Analytic Number Theory
Homework #1

(due Tuesday, February 12, 2019)

Problem 1: Let $f : \mathbb{Z} \to \mathbb{C}$ be a completely multiplicative function, i.e., $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{Z}$. Assume also that $|f(n)| \le 1$ for all $n \in \mathbb{Z}$ and that $f(1) = 1$. Prove that for $\Re(s) > 1$, we have

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \left(1 - \frac{f(p)}{p^s}\right)^{-1}$$

where the product goes over all primes p.

Problem 2: Prove that the zeta function

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s}, \qquad (\Re(s) > 1),$$

does not vanish for $\Re(s) > 1$.

Problem 3: Show that the Gamma function $\Gamma(s) = \int_0^\infty e^{-u} u^s \, \frac{du}{u}$ has simple poles at $s = 0, -1, -2, -3, \ldots$ Determine the residues at these poles.

Problem 4: (Uniqueness of Dirichlet series) For $n = 1, 2, 3, \ldots$, let a_n, b_n be complex numbers with absolute values at most one. Assume that

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$$

for all complex values of s with $\Re(s) > 1$. Prove that we must have $a_n = b_n$ for all $n = 1, 2, 3, \ldots$