1 Introduction

We begin with a theorem of Lagrange, proven in 1770:

**Theorem 1.0.1.** Every natural number \( n \in \mathbb{N} \) can be represented as a sum of four squares:

\[
    n = a^2 + b^2 + c^2 + d^2.
\]

One natural question to ask is how many such representations there are of \( n \). Denote by \( c_4(n) \) the number of ordered 4-tuples \((a, b, c, d)\) that satisfy the above formula. For example \( c_4(1) = 8 \), because we will consider all integer solutions. Lagrange’s theorem says that \( c_4(n) > 0 \) for all positive \( n \). It is often quite useful, when counting such quantities, to introduce a generating function. So let’s define

\[
    \eta(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2q + 2q^4 + 2q^9 + 2q^{16} + 2q^{25} + \cdots.
\]

The coefficient of \( q^r \) is the number of ways to write \( r \) as a sum of one square. Then

\[
    \eta^2(q) = \sum_{n,m=-\infty}^{\infty} q^{n^2 + m^2} = \sum_{r=0}^{\infty} c_2(r)q^r
\]

where \( c_2(r) \) is the number of representations of \( r \) as a sum of two squares. Similarly

\[
    \eta^4(q) = \sum_{r=0}^{\infty} c_4(r)q^r.
\]

Our goal is to compute \( \eta^4 \) in a different way so that we can equate power series coefficients and get a formula for \( c_4(r) \). But how? One beautiful way, discovered by Jacobi in 1834, is to interpret \( \eta \) as a Fourier series!

We let \( q = e^{\pi iz} \) so that \( \eta \) becomes a Fourier series

\[
    \eta(z) = \sum_{n \in \mathbb{Z}} e^{\pi in^2 z}.
\]

Recall the following:

**Definition 1.0.2.** A Fourier series is a sum \( f(z) = \sum a(n)e^{2\pi inz} \) that expresses a continuous, periodic function \( f(z) \) (with period 1 in this case) as an infinite sum of sinusoids.

Note that \( f(z+1) = \sum a(n)e^{2\pi in(z+1)} = f(z) \). The essential theorem about Fourier series is that every continuous periodic function has a unique Fourier series expansion! We may be curious in the case of \( \eta(z) \) where our Fourier series actually converges. Note that

\[
|e^{\pi in^2 z}| = e^{-\pi n^2 \text{Im}(z)}
\]

so whenever \( \text{Im}(z) > 0 \), the terms go to zero in absolute value quite quickly, and the sum converges.

**Definition 1.0.3.** The upper half-plane \( \mathbb{H} \) is the subset of complex numbers with positive imaginary part:

\[
    \mathbb{H} = \{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \}.
\]

Furthermore, \( \eta(z) \) converges to a holomorphic function on \( \mathbb{H} \), as it is an absolutely convergent sum of holomorphic functions. As mentioned above, we have that \( \eta(z) = \eta(z+2) \). What else do we know? To find another transformation that \( \eta \) satisfies, we must use the Poisson summation formula, which states the following:

**Exercise 1.0.1.** Let \( f(z) \) be a quickly decaying function on \( \mathbb{R} \). Then

\[
    \sum_{n \in \mathbb{Z}} f(n) = \sum_{m \in \mathbb{Z}} (\mathcal{F}f)(m)
\]

where \( \mathcal{F} \) denotes the Fourier transform

\[
(\mathcal{F}f)(y) = \int_{-\infty}^{\infty} e^{-2\pi i xy} f(x) \, dx.
\]
This is quite simple to prove, and we leave it as an exercise. It is well known that the Fourier transform of a Gaussian is another Gaussian, and the exact formula is a simple exercise:

\[ \mathcal{F}(e^{-ax^2}) = \sqrt{\frac{\pi}{a}} e^{-\pi^2 x^2 / a}. \]

Using this formula, we have that as a function of \( n \),

\[ \mathcal{F}(e^{\pi in^2 z}) = \sqrt{\frac{i}{z}} e^{-\pi m^2 / z}. \]

Hence the Poisson summation formula states that

\[ \eta(z) = \sum_{n \in \mathbb{Z}} e^{\pi in^2 z} = \sqrt{\frac{i}{z}} \sum_{m \in \mathbb{Z}} e^{\pi im^2 (-1/z)} = \frac{i}{z} \eta(-1/z). \]

Thus, we have two relatively simple transformation laws that \( \eta^4(z) \) satisfies

\[ \eta^4(z + 2) = \eta^4(z) \quad (1) \]
\[ \eta^4(-1/z) = -z^2 \eta^4(z) \quad (2) \]

## 2 The Modular Curve

The transformations

\[ z \mapsto z + 2 \]
\[ z \mapsto -1/z \]

are examples of so-called fractional linear transformations. In our case, the group \( SL_2(\mathbb{R}) \) of determinant 1 two-by-two matrices has a group action on \( \mathbb{H} \). They act as follows

\[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}. \]

One can check easily that this defines a group action, i.e. if we have \( \gamma \) and \( \delta \) in \( SL_2(\mathbb{R}) \), then \( \gamma \cdot (\delta \cdot z) = (\gamma \delta) \cdot z \).

In our case,

\[ \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \cdot z = z + 2 \quad \text{and} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot z = -1/z. \]

Let’s call the above matrices \( T \) and \( S \) respectively. Then \( \eta^4 \) is invariant under \( T \) and almost invariant under \( S \). Rewriting in group action notation, we have

\[ \eta^4(T \cdot z) = \eta^4(z) \quad \text{and} \quad \eta^4(S \cdot z) = -z^2 \eta^4(z) \]

There are very few holomorphic functions on the upper half-plane that satisfy both of these transformation properties. In fact \( \eta^4 \) is the only one up to scaling! Let’s prove this using some complex analysis. First, we need to see what the action of \( T \) and \( S \) look like on the upper half-plane. The transformation \( T \) is quite simple: It is a horizontal translation by 2. The transformation \( S \) is a bit more complicated: We invert about the unit circle, then reflect about the \( y \)-axis. In some sense, the place where the function \( \eta^4(z) \) naturally “lives” is the quotient of the upper half-plane by the group \( \Gamma = \langle T, S \rangle \) generated by \( T \) and \( S \). Just like the place where a periodic function on \( \mathbb{R} \) naturally lives is the circle.

**Definition 2.0.4.** We call \( \mathcal{D} \subset \mathbb{H} \) a fundamental domain for the action of \( \Gamma \) if for every orbit of \( \Gamma \), there is exactly one orbit representative in \( \mathcal{D} \), except possibly on the boundary of \( \mathcal{D} \).
Any of the regions above are a fundamental domain for the action of $\Gamma$ on $\mathbb{H}$.
The boundary of the fundamental domain glues together to construct the quotient of the action $Y(\Gamma) = \Gamma \backslash \mathbb{H}$, and we can compactify $Y(\Gamma)$ by adding a finite number of cusps: $X(\Gamma) = \overline{Y(\Gamma)}$. In our case $X(\Gamma) = \mathbb{P}^1$.

Denote the space of functions satisfying the transformation laws (1) and (2) by $M_2^{-}(\Gamma)$ and note that it is a vector space over $\mathbb{C}$.

To determine the dimension of $M_2^{-}$, we will use a contour integral around the boundary of the fundamental domain. Suppose that $f \in M_2^{-}$ and consider the contour drawn above. Then by Cauchy’s integral formula (we suppress the factor of $\frac{1}{2\pi i}$ throughout),

$$\oint f'(z) f(z) \, dz = \# \text{ zeroes of } f \text{ inside the contour} = \sum_{p \in \partial\mathbb{D}} v_p(f)$$

where $v_p(f)$ is the order of vanishing of $f$ at $p$. On the other hand, we can combine all the contributions from the individual arcs of $C$ to get a second formula:

$$\oint f'(z) f(z) \, dz = - \sum_{p \in \partial D} v_p(f) - v_1(f) - v_{\infty}(f) - \frac{1}{2}v_i(f) + \left( \int_A + \int_B + \int_C + \int_D \right) f'(z) f(z) \, dz.$$

Note that the line integrals $\int_C$ and $\int_D$ along the vertical sides of the fundamental domain cancel because $f(z+2) = f(z)$ and the line integrals are going in opposite directions. So we have used the first transformation rule to our benefit. Now, we use the second transformation rule. The arc $A$ is sent to the arc $-B$ by the
transformation $S$. By the change of variables formula,
\[
\int_A f'(z) f(z) \, dz = \int_B f'(S \cdot z) f(S \cdot z) \, d(S \cdot z) = - \int_B \frac{[z^2 f(z)]'}{z^2 f(z)} \, dz = - \int_B \frac{2 \, dz}{z} - \int_B f'(z) f(z) \, dz = \frac{1}{2} - \int_B f'(z) f(z) \, dz.
\]
The second part of this integral cancels with the remaining undetermined piece of the original contour integral, and so we combine the two formulas to get
\[
\sum_{p \in \mathbb{D}, p \neq i} v_p(f) + \frac{1}{2} \eta_i(f) = \frac{1}{2}.
\]
Since the order of vanishing is always an integer, we must have $v_i(f) = 1$ while $v_p(f) = 0$ for all $p \neq i$. In particular, this must be true of $\eta_i$. But then, consider the function
\[
h(z) = \frac{f(z)}{\eta^i(z)}.
\]
It is totally invariant under $\Gamma$ because the $-z^2$ factors cancel. But then $h(z)$ descends to a bona fide holomorphic function on $X(\Gamma) = \mathbb{P}^1$. Because $\mathbb{P}^1$ is compact, $h(z)$ attains a maximum modulus somewhere, but at the same time, by the maximum modulus principle, it must attain a maximum on the boundary, which is empty. Hence $h(z)$ is actually a constant function, and thus
\[
f(z) = c\eta(z)
\]
for some $c \in \mathbb{C}$. This proves that $\dim(M_2^-) = 1$.

3 Another Modular Form

Our goal is to now construct an element of $M_2^-$ in a new way. Once scaled properly, it must equal $\eta^4$, at which point we can equate Fourier series coefficients (by uniqueness of Fourier series) to get our formula!

Another way of constructing modular forms is with Eisenstein series. We define
\[
G_2(z) = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \frac{1}{(mz + n)^2}.
\]
Then by reindexing the second sum, we have
\[
G_2(z + 1) = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \frac{1}{(mz + m + n)^2} = G_2(z).
\]
In addition
\[
G_2(-1/z) = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \frac{1}{(-m/z + n)^2} = z^2 \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \frac{1}{(m + nz)^2} = z^2 G_2(z).
\]
Why the question mark? Because the sum is actually conditionally convergent so we can reverse the order of summation! (What madness.) In fact, one can show that there is a correction term of $2\pi iz$:
\[
G_2(-1/z) = z^2 G_2(z) + 2\pi i z.
\]
Now, we are in a position to construct an element of $M_2^-$. Let
\[
f(z) = 2G_2(2z) - \frac{1}{2} G_2(z/2).
\]
Then, because $G_2(z + 1) = G_2(z)$, we have that $f(z + 2) = f(z)$. Furthermore,
\[
f(-1/z) = 2G_2\left(-\frac{1}{z/2}\right) - \frac{1}{2} G_2\left(-\frac{1}{2z}\right)
\]
\[
= 2(z/2)^2 G_2(z/2) + 2(2\pi i)(z/2) - \frac{1}{2} (2\pi i)^2 G_2(z) - \frac{1}{2} (2\pi i)(2z)
\]
\[
= -z^2 f(z)
\]
and thus \( f \in M_c^- \). Hence there is a constant \( c \) such that \( f = c\eta^4 \).

We would like to compute the Fourier series of the periodic function

\[
\sum_{n \in \mathbb{Z}} \frac{1}{(mz + n)^2}
\]

and sum over \( m \), so as to compare coefficients with \( \eta^4 \). We can do so by using Poisson summation formula again!

**Exercise 3.0.2.** As a function of \( n \), we have

\[
\tilde{F} \left( \frac{1}{(mz + n)^2} \right) = (2\pi i)^2 u(k) e^{2\pi i kmz}
\]

where \( u(k) \) is the unit step function, which is zero for negative \( k \) and one for positive \( k \).

As hoped, the negative coefficients of the Fourier series will be zero. Summing over \( m \) gives

\[
G_2(z) = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \frac{1}{(mz + n)^2} = \sum_{n \neq 0 \in \mathbb{Z}} \frac{1}{n^2} + 2(2\pi i)^2 \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} ke^{2\pi i kmz} = 2\zeta(2) + 2(2\pi i)^2 \sum_{n=1}^{\infty} \sigma(n) e^{2\pi inz}
\]

where \( \sigma(n) \) is the sum of the divisors of \( n \). Thus, we have a formula for \( f(z) \):

\[
f(z) = 3\zeta(2) + 4(2\pi i)^2 \sum_{n=1}^{\infty} \sigma(n) e^{4\pi inz} - (2\pi i)^2 \sum_{n=1}^{\infty} \sigma(n)e^{\pi inz}.
\]

Since the constant coefficient of \( \eta^4 \) is 1, we normalize \( f(z) \) and collect like terms to show

\[
\eta^4(z) = 1 + \sum_{n=1}^{\infty} \frac{(2\pi i)^2}{3\zeta(2)} [4\sigma(n/4) - \sigma(n)] e^{\pi inz}.
\]

We can now compute our formula!

\[
c_4(n) = \frac{(2\pi i)^2}{3(\pi^2/6)} [4\sigma(n/4) - \sigma(n)]
\]

Simplifying and putting it into a nice format, we have the final result

\[
c_4(n) = \sum_{d \mid n \atop 4 \nmid d} d.
\]