Kuznetsov components and some examples of fractional Calabi-Yau categories.

Everything in this talk, unless otherwise mentioned, is taken from Alexander Kuznetsov's article: "Calabi-Yau and fractional Calabi-Yau categories".

Recall that if $X$ is a smooth projective variety of dimension $n$, then $X$ has a Serre functor, namely: $S_X = \omega_X \langle n \rangle \otimes -$ where $\omega_X$ is the canonical bundle.

In particular, if $X$ is Calabi-Yau, then $S_X = \langle \dim X \rangle$. This motivates the following definitions:

**Def:** Let $C$ be a triangulated category.

1) $C$ is a $m$-Calabi-Yau category if it has a Serre functor $S_C$ and if there exists $n \in \mathbb{Z}$ s.t. $S_C^n = \langle m \rangle$.

Such an $m$ is called the Calabi-Yau dimension of $C$.

2) $C$ is a fractional Calabi-Yau category if $C$ has a Serre functor $S_C$ and there exists $p \in \mathbb{Z}, q \in \mathbb{Z}^+$ s.t. $S_C^q \simeq \langle p \rangle$.

The main theorem of Kuznetsov's article gives several examples of $\text{CY}$-categories and fractional $\text{CY}$-categories, which will be pieces in the semi-orthogonal decomposition of some varieties.

Why are we interested in such categories?

- If $X$ is a variety with a semi-orthogonal component which is a 2-$\text{CY}$-category, then any moduli space of coherent sheaves on $X$ has a closed 2-form. In some cases this gives interesting hyper-Kähler varieties.

- More generally, some geometric properties of $X$ can be deduced from the existence of a fractional $\text{CY}$-component in its semi-orthogonal decomposition.
2. Notations and formulas:

- Let $\phi$ be a functor. We will denote by $\phi^*$ and $\phi^!$ its left, respectively right, adjoints, if they exist. Then we will denote:

$$\eta_{\phi^*}, \phi^t : \text{id} \to \phi \circ \phi^* \quad \varepsilon_{\phi^*, \phi} : \phi^* \circ \phi \to \text{id}.$$ 

Furthermore

$$(\phi \circ \varepsilon_{\phi^*, \phi}) \circ (\eta_{\phi^*}, \phi^t \circ \phi) = \text{id} \quad (1)$$

$$(\varepsilon_{\phi^*, \phi} \circ \phi^t) \circ (\phi^* \circ \eta_{\phi^*}, \phi^t) = \text{id} \quad (2)$$

- $\phi$ is fully faithful if $\varepsilon_{\phi^*, \phi}$ is an isomorphism, if $\eta_{\phi^*}, \phi^t$ is an isomorphism ("Fourier–Mukai transforms in algebraic geometry", D. Huybrechts, p.8).

- $\psi = (\eta_{\phi^*}, \phi^t) + (\phi^t \circ \eta_{\phi^*}, \phi^t) : \phi^* \circ \phi^t \to \phi \circ \phi^t \circ \phi^t$

- $\Gamma = (\phi^t \circ \varepsilon_{\phi^*, \phi}^t) + (\varepsilon_{\phi^*, \phi}^t \circ \phi^t) : \phi^* \circ \phi^t \to \phi^* \circ \phi^t$

- We have the distinguished triangles:

$$\begin{align*} 
(3) & \quad T_y \to \text{id} \to \phi \circ \phi^t \quad \eta_{\phi^*, \phi} \\
(4) & \quad \phi \circ \phi^t \xrightarrow{\varepsilon_{\phi^*, \phi}^t} \text{id} \to T_y' \\
(5) & \quad \phi^t \circ \phi \to \text{id} \to T_x \quad \phi^t \circ \phi \\
(6) & \quad T_x' \to \text{id} \to \text{id} \to \phi \circ \phi^t 
\end{align*}$$

where $\phi : D^b(X) \to D^b(Y)$ is spherical.

- $\phi_* T_x \simeq T_y \circ \phi [2] \quad (7) \quad \phi^* T_y [-2] \simeq T_x \circ \phi^* \quad (8)$

- $\rho = T_x \circ \mathcal{X}_x \quad \sigma = S_{X, T_x} \circ \mathcal{X}_x$

- $T_x \circ \mathcal{Y}_x \simeq \mathcal{Y}_x \circ T_x \quad (9)$

- $\mathcal{L}_x \circ \phi = \phi \circ \mathcal{L}_x \quad (10)$

- $T_x \circ \mathcal{L}_x = \mathcal{L}_x \circ T_x \quad (11)$

- $S_{d/c} = e^{\mathcal{M}/c} \circ d/c \quad (12)$

- $A_\chi = \{ \mathcal{F} \in D^b(X) \mid \phi_*(\mathcal{F}) \in \mathcal{B} \otimes \mathcal{L}_\Pi \}$

- $B \otimes \mathcal{Y}_x^{-1} > \mathcal{Y}_x$ 

- $\mathcal{A}_x = \{ \mathcal{F} \in \mathcal{D}_b(X) \mid \phi_*(\mathcal{F}) \in \mathcal{B} \otimes \mathcal{L}_\Pi \}$
3. Spherical functors

Let $X$ and $Y$ be smooth projective varieties.

Def. Let $\phi: D^b(X) \to D^b(Y)$ be a Fourier-Mukai functor.

In particular, $\phi$ has then a left and a right adjoint.

Then $\phi$ is called spherical if:

1) $\psi := (\eta \phi, \phi') \circ (\phi^* \eta \phi') : [\phi \oplus \phi']^*(A) \to \phi^* \phi \circ \phi' (A)$ is an isomorphism for all $A \in D^b(Y)$.

2) $\Gamma := (\phi^* \circ \phi) + (\epsilon \phi^* \circ \phi') : \phi^* \phi \circ \phi' (A) \to (\phi^* \oplus \phi') (A)$ is an isomorphism for all $A \in D^b(Y)$.

Def. Let $\phi: D^b(X) \to D^b(Y)$ be a spherical functor. Then we can define the functors $T_X, T'_X, T_Y, T'_Y$ as follows:

we have the following distinguished triangles, where we identify a Fourier-Mukai functor to its kernel (i.e. we will write $\phi_{\mathbb{A}}$ for $\mathbb{A}$), since $\text{id} = \phi_{\mathbb{A}}$:

$T_Y \to \text{id} \to \phi \circ \phi^*$

$\phi \circ \phi^*$

$\phi^* \circ \phi \to \text{id} \to T_X$

$\epsilon \phi^* \phi$

Then $T_X$ and $T_Y$ are called spherical twistings.

Prop. With the same notations:

$T_X, T'_X$ and $T_Y, T'_Y$ are mutually inverse autoequivalences of $D^b(X)$, respectively $D^b(Y)$.

Proof: Let $S: \phi \circ \phi \to T_X'\circ [1]$. Comparing (6) with $\phi^*$ on the right, one gets the distinguished triangle ($\phi^*$ is exact since it is a FM transform):

$\phi^* \to \phi \circ \phi \circ \phi^* \xrightarrow{S \circ \phi^*} T_X' \circ [1] \circ \phi^*$

This implies that $(S \circ \phi^*) \circ (\eta \phi, \phi') \circ \phi^* = 0$ (\star)

Let $\Psi = (S \circ \phi^*) \circ (\eta \phi, \phi')$. Then:
\( (a) \quad \Phi^* \rightarrow \Phi^* \oplus \Phi' \rightarrow \Phi' \)

\( \text{id} \downarrow \downarrow \psi \downarrow \downarrow \psi \downarrow \downarrow \psi \)

\( \Phi^* \rightarrow \Phi' \circ \Phi' \rightarrow T_x'[1] \circ \Phi^* \)

\( \Psi^* \rightarrow \Phi' \circ \Phi' \rightarrow S \circ \Phi^* \)

The right square commutes because of (4) and by definition of \( \psi \).
The left square commutes by definition of \( \psi \).
Therefore, (a) and (b) are distinguished triangles and \( \text{id} \) and \( \psi \) are isomorphisms, thus \( \Phi \) is an isomorphism.

Moreover, one can get the following diagram:

\[
\begin{array}{c}
\Phi' \circ \Phi \\
\xrightarrow{\Phi' \circ \Phi \circ \Phi'} \Phi' \circ \Phi \circ \Phi'
\end{array}
\]

The square commutes since we have \( \Phi' \circ \Phi \circ \Phi' \circ \Phi = \Phi \circ \Phi \), and \( \gamma = \text{id} \).

\[
S \circ \Phi \circ \Phi \circ \Phi = \Phi \circ \Phi \circ \Phi
\]

since \( \gamma = \Phi' \circ \Phi \circ \Phi \circ \Phi = \Phi' \circ \Phi \circ \Phi \circ \Phi \circ \Phi \) by (4).

As a result, \( S = S \circ \text{id} = T_x'[1] \circ \varepsilon_{\Phi'} \circ (S \circ \Phi \circ \Phi \circ \Phi \circ \Phi) \).

Therefore the right square in the following diagram is commutative:

\[
\begin{array}{c}
\Phi' \circ \Phi \\
\xrightarrow{S} T_x'[1]
\end{array}
\]

\[
\Psi^* \rightarrow \Phi' \circ \Phi' \rightarrow T_x'[1] \circ \Phi^* \rightarrow T_x'[1]
\]

Thereby \( \Psi = \Phi \circ \Phi \) is an isomorphism since \( \Phi \) is an isomorphism.
The triangles (c) and (d) are distinguished, by definition for (c) and for (d) because it is (5) composed with $T'_{x}$ on the left.

Hence, there exists a map $\psi: \text{id} \to T'_{x} \circ T_{x}$ such that the left square commutes. Finally, since $\text{id}$ and $\psi$ are isomorphisms, so is $\psi$.

Similarly, $T'_{y} \circ T_{y} = \text{id}$. Using $T$ instead of $\psi$ and a similar reasoning, one also gets $T_{x} \circ T'_{x} = \text{id}$ and $T_{y} \circ T'_{y} = \text{id}$.

**Rem:** In “Spherical DG-functors,” Rina Anno and Timothy Logvinenko give a different definition of spherical functor:

* $T_{x}$ and $T'_{x}$ are quasi-inverse autoequivalences
* $T_{y}$ and $T'_{y}$ are quasi-inverse autoequivalences
* $\phi \circ T'_{y} [-1] \to \phi \circ \phi^{*} \to \phi^{*}$ is an isomorphism

* $\phi^{*} \to \phi \circ \phi^{*} \to T'_{x} \circ \phi^{*} [-1]$ is an isomorphism.

This definition is equivalent to the one with $\psi$ and $T$.

**Prop:** With the same notation, $\phi \circ T_{x} \sim T_{y} \circ \phi [2]$ and $T_{x} \circ \phi^{*} \sim \phi \circ T_{y}^{*} [2]$.

**Proof:** Combining $\phi^{*} \sim T_{x}^{*} [1] \circ \phi^{*}$ and $T_{x} \circ T_{x} = \text{id}$ one gets:

$$T_{x} \circ \phi^{*} \sim \phi^{*} [1] \quad (\ast)$$

Similarly the triangle $\text{id} \to \phi \circ \phi^{*} \to T_{y}^{*} [1]$ is distinguished.

Composing with $\phi^{*}$ on the left, one gets the distinguished triangle:

$$\phi^{*} \circ \eta \circ \phi^{*} \to \phi^{*} \circ \phi \circ \phi^{*} \to \phi^{*} \circ T_{y}^{*} [1].$$

In particular, $(\phi^{*} \circ \tau) \circ (\phi^{*} \circ \eta \circ \phi^{*}) = 0. \quad (\ast \ast)$

Let $\psi = \phi^{*} \circ \tau \circ \eta \circ \phi^{*}$

Then

$$\phi^{*} \to \phi^{*} \circ \phi \circ \phi^{*} \to \phi^{*} \circ T_{y}^{*} [1]$$

$$\phi^{*} \circ \eta \circ \phi^{*} \to \phi^{*} \circ \phi \circ \phi^{*} \to \phi^{*} \circ T_{y}^{*} [1]$$

commutes by $(\ast \ast)$ and the definition of $\psi$. Hence $\phi^{*}$ is an isomorphism and $\phi^{*} \sim \phi^{*} \circ T_{y}^{*} [1]. \quad (\ast \ast \ast)$
6. Finally, \( \phi^* \cdot T^{-1} [-1] \cong \phi^* \cdot T^{-1} \cdot \phi^* \cdot f^! \) \((**)\) 

Composing with \( T_x \) on the left and \( T_y \cdot f^! \) on the right gives:

\[ T_x \circ T^* \cong T^* \cdot T_y \cdot f^! \] 

By Yoneda's lemma, a right adjoint is unique up to isomorphism, thus taking right adjoints on both sides one gets:

\[ T_y \circ f^! [-1] \cong \phi^* \cdot T^{-1} \Rightarrow T_y \circ f^! [2] \cong \phi^* \cdot T \]

2. Kuznetsov's Theorem

Let \( X \) be a smooth projective variety. Let \( Y \) be a smooth projective variety with a rectangular left spectral decomposition of length \( m \) with respect to \( L_X \).

Let \( \phi: D^b(X) \to D^b(Y) \) be a spherical functor.

Finally let us define \( E = T_x \circ L_x \) with \( L_x \) and \( d \) as in the following theorem, and \( E = S_x \circ T_x \circ L^m_x \).

Theorem (Kuznetsov) With the same notations:

Assume \( D^b(Y) = \langle B, B \otimes L^{i-1}_Y, \ldots, B \otimes L^{-m-1}_Y \rangle \) and:

1. \( 1 \leq d < m \) s.t. \( \forall i \in \mathbb{Z} \) \( T_Y \cdot (B \otimes L^i_Y) = B \otimes L^{i-d}_Y \).
2. \( E \) a line bundle on \( X \) s.t. \( L_Y \circ \phi \cong \phi \cdot L_X \) where \( L_Y \) here means the functor tensoring with \( L^*_Y \).
3. \( L_x \cdot L_x = L_x \cdot T_x \)

Then \( \phi^*: D^b(Y) \to D^b(X) \) is fully faithful on \( B \) and induces a semi-orthogonal decomposition:

\[ D^b(X) = \langle A_x, B_x, B_x \otimes L^1_x, \ldots, B_x \otimes L^{m-d-1}_x \rangle \]

with \( B_x = \phi^*(B) \) \( A_x = \langle B_x, \ldots, B_x \otimes L_x^{m-d-1} \rangle \)

\( A_x \) is called the Kuznetsov component.

If \( c = \text{gcd}(d, m) \) then \( A_x \) has a \( c \)-pure \( c \)-functor s.t. \( S^d/c \cdot L_x \cong L_x \).

In particular if \( c \) and \( d \) are shifts then \( A_x \) is a
7. fractional Calabi-Yau category.

Remark: if $d = m$ then $\phi^*$ is not fully faithful but if $A_X = D^b(X)$ then $S_{\phi^*} = S_{\phi} = e^{-1.6}.

3. Examples of (fractional) CY-categories given by this theorem

3.1. Let $X, \tau$ be smooth projective varieties.

Let $f: X \to \tau$ be a divisorial embedding s.t. $|f(X)| \geq |\Lambda_{11}|, 1 \leq d \leq m$.

Then $\phi = Rf^*$ is spherical.

If we will denote $f^*$ for $Rf^*$.

$\text{If} \phi^* \text{is a } \text{FM} \text{ transform of kernel } O_{\pi}^\phi$.

Then $\phi^*$ is a left adjoint of $f^*$ and $f^! = \omega_X \otimes \omega_{\tau}^* |_X [-1] \otimes \phi^*(-)$ is a right adjoint of $\phi^*$ (this comes from Grothendieck-Vardier duality).

Now the adjunction formula gives $\omega_X = f^*(\omega_{\tau} \otimes O_X)$

$$\implies f^*(\omega_{\tau} \otimes \mathbb{L}_{\tau}) = \omega_X.$$

Let us set $\mathbb{L}_X = f^*(\mathbb{L}_{\tau})$. Then $\omega_X \otimes \omega_{\tau}|_X = f^*(\omega_{\tau}) \otimes \omega_X \otimes \omega_{\tau}(-d) = \mathbb{L}_X^{-d}$.

Hence $f^! = f^*(-d) \otimes \mathbb{L}_X^{-d}[-1].$ Moreover $f^!(f_!(f^*(F))) = f^!(F \otimes f_! O_X)$.

But there is a resolution (the Koszul resolution) of $f_!(O_X)$:

$$O(-X) \xrightarrow{\psi} O_{\tau} \to f_!(O_X) \to 0$$

Therefore $f^!(f_! O_X \otimes F) = f^!(F \otimes (\mathbb{L}_{\tau} \to O_{\tau}))$

$$= f^!(F) \otimes f^!(\mathbb{L}_{\tau} \to O_{\tau}) \otimes \mathbb{L}_X^{-d}[-1] = (\ast).$$

But $\psi|_X = 0$, hence $f^*(\mathbb{L}_{\tau} \to O_{\tau}) = \mathbb{L}_X^{-d} \otimes O_X \otimes \mathbb{L}_X^d [-1]$.

Then $\ast \implies f^!(F) \otimes \mathbb{L}_X^{-d} \otimes (O_X \otimes \mathbb{L}_X^{-d} [-1])[-1]$. 
\[ \otimes = (p^*(F) \otimes \mathcal{L}_x^d [-1]) \oplus p^*(F) = \mathcal{G}^!(F) \oplus \mathcal{G}^*(F). \]

Thus \( \psi \) is an isomorphism.

Similarly, \( p^* \circ \mathcal{G}^! (F) = p^* \circ \mathcal{G}^! (p^*(F) \otimes \mathcal{L}_x^d [-1]) \]
\[ = p^* (F \otimes \mathcal{G}^! (\mathcal{L}_x^d)) [-1] \]
\[ = p^* (F \otimes \mathcal{L}_x^d \otimes \mathcal{G}^! (\mathcal{L}_x^d)) [-1] \]
\[ = p^* (F) \otimes \mathcal{L}_x^d \otimes (\mathcal{L}_x^d \otimes \mathcal{L}_x^d) [-1] \]
\[ = p^* (F) \otimes \mathcal{G}^! (F) \]

And \( \Gamma \) is an isomorphism.

We would like to apply the Theorem, for this we need:

**Prop:** With the same notations and assumptions as above:

1. \( T_x \) commutes with \( \mathcal{L}_x \)
2. \( \exists \ p \ s.t. \ p^p \) is a shift
3. \( \exists \ q \ s.t. \ \sigma^q \) is a shift.

**Proof:** There is a distinguished triangle \( \mathcal{L}_x^{-d} \to \mathcal{G}^! \to \mathcal{G}^* \)

\[ \forall F \in D^b(X) \]
\[ \mathcal{G} \otimes \mathcal{L}_x^{-d} \to \mathcal{G} \to \mathcal{G}^* \]

By definition of \( T_y \) and \( T_x \) this implies \( T_y = \mathcal{L}_y^{-d} \) and \( T_x = \mathcal{L}_x^{-d} \).

Thus \( T_x \) commutes with \( \mathcal{L}_x \) and (11) is satisfied.

\[ e = T_x \circ \mathcal{L}_x^{-d} = [2] \]
\[ \sigma = S_x \circ T_x \circ \mathcal{L}_x^m = \omega_x \cdot \text{dim} \ X \otimes \mathcal{L}_x^{-d} [-2] \otimes \mathcal{L}_x^m \]
\[ \omega_x = p^* (\omega_y \otimes \mathcal{L}_x^d) = p^* (\mathcal{L}_x^{d-m}) = \mathcal{L}_x^{d-m} \]
\[ \sigma = \text{dim} \ X + 2 = \text{dim} \ Y + 1. \]
To apply the theorem we still need to check (10) and (9).

Here $\phi = f^*$ and $f^*(L_x \otimes F) = f^*(f^*(\mathcal{L}_\pi) \otimes F) = \mathcal{L}_\pi \otimes f^*(F)$

Hence (10) holds.

$$T_\pi = \mathcal{L}_\pi^{-d} \Rightarrow T_\pi (B \otimes \mathcal{L}_\pi^i) = B \otimes \mathcal{L}_\pi^{-d} \text{ and (9) holds.}$$

In the end in such a case we need to check:
- That the decomposition of $\Pi$ exists.

Examples: 1) $X \subset \mathbb{P}^n$ a smooth hypersurface of degree $d \leq m$. Then:

* $f : X \rightarrow \mathbb{P}^n$ and $|f(x)| \leq |O_{\mathbb{P}^n}(d)| = |O_{\mathbb{P}^n}(d)|$

* $\omega_{\mathbb{P}^n} = O_{\mathbb{P}^n}(-M-1) = O_{\mathbb{P}^n}(1) = (M+1)$

* $D^b(\mathbb{P}^n) = < O_{\mathbb{P}^n}, \ldots, O_{\mathbb{P}^n}(M) >$

* $B = < O_X >, B = < O_{\mathbb{P}^n} >$

$$T_\pi (B \otimes \mathcal{L}_\pi^i) = T_\pi (\mathcal{L}_\pi^i) = \mathcal{L}_\pi^{-d}$$

Hence $D^b(X) = < O_X, O_X, \ldots, O_X(n-d) >$

$$c = \gcd(d, M+1) \quad S_{\mathcal{O}_X} = \begin{bmatrix} -2 (M+1) \quad (M+n) \quad d \\ \frac{1}{c} \end{bmatrix} = \begin{bmatrix} -2 (M+1)(d-2) \\ c \end{bmatrix}$$

If $d | M+1$ then $c = d$ and $S_{\mathcal{O}_X} = \begin{bmatrix} (M+1)(d-2) \\ c \end{bmatrix}$.

Hypersurfaces of degree $3$ in $\mathbb{P}^5$ is a particular case. $d$

2) $X \subset \mathbb{P}(w_0, \ldots, w_m) = \Pi$ a smooth hypersurface of degree $d < \omega = \sum_{i=0}^{m} w_i$

in a weighted projective space.

* $D^b(\Pi) = < 0, \ldots, O_\Pi(\omega-1) >, m = \omega$

* $f : X \rightarrow \Pi$ and $|f(x)| \leq |O_{\Pi}(d)|$

* $\omega_{\Pi} = O_{\Pi}(1) - \omega$

* $B = < O_X >, \omega_{\Pi} = O_{\Pi}(1) \text{ so it is as in the example 1}$

Therefore $D^b(X) = < O_X, O_X, \ldots, O_X(n-d-1)>$
10. and if \( c = \gcd(d, w) \) then \( S_{A_x}^{dc} = \left[ -2 \frac{w}{c} + d (m+1) \right] \)

If \( d \mid w \) then \( c = d \) and \( S_{A_x} = \left[ -2 \frac{w}{d} + m+1 \right] \)

3) Let \( Q \) be a smooth quadric of dimension \( n = 4s+2 \). Then \( D^b(Q) = \langle B, B(2s+1) \rangle \) \( m = 2 \)

where \( B = \langle 0, O(1), \ldots, O(2s), S(2s) \rangle \) and \( S \) is a spinor bundle.

Let \( X \subset Q \) be a hypersurface of degree \( 2s+1 \).

\[ \omega_X = O(2s+1)^{-2} = O(-4s-2) \quad c = \gcd(1, 2) = 1 \]

As a result, \( D^b(X) = \langle A_x, B_x \rangle \) since \( m-d-1 = 0 \), with

\( B_x = \langle O_x, O_x(1), \ldots, O_x(2s), S(2s) \rangle \)

\[ S_{A_x} = \left[ -2 \times 2 + 1 \times (4s+3) \right] = \left[ 4s-1 \right] \]

4) Assume \( \gcd(k, m) = 1 \), let \( X \subset Gr(k, m) \) be a hypersurface of degree \( d < M \).

\[ D^b(Gr(k, m)) = \langle B, B(1), \ldots, B(n-1) \rangle \quad m = m \]

\( B = \sum x^\alpha \quad \alpha_k < (n-k)(k-1)/k, \alpha_k < (n-k)(k-2)/k, \ldots, \alpha_k < (n-k)/k \)

\[ \omega_{Gr(k, m)} = O(1)^{-m} \quad c = \gcd(m, d) \]

Thus \( D^b(X) = \langle A_x, B_x, \ldots, B_x \rangle \quad m-d-1 \)

\[ S_{A_x}^{dc} = \left[ -2 \frac{m}{c} + d (k(n-k)+1) \right] \]

If \( d \mid m \) then \( c = d \) and \( S_{A_x} = \left[ -2 \frac{m}{d} + k(n-k)+1 \right] \)

3.2 Let \( f : X \to Y \) be a double covering branched in \( \Delta X \), \( \forall \leq d \leq M \).

Proposition: \( Rf_* \) is spherical.
Proof: \[ p^!(F) = p^*(F) \otimes \omega_x \otimes p^*(\omega_{\pi}) \]

\[ \omega_x = p^*(\omega_{\pi} \otimes O(X)) = p^*(\omega_{\pi}) \otimes \mathcal{L}_x \quad \text{with} \quad \mathcal{L}_x = p^*(\mathcal{L}_{\pi}) \]

\[ \Rightarrow p^!(F) = p^*(F) \otimes \mathcal{L}_x \]

\[ \Rightarrow p^!(p_*(p^*(F))) = p^!(p_*(O_x \otimes F)) \]

but since \( f \) is a covering \( p_* O_x = O_{\pi} \otimes O(-X) = O_{\pi} \otimes \mathcal{L}_{\pi}^{-d} \)

Thus \[ p^!(p_*(p^*(F))) = p^!(F \otimes (O_{\pi} \otimes \mathcal{L}_{\pi}^{-d})) \]

\[ = \mathcal{L}_x d \otimes p^*(F) \otimes (O_x \otimes \mathcal{L}_x^{-d}) \]

\[ = p^*(F) \otimes p^*(F) \otimes \mathcal{L}_x^d = p^*(F) \oplus p^!(F) \]

and \( \psi \) is an isomorphism. Similarly, \( \Gamma \) is an isomorphism. \( \blacksquare \)

Proof: 1) \( T_x \) commutes with \( \mathcal{L}_x \)

2) some power of \( e \) is a shift

3) if \( \omega_{\pi} = \mathcal{L}_{\pi}^{-m} \) there is a power of \( e \) which is a shift.

Proof: There is a distinguished triangle:

\[ O_{\pi} \rightarrow p_*(O_x) \rightarrow \mathcal{L}_{\pi}^{-d} \]

Therefore for all \( F \in D^b(\pi) \) one has a distinguished triangle:

\[ F \rightarrow F \otimes p_*(O_x) \rightarrow F \otimes \mathcal{L}_{\pi}^{-d} \]

\[ \Rightarrow p^*(F) \]

Moreover, for all \( F \in D^b(X) \) the following triangle is distinguished:

\[ \tau^* F \otimes \mathcal{L}_x^{-d} \rightarrow p^* \pi_*(F) \rightarrow F \]

where \( \tau \) is the involution of the covering \( p \).

By definition of \( T_{\pi} \) and \( T_x \) this yields \( T_{\pi} = \mathcal{L}_{\pi}^{-d} [-1] \) and

\[ T_x = T \circ \mathcal{L}_x^{-d} [1] \]

Since \( T(\mathcal{L}_x) \cong \mathcal{L}_x \) then \( T_x = T \circ \mathcal{L}_x^{-d} [1] \) commutes with \( \mathcal{L}_x \) and (11) holds.

\[ \varphi = T \circ \mathcal{L}_x^{-d} [1] \circ \mathcal{L}_x^d = \tau [1] \quad \Rightarrow \quad \varphi^2 = [2] \]
\( \sigma = \omega X \cdot [d \dim X] \cdot \omega_{d-m} \cdot \omega^\prime \cdot \omega^\prime \)

\( \omega X = J \mathfrak{g} \left( \omega^X \otimes \omega^\prime \right) = J \mathfrak{g} (\omega^X \otimes d-m) = \omega_{d-m} \)

\( \Rightarrow \sigma = \mathfrak{g} \left[ d \dim X + 1 \right] \quad \sigma^2 = \left[ 2 (d \dim X + 1) \right] = \left[ 2 (d \dim X + 1) \right] \)

As in the previous case, we would like to use the theorem and for this we still need to check (9) and (10). Again:

* \( J \mathfrak{g} (\omega X \otimes F) = J \mathfrak{g} \left( J \mathfrak{g} (\omega X \otimes F) = \omega_{d-m} \otimes \mathfrak{g} (F) \right) \) and (10) holds.

* \( T_{\mathfrak{g}} (\mathcal{B} \otimes \omega^X) = \mathcal{B} \otimes \omega_{d-m} \left[ -1 \right] \)

Since \( \mathcal{B} \) is stable under shifts, (9) holds.

So as before, we need to check that \( \omega_{\mathfrak{g}} = \omega_{d-m} \) and that the decomposition of \( \mathfrak{g} \) exists.

**Examples:**

1) Let \( X \rightarrow \mathbb{P}^m \) be a double covering ramified in a smooth hypersurface of degree \( 2d \), \( 1 \leq d \leq m \).

We have already checked that \( D^b(\mathbb{P}^m) = \omega_{\mathfrak{g}} \) in this case, and we have seen the semi-orthogonal decomposition of \( D^b(\mathbb{P}^m) \) in 3.1.

Thus \( D^b(X) = \langle A_x, B_x, \ldots, B_x (m-d) \rangle \) and if \( c = \gcd(m+1, d) \) then

\[
S_{A_x}^{d/c} = \tau \left[ \frac{(d-(m+1))/c + \frac{d}{c}(m+1)}{c} \right] = \tau \left[ \frac{(d-(m+1))}{c} \right] \left[ \frac{(m+1)(d-1)}{c} \right]
\]

If \( d \mid m+1 \) and \( (m+1)/d \) is odd then \( S_{A_x} = \left[ \frac{(m+1)(d-1)}{d} \right] \).

2) Assume \( \gcd(k, m) = 1 \) and \( X \rightarrow \mathbb{G}_r(k, m) \) is a double covering ramified in a smooth hypersurface of degree \( 2d \), \( d < m \).

As in 3.1 we can apply the theorem and:

\( D^b(X) = \langle A_x, B_x, \ldots, B_x (m-d-1) \rangle \quad c = \gcd(m,d) \)

\[
S_{A_x}^{d/c} = \tau \left[ \frac{(d-m)/c + \frac{k(m-k)}{c} \times \frac{d}{c}}{c} \right]
\]

If \( d \mid m \) and \( m/d \) is odd then \( S_{A_x} = \left[ \frac{-M}{d} + \frac{k(m-k)}{d} \right] \).
3.3 Gushel–Tucker manifolds

This part comes from "Lectures on non-commutative K3 surfaces, Bridgeland stability and moduli spaces," E. Macrì, P. Stellari.

Def: Let $\text{Cone}(\text{Gr}(2, 5)) \mathbb{C}P^1_0$ be the cone over $\text{Gr}(2, 5) \mathbb{C}P^9$ (where the inclusion is the Plücker embedding).

Let $\mathbb{P}^{n+4} \mathbb{C}P^1_0$ be a linear subspace and $Q \mathbb{C}P^{n+4}$ be a quadric hypersurface. Let $X = \text{Cone}(\text{Gr}(2, 5)) \cap \mathbb{P}^{n+4} \cap Q$ 2 ≤ $n$ ≤ 6

If $X$ is smooth of dimension $n$ then $X$ is a Gushel–Tucker manifold.

Since $X$ is smooth, it doesn't contain the vertex of the cone. Therefore one can consider the projection $f: X \to \text{Gr}(2, 5)$ called the Gushel map.

We will consider only the cases $n=4, 6$.

There are two possibilities:

1) $f$ is an embedding, its image is a quadric section of a smooth linear section of $\text{Gr}(2, 5)$. In this case $X$ is called ordinary.

2) $f$ is a double covering onto a smooth linear section of $\text{Gr}(2, 5)$ ramified along a quadric section. In this case $X$ is called special.

We will denote by $\Pi$ the involution of the covering.

In both cases we will denote by $\Pi_X$ the smooth linear section.

Lemma: Let $i: M \to \text{Gr}(2, 5)$ be a smooth linear section of dim $N \geq 3$ then $M$ has a rectangular nef-ample decomposition with respect to

$O_M \langle 1 \rangle = O_{\text{Gr}(2, 5)} | M$

$D^b(M) = \langle B_M, B_M \langle 1 \rangle, \ldots, B_M \langle N-2 \rangle \rangle$

with $B_M = \langle O_M, U_M \rangle$ $U$ = holomorphic rank 2 subbundle.

Idea of proof: by inverse induction on $N$. $\dim \text{Gr}(2, 5) = 2 \times 3 = 6$

If $N = 5$ then we can apply example 3.1.4): $d = 1 = c$.
$\mathcal{D}^b(M) = \langle \mathcal{A}_X, \mathcal{B}_X, \mathcal{B}_X(1), \ldots, \mathcal{B}_X(5-1-1) \rangle$

$3 = 5 - 2$

$S_{\mathcal{A}_X} = \left[ 2 \cdot (5-2) + 1 - 2 \times 5 \right] = [-3]$

But $\mathbb{H}^3_X(M)$ is concentrated in degree 0

* $\mathcal{A}_X$ is a $-3$ CY category $\Rightarrow \mathbb{H}^3_X \mathcal{A}_X \neq 0$

* If $\mathcal{D} = \langle \mathcal{D}_1, \ldots, \mathcal{D}_3 \rangle$ then $\mathbb{H}^i_X(\mathcal{D}) = \bigoplus_{j=1}^{n} \mathbb{H}^i_X(\mathcal{D}_j)$

$\Rightarrow \mathbb{H}^3_X(\mathcal{D}^b(M)) \cong \mathbb{H}^3_X \mathcal{A}_X \oplus \ldots$

$\Rightarrow \mathcal{A}_X$ has to be zero.

$\Rightarrow \mathcal{D}^b(M) = \langle \mathcal{B}_X, \mathcal{B}_X(1), \ldots, \mathcal{B}_X(3) \rangle$

Now let $\mathbb{H}' \subset M$ be a linear section.

Again we can apply example 3.1: $d = 1 = c$

$\mathcal{D}^b(M') = \langle \mathcal{A}'_X, \mathcal{B}'_{\mathbb{H}'}, \ldots, \mathcal{B}'_{\mathbb{H}'}, (4-1-1) \rangle$

$4 - 2 = 2$

$S_{A'_X} = [-2 \times 4 + 6 \times 1] = [-27]$ and as before $A'_X = 0$, etc.

* Let us now consider the case $M = 6$. Then $f$ is a double covering, branched in a quasiregular section. Thus the branched locus is in $O(1)^2$ and $d = 1$. We can apply example 3.2.2.

$\mathcal{D}^b(X_6) = \langle \mathcal{A}_X, \mathcal{B}_X, \ldots, \mathcal{B}_X(5-1-1) \rangle$

$3$

$c = 1$

$S_{\mathcal{A}_X} = 7^{5-1} \left[ (2 \times 3 + 1) \times 1 - 5 \right] = [27]$.

* Let us now consider the case $M = 4$. There are two possibilities:

1) $f$ is an embedding. There is a quasiregular $Q'$ s.t.

$X_4 = M_X \cap Q' \hookrightarrow \Pi_X \rightarrow \mathbb{G}(2, 5)$

by the lemma $\mathcal{D}^b(\Pi_X) = \langle \mathcal{B}_{\Pi_X}, \ldots, \mathcal{B}_{\Pi_X}(3) \rangle$

We can use the following lemma:
Lemma: \( f_* : D^b(X) \to D^b(\mathbb{P}_X) \) is spherical

\[
T_X = \mathcal{O}_X(-2)[2] \quad T_{\mathbb{P}_X} = \mathcal{O}_{\mathbb{P}_X}(-2) \quad \text{if } X \text{ ordinary}
\]

\[
T_X = \tau \circ \mathcal{O}_X(-1)[1] \quad T_{\mathbb{P}_X} = \mathcal{O}_{\mathbb{P}_X}(-1)[-1] \quad \text{if } X \text{ special}
\]

This is (3.1) and (3.2).

We can now apply the example 3.1.

\[d = 2 \text{ since } X_4 \text{ is a hypersurface of degree 2 in } \mathbb{P}_X\]
\[m = 4, \ c = 2\]
\[D^b(X_4) = \langle A_X, B_X \ldots, B_X(4-2-1) \rangle\]

\[S_{A_X} = \left[ -2 \times \frac{4}{2} + 6 \right] = [2]\]

2) \( f \) is a double covering branched in \( Q \subset \mathbb{P}_X \), a quadric.

\[|Q| \in |\mathcal{O}(1)|^2 \Rightarrow d = 1, \ c = 1, \ m = 3\]

\[D^b(\mathbb{P}_X) = \langle B_{\mathbb{P}_X} \ldots, B_{\mathbb{P}_X}(2) \rangle\]

Applying example 3.2) gives:

\[D^b(X_4) = \langle A_X, B_X \ldots, B_X(3-1-1) \rangle\]

\[S_{A_X} = \tau^{1-3} \left[ -3 + 5 \right] = \tau^2 [2]\]

\[S_{A_X} = [2]\]

Since the Debarre-Voisin varieties are hyperplane sections \( X_2 \subset G(3,10) \)

with the example 3.1 one gets:

\[d = 1, \ c = 1\]
\[D^b(X_2) = \langle A_X, B_X \ldots, B_X(10-1-1) \rangle\]

\[S_{A_X} = \left[ -2 \times 10 + (3 \times 7 + 1) \times 1 \right] = \left[ -20 + 22 \right] = [2]\]
4. Idea of the proof of the theorem

4.11 The semi-orthogonal decomposition

With the same notations as in the theorem, let \( \pi_B : B \hookrightarrow D^b(H) \). \( B \) is admissible because it is a piece of the orthogonal decomposition of \( D^b(H) \). Hence, \( \pi_B \) has a right adjoint \( \pi_B^! \). Then, \( \pi_B^! \circ \phi \) is a right adjoint to \( \phi^* \circ \pi_B^! \).

To show that \( \phi^* \) is fully faithful on \( B \) it is enough to show that \( \phi^* \circ \phi \circ \pi_B \circ \phi^* \circ \pi_B = \text{id} \), i.e. to say \( \pi_B^! \circ \phi \circ \phi^* \circ \pi_B \circ \phi^* \circ \pi_B = \text{id} \).

Composing the triangle (3) by \( \pi_B \) on the left and \( \pi_B^! \) on the right, one gets the following distinguished triangle:

\[
\pi_B^! \circ T_\pi \circ \pi_B(A) \to \pi_B^! \circ \pi_B(A) \to \pi_B^! \circ \phi \circ \phi^* \circ \pi_B(A)
\]

for all \( A \in B \) since \( \pi_B \) is exact and thus \( \pi_B^! \) too.

But \( \pi_B^! \circ \pi_B = \text{id} \) since \( \pi_B \) is fully faithful, therefore it is enough to show that \( \pi_B^! \circ T_\pi \circ \pi_B = 0 \).

Let us show that \( \ker \pi_B^! = B^\perp \).

\[
A \in \ker \pi_B^! \iff \text{Hom}(B, \pi_B^! A) = 0 \quad \forall B \in B
\]

\[
\iff \text{Hom}(\pi_B B, A) = 0 \quad \forall B \in B
\]

\[
\iff A \in B^\perp
\]

Thus it is enough to show that \( \text{Im}(T_\pi \circ \pi_B) \subset B^\perp \).

By hypothesis, \( T_\pi(B) = B \otimes L_{-d} \).

Let \( B, B_1 \in B \). Then:

\[
\text{Hom}(B, B_1 \otimes L_{-d}) \subset \text{Hom}(B \otimes L_{-d}, B_1) = 0
\]

since \( 1 \leq d < m-1 \) and by definition of the semi-orthogonal decomposition of \( D^b(H) \).

As a result, \( T_\pi(B) \subset B^\perp \) and \( \phi^* \) is fully faithful.

Now we need to show that \( B_{x_1}, \ldots, B_x \otimes L_{-d}^{m-1} \) is semi-orthogonal.
15 let $B_1, B_2 \in B_x$, $0 \leq i < j \leq m-d-1$.

$B_x = \phi^+(B)$ so $B_1 = \phi^+ B_3$, $B_2 = \phi^+ B_4$

$\text{Hom}(B_1 \otimes_x L_x^j, B_2 \otimes_x L_x^i) \cong \text{Hom}(B_1, B_2 \otimes_x L_x^{i-j})$

$\cong \text{Hom}(B_3, \phi_+^+ (B_4 \otimes_x L_x^{i-j}))$

$= \text{Hom}(B_3, \phi_+^{i-j} \circ \phi_+(B_4))$

Hence it's enough to show $\beta_\pi^{i-j} \circ \phi_+ \circ \phi_+(B_4) = 0$ for $1 + d - m < i-j < 0$.

By assumption $\phi_+ L_x = L_x^{i} \circ \phi_+ \Rightarrow L_x^{i} \circ \phi_+ = \phi_+ L_x^{i}$ taking left adjoints

$\Rightarrow \beta_\pi^{i-j} \circ \phi_+ \circ \phi_+(B_4)$

Composition (3) with $\beta_\pi^{i-j}$ on the left, $\beta_\pi^{i-j}$ on the right, one gets the distinguished triangle:

$\beta_\pi^{i-j} \circ \phi_+(B_4) \rightarrow \beta_\pi^{i-j} \circ \phi_+ \circ \phi_+(B_4) \rightarrow \beta_\pi^{i-j} \circ \phi_+ \circ \phi_+(B_4)$

$\text{Im}(\beta_\pi^{i-j} \circ \phi_+(B_4)) = B \otimes_x L_x^{i-j}$

$\text{Im}(\beta_\pi^{i-j} \circ \phi_+ \circ \phi_+(B_4)) = L_x^{i-j} \otimes_x B \otimes_x L_x^{-d} = B \otimes_x L_x^{-d}$

$\Rightarrow$ these two images are in $B^+ = \text{ker} \beta_\pi^{i-j}$

$\Rightarrow$ the triangle (4) is in fact $0 \rightarrow 0 \rightarrow \beta_\pi^{i-j} \circ \phi_+ \circ \phi_+(B_4)$

$\Rightarrow \beta_\pi^{i-j} \circ \phi_+ \circ \phi_+(B_4) = 0$.

Finally, $B_x \rightarrow D^b(X)$ has a right adjoint, namely $\beta_\pi^{i-j} \circ \phi_+$.

$B_x = \phi^+ \circ \phi_+(B)$, $\phi_+ \circ \phi_+(B_x) \cong B \rightarrow B_x$

Thus $B_x$ is admissible, and so is $<B_x, B_x \otimes_x L_x^{-m-d-1}>$

$A_x = <B_x, B_x \otimes_x L_x^{-m-d-1}>$
4.2 Let us denote \( \phi_x = \phi^* \circ \phi \) and \( \phi_x^! = \phi_! \circ \phi \).

We have now to compute \( S_x \) (and show it exists).

**A. Mutation functors**

**Def** let \( B, C, \mathcal{D}(\mathcal{M}) \) be admissible, let \( \phi : B \rightarrow \mathcal{D}(\mathcal{M}) \).

The left mutation functor associated to \( B, L_B \) is the functor defined by the distinguished triangle:

\[
\phi_! \xrightarrow{\epsilon} \text{id} \xrightarrow{\eta} L_B
\]

(whatever before each functor is a \( \mathcal{D}(\mathcal{M}) \)-transform and the triangle given by their kernels is distinguished).

The right mutation functor is given by the distinguished triangle:

\[
R_B \xrightarrow{\nu} \text{id} \xrightarrow{\mu} \phi_!^*
\]

Here are some properties of \( L_B \) which can be found in "Homological projective duality", A. Kaynuleo, p.475. (for the proof he refers to "Representation of associative algebras and coherent sheaves", A. Bondal).

**Prop**: \( L_B = \mathcal{I}_B^! \circ \mathcal{I}_B^+ \)

\( \mathcal{I}_B \) is the inclusion functor.

* \( L_B(B) = 0 \) if \( B \) is semi-orthogonal then \( L_B(B) = B^! \) is an equivalence.

* If \( \{ A_0, \ldots, A_m \} \) is semi-orthogonal then

\[
L_B(\langle A_0, \ldots, A_m \rangle) = L_B(A_0) \cdots L_B(A_m)
\]

**Lemma**: let \( C = \langle A, B \rangle \) be a semi-orthogonal decomposition of a triangulated category \( \mathcal{C} \). In particular, \( A \) and \( B \) are admissible. If \( C \) has a Serre functor then so do \( A \) and \( B \). Moreover

\[
S_B = R_A \circ S_C \quad S_B^{-1} = L_B \circ S_C^{-1}
\]

**Idea of proof**: \( \forall A, B \in \mathcal{C} \quad \text{Hom}(A, B) \simeq \text{Hom}(B, S_C(A)) \)

\( = \text{Hom}(S_C^{-1}(A), B) \simeq \text{Hom}(B, A) \) since \( S_C \) is an equivalence.
Let $A, B \subseteq A$. Since $\mathbb{p}^i A \to A \to \mathbb{p}^i A$ is distinguished, there is a long exact sequence:

$\text{Hom}(\mathbb{p}^i A, B) \to \text{Hom}(\mathbb{p}^i A, B) \to \text{Hom}(A, B) \to \text{Hom}(\mathbb{p}^i A, B)$

But $B \subseteq A$ and $\mathbb{p}^i A \subseteq B$ implies $\text{Hom}(\mathbb{p}^i A, B) = 0$ if $i = -1, 0$.

$\Rightarrow \text{Hom}(\mathbb{p}^i A, B) = \text{Hom}(A, B)$.

Since $S^\infty$ is an equivalence, we can write $A \simeq S^\infty C$. Hence

$\text{Hom}(\mathbb{p}^i A, B) \simeq \text{Hom}(\mathbb{p}^i A, S^\infty C)$,

$\Rightarrow \text{Hom}(S^\infty C, B) = \text{Hom}(B, C)$.

In the situation of the theorem, this gives that $A_x$ has a Serre functor and $S_{A_x} = \bigoplus_{i=1}^{m-d-1} B_x \otimes L_i^{-1}$.

**Lemma:** $A_x = \{ f \in D^b(X) | \phi(f) \in \bigoplus_{i=1}^{m-d-1} B \otimes L_i^{-1} \}$

**Proof:**

$A_x = \{ f \in D^b(X) | \text{Hom}(\phi^*(B), f) = \ldots = \text{Hom}(\phi^*(B \otimes L_i^{-1}), f) = 0 \}$

Since by (10) and since $\phi^*$ is fully faithful on $B$ one gets:

$\phi^* \phi \simeq \text{id}$ and $\phi^* \phi(B_x) \simeq \phi^* \phi(B_x)$

$\Rightarrow A_x = \{ f \in D^b(X), \text{Hom}(B, \phi(f)) = \ldots = \text{Hom}(B \otimes L_i^{-1}, \phi(f)) = 0 \}$

$\Rightarrow f \in D^b(X), \phi(f) \in \bigoplus_{i=1}^{m-d-1} B \otimes L_i^{-1}$

Since by inverting the semi-orthogonal decomposition of $D^b(M)$ by $L^{-d}$ one gets a new semi-orthogonal decomposition:

$D^b(M) = \langle B \otimes L^{-d}, \ldots, B \otimes L_i^{-1} \rangle$.

\[ \text{Hom}(\mathbb{p}^i A, B) \to \text{Hom}(\mathbb{p}^i A, B) \to \text{Hom}(A, B) \to \text{Hom}(\mathbb{p}^i A, B) \]

But $B \subseteq A$ and $\mathbb{p}^i A \subseteq B$ implies $\text{Hom}(\mathbb{p}^i A, B) = 0$ if $i = -1, 0$.

$\Rightarrow \text{Hom}(\mathbb{p}^i A, B) = \text{Hom}(A, B)$.

Since $S^\infty$ is an equivalence, we can write $A \simeq S^\infty C$. Hence

$\text{Hom}(\mathbb{p}^i A, B) \simeq \text{Hom}(\mathbb{p}^i A, S^\infty C)$,

$\Rightarrow \text{Hom}(S^\infty C, B) = \text{Hom}(B, C)$.

In the situation of the theorem, this gives that $A_x$ has a Serre functor and $S_{A_x} = \bigoplus_{i=1}^{m-d-1} B_x \otimes L_i^{-1}$.

**Lemma:** $A_x = \{ f \in D^b(X) | \phi(f) \in \bigoplus_{i=1}^{m-d-1} B \otimes L_i^{-1} \}$

**Proof:**

$A_x = \{ f \in D^b(X) | \text{Hom}(\phi^*(B), f) = \ldots = \text{Hom}(\phi^*(B \otimes L_i^{-1}), f) = 0 \}$

Since by (10) and since $\phi^*$ is fully faithful on $B$ one gets:

$\phi^* \phi \simeq \text{id}$ and $\phi^* \phi(B_x) \simeq \phi^* \phi(B_x)$

$\Rightarrow A_x = \{ f \in D^b(X), \text{Hom}(B, \phi(f)) = \ldots = \text{Hom}(B \otimes L_i^{-1}, \phi(f)) = 0 \}$

$\Rightarrow f \in D^b(X), \phi(f) \in \bigoplus_{i=1}^{m-d-1} B \otimes L_i^{-1}$

Since by inverting the semi-orthogonal decomposition of $D^b(M)$ by $L^{-d}$ one gets a new semi-orthogonal decomposition:

$D^b(M) = \langle B \otimes L^{-d}, \ldots, B \otimes L_i^{-1} \rangle$. 

Lemma: \( \varepsilon \circ \phi^* \cong \phi^* \circ T_{\eta} \circ \mathcal{L}_\eta^d \) \[2\]
\( \sigma \circ \phi^* = \phi^* \circ \mathcal{L}_\eta^m \circ S_{\eta} \) \[1\]

In particular \( \varepsilon \) and \( \sigma \) preserve each piece of \( \{B_x, \ldots, B_x \otimes \mathcal{L}_x^m \cdot d^{-1} \} \Rightarrow \mathcal{C} \).

**Proof:** \( \varepsilon \circ \phi^* = T_x \circ \mathcal{L}_x^d \circ \phi^* \)

\( = T_x \circ \phi^* \circ \mathcal{L}_\eta^d \) by taking left adjoint in (10).

\( = \phi^* \circ T_{\eta} \circ \mathcal{L}_\eta^d \) \[2\] by (8)

\( \sigma \circ \phi^* = S_x \circ T_x \circ \mathcal{L}_x^m \circ \phi^* = S_x \circ T_x \circ \phi^* \circ \mathcal{L}_\eta^m \)

\( = S_x \circ \phi^* \circ T_{\eta} \circ \mathcal{L}_\eta^m \) \[2\] by (8)

Moreover if \( \alpha : \mathcal{C}_1 \rightarrow \mathcal{C}_2 \) is a functor between triangulated categories with some functors, then if \( \alpha^* \) and \( \alpha' \) exists one has:

\( \alpha' \circ S_{\mathcal{C}_2} = S_{\mathcal{C}_1} \circ \alpha^*. \) \((*)\)

Applying (\( \ast \)) to \( T_{\eta} \circ \mathcal{L}_\eta^m \) one gets:

\( \mathcal{L}_\eta^m \circ T_{\eta} \circ S_{\eta} = S_{\eta} \circ \mathcal{L}_\eta^m \circ T_{\eta} \)

\( \Rightarrow S_{\eta}^{-1} \circ T_{\eta} \circ \mathcal{L}_\eta^m = T_{\eta} \circ \mathcal{L}_\eta^m \circ S_{\eta}^{-1} \Rightarrow T_{\eta} \circ \mathcal{L}_\eta^m = S_{\eta}^{-1} \circ T_{\eta} \circ \mathcal{L}_\eta^m \circ S_{\eta} \)

\( \Rightarrow \sigma \circ \phi^* = S_x \circ \phi^* \circ T_{\eta} \circ \mathcal{L}_\eta^m \) \[2\] = \( S_x \circ \phi^* \circ S_{\eta}^{-1} \circ T_{\eta} \circ \mathcal{L}_\eta^m \circ S_{\eta} \)

\( = \phi^* \circ T_{\eta} \circ \mathcal{L}_\eta^m \circ S_{\eta} \) \[2\] by (\( \ast \)) applied to \( \phi \)

\( = \phi^* \circ T_{\eta} \circ \mathcal{L}_\eta^m \circ S_{\eta} \) \[1\] by (\( \ast \ast \ast \ast \)) p. (6).

Then \( \varepsilon (B_x) = \varepsilon \circ \phi^* (B) = (\phi^* \circ T_{\eta} \circ \mathcal{L}_\eta^d) (B) \) since \( B \) is invariant under shifts

\( = \phi^* (T_{\eta} (B \otimes \mathcal{L}_\eta^d)) \)

\( = \phi^* (B) \) by (3)

\( = B_x \)

So \( \varepsilon \) preserves \( B_x \), and it commutes with \( \mathcal{L}_x \) by (11) thus it preserves each
Similarly, using that $S_n (B \otimes L_i) = B \otimes L_i^{i-m}$ (one can write
$B \otimes L_i = \langle B \otimes L_i^{i-m}, \ldots, B \otimes L_i^{i-1} \rangle$)

$\Rightarrow S_n (B \otimes L_i) = \langle B \otimes L_i^{i-m+1}, \ldots, B \otimes L_i^{i-1} \rangle \!=\! B \otimes L_i^{i-m}$ one gets:

$\sigma(B_x) = \sigma(\phi^*(\mathcal{B})) = \phi^* L_i^m \circ S_n(B)$

$\Rightarrow \phi^*(B \otimes L_i^{m-m}) = B_x$

Finally $\sigma$ commutes with $L_x$ by (ii) and (iv) applied to $S_x$, therefore it preserves each piece of $C$.

**B. Recollement functors**

Let $Y$ be a smooth projective variety with rectangular lefschetz decomposition $D^b(Y) = \langle B_1, B_2 \otimes L_y, \ldots, B_y \otimes L_y^{s-1} \rangle$

Then $\mathcal{O}_Y = L \otimes L_y$

**Lemma:** $\mathcal{O}_Y^i = L \langle B_1, \ldots, B \otimes L_y^{i-1} \rangle \otimes L_y \forall 0 \leq i \leq s$

**Proof:**

$\mathcal{O}_Y^i = (L \otimes L_y) \circ (L \otimes L_y) \circ \ldots \circ (L \otimes L_y)$

$= L \otimes (L_y \circ \otimes \circ L_y^{-1}) \circ (L_y \circ \otimes \circ L_y^{-2}) \circ \ldots \circ (L_y \circ \otimes \circ L_y \circ L_y^{i-1})

= L \otimes L_y \otimes \ldots \otimes L_y \otimes L_y \otimes L_y \otimes \ldots \otimes L_y \otimes L_y \otimes L_y

= \langle B_1, \ldots, B \otimes L_y^{i-1} \rangle \otimes L_y

since for any equivalence $T$ of $D^b(Y)$, $TL \otimes T^{-1} = L T(B)$

If $\beta : B \rightarrow D^b(Y)$ then $T \beta : T(B) \rightarrow D^b(Y)$ and $(T \beta)^{-1} = \beta^{-1} \circ T^{-1}$

$\Rightarrow T \circ \circ \circ \circ \circ \rightarrow id \rightarrow LL T(B)$

On the other hand $T (\circ \circ \circ \circ \circ \rightarrow LL \rightarrow T(B)$

**Corollary:** $Q^i_n(\langle B \otimes L_i^{i-1}, \ldots, B \otimes L_i^{i-1} \rangle) = 0$, $Q^M = 0$ where $Q_n = Q$.
**Proof:** \[ L^1 \otimes (B \otimes L_1, \ldots, B \otimes L_n) = \langle B, \ldots, B \otimes L_1 \rangle \]

and \[ \langle B, \ldots, B \otimes L_1 \rangle \langle B, \ldots, B \otimes L_1 \rangle = 0. \]

We will denote \[ O_n = O_B, \quad O_x = O_{B_x}. \]

**Lemma:** \( O_x \) commutes with \( \beta \) and \( \delta \).

**Proof:** We have already seen that \( \beta \) and \( \delta \) commute with \( L_x \), so we only have to show they commute with \( L_{B_x} \).

We have seen that \( \beta \) preserves \( B_x \), hence \( \beta \circ \beta \circ \beta = \beta \circ \beta \circ \beta \). Thus:

\[
\beta \circ \beta \circ \beta \rightarrow \beta \rightarrow L_{B_x} \circ \beta \\
\beta \circ \beta \circ \beta \rightarrow \beta \rightarrow L_{B_x} \circ \beta
\]

and \( \beta \) commutes with \( L_{B_x} \). Similarly \( \delta \) commutes with \( L_{B_x} \). \( \Box \)

**Lemma:** For all \( 0 \leq i \leq d-1 \) there is a morphism \( \phi^* : O^*_{n} \rightarrow O^*_x \) such that \( Y^i \langle B \otimes L_{d-1-n} \ldots, B \otimes L_{d-i-1} \rangle \) is an isomorphism.

**Proposition:** For all \( 0 \leq i \leq d \) there is a distinguished triangle:

\[ \phi^* \circ O^*_{n} \rightarrow O^*_x \rightarrow T_x \circ X^i \]

**Corollary:** \( \mathcal{O}^d_x |A_x \cong \mathcal{O}^d_x |A_x \).

**Proof:** We have the distinguished triangle:

\[ \phi^* \circ \mathcal{O}^d_{n} \circ \phi |A_x \rightarrow \mathcal{O}^d_x |A_x \rightarrow T_x \circ X^i |A_x \quad (\ast) \]

Since \( A_x = \phi^* |F, \phi(F) \in \langle B \otimes L_{d-1} \ldots, B \otimes L_{d-1} \rangle \), then

\[ \phi^* \circ \mathcal{O}^d_{n} \circ \phi |A_x = \phi^* \circ \mathcal{O}^d_{n} \langle B \otimes L_{d-1} \ldots, B \otimes L_{d-1} \rangle = 0 \] by the property of restriction functors we have seen.

Thus \( (\ast) \) is in fact \( 0 \rightarrow \mathcal{O}^d_x |A_x \rightarrow \mathcal{O}^d_x |A_x \) and \( \mathcal{O}^d_x |A_x \cong \mathcal{O}^d_x |A_x \). \( \Box \)
Lemma: \( S_{A^c}^{-1} = \mathcal{O}_{X}^{m-d} \circ \sigma^{-1} \)

Proof: We have seen \( S_{A^c}^{-1} = \mathcal{L}(B_{X}^{-d} \circ \mathcal{O}_{X}^{m-d} \circ \sigma^{-1}) \).

But by definition of \( \sigma \), \( S_{X}^{-1} = \mathcal{X} \circ \mathcal{T}_{X} \circ \sigma^{-1} \).

\( \Rightarrow S_{A^c}^{-1} = \mathcal{L}(B_{X}^{-d} \circ \mathcal{O}_{X}^{m-d} \circ \mathcal{T}_{X} \circ \sigma^{-1}) \).

\( = \mathcal{O}_{X}^{m-d} \circ \sigma^{-1} \).

Corollary: \( S_{A^c}^{-d/c} = \mathcal{E}^{m/c} \circ \sigma^{-d/c} \) and \( S_{A^c}^{d/c} = \mathcal{E}^{-m/c} \circ \sigma^{d/c} \).

Proof: First, \( \mathcal{E} \) and \( \sigma \) commute, since \( \mathcal{X} \) and \( \mathcal{T}_{X} \) commute. By (1) and \( S_{X} \) commutes with any equivalence of \( \mathcal{D}^{b}(X) \) by (4).

We have also seen that \( \mathcal{E} \) and \( \sigma \) commute with \( \mathcal{O}_{X} \).

Thus, if \( c = \gcd(d, m) \) one gets:

\( S_{A^c}^{-d/c} = \mathcal{O}_{X}^{(m-d)/c} \circ \mathcal{E} \circ \sigma^{-d/c} \).

\( = \mathcal{E} \circ \mathcal{O}_{X}^{(m-d)/c} \circ \sigma^{-d/c} \).

Since \( \mathcal{E} \) and \( \sigma \) commute this yields \( S_{A^c}^{d/c} = \mathcal{E}^{m/c} \circ \sigma^{d/c} \).