Ordinary Differential Equations, Spring 2015
Solutions to Homework 4

Maximal grade for HW4: 100 points

**Section 3.1:** 2. (10 points) Solve the differential equation $y'' + 3y + 2y = 0$.

**Solution:** The characteristic equation is $r^2 + 3r + 2 = (r + 1)(r + 2) = 0$, and its roots are $r_1 = -1$ and $r_2 = -2$. Therefore

$$y(t) = C_1 e^{-t} + C_2 e^{-2t}.$$

6. (10 points) Solve the differential equation $4y'' - 9y = 0$.

**Solution:** The characteristic equation is $4r^2 - 9 = 0$, and its roots are $r_1 = 3/2$ and $r_2 = -3/2$. Therefore

$$y(t) = C_1 e^{3t/2} + C_2 e^{-3t/2}.$$

12. (20 points) Solve the initial value problem $y'' + 3y' = 0$, $y(0) = -2$, $y'(0) = 3$ and describe its behaviour as $t$ increases.

**Solution:** The characteristic equation is $r^2 + 3r = 0$, and its roots are $r_1 = 0$ and $r_2 = -3$. Therefore the general solution has a form

$$y(t) = C_1 + C_2 e^{-3t}.$$

We have $y'(t) = -3C_2 e^{-3t}$, so $y(0) = C_1 + C_2 = -2$, $y'(0) = -3C_2 = 3$, and $C_2 = -1$, $C_1 = -1$.

Finally,

$$y(t) = -1 - e^{-3t}.$$

As $t$ increases, we have $\lim_{t \to +\infty} y(t) = -1$.

13. (20 points) Solve the initial value problem $y'' + 5y' + 3y = 0$, $y(0) = 1$, $y'(0) = 0$ and describe its behaviour as $t$ increases.

**Solution:** The characteristic equation is $r^2 + 5r + 3 = 0$, to find its roots we use the quadratic formula: $D = 5^2 - 4 \cdot 3 = 25 - 12 = 13$,

$$r_1 = \frac{-5 - \sqrt{13}}{2}, \quad r_2 = \frac{-5 + \sqrt{13}}{2}.$$
We have \( y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} \), so \( y'(t) = C_1 r_1 e^{r_1 t} + C_2 r_2 e^{r_2 t} \). Therefore \( y'(0) = C_1 r_1 + C_2 r_2 = 0 \), and \( C_2 = -r_1 C_1 / r_2 \). On the other hand, 

\[
1 = y(0) = C_1 - r_1 C_1 / r_2 = C_1 (r_2 - r_1) / r_2,
\]

hence 

\[
C_1 = \frac{r_2}{r_2 - r_1} = \frac{-5 + \sqrt{13}}{2\sqrt{13}}, \quad C_2 = -\frac{r_1}{r_2 - r_1} = \frac{5 + \sqrt{13}}{2\sqrt{13}}.
\]

Finally, 

\[
y(t) = \frac{-5 + \sqrt{13}}{2\sqrt{13}} e^{-\frac{5 - \sqrt{13}}{2\sqrt{13}} t} + \frac{5 + \sqrt{13}}{2\sqrt{13}} e^{-\frac{5 + \sqrt{13}}{2\sqrt{13}} t}.
\]

Since both \(-5 - \sqrt{13}\) and \(-5 + \sqrt{13}\) are negative, \( y(t) \to 0 \) as \( t \to \infty \).

24. (20 points) Determine the values of \( \alpha \) for which all solutions tend to zero as \( t \to \infty \): \( y'' + (3 - \alpha) y' - 2(\alpha - 1) y = 0 \); also determine the values of \( \alpha \) for which all nonzero solutions become unbounded as \( t \to \infty \).

**Solution:** The characteristic equation is \( r^2 + (3 - \alpha) r - 2(\alpha - 1) = (y + 2)(y + 1 - \alpha) = 0 \), its roots are \( r_1 = -2 \) and \( r_2 = \alpha - 1 \), so the general solution is

\[
y(t) = C_1 e^{-2t} + C_2 e^{(\alpha - 1)t}.
\]

Since \(-2 < 0\), all solutions tend to zero as \( t \to \infty \) if and only if \( \alpha - 1 < 0 \iff \alpha < 1 \). Since \( y = e^{-2t} \) is a solution for all \( \alpha \), the equation has bounded solutions for all \( \alpha \) and the second condition is never satisfied.

26. (20 points) (a) Solve the initial value problem \( y'' + 5y + 6y = 0, y(0) = 2, y'(0) = \beta \),

(b) Determine the coordinates \( t_m \) and \( y_m \) of the maximum point of the solution

(d) Determine the behaviour of \( t_m \) and \( y_m \) as \( \beta \to \infty \).

**Solution:** The characteristic equation is \( r^2 + 5r + 6 = (r + 1)(r + 5) = 0 \), its roots are \( r_1 = -2 \) and \( r_2 = -3 \), so the general solution has the form \( y(t) = C_1 e^{-2t} + C_2 e^{-3t} \). We have

\[
y(0) = C_1 + C_2 = 2, \quad y'(0) = -2C_1 - 3C_2 = \beta.
\]

From the first equation \( C_1 = 2 - C_2 \), so from the second equation

\[
-4 + 2C_2 - 3C_2 = \beta, \quad -4 - C_2 = \beta, \quad C_2 = -(4 + \beta), \quad C_1 = 2 - C_2 = 6 + \beta.
\]
Therefore
\[ y(t) = (6 + \beta)e^{-2t} - (4 + \beta)e^{-3t}. \]
Furthermore, at the maximum point
\[ y'(t) = -2(6 + \beta)e^{-2t} + 3(4 + \beta)e^{-3t} = 0, \quad 3(4 + \beta)e^{-3t} = 2(6 + \beta)e^{-2t}, \]
so
\[ e^{t_m} = \frac{3(4 + \beta)}{2(6 + \beta)}, \quad t_m = \ln\left(\frac{3(4 + \beta)}{2(6 + \beta)}\right). \]
Now
\[ y_m = (6 + \beta)e^{-2t_m} - (4 + \beta)e^{-3t_m} = (6 + \beta)\left(\frac{3(4 + \beta)}{2(6 + \beta)}\right)^{-2} - (4 + \beta)\left(\frac{3(4 + \beta)}{2(6 + \beta)}\right)^{-3} = \]
\[ \left(\frac{3(4 + \beta)}{2(6 + \beta)}\right)^{-3} \left( (6 + \beta)\left(\frac{3(4 + \beta)}{2(6 + \beta)}\right) - (4 + \beta) \right) = \left(\frac{2(6 + \beta)}{3(4 + \beta)}\right)^3 (3/2(4 + \beta) - (4 + \beta)) = \]
\[ \frac{8(6 + \beta)^3}{27(4 + \beta)^3} \cdot \frac{1}{2}(4 + \beta) = \frac{4(6 + \beta)^3}{27(4 + \beta)^2}. \]
As \( \beta \to \infty \), we have \( t_m \to \ln(3/2) \) and \( y_m \to +\infty \) (since the degree of the numerator is bigger than the degree of the denominator).