A Proof of Looijenga’s Conjecture via Integral-affine Geometry

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0 Introduction

A cusp singularity $(\mathcal{V}, p)$ is the germ of an isolated, normal surface singularity such that the exceptional divisor of the minimal resolution $\pi : V \rightarrow \mathcal{V}$ is a cycle of smooth rational curves meeting transversely:

$$\pi^{-1}(p) = D = D_1 + \cdots + D_n.$$ 

The analytic germ of a cusp singularity is uniquely determined by the self-intersections $D_i^2$ of the components of $D$. Cusp singularities come in naturally dual pairs $(\mathcal{V}, p)$ and $(\mathcal{V}', p')$, whose exceptional divisors $D$ and $D'$ are called dual cycles. For every pair of dual cusps, Inoue [Ino77] constructs an associated Hirzebruch-Inoue surface—a smooth, non-algebraic, complex surface whose only curves are the components of two disjoint cycles $D$ and $D'$. Contracting $D$ and $D'$ produces a surface with two dual cusp singularities and no algebraic curves.

By working out the deformation theory of the contracted Hirzebruch-Inoue surface, Looijenga [Loo81] proved that if the cusp with cycle $D'$ is smoothable, then there exists an anticanonical pair $(Y, D)$—a smooth rational surface $Y$ with an anticanonical divisor $D \in | - K_Y|$ whose components have the appropriate self-intersections. Conversely, Looijenga conjectured that the existence of such an anticanonical pair $(Y, D)$ implies the smoothability of the cusp with cycle $D'$. Recently, work of Gross, Hacking, and Keel proved Looijenga’s conjecture using methods from mirror symmetry [GHK11]. In this paper, we provide an alternative proof of Looijenga’s conjecture.

In the first section, we review foundational material on cusp singularities, Hirzebruch-Inoue surfaces, anticanonical pairs, and discuss the main result of Friedman-Miranda [FM83]: The cusp $D'$ is smoothable if there exists a simple normal crossings surface $X_0 = \bigcup V_i$ satisfying certain combinatorial conditions.

We begin the second section by defining an integral-affine structure on the dual complex $\Gamma(X_0)$ of such a surface. The existence of this integral-affine structure is not needed in the proof of Looijenga’s conjecture, but motivates the construction in the fourth section. Then, we associate an integral-affine surface, called the pseudo-fan, to any anticanonical pair $(V, D)$ and describe two surgeries on the pseudo-fan of $(V, D)$ that correspond to blowing up a point on a component of $D$ and smoothing a node of $D$.

In the third section, we define two surgeries on an integral-affine surface $(S, P, A)$ with a locally polygonal boundary $\partial S = P$ and singularities $A$—an internal blow-up and a node smoothing. Both surgeries appear in Symington’s work [Sym03] on almost toric fibrations of four-dimensional

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symplectic manifolds. Assuming certain conditions, there is a unique symplectic four-manifold \( Y \) with a Lagrangian torus fibration 
\[
(Y, D, \omega) \to (S, P, A)
\]
which attains certain allowable singular fibers over the singular points \( A \). An internal blow-up or node smoothing of \((S, P, A)\) is the base of a Lagrangian torus fibration of an internal blow-up or node smoothing of \((Y, D, \omega)\), respectively.

In the fourth section, we construct from an anticanonical pair \((Y, D)\) a surface \( X_0 \) satisfying the conditions of the theorem of Friedman-Miranda, thus proving Looijenga’s conjecture. To summarize the construction:

1. We express \((Y, D)\) as a sequence of internal blow-ups and node smoothings of a toric surface \((\mathcal{Y}, \mathcal{D})\). By performing the analogous surgeries on a moment polygon \((\mathcal{S}, \mathcal{P}, \emptyset)\) for \((\mathcal{Y}, \mathcal{D})\), we produce the base \((S, P, A)\) of a Lagrangian torus fibration of a symplectic surface \((Y, D, \omega)\).

2. We glue \((S, P, A)\) to the cone over its boundary \( C \) to form a sphere. We define an integral-affine structure on \( C \) by extending the developing map of a collar neighborhood of the boundary of \((S, P, A)\) towards the fixed point \( v_0 \) of the monodromy around this neighborhood.

3. We triangulate the sphere \( \hat{S} := S \cup C \) into lattice triangles of area \( \frac{1}{2} \). We show that a neighborhood of every vertex \( v_i \) of the triangulation with \( i \neq 0 \) is locally modeled by the pseudo-fan of a rational surface \((V_i, D_i)\). The cone point \( v_0 \in C \) is locally modeled by the pseudo-fan of the Hirzebruch-Inoue pair \((V_0, D')\). We then produce a surface \( X_0 \) whose dual complex is the chosen triangulation of \( \hat{S} \).

The construction of \( X_0 \) may be phrased solely in terms of operations on integral-affine surfaces—no results in symplectic geometry are necessary for the proof, but they provide the primary motivation. Our construction is algorithmic, providing a simple normal crossings resolution of at least one smoothing of any smoothable cusp singularity. We describe, without proof, four modifications and generalizations of the construction and conclude by giving an example of the modified construction in the charge three case \( Q(Y, D) = 3 \).

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1 Background

Let \((\mathcal{V}, p)\) be the germ of a cusp singularity. The exception divisor of the minimal resolution 
\[
\pi^{-1}(p) = D = D_1 + \cdots + D_n
\]
is a cycle of smooth rational curves meeting transversely. We define \( \ell(D) := n \) to be the length of the cycle. Whenever \( n \geq 3 \), we assume \( D_i \cdot D_{i \pm 1} = 1 \), with indices taken mod \( n \). If \( n = 1 \), then \( D \) is an irreducible, nodal rational curve. If \( n = 2 \), then \( D \) is the union of two smooth rational curves that meet transversely at two distinct points. We define 
\[
d_i := \begin{cases} 
-D_i^2 & \text{if } n > 1 \\
2 - D_i^2 & \text{if } n = 1.
\end{cases}
\]
The analytic germ of a cusp singularity is uniquely determined by the cycle $d := (d_1, \ldots, d_n)$ of negative self-intersections, well-defined up to cyclic permutation and orientation. As $D$ is contractible, Artin’s contractibility criterion implies that the intersection matrix $[D_i \cdot D_j]$ is negative-definite. No component of $D$ is an exceptional curve, because it is a minimal resolution. The negative-definite condition is then equivalent to

$$d_i \geq 2 \quad \text{for all } i,$$
$$d_i \geq 3 \quad \text{for some } i.$$

Cusp singularities arise in the classification of complex analytic surfaces. Amongst those of Type VII$_0$ are the Hirzebruch-Inoue surfaces, which have first Betti number $b_1 = 1$ and Kodaira dimension $\kappa = -\infty$. For a construction, see [Ino77]. The only curves on a Hirzebruch-Inoue surface $V$ are the components of two contractible cycles $D$ and $D'$ of rational curves satisfying $D + D' \in | - K_V|$. Both cycles can be blown down to give a surface $(\overline{V}, p, p')$ with two dual cusps. For any cusp singularity $p$, there is a construction of $\overline{V}$ as the compactification of a quotient of $\mathbb{H} \times \mathbb{C}$ by a discrete group action (hence the terminology “cusp”). Suppose that the cycle of negative self-intersections of $D$ is

$$d = (d_1, \ldots, d_n) = (a_1, 2, \ldots, 2, \ldots, a_k, 2, \ldots, 2)$$

with $a_i \geq 3$ and $b_i \geq 0$. An explicit formula for the negative self-intersections of the dual cycle $D'$ may be given from those of the original cycle $D$:

$$d' = (d'_1, \ldots, d'_n) = (b_1 + 3, 2, \ldots, 2, \ldots, b_k + 3, 2, \ldots, 2).$$

Let $(\overline{V}, p, p')$ denote the (disconnected) germ of the two cusp singularities on the doubly contracted Hirzebruch-Inoue surface $\overline{V}$. Looijenga [Loo81] proves:

**Theorem 1.1.** $\overline{V}$ has a universal deformation which is semi-universal for the germ $(\overline{V}, p, p')$.

Suppose that $p'$ is smoothable. By Theorem 1.1 there exists a deformation

$$\pi : V \rightarrow \Delta$$

over an analytic disc, with $V_0 = \overline{V}$, which keeps the germ $(\overline{V}, p)$ constant while smoothing the germ $(\overline{V}, p')$. Any fiber $V_t$ with $t \neq 0$ is a surface with a single cusp singularity $p = p_t$. Simultaneously resolving the singularities $p_t$ produces a family $V \rightarrow \Delta$ whose central fiber is the partially contracted Hirzebruch-Inoue surface with cusp singularity $p'$ and whose general fiber is a smooth surface. Any fiber $V_t$ with $t \neq 0$ is a simply connected surface with anticanonical divisor $D$, which by the classification of surfaces must be rational. Hence, the following corollary to Theorem 1.1:

**Corollary 1.2.** Suppose that $D'$ contracts to a smoothable cusp singularity. Then, $D$ is the anticanonical divisor of some rational surface.

Looijenga conjectured the converse, which by the work of Gross, Hacking, and Keel [GHK11] on mirror symmetry for anticanonical pairs, is now a theorem:

**Theorem 1.3** (Looijenga’s Conjecture). If $D$ is the anticanonical divisor of some rational surface, then the cusp singularity associated to $D'$ is smoothable.
Now we review some basic facts about rational surfaces with an anticanonical cycle $D$. Such surfaces are log generalizations of K3 surfaces: They are simply connected surfaces with a global non-vanishing meromorphic 2-form whose locus of poles is a simple normal crossings divisor $D$:

**Definition 1.4.** An anticanonical pair or simply pair $(Y, D)$ is a rational surface $Y$ with an anticanonical divisor $D$ equal to a cycle of rational curves $D = D_1 + \cdots + D_n \in |−K_Y|$ meeting transversely. A negative-definite pair satisfies the additional condition that the intersection matrix $[D_i \cdot D_j]$ is negative-definite. A toric pair is a pair where $Y$ is a toric surface and $D$ is the toric boundary.

Let $E$ be an exceptional curve on $(Y, D)$. Contracting $E$ gives a smooth anticanonical pair: 

$$\pi : (Y, D) \to (\overline{Y}, \overline{D}).$$

If $E$ is a component of $D$, then $E$ contracts to a node point of the cycle $\overline{D}$. In this case, we call $\pi$ a corner blow-up. If $E$ is not a component of $D$, then $E$ intersects $D$ at one of its smooth points. Thus, $E$ contracts to a smooth point of the cycle $\overline{D}$, in which case, we call $\pi$ an internal blow-up. Conversely, given any anticanonical pair, we can blow-up either a corner or a smooth point of the cycle to produce a new anticanonical pair. In addition to blowing up on $D$, we can smooth any node of $D$:

**Proposition 1.5.** Let $(V, D)$ be an anticanonical pair and let $p$ be a node of $D$. There exists a family of anticanonical pairs $(V, D) \to \Delta$ over the disc whose central fiber is $(V, D)$ such that $D \to \Delta$ is a smoothing of the node $p$.

**Proof.** The proposition follows from Corollary 3.5 of [Fri14], which proves the result for any subset of the nodes of $D$. Roughly, the deformations of $(V, D)$ surject onto the deformations of $D$. □

**Definition 1.6.** The charge of a cycle $D$ or of a pair $(Y, D)$ is defined by the formula

$$Q(D) = Q(Y, D) := 12 + \sum_{i=1}^{n} (d_i - 3) = 12 + \sum_{i=1}^{n} (a_i - b_i - 3).$$

The formula for the dual cusp $D'$ implies that $Q(D) + Q(D') = 24$. The charge of an anticanonical pair $(Y, D)$ is essentially a measure of how far it is from being toric: All toric pairs have charge zero, while all other anticanonical pairs have positive charge.

**Remark 1.7.** Let $(Y, D)$ be a pair. An internal blow-up on $D_i$ changes the cycle $d$ by

$$(\ldots, d_i, \ldots) \mapsto (\ldots, d_i + 1, \ldots)$$

and increases the charge by 1. A corner blow-up at $D_i \cap D_{i+1}$ changes the cycle $d$ by

$$(\ldots, d_i, d_{i+1}, \ldots) \mapsto (\ldots, d_i + 1, 1, d_{i+1} + 1, \ldots)$$

and keeps the charge constant. A node smoothing at $D_i \cap D_{i+1}$ changes the cycle $d$ by

$$(\ldots, d_i, d_{i+1}, \ldots) \mapsto (\ldots, d_i + d_{i+1} - 2, \ldots)$$

and increases the charge by 1.
Consider a smoothing family $\mathcal{Y} \to \Delta$ whose central fiber is the partially contracted Hirzebruch-Inoue surface with cusp singularity $p'$. Using the same methods as Kulikov [Kul77] and Persson and Pinkham [PP81] in their study of degenerations of K3 surfaces, Friedman and Miranda [FM83] prove that after a finite base change and bi-meromorphic modifications on $\mathcal{Y} \to \Delta$, we can produce a smooth family $\mathcal{X} \to \Delta$ such that $D \in -K_X$ is a divisor restricting to $D$ on every fiber and the central fiber is a simple normal crossings surface. In analogy with so-called Type III degenerations of K3 surfaces, the central fiber

$$X_0 = \bigcup_{i=0}^n V_i$$

of the family $\mathcal{X} \to \Delta$ satisfies the following conditions:

i. $V_0$ is the Hirzebruch-Inoue surface with cycles $D$ and $D'$. For $i > 0$, the normalization $\tilde{V}_i$ of each $V_i$ is a smooth rational surface.

ii. Let $D_{ij}$ denote an irreducible double curve of $X_0$ lying on $V_i$ and $V_j$ (if $V_i$ is not normal, we may have $i = j$). Define $D_i$ to be the union of the double curves $D_{ij}$ contained in $V_i$. Let $\tilde{D}_i$ be the inverse image of $D_i$ under the normalization map $\tilde{V}_i \to V_i$. Then

$$(\tilde{V}_i, \tilde{D}_i)$$

is an anticanonical pair for $i > 0$ and $D_0 = D'$.

iii. (Triple Point Formula) Let $D_{ij}$ be a double curve joining the surfaces $V_i$ and $V_j$. Then

$$\left(D_{ij} \mid_{\tilde{V}_i}\right)^2 + \left(D_{ij} \mid_{\tilde{V}_j}\right)^2 = \begin{cases} -2 & \text{if } D_{ij} \text{ is smooth} \\ 0 & \text{if } D_{ij} \text{ is nodal.} \end{cases}$$

iv. The dual complex of $X_0$ is a triangulation of the sphere.

**Definition 1.8.** We call a surface $X_0$ satisfying conditions i.-iv. a **Type III anticanonical pair** $(X_0, D)$.

Conditions i.-iv. are the only combinatorial conditions necessary to ensure that $X_0$ smooths to an anticanonical pair $(Y, D)$ in a family $\mathcal{X} \to \Delta$. The remaining condition, $d$-semistability, is analytic:

$$T_{X_0}^1 := Ext^1_{\mathcal{O}_{X_0}}(\Omega^1_{X_0}, \mathcal{O}_{X_0}) \cong \mathcal{O}_{\text{sing}(X_0)}.$$ 

It is easy to show that any Type III anticanonical pair has a topologically trivial deformation to one which is $d$-semistable. Motivated by the result of [Fri83] in the case of Type III K3 surfaces, [FM83] prove that a $d$-semistable Type III anticanonical pair $(X_0, D)$ smooths to an anticanonical pair $(Y, D)$. By a result of Shepherd-Barron [SB83], the union of the surfaces $V_i$ for $i > 0$ can be contracted to a point, assuming we also contract the cycle $D'$ on $V_0$. Thus, the existence of $(X_0, D)$ implies that $D'$ is smoothable. Hence, the main theorem of [FM83]:

**Theorem 1.9.** The cusp singularity associated to $D'$ is smoothable if and only if there exists a Type III anticanonical pair $(X_0, D)$.

**Notation 1.10.** To simplify the notation, we will henceforth suppress the tildes on $(\tilde{V}_i, \tilde{D}_i)$ so that $(V_i, D_i)$ denotes a smooth anticanonical pair. In addition, we introduce the convention

$$D_{ij} = D_{ij} \mid_{V_i} \quad \text{and} \quad D_{ji} = D_{ij} \mid_{V_j}$$
so that $D_{ij}$ always denotes a curve on the smooth surface $V_i$. Then $D_{ij}$ and $D_{ji}$ have equal image in $X_0$ but may not be isomorphic. In fact, the image of $D_{ij}$ in $X_0$ is nodal if and only if exactly one of $D_{ij}$ or $D_{ji}$ is nodal. We define

$$d_{ij} := \begin{cases} 
-D_{ij}^2 & \text{if } \ell(D_i) \geq 2 \\
2 - D_{ij}^2 & \text{if } \ell(D_i) = 1 
\end{cases}.$$ 

Then, the triple point formula states that $d_{ij} + d_{ji} = 2$ in all cases.

**Proposition 1.11 (Conservation of Charge).** Let $(X_0, D)$ be a Type III anticanonical pair. Then,

$$\sum Q(V_i, D_i) = 24.$$ 

*Proof.* See [FM83], Proposition 3.7. \hfill \square

## 2 Type III Anticanonical Pairs and Integral-Affine Surfaces

Conservation of charge is analogous to the Gauss-Bonnet formula, where curvature and charge are equated: The sum of the charges is a constant multiple of the Euler characteristic, as is the integral of curvature. Toric surfaces, which have charge zero, are “flat” in some sense. The analogy will take a precise form in this section; we define an integral-affine structure on the dual complex $\Gamma(X_0)$ of a Type III anticanonical pair $X_0$ which has singularities at the vertices corresponding anticanonical pairs $(V_i, D_i)$ of positive charge. Imposing a singular, integral-affine “fan” structure on the dual complex of a maximally unipotent degeneration of a Calabi-Yau manifold plays a role in the Gross-Siebert program [GS03] for proving the SYZ conjecture. The case of a Type III degeneration of K3 surfaces is specifically discussed in [GHK11].

**Definition 2.1.** A triangulated integral-affine surface with singularities $(S, P, A)$ is a triangulated real surface $S$ with boundary $\partial S = P$ and a finite, possibly empty, subset $A \subset S \setminus P$ such that

1. $S \setminus A$ has an integral-affine structure—that is, $S \setminus A$ is endowed with a set of charts into $\mathbb{R}^2$ with transition functions valued in the integral-affine transformation group $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$.
2. Every element of $A$ is a vertex of the triangulation.
3. Every triangle is integral-affine equivalent to a lattice triangle of area $\frac{1}{2}$.

An integral-affine surface with singularities has a canonical orientation induced from the standard orientation on $\mathbb{R}^2$. We define a *basis triangle* to be an oriented lattice triangle of area $\frac{1}{2}$. All labeled basis triangles are equivalent, up to a unique integral-affine transformation. Because $S$ is triangulated into basis triangles, the boundary $P$ is polygonal: There is a decomposition of the boundary $P = P_1 + \cdots + P_n$ such that each $P_i$ is integral-affine equivalent to a line segment between two lattice points. By convention, we assume that the boundary components are maximal: The union of two distinct boundary components is never integral-affine equivalent to a single line segment between two lattice points.

Let $\Gamma(X_0)$ denote the dual complex of a Type III anticanonical pair $X_0$. We will endow $\Gamma(X_0)$ with the structure of a triangulated integral-affine sphere with singularities. Choose an orientation on $\Gamma(X_0)$. We define notation for the vertices, edges, and faces of the triangulation which will hold for the rest of the paper:

1. The vertices $v_i$ of the triangulation correspond to the components $V_i$ of $X_0$.
2. The directed edges $e_{ij} = (v_i, v_j)$ of the triangulation correspond to double curves $D_{ij}$.
3. The oriented triangular faces $f_{ijk} = (v_i, v_j, v_k)$ correspond to triple points $T_{ijk}$. 

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We denote the $i$-skeleton of $\Gamma(\mathcal{X}_0)$ by $\Gamma(\mathcal{X}_0)^i$. There is a non-singular integral-affine structure on 

$$\Gamma(\mathcal{X}_0) \setminus \{v_i : Q(V_i, D_i) > 0 \text{ or } i = 0\}$$

which we now define. We declare each triangular face $f_{ijk}$ to be integral-affine equivalent to a basis triangle. We define the integral-affine structure on the union of two triangular faces $f_{ijk}$ and $f_{ik\ell}$ that share an edge $e_{ik}$ by gluing two basis triangles in the plane along their shared edge. We glue in such a way that

$$d_{ik} e_{ik} = e_{ij} + e_{i\ell},$$

where we view the directed edges $e_{ij}$, $e_{ik}$ and $e_{i\ell}$ as integral vectors. For example, if $d_{13} = -1$, we can use the chart in Figure 1 to glue together $f_{123}$ and $f_{134}$. We get an equivalent integral-affine structure on the union of $f_{123}$ and $f_{134}$ by using the value $d_{31} = 3$. In fact, using either $e_{ik}$ or $e_{ki}$ to define the integral-affine structure on the union of $f_{ijk}$ and $f_{ik\ell}$ produces the same result.

![Figure 1: The integral-affine structure on the union of $f_{123}$ and $f_{134}$ if $d_{13} = -1$.](image)

**Proposition 2.2.** Let the quadrilateral $(v_i, v_j, v_k, v_\ell)$ in $\Gamma(\mathcal{X}_0)$ be the union of two triangular faces $f_{ijk}$ and $f_{ik\ell}$ along a shared edge $e_{ik}$. The integral-affine structure on $(v_i, v_j, v_k, v_\ell)$ is well-defined.

**Proof.** There is a unique chart on $f_{ijk}$ which sends

$$v_i \mapsto (0, 0)$$

$$v_j \mapsto (1, 0)$$

$$v_k \mapsto (0, 1).$$

This chart extends uniquely to $(v_i, v_j, v_k, v_\ell)$. Because $f_{ik\ell}$ is a basis triangle, the image of $v_\ell$ must be $(-1, n)$ for some integer $n$. Recall, we define the integral-affine structure on $(v_i, v_j, v_k, v_\ell)$ by requiring that $d_{ik} e_{ik} = e_{ij} + e_{i\ell}$. Considering the edges as unbased lattice vectors, which we denote by angle brackets, we have $e_{ij} = \langle 1, 0 \rangle$, $e_{ik} = \langle 0, 1 \rangle$, and $e_{i\ell} = \langle -1, n \rangle$. Thus

$$\langle 1, 0 \rangle + \langle -1, n \rangle = d_{ik} \langle 0, 1 \rangle.$$

Hence $n = d_{ik}$ and the integral-affine structure on $(v_i, v_j, v_k, v_\ell)$ is uniquely determined by the integer $d_{ik}$. 

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Now, we must check that the integral-affine structure defined on the labeled quadrilateral \((v_k, v_{\ell}, v_i, v_j)\) is compatible with the structure we have already defined on \((v_i, v_j, v_k, v_{\ell})\). Consider the chart on \((v_i, v_j, v_k, v_{\ell})\) defined in the previous paragraph. We solve for the the integer \(m\) satisfying \(me_{ki} = e_{k\ell} + e_{kj}\):

\[
m(0, -1) = \langle -1, n - 1 \rangle + \langle 1, -1 \rangle
\]

to get \(m = 2 - n\). Since \(n = d_{ik}\), we have that \(m = 2 - d_{ik} = d_{ki}\) by the triple point formula. Thus, by the uniqueness shown in the previous paragraph, the integral-affine structure on \((v_i, v_j, v_k, v_{\ell})\) is compatible with the structure defined on \((v_k, v_{\ell}, v_i, v_j)\).

All points except the vertices \(v \in \Gamma(\mathcal{X}_0)^{[0]}\) are now contained in the interior of some chart. Hence, we have endowed \(\Gamma(\mathcal{X}_0) \setminus \Gamma(\mathcal{X}_0)^{[0]}\) with an integral-affine structure.

**Proposition 2.3.** The integral-affine structure defined above on \(\Gamma(\mathcal{X}_0) \setminus \Gamma(\mathcal{X}_0)^{[0]}\) extends to the vertices \(v_i \in \Gamma(\mathcal{X}_0)^{[0]}\) corresponding to toric pairs, but to no other vertices.

**Proof.** Consider the fan \(\mathfrak{F}\) of a toric pair \((V_i, D_i)\) in \(\mathcal{X}_0\). A boundary component \(D_{ik}\) corresponds to a one-dimensional facet of \(\mathfrak{F}\), which is the positive span of some primitive integral vector \(w_{ik}\). Let \(D_{ij}\) and \(D_{il}\) be the boundary components of \(D_i\) that intersect \(D_{ik}\) in the clockwise and counterclockwise direction, respectively. Because \(V_i\) is smooth, \((w_{ij}, w_{ik})\) and \((w_{ik}, w_{il})\) are oriented lattice bases. Furthermore, the formula for \(d_{ik} = -D_{ik}^2\) is simply

\[
d_{ik}w_{ik} = w_{ij} + w_{il},
\]

see Section 2.5 of [Ful93]. Let \(p_{ix}\) denote the endpoint of \(w_{ix}\). The self-intersection formula above implies that \((v_i, v_j, v_k, v_{\ell})\) is integral-affine equivalent to the quadrilateral \((O, p_{ij}, p_{ik}, p_{il})\).

We may define an integral-affine structure on the union of the triangular faces that contain \(v_i\) by a single chart to the polygon with vertices \(p_{ij}\). For example, if \((V_1, D_1)\) is \(\mathbb{P}^2\) with its toric boundary, and \(V_1\) intersects \(V_2, V_3,\) and \(V_4\) at double curves, a chart on the union of the triangles containing \(v_1\) is shown in Figure 2. Thus, we have defined an integral-affine structure on

\[
\Gamma(\mathcal{X}_0) \setminus \{v_i : Q(V_i, D_i) > 0 \text{ or } i = 0\}.
\]

Finally, we show that the integral-affine structure we have defined fails to extend to the vertices of \(\Gamma(\mathcal{X}_0)\) corresponding to the Hirzebruch-Inoue surface \((V_0, D_s')\) or the anticanonical pairs \((V_i, D_i)\) of positive charge. Suppose, for the sake of contradiction that the integral-affine structure extends to such a point \(v_i\). There is a chart containing \(v_i\) in its interior. This chart may be extended to the union of the triangles containing \(v_i\). Thus, the union of the triangles containing \(v_i\) is integral-affine equivalent to some planar polygon formed from the union of basis triangles. But this planar polygon corresponds to the fan of some smooth toric surface. Thus, the values \(d_{ij} = -D_{ij}^2\) are those of a toric surface.

We conclude that \(Q(V_i, D_i) = Q(D_i) = 0\). By assumption, \(Q(V_i, D_i) > 0\) unless \(i = 0\). Hence \(v_0\) is the only possible vertex to which the integral-affine structure might extend. But \((V_0, D_0)\) is the Hirzebruch-Inoue surface and \(D_0 = D_s'\) is a negative-definite cycle. No toric surface has a negative-definite anticanonical cycle because the boundary components span the Picard group, which has indefinite intersection form. Hence the integral-affine structure cannot extend to the point \(v_0\).
Figure 2: The integral-affine structure on the union of the triangles containing the vertex \(v_1\) corresponding to a toric pair \((V_1, D_1) \cong (\mathbb{P}^2, D_{12} + D_{13} + D_{14})\).

For the remainder of the paper, we will implicitly assume that an integral-affine surface has singularities and we will specify when an integral-affine surface is non-singular. In the context of the dual complex of a Type III anticanonical pair, let

\[ A = \{ v_i : Q(V_i, D_i) > 0 \text{ or } i = 0 \}. \]

We have defined a non-singular integral-affine structure on \(\Gamma(X_0) \setminus A\) that makes \((\Gamma(X_0), \emptyset, A)\) into a boundary-less triangulated integral-affine surface. Some properties of \(X_0\) can be generalized to notions that make sense for all triangulated integral-affine surfaces:

**Definition 2.4.** Let \(e_{ik}\) be a directed edge in the interior of a triangulated integral-affine surface. Let \(e_{ij}\) and \(e_{i\ell}\) be the edges emanating from \(v_i\) directly clockwise and counterclockwise to \(e_{ik}\). We define the **negative self-intersection** \(d_{ik}\) by the formula

\[ d_{ik}e_{ik} = e_{ij} + e_{i\ell}, \]

where we have implicitly applied a chart to the union of the two triangles containing \(e_{ik}\) so that the directed edges gain the structure of lattice vectors. Note that \(d_{ik}\) is an integer because \((e_{ij}, e_{ik})\) and \((e_{ik}, e_{i\ell})\) are both oriented lattice bases. The integer \(d_{ik}\) is independent of the choice of integral-affine chart.

**Proposition 2.5.** For any triangulated integral-affine surface, the negative self-intersections \(d_{ik}\) satisfy the triple point formula: \(d_{ik} + d_{ki} = 2\).

**Proof.** The argument is similar to that of Proposition 2.2. \(\square\)

**Definition 2.6.** Let \((V, D)\) be a pair with cycle components \(D = D_1 + \cdots + D_n\). The **pseudo-fan** of \((V, D)\) is a triangulated integral-affine surface PL-equivalent to the cone over the dual complex of \(D\). For each intersection point \(D_i \cap D_{i+1}\) there is an associated triangle \(t_{i,i+1}\) of this cone, which we declare to be integral-affine equivalent to a basis triangle. The directed edges \(e_i\) that originate at the cone point correspond to some component \(D_i\). We glue \(t_{i-1,i}\) and \(t_{i,i+1}\) together in the unique manner such that

\[ d_i = \begin{cases} -D_i^2 & \text{if } n > 1 \\ 2 - D_i^2 & \text{if } n = 1 \end{cases} \]
where $d_i$ is the negative self-intersection of $e_i$ in the sense of Definition 2.4. We orient the pseudo-fan is such that $e_{i+1}$ is counterclockwise to $e_i$. The imposed integral-affine structure has at most one singularity, at the cone point. By the argument of Proposition 2.3, the pseudo-fan of $(V, D)$ is non-singular if and only if $(V, D)$ is toric. Compare to Section 0.3.1 and Lemma 1.3 of [GHK11].

Note that the definition of a pseudo-fan also applies for the Hirzebruch-Inoue pair $(V_0, D')$. The pseudo-fan of a toric surface $(V, D)$ is simply a lattice polygon whose vertices are the endpoints of the primitive integral vectors that span the one-dimensional facets of the fan of $(V, D)$. In any Type III anticanonical pair, the integral-affine surface formed from the union of the triangles containing $v_i \in \Gamma(A_0)[0]$ is isomorphic to the pseudo-fan of $(V_i, D_i)$. More generally:

**Definition 2.7.** Let $(S, P, A)$ be a triangulated integral-affine surface. Let $v$ be a vertex of the triangulation. We define $\text{star}(v)$ to be the triangulated integral-affine surface formed from the union of the triangles that contain $v$. Note that $\text{star}(v)$ is singular only if $v \in A$.

**Remark 2.8.** Let $S$ be a non-singular integral-affine surface. Any small contractible open subset $U \subset S$ has a chart $\phi_U : U \to \mathbb{R}^2$ which is uniquely defined up to integral-affine transformation. We can then construct a developing map

$$
\phi : \tilde{S} \to \mathbb{R}^2
$$

from the universal cover of $S$ to $\mathbb{R}^2$ by gluing together local charts $\phi_U$ and $\phi_V$ that agree on $U \cap V$. The map $\phi$ is uniquely determined up to post-composition with an element of $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$. The developing map is equivalent to the data of the monodromy representation

$$
M : \pi_1(S) \to SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2
$$

crafted from the parallel transport of the integral-affine structure along a loop. We also make use of the less refined monodromy map $N : \pi_1(S) \to SL_2(\mathbb{Z})$ which projects onto the $SL_2(\mathbb{Z})$ part of the monodromy.

We now describe how the pseudo-fan of $(V, D)$ changes when we smooth a node of $D$:

**Proposition 2.9.** Let $(\tilde{V}, \tilde{D})$ be a deformation of an anticanonical pair $(V, D)$ such that $\tilde{D}$ is the smoothing of the node $D_{i-1} \cap D_i$. The pseudo-fan of $(\tilde{V}, \tilde{D})$ is the result of the following surgery on the pseudo-fan of $(V, D)$: Delete the triangular face $t_{i-1,i}$ and glue the triangle $t_{i,i+1}$ to the triangle $t_{i-2,i-1}$ via the gluing map

$$
e_i \mapsto e_{i-1}
$$
$$
e_{i-1} \mapsto 2e_{i-1} - e_i.
$$

**Proof.** By the definition of the pseudo-fan of $(V, D)$,

$$
e_{i-1} + e_{i+1} = d_i e_i
$$
$$
e_{i-2} + e_i = d_{i-1} e_{i-1}.
$$

After gluing, the edge clockwise to $e_{i-1}$ is simply $e_{i-2}$, whereas the edge counterclockwise to $e_{i-1}$ is the image of $e_{i+1}$ under the gluing map, which equals $(d_i - 2)e_{i-1} + e_i$. From the equation

$$
e_{i-2} + [(d_i - 2)e_{i-1} + e_i] = (d_{i-1} + d_i - 2)e_{i-1}
$$

we determine that the glued edge has negative self-intersection $d_{i-1} + d_i - 2$. By Remark 1.7, the resulting integral-affine surface is the pseudo-fan of $(\tilde{V}, \tilde{D})$. Note that in the oriented basis $(e_{i-1}, e_i)$, the gluing map is given by the matrix

$$
\begin{pmatrix}
2 & 1 \\
-1 & 0
\end{pmatrix}.
$$

10
And how an internal blow-up changes the pseudo-fan of \((V, D)\):

**Proposition 2.10.** Let \((\tilde{V}, \tilde{D})\) be an internal blow-up of an anticanonical pair \((V, D)\) on the component \(D_i\). The pseudo-fan of \((\tilde{V}, \tilde{D})\) is the result of the following surgery on the pseudo-fan of \((V, D)\): Cut open the edge \(e_i\) of the pseudo-fan of \((V, D)\) corresponding to the component \(D_i\) and re-glue triangle \(t_{i,i+1}\) to triangle \(t_{i-1,i}\) via the gluing map

\[
e_i \mapsto e_i \\
e_{i+1} \mapsto e_i + e_{i+1}.
\]

**Proof.** We note that \(e_{i-1} + e_{i+1} = d_i e_i\). After gluing, the edge clockwise to \(e_i\) is \(e_{i-1}\) and the edge counterclockwise to \(e_i\) is the image of \(e_{i+1}\) under the gluing map, which is \(e_i + e_{i+1}\). The equation

\[
e_{i-1} + [e_i + e_{i+1}] = (d_i + 1)e_i
\]

implies that the glued edge has negative self-intersection \(d_i + 1\), as is the case for the pseudo-fan of \((\tilde{V}, \tilde{D})\). Note that in the oriented basis \((e_i, e_{i+1})\), the gluing map is given by the matrix \[
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}.
\]

### 3 Two Surgeries on Integral-Affine Surfaces

This section describes two surgeries on integral-affine surfaces that prove useful in the construction of a Type III anticanonical pair \((X_0, D)\) in the next section. These surgeries are motivated by work of Symington [Sym03] on almost toric fibrations, with further details in Remark 3.1. For the remainder of this section, let \((S, P, A)\) denote a singular integral-affine surface with a polygonal boundary \(\partial S = P\) homeomorphic to a circle. That is, the boundary

\[
P = P_1 + \cdots + P_n
\]

is the union of a sequence of segments \(P_i\) put end-to-end, with each segment integral-affine equivalent to a straight line segment between two lattice points. We index the boundary components \(P_i\) such that they go counterclockwise around \(S\) as \(i\) increases. Let

\[
v_{i,i+1} := P_i \cap P_{i+1}
\]

denote a vertex of \(P\) and let \((x_i, y_i)\) denote the primitive integral vectors emanating from \(v_{i,i+1}\) along \(P_{i+1}\) and \(P_i\), respectively. Thus, \(y_{i+1} = -x_i\) in a local chart on \(S\) containing the edge \(P_i\). In this section, we assume that \((x_i, y_i)\) is an oriented lattice basis. Consequently, the interior angles at the vertices of \(P\) are less than \(\pi\) in any integral-affine chart.

**Remark 3.1.** Let \((X, \omega)\) be a symplectic, toric surface. Recall that there is a moment map

\[
\mu : (X, \omega) \to S
\]

to a convex planar polygon \(S\) (including its interior) such that the toric boundary components of \(X\) map to the components of \(\partial S\). The general fiber of \(\mu\) is a Lagrangian torus, which degenerates on the edges of \(S\) to a circle and on the vertices of \(S\) to a point. When \([\omega] \in H^2(Y, Z)\) is integral, the moment polygon can be taken to have integral vertices. Setting \(P = \partial S\), we have that \((S, P, \emptyset)\) is a non-singular integral-affine surface satisfying the assumptions of this section.

Following [Sym03], an *almost toric fibration* of \((X, \omega)\) is a Lagrangian fibration

\[
\mu : (X, \omega) \to (S, P, A)
\]
whose general fiber is a smooth 2-torus, which undergoes symplectic reduction over the boundary
$P$, but whose interior fibers may also degenerate to chains of spheres at some finite set of points
$A$. The almost toric base $(S, P, A)$ is a generalization of the moment polygon, and has a natural
integral affine-linear structure, with singularities at points $v \in A$ where the fiber $\mu^{-1}(v)$ is singular.
The inverse image of $P$ is an anticanonical divisor of $X$ in the sense of symplectic geometry.

Let $(Y, D, \omega)$ be an anticanonical pair with an almost toric fibration

$$(Y, D, \omega) \rightarrow (S, P, A)$$

which maps components of $D$ to components of $P$. Symington defines two surgeries on $(S, P, A)$: An internal blow-up and a node smoothing (in the terminology of [Sym03], an almost toric blow-up and a nodal trade, respectively). An internal blow-up of $(S, P, A)$ on the boundary component $P_i$ is the base of an almost toric fibration of a blow-up of $(Y, D, \omega)$ on the component $D_i$. Furthermore, the class of the symplectic form on the blow-up is integral whenever the surgery on $(S, P, A)$ is of the form of Definition 3.3. Analogous statements hold for the node smoothing in Definition 3.4. In addition, the negative self-intersection $d_i$ of the component $D_i$ is equal to the negative self-intersection of $P_i$ in the sense of Definition 3.2.

**Definition 3.2.** Choose a chart of $S$ containing a neighborhood of the edge $P_i$. We define the negative self-intersection $d_i$ of the boundary component $P_i$ by the formula

$$d_i y_i = y_{i-1} - x_i$$

$$= y_{i-1} + y_{i+1}.$$  

We do not define $d_i$ by the formula $d_i y_i = y_{i-1} + y_{i+1}$ because $y_{i+1}$ is not a vector based at a point on $P_i$ and thus, may not be defined in our chosen neighborhood of $P_i$. By extending our chart to a neighborhood of $P_i \cup P_{i+1}$, this definition would become valid. Note that Definition 3.2 fails when $P$ has only one boundary component, as no neighborhood of the single edge is contractible. This problem is resolved by working in a chart on the universal cover of a neighborhood of the boundary component.

**Definition 3.3.** We define an internal blow-up of $(S, P, A)$ on the boundary component $P_i$. First, delete a triangle $T \subset S$ satisfying:

i. One edge of $T$ is a proper subset of $P_i$.

ii. The remainder of $T$ lies in the interior of $S$.

iii. $T$ is integral-affine equivalent to an integer multiple of a basis triangle.

Let $v$ be the unique vertex of $T$ contained in the interior of $S$. Denote by $(e_1, e_2)$ the oriented lattice basis emanating from $v$ along the edges of $T$. See Figure 3. Glue the edge along $e_2$ of $S \setminus T$ to the edge along $e_1$ of $S \setminus T$ via the unique affine-linear map which fixes $v$, maps $e_2 \mapsto e_1$, and preserves the line containing $P_i$. The resulting integral-affine surface is an internal blow-up of $(S, P, A)$ on the boundary component $P_i$. Its singular set is $A \cup \{v\}$.

In the $(e_1, e_2)$ basis, the gluing map acts by the matrix

$$\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$$

and is a shear transformation along the line through $v$ parallel to $e_2 - e_1$. An internal blow-up does not change the number of boundary components because the lefthand and righthand pieces of $P_i$...
are glued into a single line segment. After the surgery, \((x_j, y_j)\) is still an oriented lattice basis for all \(j\) because the internal blow-up does not alter the integral-affine structure in the neighborhood of a vertex. The smallest possible triangle \(T\) removed in an internal blow-up is a basis triangle.

Figure 3: An internal blow-up on \(P_i\).

Figure 4: A node smoothing of \(v_{i,i+1}\).

**Definition 3.4.** We define a node smoothing of \((S, P, A)\) at \(P_i \cap P_{i+1}\). For some \(n \in \mathbb{N}\), make a cut along the segment from \(v_{i,i+1}\) to a point

\[
v := v_{i,i+1} + n(x_i + y_i)
\]

lying in \(S \setminus P\). See Figure 4. Glue the clockwise edge of the cut (from the perspective of \(v\)) to the counterclockwise edge of the cut by the shearing map which point-wise fixes the line containing the cut and maps \(x_i\) to \(-y_i\). The resulting integral-affine surface is a node smoothing of \((S, P, A)\) at \(P_i \cap P_{i+1}\). Its singular set is \(A \cup \{v\}\).

Note that even though the gluing fixes the cut point-wise, it alters the integral-affine structure on the clockwise edge of the cut. Let \(e_1 := -x_i - y_i\) be the primitive integral vector emanating from \(v\) along the cut, and let \(e_2\) be any vector such that \((e_1, e_2)\) is an oriented lattice basis. Then, in the \((e_1, e_2)\) basis, the gluing map is

\[
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
\]

The gluing is independent of the choice of \(e_2\) because it is a shear fixing \(e_1\). The boundary of a node smoothing of \((S, P, A)\) has one fewer edge than \((S, P, A)\): After a node smoothing, the edges \(P_i\) and \(P_{i+1}\) are straightened into a single edge because the image of \(x_i\) under the gluing map is \(-y_i\). As in the case of the internal blow-up, \((x_j, y_j)\) is still an oriented lattice basis for all \(j \neq i\) after the surgery because the node smoothing does not alter the integral-affine structure in the neighborhood of a vertex \(v_{j,j+1}\) such that \(j \neq i\). The vertex \(v_{i,i+1}\) ceases to exist after the surgery. The smallest possible cut for a node smoothing is when \(n = 1\).

**Proposition 3.5.** An internal blow-up of \((S, P, A)\) on the boundary component \(P_i\) transforms the negative self-intersections of the boundary components as follows:

\[(\ldots, d_i, \ldots) \mapsto (\ldots, d_i + 1, \ldots)\]

while smoothing the node \(P_i \cap P_{i+1}\) of \((S, P, A)\) transforms the negative self-intersections of the boundary components as follows:

\[(\ldots, d_i, d_{i+1}, \ldots) \mapsto (\ldots, d_i + d_{i+1} - 2, \ldots).\]
Proof. We omit the proof of the proposition. It follows from straightforward computations involving
the gluing matrices and the vectors \( x_i \) and \( y_i \). Also, it is implicitly proven in [Sym03], because the
negative self-intersection of \( P_i \) from Definition 3.2 is equal to the negative self-intersection \( d_i \) of the
component \( D_i \) which maps to \( P_i \) under an almost toric fibration.

4 A Proof of Looijenga’s Conjecture

In this section, we present the construction of a Type III anticanonical pair \((X_0, D)\) from an
anticanonical pair \((Y, D)\), thus proving Looijenga’s conjecture. But first, we need:

Proposition 4.1. Every anticanonical pair \((Y, D)\) can be expressed as a sequence of node smooth-
ings and internal blow-ups starting with a toric pair \((Y, D)\).

Proof. First, we express \((Y, D)\) as a sequence of corner and internal blow-ups of a minimal anti-
canonical pair \((Y_0, D_0)\). We factor the blow-down to \((Y_0, D_0)\) into maps
\[
(Y, D) \xrightarrow{\alpha} (Y_1, D_1) \xrightarrow{\beta} (Y_0, D_0)
\]
such that \(\alpha\) consists only of interior blow-ups, while \(\beta\) consists only of corner blow-ups, cf. Remark
2.6 of [Fri14]. By direct examination (see Lemma 3.2 of [FM83]), every minimal anticanonical
pair \((Y_0, D_0)\) is a node smoothing of a minimal toric anticanonical pair, i.e. there is a family of
anticanonical pairs with cycle \(D_0\) that degenerates to a toric pair. Performing all the corner blow-
ups of \(\beta\) on this family expresses \((Y_1, D_1)\) as a node smoothing of a toric pair \((Y, D)\). Thus, \((Y, D)\)
is the result of interior blow-ups and node smoothings on \((Y, D)\).

Construction: Part 1. We express \((Y, D)\) as a sequence of node smoothings and internal
blow-ups on a toric pair \((Y, D)\). Let \(S\) be a large moment polygon for \((Y, D)\) with integral vertices.
We define \(P := \partial S\). Then \((S, P, \emptyset)\) is an integral-affine surface with boundary, and no singularities,
such that \((x_i, y_i)\) is an oriented lattice basis. The components \(P_i\) of the boundary have negative
self-intersections \(-D_i^2\) for all \(i\), in the sense of Definition 3.2, and we may simply define \((S, P, \emptyset)\)
by this property, should we wish to avoid an appeal to symplectic geometry.

For each internal blow-up or node smoothing of \((Y, D)\) applied to produce \((Y, D)\), we perform
an associated internal blow-up or node smoothing on the integral-affine surface \((S, P, \emptyset)\). For
instance, we may assume that all surgeries are minimal size, i.e. internal blow-ups remove a single
basis triangle and node smoothings have cuts of minimal length, as in the example below. Since
we are applying \(Q(Y, D)\) surgeries of fixed size, but \((S, P, \emptyset)\) can be arbitrarily large, we may
choose a moment polygon large enough to accommodate all the surgeries. We denote the resulting
integral-affine surface by \((S, P, A)\).

By Proposition 3.5, the negative self-intersections of the boundary components \(P_i\) are equal to
the negative self-intersections \(d_i\) of the components of \(D\). Thus, by Remark 3.1 \((S, P, A)\) is the
base of an almost toric fibration of a symplectic rational surface \((Y, D, \omega)\). For example, the Figure
5 is a large moment polygon \((S, P, \emptyset)\) for the toric surface
\[
(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{D}_1 + \mathcal{D}_2 + \mathcal{D}_3 + \mathcal{D}_4).
\]

Figure 6 demonstrates 18 surgeries on \((S, P, \emptyset)\): Four internal blow-ups on \(\mathcal{D}_1\), two internal blow-
ups on \(\mathcal{D}_2\), six internal blow-ups on \(\mathcal{D}_3\), five internal blow-ups on \(\mathcal{D}_4\) and a node smoothing of
\(\mathcal{D}_1 \cap \mathcal{D}_2\). After the surgeries, the integral-affine surface \((S, P, A)\) is the almost toric base of a
negative-definite anticanonical pair \((Y, D, \omega)\) with \(d = (d_1, d_2, d_3) = (4, 6, 5)\). The charge of \((Y, D)\)
is 18, because each surgery increases the charge by 1.
Figure 5: A moment polygon \((\mathcal{S}, \mathcal{P}, \emptyset)\) for \(Y\).

Figure 6: An almost toric base \((S, P, A)\) for \(Y\).

**Construction: Part 2.** Let \(U \supset P\) be a small, collar neighborhood of the boundary of \((S, P, A)\), as shown in Figure 6. Note that \(\pi_1(U) = \mathbb{Z}\). Consider the developing map

\[\phi : \tilde{U} \to \mathbb{R}^2\]

from the universal cover of \(U\) to \(\mathbb{R}^2\). The universal cover \(\tilde{P}\) of the boundary \(P\) maps to an infinite lattice polygon in \(\mathbb{R}^2\). Let \((d_1, \ldots, d_n)\) denote the cycle of negative self-intersections of the components of \(D\), with indices in \(\mathbb{Z}/n\mathbb{Z}\). Each edge \(P_i\) is integral-affine equivalent to some interval \([0, m_i]\) on the \(x\)-axis, for a unique \(m_i \in \mathbb{N}\). Then \(\phi(\tilde{P})\) is an infinite sequence of vectors \(\{m_i z_i\}_{i \in \mathbb{Z}}\) put end-to-end, such that \((-z_i, z_{i+1})\) is an oriented lattice basis, and

\[d_i z_i = z_{i-1} + z_{i+1}\]

for all \(i\) (the indices of \(m_i\) and \(d_i\) are taken mod \(n\)). We call \(\phi(\tilde{P})\) a *discrete hyperbola*. The interior angles of the discrete hyperbola are less than \(\pi\) because \((-z_i, z_{i+1})\) is a lattice basis for all \(i \in \mathbb{Z}\). One possible image of \(\tilde{U}\) under the developing map is shown in Figure 7, with the lower edge forming the discrete hyperbola.

We claim that the discrete hyperbola has two asymptotic lines \(L_1\) and \(L_2\) which are the invariant lines of the monodromy transformation \(M := M(\gamma)\) associated to a counterclockwise loop \(\gamma\) around the boundary of \((S, P, A)\). The change-of-basis from \((-z_{i-1}, z_i)\) to \((-z_i, z_{i+1})\) is

\[
\begin{pmatrix}
0 & 1 \\
-1 & d_i
\end{pmatrix}
\]

Therefore, the \(SL_2(\mathbb{Z})\) part of the monodromy \(N := N(\gamma)\) is conjugate in \(SL_2(\mathbb{Z})\) to the product

\[
\prod_{i=1}^{n} \begin{pmatrix}
0 & 1 \\
-1 & d_i
\end{pmatrix}
\]

because the counterclockwise monodromy is conjugate to the change-of-basis from \((-z_0, z_1)\) to \((-z_n, z_{n+1})\). By choosing a basis properly, we may assume that \(N\) is equal to the above product.
Figure 7: The image of the developing map of a collar neighborhood of the boundary of \((S,P,A)\).

The full monodromy transformation is

\[
M \cdot v = N \cdot v + B
\]

for some \(B \in \mathbb{Z}^2\). Whenever \((d_1, \ldots, d_n)\) is negative-definite, \(\text{tr} \ N > 2\) and therefore \(N\) has two distinct, irrational positive eigenvalues. We solve the equation \(v = N \cdot v + B\) to find the unique, rational fixed point

\[
v_0 := (I - N)^{-1}B
\]

of \(M\). Then \(L_1\) and \(L_2\) are the lines going through \(v_0\) parallel to the eigenvectors of \(N\). The invariant line associated to the eigenvalue greater than one is stable, while the other invariant line is unstable. To prove that the discrete hyperbola is asymptotic to \(L_1\) and \(L_2\), we note that the monodromy transformation \(M\) sends the discrete hyperbola to itself by mapping

\[
M : \phi(\tilde{P}_i) \mapsto \phi(\tilde{P}_{i+n}).
\]

Thus, the edges \(\phi(\tilde{P}_i)\) of the discrete hyperbola approach the stable and unstable invariant lines of \(M\) as the index \(i\) approaches positive and negative infinity, respectively. The discrete hyperbola bounds a convex region because its interior angles are less than \(\pi\). Any line going through \(v_0\) between \(L_1\) and \(L_2\) eventually intersects this region, because the discrete hyperbola approaches the eigenlines of \(M\). Then, convexity implies that the complement of this convex region is star-shaped at \(v_0\).

Let \(R\) denote the region bounded by \(L_1, L_2,\) and the discrete hyperbola. Let \(L\) be any line going through \(v_0\) between \(L_1\) and \(L_2\) (for instance, we may assume \(L\) is a line through \(v_0\) and a vertex of the discrete hyperbola). Then the region bounded by \(L, M \cdot L,\) and \(\phi(\tilde{P})\) is a fundamental domain for the action of \(M\) on \(R\), see figure 8. The integral-affine quotient

\[
(C, P^{op}, \{v_0\}) := \{M^n : n \in \mathbb{Z}\}\setminus R
\]

is a cone over \(P\) with the opposite orientation, which we denote \(P^{op}\). It is the result of gluing the edge of the fundamental domain bounded by \(L\) to the edge bounded by \(M \cdot L\).

We can in fact form a quotient

\[
\{M^n : n \in \mathbb{Z}\}\setminus (\phi(\tilde{U}) \cup R) = U \cup C
\]
Figure 8: A fundamental domain for the action of $M$ on $R$.

and thus the cone $C$ glues to the boundary of $S$ to produce an integral-affine sphere

$$(\hat{S}, \emptyset, A \cup \{v_0\}) = (S, P, A) \cup (C, P_{\text{op}}, \{v_0\}).$$

The only extra singularity introduced is that at $v_0$. But $v_0$ may have rational coordinates. We therefore define the order $k$ refinement $(S[k], P[k], A)$ of an integral-affine surface $(S, P, A)$ by post-composing the charts on $S$ with multiplication by $k$ (the order $k$ refinement of $\Gamma(X_0)$ corresponds to an order $k$ base change on $X \to \Delta$). Let $k$ be an integer such that $v_0$ is an integral point of $(\hat{S}[k], \emptyset, A \cup \{v_0\})$.

**Construction: Part 3.** The fundamental domains of $(S[k], P[k], A)$ and $(C[k], P_{\text{op}}[k], \{v_0\})$, shown (up to refinement) in Figures 6 and 8, are simply lattice polygons. We view a cut from a node smoothing as two overlapping-edges. We may triangulate these fundamental domains into basis triangles, and glue them together to give $(\hat{S}[k], \emptyset, A \cup \{v_0\})$ the structure of a triangulated integral-affine surface with singularities. Now that we have established the existence of such a triangulation of $(\hat{S}[k], \emptyset, A \cup \{v_0\})$, we choose amongst all of them one which attains the minimal possible number of edges emanating from $v_0$.

Let $v_i$ be a vertex of the triangulation of $\hat{S}[k]$ which is non-singular. Then $\text{star}(v_i)$ is the pseudo-fan of some toric surface $(V_i, D_i)$. Now suppose that $v_i \in A$ is a singular point not equal to $v_0$. Each such singularity $v_i$ is introduced by a surgery on $\overline{S}$. Let $\overline{v}_i \in \overline{S}$ denote the pre-image of $v_i \in S$. In the case of an internal blow-up, one of the triangular faces of $\text{star}(\overline{v}_i)$ is along the triangle removed for the surgery. Then $\text{star}(\overline{v}_i)$ is the pseudo-fan of a toric surface. Note:

i. An internal blow-up on $\overline{S}$ corresponds to a node smoothing on $\text{star}(\overline{v}_i)$ by Proposition 2.9.

ii. A node smoothing on $\overline{S}$ corresponds to an internal blow-up on $\text{star}(\overline{v}_i)$ by Proposition 2.10.

We conclude that there is an anticanonical pair $(V_i, D_i)$ whose pseudo-fan is $\text{star}(v_i)$ for all $i \neq 0$.

Finally, we consider $v_0$. The monodromy $N = N(\gamma)$ of a counterclockwise loop around the boundary $P$ is equal to the monodromy of a clockwise loop around $v_0$. Thus, the monodromy of a counterclockwise loop around $v_0$ is $N^{-1}$.

**Lemma 4.2.** The pseudo-fan of $(V_0, D')$ is isomorphic to $\text{star}(v_0)$.  

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Proof. Let $d_0 = (d_{01}, \ldots, d_{0r})$ denote the negative self-intersections of the edges $(e_{01}, \ldots, e_{0r})$ emanating from $v_0$. We claim that $d_{0i} \geq 2$ for all $i$. First, we show that $d_{0i} \leq 0$ is impossible. Suppose for the sake of contradiction that $d_{0i} \leq 0$. Then the formula

$$d_{0i} e_{0i} = e_{0(i-1)} + e_{0(i+1)}$$

implies that the angle $\angle(v_{i-1}v_0v_{i+1})$ subtended by $\text{star}(v_0)$ between $e_{0(i-1)}$ and $e_{0(i+1)}$ is at least $\pi$ in any integral-affine chart. But, by the definition of the integral-affine structure on $C$, the image of the developing map of $\text{star}(v_0)$ lies within the region $R$. Thus $d_{0i} \leq 0$ is impossible, because the image of the developing map of $\text{star}(v_0)$ subtends an angle less than $\pi$—it subtends the angle formed at $v_0$ by $L_1$ and $L_2$.

If $d_{0i} = 1$, then the union of the two triangles containing $e_{0i}$ is integral-affine equivalent to the unit square. But then we may alter the triangulation by flipping the diagonal of this square, thus decreasing the total number of edges emanating from $v_0$. This contradicts our assumption that the number of edges emanating from $v_0$ is minimal. Hence $d_{0i} \geq 2$ for all $i$. If $d_{0i} = 2$ for all $i$, then the image of developing map subtends an angle of exactly $\pi$, which is also impossible. Hence $d_{0i} \geq 3$ for some $i$. Thus $d_0$ is negative-definite. We remark that similar ideas arise when constructing minimal resolutions of non-smooth toric surfaces, see Section 2.6 of [Ful93].

The developing map, when restricted to the boundary of $\text{star}(v_0)$, maps to an infinite lattice polygon lying in $R$ and bounded by $L_1$ and $L_2$. Because $d_0$ is negative-definite, the angles of this infinite lattice polygon are less than $\pi$, and thus, it bounds a convex region. Furthermore, the image of the developing map of $\text{star}(v_0)$ contains no lattice points in its interior because it is a union of basis triangles containing $v_0$. See Figure 9. This uniquely characterizes the image of the developing map of $\text{star}(v_0)$: It is the region between $L_1$ and $L_2$ in the complement of the convex hull of the lattice points between $L_1$ and $L_2$. We say $\text{star}(v_0)$ has property $(\ast)$.

Let $d = (d_1, \ldots, d_n)$ and $d' = (d'_1, \ldots, d'_s)$. The following matrices are conjugate in $SL_2(\mathbb{Z})$:

$$\prod_{i=1}^r \begin{pmatrix} 0 & -1 \\ 1 & d_{0i} \end{pmatrix} \sim N^{-1} = \left[ \prod_{i=1}^n \begin{pmatrix} 0 & 1 \\ -1 & d_i \end{pmatrix} \right]^{-1} \sim \prod_{i=1}^s \begin{pmatrix} 0 & -1 \\ 1 & d'_i \end{pmatrix}$$

where the last similarity is a general fact about dual cycles, see [Ino77]. That is, the monodromy of $\text{star}(v_0)$ is conjugate to the monodromy of the pseudo-fan of $(V_0, D')$. By post-composing with
an integral-affine transformation, we may assume that these monodromies are equal and that the
developing map of the pseudo-fan of \((V_0, D')\) maps into the region between \(L_1\) and \(L_2\). Since \(d'\) is
negative-definite, the image of the developing map of the pseudo-fan of \((V_0, D')\) is also characterized
by property \((\ast)\). Since the monodromy acts the same on these images, the pseudo-fan of \((V_0, D')\)
and \(\text{star}(v_0)\) are isomorphic, as triangulated integral-affine surfaces.

For every vertex \(v_i\) with \(i \neq 0\) of the triangulation of \((\hat{S}[k], \emptyset, A \cup \{v_0\})\), we have found an
anticanonical pair \((V_i, D_i)\) whose pseudo-fan is \(\text{star}(v_i)\). In addition, we have proved that the
pseudo-fan of \((V_0, D')\) is \(\text{star}(v_0)\). Consider the union of the surfaces
\[ X_0 := \bigcup_{v_i \in \hat{S}[k]^{[0]} (V_i, D_i) } \]
where we identify \(D_{ij}\) with \(D_{ji}\) so that nodes of \(D_i\) are identified with nodes of \(D_j\). By Remark 2.5,
\(X_0\) satisfies the triple point formula. So \(X_0\) satisfies all the assumptions of a Type III anticanonical pair
\((X_0, D)\).

**Theorem 4.3** (Looijenga’s Conjecture). If \(D\) is the anticanonical divisor of some rational surface,
then the cusp singularity associated to \(D'\) is smoothable.

**Proof.** Let \((Y, D)\) be an anticanonical pair. Apply parts 1, 2, and 3 of the construction to produce
a Type III anticanonical pair \((X_0, D)\). Theorem 1.9 implies that the cusp \(D'\) is smoothable.

**Modifications and Generalizations of the Construction.**

1. We need not assume that the singularities introduced in the surgeries on \((\overline{S}, \overline{P}, \emptyset)\) are distinct.
The number of surgeries that the vertex \(v_i\) is involved in (either as the vertex of a triangle
removed from \(\overline{S}\) for an internal blow-up, or as the end of a cut for a node smoothing) is equal
to the charge \(Q(V_i, D_i)\) of the anticanonical pair whose pseudo-fan is \(\text{star}(v_i)\). In addition,
the edges of the triangles removed for internal blow-ups may overlap, and may also overlap
cuts for node smoothings.

2. We assumed that for every component of \(\overline{D}\), the length of the associated boundary component
of \(\overline{P}\) was positive. This assumption is unnecessary—some may have length zero (but at least
three edges must have positive length, because \(\overline{S}\) must have nonempty interior).

3. The internal blow-ups on a boundary component of \((\overline{S}, \overline{P}, \emptyset)\) decrease its length. After the
surgeries are performed, we may allow some of the boundary components of \((\overline{S}, \overline{P}, A)\) to have
length zero. When only some of the boundary components have length zero, we continue with
the construction by applying the developing map to a collar neighborhood of the boundary.

4. If, after the surgeries, \textit{all} the boundary components have length zero, then \((S, P, A)\) has no
boundary, is already homeomorphic to a sphere, and we may directly apply part 3 of the
construction.

**Remark 4.4.** A more careful analysis shows that the third modification is always possible for
the sub-class of anticanonical pairs \((Y, D)\) that do not blow down to any other negative-definite
pair. This gives a simple normal crossings resolution of a smoothing of the cusp associated to \(D'\).
Looijenga’s conjecture follows by observing that whenever \((Z, E)\) blows down to \((Y, D)\), there is a
Wahl cusp adjacency \([\text{Wah80}]\) from \(E'\) to \(D'\). That is, the cusp associated to \(E'\) partially smooths
to the cusp associated to \(D'\). By the openness of versality, smoothability of the cusp associated \(D'\)
implies smoothability of the cusp associated to \(E'\).
We now present an example of the third modification of the construction:

**Example 4.5.** Let \((Y,D)\) be a negative-definite anticanonical pair such that \(Q(Y,D) = 3\). It can be shown that all negative-definite anticanonical pairs with \(Q(Y,D) = 3\) have three disjoint internal exceptional curves, which can be blown down to a toric surface

\[ \pi : (Y,D) \to (\overline{Y}, \overline{D}) . \]

We call \((\overline{Y}, \overline{D})\) a toric model. Let \(D_{a_1}, D_{a_2}, \text{ and } D_{a_3}\) be the three components of \(D\) that receive the internal blow-ups. Up to scaling by \(\mathbb{Z}\), there is a unique moment polygon \((\overline{S}, \overline{P}, \emptyset)\) of the toric model in which \(\overline{P}_{a_1}, \overline{P}_{a_2}, \text{ and } \overline{P}_{a_3}\) are the only edges with nonzero length.

To construct \((S, P, A)\), we perform internal blow-ups on the three boundary components of \((\overline{S}, \overline{P}, \emptyset)\). To do so, we must delete a multiple of a basis triangle from each of the edges. For any anticanonical pair \((Y,D)\) of charge three, it can be proven that \((\overline{S}, \overline{P}, \emptyset)\) has room to perform internal blow-ups that decrease the lengths of all three edges to zero. So \(P\) is empty, and \((S, P, A)\) may immediately be triangulated into basis triangles, from which we construct \(X_0\).

Consider the cusp singularity \(D'\) with cycle \(d' = (6, 9)\). The dual cycle is given by the formula

\[ d = (3, 2, 2, 3, 2, 2, 2, 2, 2, 2, 2) . \]

There are two distinct deformation families of pairs with anticanonical cycle \(D\). The deformations preserve the classes of exceptional curves, so each deformation family is associated to a different toric model. Let \((iY, D)\) with \(i = 1, 2\) be two anticanonical pairs representing these two deformation families. The cycles of negative self-intersections of the two toric models are

\[ ^1d = (3, 2, 1, 2, 3, 1, 2, 2, 2, 2, 1) \]
\[ ^2d = (3, 2, 2, 1, 3, 2, 1, 2, 2, 2, 1) \]

By blowing down exceptional curves in \(^iD\), we can draw fans for \((iY, ^iD)\), from which we can construct moment polygons \((^iS, ^iP, \emptyset)\) for \(i = 1, 2\). We choose moment polygons such that only components of \(^iP\) of positive length are \(^1P_3, ^1P_6, \text{ and } ^1P_{11}\), while the only components of \(^2P\) of positive length are \(^2P_4, ^2P_7, \text{ and } ^2P_{11}\) as in Figure 10.

![Figure 10: Moment polygons for \((1Y, 1D)\).](image1)

![Figure 11: Almost toric bases for \((1Y, D)\).](image2)
We perform three internal blow-ups on $\langle S, P, A \rangle$ by deleting a multiple of a basis triangle resting on each of the three edges, then gluing the remaining two edges of each triangle. Furthermore, we choose surgeries large enough to reduce the length of the boundary to zero. The resulting integral affine surfaces $\langle S, P, A \rangle$ are shown Figure 11. Because $S$ has no boundary, we may immediately triangulate it into basis triangles, and construct the simple normal crossings surface $\mathcal{X}_0$ whose dual complex is $S$. The triangulations of $S$ and the surfaces $\mathcal{X}_0$ are shown in Figure 12.

The Hirzebruch-Inoue pair $(V_0, D')$ is the outer face in both illustrations.

The surface $\mathcal{X}_0$ smooths to give a family $\mathcal{X} \to \Delta$ of surfaces over the disc whose general fiber is a pair with anticanonical cycle $D$. It is a natural question to ask whether the general fiber of $\mathcal{X}$ is deformation-equivalent to $(\mathcal{Y}, D)$. An affirmative answer in the case $Q(Y, D) = 3$ would verify Conjecture 6.1 of [FM83]. To prove smoothability of the cusp $D' = (6, 9)$, Friedman and Miranda exhibit the special fiber $\mathcal{X}_0$. The surface $\mathcal{X}_0$ is likely to be the alternative special fiber with 50 triple points that they conjecture to exist.

More generally, it is natural to conjecture that if we begin with an anticanonical pair $(Y, D)$, apply parts 1, 2, and 3 of the construction to produce a Type III anticanonical pair $(\mathcal{X}_0, D)$, and smooth $\mathcal{X}_0$ to an anticanonical pair with cycle $D$, we land in the same deformation family as the original pair $(Y, D)$. Perhaps, this says something to the effect of “$(Y, D)$ is its own mirror.” If, in addition, one may prove that every SNC model of a smoothing arises from the construction (or a generalization of it), a correspondence between smoothing components of the cusp with resolution $D'$ and deformation families of anticanonical pairs $(Y, D)$ might be established. Even more speculatively, it may be possible to construct a compactification of the moduli space of anticanonical pairs deformation-equivalent to $(Y, D)$ which explicitly produces a smoothing component of the cusp associated to $D'$.
Figure 12: Two Type III anticanonical pairs \((\mathcal{X}_0, D)\).
References


