Analytic Number Theory
Homework #4
(due Thursday, April 25, 2019)

Problem 1: Let \( \mathfrak{h} = \{ z = x + iy \mid x \in \mathbb{R}, y > 0 \} \) and let \( f : \mathfrak{h} \to \mathbb{C} \) be a holomorphic modular form of weight 0 for \( SL(2, \mathbb{Z}) \).

(a) Show that every element in \( \mathfrak{h} \) is \( SL(2, \mathbb{Z}) \) equivalent to some \( z = x + iy \) with \( y \geq \frac{\sqrt{3}}{2} \). Hint: Use the known fundamental domain for \( SL(2, \mathbb{Z}) \backslash \mathfrak{h} \).

(b) Deduce that \( |f| \) attains a maximum on \( \mathfrak{h} \).

(c) Conclude that the only holomorphic modular forms of weight zero are the constant functions. Hint: maximum modulus principle.

Problem 2: Fix a prime \( p \). Show that \( \Gamma_0(p) := \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \mid c \equiv 0 \pmod{p} \} \) is a subgroup of \( SL(2, \mathbb{Z}) \).

Problem 3: Define \( P_k(z) := \sum_{c \in \mathbb{Z}} \sum_{d \in \mathbb{Z}} \frac{e^{2\pi i \frac{ax + by}{cz + d}}}{(cz + d)^k} \) for \( c, d \) coprime.

Here, for every pair of coprime integers \( c, d \) we choose integers \( a, b \) so that \( ad - bc = 1 \). Show that the above series is independent of the choice of \( a, b \). Also show that the above series converges absolutely for \( k > 2 \).

Problem 4: Rewrite \( P_k(z) \) as a sum involving \( j(\gamma, z) = cz + d \) for \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). If \( k > 2 \) is an integer, prove that \( P_k(z) \) is a holomorphic modular form of weight \( k \) for \( SL(2, \mathbb{Z}) \). Hint: use the cocycle relation for \( j \).

Problem 5: For \( s \in \mathbb{C} \) with \( \Re(s) > 8 \), let \( L(s) := \sum_{n=1}^{\infty} a(n)n^{-s} \), where \( a(1) = 1 \) and \( |a(n)| \ll n^7 \) (for \( n = 1, 2, \ldots \)).

Assume that the function \( \Phi(s) := (2\pi)^{-s} \Gamma(s)L(s) \) is an entire function which is bounded in any fixed vertical strip \( \{ s \in \mathbb{C} \mid a < \Re(s) < b \} \) and satisfies the functional equation \( \Phi(s) = \Phi(12 - s) \) for all \( s \in \mathbb{C} \). Using the inverse Mellin transform, prove that \( f(z) := \sum_{n=1}^{\infty} a(n)e^{2\piinz} \) is the Ramanujan cusp form of weight 12. Hint: It is enough to show that \( f \) satisfies \( f(-1/z) = z^{12}f(z) \).