Maximal grade for HW6: 50 points

Section 3.4 78. (10 points) Find the 1000-th derivative of \( f(x) = xe^{-x} \).

Solution: We have

\[
    f'(x) = e^{-x} - xe^{-x},
    f''(x) = -e^{-x} - e^{-x} + xe^{-x} = -2e^{-x} + xe^{-x},
    f'''(x) = 2e^{-x} + e^{-x} - xe^{-x} = 3e^{-x} - xe^{-x},
\]

etc. From these first three derivatives one can deduce the general pattern: \( f^{(n)}(x) = (-1)^{n+1}(ne^{-x} - xe^{-x}) \), so

\[
    f^{(1000)}(x) = -1000e^{-x} + xe^{-x}.
\]

Section 4.1: 54. (10 points) Find the maximal and the minimum value of the function \( f(x) = \frac{x}{x^2-x+1} \) on the interval \([0, 3]\).

Solution: We have

\[
    f'(x) = \frac{1(x^2 - x + 1) - x(2x - 1)}{(x^2 - x + 1)^2} = \frac{x^2 - x + 1 - 2x^2 + x}{(x^2 - x + 1)^2} = \frac{1 - x^2}{(x^2 - x + 1)^2}.
\]

The critical numbers of this function are \( x = 1 \) and \( x = -1 \), but only \( x = 1 \) belongs to the interval.

Furthermore,

\[
    f(0) = 0, \quad f(1) = 1, \quad f(3) = \frac{3}{9 - 3 + 1} = \frac{3}{7},
\]

so the minimal value equals 0 and the maximal value equals 1.
60. (10 points) Find the maximal and the minimum value of the function \( f(x) = x - \ln x \) on the interval \([\frac{1}{2}, 2]\).

**Solution:** We have \( f'(x) = 1 - \frac{1}{x} \), so the only critical number is \( x = 1 \).

Now
\[
f(1/2) = 1/2 - \ln(1/2) \approx 1.19, \quad f(1) = 1, \quad f(2) = 2 - \ln(2) \approx 1.31.
\]
Therefore the maximal value equals 2 – \( \ln 2 \) and the minimal value equals 1.

Section 4.3: 16. (10 points) For the function \( f(x) = x^2 \ln x \) find the intervals where it is increasing or decreasing, local maximum and minimum values and the intervals of concavity.

**Solution:** The function is defined for \( x > 0 \). Its derivative has the following form:
\[
f'(x) = 2x \ln x + x^2 \cdot \frac{1}{x} = 2x \ln x + x = x(2 \ln x + 1),
\]
so \( f'(x) > 0 \) if \( 2 \ln x + 1 > 0 \), that is,
\[
2 \ln x > -1, \quad \ln x > -1/2, \quad x > e^{-1/2}.
\]
Therefore the function increases on \((e^{-1/2}, +\infty)\) and decreases on \((0, e^{-1/2})\).
The point \( x = e^{-1/2} \) is a local minimum, and the corresponding value equals
\[
f(e^{-1/2}) = e^{-1} \cdot (-1/2) = -\frac{1}{2e}.
\]
Furthermore,
\[
f''(x) = 2 \ln x + 2x \cdot 1x + 1 = 2 \ln x + 2 + 1 = 2 \ln x + 3,
\]
so \( f''(x) > 0 \) if \( 2 \ln x > -3 \), that is, \( \ln x > -3/2 \) or \( x > e^{-3/2} \). Therefore \( f(x) \) is concave up on \((e^{-3/2}, +\infty)\) and concave down on \((0, e^{-3/2})\). The point \( x = e^{-3/2} \) is an inflection point.

48. (10 points) For the function \( f(x) = \frac{e^x}{1-e^x} \):

a) Find the vertical and horizontal asymptotes
b) Find the intervals of increase and decrease
c) Find the local minimum and maximum values
d) Find the intervals of concavity and the inflection points
e) Sketch the graph

**Solution:** The function is defined if \( e^x \neq 1 \), so \( x \neq 0 \). Since \( \lim_{x \to 0} f(x) = \infty \), \( f(x) \) has a vertical asymptote at \( x = 0 \). Furthermore,

\[
\lim_{x \to +\infty} \frac{e^x}{1 - e^x} = \lim_{x \to +\infty} \frac{1}{1/e^x - 1} = \frac{1}{0 - 1} = -1,
\]

so \( f(x) \) has two horizontal asymptotes: \( y = 0 \) at \( x \to -\infty \) and \( y = -1 \) at \( x \to +\infty \).

We have

\[
f'(x) = \frac{e^x(1 - e^x) - e^x(-e^x)}{(1 - e^x)^2} = \frac{e^x}{(1 - e^x)^2}.
\]

Since \( f'(x) > 0 \) for all \( x \), the function is increasing both on \( (-\infty, 0) \) and on \( (0, +\infty) \). There are no minimal or maximal points.

Finally,

\[
f''(x) = \frac{e^x(1 - e^x)^2 - e^x \cdot 2(1 - e^x)(-e^x)}{(1 - e^x)^4} = \frac{e^x(1 - e^x) - e^x \cdot 2(-e^x)}{(1 - e^x)^3} = \frac{e^x - e^{2x} + 2e^{2x}}{(1 - e^x)^3} = \frac{e^x + e^{2x}}{(1 - e^x)^3},
\]

so \( f''(x) > 0 \) when \( e^x < 1 \), that is, \( x < 0 \). Therefore \( f(x) \) is concave up for \( x < 0 \) and concave down for \( x > 0 \), but there are no inflection points.
Figure 1: The graph of $f(x) = \frac{e^x}{1-e^x}$