Maximal grade for HW8: 40 points

Section 5.1: 30. (10 points) Let $A_n$ be the area of a polygon with $n$ equal sides inscribed in a circle with radius $r$. a) Show that

$$A_n = \frac{1}{2}nr^2 \sin \left(\frac{2\pi}{n}\right)$$

b) Show that $\lim_{n \to \infty} A_n = \pi r^2$.

Solution: The polygon can be decomposed into $n$ equal triangles with two sides of length $r$ and the (central) angle between them $\frac{2\pi}{n}$. Therefore the area of each triangle equals $\frac{1}{2}r^2 \sin \left(\frac{2\pi}{n}\right)$, and the area of the whole polygon in $n$ times bigger:

$$A_n = \frac{1}{2}nr^2 \sin \left(\frac{2\pi}{n}\right)$$

To find the limit of $A_n$, let us denote $x = \frac{1}{n}$, then

$$\lim_{n \to \infty} A_n = \lim_{x \to 0} \frac{1}{2}r^2 \sin (2\pi x)$$

By applying the L’Hôpital’s rule, we get

$$\lim_{x \to 0} \frac{1}{2}r^2 \sin (2\pi x) = \lim_{x \to 0} \frac{1}{2}r^2 \frac{2\pi \cos (2\pi x)}{1} = \frac{1}{2}r^2 \cdot 2\pi = \pi r^2.$$ 

Section 5.2: 6. (10 points) The graph of $g$ is shown. Estimate $\int_{-2}^{4} g(x) \, dx$ with six subintervals using (a) right endpoints, (b) left endpoints, (c) midpoints.
Solution: From the graph we can construct the following table of values of $g(x)$:

<table>
<thead>
<tr>
<th>$x$</th>
<th>-2</th>
<th>-1.5</th>
<th>-1</th>
<th>-0.5</th>
<th>0</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
<th>3.5</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g(x)$</td>
<td>0</td>
<td>-1</td>
<td>-1.5</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>1.5</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>-0.5</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Therefore:

\[
S_{left} = 1(0 - 1.5 + 0 + 1.5 + 0.5 - 1) = -0.5,
\]

\[
S_{right} = 1(-1.5 + 0 + 1.5 + 0.5 - 1 + 0.5) = 0,
\]

\[
S_{mid} = 1(-1 - 1 + 1 + 1 + 0 - 0.5) = -0.5.
\]

40. (10 points) $\int_{0}^{10} |x - 5|dx$

Solution: The integral is given by the sum of the areas of two right triangles with sides 5 and 5, so:

\[
\int_{0}^{10} |x - 5|dx = 5 \cdot 5/2 + 5 \cdot 5/2 = 25.
\]

50. (10 points) $\int_{0}^{5} f(x)dx$, where

\[
f(x) = \begin{cases} 
  3 & \text{for } x < 3 \\
  x & \text{for } x \geq 3 
\end{cases}
\]
Solution: The integral is given by the sum of the area of a rectangle with height 3 and width 5, and the area of a triangle with sides 2 and 2:

\[
\int_{0}^{5} f(x)dx = 3 \cdot 5 + 2 \cdot 2/2 = 15 + 2 = 17.
\]