

Analytic Number Theory
Homework #3

(due Tuesday, March 23, 2016)

Problem 1: By the functional equation for

$$\xi(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s),$$

the function $s(1-s)\xi(s)$ can be regarded as an entire function of $s^2 - s$; what is the order of this function? Use this to obtain the alternative infinite product

$$\xi(s) = \frac{\xi(1/2)}{4(s-s^2)} \prod_{\rho} \left(1 - \left(\frac{s - \frac{1}{2}}{\rho - \frac{1}{2}}\right)^2\right)$$

the product extending over zeros ρ of $\xi(s)$ whose imaginary part is positive. [*This symmetrical form eliminates the exponential factors e^{A+Bs} and $e^{s/\rho}$ occurring in the usual Hadamard factorization of $\xi(s)$.*]

Problem 2: Calculate the mean value

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |\zeta(\sigma + it)|^2 dt$$

provided $\sigma > 1$.

Problem 3: Explicitly construct all Dirichlet characters (mod 15). Each such character is a completely multiplicative function $\chi : \mathbb{Z} \rightarrow \mathbb{C}$ satisfying $\chi(n+15) = \chi(n)$ for all $n \in \mathbb{Z}$.

Problem 4: Let q be an integer which has the property that every Dirichlet character $\chi \pmod{q}$ is real valued (takes on only the values $0, \pm 1$). Show that q must divide 24.