dz.$$

With these definitions, it is not too hard to then show that $F'(z) = f(z)$.

We will prove that

$$f \text{ holomorphic} \Leftarrow \oint_C f(z)\, dz = 0$$

or, better,

$$f \text{ differentiable} \Leftarrow f \text{ integrable}.$$

Actually, the implication goes both ways.

Exercise 5.5. Prove the other direction of the implication, based on the computation we did last time.

5.2. Loop integral of holomorphic functions.

Theorem 5.6. *Let f be a function that is holomorphic on an open set including a neighborhood of a closed, axis-aligned rectangle R. Let ∂R be the path counterclockwise around the boundary of R. Then*

$$\oint_{\partial R} f(z)\, dz = 0.$$

Before we prove this theorem, let's give an overview of the proof. Consider chopping up the rectangle with N equally-spaced cuts both vertically and horizontally into N^2 little rectangles R_i. Then

$$\oint_{\partial R} f(z)\, dz = \sum_i \oint_{\partial R_1} f(z)\, dz.$$

Reason: For any internal segment in this division, the integral over that segment occurs twice in the sum on the right, once from each neighboring rectangle. These two integrals cancel each other since they are in opposite directions.

Now, our computation from the end of last time can be summarized by saying that, for a small rectangle r around z and any function f,

$$\oint_{\partial r} f(z)\, dz = \text{Area}(r) \cdot (\text{failure of CR equations for } f \text{ at } z) + (\text{lower terms}).$$

In particular, if f satisfies the Cauchy-Riemann equations at z, we expect the integral to go to zero faster than the area as the rectangle shrinks.

But we chopped our big rectangle R into N^2 rectangles, each of area $\text{Area}(R)/N^2$. If the integral around each little rectangle gets smaller than its area, the whole thing must tend to zero.

Let's formalize this.

Lemma 5.7. *For $f : \mathbb{C} \to \mathbb{C}$ any continuous function and C a piecewise smooth curve,*

$$\left| \int_C f(z)\, dz \right| \leq \left(\max_{z \in C} |f(z)| \right) \cdot (\textit{length of } C).$$

This is the M–L lemma, for "maximum–length". We will use it frequently!

Proof. In general, for real integrals,

$$\left| \int_a^b g(t)h(t) \right| dt \leq \left(\max_{t \in [a,b]} |g(t)| \right) \cdot \int_a^b |h(t)|\, dt.$$

Apply this to the line integral to find

$$\left| \int_C f(z)\, dz \right| \leq \left(\max_{z \in C} |f(z)| \right) \cdot \int_a^b |z'(t)|\, dt.$$

The last integral can be taken as the definition of the length of C. □

Lemma 5.8. *If f is differentiable at z_0 and R_i is a sequence of rectangles containing z_0 with side length going to 0, then*

$$\lim_{i\to\infty} \frac{\oint_{\partial R_i} f(z)\,dz}{\operatorname{Area}(R_i)} = 0.$$

Proof. Use $f(z) = f(z_0) + f'(z)\cdot(z - z_0) + \epsilon(z)\cdot(z - z_0)$. First two terms give 0, last term is estimated by M–L Lemma. □

Proof of Theorem 5.6. Chop up into 4^n rectangles. At least one must have integral at least as large as area. Use compactness. □

5.3. Entire functions are integrable.

Theorem 5.9. *An entire function f has an anti-derivative F.*

Proof. Define $F(z)$ by integral along any rectilinear path from 0 to z. Independent of path by Rectangle Theorem. To show: $F(z)$ satisfies the Cauchy-Riemann equations, derivative is f. But this is easy by picking the paths carefully. □

Corollary 5.10. *For any closed curve C contained in a rectangle R and a function f that is holomorphic on R, $\oint_C f\,dz = 0$.*

Proof. This follows from Theorem 5.9 and Proposition 5.1. (We need to extend Theorem 5.9 slightly to allow functions defined only inside a rectangle. □

5.4. Green's Theorem and another proof of the Rectangle Theorem.

There is another point of view on the Rectangle Theorem, that avoids some of the technicalities of the earlier proof. This also provides some insight into what the line integral means.[1]

First we need some definitions. Instead of integrating with respect to "dt" or "dz" as we have earlier, we can integrate with respect to "dx".

Definition 5.11. For $f(x, y)$ and $g(x, y)$ real- or complex-valued function and $C = ([a, b], z)$ a piecewise-smooth curve as before, define

$$\int_C f(x,y)\,dx = \int_a^b f(x,y)x'(t)\,dt$$

$$\int_C g(x,y)\,dy = \int_a^b f(x,y)y'(t)\,dt$$

$$\int_C (f(x,y)\,dx + g(x,y)\,dy) = \int_C f(x,y)\,dx + \int_C g(x,y)\,dy.$$

(Here $z(t) = x(t) + iy(t)$.)

We can repeat the proof of Propositions 4.9 and 4.10 in this context to show that these integrals are invariant under smooth reparametrization and are negated when the curve is reversed.

You can think of a combination $f(x, y)\,dx + g(x, y)\,dy$ as a vector field, with a vector (f, g) at each point in the plane. See Figure 1 for a few examples.[2]

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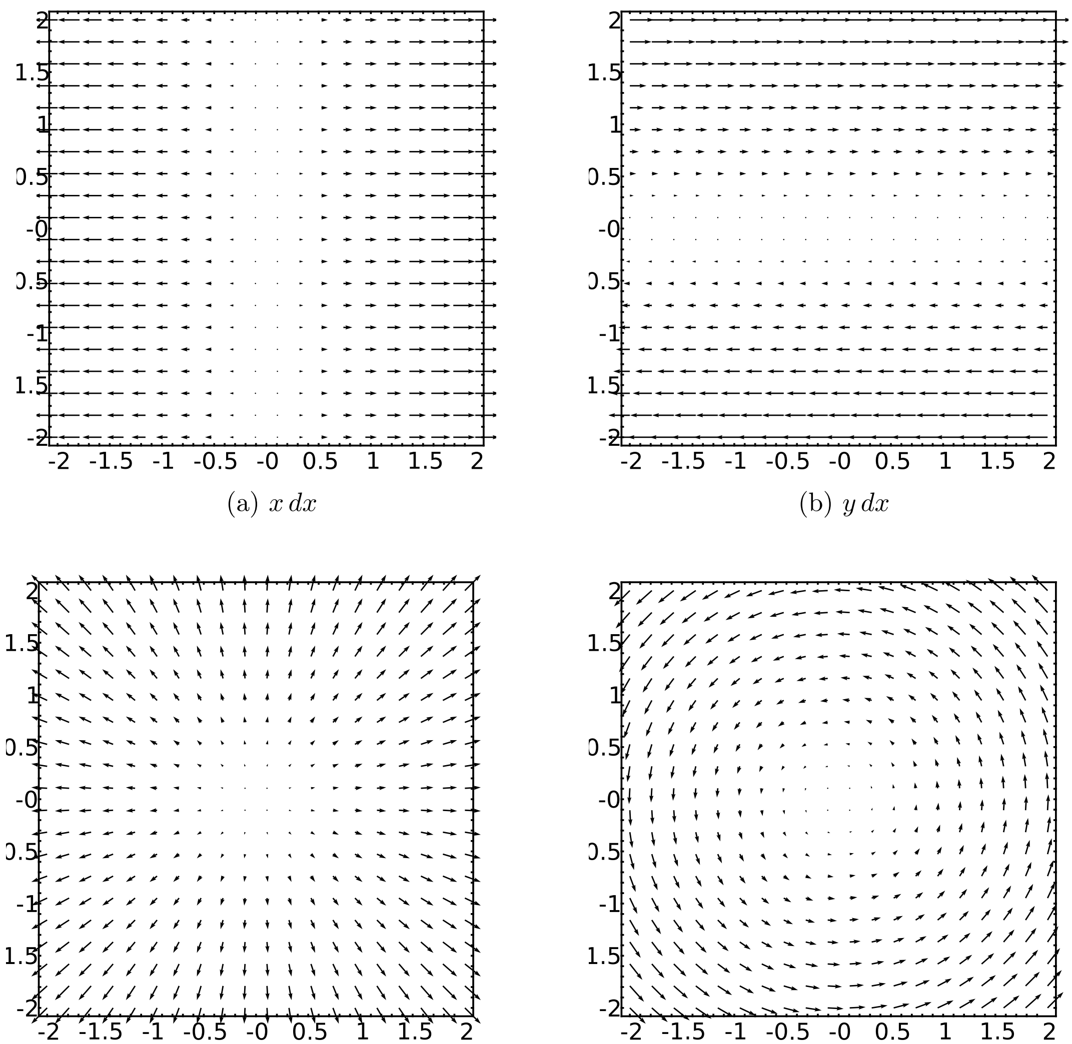
[1]Thanks to Rick Mardiat for noticing this connection.

[2]Strictly speaking, these are *covector* or *dual vector* fields. The distinction is not important at the moment.



(a) $x\,dx$ (b) $y\,dx$

(c) $x\,dx + y\,dy$ (d) $-y\,dx + x\,dy$

Figure 1: Some sample vector fields

The integral $\int_C (f\,dx + g\,dy)$ has a physical interpretation: you can think of it as integrating the total amount of "tailwind" felt as you travel along the curve C. For instance, it is zero if the wind is always perpendicular to your path, as when travel vertically in a vector field of the form $f(x, y)\,dx$. (More precisely, it is the total amount of work done by a wind with the given pattern on you as you travel along C.)

Theorem 5.12 (Green's Theorem, special case)**.** *For R a rectangle in $\mathbb{R}^2$, ∂R the counterclockwise path around its boundary, and $f\,dx + g\,dy$ a vector field so that f and g have continuous partial derivatives,*

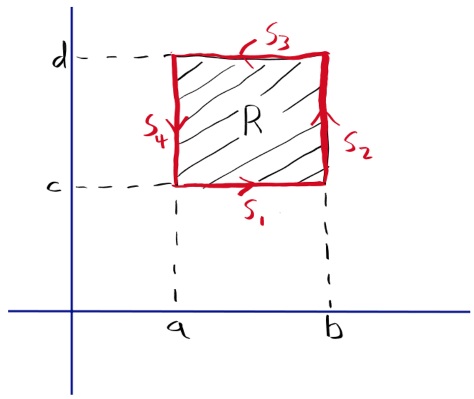
$$\oint_{\partial R} f\,dx + g\,dy = \int_R (g_x - f_y)\,dA.$$

(The integral $\int_R dA$ is the integral with respect to the area on R.)

More generally, Green's theorem says that the same thing is true for an arbitrary simple closed curve and its interior.

Exercise 5.13. Theorem 5.12 says that the integral around rectangles of some of the sample vector fields in Figure 1 is equal to 0. Which ones?

Proof sketch. Suppose that the rectangle R covers the interval $[a, b]$ along the x-axis and $[c, d]$ along the y-axis, and call the four sides S_1, S_2, S_3, and S_4.



It's enough to consider $f\,dx$ and $g\,dy$ separately. Let's start with $g\,dy$, starting with the right hand side. The integral $\int g_x\,dA$ can be written as a double integral, first integrating with respect to x and then with respect to y. Then the integral with respect to x is easy, as we are integrating a derivative:

$$\int_R g_x\,dA = \int_c^d \left(\int_a^b g_x\,dx\right) dy$$
$$= \int_c^d (g(b,y) - g(a,y))\,dy.$$

But $\int_c^d g(b,y)\,dy = \int_{S_2} g(x,y)\,dy$, by definition of the integral, as S_2 is a vertical interval from (b,c) to (b,d). For similar reasons, $\int_c^d g(a,y)\,dy = -\int_{S_4} g(x,y)\,dy$. (We get an extra negative since S_4 is traversed in the downwards direction.) Thus

$$\int_R g_x\,dA = \left(\int_{S_2} + \int_{S_4}\right) g(x,y)\,dy = \int_{\partial R} g(x,y)\,dy,$$

since the integrals on the remaining two sides vanish. (The vector field is perpendicular to the path.)

We similarly find

$$\int_R f_y\,dA = \int_a^b \left(\int_c^d f_y\,dy\right) dx$$
$$= \int_a^b (f(x,d) - f(x,c))\,dx$$
$$= \left(-\int_{S_3} - \int_{S_1}\right) f(x,y)\,dx = -\int_{\partial R} f(x,y)\,dx.$$

Combining these gives the stated result. □

The main point that is not rigorous in this proof is the definition of the integral $\int_R dA$ and its basic properties. In particular, Fubini's theorem tells us that in appropriate circumstances we can do the integral first with respect to x and then with respect to y, or vice versa.

Definition 5.14. The *curl* of a vector field $f(x,y)\,dx + g(x,y)\,dy$ is the combination appearing in Green's theorem:

$$\operatorname{curl}(f\,dx + g\,dy)(x,y) = g_x(x,y) - f_x(x,y).$$

It is a function on $\mathbb{R}^2$.

We want to do a similar thing, but with *complex-valued* vector fields, where the functions f and g in $f\,dx + g\,dy$ can take complex values. This is a little confusing, since we were already getting two-dimensional pictures when f and g were real-valued, and now we're adding some extra dimensions. I recommend thinking about the real and imaginary parts separately; each one gives an ordinary vector field.

One basic building block is dz, defined to be $dx + i\,dy$. (Think about differentiating $z = x + iy$.) For instance, consider $\overline{z}\,dz$; explicitly, this is

$$\begin{aligned}\overline{z}\,dz &= (x - iy)(dx + idy)\\ &= (x\,dx + y\,dy) + i(-y\,dx + x\,dy).\end{aligned}$$

Thus the real and imaginary parts of this vector field are the two vector fields in Figure 1c and Figure 1d, respectively.

Exercise 5.15. Show that in general, for an arbitrary complex-valued function $f(z)$ (not necessarily holomorphic), the real and imaginary parts of $f(z)\,dz$ differ by a 90° rotation.

We can integrate a complex-valued vector field along a curve; this just means that we integrate the real and imaginary parts separately. For instance,

$$\int_C f\,dz = \int_C f\,dx + i\int_C f\,dy$$

or, splitting up $f = u + iv$ into real and imaginary parts,

$$\begin{aligned}\int_C f\,dz &= \int_C (u + iv)\,dx + i\int_C (u + iv)\,dy\\ &= \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy).\end{aligned}$$

In this last expression we see two ordinary, real-valued vector fields.

Exercise 5.16. Check that this last expression agrees with the earlier definition of $\int_C f\,dz$.

If the function f is holomorphic inside R, Theorem 5.12 then tells us that

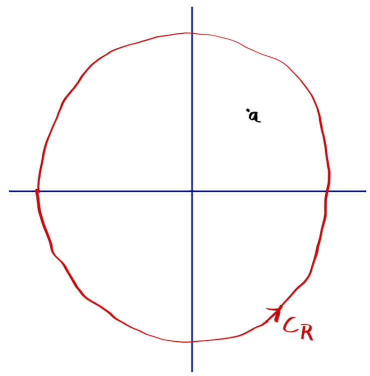
$$\oint_{\partial R} f\,dz = \oint_{\partial R} (f\,dx + if\,dy) = \int_R (if_x - f_y)\,dA = 0.$$

At the last step we used the Cauchy-Riemann equations: $f_y = if_x$. This then completes the alternate proof sketch for Theorem 5.6.

5.5. **Fourier transforms and complex power series.** Our next major goal is to prove *Cauchy's integral theorem*, which for simplicity we state for entire functions.

Theorem 5.17. *If f is an entire function, $a \in \mathbb{C}$ is any point, and C_R is a circle around* 0 *of radius R with $R > |a|$, then*

$$\oint_{C_R} \frac{f(z)}{z-a}\, dz = 2\pi i f(a).$$



In particular, $f(a)$ is determined by the values of f on the circle C_R, which need not come anywhere near the point a! You can think of the integral in Theorem 5.17 as expressing $f(a)$ as a kind of weighted average of the values $f(z)$ for z on the circle, with weights given by $1/(z-a)$. In particular, the weight is larger for points on the circle closer to a.

Theorem 5.17 says that an entire function is determined by its values on sufficiently large circle. You might suspect that this can go backwards as well: that you can find a function on the disk with any given behavior on the circle. More precisely, pick a function $f(e^{i\theta})$ as a function of θ and define

$$g(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\, dz.$$

That gives a well-defined function of $g(a)$, and we will see shortly that g is holomorphic in the disk. But this turns out not to be true; the reason is that, for z_0 on the boundary of the circle, we may have

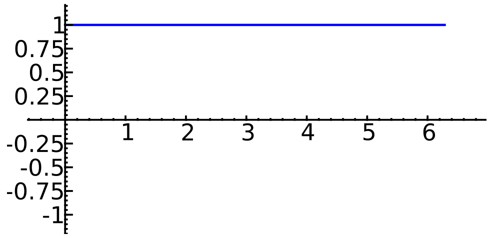
$$\lim_{a \to z_0} g(a) \neq f(z_0).$$

Exercise 5.18. Carry out this procedure for $f(e^{i\theta}) = \cos\theta$. What is the resulting function $g(z)$?

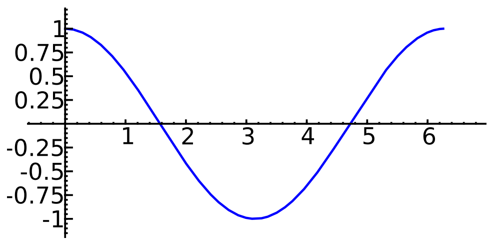
To see this better, we need the basics of Fourier analysis. We will state one basic result without proof.

Theorem 5.19. *For $f : \mathbb{R} \to \mathbb{C}$ a continuous function with period 2π (i.e., $f(t+2\pi) = f(t)$, there is are unique constants a_k giving a uniformly convergent series*

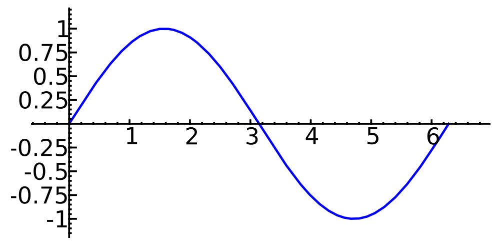
$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{ik\theta}$$



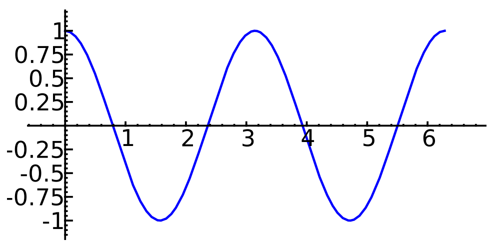
(a) $\cos(0t)$



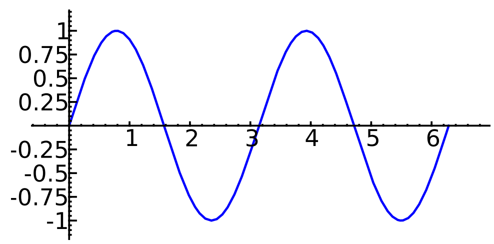
(b) $\cos(1t)$



(c) $\sin(1t)$



(d) $\cos(2t)$



(e) $\sin(2t)$

Figure 2: The first few basic periodic waves

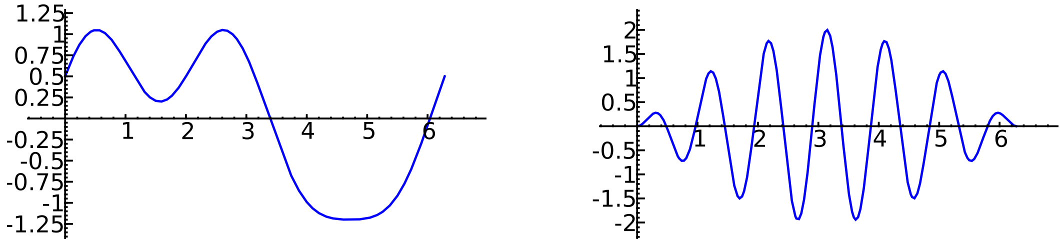
or, alternatively, constant c_k and s_k yielding

$$f(t) = \sum_{k=0}^{\infty} c_k \cos(k\theta) + \sum_{k=1}^{\infty} s_k \sin(k\theta).$$

The intuition here is that all of the terms on the right are periodic with period 2π, and in some sense they give a basis for the periodic functions: *every* periodic function can be written as a linear combination of the basic waves. (This is not strictly a basis in the linear algebra since, since we are allowing infinite sums and need to worry about convergence.) See Figure 2 for the first few basic waves in the cos and sin representation of the Fourier series. (Note that $\sin(0t) \equiv 0$, so there is no $\sin(0t)$.)

By superimposing these waves, in with fairly small combinations, you can get quite intricate combinations; see Figure 3 for some examples. Theorem 5.19 asserts that this is possible for any continuous, periodic function, either in terms of e^{ikx} or in terms of $\cos(kx)$ and $\sin(kx)$. (In fact this is possible for many more functions.)

Exercise 5.20. Use the expressions for $\cos(x)$ and $\sin(x)$ in terms of $e^{\pm ix}$ to express the constants s_k and c_k in terms of a_k in a Fourier series.



(a) $\sin(t) + 0.5 \cdot \cos(2t) + 0.3 \cdot \sin(3t)$ (b) $\cos(6t) - \cos(7t)$

Figure 3: Some superpositions of waves

If we assume that a Fourier expansion as in Theorem 5.19 exists, it is easy to find the constants:

Lemma 5.21. *If $f(x)$ has a uniformly convergent Fourier series $\sum_{k=-\infty}^{\infty} a_k e^{ikx}$, then*

$$a_k = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ikx}\, dx.$$

Proof. Expand out the sum:

$$\begin{aligned}\frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ikx}\, dx &= \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{l=-\infty}^{\infty} a_l e^{ilx} \right) e^{-ikx}\, dx \\ &= \frac{1}{2\pi} \sum_{l=-\infty}^{\infty} \left(\int_0^{2\pi} a_l e^{i(l-k)x}\, dx \right).\end{aligned}$$

(We can interchange the order of summation and integration since the Fourier series converges uniformly by hypothesis.) But

$$\int_0^{2\pi} e^{i(l-k)x}\, dx = \begin{cases} 0 & l \neq k \\ 2\pi & l = k. \end{cases}$$

(If $l \neq k$, then $e^{i(l-k)x}$ rotates some even number of times around 0 as x goes from 0 to 2π; this rotating vector has average 0.) This then gives the stated result. □

Now, how does this relate to complex analysis? Suppose that we have a holomorphic function $f(z)$ that is given by a power series with radius of convergence $R > 1$:

$$f(z) = \sum_{k=0}^{\infty} a_k z^k.$$

Now look at the values of f at a point $e^{i\theta}$ on the unit circle. We find

$$f(e^{i\theta}) = \sum_{k=0}^{\infty} a_k e^{ik\theta}. \tag{5.22}$$

This looks very similar to the Fourier expansion in Theorem 5.19. The one difference is that the range of summation: the terms with k negative do not appear in Equation (5.22). We have therefore proved the following lemma.

Lemma 5.23. *If f is a holomorphic function given by a power series with radius of convergence greater than 1, then all negative Fourier coefficients of $f(e^{i\theta})$ are 0.*

Corollary 5.24. *The function $\cos(\theta)$ cannot appear as the boundary value of any holomorphic function on the unit disk.*

Proof. We have $\cos(\theta) = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$. In particular, $a_{-1} = \frac{1}{2} \neq 0$. □

Exercise 5.25. Use Lemma 5.23 to show that if f has a convergent power series an the unit disk and takes real values on the circle, then f is constant.

6. Local behavior of holomorphic functions at singularities

We now study how holomorphic functions behave near a point, including when there is a singularity at the point.

6.1. **Zeroes of order k.**

First let's look at how holomorphic functions behave when there is no singularity.

Suppose that f is holomorphic in a neighborhood of $z_0 \in \mathbb{C}$. We have seen that f has a power series near z:

$$f(z) = c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \cdots + c_k(z - z_0)^k + \cdots$$

This power series converges on some disk around z_0.

Definition 6.1. The function $f(z)$ has a *zero of order k at z_0* if, in the power series expansion above, the first k terms vanish:

$$c_0 = c_1 = \cdots = c_{k-1} = 0.$$

We say $f(z)$ has a zero of order *exactly* k if in addition $c_k \neq 0$.

Lemma 6.2. *The function $f(z)$ has a zero of order k at z_0 if and only if $g(z) = \frac{f(z)}{(z-z_0)^k}$ is holomorphic near z_0. Furthermore, $f(z)$ has a zero of order exactly k iff $g(0) \neq 0$.*

Proof. $\Leftarrow$: Clear from power series expansion.
$\Rightarrow$: Look at power series for $g(z)$ and multiply. □

This is the main local data of a holomorphic function.

As an application of this point of view, we prove a surprising related result to the extended Liouville Theorem (which says that if a function grows less fast than $|z|^k$ at ∞, then it is a polynomial).

Theorem 6.3. *If $f(z)$ is entire, $\lim_{z\to\infty}|f(z)| = \infty$, then $f(z)$ is a polynomial.*

(Expand out statement.)

Proof. There are a finite number of zeroes $a_1, \ldots, a_k$, counted with multiplicity (explain).

$g(z) = \frac{f(z)}{(z-a_1)\ldots(z-a_k)}$ is holomorphic (by repeated applications of Lemma 6.2) and never zero (by definition of zeroes).

Then $1/g(z)$ is holomorphic and bounded, and so constant. □

6.2. Singularities.

Definition 6.4. An *isolated singularity* of a holomorphic function f is a point $z_0 \in \mathbb{C}$ so that f is defined on a punctured disk $D' = D \setminus \{z_0\}$ around z_0, but not at z_0.

We distinguish three types of isolated singularities.

Definition 6.5. An isolated singularity of f at z_0 is *removable* if there is a holomorphic function $\tilde{f}$ defined on the entire disk around z_0.

It is a *pole* if $f(z) = \frac{p(z)}{q(z)}$ where $p(z_0) \neq 0$ and $q(z_0) = 0$.

It is *essential* if it is neither removable nor a pole.

(Of course, the last definition seems like a cop-out. . . . But we will give other, positive characterizations.)

Example 6.6. The function

$$f(z) = \begin{cases} z^2 & z \neq 2 \\ \text{undefined} & z = 2 \end{cases}$$

has a removable singularity at 2. (This may seem unnatural, but perhaps we were looking at $\frac{z^3-2z^2}{z-2}$.)

Example 6.7. The function

$$f(z) = \frac{z^2-1}{z^2+1}$$

has poles at $z = \pm i$.

Example 6.8. The function $e^{1/z}$ has an essential singularity at 0. (Proof: Exercise, using tools to come.)

Non-example 6.9. The functions $\sqrt{z}$ and $\log(z)$ do not have isolated singularities at 0, as they are not well-defined on the punctured disk.

First, a little closer look at removable singularities and poles.

Proposition 6.10. *If $f(z)$ is continuous on a disk D and holomorphic on the punctured disk $D' = D \setminus \{z_0\}$, then f is holomorphic on D.*

Proof. Estimate as in previous techniques to see that $\oint_{\partial R} f\,dz = 0$ for any rectangle R. (This is only hard for rectangles containing z_0, and for small rectangles we get good bounds.) Use theorem below. □

Theorem 6.11 (Morera)**.** *If f is a function defined on a domain U so that $\oint_R f\,dz = 0$ for every [small] rectangle R in U, then f is holomorphic.*

Proof. Enough to restrict to a disk.

Can find an anti-derivative F in the disk.

The anti-derivative F is differentiable, and so its derivative f is as well. □

Proposition 6.12. *$f(z)$ has a pole at $z_0 \Leftrightarrow f(z) = g(z)/z^k$ for some g with $g(z_0) \neq 0$ and some (unique) $k > 0$.*

Proof. Look at the power series expansion of p, q in definition of a pole. □

Definition 6.13. The number k from Proposition 6.12 is called the *order* of the pole of f at z_0.

In fact, the classification of the singularity can be determined entirely by the behavior of $|f(z)|$ as $z \to z_0$.

Theorem 6.14. *If f has an isolated singularity at z_0 and if $\lim_{z\to z_0} f(z)$*

- *exists and is finite, then the singularity is removable;*
- *exists and is infinite, then the singularity is a pole;*
- *does not exist, then the singularity is essential.*

In fact, in each case we prove a stronger criterion. In what follows, suppose f has an isolated singularity at z_0.

Theorem 6.15 (Riemann's criterion)**.** *The singularity is removable $\Leftrightarrow \lim_{z\to z_0} f(z)(z-z_0) = 0$.*

Proof. $\Rightarrow$: Easy from power series (or just continuity of $\tilde{f}$)

$\Leftarrow$: Let $h(z)$ be $f(z)(z-z_0)$ for $z \neq z_0$, and 0 for $z = z_0$. Then by Proposition 6.10, $h(z)$ is holomorphic, and thus so is $f = \frac{h}{z-z_0}$. □

Corollary 6.16. *If f is bounded near an isolated singularity, the singularity is removable.*

Theorem 6.17. *If there is a positive integer k so that $\lim_{z\to z_0}(z-z_0)^{k+1}f(z) = 0$, then f has a pole of order less than or equal to k at z_0.*

Proof. Similar to above: consider $(z-z_0)^{k+1}f(z)$. This is holomorphic as above. Divide out. □

Perhaps one of the most remarkable results is the following.

Theorem 6.18 (Casorati-Weierstrass)**.** *If f has an essential singularity at z_0, then in any punctured disk D' around z_0 the image of f is dense in $\mathbb{C}$.*

In fact, something even stronger is true:

Theorem 6.19 (Picard)**.** *If f has an essential singularity at z_0, then f assumes every value in $\mathbb{C}$, with at most one exception, infinitely often on the punctured disk around z_0.*

(The "infinitely often" statement follows from the rest, as you can just look at smaller and smaller disks.)

We will not prove Picard's Theorem in this class, although it isn't far from what we will do and would make a good paper topic.

For example, e^z takes every value except 0 on every strip of height 2π in the imaginary direction in $\mathbb{C}$. In $e^{1/z}$, we invert the argument first; these strips become little crescent-like shapes near the origin.

Proof of Casorati-Weierstrass Theorem. Suppose that f is holomorphic on a punctured disk D' around z_0, and suppose that the image of f misses some disk $D_2 = D(R; w)$ in $\mathbb{C}$. We must show that f has a pole or removable singularity at z_0. Let $g(z) = 1/(f(z) - w)$. Then

$|g(z)| < 1/R$ on D'. But then by Corollary 6.16, the singularity of $g(z)$ is removable, so we can extend $g(z)$ to all of D. Therefore $f(z) = w + \frac{1}{g(z)}$ has a pole or removable singularity at z_0, depending whether $g(z_0)$ is 0 or not. □

6.3. Meromorphic functions.

We now focus a little more on the case of pole singularities.

Definition 6.20. A function $f(z)$ is *meromorphic* on a domain U if its singularities are only poles: there is a discrete set of points $\mathbf{z} = \{z_i\}$ so that f is holomorphic on $U \setminus \mathbf{z}$ and f has a pole at each z_i.

Examples include *rational functions*, functions of the form $\frac{p(z)}{q(z)}$ where p and q are polynomials, and functions like $\frac{sin(z)}{\cos(z)}$.

We can reinterpret meromorphic functions in terms of continuity.

Definition 6.21. The *extended complex plane* $\hat{\mathbb{C}}$ is the set $\mathbb{C} \cup \infty$. We give it some more structure (topological space) by declaring that a sequence (z_i) converges to ∞ if it converges to ∞ in the ordinary sense: for all R, there is an N so that for $n > N$, $|z_i| > R$. (The "small" neighborhoods around ∞ are the large regions $|z| > R$.)

Lemma 6.22. *A function $f(z)$ is meromorphic on U iff it extends to a continuous function $f : U \to \hat{\mathbb{C}}$ and is holomorphic where it is not infinity.*

There is no need for additional condition (like the Cauchy-Riemann equations) when $f(z) = \infty$; this is related to Proposition 6.10.

A more visual interpretation is the *Riemann sphere*. One way to add the point ∞ to the plane is to wrap the plane around a sphere, covering all but one point; that missing point is ∞. There is a beautiful, explicit way to do this, called *stereographic projection*. One virtue of this map is that it is *conformal*: a little circle on the sphere gets transformed to a little circle in the plane. (Little picture for argument)

Exercise 6.23. What is the map $1/z$ considered as a map of the Riemann sphere to itself?

Can talk about *singularity at infinity*

- bounded $\Leftrightarrow$ removable (e.g., constants)
- growing $\Leftrightarrow$ pole (e.g., polynomials)
- no limit $\Leftrightarrow$ singularity

6.4. **Interlude: Möbius transformations.**

Definition 6.24. A *Möbius transformation* is a meromorphic function on $\mathbb{C}$ that gives an invertible map from the extended complex plane $\hat{\mathbb{C}}$ to itself.

A Möbius transformation $f(z)$ has at most one pole (otherwise it would not be invertible), and approaches a definite limit at infinity. Thus it can be written as a ratio of two polynomials:

$$f(z) = \frac{P(z)}{Q(z)}$$

where $P(z)$ and $Q(z)$ are both polynomials. In fact, $P(z)$ and $Q(z)$ must both be degree at most 1, or there would be too many zeros (respectively, too many poles).

Proposition 6.25. *A Möbius transformation takes the form* $f(z) = \frac{az+b}{cz+d}$.

(This is often taken as a definition.)
Start with a warm-up:

Lemma 6.26. *An invertible holomorphic map* $f : \mathbb{C} \to \mathbb{C}$ *has the form* $f(z) = az + b$.

Exercise 6.27. Complete the proof of Proposition 6.25, making sure to take care of the case when P or Q has a multiple root.

That's a little abstract. What are some examples?

Translation: $f(z) = z + k$, for some $k \in \mathbb{C}$.
Rotation: $f(z) = e^{i\theta} z$, for $\theta \in \mathbb{R}$.
Dilation: $f(z) = cz$, for $c \in \mathbb{R}$, $c > 0$.
"Inversion": What about something with a denominator? $f(z) = 1/z$.

All of these have interpretations in terms of projecting onto the Riemann sphere, moving the sphere around, and projecting back.

One note: the last operation is not quite inversion, despite what the movie said.

Definition 6.28. The map of *inversion* in a circle centered at z_0 with radius R takes any other point $z \in \mathbb{C}$, at distance r from z_0 to the point on the same ray from z_0 at distance R^2/r from z_0. It takes the inside of the circle to the outside, and vice versa.

Proposition 6.29. *Inversion takes circles and straight lines to circles and straight lines, and preserves angles (reversing their sense).*

Inversion is not conformal! In terms of Riemann sphere, inversion is reflection in the horizontal plane; in $\mathbb{C}$, inversion is $z \mapsto \frac{1}{\bar{z}}$. (Verify explicitly.)

Exercise 6.30. Every Möbius transform can be written as a composition of the 4 basic types.

What happens if we compose Möbius transforms? Compute.

Note that this is like matrix multiplication. This is strange, since Möbius transforms are definitely not linear!

Define projective space of a vector space, projective line. Rep of projective line as a single number.

7. Homotopy and deforming complex integrals

We have already seen that under many circumstances the complex integral of an analytic function along a path does not depend on the specific path taken, but only on the endpoints of the path, and correspondingly the integral around a closed curve is 0. Intuitively, the complex integrals along two different paths are equal if we can deform one path to the other smoothly, without getting caught up on singularities where the function is not defined. Let us formalise these intuitions.

7.1. **Deforming a curve.**

Intuitively, a "deformation" of a curve in a region R should be a continuous path in the space of paths: if I is the unit interval $[0,1]$ and $I \rightsquigarrow \mathbb{C}$ is the space of piecewise-smooth paths from $z \in R$ to $w \in R$, then we are interested in continuous maps

$$I \to (I \rightsquigarrow R)$$

from the interval, to the space of paths. To make this precise, we would need to put a topology on the space of piecewise-smooth paths, which would take us too far afield. Instead, we will use a general fact: the set of functions from A to (functions from B to C),

$$A \to (B \to C),$$

is isomorphic to the set of functions from $A \times B$ to C,

$$A \times B \to C.$$

We can therefore define deformations in the space of paths in terms of maps from the square $I \times I$ to R. The formal name for this type of deformation is a *homotopy.*

Definition 7.1. A homotopy between two piecewise smooth paths Γ_0 and Γ_1, each starting at $z \in R$ and ending at $w \in R$, is a map

$$\Phi : I \times I \to R$$

so that:

- Φ is continuous;
- The map $s \mapsto \Phi(0, s)$ agrees with the path Γ_0;
- The map $s \mapsto \Phi(1, s)$ agrees with the path Γ_1; and
- For each fixed $t \in I$, the map $s \mapsto \Phi(t, s)$ is a piecewise smooth path Γ_t from z to w.

If there is a homotopy between Γ_0 and Γ_1, they are said to be *homotopic within R.*

That is, on each horizontal line through the square, we see a path from z to w; as we move the line from top to bottom in the square, the path is deformed from Γ_0 to Γ_1.

(It is a little asymmetric that we require the paths, the horizontal lines in the square, to be piecewise smooth, but only require the map to be continuous in the vertical direction. We could require the map Φ itself to be piecewise smooth in some sense, but as we will see, that will not be necessary.)

The key fact for our purposes is that if Γ_0 and Γ_1 are homotopic, then the complex integral along Γ_0 equals that along Γ_1.

Proposition 7.2. *If $f(z)$ is an analytic function defined on a region R, and Γ_0 and Γ_1 are homotopic within R, then*

$$\int_{\Gamma_0} f(z)\,dz = \int_{\Gamma_1} f(z)\,dz.$$

The proof of Proposition 7.2 will rely on covering the square defining the homotopy by little rectangles, so that the image of each rectangle is contained inside a rectangle; since we have already proved that complex integrals do not depend on the path within a rectangle, we will be able to succesively push the paths across the rectangles. But first, let's look at a somewhat easier problem, one dimension down.

Proposition 7.3. *If Γ is a piecewise-smooth path from z to w within a region R, then there is a piecewise-linear path Γ' (with the same endpoints as Γ) so that, for all analytic functions $f(z)$ defined on R,*

$$\int_{\Gamma} f(z)\,dz = \int_{\Gamma'} f(z)\,dz.$$

(A *piecewise-linear* path is a concatenation of straight line segments.)

Proof. Since R is a region, each point $z(t)$ on Γ is contained in a disk of some positive radius $r(t)$ which is contained inside R. Since the map $t \mapsto z(t)$ defining Γ is a continuous map, for each t there is some interval $I(t)$ centered on t so that $I(t)$ is mapped inside the disk of radius $r(t)$. These intervals give a covering of the interval defining Γ, which is compact, so there is a finite sub-covering. Order the intervals

$$0 \in I_0, I_1, \ldots, I_n \ni 1$$

so that $I_{k-1} \cap I_k \neq \emptyset$.

Pick a point t_k inside each $I_{k-1} \cap I_k$; set $t_0 = 0$ and $t_{n+1} = 1$. Set $z_k = z(t_k)$ (so that $z_0 = z$ and $z_{n+1} = w$). Then t_k and t_{k+1} are both contained inside I_k, and so by the choice of I_k, we see that z_k, z_{k+1}, and the portion of Γ between them are all contained inside a single disk. Since the disk is convex, the straight line segment between them is also contained inside the disk. But we have already seen that inside a disk (on which $f(z)$ is defined) the complex integral between the points z_k and z_{k+1} does not depend on the path we take, so we can replace the portion of the piecewise-smooth path Γ between z_k and z_{k+1} by this straight line segment. Repeating this for each sub-interval, we replace Γ by a piecewise linear path Γ' which runs through the points

$$z_0 — z_1 — \cdots — z_{n-1} — z_n$$

in sequence. □

As an aside, we can use the ideas in Proposition 7.3 to *define* the complex integral along an arbitrary continuous curve, not necessarily piecewise-smooth, even though the naïve Riemann integral does not necessarily converge along such a curve.

We are now ready to prove Proposition 7.2.

Proof. For a fixed t, consider the curve Γ_t. As in Proposition 7.3, we can cover the interval I with an overlapping sequence of intervals I_k so that each I_k is mapped by $\Phi(t, \cdot)$ into a disk $D_k \subset R$. Let $z_k \in D_k \cap D_{k+1}$ be a point on Γ_t.

Note that, since the map Φ is a continuous function, there will be some open interval $(t - \varepsilon, t + \varepsilon)$ so that for each $t' \in (t - \varepsilon, t + \varepsilon)$, $\Gamma_{t'}$ satisfies the same conditions as Γ_t: each

I_k is mapped by $\Phi(t', \cdot)$ into the same disk D_k. Let $z'_k \in D_k \cap D_{k+1}$ be the corresponding points on $\Gamma_{t'}$.

Now we can use the invariance of complex integrals within a disk to show that the integral along Γ_t equals the integral along $\Gamma_{t'}$: the integral along Γ_t is, as before, equal to the integral along the piecewise linear path

$$z = z_0 — z_1 — z_2 — \cdots — z_{n-1} — z_n = w.$$

Since $z_0 = z'_0$, z_1, and z'_1 are all in the same disk D_1, this is also equal to the integral along the piecewise linear path

$$z = z'_0 — z'_1 — z_1 — z_2 — \cdots — z_{n-1} — z_n = w.$$

Since z_1, z'_1, z_2, and z'_2 are all in the same disk D_2, this is equal to the integral along the path

$$z = z'_0 — z'_1 — z'_2 — z_2 — \cdots — z_{n-1} — z_n = w.$$

Continuing in this fashion, we eventually show that the integral along Γ_t is the same as the integral along

$$z = z'_0 — z'_1 — z'_2 — \cdots — z'_{n-1} — z'_n = w$$

which is also equal to the integral along $\Gamma_{t'}$.

Therefore the integral along Γ_t is equal to the integral along $\Gamma_{t'}$ for all t' in a neighborhood of t. But since this was true for all t and the interval is connected, the integral along all of the different Γ_t must be equal; in particular,

$$\int_{\Gamma_0} f(z)\, dz = \int_{\Gamma_1} f(z)\, dz.$$

□

7.2. **Simple connectivity.** Of particular interest are those regions R so that the complex integral between two points is completely independent of the path. Based on the above discussion, there is a clear condition for this to be true.

Definition 7.4. A region R is said to be *simply connected* if, for every pair of points z, w in R and every pair of piecewise-smooth paths Γ_0, Γ_1 between z and w, the paths Γ_0 and Γ_1 are homotopic.

The intuition behind the name is that an open subset of $\mathbb{C}$ is *connected* if there is a path between any pair of points z, w; it is *simply connected* if there is essentially only one way to connect z and w (up to homotopy).

A complete discussion of simple connectivity is outside the scope of this course. Let us content ourselves with a few observations. The first is the reason we defined simple connectivity:

Proposition 7.5. *In a simply connected region R, the integral $\int_\Gamma f(z)\, dz$ of an analytic function along a path Γ depends only on the endpoints of Γ. In particular, if Γ is a closed curve, the integral along Γ is 0.*

The proof is immediate from the definitions.

With a little bit of thought, you should be able to convince yourself that the number of ways to connect a pair of points z and w is independent of z and w; that is, in Definition 7.4 we can pick the pair of points z and w rather than quantifying over all pairs. A standard choice is to pick $z = w$. In this case, there is a canonical path from z to z: the constant path. (Strictly speaking, this is not a piecewise-smooth path according to our earlier definition, since the derivative is not 0. We need to modify the notion of homotopy slightly: we need to drop the requirement that $\Phi(t, \cdot)$ is piecewise-smooth when $t = 1$. In fact, we could equally well drop this requirement for all t and get an equivalent definition of simple-connectivity, since we have already seen how to "approximate" an arbitrary continuous curve by a piecewise-linear curve.)

Definition 7.6. A region R is said to be simply-connected if, for some point z in R, every path from z to itself is homotopic to the constant path. In other words, R is simply-connected if, for every Γ from z to z, there is a map

$$\Phi : I \times I \to R$$

so that

- Φ is continuous;
- $\Phi(0, \cdot)$ coincides with the path Γ;
- $\Phi(t, 0) = \Phi(t, 1) = z$ (in other words, $\Phi(t, \cdot)$ is a path from z to itself);
- $\Phi(1, s) = z$; and
- (optional) $\Phi(t, \cdot)$ is piecewise-smooth for each t.

The definition is true for the same regions R whether or not we include the last condition.

(The difference between Definition 7.6 and Definition 7.4 is precisely the difference between showing that the integral along a path is independent of the path, and showing that the integral along any closed curve is 0.)

The advantage of this definition over the earlier one is that it is easier to show that certain regions are simply-connected.

Proposition 7.7. *Any convex region R is simply connected.*

Proof. Take any path $z(s)$ from z_0 to itself, and define the homotopy Φ by

$$\Phi(t, s) = (1 - t)z(s) + tz_0$$

That is, the path $\Phi(t, \cdot)$ interpolates linearly between the path $z(s)$ at $t = 0$ and the constant path at z_0 at $t = 1$. Since R is convex, the line between each point $z(s)$ and z_0 is contained inside of R, so Φ is a continuous map from $I \times I$ to R. The remaining properties of Φ are immediate. □

This proof applies to a slightly more general class of regions: those which are *star-shaped*. A region is said to be star-shaped if there is a point $z_0 \in R$ so that, for every $w \in R$, the straight line segment between z_0 and w is contained inside R. But many more regions are simply-connected than just the star-shaped ones.

In the Bak and Newman text there is a completely different definition of simply connected for regions in $\mathbb{C}$. There definition turns out to be equivalent to the one above, although that is not obvious and the definition above generalises to more general spaces, while the one in Bak and Newman does not.

Finally, let me mention the following result, the converse to Proposition 7.5:

Theorem 7.8. *Let R be a connected region in $\mathbb{C}$. If, for every analytic function $f(z)$ defined on R and every closed curve Γ in R, $\int_\Gamma f(z)\,dz = 0$, then R is simply connected.*

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