MATH 1101, Sections 2 and 3
Midterm 1, **Variant B**

Solutions

1. (15 points) Compute the following limits:

   a)(3 points)
   \[ \lim_{x \to 1} \sin(x^2 - 1). \]

   **Solution:** As \( x \to 1 \) we have \( x^2 - 1 \to 0 \), so
   \[ \lim_{x \to 1} \sin(x^2 - 1) = \sin(0) = 0. \]

   b)(3 points)
   \[ \lim_{x \to -3} \frac{x^2 + 3x}{x^2 + 4x + 3}. \]

   **Solution:** We have
   \[
   \lim_{x \to -3} \frac{x^2 + 3x}{x^2 + 4x + 3} = \lim_{x \to -3} \frac{x(x + 3)}{(x + 1)(x + 3)} = \lim_{x \to -3} \frac{x}{x + 1} = \frac{-3}{-2} = \frac{3}{2}.
   \]

   c)(3 points)
   \[ \lim_{x \to +\infty} \frac{\cos(x)}{x^2}. \]

   **Solution:** Since \(-1 \leq \cos(x) \leq 1\), one has
   \[
   -\frac{1}{x^2} \leq \frac{\cos(x)}{x^2} \leq \frac{1}{x^2}.
   \]

   Since \( \lim_{x \to +\infty} \left(-\frac{1}{x^2}\right) = \lim_{x \to +\infty} \frac{1}{x^2} = 0 \), by Squeeze Theorem
   \[
   \lim_{x \to +\infty} \frac{\cos(x)}{x^2} = 0.
   \]

   d)(3 points)
   \[ \lim_{x \to 2} \frac{\sqrt{x + 1}}{\ln(x - 1)}.
   \]
Solution: At $x \to 2$ $\frac{\sqrt{x+1}}{\ln(x-1)} = \sqrt{3}$ and $\ln(x-1) \to \ln(1) = 0$, so $\lim_{x \to 2} \frac{\sqrt{x+1}}{\ln(x-1)} = \infty$.

e)(3 points)

$$\lim_{x \to +\infty} \frac{5x^3 + x^2 - x + 1}{2x^4 - x^2 + x - 1}.$$ 

Solution: Let us divide the top and the bottom by the highest power of $x$:

$$\lim_{x \to +\infty} \frac{5x^3 + x^2 - x + 1}{2x^4 - x^2 + x - 1} = \lim_{x \to +\infty} \frac{5/x + 1/x^2 - 1/x^2 + 1/x^4}{2 - 1/x^2 + 1/x^3 - 1/x^4} = \frac{0}{2} = 0.$$

2.(15 points) Consider the function

$$f(x) = \frac{x\sqrt{x+1}}{(x-4)\sqrt{5-x}}$$

a) (3 points) Determine the domain of this function.

b) (3 points) Write the equations of its vertical asymptotes.

c) (3 points) Write the equations of its horizontal asymptotes.

d) (3 points) Determine all intervals where $f(x) > 0$

e) (3 points) Sketch the graph of $f(x)$.

Solution: The function is defined if $x \neq 4$, $x \neq 5$, $x \geq -1$ and $x \leq 5$, so the domain is $[-1, 4) \cup (4, 5)$. Since $\lim_{x \to 4} f(x) = \lim_{x \to 5} f(x) = \infty$, the graph has vertical asymptotes $x = 4$ and $x = 5$. The function is not defined for large values of $x$, so the graph does not have horizontal asymptotes. Finally, on the domain both square roots are positive; on $(4,5)$ $x > 0$ and $x - 4 > 0$, so $f(x) > 0$; on $(0,4)$ $x > 0$ and $x - 4 < 0$, so $f(x) < 0$; on $(-1,0)$ $x < 0$ and $x - 4 < 0$, so $f(x) > 0$. 

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3. (5 points) Is the following function continuous everywhere?

\[ f(x) = \begin{cases} 
  x^2, & \text{if } |x| \leq 3 \\
  2x + 3, & \text{if } |x| > 3 
\end{cases} \]

**Solution:** We have

\[ \lim_{x \to 3^-} f(x) = \lim_{x \to 3} x^2 = 9, \quad \lim_{x \to 3^+} f(x) = \lim_{x \to 3} 2x + 3 = 9, \]

\[ \lim_{x \to (-3)^-} f(x) = \lim_{x \to -3} 2x + 3 = -3, \quad \lim_{x \to (-3)^+} f(x) = \lim_{x \to -3} x^2 = 9, \]

hence \( f(x) \) is discontinuous at \( x = -3 \).

4. (5 points) Find the limit:

\[ \lim_{x \to +\infty} \frac{ax + \sqrt{x^4 + 1}}{2x^2 - 5} \]

for \( a = -1, 0, 1 \). What happens for general \( a \)?

**Solution:** Let us divide the top and the bottom by \( x \):

\[ \lim_{x \to +\infty} \frac{ax + \sqrt{x^4 + 1}}{2x^2 - 5} = \lim_{x \to +\infty} \frac{a/x + \sqrt{1 + 1/x^4}}{2 - 5/x^2} = \frac{1}{2}. \]

This answer does not depend on \( a \).

5*. (5 points) Use Intermediate Value Theorem to show that the equation \( x^3 - 5x^2 + 6x - 1 = 0 \) has exactly three solutions on the interval \([0, 4]\).
Solution: One can check that \( f(0) = -1, f(1) = 1 - 5 + 6 - 1 = 1, f(2) = 8 - 20 + 12 - 1 = -1, f(3) = 27 - 45 + 18 - 1 = -1, f(4) = 64 - 80 + 24 - 1 = 7. \)

Since \( f(0) < 0 \) and \( f(1) > 0 \), by Intermediate Value Theorem there is a solution on \([0, 1]\). Similarly, there should be solutions on \([1, 2]\) and on \([3, 4]\), so there are at least three distinct solutions. On the other hand, an equation of degree 3 cannot have more than 3 solutions, so there are exactly 3 of them.