Analytic Number Theory
Homework #2

(due Thursday, February 18, 2016)

Problem 1: Let $f : \mathbb{R} \to \mathbb{C}$ be in the Schwarz space of functions with rapid decay at infinity. Prove that the function

$$F(s) := \int_1^\infty \sum_{n=1}^\infty f(ny) \, y^s \, \frac{dy}{y}$$

is holomorphic for all $s \in \mathbb{C}$.

Problem 2: Evaluate the integral

$$\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{x^s}{s(s+1)\cdots(s+k)} \, ds$$

for any $\sigma > 0$ and any integer $k = 1, 2, 3, \ldots$ Give a rigorous proof of your evaluation.

Problem 3: Fix $u > 0$. Show that the Fourier transform of $\frac{u}{\pi(x^2+u^2)}$ is $e^{-2\pi|x|u}$. Plug this into the Poisson summation formula to deduce that

$$\sum_{n=-\infty}^\infty \frac{1}{n^2 + u^2} = \frac{\pi}{u} \sum_{n=-\infty}^\infty e^{-2\pi|n|u}.$$

By letting $u \to 0$ prove that $\zeta(2) = \pi^2/6$.

Problem 4: Let $f(x) = e^{-A|x|}$. Show that

$$\widehat{f}(y) = \frac{2A}{A^2 + 4\pi^2 y^2},$$

$$\widetilde{f}(s) = \Gamma(s) A^{-s}, \qquad (\Re(s) > 0),$$

$$\widetilde{\widehat{f}}(s) = \frac{1}{2\pi} \left(\frac{2\pi}{A}\right)^{1-s} \Gamma\left(\frac{s}{2}\right) \Gamma\left(1 - \frac{s}{2}\right), \qquad (0 < \Re(s) < 2).$$