Section 1.1 23. (20 points) Newton’s law of cooling states that the temperature of an object changes at a rate proportional to the difference between the temperature of the object itself and the temperature of the surroundings. Suppose that the ambient temperature is 70 °F and the rate constant is \(0.05 \text{min}^{-1}\). Write a differential equation for the temperature of the object at any time. Note that the differential equation is the same whether the temperature of the object is above or below the ambient temperature.

Solution: Let \(T(t)\) be the temperature of the object at moment \(t\) and let \(T_0 = 70\) denote the ambient temperature. The rate of change of the temperature is given by the derivative \(T'(t)\), so we have
\[
T'(t) = k(T_0 - T(t)), \quad k > 0
\]
Indeed, if \(T(t) < T_0\) then the temperature of the object will increase and \(T'(t) > 0\). Similarly, if \(T(t) > T_0\) then the temperature of the object will decrease and \(T'(t) < 0\). By plugging in \(T_0 = 70\) and \(k = 0.05\) one gets
\[
T'(t) = 0.05(70 - T(t)).
\]

Section 1.2 7. (20 points) The field mouse population satisfies the differential equation
\[
\frac{dp}{dt} = 0.5p - 450.
\]
(a) Find the time when the population becomes extinct if \(p(0) = 850\).
(b) Find the time of extinction if \(p(0) = p_0\), where \(0 < p_0 < 900\).
(c) Find the initial population \(p_0\) if the population is to become extinct in 1 year.

Solution: Let us solve the differential equation with the initial condition \(p(0) = p_0\). We have \(p'(t) = 0.5(p - 900)\), so
\[
\frac{p'(t)}{p(t) - 900} = 0.5, \quad (\ln |p(t) - 900|)' = 0.5,
\]
so

\[ \ln |p(t) - 900| = 0.5t + C, \quad p(t) - 900 = \pm e^{0.5t+C} = \pm e^C \cdot e^{0.5t} = Ae^{0.5t} \]

where \( A = \pm e^C \). Therefore \( p(t) = 900 + Ae^{0.5t} \). Furthermore, \( p(0) = 900 + A = p_0 \), so \( A = p_0 - 900 \) and

\[ p(t) = 900 + (p_0 - 900)e^{0.5t}. \]

(b) The population becomes extinct if \( 900 + (p_0 - 900)e^{0.5t} = 0 \), so

\[ e^{0.5t} = -\frac{900}{p_0 - 900} = \frac{900}{900 - p_0}, \quad 0.5t = \ln \left( \frac{900}{900 - p_0} \right), \]

and

\[ t_{\text{extinct}} = 2 \ln \left( \frac{900}{900 - p_0} \right). \]

(a) If \( p_0 = 850 \), we get \( t_{\text{extinct}} = 2 \ln(\frac{900}{50}) = 2 \ln(18) \).

(c) If \( t_{\text{extinct}} = 12 \) (recall that the rime is measured in months), then \( e^{6} = \frac{900}{900 - p_0} \), so \( 900 - p_0 = 900/e^{6} \) and \( p_0 = 900 - 900/e^{6} \).

8. (20 points) Consider a population \( p \) of field mice that grows at a rate proportional to the current population, so \( \frac{dp}{dt} = rp \).

(a) Find the rate constant \( r \) if the population doubles in 30 days.

(b) Find \( r \) if the population doubles in \( N \) days.

**Solution:** Since \( \frac{dp}{dt} = rp \), we have \( p(t) = Ae^{rt} \). At \( t = 0 \) we have \( p(0) = A \). The population doubles if

\[ Ae^{rt} = 2 \Rightarrow e^{rt} = 2 \Rightarrow rt = \ln(2) \Rightarrow r = \frac{\ln(2)}{t}. \]

Therefore in (a) the population doubles in \( \frac{\ln(2)}{30} \) days and in (b) the population doubles in \( \frac{\ln(2)}{N} \) days.

15. According to Newton’s law of cooling, the temperature \( u(t) \) of an object satisfies the differential equation

\[ \frac{du}{dt} = -k(u - T), \]
where $T$ is the constant ambient temperature and $k$ is a positive constant. Suppose that $u(0) = u_0$.

(a) Find the temperature of the object of any time

(b) Find the time when the temperature difference $u(t) - T$ reduces by half.

**Solution:** (a) We have

$$\frac{u'(t)}{u(t) - T} = -k, \quad (\ln |u(t) - T|)' = -k, \quad \ln |u(t) - T| = -kt + C,$$

so

$$u(t) - T = \pm e^{-kt+C} = \pm e^C \cdot e^{-kt} = A e^{-kt}.$$  

At $t = 0$ we have $A = u_0 - T$, so

$$u(t) = T + (u_0 - T) e^{-kt}.$$

(b) The difference reduces by half if $u(t) - T = \frac{1}{2}(u_0 - T)$. Since

$$u(t) - T = (u_0 - T) e^{-kt},$$

we need $e^{-kt} = \frac{1}{2}$, so $kt = \ln(2)$ and $t = \frac{\ln(2)}{k}$.

16. (20 points) Suppose that a building loses heat in accordance with Newton’s law of cooling and the rate constant $k$ has the value 0.15$h^{-1}$. Assume that the internal temperature is 70°F when the heating system fails. If the external temperature is 10°F, how long will it take for the interior temperature to fall to 32°F?

**Solution:** In the notations of the previous problem, we have $k = 0.15$, $u_0 = 70$ and $T = 10$, so

$$u(t) = 10 + (70 - 10)e^{-0.15t} = 10 + 60e^{-0.15t}.$$

We have

$$10 + 60e^{-0.15t} = 32, \quad 60e^{-0.15t} = 22, \quad e^{-0.15t} = \frac{11}{30},$$

so $-0.15t = \ln(\frac{11}{30}) = \ln(11) - \ln(30)$ and $t = \frac{\ln(30) - \ln(11)}{0.15} \approx 6.69$ hours.