1. Let $X, Y$ be metric spaces.
   (a) Show that the identity map $f : X \to X$ defined by $f(x) = x$ is continuous.
   (b) For a point $q$ of $Y$, show that the constant map $g : X \to Y$ defined by $g(x) = q$ is continuous.

2. Let $f : \mathbb{R}^k \to \mathbb{R}$ denote the function determined by the norm $f(x) = \|x\|$. Show that $f$ is continuous.

3. Let $X, Y$ be metric spaces. Suppose $X$ is equipped with the discrete metric
   \[ d(p, q) = \begin{cases} 
   1 & p \neq q \\
   0 & p = q.
   \end{cases} \]
   Show that any function $f : X \to Y$ is continuous.

4. Let $X, Y$ be metric spaces. Suppose $X$ is connected and $Y$ satisfies the property that every singleton set $\{y\}$ is open in $Y$. Show that a function $f : X \to Y$ is continuous if and only if $f$ is constant. Deduce that any continuous function $\mathbb{R} \to \mathbb{N}$ is constant.

5. Let $f : X \to Y$ be continuous.
   (a) For any subset $E \subset X$, show that $f(E) \subset f(E)$.
   Also, find an example where the inclusion is strict.
   (b) If $E$ is dense in $X$ and $f$ is surjective, show that $f(E)$ is dense in $Y$.
   (c) Let $g : X \to Y$ be another continuous function, and let $X_0 \subset X$ be a dense subset of $X$. Show that if $f(x) = g(x)$ for each $x \in X_0$, then $f(x) = g(x)$ for each $x \in Y$.

6. Let $I$ denote the unit interval $I = [0, 1]$ of $\mathbb{R}$. Show that any continuous map $f : I \to I$ has a fixed point, that is, a point $x_0 \in I$ such that $f(x_0) = x_0$.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = (x + 1)^2$. Let $\epsilon > 0$ be given.
   (a) Find a $\delta > 0$ such that if $x$ satisfies $|x - 3| < \delta$, then $|f(x) - f(3)| < \epsilon$.
   (b) Find a function $\delta : \mathbb{R} \to \mathbb{R}_{>0}$ such that if $x, p \in \mathbb{R}$ satisfy $|x - p| < \delta(p)$, then $|f(x) - f(p)| < \epsilon$. (Note that part (a) determines a possible value for $\delta(3)$.)
   (c) Is it possible to choose $\delta(p)$ from (b) to be independent of $p$, that is, to be a constant function? Why or why not?
(d) What if the domains of \( f \) and \( \delta \) are restricted to \([-2, 0]\)? Then is it possible to make \( \delta(p) \) constant? Why or why not?

8. Let \( f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) be the square root function \( f(x) = \sqrt{x} \). Show that \( f \) is uniformly continuous (even though the domain of \( f \) is not compact).

9. Let \( f : \mathbb{R} \to \mathbb{R} \) denote the function defined by

\[
f(x) = \begin{cases} 
\frac{1}{n} & x = m/n \text{ for } m, n \text{ relatively prime integers with } n > 0 \\
0 & x \text{ is irrational}
\end{cases}
\]

(And when \( x = 0 \), take \( n = 1 \).) Prove that \( f \) is continuous at every irrational number and discontinuous at every rational number.

10. Let \( \alpha \) be a positive irrational number. Let \( E \) denote the subset of \( \mathbb{R} \) given by

\[
E = \{m + n\alpha : m, n \in \mathbb{Z}\}.
\]

The goal of this problem is to show that \( E \) is dense in \( \mathbb{R} \).

(a) Show that if \( e \in E \), then \( -e \in E \).

(b) Let \( \lfloor \alpha \rfloor \) denote the largest nonnegative integer smaller than \( \alpha \). In other words,

\[
\lfloor \alpha \rfloor = \sup(\mathbb{Z} \cap (-\infty, \alpha]).
\]

Note that \( 0 \leq \alpha - \lfloor \alpha \rfloor < 1 \). For each positive integer \( k \), let

\[
\beta_k = k\alpha - \lfloor k\alpha \rfloor.
\]

Show that if \( j \neq k \), then \( \beta_j \neq \beta_k \).

(c) Let \( N \) be an integer satisfying \( N \geq 2 \). For each integer \( \ell \), let

\[
A_\ell = \left[ \frac{\ell}{N}, \frac{\ell + 1}{N} \right).
\]

Show that there is an integer \( \ell \) satisfying \( 0 \leq \ell \leq N - 1 \) and integers \( j, k \) satisfying \( 1 \leq j < k \leq N + 1 \) such that

\[
\beta_j, \beta_k \in A_\ell.
\]

(d) Use (b) to show that there is an element \( e \in E \) such that \( 0 < e < \frac{1}{N} \).

(e) For each integer \( \ell \geq 0 \), show that there is an element of \( E \) in \( A_\ell \). Deduce from (a) that there is also an element in \( A_{-\ell} \).

(f) For each point \( x \in \mathbb{R} \) and each \( \epsilon > 0 \), show that there is a point in the intersection \( E \cap B_\epsilon(x) \).

(g) Deduce that \( E \) is dense in \( \mathbb{R} \).