1. For the differential equation \( y' = y^2 - 3y + 2 \):
   a) Find all equilibrium solutions
   b) Sketch the graphs of various types of solutions
   c) Find the limit \( \lim_{t \to +\infty} y(t) \) depending on the initial condition \( y(0) = y_0 \)
   d) Solve the equation explicitly

   **Solutions:** We have \( y' = (y - 1)(y - 2) \). For an equilibrium solution \( y = \text{const} \) we have \( y' = 0 \), so \( y = 1 \) or \( y = 2 \). The derivative \( y' \) is positive for \( y > 2 \) and \( y < 1 \), and negative for \( 1 < x < 2 \), so \( y(x) \) is decreasing for \( 1 < y < 2 \) and increasing for \( y > 2 \) or \( y < 1 \).

   We have
   \[
   \lim_{t \to +\infty} y(t) = \begin{cases} 
   +\infty & \text{if } y_0 > 2, \\
   2 & \text{if } y_0 = 2, \\
   1 & \text{if } y_0 < 2.
   \end{cases}
   \]
To solve the equation explicitly, remark that it is separable:

\[ y' = (y - 1)(y - 2) \Rightarrow \int \frac{dy}{(y - 1)(y - 2)} = \int dt. \]

To compute the integral in the left hand side, we use partial fractions:

\[ \frac{1}{(y - 1)(y - 2)} = \frac{1}{y - 2} - \frac{1}{y - 1}, \int \frac{dy}{(y - 1)(y - 2)} = \ln |y - 2| - \ln |y - 1| = \ln \frac{y - 2}{y - 1}. \]

Therefore

\[ \ln \left| \frac{y - 2}{y - 1} \right| = t + C, \quad \frac{y - 2}{y - 1} = \pm e^{t+C} = Ae^t, \quad y - 2 = (y - 1)Ae^t = Ae^ty - Ae^t, \]

so

\[ y(1 - Ae^t) = 2 - Ae^t \Rightarrow y(t) = \frac{2 - Ae^t}{1 - Ae^t}. \]

2. Solve the differential equations:

a) \( xy' = 2y + 1 \)

**Solution:** This is a separable equation:

\[ \int \frac{dy}{2y + 1} = \int \frac{dx}{x} \Rightarrow \frac{1}{2} \ln |2y + 1| = \ln |x| + C, \quad \ln |2y + 1| = 2\ln |x| + 2C, \]

\[ 2y + 1 = \pm e^{2\ln |x|+2C} = Ax^2 \Rightarrow y(x) = \frac{Ax^2 - 1}{2}. \]

b) \((1 + t^2)y' + 2ty = 1 \)

**Solution:** This is an exact equation:

\[ ((1 + t^2)y') = 1, \quad (1 + t^2)y = t + C, \quad y(t) = \frac{t + C}{t^2 + 1}. \]

3. Find all values of the parameter \( a \) such that the differential equation

\[ (x + y + 1) + (ax + y + 2)y' = 0. \]

is exact. Solve the equation for these values of \( a \).

**Solution:** We have \( M = x + y + 1, N = ax + y + 2 \), so \( M_y = 1, N_x = a \), and the equation is exact if \( a = 1 \).
In this case, we need to find a function $F$ such that

$$F_x = x + y + 1, \quad F_y = x + y + 2.$$  

From the first equation we have $F = \frac{x^2}{2} + xy + x + h(y)$, so $F_y = x + h'(y) = x + y + 2$, and we can pick $h(y) = \frac{y^2}{2} + 2y$. Therefore:

$$F(x, y) = \frac{y^2}{2} + xy + 2y + \frac{x^2}{2} + x = C.$$  

This is a quadratic equation for $y$:

$$\frac{y^2}{2} + (x + 2)y + (\frac{x^2}{2} + x - C) = 0,$$

so

$$y(x) = -x - 2 \pm \sqrt{(x + 2)^2 - 4 \frac{1}{2} (\frac{x^2}{2} + x - C)} = -x - 2 \pm \sqrt{x^2 + 4x + 4 - x^2 - 2x + 2C} = -x - 2 \pm \sqrt{2x + 4 + 2C}.$$  

4. Solve the initial value problems:

a) $y' = \sqrt{y}, \ y(0) = 1$

**Solution:** This is a separable equation:

$$\int \frac{dy}{\sqrt{y}} = \int dx, \ 2\sqrt{y} = x + C.$$  

Since $y(0) = 1$, we have $C = 2$, so

$$2\sqrt{y} = x + 2, \ y(x) = \left(\frac{x}{2} + 1\right)^2.$$  

b) $ty' + 3y = 2t^2 + 1, \ y(1) = 1$

**Solution:** This is a linear equation:

$$y' + \frac{3}{t} y = 2t + \frac{1}{t}.$$  

Let us find an integrating factor:

$$\mu'(t) = \frac{3}{t^2} \mu(t), \ \int \frac{d\mu}{\mu} = \int \frac{3}{t} dt.$$
\[ \ln |\mu| = 3 \ln |t| + C \Rightarrow \mu(t) = \pm e^{3 \ln |t| + C} = At^3. \]

We can pick \( A = 1 \) and get:

\[
\begin{align*}
(t^3 y(t))' &= t^3 y'(t) + 3t^2 y(t) = 2t^4 + t^2, \\
t^3 y(t) &= \frac{2t^5}{5} + \frac{t^3}{3} + C, \\
y(t) &= \frac{2t^2}{5} + \frac{1}{3} + \frac{C}{t^3}.
\end{align*}
\]

Since \( y(1) = 1 \), we have

\[ C = 1 - \frac{2}{5} - \frac{1}{3} = \frac{4}{15}. \]

and

\[ y(t) = \frac{2t^2}{5} + \frac{1}{3} + \frac{4}{15t^3}. \]

5. Find all values of the parameter \( r \) such that all solutions of the differential equation

\[ y' = ry + r^2 t \]

are bounded for \( t > 0 \).

**Solution:** Let us find a general solution first. This is a linear equation: \( y' = ry + r^2 t \), and the integrating factor equals \( \mu = e^{-rt} \), so

\[
\begin{align*}
(e^{-rt} y)' &= r^2 e^{-rt} t, \\
e^{-rt} y(t) &= r^2 \int e^{-rt} t dt = \\
- r \int t d(e^{-rt}) &= -rte^{-rt} + r \int e^{-rt} dt = -rte^{-rt} - e^{-rt} + C, \\
y(t) &= -rt - 1 + Ce^{rt}.
\end{align*}
\]

For \( r > 0 \) and \( C \neq 0 \) the term \( Ce^{rt} \) tends to infinity much faster then \( rt \), so \( y(t) \to \infty \). For \( r < 0 \) we have \( Ce^{rt} \to 0 \), but \( rt \to \infty \), so \( y(t) \to \infty \). Finally, for \( r = 0 \) we get \( y(t) = C - 1 \). Therefore the solutions are bounded if and only if \( r = 0 \).

6*. Show that the solution of the initial value problem

\[ y' = x^2 + y^2 - 1, \quad y(0) = 0.5 \]
has an inflection point. *Hint: sketch the graph of this solution first*

**Solution:** Note that \( x^2 + y^2 - 1 \) is negative inside the unit circle and positive outside the unit circle. Therefore the solution is increasing outside the circle and has both a local minimum and a local maximum on a circle (since it changes from increasing to decreasing and back). At local maximum, we have \( y'' \leq 0 \), and at local minimum \( y'' \geq 0 \), so by Intermediate Value Theorem there is a point inside the circle where \( y'' = 0 \). This is an inflection point.