Maximal grade for HW10: 40 points

Section 6.1: 22. (10 points) Find the area of the region bounded by the curves \( y = x^3 \) and \( y = x \).

Solution: Let us find the intersection points: \( x^3 = x \), so

\[
x^3 - x = x(x^2 - 1) = 0, \quad x = -1, 0, 1.
\]

On \([-1, 0]\) one has \( x^3 > x \) and on \([0, 1]\) one has \( x^3 < x \), so the area equals

\[
A = \int_{-1}^{0} (x^3 - x) \, dx + \int_{0}^{1} (x - x^3) \, dx = (\frac{x^4}{4} - \frac{x^2}{2})|^0_{-1} + (\frac{x^2}{2} - \frac{x^4}{4})|^1_0 =
\]

\[
0 - (1/4 - 1/2) + (1/2 - 1/4) - 0 = 1/2.
\]

32. (10 points) Evaluate the integral \( \int_{-1}^{1} |3^x - 2^x| \, dx \).

Solution: On \([-1, 0]\) we have \( 2^x > 3^x \) and on \([0, 1]\) we have \( 2^x < 3^x \), so

\[
\int_{-1}^{1} |3^x - 2^x| \, dx = \int_{-1}^{0} (2^x - 3^x) \, dx + \int_{0}^{1} (3^x - 2^x) \, dx =
\]

\[
= (\frac{2^x}{\ln 2} - \frac{3^x}{\ln 3})|^0_{-1} + (\frac{3^x}{\ln 3} - \frac{2^x}{\ln 2})|^1_0 =
\]

\[
= (\frac{1}{\ln 2} - \frac{1}{\ln 3}) - (\frac{1}{2\ln 2} - \frac{1}{3\ln 3}) + (\frac{3}{\ln 3} - \frac{2}{\ln 2}) - (\frac{1}{\ln 3} - \frac{1}{\ln 2}) =
\]

\[
= \frac{4}{3\ln 3} - \frac{1}{2\ln 2}.
\]
Section 6.2: 2. (10 points) Find the volume of the solid obtained by the rotation of the parabola \( y = 1 - x^2 \) about the x-axis.

**Solution:** The section of this solid at a point \( x \) is a circle with radius \( 1 - x^2 \), so the volume equals

\[
V = \int_{-1}^{1} \pi (1 - x^2)^2 \, dx = 2\pi \int_{0}^{1} (1 - x^2)^2 \, dx = 2\pi \int_{0}^{1} (1 - 2x^2 + x^4) \, dx = 2\pi (x - 2x^3/3 + x^5/5)|_{0}^{1} = 2\pi (1 - 2/3 + 1/5) = \frac{16\pi}{15}.
\]

69. (10 points) Find the volume of the solid obtained by rotation of the parabola \( y = R - cx^2, \; -h/2 \leq x \leq h/2 \) about the x-axis and show that it equals

\[
V = \frac{1}{3}\pi h(2R^2 + r^2 - \frac{2}{5}d^2),
\]

where \( d = ch^2/4 \) and \( r = R - d \).

**Solution:** The section of this solid at a point \( x \) is a circle with radius \( R - cx^2 \), so the volume equals

\[
V = \int_{-h/2}^{h/2} \pi (R-cx^2)^2 \, dx = 2\pi \int_{0}^{h/2} (R-cx^2)^2 \, dx = 2\pi \int_{0}^{h/2} (R^2 - 2Rcx^2 + c^2x^4) \, dx = 2\pi \left( R^2 \frac{x}{2} - \frac{2RC}{3} \left( \frac{h}{2} \right)^3 + \frac{c^2}{5} \left( \frac{h}{2} \right)^5 \right) = h\pi \left( R^2 - \frac{2Rc}{3} \left( \frac{h}{2} \right)^2 + \frac{c^2}{5} \left( \frac{h}{2} \right)^4 \right).
\]

Since \( d = ch^2/4 = c(\frac{h}{2})^2 \), we can write

\[
V = h\pi(R^2 - \frac{2Rd}{3} + \frac{d^2}{5}).
\]

On the other hand,

\[
\frac{1}{3}\pi h(2R^2 + r^2 - \frac{2}{5}d^2) = \frac{1}{3}\pi h(2R^2 + (R - d)^2 - \frac{2}{5}d^2) = \frac{1}{3}\pi h(2R^2 + R^2 - 2Rd + d^2 - \frac{2}{5}d^2) = \frac{1}{3}\pi h(3R^2 - 2Rd + \frac{3}{5}d^2) = h\pi(R^2 - \frac{2Rd}{3} + \frac{d^2}{5}).
\]