Analytic Number Theory
Homework #3

(due Thursday, March 28, 2019)

Problem 1: By the functional equation for

$$\xi(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

the function $s(1-s)\xi(s)$ can be regarded as an entire function of s^2-s; what is the order of this function? Use this to obtain the alternative infinite product

$$\xi(s) = \frac{\xi(1/2)}{4(s-s^2)}\prod_{\rho}\left(1-\left(\frac{s-\frac{1}{2}}{\rho-\frac{1}{2}}\right)^2\right)$$

the product extending over zeros ρ of $\xi(s)$ whose imaginary part is positive. [*This symmetrical form eliminates the exponential factors* e^{A+Bs} *and* $e^{s/\rho}$ *occurring in the usual Hadamard factorization of* $\xi(s)$.]

Problem 2: Explicitly construct all Dirichlet characters (mod 15). Each such character is a completely multiplicative function $\chi : \mathbb{Z} \to \mathbb{C}$ satisfying $\chi(n+15) = \chi(n)$ for all $n \in \mathbb{Z}$.

Problem 3: Let q be an integer which has the property that every Dirichlet character χ (mod q) is real valued (takes on only the values $0, \pm 1$). Show that q must divide 24.

Problem 4: Let χ be a Dirichlet character (mod q) with $q > 1$. Let $s \in \mathbb{C}$ with $\Re(s) > 1$. Show that

$$L(s,\chi) = q^{-s}\sum_{c=1}^{q}\chi(c)\zeta\left(s, \frac{c}{q}\right)$$

where $\zeta(s,\alpha) = \sum\limits_{n=0}^{\infty}\frac{1}{(n+\alpha)^s}$ is the Hurwitz zeta function.