MATH W4065: HONORS COMPLEX VARIABLES
FINAL PAPER

For the paper in this class, you are to learn a new topic, related to this course in complex analysis, and write an exposition with your classmates as a target audience. The paper should be no longer than 10 pages, and shorter papers are encouraged.

Some guidelines for the papers:

- Cover a topic well. Depending on your mathematical writing style, this could take 5 pages, or it could be 10. (You can see this from our different textbooks; for instance, Needham is much more verbose than Bak and Newman.) Note that it is not necessarily easier to write a short paper: it takes good judgement to distill a subject to its essence.
- Pick a small, focused topic: the smaller your topic, the more you'll be able to master it fully before writing about it. Failing to do this is by far the most common problem.
- Write something that you would enjoy reading. Try reading your work out loud to see what it sounds like.
- You should aim your paper at your classmates, who will naturally have forgotten some of the definitions from class.
- Make sure you understand what you write about. If you don't quite understand some part of the topic, it's usually better to omit it from your paper.

Time line.

- By Wednesday, November 11: One-page problem statement due, including the topic you will write about, the point of view you will take, and the references you intend to use. Please explain why the topic is interesting, and what your particular take will be.
- Monday, November 23: Complete first version due. Hand in three copies, one for me and one for your classmates; in groups of three, you will read each others papers and provide feedback on them.
- Tuesday–Wednesday, November 24–25: Revised first versions returned with comments in one-on-one sessions.
- Thursday, December 3: Final version due. Hand in the revised version and all draft versions with their comments.

Please note that your grade on the paper is based on *all* components of the assignment, including the problem statement and the first version. The first version is *not* a draft; it should be as good as you can make it by yourself.

Topics. One good source for topics is to take something mentioned in class and explore it further. A few other possible topics are listed below. Several of them are a bit too large in scope; you made need to work to limit the scope a little. These are only suggestions to get you started thinking; you should feel free to come up with your own topic! In any case, please come talk to me for further references or suggestions on your topics.

Date: October 28, 2009.

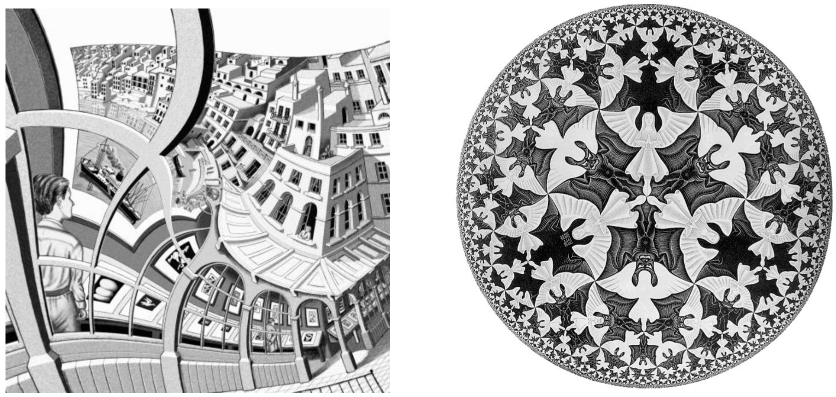

FIGURE 1. Artistic conformal mapping

Conformal maps of the brain: Complex analysis has recently been used to construct conformal maps of the brain for use in medicine. This relies on a theorem, the Uniformization Theorem, which we will not prove in class, but which is related to the Riemann Mapping Theorem, which we will prove. Explain the Uniformization Theorem and either give a proof or the outline of a proof, or explain how to compute such maps in practice.

Artistic conformal mapping: Conformal mappings can also be used to transform paintings with minimal distortion. This was used (implicitly) by Escher in the painting "Print Gallery", as you can see on the left of Figure 1. This is nicely explained at `http://escherdroste.math.leidenuniv.nl/`. See if you can find another example of conformal mappings or complex analysis in art (or create your own work of art!) and explain the mathematics behind it. There are several other works of Escher that would make a good starting point, like the Circle Limit picture mentioned at the beginning of the course.

Harmonic functions and minimal surfaces: We mentioned harmonic functions in class: they are essentially the real parts of conformal functions. They arise in a large number of contexts, including in the construction of minimal surfaces, surfaces in $\mathbb{R}^3$ that (locally) minimize area, like a flexible piece of fabric pulled tight. Explain this connection and give some examples.

Fluid flow over an airplane wing: It turns out that conformal maps and holomorphic functions are closely related to fluid flow in 2 dimensions, for instance the flow of air over an airplane wing. Explain how to compute this in practice.

Orbits in central force field: A particle moving in a central force field with force $-1/r^2$ (i.e., according to Newtonian gravity) traces out an elliptical orbit, with one focus of the ellipse at the origin of the force field.One unusual proof of this fact uses complex analysis: the map $z \mapsto z^2$ takes orbits for the central force field $-r$ (i.e., a simple harmonic oscillator in two dimensions, whose orbits are ellipses with center at the origin) to orbits for the field $-1/r^2$. There is a discussion of this in Needham, Chapter 5.X, and in the reference mention therein. Come to your own understanding of this fact and explain it.

Taylor series and Fourier analysis: As we have seen, the Taylor series of complex analytic functions are intimately connected with Fourier series when we look at the values of the Taylor series on the unit circle. Show how to use this connection to calculate Fourier series for some common functions. Section 2.III.7 (pages 77–79) in Needham is a good starting point, and Stein and Shakarchi has some more material, but you should find your own examples.

Elliptic functions: One of the main subjects in 19th century mathematics was elliptic functions, which can be thought of as generalizations of trigonometric functions; where the trigonometric functions are periodic with period 2π, the elliptic functions have two independent periods $p_1, p_2 \in \mathbb{C}$, and so take the same values in a lattice in the plane. Either explain some of the geometry of elliptic functions and how they relate to complex tori, or explain how to use them to do practical integrals.

Brownian motion and percolation: There are a number of 2-dimensional physical models that have recently been shown to be conformally invariant, including Brownian motion (a.k.a. random walks in the plane) and percolation. Show how knowing that a model is conformally invariant allows you to compute physical quantities in practice. (The actual proofs of conformal invariance are probably too hard, except possibly for Brownian motion.)

Practical conformal mapping: Explain how to construct explicit conformal maps between a variety of different regions. Draw pictures of the results.

Branched coverings and Riemann surfaces: The branches of a multi-valued function can be glued together to construct a *Riemann surface*: a surface that "locally looks like" $\mathbb{C}$. Learn about and explain this connection.

Format. Your papers should be neatly typeset in a 12pt font with 1 inch margins (like this document). Graphics are encouraged, even hand-drawn ones!

For my own papers, I use LaTeX, which I also recommend to you; although the commands take a little getting used to at first, the output it produces in the end is quite good, and much better than any other system I have seen. (However, you may use whatever technology you wish.)

Attached is an article from the Notices of the AMS (March 2009), a short introduction to what TeX, LaTeX and related packages are. In particular it mentions some more user-friendly front-ends. In addition to the ones mentioned there, you might consider LYX (`http://www.lyx.org`) or Texmacs (`http://www.texmacs.org`) if you want a WYSIWYG front end.

Once you have a TeX system installed, *The Not So Short Introduction to LaTeX* is a good all-around introduction to how to produce text. It is available from, for instance, `http://www.ctan.org/tex-archive/info/lshort/english/`, and is also included in many TeX systems.